MS-E1600 - Probability theory, 2019/III

Problem set 1

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1. Definition of a sigma-algebra. A collection $\mathscr{F}$ of subsets of a set is a sigma algebra if it satisfies the following conditions:

(Σ-1) : $\Omega \in \mathscr{F}$.

(Σ-2) : if $E \in \mathscr{F}$ then $E^c \in \mathscr{F}$.

(Σ-3) : If $E_1, E_2, \cdots \in \mathscr{F}$ then $\bigcup_{n=1}^{\infty} E_n \in \mathscr{F}$

Assume that $\mathscr{F}$ is a sigma-algebra on Ω.

(a) Prove that $\emptyset \in \mathscr{F}$.

Proof. Assume $\Omega \in \mathscr{F}$, by (Σ-2) then $\Omega^c = \Omega \setminus \Omega = \emptyset \in \mathscr{F}$ □

(b) Prove that if $E_1, E_2, \cdots \in \mathscr{F}$, then also $\bigcap_{n=1}^{\infty} E_n \in \mathscr{F}$.

Proof. Assume $E_1, E_2, \cdots \in \mathscr{F}$, then by ($\Sigma$-2), we have their complements also belong to $\mathscr{F}$,say $E_1^c, E_2^c, \cdots \in \mathscr{F}$ and by (Σ-3), this implies $\bigcup_{n=1}^{\infty} E_n^c \in \mathscr{F}$. We will use property ($\Sigma$-2) again and De Morgan's law to conclude $\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c \in \mathscr{F}$. □

2. Sigma-algebras on small finite sets. Let a, b, c be three distinct points.

(a) Write down all sigma-algebras on $\Omega = \{a, b\}$.

Proof. The possible sigma-algebras on $\Omega = \{a, b\}$ are

$$\mathscr{F}_1 = \{\emptyset, \Omega\}.$$
$$\mathscr{F}_2 = \{\emptyset, \Omega, \{a\}, \{b\}\} = \mathscr{P}(\Omega).$$

□

(b) Write down all sigma-algebras on $\Omega' = \{a, b, c\}$.

Proof. The possible sigma-algebras on $\Omega = \{a, b, c\}$ are

$$\begin{aligned}
\mathscr{F}_1' &= \{\emptyset, \Omega'\},\\
\mathscr{F}_2' &= \{\emptyset, \Omega', \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\} = \mathscr{P}(\Omega'),\\
\mathscr{F}_3' &= \{\emptyset, \Omega', \{a\}, \{b, c\}\},\\
\mathscr{F}_4' &= \{\emptyset, \Omega', \{b\}, \{a, c\}\},\\
\mathscr{F}_5' &= \{\emptyset, \Omega', \{c\}, \{a, b\}\}.
\end{aligned}$$

□

(c) Give an explicit counterexample which shows that the union of two sigma-algebras is not necessarily a sigma-algebra.

Proof. Let $\Omega = \{a, b\}$ whose sigma-algebra is $\mathscr{F} = \{\emptyset, \{a, b\}, \{a\}, \{b\}\}$ and $\Omega' = \{c\}$ whose sigma-algebra is $\mathscr{F}' = \{\emptyset, \{c\}\}$. The union of those two signma-algebras is $\mathscr{S} := \mathscr{F} \cup \mathscr{F}' = \{\emptyset, \{a, b\}, \{a\}, \{b\}, \{c\}\}$. However, $\mathscr{S}$ is not a sigma-algebra as the union of two elements $\{a\} \cup \{c\} \notin \mathscr{S}$. □

3. Probability distributions on countable spaces. Let Ω be a finite or a countably infinite set, and denote by $\mathscr{P}(\Omega)$ the collection of all subsets of Ω. A function $p : \Omega \to \mathbb{R}$ is called a probability mass function (pmf) if $p(\omega) \geq 0$ for all ω and $\sum_{\omega\in\Omega} p(\omega) = 1$.

   (a) Show that if p is a pmf on Ω, then the set function $\mu[E] = \sum_{\omega\in E} p(\omega)$ is a probability measure on $(\Omega, \mathscr{P}(\Omega))$.

   Proof. Assume p is a pmf on Ω, then μ is a measure since

   (i) $\mu[\emptyset] = \sum_{\omega\in\emptyset} p(\omega) = 0$, since there is no elements in an empty set.

   (ii) Let $E_1, E_2, \dots \in \mathscr{P}(\Omega)$ be disjoint sets then

   $$\mu[\cup_{i=1}^{\infty} E_i] = \sum_{\omega\in\cup_{i=1}^{\infty} E_i} p(\omega) = \sum_{i=1}^{\infty}\sum_{\omega\in E_i} p(\omega) = \sum_{i=1}^{\infty} \mu(E_i).$$

   and μ is a probability measure since

   (iii)
   $$\mu[\Omega] = \sum_{\omega\in\Omega} p(\omega) = 1,$$

   and for any $E \in \mathscr{P}(\Omega)$,

   $$0 \leq \mu[E] = \sum_{\omega\in E} p(\omega) \leq \sum_{\omega\in\Omega} p(\omega) = 1.$$

$\square$

(b) Show that if μ is a probability measure on $(\Omega, \mathscr{P}(\Omega))$, then the function $p(\omega) = \mu[\{\omega\}]$ is a pmf on Ω.

Proof. Assume μ is a probability measure on $(\Omega, \mathscr{P}(\Omega))$, define $p : \Omega \to \mathbb{R}$ such that $p(\omega) = \mu[\{\omega\}]$ then

$$p(\omega) = \mu[\{\omega\}] \geq 0 \text{ for all } \omega \in \Omega,$$

and since Ω is at most countably infinite so it can be written as a countable union of its elements, then

$$\sum_{\omega \in \Omega} p(\omega) = \sum_{\omega \in \Omega} \mu[\{\omega\}] = \mu[\cup_{\omega \in \Omega} \omega] = \mu[\Omega] = 1.$$

$\square$

4. Suppose that $(\Omega, \mathscr{F}, \mu)$ is a probability space, that is μ is a probability measure on $(\Omega, \mathscr{F})$. Let $F_1, F_2, \dots \in \mathscr{F}$ be a sequence of events.

(a) Prove that $F_n \uparrow F \implies \mu[F_n] \uparrow \mu[F]$.

Proof. Assume $F_n \uparrow F$ which means $F_1 \subset F_2 \subset F_3 \subset \dots$ and let $F := \cup_{n \geq 1} F_n$. Define $E_1 = F_1, E_2 = F_2 \backslash F_1, \dots E_n = F_n \backslash F_{n-1}$. Thus the sets $E_n, n \in \mathbb{N}$ are disjoint and $F_n = \cup_j^n E_j$. By property of disjointness we have

$$\mu[F_n] = \mu[E_1 \cup E_2 \cup \dots \cup E_n] = \sum_{j=1}^{n} \mu[E_j].$$

Since the right hand sides are the partial sums of an infinite sum with non-negative terms, so they form a sequence increasing to that infinite sum, thus

$$\mu[F_n] \uparrow = \sum_{j=1}^{\infty} \mu[E_j] = \mu[\cup_{j=1}^{\infty} E_j] = \mu[\cup_{j=1}^{\infty} F_j] = \mu[F].$$

$\square$

(b) Prove that $F_n \downarrow F \implies \mu[F_n] \downarrow \mu[F]$.

Proof. Assume $F_n \downarrow F$ which means $F_1 \supset F_2 \supset \dots$ and let $F := \cap_{n \geq 1} F_n$. Observe that $F_1^c \subset F_2^c \subset F_3^c \subset \dots$, then by result from part (a) we have

$$\mu[F_n^c] \uparrow \mu[\cup_{j=1}^{\infty} F_j^c] = \mu[F^c].$$

Since $\mu[\Omega] = 1$, then

$$\mu[F_n] = 1 - \mu[F_n^c] \quad \text{and} \quad \mu[F] = 1 - \mu[F^c],$$

which implies $\mu[F_n] \downarrow \mu[F]$. $\square$

(c) Are the statements (a) and (b) still true in the case when μ is just assumed to be a measure on $(\Omega, \mathscr{F})$, but not necessarily a probability measure?

Proof. When μ is just assumed to be a measure on $(\Omega, \mathscr{F})$, statement (a) still holds since we did not need the fact that the measure of the whole space is finite. However, statement (b) may not hold since it requires measure of the whole space to be finite $\mu[\Omega] < \infty$.

□

5. Borel sets of the two-dimensional Euclidean space. The Borel sigma-algebra $\mathscr{B}(\mathbb{R}^2)$ is defined as the smallest sigma-algebra on $\mathbb{R}^2$ which contains all open sets in $\mathbb{R}^2$. Denote by

$$\pi(\mathbb{R}^2) = \Big\{(-\infty, x] \times (-\infty, y] \,\Big|\, x \in \mathbb{R}, y \in \mathbb{R}\Big\}.$$

the collection of closed south-west quadrants in $\mathbb{R}^2$. Prove that the collection of closed south-west quadrants generates $\mathscr{B}(\mathbb{R}^2)$, that is, show that $\mathscr{B}(\mathbb{R}^2) = \sigma(\pi(\mathbb{R}^2))$.

Proof. Define

$$E_x = \cup_{n=1}^{\infty}(-\infty, x] \times (-\infty, n] = (-\infty, x] \times \mathbb{R},$$

and

$$F_y = \cup_{n=1}^{\infty}(-\infty, n] \times (-\infty, y] = \mathbb{R} \times (-\infty, y].$$

Since the Borel sigma-algebra is closed under countable union, E_x and F_y belongs to the generated sigma-algebra. Furthermore, we can write

$$(x, \infty) \times (y, \infty) = \mathbb{R}^2 \backslash (-\infty, x] \times (-\infty, y] \cup E_x \cup F_y,$$

which implies $(x, \infty) \times (y, \infty)$ belongs to Borel sigma-algebra.
For any $a, b \in \mathbb{R}$ such that $a > x$ and $b > y$, we have

$$(x, a] \times (y, b] = (x, \infty) \times (y, \infty) \cap (-\infty, y] \cap (-\infty, z],$$

and if $a_n \uparrow a$ and $b_n \uparrow b$, the open rectangle $(x, a) \times (y, b)$ can be written as

$$(x, a) \times (y, b) = \bigcup_{n=1}^{\infty} (x, a - \frac{1}{n}] \times (y, b - \frac{1}{n}],$$

this shows that the open rectangle is contained in the Borel sigma-algebra. Since every open set is the countable union of open rectangles it implies $\mathscr{B}(\mathbb{R}^2) \subset \sigma(\pi(\mathbb{R}^2))$.
Since $(-\infty, x] \times (-\infty, y]$ is the complement of an open set thus belongs to the Borel sigma-algebra, which implies $\sigma(\pi(\mathbb{R}^2)) \subset \mathscr{B}(\mathbb{R}^2)$.

□