MS-E1600 - Probability theory, 2019/III

Problem set 2

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1. Truncation and conditioning.

   (a) Let ν be a measure on $(S, \mathscr{S})$ and let $B \in \mathscr{S}$. Show that also $A \to \nu[A \cap B]$ defines a measure on $(S, \mathscr{S})$.

   Proof. In order to show that $A \to \nu[A \cap B]$ defines a measure on $(S, \mathscr{S})$, we need to check if $A \to \nu[A \cap B]$ satisfy the measure definition

   i. Measure of an empty set

   $$\emptyset \to \nu[\emptyset \cap B] = \nu[\emptyset] = 0$$

   ii. (Close under union of countably many sets) Assume $E_1, E_2, \dots \in \mathscr{S}$ are disjoint sets,

   $$\Big(\cup_{j=1}^{\infty} E_j \Big) \to \nu[\Big(\cup_{j=1}^{\infty} E_j \Big) \cap B] = \nu[\cup_{j=1}^{\infty} (E_j \cap B)] = \sum_{j=1}^{\infty} \nu[E_j \cap B]$$

   which shows that $A \to \nu[A \cap B]$ defines a measure on $(S, \mathscr{S})$.

   □

   (b) Let P be a probability measure on $(\Omega, \mathscr{F})$, and let B be an event such that $P[B] > 0$. Show that the conditional probability $A \to P[A|B] := \frac{P[A \cap B]}{P[B]}$ is a probability measure.

   Proof. Similarly, we will check if $A \to P[A|B] := \frac{P[A \cap B]}{P[B]}$ satisfies the necessary condition of a probability measure.

   i.
   $$\emptyset \to P[\emptyset|B] := \frac{P[\emptyset \cap B]}{P[B]} = \frac{P[\emptyset]}{P[B]} = \frac{0}{P[B]} = 0.$$

ii. Assume $E_1, E_2, \dots \in \mathscr{S}$ are disjoint,

$$
\begin{aligned}
\left(\cup_{j=1}^{\infty} E_j\right) \to P[\left(\cup_{j=1}^{\infty} E_j\right)|B] :&= \frac{P[\left(\cup_{j=1}^{\infty} E_j\right) \cap B]}{P[B]} \\
&= \frac{P[\cup_{j=1}^{\infty}(E_j \cap B)]}{P[B]} = \frac{\sum_{j=1}^{\infty} P[(E_j \cap B)]}{P[B]} \\
&= \sum_{j=1}^{\infty} \frac{P[(E_j \cap B)]}{P[B]}.
\end{aligned}
$$

iii. Measure of the whole space

$$
\Omega \to P[\Omega|B] := \frac{P[\Omega \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1,
$$

which show that the conditional probability $A \to P[A|B] := \frac{P[A \cap B]}{P[B]}$ is a probability measure. □

2. Indicators of sets. The indicator function of a set $A \subset S$ is the function $1_A : S \to \mathbb{R}$ defined by

$$
1_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}
$$

(a) Let $\mathscr{S}$ be a sigma-algebra on S. Show that the indicator function 1_A of $A \subset S$ is a $\mathscr{S}$ measurable function if and only if A is a $\mathscr{S}$-measurable set.

Proof. " $\implies$ " Suppose that the indicator function 1_A of $A \subset S$ is a $\mathscr{S}$ measurable function □

(b) Show that $1_{A \cap B} = 1_A 1_B$.

Proof. Let $A, B \in \mathscr{S}$,

$$
\begin{aligned}
1_{A \cap B}(x) &= \begin{cases} 1, & \text{if } x \in A \cap B, \\ 0, & \text{otherwise.} \end{cases} \\
1_A(x) 1_B(x) &= \begin{cases} 1, & \text{if } x \in A \text{ and } x \in B \text{ or} x \in A \cap B \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

which show $1_{A \cap B} = 1_A 1_B$. □

(c) When is it true that $1_{A \cup B} = 1_A + 1_B$?

Proof. We observe that

$$1_{A\cup B}(x) = \begin{cases} 1, & \text{if } x \in A \cup B, \\ 0, & \text{otherwise.} \end{cases}$$

$$1_A(x) + 1_B(x) = \begin{cases} 1, & \text{if } x \in A \text{ or } x \in B \\ 0, & \text{if } x \notin A \text{ and } x \notin B. \end{cases}$$

In order to have $1_{A\cup B} = 1_A + 1_B$? , we require $A \cap B = \emptyset$. In other words, A and B are disjoint. □

3. The sigma-algebra generated by a random variable. Let Y be a real-valued random variable defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$.

   (a) Show that the σ-algebra $\sigma(Y)$ generated by the random variable Y coincides with the σ-algebra $\sigma(Y^{-1}(\mathscr{B}))$ generated by the collection of events $Y^{-1}(\mathscr{B}) = \{Y^{-1}(B) \,|\, B \in \mathscr{B}\}$.

   Proof. Inclusion $\sigma(Y) \subset \sigma(Y^{-1}(\mathscr{B}))$. $\sigma(Y)$ is a collection of
   By assumption, Y is a real-valued random variable defined on a probability space $(\Omega, \mathscr{F}, \boldsymbol{P})$, thus Y is $\mathscr{F}/\mathscr{B}$- measurable.

   □

   (b) Show that we in fact have the equality $\sigma(Y) = Y^{-1}(\mathscr{B})$.

   (c) Show that for any Borel function $h : \mathbb{R} \to \mathbb{R}$ the random variable $h(Y)$ is $\sigma(Y)$-measurable.

4. Limsup and liminf.

   (a) Show that $\liminf_n x_n \leq \limsup_n x_n$.

   Proof. Define $x_m^+ := \sup(x_n)_{n=m}^{\infty}$ and $x_m^- := \inf(x_n)_{n=m}^{\infty}$, thus we have $x_m^+ \leq x_m^-$ □

   (b) For any $A \subset S$, denote by $1_A : S \to R$ the indicator function of A as in Exercise 2.2. Show that for all $s \in S$

   $$\limsup_n 1_{A_n}(s) = 1_{\limsup A_n}(s) \quad \text{and} \quad \liminf_n 1_{A_n}(s) = 1_{\liminf A_n}(s)$$

   (c) Use (a) and (b) to conclude that $\liminf_n A_n \leq \limsup_n A_n$.

5. Let $U : \Omega \to R$ be a random variable on some probability space $(\Omega, \mathscr{F}, P)$ following the uniform distribution on the unit interval, so that $P[\{\omega in \Omega \,|\, a < U(\omega) < b\}] = b - a$ whenever $0 \leq a \leq b \leq 1$. Define $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \inf\{z \in \mathbb{R} \,|\, F(z) \geq U(\omega)\}.$$

   (a) Show that for any $c \in \mathbb{R}$ we have $X(\omega) \leq c$ if and only if $U(\omega) \leq F(c)$.

   (b) Using (a), show that for any $c \in \mathbb{R}$ we have $X^{-1}\Big((-\infty, c]\Big) \in \mathscr{F}$

   (c) Conclude that X is a $\mathscr{F}$-measurable random variable.

   (d) Using (a) and the distribution of U, show that $P[X \leq c] = F(c)$, i.e., the cumulative distribution function of the random variable X is F.