Notes on removable sets for $\mathcal{A}$-harmonic functions

Lien Tran *

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Abstract

This monograph is primarily concerned with the removability of α-Hölder continuous $\mathcal{A}$-harmonic functions on sets of $(n - p + \alpha(p - 1))$-Hausdorff measure zero.

Contents

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1 Introduction

1.1 Second order quasilinear elliptic equation

Partial differential equations (PDEs) are well known as a fundamental tool to model relevant phenomenon in physics, geology, astrophysics, mechanics, geophysics, as well as in most engineering disciplines. Often, they are employed to describe the dependence of phenomena under investigation on large number of parameters of various kinds. In this work, we pay attention in the class of elliptic equations. In particular, we consider the second order quasilinear elliptic equations of divergence form in Ω,

$$-\ \operatorname{div} \mathcal{A}(x, \nabla u) = 0, \tag{1.1}$$

where Ω denotes a bounded open set in $\mathbb{R}^n$ and where $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a mapping satisfying the certain structural assumptions. $\mathcal{A}$ is a typical given vector field (called the principal part and the inhomogeneity) that satisfies the following structural assumptions with $0 < \lambda < \Lambda < \infty$ [3, p.56]:

(i) Carathéodory condition

$$\begin{aligned}&\text{the function } x \mapsto \mathcal{A}(x, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^n, \text{ and}\\ &\text{the function } \xi \mapsto \mathcal{A}(x, \xi) \text{ is continuous for a.e. } x \in \mathbb{R}^n;\end{aligned} \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \mathbb{R}^n$

(ii) ellipticity degeneracy

$$\mathcal{A}(x, \xi) \cdot \xi \geq \lambda \left|\xi\right|^p, \tag{1.3}$$

(iii) growth assumption

$$\left|\mathcal{A}(x, \xi)\right| \leq \Lambda \left|\xi\right|^{p-1}, \tag{1.4}$$

(iv) monotonicity assumption

$$\big(\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)\big) \cdot (\xi_1 - \xi_2) > 0 \tag{1.5}$$

whenever $\xi_1 \neq \xi_2$. And

(v) homogeneity assumption

$$\mathcal{A}(x, \alpha\xi) = \alpha \left|\alpha\right|^{p-2} \mathcal{A}(x, \xi) \tag{1.6}$$

whenever $\alpha \in \mathbb{R}, \alpha \neq 0$.

The Carathéodory condition (i) is crucially important since it guarantees that the composition of a Carathéodory function with a measurable function is again measurable (note that measurability is not necessarily the case for the composition of two functions which are merely Lebesgue-measurable). The ellipticity assumption (ii) of $\mathcal{A}$ is compatible with the integrability assumptions. Intuitively, ellipticity tells us how degenerate the diffusion determined by the diffusion coefficients in each direction can be: diffusion does not vanish nor blows up. The principal prototype of (1.1) is the quasilinear second-order partial differential equation

$$-\ \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = 0, \quad 1 < p < \infty. \tag{1.7}$$

Equation (1.7) is called p-Laplace equation with the principal part in divergence form. When p = 2, equation (1.7) reduces to the well-known Laplace equation whose solutions are called harmonic functions,

$$\Delta u = 0, \tag{1.8}$$

where Δ denotes Laplace operator. When $p \neq 2$, equation (1.7) behaves as nonlinear and degenerates whenever $\nabla u = 0$. In this case, its solutions are commonly referred to as p-harmonic functions. This class of functions contains functions that satisfy equation (1.7) in a weak sense and do not need to be smooth, nor even $C^2(\Omega)$. The study of equation (1.7) as a prototype of equation (1.1) is motivated by the fact that it is the Euler equation for the variational integral, or the p-Dirichlet integral,

$$I(u) = \int_\Omega |\Delta u|^p \ dx \qquad 1 < p < \infty. \tag{1.9}$$

In addition, equation (1.9) is the simplest variational functional of nonquadratic growth. Conversely, equation (1.1) itself plays as a measurable perturbation of (1.7).

In general, solutions to (1.1) fail to be continuous and naturally associated with the first order Sobolev space $W^{1,p}(\Omega)$. The properties of this class of functions allow their elements to be modified in a set of Lebesgue measure zero so that the new function is locally Hölder continuous with an exponent $\alpha = \alpha(n,p)$. Thus, the weak solutions of (1.1) are also of class $C^\alpha_{loc}(\Omega)$ of locally Hölder continuous functions. In fact, the weak solutions to the general quasilinear equations of type (1.1) are proven to exhibit many features in common with harmonic functions. A significant characteristic property thereof is the comparison principle which means that the Dirichlet solutions are order preserving. In other words, if u and v are two solutions in $\Omega \subset \mathbb{R}^n$ with $u \leq v$ on the boundary $\partial\Omega$, then $u \leq v$ in the interior of Ω.

In this work, our primary concern is removability of the Hölder continuous solutions of (1.1). Roughly speaking, given a solution and an exceptional set with extra assumptions on both, the removability theorem allows us to extend the solution to be a solution in the exceptional set.

1.2 Physical Interpretation

In Cartesian coordinates, the Laplace operator or Laplacian takes the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2},$$

and the p-Laplace equation is defined as

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-4}\left\{|\nabla u|^2\Delta u + (p-2)\sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}\right\}. \tag{1.10}$$

The Laplace operator is invariant under translations meaning its coefficients are constant under orthogonal transformations in $\mathbb{R}^n$. The Laplace equation (1.8) is a linear, scalar equation. The non-homogeneous version of the Laplace equation is called Possion's equation

$$-\Delta u(x) = f(x), \tag{1.11}$$

where $u : \overline{\Omega} \to \mathbb{R}$ is the unknown function, $\Omega \subset \mathbb{R}^n$ is a given open set, and $f : \Omega \to \mathbb{R}$ is a given function.

Definition 1.1. *A $C^2(\Omega)$ function u that satisfies (1.8) is called harmonic function.*

Laplace and Possion equations appear in a wide variety of physical contexts, for instant potential theory, Riemannian geometry, stochastic processes, complex analysis and so forth. Physically, the Laplace equation describes an equilibrium state obtained from a time-dependent evolution in the limit of infinite time. A typical physical presentation is that u denotes density of some quantity, such as heat flow in a body at equilibrium attained as the result of thermal diffusion across the entire domain. If V is any smooth subregion within the body Ω, then the net flux of u through the boundary of V, ∂V, is zero

$$0 = \int_{\partial V} F \cdot \nu \, dS = \int_V \operatorname{div} F \, dx, \tag{1.12}$$

where $F = (F_1, F_2, \cdots, F_n)$ is the flux density and ν is the unit outer normal of ∂V. The last equality is the consequence of the Gauss-Green theorem. Since V is an arbitrary subdomain of Ω, we conclude

$$\operatorname{div} F = 0 \quad \text{in } \Omega. \tag{1.13}$$

Physically, the flux F is assumed to be proportional to the gradient ∇u but to point in the opposite direction

$$\operatorname{div} F = -a\nabla u, \quad a > 0. \tag{1.14}$$

By substituting the above result into (1.13), we obtain Laplace's equation

$$\operatorname{div}(\nabla u) = \Delta u = 0. \tag{1.15}$$

In most cases, we are interested in finding solutions of the Laplace or Possion equation that satisfy additional conditions. As an example, we consider the classical Dirichlet problem Poissons equation is to find a function $u : \Omega \to R$, $\Omega \subset \mathbb{R}^n$ a bounded open set, such that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.16}$$

where $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ are given functions. In this context, the term 'classical' refers to the requirement that the functions u and its derivative are defined as pointwise continuous functions.

2 Preliminaries

Throughout the work we denote by c the positive constant whose value may vary at each time occurrence, even in the same line. We denote the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ as

$$B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}.$$

The symbol $\omega(n)$ denotes the volume of the ball of radius 1 in $\mathbb{R}^n$. We denote $C^0(\Omega)$ the space of continuous functions on an open set $\Omega \subset \mathbb{R}^n$. Moreover, if k is a nonnegative integer we denote

$$\begin{aligned} C^k(\Omega) &= \{u \in C(\Omega) : u \text{ is } k \text{ times continuously differentiable}\}. \\ C^\infty &= \bigcap_{k=1}^{\infty} C^k(\Omega) \text{ smooth functions.} \\ \text{supp } u &= \overline{\{x \in \Omega : u(x) \neq 0\}} = \text{ The support of u.} \\ C_0^\infty(\Omega) &= \{\varphi \in C^\infty(\Omega) : \text{ supp } \varphi \Subset \Omega\}. \end{aligned}$$

Since Ω is open, a function $u \in C^k(\Omega)$ may be unbounded on Ω. However, if u is bounded and uniformly continuous on Ω, then u can be uniquely extended to a continuous function on $\overline{\Omega}$. For a function $\varphi \in C^\infty(\Omega)$ we write

$$\nabla\varphi = (\partial_1\varphi, \partial_2\varphi, \cdots, \partial_n\varphi)$$

for the gradient of φ. Furthermore, we denote

$$u_E = \frac{1}{|E|}\int_E u\,dx = \fint_E u\,dx.$$

for a measurable $E \subset \Omega$ and a measurable function $u : \Omega \to \mathbb{R}$, and $|E|$ denotes Lebesgue measure of E. If $u : E \to \mathbb{R}$ is a function, then

$$\operatorname{osc}(u, E) = \sup_E u - \inf_E u$$

is the oscillation of u in E.

2.1 Young's inequality

$$ab \leq \epsilon a^p + \frac{b^{p/(p-1)}}{\epsilon^{1/(p-1)}}, \qquad \epsilon > 0, \text{ and } p > 1. \tag{2.1}$$

2.2 Hausdorff measure

The concept of Hausdorff measure arises from the need of measuring the size of the lower dimensional subsets of a metric space.

Definition 2.1. *(Hausdorff measure) Let E be any subset of X and $\delta > 0$ be a real number. The Hausdorff outer measure of dimensional s bounded by δ (written by $\mathcal{H}^s_\delta$)*

$$\mathcal{H}^s_\delta(E) = \inf\Big\{\sum_{i=1}^{\infty}(diam(U_i))^s : \bigcup_{i=1}^{\infty} U_i \supseteq E, diam(U_i) < \delta\Big\}, \tag{2.2}$$

where the infimum is taken over all countable covers of E by sets $U_i \subseteq X$ satisfying $U_i < \delta$.

In the limit δ approaches zero, the infimum is taken over a decreasing collection sets, and is therefore increasing.

$$\lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \mathcal{H}^s(E).$$

The set $\mathcal{H}_{(\infty)}$ denotes the Hausdorff capacity or Hausdorff content.

Proposition 2.2. *[1, p. 134] $\mathcal{H}_{(\infty)}(E) = 0$ if and only if $\mathcal{H}^s(E) = 0$.*

Lemma 2.3. *(Frostman's lemma) [1, p. 136] Let s be an increasing function on $[0, \infty)$ such that $s(0) = 0$ and let $E \subset \mathbb{R}^n$ be a compact set. Then*

$$\mu(E) \leq \mathcal{H}^s_{(\infty)}(E), \tag{2.3}$$

for all $\mu \in \mathcal{M}^+(E)$, where $\mathcal{M}^+(E)$ is the set of positive measures on E, such that $\mu(B(x, r)) \leq s(r)$ for all balls $B(x, r)$. Furthermore, there is a constant $A > 0$, depending only on n, and a $\mu \in \mathcal{M}^+(E)$, satisfying $\mu(B(x, r)) \leq s(r)$ for all $B(x, r)$, such that

$$\mathcal{H}^s_\infty(E) \leq A\,\mu(E). \tag{2.4}$$

2.3 Hölder continuity

Definition 2.4. *(Hölder continuous functions) Let $\Omega \subset \mathbb{R}^n$ be open and bounded.A function $u : \Omega \to \mathbb{R}$ is Hölder continuous with the exponent $0 < \alpha \leq 1$, if there exists a constant c, called Hölder constant, such that*

$$|u(x) - u(y)| \leq c\,|x - y|^\alpha \tag{2.5}$$

for every $x, y \in \Omega$. Then $u \in C(\overline{\Omega})$ and in particular u is bounded. The set $C^{0,\alpha}(\Omega)$ of all Hölder continuous functions on Ω with exponent α, equipped with the norm

$$|u|_{C^{0,\alpha}(\Omega)} := |u|_{L^\infty(\Omega)} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \tag{2.6}$$

is a Banach space. Functions in $C^{1,\alpha}(\Omega)$ are also called Lipschitz continuous.

2.4 Sobolev Spaces

Definition 2.5. *Assume that Ω is an open subset of $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| < k$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$. Then the Sobolev space is defined by*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\},$$

and is equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u| \Big)^{1/p}, \quad 1 \leq p < \infty, \tag{2.7}$$

and

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leq k} \operatorname*{ess\,sup}_\Omega |D^\alpha u| \,. \tag{2.8}$$

The Sobolev space $W^{k,p}(\Omega)$ consists of functions in $L^p(\Omega)$ those have weak partial derivatives up to order k and belong to $L^p(\Omega)$. Notice that weak derivatives have the same properties as classical derivative of smooth functions. In this work, we shall denote $Du = \nabla u$ the first order weak derivative of function u.

Theorem 2.6. *[2, p.64] (Dirichlet growth theorem) Let $u \in W^{1,p}(B(x_0, R))$, $1 \leq p \leq n$. Suppose that for all $x \in B(x_0, R)$, all $r, 0 < r <\leq \delta(x) = R - |x - x_0|$*

$$\int_{B(x,r} |\nabla u|^p \; dx \leq L^p \Big(\frac{r}{\delta}\Big)^{n-p+p\alpha} \tag{2.9}$$

holds with $0 < \alpha \leq 1$. Then $u \in C^{0,\alpha}(B(x_0, \rho))$ for all $\rho < R$; moreover if $|x - y| < \frac{\delta(x)}{2}$ the following estimate holds

$$|u(x) - u(y)| \leq c(n, p, \alpha) L \, \delta^{1-\frac{n}{p}} \Big(\frac{|x - y|}{\delta}\Big)^\alpha. \tag{2.10}$$

2.5 Riesz Potential of a function

The Riesz kernel , $I_\alpha, 0 < \alpha < n$, is defined by

$$I_\alpha(x) = \gamma(\alpha)^{-1} |x|^{\alpha - n} \,, \tag{2.11}$$

where

$$\gamma(\alpha) = \frac{\pi^{n/2} 2^{\alpha} \Gamma(\alpha/2)}{\Gamma(n/2 - \alpha/2)}.$$

The Riesz potential of a function f is defined as the convolution

$$I_\alpha * f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha}}. \tag{2.12}$$

Next, we recap the representation formula for a compactly supported continuously differentiable function in terms of its gradient. The formula suggests that a function can be integrated back from its derivative.

Lemma 2.7. *(Representation formula) If $u \in \mathbb{C}_0^1(\Omega)$, then*

$$u(x) = \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\nabla u(y) \cdot (x-y)}{|x-y|}^n dy \quad \textit{for every} \quad x \in \mathbb{R}^n,$$

where ω_{n-1} is the $(n-1)$-dimensional measure of $\partial B(0,1)$.

Theorem 2.8. *Assume that $|\Omega| < \infty$ and $1 \leq p < \infty$. Then*

$$\|I_1 * (|f|\, \chi_\Omega)\|_{L^p(\Omega)} \leq c(n,p)\, |\Omega|^{1/n}\, \|f\|_{L^p(\Omega)}\,. \tag{2.13}$$

3 $\mathcal{A}$-harmonic functions

3.1 Definitions and Properties

3.1.1 $\mathcal{A}$-harmonic functions

In this section, we will study the definition of solutions of equation (1.1) and how they relate to harmonic functions. Since subharmonic and superharmonic functions are used as a tool to study harmonic functions, they shall be explored extensively. The key property of bounded superharmonic functions is that they are exactly the bounded lower semicontinuously regularized supersolutions.

Definition 3.1. *(Weak solution) A function $u \in W^{1,p}_{loc}(\Omega)$ is a solution of (1.1) in Ω if*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = 0, \tag{3.1}$$

for all compactly supported function $\varphi \in C_0^\infty(\Omega)$.

In a similar fashion, we define the (weak) supersolution of (1.1).

Definition 3.2. *(Supersolution) A function $u \in W^{1,p}_{loc}(\Omega)$ is a (weak) supersolution of (1.1) in Ω if*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0, \tag{3.2}$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$.

Analogously, function $v \in W^{1,p}_{loc}(\Omega)$ is a weak subsolution of (1.1) in Ω if (3.2) holds for all nonpositive compactly supported smooth function $\varphi \in C_0^\infty(\Omega)$. In other words, a function v is a subsolution of (1.1) if $-v$ is a supersolution of (1.1). It is important to note that a function $u \in W^{1,p}_{loc}(\Omega)$ is a solution of (1.1) if and only if it is both a supersolution and a subsolution. The claim follows from the fact that the positive and the negative parts of a test function $\varphi \in C_0^\infty(\Omega)$ both belong to $W_0^{1,p}(\Omega)$ and have compact support, thus the integral in (3.1) is nonnegative for all test functions $\varphi \in C_0^\infty(\Omega)$. The other important property of (super) solutions is that if u is a (super) solution and $\lambda, \tau \in \mathbb{R}, \lambda > 0$, then $\lambda u + \tau$ is again a (super) solution.

It is worth noticing that being a solution or a supersolution to (1.1) is a local property by appealing to a partition of unity argument. In particular, a function u is a (super) solution in Ω if and only if Ω can be covered by open sets on which u is a (super) solution.

Observe that the zero boundary, first order Sobolev space is the closure of the compactly supported smooth functions. This property allows us to work with a larger class of test functions than $C_0^\infty(\Omega)$.

Lemma 3.3. *If $u \in L^p(\Omega)$ is a solution (respectively, a supersolution) of (1.1) in Ω, then*

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = 0 \qquad \textit{(respectively, } \geq 0\textit{)}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ (respectively, for all nonnegative $\varphi \in W_0^{1,p}(\Omega)$).

Proof. Let $\varphi \in W_0^{1,p}(\Omega)$ and a sequence of functions $\varphi_i \in C_0^\infty(\Omega)$, for $i = 1, 2, \cdots$, such that $\varphi_i \to \varphi$ in $W_0^{1,p}(\Omega)$ as $i \to \infty$. Note that if φ is nonnegative, we can choose sequence of nonnegative functions $\varphi_i \in C_0^\infty(\Omega)$ by . Thus we have

$$\begin{aligned}
&\left|\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx - \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi_i \, dx\right| \\
&\leq \Lambda \int_\Omega |\nabla u|^{p-1} |\nabla \varphi - \nabla \varphi_i| \, dx \quad \text{(The grow assumption in 1.4)} \\
&\leq \Lambda \Big(\int_\Omega |\nabla u|^p \, dx\Big)^{(p-1)/p} \Big(\int_\Omega |\nabla \varphi - \nabla \varphi_i|^p \, dx\Big)^{1/p} \text{(Hölder's inequality)} \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

Thus we concludes

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = \lim_{i \to \infty} \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi_i \, dx \geq 0$$

and the lemma follows. □

The argument in the proof of Lemma 3.3 implies that if u is any (super)solution in Ω, then

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx = 0 \tag{3.3}$$

holds for all (nonnegative) $\varphi \in W_0^{1,p}(\Omega)$ with compact support. In addition, if $u \in L^{1,p}(\Omega)$, the Dirichlet spaces, is the solution of (1.1), where

$$L^{1,p}(\Omega) = \{u \in W_{loc}^{1,p}(\Omega) : \nabla u \in L^p(\Omega)\},$$

then (3.3) holds for all $\varphi \in L_0^{1,p}(\Omega)$, a closure of $C_0^\infty(\Omega)$. Further more if $u \in L^{1,p}(\Omega)$ is a supersolution, then

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0 \tag{3.4}$$

whenever $\varphi \in L_0^{1,p}(\Omega)$ such that there is a sequence of nonnegative functions $\varphi_i \in C_0^\infty(\Omega)$ with $\nabla \varphi \to \nabla \varphi_i$ in $L^p(\Omega)$.

The leading property of solutions of (1.1) is that they are quasiminimizers of the p-Dirichlet integral (1.9). In particular, if $u \in W_{loc}^{1,p}(\Omega)$ is a solution of (1.1), u has the least weighted p-Dirichlet integral, amongst all functions v having same boundary values as u, i.e. $u - v \in W_0^{1,p}(\Omega)$, up to a factor $(\frac{\Lambda}{\lambda})^p$. This follows straightforwardly from the elliptic degeneracy property (1.3) and property (1.4) of $\mathcal{A}$

$$\begin{aligned}\int_\Omega |\nabla u|^p \, d\mu &\le \lambda^{-1} \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla u \, dx \\ &\le \lambda^{-1} \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla v \, dx \\ &\le \frac{\Lambda}{\lambda} \int_\Omega |\nabla u|^{(p-1)} |\nabla v| \, dx \\ &\le \frac{\Lambda}{\lambda} (\int_\Omega |\nabla u|^p \, d\mu)^{(p-1)/p} (\int_\Omega |\nabla v|^p \, d\mu)^{1/p} \text{ (Hölder inequality)}\end{aligned}$$

and thus

$$\int_\Omega |\nabla u|^p \, d\mu \le (\frac{\Lambda}{\lambda})^p \int_\Omega |\nabla v|^p \, d\mu. \tag{3.5}$$

In the special case of p-Laplace equation, then $\lambda = \Lambda = 1$ and u has the least weighted p energy among all functions v with $u - v \in W_0^{1,p}(\Omega)$. Analogously, the supersolution $u \in W_{loc}^{1,p}(\Omega)$ is a quasisuperminimizer , that is (3.5) holds for all functions φ with $u - \varphi \in W_0^{1,p}(\Omega)$ and $\varphi \ge u$.

Furthermore, the nonnegative supersolution satisfies the weak Hanack's inequality.

Corollary 3.4. *(Weak Harnack's inequality) Suppose that $v \in W_{loc}^{1,p}(\Omega)$ is a non-negative supersolution. Then*

$$\left(\fint_{B_r} v^\beta dx \right)^{1/\beta} \le c(n, p, \beta) \operatorname*{ess\,inf}_{B_r} v, \qquad \beta < \frac{n(p-1)}{n-1}, \tag{3.6}$$

whenever $B_{2r} \subset \Omega$.

In general, supersolution of (1.1) fail to be continuous, thus the above definition is not adequate for us to have pointwise estimates, it is more useful to work with the following class of functions called p-superharmonic functions.

Definition 3.5. *A function $u : \Omega \to (-\infty, \infty]$ is p-superharmonic in Ω if*

(i) u is lower semicontinuous and is not identically infinite on any component of Ω,

(ii) u belongs to $L_{loc}^q(\Omega)$ for some $q > 0$, and

(iii) for all nonempty open $V \subset\subset \Omega$ with $V \ne \Omega$ and all function $v \in C(\bar{V})$ such that v is p-harmonic in V and $v \le u$ on ∂V, we have $v \le u$ in V.

It is shown that each weak solution of equation (1.1) can be defined in a set of measure zero so that it becomes continuous and These continuous weak solution of (1.1) form a class of function called $\mathcal{A}$-harmonic functions.

Definition 3.6. *[3, p.110] A function $u : \Omega \to \mathbb{R}$ is said to be $\mathcal{A}$-harmonic in Ω if it is a continuous weak solution of (1.1) in Ω.*

The set of $\mathcal{A}$-harmonic functions does not form a linear space. However, due to the fact that $\mathcal{A}$ depends only on x and ∇u, and $\mathcal{A}$ is homogeneous in ∇u, $\mathcal{A}$-harmonic functions do enjoy the property that for every $\mathcal{A}$-harmonic function u and a real constant τ, τu and $u + \tau$ are $\mathcal{A}$-harmonic. In addition, they satisfy the strong form of Harnack's inequality.

Theorem 3.7. *(Harnack's Inequality) Let h be a nonnegative $\mathcal{A}$-harmonic function in Ω. Then there is a constant $c = c(n, p, \lambda, \Lambda, c_\mu)$ such that*

$$\sup_B h \leq c \inf_B h \tag{3.7}$$

whenever B is a ball in Ω such that $2B \subset \Omega$.

3.1.2 Obstacle problem

In the Calculus of Variations, p-harmonic functions acts as a minimizer of the Dirichlet integral. A natural question arise that what if we add a restriction on the admissible function when minimizing? In particular, the restrictive condition is the solution must lie above a given function, which acts as a fixed obstacle. In fact, this leads to the weak supersolutions of the p-harmonic equation.

Suppose Ω is a bounded domain in $\mathbb{R}^n$. Given a function $\psi \in C(\Omega) \cap W^{1,p}_{loc}(\Omega)$ and $\vartheta \in W^{1,p}_{loc}(\Omega)$ we consider the problem of minimizing the integral

$$\int_\Omega |\nabla u|^p \, dx \tag{3.8}$$

among all functions in the class

$$\mathcal{K}_{\psi,\vartheta} = \mathcal{K}_{\psi,\vartheta}(\Omega) = \left\{ u \in W^{1,p}_{loc}(\Omega) : u \geq \psi \text{ a.e. in } \Omega, v - \vartheta \in W^{1,p}_0(\Omega) \right\} \tag{3.9}$$

If $\psi = \vartheta$, we write $\mathcal{K}_{\psi,\vartheta} = \mathcal{K}_\psi$. This is the obstacle problem with function ψ acting as an obstacle from below and the boundary values is described by ϑ. The solution u to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}$ has the minimal 'energy' among all the functions that lie above the obstacle ψ and have the boundary values ϑ. In addition, u also satisfies the inequality

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla(v - u) \geq 0 \tag{3.10}$$

whenever $v \in \mathcal{K}_{\psi,\vartheta}$. The function ψ is called an obstacle.

Definition 3.8. *[3, p.60] A function u in $\mathcal{K}_{\psi,\vartheta}(\Omega)$ that satisfies (3.10) for all $v \in \mathcal{K}_{\psi,\vartheta}(\Omega)$ is called a solution to the obstacle problem with obstacle ψ and boundary values ϑ or a solution to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}(\Omega)$.*

Observe that $u + \varphi \in \mathcal{K}_{\psi,\vartheta}$ for all nonnegative $\phi \in C^\infty_0(\Omega)$, then the solution u to the obstacle problem is always a supersolution of (**??**) in Ω. Conversely, a supersolution is always a solution to the obstacle problem in $\mathcal{K}_{u,u}$ for all open sets D which is compactly contained in Ω, i.e. $D \Subset \Omega$.

If $\mathcal{K}_{\psi,\vartheta}(\Omega)$ is nonempty, there is a unique solution to the obstacle problem in $\mathcal{K}_{\psi,\vartheta}(\Omega)$. The existence and uniqueness of the solution to the obstacle $\mathcal{K}_{\psi,\vartheta}(\Omega)$ is established in [3].

3.2 Balayage

In this section, we study the theory of balayage (sweeping) of a function which acn be thought of as a basic form of the obstacle problem. Recall that the lower semicontinuous regularization $\hat{u}$ of any function $u : E \to [-\infty, \infty]$ is defined by

$$\hat{u}(x) = \lim_{r\to 0} \inf_{E\cap B(x,r)} u.$$

Then $\hat{u} \leq u$ on E. We say $\hat{u}$ is lower semicontinuous if u is locally bounded below.

Definition 3.9. *Let $\psi : \Omega \to (-\infty, \infty]$ be a function that is locally bounded, and let*

$$\begin{aligned}\Phi^\psi &= \Phi^\psi(\Omega) = \Phi^\psi(\Omega; \mathcal{A}) \\ &= \{u : u \text{ is } \mathcal{A}\text{-superharmonic in } \Omega \text{ and } u \geq \psi \text{ in } \Omega\}.\end{aligned}$$

Then the function

$$R^\psi = R^\psi(\Omega) = R^\psi(\Omega, \mathcal{A}) = \inf \Phi^\psi$$

is called the réduite and its lower semicontinuous regularization

$$\hat{R}^\psi = \hat{R}^\psi(\Omega) = \hat{R}^\psi(\Omega, \mathcal{A})$$

is the balayage of ψ in Ω.

If Φ^ψ is empty, then $\hat{R}^\psi(\Omega) \equiv \infty$.

3.3 Estimates

In this section we present some technical estimates that will be needed in the next section. We start with the famous Caccioppoli estimate.

Lemma 3.10. *[3, p.63](Caccioppoli estimates) Suppose that $\eta \in C_0^\infty(\Omega)$ is nonnegative and $q \geq 0$.*

(i) If u is a solution to the obstacle problem in $\mathcal{K}_{\psi,u}(\Omega)$ with nonpositive obstacle ψ, then

$$\int_\Omega \left|u^+\right|^q \left|\nabla u^+\right|^p \eta^p dx \leq \int_\Omega \left|u^+\right|^{p+q} |\nabla\eta|^p \, dx. \tag{3.11}$$

(ii) If u is a supersolution of (1.1) in Ω, then

$$\int_\Omega \left|u^-\right|^q \left|\nabla u^-\right|^p \eta^p dx \leq \int_\Omega \left|u^-\right|^{p+q} |\nabla\eta|^p \, dx. \tag{3.12}$$

Here $c = p^p(\Lambda/\lambda)$.

Proof. We prove (i). Without loss of generality we assume $0 \leq \eta \leq 1$ in Ω. Choose the test function to be $\varphi = -u^+\eta^p$. Then $\varphi \in W_0^{1,p}(\Omega)$ has compact support in Ω and

$$\nabla\varphi = -\nabla u^+\eta^p - pu^+\eta^{p-1}\nabla\eta$$

Since u is assumed to be a solution to the obstacle problem $\mathcal{K}_{\psi,u}(\Omega)$ which implies $u + \varphi \in \mathcal{K}_{\psi,u}(\Omega)$, we have

$$\begin{aligned}
0 &\leq \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \\
&= \int_\Omega \mathcal{A}(x,\nabla u)\cdot(-\nabla u^+)\eta^p\,dx + \int_\Omega \mathcal{A}(x,\nabla u)\cdot(-pu^+\eta^{p-1}\nabla\eta))dx \\
&\leq -\int_\Omega \mathcal{A}(x,\nabla u^+)\cdot(\nabla u^+)\eta^p\,dx + p\int_\Omega \left|\mathcal{A}(x,\nabla u^+)\right|\left|u^+\right|\left|\eta^{p-1}\right||\nabla\eta|\,dx \\
&\leq -\lambda\int_\Omega \left|\nabla u^+\right|^p\eta^p\,dx + p\Lambda\int_\Omega \left|\nabla u^+\right|\left|u^+\right|\left|\eta^{p-1}\right||\nabla\eta|\,dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\lambda\int_\Omega \left|\nabla u^+\right|^p\eta^p\,dx &\leq p\frac{\Lambda}{\lambda}\int_\Omega \left|\nabla u^+\right|\left|u^+\right|\left|\eta^{p-1}\right||\nabla\eta|\,dx \\
\text{(By Hölder's inequality)} &\leq p\frac{\Lambda}{\lambda}(\int_\Omega \left|u^+\right|^p|\nabla\eta|^p\,dx)^{1/p}(\int_\Omega \left|\nabla u^+\right|^p\eta^p dx)^{(p-1)/p}.
\end{aligned}$$

Since $\int_\Omega \left|\nabla u^+\right|^p\eta^p dx < \infty$ we absorb the right-hand side to the left-hand side to get

$$\int_\Omega \left|\nabla u^+\right|^p\eta^p dx \leq p^p(\Lambda/\lambda)^p\int_\Omega \left|u^+\right|^p|\nabla\eta|^p\,dx.$$

Next we observe that the function $u - t$, where t is a real constant, is a solution to the obstacle problem $\mathcal{K}_{\psi-t,u-t}$; thus for $t > 0$ we have

$$\int_\Omega \left|\nabla(u-t)^+\right|^p\eta^p dx \leq c\int_\Omega \left|(u-t)^+\right|^p|\nabla\eta|^p\,dx, \tag{3.13}$$

where $c = p^p(\Lambda/\lambda)^p$. Finally, let

$$\nu(A) := \int_A \left|\nabla u^+\right|^p\eta^p dx$$

we can easily check that ν is a Radon measure. Thus, by Cavalieri's principle for some $0 < q < \infty$ we have the estimate

$$\begin{aligned}
\int_\Omega \left|u^+\right|^q \left|\nabla u^+\right|^p \eta^p dx &= \int_\Omega \left|u^+\right|^q \, d\nu \\
&= q\int_0^\infty t^{q-1}\nu(\{x : u \geq t\})dt \\
&= q\int_0^\infty t^{q-1}\int_{\{u>t\}} \left|\nabla u^+\right|^p \eta^p dx\, dt \\
&= q\int_0^\infty t^{q-1}\int_{\{u>t\}} \left|\nabla (u-t)^+\right|^p \eta^p dx\, dt \\
&\leq cq\int_0^t t^{q-1}\int_{\{u>t\}} \left|(u-t)^+\right|^p |\nabla\eta|^p\, dx\, dt \\
&\leq cq\int_0^t t^{q-1}\int_{\{u>t\}} \left|u^+\right|^p |\nabla\eta|^p\, dx\, dt \\
&= c\int_\Omega \left|u^+\right|^{q+p} |\nabla\eta|^p\, dx,
\end{aligned}$$

as desired. □

Now we eill show that $\mathcal{A}$-harmonic functions are Hölder continuous locally.

Theorem 3.11. *[3, p.111] (Local Hölder continuity estimate) Suppose h is $\mathcal{A}$-harmonic in Ω. If $0 < r < R < \infty$ are such that $B(x_0, R) \subset \Omega$, then*

$$osc(h, B(x_0, r)) \leq 2^\kappa \left(\frac{r}{R}\right)^\kappa osc(h, B(x_0, R)), \tag{3.14}$$

where $\kappa \in (0, 1]$ depending on $n, p, \Lambda/\lambda$, and c_μ.

Proof. Let $0 < \rho \leq R$ and fix $x_0 \in \Omega$. We write $M(r) = \sup_{B(x_0,r)} h$ and $m(r) = \inf_{B(x_0,r)} h$. We apply the Harnack's inequality to the $\mathcal{A}$-harmonic function $(h - \inf_{B_\rho} h)$ to obtain

$$M(\rho/2) - m(\rho) = \sup_{B(x_0,\rho)} (h - m(\rho)) \leq c_0(m(\rho/2) - m(\rho)), \tag{3.15}$$

where $c_0 = c_0(n, p, \lambda/\lambda, c_\mu) \geq 1$. Set $\delta = (c_0 - 1)/c_0$, we look at two cases induced by (3.15). First we consider the case

$$m(\rho/2) - m(\rho) \leq c_0^{-1}(M(\rho) - m(\rho),$$

then

$$\begin{aligned}
\mathrm{osc}(h, B(x_0, \frac{\rho}{2})) &= M(\frac{\rho}{2}) - m(\rho) + m(\rho) - m(\frac{\rho}{2}) \\
&\leq (c_0 - 1)(m(\frac{\rho}{2}) - m(\rho) \leq \delta\, \mathrm{osc}(h, B(x_0, \rho)).
\end{aligned} \tag{3.16}$$

And if

$$m(\rho/2) - m(\rho) > c_0^{-1}(M(\rho) - m(\rho),$$

we have

$$\begin{aligned}\operatorname{osc}(h, B(x_0, \frac{\rho}{2})) &= M(\frac{\rho}{2}) - m(\rho) - (m(\frac{\rho}{2}) - m(\rho)) \\ &\le \delta\,(M(\rho) - m(\rho) = \delta \operatorname{osc}(h, B(x_0, \rho)).\end{aligned} \tag{3.17}$$

Thus

$$\operatorname{osc}(h, B(x_0, \frac{\rho}{2})) \le \delta \operatorname{osc}(h, B(x_0, \rho)) \tag{3.18}$$

always holds. Next we choose the integer $m \ge 1$ such that $2^{m-1} \le R/r < 2^m$. Then (3.18) implies

$$\operatorname{osc}(h, B(x_0, r)) \le \delta^{m-1} \operatorname{osc}(h, B(x_0, 2^{m-1}r)) \le \delta^{m-1} \operatorname{osc}(h, B(x_0, R)).$$

Let $\kappa = (-\log \delta)/(\log 2) \le 1$ and obtain

$$(\frac{r}{R})^{\kappa} \ge 2^{-\kappa}(2^{m-1})^{-\kappa} = 2^{-\kappa}\delta^{m-1},$$

and thus

$$\operatorname{osc}(h, B(x_0, r)) \le 2^{\kappa}\big(\frac{r}{R}\big)^{\kappa} \operatorname{osc}(h, B(x_0, R)),$$

as desired. □

Lemma 3.12. *[4, p.3] There exists a number $\kappa = \kappa(n, p, \lambda, \Lambda) > 0$ such that for each $0 < r < R$ and $\mathcal{A}$-harmonic h in $B(x_0, R)$ it holds that*

$$\int_{B(x_0,r)} |\nabla h|^p \, dx \le c\big(\frac{r}{R}\big)^{n-p+p\kappa} \int_{B(x_0,R)} |\nabla h|^p \, dx, \tag{3.19}$$

where $c = c(n, p, \lambda, \Lambda) > 0$.

Proof. Let $\kappa = \kappa(n, p, \lambda, \Lambda)$ be the number as described in Lemma 3.11. First, we assume $r < R/4$ and using the gradient estimate for p-harmonic functions and the Caccioppoli estimate we infer that

$$\begin{aligned}\int_{B(x_0,r)} |\nabla h|^p \, dx &= \int_{B(x_0,r)} \Big|\nabla(h - \inf_{B(x_0,2r)} h)\Big|^p \, dx \\ &\le p^p r^{-p} \int_{B(x_0,2r)} \Big|h - \inf_{B(x_0,2r)} h\Big|^p \, dx \\ &\le p^p r^{-p} \int_{B(x_0,2r)} \Big|\sup_{B(x_0,2r)} h - \inf_{B(x_0,2r)} h\Big|^p \, dx \\ &= p^p r^{-p}\, |B(x_0, 2r)|\, (\operatorname{osc}(h, B(x_0, 2r)))^p \\ &\le c\, r^{n-p+\kappa p} R^{-p\kappa} \operatorname{osc}(h, B(x_0, R/2))^p \qquad \text{(by Theorem 3.11)}.\end{aligned} \tag{3.20}$$

By weak Harnack's inequality, we have

$$h_R := \fint_{B(x_0,R)} h \, dx \le c \operatorname*{ess\,inf}_{B(x_0,R)} h.$$

Then the last term can be estimated by

$$\begin{aligned}
\operatorname{osc}(h, B(x_0, R/2))^p &= (\sup_{B(x_0,R/2)} h - \inf_{B(x_0,R/2)} h)^p \le c\,(\sup_{B(x_0,R/2)} h - h_R)^p \\
&= c \sup_{B(x_0,R/2)} |h - h_R|^p \\
&\le \frac{c}{|B(x_0,R)|} \int_{B(x_0,R)} |h - h_R|^p \, dx \quad \text{(by weak Harnack's inequality)} \\
&\le cR^{p-n} \int_{B(x_0,R)} |\nabla h|^p \, dx \quad \text{(by Poincaré's inequality).}
\end{aligned} \tag{3.21}$$

This completes the proof. □

Theorem 3.13. *[7, p.199] Let μ be a Radon measure on $\mathbb{R}^n$ such that for all $x \in \mathbb{R}^n$ and $0 < r < \infty$, there is a constant M with the property*

$$\mu(B(x,r)) \le M\, r^{\alpha},$$

where $\alpha = q/p(n-kp), k > 0, 1 < p < q < \infty$, and $kp < n$. If $u \in L^p(\mathbb{R}^n)$ then

$$\Big(\int_{\mathbb{R}^n} |I_k * u|^q \, d\mu \Big)^{1/q} \le C\, M^{1/q} \, \|u\|_p \,,$$

where $C = C(k,p,q,n)$.

Proof. First, we denote $\mu_t = \mu|_{A_t}$ andfor $t > 0$ we define the level set

$$E_t = \{y : I_k * |u|\,(y) > t.\}$$

Then by Fubini's theorem we have

$$\begin{aligned}
t\mu(A_t) &\le \int_{A_t} I_k * |u| \, d\mu = \int_{\mathbb{R}^n} I_k * |u| \, d\mu_t \\
&= \int_{\mathbb{R}^n} I_k * \mu_t(x) \, |u| \, dx
\end{aligned}$$

Referring to the Cavalieri's principle, it follows that

$$\begin{aligned}
I_k * \mu_t(x) &= \frac{1}{\gamma(k)} \int_0^{\infty} \mu_t(B(x, r^{1/(k-n)}))dr \\
&= \frac{(n-k)}{\gamma(k)} \int_0^{\infty} \mu_t(B(x,r)) r^{k-n-1} dr \\
&= I_1 + I_2.
\end{aligned}$$

By the hypothesis we have

$$\mu_t(B(x,r)) \le \mu(B(x,r))^{1-1/p} (M r^{a})^{1/p},$$

and then by Hölder's inequality we have I_1 estimated as follow

$$I_1 \leq \frac{(n-k)}{\gamma(k)} \|u\|_p \, M^{1/p} \int_0^R \Big(\int_{\mathbb{R}^n} \mu_t(B(x,r)) \, dx \Big)^{1/(p-1)} r^{k-n-1+(\alpha/p)} dr.$$

In order to evaluate the integral $\int_{\mathbb{R}^n} \mu_t(B(x,r)) \, dx$, we consider the diagonal

$$D = (\mathbb{R}^n \times \mathbb{R}^n) \cap \{(x,y) : x = y\}$$

and we define for $r > 0$,

$$D_r = (\mathbb{R}^n \times \mathbb{R}^n) \cap \{(x,y) : |x-y| < r\}.$$

Now, we let $F = \chi_{D_r}$ and by the Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mu_t(B(x,r)) \, dx &= \int_{\mathbb{R}^n} \int_{(B(x,r))} d\mu_t(y) \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x,y) \, d\mu_t(y) \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(x,y) \, dx \, d\mu_t(y) \\
&= \int_{\mathbb{R}^n} |B(y,r)| \, d\mu_t(y) \\
&= \omega(n) \, r^n \mu_t(E_t).
\end{aligned}$$

Hence,

$$I_1 \leq \frac{p(n-k)}{\gamma(k)\big(kp-(n-1)\big)} \|u\|_p \, M^{1/p} \omega(n)^{1/p'} R^{k-(n-\alpha)p}.$$

Analogously, by employing Hölder's inequality, we have

$$\begin{aligned}
I_2 &\leq \frac{(n-k)}{\gamma(k)} \|u\|_p \, \mu(E_t)^{1/p} \int_R^\infty \Big(\int_{\mathbb{R}^n} \mu_t(B(x,r)) dx \Big)^{1/p'} r^{k-n-1} dr \\
&\leq \frac{p(n-k)}{(n-kp)\gamma(k)} \|u\|_p \, \mu(E_t) \omega(n)^{1p'} R^{k-n/p}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 + I_2 &\leq \frac{p(n-k)}{\gamma(k)} \omega(n)^{1p'} \|u\|_p \left(\frac{M^{1/p} \mu(E_t)^{1/p'} R^{k-(n-\alpha)/p}}{kp-(n-\alpha)} + \frac{\mu(E_t) R^{k-n/p}}{n-kp} \right) \\
&\leq \frac{p(n-k)\alpha}{\gamma(k)(n-kp)(kp-n+\alpha)} \omega(n)^{1/p'} M^{1/q} \mu(E_t)^{1-1/q} \|u\|_q
\end{aligned} \tag{3.22}$$

The last inequality is obtained by substituting the following value of R in order to achieve the maximum effectiveness of the above inequality

$$R = \Big(\frac{\mu(A_t)}{M} \Big)^{1/\alpha}$$

Consequently, we obtain

$$
\begin{aligned}
t\mu(E_t)^{1/q} &= t\mu(E_t)\,\mu(E_t)^{1/(q-1)} \\
&\le \Big(\frac{p(n-k)\alpha}{\gamma(k)(n-kp)(kp-n+\alpha)}\omega(n)^{1/p'}M^{1/q}\Big)\cdot\Big(\mu(E_t)^{1-1/q}\left\|u\right\|_p\Big) \\
&\le \frac{pq}{\gamma(k)(n-kp)(q-p)}\omega(n)^{1/p'}M^{1/q}.
\end{aligned}
\tag{3.23}
$$

The above estimate states that Riesz potential operator I_k is of weak type (p,q) whenever p and q are numbers such that

$$
1<p<q<\infty, \qquad kp<n. \tag{3.24}
$$

Thus, if $(p_0,q_0),(p,q)$ and (p_1,q_1) are pairs of numbers such that $(p_0,q_0),(p_1,q_1)$ satisfy (3.24) and for $0\le\theta<1$,

$$
\begin{aligned}
1/p &= \frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \\
1/q &= \frac{1-\theta}{q_0}+\frac{\theta}{p_1},
\end{aligned}
\tag{3.25}
$$

then by the Marcinkiewicz Interpolation Theorem, we conclude that I_k is of type (p,q), with

$$
\left\|I_k * u\right\|_{q;\mu}\le C\,M^{1/q}\left\|u\right\|_p.
$$

□

Lemma 3.14. *[5, p.6] Let $u\in W^{1,p}(B(x_0,R))$ be a solution of*

$$
-\ div\ \mathcal{A}(x,\nabla u)=\mu
$$

where μ is a nonnegative Radon measure such that

$$
\mu(B(x_0,r))\le c_0 r^{n-p+\alpha(p-1)}
$$

for all $0<r\le R$. Then for each $0<r<R$ and $\epsilon>0$ we have

$$
\int_{B(x_0,r)}|\nabla u|^p\,dx\le c_1\Big(\big(\frac{r}{R}\big)^{n-p+p\kappa}+\epsilon\Big)\int_{B(x_0,R)}|\nabla u|^p+c_2R^{n-p+p\alpha},
$$

where $c_1=c_1(n,p,\lambda,\Lambda)>0$ and $c_2=c_2(n,p,\lambda,\Lambda,\alpha,c_0,\epsilon)$.

Proof. Without loss of generality we can assume that $r<R/2$. Let h be the $\mathcal{A}$-harmonic function in $B(x_0,R)$ with $u-h\in W_0^{1,p}(B(x_0,R))$. Then

$$
\begin{aligned}
\lambda\int_{B(x_0,r)}|\nabla u|^p\,dx &\le \int_{B(x_0,r)}\mathcal{A}(x,\nabla u)\cdot\nabla u\,dx \\
&= \int_{B(x_0,r)}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla h)\big)\cdot(\nabla u-\nabla h)dx \\
&\quad+\int_{B(x_0,r)}\mathcal{A}(x,\nabla h)\cdot(\nabla u-\nabla h)dx+\int_{B(x_0,r)}\mathcal{A}(x,\nabla u)\cdot\nabla h\,dx.
\end{aligned}
\tag{3.26}
$$

First, we define the Hölder exponent as

$$q = \frac{p(n-p+\alpha p(p-1))}{n-p}, \text{ and thus we have } \quad 1-1/q = \frac{(p-1)(n-p+\alpha p)}{p(n-p+\alpha(p-1))},$$

And now for the first term, since h is $\mathcal{A}$-harmonic with $h-u \in W_0^{1,p}(B(x_0,R))$ and hence quasiminimizers the p-Dirichlet integral, we have

$$\begin{aligned}
&\int_{B(x_0,r)} \big(\mathcal{A}(x,\nabla u) - \mathcal{A}(x,\nabla h)\big)\cdot(\nabla u - \nabla h)\,dx \\
&\qquad = \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot(\nabla u - \nabla h)\,dx + \int_{B(x_0,r)} \mathcal{A}(x,\nabla h)\cdot(\nabla(u-h))\,dx \\
&\qquad = \int_{B(x_0,R)} (u-h)\,d\mu \\
&\qquad \le \mu(B(x_0,R))^{1-1/q}(\int_{B(x_0,R)} |u-h|^q \; d\mu)^{1/q} \quad \text{(Hölder's inequality)} \\
&\qquad \le c\,\mu(B(x_0,R))^{1-1/q}(\int_{B(x_0,R)} |I_1 * |\nabla u - \nabla h||^q \; d\mu)^{1/q} (\text{ Representation Theorem}) \\
&\qquad \le c\,R^{(p-1)(n-p+\alpha p)/p}(\int_{B(x_0,R)} |\nabla u - \nabla h|^p \,dx\,)^{1/p} \quad \text{(Theorem 2.8)} \\
&\qquad \le c\,R^{n-p+\alpha p} + \frac{\lambda}{2}\epsilon \int_{B(x_0,R)} |\nabla u|^p \; dx.
\end{aligned} \tag{3.27}$$

The last inequality is obtained by applying the Young's inequality (2.1). The remaining terms

$$\begin{aligned}
&\int_{B(x_0,r)} \mathcal{A}(x,\nabla h)\cdot(\nabla u - \nabla h)\,dx + \int_{B(x_0,r)} \mathcal{A}(x,\nabla u)\cdot\nabla h\,dx \\
&\qquad \le \int_{B(x_0,r)} |\mathcal{A}(x,\nabla h)|\,|\nabla u - \nabla h|\,dx + \int_{B(x_0,r)} |\mathcal{A}(x,\nabla u)|\,|\nabla h| \; dx \\
&\qquad \le \Lambda \int_{B(x_0,r)} |\nabla h|^{p-1}\,|\nabla u - \nabla h|\,dx + \Lambda \int_{B(x_0,r)} |\nabla u|^{p-1}\,|\nabla h| \; dx \\
&\qquad \le \Lambda \int_{B(x_0,r)} |\nabla h|^{p-1}\,|\nabla u| + |\nabla h|\,|\nabla u|^{p-1} \; dx \\
&\qquad \le \frac{\lambda}{2}\epsilon \int_{B(x_0,r)} |\nabla u|^p \; dx + c \int_{B(x_0,r)} |\nabla h|^p \; dx \\
&\qquad \le \frac{\lambda}{2}\epsilon \int_{B(x_0,r)} |\nabla u|^p \; dx + c\big(\frac{r}{R}\big)^{n-p+p\kappa} \int_{B(x_0,R)} |\nabla h|^p \; dx \\
&\qquad \le \frac{\lambda}{2}\epsilon \int_{B(x_0,r)} |\nabla u|^p \; dx + c\big(\frac{r}{R}\big)^{n-p+p\kappa} \int_{B(x_0,R)} |\nabla u|^p \; dx.
\end{aligned} \tag{3.28}$$

Here, we apply the Young's inequality with $\epsilon > 0$ again and the fact that we employ the fact that the $\mathcal{A}$-harmonic function h minimizers the p-Dirichlet integral and the result from Lemma 3.12. The above estimates allow us to conclude the claim.

$\square$

Theorem 3.15. *[5, p.3] Let κ be the number given by in Theorem 3.11. Suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution of*

$$- \ div \ \mathcal{A}(x, \nabla u) = \mu,$$

where μ is a nonnegative Radon measure such that there are constants $M > 0$ and $0 < \alpha < \kappa$ with

$$\mu(B(x,r)) \le M r^{n-p+\alpha(p-1)}$$

whenever $B(x, 3r) \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$. Morever, in the case of the p-Laplace the claim holds for $\kappa(n, p, 1, 1)$ and any $\alpha < 1$.

Proof. If $B(x_0, 4R) \subset \Omega$, then by appealing to Lemma 3.12 we get

$$\int_{B(x_0,r)} |\nabla u|^p \ dx \le c\Big(\frac{r}{R}\Big)^{n-p+p\alpha}$$

for $r < R$. Thus, by the Dirichlet growth theorem 2.6, we conclude $u \in C^{0,\alpha}(\Omega)$.

$\square$

3.4 Removability

Definition 3.16. *(Removable sets) A closed set $E \subset \Omega$ is said to be removable for $\mathcal{A}$- harmonic functions in $\mathcal{F}$, if every $u \in \mathcal{F}$ that is $\mathcal{A}$- harmonic in $\Omega \backslash E$ is A-harmonic in the whole of Ω.*

Lemma 3.17. *[6, p.615] If u is nonnegative, continuous function on Ω, $I \subset \Omega$ is a closed set such that $u = 0$ on I and u is a p-subsolution in $\Omega \setminus I$, then u us a p-subsolution in Ω.*

Lemma 3.18. *[5, p.4] Let $K \subset \Omega$ be compact. Suppose that ψ is a continuous function with*

$$|\psi(x) - \psi(y)| \le M\, |x - y|^{\alpha} \ \text{for all } x \in K \text{ and } y \in \Omega, \tag{3.29}$$

where $M > 0$ and $\alpha > 0$. Let $u = \hat{R}^{\psi}$ and

$$\mu = - \ div \ \mathcal{A}(x, \nabla x) \tag{3.30}$$

Then

$$\mu(B(x,r)) \le c r^{n-p+\alpha(p-1)}$$

for all $r < r_0 = \frac{1}{64}\ dist(K, \partial\Omega)$ and $x \in K$, here $c = c(n, p, \lambda, \Lambda, M, \alpha) > 0$.

Proof. We consider the contact set

$$I = \{x \in \Omega : \psi(x) = u(x)\}$$

Let $x_0 \in I$. We may assume that $u(x_0) = 0 = \psi(x_0)$ by adding constant. Consider if a radius $r \le \frac{1}{8}\operatorname{dist}(x_0, \partial\Omega)$ and

$$\gamma_0 = \operatorname{osc}(\psi, B(x_0, 8r)) \ge 0,$$

(i.) First, we observe that $u + \gamma_0$ is a nonnegative supersolution in $B(x_0, 8r)$ and $(u-\gamma_0)^+$ is a subsolution in $B(x_0, 8r)$. The second part of the claim follows as $\psi - \gamma_0 \le 0$ on $I \cap B(x_0, 8r)$ implies $u - \gamma_0 \le 0$ on $I \cap B(x_0, 8r)$ as $u = \psi$ on I. The fact that u is a weak solution in $B(x_0, 8r)\backslash I$ infers that $u - \gamma_0$ is a weak solution in $B(x_0, 8r)\backslash I$, and thus, $(u-\gamma_0)_+$ is also a weak solution in $B(x_0, 8r)\backslash I$. Since $u - \gamma_0 = 0$ on $I \cup B(x_0, 8r)$, by Lemma 3.17, we conclude $(u-\gamma_0)^+$ is a subsolution in $B(x_0, 8r)$.

(ii.) Following from the previous result, we apply the weak Harnack inequalities to a positive subsolution $(u-\gamma_0)^+$ to get

$$\begin{aligned} \sup_{B(x_0,r)}(u-\gamma_0) = \sup_{B(x_0,r)}(u-\gamma_0)^+ &\le c\Big(\fint_{B(x_0,2r)} \big|(u-\gamma_0)^+\big|^{p-1}\,dx\Big)^{1/(p-1)} \\ &\le c\Big(\fint_{B(x_0,2r)} (u+\gamma_0)^{p-1}\Big)^{1/(p-1)} \\ &\le c\inf_{B(x_0,r)}(u+\gamma_0) = c\,(\inf_{B(x_0,r)} u + \gamma_0) \le c\gamma_0. \end{aligned} \tag{3.31}$$

The last inequality follows from the fact that $\inf_{B(x_0,r)} u \le u(x_0) = 0$. Since $u \ge \psi \ge -\gamma_0$ in $B(x_0, 8r)$ we conclude

$$\begin{aligned} \operatorname{osc}(u, B(x_0, r)) = \sup_{B(x_0,r)} u - \inf_{B(x_0,r)} u &\le c\gamma_0 + \gamma_0 + \sup_{B(x_0,r)}(-u) \\ &\le c\gamma_0 = c\operatorname{osc}(\psi, B(x_0, 8r)). \end{aligned} \tag{3.32}$$

The last inequality holds since $-u \le \gamma_0$ in $B(x_0, 8r)$ implies $\sup_{B(x_0,r)}(-u) \le \gamma_0$.

(iii.) Next, we estimate $\mu(B(x_0, r))$. Let $r \le \frac{1}{32}\operatorname{dist}(x_0, \partial\Omega)$ and let $\eta \in C_0^\infty(B(x_0, 2r))$ be a nonnegative cut-off function with $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, r)$ and $|\nabla\eta| \le 2/r$. Applying the Caccioppoli estimate to

$(u - \sup_{B(x_0,2r)} u)$, we obtain

$$
\begin{aligned}
\mu(B(x_0,r)) &= \int_{B(x_0,r)} 1\, d\mu \le \int_{B(x_0,2r)} \eta^p\, d\mu = p \int_{B(x_0,2r)} \eta^{p-1} \mathcal{A}(x,\nabla u)\cdot \nabla\eta\, dx \\
&\le p \int_{B(x_0,2r)} \eta^{p-1}\, |\mathcal{A}(x,\nabla u)|\, |\nabla\eta|\; dx \\
&\le p\Lambda \int_{B(x_0,2r)} \eta^{p-1}\, |\nabla u|^{p-1}\, |\nabla\eta|\; dx \\
&\le c\Big(\int_{B(x_0,2r)} \eta^p\, |\nabla u|^p\, dx\Big)^{(p-1)/p} \Big(\int_{B(x_0,2r)} |\nabla\eta|^p\, dx\Big)^{1/p} \\
&\quad \text{(by Hölder's inequality)} \\
&= c\Big(\int_{B(x_0,2r)} \eta^p |\nabla(u - \sup_{B(x_0,2r)} u)|^p dx\Big)^{(p-1)/p} \Big(\int_{B(x_0,2r)} |\nabla\eta|^p\, dx\Big)^{1/p} \\
&\quad \text{(by Caccioppoli estimate)} \\
&\le c\Big(p^p(\int_{B(x_0,2r)} \Big| u - \sup_{B(x_0,2r)} u\Big|^p\, |\nabla\eta|^p\, dx)\Big)^{(p-1)/p} \Big(\int_{B(x_0,2r)} |\nabla\eta|^p\, dx\Big)^{1/p} \\
&\le c \operatorname{osc}(u, B(x_0,2r))^{p-1} \int_{B(x_0,2r)} |\nabla\eta|^p\, dx \\
&\le c \operatorname{osc}(u, B(x_0,2r))^{p-1}\, |B(x_0,2r)|\, \big(\frac{2}{r}\big)^p \\
&\le c\, r^{n-p} \operatorname{osc}(u, B(x_0,2r))^{p-1} \\
&\le c\, r^{n-p} \operatorname{osc}(\psi, B(x_0,16r))^{p-1}.
\end{aligned}
$$

(iv.) Now if $x_0 \in I$ such that

$$\operatorname{dist}(x_0, K) \le r \le 2r_0,$$

we have the estimate

$$
\begin{aligned}
\mu(B(x_0,r)) &\le c\, r^{n-p} \operatorname{osc}(\psi, B(x_0,16r))^{p-1} \\
&\le c\, r^{n-p} (\sup_{B(x_0,16r)} \psi - \inf_{B(x_0,16r)} \psi)^{p-1} \\
&\le c\, M^{p-1}\, r^{n-p+\alpha(p-1)}.
\end{aligned}
\tag{3.33}
$$

(v.) Finally, for any point $x_0 \in K$ and $r \le r_0$, there happen two possible cases. If $B(x_0,r) \cap I = \emptyset$, due to the fact that $B(x_0,r) \subset \{u > \psi\}$ and u is a p-solution in $\{u > \psi\}$, we conclude $\mu(B(x_0,r)) = 0$. Otherwise, there is a point $y \in B(x_0,r) \cap I$ and by by step (iv.) we have the estimate

$$\mu(B(x_0,r)) \le B(y,2r) \le c\, r^{n-p+\alpha(p-1)}$$

This completes the proof.

$\square$

Theorem 3.19. *[5, p.2] Let $E \subset \Omega$ be closed and $s > 0, s > n-p$. Suppose that u is a continuous function in $\Omega, \mathcal{A}$-harmonic in $\Omega \backslash E$ such that*

$$|u(x_0) - u(y)| \leq C\,|x_0 - y|^{(s+p-n)/(p-1)} \tag{3.34}$$

for all $y \in \Omega$ and $x_0 \in E$. If E is of s-Hausdorff measure zero, then u is $\mathcal{A}$-harmonic in Ω.

Proof. First, we fix a set $D \Subset \Omega$. Let $v = \hat{R}^u(D)$ and let

$$\mu = -\operatorname{div} \mathcal{A}(x, \nabla u)$$

Now let $K \subset E$ be compact. Since K is of $s = n - p + \alpha(p-1)$ Hausforff measure zero, we may cover K by balls $B(x_j, r_j)$ so that

$$\mu(K) \leq \sum_j \mu(B(x_j, r_j)) \leq c \sum_j r_j^s \leq \varepsilon,$$

where $\varepsilon > 0$ is given. By sending ε to zero, we obtain $\mu(K) = 0$. Consequently, $\mu(E) = 0$ and therefore $\mu = 0$, which means that v is $\mathcal{A}$-harmonic in D. To this end, let

$$w = \underline{\hat{R}}^u(D).$$

Analogously, we find that w is $\mathcal{A}$-harmonic in D. Since on the boundary ∂D, we have $v = u = w$, and thus, $v = w$ in D by the uniqueness of $\mathcal{A}$-harmonic functions. Consequently, we conclude u is $\mathcal{A}$-harmonic in D since $w \leq u \leq v = w$, and the theorem follows. $\square$

Corollary 3.20. *[5, p.2] Suppose that $u \in C^{0,\alpha}(\Omega), 0 < \alpha \leq 1$, is $\mathcal{A}$-harmonic in $\Omega \setminus E$. If E is a closed set of $n - p + \alpha(p-1)$ Hausdorff measure zero, then u is $\mathcal{A}$-harmonic in Ω.*

Theorem 3.21. *[5, p.2] Let κ be as above and $0 < \alpha < \kappa$. Suppose that $E \subset \Omega$ is a closed set with positive $n - p + \alpha(p-1)$ Hausdorff measure. The there is $u \in C^{0,\alpha}(\Omega)$, but does not have an $\mathcal{A}$-harmonic extension to Ω.*

Proof. Let κ be the number in Theorem 3.11 and $K \subset E$ be compact with $\mathcal{H}^{n-p+\alpha(p-1)}(K) > 0$. The Frostman's Lemma 2.3 provides us a nonnegative Radon measure μ living on K with $\mu(K) > 0$ and $\mu(B(x,r)) \leq r^{n-p+\alpha(p-1)}$. Any solutions $u \in W^{1,p}_{loc}(\Omega)$ to

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu$$

is $\mathcal{A}$-harmonic in $\Omega \setminus E$ and $u \in C^{0,\alpha}(\Omega)$ by Theorem 3.15, but u fails to have an $\mathcal{A}$-harmonic extension to Ω, since $\mu(K) > 0$. $\square$

Corollary 3.22. *[5, p.2] Let $0 < \alpha < 1$. A closed set E is removable for α-Hölder continuous p-harmonic functions if and only if E is of $n - p + \alpha(p-1)$ Hausdorff measure zero.*

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