Assignment 1

Consider a 3D domain in which a Sphere and a Torus exist. The equation of the sphere is given by $x^2 + y^2 + z^2 \leq k^2$. $k > 0$ is the radius of the sphere. The equation of the Torus is $\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 \leq r^2$. R>0 is the major radius, and r>0 is the minor radius of the torus.

1. If both geometries have their center at origin (0,0,0), using Monte-Carlo Integration and uniform random sampling, estimate the volume of intersection of sphere and torus for the following cases:

   Case a: k=1, R = 0.75 and r = 0.4

   Case b: k= 1, R = 0.5 and r = 0.5

   Use any bounding box 'B' that fully contains the intersection; report the box you used.

2. Considering both case a and b, how would the estimate and error change, if a deterministic sequence such as $X_{n+1} = mX_n(1 - X_n)$ were used for sampling. Implement this approach and critically evaluate its validity for Monte Carlo integration. Here, m = 3.8, you can use different X_0 for each coordinate to generate three independent sequences.
3. Now let us consider an off-center Torus, with equation $\left(\sqrt{(x - x_c)^2 + (y - y_c)^2} - R\right)^2 + (z - z_c)^2 \leq r^2$ existing in the same domain as the sphere $x^2 + y^2 + z^2 \leq k^2$. x_c, y_c, z_c are the centers of the Torus. x_c, y_c, z_c are 0, 0, 0.1 respectively, k=1, R = 0.75 and r = 0.4.
   a. Using Monte-Carlo Integration and uniform random sampling, estimate the volume of intersection of sphere and torus.
   b. Based on the geometric knowledge of off-centered nature of the Torus (along z-axis), propose and implement a mixture proposal by sampling from B with probability p and from a smaller box S centered at (0, 0, 0.1) with probability 1 – p to estimate the volume of intersection and error. What observations did you make? Did you identify any flaws in the mixed proposal?

Support your answers with figures, equations and illustrations wherever possible. In the video, clearly walk us through assumptions, what has been done, its intention and purpose. Note: Do not use Latin Hypercube, Sobol/Halton, or stratification in this assignment. Applying importance weights is not required, but feel free to discuss the implications. Recommended sample size is 100000.