Bayesian Semiparametric Regression

A Tutorial

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1 Introduction

1.1 Aims and Scope

- Markov chain Monte Carlo simulations (MCMC) are the de facto standard of Bayesian inference in applied statistics (although approximate forms such as variational inference are gaining in popularity).
- MCMC was also the foundation for the success of Bayesian inference since the 1990s.
- It comes with a number of specific challenges but also has distinct advantages for complex statistical models.
- Advantages are particularly beneficial for semiparametric regression models that can be represented as directed acyclic graphs due to their hierarchical model formulation.
- This representation can also be exploited in computations and software, as we will demonstrate the in the practical parts of this tutorial using the Liesel software package.

1.2 Instructors for the Course

- Johannes Brachem (University of Göttingen)
- Gianmarco Callegher (University of Göttingen)
- Thomas Kneib (University of Göttingen)
- Hannes Riebl (University of Göttingen)
- Paul F. V. Wiemann (University of Wisconsin-Madison)

1.3 Outline

Wednesday morning: Bayesian Inference with Markov Chain Monte Carlo Simulations

- 9:00 - 10:00 Lecture: Bayesian Inference with MCMC I (Thomas)
- 10:00 - 11:00 Exercises: An Introduction to Scientific Computing with Python (Paul)
- 11:00 - 11:30 Coffee break
- 11:30 - 12:30 Lecture: Bayesian Inference with MCMC II (Thomas)
- 12:30 - 13:30 Exercises: MCMC with Liesel-Goose (Paul)

Wednesday afternoon: Bayesian Additive Regression

- 15:00 - 16:00 Lecture: Bayesian Additive Regression (Thomas)
- 16:00 - 16:30 Exercises: Model development with Liesel I (Johannes)
- 16:30 – 17:00 Coffee break
- 17:00 – 19:30 Exercises: Model development with Liesel II (Johannes)

Thursday morning: Bayesian Distributional Regression

- 9:00 - 10:00 Lecture: Bayesian Distributional Regression (Thomas)
- 10:00 - 11:00 Exercises: MCMC for Distributional Regression with Liesel-Goose (Johannes)
- 11:00 - 11:30 Coffee break
- 11:30 - 13:30 Exercises: Integrating Python and R with Quarto and Reticulate (Hannes)

2 Bayesian Inference with Markov Chain Monte Carlo Simulations

2.1 Bayesian Inference with MCMC I

Aims of this section:

- Introduce the foundations of Bayesian inference and compare it to frequentist maximum likelihood.
- Motivate how Markov chain Monte Carlo (MCMC) simulations provide numerical access to the posterior distributions.
- Discuss practical aspects of working with MCMC simulations.

Bayes' theorem:

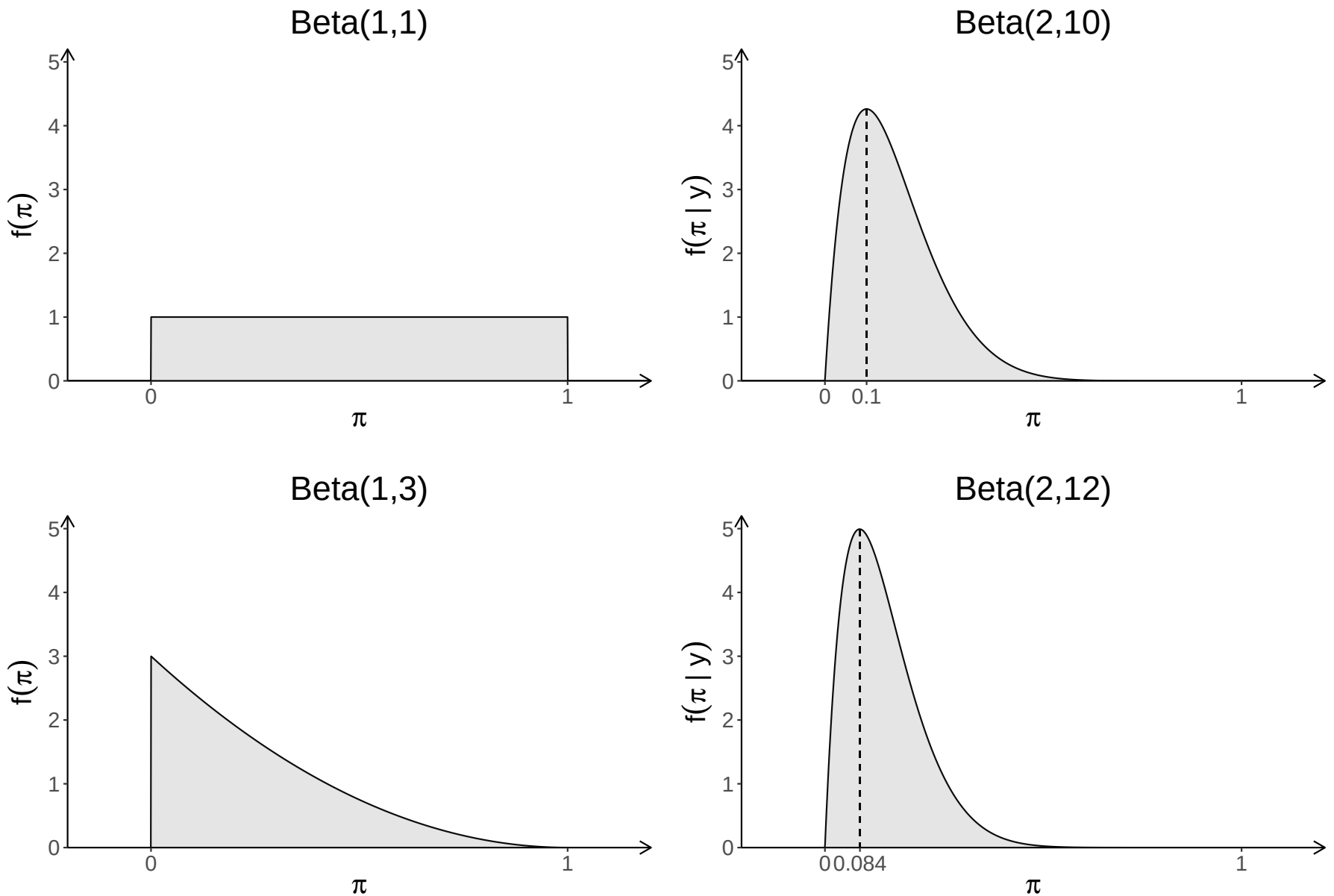
- Two central components of a Bayesian model formulation:
  - Observation model $f(\boldsymbol{y}|\boldsymbol{\vartheta})$ which describes how the data $\boldsymbol{y}$ are generated for given model parameters $\boldsymbol{\vartheta}$.
  - Prior distribution $f(\boldsymbol{\vartheta})$ representing prior beliefs about the parameter vector $\boldsymbol{\vartheta}$
- Bayesian learning updates prior beliefs on $\boldsymbol{\vartheta}$ based on information in the data $\boldsymbol{y}$ using Bayes' theorem

$$f(\boldsymbol{\vartheta}|\boldsymbol{y}) = \frac{f(\boldsymbol{y}|\boldsymbol{\vartheta})f(\boldsymbol{\vartheta})}{f(\boldsymbol{y})} = \frac{f(\boldsymbol{y}|\boldsymbol{\vartheta})f(\boldsymbol{\vartheta})}{\int f(\boldsymbol{y}|\boldsymbol{\vartheta})f(\boldsymbol{\vartheta})d\boldsymbol{\vartheta}}$$

where $f(\boldsymbol{y})$ is the marginal density of the data.

Example: Bayesian inference for the success probability in a Bernoulli trial:

- Data $y_i \overset{\text{i.i.d.}}{\sim} \text{Be}(\pi)$ with unknown success probability $\pi \in (0, 1)$.
- We consider $n = 10$ trials with one success and nine failures such that the maximum likelihood estimate is
$$\hat{\pi} = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{10}.$$
- As a prior distribution, we choose the beta distribution with parameters $a > 0$ and $b > 0$.
- In this case, the posterior an be worked out analytically and turns out to be a beta distribution with parameters
$$\tilde{a} = a + \sum_{i=1}^{n} y_i \qquad \tilde{b} = b + n - \sum_{i=1}^{n} y_i.$$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f(π)
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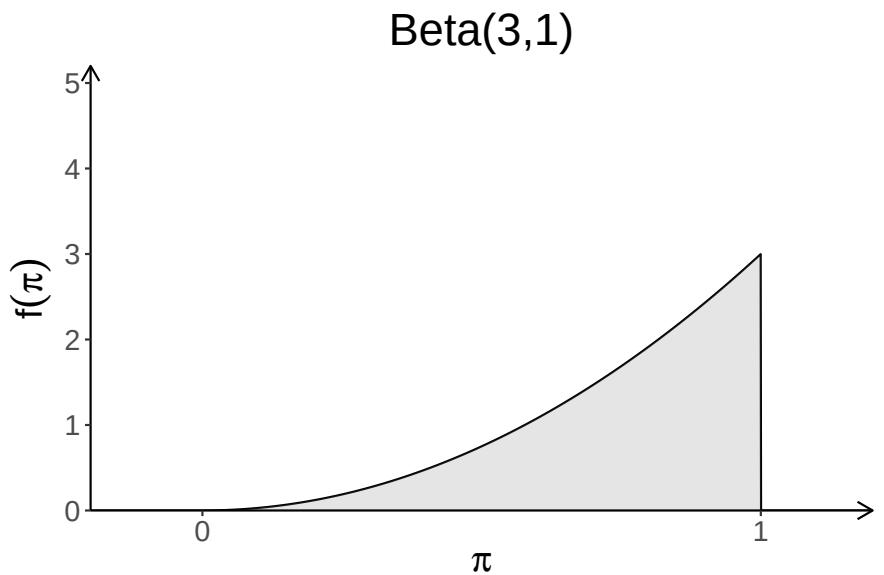
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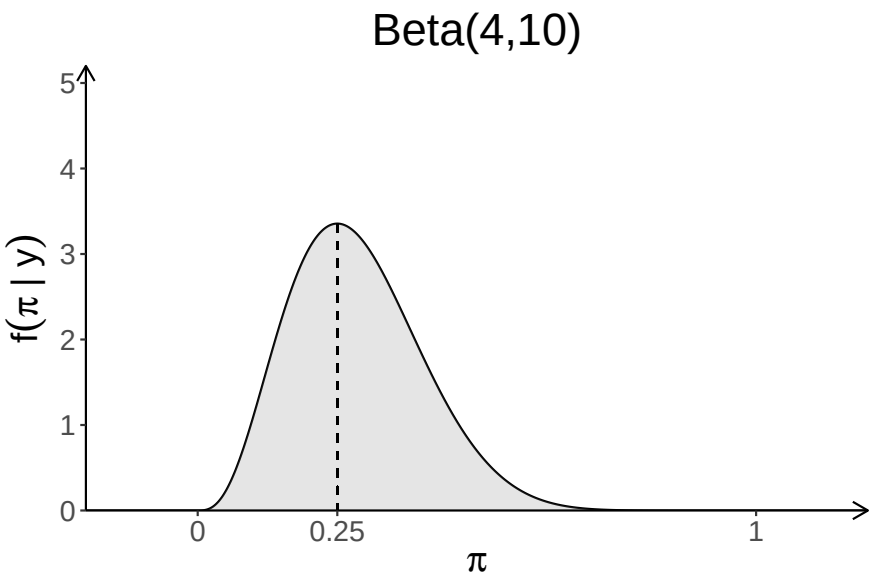
Beta(4,10)
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π

Relation to maximum likelihood estimation:

- If the prior distribution is flat, i.e.

$$f(\boldsymbol{\vartheta}) \propto \text{const},$$

the posterior is proportional to the likelihood:

$$f(\boldsymbol{\vartheta}|\boldsymbol{y}) = \frac{f(\boldsymbol{y}|\boldsymbol{\vartheta})f(\boldsymbol{\vartheta})}{f(\boldsymbol{y})} \propto f(\boldsymbol{y}|\boldsymbol{\vartheta})f(\boldsymbol{\vartheta}) \propto f(\boldsymbol{y}|\boldsymbol{\vartheta}).$$

- Hence the mode of the posterior coincides with the maximum likelihood estimate.
- In general,
  - the likelihood is a central part of Bayes' theorem that quantifies the information coming from the data and
  - the posterior forms a compromise between data (likelihood) and prior beliefs (prior).

Priof beliefs and prior elicitation:

- Main conceptual difference between likelihood-based and Bayesian inference: Coming up with a sensible prior distribution.
- The prior should reflect your prior beliefs about the parameter of interest.
- Very common practice:
  - Pick a mathematically convenient class of distributions for the prior and
  - only decide on the parameter of this prior distribution.
- For example, one can formulate belief statements such as

$$\mathbb{P}(c_1 \leq \vartheta \leq c_2) = 1 - \alpha$$

  where c_1 and c_2 are pre-specified constants from which the prior parameters are determined.

- There are a variety of other approaches to decide on the prior, such as
  - flat priors where
$$f(\boldsymbol{\vartheta}) \propto \text{const},$$
  - noninformative priors where the data should alone determine the posterior,
  - reference priors,
  - etc.
- It is also very common to run analyses for a variety of different priors to study prior sensitivity.

- A typical discussion on Bayesian inference is that
  - frequentist inference assumes a true, fixed parameter value whereas
  - Bayesian inference assumes the parameter to be a random variable.
- This is, in general, misleading since the prior is merely used to reflect prior (un)certainty about the parameter of interest.

Numerically assessing the posterior:

- The ultimate outcome of a Bayesian data analysis is the posterior, reflecting posterior beliefs about the parameter of interest.
- This is often reduced to point estimates, credible intervals, etc.
- Unfortunately, in most models of reasonable complexity, the posterior is not analytically accessible.
- In particular, the normalizing constant

$$f(\boldsymbol{y}) = \int f(\boldsymbol{y}|\boldsymbol{\vartheta}) f(\boldsymbol{\vartheta}) d\boldsymbol{\vartheta}$$

  is unknown and for models of at least moderate complexity it can also not easily be numerically determined.

- If we could obtain random samples $\boldsymbol{\vartheta}^{[t]}$, $t = 1, \ldots, T$ from the posterior, we could empirically estimate any quantity of interest at any desired level of precision:
  - Posterior expectations can be determined based on the law of large numbers via

$$\frac{1}{T} \sum_{t=1}^{T} g(\boldsymbol{\vartheta}^{[t]}) \rightarrow \mathbb{E}(g(\boldsymbol{\vartheta})|\boldsymbol{y}).$$

  - Similar statements exist for empirical quantiles.
  - Even the complete posterior could be estimated based on histograms or kernel density estimates.
- Markov chain Monte Carlo simulations are a way of simulating from the unknown and numerically untractable posterior!

Basic principles of MCMC:

- Generate a Markov chain that iteratively samples new values $\boldsymbol{\vartheta}^{[t]}$ given current values $\boldsymbol{\vartheta}^{[t-1]}$.
- The transition probabilities are chosen such that the Markov chain converges to the posterior as its stationarity distribution.
- Important consequences:
  - The samples $\boldsymbol{\vartheta}^{[t]}$, $t = 1, \ldots, T$ are not independent but feature serial correlation.
  - The Markov chain has to converge such that early values with small index t are not yet realisations from the posterior.

- Consider a discrete time Markov chain with a discrete state space $\mathcal{S}$ of size $S = |\mathcal{S}|$
- Transitions from the current state to future states can then be characterized by a transition probability matrix $\boldsymbol{P}$.
- Under some regularity conditions, one can show that

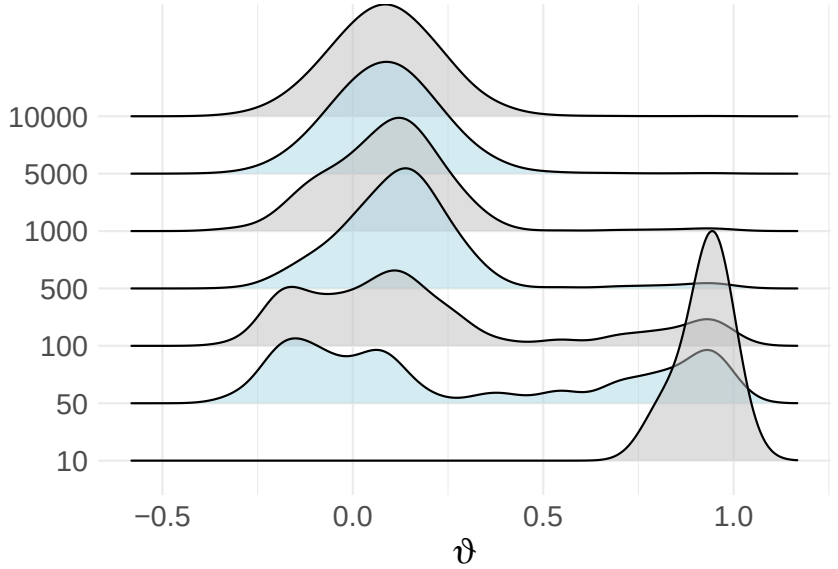
$$\lim_{t \to \infty} \boldsymbol{P}^t = \boldsymbol{P}^\infty$$

  i.e. the repeated application of the transition probability matrix converges to a limiting matrix and each row in this matrix has exactly the same entries such that

$$\boldsymbol{P}^\infty = \begin{pmatrix} \boldsymbol{\pi} \\ \vdots \\ \boldsymbol{\pi} \end{pmatrix}$$

  where $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_S)$ is the stationary distribution of the Markov chain.
- In Bayesian inference, $\boldsymbol{\pi}$ should be the posterior distribution.

- Mathematical theory ensures convergence towards the stationary distribution in the limit, but in practice convergence has to be monitored appropriately.
- The convergence behaviour also depends on the starting values
  $\Rightarrow$ Remove burn in period.
- Samples from a Markov chain exhibit serial dependence that has to be accounted for
  $\Rightarrow$ thin out the Markov chain to achieve approximate independence.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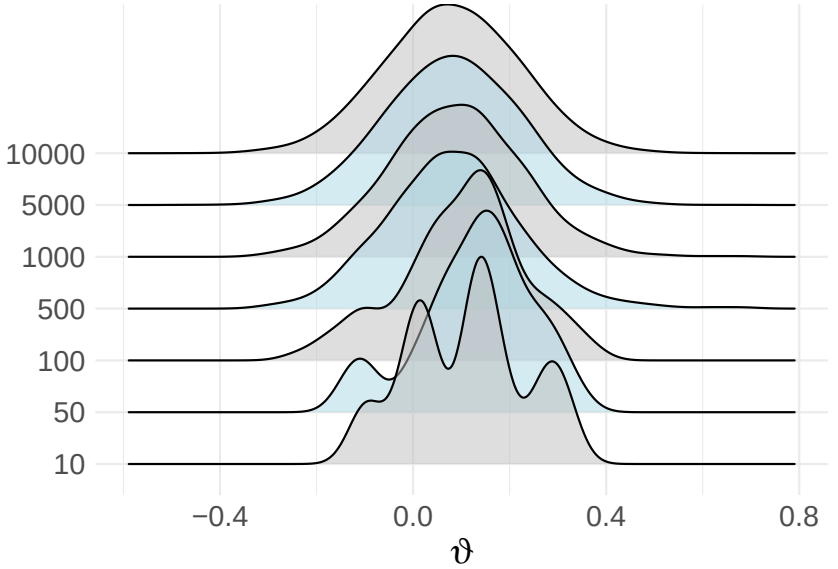
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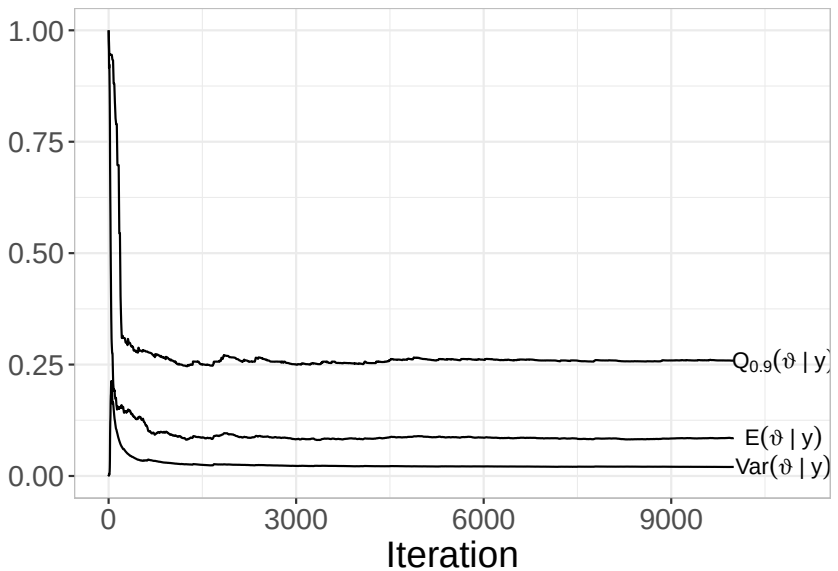
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Q0.9(ϑ | y)
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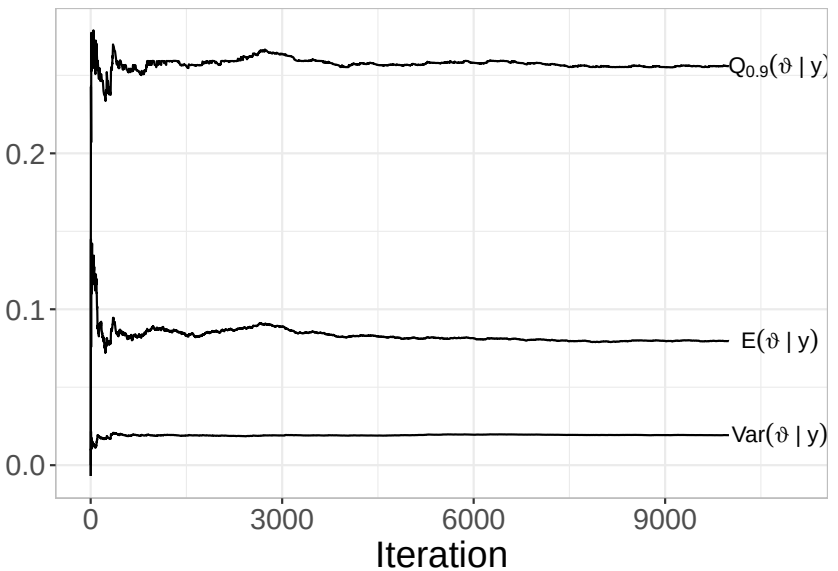
Iteration


0.2
0.1
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Q0.9(ϑ | y)
E(ϑ | y)
Var(ϑ | y)
0
3000
6000
9000
Iteration

- A generic MCMC algorithm:
  - generate proposals for a new value of $\boldsymbol{\vartheta}^*$ from a so-called proposal density that depends on the data and the current state of all parameters.
  - accept the proposal only with a certain probability that depends on the posterior as well as proposal density. If the proposal is not accepted, the parameter remains in its current state.
- Instead of considering all parameters simultaneously, this is typically done in turn for sub-blocks of parameters, i.e. we split $\boldsymbol{\vartheta}$ into

$$\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}_1', \ldots, \boldsymbol{\vartheta}_s', \ldots, \boldsymbol{\vartheta}_S')'.$$

- The blocks typically reflect structures from the model formulation.

- A generic MCMC algorithm:
  - propose a new value for $\boldsymbol{\vartheta}_s$ from a proposal density

$$q_s(\boldsymbol{\vartheta}_s^*|\boldsymbol{\vartheta}_1^{[t]},\ldots,\boldsymbol{\vartheta}_{s-1}^{[t]},\boldsymbol{\vartheta}_s^{[t-1]},\ldots,\boldsymbol{\vartheta}_S^{[t-1]},\boldsymbol{y})$$

  - accept the proposed new value $\boldsymbol{\vartheta}_s^*$ with probability

$$\alpha(\boldsymbol{\vartheta}_s^*|\boldsymbol{\vartheta}_s^{[t-1]}) = \min\left\{\frac{f(\boldsymbol{\vartheta}_s^*|\boldsymbol{\vartheta}_{-s}^{[t-1]},\boldsymbol{y})q_s(\boldsymbol{\vartheta}_s^{[t-1]}|\boldsymbol{\vartheta}_1^{[t]},\ldots,\boldsymbol{\vartheta}_{s-1}^{[t]},\boldsymbol{\vartheta}_s^*,\ldots,\boldsymbol{\vartheta}_S^{[t-1]})}{f(\boldsymbol{\vartheta}_s^{[t-1]}|\boldsymbol{\vartheta}_{-s}^{[t-1]},\boldsymbol{y})q_s(\boldsymbol{\vartheta}_s^*|\boldsymbol{\vartheta}_1^{[t]},\ldots,\boldsymbol{\vartheta}_{s-1}^{[t]},\boldsymbol{\vartheta}_s^{[t-1]},\ldots,\boldsymbol{\vartheta}_S^{[t-1]})},1\right\}$$

    otherwise set $\boldsymbol{\vartheta}_s^{[t]} = \boldsymbol{\vartheta}_s^{[t-1]}$.

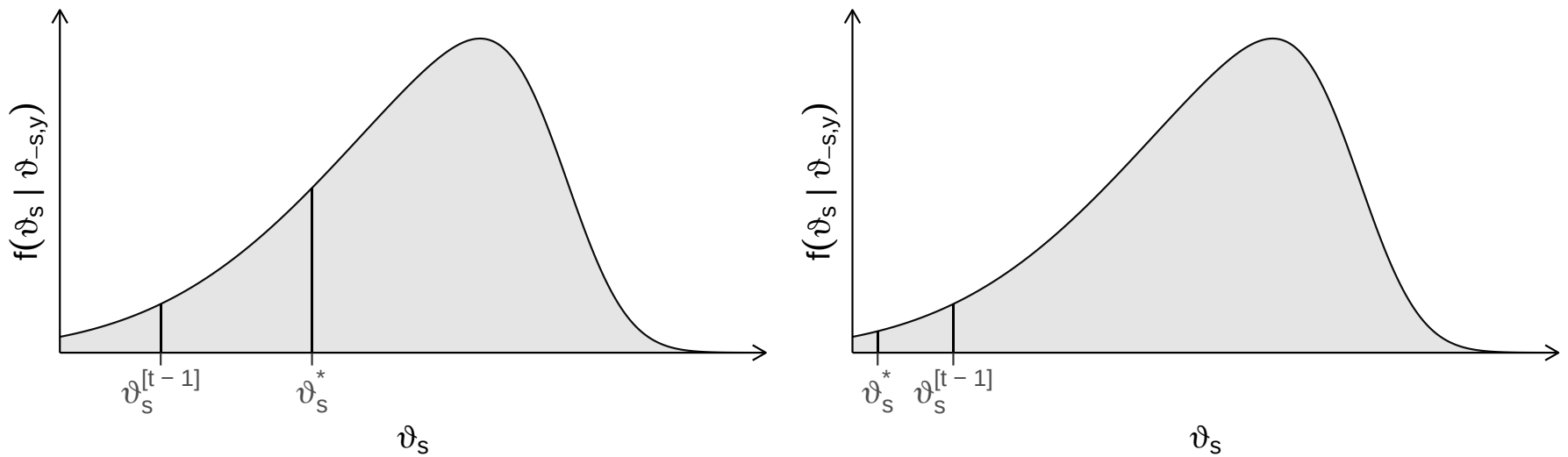
- The full conditional distribution

$$f(\boldsymbol{\vartheta}_s|\boldsymbol{\vartheta}_{-s}^{[t-1]}, \boldsymbol{y}) = f(\boldsymbol{\vartheta}_s|\boldsymbol{\vartheta}_1^{[t]}, \ldots, \boldsymbol{\vartheta}_{s-1}^{[t]}, \boldsymbol{\vartheta}_{s+1}^{[t-1]}, \ldots, \boldsymbol{\vartheta}_S^{[t-1]}, \boldsymbol{y})$$

  is proportional to the posterior, i.e.

$$f(\boldsymbol{\vartheta}_s|\boldsymbol{\vartheta}_{-s}^{[t-1]}, \boldsymbol{y}) \propto f(\boldsymbol{\vartheta}|\boldsymbol{y}).$$

- Intuition for the acceptance probability:



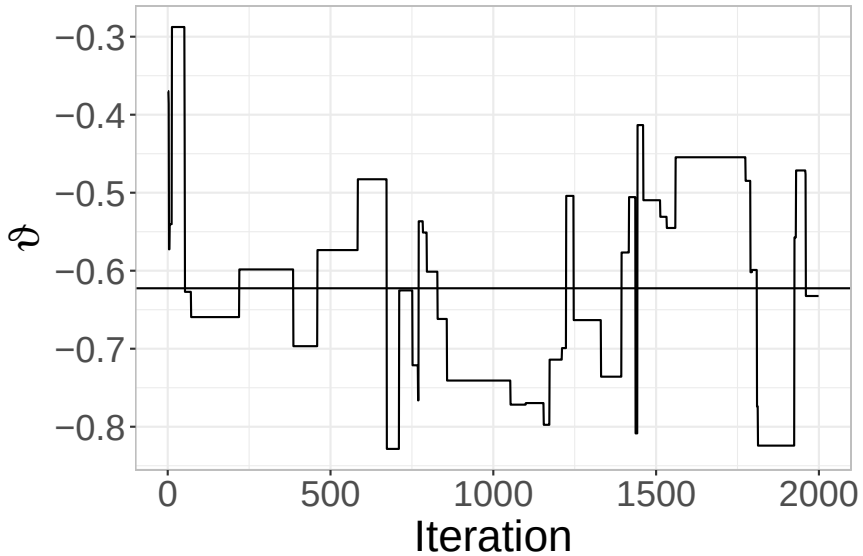
Monitoring MCMC:

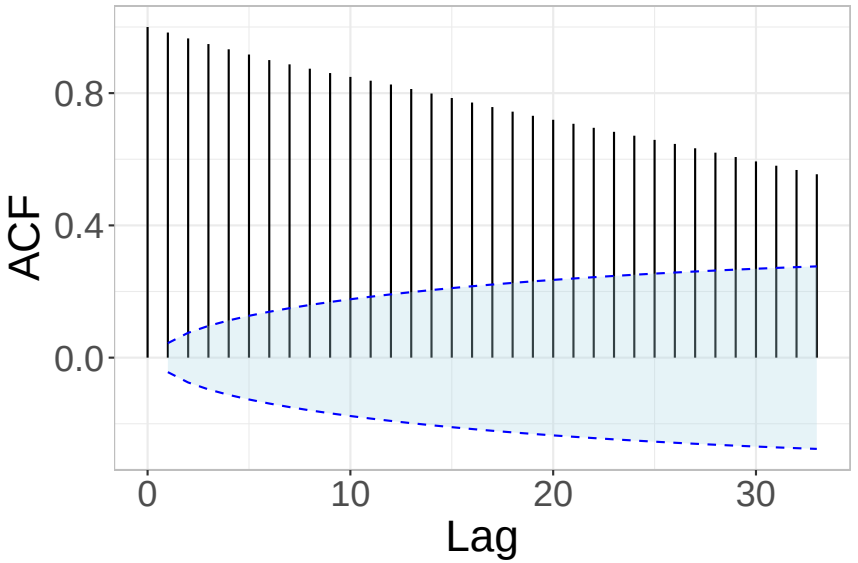
- Early versions of MCMC often relied on random walk proposals

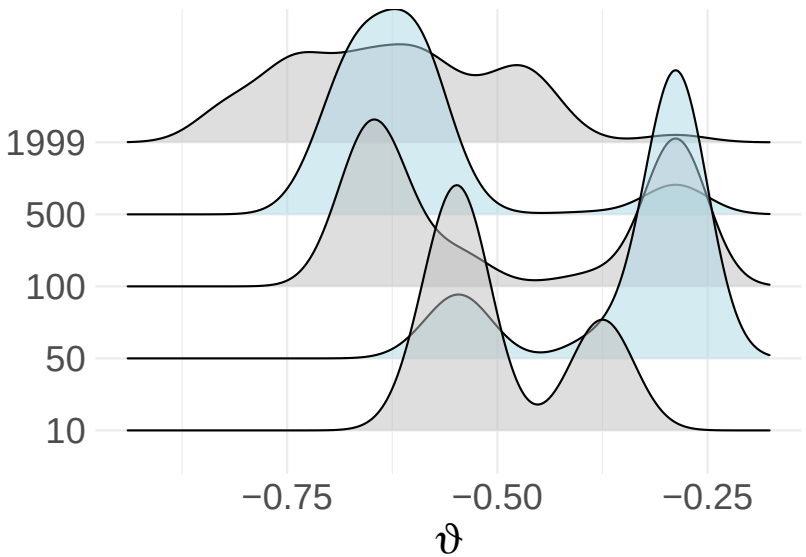
$$\boldsymbol{\vartheta}_s^* = \boldsymbol{\vartheta}_s^{[t-1]} + \boldsymbol{u}_t, \qquad \boldsymbol{u}_t \sim \mathrm{N}(\mathbf{0}, \tau_u^2 \boldsymbol{I})$$

  where the variance τ_u^2 is a tuning parameter that determines the mixing and convergence behaviour.

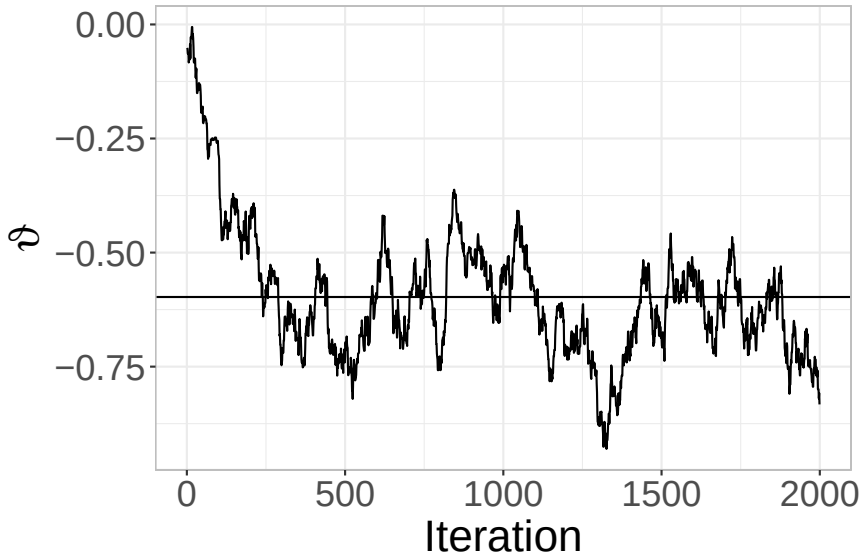
- Random walk proposal with a large variance τ_u^2:

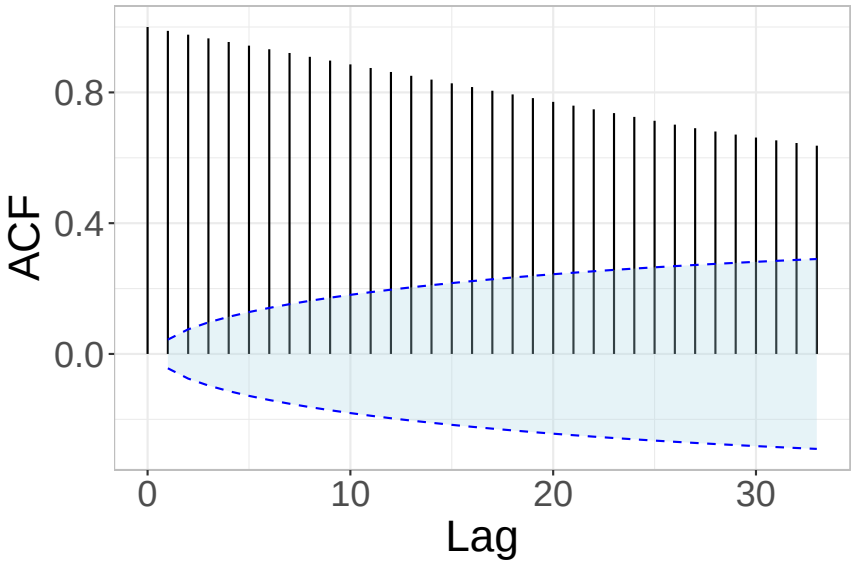


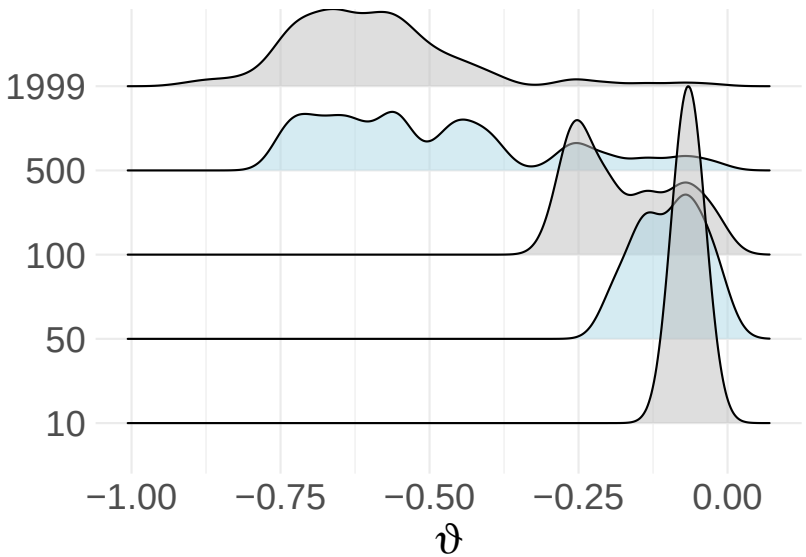




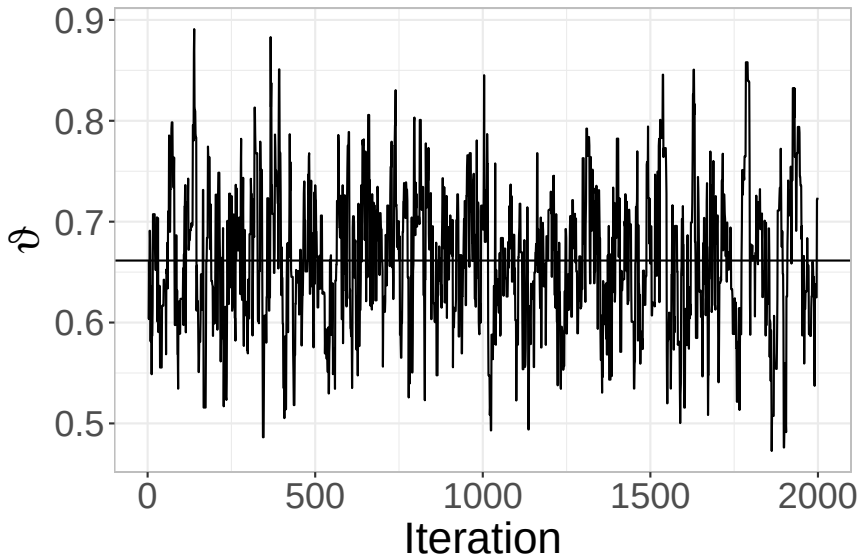
- Random walk proposal with a small variance τ_u^2:

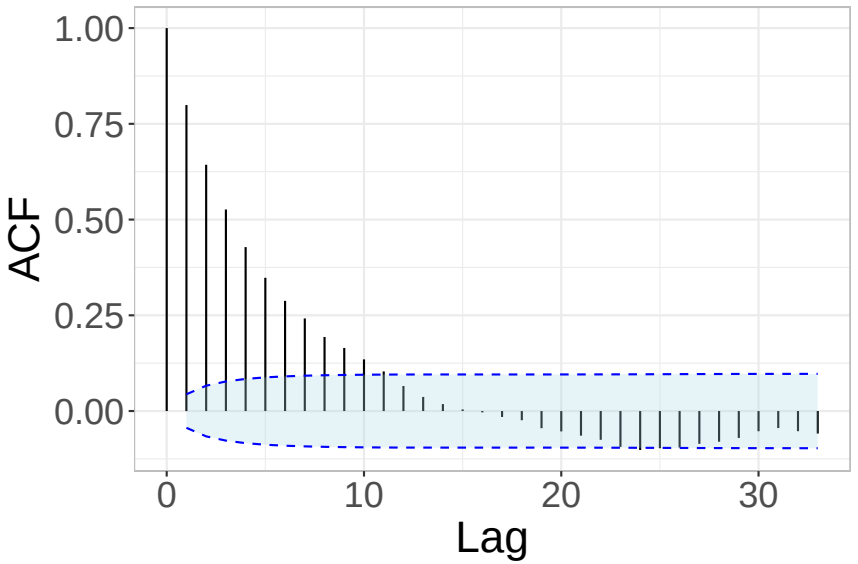


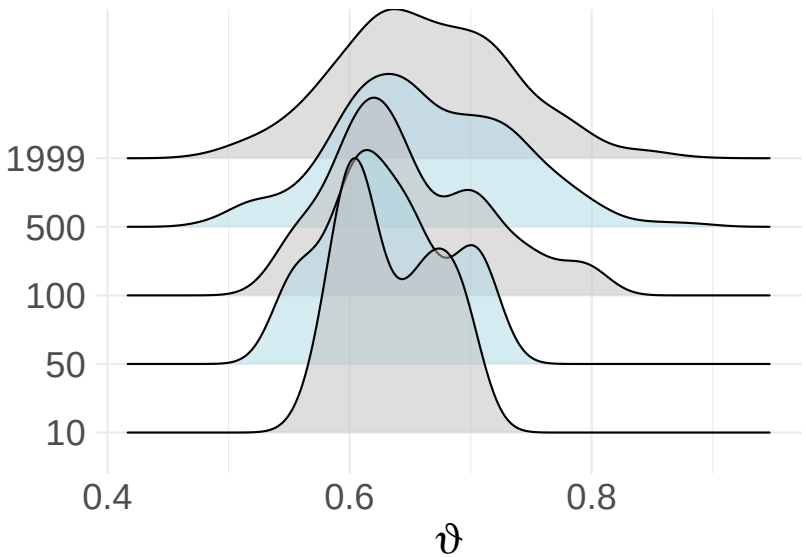




- Locally quadratic approximation of the log-full conditional:







Advantages of MCMC:

- Access to the complete posterior distribution without requiring asymptotic considerations.
- Divide and conquer approach based on updating blocks of parameters separately allows handling very complex models having hundreds or thousands of parameters.
- Modular representation of hierarchically formulated statistical models where certain parts of the model can be replaced without affecting the other model components
- From the samples of the model parameters, we can determine not only inferences about these parameters themselves, but also inference for complex functionals of these parameters.

2.2 Bayesian Inference with MCMC II

Posterior summaries:

- While the ultimate outcome of Bayesian inference is the posterior, this is often compressed into posterior summaries, in particular
  - posterior point estimates and
  - posterior measures of uncertainty.
- Typical point estimates:
  - posterior mean (estimated by averages of samples),
  - posterior median (estimated by empirical median),
  - posterior mode (difficult to determine from samples).

- Typical measures of uncertainty:
  - posterior variance / standard deviation (estimated by empirical analogues),
  - posterior quantiles.

Credible intervals and bands:

- A pointwise Bayesian credible interval $[\vartheta_{s,\text{low}}, \vartheta_{s,\text{upp}}]$ for a scalar parameter ϑ_s is characterized by the posterior coverage probability

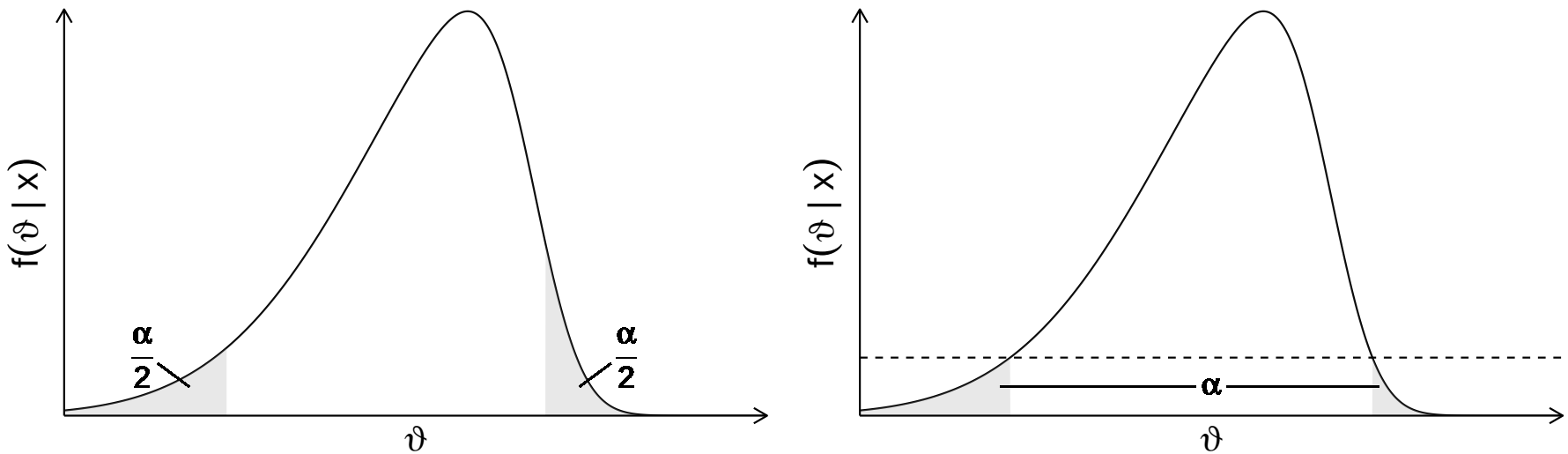
$$\mathbb{P}(\vartheta_{s,\text{low}} \leq \vartheta_s \leq \vartheta_{s,\text{upp}} | \boldsymbol{y}) \geq 1 - \alpha$$

  where $1 - \alpha$ denotes the desired coverage level.

- A simultaneous band for multiple parameters $\{\vartheta_s, s \in \mathcal{S}\}$ should have

$$\mathbb{P}(\vartheta_{s,\text{low}} \leq \vartheta_s \leq \vartheta_{s,\text{upp}}, s \in \mathcal{S} | \boldsymbol{y}) \geq 1 - \alpha$$

- Symmetric and highest posterior density credible intervals:



Inference for derived quantities:

- Goal: Conduct Bayesian inference for a derived quantity

$$\boldsymbol{\eta} = g(\boldsymbol{\vartheta}).$$

- Convenient feature of MCMC: If $\boldsymbol{\vartheta}^{[1]}, \ldots, \boldsymbol{\vartheta}^{[T]}$ is a sample from the posterior of $\boldsymbol{\vartheta}$, $g(\boldsymbol{\vartheta}^{[1]}), \ldots, g(\boldsymbol{\vartheta}^{[T]})$ will be a sample from the posterior of the transformed parameter.
- No restrictions on the transformation $g(\cdot)$ and no need to deal with asymptotic considerations

The Bayes factor for model comparison:

- If there are L competing models $M_1, \ldots, M_L$ with associated parameters $\boldsymbol{\vartheta}_1, \ldots, \boldsymbol{\vartheta}_L$, the posterior for $\boldsymbol{\vartheta}_l$ given the model M_l is

$$f(\boldsymbol{\vartheta}_l|\boldsymbol{y}, M_l) = \frac{f(\boldsymbol{y}|\boldsymbol{\vartheta}_l, M_l) f(\boldsymbol{\vartheta}_l|M_l)}{f(\boldsymbol{y}|M_l)},$$

  where $f(\boldsymbol{y}|\boldsymbol{\vartheta}_l, M_l)$ and $f(\boldsymbol{\vartheta}_l|M_l)$ are the likelihood and the prior of $\boldsymbol{\vartheta}_l$ under model M_l, respectively, and

$$f(\boldsymbol{y}|M_l) = \int f(\boldsymbol{y}|\boldsymbol{\vartheta}_l, M_l) f(\boldsymbol{\vartheta}_l|M_l) d\boldsymbol{\vartheta}_l$$

  is the marginal likelihood of model M_l.

- For model selection, we assign prior distributions to the competing models, $f(M_l)$, and compare models through their marginal posteriors

$$f(M_l|\boldsymbol{y}) = \frac{f(\boldsymbol{y}|M_l)f(M_l)}{f(\boldsymbol{y})}, \qquad l = 1, \ldots, L,$$

  where $f(\boldsymbol{y}) = \sum_{l=1}^{L} f(\boldsymbol{y}|M_l)f(M_l)$.

- Prefer model M_l against model M_s, $s \neq l$ if

$$f(M_l|\boldsymbol{y}) > f(M_s|\boldsymbol{y})$$

  or in other words if

$$\frac{f(M_l|\boldsymbol{y})}{f(M_s|\boldsymbol{y})} = \frac{f(M_l)f(\boldsymbol{y}|M_l)}{f(M_s)f(\boldsymbol{y}|M_s)} > 1.$$

- The ratio of marginal likelihoods is referred to as the Bayes factor

$$\text{BF}_{ls} = \frac{f(\boldsymbol{y}|M_l)}{f(\boldsymbol{y}|M_s)} \ .$$

which reflects model preference under equal prior probabilities

$$f(M_1) = \ldots = f(M_L)$$

- The marginal likelihoods can be estimated as arithmetic mean

$$\hat{f}(\boldsymbol{y}) = \frac{1}{T}\sum_{t=1}^{T} f(\boldsymbol{y}|\boldsymbol{\vartheta}^{[t]})$$

or harmonic mean

$$\hat{f}(\boldsymbol{y}) = \left(\frac{1}{T}\sum_{t=1}^{T} \frac{1}{f(\boldsymbol{y}|\boldsymbol{\vartheta}^{[t]})}\right)^{-1}$$

with samples $\boldsymbol{\vartheta}^{[t]}$, $t = 1, \ldots, T$ from the posterior distribution.

Bayesian information criteria:

- Bayesian information criterion (BIC)

$$\mathrm{BIC}(M_l) = -2l(\hat{\boldsymbol{\vartheta}}_l) + \log(n)p_l$$

where p_l is the number of parameters in model l.

- Deviance information criterion (DIC)

$$\mathrm{DIC} = \overline{D(\boldsymbol{\vartheta})} + p_{\mathrm{DIC}}$$

where

$$D(\boldsymbol{\vartheta}) = -2\log(f(\boldsymbol{y}|\boldsymbol{\vartheta})) = \frac{1}{T}\sum_{t=1}^{T} D(\boldsymbol{\vartheta}^{[t]})$$

denotes the model deviance and

$$p_{\mathrm{DIC}} = \overline{D(\boldsymbol{\vartheta})} - D(\overline{\boldsymbol{\vartheta}}) = \frac{1}{T}\sum_{t=1}^{T} D(\boldsymbol{\vartheta}^{[t]}) - D\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\vartheta}^{[t]}\right)$$

provides an estimate for the effective parameter count.

- Widely applicable information criterion (WAIC)

$$\text{WAIC} = 2\left(D_{\text{WAIC}} + p_{\text{WAIC}}\right)$$

with

$$D_{\text{WAIC}} = -\sum_{i=1}^{n} \log\left(\frac{1}{T}\sum_{t=1}^{T} p\left(y_i|\boldsymbol{\vartheta}^{[t]}\right)\right)$$

as the measure of model fit,

$$p_{\text{WAIC}} = \sum_{i=1}^{n} \widehat{\text{Var}}\left(\log\left(p\left(y_i|\boldsymbol{\vartheta}\right)\right)\right)$$

as the measure of model complexity, and the empirical variance

$$\widehat{\text{Var}}\left(a\right) = \frac{1}{T-1}\sum_{t=1}^{T}\left(a_t - \bar{a}\right)^2.$$

3 Bayesian Additive Regression

3.1 Introduction

Linear models:

- The work horse of statistical modelling and analysis is the linear model where

$$y_i = \boldsymbol{x}_i'\boldsymbol{\beta} = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik} + \varepsilon_i, \quad \varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2).$$

- The parameters β_j can be related to the expected change in the response associated with differences in x_j.

  $\Rightarrow$ Parameters have a specific meaning and purpose.

- Statistical inference is facilitated by the distributional assumptions on the error terms.
- However, in many practical situations the linear model is not flexible enough and/or assumptions may be questionable.

Nonlinear effects:

- Common practice if the linearity of the effect of x_j is questionable: Include low-order polynomials, e.g. replace $x_j\beta_j$ by

$$x_j\beta_j + x_j^2\beta_{j+1} + x_j^3\beta_{j+2}.$$

- Imposes strong assumptions on the form of the effect and is not very flexible.
- Ideally, the form of an effect should be left unspecified and should be determined by the data (under mild, qualitative assumptions).
- Additive model:

$$y_i = \beta_0 + f_1(x_{i1}) + \ldots + f_k(x_{ik}) + \varepsilon_i, \quad \varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2).$$

- We will use penalized splines to represent the effects $f_j(x_{ij})$.

Clustered data:

- For longitudinal data $(y_{it}, \boldsymbol{x}_{it})$, $i = 1 \ldots, n$, $t = 1, \ldots, T$, a classical model of the form

$$y_{it} = \boldsymbol{x}_{it}'\boldsymbol{\beta} + \varepsilon_{it}$$

  may be questionable for a number of reasons:

  - Unobserved heterogeneity due to individual-specific, unobserved confounders that have not been included in the model,
  - Dependence between observations on one individual, or
  - Individual-specific regression coefficients.

- Similarly applies to other grouping structures (families, geographical regions, school classes, . . .)

- Random effects models are then often considered, e.g. random intercepts

$$y_{it} = \gamma_{0i} + \boldsymbol{x}_{it}'\boldsymbol{\beta} + \varepsilon_{it}$$

  with $\gamma_{i0} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \tau^2)$.

- More complex models can also have individual-specific random slopes or other additional structures.

Spatial dependence:

- For spatial regression data $(y(s), \boldsymbol{x}(s))$, one may similarly question whether linear models take unobserved spatial heterogeneity and/or dependence into account.
- Include spatially correlated random effects, leading to

$$y(s) = \gamma(s) + \boldsymbol{x}(s)'\boldsymbol{\beta} + \varepsilon(s)$$

with $\gamma(s)$ being an appropriately specified spatial stochastic process.

Bayesian additive regression:

- Bayesian additive regression provides a unifying framework for dealing with the challenges discussed so far.
- The model also supports other effect types, e.g. varying coefficients or interaction surfaces.
- The models can be conveniently represented in a hierarchical fashion that enables us to benefit from the flexibility of Bayesian inference.
- Tomorrow, we will discuss Bayesian distributional regression that allows us to overcome the normality assumption for the error terms.

3.2 Penalized Spline Smoothing

Scatterplot smoothing:

- Start from scatterplot smoothing

$$y_i = s(z_i) + \varepsilon_i, \quad \varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2)$$

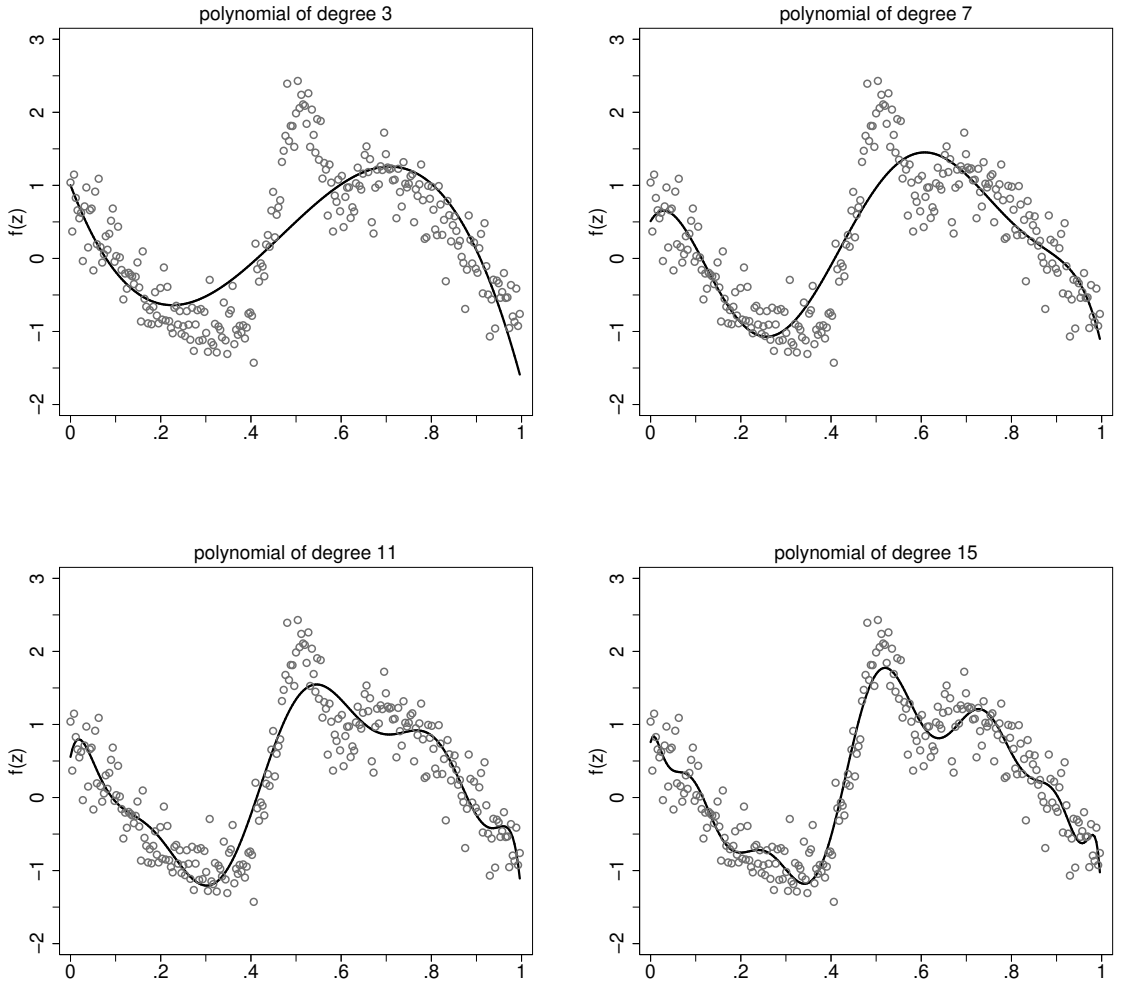
  where $s(z)$ should be inferred based on observations (z_i, y_i), $i = 1, \ldots, n$, for a continuous covariate z and response y.

- Common approach: Approximate $s(z)$ by a low-order polynomial

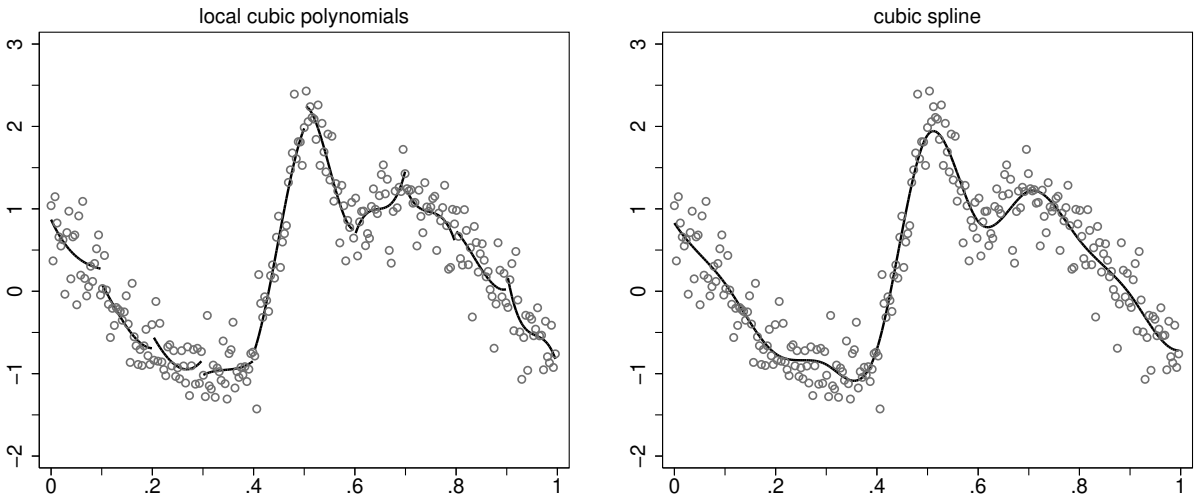
$$s(z_i) = \gamma_0 + \gamma_1 z_i + \ldots + \gamma_l z_i^l$$

  since any smooth function f can be approximated arbitrarily accurately if the degree l is chosen large enough.

- In statistics, the problem of estimating the coefficients $\gamma_0, \ldots, \gamma_l$ limits the applicability of high polynomial degrees:



- Moreover, polynomials have the following disadvantages:
  - A polynomial assumes a global amount of smoothness for the function s.
  - Polynomial estimates tend to be unstable at the boundaries of the covariate space.
  - Low order polynomials induce very specific types of functional forms.
- A possibility to overcome some of these problems is to define polynomials piecewise on partial intervals of the covariate domain.



- Advantages:
  - Localized fits instead of global smoothness, and
  - high flexibility.
- Disadvantages:
  - Potentially large number of regression coefficients.
  - The resulting function is no longer smooth since the function pieces on the intervals are fitted separately.

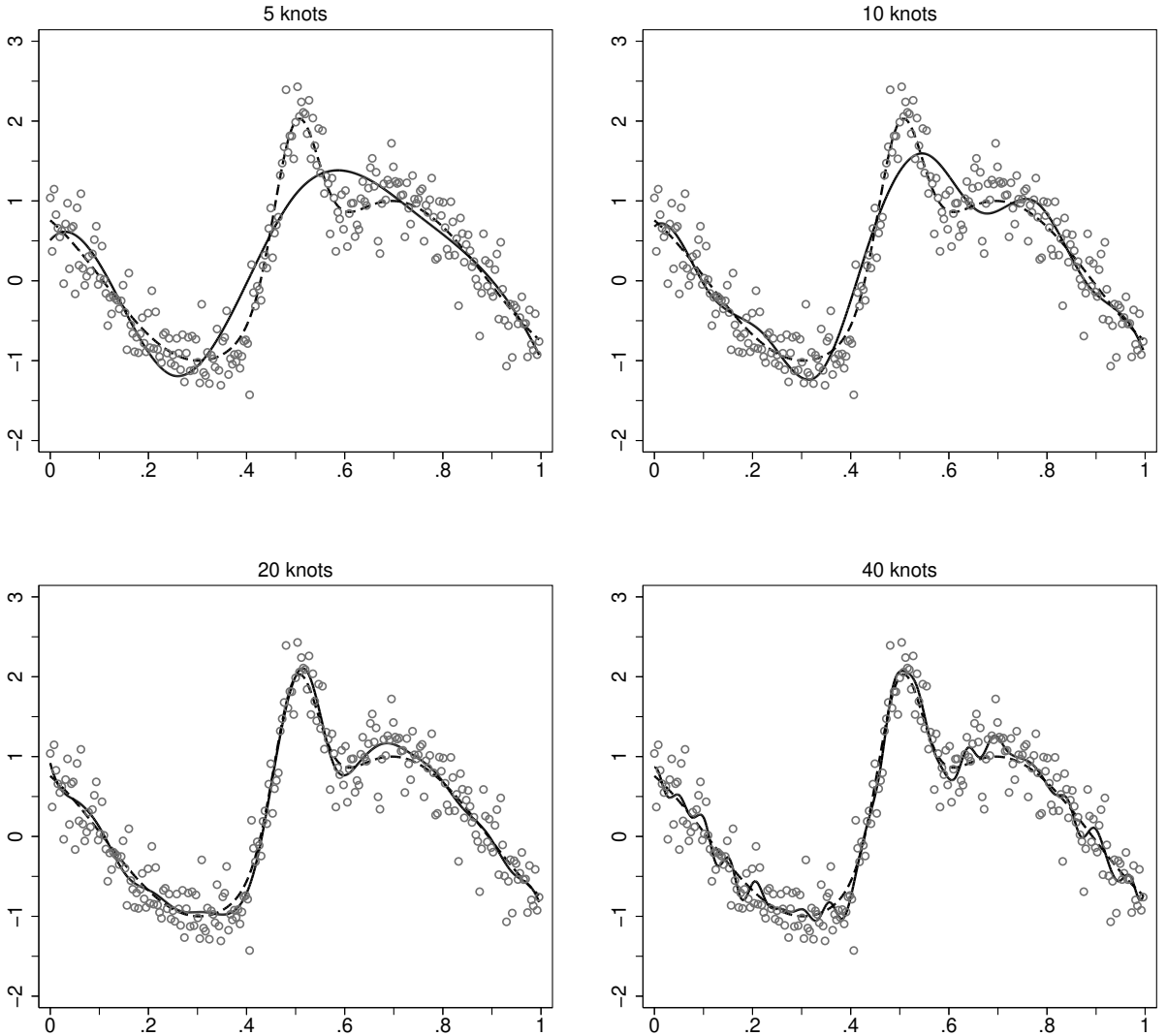
$\Rightarrow$ Combine global polynomials and piecewise polynomials to obtain polynomial splines.

Polynomial splines:

- A function s is a polynomial spline of degree l with knots $a = \kappa_1 < \ldots < \kappa_m = b$ if
  - $s(z)$ is $(l-1)$ times continuously differentiable,
  - $s(z)$ is polynomial of degree l on each of the intervals $[\kappa_j, \kappa_{j+1})$.

  $\Rightarrow$ Piecewise polynomial with global smoothness.

- The flexibility of a polynomial spline is determined by the number of knots.



- Polynomial splines form a vector space such that they can be represented in terms of $L = m + l - 1$ basis functions, i.e.

$$s(z) = \sum_{l=1}^{L} \gamma_l B_l(z).$$

- Different basis representations exist, but we focus on B-splines due to their advantageous numerical properties.

- The model can now be represented in matrix notation as

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\gamma} + \boldsymbol{\varepsilon},$$

where

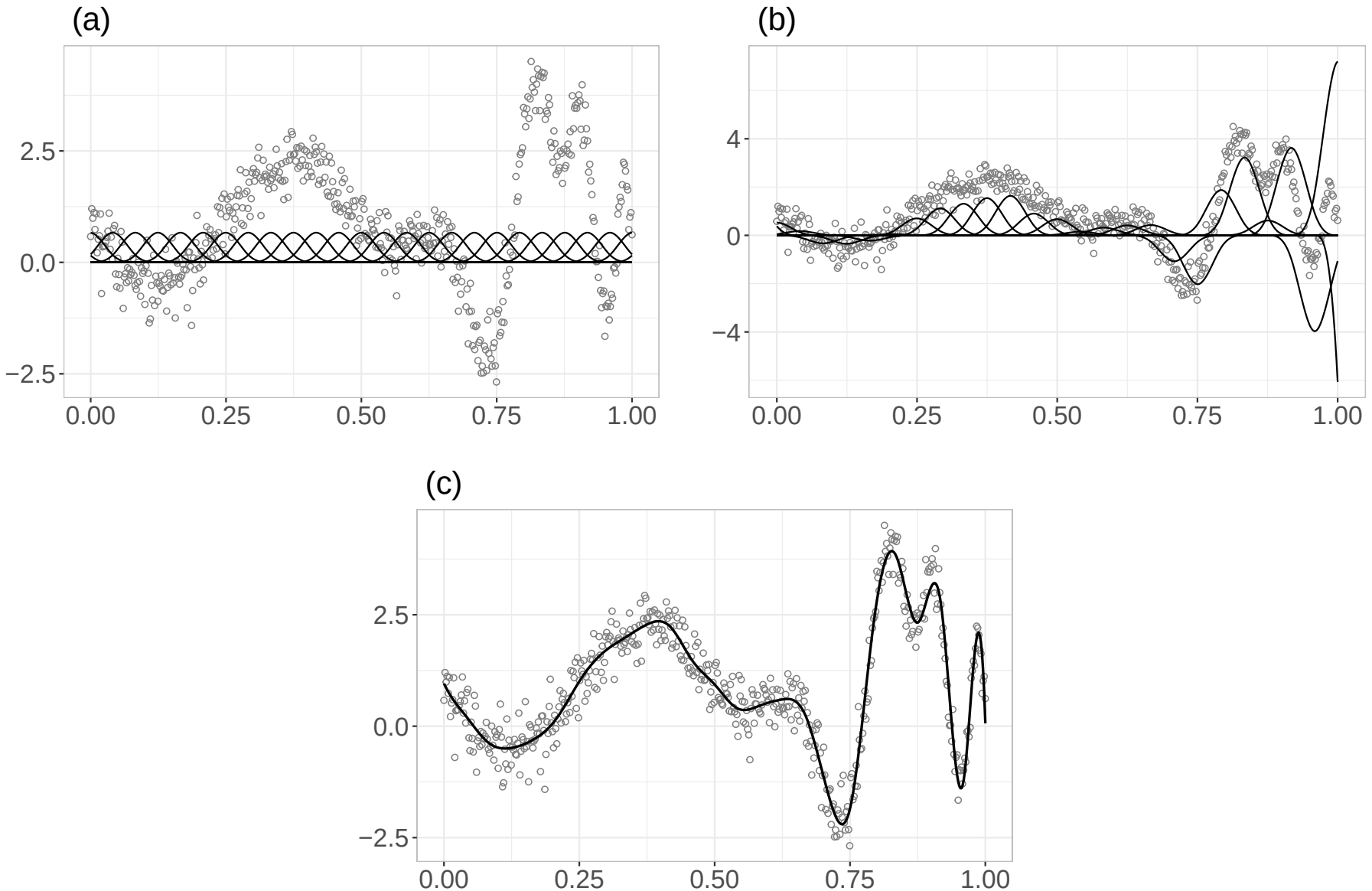
$$\boldsymbol{B} = \begin{pmatrix} B_1(z_1) & \dots & B_L(z_1) \\ \vdots & & \vdots \\ B_1(z_n) & \dots & B_L(z_n) \end{pmatrix}.$$

- Estimate the basis coefficients via least squares as

$$\hat{\boldsymbol{\gamma}} = (\boldsymbol{B}'\boldsymbol{B})^{-1}\boldsymbol{B}'\boldsymbol{y}$$

and the function evaluations as

$$\hat{\boldsymbol{s}} = \boldsymbol{B}\hat{\boldsymbol{\gamma}}.$$


(a)
2.5
0.0
−2.5
0.00
0.25
0.50
0.75
1.00
(b)
4
0
−4
0.00
0.25
0.50
0.75
1.00
(c)
2.5
0.0
−2.5
0.00
0.25
0.50
0.75
1.00

Penalized splines:

- To avoid the need to optimize the number and location of the knots for polynomial splines, we
  - approximate $s(z)$ based on a rich spline basis (usually about 20 to 40 basis functions) to ensure enough flexibility of the estimate, and
  - regularize estimation by adding a penalty term to the least squares fit criterion to ensure smoothness of the estimate.
- From a Bayesian perspective, regularisation is achieved by assigning an informative prior that encourages smoothness.

- More precisely, use random walk priors of order k, e.g.

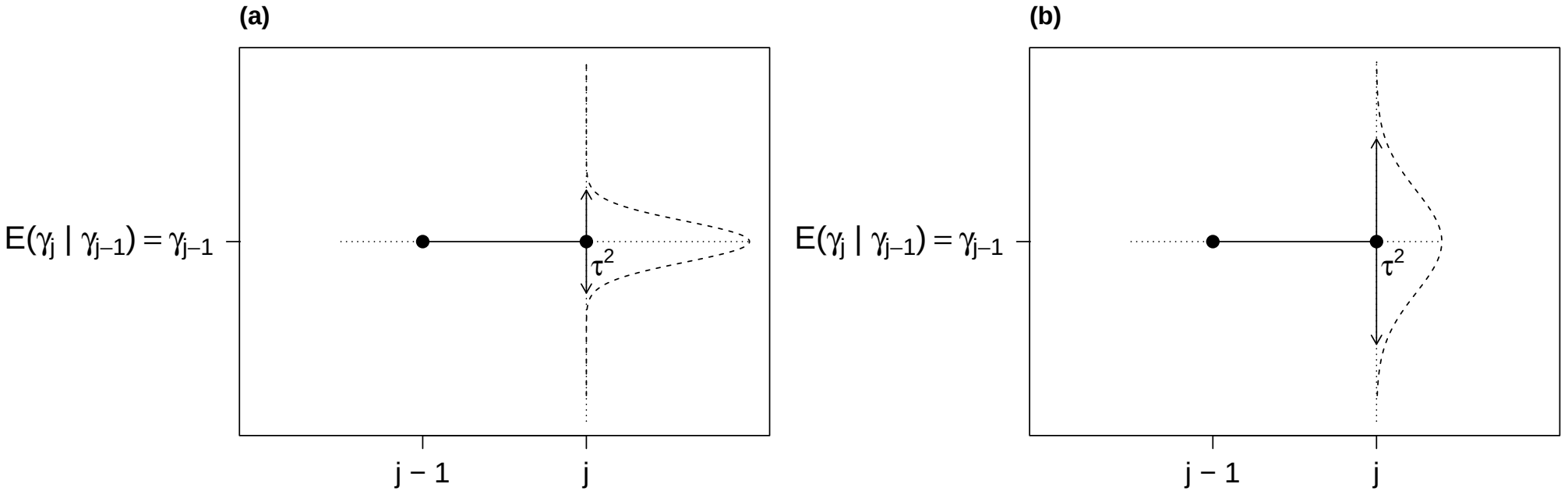
$$\gamma_j = \gamma_{j-1} + u_j, \quad u_j \sim \mathrm{N}(0, \tau^2), \quad j = 2, \ldots, d,$$

  for a first order random walk and

$$\gamma_j = 2\gamma_{j-1} - \gamma_{j-2} + u_j, \quad u_j \sim \mathrm{N}(0, \tau^2), \quad j = 3, \ldots, d,$$

  for a second order random walk with noninformative priors $f(\gamma_1) \propto const$ and $f(\gamma_1, \gamma_2) \propto const$ for initial values.

- The random walk variance τ^2 determines the impact of the prior on the posterior



- The random walk assumptions imply a (partially improper) multivariate Gaussian prior

$$f(\boldsymbol{\gamma}|\tau^2) \propto \left(\frac{1}{\tau^2}\right)^{(L-k)/2} \exp\left(-\frac{1}{2\tau^2}\boldsymbol{\gamma}'\boldsymbol{K}\boldsymbol{\gamma}\right),$$

  where k is the order of the random walk and $\boldsymbol{K}$ is the prior precision matrix.

- Penalized splines can be modified, for example
  - to obtain cyclic effects when smoothing over temporal domains,
  - to add shape constraints such as monotonicity or concavity, or
  - to make the amount of smoothness adaptive to the covariate space.
- They can also be extended to bivariate (or higher order) smoothing.

3.3 Generic Basis Function Framework

- Penalized splines are one representative for a variety of effects that can be cast into a generic basis function framework.
- Let ν_i denote some generic type of covariate information and assume

$$s(\nu_i) = \sum_{l=1}^{L} \gamma_l B_l(\nu_i)$$

  with L basis functions $B_l(\nu_i)$.
- The vector of function evaluations $\boldsymbol{s} = (s(\boldsymbol{x}_1), \ldots, s(\boldsymbol{x}_n))'$ at the observed covariate values is then given by

$$\boldsymbol{s} = \boldsymbol{B}\boldsymbol{\gamma},$$

  where $\boldsymbol{B}$ is the design matrix obtained from the basis function evaluations and $\boldsymbol{\gamma}$ is the corresponding vector of basis coefficients.

- To regularize estimation, assign the prior

$$f(\boldsymbol{\gamma}|\tau^2) \propto \left(\frac{1}{\tau^2}\right)^{0.5\,\mathrm{rg}(\boldsymbol{K})} \exp\left(-\frac{1}{2\tau^2}\boldsymbol{\gamma}'\boldsymbol{K}\boldsymbol{\gamma}\right)$$

  with positive semidefinite prior precision matrix $\boldsymbol{K}$ and prior variance parameter τ^2.

- An effect is then characterized by the chosen basis functions $B_l(\nu)$, the structure of the precision matrix $\boldsymbol{K}$ and the hyperprior assumed for τ^2

- For the latter, it is common to use conjugate inverse gamma hyperpriors.

3.4 Special Cases

Spatial effects for regional data:

- Each observation i is assumed to belong to one of the spatial regions represented by a spatial indicator $r_i \in \{1, \ldots, L\}$.
- Assign separate regression coefficients γ_l, $l = 1, \ldots, L$, to each of the L regions.
- The spatial effect $\gamma_{r_i} = s(r_i)$ of an individual observation i collected in region r_i can then be expressed as

$$s(r_i) = \sum_{l=1}^{L} \gamma_l B_l(r_i),$$

where

$$B_l(r_i) = \begin{cases} 1 & \text{if } r_i = l \\ 0 & \text{otherwise.} \end{cases}$$

- In matrix notation this yields the $(n \times L)$ design matrix $\boldsymbol{B}$ with entries

$$\boldsymbol{B}[i, l] = \begin{cases} 1 & \text{if } r_i = l \\ 0 & \text{otherwise} \end{cases}$$

  and the complete vector of spatial effects is given by $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_L)'$.

- Assume a Gaussian Markov random field prior, where the conditional distribution of γ_l given all the neighboring effects is specified as

$$\gamma_l \,|\, \gamma_r, r \neq l \sim \mathrm{N}\left(\frac{1}{|N(l)|} \sum_{r:r\sim l} \gamma_r, \frac{\tau^2}{|N(l)|}\right),$$

  where $l \sim r$ indicates that l and r are neighbors and $|N(l)|$ is the number of neighbors of region l.

- This implies that
  - the prior expectation for the spatial effect in region l is given by the average of all spatial effects of neighboring regions,
  - the effect in region l is conditionally independent of all non-neighbors, and
  - the variance of the conditional prior distribution in region l is inversely proportional to the number of neighbors.

- The conditional distributions yield a multivariate Gaussian joint distribution for γ given by

$$f(\boldsymbol{\gamma} \,|\, \tau^2) \propto \left(\frac{1}{\tau^2}\right)^{(\text{rg}(\boldsymbol{K}))/2} \exp\left(-\frac{1}{2\tau^2}\boldsymbol{\gamma}'\boldsymbol{K}\boldsymbol{\gamma}\right),$$

  with prior precision matrix

$$\boldsymbol{K}[l, r] = \begin{cases} -1 & l \neq r, l \sim r, \\ 0 & l \neq r, l \nsim r, \\ |N(l)| & l = r, \end{cases}$$

- If each region has at least one neighbor and the map is fully connected, the rank of the spatial adjacency matrix is given by $\text{rg}(\boldsymbol{K}) = L - 1$.

Random effects:

- Assume that the data are grouped into L disjoint sets of observations and define group-specific regression coefficients γ_l, $l = 1, \ldots, L$.
- If group membership is represented by the indicator $g_i \in \{1, \ldots, L\}$, the group-specific effects can be represented as

$$s(g_i) = \sum_{l=1}^{L} \gamma_l B_l(g_i)$$

with indicator basis functions

$$B_l(g_i) = \begin{cases} 1 & \text{if } g_i = l \\ 0 & \text{otherwise.} \end{cases}$$

- In matrix notation, the indicator basis functions imply a dummy-coded design matrix $\boldsymbol{B}$ with elements

$$\boldsymbol{B}[i, l] = \begin{cases} 1 & \text{if } s_i = l \\ 0 & \text{otherwise.} \end{cases}$$

- The assumption of i.i.d. random intercepts translates to $\boldsymbol{\gamma} \sim \mathrm{N}(\mathbf{0}, \tau^2 \boldsymbol{I}_L)$ for the complete vector of random effects with prior precision matrix $\boldsymbol{K} = \boldsymbol{I}_L$.

Other effect types:

- The framework support a number of further effect types such as
  - Spatial effects $s(x_1, x_2)$ for spatial domains that are continuously indexed by coordinates (x_1, x_2),
  - varying coefficients $\nu_1 s(\nu_2)$ with effect modifier ν_2 and interaction variable ν_1 (e.g. random slopes), ot
  - interaction surfaces $s(z_1, z_2)$ with two continuous covariates z_1 and z_2.

3.5 Hyperprior Specifications

Inverse gamma prior:

- The smoothing variance τ^2 determines how strongly estimates $\hat{s}(\nu)$ are affected by the smoothness properties induced by the precision matrix $\boldsymbol{K}$.
- Inference for τ^2 is therefore extremely important (similar as determining the regularisation parameter for, e.g., the LASSO).
- Bayesian approach: Assign a hyperprior $f(\tau^2)$ and include τ^2 as a hyperparameter.
- The de factor standard are inverse gamma priors $\tau^2 \sim \text{IG}(a, b)$, $a > 0$, $b > 0$ since this is conjugate to the multivariate normal prior $f(\boldsymbol{\gamma}|\tau^2)$.
- Allows updating τ^2 in a simple Gibbs step without requiring a proposal density and/or acceptance step.

- How should the parameters a and b of the inverse gamma distribution $\tau^2 \sim \text{IG}(a, b)$ be chosen?
- Limiting cases:
  - $a \to 0$, $b \to 0$ leads to a flat prior for $\log(\tau^2)$ (this is also Jeffreys' prior).
  - $a = 1$, $b \to 0$ leads to a flat prior for the precision $1/\tau^2$.
  - $a = -1$, $b = 0$ leads to a flat prior for τ^2.
  - $a = -0.5$, $b = 0$ leads to a flat prior for the standard deviation τ.

- In practice, the limit of 0 is then often approximated by a small constant ϵ and different values are tried out to study prior sensitivity.
- $a < 0$ leads to improper priors (prior does not integrate to one), such that propriety of the posterior has to be ensured.
- Especially in situations with little information per parameter (e.g. random effects models with small groups), there has been considerable debate about prior sensitivity and the suitability of the inverse gamma distribution.

Alternatives to the conjugate inverse gamma:

- Alternative prior distributions that have been considered:
  - Half-normal $\tau^2 \sim \mathrm{HN}(0, \theta^2)$
  - Half-Cauchy $\tau^2 \sim \mathrm{HC}(0, \theta^2)$
  - Uniform $\tau^2 \sim \mathrm{U}(0, \theta)$
- The hyperparameter θ has to be chosen with respect to the prior beliefs.

Scale-dependent hyperpriors

- Goal: Determine a prior based on a simple set of principles and derive an intuitive way of eliciting hyperparameters.
- Principle 1: Occam's Razor.
  - The hyperprior should invoke the principle of parsimony.
  - Simple base model for each effect is preferred unless the data provide convincing evidence for more complex modelling.
  - For structured additive regression terms, $\tau^2 \to 0$ results in the base model $f_b(\boldsymbol{\gamma}|\tau^2 = 0)$ determined by the nullspace of $\boldsymbol{K}$.

- Principle 2: Measure of Complexity.
  - The increased complexity is measured by the Kullback-Leibler divergence

$$\text{KLD}(f||f_b) = 2\int f(u)\log\left(\frac{f(u)}{f_b(u)}\right)\mathrm{d}u$$

    for the base model f_b and an alternative flexible model f.
  - Gives a measure of the information loss when the base model is used to approximate the more flexible models.
  - Define

$$d(f||f_b) = \sqrt{2\text{KLD}(f||f_b)}$$

    as the unidirectional 'distance' from the flexible model to the base model.

- Principle 3: Constant Rate Penalisation.
  - Constant rate penalisation implies an exponential prior on the distance scale, i.e.

$$f_d(d) = \lambda \exp(-\lambda d), \quad \lambda > 0$$

    with mode at $d = 0$.
  - Constant rate of decay in the distance prior from f_b to stronger deviations from f_b.
  - λ determines the rate of penalisation.

- The change of variable theorem gives

$$f(\tau^2) = \lambda \exp(-\lambda d(\tau^2)) \left| \frac{\partial d(\tau^2)}{\partial \tau^2} \right| \text{ with } d(\tau^2) = \sqrt{2\text{KLD}}.$$

- For structured additive regression terms, this induces a Weibull prior $\tau^2 \sim \text{We}(0.5, \theta)$, i.e.

$$f(\tau^2|\theta) = \frac{1}{2\theta} \left(\frac{\tau^2}{\theta} \right)^{-1/2} \exp\left(- \left(\frac{\tau^2}{\theta} \right)^{1/2} \right).$$

- By construction, the scale-dependent prior is invariant under transformations, i.e. we obtain equivalent priors for τ, τ^2, $1/\tau^2$, etc.

Hyperprior elicitation

- The decay rate $\exp(-\lambda)$ can be controlled by the condition

$$\mathbb{P}(q(\tau^2) \leq c) = 1 - \alpha$$

  for a suitable transformation $q(\cdot)$ of τ^2 and user-defined values c and α.

- Alternative: Prior knowledge about the scale of functional effects $s(\boldsymbol{x})$ allows to specify a certain interval with high marginal probability:

$$\mathbb{P}(|s(\nu)| \leq c\,; \forall \nu \in \mathcal{D}) \geq 1 - \alpha.$$

- The marginal density of $s(\nu) = \boldsymbol{b}'\boldsymbol{\gamma}$ is

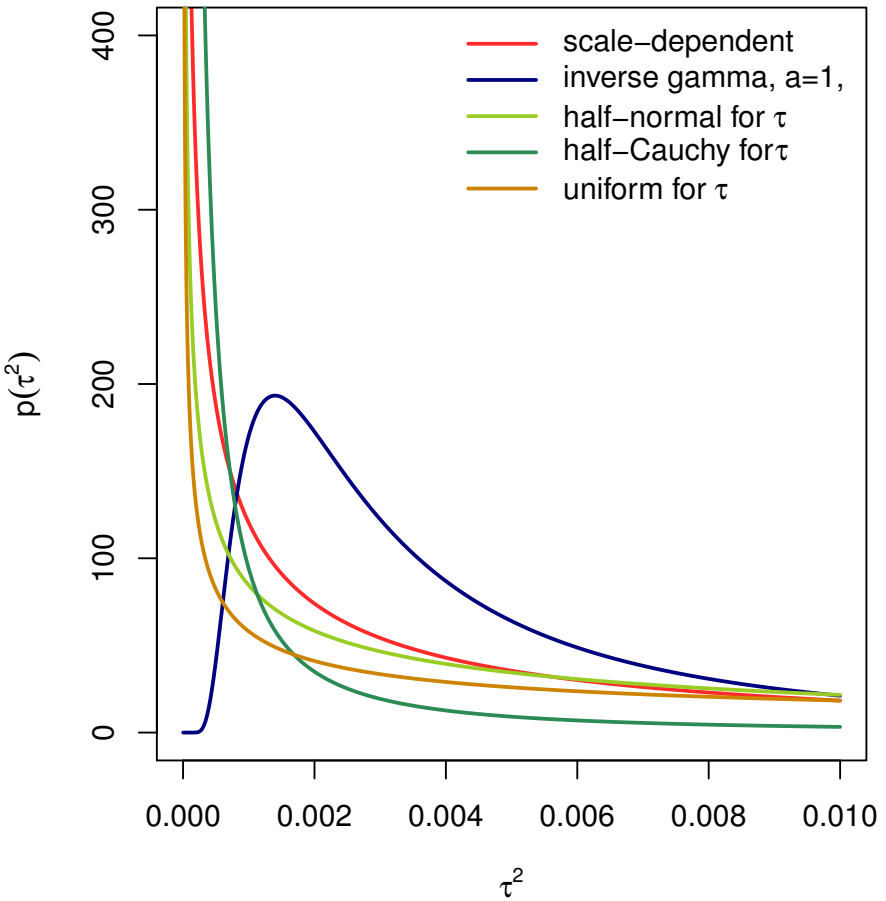
$$f(\boldsymbol{b}'\boldsymbol{\gamma}) = \int_0^\infty f(\boldsymbol{b}'\boldsymbol{\gamma}, \tau^2)\mathrm{d}\tau^2 = \int_0^\infty f(\boldsymbol{b}'\boldsymbol{\gamma}|\tau^2) f(\tau^2|\theta)\mathrm{d}\tau^2$$

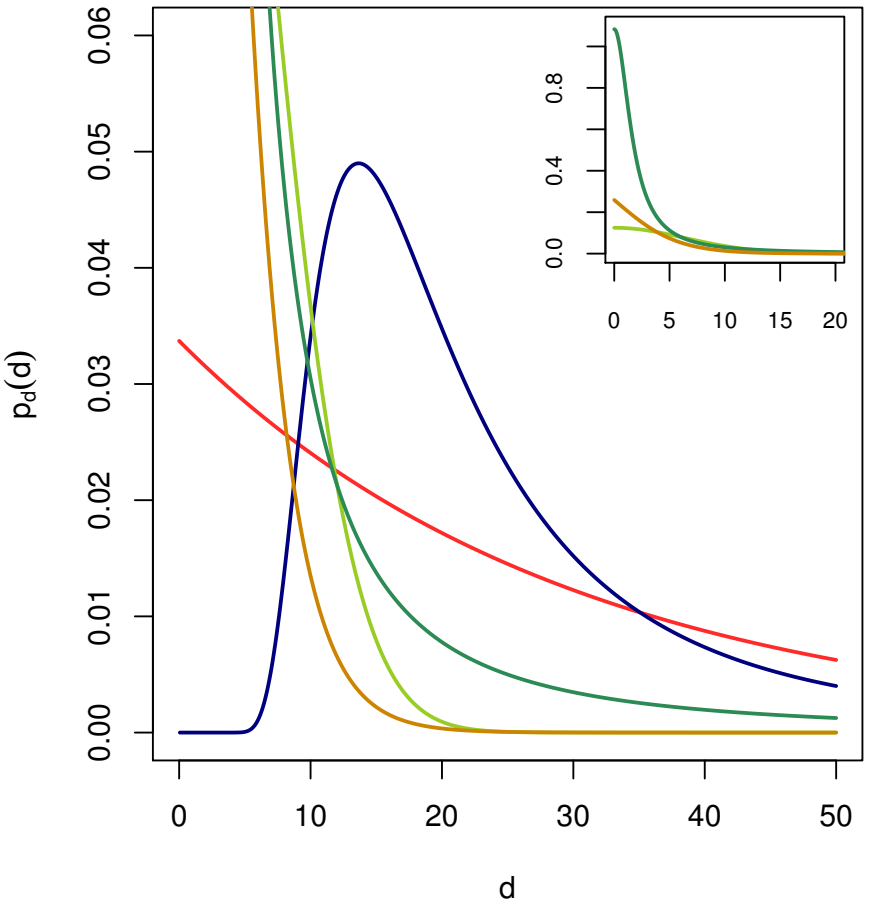
- θ can be chosen such that

$$\left(1 - \int_{-c}^{c}\int_0^\infty f_{\boldsymbol{b}'\boldsymbol{\gamma}}(u|\tau^2) f(\tau^2|\theta)\mathrm{d}\tau^2\mathrm{d}u\right) = \alpha.$$

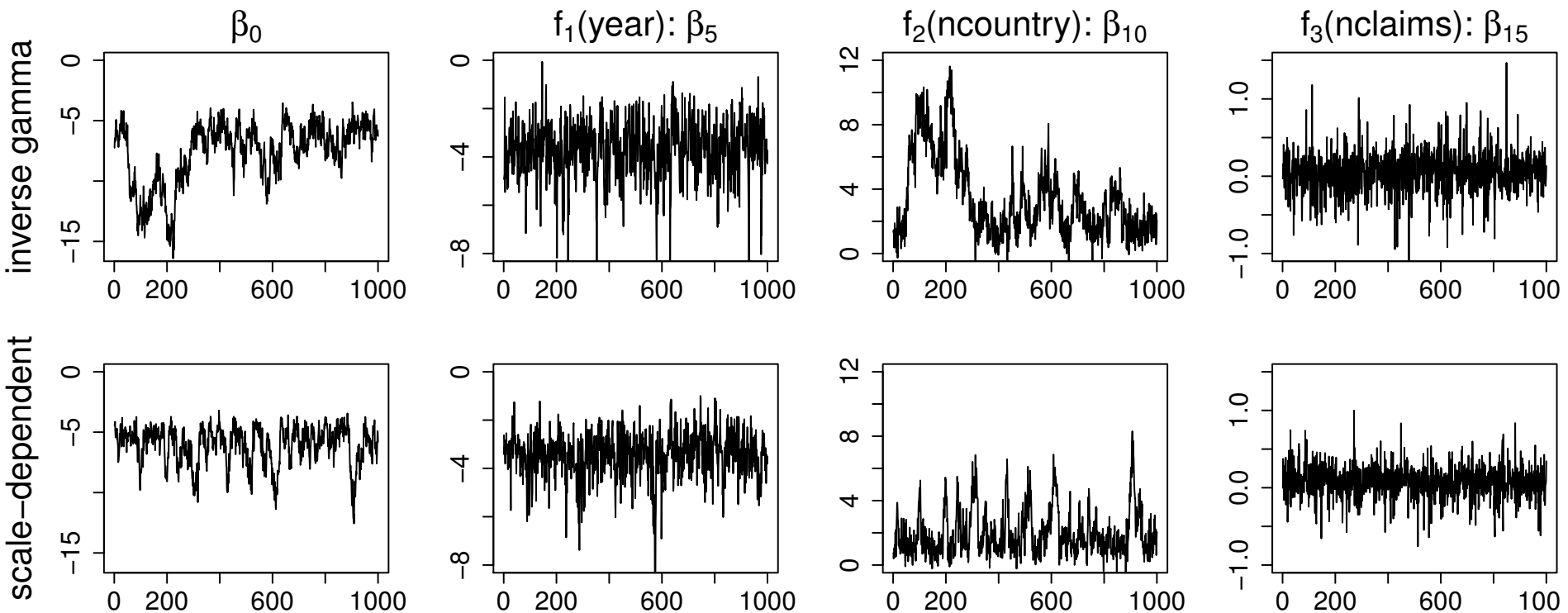
- The integral can be approximated by Monte Carlo sampling from the prior.

- The scaling criterion can also be employed for other prior structures.
- Comparison of resulting priors:





- The prior can make a difference:



3.6 Posterior Inference

- For Gaussian responses and inverse gamma hyperpriors, a Gibbs sampler can be derived.
- For other response types or non-conjugate priors, more general steps such as Metropolis-Hastings have to be included.
- Bayesian additive models can be conveniently expressed in a hierarchical fashion that can be exploited in the implementation of MCMC iterations.

- Stylized model hierarchy:

$$
\begin{aligned}
y \mid \boldsymbol{\nu}, \sigma^2 &\sim \mathrm{N}(\mu(\boldsymbol{\nu}), \sigma^2), \\
\mu(\boldsymbol{\nu}) &= s_1(\boldsymbol{\nu}) + \cdots + s_J(\boldsymbol{\nu}), \\
s_j(\boldsymbol{\nu}) &= \sum_{l=1}^{L} B_{lj}(\nu)\gamma_{lj} \\
\boldsymbol{\gamma}_j \mid \tau_j^2 &\sim \mathrm{N}(0, \tau_j^2 \boldsymbol{K}_j^-), \\
\tau_j^2 &\sim \mathrm{IG}(a, b).
\end{aligned}
$$

4 Bayesian Distributional Regression

4.1 Introduction

- Classical regression has focused on relating the conditional mean of a response y_i to covariate information x_i for observations $(x_1, y_1), \ldots, (x_n, y_n)$.

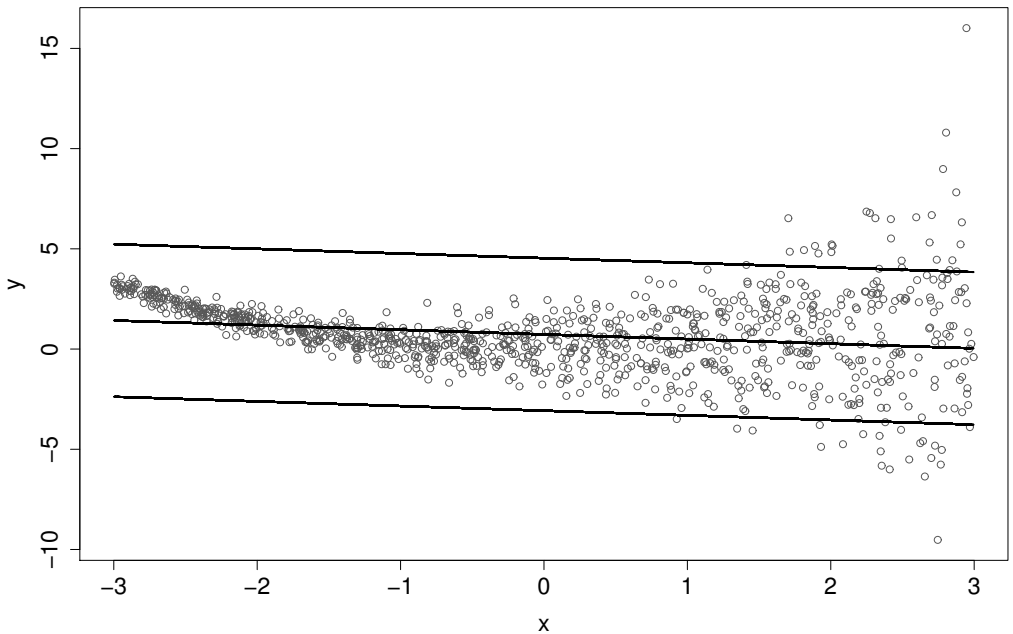
- Linear model:

$$y_i = \beta_0 + \beta x_i + \varepsilon_i$$

with ε_i i.i.d. $\mathrm{N}\left(0, \sigma^2\right)$

$$\Rightarrow \quad \mathrm{E}(y_i|x_i) = \mu_i(x_i) = \beta_0 + \beta x_i$$

$$\Rightarrow \quad \mathrm{Var}(y_i|x_i) = \sigma^2$$



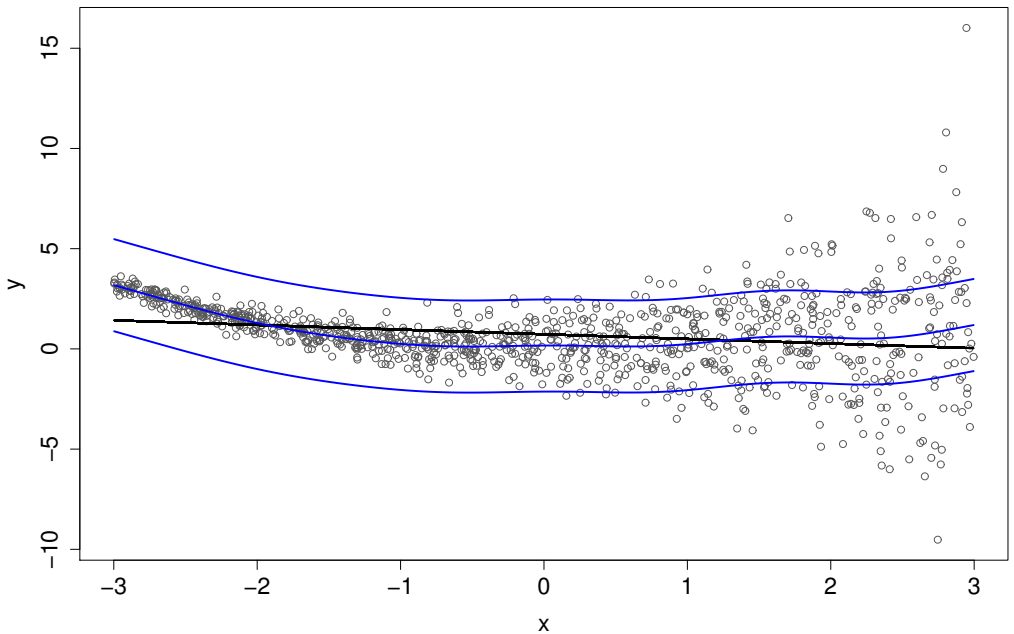
- Classical regression has focused on relating the conditional mean of a response y_i to covariate information x_i for observations $(x_1, y_1), \ldots, (x_n, y_n)$.

- Nonparametric model:

$$\mathrm{E}(y_i|x_i) = \mu_i(x_i) = \beta_0 + f(x_i)$$

with $y_i|x_i \sim \mathrm{N}\left(\mu_i(x_i), \sigma^2\right)$

σ^2 fixed

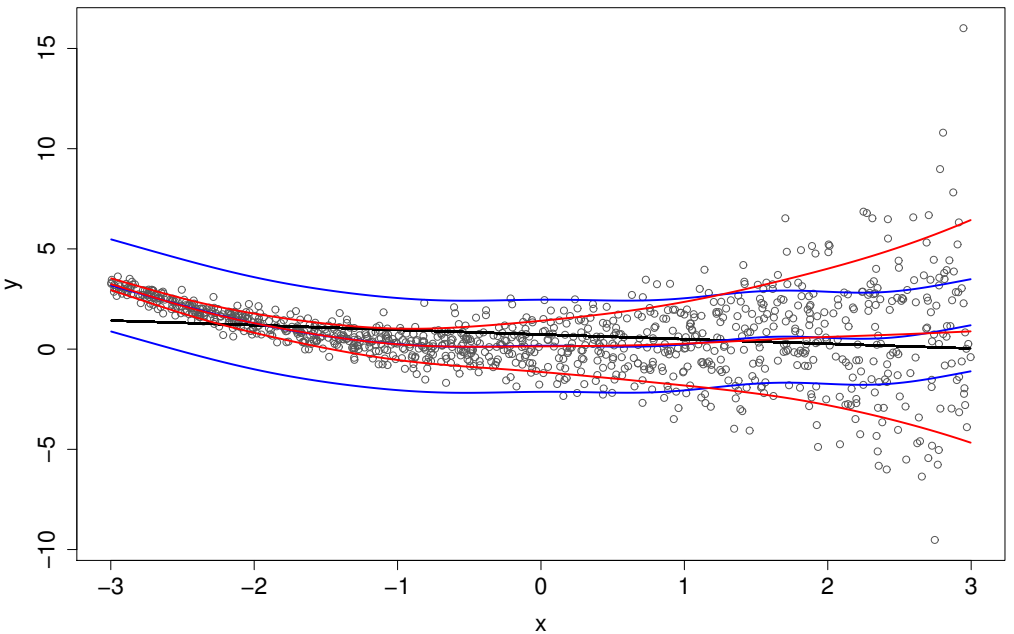


- Nonparametric model for location and scale:

$$\mathrm{E}(y_i|x_i) = \mu_i(x_i) = \beta_0^\mu + f^\mu(x_i)$$

$$\mathrm{Var}(y_i|x_i) = \sigma_i^2(x_i)$$
$$= \exp\left(\beta_0^{\sigma^2} + f^{\sigma^2}(x_i)\right)$$

with $y_i|x_i \sim \mathrm{N}\left(\mu_i(x_i), \sigma_i^2(x_i)\right)$



- So why should we focus on the mean alone if this gives only such an incomplete picture about the (conditional) distribution of the response y_i?

4.2 Bayesian Distributional Regression

- Assume a parametric specification for the conditional distribution of the responses y_i given covariates $\boldsymbol{\nu}_i$ such that

$$f(y_i|\boldsymbol{\nu}_i) = f(y_i|\boldsymbol{\vartheta}(\boldsymbol{\nu}_i)),$$

  where $\boldsymbol{\vartheta}(\boldsymbol{\nu}_i) = (\vartheta_1(\boldsymbol{\nu}_i), \ldots, \vartheta_K(\boldsymbol{\nu}_i))^\top$ is a K-dimensional vector of distributional parameters.

- Each parameter $\vartheta_k(\boldsymbol{\nu}_i)$ is linked to a regression predictor $\eta_k(\boldsymbol{\nu}_i)$ based on a response function $h_k(\cdot)$:

$$\vartheta_k(\boldsymbol{\nu}_i) = h_k(\eta_k(\boldsymbol{u}_i)) \quad \text{and} \quad \eta_k(\boldsymbol{\nu}_i) = h_k^{-1}(\vartheta_k(\boldsymbol{\nu}_i)).$$

- Flexibility attained by specifying additive predictors

$$\eta_k(\boldsymbol{\nu}_i) = s_{k1}(\boldsymbol{\nu}_i) + \ldots + s_{kJ_k}(\boldsymbol{\nu}_i)$$

  for each parameter of interest, comprising

  – flexible nonlinear effects of continuous covariates where the amount of smoothness is determined based on the data.

  – spatial effects to capture unobserved spatial heterogeneity and spatial correlations.

  – interaction terms such as varying coefficients or interaction surfaces.

  – cluster-specific random effects.

- The normal location-scale model

$$y_i|\boldsymbol{\nu}_i \sim \mathrm{N}(\mu(\boldsymbol{\nu}_i), \sigma^2(\boldsymbol{\nu}_i))$$

with

$$\begin{aligned} \mu(\boldsymbol{\nu}_i) &= \eta_\mu(\boldsymbol{\nu}_i) \\ \sigma^2(\boldsymbol{\nu}_i) &= \exp(\eta_{\sigma^2}(\boldsymbol{\nu}_i)) \end{aligned}$$

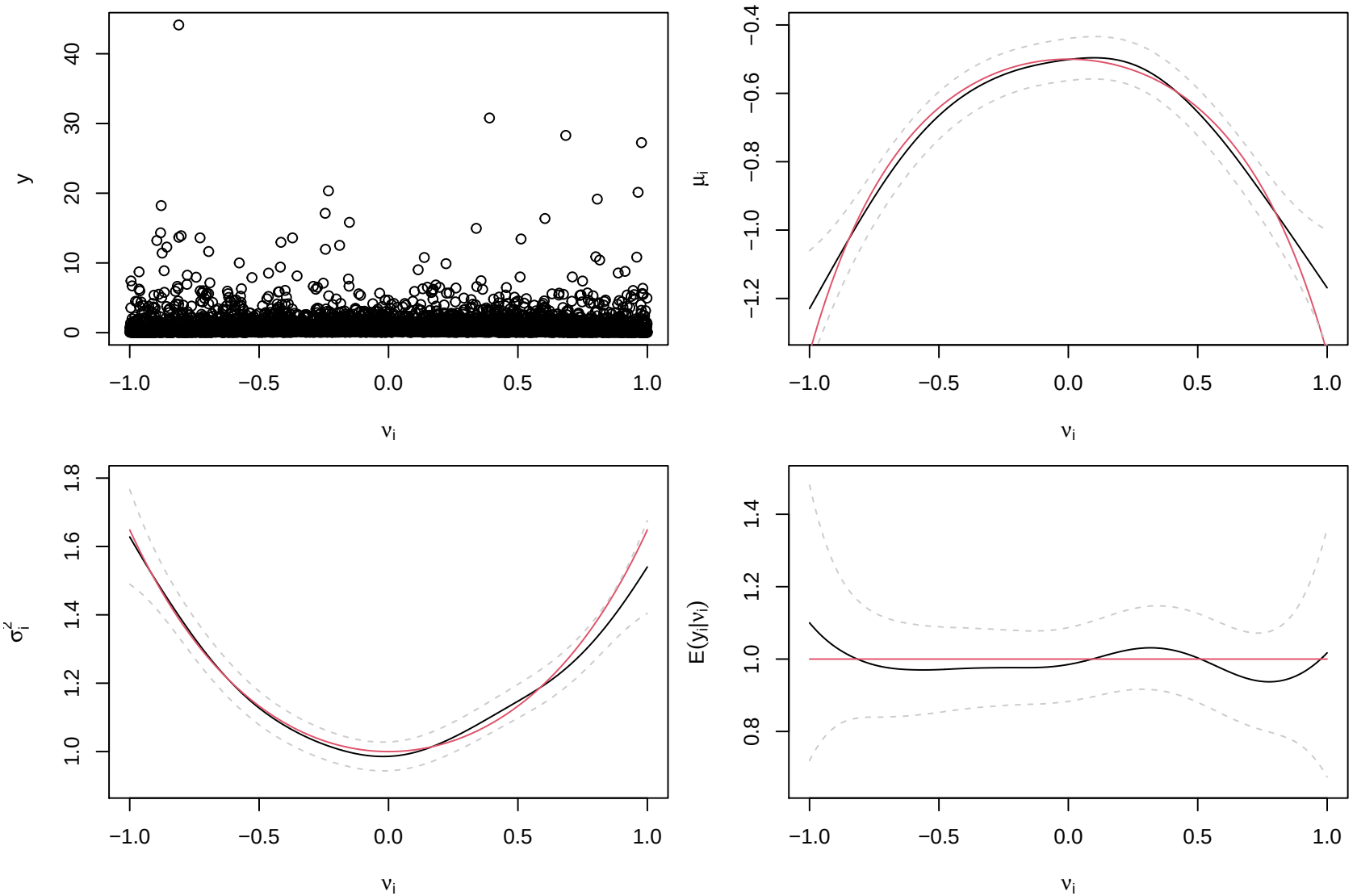
is the most commonly known distributional regression model.

- However, there is a huge variety of models to pick from!

- Examples for interesting response structures:
  - Zero-inflated and / or overdispersed count data, i.e. responses with an excess of zeros and / or variances exceeding the expectation.
  - Responses with heteroscedastic or skewed distribution.
  - Continuous data with a spike in zero.
  - Fractional responses restricted to [0,1] (possibly with inflation in 0 and 1).
  - Multivariate responses with regression effects on the dependency parameters for example based on copula specifications.

- Conceptually, distributional regression is very appealing and intuitive, but it comes with a number of challenges:
  - No conjugate priors for the observation models such that MCMC requires more care.
  - Interpretation of the estimated effects more difficult due to link functions and multi-parameter setup.
  - Model choice and checking to avoid model miss-specification.

- Illustration for simulated log-normal data:



- Quantile residuals
  - For a continuous random variable $Y \sim F$ with cumulative distribution function F, we have $F(Y) \sim \mathrm{U}(0,1)$ (probabilitiy integral transform) or

$$\Phi^{-1}(F(Y)) \sim \mathrm{N}(0,1).$$

  - For a correctly specified distributional regression model, we should therefore have

$$u_i = \Phi^{-1}(F(y_i|\hat{\boldsymbol{\vartheta}}(\nu_i))) \overset{a}{\sim} \mathrm{N}(0,1)$$

    and the quantile residuals u_i can, e.g., be visualized in a quantile-quantile plot.
  - For discrete or multivariate data, appropriate generalisations are needed.

- Information criteria:
  - DIC or WAIC can straightforwardly be used in the context of distributional regression.
- Proper scoring rules:
  - In a distributional setting, typical predictive measures such as the mean squared error of prediction or the mean absolute error of prediction are not adequate.
  - Proper scoring rules provide a framework for evaluating predictive distributions rather than point predictions.
  - Underlying theory ensures that proper scores encourage the analyst to honestly report their uncertainty in terms of the predictive distribution.
  - The cross-validated log-likelihood is the most commonly used proper score.

- Scoring rules for real-valued outcomes with predictive density $f(y)$ and observed outcome y_0:
  - Spherical score
$$\text{SPS}(f(y), y_0) = -\frac{f(y_0)}{\left(\int f^2(t)dt\right)^{1/2}}.$$
  - Logarithmic score
$$\text{LS}(f(y), y_0) = -\log(f(y_0)).$$
  - Continuously ranked probability score
$$\text{CRPS}(f(y), y_0) = \int \left[F(t) - \mathbb{1}_{[y_0,\infty)}(t)\right]^2 dt.$$
  - Note: All scores are negatively oriented, i.e. smaller values indicate a better agreement between the predictive distribution and the observed values.

Case Study: Conditional Income Distributions

- Utilise information from the German Socio-Economic Panel to study real gross annual personal labour income in Germany for the years 2001 to 2010.
- Specific focus on changes in spatial differences in income inequality.
- Response: income of males in full time employment in the age range 20–60.
- Information available on $7,216$ individuals with a total of $n = 40,965$ observations.
- Potential response distributions:
  - Log-normal $\mathrm{LN}(\mu, \sigma^2)$.
  - Gamma $\mathrm{Ga}(\mu, \sigma)$.
  - Inverse Gaussian $\mathrm{IG}(\mu, \sigma^2)$.
  - Dagum $\mathrm{Da}(a, b, c)$.

  with covariate effects on potentially all distributional parameters.

- Covariates:
  - $educ$: Educational level measured as a binary indicator for completed higher education (according to the UNESCO International Standard Classification of Education 1997).
  - age: age in years.
  - $lmexp$: previous labour market experience in years.
  - t calendar time.
  - s: area of residence in terms of geographical district ($Raumordnungsregion$).
  - $east$: indicator in effect coding for districts belonging to the eastern part of Germany.

- Hierarchical predictor structure:

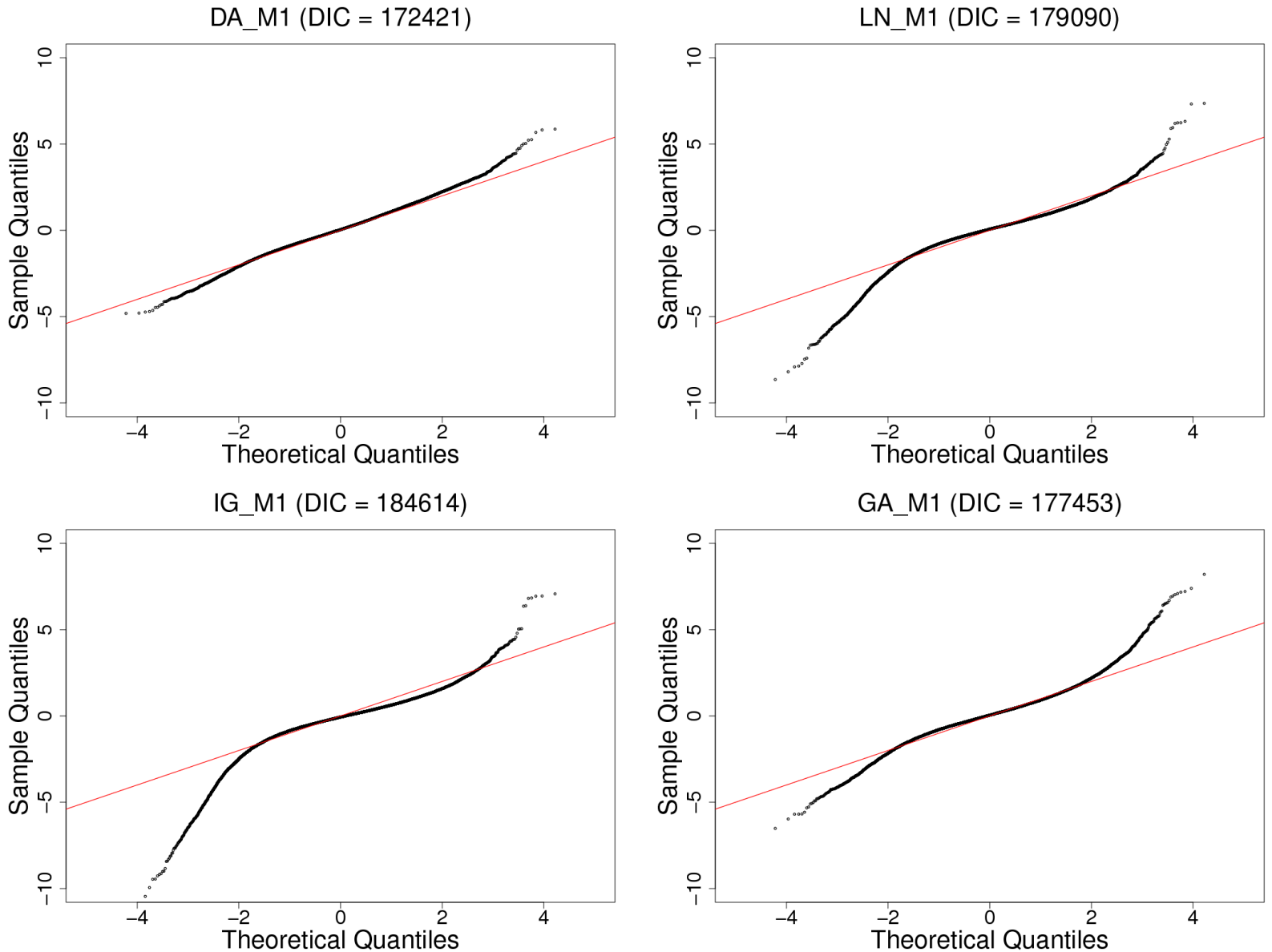
$$\eta_i = \beta_0 + educ_i\beta_1 + f_1(age_i) + educ_i f_2(age_i) + f_3(lmexp_i) + f_{spat}(s_i) + f_{time}(t_i)$$

where the spatial effects is decomposed as

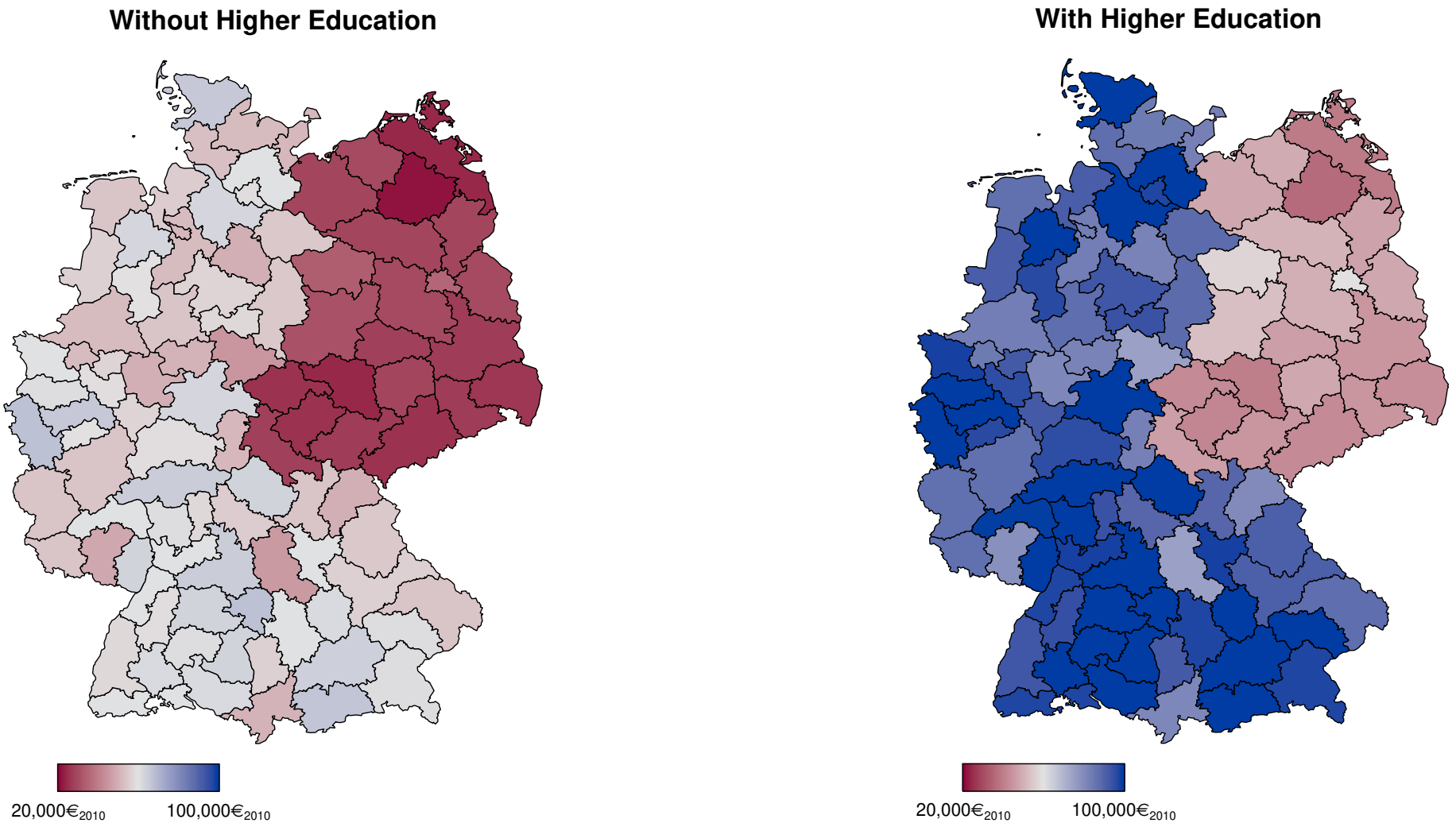
$$f_{spat}(s) = east_s\gamma_1 + g_{str}(s) + g_{unstr}(s)$$

- DIC and scoring rules:

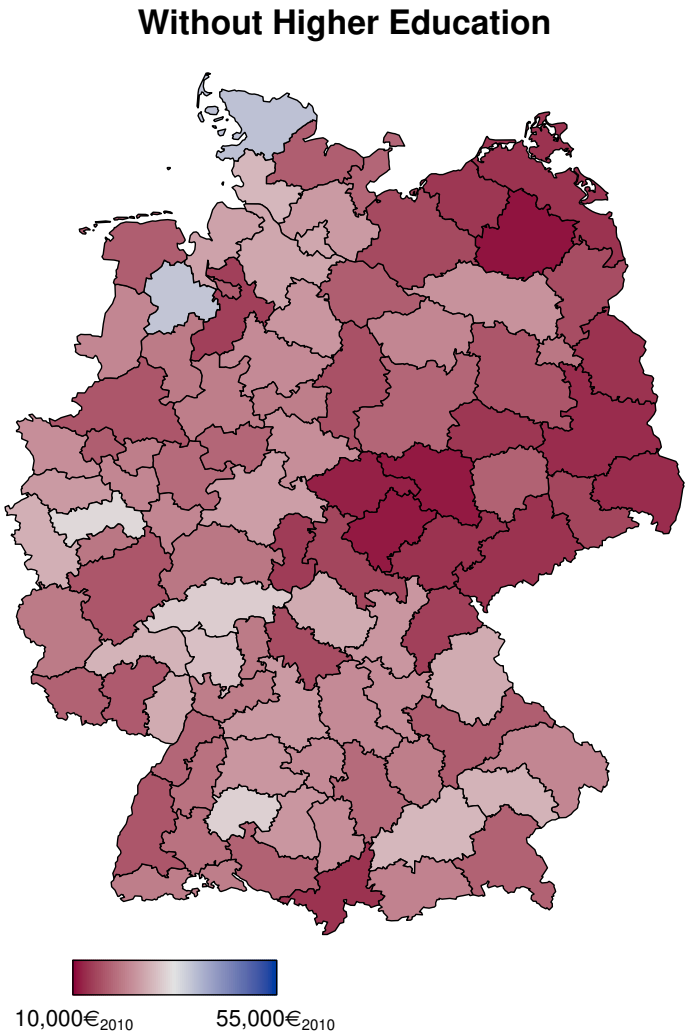
Distribution	DIC	Quadratic	Logarithmic	Spherical	CRPS
LN	179,090	0.1304	-2.4363	0.3621	-2.1581
IG	184,614	0.1464	-2.2741	0.3777	-1.6195
GA	177,453	0.1609	-2.1715	0.3963	-1.2735
DA	**172,421**	**0.1684**	**-2.1034**	**0.4053**	**-1.2662**

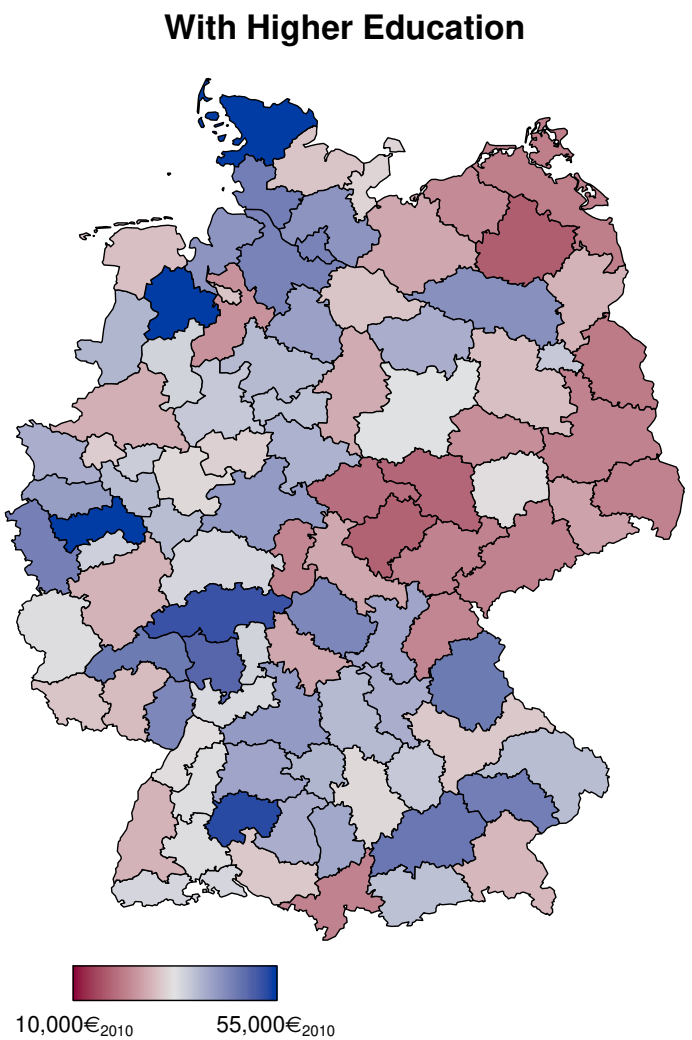

DA_M1 (DIC = 172421)
LN_M1 (DIC = 179090)
IG_M1 (DIC = 184614)
GA_M1 (DIC = 177453)
Sample Quantiles
Theoretical Quantiles
10
5
0
−5
−10
−4
−2
0
2
4

- Expected income for an "average man" with / without higher education:



- Income standard deviation for an "average man":





- Income inequality (measured by the Gini coefficient) for an "average man":