Day 1: Introduction to Bayesian Inference

Exercise 1 (Gibbs Sampler for the Normal Distribution)

Assume the following Bayesian model for $i = 1, \dots, n$:

$$y_i \overset{iid}{\sim} \mathrm{N}(\mu, \sigma^2) \qquad \mu \sim \mathrm{N}(m, s^2) \qquad \sigma^2 \sim \mathrm{IG}(a, b)$$

The probability density function of an $\mathrm{IG}(a, b)$ distribution is

$$f(\sigma^2) = \frac{b}{\Gamma(a)} \left(\frac{1}{\sigma^2}\right)^{a+1} \exp\left(-\frac{b}{\sigma^2}\right).$$

- Identify the individual model parts: i) observation model, ii) priors, iii) parameters, iv) hyperparameters.
- Sketch the generic approach for Gibbs sampling from the joint posterior distribution of μ and σ^2.
- Derive the full conditional for μ.
- Derive the full conditional for σ.

Exercise 2 (Metropolis-Hastings with Random Walk Proposals)

Let $y_1, \dots, y_n$ be a sample from a normal distribution such that $y_i \overset{ind.}{\sim} \mathrm{N}(\mu_i, \sigma^2)$. Given covariate observations $x_1, \dots, x_n$, assume the following linear regression model for the mean:

$$\mu_i = \beta_0 + \beta_1 x_i,$$

Assume a constant prior for β_0 and a normal prior with mean b and variance s^2 for β_1. The regression coefficients can be collected in a vector $\boldsymbol{\beta} = (\beta_0, \beta_1)'$.

- Derive the natural logarithm of the acceptance probability for a proposal $\boldsymbol{\beta}^*$ in a Metropolis-Hastings step, using a general proposal density $h(\boldsymbol{\beta}^* | \boldsymbol{\beta}^{[t-1]})$, where $t = 1, \dots, T$ indicates the MCMC iteration.
- Implement the log posterior for $\boldsymbol{\beta}$ in `R` and set up a Metropolis-Hastings-Algorithm to sample from the posterior distribution of $\boldsymbol{\beta}$.
- The file `linreg.dat` contains observations of the response and the covariate. Apply your implemented sampler for these data with the following settings:
  - Use $b = 0$ and $s^2 = 100$ for the prior of $\boldsymbol{\beta}_1$.
  - Use a random walk proposal $\boldsymbol{\beta}^* \sim \mathrm{N}(\boldsymbol{\beta}^{[t-1]}, 1)$.
  - Use $T = 1,000$, $T = 10,000$, and $T = 100,000$ posterior samples.
  - Use a starting value of $\boldsymbol{\beta}^{[0]} = \mathbf{0}$.
  - Assume $\sigma^2 = 3$ to be known.
- Use `plot()` or `matplot(..., type = "l")` to plot trace plots of your samples and `acf()` to plot the autocorrelation.

- Discard burn in samples and apply thinning, then compute the posterior mean parameter estimates and the posterior standard deviations for $\boldsymbol{\beta}$.

Exercise 3 (Hamiltonian Monte Carlo)

We are assuming the same setting as in the previous exercise but now consider Hamiltonian Monte Carlo.

- Sketch a Hamiltonian Monte Carlo algorithm for drawing samples from the posterior of $\boldsymbol{\beta}$.
- Find the derivative of the log posterior with respect to $\boldsymbol{\beta}$. You can assume a prior mean of $b = 0$ for β_1.
- Implement HMC for $\boldsymbol{\beta}$ in `R`.
- Apply your sample to the data in `linreg.dat` with the following settings:
  - Use $b = 0$ and $s^2 = 100$ for the prior of β_1.
  - Use the identity matrix of appropriate dimensions as the mass matrix.
  - Use a step size of $\epsilon = 0.1$.
  - Use $L = 10$ leapfrog steps, such that $\epsilon L = 1$.
  - Use $T = 1,000$ posterior samples.
  - Use a starting value of $\boldsymbol{\beta}^{[0]} = \mathbf{0}$.
  - Assume $\sigma^2 = 3$ to be known.
- Use `plot()` or `matplot(..., type = "l")` to plot a trace plot of your samples and `acf()` to plot the autocorrelation.
- Compute the posterior mean parameter estimates and the posterior standard deviations.

Exercise 4 (IWLS)

The iteratively weighted least squares (IWLS) sampler can be seen as a variant of Metropolis-Hastings using the proposal density

$$\boldsymbol{\vartheta}_s^* \sim \mathrm{N}(\boldsymbol{m}_s, \boldsymbol{F}_s^{*-1}),$$

for a proposal $\boldsymbol{\vartheta}_s^*$ with mean

$$\boldsymbol{m}_s = \boldsymbol{\vartheta}_s^{[t-1]} + \boldsymbol{F}_s^{*-1}\boldsymbol{s}_s,$$

expected Fisher information

$$\boldsymbol{F}_s^* = \mathbb{E}\left(-\frac{\partial^2}{\partial\boldsymbol{\vartheta}_s\partial\boldsymbol{\vartheta}_s'}\log f(\boldsymbol{\vartheta}_s^{[t-1]}|\vartheta_{-s}^{[t]}, \boldsymbol{y})\right)$$

and score function

$$\boldsymbol{s}_s = \frac{\partial}{\partial\boldsymbol{\vartheta}_s}\log f(\boldsymbol{\vartheta}_s^{[t-1]}|\boldsymbol{\vartheta}_{-s}^{[t]}, \boldsymbol{y})$$

In this formulation, $t = 1, \ldots, T$ indicates the MCMC iteration, $\boldsymbol{y}$ is a vector of response observations (the data), $\boldsymbol{\vartheta}_s$ is a parameter block and $f(\boldsymbol{\vartheta}_s|\boldsymbol{\vartheta}_{-s}, \boldsymbol{y})$ is the full conditional of the parameter block.

- Derive the quantities $\boldsymbol{F}_s^*$ and $\boldsymbol{s}_s$ for $\boldsymbol{\beta}$ using the model from the previous exercises.
- Implement an IWLS sampler for $\boldsymbol{\beta}$ using this model in `R`.
- The file `linreg.csv` contains observations of the response and the covariate. Apply your implemented sampler for these data. Use the following settings:

- Use $b = 0$ and $s^2 = 100$ for the prior of β_1.
- Use $T = 1,000$ posterior samples.
- Use a starting value of $\boldsymbol{\beta}^{[0]} = \mathbf{0}$.
- Assume $\sigma^2 = 3$ to be known.

- Use `plot()` or `matplot(..., type = "l")` to plot a trace plot of your samples and `acf()` to plot the autocorrelation.
- Compute the posterior mean parameter estimates and the posterior standard deviations.