Controlled system

$$\boldsymbol{M}\vec{a} + \widetilde{\boldsymbol{C}}(\vec{r},\vec{v}) = \widetilde{\boldsymbol{J}}(\vec{r},\vec{v})\,\boldsymbol{\tau} + \widetilde{\boldsymbol{G}}(\vec{r})$$

ODE formulation of controlled system

$$\boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix} \;;\quad \boldsymbol{x}_1 = \begin{bmatrix}x_1 & \dots & x_n\end{bmatrix}^T = \vec{r}\,,\quad \boldsymbol{x}_2 = \begin{bmatrix}x_{n+1} & \dots & x_r\end{bmatrix}^T = \vec{v}\,;\quad \boldsymbol{u} = \boldsymbol{\tau}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f} = \begin{bmatrix}\boldsymbol{x}_2\\ \boldsymbol{w}\end{bmatrix}\,,\quad \boldsymbol{w} = \boldsymbol{M}^{-1}\big(\boldsymbol{J}(\boldsymbol{x})\boldsymbol{u} + \boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{C}(\boldsymbol{x})\big)$$

Cost function

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x},\boldsymbol{u})\,dt\,,\quad L(\boldsymbol{x},\boldsymbol{u}) = \boldsymbol{x}^T\boldsymbol{Q}\,\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\,\boldsymbol{u}$$

Hamiltonian

$$H = -L(\boldsymbol{x},\boldsymbol{u}) + \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})^T\boldsymbol{p}$$

ODE of conjugated parameters

$$\dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{x}} = 2\boldsymbol{Q}\boldsymbol{x} - \frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}}\boldsymbol{p}\,,\quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix}\boldsymbol{0} & \boldsymbol{E}_s\\ \multicolumn{2}{c}{\partial\boldsymbol{w}/\partial\boldsymbol{x}}\end{bmatrix}$$

Condition for optimal control

$$\frac{\partial H}{\partial \boldsymbol{u}} = -2\boldsymbol{R}\boldsymbol{u} + \frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{u}}\boldsymbol{p} = \boldsymbol{0}$$

Optimal input

$$\boldsymbol{u} = \frac{1}{2}\boldsymbol{R}^{-1}\frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{u}}\boldsymbol{p}\,,\quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{bmatrix}\boldsymbol{0}_{s\times s}\\ \boldsymbol{M}^{-1}\boldsymbol{J}\end{bmatrix}$$

Boundary Value Problem (BVP) of Pontryagin's maximum principle

$$\dot{\boldsymbol{y}} = \begin{bmatrix}\dot{\boldsymbol{y}}_1\\ \dot{\boldsymbol{y}}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}(\boldsymbol{x},\boldsymbol{p}))\\ \dot{\boldsymbol{p}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u}(\boldsymbol{x},\boldsymbol{p}))\end{bmatrix}\;;\quad \boldsymbol{y}_1(t_0) = \boldsymbol{x}_0\,,\ \boldsymbol{y}_1(t_f) = \boldsymbol{x}_f$$