Controlled system

$$oldsymbol{M}ec{a}+\widetilde{oldsymbol{C}}(ec{r},ec{v})=\widetilde{oldsymbol{J}}(ec{r},ec{v})\,oldsymbol{ au}+\widetilde{oldsymbol{G}}(ec{r})$$

ODE formulation of controlled system

$$egin{aligned} oldsymbol{x} &= egin{bmatrix} oldsymbol{x}_1 \\ oldsymbol{x}_2 \end{bmatrix} \;; \quad oldsymbol{x}_1 &= ig[x_1 & \dots & x_nig]^T = ec{r} \;, \quad oldsymbol{x}_2 &= ig[x_{n+1} & \dots & x_rig]^T = ec{v} \;; \quad oldsymbol{u} &= oldsymbol{ au} \ \dot{oldsymbol{x}} &= oldsymbol{f} = ig[x_2 \\ oldsymbol{w} \end{bmatrix} \;, \quad oldsymbol{w} &= oldsymbol{M}^{-1} ig(oldsymbol{J}(oldsymbol{x}) oldsymbol{u} + oldsymbol{G}(oldsymbol{x}) - oldsymbol{C}(oldsymbol{x}) ig) \end{aligned}$$

Cost function

$$J = \int_{t_0}^{t_f} L(\boldsymbol{x}, \boldsymbol{u}) dt , \quad L(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}^T \boldsymbol{Q} \, \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \, \boldsymbol{u}$$

Hamiltonian

$$H = -L(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})^T \boldsymbol{p}$$

ODE of conjugated parameters

$$\dot{\boldsymbol{p}} = -rac{\partial H}{\partial \boldsymbol{x}} = 2\boldsymbol{Q}\boldsymbol{x} - rac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{x}}\boldsymbol{p} \;, \quad rac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = egin{bmatrix} \mathbf{0} & \boldsymbol{E}_s \\ \partial \boldsymbol{w}/\partial \boldsymbol{x} \end{bmatrix}$$

Condition for optimal control

$$\frac{\partial H}{\partial \boldsymbol{u}} = -2\boldsymbol{R}\boldsymbol{u} + \frac{\partial \boldsymbol{f}^T}{\partial \boldsymbol{u}}\boldsymbol{p} = \boldsymbol{0}$$

Optimal input

$$u = \frac{1}{2} R^{-1} \frac{\partial f^T}{\partial u} p$$
, $\frac{\partial f}{\partial u} = \begin{bmatrix} \mathbf{0}_{s \times s} \\ M^{-1} J \end{bmatrix}$

Boundary Value Problem (BVP) of Pontryagin's maximum principle

$$\dot{oldsymbol{y}} = egin{bmatrix} \dot{oldsymbol{y}}_1 \ \dot{oldsymbol{y}}_2 \end{bmatrix} = egin{bmatrix} oldsymbol{f}(oldsymbol{x}, oldsymbol{u}(oldsymbol{x}, oldsymbol{p}, oldsymbol{u}(oldsymbol{x}, oldsymbol{p}) \end{bmatrix} \; ; \quad oldsymbol{y}_1(t_0) = oldsymbol{x}_0 \, , \; oldsymbol{y}_1(t_f) = oldsymbol{x}_f$$