Foundations of Statistical Inference, BS2a, Exercises 2

1. Let $X_1, \ldots, X_n$ be a random sample from the uniform distribution $U[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$. Show that the MLE of θ is any value $\widehat{\theta}$ in the interval $[\max(X_i) - \frac{1}{2}, \min(X_i) + \frac{1}{2}]$. What is the method of moments estimator?

2. The random variable X has a discrete distribution such that $P(X = r) = \theta^{-1}$ for $r = 1, 2, \ldots, \theta$, where θ is an unknown positive integer. Show that Y, the maximum of a sample of n independent observations of X, is a complete sufficient statistic for θ, and hence verify that

$$\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}$$

is a minimum-variance unbiased estimator for θ.

3. Consider a binomial experiment with probability of success p in which m fixed trials are conducted, resulting in R successes; a further set of trials is then conducted until s (fixed) further successes have occurred. The number of trials necessary in the second set is a random variable N. By considering the function

$$U(R, N) = \frac{R}{m} - \frac{s-1}{N-1}$$

show that (R, N) are jointly sufficient for p, but not complete.

4. (GJJ 2.19) The random variables $X_1, X_2, ..., X_n$ are iid with density $f(x;\theta) = \theta x^{\theta-1}$ for $0 < x < 1$ and $\theta > 0$ unknown.

(i) Find a sufficient statistic T for θ.

(ii) Given that $-\log(X_1)$ is unbiased for θ^{-1}, find another unbiased estimator with smaller variance.Give a simple expression of this estimator involving T.

5. Let $X = (X_1, \ldots, X_n)$ be a random sample from a density $f_X(x;\theta)$ belonging to a parametric family $\mathcal{F}$. Let $T = t(X)$ be a function of X and denote the density of T by $f_T(t;\theta)$. Assuming statistical regularity, define $i_X(\theta)$ to be the Fisher information about θ in X. Finally, let $i_{X|t}(\theta)$ denote the Fisher information conditional on $T = t$ and define

$$i_{X|T}(\theta) = \int i_{X|t}(\theta) f_T(t;\theta) dt$$

(a) Show that

$$i_X(\theta) = i_{X|T}(\theta) + i_T(\theta)$$

(b) Show that

$$i_X(\theta) \geq i_T(\theta),$$

with equality for all θ if and only if $T = t(X)$ is sufficient for θ.

Hint: Use the factorization theorem for the density

$$f_X(x;\theta) = f_{X|T}(x \mid t;\theta) f_T(t;\theta).$$

(c) Hence, or otherwise, determine the Fisher information about θ in the first r order statistics

$$X_{(1)} < X_{(2)} < \cdots < X_{(r)}$$

of a sample of size n from the density

$$f(x;\theta) = \theta \exp(-\theta x),\ x > 0$$

6. Suppose $T(x)$ is complete sufficient for θ given data x. Show that if a minimal sufficient statistic $S(x)$ for θ exists, then $T(x)$ is also minimal sufficient.