Attn : Prof Alex Scott.

Real Analysis

TT W8

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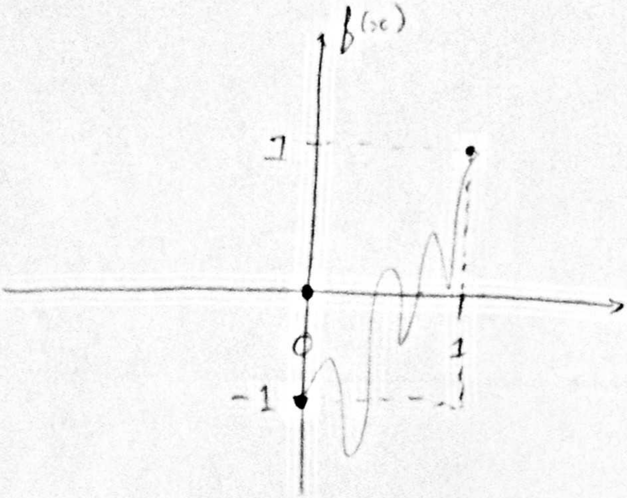
2. Show that the equation $x^{16}+x^7-1=0$ has a solution $\xi \in (0,1)$.

$\Rightarrow$ $x^{16}+x^7-1$ is a polynomial $\Rightarrow$ $f(x)$ is continuous.



Consider $f(0)=-1$. Now consider $f(1)=1$.

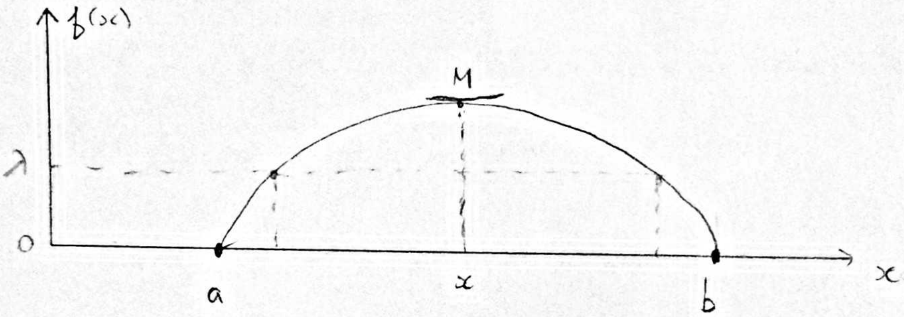
Now consider the function over the closed interval $[0,1]$. $f(0)=-1$ and $f(1)=1$. By the Intermediate Value Theorem, since 0 lies between -1 and 1, we can find a ξ between 0 and 1 such that $0=f(\xi)$.

"But this is exactly what we needed to prove."* QED$_\square$

↳ *technically $\xi \in [0,1]$, not $(0,1)$, but since $f(0)\neq 0$ and $f(1)\neq 0$, $\xi \in (0,1)$.

3. $f : [a,b] \to \mathbb{R}$ has the property that $f(x) \geq 0$ for $a \leq x \leq b$ and $f(a) = 0$, $f(b) = 0$. If for each $x \in [a,b]$ there exists exactly one distinct $y \in [a,b]$ such that $f(x) = f(y)$, prove that f cannot be continuous on $[a,b]$.

We prove by contradiction.



If f is continuous on $[a,b]$, it must obtain a maximum on the interval $[a,b]$.

Let this maximum M be reached at point x.

Now consider the function on the intervals $[a,x]$ and $[x,b]$. Of course, f must be continuous on both intervals, and go from $[0,M]$ from $[a,x]$ and $[M,0]$ from x to b.

psch me!

Consider the value $\lambda \in$ that lies between 0 and M.

By the Intermediate Value Theorem, we can find a ξ_1 between a and x such that $\lambda = f(\xi_1)$.

But similarly, we can also find a ξ_2 between x and b such that $\lambda = f(\xi_2)$. But this contradicts our distinctness assumption, as $\xi_2 > \xi_1$ and yet $f(\xi_2) = f(\xi_1)$.

Hence f cannot be continuous on $[a,b]$.

5. Suppose $a < c < d < b$ and $f:(a,b) \to \mathbb{R}$ is differentiable on (a,b). If $f'(c) > 0$ and $f'(d) < 0$, prove there exists $\xi \in (c,d)$, such that $f'(\xi) = 0$.

~~Consider the~~ ~~interval (c,d).~~

~~As (c,d) is contained within (a,b)~~

~~$f(c,d) \to \mathbb{R}$ is differentiable on the interval~~

~~It follows that it is continuous.~~

Consider the closed interval $[c,d]$. By continuity, it must have a maximum M.

Claim:

$f(c)$ and $f(d)$ are not maximum M.

Proof:

$f'(c) > 0$.

Let $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \varepsilon > 0$. $\exists \delta > 0$ s.t. $\forall |x-c| < \delta$,

$\left| \frac{f(x)-f(c)}{x-c} - \varepsilon \right| < \frac{\varepsilon}{2}$. This implies that

$\Rightarrow \frac{f(x)-f(c)}{x-c} > \frac{\varepsilon}{2}$. As $x > c$,

$\Rightarrow f(x) - f(c) > \frac{\varepsilon}{2}(x-c) > 0$. Therefore, $f(c)$ cannot be a maximum.

Similarly,

$\lim_{x \to d} \frac{f(x)-f(d)}{x-d} < 0$, $\Rightarrow \frac{f(d)-f(x)}{x-d} > 0$.

By a similar argument

$f(d) - f(x) < \frac{\varepsilon}{2}(x-d) < 0$; $f(d)$ cannot be a max.

$\Rightarrow$ Since $f(c)$ and $f(d)$ are not maximum, this implies that there exists $\xi \in (c,d)$ s.t. $f(\xi) = M$.

But if $\exists \xi \in (c,d)$ s.t. $f(\xi) = M$, then this implies (by theorem 11.2) that $f'(\xi) = 0$. $\square$ ✓

Deduce that the image of (a,b) under f' is an interval.

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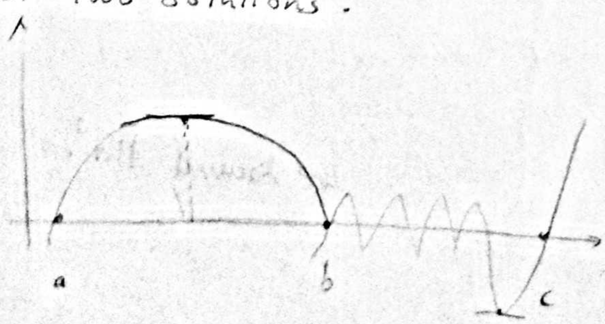
6. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable at every point and satisfy $f'(x) > 0$ for all values of x. Prove that the equation $f(x) = 0$ can have at most one solution.

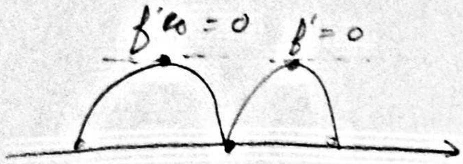
We prove by contradiction. Suppose that there were exactly two solutions of $f(x) = 0$, call them a and b. Then, by Rolle's Theorem, there must exist a stationary point $\xi \in (a, b)$ s.t. $f'(\xi) = 0$. But this gives us a contradiction as $f'(x) > 0$ $\forall x$. With any number of solutions greater or equal to two (e.g. $f(a) = f(b) = f(c) = 0$), applying Rolle's Theorem to any pair gives us a contradiction. Hence, it can have at most one soln. $\square$

✓

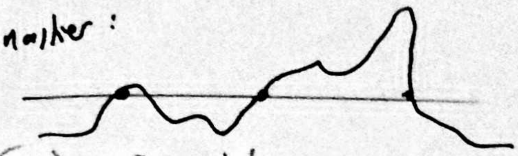
If $f''(x) > 0$ $\forall x$ show $f(x) = 0$ can have at most two solutions.



If $f(x)$ has more than two solutions, there must be at least two stationary points.



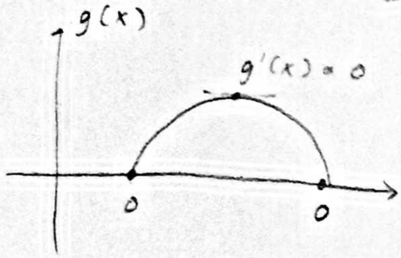
← right idea, but you should use Rolle's Theorem, as the picture could be another:



Let $g = f'$. Then $g'(x) > 0$ $\forall x$,

and we have $g(x) = 0$ on two occasions.

But by a similar application of Rolle's Theorem,



this leads to a contradiction.

For derivatives of order n: if $f: R \to R$ is differentiable at every point and $f^{(n)}(x) > 0$ for all x, $f(x)$ can have at most $n-1$ solutions.

Proof ... ?

We discussed this!