# Assignment One

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**1. For each of the following functions, indicate how much the function's value will change if its argument is increased fourfold.**

a. $log_2 n$ b.$\sqrt{n}$ c.$n$ d.$n^2$ e.$n^3$ f.$2^n$

**Solution:**

a. $log_2(4n) = log_2 n + log_2 4 = log_2 n + 2$,then $log_2(4n) - log_2 n = 2$.
when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{log_2(4n)}{log_2 n} = \lim\limits_{n\to\infty} \frac{log_2 n+2}{log_2 n} = 1$.

b. $\frac{\sqrt{4n}}{\sqrt{n}} = \sqrt{4} = 2$, when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{\sqrt{4n}}{n} = 2$.

c. $\frac{4n}{n} = 4$, when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{4n}{n} = 4$.

d. $\frac{(4n)^2}{n^2} = 4^2 = 16$, when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{(4n)^2}{n^2} = 16$.

e. $\frac{(4n)^3}{n^3} = 4^3 = 64$, when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{(4n)^3}{n^3} = 64$.

f. $\frac{2^{4n}}{2^n} = 2^{3n}$, when $n \to \infty$, $\lim\limits_{n\to\infty} \frac{2^{4n}}{2^n} = \infty$.

**2. Prove (by using the definitions of the notations involved) or disprove (by giving a specific counterexample) the following assertions.**

**Solution:**

a. if $t(n) \in O(g(n))$, then $g(n) \in \Omega(t(n))$.

The assertion is true. **Prove:**

According to the definition:

$$\begin{aligned} t(n) \in O(g(n)) &\Rightarrow t(n) \leq cg(n)) \text{ for all } n > n_0 \ (n_0 > 0), \text{where } c > 0 \\ &\Rightarrow (\frac{1}{c})t(n) \leq g(n) \text{ for all } n > n_0 \ (n_0 > 0), \text{where } c > 0 \\ &\Rightarrow g(n) \in \Omega(t(n)) \end{aligned}$$

b. $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

The assertion is true. **Prove:**

According to the definition:

$\Theta(\alpha g(n)) = \Theta(g(n)) \Leftrightarrow \Theta(\alpha g(n)) \subseteq \Theta(g(n))$ and $\Theta(g(n)) \subseteq \Theta(\alpha g(n))$

I. Let $t(n) \in \Theta(\alpha g(n))$:

$$\begin{aligned} t(n) \in \Theta(\alpha g(n)) &\Rightarrow c_1 \alpha g(n) \leq t(n) \leq c_2 \alpha g(n) \text{ for all } n > n_0, \text{ where } c_1 > 0, c_2 > 0 \\ &\Rightarrow c_{10} g(n) \leq t(n) \leq c_{20} g(n) \text{ for all } n > n_0, \text{ where } c_{10} = c_1 \alpha > 0, c_{20} = c_2 \alpha > 0 \\ &\Rightarrow t(n) \in \Theta(g(n)) \end{aligned}$$

II. let now $t(n) \in \Theta(g(n))$:

$$\begin{aligned} t(n) \in \Theta(g(n)) &\Rightarrow c_1 g(n) \leq t(n) \leq c_2 g(n) \text{ for all } n > n_0, \text{ where } c_1 > 0, c_2 > 0 \\ &\Rightarrow c_{10} \alpha g(n) \leq t(n) \leq c_{20} \alpha g(n) \text{ for all } n > n_0, \text{ where } c_{10} = c_1/\alpha > 0, c_{20} = c_2/\alpha > 0 \\ &\Rightarrow t(n) \in \Theta(\alpha g(n)) \end{aligned}$$

According to I and II: $\Theta(\alpha g(n)) = \Theta(g(n))$, where $\alpha > 0$.

c. $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$.

The assertion is true. **Prove:**

According to the definition:

I. Let $t(n) \in \Theta(g(n))$:

$$t(n) \in \Theta(g(n) \Rightarrow c_1 g(n) \leq t(n) \leq c_2 g(n) \text{ for all } n > n_0, \text{where } c_1 > 0, c_2 > 0$$

II. Let $t(n) \in O(g(n))$:

$$t(n) \in O(g(n)) \Rightarrow t(n) \leq bg(n) \text{ for all } n > n_0, \text{where } b > 0$$

III. Let $t(n) \in \Omega(g(n))$:

$$t(n) \in \Omega(g(n)) \Rightarrow ag(n) \leq t(n) \text{ for all } n > n_0, \text{where } a > 0$$

IV. According to II and III:

$$t(n) \in O(g(n)) \cap \Omega(g(n)) \Rightarrow ag(n) \leq t(n) \leq bg(n) \text{ for all } n > n_0, \text{where } a > 0, b > 0$$

According to I and IV, because the constants$(c_1, c_2, a, b)$ are arbitrary, hence:

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

d. For any two nonnegative functions $t(n)$ and $g(n)$ defined on the set of non-negative integers, either $t(n) \in O(g(n))$ , or $t(n) \in \Omega(g(n))$ , or both.

The assertion is false. **Disprove:**

eg:

$$t(n) = \begin{cases} n, \text{if n is odd} \\ 2^n, \text{if n is even} \end{cases}$$

$$g(n) = \begin{cases} 2^n, \text{if n is odd} \\ n, \text{if n is even} \end{cases}$$

Obviously, there's no $n_0$ that makes $t(n) \leq cg(n)$, or $cg(n) \leq t(n)$ or both for all $n > n_0$ true.

**3. Solve the following recurrence relations.**

a. $x(n) = 3x(n-1)$ for $n > 1, x(1) = 4$

**Solution:**

According to the geometric progression:

$$x(n) = 4 \cdot 3^{n-1}$$

b. $x(n) = x(n-1) + n$ for $n > 0, x(0) = 0$

**Solution:**

According to Method of Backward Substitution:

$$\begin{aligned}
x(n) &= x(n-1) + n \\
&= (x(n-2) + n - 1) + n \\
&= (x(n-3) + n - 2) + n - 1) + n \\
&= ... \\
&= x(0) + 1 + 2 + 3 + ... + n \\
&= \frac{n(n-1)}{2}
\end{aligned}$$

c. $x(n) = x(\frac{n}{2}) + n$ for $n > 1, x(1) = 1$ (solve for $n = 2^k$).

**Solution:**

Let $n = 2^k$:

$$\begin{aligned}
x(n) = x(2^k) &= x(2^{k-1}) + 2^k \\
&= (x(2^{k-2}) + 2^{k-1}) + 2^k \\
&= ... \\
&= x(2^0) + 2^1 + 2^2 + ... + 2^k \\
&= 2^0 + 2^1 + 2^2 + ... + 2^k \\
&= \frac{2^0(1 - 2^{k+1})}{1-2} \\
&= 2^{k+1} - 1 \\
&= 2 \cdot 2^k - 1 \\
&= 2n - 1
\end{aligned}$$