第一次习题课

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预备知识

1. 爱因斯坦求和规则：公式中重复指标自动求和，略去求和号

例: $$\sum_{i=1}^{3} A_i B_i = A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$

$$= \vec{A} \cdot \vec{B} = A_i B_j \hat{e_i} \cdot \hat{e_j}$$

$$\sigma_{ij}\varepsilon_{ij} = \sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{13}\varepsilon_{13} + \sigma_{21}\varepsilon_{21}$$

$$+ \sigma_{22}\varepsilon_{22} + \sigma_{23}\varepsilon_{23} + \sigma_{31}\varepsilon_{31} + \sigma_{32}\varepsilon_{32} + \sigma_{33}\varepsilon_{33}$$

$$a_{ij}x_j = b_i \text{即为:} \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array}$$

预备知识

2. Kronecker delta函数δ_{ij}(二阶完全对称张量)

$$\delta_{ij} = \delta_{ji} = \hat{e_i} \cdot \hat{e_j} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

$$\delta_{im} \cdot \delta_{mj} = \delta_{ij}$$

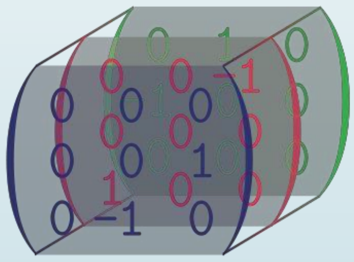
预备知识

3. Levi-Civita 张量 ϵ_{ijk}

$$\epsilon_{ijk} = \begin{cases} 1 & ijk = 123, 231, 312 \\ -1 & ijk = 213, 132, 321 \\ 0 & otherwise \end{cases}$$

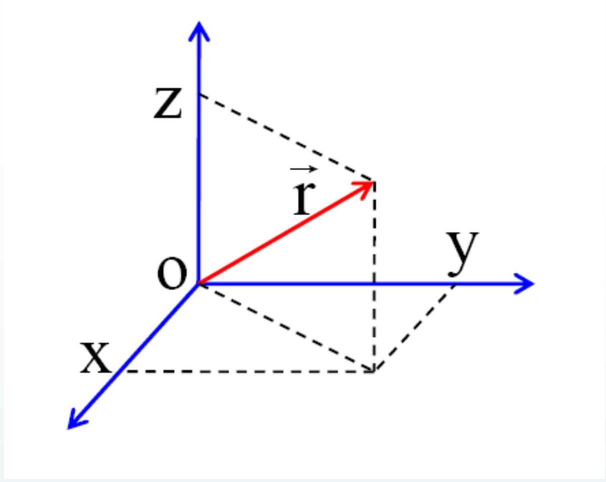
$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{jl} & \delta_{ml} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$\epsilon_{ijk} =$



矢量的表示

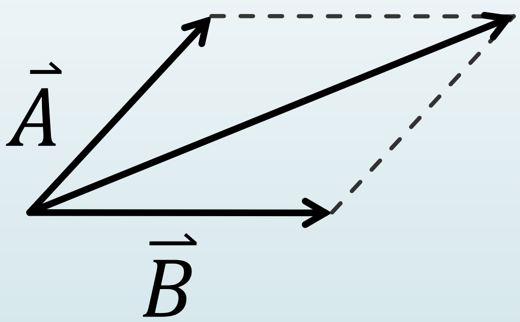
1、直接描述其大小与方向



2、坐标系表示

矢量的加法

1、



$$\vec{V} = \vec{A} + \vec{B}$$

以A, B 邻边所构成的平行四边形的对角线V,即为矢量A , B 的合矢量，记为

$$\vec{V} = \vec{A} + \vec{B}$$

2、坐标系：

$$\vec{A} + \vec{B} = (A_x\widehat{e_x} + A_y\widehat{e_y} + A_z\widehat{e_z}) + (B_x\widehat{e_x} + B_y\widehat{e_y} + B_z\widehat{e_z})$$

矢量代数

- 1. 矢量点积

$$\vec{A}\cdot\vec{B} = AB\cos(\vec{A},\vec{B})$$

$$\vec{A}\cdot\vec{B} = AiBj\hat{e_i}\cdot\hat{e_j} = A_iB_j\delta_{ij}$$

$$= A_iB_i$$

矢量代数

2. 矢量叉积

两矢量$\vec{A},\vec{B}$的矢量积是一矢量$\vec{C}$。

$\vec{C}$的模是$\vec{A},\vec{B}$的模与两矢量夹角的正弦$\sin\theta$的乘积，$\vec{C}$垂直于$\vec{A},\vec{B}$的平面，且$\vec{A},\vec{B},\vec{C}$构成右手关系

$$C = AB\sin(\vec{A},\vec{B})$$

$$\vec{C} = \vec{A}\times\vec{B}$$

$$= \begin{vmatrix} \widehat{e_x} & \widehat{e_y} & \widehat{e_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \epsilon_{ijk}A_iB_j\widehat{e_k}$$

矢量代数

- 3. 三重标积（平行六面体体积）

$$\vec{A}\cdot(\vec{B}\times\vec{C})=A_m\widehat{e_m}\cdot\epsilon_{ijk}B_iC_j\widehat{e_k}=A_m\epsilon_{ijk}B_iC_j\widehat{e_m}\cdot\widehat{e_k}$$
$$=A_m\epsilon_{ijk}B_iC_j\delta_{mk}=\epsilon_{ijk}A_kB_iC_j$$

证明：$\vec{A}\cdot(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{C}\times\vec{A})=\vec{C}\cdot(\vec{A}\times\vec{B})$

法一：

$$\vec{A}\cdot(\vec{B}\times\vec{C})=\begin{vmatrix}A_x & A_y & A_z\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}$$

法二：

$$\vec{A}\cdot(\vec{B}\times\vec{C})=A_m\widehat{e_m}\cdot\epsilon_{ijk}B_iC_j\widehat{e_k}$$

矢量代数

- 4. 三重矢积

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}\cdot(\vec{A}\cdot\vec{C})\text{-}\vec{C}\cdot(\vec{A}\cdot\vec{B})$$

$$\begin{aligned}\vec{A}\times(\vec{B}\times\vec{C})&=A_m\widehat{e_m}\times\epsilon_{ijk}B_iC_j\widehat{e_k}\\&=\epsilon_{ijk}\epsilon_{mkn}A_mB_iC_j\widehat{e_n}\\&=(\delta_{in}\delta_{jm}-\delta_{im}\delta_{jn})A_mB_iC_j\widehat{e_n}\\&=A_jB_iC_j\widehat{e_i}-A_iB_iC_j\widehat{e_j}\\&=\vec{B}(\vec{A}\cdot\vec{C})\text{-}\vec{C}(\vec{A}\cdot\vec{B})\end{aligned}$$

Delta算符：∇

∇算符既有矢量性，又有微分性：

$$\nabla = \widehat{e_x}\frac{\partial}{\partial x} + \widehat{e_y}\frac{\partial}{\partial y} + \widehat{e_z}\frac{\partial}{\partial z} = \widehat{e_i}\partial_i$$

梯度：$\nabla T = \widehat{e_x}\frac{\partial T}{\partial x} + \widehat{e_y}\frac{\partial T}{\partial y} + \widehat{e_z}\frac{\partial T}{\partial z} = \widehat{e_i}\partial_i T$

散度：$\nabla \cdot \vec{V} = \left(\widehat{e_x}\frac{\partial f}{\partial x} + \widehat{e_y}\frac{\partial f}{\partial y} + \widehat{e_z}\frac{\partial f}{\partial z}\right) \cdot \left(V_x\widehat{e_x} + V_y\widehat{e_y} + V_z\widehat{e_z}\right)$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \partial_i V_i$$

旋度：$\nabla \times \vec{V} = \begin{vmatrix} \widehat{e_x} & \widehat{e_y} & \widehat{e_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \epsilon_{ijk}\partial_i V_j \widehat{e_k}$

Eg.求$\nabla r, \nabla \vec{r}, \nabla \cdot \vec{r}$

一些常用的微分运算（书后附录）

$$\nabla \times (\nabla f)=\epsilon_{ijk}\partial_i\hat{e_i} \times \left(\partial_j f \hat{e_j}\right) = \epsilon_{ijk}\partial_i\partial_j f \widehat{e_k} = 0$$

$$\nabla \cdot (\nabla \times \vec{a}) = \partial_k \epsilon_{ijk}\partial_i a_j \widehat{e_k} = 0$$

$$\begin{aligned}\nabla \times (\nabla \times \vec{a}) &= \nabla \times (\epsilon_{mnk}\partial_m a_n \widehat{e_k}) = \epsilon_{ijk}\partial_j \epsilon_{mnk}\partial_m a_n \hat{e_i} \\ &= \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\partial_j\partial_m a_n \hat{e_i} \\ &= (\partial_i\partial_j a_j - \partial_j\partial_j a_i)\ \hat{e_i} \\ &= \nabla(\nabla \cdot \vec{a}) - (\nabla \cdot \nabla)\vec{a}\end{aligned}$$

定义：$\nabla^2 \vec{a} \equiv (\nabla \cdot \nabla)\vec{a}$

∇算符的矢量性、微分性

计算∇算符的运算时，可以先利用它的微分性，将算式展开。之后直接将它当做一个矢量来运算即可。

例：

$$\nabla\cdot(\varphi\vec{a}) = \nabla_\varphi\cdot(\varphi\vec{a}) + \nabla_{\vec{a}}\cdot(\varphi\vec{a})$$
$$= \nabla_\varphi\varphi\cdot\vec{a} + \varphi(\nabla_{\vec{a}}\cdot\vec{a})$$
$$= \nabla\varphi\cdot\vec{a} + \varphi(\nabla\cdot\vec{a})$$

$$\nabla\cdot\left(\vec{a}\times\vec{b}\right) = \nabla_{\vec{a}}\cdot\left(\vec{a}\times\vec{b}\right) + \nabla_{\vec{b}}\cdot\left(\vec{a}\times\vec{b}\right)$$
$$= \nabla_{\vec{a}}\cdot\left(\vec{a}\times\vec{b}\right) - \nabla_{\vec{b}}\cdot\left(\vec{b}\times\vec{a}\right)$$
$$= \vec{b}\cdot(\nabla_{\vec{a}}\times\vec{a}) - \vec{a}\cdot\left(\nabla_{\vec{b}}\times\vec{b}\right)$$

$$Remind\ of:\ \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = \vec{b}\cdot(\vec{c}\times a) = \vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$$

$$= \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot\left(\nabla\times\vec{b}\right)$$

∇算符的矢量性、微分性

$$\nabla\left(\vec{a}\cdot\vec{b}\right)=\nabla_{\vec{a}}\left(\vec{a}\cdot\vec{b}\right)+\nabla_{\vec{b}}\left(\vec{a}\cdot\vec{b}\right)$$

$$=\nabla_{\vec{a}}\left(\vec{b}\cdot\vec{a}\right)+\nabla_{\vec{b}}\left(\vec{a}\cdot\vec{b}\right)$$

$$Remind\ of：\ \vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}\left(\vec{a}\cdot\vec{b}\right)$$

$$=\vec{b}\times(\nabla_{\vec{a}}\times\vec{a})+(\vec{b}\cdot\nabla_{\vec{a}})\,\vec{a}+\vec{a}\times\left(\nabla_{\vec{b}}\times\vec{b}\right)+(\vec{a}\cdot\nabla_{\vec{b}})\,\vec{b}$$

$$=\vec{b}\times(\nabla\times\vec{a})+(\vec{b}\cdot\nabla)\,\vec{a}+\vec{a}\times\left(\nabla\times\vec{b}\right)+(\vec{a}\cdot\nabla)\,\vec{b}$$

$\nabla\times\left(\vec{a}\times\vec{b}\right)$证明方法同上

柱坐标系下的梯度、散度和旋度

笛卡尔坐标：$\widehat{e_x}$、$\widehat{e_y}$、$\widehat{e_z}$。

柱坐标：$\widehat{e_\rho}$、$\widehat{e_\varphi}$、$\widehat{e_x}$。

$$\nabla f = \widehat{e_x}\frac{\partial f}{\partial x} + \widehat{e_y}\frac{\partial f}{\partial y} + \widehat{e_z}\frac{\partial f}{\partial z}$$

$$= \frac{\partial f}{\partial \rho}\widehat{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\widehat{e_\varphi} + \widehat{e_z}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

$$\nabla \times \vec{a} = \begin{vmatrix} \widehat{e_x} & \widehat{e_y} & \widehat{e_z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

柱坐标下的梯度

Notice: $\widehat{e_x} = \widehat{e_\rho}\cos(\varphi) + \widehat{e_\varphi}\sin(\varphi)$

$\widehat{e_y} = \widehat{e_\rho}\sin(\varphi) - \widehat{e_\varphi}\cos(\varphi)$

$$\frac{\partial f}{\partial x}\widehat{e_x} = \left(\frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \varphi}\frac{\partial \varphi}{\partial x}\right)\cdot\left[\widehat{e_\rho}\cos(\varphi) + \widehat{e_\varphi}\sin(\varphi)\right]$$

$\because \quad \rho^2 = x^2 + y^2$

$\therefore \quad 2\rho\frac{\partial \rho}{\partial x} = 2x$

又$\because \quad x = \rho\cos(\varphi)$

$\therefore \quad \frac{\partial \rho}{\partial x} = \cos(\varphi)$

$\because \quad x = \rho\cos(\varphi)$

$\therefore \quad 1 = \cos(\varphi)\frac{\partial \rho}{\partial x} - \rho\sin(\varphi)\frac{\partial \varphi}{\partial x}$

即: $\quad \rho\sin(\varphi)\frac{\partial \varphi}{\partial x} = \sin^2(\varphi)$

$\frac{\partial \varphi}{\partial x} = \frac{\sin(\varphi)}{\rho}$

$$\therefore \quad \frac{\partial f}{\partial x}\widehat{e_x} = \left(\frac{\partial f}{\partial \rho}\cos(\varphi) + \frac{\partial f}{\partial \varphi}\frac{\sin(\varphi)}{\rho}\right)\cdot\left[\widehat{e_\rho}\cos(\varphi) + \widehat{e_\varphi}\sin(\varphi)\right]$$

$$= \cos^2(\varphi)\frac{\partial f}{\partial \rho}\widehat{e_\rho} + \frac{1}{\rho}\sin^2(\varphi)\frac{\partial f}{\partial \varphi}\widehat{e_\varphi}$$

同理: $\frac{\partial f}{\partial y}\widehat{e_y} = \sin^2(\varphi)\frac{\partial f}{\partial \rho}\widehat{e_\rho} + \frac{1}{\rho}\cos^2(\varphi)\frac{\partial f}{\partial \varphi}\widehat{e_\varphi}$

整理可得: $\nabla f = \frac{\partial f}{\partial \rho}\widehat{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\widehat{e_\varphi} + \widehat{e_z}\frac{\partial f}{\partial z}$

柱坐标下的散度

$$\nabla\cdot\vec{a}=(\widehat{e_\rho}\frac{\partial}{\partial\rho}+\widehat{e_\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi}+\hat{e_z}\frac{\partial}{\partial z})\cdot\left(a_\rho\widehat{e_\rho}+a_\varphi\widehat{e_\varphi}+a_z\hat{e_z}\right)$$

$$=\frac{\partial a_\rho}{\partial\rho}+\widehat{e_\varphi}\frac{1}{\rho}\frac{\partial\widehat{e_\rho}}{\partial\varphi}a_\rho+\widehat{e_\varphi}\frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi}\widehat{e_\varphi}+\widehat{e_\varphi}\frac{1}{\rho}\frac{\partial\widehat{e_\varphi}}{\partial\varphi}a_\varphi+\frac{\partial a_z}{\partial z}$$

Note: $\widehat{e_\rho}=\widehat{e_x}\cos(\varphi)+\widehat{e_y}\sin(\varphi)$

$\widehat{e_\varphi}=-\widehat{e_x}\sin(\varphi)+\widehat{e_y}\cos(\varphi)$

$\therefore$ $\frac{\partial\widehat{e_\rho}}{\partial\varphi}\,\&\,\frac{\partial\widehat{e_\varphi}}{\partial\varphi}\neq 0$

Check: $\frac{\partial\widehat{e_\rho}}{\partial\varphi}=-\widehat{e_x}\sin(\varphi)+\widehat{e_y}\cos(\varphi)=\widehat{e_\varphi}$

$\frac{\partial\widehat{e_\varphi}}{\partial\varphi}=-\widehat{e_x}\cos(\varphi)-\widehat{e_y}\sin(\varphi)=-\widehat{e_\rho}$

$$\therefore\quad \nabla\cdot\vec{a}=\frac{\partial a_\rho}{\partial\rho}+\frac{1}{\rho}a_\rho+\frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi}+\frac{\partial a_z}{\partial z}$$

$$=\frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial a_\varphi}{\partial\varphi}+\frac{\partial a_z}{\partial z}$$

柱坐标下的旋度

$$\nabla \times \vec{a} = (\widehat{e_\rho}\frac{\partial}{\partial\rho} + \widehat{e_\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{e_z}\frac{\partial}{\partial z}) \times \left(a_\rho\widehat{e_\rho} + a_\varphi\widehat{e_\varphi} + a_z\hat{e_z}\right)$$

同散度相同的计算方法，我们可以得到：

$$\nabla \times \vec{a} = \begin{vmatrix} \widehat{e_\rho} & \widehat{e_\varphi} & \hat{e_z} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\varphi & a_z \end{vmatrix}$$

总结：$\nabla = \widehat{e_\rho}\frac{\partial}{\partial\rho} + \widehat{e_\varphi}\frac{1}{\rho}\frac{\partial}{\partial\varphi} + \hat{e_z}\frac{\partial}{\partial z}$

散度和旋度可现场计算，不需要死记硬背，只需要注意单位矢量随方向变化的关系。

球坐标系下的梯度、散度和旋度

- 用同样的方式坐标变换得到
- 注意比较：
  - 球坐标下r、φ、θ
  - 柱坐标下ρ、φ、z
  - 笛卡尔坐标下x、y、z
- 同样注意单位矢量随方向变化的关系。

并矢(二阶张量)：

定义：$\vec{a}\,\vec{b} = a_i b_j\,\hat{e}_i\,\hat{e}_j$

单位张量：$\overleftrightarrow{I} = \delta_{ij}\,\hat{e}_i\,\hat{e}_j = \hat{e}_i\,\hat{e}_i$

双点积：
$$\begin{aligned}\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) &= (a_i\,\hat{e}_i)\cdot[(b_j\,\hat{e}_j)\cdot(C_{kl}\,\hat{e}_k\,\hat{e}_l)]\\ &= a_i b_j C_{kl}\,[(\hat{e}_j\cdot\hat{e}_k)(\hat{e}_i\cdot\hat{e}_l)]\\ &= a_i b_j C_{kl}\,[(\hat{e}_i\,\hat{e}_j):(\hat{e}_k\,\hat{e}_l)]\end{aligned}$$

令 $\vec{a}\cdot(\vec{b}\cdot\overleftrightarrow{C}) = (\vec{a}\vec{b}):\overleftrightarrow{C}$

$$\boxed{(\hat{e}_i\,\hat{e}_j):(\hat{e}_k\,\hat{e}_l) \equiv (\hat{e}_j\cdot\hat{e}_k)(\hat{e}_i\cdot\hat{e}_l)}$$

张量与张量的双点积：

$$\begin{aligned}\overleftrightarrow{A}:\overleftrightarrow{B} &= (A_{ij}\,\hat{e}_i\,\hat{e}_j):(B_{kl}\,\hat{e}_k\,\hat{e}_l)\\ &= A_{ij}B_{kl}\,[(\hat{e}_i\,\hat{e}_j):(\hat{e}_k\,\hat{e}_l)]\\ &= A_{ij}B_{kl}\,[(\hat{e}_j\cdot\hat{e}_k)(\hat{e}_i\cdot\hat{e}_l)]\\ &= A_{ij}B_{kl}\,\delta_{jk}\delta_{il}\\ &= A_{ij}B_{ji}\end{aligned}$$

即矩阵相乘以后再取迹

并矢与矢量、标量间的运算——分量形式计算，矢量运算　法则基本都可以推广但要注意很多时候交换律往往不成立。