1.1 Introduction

Optimization is the act of achieving the best possible result under given circumstances. In design, construction, maintenance, ..., engineers have to take decisions. The goal of all such decisions is either to minimize effort or to maximize benefit.
The effort or the benefit can be usually expressed as a function of certain design variables. Hence, optimization is the process of finding the conditions that give the maximum or the minimum value of a function.
It is obvious that if a point $x^\star$ corresponds to the minimum value of a function $f(x)$, the same point corresponds to the maximum value of the function $-f(x)$. Thus, optimization can be taken to be minimization.
There is no single method available for solving all optimization problems efficiently. Hence, a number of methods have been developed for solving different types of problems.
Optimum seeking methods are also known as mathematical programming techniques, which are a branch of operations research. Operations research is *coarsely* composed of the following areas.

- Mathematical programming methods. These are useful in finding the minimum of a function of several variables under a prescribed set of constraints.

- Stochastic process techniques. These are used to analyze problems which are described by a set of random variables of known distribution.

- Statistical methods. These are used in the analysis of experimental data and in the construction of empirical models.

These lecture notes deal mainly with the theory and applications of mathematical programming methods. Mathematical programming is a vast area of mathematics and engineering. It includes

- calculus of variations and optimal control;
- linear, quadratic and non-linear programming;
- geometric programming;
- integer programming;
- network methods (PERT);
- game theory.

The foundations of optimization can be traced back to Newton, Lagrange and Cauchy. The development of differential methods for optimization was possible because of the contribution of Newton and Leibnitz. The foundations of the calculus of variations were laid by Bernoulli, Euler, Lagrange and Weierstrasse. Constrained optimization was first studied by Lagrange and the notion of descent was introduced by Cauchy.

Despite these early contributions, very little progress was made till the 20th century, when computer power made the implementation of optimization procedures possible and this in turn stimulated further research methods.
The major developments in the area of numerical methods for unconstrained optimization have been made in the UK. These include the development of the simplex method (Dantzig, 1947), the principle of optimality (Bellman, 1957), necessary and sufficient conditions of optimality (Kuhn and Tucker, 1951).
Optimization in its broadest sense can be applied to solve any engineering problem, *e.g.*

- design of aircraft for minimum weight;
- optimal (minimum time) trajectories for space missions;
- minimum weight design of structures for earthquake;
- optimal design of electric networks;
- optimal production planning, resources allocation, scheduling;
- shortest route;
- design of optimum pipeline networks;
- minimum processing time in production systems;
- optimal control.

1.2 Statement of an optimization problem

An optimization, or a mathematical programming problem can be stated as follows.
Find

$$x = (x^1, x^2, ..., x^n)$$

which minimizes

$$f(x)$$

subject to the constraints

$$g_j(x) \leq 0 \tag{1.1}$$

for $j = 1, \ldots, m$, and

$$l_j(x) = 0 \tag{1.2}$$

for $j = 1, \ldots, p$.
The variable x is called the design vector, $f(x)$ is the objective function, $g_j(x)$ are the inequality constraints and $l_j(x)$ are the equality constraints. The number of variables n and the number of constraints $p + m$ need not be related. If $p + m = 0$ the problem is called an unconstrained optimization problem.

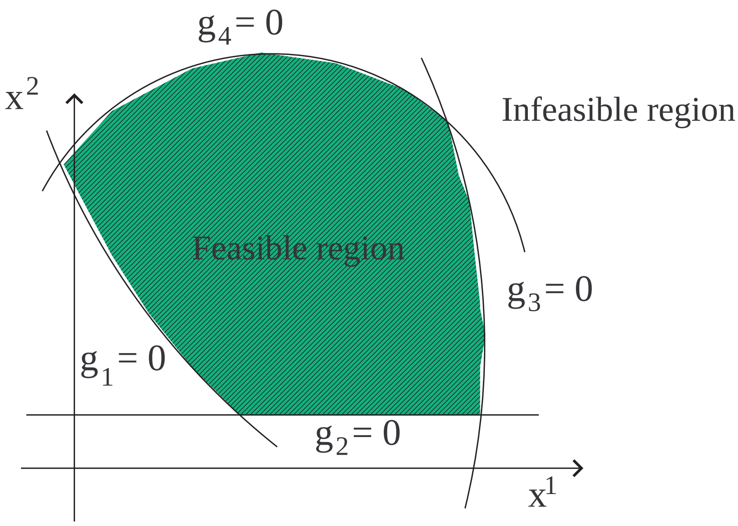

Figure 1.1: Feasible region in a two-dimensional design space. Only inequality constraints are present.

1.2.1 Design vector

Any system is described by a set of quantities, some of which are viewed as variables during the design process, and some of which are preassigned parameters or are imposed by the *environment*. All the quantities that can be treated as variables are called design or decision variables, and are collected in the design vector x.

1.2.2 Design constraints

In practice, the design variables cannot be selected arbitrarily, but have to satisfy certain requirements. These restrictions are called design constraints. Design constraints may represent limitation on the performance or behaviour of the system or physical limitations. Consider, for example, an optimization problem with only inequality constraints, *i.e.* $g_j(x) \leq 0$. The set of values of x that satisfy the equations $g_j(x) = 0$ forms a hypersurface in the design space, which is called constraint surface. In general, if n is the number of design variables, the constraint surface is an $n - 1$ dimensional surface. The constraint surface divides the design space into two regions: one in which $g_j(x) < 0$ and one in which $g_j(x) > 0$. The points x on the constraint surface satisfy the constraint critically, whereas the points x such that $g_j(x) > 0$, for some j, are infeasible, *i.e.* are unacceptable, see Figure 1.1.

1.2.3 Objective function

The classical design procedure aims at finding an acceptable design, *i.e.* a design which satisfies the constraints. In general there are several acceptable designs, and the purpose

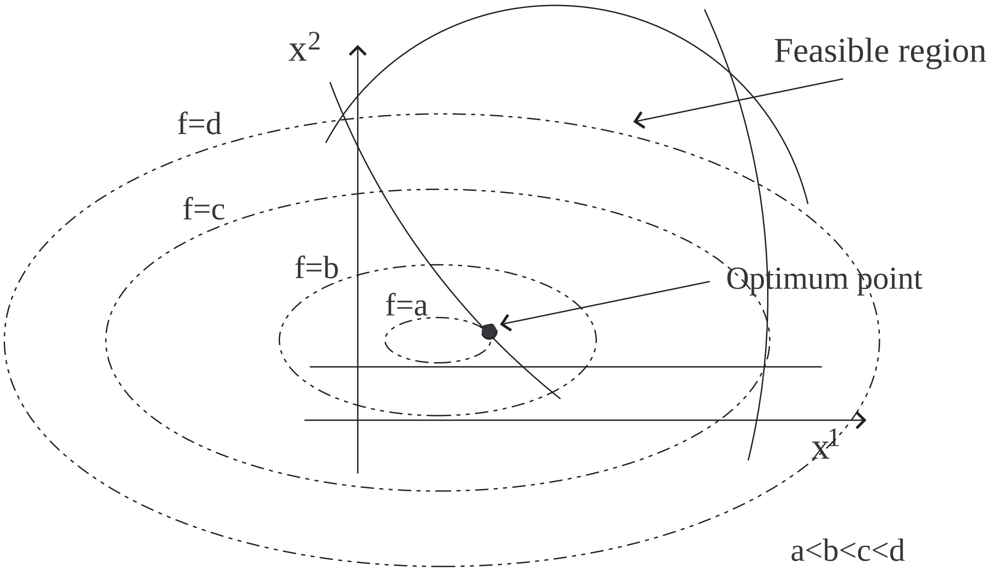

Figure 1.2: Design space, objective functions surfaces, and optimum point.

of the optimization is to single out the best possible design. Thus, a criterion has to be selected for comparing different designs. This criterion, when expressed as a function of the design variables, is known as objective function. The objective function is in general specified by physical or economical considerations. However, the selection of an objective function is not trivial, because what is the optimal design with respect to a certain criterion may be unacceptable with respect to another criterion. Typically there is a trade off performance–cost, or performance–reliability, hence the selection of the objective function is one of the most important decisions in the whole design process. If more than one criterion has to be satisfied we have a multiobjective optimization problem, that may be approximately solved considering a cost function which is a weighted sum of several objective functions.

Given an objective function $f(x)$, the locus of all points x such that $f(x) = c$ forms a hypersurface. For each value of c there is a different hypersurface. The set of all these surfaces are called objective function surfaces.

Once the objective function surfaces are drawn, together with the constraint surfaces, the optimization problem can be easily solved, at least in the case of a two dimensional decision space, as shown in Figure 1.2. If the number of decision variables exceeds two or three, this graphical approach is not viable and the problem has to be solved as a mathematical problem. Note however that more general problems have similar geometrical properties of two or three dimensional problems.

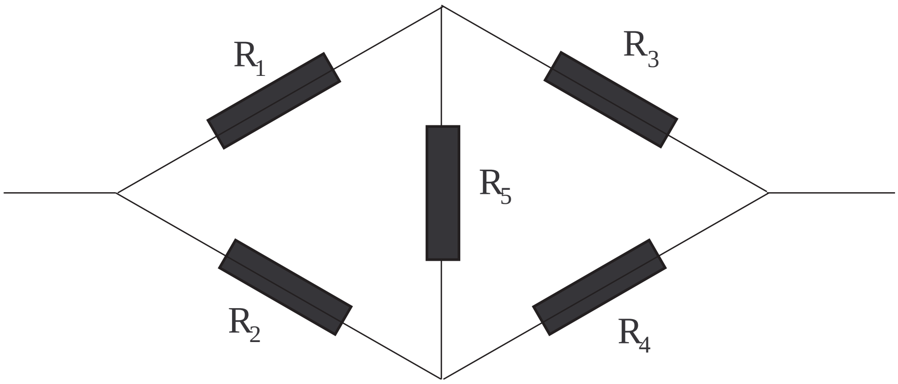

Figure 1.3: Electrical bridge network.

1.3 Classification of optimization problems

Optimization problem can be classified in several ways.

- Existence of constraints. An optimization problem can be classified as a constrained or an unconstrained one, depending upon the presence or not of constraints.
- Nature of the equations. Optimization problems can be classified as linear, quadratic, polynomial, non-linear depending upon the nature of the objective functions and the constraints. This classification is important, because computational methods are usually selected on the basis of such a classification, *i.e.* the nature of the involved functions dictates the type of solution procedure.
- Admissible values of the design variables. Depending upon the values permitted for the design variables, optimization problems can be classified as integer or real valued, and deterministic or stochastic.

1.4 Examples

Example 1 A travelling salesman has to visit n towns. He plans to start from a particular town numbered 1, visit each one of the other $n-1$ towns, and return to the town 1. The distance between town i and j is given by d_{ij}. How should he select the sequence in which the towns are visited to minimize the total distance travelled?

Example 2 The bridge network in Figure 1.3 consists of five resistors R_i, $i = 1, \ldots, 5$. Let I_i be the current through the resistance R_i, find the values of R_i so that the total dissipated power is minimum. The current I_i can vary between the lower limit $\underline{I}_i$ and the upper limit $\bar{I}_i$ and the voltage drop $V_i = R_i I_i$ must be equal to a constant c_i.

Example 3 A manufacturing firm produces two products, A and B, using two limited resources, 1 and 2. The maximum amount of resource 1 available per week is 1000 and the

Article type	w_i	v_i	c_i
1	4	9	5
2	8	7	6
3	2	4	3

Table 1.1: Properties of the articles to load.

maximum amount of resource 2 is 250. The production of one unit of A requires 1 unit of resource 1 and 1/5 unit of resource 2. The production of one unit of B requires 1/2 unit of resource 1 and 1/2 unit of resource 2. The unit cost of resource 1 is $1 - 0.0005u_1$, where u_1 is the number of units of resource 1 used. The unit cost of resource 2 is $3/4 - 0.0001u_2$, where u_2 is the number of units of resource 2 used. The selling price of one unit of A is

$$2 - 0.005x_A - 0.0001x_B$$

and the selling price of one unit of B is

$$4 - 0.002x_A - 0.01x_B,$$

where x_A and x_B are the number of units of A and B sold. Assuming that the firm is able to sell all manufactured units, maximize the weekly profit.

Example 4 A cargo load is to be prepared for three types of articles. The weight, w_i, volume, v_i, and value, c_i, of each article is given in Table 1.1.
Find the number of articles x_i selected from type i so that the total value of the cargo is maximized. The total weight and volume of the cargo cannot exceed 2000 and 2500 units respectively.

Example 5 There are two types of gas molecules in a gaseous mixture at equilibrium. It is known that the Gibbs free energy

$$G(x) = c_1x^1 + c_2x^2 + x^1 log(x^1/x_T) + x^2 log(x^2/x_T),$$

with $x_T = x^1 + x^2$ and c_1, c_2 known parameters depending upon the temperature and pressure of the mixture, has to be minimum in these conditions. The minimization of $G(x)$ is also subject to the mass balance equations:

$$x^1 a_{i1} + x^2 a_{i2} = b_i,$$

for $i = 1, \ldots, m$, where m is the number of atomic species in the mixture, b_i is the total weight of atoms of type i, and a_{ij} is the number of atoms of type i in the molecule of type j. Show that the problem of determining the equilibrium of the mixture can be posed as an optimization problem.