Cryptology

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Lecture 04

Computational Ciphers and Semantic Security

A Computational Approach

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- Modern Cryptography moved from perfect security to computational security.
- Computational security is weaker notion of security than Perfect security.

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- It suffices to use a scheme that cannot be broken in reasonable time with any reasonable probability of success.

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- Asymptotic Approach.

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  - Let $t = 2^{80}$. Then $t = 9 \times 2^{20}$ years with success probability 1.

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According to physicists' estimates the number of seconds since the big bang is on the order of 2^{58}.

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  - Does this take into account future advances in computing power (e.g., Moore's Law)?
  - Does the estimate assume the use of off-the-shelf algorithms, or dedicated software implementations optimized for the attack?
  - Says little about the success probability of an adversary running for 2 years (other than the fact that it can be at most ϵ) and says nothing about the success probability of an adversary running for 10 years.

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    1. Efficient adversaries: Randomized (i.e., probabilistic) algorithms running in time polynomial in n (PPT algorithms).
    2. For real-world efficiency: It is required that honest parties run in polynomial time. For example, algorithms $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{D}$ all run in polynomial time.
    3. Small probabilities of success: Probabilities that are called negligible that may as well be regarded as being equal to zero for all practical purposes.

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  - $n = 500$**:** But an adversary running for 200 years breaks the scheme only with probability 2^{-460}.

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- The longer the key, the higher the security.
- Enables honest parties to defend against increases in computing power as well as algorithmic advances.

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  - All parties upgrades to 16 GHz machines.
    - Honest parties upgrade the security parameter, and let now $n = 100$.
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- The effect of faster computer made the adversary's job harder.

Mathematical details

Negligible Function

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Negligible Function (Alternate Definition)

A function $f : \mathbb{N} \to \mathbb{R}$ is called negligible if for all polynomial $p(n) > 0$ there exists a $n_0 \in \mathbb{N}$ such that or all integers $n \geqslant n_0$, we have $| f(n) | < \frac{1}{p(n)}$.

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  - $n^{-\log n} \leqslant 10^{-6}$ when $n \geqslant 32$, that is $32^5 = 33554432$.
- Does $n^{-\log n}$ approach zero more quickly than $2^{-\sqrt{n}}$?
  - It seems so.
  - In reality, $2^{-\sqrt{n}} < n^{-\log n}$ for all $n \geqslant 65536$.

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Example

$$\begin{aligned} f \quad &: \quad \mathbb{N} \rightarrow \mathbb{R} \\ f \quad &= \quad \begin{cases} \frac{1}{n}, & n \text{ is even} \\ 2^{-n}, & n \text{ is odd} \end{cases} \end{aligned}$$

- f is neither negligible nor $1/f$ is poly-bounded or super-poly.

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Proof

Exercise.

Efficient Algorithm

Let A be an algorithm (possibly probabilistic) that takes as input a security parameter $n \in \mathbb{N}$, as well as other parameters encoded as a bit string $x \in \{0,1\}^{\leqslant p(n)}$ for some fixed polynomial p. We call A an efficient algorithm if there exist a poly-bounded function t and a negligible function ϵ such that for all $n \in \mathbb{N}$, and all $x \in \{0,1\}^{\leqslant p(n)}$, the probability that the running time of A on input (n, x) exceeds $t(n)$ is at most $\epsilon(n)$.

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Note: Nothing is guaranteed for values $n \leq n_0$.

Computational Cipher

A Computational Cipher with security parameter $n \in \mathbb{N}$ is a tuple of PPT algorithms $(\mathcal{G}, \mathcal{E}, \mathcal{D})$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ such that:

- $\mathcal{M}$: A finite message space, like all finite length strings from $\{0,1\}^*$.
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- $\mathcal{D}$: We assume that $\mathcal{D}$ is deterministic.

  $$m := \mathcal{D}(k, c), \text{ and returns } \perp \text{ on error.}$$

Private-key Encryption Scheme: Definition

Correctness

For all $k \in \mathcal{K}$ and $m \in \mathcal{M}$, we have

$$c \xleftarrow{R} \mathcal{E}(m, k); \;\; m' := \mathcal{D}(k, c),$$

and $m = m'$ with probability 1.

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- Basic notion of security: Security against a *ciphertext-only attack* where the adversary observes only *a single ciphertext*.
- Equivalently, security when a given key is used to encrypt just a *single message*.
- **Assumptions about the adversary's capabilities:**
  - It only eavesdrops.
  - It runs in polynomial time.

The Basic Assumptions

- Basic notion of security: Security against a *ciphertext-only attack* where the adversary observes only *a single ciphertext*.
- Equivalently, security when a given key is used to encrypt just a *single message*.
- **Assumptions about the adversary's capabilities:**
  - It only eavesdrops.
  - It runs in polynomial time.
- No assumptions are made about the adversary's *strategy* in trying to decipher the ciphertext it observes.

Indistinguishability

- Defined as an attack game played between two parties,
  - a challenger, and
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- The attack game also defines a probability space, and this in turn defines the adversary's advantage, which is determined by the probability of one or more events.
- In Indistinguishability game, there are two alternative experiments.
  - In both experiments, the adversary follows the same protocol.
  - The challenger behaves differently in each experiment.

Indistinguishability Game

For a given cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ with security parameter $n \in \mathbb{N}$, and for a given adversary $\mathcal{A}$, we define two experiments: Experiment 0 and Experiment 1. For $b = 0, 1$, we define Experiment b as:

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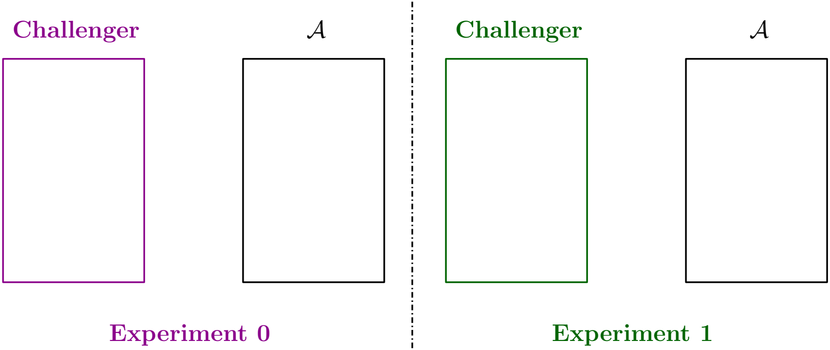
Experiment b, where $b \in \{0, 1\}$



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Challenger	$\mathcal{A}$	**Challenger**	$\mathcal{A}$
	$m_0, m_1 \in \mathcal{M}$		$m_0, m_1 \in \mathcal{M}$

Experiment 0 | **Experiment 1**

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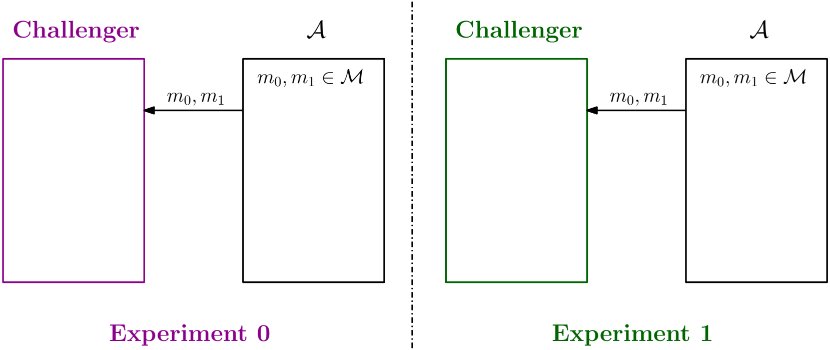

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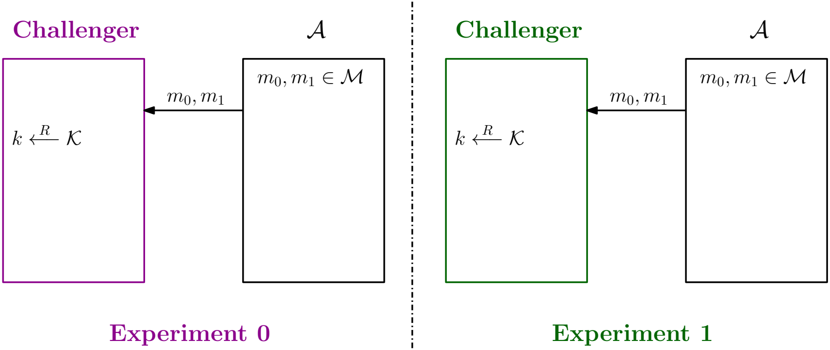

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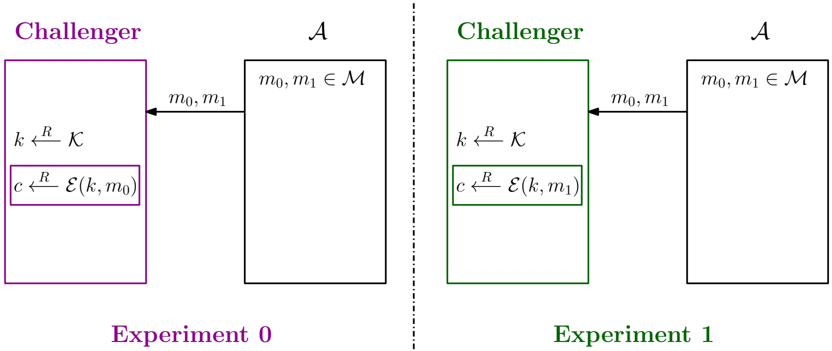

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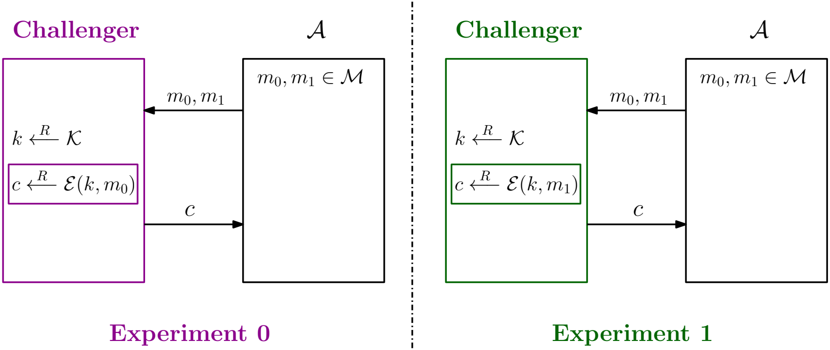

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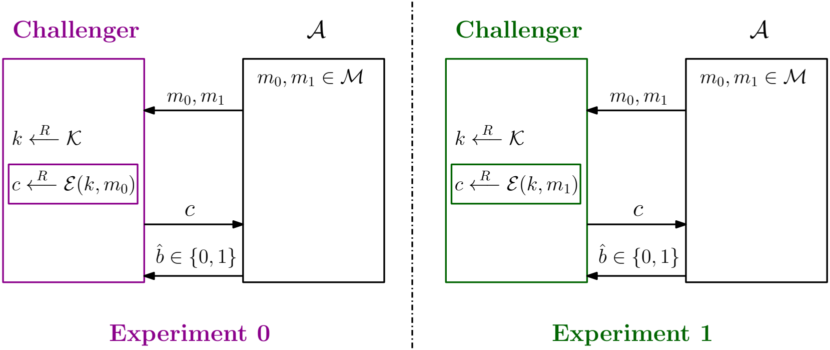

Indistinguishability Advantage

For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define the advantage of $\mathcal{A}$ in the Indistinguishability attack with respect to $\mathfrak{E}$ as

$$\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = | \Pr[W_0] - \Pr[W_1] | .$$

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$$0 \leqslant \mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] \leqslant 1.$$

Indistinguishable

We say $\mathfrak{E}$ is (computationally) indistinguishable in the presence of an eavesdropper if for all PPT adversaries $\mathcal{A}$ there exists a negligible function ϵ such that

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- We achieve perfect security if $\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 0$.

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- An encryption scheme should be able to encrypt arbitrary-length messages.
- This restriction captures the notion that an encryption of a message is allowed to leak the length of the message (but nothing else).

Message Recovery Attack

For a given cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ with security parameter $n \in \mathbb{N}$, and for a given adversary $\mathcal{A}$, the attack game proceeds as follows:

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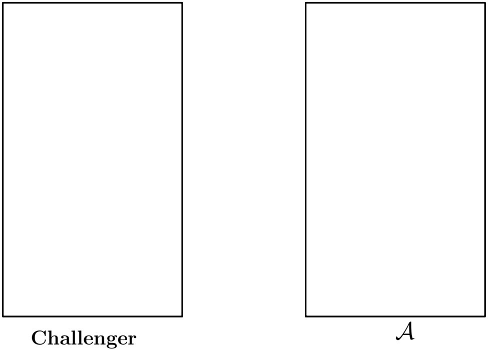
Let W be the event that $\hat{m} = m$ and $\mathcal{A}$ wins. We define the advantage of $\mathcal{A}$ in message recovery attack with respect to $\mathfrak{E}$ as

$$\mathsf{MRadv}[\mathcal{A}, \mathfrak{E}] = \left| \Pr[W] - \frac{1}{|\mathcal{M}|} \right|.$$

Message Recovery Attack



Message Recovery Attack

$m \xleftarrow{R} \mathcal{M}$

Challenger

$\mathcal{A}$

Message Recovery Attack

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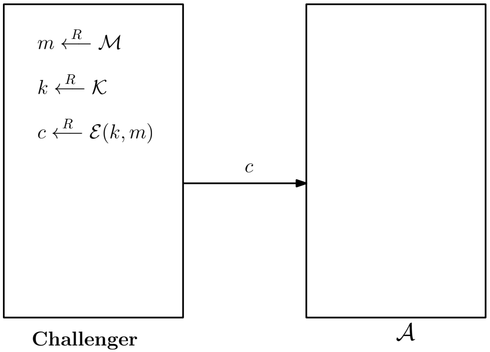

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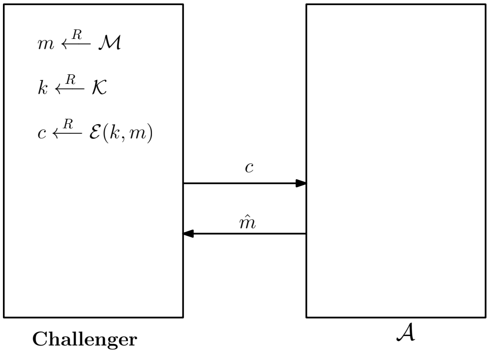

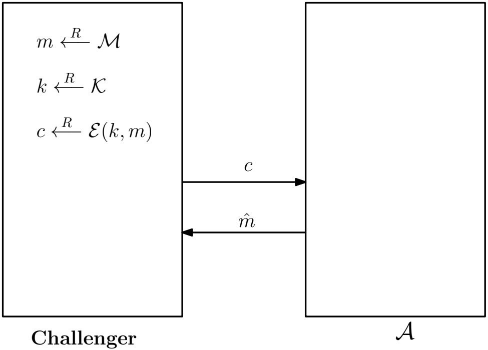

$\mathcal{A}$ **wins if** $\hat{m} = m$

Message Recovery Attack

Security against Message Recovery

A cipher $\mathfrak{E}$ is secure against message recovery if for all PPT adversaries $\mathcal{A}$ there exists a negligible function ϵ such that

$$\mathsf{MRadv}[\mathcal{A}, \mathfrak{E}] \leqslant \epsilon(n).$$

Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ with security parameter $n \in \mathbb{N}$. If $\mathfrak{E}$ is secure against indistinguishability attack, then $\mathfrak{E}$ is secure against message recovery attack.

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- Then we have,

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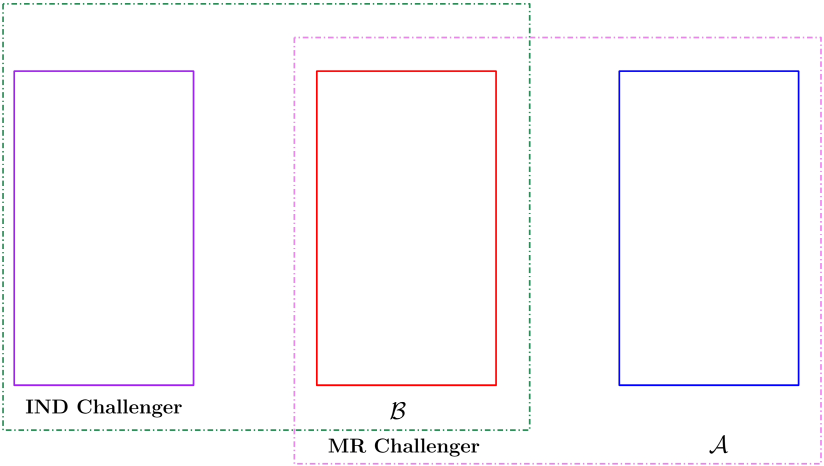
Message Recovery Attack



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  6. $\mathcal{B}$ outputs $\hat{b} = 1$ if $\hat{m} = m_1$, else $\hat{b} = 0$ in the indistinguishability game.

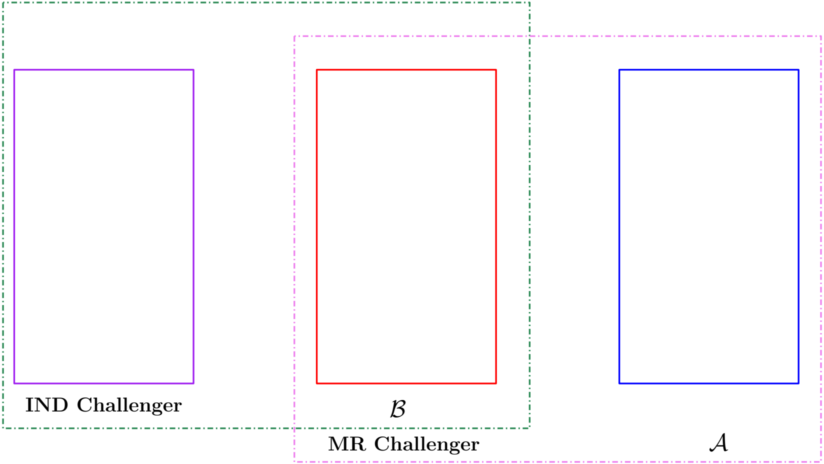
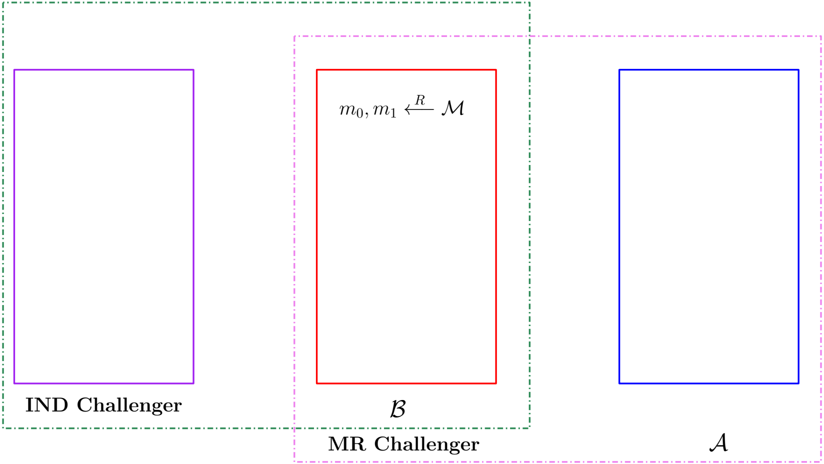


Figure: Experiment b

Message Recovery Attack



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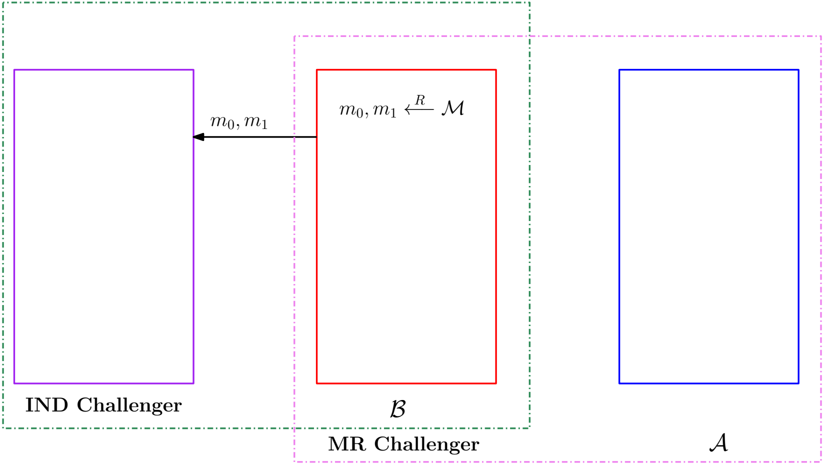

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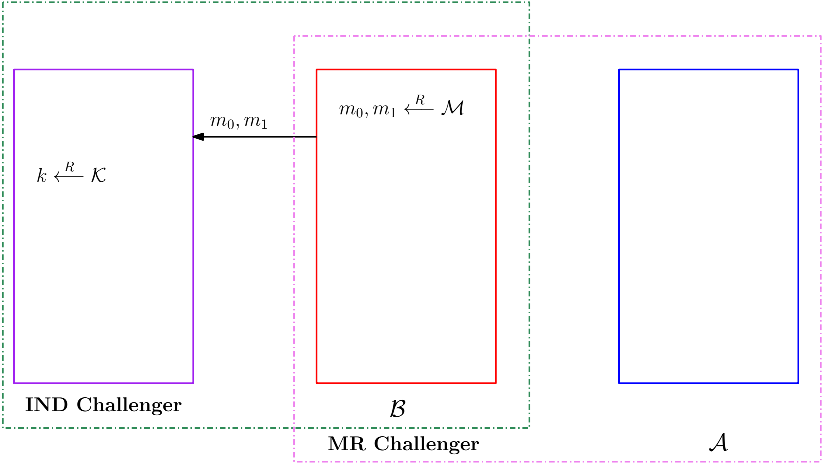

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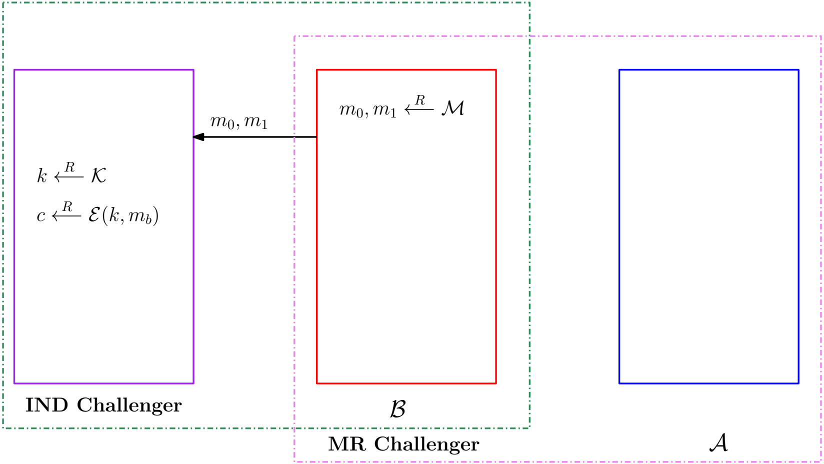

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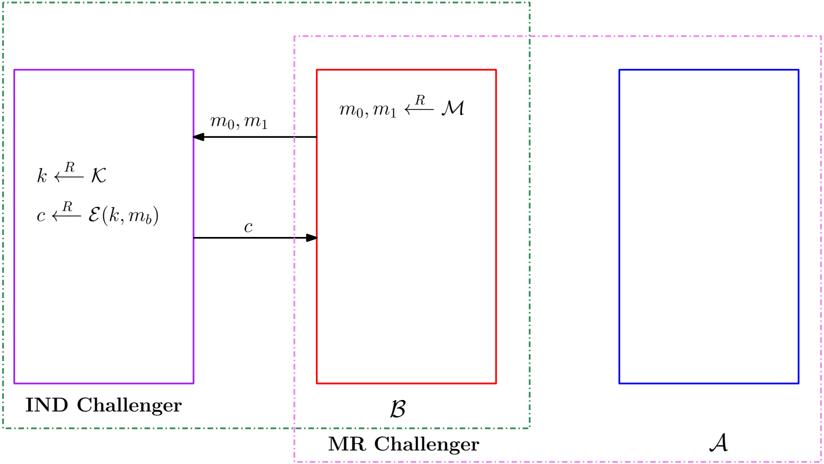

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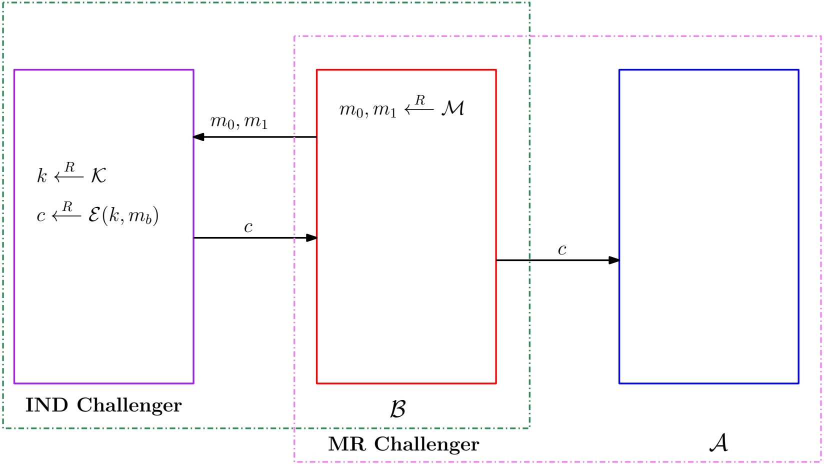

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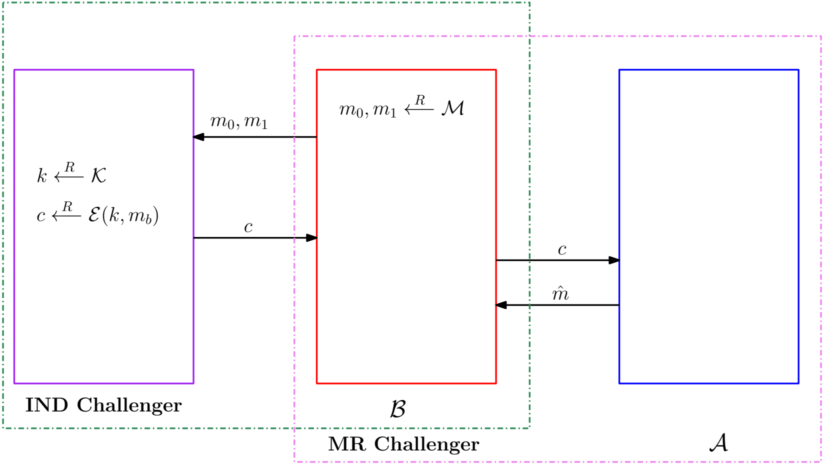

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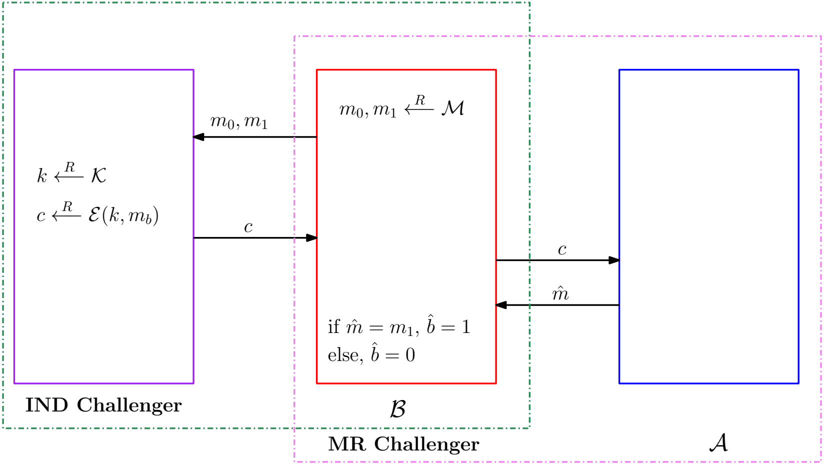

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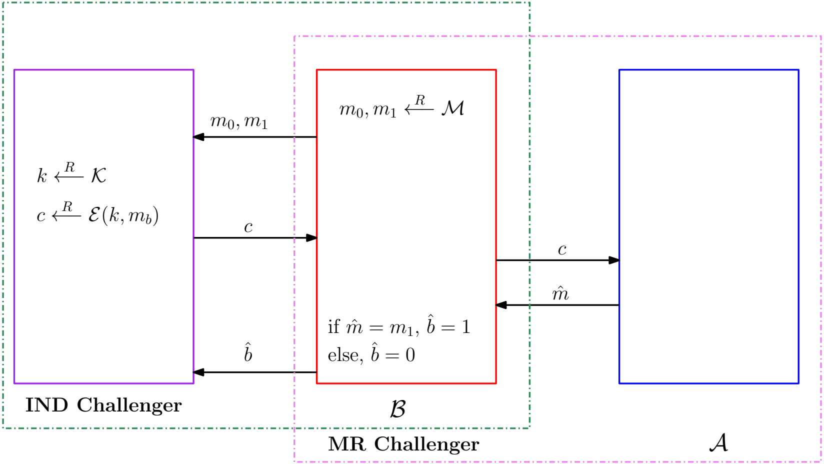

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Message Recovery Attack

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- We have to show:

$$\text{MRadv}[\mathcal{A}, \mathfrak{E}] \leqslant \text{INDadv}[\mathcal{B}, \mathfrak{E}]$$

Proof

$$p_1 = \Pr[W_1]$$

Message Recovery Attack

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\begin{aligned}
p_1 &= \Pr[W_1] \\
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&= \Pr[\mathcal{B} \text{ outputs } \hat{b} = 1 \text{ in Experiment } 1] \\
&= \Pr[\mathcal{A} \text{ outputs } \hat{m} = m_1 \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)] \\
&= \Pr[\mathcal{A} \text{ wins in Message Recovery attack}]
\end{aligned}
$$

Proof

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\begin{aligned}
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&= \Pr[\mathcal{A} \text{ outputs } \hat{m} = m_1 \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)] \\
&= \Pr[\mathcal{A} \text{ wins in Message Recovery attack}] \\
&= p.
\end{aligned}
$$

Message Recovery Attack

Proof

- Let $\mathbf{m}_0$ and $\mathbf{m}_1$ be two random variables by takes value from $\mathcal{M}$ uniformly at random.
    - $\mathbf{m}_0$ and $\mathbf{m}_1$ are independent.

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  - $\mathbf{k}$ is independent of $\mathbf{m}_0$ and $\mathbf{m}_1$.
- Define random variable $\mathbf{c} = \mathcal{E}(\mathbf{k}, \mathbf{m}_0)$.
- $\mathbf{k}, \mathbf{m}_0$ are independent of $\mathbf{m}_1 \Rightarrow \mathbf{c}$ is independent of $\mathbf{m}_1$.

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- Let $\hat{\mathbf{m}}$ be a random variable that defines the output of $\mathcal{A}$.

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  - Output of $\mathcal{A}$ depends on $\mathbf{c}$ (and may be on some internal randomness of $\mathcal{A}$).

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- Let $\hat{\mathbf{m}}$ be a random variable that defines the output of $\mathcal{A}$.
    - Output of $\mathcal{A}$ depends on $\mathbf{c}$ (and may be on some internal randomness of $\mathcal{A}$).
    - Therefore, $\hat{\mathbf{m}}$ is independent of $\mathbf{m}_1$.

$$p_0 = \Pr[W_0]$$

Message Recovery Attack

$$
\begin{aligned}
p_0 &= \Pr[W_0] \\
&= \Pr[\mathcal{B} \text{ outputs } \hat{b} = 1 \text{ in Experiment } 0]
\end{aligned}
$$

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p_0 &= \Pr[W_0] \\
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&= \Pr[\mathcal{A} \text{ outputs } m_1 \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_0)]
\end{aligned}$$

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p_0 &= \Pr[W_0] \\
&= \Pr[\mathcal{B} \text{ outputs } \hat{b} = 1 \text{ in Experiment } 0] \\
&= \Pr[\mathcal{A} \text{ outputs } m_1 \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_0)] \\
&= \sum_{m \in \mathcal{M}} \Pr[\mathbf{m}_1 = m \wedge \hat{\mathbf{m}} = m]
\end{aligned}
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Message Recovery Attack

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\begin{aligned}
p_0 &= \Pr[W_0] \\
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&= \sum_{m \in \mathcal{M}} \Pr[\mathbf{m}_1 = m \wedge \hat{\mathbf{m}} = m] \\
&= \sum_{m \in \mathcal{M}} \Pr[\mathbf{m}_1 = m] \Pr[\hat{\mathbf{m}} = m], \text{ by independence of } \mathbf{m}_1 \text{ and } \hat{\mathbf{m}}
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Message Recovery Attack

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\end{aligned}
$$

$$
\begin{aligned}
p_0 &= \Pr[W_0] \\
&= \Pr[\mathcal{B} \text{ outputs } \hat{b} = 1 \text{ in Experiment 0}] \\
&= \Pr[\mathcal{A} \text{ outputs } m_1 \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_0)] \\
&= \sum_{m \in \mathcal{M}} \Pr[\mathbf{m}_1 = m \wedge \hat{\mathbf{m}} = m] \\
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&= \sum_{m \in \mathcal{M}} \frac{1}{|\mathcal{M}|} \Pr[\hat{\mathbf{m}} = m], \ \mathbf{m}_1 \text{ is uniform over } \mathcal{M} \\
&= \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \Pr[\hat{\mathbf{m}} = m] \\
&= \frac{1}{|\mathcal{M}|}, \text{ As } \sum_{m \in \mathcal{M}} \Pr[\hat{\mathbf{m}} = m] = 1.
\end{aligned}
$$

Proof

$$\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] \quad = \quad |\Pr[W_0] - \Pr[W_1]|$$

Message Recovery Attack

Proof

$$\begin{aligned} \mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\ &= |p_0-p_1| \end{aligned}$$

Message Recovery Attack

Proof

$$
\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\
&= |p_0-p_1| \\
&= \left| \frac{1}{|\mathcal{M}|}-p \right|
\end{aligned}
$$

Proof

$$
\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= \mid \Pr[W_0] - \Pr[W_1] \mid \\
&= \mid p_0 - p_1 \mid \\
&= \left| \frac{1}{\mid \mathcal{M} \mid} - p \right| \\
&= \left| p - \frac{1}{\mid \mathcal{M} \mid} \right|
\end{aligned}
$$

Proof

$$
\begin{aligned}
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&= \mathsf{MRadv}[\mathcal{A},\mathfrak{E}]
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Proof

$$\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\
&= |p_0 - p_1| \\
&= \left| \frac{1}{|\mathcal{M}|} - p \right| \\
&= \left| p - \frac{1}{|\mathcal{M}|} \right| \\
&= \mathsf{MRadv}[\mathcal{A},\mathfrak{E}]
\end{aligned}$$

Therefore, we have

$$\mathsf{MRadv}[\mathcal{A},\mathfrak{E}] = \mathsf{INDadv}[\mathcal{B},\mathfrak{E}].$$

Parity Prediction

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  - It should be hard to recover the whole message,

Parity Prediction

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- Let m be a message.
- Parity = XOR of all bits of m

$$\mathsf{Parity}(m) = \begin{cases} 1, & \text{if number of 1's in } m \text{ is odd} \\ 0, & \text{if number of 1's in } m \text{ is even} \end{cases}$$

Parity Prediction

For a given cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ with security parameter $n \in \mathbb{N}$, and for a given adversary $\mathcal{A}$, the attack game proceeds as follows:

- The challenger computes $m \xleftarrow{R} \mathcal{M}$, $k \xleftarrow{R} \mathcal{K}$ and $c \xleftarrow{R} \mathcal{E}(k, m)$.
- The challenger then sends c to adversary.

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Let W be the event that $\hat{b} = \mathsf{Parity}(m)$ and $\mathcal{A}$ wins.

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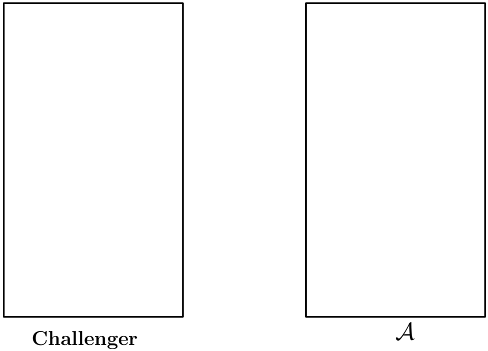
- The challenger then sends c to adversary.
- The adversary outputs a message $\hat{b} \in \{0, 1\}$.

Let W be the event that $\hat{b} = \mathsf{Parity}(m)$ and $\mathcal{A}$ wins. We define the advantage of $\mathcal{A}$ in parity prediction attack with respect to $\mathfrak{E}$ as

$$\mathsf{PARITYadv}[\mathcal{A}, \mathfrak{E}] = \left| \Pr[W] - \frac{1}{2} \right|.$$

Parity Prediction



Parity Prediction

$m \xleftarrow{R} \mathcal{M}$

Challenger

$\mathcal{A}$

Parity Prediction

$m \xleftarrow{R} \mathcal{M}$

$k \xleftarrow{R} \mathcal{K}$

Challenger

$\mathcal{A}$

Parity Prediction

$m \xleftarrow{R} \mathcal{M}$

$k \xleftarrow{R} \mathcal{K}$

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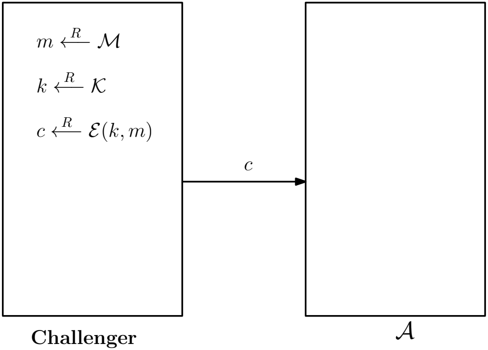
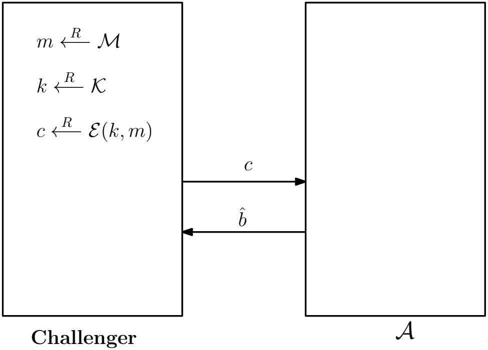
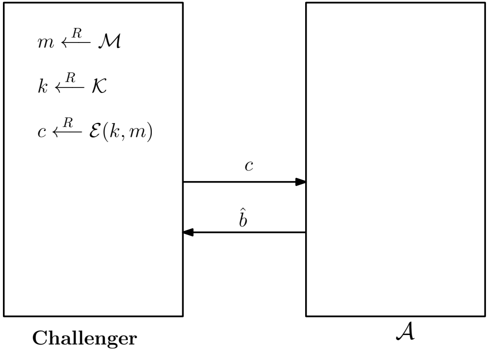
Challenger

$\mathcal{A}$

Parity Prediction




$m \xleftarrow{R} \mathcal{M}$
$k \xleftarrow{R} \mathcal{K}$
$c \xleftarrow{R} \mathcal{E}(k, m)$
c
$\hat{b}$
Challenger
$\mathcal{A}$



$\mathcal{A}$ **wins if** $\hat{b} = \mathsf{Parity}(m)$

Security against Parity Prediction

A cipher $\mathfrak{E}$ is secure against parity prediction attack if for all PPT adversaries $\mathcal{A}$, there exists a negligible function ϵ such that

$$\mathsf{PARITYadv}[\mathcal{A}, \mathfrak{E}] \leqslant \epsilon(n).$$

Parity Prediction

Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, $\mathcal{M} = \{0,1\}^L$ with security parameter $n \in \mathbb{N}$. If $\mathfrak{E}$ is secure against indistinguishability attack, then $\mathfrak{E}$ is secure against parity prediction attack.

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Theorem

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Proof

- Let $\mathfrak{E}$ is secure against indistinguishability attack.

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Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, $\mathcal{M} = \{0,1\}^L$ with security parameter $n \in \mathbb{N}$. If $\mathfrak{E}$ is secure against indistinguishability attack, then $\mathfrak{E}$ is secure against parity prediction attack.

Proof

- Let $\mathfrak{E}$ is secure against indistinguishability attack.
- Goal: For all PPT adversaries, the parity prediction advantage of $\mathcal{A}$ is negligible.

Parity Prediction

Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, $\mathcal{M} = \{0,1\}^L$ with security parameter $n \in \mathbb{N}$. If $\mathfrak{E}$ is secure against indistinguishability attack, then $\mathfrak{E}$ is secure against parity prediction attack.

Proof

- Let $\mathfrak{E}$ is secure against indistinguishability attack.
- Goal: For all PPT adversaries, the parity prediction advantage of $\mathcal{A}$ is negligible.
- Let there be an efficient adversary $\mathcal{A}$ that can win the parity prediction game with probability p.
- In the parity prediction game, $\Pr[W] = p$.

Parity Prediction

Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, $\mathcal{M} = \{0,1\}^L$ with security parameter $n \in \mathbb{N}$. If $\mathfrak{E}$ is secure against indistinguishability attack, then $\mathfrak{E}$ is secure against parity prediction attack.

Proof

- Let $\mathfrak{E}$ is secure against indistinguishability attack.
- Goal: For all PPT adversaries, the parity prediction advantage of $\mathcal{A}$ is negligible.
- Let there be an efficient adversary $\mathcal{A}$ that can win the parity prediction game with probability p.
- In the parity prediction game, $\Pr[W] = p$.
- Then we have,

$$\mathsf{PARITYadv}[\mathcal{A}, \mathfrak{E}] = \left| \, p - \frac{1}{2} \, \right|.$$

Proof

- Construct an efficient adversary $\mathcal{B}$ of the indistinguishability attack.
  - $\mathcal{B}$ uses $\mathcal{A}$ as a sub-routine.

Parity Prediction

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Proof

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  - $\mathcal{B}$ uses $\mathcal{A}$ as a sub-routine.
  - $\mathcal{A}$ is a black-box to $\mathcal{B}$.
  - $\mathcal{B}$ behaves as the challenger to $\mathcal{A}$.

Parity Prediction

Proof

- Indistinguishability Experiment b:

  1. $\mathcal{B}$ computes $m_0 \xleftarrow{R} \mathcal{M}$

Proof

- Indistinguishability Experiment b:

  1. $\mathcal{B}$ computes $m_0 \xleftarrow{R} \mathcal{M}$ and sets $m_1 \longleftarrow m_0 \oplus (0^{L-1} \| 1)$, and sends m_0, m_1 to indistinguishability challenger.

Proof

- Indistinguishability Experiment b:

  1. $\mathcal{B}$ computes $m_0 \xleftarrow{R} \mathcal{M}$ and sets $m_1 \longleftarrow m_0 \oplus (0^{L-1} \| 1)$, and sends m_0, m_1 to indistinguishability challenger.
  2. Indistinguishability challenger computes $k \xleftarrow{R} \mathcal{K}$ and $c \xleftarrow{R} \mathcal{E}(k, m_b)$.

Proof

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1. $\mathcal{B}$ computes $m_0 \xleftarrow{R} \mathcal{M}$ and sets $m_1 \longleftarrow m_0 \oplus (0^{L-1} \| 1)$, and sends m_0, m_1 to indistinguishability challenger.
2. Indistinguishability challenger computes $k \xleftarrow{R} \mathcal{K}$ and $c \xleftarrow{R} \mathcal{E}(k, m_b)$.
3. Indistinguishability challenger sends c to $\mathcal{B}$.
4. $\mathcal{B}$ then forwards c to $\mathcal{A}$.

Proof

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1. $\mathcal{B}$ computes $m_0 \xleftarrow{R} \mathcal{M}$ and sets $m_1 \longleftarrow m_0 \oplus (0^{L-1} \| 1)$, and sends m_0, m_1 to indistinguishability challenger.
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5. $\mathcal{A}$ returns $\hat{b}$ to $\mathcal{B}$.

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3. Indistinguishability challenger sends c to $\mathcal{B}$.
4. $\mathcal{B}$ then forwards c to $\mathcal{A}$.
5. $\mathcal{A}$ returns $\hat{b}$ to $\mathcal{B}$.
6. $\mathcal{B}$ outputs $\bar{b} = 1$ if $\hat{b} = \mathsf{Parity}(m_0)$, else $\bar{b} = 0$ in the indistinguishability game.

Parity Prediction

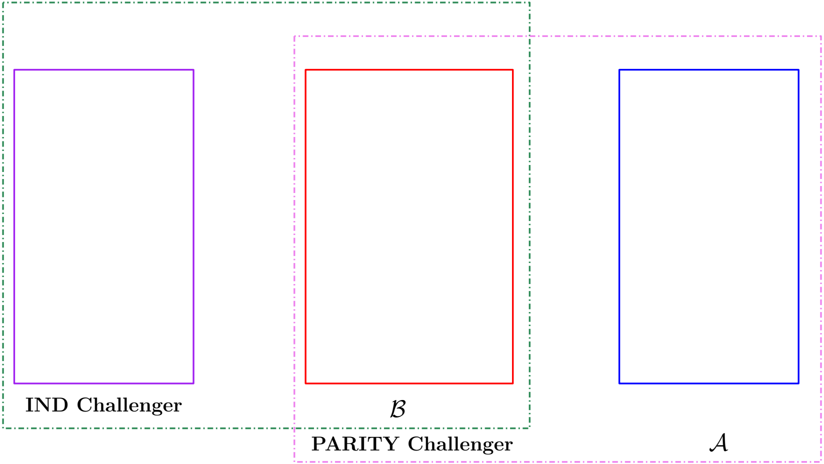
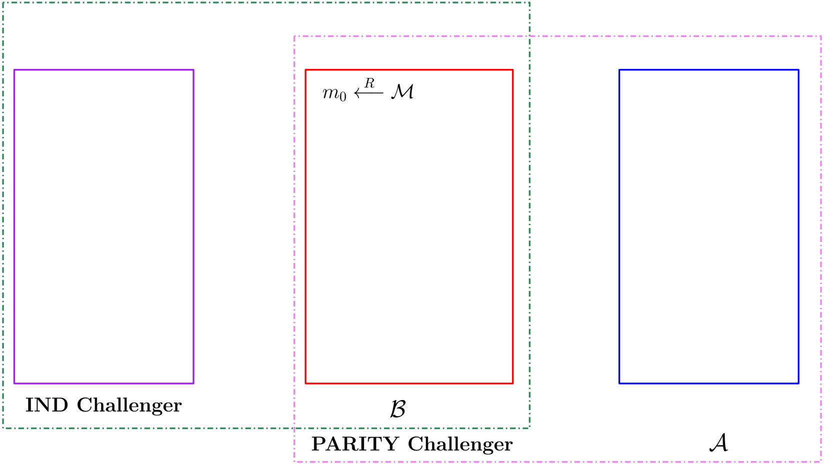


Figure: Experiment b

Parity Prediction



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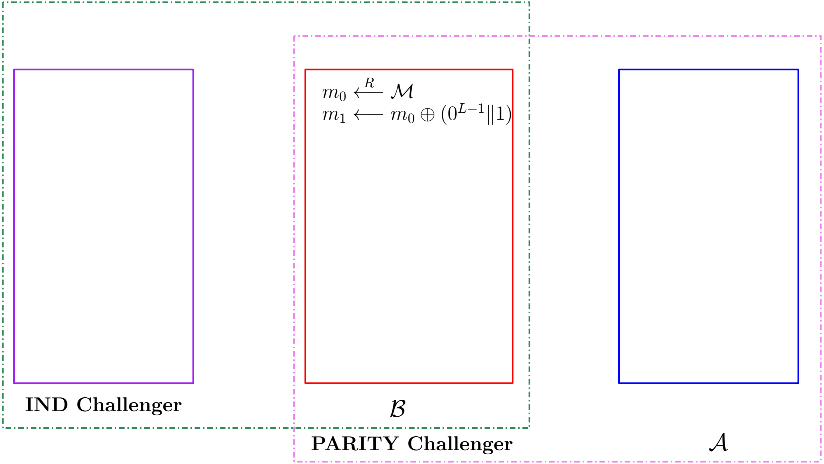

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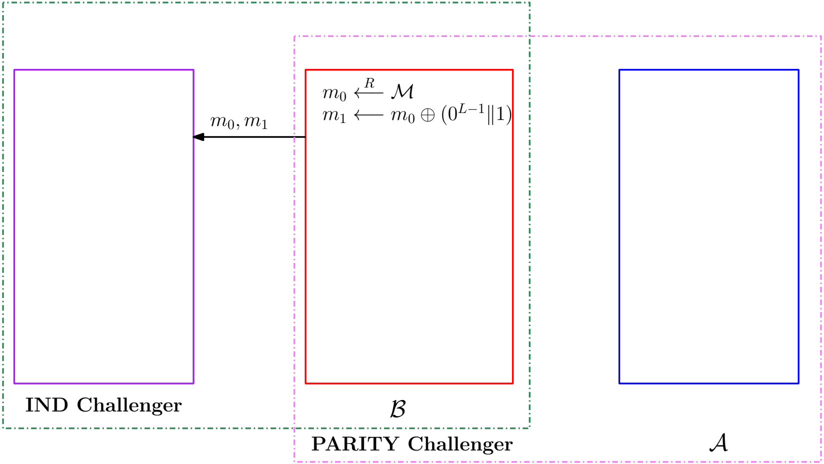

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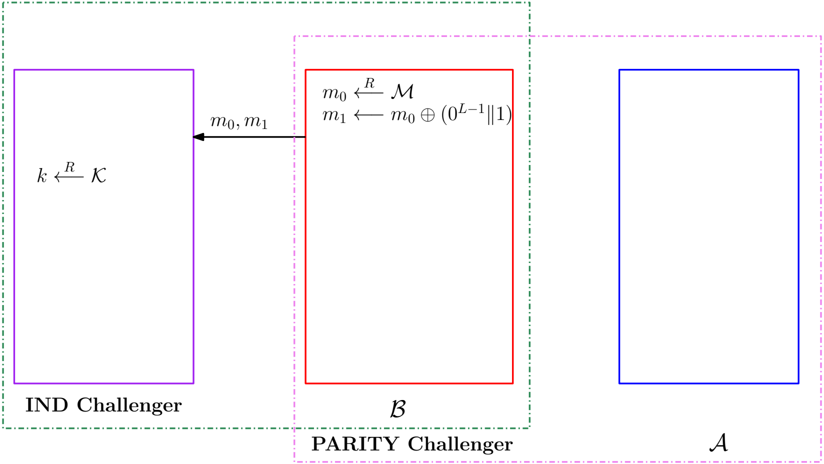

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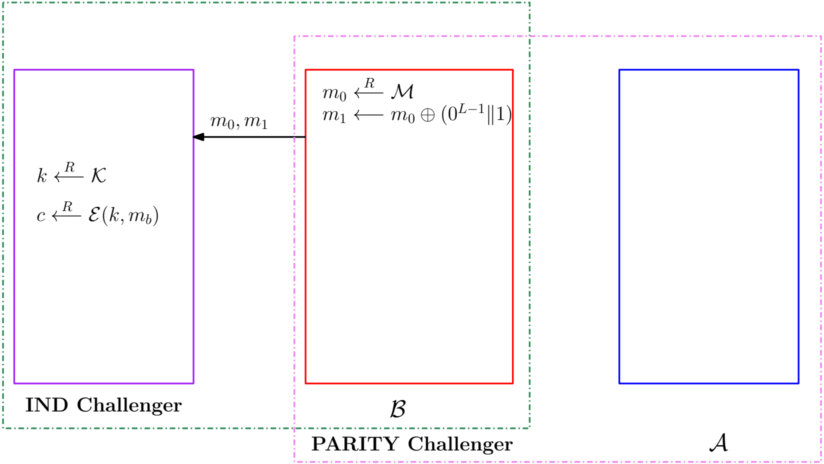

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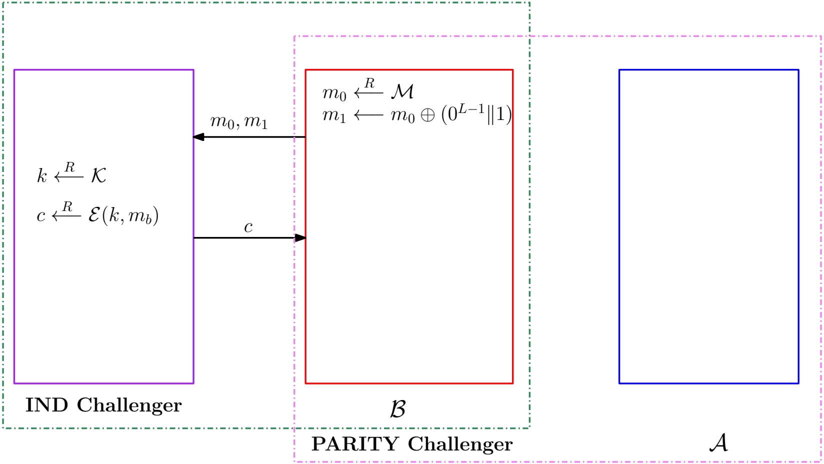

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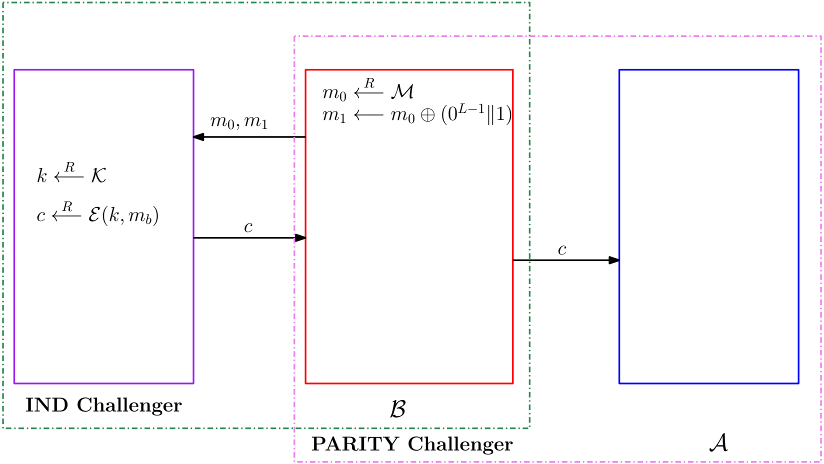

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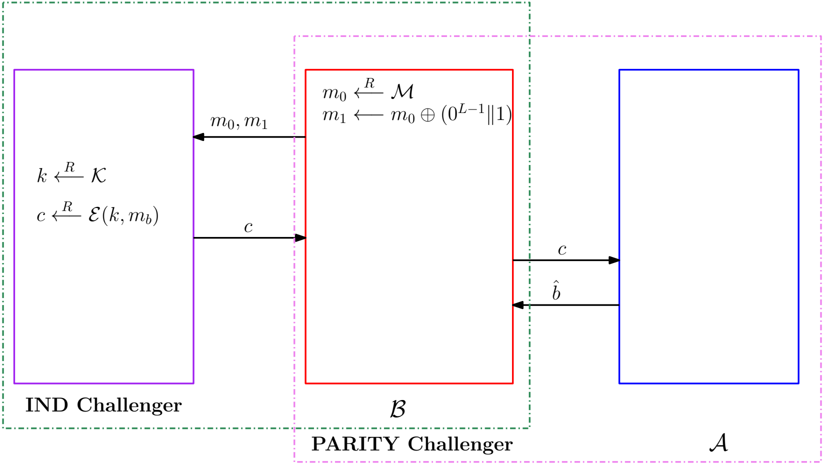


Figure: Experiment b

Parity Prediction

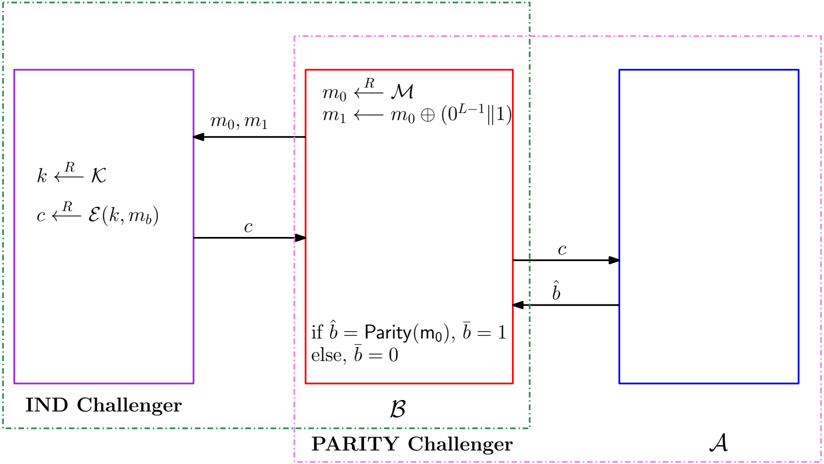
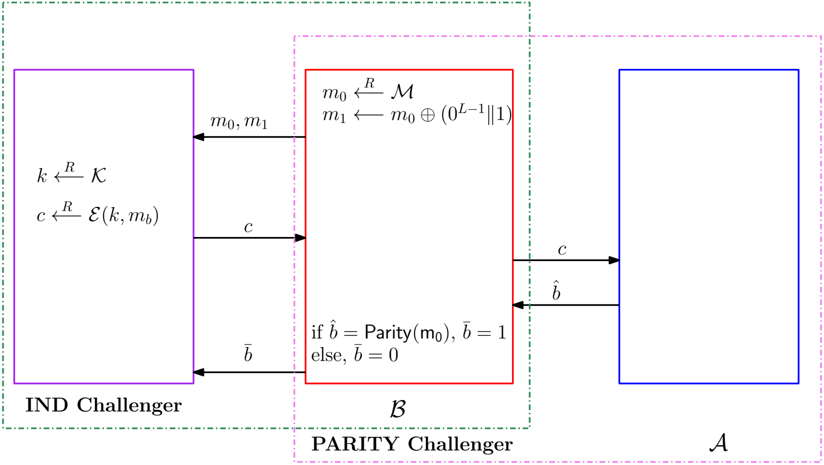


Figure: Experiment b

Parity Prediction



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- We have to show:

$$\mathsf{PARITYadv}[\mathcal{A}, \mathfrak{E}] \leqslant \mathsf{INDadv}[\mathcal{B}, \mathfrak{E}]$$

Proof

- Regardless of whether m_0 or m_1 is encrypted, the distribution of m_b is uniform over $\mathcal{M}$ for $\mathcal{A}$.

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p_1 &= \Pr[W_1] \\
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- Regardless of whether m_0 or m_1 is encrypted, the distribution of m_b is uniform over $\mathcal{M}$ for $\mathcal{A}$.

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p_0 &= \Pr[W_0] \\
&= \Pr[\mathcal{B} \text{ outputs } \bar{b} = 1 \text{ in Experiment } 0] \\
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&= p.
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p_1 &= \Pr[W_1] \\
&= \Pr[\mathcal{B} \text{ outputs } \bar{b} = 1 \text{ in Experiment } 1] \\
&= \Pr[\mathcal{A} \text{ outputs } \hat{b} = \mathsf{Parity}(m_0) \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)] \\
&= 1 - \Pr[\mathcal{A} \text{ outputs } \hat{b} = \mathsf{Parity}(m_1) \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)]
\end{aligned}
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Proof

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&= \Pr[\mathcal{A} \text{ outputs } \hat{b} = \mathsf{Parity}(m_0) \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)] \\
&= 1 - \Pr[\mathcal{A} \text{ outputs } \hat{b} = \mathsf{Parity}(m_1) \text{ given } c \xleftarrow{R} \mathcal{E}(k, m_1)] \\
&= 1 - p.
\end{aligned}
$$

Parity Prediction

Proof

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\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0] - \Pr[W_1]| \\
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Proof

$$\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\
&= |p_0 - p_1| \\
&= |\ p-(1-p)\ |
\end{aligned}$$

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\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\
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&= 2\left|\ p-\frac{1}{2}\ \right|
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\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0]-\Pr[W_1]| \\
&= |\,p_0-p_1\,| \\
&= |\;p-(1-p)\;| \\
&= |\;2p-1\;| \\
&= 2\left|\;p-\frac{1}{2}\;\right| \\
&= 2\cdot\mathsf{PARITYadv}[\mathcal{A},\mathfrak{E}]
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$$

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$$
\begin{aligned}
\mathsf{INDadv}[\mathcal{B},\mathfrak{E}] &= |\Pr[W_0] - \Pr[W_1]| \\
&= |p_0 - p_1| \\
&= |\ p - (1-p)\ | \\
&= |\ 2p - 1\ | \\
&= 2\left|\ p - \frac{1}{2}\ \right| \\
&= 2 \cdot \mathsf{PARITYadv}[\mathcal{A},\mathfrak{E}]
\end{aligned}
$$

Therefore, we have

$$\mathsf{PARITYadv}[\mathcal{A},\mathfrak{E}] = \frac{1}{2} \cdot \mathsf{INDadv}[\mathcal{B},\mathfrak{E}].$$

Bit Guessing: an alternative characterization of Indistinguishability

- This alternative characterization needs a new attack game.

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- This alternative characterization needs a new attack game.
- The role played by the adversary is exactly the same as before.
- Instead of having two different experiments, there is just a single experiment.
- The challenger chooses $b \in \{0, 1\}$ at random and runs Experiment b of previous indistinguishability game.

Bit Guessing: an alternative characterization of Indistinguishability

Bit Guessing Indistinguishability Game

For a given cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ with security parameter $n \in \mathbb{N}$, and for a given adversary $\mathcal{A}$, we define the Experiment as:

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- The challenger computes $b \xleftarrow{R} \{0, 1\}$.
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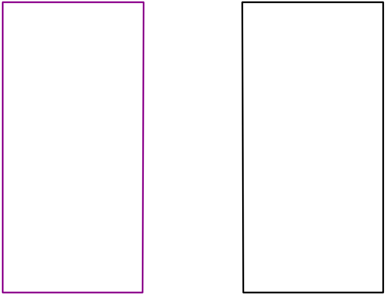
- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.

Bit Guessing Indistinguishability Experiment

Challenger $\mathcal{A}$



Experiment

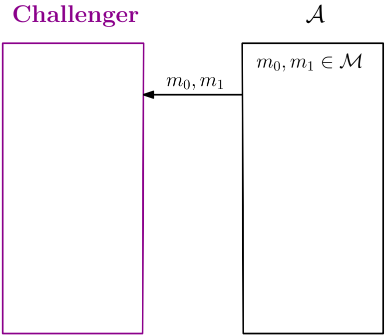
Bit Guessing Indistinguishability Experiment

Challenger $\mathcal{A}$

$m_0, m_1 \in \mathcal{M}$

Experiment

Bit Guessing Indistinguishability Experiment



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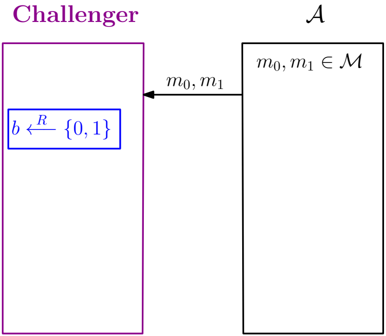

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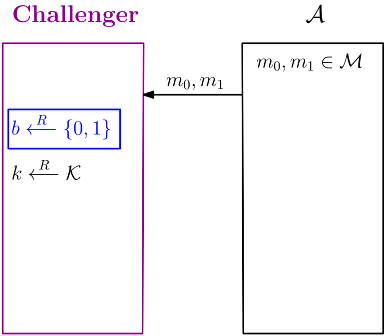


Experiment

Bit Guessing Indistinguishability Experiment

Challenger | $\mathcal{A}$

$m_0, m_1 \in \mathcal{M}$

$\xleftarrow{m_0, m_1}$

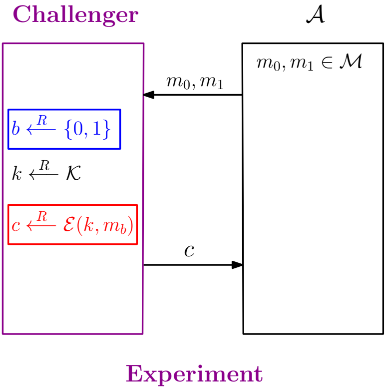
$\boxed{b \xleftarrow{R} \{0,1\}}$

$k \xleftarrow{R} \mathcal{K}$

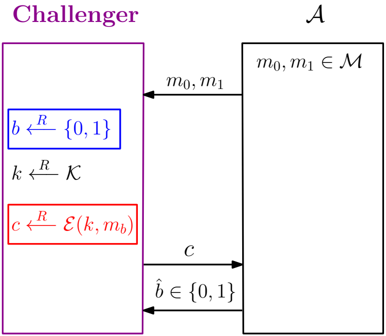
$\boxed{c \xleftarrow{R} \mathcal{E}(k, m_b)}$

Experiment

Bit Guessing Indistinguishability Experiment



Bit Guessing Indistinguishability Experiment



Experiment

Indistinguishability Advantage

Let W be the event that $\mathcal{A}$ outputs $\hat{b} = b$ in the Experiment. We define advantage of $\mathcal{A}$ in the Indistinguishability attack with respect to $\mathfrak{E}$ as

$$\mathsf{INDadv}^*[\mathcal{A}, \mathfrak{E}] = \left| \Pr[W] - \frac{1}{2} \right|.$$

Bit Guessing: an alternative characterization of Indistinguishability

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Source of Randomness

Event W is defined with respect to the probability space determined by:

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Source of Randomness

Event W is defined with respect to the probability space determined by:

- the random choice of b,
- the random choice of k,
- random choices made (if any) by the encryption algorithm, and
- the random choices made (if any) by the adversary.

Theorem

For every cipher $\mathfrak{E}$ and every PPT adversary $\mathcal{A}$, we have

$$\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 2 \cdot \mathsf{INDadv}^*[\mathcal{A}, \mathfrak{E}].$$

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Proof

- Let p_0 be the probability that the adversary outputs 1 in Experiment 0 of Attack Game of INDadv.

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- Let p_0 be the probability that the adversary outputs 1 in Experiment 0 of Attack Game of INDadv.
- Let p_1 be the probability that the adversary outputs 1 in Experiment 1 of Attack Game of INDadv.

Bit Guessing: an alternative characterization of Indistinguishability

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For every cipher $\mathfrak{E}$ and every PPT adversary $\mathcal{A}$, we have

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Proof

- Let p_0 be the probability that the adversary outputs 1 in Experiment 0 of Attack Game of INDadv.
- Let p_1 be the probability that the adversary outputs 1 in Experiment 1 of Attack Game of INDadv.
- In Attack Game of INDadv^*, the Experiment 0 and Experiment 1 are conditioned over the choice of b.

Proof

- From Experiment 0: $\Pr[\hat{b} = 1 \mid b = 0] = p_0$.
- From Experiment 1: $\Pr[\hat{b} = 1 \mid b = 1] = p_1$.

Bit Guessing: an alternative characterization of Indistinguishability

Proof

- From Experiment 0: $\Pr[\hat{b} = 1 \mid b = 0] = p_0$.
- From Experiment 1: $\Pr[\hat{b} = 1 \mid b = 1] = p_1$.

$$\Pr[\hat{b} = b] = \Pr[\hat{b} = 0 \mid b = 0]\Pr[b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\Pr[b = 1]$$

Proof

- From Experiment 0: $\Pr[\hat{b} = 1 \mid b = 0] = p_0$.
- From Experiment 1: $\Pr[\hat{b} = 1 \mid b = 1] = p_1$.

$$
\begin{aligned}
\Pr[\hat{b} = b] &= \Pr[\hat{b} = 0 \mid b = 0]\Pr[b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\Pr[b = 1] \\
&= \Pr[\hat{b} = 0 \mid b = 0] \cdot \frac{1}{2} + \Pr[\hat{b} = 1 \mid b = 1] \cdot \frac{1}{2}, \\
&\quad \text{as } b \text{ is chosen uniformly from } \{0, 1\}
\end{aligned}
$$

Bit Guessing: an alternative characterization of Indistinguishability

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&= \Pr[\hat{b} = 0 \mid b = 0] \cdot \frac{1}{2} + \Pr[\hat{b} = 1 \mid b = 1] \cdot \frac{1}{2}, \\
&\quad \text{as } b \text{ is chosen uniformly from } \{0, 1\} \\
&= \frac{1}{2}\left[\Pr[\hat{b} = 0 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\right]
\end{aligned}
$$

Bit Guessing: an alternative characterization of Indistinguishability

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- From Experiment 0: $\Pr[\hat{b} = 1 \mid b = 0] = p_0$.
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&\quad \text{as } b \text{ is chosen uniformly from } \{0, 1\} \\
&= \frac{1}{2}\left[\Pr[\hat{b} = 0 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\right] \\
&= \frac{1}{2}\left[1 - \Pr[\hat{b} = 1 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\right]
\end{aligned}
$$

Bit Guessing: an alternative characterization of Indistinguishability

Proof

- From Experiment 0: $\Pr[\hat{b} = 1 \mid b = 0] = p_0$.
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$$
\begin{aligned}
\Pr[\hat{b} = b] &= \Pr[\hat{b} = 0 \mid b = 0]\Pr[b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\Pr[b = 1] \\
&= \Pr[\hat{b} = 0 \mid b = 0] \cdot \frac{1}{2} + \Pr[\hat{b} = 1 \mid b = 1] \cdot \frac{1}{2}, \\
&\quad \text{as } b \text{ is chosen uniformly from } \{0,1\} \\
&= \frac{1}{2}\left[\Pr[\hat{b} = 0 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\right] \\
&= \frac{1}{2}\left[1 - \Pr[\hat{b} = 1 \mid b = 0] + \Pr[\hat{b} = 1 \mid b = 1]\right] \\
&= \frac{1}{2}(1 - p_0 + p_1)
\end{aligned}
$$

Proof

$$\mathsf{INDadv}^*[\mathcal{A}, \mathfrak{E}] = \left| \Pr[W] - \frac{1}{2} \right|$$

Bit Guessing: an alternative characterization of Indistinguishability

Proof

$$\begin{aligned}\mathsf{INDadv}^*[\mathcal{A},\mathfrak{E}] &= \left|\ \Pr[W] - \frac{1}{2}\ \right| \\ &= \left|\ \Pr[\hat{b} = b] - \frac{1}{2}\ \right|\end{aligned}$$

Proof

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\mathsf{INDadv}^*[\mathcal{A},\mathfrak{E}] &= \left| \Pr[W] - \frac{1}{2} \right| \\
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&= \left| \frac{1}{2} + \frac{1}{2}(p_1 - p_1) - \frac{1}{2} \right|
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&= \left|\ \frac{1}{2} + \frac{1}{2}(p_1 - p_1) - \frac{1}{2}\ \right| \\
&= \frac{1}{2}\left|\ p_1 - p_0\ \right|
\end{aligned}
$$

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Proof

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\begin{aligned}
\mathsf{INDadv}^*[\mathcal{A},\mathfrak{E}] &= \left| \Pr[W] - \frac{1}{2} \right| \\
&= \left| \Pr[\hat{b} = b] - \frac{1}{2} \right| \\
&= \left| \frac{1}{2}(1 - p_0 + p_1) - \frac{1}{2} \right| \\
&= \left| \frac{1}{2} + \frac{1}{2}(p_1 - p_1) - \frac{1}{2} \right| \\
&= \frac{1}{2} \mid p_1 - p_0 \mid \\
&= \frac{1}{2}\mathsf{INDadv}[\mathcal{A},\mathfrak{E}].
\end{aligned}
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- Indistinguishability means that no PPT adversary can learn any function of the plaintext given the ciphertext, regardless of the distribution of the message being sent.
  - If there exists any adversary who correctly computes $f(m)$ with some probability when given $\mathcal{E}(k,m)$, then there exists another adversary that can correctly compute $f(m)$ with the same probability without being given the ciphertext at all (and only knowing the distribution on m).

Semantic Security

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- Plaintext is chosen uniformly from $\mathcal{S}_n \stackrel{def}{=} \mathcal{S} \cap \{0,1\}^n$.
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- The functions f that can be computed in polynomial time.

Theorem

Let $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ be a fixed-length computational cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$ for security parameter n that is computationally indistinguishable in the presence of an eavesdropper. Then for any PPT algorithm $\mathcal{A}$, there is a PPT algorithm $\mathcal{A}'$ such that for any efficiently-sampleable set $\mathcal{S}$ and any function f, there is a negligible function ϵ such that:

$$\left|\Pr[\mathcal{A}(1^n, \mathcal{E}(k, m)) = f(m)] - \Pr[\mathcal{A}'(1^n) = f(m)]\right| \leq \mathsf{negl}(n),$$

where the where m is chosen uniformly at random from $\mathcal{S}_n \stackrel{def}{=} \mathcal{S} \cap \{0, 1\}^n$ and the probabilities are taken over the choice of m and the key k, and any random coins used by $\mathcal{A}, \mathcal{A}'$, and the encryption process.

Semantic Security

Proof (Sketch)

- $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ is computationally indistinguishable in the presence of an eavesdropper.
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- Let $\mathcal{A}$ successfully computes $f(m)$ given $\mathcal{E}(k, m)$.
- **Claim:** $\mathcal{A}$ computes $f(m)$ given $\mathcal{E}(k, 1^n)$ with almost same probability.
- Otherwise, $\mathcal{A}$ could be used to distinguish between $\mathcal{E}(k, m)$ and $\mathcal{E}(k, 1^n)$.

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- This observation, gives us the idea of $\mathcal{A}'$.

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Semantic Security

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- By the above, $\mathcal{A}$ outputs $f(m)$ when run as a sub-routine by $\mathcal{A}'$ with almost the same probability as when it receives $\mathcal{E}(k,m)$.
- Thus, $\mathcal{A}'$ fulfills the property required by the claim.

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  - Leaked information $h(m)$ given to adversary.
  - Adversary computes $f(m)$.

Definition (Semantic Security)

A computational cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ is semantically secure in the presence of an eavesdropper if for every PPT adversary $\mathcal{A}$ there exists a PPT adversary $\mathcal{A}'$ such that for all efficiently-sampleable distributions $X = (X_1, X_2, \ldots)$ and all polynomial-time computable functions f and h, there exists a negligible function ϵ such that

$$\left|\Pr[\mathcal{A}(1^n, \mathcal{E}(k,m), h(m)) = f(m)] - \Pr[\mathcal{A}'(1^n, h(m)) = f(m)]\right| \leqslant \epsilon(n),$$

where m is chosen according to distribution X_n, and the probabilities are taken over the choice of m and the key k, and any random coins used by $\mathcal{A}, \mathcal{A}'$, and the encryption process.

Theorem

A computational cipher has indistinguishable encryptions in the presence of an eavesdropper if and only if it is semantically secure in the presence of an eavesdropper.

Computational ciphers: the formalities

System Parameterization

- In the mathematical model (though not always in real-world systems), we associate with $\mathfrak{E}$ families of key, message, and ciphertext spaces, indexed by
  - a security parameter, which is a positive integer, and is denoted by n, and
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1. First choose $n \in \mathbb{N}$ as security parameter.
2. A system parameter ν is generated using an algorithm specific to the cipher.
3. The system parameter ν together with n, gives a detailed description of a fixed instance of the cipher, with

$$(\mathcal{K}, \mathcal{M}, \mathcal{C}) = (\mathcal{K}_{n,\nu}, \mathcal{M}_{n,\nu}, \mathcal{C}_{n,\nu}).$$

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- A collection $\mathcal{S} = \{\mathcal{S}_{n,\nu}\}_{n,\nu}$ of finite sets of bits strings, where n runs over $\mathbb{N}$ and ν runs over $\mathsf{Supp}(P(n))$, is called a family of spaces with system parameterization P, provided the lengths of all the strings in each of the sets $\mathcal{S}_{n,\nu}$ are bounded by some polynomial p in n.

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- We say that $\mathcal{S}$ is efficiently recognizable if there is an efficient deterministic algorithm that on input $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, and $s \in \{0,1\}^{\leqslant p(n)}$, determines if $s \in \mathcal{S}_{n,\nu}$.

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- We say that $\mathcal{S}$ is efficiently sampleable if there is an efficient probabilistic algorithm that on input $n \in \mathbb{N}$ and $\nu \in \mathsf{Supp}(P(n))$, outputs an element uniformly distributed over $\mathcal{S}_{n,\nu}$.

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- We say that $\mathcal{S}$ has an effective length function if there is an efficient deterministic algorithm that on input $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, and $s \in \mathcal{S}_{n,\nu}$, outputs a non-negative integer, called the length of s.

Definition (Computational ciphers)

A computational cipher $\mathfrak{E}$ consists of a tuple of algorithms $(\mathcal{G}, \mathcal{E}, \mathcal{D})$, along with three families of spaces with system parameterization P:

$$\mathbf{K} = \{\mathcal{K}_{n,\nu}\}_{n,\nu}, \ \mathbf{M} = \{\mathcal{M}_{n,\nu}\}_{n,\nu} \ \text{ and } \ \mathbf{C} = \{\mathcal{C}_{n,\nu}\}_{n,\nu}.$$

1. $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are efficiently recognizable.

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2. $\mathbf{K}$ is efficiently sampleable and $\mathcal{G}$ is such a sampling algorithm.
3. $\mathbf{M}$ has an effective length function.
4. Algorithm $\mathcal{E}$ is an efficient probabilistic algorithm that on input n, ν, k, m, where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, and $m \in \mathcal{M}_{n,\nu}$, always outputs an element of $\mathcal{C}_{n,\nu}$.

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1. $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are efficiently recognizable.
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4. Algorithm $\mathcal{E}$ is an efficient probabilistic algorithm that on input n, ν, k, m, where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, and $m \in \mathcal{M}_{n,\nu}$, always outputs an element of $\mathcal{C}_{n,\nu}$.
5. Algorithm $\mathcal{D}$ is an efficient deterministic algorithm that on input n, ν, k, m, where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, and $c \in \mathcal{C}_{n,\nu}$, outputs either an element of $\mathcal{M}_{n,\nu}$, or a special symbol $\mathsf{reject} \in \mathcal{M}_{n,\nu}$.

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Definition (Computational ciphers)

A computational cipher $\mathfrak{E}$ consists of a tuple of algorithms $(\mathcal{G}, \mathcal{E}, \mathcal{D})$, along with three families of spaces with system parameterization P:

$$\mathbf{K} = \{\mathcal{K}_{n,\nu}\}_{n,\nu}, \ \mathbf{M} = \{\mathcal{M}_{n,\nu}\}_{n,\nu} \ \text{ and } \ \mathbf{C} = \{\mathcal{C}_{n,\nu}\}_{n,\nu}.$$

1. $\mathbf{M}, \mathbf{C}$ and $\mathbf{K}$ are efficiently recognizable.
2. $\mathbf{K}$ is efficiently sampleable and $\mathcal{G}$ is such a sampling algorithm.
3. $\mathbf{M}$ has an effective length function.
4. Algorithm $\mathcal{E}$ is an efficient probabilistic algorithm that on input n, ν, k, m, where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, and $m \in \mathcal{M}_{n,\nu}$, always outputs an element of $\mathcal{C}_{n,\nu}$.
5. Algorithm $\mathcal{D}$ is an efficient deterministic algorithm that on input n, ν, k, m, where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, and $c \in \mathcal{C}_{n,\nu}$, outputs either an element of $\mathcal{M}_{n,\nu}$, or a special symbol $\mathsf{reject} \in \mathcal{M}_{n,\nu}$.
6. For all n, ν, k, m, c where $n \in \mathbb{N}$, $\nu \in \mathsf{Supp}(P(n))$, $k \in \mathcal{K}_{n,\nu}$, $m \in \mathcal{M}_{n,\nu}$, and $c \in \mathsf{Supp}(\mathcal{E}(n, \nu; k, m))$, we have $\mathcal{D}(n, \nu; k, c) = m$.

End