Cryptology

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Lecture 12

Public Key Encryption

Introduction

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- Public key is known by every one even by the adversary.

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    - Sends (c_i, c_f) to friend i.

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- We assume secure key distribution.

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- Messages are assumed to lie in some finite message space $\mathcal{M}$, and ciphertexts in some finite ciphertext space C. We say that $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ is defined over $(\mathcal{M}, C)$.

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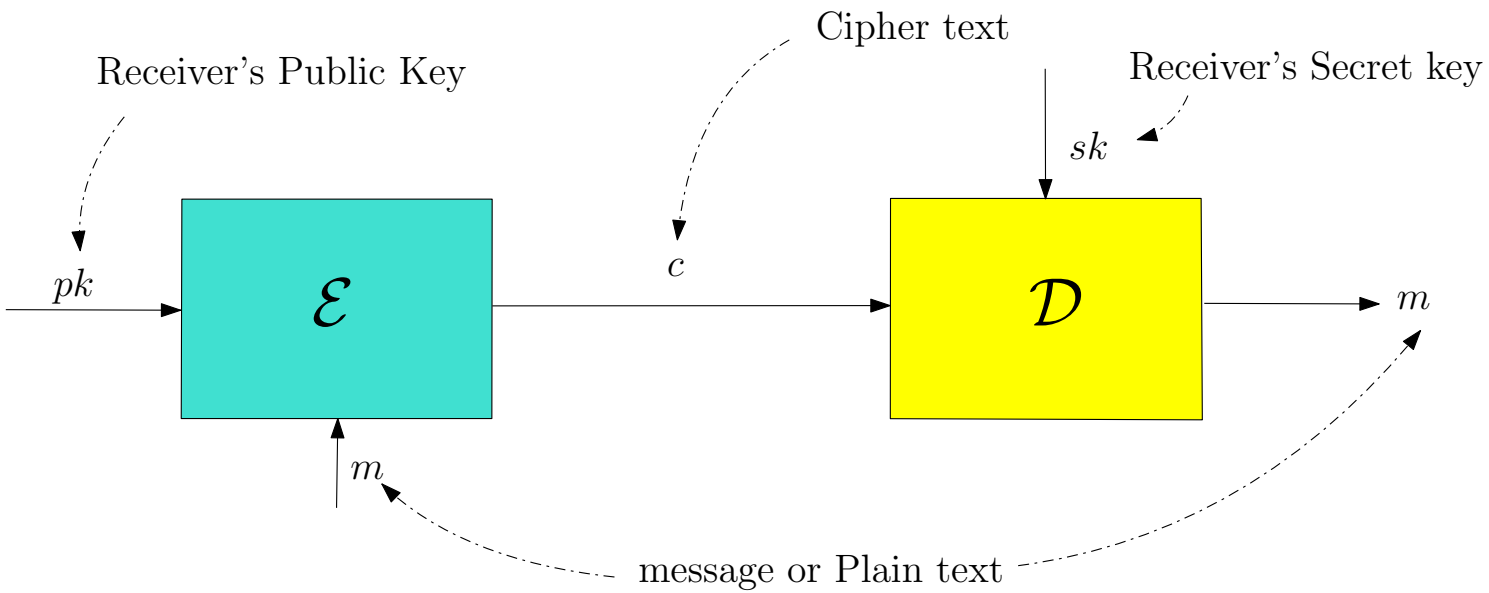

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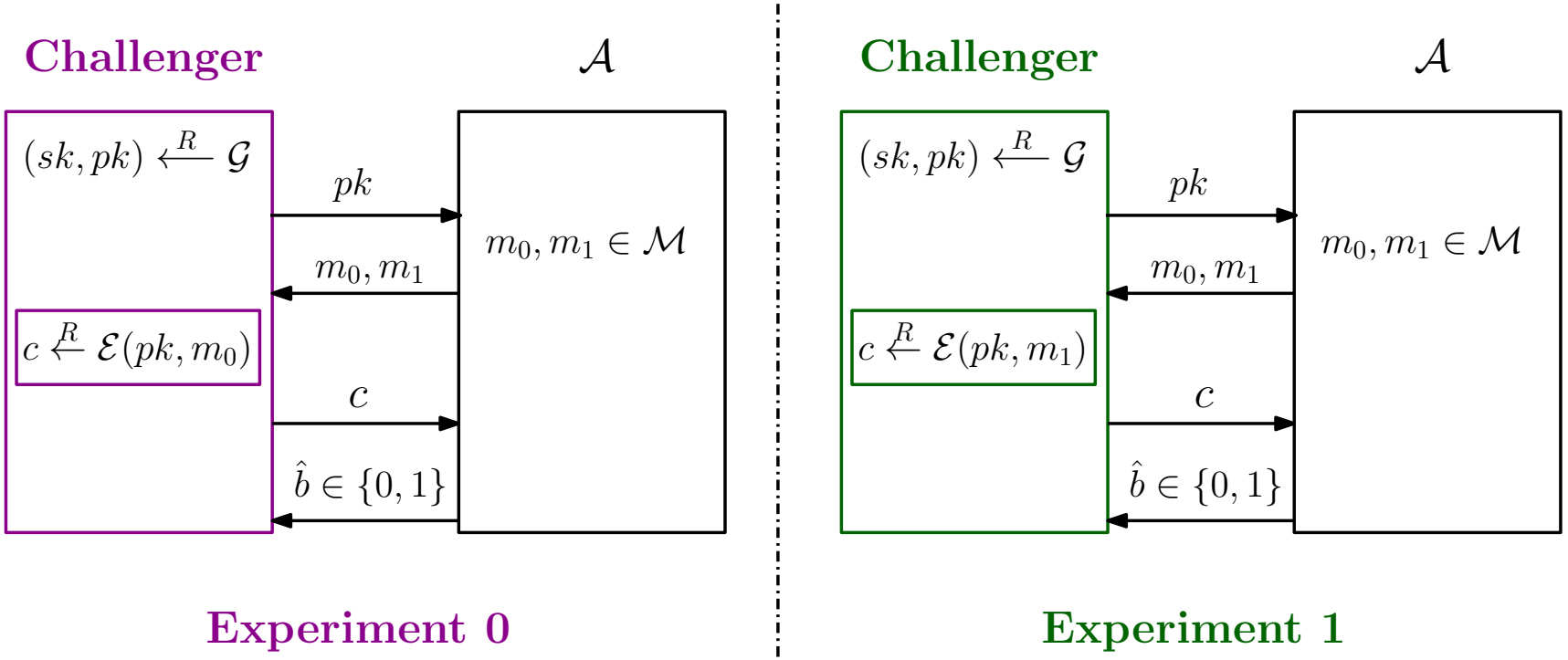

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For $b = 0, 1$, let W_b be the event that $\mathcal{A}$ outputs 1 in Experiment b. We define the advantage of $\mathcal{A}$ in the Indistinguishability attack with respect to $\mathfrak{E}$ as

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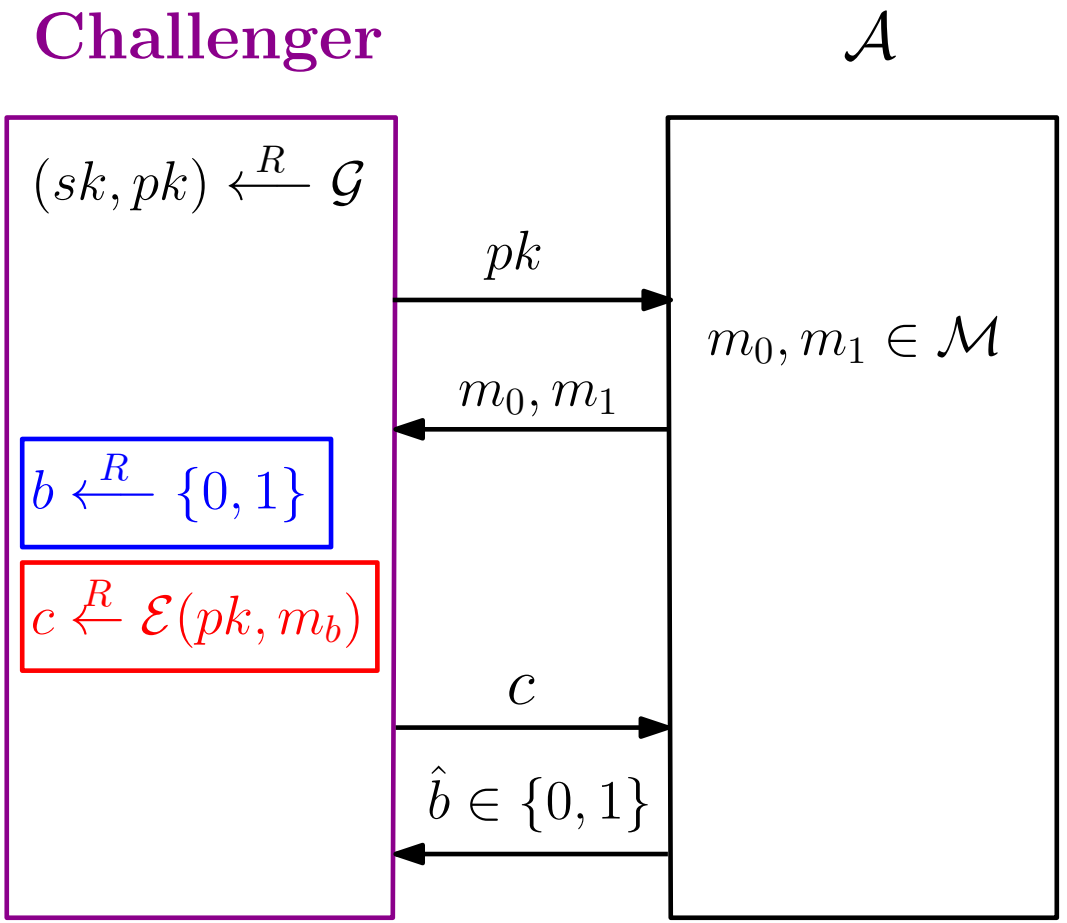

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For every public-key encryption scheme $\mathfrak{E}$ and every PPT adversary $\mathcal{A}$, we have

$$\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 2 \cdot \mathsf{INDadv}^*[\mathcal{A}, \mathfrak{E}].$$

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In particular, for every CPA adversary $\mathcal{A}$ that plays CPA Attack Game with respect to $\mathfrak{E}$, and which makes at most Q queries to its challenger, there exists an Indistinguishability adversary $\mathcal{B}$, where $\mathcal{B}$ is an elementary wrapper around $\mathcal{A}$, such that

$$\mathsf{CPAadv}[\mathcal{A}, \mathfrak{E}] = Q \cdot \mathsf{INDadv}[\mathcal{B}, \mathfrak{E}].$$

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- Compute $m \pmod N$ from equations $m = m_p \pmod p$ and $m = m_q \pmod q$ using CRT.

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- Now apply previous attack.

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- (sk, pk) for i-th user $((N, d_i), (N, e_i))$.
- r-th user can compute factor of N from (N, e_r, d_r) in polynomial time.
- Then r-th user can compute d_i of i-th user from (N, e_i), and decrypt any message intended for i-th user.

Common Modulus N: Case 2

- $pk_i = (N, e_i)$ for $i = 1, 2$ and $\gcd(e_1, e_2) = 1$.
- Let the same message m has been encrypted: $c_i := m^{e_i} \pmod N$ for $i = 1, 2$.
- $\gcd(e_1, e_2) = 1 \Leftrightarrow Xe_1 + Ye_2 = 1$, and computing X and Y are easy.
- By CRT, solve

$$c_1^X \cdot c_2^Y = m^{Xe_1} \cdot m^{Ye_2} = m^{Xe_1 + Ye_2} = m^1 = m \pmod N.$$

Padded RSA

- Let ℓ be a function s.t. $\ell(n) \leqslant 2n-2, \forall n$.

Key Generation

1. $(N, e, d) \longleftarrow \mathsf{GenRSA}(1^n)$

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$\mathcal{E}(pk, m \in \{0,1\}^{\ell(n)})$

1. Choose a random string $r \xleftarrow{R} \{0,1\}^{|N|-\ell(n)-1}$
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$\mathcal{D}(sk, c \in \mathbb{Z}_N)$

1. Compute $(r \| m) \longleftarrow c^d \pmod N$.
2. Output least significant $\ell(n)$ bits.

Padded RSA

Theorem

If the RSA problem is hard relative to GenRSA then padded RSA with $\ell(n) = O(\log n)$ has indistinguishable encryptions under a chosen-plaintext attack.

ElGamal Public-Key Encryption Scheme

Domain parameter

- Let G be a cyclic multiplicative group.

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Key Generation

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- Compute $Q \equiv g^d$.
- $sk = (G, g, n, d)$ and $pk = (G, g, n, Q)$.

ElGamal Public-Key Encryption Scheme

$\mathcal{E}(pk, m)$

1. Choose $k \xleftarrow{R} \mathbb{Z}_n \setminus \{0, 1\}$.
2. Compute $r \longleftarrow g^k$.
3. Compute $s \longleftarrow mQ^k$.
4. $c = (r, s)$.

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1. Compute $m = sr^{-d}$.

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Correctness

$$m' = sr^{-d} = mQ^k(g^k)^{-d} = m(g^d)^k(g^k)^{-d} = mg^{kd-kd} = m.$$

ElGamal Public-Key Encryption Scheme

Theorem

If the DDH problem is hard relative to G, then the ElGamal encryption scheme has indistinguishable encryptions under a chosen-plaintext attack.

End