Cryptology

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Lecture 05

Pseudo-Random Generator and Stream Cipher

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  - A distribution $\mathfrak{D}$ over strings of length ℓ is pseudorandom means that $\mathfrak{D}$ is indistinguishable from the uniform distribution over strings of length ℓ.
  - More precisely, it is infeasible for any polynomial-time algorithm to tell whether it is given a string sampled according to $\mathfrak{D}$ or an ℓ-bit string chosen uniformly at random.

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- This is the exact intuition that lies behind the perfect secrecy of the one-time pad (OTP).
- In OTP, the ciphertext is uniformly distributed.
- The OTP worked by computing the XOR of a random string (the key) with the plaintext.
- If a pseudorandom string were used instead, this should not make any noticeable difference to a polynomial-time observer.
- Thus, security should still hold for polynomial-time adversaries.

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- PRG uses a small amount of true randomness in order to generate a large amount of pseudorandomness.
- Useful whenever a large number of random(-looking) bits are needed, since generating true random bits is difficult and slow.

Pseudo-Random Generator (PRG)

Definition (Pseudo-Random Generator)

Let $\ell(\cdot)$ be a polynomial function with security parameter $n \in \mathbb{N}$, and $\mathcal{S} = \{0,1\}^n$ and $\mathcal{R} = \{0,1\}^{\ell(n)}$. Let $G : \mathcal{S} \to \mathcal{R}$ be a deterministic, polynomial-time algorithm in n such that for any n and any input (*seed*) $s \in \mathcal{S}$, the result of $G(s)$ is a string of length $\ell(n)$ with $\ell(n) \gg n$ for all n. We say that G is a Pseudo-Random Generator (PRG), and is defined over $(\mathcal{S}, \mathcal{R})$. The function $\ell(\cdot)$ is called the expansion factor of G.

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For a given PRG G defined over $(\mathcal{S}, \mathcal{R})$ with security parameter $n \in \mathbb{N}$, and for a given PPT adversary $\mathcal{A}$, we define two experiments: Experiment 0 and Experiment 1. For $b = 0, 1$, we define Experiment b as:

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- The adversary outputs a bit $\hat{b} \in \{0, 1\}$.
- Assume $\hat{b} = 0 \Rightarrow$ Not random and $\hat{b} = 1 \Rightarrow$ Random.

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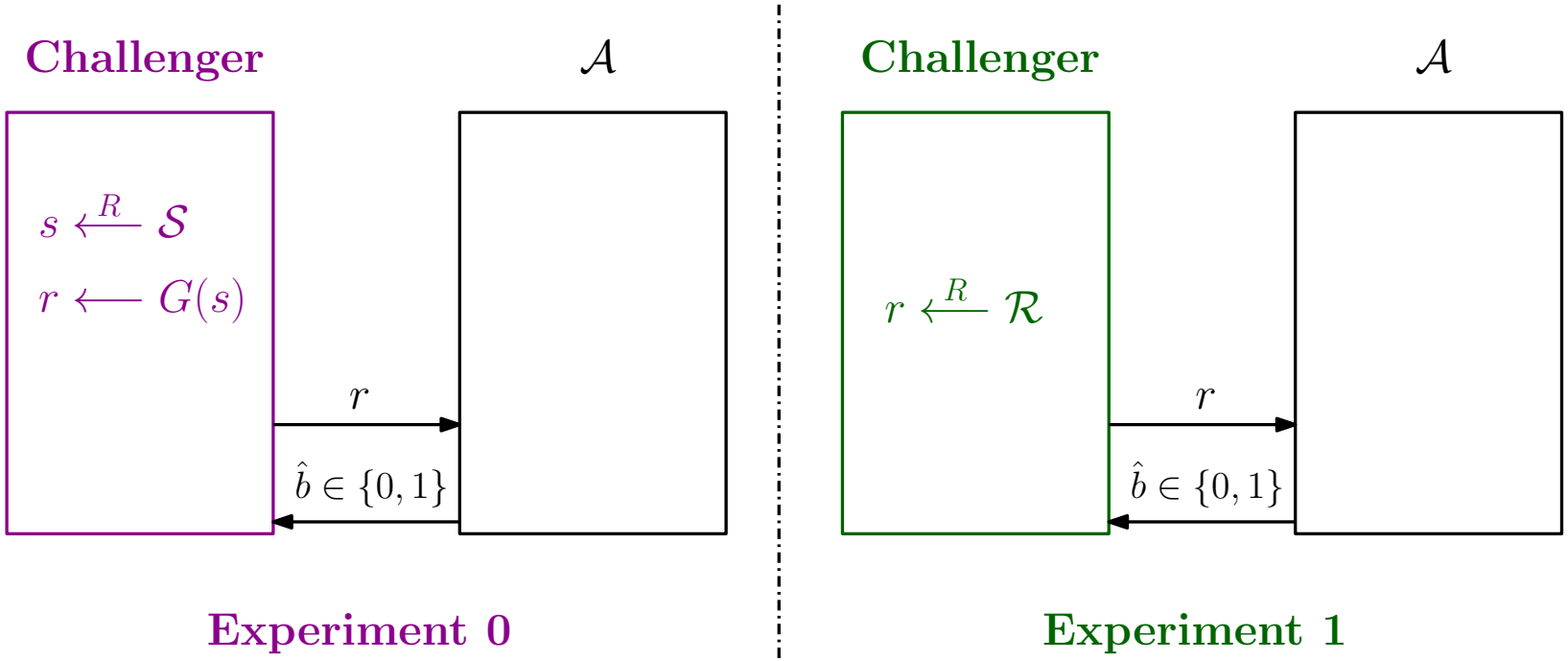

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Events W_0 and W_1 are defined with respect to the probability space determined by:

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Secure PRG

A PRG G is secure if for all PPT adversaries (or distinguisher) $\mathcal{A}$ there exists a negligible function ϵ such that

$$\mathsf{PRGadv}[\mathcal{A}, G] \leqslant \epsilon(n).$$

PRG Advantage: Bit Guessing version

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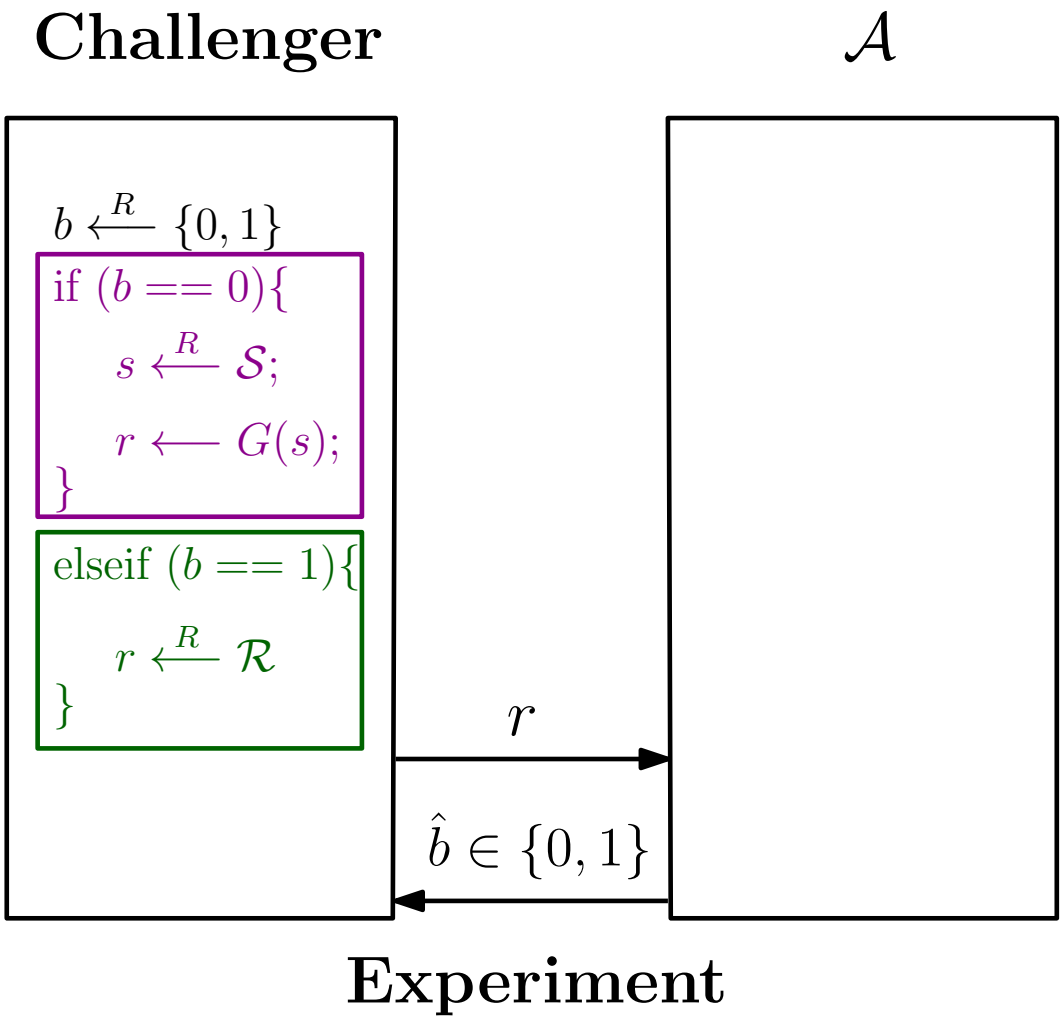

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Theorem

For every PRG G and every PPT adversary $\mathcal{A}$, we have

$$\mathsf{PRGadv}[\mathcal{A}, G] = 2 \cdot \mathsf{PRGadv}^*[\mathcal{A}, G].$$

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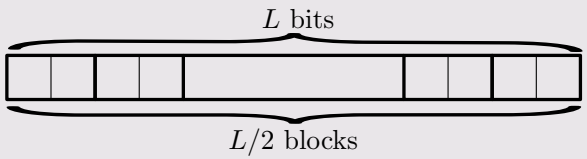

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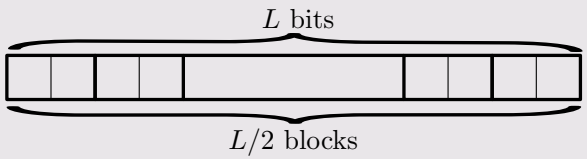

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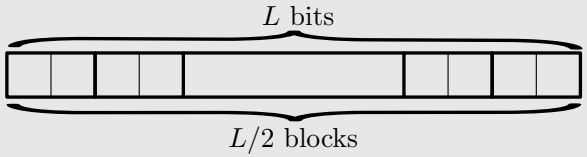

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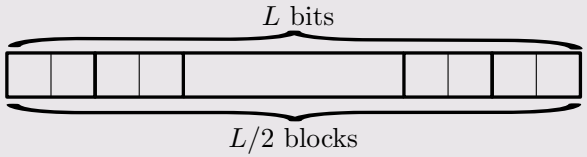

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- Conceptually, if PRG G is secure then there must be no efficient statistical test.

Discussion

- PRG is far from random.
- Let PRG G be defined over $\mathcal{S} = \{0,1\}^n$ and $\mathcal{R} = \{0,1\}^L$ where $L = \ell(n)$ and $L >> n$.
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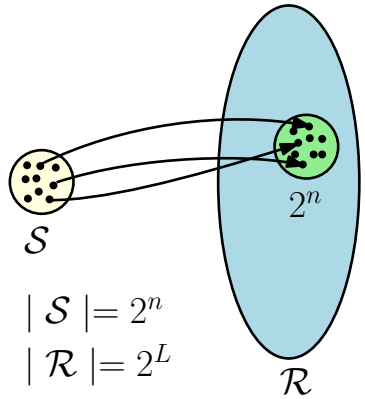

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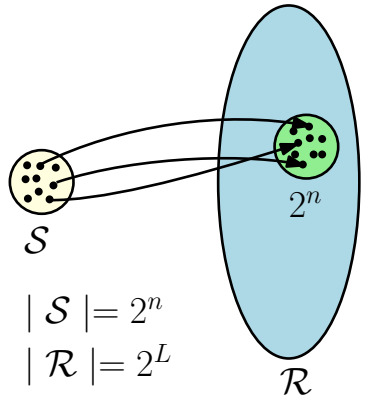

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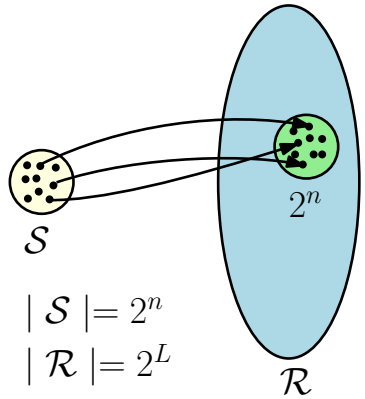

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- $\Pr[x \text{ is in range of } G] = 2^{n-L}$.

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- This attack is commonly known as Brute Force attack.
- If $\mathcal{A}$ is a poly-time distinguisher, it does not have sufficient time.

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- For seed space $\mathcal{S} = \{0,1\}^n$, n must be long enough such that brute-forced attack is not possible.

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- $\mathcal{G} : s \xleftarrow{R} \mathcal{S}$.
- $\mathcal{E}(s, m)$: If $| m | = v \leqslant L$, then

$$c := G(s)[0 \ldots (v-1)] \oplus m.$$

Stream Cipher: encryption with a PRG

Stream Cipher

- Let G be a PRG defined over $(\mathcal{S}, \mathcal{R}) = (\{0,1\}^n, \{0,1\}^L)$.
- The stream cipher $\mathfrak{E} = (\mathcal{G}, \mathcal{E}, \mathcal{D})$ constructed from G is defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where
  - $\mathcal{K} = \{0,1\}^n, \mathcal{M} = \{0,1\}^{\leqslant L}$ and $\mathcal{C} = \{0,1\}^{\leqslant L}$.
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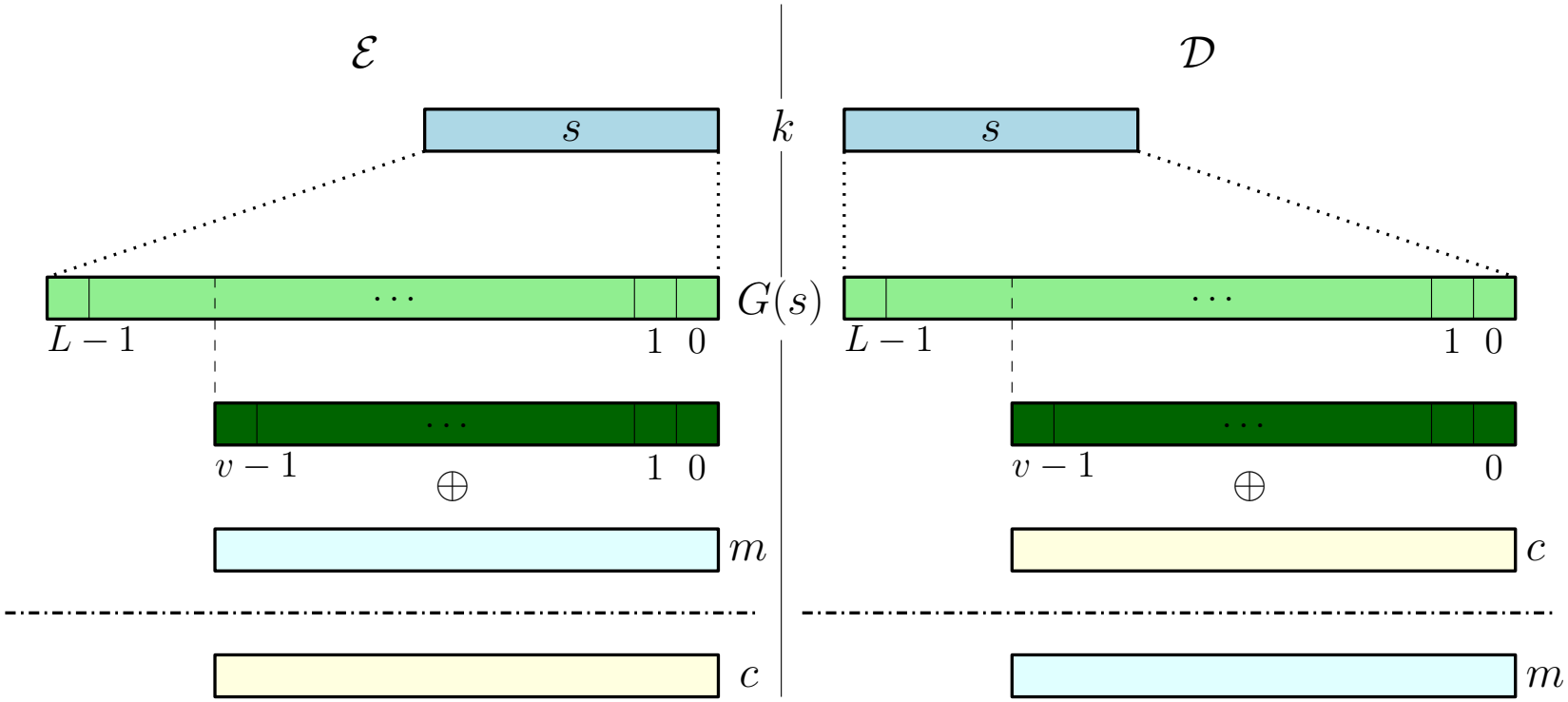
Stream Cipher



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If G is a secure PRG, then the stream cipher $\mathfrak{E}$ constructed from G is a semantically secure cipher.
In particular, for every computational indistinguishing adversary $\mathcal{A}$ that attacks $\mathfrak{E}$ as in Symmetric key Encryption Indistinguishability Attack Game, there exists a PRG adversary $\mathcal{B}$ that attacks G as in PRG Indistinguishability Attack Game, such that

$$\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 2 \cdot \mathsf{PRGadv}[\mathcal{B}, G].$$

Proof

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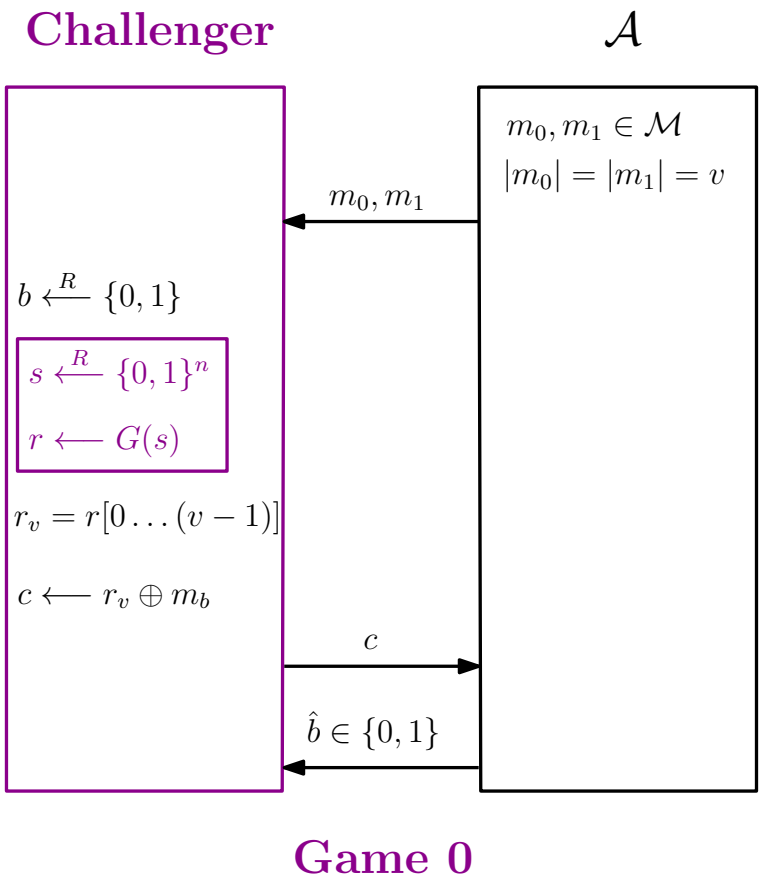
- Therefore, any PPT adversary $\mathfrak{D}$, there exists a negligible function ϵ such that

$$\mathsf{PRGadv}[\mathfrak{D}, G] \leqslant \epsilon(n).$$

Stream Cipher

Proof



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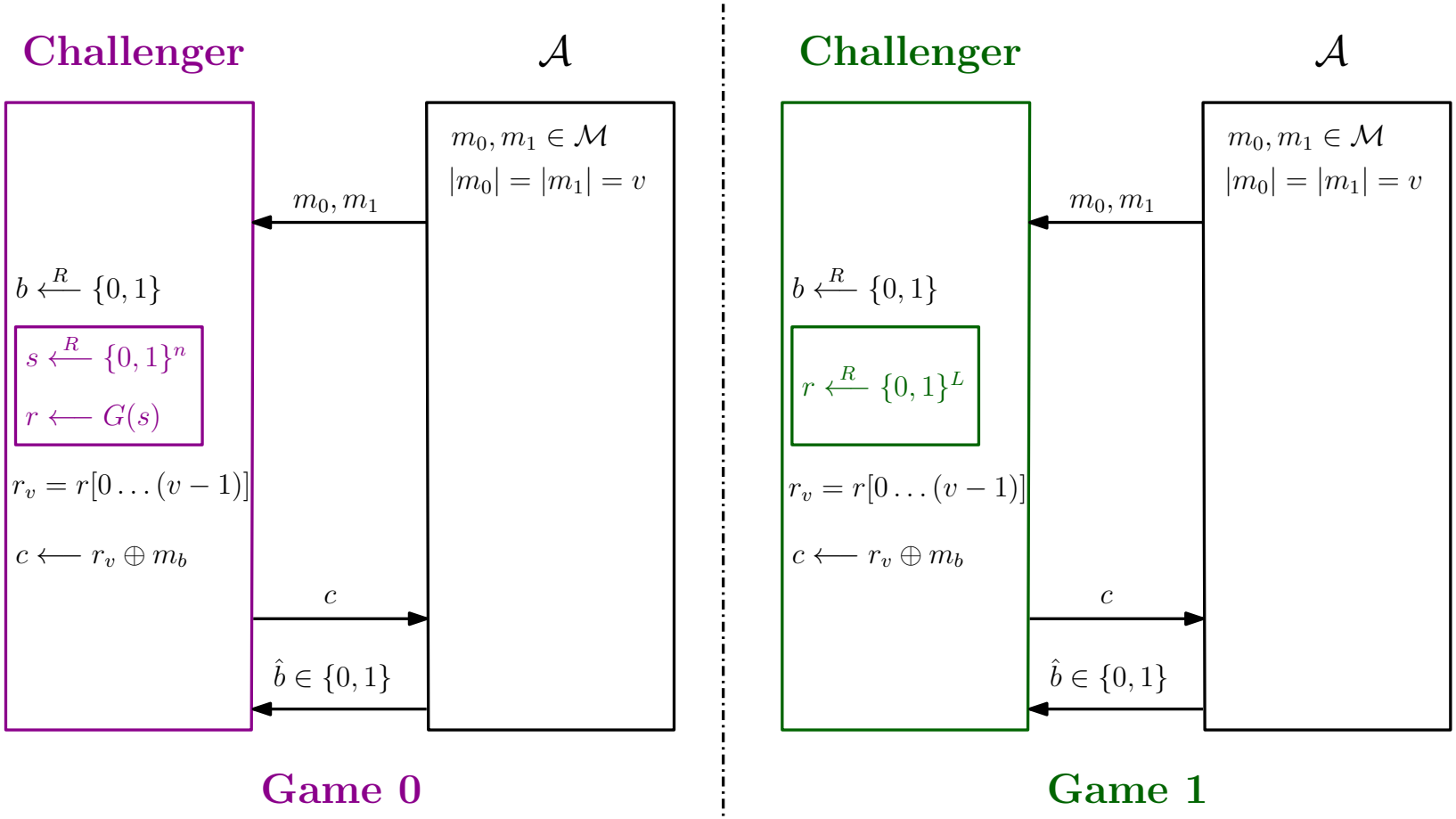

Proof

- Game 0:
  - Let W_0 be the event that $\hat{b} = b$ in Game 0.
  - By definition, we have

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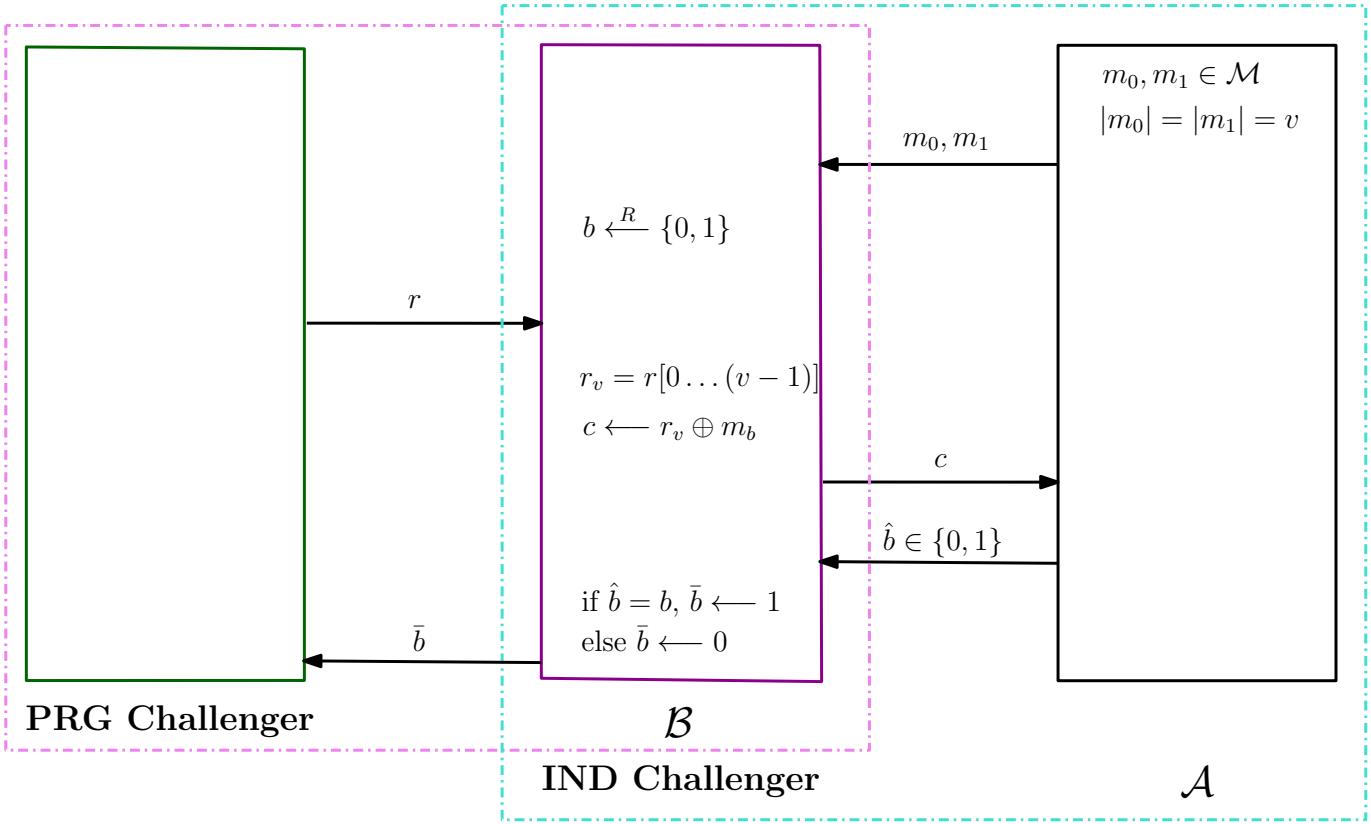
- Game 1:
  - Let W_1 be the event that $\hat{b} = b$ in Game 1.
  - $\mathcal{A}$ is attacking the variable length one-time pad.
  - the challenger's hidden bit b and $\mathcal{A}$'s output $\hat{b}$ are independent.
  - Therefore, we have

$$\Pr[W_1] = \frac{1}{2}.$$

Stream Cipher

Proof

- If $\beta = 0$, we are in Experiment 0 of PRG indistinguishability attack game.
- If $\beta = 1$, we are in Experiment 1 of PRG indistinguishability attack game.



Experiment β

Proof

- Let p_0 be the probability $\mathcal{B}$ outputs 1 in Experiment 0.
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Proof

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Proof

$$
\begin{aligned}
\mathsf{PRGadv}[\mathcal{B}, G] &= |\, p_0 - p_1 \,| \\
&= |\, \Pr[W_0] - \Pr[W_1] \,| \\
&= \left| \Pr[W_0] - \frac{1}{2} \right| \\
&= \mathsf{INDadv}^{*}[\mathcal{A}, \mathfrak{E}]
\end{aligned}
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\end{aligned}$$

As $\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 2 \cdot \mathsf{INDadv}^*[\mathcal{A}, \mathfrak{E}]$, we have

$$\mathsf{INDadv}[\mathcal{A}, \mathfrak{E}] = 2 \cdot \mathsf{PRGadv}[\mathcal{B}, G] \leqslant 2\epsilon(n).$$

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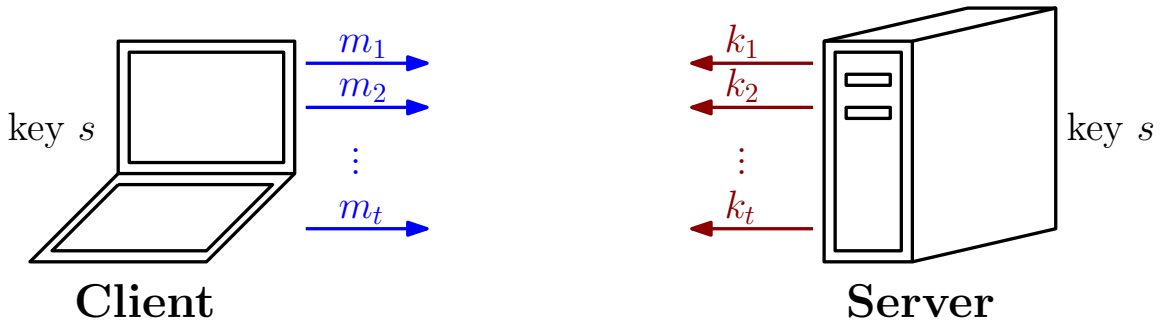
  - English text contains enough redundancy.
  - Given $\Delta = m_0 \oplus m_1$, one can recover both the messages m_0 and m_1.
  - This construction known as Two-time pad.
- A stream cipher key should never be used to encrypt more than one message.

Example of Two-time pad

- Project Venona (1941-1946):
  - During WWII, Russia was using one time pad.
  - Human used to throw dice and the results were used as pad.
  - It was hard for human being to generate all the pads.
  - They started to encrypt two messages using one pad.
  - US intelligence was able, over the next 25 years, to break some 2900 messages just from intercepted ciphers.

Example of Two-time pad

- MS-PPTP (Windows NT):
  - PPTP stands for Point to Point Transfer Protocol.



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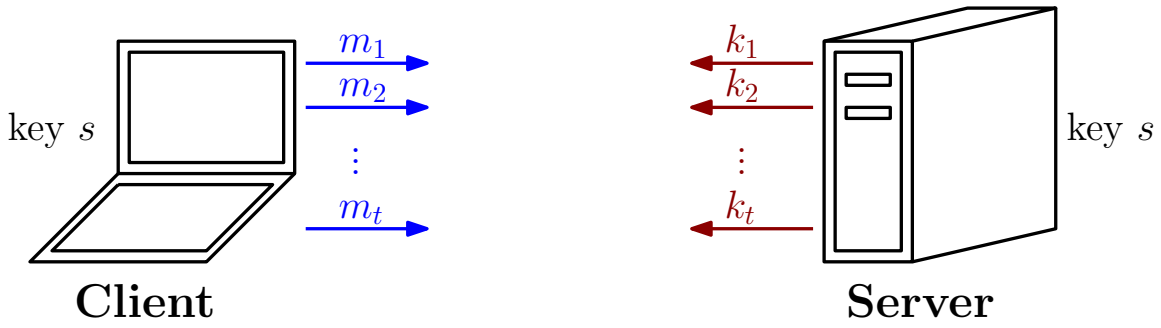

  - $m := (m_1 \| m_2 \| \cdots \| m_t)$ and $c_1 := m \oplus G(s)$.
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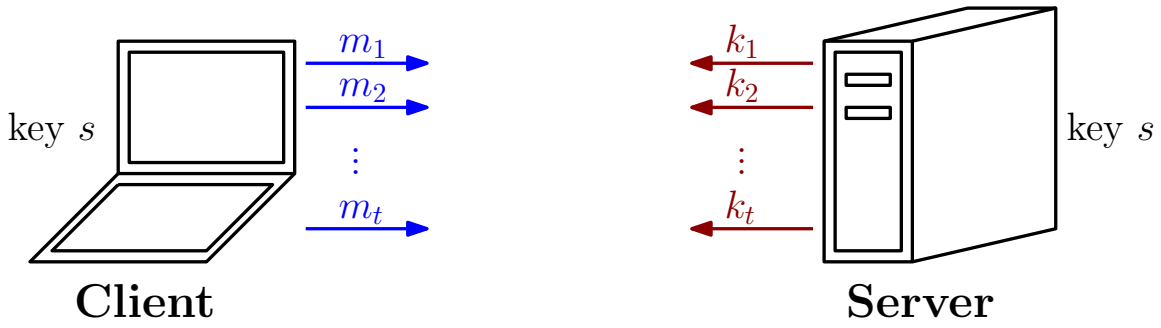

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  - Solution: $s = (s_{C \to S}, s_{S \to C})$, and both know the key.

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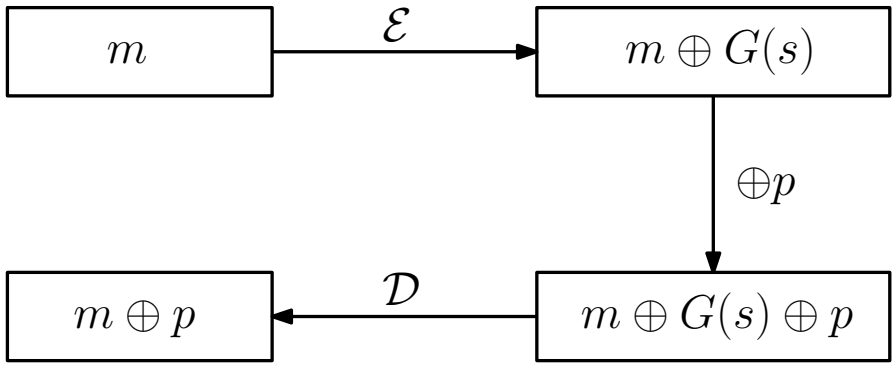

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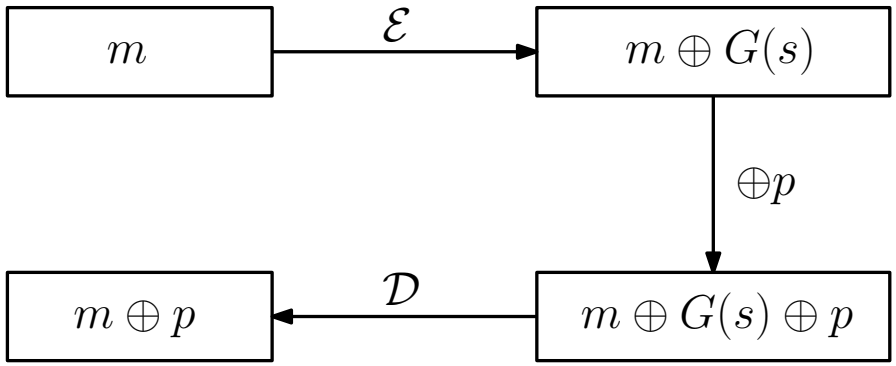

- The modification will never be detected by the decryptor.
- Malleable since an attacker can cause predictable changes to the plaintext.

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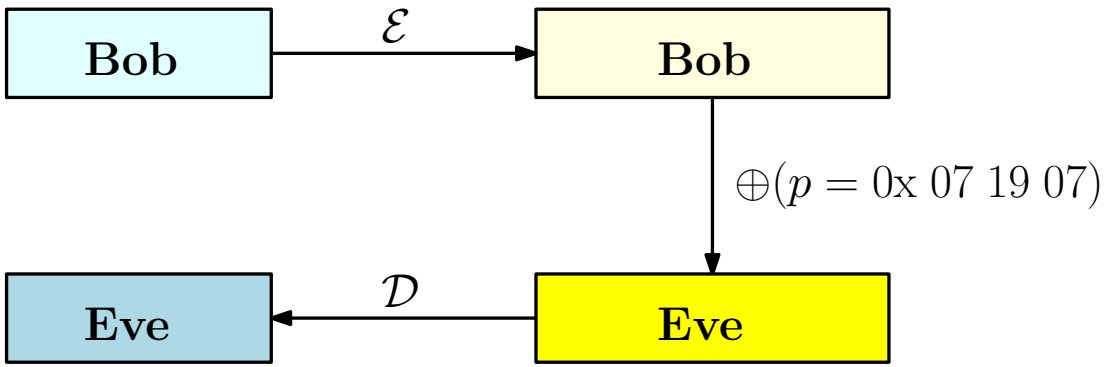

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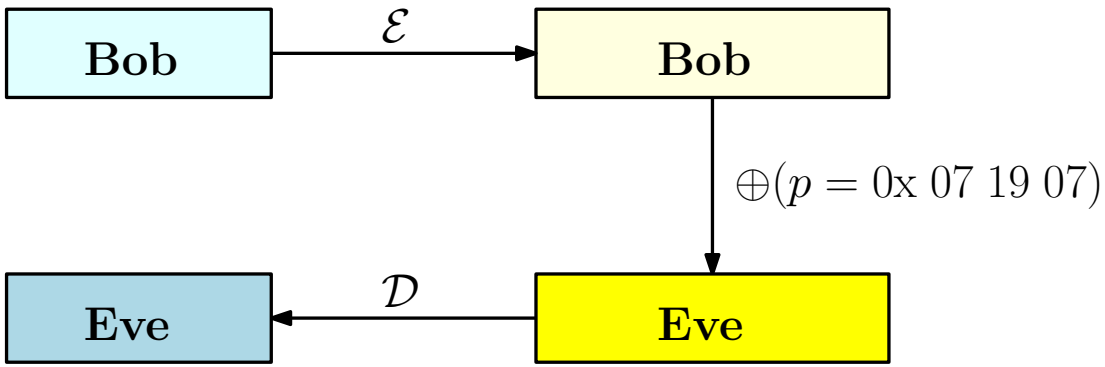

- Bob = 0x 42 6F 62.
- Eve = 0x 45 76 65.
- p =Bob$\oplus$Eve = 0x 07 19 07

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- To increase the size of the output space of the original PRG while preserving its security.

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- There are two basic constructions:
  - A Parallel Construction, and
  - A Sequential construction.

A Parallel Construction

Let G be a PRG defined over $(\mathcal{S}, \mathcal{R})$. We construct a new PRG G' that applies G to γ seeds, and is defined over $(\mathcal{S}^\gamma, \mathcal{R}^\gamma)$, and for $s_1, s_2, \ldots, s_\gamma \in \mathcal{S}$,

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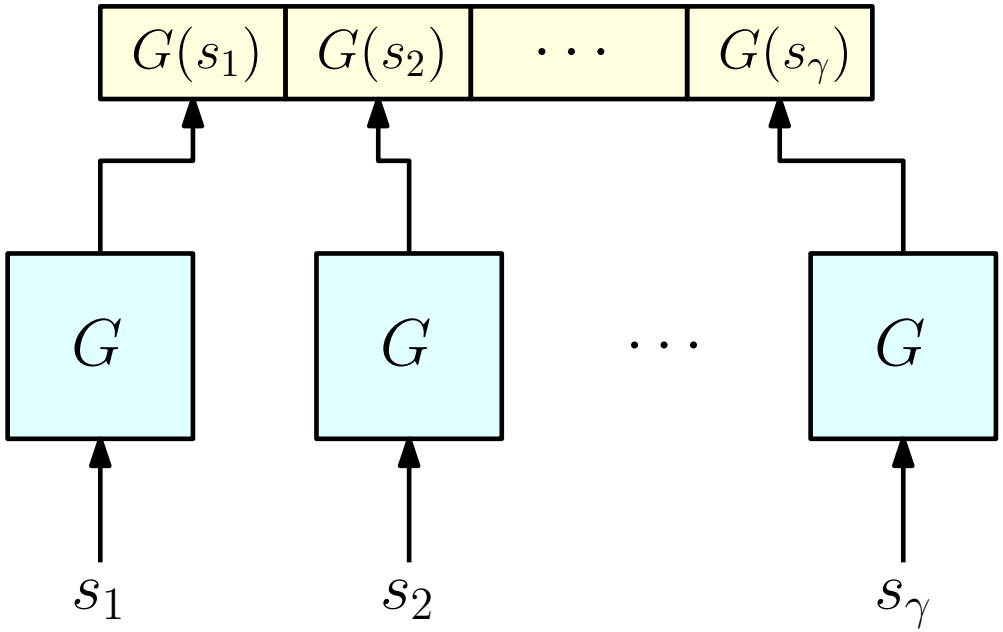
- γ is called a repetition parameter, and
- γ needs to be a poly-bounded value.

A Parallel Construction



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In particular, for every PRG distinguisher $\mathcal{A}$ that attacks G' as in PRG Indistinguishability Game, there exists a PRG distinguisher $\mathcal{B}$ that attacks G as in PRG Indistinguishability Game, where $\mathcal{B}$ uses $\mathcal{A}$ as sub-routine, such that

$$\mathsf{PRGadv}[\mathcal{A}, G'] = \gamma \cdot \mathsf{PRGadv}[\mathcal{B}, G].$$

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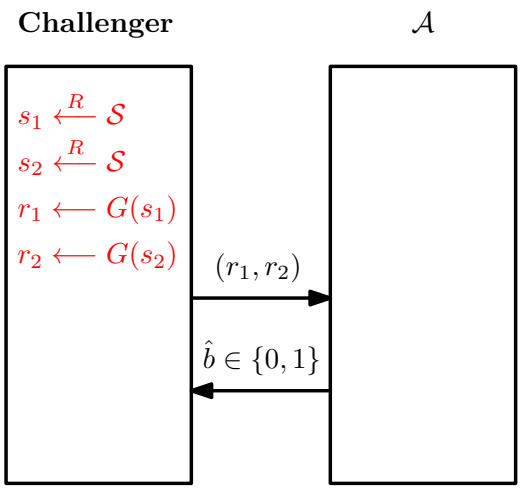

Hybrid 0

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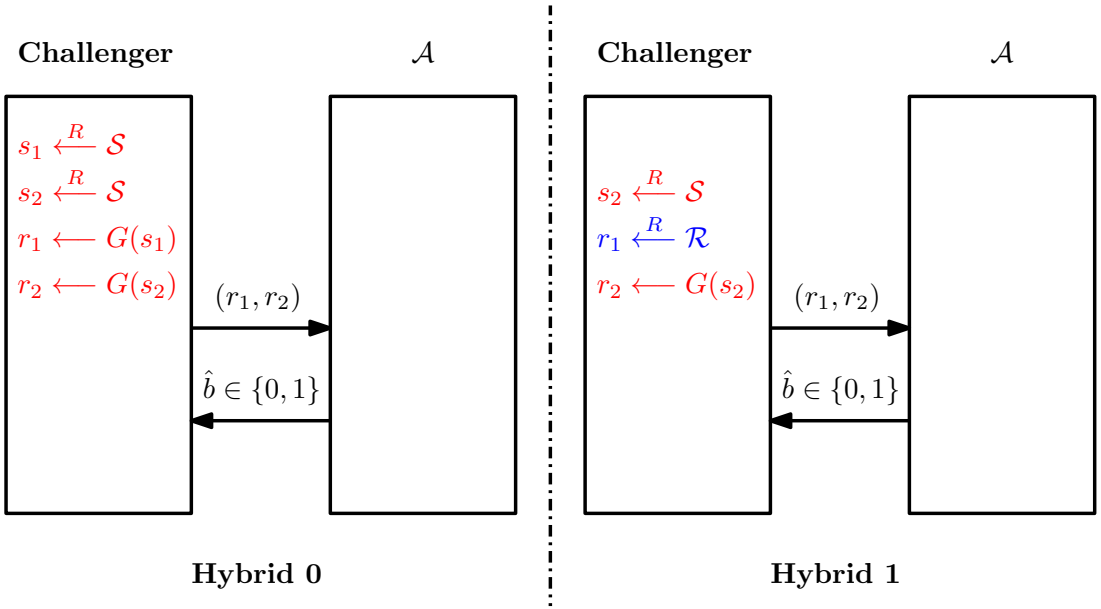

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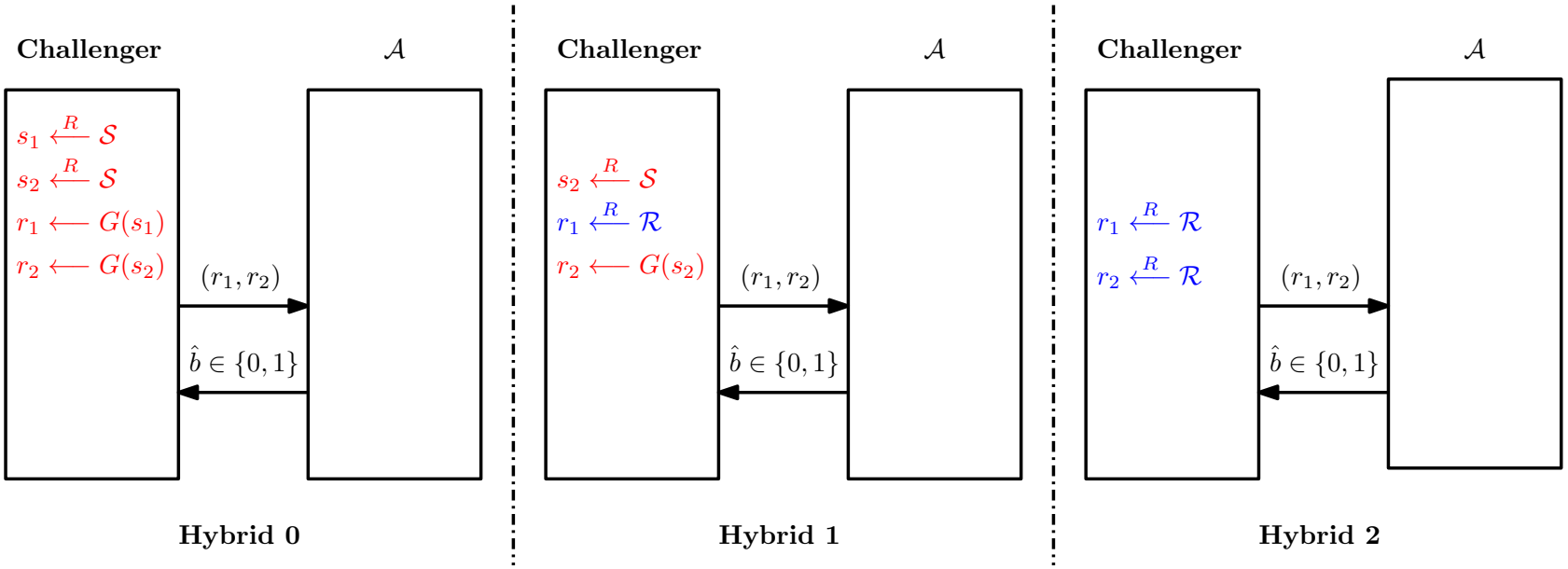

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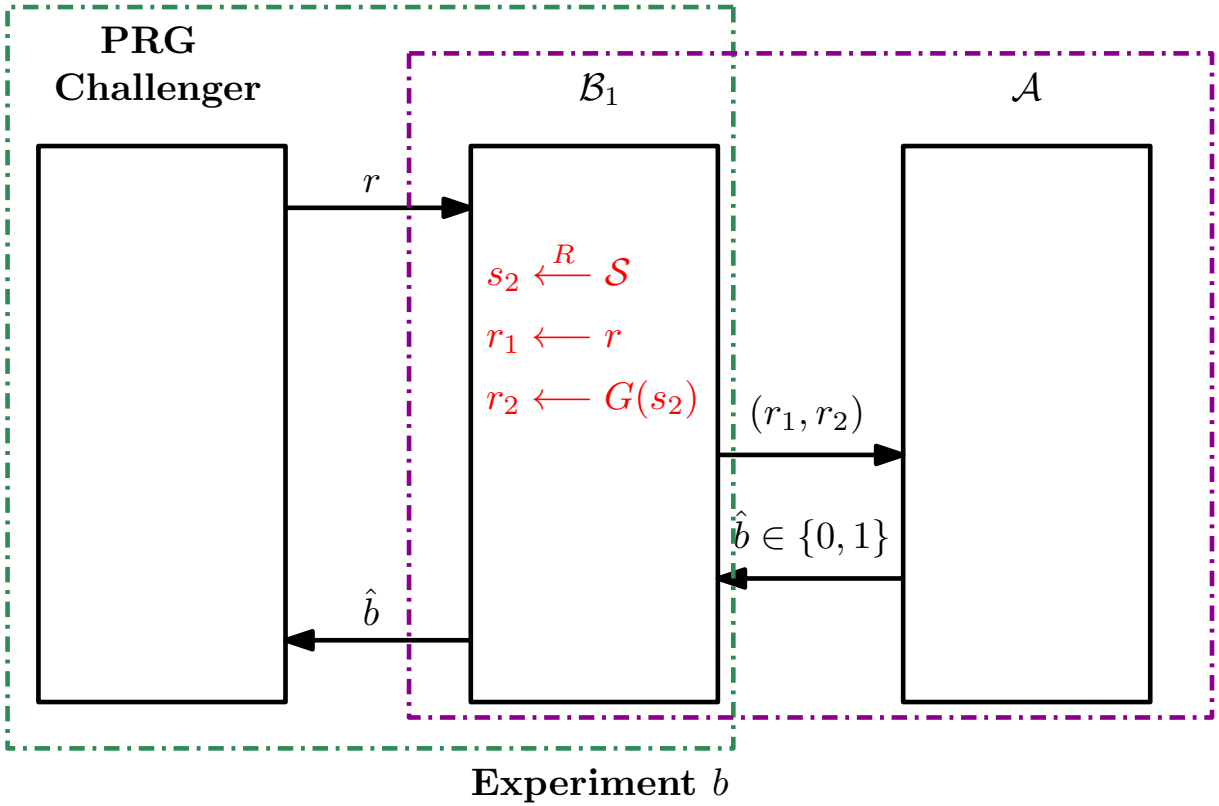
Proof

- Let p_0 = Probability that $\mathcal{A}$ outputs 1 in attack game Hybrid 0.
- Let p_1 = Probability that $\mathcal{A}$ outputs 1 in attack game Hybrid 1.
- Let p_2 = Probability that $\mathcal{A}$ outputs 1 in attack game Hybrid 2.

A Parallel Construction

$\gamma = 2$ and First Part:

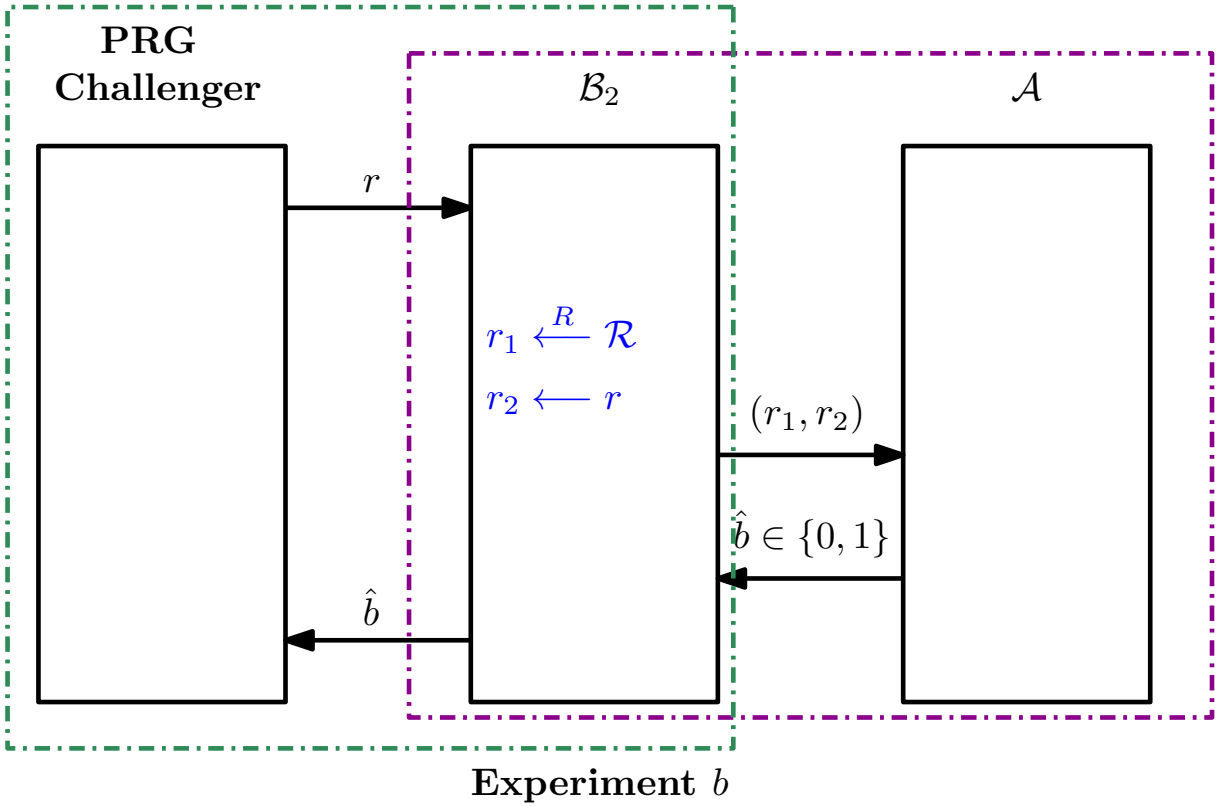
- Construction of Adversary $\mathcal{B}_1$.



A Parallel Construction

$\gamma = 2$ and First Part:

- Construction of Adversary $\mathcal{B}_2$.



A Parallel Construction

$\gamma = 2$ and First Part:

- Adversary $\mathcal{B}_1$:
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- Adversary $\mathcal{B}_2$:
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A Parallel Construction

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A Parallel Construction

$\gamma = 2$ and First Part:

- Adversary $\mathcal{A}$:

$$\begin{aligned} \mathsf{PRGadv}[\mathcal{A}, G'] &= |\, p_2 - p_0 \,| \\ &= |\, p_2 - p_1 + p_1 - p_0 \,| \end{aligned}$$

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$\gamma = 2$ and First Part:

- Adversary $\mathcal{A}$:

$$\begin{aligned} \mathsf{PRGadv}[\mathcal{A}, G'] &= |\, p_2 - p_0 \,| \\ &= |\, p_2 - p_1 + p_1 - p_0 \,| \\ &\leqslant |\, p_2 - p_1 \,| + |\, p_1 - p_0 \,| \\ &\leqslant \epsilon_1(n) + \epsilon_2(n). \end{aligned}$$

A Parallel Construction

$\gamma = 2$ and Second Part:

- Construction of Adversary $\mathcal{B}$:
  - Combine $\mathcal{B}_1$ and $\mathcal{B}_2$.

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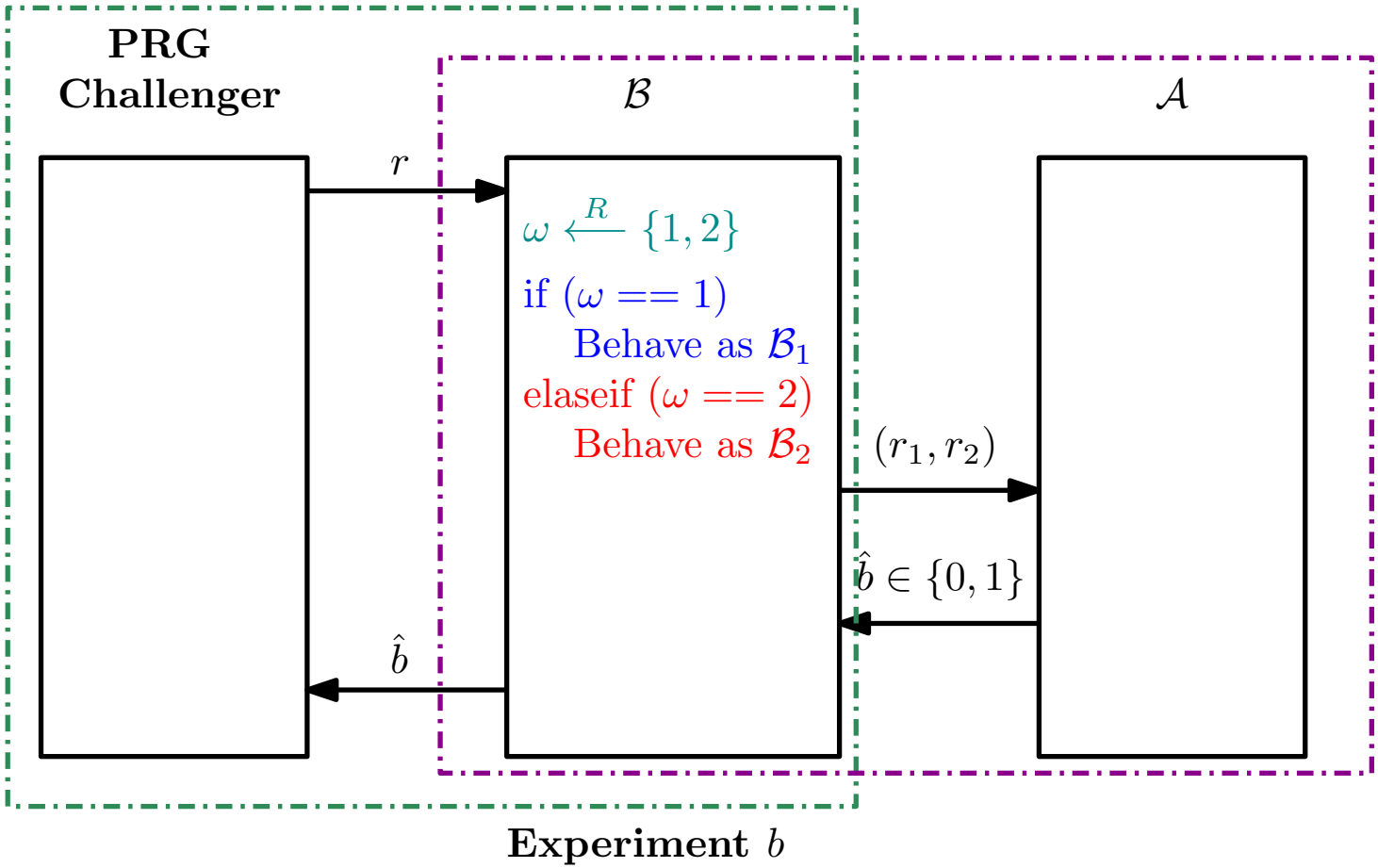
A Parallel Construction



A Parallel Construction

$\gamma = 2$ and Second Part:

$$\begin{aligned}\Pr[W_0] &= \Pr[W_0 \wedge \omega = 1] + \Pr[W_0 \wedge \omega = 2] \\ &= \Pr[W_0 \mid \omega = 1]\Pr[\omega = 1] + \Pr[W_0 \mid \omega = 2]\Pr[\omega = 2]\end{aligned}$$

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$\gamma = 2$ and Second Part:

$$\begin{aligned} \mathsf{PRGadv}[\mathcal{B}, G] &= |\Pr[W_1] - \Pr[W_0]| \\ &= \left| \frac{1}{2}(p_1 + p_2) - \frac{1}{2}(p_0 + p_1) \right| \end{aligned}$$

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Therefore,

$$\mathsf{PRGadv}[\mathcal{A}, G'] = 2 \cdot \mathsf{PRGadv}[\mathcal{B}, G].$$

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For polynomially bounded γ:

- There will be $\gamma + 1$ hybrid attack games: Hybrid 0, Hybrid 1, …, Hybrid γ.

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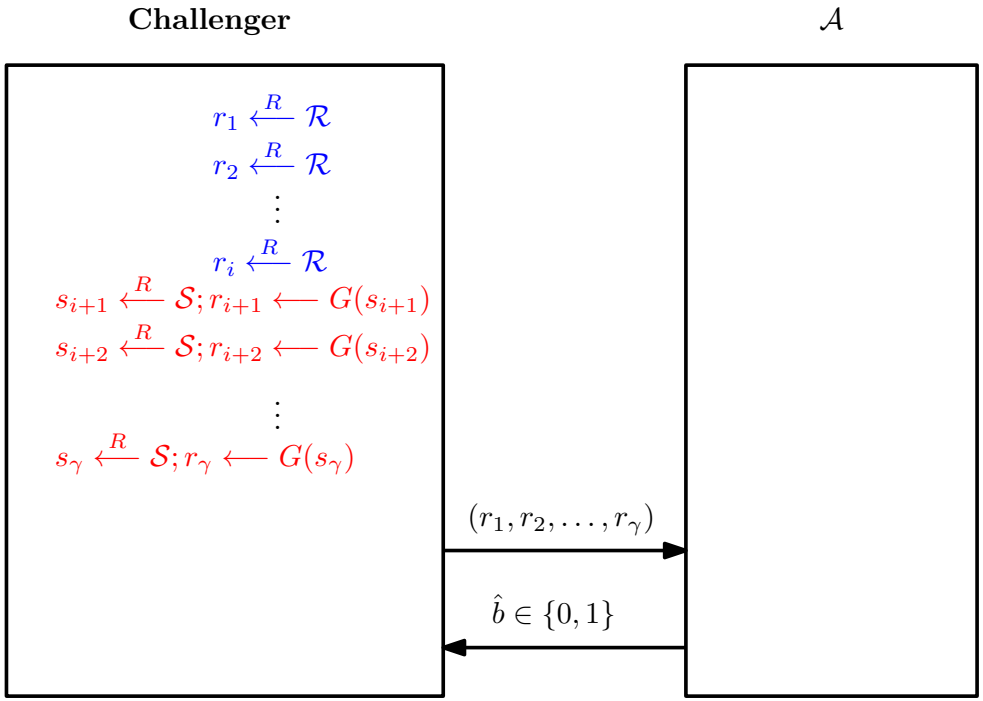

Hybrid i

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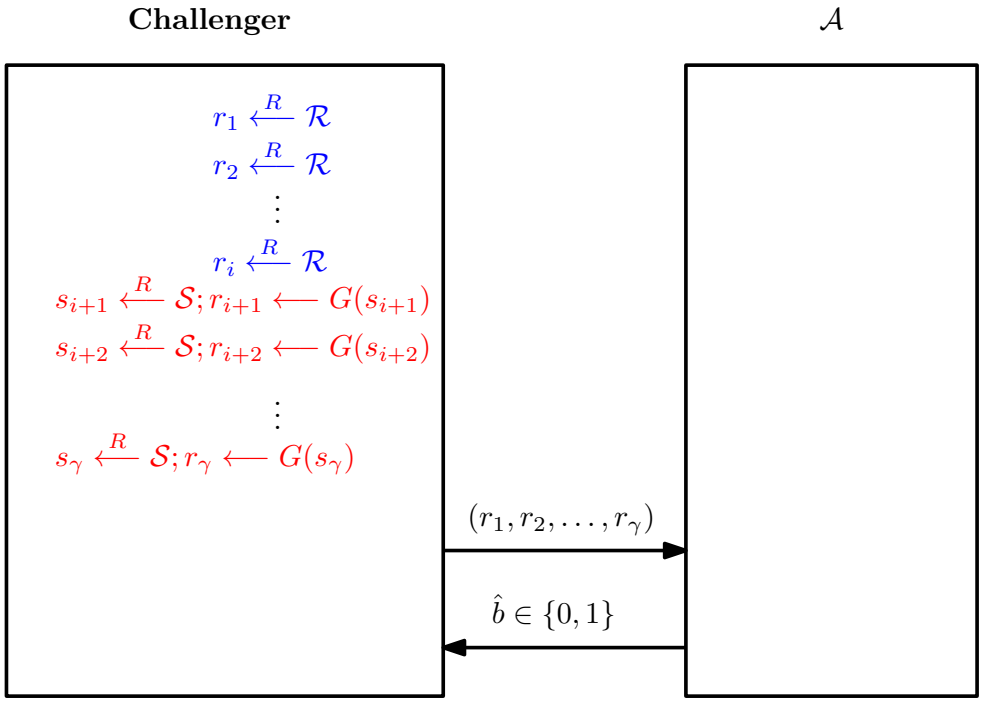

Hybrid i

- Let p_i = Probability that $\mathcal{A}$ outputs 1 in attack game Hybrid i.

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Hybrid 0	$G(s_1)$	$G(s_2)$	$G(s_3)$	$\cdots$	$G(s_{\gamma-1})$	$G(s_\gamma)$

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Hybrid 0	$G(s_1)$	$G(s_2)$	$G(s_3)$	$\cdots$	$G(s_{\gamma-1})$	$G(s_\gamma)$
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Hybrid 2	r_1	r_2	$G(s_3)$	$\cdots$	$G(s_{\gamma-1})$	$G(s_\gamma)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Hybrid $\gamma-1$	r_1	r_2	r_3	$\cdots$	$r_{\gamma-1}$	$G(s_\gamma)$

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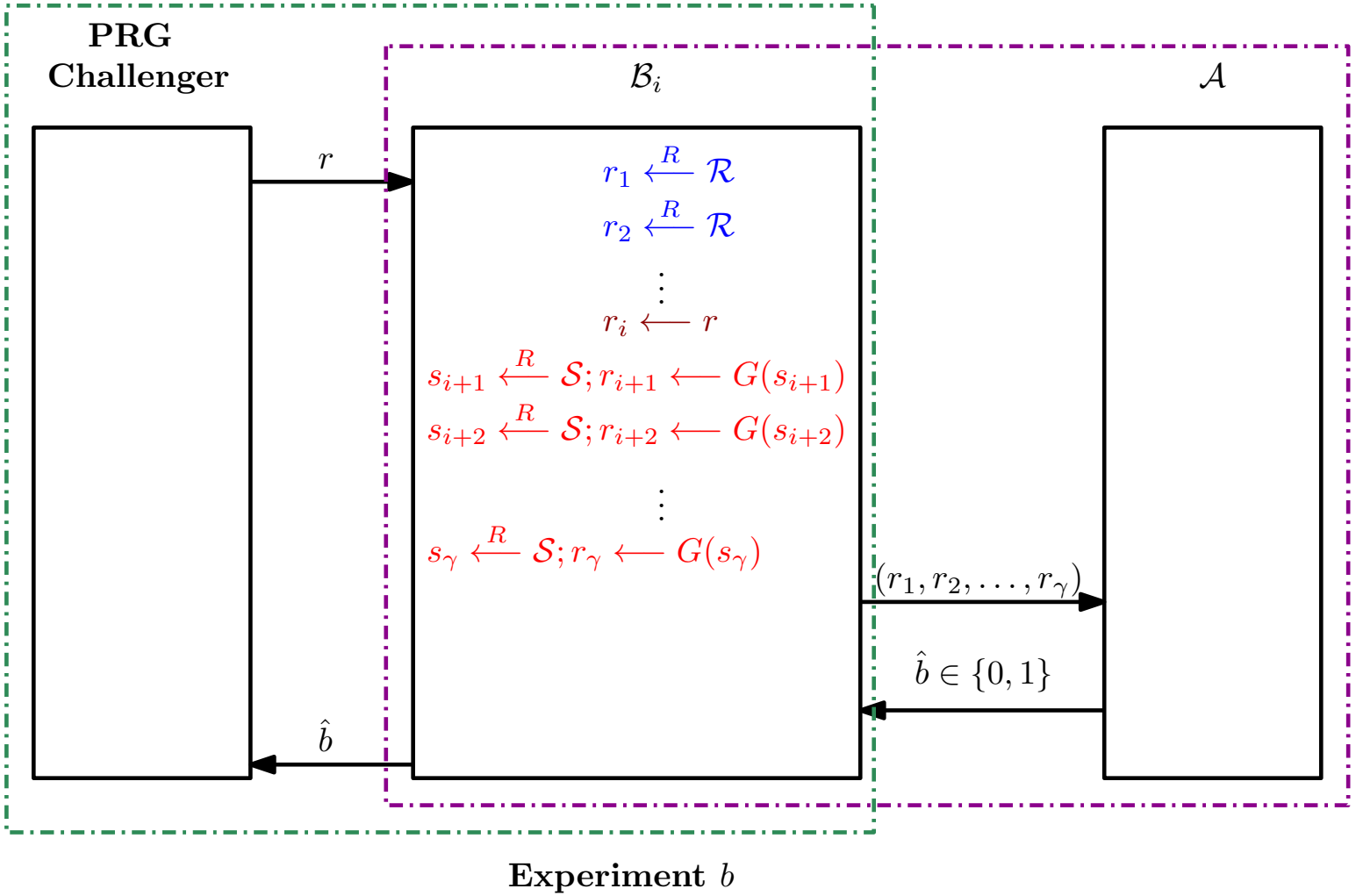
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
Hybrid $\gamma-1$	r_1	r_2	r_3	$\cdots$	$r_{\gamma-1}$	$G(s_\gamma)$
Hybrid γ	r_1	r_2	r_3	$\cdots$	$r_{\gamma-1}$	r_γ

A Parallel Construction



Experiment b

A Parallel Construction

For polynomially bounded γ:

- Construction of Adversary $\mathcal{B}$:
  - Combine $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\gamma$.

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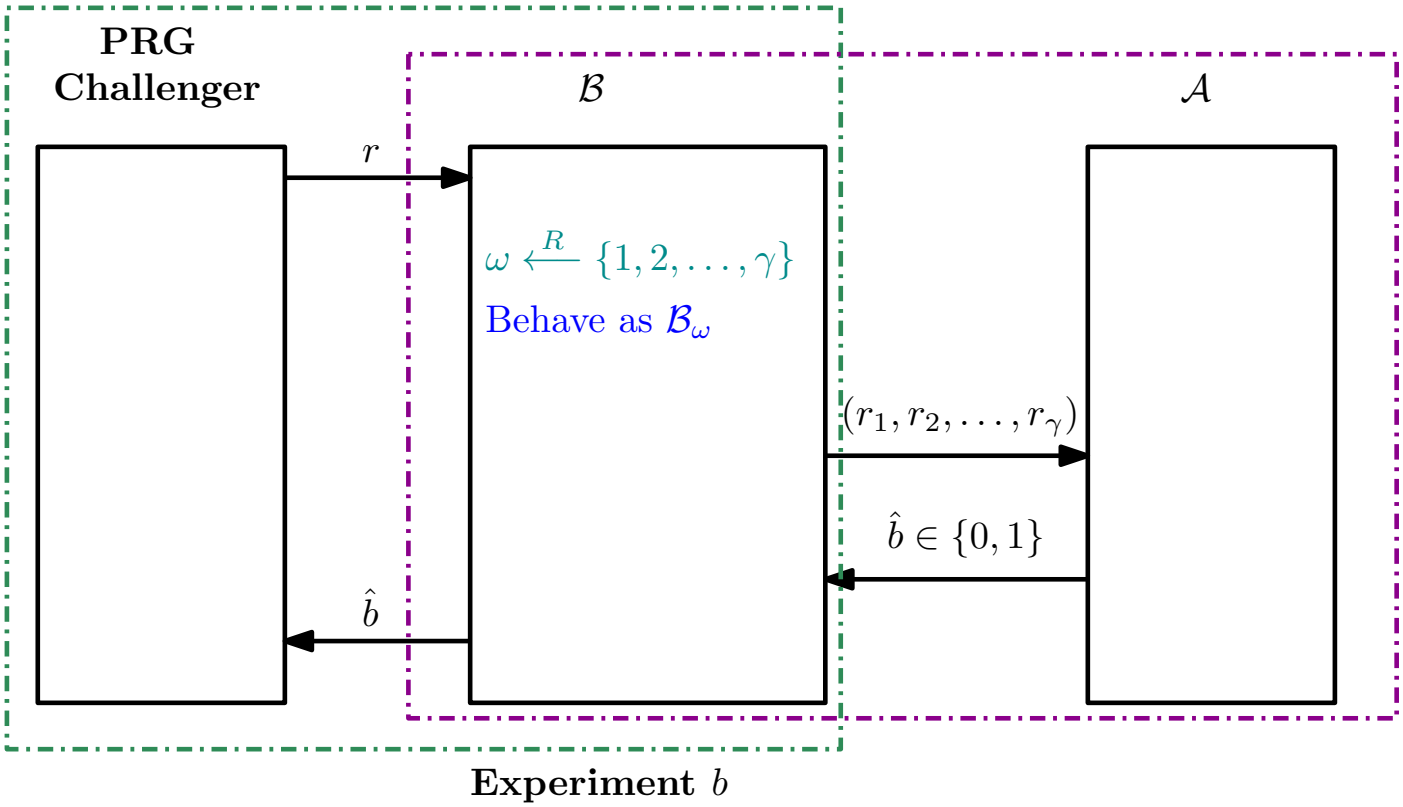

A Parallel Construction

For polynomially bounded γ:

$$
\begin{aligned}
\Pr[W_0] &= \sum_{i=1}^{\gamma} \Pr[W_0 \wedge \omega = i] \\
&= \sum_{i=1}^{\gamma} \Pr[W_0 \mid \omega = i] \Pr[\omega = i] \\
&= \sum_{i=1}^{\gamma} \Pr[W_0 \mid \omega = i] \frac{1}{\gamma} \\
&= \frac{1}{\gamma} \sum_{i=1}^{\gamma} p_{i-1}
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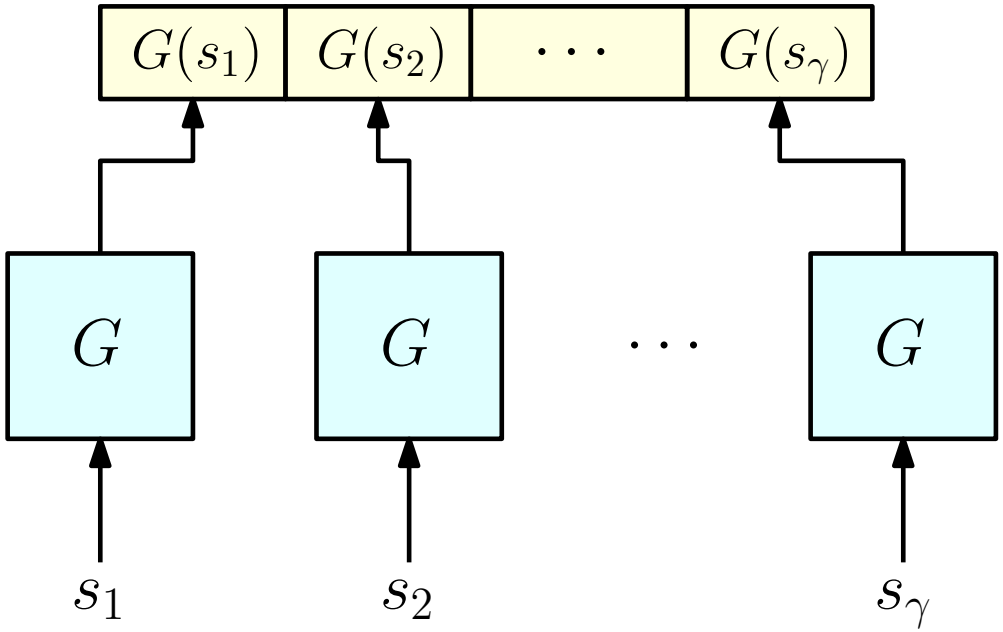

A Sequential construction

Blum-Micali method

Let G be a PRG defined over $(\mathcal{S}, \mathcal{R} \times \mathcal{S})$. We construct a new PRG G' that is defined over $(\mathcal{S}, \mathcal{R}^\gamma \times \mathcal{S})$, and for $s \in \mathcal{S}$,

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\begin{aligned}
G'(s) \quad := \quad & \{s_0 \longleftarrow s \\
& \text{For } i = 1, 2, \ldots, \gamma \text{ do} \\
& \quad (r_i, s_i) \longleftarrow G(s_{i-1}) \\
& \text{Output } (r_1, r_2, \ldots, r_\gamma, s_\gamma)\}
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- G' is called the γ-wise sequential composition of G.

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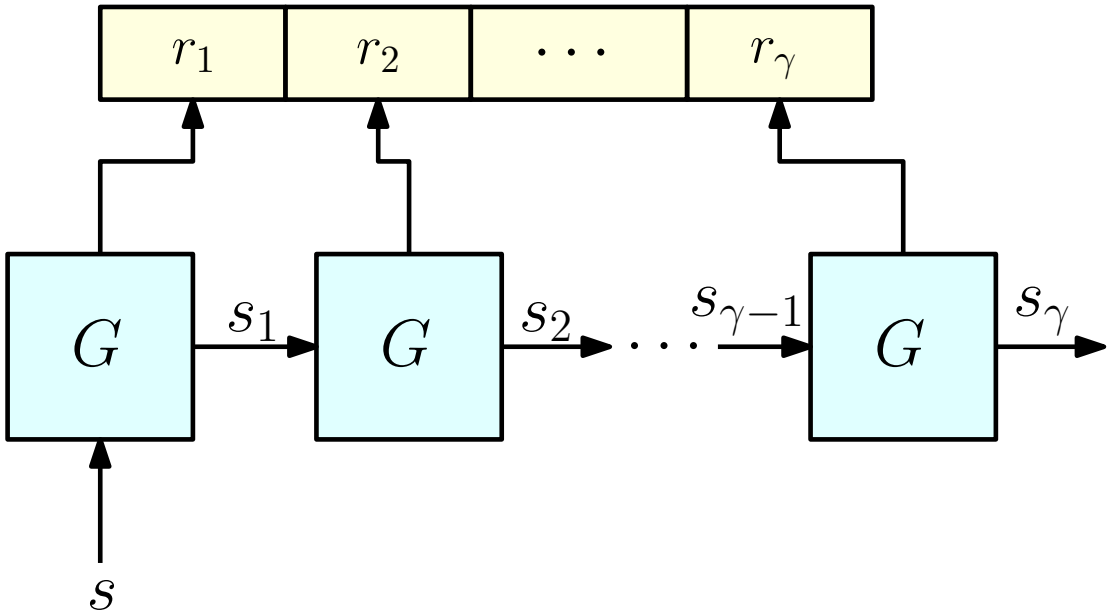
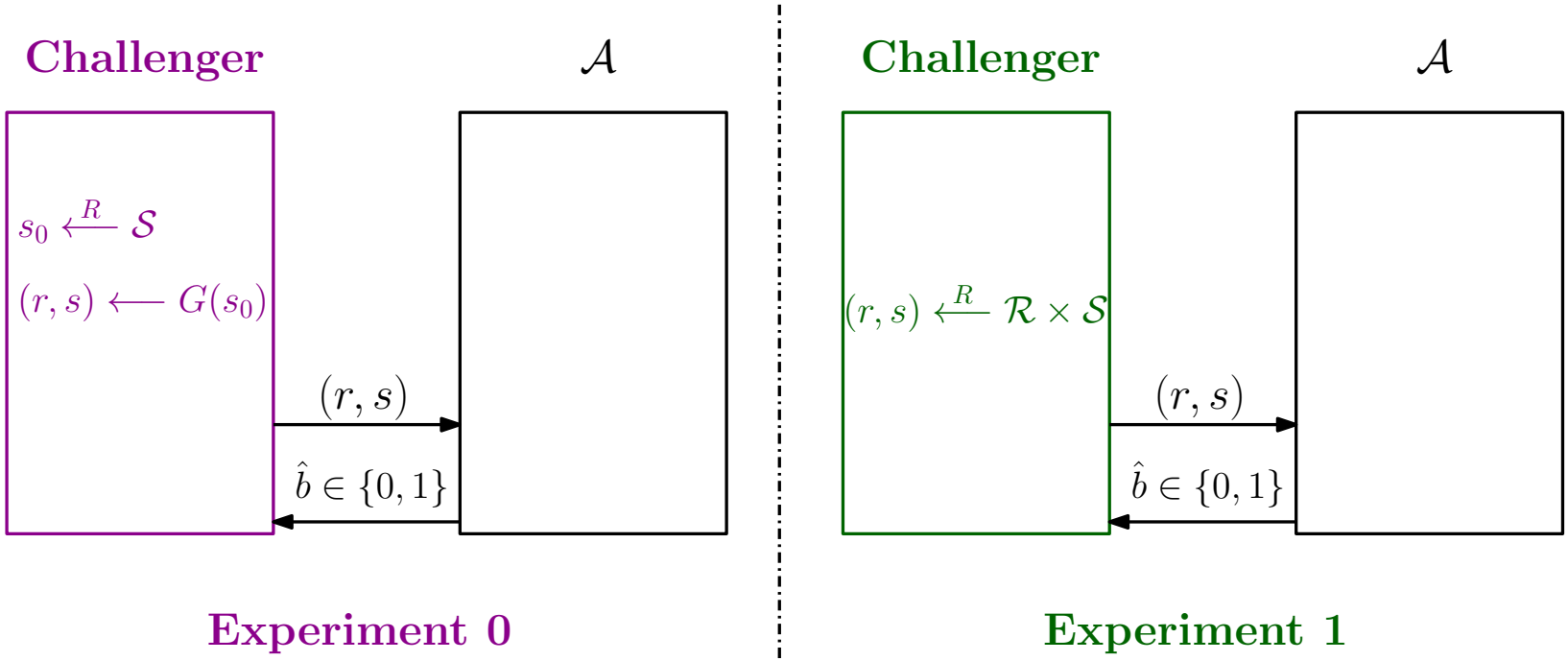
- G' is called the γ-wise sequential composition of G.
- γ needs to be a poly-bounded value.

A Sequential Construction



Pseudo-Random Generator Advantage



A Sequential Construction

Theorem

If G is a secure PRG, then the γ-wise sequential composition G' of G is also a secure PRG.

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If G is a secure PRG, then the γ-wise sequential composition G' of G is also a secure PRG.
In particular, for every PRG distinguisher $\mathcal{A}$ that attacks G' as in PRG Indistinguishability Game, there exists a PRG distinguisher $\mathcal{B}$ that attacks G as in PRG Indistinguishability Game, where $\mathcal{B}$ uses $\mathcal{A}$ as sub-routine, such that

$$\mathsf{PRGadv}[\mathcal{A}, G'] = \gamma \cdot \mathsf{PRGadv}[\mathcal{B}, G].$$

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For polynomially bounded γ:

- There will be $\gamma + 1$ hybrid attack games: Hybrid 0, Hybrid 1, …, Hybrid γ.

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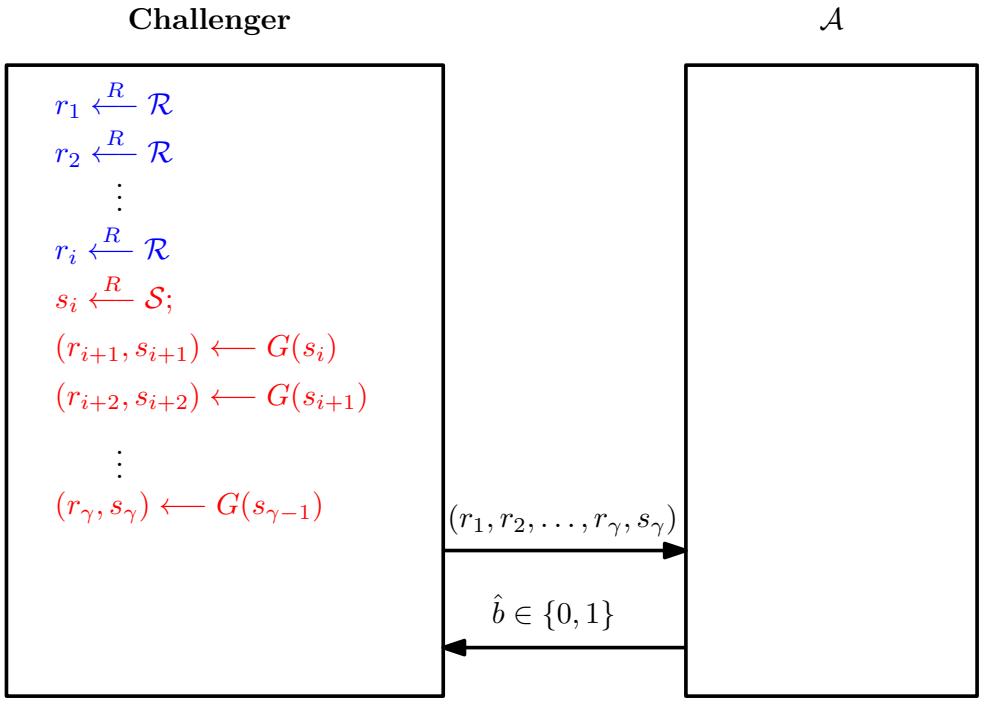

Hybrid i

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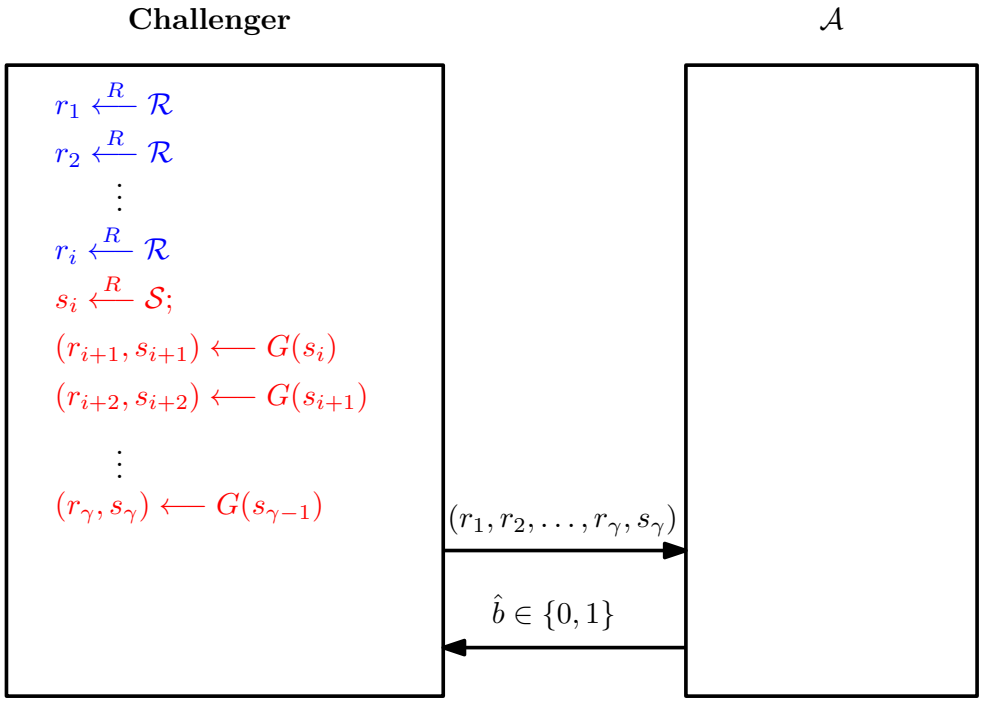

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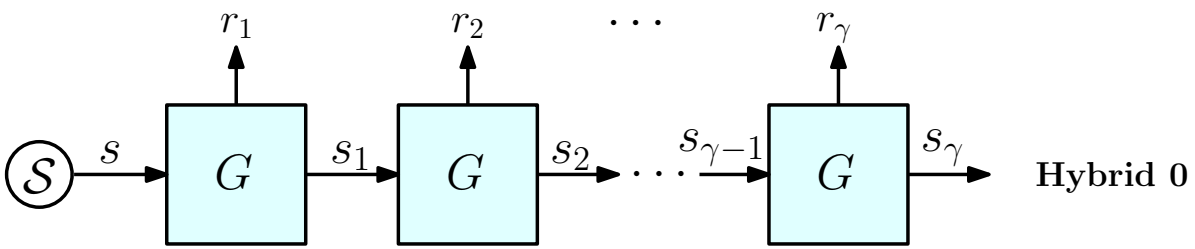

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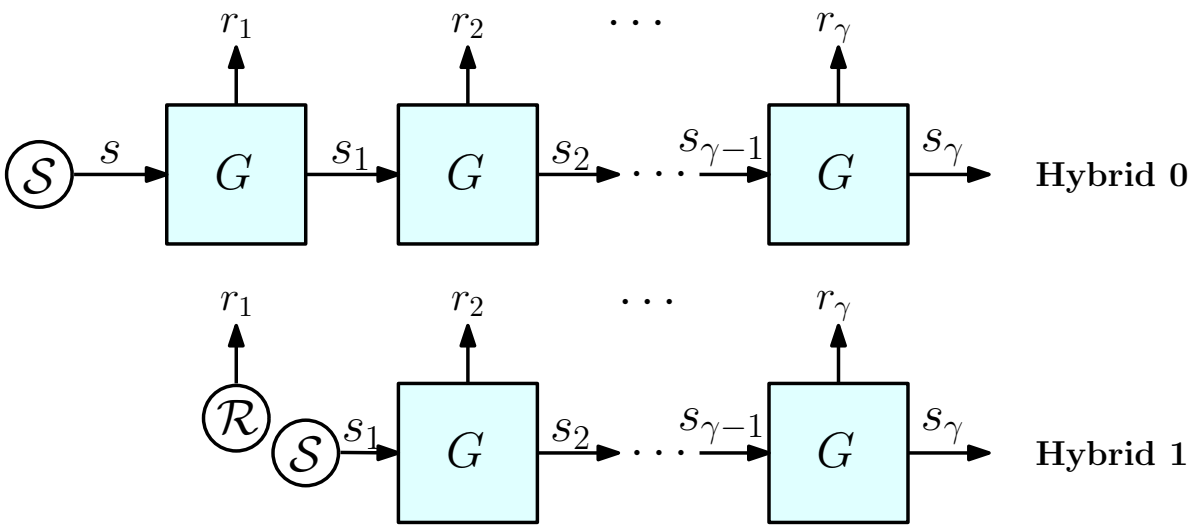

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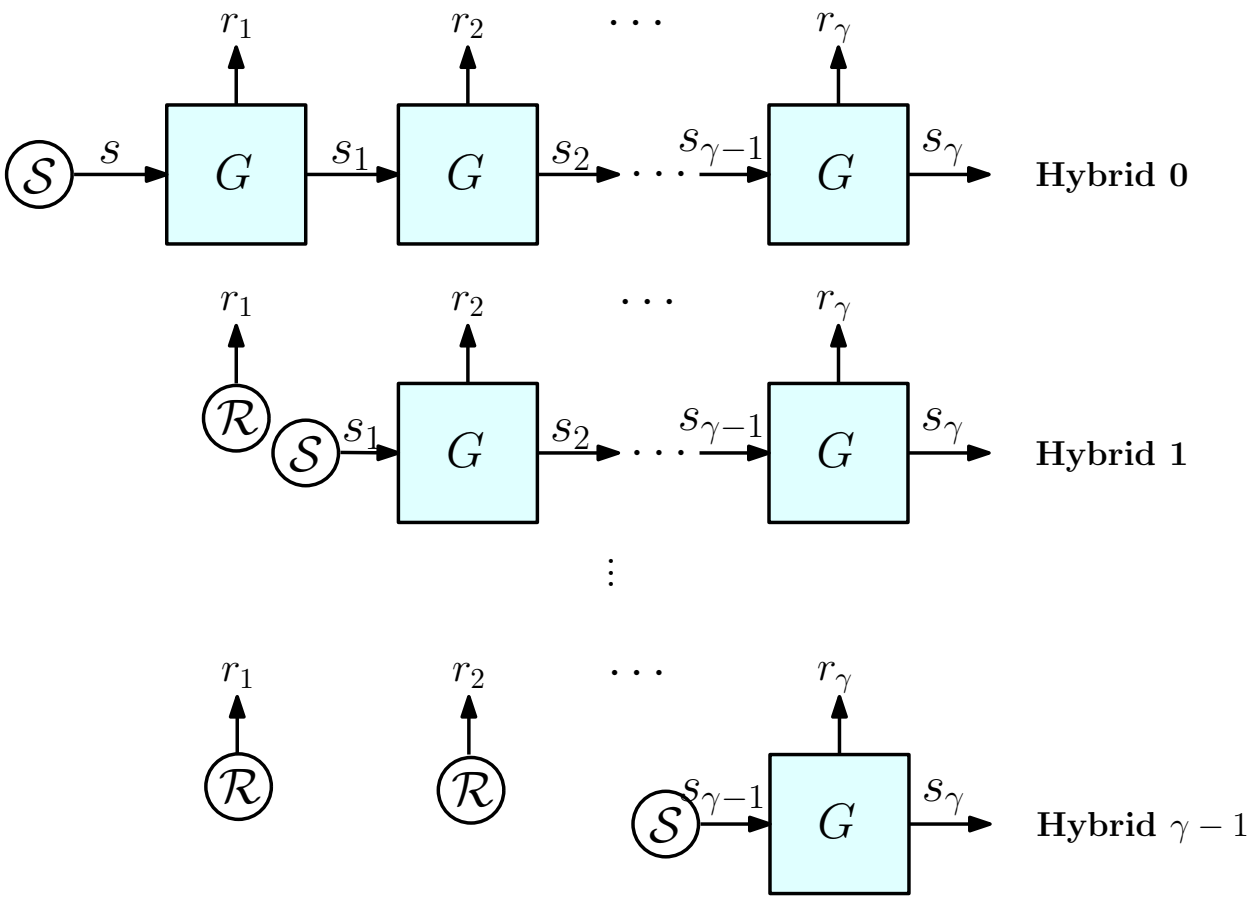

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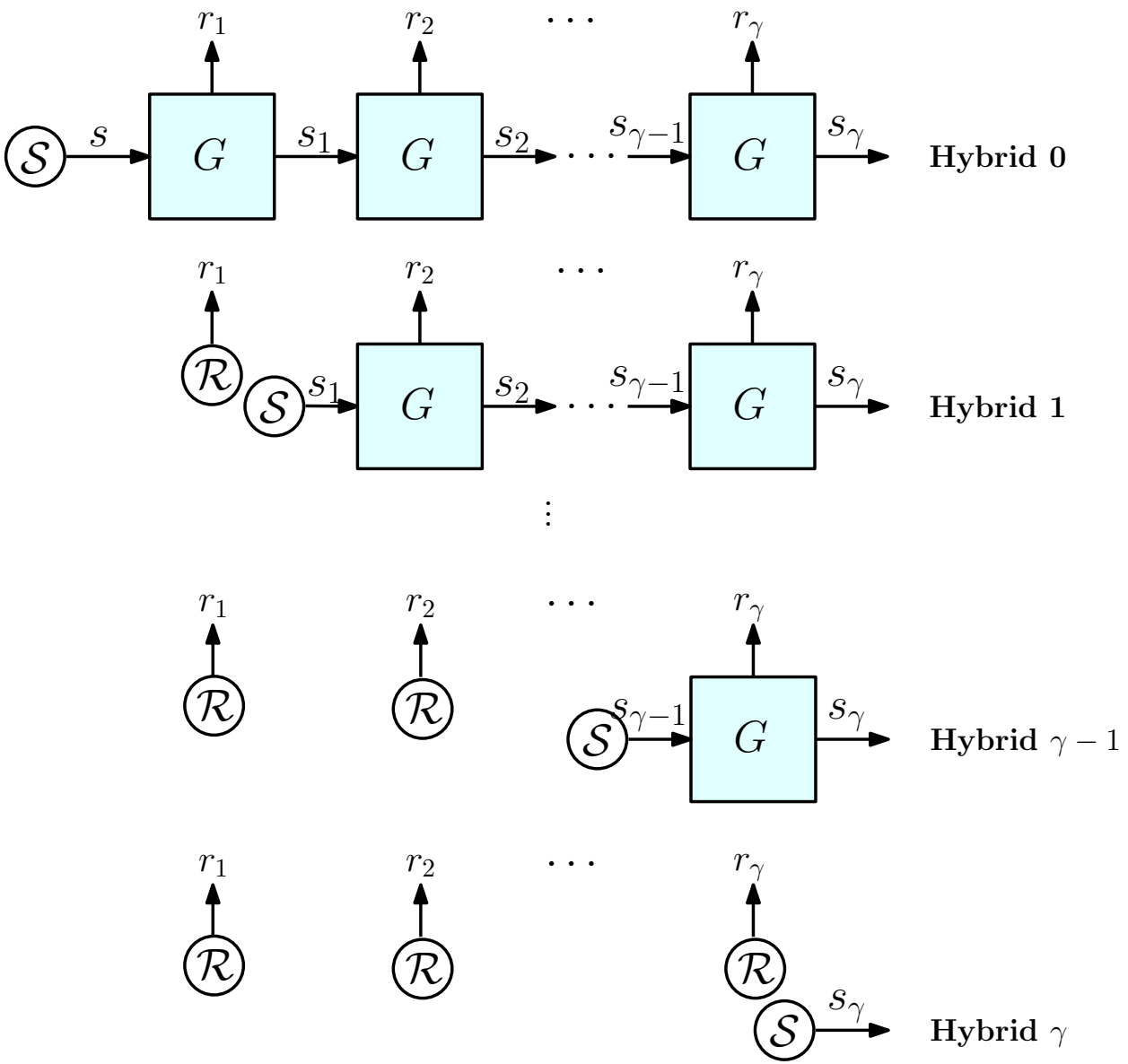

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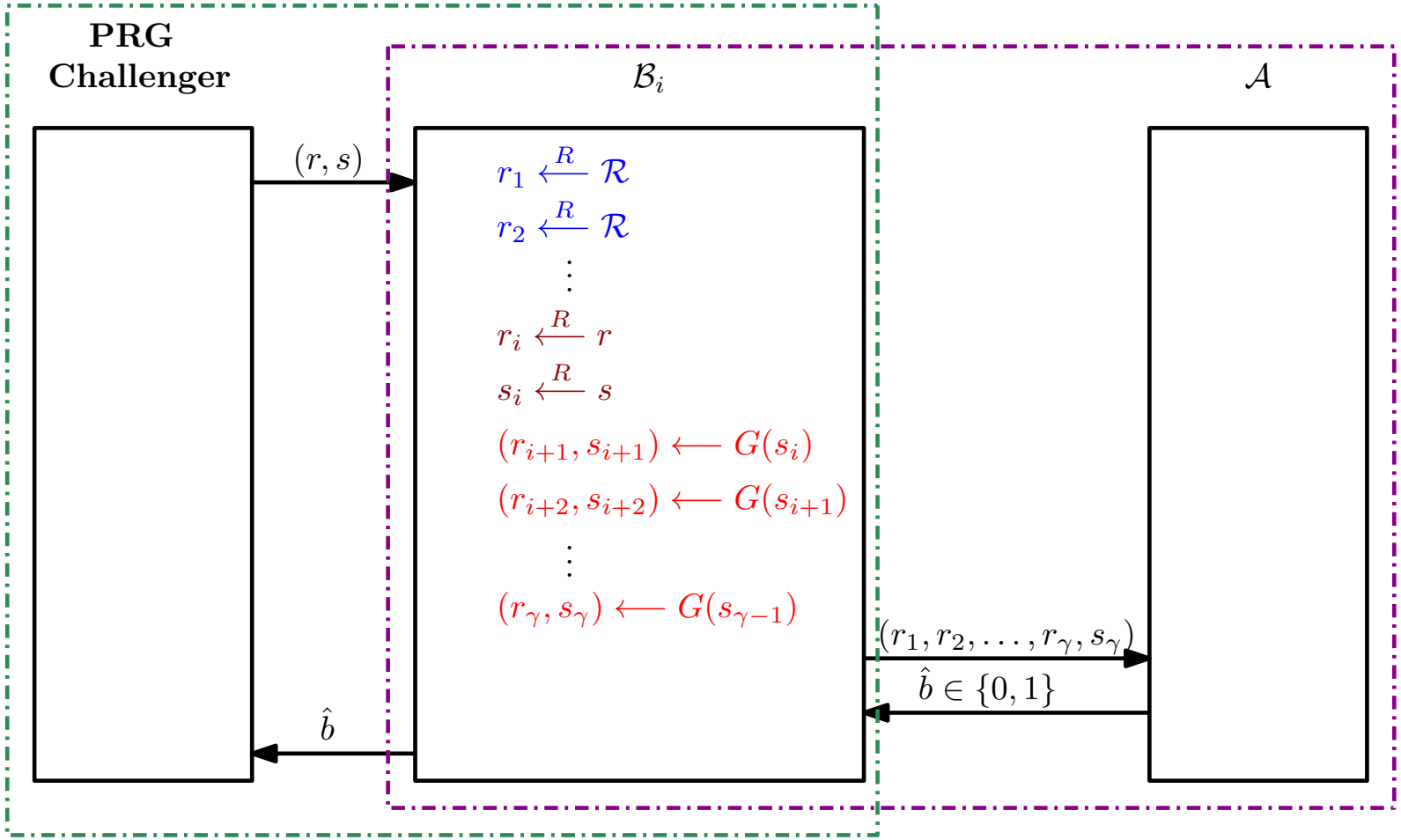

Experiment b

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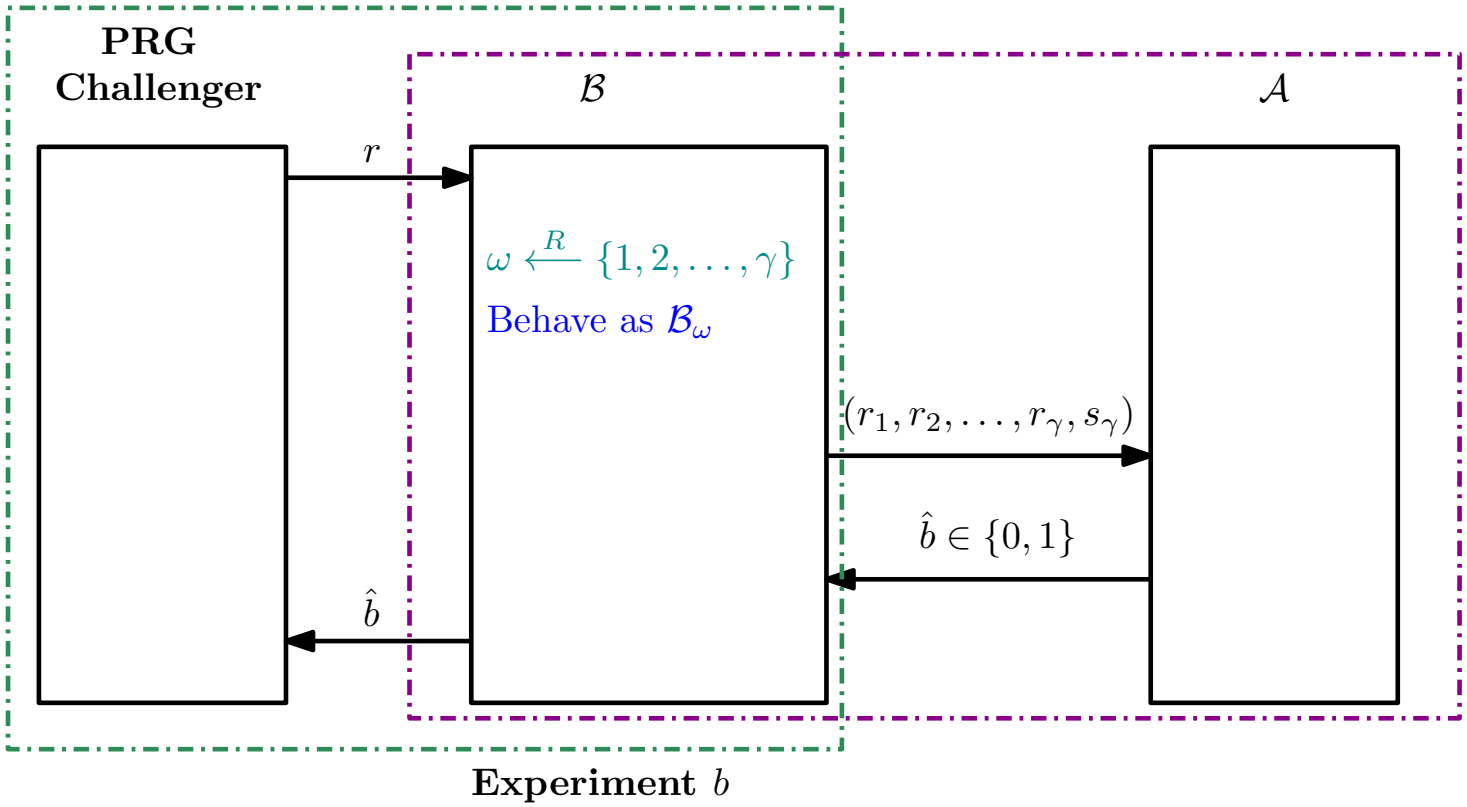
For polynomially bounded γ:

- Construction of Adversary $\mathcal{B}$:
  - Combine $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\gamma$.
  - $\mathcal{B}$ computes $\omega \xleftarrow{R} \{1, 2, \ldots, \gamma\}$.
  - Follow the behavior of $\mathcal{B}_\omega$.
- Let W_b be the event where $\mathcal{B}$ outputs 1 in Experiment b.

A Sequential Construction



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Therefore,

$$\mathsf{PRGadv}[\mathcal{A}, G'] = \gamma \cdot \mathsf{PRGadv}[\mathcal{B}, G].$$

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- Let G be a PRG defined over $(\{0,1\}^n, \{0,1\}^L)$.
- Let there be an adversary $\mathcal{A}$ that can predict the the last bit of the output of G given first $L-1$ bits.
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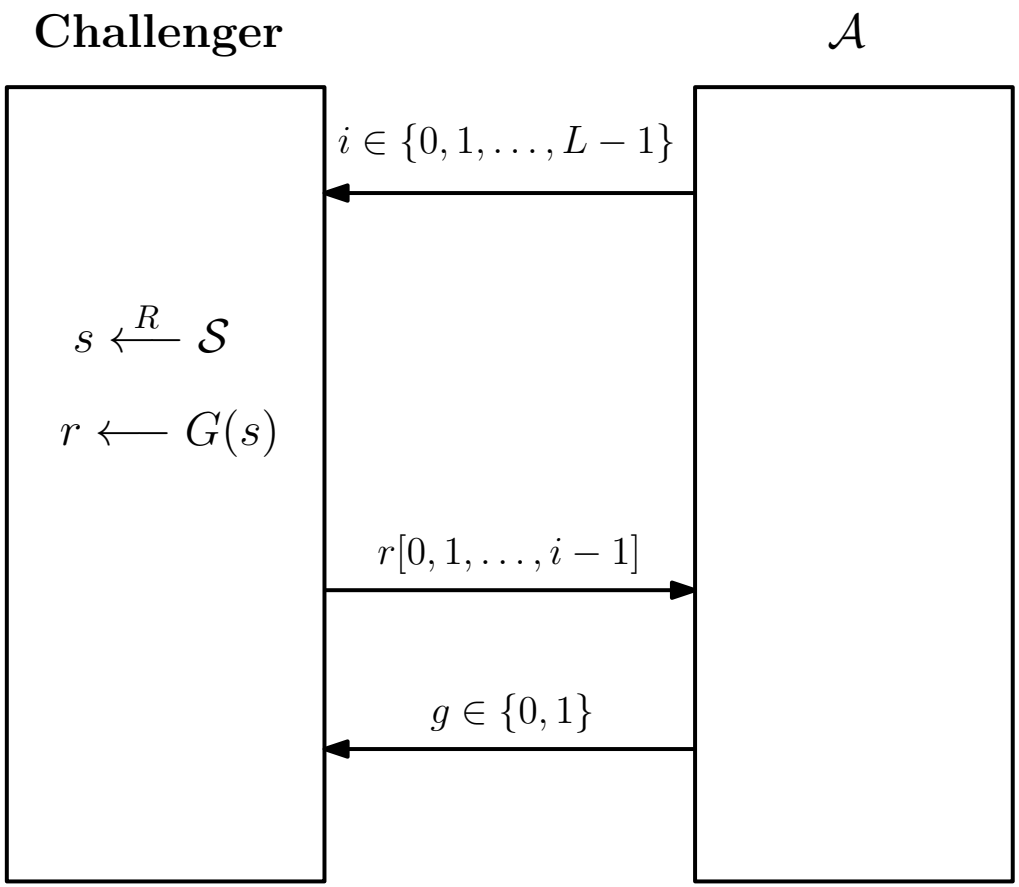
Unpredictability

Let G be a PRG defined over $(\{0,1\}^n, \{0,1\}^L)$. We say G is unpredictable if given the first i bits of G's output, it is hard to predict the next bit (that is, the $(i+1)$-st bit) with probability significantly better that 1/2 (here, i is an adversarially chosen index).

Unpredictable Attack Game



Unpredictable Attack Game

Unpredictability Advantage

Let W be be the event $\mathcal{A}$ wins if $r[i] = g$. We define the advantage of $\mathcal{A}$ in the attack game with respect to G as

$$\mathsf{PREDadv}[\mathcal{A}, G] = \left| \Pr[W] - \frac{1}{2} \right|.$$

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Unpredictable PRG

A PRG G is unpredictable if for all PPT adversaries (or distinguisher) $\mathcal{A}$ there exists a negligible function ϵ such that

$$\mathsf{PREDadv}[\mathcal{A}, G] \leqslant \epsilon(n).$$

Unpredictability

Theorem

Let G be a PRG, defined over $(\mathcal{S}, \mathcal{R})$ where $\mathcal{S} = \{0,1\}^n, \mathcal{R} = \{0,1\}^L$. If G is secure, then G is unpredictable.
In particular, for every adversary $\mathcal{A}$ breaking the unpredictability of G, as in Unpredictable Attack Game, there exists an adversary $\mathcal{B}$ breaking the security of G, as in PRG Indistinguishability Attack Game, where $\mathcal{B}$ uses $\mathcal{A}$ as a subroutine, such that

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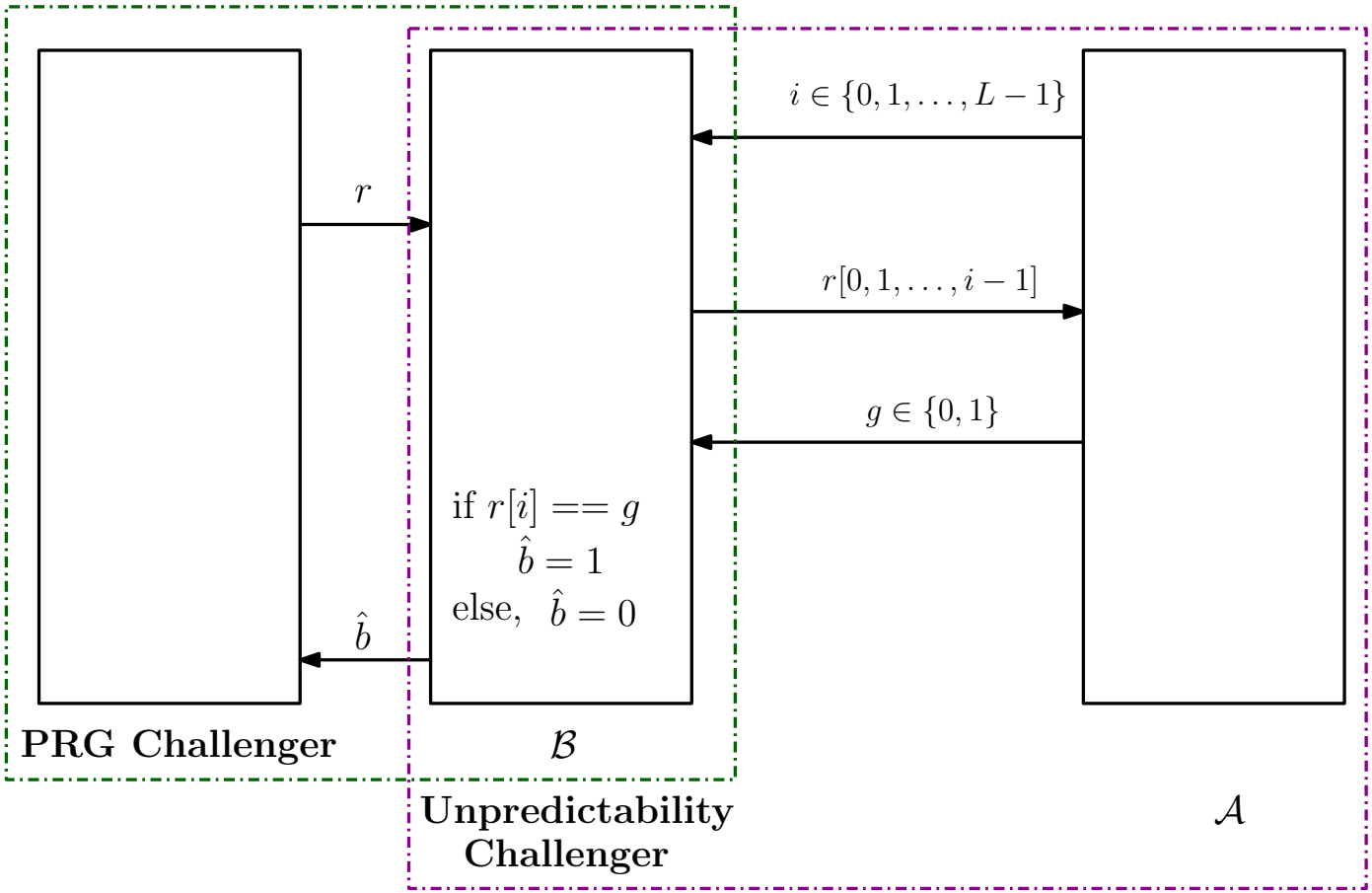
Unpredictability

Proof



Experiment b

Unpredictability

Proof

- For $b = 0, 1$, let W_b be the event $\mathcal{B}$ outputs 1 in Experiment b. Then advantage of $\mathcal{B}$ against G in PRG indistinguishability game is

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$$\begin{aligned} \Pr[W_0] &= \Pr[\mathcal{B} \text{ outputs 1 in Experiment 0}] \\ &= \Pr[\mathcal{A} \text{ wins in unpredictability game}] \\ &= p. \end{aligned}$$

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  - r is a random bit.
- $\mathcal{B}$ can distinguish the pseudo-random bit x_i from the random bit r, given the side information $x_0 \cdots x_{i-1}$.

Unpredictability

Proof Sketch

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- $\Pr[\mathbf{b}' = \mathbf{b}] = \frac{1}{2} + \epsilon$.

Unpredictability

Distinguisher/predictor Lemma

Let $\mathbf{x}$ be a random variable taking values in some set S, and let $\mathbf{b}$ and $\mathbf{r}$ be a 0/1-valued random variables, where $\mathbf{r}$ is uniformly distributed over $\{0,1\}$ and is independent of $(\mathbf{x},\mathbf{b})$. Let $d : S \times \{0,1\} \longrightarrow \{0,1\}$ be an arbitrary function, and let

$$\epsilon := \Pr[d(\mathbf{x},\mathbf{b}) = 1] - \Pr[d(\mathbf{x},\mathbf{r}) = 1].$$

Define the random variable $\mathbf{b}'$ as follows:

$$\mathbf{b}' := \begin{cases} \mathbf{r}, & \text{if } d(\mathbf{x},\mathbf{r}) = 1 \\ \bar{\mathbf{r}}, & \text{otherwise.} \end{cases}$$

Then

$$\Pr[\mathbf{b}' = \mathbf{b}] = \frac{1}{2} + \epsilon.$$

Unpredictability

Proof of Distinguisher/predictor Lemma

$$\Pr[\mathbf{b}' = \mathbf{b}] \quad = \quad \Pr[\mathbf{b}' = \mathbf{b} \mid \mathbf{b} = \mathbf{r}]\Pr[\mathbf{b} = \mathbf{r}] + \Pr[\mathbf{b}' = \mathbf{b} \mid \mathbf{b} = \bar{\mathbf{r}}]\Pr[\mathbf{b} = \bar{\mathbf{r}}]$$

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&= \Pr[d(\mathbf{x}, \mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}]\frac{1}{2} + \Pr[d(\mathbf{x}, \mathbf{r}) = 0 \mid \mathbf{b} = \bar{\mathbf{r}}]\frac{1}{2}
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&= \frac{1}{2}\left(\Pr[d(\mathbf{x}, \mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}] + (1 - \Pr[d(\mathbf{x}, \mathbf{r}) = 1 \mid \mathbf{b} = \bar{\mathbf{r}}])\right)
\end{aligned}
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&= \frac{1}{2} + \frac{1}{2}(\alpha - \beta), \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= \Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}], \\
\beta &= \Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \bar{\mathbf{r}}].
\end{aligned}
$$

Proof of Distinguisher/predictor Lemma

$$\begin{aligned}
\Pr[\mathbf{b}' = \mathbf{b}] &= \Pr[\mathbf{b}' = \mathbf{b} \mid \mathbf{b} = \mathbf{r}]\Pr[\mathbf{b} = \mathbf{r}] + \Pr[\mathbf{b}' = \mathbf{b} \mid \mathbf{b} = \bar{\mathbf{r}}]\Pr[\mathbf{b} = \bar{\mathbf{r}}] \\
&= \Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}]\frac{1}{2} + \Pr[d(\mathbf{x},\mathbf{r}) = 0 \mid \mathbf{b} = \bar{\mathbf{r}}]\frac{1}{2} \\
&= \frac{1}{2}\left(\Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}] + (1 - \Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \bar{\mathbf{r}}])\right) \\
&= \frac{1}{2} + \frac{1}{2}(\alpha - \beta), \text{ where}
\end{aligned}$$

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By independence,

$$\alpha = \Pr[d(\mathbf{x},\mathbf{r}) = 1 \mid \mathbf{b} = \mathbf{r}]$$

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Unpredictability

Proof of Distinguisher/predictor Lemma

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&= \frac{1}{2} + \frac{1}{2}(\alpha - \beta), \text{ where}
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Unpredictability

Proof of Distinguisher/predictor Lemma

$$\epsilon \quad = \quad \Pr[d(\mathbf{x}, \mathbf{b}) = 1] - \Pr[d(\mathbf{x}, \mathbf{r}) = 1]$$

Unpredictability

Proof of Distinguisher/predictor Lemma

$$\begin{aligned}\epsilon &= \Pr[d(\mathbf{x},\mathbf{b})=1]-\Pr[d(\mathbf{x},\mathbf{r})=1] \\ &= \alpha-(\Pr[d(\mathbf{x},\mathbf{r})=1 \mid \mathbf{b}=\mathbf{r}]\Pr[\mathbf{b}=\mathbf{r}]+\Pr[d(\mathbf{x},\mathbf{r})=1 \mid \mathbf{b}=\bar{\mathbf{r}}]\Pr[\mathbf{b}=\bar{\mathbf{r}}])\end{aligned}$$

Unpredictability

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\end{aligned}
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Unpredictability

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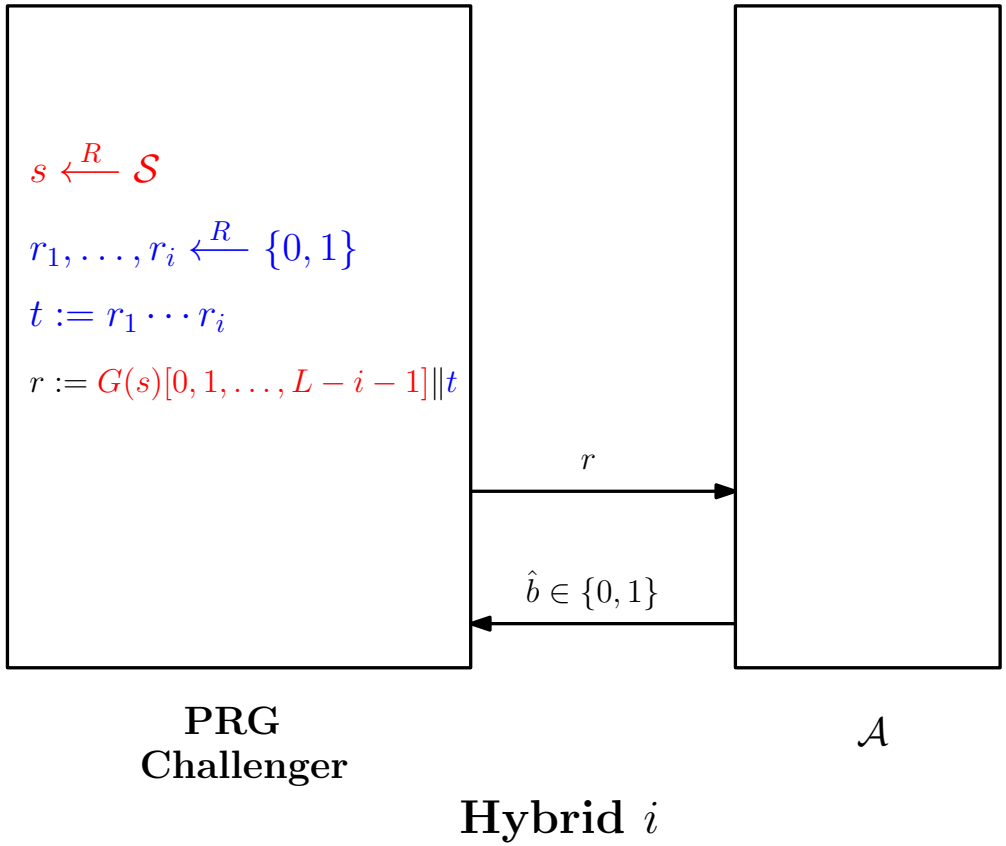
&= \alpha - \frac{1}{2}(\alpha+\beta) \\
&= \frac{1}{2}(\alpha-\beta).
\end{aligned}
$$

Hence proved.

Unpredictability

Proof



Unpredictability

Proof

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Unpredictability

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Unpredictability

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  - Let the probability $\mathcal{A}$ outputs 1 be p_i.
- Hybrid 0 represents the Experiment 0 of PRG indistinguishability attack game.
- Hybrid L represents the Experiment 1 of PRG indistinguishability attack game.
- Then the advantage of $\mathcal{A}$ against G is

$$\mathsf{PRGadv}[\mathcal{A}, G] = | p_0 - p_L | .$$

Unpredictability

Proof

- Construction of $\mathcal{B}$:
  - Choose $\omega \in \{1, \ldots, L\}$ at random.

Unpredictability

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  - Choose $\omega \in \{1, \ldots, L\}$ at random.
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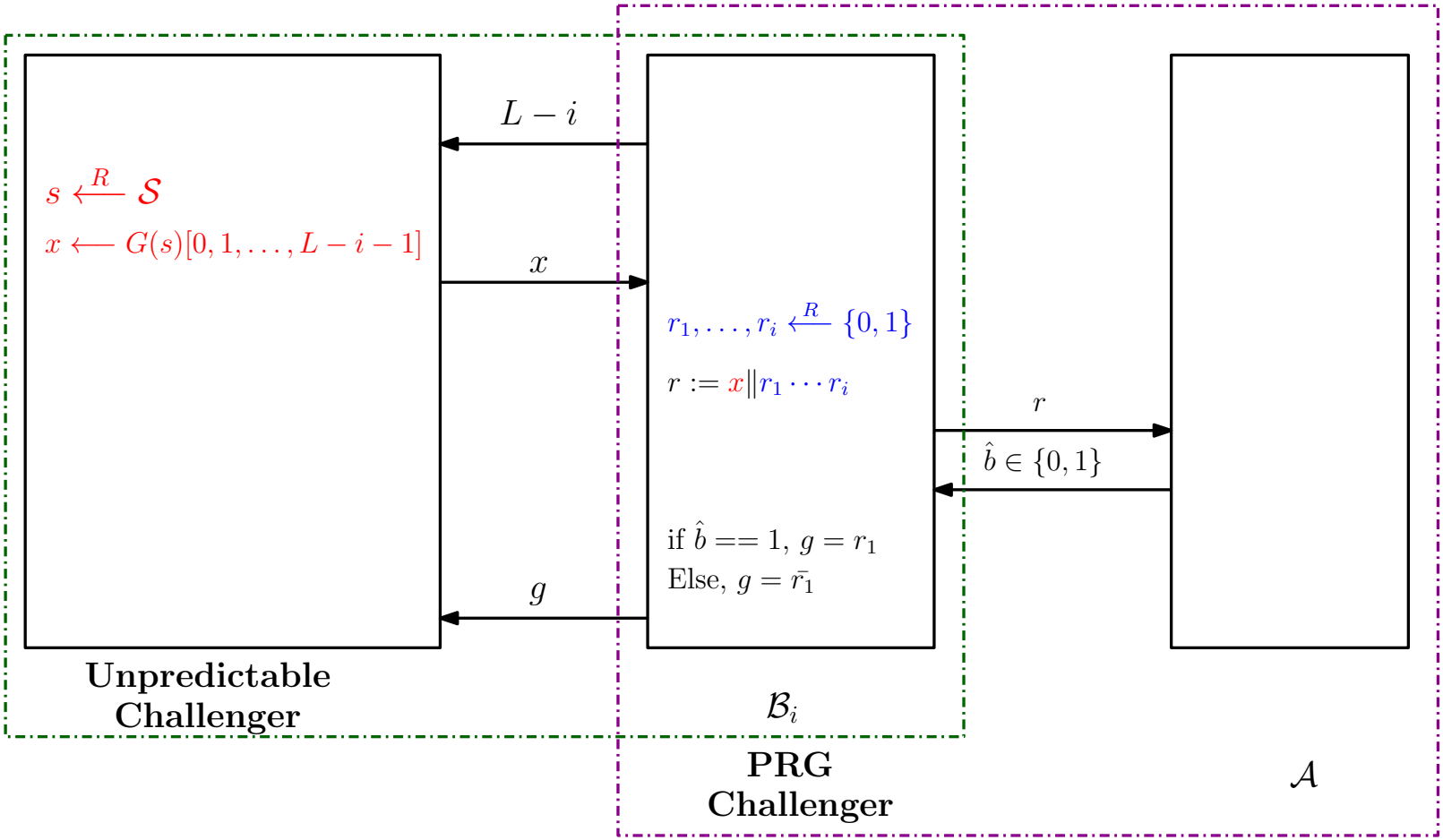
  - If $\mathcal{A}$ outputs 1, then $\mathcal{B}$ output r_1; otherwise, $\mathcal{B}$ output $\bar{r_1}$.

Unpredictability

Proof



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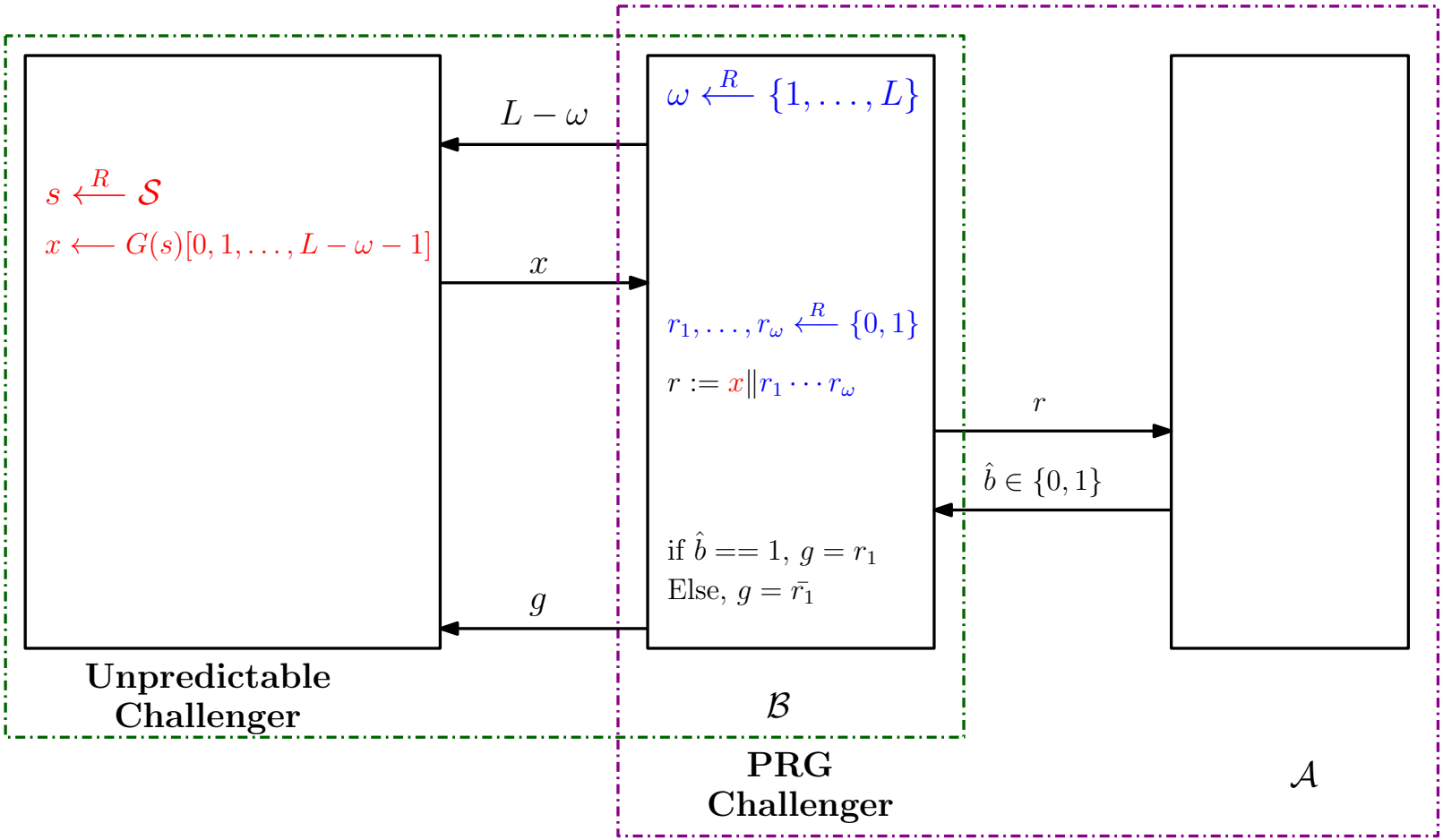
Proof



Unpredictability

Proof

- Let W be the event where $\mathcal{B}$ wins that is $g = G(s)[L - \omega]$.

$$\Pr[W] = \sum_{i=1}^{L} \Pr[W \wedge \omega = i]$$

Unpredictability

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&= \sum_{i=1}^{L} \Pr[W \mid \omega = i] \Pr[\omega = i] \\
&= \frac{1}{L} \sum_{i=1}^{L} \Pr[W \mid \omega = i]
\end{aligned}$$

Unpredictability

Proof

- **By Distinguisher/predictor Lemma:** $\Pr[W \mid \omega = i] = \left(\frac{1}{2} + p_{i-1} - p_i\right)$.

$$\Pr[W] = \frac{1}{L}\sum_{i=1}^{L} \Pr[W \mid \omega = i]$$

Proof

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&= \frac{1}{2} + \frac{1}{L}(p_0 - p_L)
\end{aligned}$$

Unpredictability

Proof

$$\mathsf{PREDadv}[\mathcal{B}, G] = \left| \Pr[W] - \frac{1}{2} \right|$$

Unpredictability

Proof

$$\begin{aligned} \mathsf{PREDadv}[\mathcal{B}, G] &= \left| \Pr[W] - \frac{1}{2} \right| \\ &= \left| \frac{1}{L} \cdot (p_0 - p_L) \right| \end{aligned}$$

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\mathsf{PREDadv}[\mathcal{B}, G] &= \left| \Pr[W] - \frac{1}{2} \right| \\
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&= \frac{1}{L} \cdot | \ (p_0 - p_L) \ |
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Unpredictability

Proof

$$
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&= \frac{1}{L} \cdot \mathsf{PRGadv}[\mathcal{A}, G].
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$$

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\mathsf{PREDadv}[\mathcal{B}, G] &= \left| \Pr[W] - \frac{1}{2} \right| \\
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&= \frac{1}{L} \cdot | \ (p_0 - p_L) \ | \\
&= \frac{1}{L} \cdot \mathsf{PRGadv}[\mathcal{A}, G].
\end{aligned}$$

Therefore,

$$\mathsf{PRGadv}[\mathcal{A}, G] = L \cdot \mathsf{PREDadv}[\mathcal{B}, G] \leqslant L \cdot \epsilon(n).$$

Modes of Operation

Two Modes

- Synchronized mode, and
- Unsynchronized mode.

Synchronized mode

Synchronized mode

- Stateful, and the sender and receiver must be synchronized in the sense that they know how much plaintext has been encrypted (resp., decrypted) so far.
- They can use the same key to encrypt multiple messages.
- Sender and Receiver has the same (IV, key) pair.

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- They can use the same key to encrypt multiple messages.
- Sender and Receiver has the same (IV, key) pair.
- Treat multiple messages $m_1, m_2, m_3, \ldots$ as a single, long message.
- Let the length of m_i be l_i.

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- Sender and Receiver has the same (IV, key) pair.
- Treat multiple messages $m_1, m_2, m_3, \ldots$ as a single, long message.
- Let the length of m_i be l_i.
- Divide the output stream into multiple blocks $pad_1, pad_2, pad_3, \ldots$ such that length of pad_i is l_i.
- $c_i \longleftarrow m_i \oplus pad_i$.

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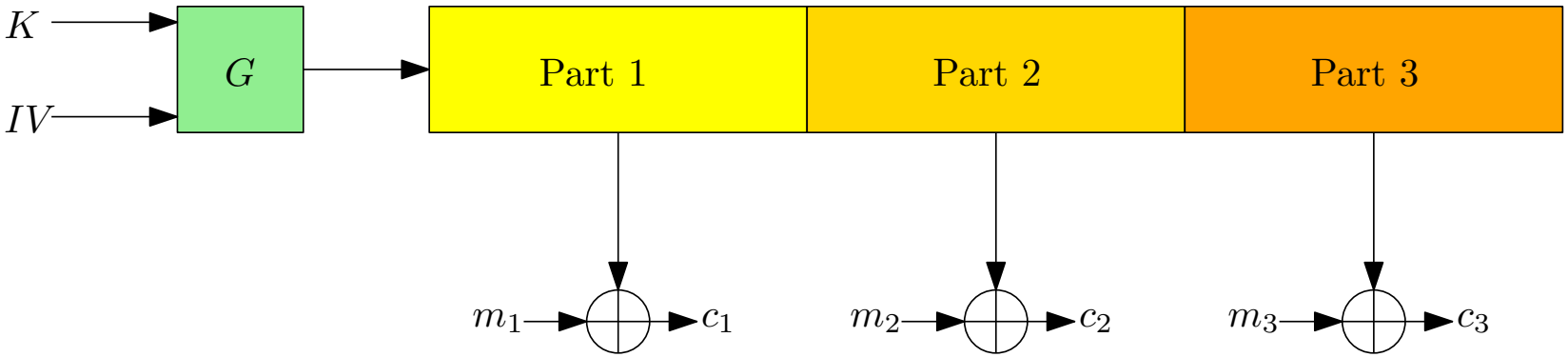
- Divide the output stream into multiple blocks $pad_1, pad_2, pad_3, \ldots$ such that length of pad_i is l_i.
- $c_i \longleftarrow m_i \oplus pad_i$.
- Upon receiving the cipher texts, receiver also computes the output stream from the same (IV, key).
- Divide the output stream into multiple blocks $pad_1, pad_2, pad_3, \ldots$ such that length of pad_i is l_i.
- $m_i \longleftarrow c_i \oplus pad_i$.

Synchronized mode



Synchronized mode

Synchronized mode

- Appropriate when two parties are communicating within a single session.
- Does not work well for sporadic communication or when a party might, over time, communicate from different devices.
  - easy to keep copies of a fixed key in different locations;
  - harder to maintain synchronized state across multiple locations
- If the parties ever get out of sync decryption will return an incorrect result.
- Resynchronization is possible, but adds additional overhead.

Unsynchronized mode

- Stateless.
- They can use the same key to encrypt multiple messages.
- Each message has different IV.

Unsynchronized mode

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Unsynchronized mode

Unsynchronized mode

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- They can use the same key to encrypt multiple messages.
- Each message has different IV.
- Let i-th message be m_i.
- Let the length of m_i be l_i.
- Generate a new IV, say IV_i.
- Generate key stream of (IV_i, key) say pad_i.
- $c_i \longleftarrow m_i \oplus pad_i[0, 1, \ldots, l_i]$.
- Send (IV_i, c_i).

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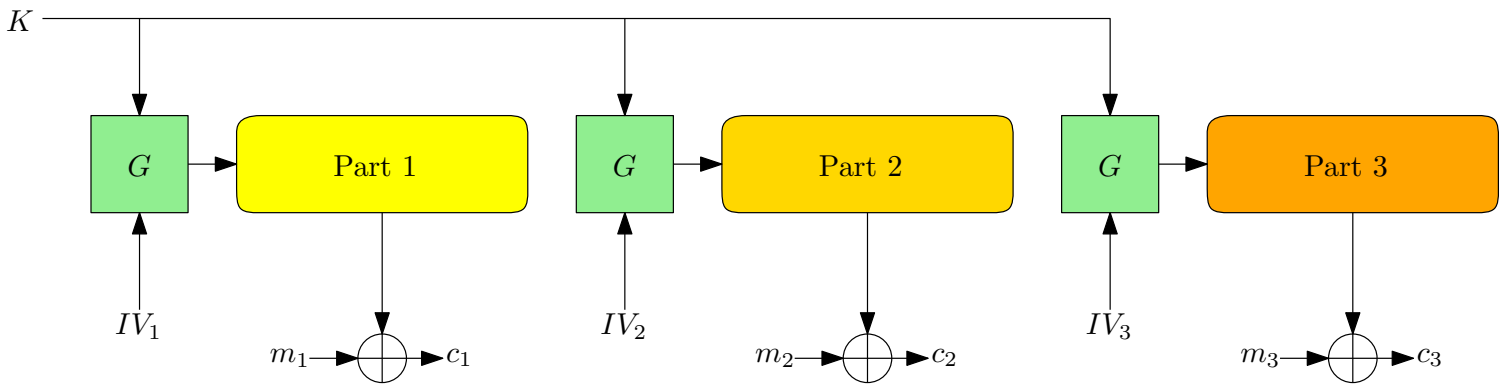
- Upon receiving the cipher text, receiver also computes pad_i from the same (IV_i, key).
- $m_i \longleftarrow c_i \oplus pad_i[0, 1, \ldots, l_i]$.

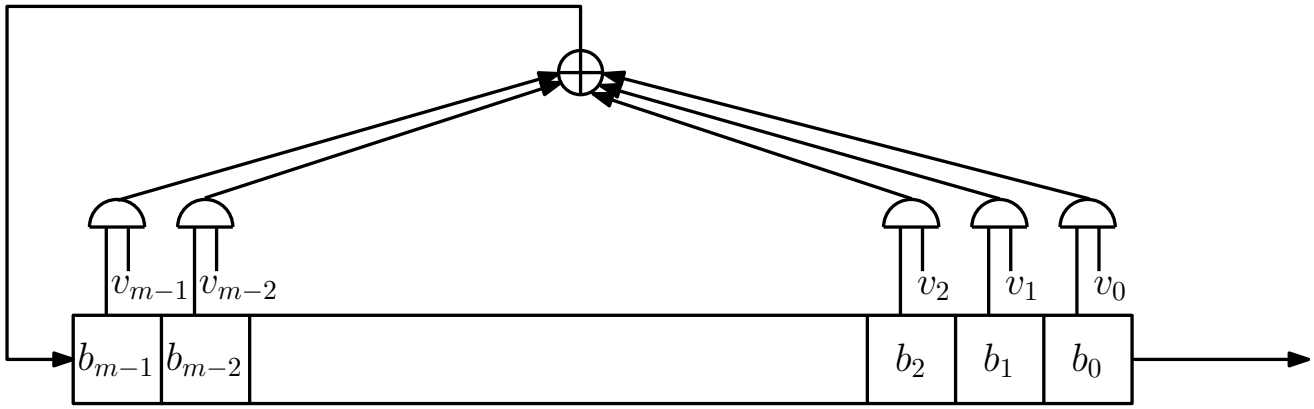
Unsynchronized mode



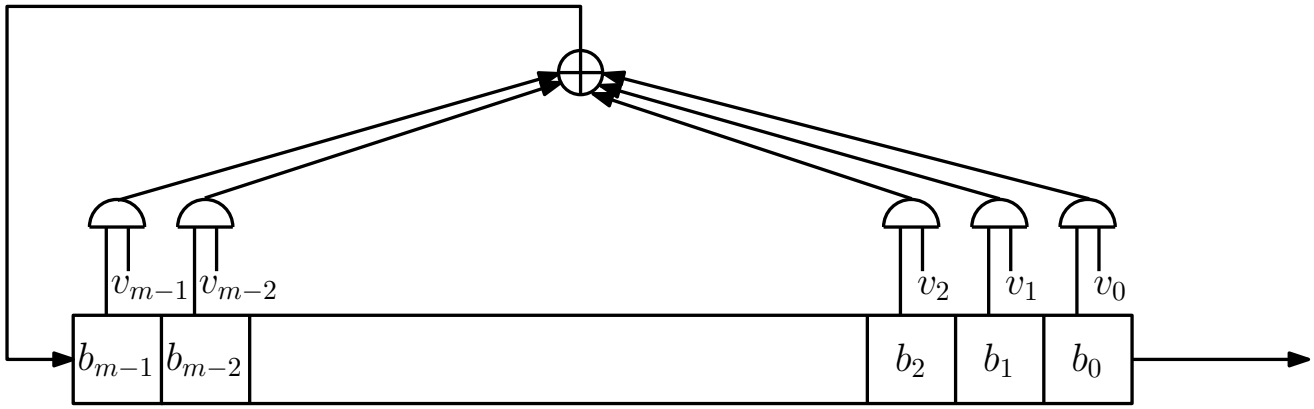
Case Study

- LFSR based PRG
  - CSS
  - Trivium
- RC4
- Salsa and ChaCha

Linear Feedback Shift Register (LFSR)



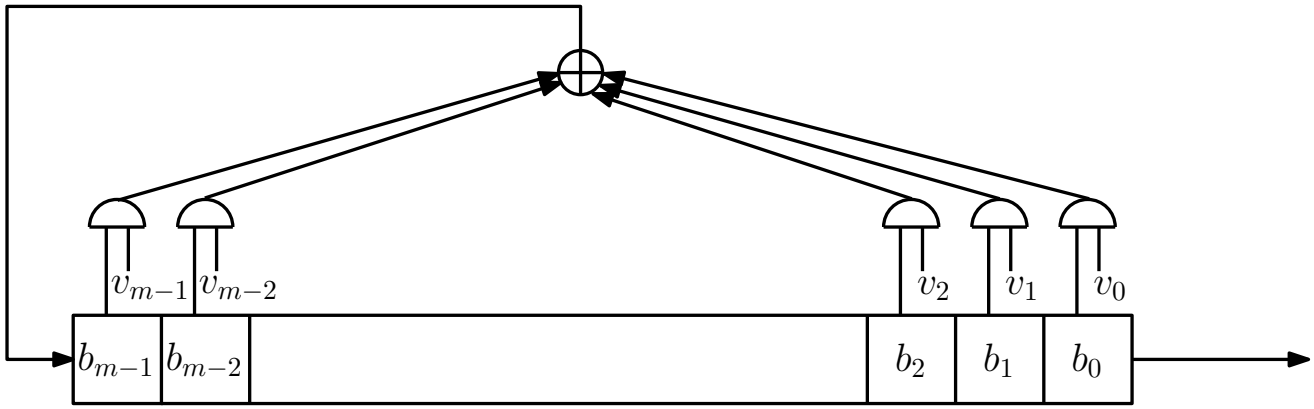
Linear Feedback Shift Register (LFSR)



LFSR

- Let the register be *m*-bit long.

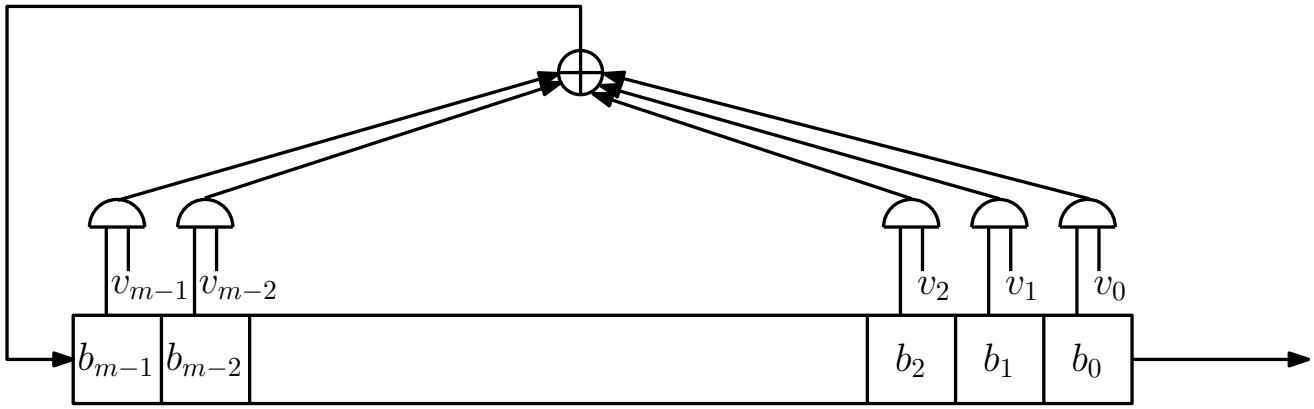
Linear Feedback Shift Register (LFSR)



LFSR

- Let the register be m-bit long.
- Tap Position $V = \{v_i \mid v_i = 1\}$.

Linear Feedback Shift Register (LFSR)



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$$b_m = b_0 v_0 + b_1 v_1 + \cdots + b_{m-1} v_{m-1} \pmod 2.$$

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Linear Feedback Shift Register (LFSR)

LFSR

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Output: $y \in \{0,1\}^\ell$ where $\ell > m$

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    $s \longleftarrow (b, b_{m-1}, \ldots, b_1)$ //shift register bits to the right

Linear Feedback Shift Register (LFSR)

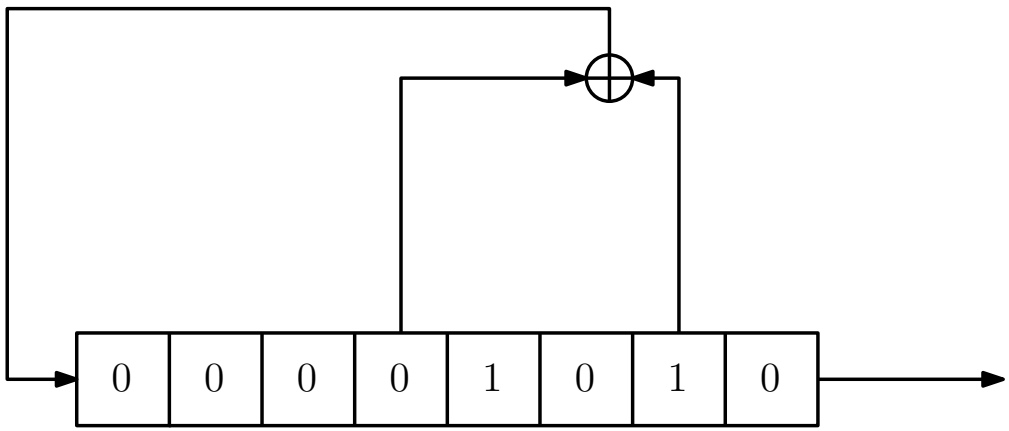
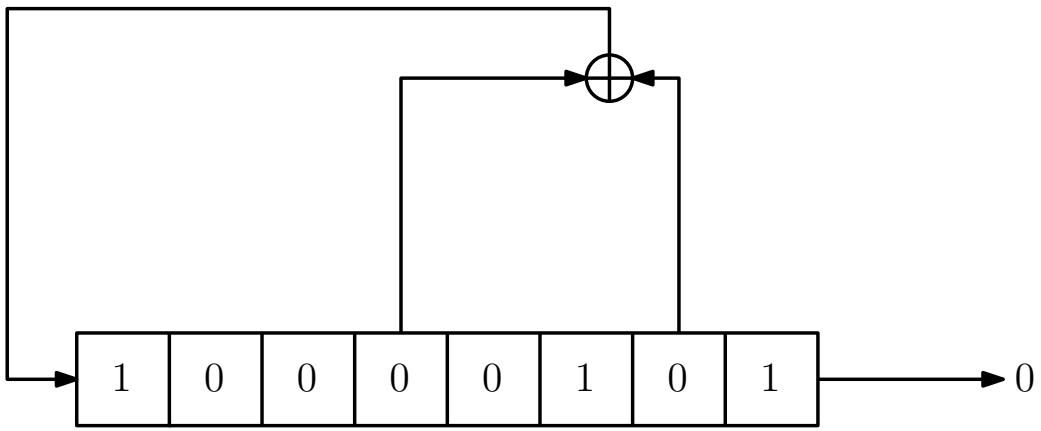


Figure: LFSR Example

Linear Feedback Shift Register (LFSR)



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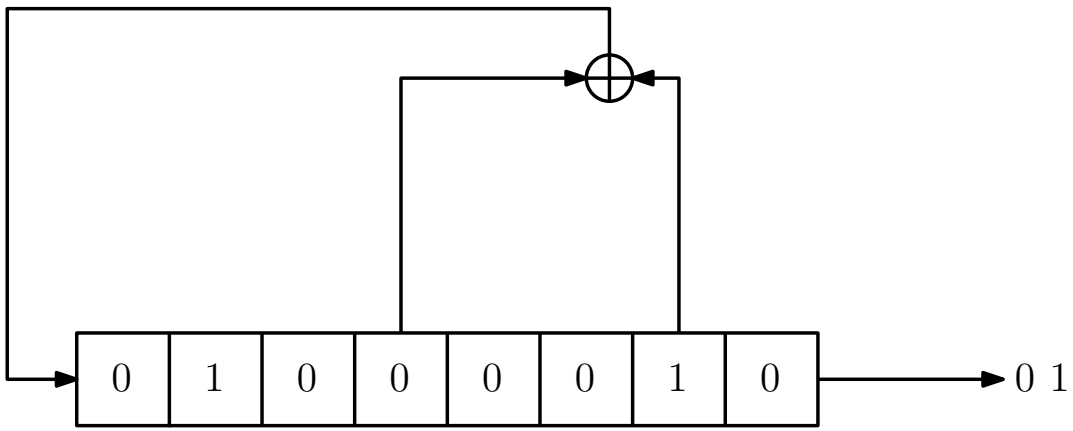

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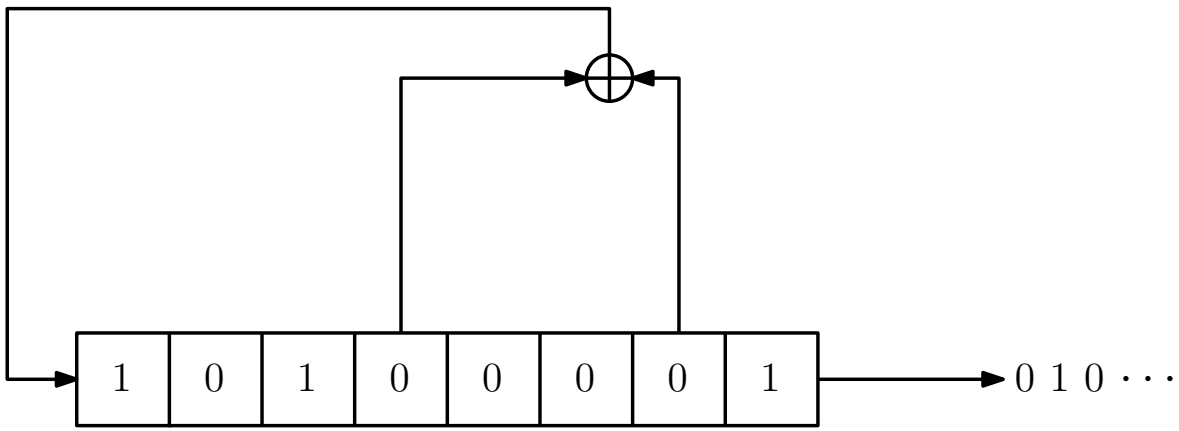
Linear Feedback Shift Register (LFSR)



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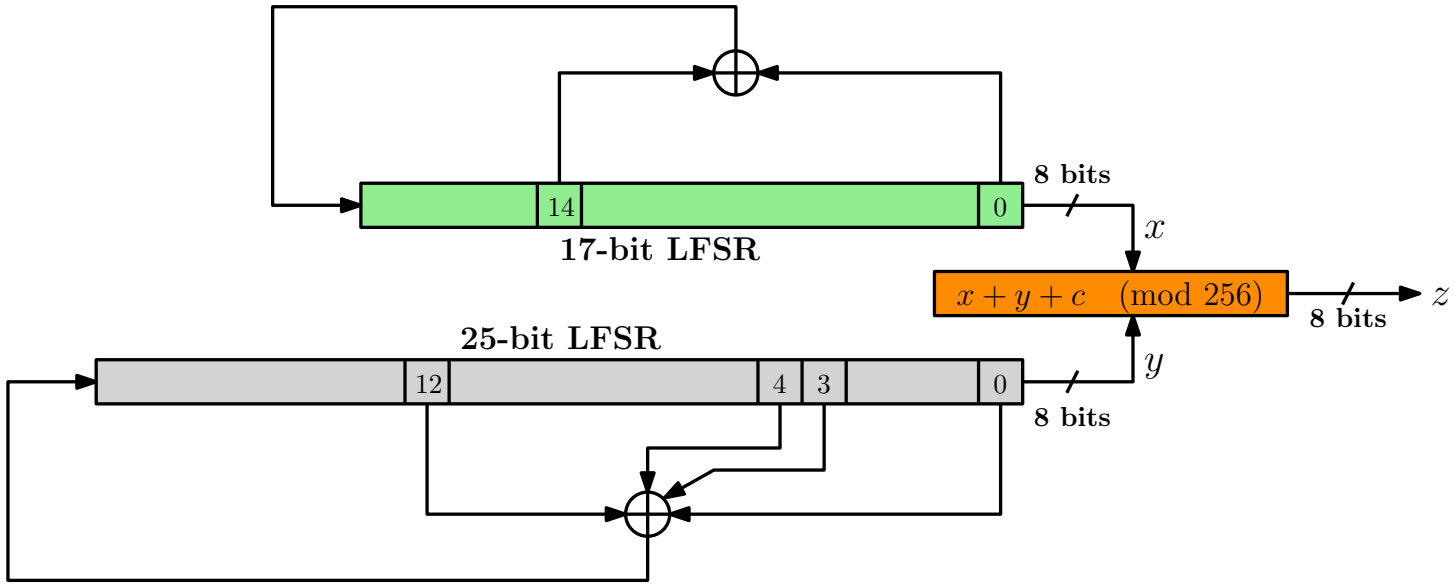

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write $s = s_1 \| s_2$ where $s_1 \in \{0,1\}^{16}$ and $s_1 \in \{0,1\}^{24}$
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/*Computational Stage*/
for $i \longleftarrow 1, \ldots, \ell$ do
    run both LFSRs for eight cycles to obtain $x_i, y_i \in \{0,1\}^8$
    treat x_i and y_i as integers in $0, \ldots, 255$
    output $z_i := x_i + y_i + c \pmod{256}$
    if $x_i + y_i > 255$ then $c \longleftarrow 1$ else $c \longleftarrow 0$ //carry bit

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4. Test the correctness, and continue until the correct guess.

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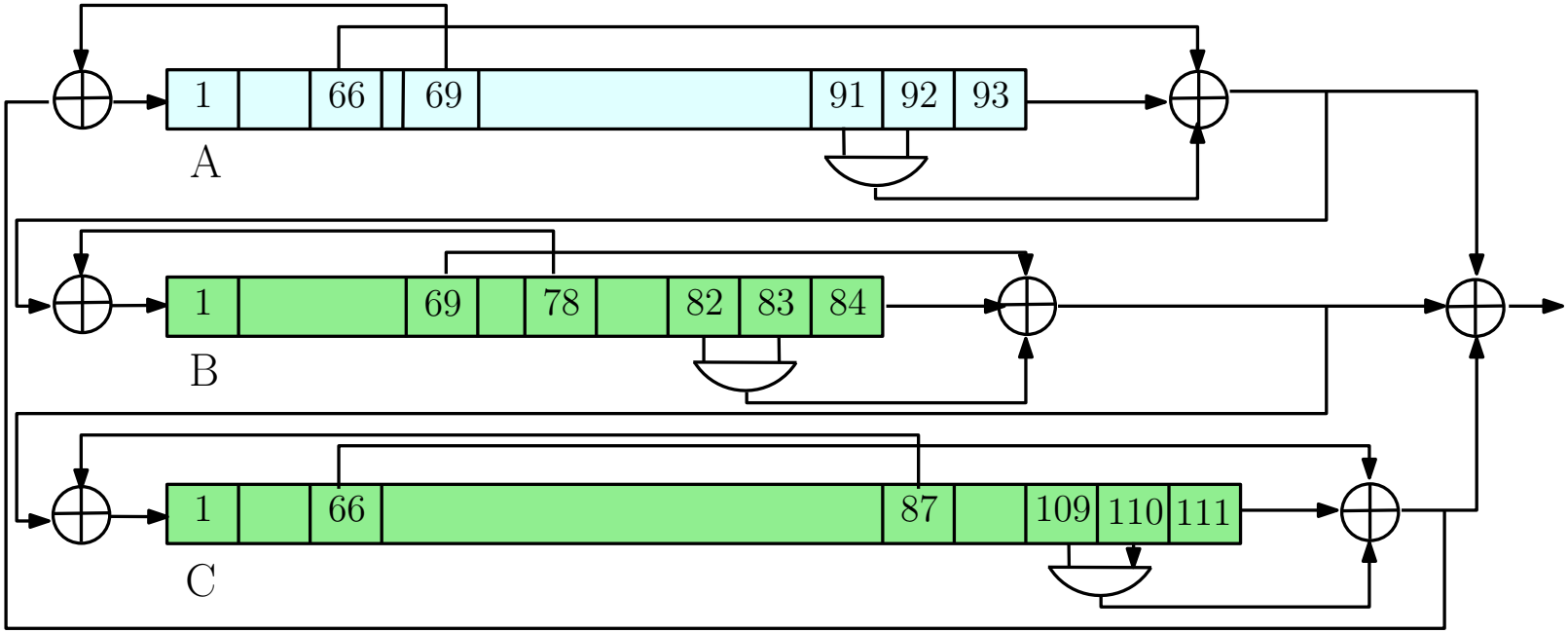

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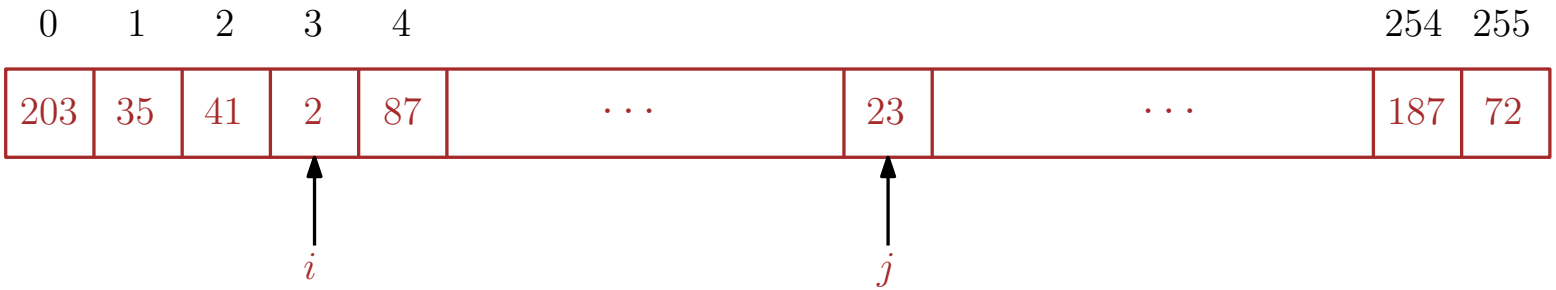

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for $i \longleftarrow 0, \ldots, 255$ do
    $S[i] \longleftarrow i$

$j \longleftarrow 0$
for $i \longleftarrow 0, \ldots, 255$ do
    $k \longleftarrow s[i \pmod{|s|}]$//extract one byte from seed
    $j \longleftarrow (j + S[i] + k) \pmod{256}$
    swap$(S[i], S[j])$

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  $j \longleftarrow (j+S[i]) \pmod{256}$
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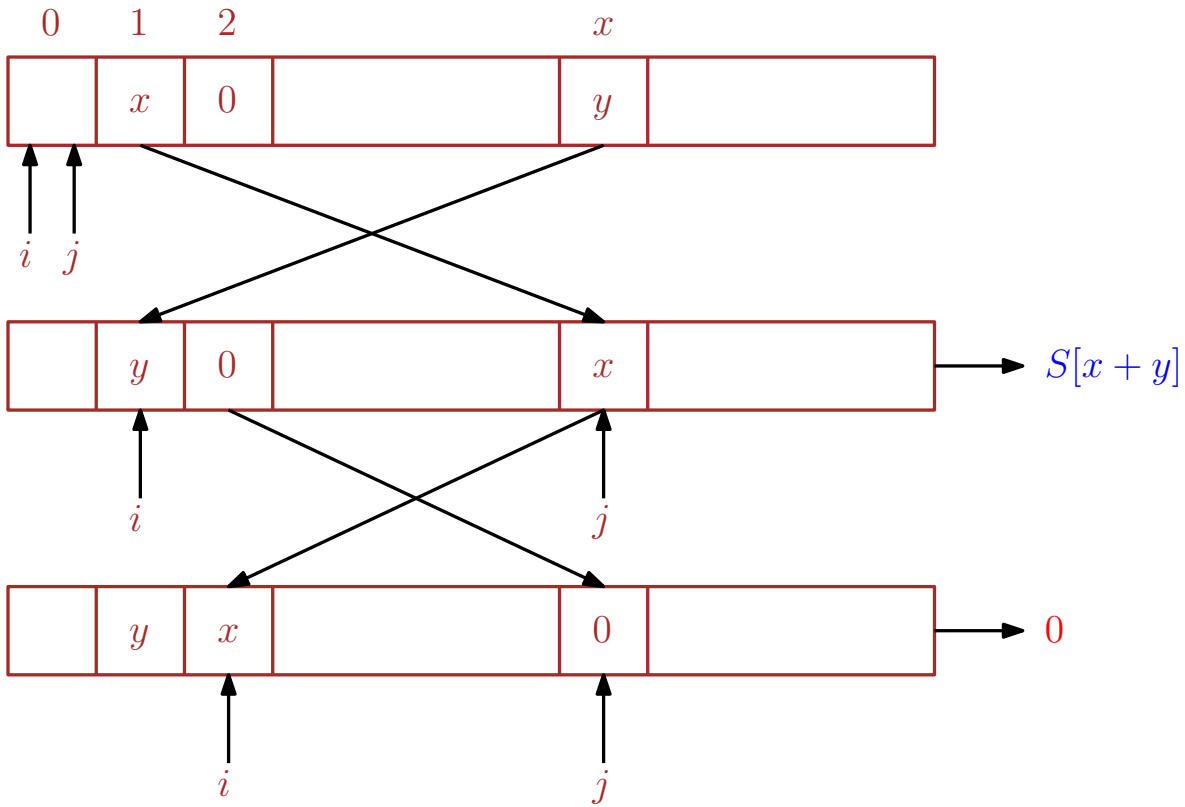
  output $S[S[i]+S[j] \pmod{256}]$
forever

RC4 Weaknesses

Lemma (Mantin-Shamir)

Suppose the array S is set to random permutation of $0, \ldots, n-1$ and that i, j are set to 0. Then the probability that the second byte of the output of RC4 is equal to 0 is $\frac{2}{n}$.



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- Recommendation: Discard first 256 bytes.

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- Another weakness of RC4 is related key attack.

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  - $\mathcal{S} = \{0,1\}^{128 \text{ or } 256}$
  - $\mathcal{R} = \{0,1\}^{512 \cdot L}$ where $L < 2^{64}$, and
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- The key can be repeated with different nonce.

Salasa20

$$G(k,n) = \Pi(k,(0,n))||\Pi(k,(1,n))||\Pi(k,(2,n))||\cdots$$

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- Counter $i \in \{0,1\}^{64}$
  - $i_0, i_1 \in \{0,1\}^{32}$.

padding(k, i, n)

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 \\ k_0 & k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 & k_7 \\ i_0 & i_1 & n_0 & n_1 \end{pmatrix}$$

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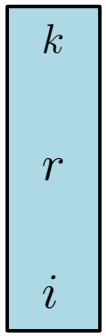
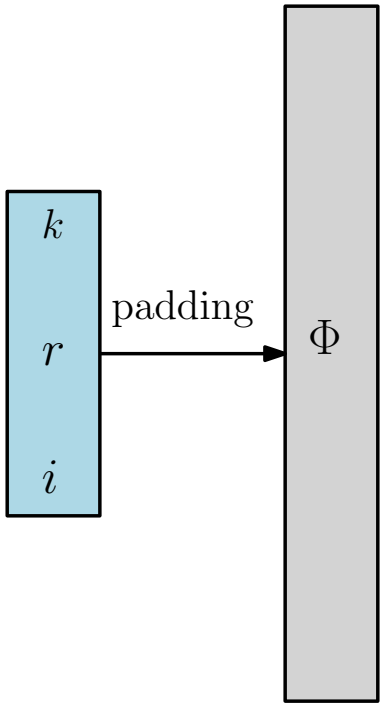
- $c_0, c_1, c_2, c_4 \in \{0,1\}^{32}$ are some specific constant.
- Let $\pi : \{0,1\}^{512} \longrightarrow \{0,1\}^{512}$ an invertible function.

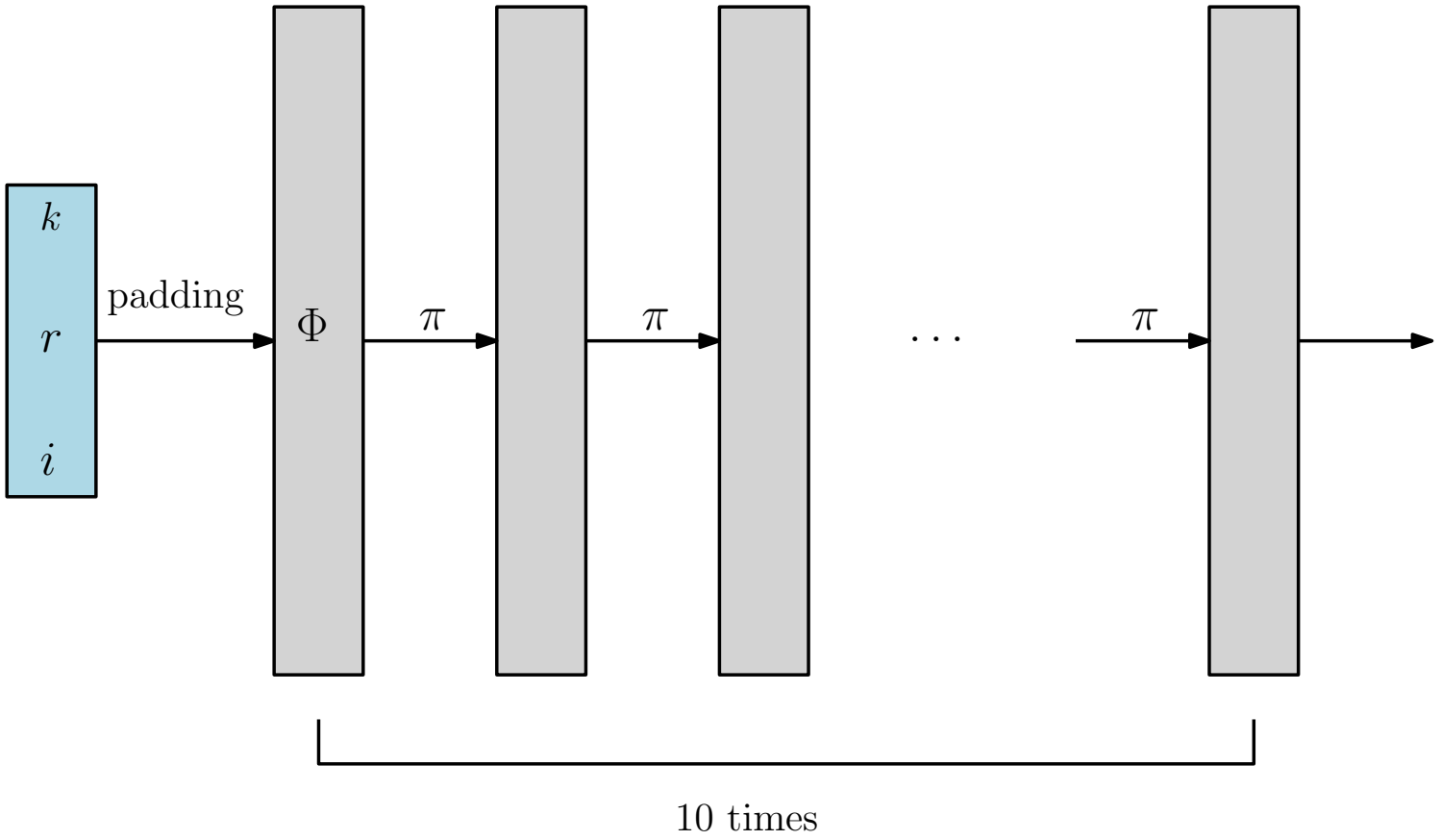
Salasa20



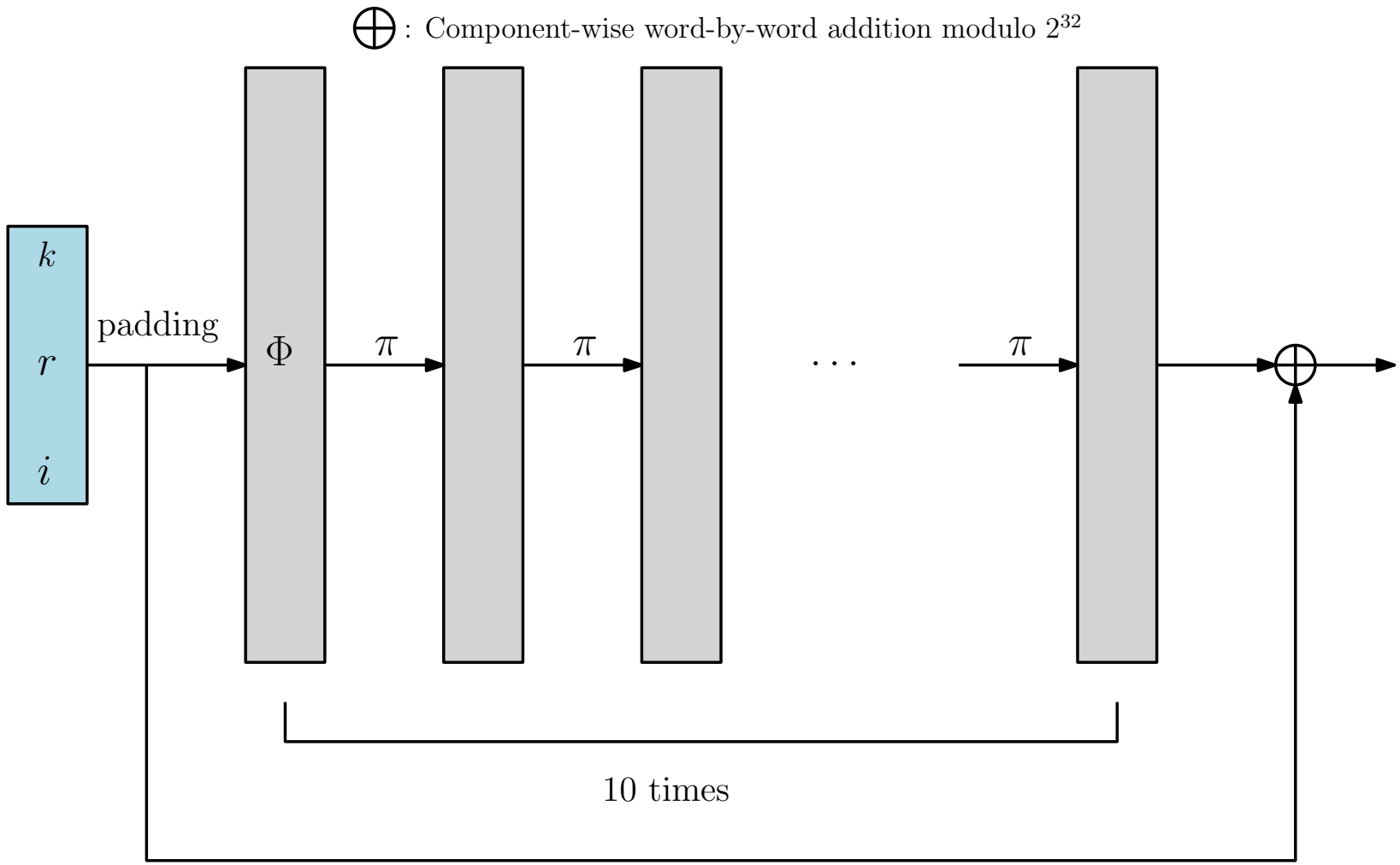
Salasa20



Salasa20



Salasa20


⊕ : Component-wise word-by-word addition modulo 2^{32}
k
r
i
padding
Φ
π
π
$\cdots$
π
10 times

π

1. QuarterRound(x_0, x_4, x_8, x_{12})
2. QuarterRound(x_1, x_5, x_9, x_{13})
3. QuarterRound$(x_2, x_6, x_{10}, x_{14})$
4. QuarterRound$(x_3, x_7, x_{11}, x_{15})$
5. QuarterRound$(x_0, x_5, x_{10}, x_{15})$
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QuarterRound(a, b, c, d)

#define ROTL(a, b) ($((a) \ll (b))$ | $((a) \gg (32 - (b)))$)

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$$
\begin{array}{lll}
a+ = b; & d\hat{} = a; & \text{ROTL}(d, 16); \\
c+ = d; & b\hat{} = c; & \text{ROTL}(b, 12); \\
a+ = b; & d\hat{} = a; & \text{ROTL}(d, 8); \\
c+ = d; & b\hat{} = c; & \text{ROTL}(b, 7);
\end{array}
$$

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$$

Advantage

- Highly parallelizable.
- Performs good in hardware or software implementation.

End