# Cryptology

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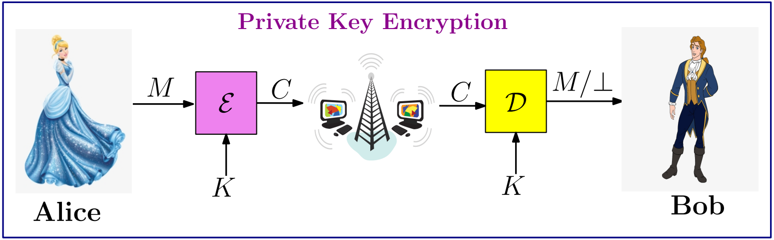
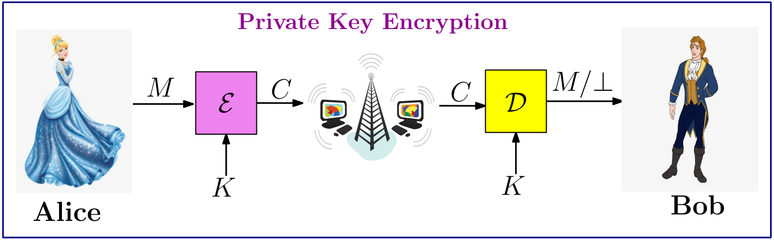
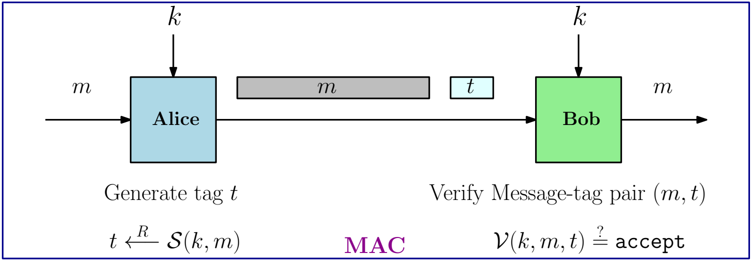
**Lecture 11**

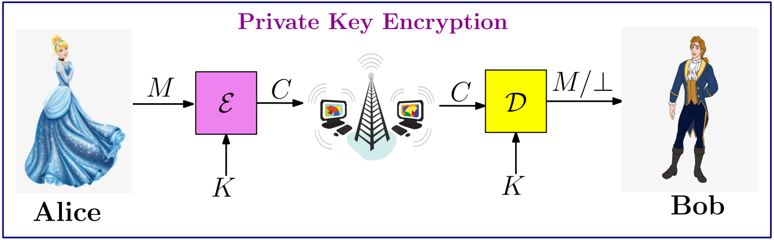
# Public Key Cryptography and Key Exchange Protocol


Private Key Encryption
$M$
$\mathcal{E}$
$C$
$K$
$C$
$\mathcal{D}$
$M/\perp$
$K$
Alice
Bob

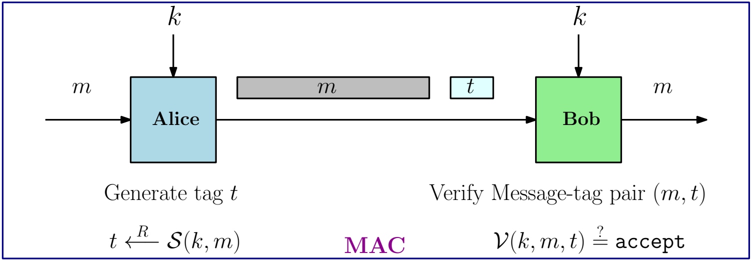
# Introduction






**How to setup the shared Key?**



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  - For average persons, this may cause severe travel cost.

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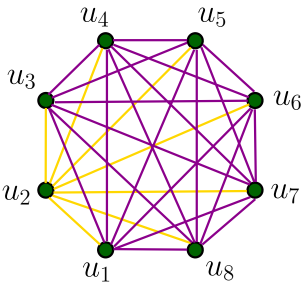

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     - The more keys there are the harder it is to protect them.
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     - Computer systems are often infected by viruses, worms, and other forms of malicious software.

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1. Key distribution,
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3. Inapplicability of private-key cryptography in open systems

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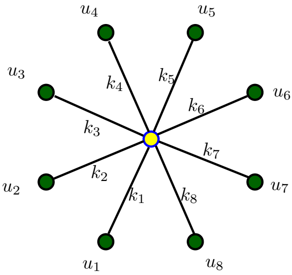

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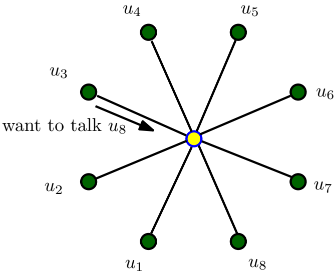


## Key distribution centers

1. $u_3$ sends a request to KDC: want to talk to $u_8$

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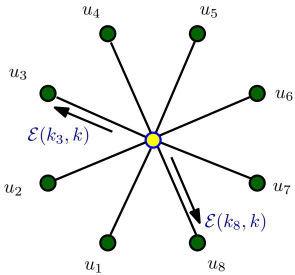


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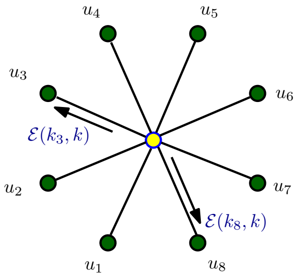


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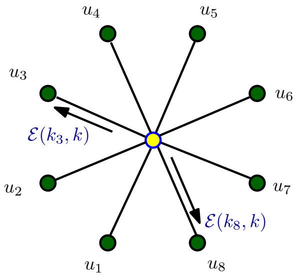


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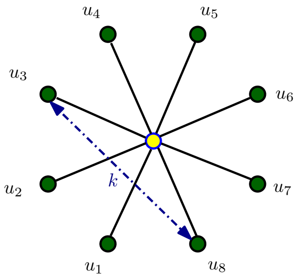


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1. $u_3$ and $u_8$ will use $k$ as shared key.
2. Must delete the key $k$ after use.

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  - The KDC is a single point of failure.

# New Directions in Cryptography (1976): Whitfield Diffie and Martin Hellman

## First two paragraphs of the paper

We stand today on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of mechanical computing and brought the cost of high grade cryptographic devices down to where they can be used in such commercial applications as remote cash dispensers and computer terminals.

In turn, such applications create a need for new types of cryptographic systems which minimize the necessity of secure key distribution channels and supply the equivalent of a written signature. At the same time, theoretical developments in information theory and computer science show promise of providing provably secure cryptosystems, changing this ancient art into a science.

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- Introduces three distinct public-key primitives:
    - Interactive Key Exchange.
    - Public Key Encryption.
    - Digital Signatures.

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- Formally, the transcript $T_P$ of protocol $P$ is a random variable, which is a function of the random bits generated by $A$ and $B$.

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Let $W$ be the event that $\mathcal{A}$ outputs $\hat{k} = k$ in the Experiment. We define advantage of $\mathcal{A}$ in the attack game with respect to $P$ as

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# Diffie-Hellman (DH) Key-Exchange Algorithm (DHKE)

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  - it must be computationally hard to guess $d$ from $e$.

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- $\{G, g, n, m\}$ are public elements.

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| Computes $k = (e_B)^{d_A}$ | Computes $k = (e_A)^{d_B}$ |

**Correctness**

$$(e_B)^{d_A} = \left(g^{d_B}\right)^{d_A} = (g)^{d_A d_B} = \left(g^{d_A}\right)^{d_B} = (e_A)^{d_B}.$$

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## Failure

If $k = 1$, then we have a Failure.

### Discrete Logarithm Problem (DLP)

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We define $\mathcal{A}$'s advantage in solving the discrete logarithm problem for $G$, denoted $\mathsf{DLadv}[\mathcal{A}, G]$, as the probability that $x = \hat{x}$.

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Let $G$ be a cyclic group of order $n$ generated by $g \in G$. For a given adversary $\mathcal{A}$, define the following attack game:

- The challenger and the adversary $\mathcal{A}$ take a description of $G$ as input. The description includes the order $n$ and a generator $g \in G$.
- The challenger computes $x \xleftarrow{R} \mathbb{Z}_n$, $u \longleftarrow g^x$, and sends $u \in G$ to $\mathcal{A}$.
- The adversary outputs some $\hat{x} \longleftarrow \mathbb{Z}_n$.

We define $\mathcal{A}$'s advantage in solving the discrete logarithm problem for $G$, denoted $\mathsf{DLadv}[\mathcal{A}, G]$, as the probability that $x = \hat{x}$.

## Discrete logarithm assumption

We say that the discrete logarithm (DL) assumption holds for $G$ if for all efficient adversaries $\mathcal{A}$ the quantity $\mathsf{DLadv}[\mathcal{A}, G]$ is negligible.

# Discrete Logarithm Problem (DLP)

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## DLP and Diffie-Hellman Key Exchange Protocol

- If DLP is easy to solve in a group $G$, then
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  - Can compute $k = (e_B)^{d_A}$ (or $k = (e_A)^{d_B}$).
- Hardness of DLP is the necessary condition for DH key Exchange, but not sufficient.

# Computational Diffie-Hellman problem (CDHP)

**Computational Diffie-Hellman Problem (CDHP)**

Let $G$ be a finite cyclic group of order $n$ and $g$ be a generator. Given $(G, g, n)$, and $g^x$ and $g^y$ for some (unknown) integers $x$ and $y$, compute $g^{xy}$.

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- The adversary outputs some $\hat{w} \longleftarrow G$.

We define $\mathcal{A}$'s advantage in solving the computational Diffie-Hellman problem for $G$, denoted $\mathsf{CDHadv}[\mathcal{A}, G]$, as the probability that $w = \hat{w}$.

# Computational Diffie-Hellman problem (CDHP)

**Computational Diffie-Hellman assumption**

We say that the computational Diffie-Hellman (CDH) assumption holds for $G$ if for all efficient adversaries $\mathcal{A}$ the quantity $\mathsf{CDHadv}[\mathcal{A}, G]$ is negligible.

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**DHKE and CDHP**

$\mathsf{KEadv}_c[\mathcal{A}, \mathrm{DHKE}] = \mathsf{CDHadv}[\mathcal{A}, G]$.

**Decisional Diffie-Hellman Problem (DDHP)**

Let $G$ be a finite cyclic group of order $n$ and $g$ be a generator. Distinguish between $(G, g, n, g^x, g^y, g^{xy})$ and $(G, g, n, g^x, g^y, g^z)$ for some (unknown) integers $x$, $y$ and $z$.

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- The adversary outputs some $\hat{b} \in \{0, 1\}$.

# Decisional Diffie-Hellman problem (DDHP)

**Advantage**

For $b = 0, 1$, let $W_b$ be the event that $\mathcal{A}$ outputs 1 in Experiment $b$. We define the advantage of $\mathcal{A}$ in the DDH attack with respect to $G$ as

$$\mathsf{DDHadv}[\mathcal{A}, G] = |\Pr[W_0] - \Pr[W_1]|.$$

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We say that the decisional Diffie-Hellman (DDH) assumption holds for $G$ if for all efficient adversaries $\mathcal{A}$ the quantity $\mathsf{DDHadv}[\mathcal{A}, G]$ is negligible.

# Decisional Diffie-Hellman problem (DDHP)

## DHKE and DDHP

$$\Pr[W] \quad = \quad \Pr[\hat{b} = 0 \wedge b = 0] + \Pr[\hat{b} = 1 \wedge b = 1]$$

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$$\mathsf{KEadv}_i[\mathcal{A},\mathrm{DHKE}] = \left|\Pr[W]-\frac{1}{2}\right| \leqslant \mathsf{DDHadv}[\mathcal{A},G].$$

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  - Merely means that the problem becomes easier.
- Every element is a generator.
- Multiplicative inverse exists.
- A final reason for working with prime-order groups applies in situations when the DDHP should be hard.
  - We stress that using a group of prime order is neither necessary nor sufficient for the DDH problem to be hard.

# Choice of Underlying Group for DHKE

**(Subgroup of) $\mathbb{Z}_p^*$**

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  - DDHP is simply not hard in such groups.
  - All the quadratic residues of $\mathbb{Z}_p^*$ forms a subgroup $H$ of order $q = (p-1)/2$.

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  - DDHP is simply not hard in such groups.
  - All the quadratic residues of $\mathbb{Z}_p^*$ forms a subgroup $H$ of order $q = (p-1)/2$.
  - It is desirable to have $q$ as prime and we work with $H$.

# Choice of Underlying Group for DHKE

## (Subgroup of) $\mathbb{Z}_p^*$

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  - It is desirable to have $q$ as prime and we work with $H$.
  - If not, then it is desirable to have a subgroup of $\mathbb{Z}_p^*$ of large prime order.

# Small Subgroup Attack

- Let $\{G, g, n, m\}$ are public elements,
- Let $h = \dfrac{n}{m}$ and
- $g' \in G$ has order $h$.

# Small Subgroup Attack

| Alice | Malice | Bob |
|---|---|---|
| Generates a random $d_A \in \{2, \ldots, m-1\}$ | | Generates a random $d_B \in \{2, \ldots, m-1\}$ |
| Computes $e_A = g^{d_A}$ | | Computes $e_B = g^{d_B}$ |

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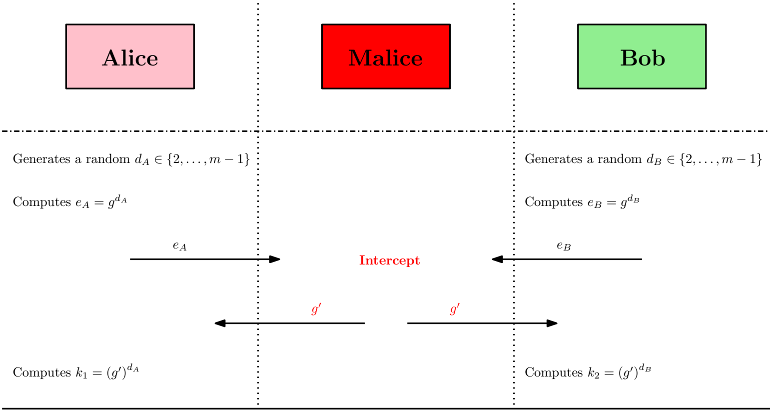
| $\xrightarrow{e_A}$ | **Intercept** | $\xleftarrow{e_B}$ |
| $\xleftarrow{g'}$ | | $\xrightarrow{g'}$ |


Alice
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$e_A$
Intercept
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$g'$
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# Small Subgroup Attack

**If $d_A \equiv d_B \equiv d \pmod{h}$**

- Let $d_A = q_A h + d$ and $d_B = q_B h + d$

$$(g')^{d_A} = (g')^{q_A h + d} = \left(g'^h\right)^{q_A} (g')^d = 1 \cdot (g')^d = (g')^d$$

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## If $d_A \not\equiv d_B \pmod{h}$

- Let $d_A = q_A h + d_1$ and $d_B = q_B h + d_2$

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- As $h$ is small, Malice can find $d_1$ and $d_2$ by exhaustive search.

# Small Subgroup Attack resistant DH Protocol

| **Alice** | **Bob** |
| --- | --- |
| Generates a random $d_A \in \{2, \ldots, m-1\}$ | Generates a random $d_B \in \{2, \ldots, m-1\}$ |
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| Alice | Bob |
| --- | --- |
| Computes $k = (e_B)^{hd_A}$ | Computes $k = (e_A)^{hd_B}$ |

# Unknown Key Share

| Alice | Malice | Bob |
|---|---|---|
| Generates a random $d_A \in \{2, \ldots, m-1\}$ | Generates a random $d_M \in \{2, \ldots, m-1\}$ | Generates a random $d_B \in \{2, \ldots, m-1\}$ |
| Computes $e_A = g^{d_A}$ | Computes $e_M = g^{d_M}$ | Computes $e_B = g^{d_B}$ |
| $e_A \longrightarrow$ | | $\longleftarrow e_B$ |

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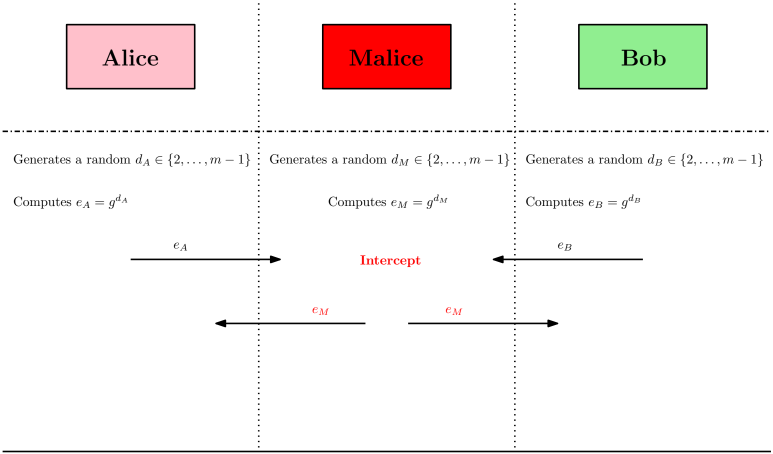

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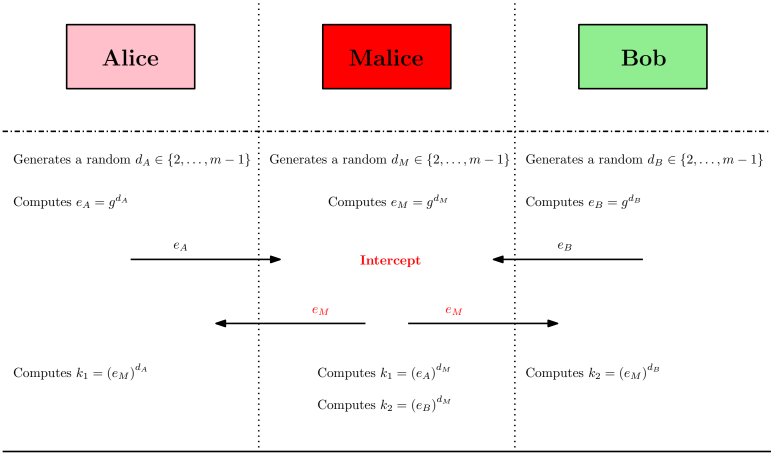

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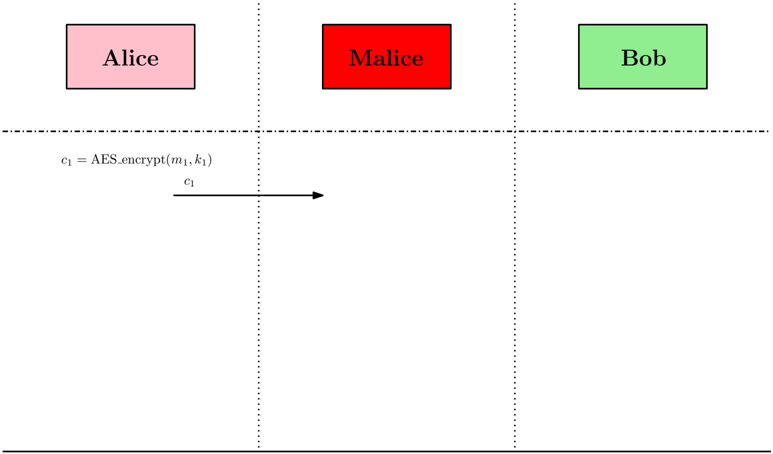

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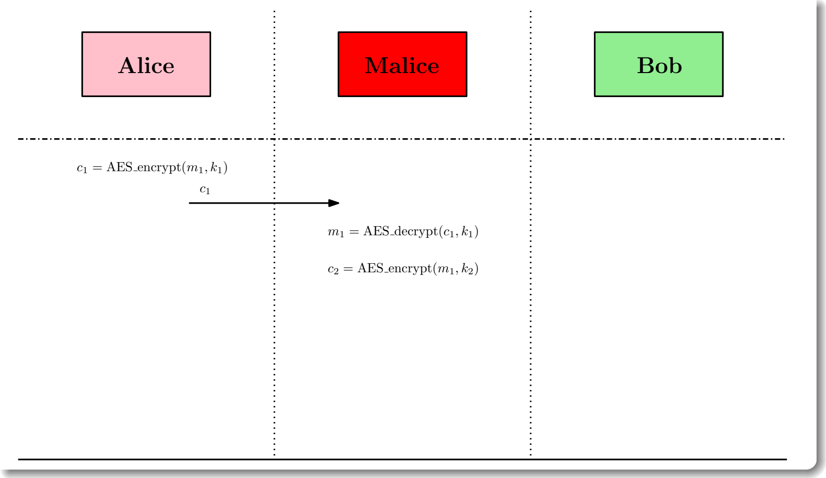

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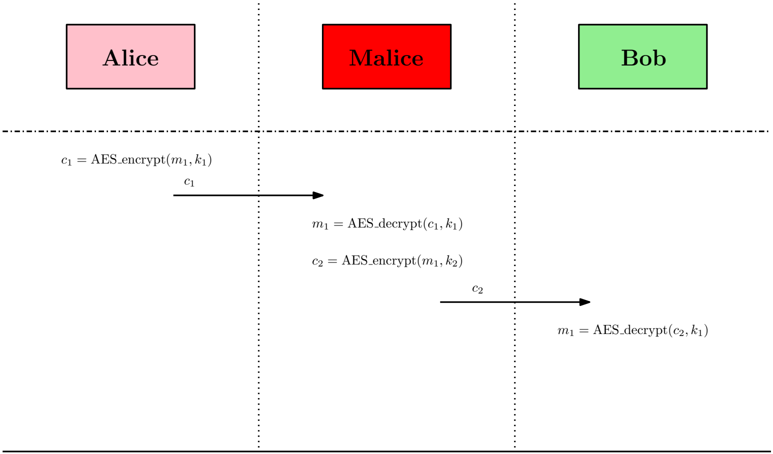

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- The ephemeral key is validated using the static private key.
- $l$ : bit-length of $m$.
- Publicly known function: $f : G \longrightarrow \mathbb{N}_0$ such that for $a \in G$

$$\hat{a} = 2^{\lceil l/2 \rceil} + f(a) \operatorname{rem} 2^{\lceil l/2 \rceil}.$$

  In particular, $\hat{a} \not\equiv 0 \pmod{m}$.

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| **Alice** | **Bob** |
|---|---|
| Computes $\sigma_A = d_A + \hat{e}_A D_A$ | Computes $\sigma_B = d_B + \hat{e}_B D_B$ |
| Computes $k = \left(e_B E_B^{\hat{e}_B}\right)^{h\sigma_A}$ | Computes $k = \left(e_A E_A^{\hat{e}_A}\right)^{h\sigma_B}$ |

## Correctness of MQV

$$\left(e_B E_B^{\hat{e}_B}\right)^{h\sigma_A} = \left(g^{d_B}\left(g^{D_B}\right)^{\hat{e}_B}\right)^{h\sigma_A}$$

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# End