Cryptology

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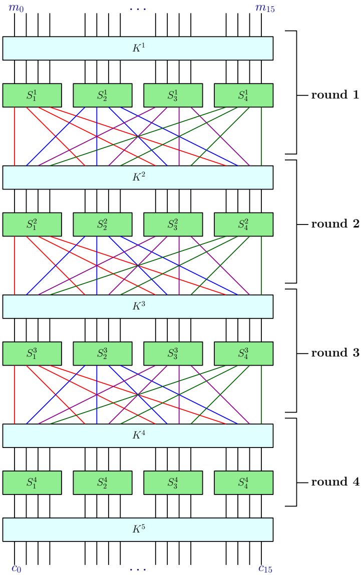
Lecture 07

Linear Cryptanalysis

SPN Network



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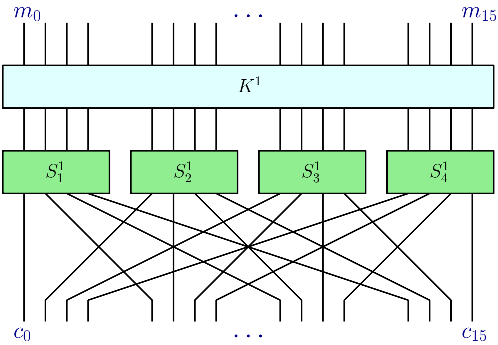
- The strength of a cipher $\mathcal{E}$ constructed in this way depends heavily on the *# rounds*.

A Trivial Case

- Assume that $\mathcal{E}$ consists of one full round and no final key-mixing step.



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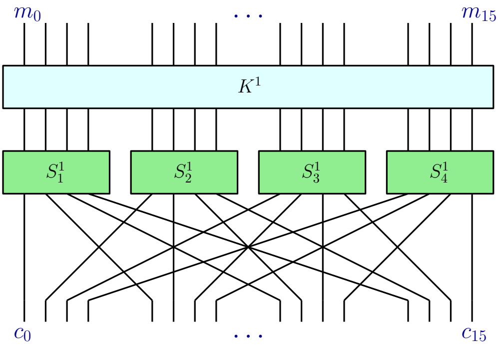

- Attack: An adversary given only a single input/output pair (m, c) can easily learn the secret key k for which $c = \mathcal{E}(k, x)$.

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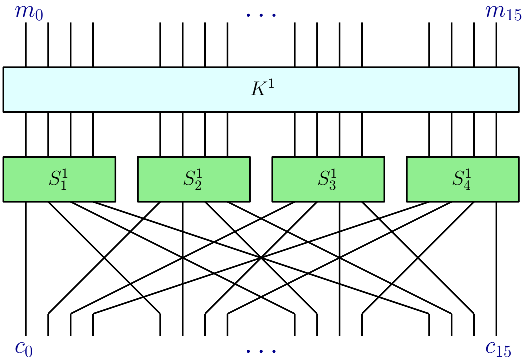

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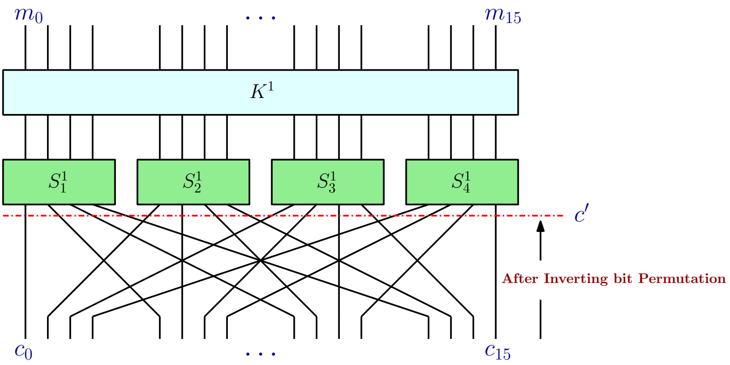

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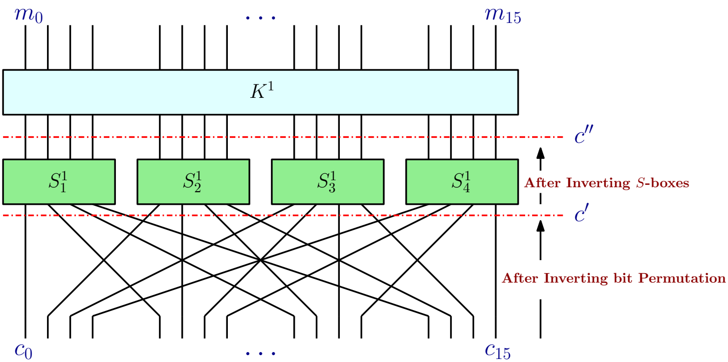

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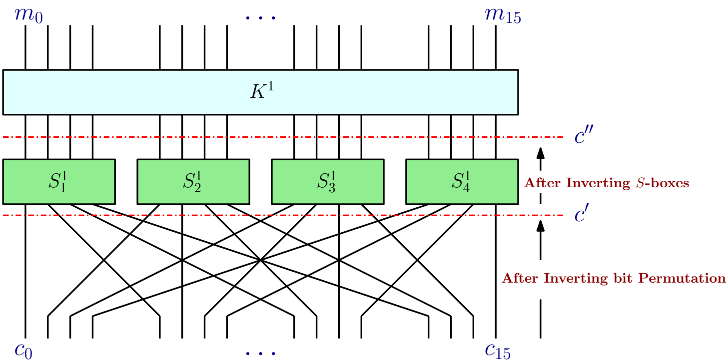

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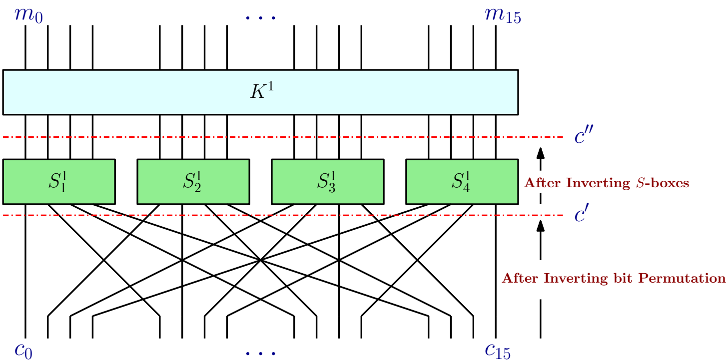
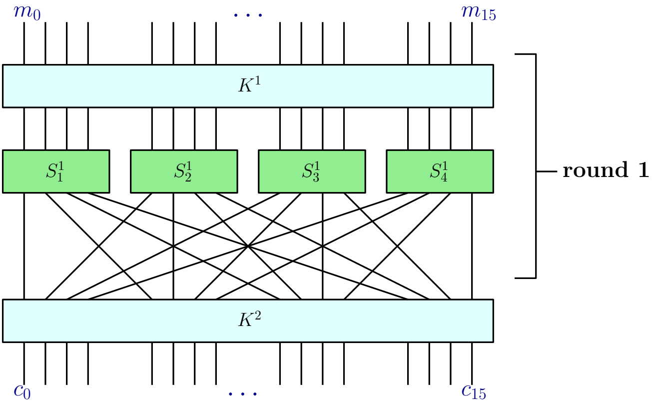


- $k = K^1 = m \oplus c''$.
- No security gained by performing S-box substitution or applying a mixing permutation after the final round-key mixing.

Attacking a 1-round SPN

- Consider one full round followed by a key-mixing step.



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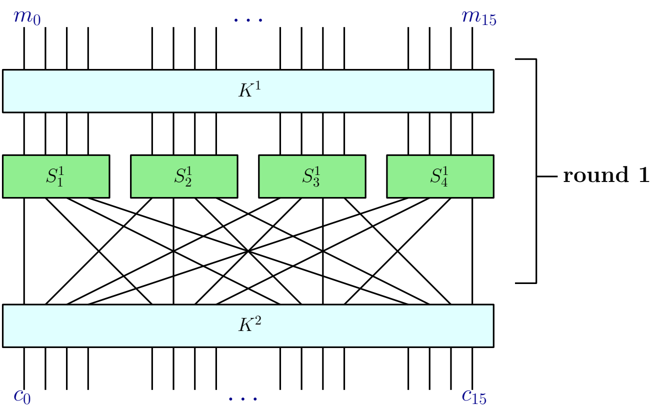

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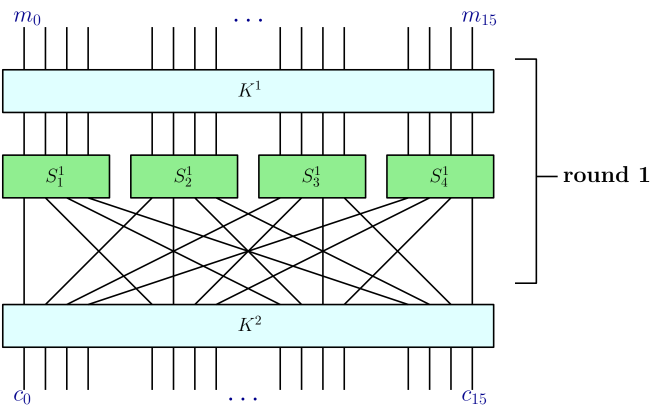

- **For concreteness:** Assume a 16-bit block length, 4×4 S-boxes and independent 16-bit round-keys K^1, k^2.
- Therefore the master key $K^1 \| K^2$ is 32 bits long.

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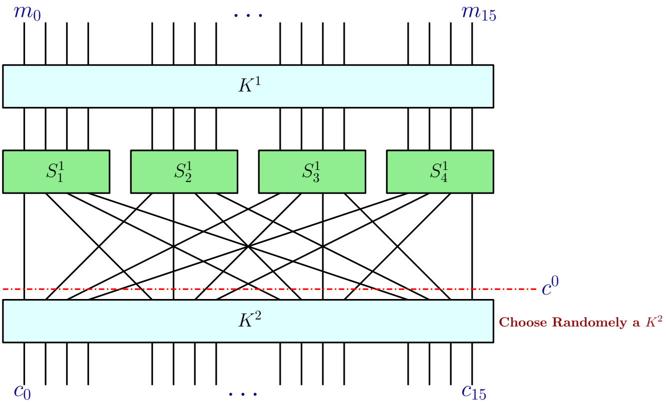

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- Observation 1: The trivial attack can be extended to give a key-recovery attack using much less than 2^{32} work.
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  - Use an additional input/output pairs to find the key in roughly 2^{16} additional time.

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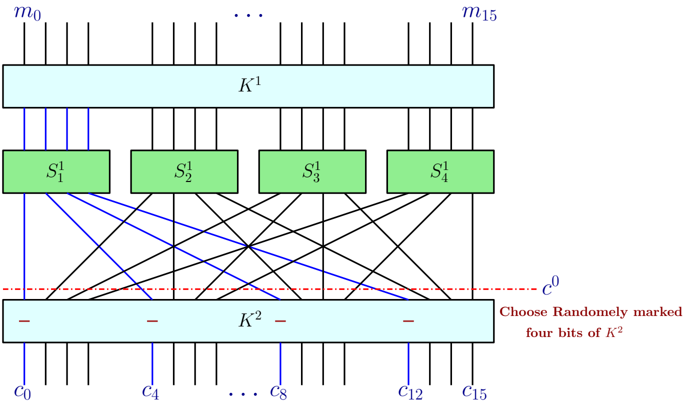

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  - Invert the S-box, to get the input to that S-box.
  - Since the input to that S-box is the XOR of first 4 bits of m and first 4 bits of K^1, this yields a candidate value for 4 bits of K^1.

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- Total time: $4 \cdot 2^4 = 2^6$, a dramatic improvement!

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- Attacking a two-round SPN: The above ideas can be extended to give a better-than-brute-force attack on a two-round SPN using independent round-keys.

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- Algebraic Attacks
  - Buchberger's Algorithm
  - Linearization Technique
  - Relinearization Technique
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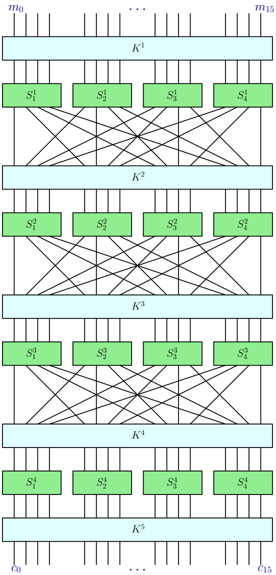
  - Differential Cryptanalysis, and variants like
    - Higher Order Differentials
    - Truncated Differential Cryptanalysis
    - Impossible Differential Cryptanalysis
    - Improbable Differential Cryptanalysis
    - Boomerang Attack
    - Cube Attack
- Other Attacks
  - Differential-linear attack
  - The Integral or Square attack and so on.

Linear Cryptanalysis



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- Count frequencies of all the candidate keys that satisfies the linear relationship.
- Hope: The candidate key that has a frequency count furthest from 1/2 times the number of plaintext-ciphertext pairs contains the correct values for these key bits.

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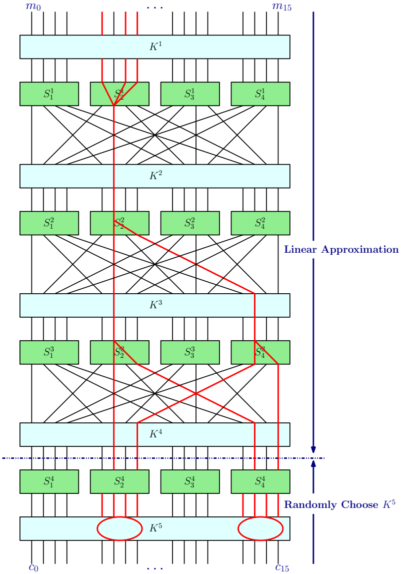

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- Main Aim: To find *linear approximations* of the form

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- It is a Known Plaintext Attack.

The Piling-up Lemma

- Let $\mathbf{X}_1, \mathbf{X}_2, \ldots$ be independent random variables taking on values from the set $\{0, 1\}$.
- Let $p_1, p_2, \ldots$ be real numbers such that $0 \leqslant p_i \leqslant 1$ for all i.
- Let $\forall i = 1, 2, \ldots$

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- For $i \neq j$

$$\begin{aligned} \Pr[\mathbf{X}_i = 0 \wedge \mathbf{X}_j = 0] &= p_i p_j, \\ \Pr[\mathbf{X}_i = 0 \wedge \mathbf{X}_j = 1] &= p_i(1 - p_j), \\ \Pr[\mathbf{X}_i = 1 \wedge \mathbf{X}_j = 0] &= (1 - p_i)p_j, \text{ and} \\ \Pr[\mathbf{X}_i = 1 \wedge \mathbf{X}_j = 1] &= (1 - p_i)(1 - p_j). \end{aligned}$$

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$$\begin{aligned}
\Pr[\mathbf{X}_i \oplus \mathbf{X}_j = 0] &= \Pr[\mathbf{X}_i = 0 \wedge \mathbf{X}_j = 0] + \Pr[\mathbf{X}_i = 1 \wedge \mathbf{X}_j = 1] \\
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- $-\frac{1}{2} \leqslant \epsilon_i \leqslant \frac{1}{2}$.

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Let $\epsilon_{i_1,i_2,\ldots,i_k}$ denote the bias of the random variable $\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_k}$. Then

$$\epsilon_{i_1,i_2,\ldots,i_k} = 2^{k-1}\Pi_{j=1}^{k}\epsilon_{i_j}.$$

Proof

- Basis:

$$\Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} = 0] = \left(\frac{1}{2} + \epsilon_{i_1}\right)\left(\frac{1}{2} + \epsilon_{i_2}\right) + \left(\frac{1}{2} - \epsilon_{i_1}\right)\left(\frac{1}{2} - \epsilon_{i_2}\right) = \frac{1}{2} + 2\epsilon_{i_1}\epsilon_{i_2}$$

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- Induction:

$$\begin{aligned} & \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_k} = 0] \\ = \quad & \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 0 \wedge \mathbf{X}_{i_k} = 0] \\ & + \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 1 \wedge \mathbf{X}_{i_k} = 1] \end{aligned}$$

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- Basis:

$$\Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} = 0] = \left(\frac{1}{2} + \epsilon_{i_1}\right)\left(\frac{1}{2} + \epsilon_{i_2}\right) + \left(\frac{1}{2} - \epsilon_{i_1}\right)\left(\frac{1}{2} - \epsilon_{i_2}\right) = \frac{1}{2} + 2\epsilon_{i_1}\epsilon_{i_2}$$

- Induction:

$$\begin{aligned}
& \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_k} = 0] \\
= \; & \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 0 \wedge \mathbf{X}_{i_k} = 0] \\
& + \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 1 \wedge \mathbf{X}_{i_k} = 1] \\
= \; & \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 0]\Pr[\mathbf{X}_{i_k} = 0] \\
& + \Pr[\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \oplus \mathbf{X}_{i_{k-1}} = 1]\Pr[\mathbf{X}_{i_k} = 1]
\end{aligned}$$

Proof

$$= \left(\frac{1}{2} + \epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2} + \epsilon_{i_k}\right) + \left(\frac{1}{2} - \epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2} - \epsilon_{i_k}\right)$$

Proof

$$\begin{aligned} &= \left(\frac{1}{2}+\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}+\epsilon_{i_k}\right)+\left(\frac{1}{2}-\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}-\epsilon_{i_k}\right) \\ &= \frac{1}{2}+2\epsilon_{i_1,i_2,\ldots,i_{k-1}}\epsilon_{i_k} \end{aligned}$$

Proof

$$
\begin{aligned}
&= \left(\frac{1}{2}+\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}+\epsilon_{i_k}\right)+\left(\frac{1}{2}-\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}-\epsilon_{i_k}\right) \\
&= \frac{1}{2}+2\epsilon_{i_1,i_2,\ldots,i_{k-1}}\epsilon_{i_k} \\
&= \frac{1}{2}+2\left(2^{k-2}\Pi_{j=1}^{k-1}\epsilon_{i_j}\right)\epsilon_{i_k}
\end{aligned}
$$

Proof

$$
\begin{aligned}
&= \left(\frac{1}{2}+\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}+\epsilon_{i_k}\right)+\left(\frac{1}{2}-\epsilon_{i_1,i_2,\ldots,i_{k-1}}\right)\left(\frac{1}{2}-\epsilon_{i_k}\right)\\
&= \frac{1}{2}+2\epsilon_{i_1,i_2,\ldots,i_{k-1}}\epsilon_{i_k}\\
&= \frac{1}{2}+2\left(2^{k-2}\Pi_{j=1}^{k-1}\epsilon_{i_j}\right)\epsilon_{i_k}\\
&= \frac{1}{2}+2^{k-1}\Pi_{j=1}^{k}\epsilon_{i_j}.
\end{aligned}
$$

The Piling-up Lemma

Corollary. Let $\epsilon_{i_1}, \epsilon_{i_2}, \ldots$ denote the bias of the random variable $\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \cdots$. Suppose that $\epsilon_{i_j} = 0$ for some j. Then $\epsilon_{i_1, i_2, \ldots, i_k} = 0$.

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  - $(\mathbf{X}_1 \oplus \mathbf{X}_2 = 0)$ and $(\mathbf{X}_2 \oplus \mathbf{X}_3 = 0)$ are analogus to linear approximations of S-boxes.

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- In Linear Cryptanalysis:
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  - $(\mathbf{X}_1 \oplus \mathbf{X}_3 = 0)$ is analogous to a cipher approximation where the intermediate bit $\mathbf{X}_2$ is eliminated.

Linear Approximations of S-boxes

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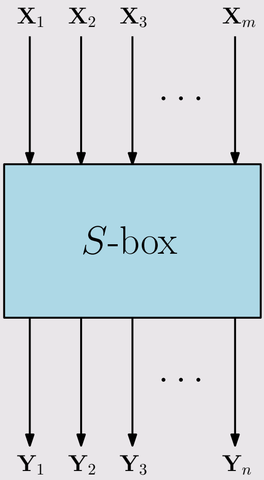
- Possible to concatenate linear approximations of the S-boxes.

Linear Approximations of S-boxes



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- $\mathbf{Y}_j$s are not independent from other $\mathbf{Y}_j$s and $\mathbf{X}_i$s.
- If $(y_1 \cdots y_n) \neq \pi_S(x_1 \cdots x_m)$,

$$\Pr[\mathbf{X}_1 = x_1, \ldots, \mathbf{X}_m = x_m, \mathbf{Y}_1 = y_1, \ldots, \mathbf{Y}_n = y_n] = 0.$$

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- if $(y_1 \cdots y_n) = \pi_S(x_1 \cdots x_m)$,

$$\Pr[\mathbf{X}_1 = x_1, \ldots, \mathbf{X}_m = x_m, \mathbf{Y}_1 = y_1, \ldots, \mathbf{Y}_n = y_n] = 2^m.$$

Linear Approximations of S-boxes

input	0	1	2	3	4	5	6	7
output	E	4	D	1	2	F	B	8

input	8	9	A	B	C	D	E	F
output	3	A	6	C	5	9	0	7

Linear Approximations of S-boxes

input	0	1	2	3	4	5	6	7
output	E	4	D	1	2	F	B	8
input	8	9	A	B	C	D	E	F
output	3	A	6	C	5	9	0	7

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4
0	0	0	0	1	1	1	0
0	0	0	1	0	1	0	0
0	0	1	0	1	1	0	1
0	0	1	1	0	0	0	1
0	1	0	0	0	0	1	0
0	1	0	1	1	1	1	1
0	1	1	0	1	0	1	1
0	1	1	1	1	0	0	0
0	0	0	0	0	0	1	1
0	0	0	1	1	0	1	0
0	0	1	0	0	1	1	0
0	0	1	1	1	1	0	0
0	1	0	0	0	1	0	1
0	1	0	1	1	0	0	1
0	1	1	0	0	0	0	0
0	1	1	1	0	1	1	1

Linear Approximations of S-boxes

- Straightforward to compute the bias of a random variable of the form

$$\mathbf{X}_{i_1} \oplus \mathbf{X}_{i_2} \oplus \cdots \mathbf{X}_{i_u} \oplus \mathbf{Y}_{j_1} \oplus \mathbf{Y}_{j_2} \oplus \cdots \oplus \mathbf{Y}_{j_v}.$$

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- A linear cryptanalytic attack can potentially be mounted when a random variable of this form has a bias that is bounded away from zero.

Linear Approximations of S-boxes

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	$X_2 \oplus X_3$	$Y_1 \oplus Y_3 \oplus Y_4$	$X_1 \oplus X_4$	Y_2	$X_3 \oplus X_4$	$Y_1 \oplus Y_4$
0	0	0	0	1	1	1	0	0	0	0	1	0	1
0	0	0	1	0	1	0	0	0	0	1	1	1	0
0	0	1	0	1	1	0	1	1	0	0	1	1	0
0	0	1	1	0	0	0	1	1	1	1	0	0	1
0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	1	1	1	1	1	1	1	1	1	1	0
0	1	1	0	1	0	1	1	0	1	0	0	1	0
0	1	1	1	1	0	0	0	0	1	1	0	0	1
0	0	0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	1	1	0	1	0	0	0	0	0	1	1
0	0	1	0	0	1	1	0	1	1	1	1	1	0
0	0	1	1	1	1	0	0	1	1	0	1	0	1
0	1	0	0	0	1	0	1	1	1	1	1	0	1
0	1	0	1	1	0	0	1	1	0	0	0	1	0
0	1	1	0	0	0	0	0	0	0	1	0	1	0
0	1	1	1	0	1	1	1	0	0	0	1	0	1

Linear Cryptanalysis

Linear Approximations of S-boxes

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	$X_2 \oplus X_3$	$Y_1 \oplus Y_3 \oplus Y_4$	$X_1 \oplus X_4$	Y_2	$X_3 \oplus X_4$	$Y_1 \oplus Y_4$
0	0	0	0	1	1	1	0	0	0	0	1	0	1
0	0	0	1	0	1	0	0	0	0	1	1	1	0
0	0	1	0	1	1	0	1	1	0	0	1	1	0
0	0	1	1	0	0	0	1	1	1	1	0	0	1
0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	1	1	1	1	1	1	1	1	1	1	0
0	1	1	0	1	0	1	1	0	1	0	0	1	0
0	1	1	1	1	0	0	0	0	1	1	0	0	1
0	0	0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	1	1	0	1	0	0	0	0	0	1	1
0	0	1	0	0	1	1	0	1	1	1	1	1	0
0	0	1	1	1	1	0	0	1	1	0	1	0	1
0	1	0	0	0	1	0	1	1	1	1	1	0	1
0	1	0	1	1	0	0	1	1	0	0	0	1	0
0	1	1	0	0	0	0	0	0	0	1	0	1	0
0	1	1	1	0	1	1	1	0	0	0	1	0	1

- $\Pr[\mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{Y}_1 \oplus \mathbf{Y}_3 \oplus \mathbf{Y}_4 = 0] = \frac{3}{4}$

Linear Approximations of S-boxes

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	$X_2 \oplus X_3$	$Y_1 \oplus Y_3 \oplus Y_4$	$X_1 \oplus X_4$	Y_2	$X_3 \oplus X_4$	$Y_1 \oplus Y_4$
0	0	0	0	1	1	1	0	0	0	0	1	0	1
0	0	0	1	0	1	0	0	0	0	1	1	1	0
0	0	1	0	1	1	0	1	1	0	0	1	1	0
0	0	1	1	0	0	0	1	1	1	1	0	0	1
0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	1	1	1	1	1	1	1	1	1	1	0
0	1	1	0	1	0	1	1	0	1	0	0	1	0
0	1	1	1	1	0	0	0	0	1	1	0	0	1
0	0	0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	1	1	0	1	0	0	0	0	0	1	1
0	0	1	0	0	1	1	0	1	1	1	1	1	0
0	0	1	1	1	1	0	0	1	1	0	1	0	1
0	1	0	0	0	1	0	1	1	1	1	1	0	1
0	1	0	1	1	0	0	1	1	0	0	0	1	0
0	1	1	0	0	0	0	0	0	0	1	0	1	0
0	1	1	1	0	1	1	1	0	0	0	1	0	1

- $\Pr[\mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{Y}_1 \oplus \mathbf{Y}_3 \oplus \mathbf{Y}_4 = 0] = \frac{3}{4}$
- $\Pr[\mathbf{X}_1 \oplus \mathbf{X}_4 \oplus \mathbf{Y}_2 = 0] = \frac{1}{2}$

Linear Approximations of S-boxes

X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Y_4	$X_2 \oplus X_3$	$Y_1 \oplus Y_3 \oplus Y_4$	$X_1 \oplus X_4$	Y_2	$X_3 \oplus X_4$	$Y_1 \oplus Y_4$
0	0	0	0	1	1	1	0	0	0	0	1	0	1
0	0	0	1	0	1	0	0	0	0	1	1	1	0
0	0	1	0	1	1	0	1	1	0	0	1	1	0
0	0	1	1	0	0	0	1	1	1	1	0	0	1
0	1	0	0	0	0	1	0	1	1	0	0	0	0
0	1	0	1	1	1	1	1	1	1	1	1	1	0
0	1	1	0	1	0	1	1	0	1	0	0	1	0
0	1	1	1	1	0	0	0	0	1	1	0	0	1
0	0	0	0	0	0	1	1	0	0	1	0	0	1
0	0	0	1	1	0	1	0	0	0	0	0	1	1
0	0	1	0	0	1	1	0	1	1	1	1	1	0
0	0	1	1	1	1	0	0	1	1	0	1	0	1
0	1	0	0	0	1	0	1	1	1	1	1	0	1
0	1	0	1	1	0	0	1	1	0	0	0	1	0
0	1	1	0	0	0	0	0	0	0	1	0	1	0
0	1	1	1	0	1	1	1	0	0	0	1	0	1

- $\Pr[\mathbf{X}_2 \oplus \mathbf{X}_3 \oplus \mathbf{Y}_1 \oplus \mathbf{Y}_3 \oplus \mathbf{Y}_4 = 0] = \frac{3}{4}$
- $\Pr[\mathbf{X}_1 \oplus \mathbf{X}_4 \oplus \mathbf{Y}_2 = 0] = \frac{1}{2}$
- $\Pr[\mathbf{X}_3 \oplus \mathbf{X}_4 \oplus \mathbf{Y}_1 \oplus \mathbf{Y}_4 = 0] = \frac{1}{8}$

Linear Approximations of S-boxes

- Any random variable can be written as

$$\left(\bigoplus_{i=1}^{m} a_i \mathbf{X}_i\right) \oplus \left(\bigoplus_{j=1}^{n} b_j \mathbf{Y}_j\right).$$

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- Input sum a is the binary vector $(a_1, a_2, \ldots, a_m)$.

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- Input sum a is the binary vector $(a_1, a_2, \ldots, a_m)$.
- Output sum b is the binary vector $(b_1, b_2, \ldots, b_n)$.

Linear Cryptanalysis

Linear Approximations Table of S-boxes $N_L(a,b)$

a \ b	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	+8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-2	-2	0	0	-2	+6	+2	+2	0	0	+2	+2	0	0
2	0	0	-2	-2	0	0	-2	-2	0	0	+2	+2	0	0	-6	+2
3	0	0	0	0	0	0	0	0	+2	-6	-2	-2	+2	+2	-2	-2
4	0	+2	0	-2	-2	-4	-2	0	0	-2	0	+2	+2	-4	+2	0
5	0	-2	-2	0	-2	0	+4	+2	-2	0	-4	+2	0	-2	-2	0
6	0	+2	-2	+4	+2	0	0	+2	0	-2	+2	+4	-2	0	0	-2
7	0	-2	0	+2	+2	-4	+2	0	-2	0	+2	0	+4	+2	0	+2
8	0	0	0	0	0	0	0	0	-2	+2	+2	-2	+2	-2	-2	-6
9	0	0	-2	-2	0	0	-2	-2	-4	0	-2	+2	0	+4	+2	-2
A	0	+4	-2	+2	-4	0	+2	-2	+2	+2	0	0	+2	+2	0	0
B	0	+4	0	-4	+4	0	+4	0	0	0	0	0	0	0	0	0
C	0	-2	+4	-2	-2	0	+2	0	+2	0	+2	+4	0	+2	0	-2
D	0	+2	+2	0	-2	+4	0	+2	-4	-2	+2	0	+2	0	0	+2
E	0	+2	+2	0	-2	-4	0	+2	-2	0	0	-2	-4	+2	-2	0
F	0	-2	-4	-2	-2	0	+2	0	0	-2	+4	-2	-2	0	+2	0

Linear Approximations Table of S-boxes $N_L(a,b)$

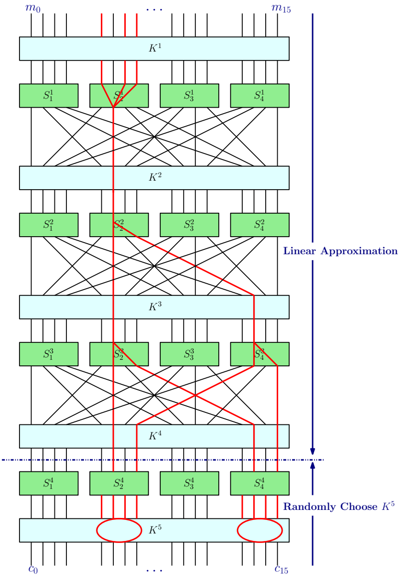
	b															
a	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	+8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	-2	-2	0	0	-2	+6	+2	+2	0	0	+2	+2	0	0
2	0	0	-2	-2	0	0	-2	-2	0	0	+2	+2	0	0	-6	+2
3	0	0	0	0	0	0	0	0	+2	-6	-2	-2	+2	+2	-2	-2
4	0	+2	0	-2	-2	-4	-2	0	0	-2	0	+2	+2	-4	+2	0
5	0	-2	-2	0	-2	0	+4	+2	-2	0	-4	+2	0	-2	-2	0
6	0	+2	-2	+4	+2	0	0	+2	0	-2	+2	+4	-2	0	0	-2
7	0	-2	0	+2	+2	-4	+2	0	-2	0	+2	0	+4	+2	0	+2
8	0	0	0	0	0	0	0	0	-2	+2	+2	-2	+2	-2	-2	-6
9	0	0	-2	-2	0	0	-2	-2	-4	0	-2	+2	0	+4	+2	-2
A	0	+4	-2	+2	-4	0	+2	-2	+2	+2	0	0	+2	+2	0	0
B	0	+4	0	-4	+4	0	+4	0	0	0	0	0	0	0	0	0
C	0	-2	+4	-2	-2	0	+2	0	+2	0	+2	+4	0	+2	0	-2
D	0	+2	+2	0	-2	+4	0	+2	-4	-2	+2	0	+2	0	0	+2
E	0	+2	+2	0	-2	-4	0	+2	-2	0	0	-2	-4	+2	-2	0
F	0	-2	-4	-2	-2	0	+2	0	0	-2	+4	-2	-2	0	+2	0

$$\Pr\left[\left(\bigoplus_{i=1}^{m} a_i \mathbf{X}_i\right) \oplus \left(\bigoplus_{j=1}^{n} b_j \mathbf{Y}_j\right)\right] = \frac{8 + N_L(a,b)}{16}.$$

A Linear Attack on SPN

- $\mathbf{U}^i$: The input to the i^{th} round S-box.
- $\mathbf{U}^i_j$: The j^{th} bit of block $\mathbf{U}^{(i)}$.
- $\mathbf{V}^i$: The output of the i^{th} round S-box.
- $\mathbf{V}^i_j$: The j^{th} bit of block $\mathbf{V}^{(i)}$.
- $\mathbf{K}^i$: The i^{th} round key.

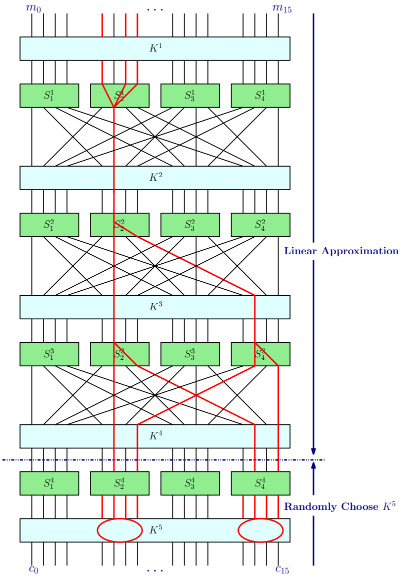
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

- S_2^1: $\mathbf{T}_1 = \mathbf{U}_4^1 \oplus \mathbf{U}_6^1 \oplus \mathbf{U}_7^1 \oplus \mathbf{V}_5^1$, bias = $\frac{1}{4}$

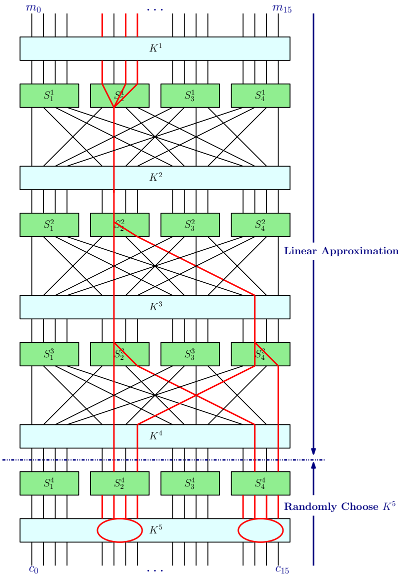
Linear Cryptanalysis



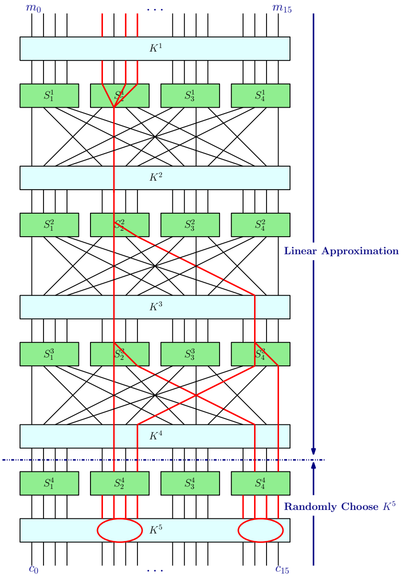
$\ell = 4$, $l = 4$, and $m = 4$

- S_2^1: $\mathbf{T}_1 = \mathbf{U}_4^1 \oplus \mathbf{U}_6^1 \oplus \mathbf{U}_7^1 \oplus \mathbf{V}_5^1$, bias = $\frac{1}{4}$
- S_2^2: $\mathbf{T}_2 = \mathbf{U}_5^2 \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$, bias = $-\frac{1}{4}$

Linear Cryptanalysis



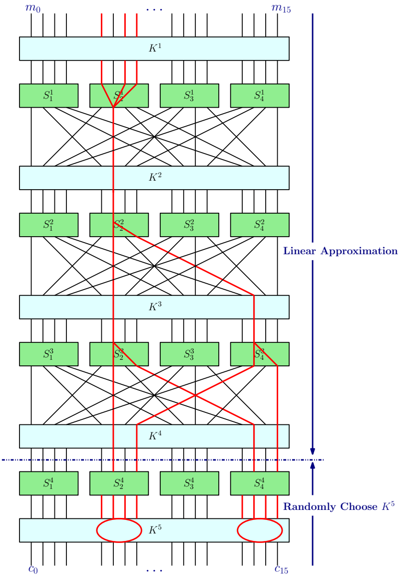
- S_2^1: $\mathbf{T}_1 = \mathbf{U}_4^1 \oplus \mathbf{U}_6^1 \oplus \mathbf{U}_7^1 \oplus \mathbf{V}_5^1$, bias = $\frac{1}{4}$
- S_2^2: $\mathbf{T}_2 = \mathbf{U}_5^2 \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$, bias = $-\frac{1}{4}$
- S_2^3: $\mathbf{T}_3 = \mathbf{U}_5^3 \oplus \mathbf{V}_5^3 \oplus \mathbf{V}_7^3$, bias = $-\frac{1}{4}$



$\ell = 4$, $l = 4$, and $m = 4$

- S_2^1: $\mathbf{T}_1 = \mathbf{U}_4^1 \oplus \mathbf{U}_6^1 \oplus \mathbf{U}_7^1 \oplus \mathbf{V}_5^1$, bias = $\frac{1}{4}$
- S_2^2: $\mathbf{T}_2 = \mathbf{U}_5^2 \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$, bias = $-\frac{1}{4}$
- S_2^3: $\mathbf{T}_3 = \mathbf{U}_5^3 \oplus \mathbf{V}_5^3 \oplus \mathbf{V}_7^3$, bias = $-\frac{1}{4}$
- S_4^3: $\mathbf{T}_4 = \mathbf{U}_{13}^3 \oplus \mathbf{V}_{13}^3 \oplus \mathbf{V}_{15}^3$, bias = $-\frac{1}{4}$

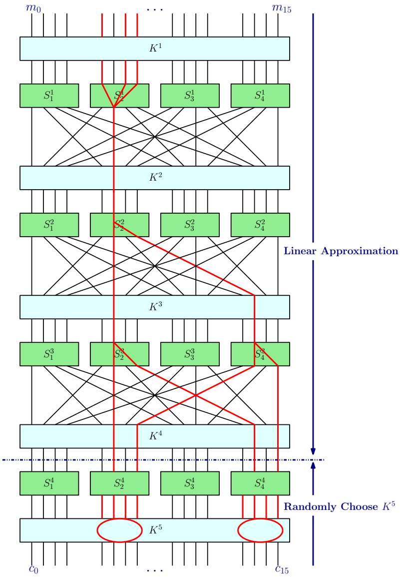
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

- $\mathbf{T}_1 = \left(m_4 \oplus \mathbf{K}^1_4\right) \oplus \left(m_6 \oplus \mathbf{K}^1_6\right) \oplus \left(m_7 \oplus \mathbf{K}^1_7\right) \oplus \mathbf{V}^1_5$

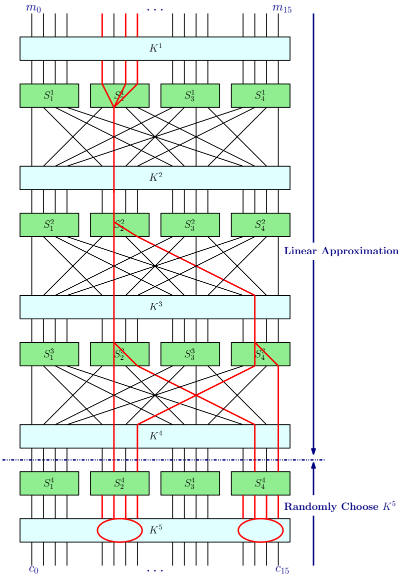
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

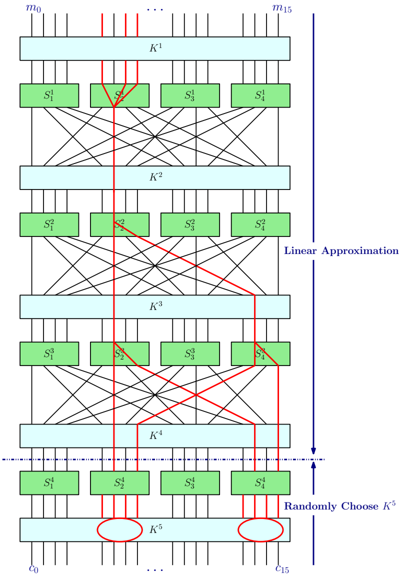
- $\mathbf{T}_1 = \left(m_4 \oplus \mathbf{K}_4^1\right) \oplus \left(m_6 \oplus \mathbf{K}_6^1\right) \oplus \left(m_7 \oplus \mathbf{K}_7^1\right) \oplus \mathbf{V}_5^1$
- $\mathbf{T}_2 = \left(\mathbf{V}_5^1 \oplus \mathbf{K}_5^2\right) \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$

Linear Cryptanalysis



- $\mathbf{T}_1 = \left(m_4 \oplus \mathbf{K}_4^1\right) \oplus \left(m_6 \oplus \mathbf{K}_6^1\right) \oplus \left(m_7 \oplus \mathbf{K}_7^1\right) \oplus \mathbf{V}_5^1$
- $\mathbf{T}_2 = \left(\mathbf{V}_5^1 \oplus \mathbf{K}_5^2\right) \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$
- $\mathbf{T}_3 = \left(\mathbf{V}_5^2 \oplus \mathbf{K}_5^3\right) \oplus \mathbf{V}_5^3 \oplus \mathbf{V}_7^3$

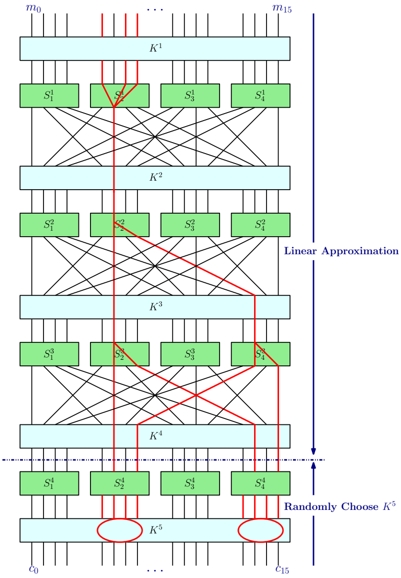
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

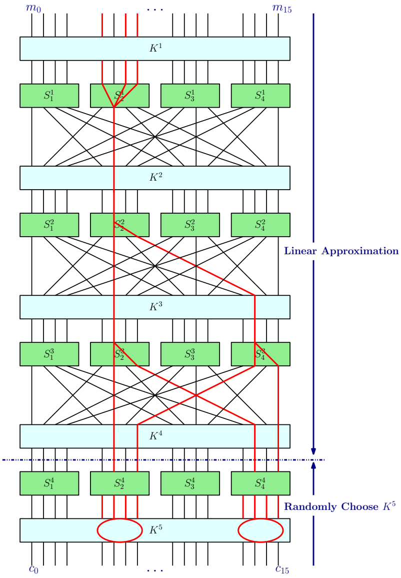
- $\mathbf{T}_1 = \left(m_4 \oplus \mathbf{K}_4^1\right) \oplus \left(m_6 \oplus \mathbf{K}_6^1\right) \oplus \left(m_7 \oplus \mathbf{K}_7^1\right) \oplus \mathbf{V}_5^1$
- $\mathbf{T}_2 = \left(\mathbf{V}_5^1 \oplus \mathbf{K}_5^2\right) \oplus \mathbf{V}_5^2 \oplus \mathbf{V}_7^2$
- $\mathbf{T}_3 = \left(\mathbf{V}_5^2 \oplus \mathbf{K}_5^3\right) \oplus \mathbf{V}_5^3 \oplus \mathbf{V}_7^3$
- $\mathbf{T}_4 = \left(\mathbf{V}_7^2 \oplus \mathbf{K}_{13}^3\right) \oplus \mathbf{V}_{13}^3 \oplus \mathbf{V}_{15}^3$

Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

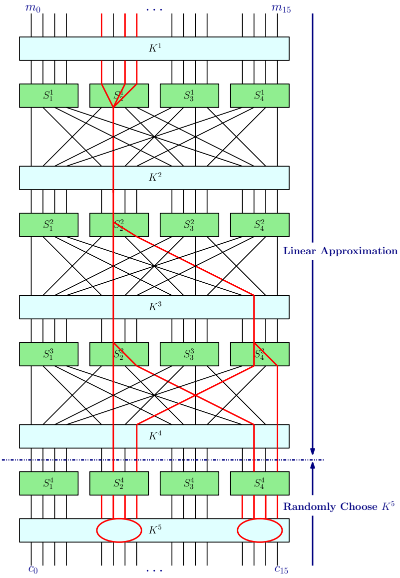
$$\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$$



$\ell = 4$, $l = 4$, and $m = 4$

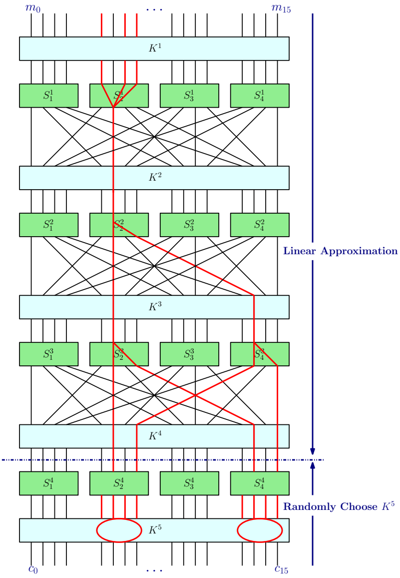
$$
\begin{aligned}
\mathbf{T} &= \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \mathbf{V}_5^3 \oplus \mathbf{V}_7^3 \oplus \mathbf{V}_{13}^3 \oplus \mathbf{V}_{15}^3 \\
&\quad \oplus \mathbf{K}_4^1 \oplus \mathbf{K}_6^1 \oplus \mathbf{K}_7^1 \\
&\quad \oplus \mathbf{K}_5^2 \oplus \mathbf{K}_5^3 \oplus \mathbf{K}_{13}^3
\end{aligned}
$$

Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

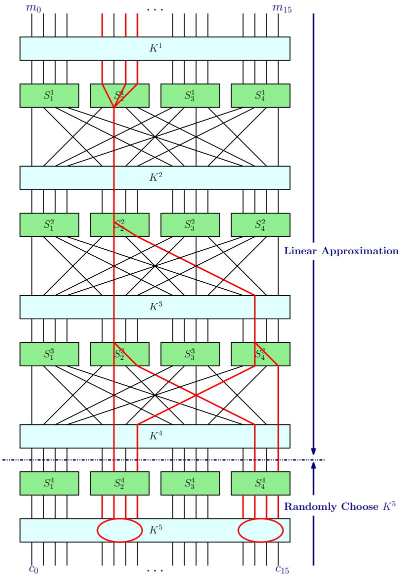
$$\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$$
$$= m_4 \oplus m_6 \oplus m_7$$



$\ell = 4$, $l = 4$, and $m = 4$

$$
\begin{aligned}
\mathbf{T} &= \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \left(\mathbf{U}_5^4 \oplus \mathbf{K}_5^4\right) \oplus \left(\mathbf{U}_7^4 \oplus \mathbf{K}_7^4\right) \\
&\quad \oplus \left(\mathbf{U}_{13}^4 \oplus \mathbf{K}_{13}^4\right) \oplus \left(\mathbf{U}_{15}^4 \oplus \mathbf{K}_{15}^4\right)
\end{aligned}
$$

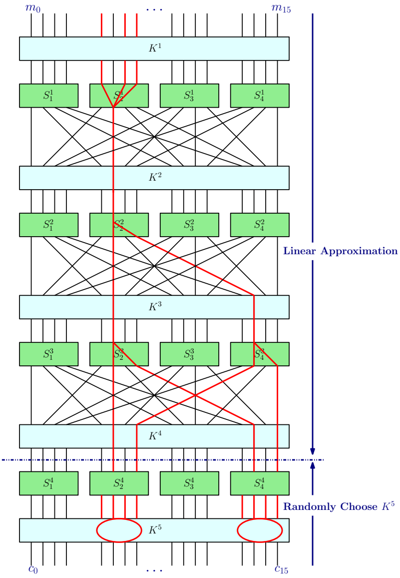
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

$$
\begin{aligned}
\mathbf{T} &= \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\oplus \left(\mathbf{U}^4_5 \oplus \mathbf{K}^4_5\right) \oplus \left(\mathbf{U}^4_7 \oplus \mathbf{K}^4_7\right) \\
&\oplus \left(\mathbf{U}^4_{13} \oplus \mathbf{K}^4_{13}\right) \oplus \left(\mathbf{U}^4_{15} \oplus \mathbf{K}^4_{15}\right) \\
&\oplus \mathbf{K}^1_4 \oplus \mathbf{K}^1_6 \oplus \mathbf{K}^1_7 \\
&\oplus \mathbf{K}^2_5 \oplus \mathbf{K}^3_5 \oplus \mathbf{K}^3_{13}
\end{aligned}
$$

Linear Cryptanalysis



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$$\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4$$
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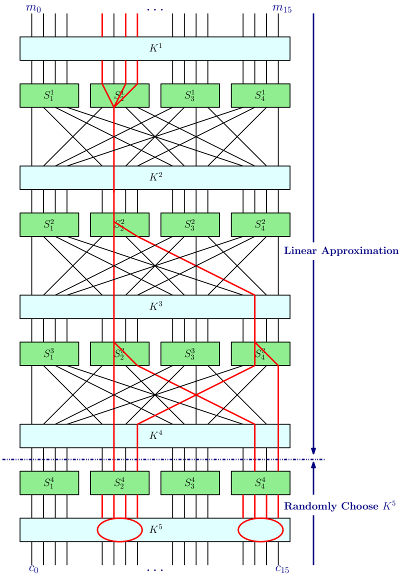
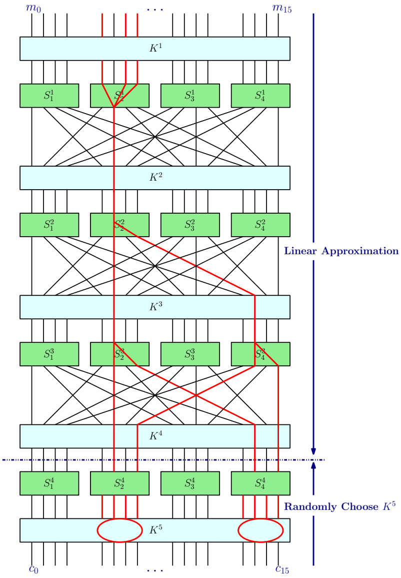
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

$$\begin{aligned}
\mathbf{T} &= \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \mathbf{U}_5^4 \oplus \mathbf{U}_7^4 \oplus \mathbf{U}_{13}^4 \oplus \mathbf{U}_{15}^4
\end{aligned}$$

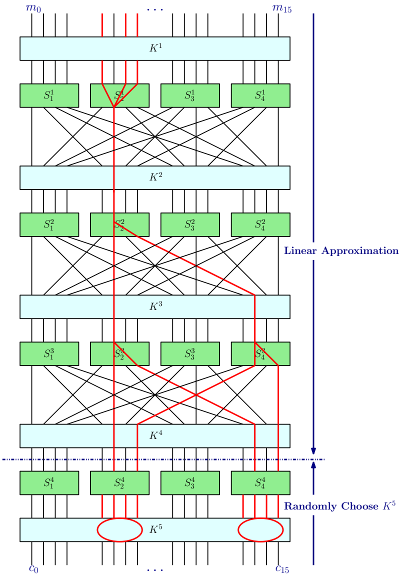
Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

$$
\begin{aligned}
\mathbf{T} &= \mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \mathbf{T}_3 \oplus \mathbf{T}_4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \mathbf{U}_5^4 \oplus \mathbf{U}_7^4 \oplus \mathbf{U}_{13}^4 \oplus \mathbf{U}_{15}^4 \\
&\quad \oplus \mathbf{K}_4^1 \oplus \mathbf{K}_6^1 \oplus \mathbf{K}_7^1 \\
&\quad \oplus \mathbf{K}_5^2 \oplus \mathbf{K}_5^3 \oplus \mathbf{K}_{13}^3 \\
&\quad \oplus \mathbf{K}_5^4 \oplus \mathbf{K}_7^4 \oplus \mathbf{K}_{13}^4 \oplus \mathbf{K}_{15}^4
\end{aligned}
$$

Linear Cryptanalysis



$\ell = 4$, $l = 4$, and $m = 4$

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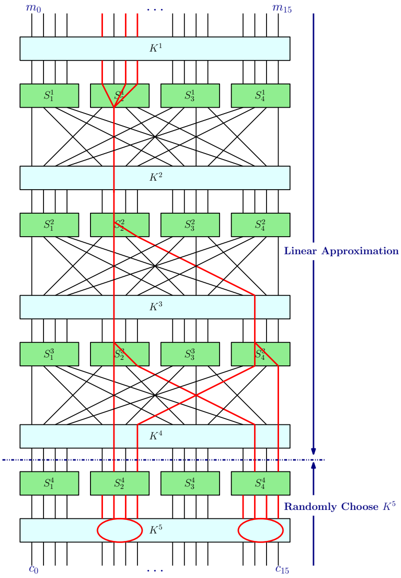
&\quad \oplus \mathbf{K}_4^1 \oplus \mathbf{K}_6^1 \oplus \mathbf{K}_7^1 \\
&\quad \oplus \mathbf{K}_5^2 \oplus \mathbf{K}_5^3 \oplus \mathbf{K}_{13}^3 \\
&\quad \oplus \mathbf{K}_5^4 \oplus \mathbf{K}_7^4 \oplus \mathbf{K}_{13}^4 \oplus \mathbf{K}_{15}^4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \mathbf{U}_5^4 \oplus \mathbf{U}_7^4 \oplus \mathbf{U}_{13}^4 \oplus \mathbf{U}_{15}^4
\end{aligned}
$$

Linear Cryptanalysis



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&\quad \oplus \mathbf{K}_4^1 \oplus \mathbf{K}_6^1 \oplus \mathbf{K}_7^1 \\
&\quad \oplus \mathbf{K}_5^2 \oplus \mathbf{K}_5^3 \oplus \mathbf{K}_{13}^3 \\
&\quad \oplus \mathbf{K}_5^4 \oplus \mathbf{K}_7^4 \oplus \mathbf{K}_{13}^4 \oplus \mathbf{K}_{15}^4 \\
&= m_4 \oplus m_6 \oplus m_7 \\
&\quad \oplus \mathbf{U}_5^4 \oplus \mathbf{U}_7^4 \oplus \mathbf{U}_{13}^4 \oplus \mathbf{U}_{15}^4 \\
&\quad \oplus \Sigma_K
\end{aligned}
$$

Bias of bT

- Bias of $\mathbf{T} = 2^{4-1} \times \frac{1}{4} \times (-\frac{1}{4})^2 = -\frac{1}{32}$

Bias of bT

- Bias of $\mathbf{T} = 2^{4-1} \times \frac{1}{4} \times (-\frac{1}{4})^2 = -\frac{1}{32}$
- Assume that all sub-key bits are fixed.
- $\Sigma_K = 0$ or 1.

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- Therefore, bias of $\mathbf{T}$ actually $\pm\frac{1}{32}$.

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- We assume that $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ are independent random variable.

Bias of bT

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- Assume that all sub-key bits are fixed.
- $\Sigma_K = 0$ or 1.
- Therefore, bias of $\mathbf{T}$ actually $\pm\frac{1}{32}$.
- We assume that $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ are independent random variable.
- But it reality they are not.

Bias of bT

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- Assume that all sub-key bits are fixed.
- $\Sigma_K = 0$ or 1.
- Therefore, bias of $\mathbf{T}$ actually $\pm\frac{1}{32}$.
- We assume that $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ are independent random variable.
- But it reality they are not.
- This approximation actually works in practice.

Attack

Input: A list $\mathcal{T}$ that contains T (m, c) pairs.

1. for $(L_1, L_2) \longleftarrow (0, 0)$ to (F, F) do
2. $\quad C[L_1, L_2] \longleftarrow 0$

Attack

Input: A list $\mathcal{T}$ that contains T (m,c) pairs.

1. for $(L_1, L_2) \longleftarrow (0,0)$ to (F,F) do
2. $\quad C[L_1, L_2] \longleftarrow 0$
3. for each $(m,c) \in \mathcal{T}$ do
4. $\quad v^4_{\langle 2 \rangle} \longleftarrow L_1 \oplus c_{\langle 2 \rangle}$
5. $\quad v^4_{\langle 4 \rangle} \longleftarrow L_2 \oplus c_{\langle 4 \rangle}$

Attack

Input: A list $\mathcal{T}$ that contains T (m,c) pairs.

1. for $(L_1, L_2) \longleftarrow (0,0)$ to (F,F) do
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3. for each $(m,c) \in \mathcal{T}$ do
4. $\quad v^4_{\langle 2 \rangle} \longleftarrow L_1 \oplus c_{\langle 2 \rangle}$
5. $\quad v^4_{\langle 4 \rangle} \longleftarrow L_2 \oplus c_{\langle 4 \rangle}$
6. $\quad u^4_{\langle 2 \rangle} \longleftarrow \pi_S^{-1}\left(v^4_{\langle 2 \rangle}\right)$
7. $\quad u^4_{\langle 4 \rangle} \longleftarrow \pi_S^{-1}\left(v^4_{\langle 4 \rangle}\right)$

Attack

Input: A list $\mathcal{T}$ that contains T (m,c) pairs.

1. for $(L_1,L_2) \longleftarrow (0,0)$ to (F,F) do
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6. $\quad u^4_{\langle 2\rangle} \longleftarrow \pi_S^{-1}\left(v^4_{\langle 2\rangle}\right)$
7. $\quad u^4_{\langle 4\rangle} \longleftarrow \pi_S^{-1}\left(v^4_{\langle 4\rangle}\right)$
8. $\quad z \longleftarrow m_4 \oplus m_6 \oplus m_7 \oplus u^4_5 \oplus u^4_7 \oplus u^4_{13} \oplus u^4_{15}$

Attack

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1. for $(L_1, L_2) \longleftarrow (0,0)$ to (F, F) do
2. $\quad C[L_1, L_2] \longleftarrow 0$
3. for each $(m, c) \in \mathcal{T}$ do
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12. Let for (l_1, l_2), $C[l_1, l_2]$ has maximum. Then return (l_1, l_2).

End