Exercise

1. Given $\mathbb{F}_2 = \{0, 1\}$ we have it to be a field because $a + 0 = 0 + a = 1 \cdot a = a \cdot 1 = a$ for any $a \in \mathbb{F}_2$, which satisfies both additive and multiplicative identities, and $a + a = a + a = 0$; $1 \cdot 1 = 1$, where $1 \neq 0$, which satisfies both additive and multiplicative inverses.

2. a) Given $g(x,y) = x^2y + y^2x \in \mathbb{F}_2[x,y]$. Then
$$g(x,y) = xy(x+y),$$
since $(1,1) \in \mathbb{F}_2^2$, we have
$$\begin{aligned} xy(x+y) &= 1 \cdot 1(1+1) \\ &= 1 \cdot 1 \cdot 0 = 1 \cdot 0 = 0, \end{aligned}$$
which is a nonzero polynomial. Hence our proposition wasn't violated since $\mathbb{F}_2$ is a finite field.

b) Consider $h(x,y,z) = xyz(y+z) \in \mathbb{F}_2[x,y,z]$. Then for any $(x,y,z) \in \mathbb{F}_2^3$ we have $h(x,y,z) = 0$.

c) Consider the polynomial
$$f(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n (dx_1 + x_2 + \cdots + x_n) \in \mathbb{F}_2[x_1, \dots, x_n],$$
where $d = 1$ if n is even, otherwise $d = 0$. Then for any $(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n$ we have $f(x_1, x_2, \dots, x_n) = 0$.

5. Given that
$$f(x,y,z) = x^5y^2z - x^4y^3 + y^5 + x^2z - y^3z + xy + 2x - 5z + 3.$$
Then

a) we have $(-5z + 3 - y^3z + y^5)x^0 + (2 + y)x + z \cdot x^2 + 0 \cdot x^3 + (-y)^3 x^4 + (y^2z)x^5$.

b) we have $(3 - 5z + 2x + x^2z)y^0 + xy + (x^5z)y^2 + (-x^4 - z)y^3 + 0y^4 + y^5$.

c) we have $(3 + 2x + y^5 - x^4y^3 + xy)z^0 + (x^2 - y^3 - 5 + x^5y^2)z$.

6.

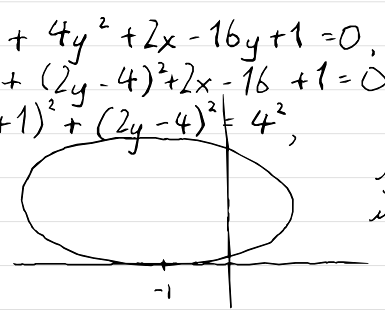
(a) *Proof.* We prove by induction on n. Suppose that $n = 1$ where $f \in \mathbb{C}[x_1]$ vanishes at every point of $\mathbb{Z}$. Then we have at most $\deg(f)$ distinct roots. Thus since $\mathbb{C}$ is infinite, we have f to have infinitely many roots, so f is a zero polynomial. Now suppose that $n > 1$, and $f \in \mathbb{C}[x_1, \dots, x_{n-1}]$ vanishes at every point of $\mathbb{Z}^{n-1}$ implies f is a zero polynomial. Next consider $f \in \mathbb{C}[x_1, \dots, x_n]$ to vanish at every point of $\mathbb{Z}^n$, and that
$$f = \sum_{i=0}^{N} g_i(x_1, \dots, x_{n-1}) x_n^i$$
where $g_i \in \mathbb{C}[x_1, \dots, x_{n-1}]$ for all i. Now consider an arbitrary point $(a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$. Then we have
$$f(a_1, \dots, a_{n-1}, x_n) = \sum_{i=0}^{N} g_i(a_1, \dots, a_{n-1}) x_n^i.$$
Thus by our assumption on f, we have f to vanishes at every $a_n \in \mathbb{C}$. Hence it follows from our base case $n = 1$ that $f(a_1, \dots, a_{n-1}, x_n)$ is a zero polynomial in $\mathbb{C}[x_n]$. Hence it follows $g_i(a_1, \dots, a_{n-1}) = 0$ for all i. Thus since $(a_1, \dots, a_{n-1}) \in \mathbb{Z}^{n-1}$ is arbitrary, we have $g_i \in \mathbb{C}[x_1, \dots, x_{n-1}]$ to vanish at all point of $\mathbb{Z}^{n-1}$ for each i. Thus from the inductive hypothesis of $n-1$ we have g_i to be a zero polynomial for each i. Therefore this meant f has to be a zero polynomial. ∎

b) *Proof.* From (a) note that no property relating of $\mathbb{Z}^n$ being infinite was used but rather having more roots than degree, so f is a zero polynomial anyway. ∎

Exercise

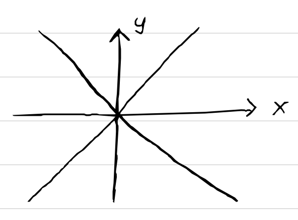
1.

a. $x^2 + 4y^2 + 2x - 16y + 1 = 0$,
$x^2 + (2y-4)^2 + 2x - 16 + 1 = 0$,
$(x+1)^2 + (2y-4)^2 = 4^2$,



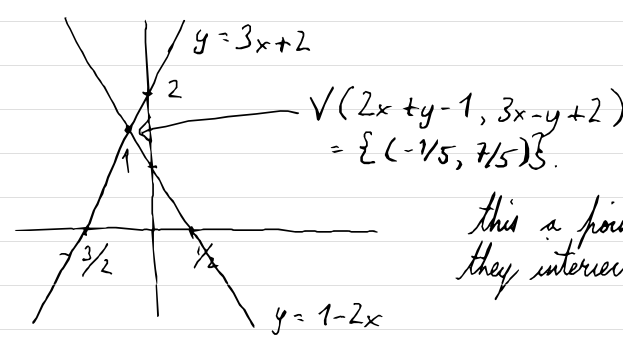
this is 1D since it's an ellipse - a line.

b. Note that $x^2 - y^2 = 0$, so
$x^2 = y^2$,
so $y = \pm x$. Thus



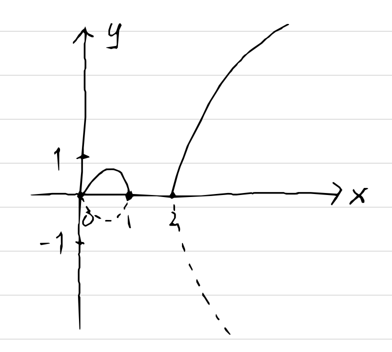
this is also 1D since we have two lines with a singular point.

c. Note that $V(2x+y-1, 3x-y+2)$
$= V(2x+y-1) \cap V(3x-y+2)$.
Then this gives us $y = 1-2x$, $y = 3x+2$, so
$3x+2 = 1-2x$,
$5x = -1$,
$x = -1/5$, so $y = 1 - 2(-1/5) = 1 + 2/5 = 7/5$



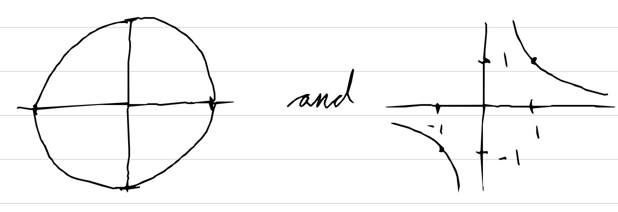
this a point since they intersect, so 0D.

2. Since
$V(y^2 - x(x-1)(x-2))$, we
have $y^2 - x(x-1)(x-2) = 0$,
so $y^2 = x(x-1)(x-2)$.
Thus we have $y = \pm\sqrt{x(x-1)(x-2)}$.
Note that $1 < x < 2$ doesn't exists since $y \notin \mathbb{C}$,
so



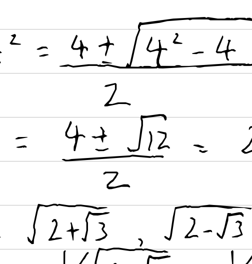
Thus it has reflectional symmetry at the x-axis.

3. We sketch to show
$V(x^2+y^2-4) \cap V(xy-1) = V(x^2+y^2-4, xy-1)$.
Note we have
$x^2 + y^2 = 2^2$ and $y = 1/x$



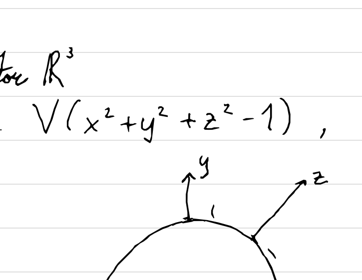
so we have $x^2 + (1/x)^2 = 2^2$,
$x^4 + 1 - 4x^2 = 0$,
$(x^2)^2 - 4(x^2) + 1 = 0$,
gives us
$$x^2 = \frac{4 \pm \sqrt{4^2 - 4}}{2},$$
$$= \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}, \quad x = \pm\sqrt{2 \pm \sqrt{3}}$$
$x = \sqrt{2+\sqrt{3}}, \sqrt{2-\sqrt{3}}, -\sqrt{2+\sqrt{3}}, -\sqrt{2-\sqrt{3}}$,
$y = 1/\sqrt{2+\sqrt{3}}, 1/\sqrt{2-\sqrt{3}}, -1/\sqrt{2+\sqrt{3}}, -1/\sqrt{2-\sqrt{3}}$,

so we have 4 intersection points which are
$(\sqrt{2+\sqrt{3}}, 1/\sqrt{2+\sqrt{3}})$, $(\sqrt{2-\sqrt{3}}, 1/\sqrt{2-\sqrt{3}})$, $(-\sqrt{2+\sqrt{3}}, -1/\sqrt{2+\sqrt{3}})$,
$(-\sqrt{2-\sqrt{3}}, -1/\sqrt{2-\sqrt{3}})$. Thus giving us



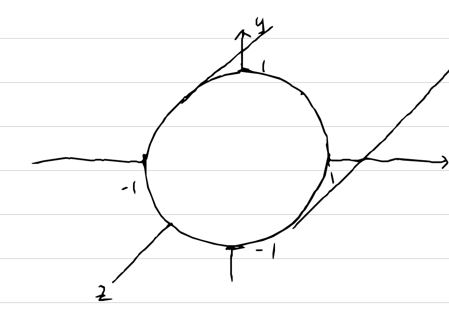
4. For $\mathbb{R}^3$

a. $V(x^2+y^2+z^2-1)$, so $x^2+y^2+z^2 = 1^2$



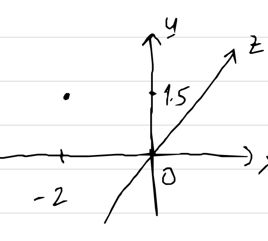
this is a sphere, so it's a surface which 2D.

b. $V(x^2+y^2-1)$



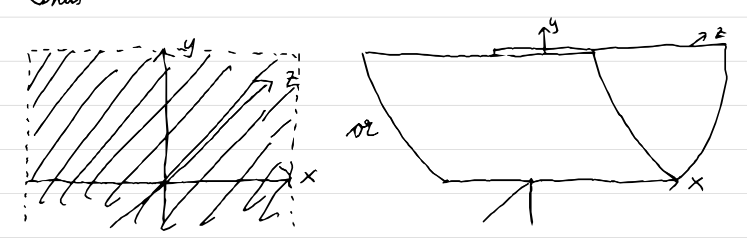
this is a cylinder, so it's a surface which 2D.

c. $V(x+2, y-1.5, z) = \{(-2, 1.5, 0)\}$, so



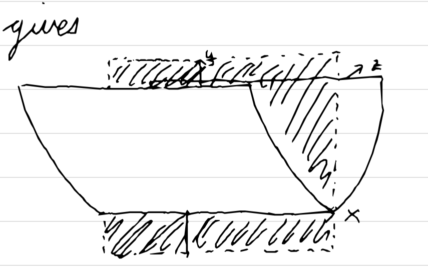
this is a point, so it's 0D.

d. $V(xz^2 - xy) = V(x(z^2-y))$
$= V(x) \cup V(z^2-y)$ by lemma 2.
Thus



,

which gives

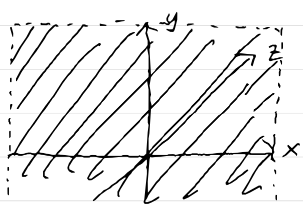


which are surfaces, so 2D.

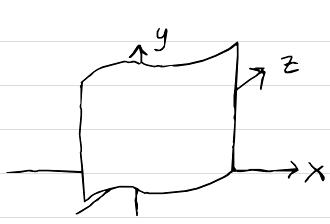
e. $V(x^4 - zx, x^3 - yx) = V(x) \cup V(x^3 - z, x^2 - y)$
$= V(x) \cup (V(x^3 - z) \cap V(x^2 - y))$.

Thus we have

$x^3 = z$



or,



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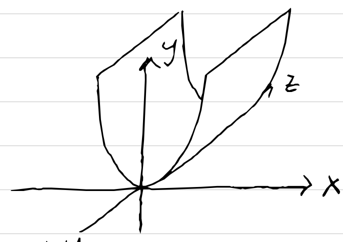
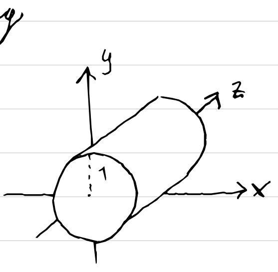


Thus we have either a surface or a line, so either 2D or 1D.

f. $V(x^2 + y^2 - 1, x^2 + y^2 + (z-1)^2 - 1)$
$= V(x^2 + y^2 - 1) \cap V(x^2 + y^2 + (z-1)^2 - 1)$.
Thus we have $x^2 + y^2 = 1^2$, and $x^2 + y^2 + (z-1)^2 = 1^3$.
Hence giving



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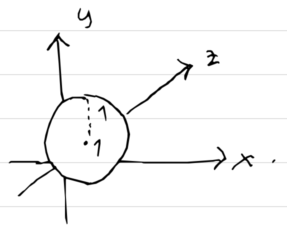
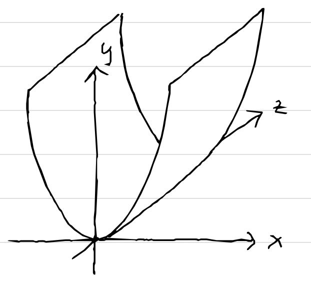


Thus giving us a circle, since we intersect their outlines, so it's 1D.

5. We have
$V((x-2)(x^2 - y), y(x^2 - y), (z+1)(x^2 - y))$
$= V(x^2 - y) \cup V(x - 2, y, z + 1)$
$= V(x^2 - y) \cup (V(x-2) \cap V(y) \cap V(z+1)) \in \mathbb{R}^3$.

Thus



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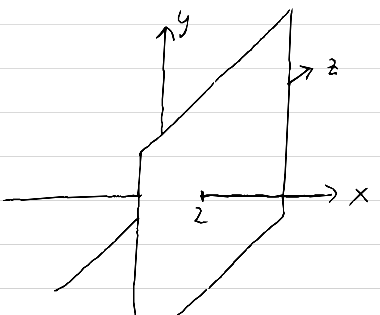

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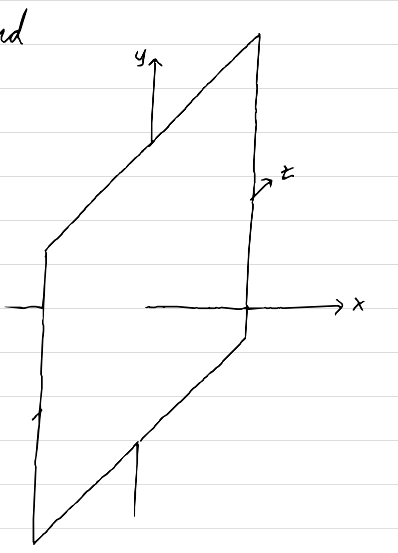

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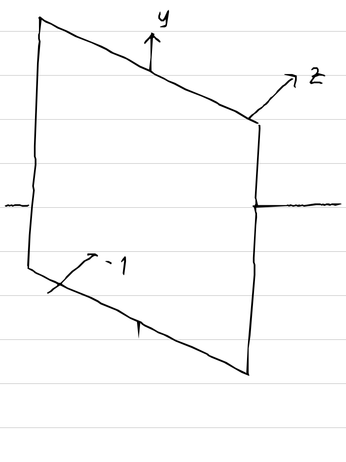


Hence it follows we have a point $(2, 0, -1)$. Thus we have either surface or a point, so 2D or 1D.

6. (a) *Proof.* Consider an arbitrary point $(a_1, \dots, a_n) \in k^n$. Let f_i be a set of polynomials in $k[x_1, \dots, x_n]$ where $f_i = x_i - a_i$ and $1 \le i \le n$. Then we have an affine variety $V(f_1, \dots, f_n) = \{(a_1, \dots, a_n)\}$. Thus since $V(f_1, \dots, f_n) \subseteq k^n$, the point $(a_1, \dots, a_n) \in k^n$ is an affine variety. ∎

(b) *Proof.* From (a) consider affine points of $(a_1, \dots, a_n)$, $(b_1, \dots, b_n) \in k^n$. Then we have an affine variety, where $V = \{(a_1, \dots, a_n)\}$ and $W = \{(b_1, \dots, b_n)\}$. Hence by lemma 2 we have $V \cap W$, $V \cup W \subseteq k^n$ to also be affine varieties. ∎

7. (a) Given $r = \sin\theta = 2\sin\theta\cos\theta$, we have

$$0 = r - 2\sin\theta\cos\theta,$$
$$0 = (r - 2\sin\theta\cos\theta)(r + 2\sin\theta\cos\theta)$$
$$= r^2 - 4\sin^2\theta\cos^2\theta,$$
$$0 = r^4(r^2 - 4\sin^2\theta\cos^2\theta)$$
$$= r^6 - 4r^4\sin^2\theta\cos^2\theta$$
$$= ((r\cos\theta)^2 + (r\sin\theta)^2)^3 - 4(r\cos\theta)^2(r\sin\theta)^2$$
$$= (x^2 + y^2)^3 - 4x^2y^2.$$

Hence four-leaved rose is contained under our affine variety.

(b) We have

$$0 = (x^2 + y^2)^3 - 4x^2y^2$$
$$= ((r\cos\theta)^2 + (r\sin\theta)^2)^3 - 4(r\cos\theta)^2(r\sin\theta)^2$$
$$= r^6 - 4r^4\cos^2\theta\sin^2\theta$$
$$= r^4(r^2 - (2\cos\theta\sin\theta)^2)$$
$$= r^4(r^2 - \sin^2 2\theta)$$
$$= r^4(r - \sin 2\theta)(r + \sin 2\theta).$$

Thus we have $r = 0$, $r = \sin 2\theta$, or $r = -\sin 2\theta$.

8 - 11, 13, 14

8. Given $X = V(f_1, \dots, f_s)$ and that $X = \{(x,x) \mid x \in \mathbb{R}, x \neq 1\}$. Suppose that $f \in \mathbb{R}[x,y]$ vanishes at X. Now consider a polynomial g in $\mathbb{R}[t]$ where $g(t) = f(t,t)$. Thus since $\mathbb{R}$ is an infinite field, we have g to have infinitely many roots, so g is a zero polynomial. Hence by proposition 5 we have $g : \mathbb{R} \to \mathbb{R}$ to be a zero function, so $g(1) = f(1,1) = 0$, which contradicts X being an affine variety. ∎

9. Proof. By contradiction suppose that $R = V(f_1, \dots, f_s)$. Then each f_i vanishes on R. Now consider a particular $f \in \mathbb{R}[x,y]$ that vanishes at R. Let $g \in \mathbb{R}[t]$ be a poly-nomial where $g(t) = f(t,t)$. Then since $\mathbb{R}$ is an infinite field, g has infinitely roots, so g is a zero polynomial, so $g = 0$. Hence by proposition 5 we have g to be a zero function. Hence $0 = g(y) = f(y,y)$ where $y < 0$. However R has the property $y > 0$, which contradicts our assumption. Therefore we have $R \neq V(f_1, \dots, f_s)$, so R isn't an affine variety. ∎

10. Proof. By contradiction suppose that $\mathbb{Z}^n = V(f_1, \dots, f_s)$. Then each f_i vanishes on $\mathbb{Z}^n$. Now consider a function $f \in \mathbb{C}[x_1, \dots, x_n]$ that vanishes at every point of $\mathbb{Z}^n$. Then by Exercise 1.6 (a) we have f to be our zero function. Then we would also have f vanishing at every point in $\mathbb{C}^n$, which contradicts our assumption. Therefore we have $\mathbb{Z}^n$ to not be our affine variety. ∎

11. (a) If n is odd then we have

$$1^n + 0^n = 1^n, \text{ and}$$
$$0^n + 1^n = 1^n;$$

if n is even then

$$(-1)^n + 0^n = 1^n + 0^n = 1^n,$$
$$0 + (-1)^n = 0 + 1^n = 1^n,$$
$$1^n + 0^n = 1^n, \text{ and}$$
$$0^n + 1^n = 1^n.$$

(b) $(\Rightarrow)$ Suppose that F_n has a nontrivial solution for some $n \geq 3$. Then we have

$$1 = x^n + y^n$$
$$= \left(\frac{a}{b}\right)^n + \left(\frac{c}{d}\right)^n \quad \text{for } a,b,c,d \in \mathbb{Z}$$
$$= \frac{a^n}{b^n} + \frac{c^n}{d^n} = \frac{(ad)^n + (bc)^n}{(bd)^n},$$

so $(bd)^n = (ad)^n + (bc)^n$. Thus Fermat's Last Theorem is false. $(\Leftarrow)$ Now suppose that Fermat's Last Theorem were false. Then we have for $n \geq 3$ satisfying the equation

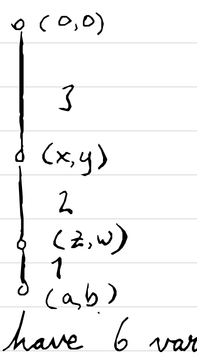
$$x^n + y^n = z^n \quad \text{where } x,y,z \in \mathbb{Z} \setminus \{0\}.$$

Thus we have

$$1 = \frac{x^n}{z^n} + \frac{y^n}{z^n}$$
$$= \left(\frac{x}{z}\right)^n + \left(\frac{y}{z}\right)^n$$
$$= a^n + b^n \quad \text{for } a = \frac{x}{z},\ b = \frac{y}{z},$$

so $a, b \in \mathbb{Q}$. ∎

13. (a)



(b) We have 6 variables $(x,y,z,w,a,b) \in \mathbb{R}^6$.

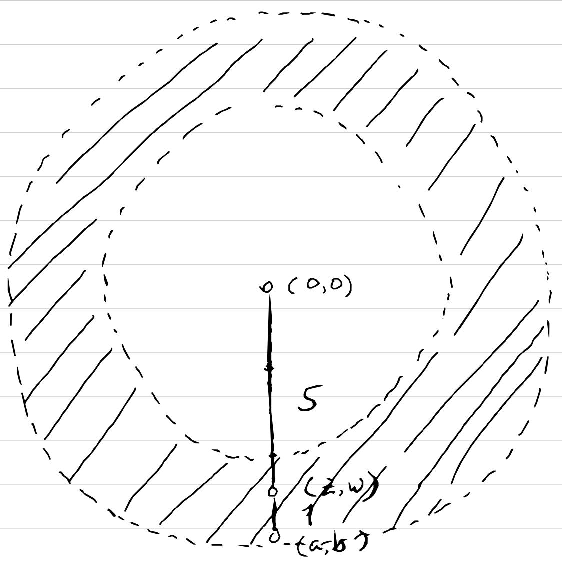
(c) We would have

$$x^2+y^2=9,$$
$$(x-z)^2+(y-w)^2=4,$$
$$(z-a)^2+(w-b)^2=1.$$

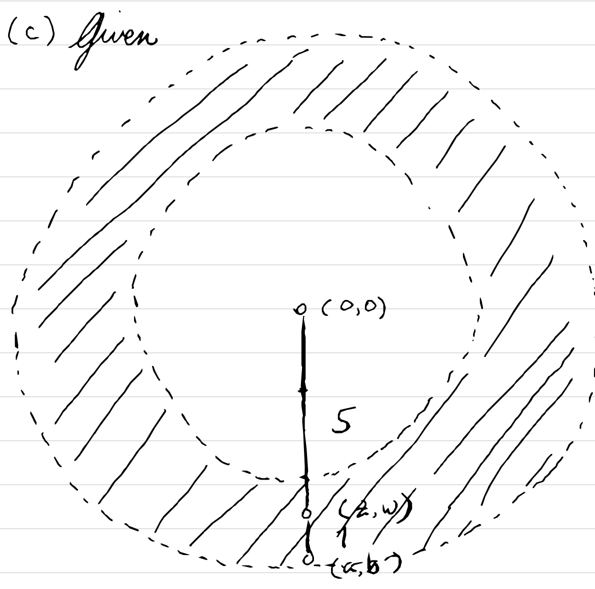
(d) We would have 3 dimensions, since we have 3 equations.

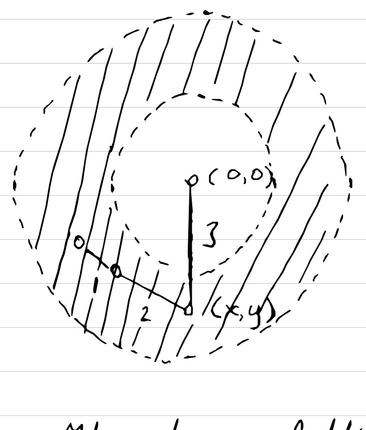
14. (a) Because the length of our arm altogether would be 6, which our hand shouldn't surpass.

(b)



(c) Given





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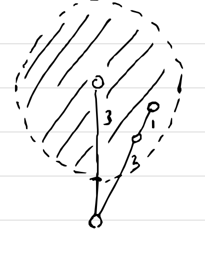
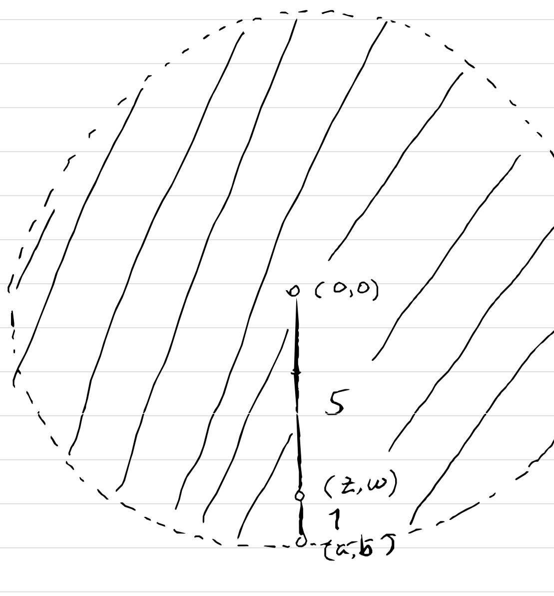


Then by overlapping the disk and two annuluses together we get a disk.



Thus we have $u^2+v^2 \leq 36$.

15. a) We prove that if $V_i \subseteq k^n$ are affine varieties where $1 < i \leq s$ then $\bigcup^s V_i$ and $\bigcap^s V_i$ are varieties.

Proof. Suppose that $V_i \subseteq k^n$ are affine varieties. Let $s=2$. Then we have $\bigcup^2 V_i = V_1 \cup V_2$ and $\bigcap^2 V_i = V_1 \cap V_2$. Hence by Lemma 2 we have $V_1 \cup V_2$ and $V_1 \cap V_2$ to be affine varieties. Now suppose that $\bigcup^{s-1} V_i$ and $\bigcap^{s-1} V_i$ are varieties. Then we have $(\bigcup^{s-1} V_i) \cup V_s = \bigcup^s V_i$ and $(\bigcap^{s-1} V_i) \cap V_s = \bigcap^s V_i$. Hence using Lemma 2 we have $\bigcup^s V_i$ and $\bigcap^s V_i$ to be affine varieties. $\blacksquare$

b) Proof. By question 6 we consider the finite affine variety

$$\{-k, \dots, -2, -1, 0, 1, 2, \dots, k\}^n \subseteq \mathbb{C}^n$$

for some $k \in \mathbb{N}$. Then by taking its infinite union we have $\mathbb{Z}^n \subseteq \mathbb{C}^n$, which is not an affine variety by question 10. $\blacksquare$

c) We consider the affine varieties $\{(x,x) \mid x \in \mathbb{R}\}$ and $\{(1,1)\}$. Then we have $\{(x,x) \mid x \in \mathbb{R}\} \setminus \{(1,1)\} = \{(x,x) \mid x \in \mathbb{R}, x \neq 1\}$, which isn't an affine variety due to question 8.

d) Proof. Suppose that $V = V(f_1, \dots, f_s)$ and $W = V(g_1, \dots, g_t)$. Then

$$V \times W = \{(x_1, \dots, x_n, y_1, \dots, y_m) \in k^{n+m} \mid (x_1, \dots, x_n) \in V, (y_1, \dots, y_m) \in W\}$$

Thus we have $f_i(x_1, \dots, x_n) = 0$ for all $1 \leq i \leq s$ and $g_k(y_1, \dots, y_m) = 0$ for all $1 \leq k \leq t$. Thus we have $f_i, g_k \in [x_1, \dots, x_n, y_1, \dots, y_m]$ and both are zero functions. Thus by definition we have $V \times W$ to be an affine variety.

$$m = \frac{t-0}{0-(-1)} = \frac{y-0}{x-(-1)},$$

$$m = t = \frac{y}{x+1},$$

$$\therefore \quad y = t(x+1).$$

Since $x^2+y^2=1$,

$$x^2 + t^2(x+1)^2 = 1$$
$$x^2 + t^2x^2 + 2t^2x + t^2 = 1,$$
$$(1+t^2)x^2 + 2t^2x + (t^2-1),$$

so $$x = \frac{-2t^2 \pm \sqrt{(2t^2)^2 - 4(1+t^2)(t^2-1)}}{2(1+t^2)}$$

$$= \frac{-2t^2 \pm \sqrt{(2t^2)^2 - 4(t^4-1)}}{2(t^2+1)}$$

$$= \frac{-2t^2 \pm \sqrt{\cancel{4t^4} - \cancel{4t^4} + 4}}{2(t^2+1)}$$

$$= \frac{-2t^2 \pm 2}{2(t^2+1)} = \frac{-t^2 \pm 1}{t^2+1}$$

$$\therefore \quad x = \frac{-t^2+1}{t^2+1} \quad \text{or} \quad x = \frac{-t^2-1}{t^2+1} = \frac{-(t^2+1)}{t^2+1}$$
$$= -1$$

$$y = t(x+1)$$

$$= t\left(\frac{-t^2+1}{t^2+1} + 1\right)$$

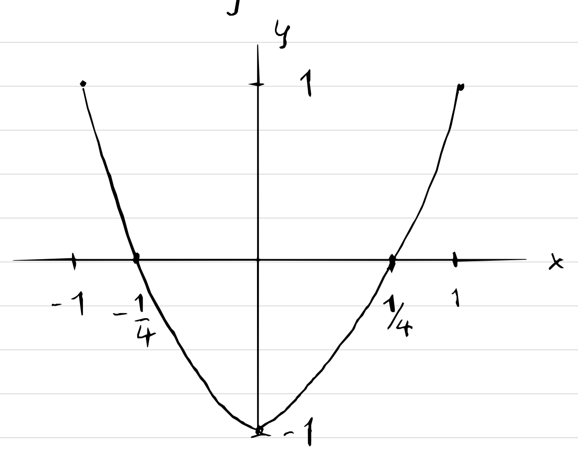
$$= t\left(\frac{2}{t^2+1}\right) = \frac{2t}{t^2+1}$$

Exercise 2 -6, 8-12

2. Given

$$x = \cos(t),$$
$$y = \sin(2t).$$

Then $y = \sin(2t) = 2\cos^2(t) - 1 = 2x^2 - 1$.
Thus we have a parabola of
$y = 2x^2 - 1$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.



3. Given $V(y - f(x))$. Then we have $y = f(x)$.
Hence $y = f(t)$,
$x = t$.

4. (a) Given

$$x = \frac{t}{1+t}, \quad y = 1 - \frac{1}{t^2}$$

we have
$$x(1+t) = t,$$
$$x + xt = t,$$
$$x = t(1-x),$$
$$t = \frac{x}{1-x}.$$

Then $y = 1 - 1/\left(\frac{x}{1-x}\right)^2$

$$= 1 - \frac{(1-x)^2}{x^2} = \frac{x^2 - (1-2x+x^2)}{x^2}$$

$$= \frac{\cancel{x^2} - 1 + 2x - \cancel{x^2}}{x^2} = \frac{-1+2x}{x} = 2 - \frac{1}{x}$$

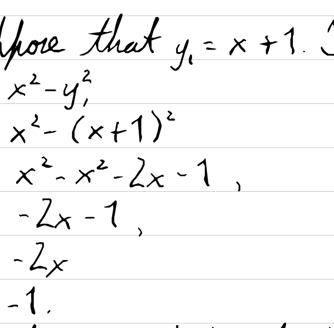
(b) For any points (x,y)

$$x = t/(1+t), \qquad y = 1 - 1/t^2,$$
$$x(1+t) = t, \qquad yt^2 = t^2 - 1,$$
$$x + xt = t, \qquad 1 = t^2 - yt^2,$$
$$x = t - xt, \qquad 1 = t^2(1-y),$$
$$= t(1-x), \qquad t^2 = 1/(1-y),$$
$$t = x/(1-x). \qquad t = \pm 1/\sqrt{(1-y)}.$$

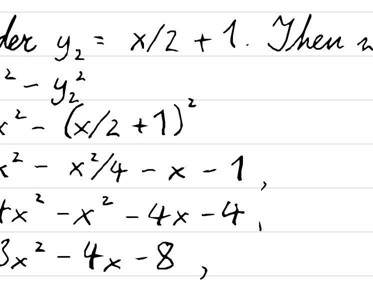
Thus $x \neq 1$ and $y \neq 1$, so the parameter satify all points of (x,y) except $(1,1)$.

5. (a) We have
$$1 = (\cosh t)^2 - (\sinh t)^2$$
$$= x^2 - y^2.$$
The hyperbola covers only $\cosh(t) \geq 1$, so $x \geq 1$ side of the graph.

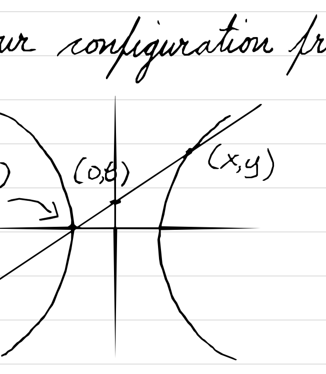
(b) We consider y_n where $0 \leq n \leq 2$ is the number of points a straight line meets.
i) Consider the line $y_0 = x$. Then
$$1 = x^2 - y_0^2$$
$$= x^2 - x^2 = 0,$$
but $1 \neq 0$, so our line $y_0 = x$ never touched the function, hence



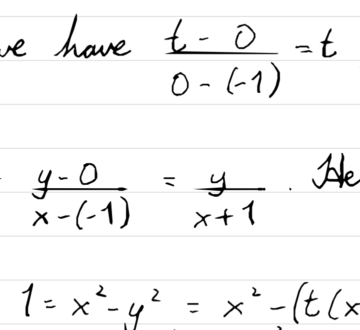
ii) Next suppose that $y_1 = x + 1$. Then we have
$$1 = x^2 - y_1^2$$
$$= x^2 - (x+1)^2$$
$$= x^2 - x^2 - 2x - 1,$$
$$1 = -2x - 1,$$
$$2 = -2x$$
$$x = -1.$$
Thus we have an intersect point at $x = -1$, so



iii) We consider $y_2 = x/2 + 1$. Then we have
$$1 = x^2 - y_2^2$$
$$= x^2 - (x/2 + 1)^2$$
$$= x^2 - x^2/4 - x - 1,$$
$$4 = 4x^2 - x^2 - 4x - 4,$$
$$0 = 3x^2 - 4x - 8,$$
hence
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-8)}}{2(3)}$$
$$= \frac{4 \pm \sqrt{112}}{6}.$$
Thus $y_2 = \frac{16 + \sqrt{112}}{12}$, or $y_2 = \frac{16 - \sqrt{112}}{12}$,
hence



c) Using our configuration from (b) we have



,

so we have $\frac{t - 0}{0 - (-1)} = t$, so

$$t = \frac{y - 0}{x - (-1)} = \frac{y}{x+1}.$$ Hence $y = t(x+1)$.

Thus $1 = x^2 - y^2 = x^2 - (t(x+1))^2$
$$= x^2 - t^2(x+1)^2$$
$$= x^2 - t^2(x^2 + 2x + 1)$$
$$= x^2 - t^2x^2 - 2t^2x - t^2,$$
$$(1-t^2)x^2 - 2t^2x + (-1-t^2) = 0.$$
Thus
$$x = \frac{2t^2 \pm \sqrt{(-2t^2)^2 - 4(1-t^2)(-1-t^2)}}{2(1-t^2)}$$

$$= \frac{2t^2 \pm \sqrt{4t^4 + 4 - 4t^4}}{2(1-t^2)}$$

$$= \frac{2t^2 \pm 2}{2(1-t^2)} = \frac{t^2 \pm 1}{-t^2 + 1}.$$

Hence $x = -1$ or $x = \frac{t^2 + 1}{-t^2 + 1} = \frac{-t^2 - 1}{t^2 - 1}$

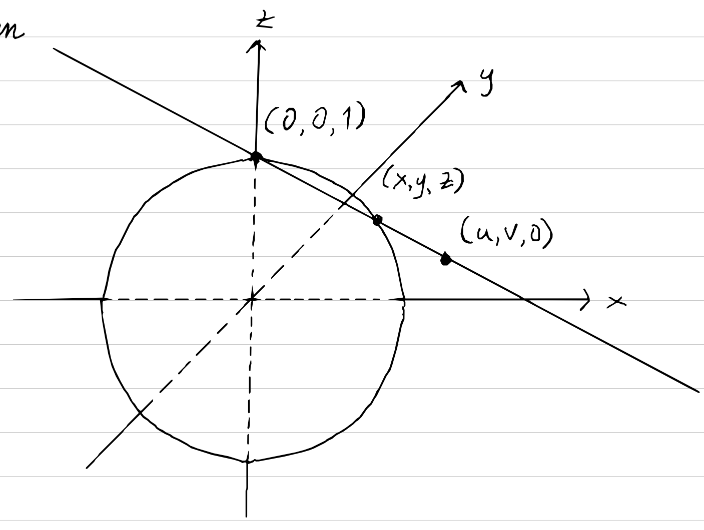
and $y = t\left(\frac{-t^2 - 1}{t^2 - 1} + 1\right)$

$$= t\left(\frac{\cancel{-t^2} - 1 + \cancel{t^2} - 1}{t^2 - 1}\right) = \frac{-2t}{t^2 - 1}.$$

d) When $t = 1$ we get the line
$y = x + 1$, which is
parallel to $y = x$, which is our asymptote.
Hence going through only one point $(-1,0)$.

6.

(a) Given



(b) As we move along the line between $(0,0,1)$ to $(u,v,0)$ via its parameterised line $(tu, tv, 1-t)$, we have $t=0$ to give $(0,0,1)$ and $t=1$ to give $(u,v,0)$.

(c) Let $x = tu$, $y = tv$, $z = 1-t$. Then

$$1 = x^2+y^2+z^2$$
$$= (tu)^2+(tv)^2+(1-t)^2$$
$$= t^2u^2+t^2v^2+1-2t+t^2,$$
$$t^2u^2+t^2v^2-2t+t^2 = 0,$$
$$t^2(u^2+v^2+1)-2t = 0,$$
$$t^2(u^2+v^2+1) = 2t,$$

assuming $t \neq 0$,

$$u^2+v^2+1 = 2t^{-1},$$
$$\frac{t}{2} = \frac{1}{u^2+v^2+1},$$

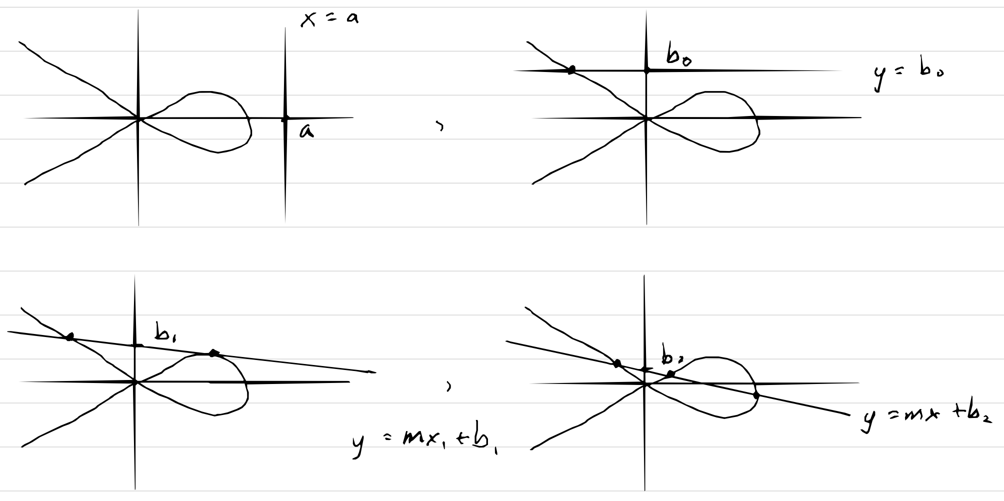
$$t = \frac{2}{u^2+v^2+1}.$$

Thus $x = tu = \dfrac{2u}{u^2+v^2+1}$, $y = tv = \dfrac{2v}{u^2+v^2+1}$,

and $z = 1 - \dfrac{2}{u^2+v^2+1} = \dfrac{u^2+v^2+1-2}{u^2+v^2+1}$

$$= \frac{u^2+v^2-1}{u^2+v^2+1}. \quad //$$

8. (a) We have



(b) We have $y = mx$ and $y^2 = cx^2 - x^3$, so

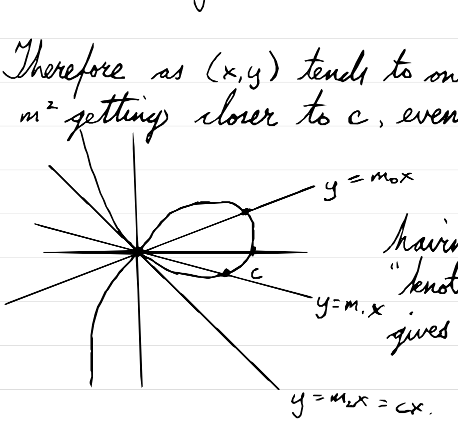
$$(mx)^2 = cx^2-x^3,$$
$$m^2x^2 = cx^2-x^3,$$
$$m^2x^2-cx^2+x^3 = 0,$$
$$x^2(m^2-c+x) = 0.$$

Thus $x = 0$ or $x = c-m^2$ where $m^2 \neq c$.

Hence
$$y^2 = c(c-m^2)^2-(c-m^2)^3,$$
$$y^2 = c(c^2-2cm^2+m^4)-(c^3-3c^2m^2+3cm^4-m^6)$$
$$= \cancel{c^3}-2c^2m^2+cm^4-\cancel{c^3}+3c^2m^2-3cm^4+m^6$$
$$= c^2m^2-2cm^4+m^6$$
$$= m^2(c^2-2cm^2+m^4),$$

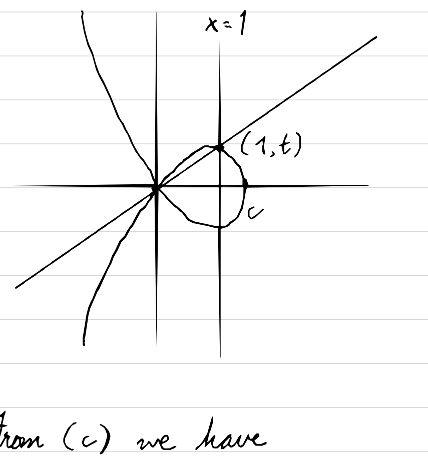
so we have
$$y = \pm\sqrt{m^2(c^2-2cm^2+m^4)}.$$

Therefore as (x,y) tends to only one solution $(0,0)$ we have m^2 getting closer to c, eventually giving $y = cx$.



having a tangent on the "knot" part at the origin $(0,0)$ gives a single point.

(c)



We can modify t in a way that $(1,t)$ will no longer touches the curve via its gradient $y = mx$.

(d) From (c) we have

$$m = \frac{t-0}{1-0} = t,$$

so $y = tx$. Hence

$$(tx)^2 = cx^2-x^3,$$
$$t^2x^2 = cx^2-x^3,$$
$$t^2 = c-x,$$
$$x = c-t^2,$$

it follows

$$y^2 = c(c-t^2)^2-(c-t^2)^3$$
$$= (c-t^2)^2(c-(c-t^2))$$
$$= (c-t^2)^2(c-c+t^2)$$
$$= [t(c-t^2)]^2.$$

Thus $y = \pm t(c-t^2)$, so
$$y = t(c-t^2).$$

Therefore
$$x = c-t^2,$$
$$y = t(c-t^2).$$

9.

(a) Given $x = a\sin(t)$
$y = a\tan(t)(1+\sin(t))$.

Then

$$
\begin{aligned}
y &= a\tan(t)(1+\sin(t)) \\
&= \tan(t)(a + a\sin(t)) \\
&= \tan(t)(a+x) \\
&= \frac{\sin(t)}{\cos(t)}(a+x) \\
&= \frac{x + \frac{x^2}{a}}{\pm\sqrt{\cos^2(t)}} \\
&= \frac{x + \frac{x^2}{a}}{\pm\sqrt{1-\sin^2(t)}} \\
&= \frac{x + \frac{x^2}{a}}{\pm\sqrt{1-\frac{x^2}{a^2}}} \\
&= \frac{\left(\frac{ax+x^2}{a}\right)}{\pm\sqrt{\left(\frac{a^2-x^2}{a^2}\right)}},
\end{aligned}
$$

so
$$
\begin{aligned}
y^2 &= \frac{\left(\frac{ax+x^2}{a}\right)^2}{\left(\frac{a^2-x^2}{a^2}\right)} \\
&= \frac{(ax+x^2)^2}{a^2}\cdot\frac{a^2}{(a-x^2)} = \frac{(ax+x^2)^2}{(a-x^2)},
\end{aligned}
$$

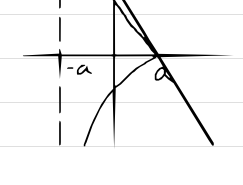
so $y^2(a-x^2) = (ax+x^2)^2$

(b) Using 8(d) we have
$x = -(c-t^2)$
$y = -t(c-t^2)$, so

$x = t^2 - c$
$y = t(t^2-c)$.

10.

(a) We first a line on our curve for
$y^2(a+x) = (a-x)^3$.
Then we have



Hence
$$m = \frac{t-0}{-1-a} = \frac{-t}{1+a}, \text{ so}$$

$$y = \frac{-tx}{1+a} + c,$$

$$0 = \frac{-at}{1+a} + c,$$

$$c = \frac{at}{1+a}.$$

Hence we have

$$y = \frac{t}{1+a}(-x+a) = \frac{t}{1+a}(a-x).$$

Hence
$$
\begin{aligned}
y^2(a+x) &= (a-x)^3, \\
\left[\frac{t^2}{(1+a)^2}(a-x)^2\right](a+x) &= (a-x)^3, \\
\frac{t^2}{(1+a)^2}(a+x) &= a-x,
\end{aligned}
$$

$$
\begin{aligned}
t^2(a+x) &= (a-x)(1+a)^2, \\
at^2 + xt^2 &= a + 2a^2 + a^3 - x - 2ax - a^2x, \\
xt^2 + 2ax + a^2x + x &= -at^2 + a + 2a^2 + a^3, \\
x(t^2+2a+a^2+1) &= a(-t^2+1+2a+a^2) \\
x &= \frac{a(-t^2+2a+a^2+1)}{t^2+2a+a^2+1}
\end{aligned}
$$

Hence
$$
\begin{aligned}
y &= \frac{t}{1+a}\left[a - \frac{a(-t^2+2a+a^2+1)}{t^2+2a+a^2+1}\right] \\
&= \frac{t}{1+a}\left(\frac{at^2-2a-a^2}{t^2+2a+a^2+1}\right) \\
&= \frac{at^3-2at-a^2t}{t^2+3a+3a^2+a^3+1+at^2}
\end{aligned}
$$

$$\cancel{t^2} + \cancel{2a} + \cancel{a^2+1} + at^2 + \cancel{2a^2} + \cancel{a^3} + \cancel{a}$$

(b) Given $P = (-x, \sqrt{a^2-x^2})$, $Q = (x,y)$, and $(a,0)$ all lie on L.
Then we have

$$m = \frac{\sqrt{a^2-x^2} - 0}{-x-a} = -\frac{\sqrt{a^2-x^2}}{x+a},$$

so
$$Y = \left(-\frac{\sqrt{a^2-x^2}}{x+a}\right)X + c,$$

with point $(a,0)$ we have
$$Y = \left(-\frac{\sqrt{a^2-x^2}}{x+a}\right)X + c$$

$$0 = -\frac{a\sqrt{a^2-x^2}}{x+a} + c,$$

$$c = \frac{a\sqrt{a^2-x^2}}{x+a}.$$

Thus we have

$$(x+a)Y = (-\sqrt{a^2-x^2})X + a\sqrt{a^2-x^2},$$

so
$$0 = (\sqrt{a^2-x^2})X + (x+a)Y - a\sqrt{a^2-x^2}.$$

Thus given $Q = (x,y)$ we have

$$0 = (x-a)\sqrt{a^2-x^2} + (x+a)y,$$

$$
\begin{aligned}
y &= \frac{(a-x)\sqrt{a^2-x^2}}{(x+a)}. \\
&= \frac{(a-x)\sqrt{(a-x)(a+x)}}{(a+x)} \\
&= \frac{(a-x)\sqrt{a-x}}{\sqrt{a+x}},
\end{aligned}
$$

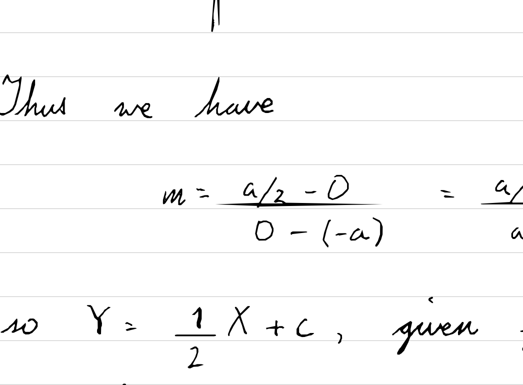
hence we have

$$y\sqrt{a+x} = (a-x)\sqrt{a-x},$$

$$y^2(a+x) = (a-x)^3.$$

Thus it's our cissoid for the locus of all points Q.

(c) Given our cissoid with our line passing points $(-a,0)$ and $(0, a/2)$ we have point (x,y) touching our cissoid below.



Thus we have

$$m = \frac{a/2 - 0}{0-(-a)} = \frac{a/2}{a} = \frac{1}{2},$$

so $Y = \frac{1}{2}X + c$, given point $(0, a/2)$ it follows $c = a/2$, so

$$Y = \frac{X}{2} + \frac{a}{2},$$

$$2Y = X + a.$$

Thus at point (x,y) we have

$$2y = x + a,$$

note that $y^2(a+x) = (a-x)^3$ implies $a + x = \frac{(a-x)^3}{y^2}$, so we have

$$2y = \frac{(a-x)^3}{y^2},$$

$$
\begin{aligned}
2 &= \frac{(a-x)^3}{y^3} \\
&= \left(\frac{a-x}{y}\right)^3. \quad \blacksquare
\end{aligned}
$$

11. (a) We consider the curve

$$x^2 = cz^2 - z^3.$$

Then the line at $x = 1$ we have points $(1, t)$ and $(0,0)$ for all points (z, x).
Thus we have

$$m = \frac{x_2 - x_1}{z_2 - z_1} = \frac{t - 0}{1 - 0} = t,$$

so $x = tz + c$, hence with point $(0,0)$ we have $c = 0$, it follows $x = tz$.
Thus

$$\begin{aligned}(tz)^2 &= x^2 \\ &= cz^2 - z^3, \\ t^2 z^2 &= cz^2 - z^3, \\ t^2 &= c - z, \\ z &= c - t^2, \text{ and } x = tz = t(c - t^2).\end{aligned}$$

(b) We have $x^2 = y^2 z^2 - z^3$, so

$$x^2 - y^2 z^2 - z^3 = 0.$$

Hence since t is our parameter for (a) and $c = y^2$ it follows

$$\begin{aligned} z &= c - t^2 \\ &= y^2 - t^2, \text{ and} \\ x &= t(c - t^2) \\ &= t(y^2 - t^2). \end{aligned}$$

(c) We modify our parameter t in a way all of $x = tz$ will no longer touches the curve at $x^2 = cz^2 - z^3$, and if $c = y^2$ then we have $V(x^2 - y^2 z^2 + z^3)$, which t also fulfills.