Counting 1, 2, 4-6, 10, 14-16, 20, 21, 24, 29, 31, 39
41, 42, 44, 46, 47, 50, 53, 55, 58.

1. $\binom{11}{4}\binom{7}{4}\binom{3}{2} = \frac{11!}{4!4!2!} = 34650.$

2. (a) $8 \cdot 10^6 = 8{,}000{,}000$ possible numbers.

(b) We first count the possible numbers that start with 911. Note that that will be 10^4 possible numbers,
so $8 \cdot 10^6 - 10^4 = 10^4(8 \cdot 10^2 - 1) = 10^4 \cdot 799$
$= 7{,}990{,}000$ possible numbers.

4. (a) We have $2^{\binom{n}{2}}$ possible outcomes for those who won and lost.

(b) We have $\binom{n}{2} = \frac{n!}{2!(n-2)!}$ games total.

5. (a) n rounds.
(b) $2^{n-1} + 2^{n-2} + \cdots + 2^0$ games
(c) $2^n - 1$ games played.

$2^4 = 16$ players

Rounds	1	2	3	4	
Games	8	4	2	1	$= 15$
Winners	0	8	4	2	
Eliminated	8	12	14	15	$= 2^n - 1.$

6. We consider the following labels as players $1, 2, \ldots, 20$. the match possibilities are $\boxed{12}, \boxed{13}, \ldots, \boxed{119}, \boxed{120}, \boxed{21}, \boxed{23}, \ldots, \boxed{220}, \ldots, \boxed{201}, \boxed{202}, \ldots, \boxed{2019}$. Since order matters where the left side is black and right side is white, giving us $20 \cdot 19 = 380$ match ups.

$n = 5$

$\boxed{12}\ \boxed{13}\ \boxed{14}\ \boxed{15}$
$\boxed{21}\ \boxed{23}\ \boxed{24}\ \boxed{25}$ $\quad 5 \cdot 4 = 20$
$\boxed{31}\ \boxed{32}\ \boxed{34}\ \boxed{35}$
$\boxed{41}\ \boxed{42}\ \boxed{43}\ \boxed{45}$
$\boxed{51}\ \boxed{52}\ \boxed{53}\ \boxed{54}$

$n = 4$

$\boxed{12}\ \boxed{13}\ \boxed{14}$
$\boxed{21}\ \boxed{23}\ \boxed{24}$ $\quad 4 \cdot 3 = 12$
$\boxed{31}\ \boxed{32}\ \boxed{34}$
$\boxed{41}\ \boxed{42}\ \boxed{43}$

$n = 3$

$\boxed{12}\ \boxed{13}$ $\quad 3 \cdot 2$
$\boxed{21}\ \boxed{23}$
$\boxed{31}\ \boxed{32}$

$n = 2$

$\boxed{12}$ $\quad 2 \cdot 1$
$\boxed{21}$

10.

(a) $\boxed{\begin{matrix} S1 \\ S2 \end{matrix}} \quad \underline{3} \quad \boxed{\begin{matrix} E & B \\ M & H \end{matrix}}$ $\quad$ 6 courses, 2 stats

S1 S2 E
S1 S2 M $\quad$ 4
S1 S2 B
S1 S2 H

S1 E B	S2 E B	
S1 E M	S2 E M	
S1 E H	S2 E H	$12 = \binom{2}{1} \cdot \binom{4}{2}$
S1 B M	S2 B M	
S1 B H	S2 B H	
S1 H M	S2 H M	

$\therefore \quad 4 + 12 = 16.$

Thus $\binom{5}{1} \cdot \binom{15}{6} + \binom{5}{2} \cdot \binom{15}{5} + \binom{5}{3} \cdot \binom{15}{4} + \binom{5}{4}\binom{15}{3} +$

$\binom{5}{5}\binom{15}{2} = 7085$ possibilities.

(b) Because we choose only 1 of the 5 stats or all 5 of the 5 stats, then we choose the remaining 15 subjects.

14. We first calulate the combination of toppings, so

$$\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} +$$

$$\binom{8}{7} + \binom{8}{8} = 1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1$$
$$= 256.$$

Thus with 4 sizes so $4 \cdot 256 = 1024$ combinations, so

$1024^2 = 1048576$ combinations for two pizzas.

15. We have
$$\begin{aligned} 2^n &= (1+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k}(1)(1) \quad \text{binomial theorem} \\ &= \sum_{k=0}^{n} \binom{n}{k}. \quad \blacksquare \end{aligned}$$

Legolas climbs up the Oliphaunt then shoots k arrows killing n men. After killing the Oliphaunt, he reruns the possibilities in his head, shooting 0 arrows killing n men, due to the presence of the Oathbreakers, 1 arrow, 2, ..., all the way to all of his n arrows. He realised all senarios added up to 2^n ways of killing the n number of Haradrim.

16 (a)
$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\ &= \frac{n!(n-k+1) + n!k}{k!(n-k+1)!} \\ &= \frac{n!(n-k+k+1)}{k!(n+1-k)!} \\ &= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}. \quad \blacksquare \end{aligned}$$

(b) In the land of Mordor there are $n+1$ creatures, including the dark necromancer himself, Sauron. With n creatures preparing for war the remaining k troops, the Nazgûl, are tasked to hunt for the One Ring, with Sauron appearing or not appearing during the hunt. This meant $\binom{n}{k-1}$ possibilities with Sauron's aid, so $k-1$ left to choose, and $\binom{n}{k}$ for the Nazgûls without Sauron's watchful eye. Thus this gave us $n+1$ inhabitants, including Sauron, and we choose k of them giving $\binom{n+1}{k}$ ways of arranging their formations.

20. (a) We first prove by induction. Note that for $n = 1$, we have $\binom{0}{0} + \binom{1}{0} = 1 + 1 = 2 = \binom{2}{1} = \binom{1+1}{0+1}$ to hold. Now suppose that
$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}.$$
Then we have
$$\left[\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k}\right] + \binom{n+1}{k}$$
$$= \binom{n+1}{k+1} + \binom{n}{k} = \binom{n+2}{k+1} \text{ by 16 (a) holds. } \blacksquare$$

Given we are choosing the Fellowship of $k+1$ people out of $n+1$ people at the council. Consider Gandalf the oldest in the Fellowship. If he is the oldest in the council, then it's $\binom{n}{k}$ choices of the Fellowship. If Legolas is second oldest in the council then $\binom{n-1}{k}$ choices to form the Fellowship, so giving in total
$$\binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k} = \binom{n+1}{k+1}$$
choices to form the Fellowship.

(b) Consider the following scenario

clear	red	orange	green	yellow
1	2	10	10	7

Then we have $\binom{30+5-1}{5} = \binom{34}{5}$. Thus when we have $n = 30, 31, \ldots, 49, 50$ gummy bears and 5 flavours with replacement, since order doesn't matter the total possibilities is

$$\binom{54}{5} + \cdots + \binom{34}{5}$$
$$= \left[\binom{54}{5} + \cdots + \binom{5}{5}\right] - \left[\binom{33}{5} + \cdots + \binom{5}{5}\right]$$
$$= \binom{55}{6} - \binom{34}{6} = 27644771.$$

, 31, 33, 43 - 45, 48, 49, 55, 57

23. Given 2 - 10 floors with repeat we have the following

$(2,2,2)$
$(2,2,3)$
$(2,2,4)$
$\vdots$
$(2,2,10)$
$(2,3,2)$
$\vdots$
$(10,10,10)$, so $9^3 = 729$ possibilities.

Thus for 3 consecutive floors we have
$(2,3,4), (3,4,5), (4,5,6), (5,6,7), (6,7,8), (7,8,9),$
$(8,9,10)$
so we have the probability to be $7/729 \approx 0.96\%$.

26. (a) With replacement is $1000{,}000^{1000}$, without replacement is $1000{,}000 \cdot 999{,}999 \cdot \cdots \cdot 999{,}000$, so using the same idea of the birthday problem instead of no one having the same birthday we have no same person being chosen. Thus

$$P(\text{no same person chosen}) = \frac{1000{,}000 \cdot 999{,}999 \cdot \cdots \cdot 999{,}000}{1000{,}000^{1000}}.$$

(b) $P(\text{same person chosen at least once})$

$$= 1 - \frac{1000{,}000 \cdot 999{,}999 \cdot \cdots \cdot 999{,}000}{1000{,}000^{1000}}.$$

29. (a) Total possibilities for 4 dice 6^4. The sum of 21 are
$(3,6,6,6), (4,5,6,6), (4,6,5,6), (4,6,6,5),$
$(5,4,6,6), (5,6,4,6), (5,6,6,4), (6,3,6,6),$
$(6,5,5,5), (6,6,3,6), (6,6,6,3).$

while for 22 we have
$(4,6,6,6), (5,5,6,6), (5,6,5,6), (5,6,6,5),$
$(6,4,6,6), (6,5,5,6), (6,5,6,5), (6,6,4,6),$
$(6,6,5,5), (6,6,6,4).$

Thus $\quad 11/6^4 > 10/6^4$

so $\quad P(\text{sum } 21) > P(\text{sum } 22).$

(b)

$$\left.\begin{array}{c} aa \\ bb \\ cc \\ ee \\ \vdots \\ zz \end{array}\right\} 26 \qquad \begin{array}{c} \left.\begin{array}{c} aaa \\ aba \\ aca \\ \vdots \\ aza \end{array}\right\} 26 \\ \vdots \\ \left.\begin{array}{c} bab \\ bbb \\ \vdots \\ bzb \end{array}\right\} 26 \\ \vdots \\ zzz \end{array}$$

Given 26^2 and 26^3

Thus there $26/26^2 = 1/26$, while we have $26 + 26 + \cdots 26$; 26 times giving $26^2/26^3 = 1/26$, so $P(\text{2-letter pallindrome}) = P(\text{3-letter pallindrome})$.

31. We $\binom{n}{k}$ sets of exactly k recaptured elks, $\binom{N-n}{m-k}$ sets of non-captured elks, and finally $\binom{N}{m}$ sets of second size of m elks.
Therefore

$$P(\text{exactly } k \text{ elks of } m \text{ were already tagged}) = \frac{\binom{n}{k} \cdot \binom{N-n}{m-k}}{\binom{N}{m}}.$$

43

$$P(A)+P(B)-1 \leq P(A\cap B) \leq P(A\cup B) \leq P(A)+P(B).$$

i) $P(A)+P(B)-1 = P(A\cap B)$ iff $P(A\cup B)=1$.

Proof. ($\rightarrow$) Suppose that $P(A)+P(B)-1 = P(A\cap B)$.
Then $1 = P(A)+P(B)-P(A\cap B) = P(A\cup B)$.
($\leftarrow$) Suppose that $P(A\cup B)=1$. Then

$$1 = P(A\cup B) = P(A)+P(B)-P(A\cap B),$$
$$P(A\cup B) = P(A)+P(B)-1. \blacksquare$$

ii) $P(A\cap B) = P(A\cup B)$ iff $P(A\setminus B)=0$ and $P(B\setminus A)=0$.

Proof. ($\rightarrow$) Suppose that $P(A\cap B) = P(A\cup B)$.
Then we have

$$P(A\cup B) = P(A\setminus B)+P(B\setminus A)+P(A\cap B)$$
$$= P(A\setminus B)+P(B\setminus A)+P(A\cup B),$$
$$0 = P(A\setminus B)+P(B\setminus A).$$

Thus we have $P(A\setminus B) = 0 = P(B\setminus A)$.
($\leftarrow$) Suppose that $P(A\setminus B)=0$ and $P(B\setminus A)=0$.
Then

$$0 = 0+0 = P(A\setminus B)+P(B\setminus A)$$
$$= P(A)-P(A\cap B)+P(B)-P(A\cap B)$$
$$= P(A\cup B)-P(A\cap B),$$
$$P(A\cap B) = P(A\cup B). \blacksquare$$

iii) $P(A\cup B)=P(A)+P(B)$ iff $P(A\cap B)=0$.

Proof. ($\rightarrow$) Suppose that $P(A\cup B)=P(A)+P(B)$.
Then since

$$P(A\cup B) = P(A)+P(B)-P(A\cap B),$$
$$P(A)+P(B) = P(A)+P(B)-P(A\cap B),$$
$$0 = -P(A\cap B),$$
$$P(A\cap B)=0.$$

($\leftarrow$) Suppose that $P(A\cap B)=0$. Then immediately

$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$
$$= P(A)+P(B)-0 = P(A)+P(B). \blacksquare$$

44. *Proof.* Suppose that $A\subseteq B$. Then we have

$$P(B)-P(A)$$
$$= P(A\cup(B\cap A^c))-P(A)$$
$$= P(A)+P(B\cap A^c)-P(A)$$
$$= P(B\cap A^c) = P(B-A). \blacksquare$$

45. *Proof.* Given $A\Delta B = (A\cup B)-(A\cap B)$.
Then we have

$$P(A\Delta B) = P((A\cup B)-(A\cap B)).$$

Thus since $A\cap B \subseteq A\cup B$, by exercise 44 we have

$$P((A\cup B)-(A\cap B)) = P(A\cup B)-P(A\cap B)$$
$$= (P(A)+P(B)-P(A\cap B))-P(A\cap B)$$
$$= P(A)+P(B)-2P(A\cap B). \blacksquare$$

48. Given that Arby is willing to pay $1000\cdot P(A)$ dollars for any event A of an A type certificate we consider the event $A\cup B$. Then since Arby believes

$$P'(A\cup B) \neq P'(A)+P'(B)$$

for $A\cap B=\emptyset$. Then we have the following transaction:

- We sold Arby a $A\cup B$-certificate, and buy both certificates A and B.
- Thus Arby now has $\$1000\cdot(P(A)+P(B)-P(A\cap B))$.
- Either A or B happens, but not both, so Arby lost $\$1000$.
- But $A\cup B$ happenend, so Arby gained $\$1000$.
- So in total Arby has $\$1000\cdot(P(A)+P(B)-P(A\cap B))$, but if $P(A\cup B) > P(A)+P(B)$ then Arby lost money.

Then on another transaction:

- We sold both A and B certificates, and buy $A\cup B$.
- Thus Arby now has $\$1000\cdot(P(A\cup B)-P(A)-P(B))$.
- Similar to earlier Arby gained and lost $\$1000$.
- So Arby has $\$1000\cdot(P(A\cup B)-P(A)-P(B))$, but if $P(A\cup B) < P(A)+P(B)$ then Arby still lost money.

49.

Let A_i be a die we don't roll an i with n rolls, so its probability is $P(A_i) = (5/6)^n$. Then we say that we don't roll an i,j with n rolls is $P(A_i \cap A_j) = (4/6)^n$, then i,j,k $P(A_i \cap A_j \cap A_k) = (3/6)^n$, eventually giving $(1/6)^n$ for any 5 elements in n rolls.

Thus to find at least 1 to 6 never appears we have

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i \right) &= \sum_1 P(A_i) - \sum_{i<j} P(A_i \cap A_j) \\ &+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \\ &- \sum_{i<j<k<l} P(A_i \cap A_j \cap A_k \cap A_l) \\ &+ \sum_{i<j<k<l<m} P(A_i \cap A_j \cap A_k \cap A_l \cap A_m) \end{aligned}$$

$$= \binom{5}{1}(5/6)^n - \binom{5}{2}(4/6)^n + \binom{5}{3}(3/6)^n - \binom{5}{4}(2/6)^n + \binom{5}{5}(1/6)^n$$

$$= 5(5/6)^n - 10(4/6)^n + 10(3/6)^n - 5(2/6)^n + (1/6)^n$$

55.

(a) There are $\binom{15}{3}$ ways to have 3 sophmores in a commitee, so $\binom{10+12}{2} = \binom{22}{2}$ for the remaining one.

$$\binom{15}{3}\binom{10+12}{2} = 105105,$$

so $\binom{37}{5} = 435897$ total.

Thus $P(\text{there are exactly 3 sophomores}) = 105105/435897 = 0.2411$ (4 d.p.)

(b) We have

$$\binom{15}{1}\binom{12}{2}\binom{10}{2} = 44550,$$
$$\binom{15}{2}\binom{12}{1}\binom{10}{2} = 56700,$$
$$\binom{15}{2}\binom{12}{2}\binom{10}{1} = 69300,$$
$$\binom{15}{1}\binom{12}{1}\binom{10}{3} = 21600,$$
$$\binom{15}{1}\binom{12}{3}\binom{10}{1} = 33000,$$
$$\binom{15}{3}\binom{12}{1}\binom{10}{1} = 54600.$$

Thus $P(\text{at least one representative in each class}) = 0.6417$ (4 d.p).

57. Note that the probability of none of our molecules shared with Cesar's last breath is

$$\left(1 - \frac{10^{22}}{10^{44}}\right)^n, \text{ given } n = 10^{22} \text{ molecules}$$

in his last breath is

$$\left(1 - \frac{10^{22}}{10^{44}}\right)^{10^{22}} = \left(1 - \frac{1}{10^{22}}\right)^{10^{22}}.$$

so the probability at least one molecule was shared is

$$1 - \left(1 - \frac{1}{10^{22}}\right)^{10^{22}} = 1 - \left(1 + \frac{-1}{10^{22}}\right)^{10^{22}}$$

since $\left(1 + \frac{x}{n}\right)^n \approx e^x$, we have $1 - e^{-1} = 1 - \frac{1}{e}$.

1, 2, 4, 6, 7, 12 - 18, 23, 26, 31, 32, 36 - 39, 51, 52
59, 68 - 70

1. Let S be the event that the email is a spam and let F be the event where "free money" is used. Then

$$P(S|F) = \frac{P(F|S)P(S)}{P(F)}$$

$$= \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|S^c)P(S^c)}$$

Note that $P(S) = 0.8$, $P(F|S) = 0.1$, $P(F|S^c) = 0.01$.
Thus

$$P(S|F) = \frac{0.1 \cdot 0.8}{0.1 \cdot 0.8 + 0.01 \cdot 0.2} \approx 0.976 \text{ (3 d.p.)}.$$

2. Let I be the event that both boys are identical twins,

$$P(I|B,B) = \frac{P(B,B|I)P(I)}{P(B,B)}$$

$$= \frac{P(B,B|I)P(I)}{P(B,B|I)P(I) + P(B,B|I^c)P(I^c)}$$

Note that
$P(I) = 1/3$
$P(B,B|I) = 1/2$
$P(B,B|I^c) = P(B,B)$
$= P(B)P(B) = 1/4.$

Thus we have $P(I|B,B) = \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 1/4 \cdot 2/3}$

$$= \frac{1/6}{1/6 + 1/6} = \frac{1}{2}.$$

4. (a) $P(K|R) = \frac{P(R|K)P(K)}{P(R)}$

$$= \frac{P(R|K)P(K)}{P(R|K)P(K) + P(R|K^c)P(K^c)}$$

$$= \frac{1 \cdot p}{1 \cdot p + (1/n)(1-p)} = \frac{p}{p + 1/n - p/n}$$

(b) By contradiction suppose that $P(K|R) < p$.
Let $n \neq 0$. Then by (a) we have

$$\frac{p}{p + 1/n - p/n} < p,$$

$$p < p^2 + p/n - p^2/n,$$

$$pn < p^2 n + p - p^2,$$

$$p^2 - p < n(p^2 - p).$$

Hence assuming $0 < p < 1$. Then $p^2 - p < 0$. Thus $-(p^2 - p) > 0$, so $p^2 - p < n(p^2 - p)$ is false. Hence we have $P(K|R) > p$. Next let $n = 1$. Then

$$P(K|R) = \frac{p}{p + 1/n - p/n}$$

$$= \frac{p}{p + 1 - p} = p.$$

Thus it makes sense since you have only one choice!

6. Let D be an event where the coin is double-headed, and H be an event where the coin lands heads.
Then we have

$$P(D|H,H,H,H,H,H,H)$$
$$= \frac{P(H,H,H,H,H,H,H|D)P(D)}{P(H,H,H,H,H,H,H)}$$

$$= \frac{P(\bigcap^7 H|D)P(D)}{P(\bigcap^7 H|D)P(D) + P(\bigcap^7|D^c)P(D^c)}$$

$$= \frac{1 \cdot 1/100}{1 \cdot 1/100 + (1/2)^7 \cdot 99/100}$$

$$= \frac{1/100}{227/1280} \approx 0.564.$$

7. Let D be an event where "there is one coin and ninety-nine are fair", D^c be an event where "all one hunred coins are fair", and

(a) Note that E is the event where we choose a double-sided coin and

$$P(D|H_7) = \frac{P(H_7|D)P(D)}{P(H_7)}.$$

Given that

$$P(H_7|D) = P(H_7|D,E)P(E) + P(H_7|D,E^c)P(E^c)$$
$$= 1 \cdot 1/100 + (1/2)^7 \cdot 99/100$$
$$= 0.0177 \text{ (4 d.p)},$$

$$P(H_7|D^c) = P(H_7|D^c,E)P(E) + P(H_7|D^c,E^c)P(E^c)$$
$$= 0 \cdot 1/100 + (1/2)^7 \cdot 99/100$$
$$= 99/12800 = 0.0077 \text{ (4 d.p.)},$$

and

$$P(H_7) = P(H_7|D)P(D) + P(H_7|D^c)P(D^c)$$
$$= 0.0177 \cdot 1/2 + 0.0077 \cdot 1/2$$
$$= 0.0127$$

Then we have

$$P(D|H_7) = \frac{0.0177 \cdot 1/2}{0.0127}$$

$$\approx 0.6969 \text{ (4 d.p.)}$$

(b) $P(E|H_7) = \frac{P(H_7|E)P(E)}{P(H_7)}.$

$$= \frac{1 \cdot 1/100}{1 \cdot 1/100 + (1/2)^7 \cdot 99/100}$$

$$= \frac{1/100}{1/100 + 99/12800} = 0.5639 \text{ (4 d.p.)}$$

12. (a) Let E be an event when an error occurred, A_n where Alice sends n, B_m where Bob receives m, where $n, m \in \{0, 1\}$.

Then
$$P(A_1|B_1) = P(A_1|B_1, E)P(B_1|E) + P(A_1|B_1, E^c)P(B_1|E^c)$$
$$= 95\% \cdot 1 + 1 \cdot 0 = 95\%.$$

(b) Since
$$P(A_1|B_0) = P(A_1|B_0, E)P(B_0|E) + P(A_1|B_0, E^c)P(B_0, E^c)$$
$$= 90\% \cdot 1 + 1 \cdot 0 = 90\%,$$
so
$$P(A_1|B_1, B_1, B_0) = P(A_1|B_1)P(A_1|B_1)P(A_1|B_0)$$
$$= 95\% \cdot 95\% \cdot 90\% \approx 81.23\%.$$

13. $P(D) = 1\%$, $P(T|D) = P(T^c|D^c) = 0.95$, where $P(T|D)$ is sensitivity (true positive) and $P(T^c|D^c)$ is specificity (true negative).

(a) Let B be an event where Company B diagnose correctly. Then
$$P(B) = P(T|D)P(D) + P(T^c|D^c)P(D^c)$$
$$= 0.95 \cdot 0.01 + 0.95 \cdot 0.99$$
$$= 0.95,$$
so Company B's diagnosis is 95% overall success rate.

(b) Company A may still have a better method for diagnosis, since Company B has a useless measure.

(c) If $P(T|D) = 1$ then to have $P(B) > 0.95$ we have $1 \cdot 0.01 + P(T^c|D^c) \cdot 0.99 > 0.95$, so $P(T^c|D^c) > \frac{0.94}{0.99} \approx 0.95$. Hence the specificity needs to be above 0.95.

If $P(T^c|D^c) = 1$ then to have $P(B) > 0.95$ we have $P(T|D) \cdot 0.01 + 1 \cdot 0.99 > 0.95$, so $P(T|D) > -4$, hence $P(T|D) \geq 0$. Hence the sensitivity can be 0 or above.

14. (a) We would have $P(A|B) > P(A|B^c)$, since we would be more weary of burglars after it happened.

(b) We would have $P(B|A) > P(B|A^c)$, since burglars want to steal to someone who is careless.

(c) ($\rightarrow$) Suppose $P(A|B) > P(A|B^c)$. Then we have

$$\frac{P(B|A)P(A)}{P(B)} > \frac{P(B^c|A)P(A)}{P(B^c)},$$

$$P(B|A)P(B^c) > P(B^c|A)P(B),$$
$$P(B|A) - \cancel{P(B|A)P(B)} > P(B) - \cancel{P(B)P(B|A)},$$
$$P(B|A) > P(B), \text{ so}$$
$$P(B|A) > P(B|A)P(A) + P(B|A^c)P(A^c),$$
$$P(B|A)(1-P(A)) > P(B|A^c)P(A^c),$$
$$P(B|A)P(A^c) > P(B|A^c)P(A^c).$$

Thus $P(B|A) > P(B|A^c)$. Hence in a similar manner we have its converse to hold too. ∎

(d) It's likely $P(B|A) < P(B|A^c)$ was interpreted due to the presence of an alarm system says there are more burgulars present, which is false.

15. Note that $P(A \cup B)$ isn't our best case to see both events, since either A or B happens but not necessarily both. So consider $0 < P(A \cap B) < P(A) < P(B) < 1$. Then $0 < P(A \cap B)/P(A) < 1$ and $0 < P(A \cap B)/P(B) < 1$. Thus since $P(A) < P(B)$, we have $P(B)^{-1} < P(A)^{-1}$, so $0 < P(B|A) < P(A|B)$. Thus by lotp we have $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$, so $P(A)$ is our best bet.

16. *Proof.* Consider $P(A|B) \leq P(A)$. Then we have

$$P(A|B)P(B) + P(A|B^c)P(B^c) \leq P(A)P(B) + P(A|B^c)P(B^c),$$
$$P(A) \leq P(A)P(B) + P(A|B^c)P(B^c),$$
$$P(A)(1-P(B)) \leq P(A|B^c)P(B^c),$$
$$P(A)P(B^c) \leq P(A|B^c)P(B^c),$$
$$P(A) \leq P(A|B^c). \blacksquare$$

We can intuitively think that since the prior of A is more likely to happen than posterior given B already happenend, then B^c is everything else that has ever happened, except B, that already occured.

17. (a) Suppose that $P(B|A) = 1$. Then

$$1 = P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$
$$P(A) = P(A|B)P(B),$$
$$P(A|B)P(B) + P(A|B^c)P(B^c) = P(A|B)P(B),$$
$$P(A|B^c)P(B^c) = 0,$$
$$[1 - P(A^c|B^c)]P(B^c) = 0,$$
$$P(B^c) - P(A^c|B^c)P(B^c) = 0,$$
$$P(B^c) = P(A^c|B^c)P(B^c),$$
$$P(A^c|B^c) = 1.$$

(b) We consider a sample space S where we roll two dice. Let A be an event for rolling a 1 on die 1 while we let B be an event for rolling for numbers greater than 1, so 2, 3, 4, 5, 6, on die 2.

Then $$P(B|A) = \frac{P(B)P(A)}{P(A)}$$

$$= \frac{5/6 \cdot 1/6}{1/6} = \frac{5}{6},$$

while $$P(A^c|B^c) = \frac{P(A^c) \cdot P(B^c)}{P(B^c)}$$

$$= \frac{1/6 \cdot 5/6}{5/6} = \frac{1}{6}.$$

Thus $P(B|A)$ is closer to 1 whereas $P(A^c|B^c)$ is closer to 0.

18. Given $P(A)=1$ we then have

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(A\cap B)}{P(B\cap A)+P(B\cap A^c)}$$

Note that $P(B\cap A^c) = P(B|A^c)P(A^c)$
$= P(B|A^c)\cdot 0 = 0.$

Thus

$$P(A|B) = \frac{P(A\cap B)}{P(B\cap A)} = 1.$$

23. If we consider two damming evidences then the chance of being guilty to increase, so $P(G|E_1) > P(G)$ and $P(G|E_2) > P(G)$. Moreover if two evidences conflict each other then he can't be guilty, so $P(G|E_1, E_2) = 0$, so $P(G) > P(G|E_1E_2)$.

26. (a) We find $P(L|M_1) = \dfrac{P(M_1|L)P(L)}{P(M_1)}$

$$= \frac{P(M_1|L)P(L)}{P(M_1|L)P(L) + P(M|L^c)P(L^c)}$$

$$= \frac{9/10 \cdot 1/10}{9/10\cdot 1/10 + (1-P(M^c|L^c))\cdot 9/10}$$

$$= \frac{9/100}{9/100 + 9/100} = \frac{1}{2}.$$

(b) We find

$$P(L|M_1, M_2) = \frac{P(M_1, M_2|L)P(L)}{P(M_1|M_2)}$$

$$= \frac{P(M_1|L)P(M_2|L)P(L)}{P(M_1)}$$

$$= \frac{(1/2)^2\cdot 1/10}{18/100}$$

$$= \frac{1}{\cancel{40}}\cdot\frac{\overset{5}{\cancel{100}}}{18} = \frac{5}{18}$$

(c) Given M_1 had already happened. If M_2 also occurred then the probability the email is legitimate given M_1 and M_2 occurred

31. Yes. Consider the event A independent to itself. Then $P(A) = P(A \cap A) = P(A)^2$, so either $P(A) = 1$ or $P(A) = 0$, so event A is either to guarantee to happen or it's impossible.

32. (a) $P(A > B) = P(A \text{ is } 4) = 4/6 = 2/3$,

$P(B > C) = P(C \text{ is } 2) = 4/6 = 2/3$

$$P(C > D) = P(C \text{ is } 6) + P(D \text{ is } 1 \text{ and } C \text{ is } 2)$$
$$= 2/6 + P(D \text{ is } 1 \mid C \text{ is } 2) P(C \text{ is } 2)$$
$$= 2/6 + 1/2 \cdot 4/6$$
$$= 2/6 + 2/6 = 2/3$$

$$P(D > A) = P(D \text{ is } 5) + P(A \text{ is } 0 \text{ and } D \text{ is } 1)$$
$$= 3/6 + P(D \text{ is } 1 \mid A \text{ is } 0) P(A \text{ is } 0)$$
$$= 3/6 + 3/6 \cdot 2/6$$
$$= 3/6 + 1/6 = 2/3.$$

(b) Events $A > B$ and $B > C$ are independent as their results don't interfere. However, $C > D$ and $D > A$ are since $P(D \text{ is } 1 \text{ and } C \text{ is } 2) = P(D \text{ is } 1) P(C \text{ is } 2)$ and $P(A \text{ is } 0 \text{ and } D \text{ is } 1) = P(A \text{ is } 0) P(D \text{ is } 1)$.

36. (a) Having a good maths score meant investing all time and resources into improving or maintaining it, which negatively affects any time for baseball.

(b) We have

$$P(A \mid BC) = \frac{P(ABC)}{P(BC)}$$

$$= \frac{P((A \cap B) \cap (A \cup B))}{P(B \cap (A \cup B))}$$

$$= \frac{P(A \cap B)}{P(B)} = P(A), \text{ and}$$

$$P(A \mid C) = \frac{P(AC)}{P(C)}$$

$$= \frac{P(A)}{P(C)}, \text{ so}$$

$$P(A \mid C) P(C) = P(A), \text{ hence}$$

$$P(A \mid C) P(C) = P(A \mid BC).$$

Therefore since $1 > P(C)$, we have
$$P(A \mid C) \cdot 1 > P(A \mid C) P(C), \text{ so}$$
$$P(A \mid C) > P(A \mid BC).$$

37. (a) Consider $C = D_1 \cup D_2$. Then via LOTP

$$P(W) = P(W \mid C)P(C) + P(W \mid C^c)P(C^c)$$

$$= P(W \mid C)P(C) + P(W \mid C^c)\, P(D_1^c \cap D_2^c)$$
$$= 1 \cdot P(C) + w_0 \left(P(D_1^c) P(D_2^c) \right)$$
$$= P(C) + w_0 (1 - P(D_1))(1 - P(D_2))$$
$$= P(C) + w_0 q_1 q_2 .$$

Hence since

$$P(C) = P(D_1 \cup D_2)$$
$$= P(D_1) + P(D_2) - P(D_1 \cap D_2)$$
$$= p_1 + p_2 - p_1 p_2 ,\text{ so}$$

we have $P(W) = p_1 + p_2 - p_1 p_2 + w_0 q_1 q_2$
$$= p_1 + p_2 q_1 + w_0 q_1 q_2 .$$

(b) $$P(D_1 \mid W) = \frac{P(W \mid D_1) P(D_1)}{P(W)}$$

$$= \frac{1 \cdot p_1}{p_1 + p_2 q_1 + w_0 q_1 q_2} = \frac{p_1}{p_1 + p_2 q_1 + w_0 q_1 q_2},$$

similarly $P(D_2 \mid W) = \dfrac{p_2}{p_1 + p_2 q_1 + w_0 q_1 q_2}$.

Finally, $$P(D_1, D_2 \mid W) = \frac{P(W \mid D_1, D_2) P(D_1, D_2)}{P(W)}$$

$$= \frac{1 \cdot P(D_1) P(D_2)}{p_1 + p_2 q_1 + w_0 q_1 q_2}$$

$$= \frac{p_1 p_2}{p_1 + p_2 q_1 + w_0 q_1 q_2}.$$

(c) Assuming $P(D_1, D_2 \mid W)$ is conditionally independent. Then we have

$$P(D_1, D_2 \mid W) = P(D_1 \mid W) P(D_2 \mid W)$$
$$= \left(\frac{p_1}{p_1 + p_2 q_1 + w_0 q_1 q_2} \right) \left(\frac{p_2}{p_1 + p_2 q_1 + w_0 q_1 q_2} \right)$$

$$= \frac{p_1 p_2}{(p_1 + p_2 q_1 + w_0 q_1 q_2)^2}.$$

However, this is false due to (b).

(d) Let $w_0 = 0$. Then neither of the disease will not develop any symptom, so D_1, D_2 are conditionally independent.

38. Given

$p = P(\text{spam})$, $p_j = P(W_j \mid \text{spam})$, and $r_j = P(W_j \mid \text{not spam})$.

Let $D = W_1^c, \dots W_{22}^c, W_{23}, W_{24}^c, \dots, W_{63}^c, W_{64}, W_{65}, W_{66}^c, \dots W_{100}^c$.

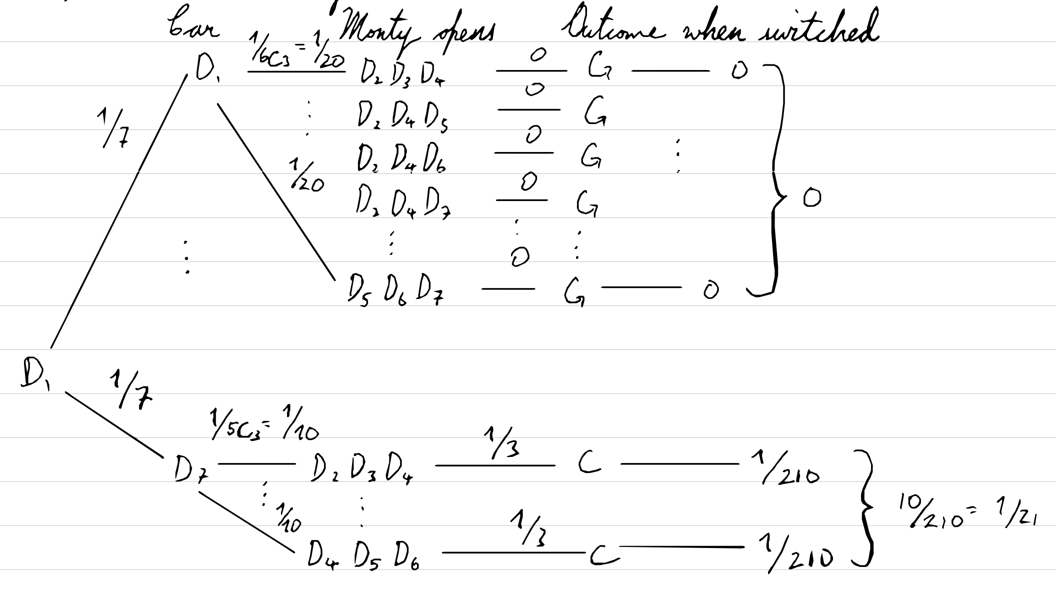
Then

$$P(\text{spam} \mid D) = \frac{P(D \mid \text{spam}) P(\text{spam})}{P(D)}$$

$$= \frac{P(D \mid \text{spam}) P(\text{spam})}{P(D \mid \text{spam}) P(\text{spam}) + P(D \mid \text{not spam}) P(\text{not spam})}$$

$$= \frac{\prod_1^{22}(1-p_i)\, p_{23} \prod_{24}^{63}(1-p_i)\, p_{64} p_{65} \prod_{66}^{100}(1-p_i)\, p}{\prod_1^{22}(1-p_i)\, p_{23} \prod_{24}^{63}(1-p_i)\, p_{64} p_{65} \prod_{66}^{100}(1-p_i)\, p + \prod_1^{22}(1-r_i)\, r_{23} \prod_{24}^{63}(1-r_i)\, r_{64} r_{65} \prod_{66}^{100}(1-r_i)(1-p)}$$

39. (a) When switching we have

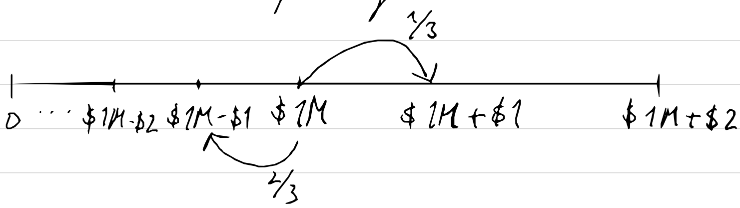


$\therefore$ $P(\text{Car when switching}) = 6/21 = 2/7$

(b) $P(\text{Car when switching}) = (n-1) / (n \cdot {}^{n-2}C_m)$

59, 68 - 70

51. We consider the following:



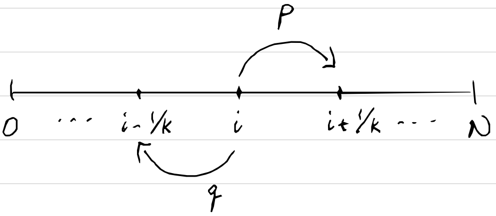
Let W be the event where we win the game.
Then we have

$$p_{\$1M} = \frac{1-\left(\frac{2/3}{1/3}\right)^{\$1M}}{1-\left(\frac{2/3}{1/3}\right)^{\$1M+2}}$$

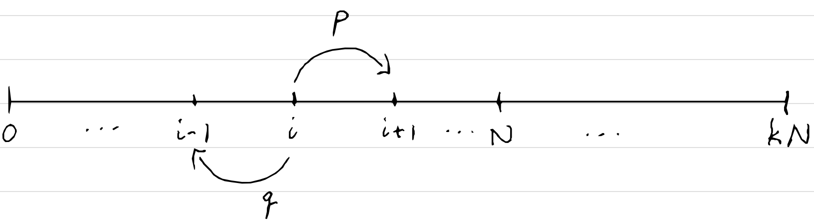
$$= \frac{1-2^{1M}}{1-2^{1M+2}} = \frac{2^{1M}-1}{2^{1M+2}-1}$$

$$> \frac{2^{1M}}{2^{1M+2}} = \frac{\cancel{2^{1M}}}{\cancel{2^{1M}}\,2^2} = \frac{1}{4},$$

52. We first consider the following



Hence we have



Then since $0 < p < 1/2$, we have

$$p_i = \frac{1-(q/p)^i}{1-(q/p)^{kN}} \approx \left(\frac{q}{p}\right)^{i-kN}.$$

Hence $\lim\limits_{k\to\infty} p_i \approx \lim\limits_{k\to\infty}\left(\frac{q}{p}\right)^{i-kN}$

$$= \lim_{k\to\infty} \frac{q/p^{\,i}}{(q/p)^{kN}}.$$

Thus since $q > p$, we have $(q/p)^{kN} \to \infty$,
so we have $p_i \to 0$.

68-70

59.

(a)

	Rich	Poor	Total
Dem	87	13	100
Rep	70	30	100
Total	157	43	200

(b) Given old using value from (a)

	Red	Blue	Total
Dem	87	13	100
Rep	70	30	100
Total	157	43	200

then if 10 people move from Blue to Red we have

	Red	Blue	Total
Dem	87	13	100
Rep	80	20	100
Total	167	33	200

Hence since $P_{old}(D) = P_{new}(D)$, we have

$P_{new}(D|B) = 13/33$, $P_{new}(D|B^c) = 87/167$

$P_{old}(D|B) = 13/43$, $P_{old}(D|B^c) = 87/157$

so we have

$P_{new}(D|B) > P_{old}(D|B)$ and

$P_{new}(D|B^c) > P_{old}(D|B^c)$, so yes it's possible.

Note that

$$\begin{aligned} P_{new}(D) &= P_{new}(D|B)P_{new}(B) + P_{new}(D|B^c)P_{new}(B^c) \\ &= 13/200 + 87/200 = 1/2. \end{aligned}$$

$$\begin{aligned} P_{old}(D) &= P_{old}(D|B)P_{old}(B) + P_{old}(D|B^c)P_{old}(B^c) \\ &= 13/200 + 87/200 = 1/2, \end{aligned}$$

so $P_{new}(D) = P_{old}(D)$ which means we didn't violate LOTP as B^c helps adjust.

68. (a) Suppose that getting the diseases is rare. Then not getting the disease is equally unlikely regardless of substance. Then we have

$$OR = \frac{odds(D|C)}{odds(D|C^c)}$$

$$= \frac{P(D|C)}{P(D^c|C)} \cdot \frac{P(D^c|C^c)}{P(D|C^c)}.$$

Hence since $P(D^c|C) = P(D^c|C^c)$, we have

$$OR = \frac{P(D|C)}{P(D|C^c)} = RR$$

(b) We have

$$OR = \frac{odds(D|C)}{odds(D|C^c)}$$

$$= \frac{P(D|C)}{P(D^c|C)} \cdot \frac{P(D^c|C^c)}{P(D|C^c)}$$

$$= \frac{P(D,C)\,P(C)^{-1}}{P(D^c,C)\,P(C)^{-1}} \cdot \frac{P(D^c,C^c)\,P(C^c)^{-1}}{P(D,C^c)\,P(C^c)^{-1}}$$

$$= \frac{P(D,C)}{P(D^c,C)} \cdot \frac{P(D^c,C^c)}{P(D,C^c)} = \frac{p_{11}\,p_{00}}{p_{10}\,p_{01}}.$$

(c) We show

$$OR = \frac{odds(C|D)}{odds(C|D^c)}.$$

From (b) we have

$$OR = \frac{p_{11}\,p_{00}}{p_{10}\,p_{01}} = \frac{p_{11}}{p_{01}} \cdot \frac{p_{00}}{p_{10}}$$

$$= \frac{P(D,C)}{P(D,C^c)} \cdot \frac{P(D^c,C^c)}{P(D^c,C)}$$

$$= \frac{P(C,D)}{P(C^c,D)} \cdot \frac{P(C^c,D^c)}{P(C,D^c)}$$

$$= \frac{P(C,D)\,P(D)^{-1}}{P(C^c,D)\,P(D)^{-1}} \cdot \frac{P(C^c,D^c)\,P(D^c)^{-1}}{P(C,D^c)\,P(D^c)^{-1}}$$

$$= \frac{P(C|D)}{P(C^c|D)} \cdot \frac{P(C^c|D^c)}{P(C|D^c)}$$

$$= \frac{odds(C|D)}{odds(C|D^c)}.$$

69.

(a) $P(\text{Yes}) = P(\text{Yes} \mid \text{Used illegal drugs slip})P(\text{Used illegal drugs slip}) + P(\text{Yes} \mid \text{Not used illegal drugs slip})P(\text{Not used illegal drugs slip})$.

Hence it follows

$$\begin{aligned} y &= d \cdot p + (1-d)(1-p) \\ &= dp + 1 - d - p + dp \\ &= 2dp - d - p + 1. \end{aligned}$$

(b) If $p = 0$ then we have $1-d$, which means those who says yes are not drug users.

(c) We have

$P(\text{Yes}) = P(\text{Yes} \mid \text{Used illegal drugs slip})P(\text{Used illegal drugs slip}) + P(\text{Yes} \mid \text{I'm born in winter slip})P(\text{I'm born in winter slip})$.

Hence it follows

$$\begin{aligned} y &= d \cdot p + 1/4 \cdot (1-p) \\ &= dp + 1/4 - p/4. \end{aligned}$$

70.

(a) Fred's friend is correct. Unless Fred predicted all 92 coins are heads right before its entire outcome, with its outcome being the same as Fred's prediction, then either Fred rigged the game or has unusually high luck.

(b) $$P(\text{Fair} \mid 92 \text{ heads}) = \frac{P(92 \text{ heads} \mid \text{Fair})P(\text{Fair})}{P(92 \text{ heads})}$$

$$= \frac{P(92 \text{ heads} \mid \text{Fair})P(\text{Fair})}{P(92 \text{ heads} \mid \text{Fair})P(\text{Fair}) + P(92 \text{ heads} \mid \text{Double sided})P(\text{Double sided})}$$

$$= \frac{1/2^{92} \cdot p}{1/2^{92} \cdot p + 1 \cdot (1-p)} = \frac{p/2^{92}}{p/2^{92} + 1 - p} = \frac{p/2^{92}}{(p + 2^{92} - 2^{92}p)/2^{92}}$$

$$= \frac{p}{p + 2^{92} - 2^{92}p}.$$

(c) Assuming it's $P(\text{Fair} \mid 92 \text{ heads}) \geq 0.5$

$$\frac{p}{p + 2^{92} - 2^{92}p} \geq \frac{1}{2},$$

$$\begin{aligned} 2p &\geq p + 2^{92} - 2^{92}p \\ p + 2^{92}p &\geq 2^{92} \\ p &\geq \frac{2^{92}}{1 + 2^{92}} \approx \frac{2^{92}}{2^{92}} = 1, \end{aligned}$$

so $p \geq 1$.

In addition, $P(\text{Fair} \mid 92 \text{ heads}) < 0.05$. Then

$$\frac{p}{p + 2^{92} - 2^{92}p} < \frac{1}{20},$$

$$\begin{aligned} 20p &< p + 2^{92} - 2^{92}p, \\ 19p + 2^{92}p &< 2^{92}, \\ p &< \frac{2^{92}}{19 + 2^{92}}, \end{aligned}$$

$p < 1$.

$$S = \{(0,0), (0,1), (1,0), (1,1)\}$$

$\therefore$ $X(s) = X(s_1, s_2) = s_1 + s_2$ ← heads

$Y(s) = 2 - X(s) = 2 - s_1 - s_2$ ← tails

$P(X=0)$, $P(X=1)$, $P(X=2)$

$\Updownarrow$

$P(\{s \in S \mid X(s) = 0\})$
$= P(\{(0,0)\}) = 1/4,$

$P(\{s \in S \mid X(s) = 420\}) =$
$P(\emptyset) = 0.$

$\forall x P(x)$

$\sim \forall x P(x) \leftrightarrow \exists x \sim P(x)$

$\sim \exists n \in \mathbb{N} (s(n) = 1) \leftrightarrow \forall n \in \mathbb{N} (s(n) = 1)$

$\exists n \in \mathbb{N} (s(n) \neq 1)$

$\forall n, m \in \mathbb{N} (s(n) = s(m) \rightarrow n = m)$