Motivation

2. Given points $P(2,-1,3)$, $Q(3,4,1)$, $R(4,-3,4)$, and $S(5,2,2)$, we have

$$\overrightarrow{PQ} = [1,5,-2]$$

$$\overrightarrow{PR} = [2,-2,1]$$

$$\overrightarrow{PS} = [3,3,-1]$$

Hence
$$\overrightarrow{QS} = [2,-2,1] = \overrightarrow{PR}$$
$$\overrightarrow{RS} = [1,5,-2] = \overrightarrow{PQ},$$
so we have
$$\overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PR} + \overrightarrow{RS}$$
$$= \overrightarrow{PQ} + \overrightarrow{PR}$$
$$= [3,3,-1] = \overrightarrow{PS},$$
so it's a parallelogram.

5. Given points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$, we have
$$\overrightarrow{OM} = \tfrac{1}{2}\cdot\overrightarrow{OP_1} + \tfrac{1}{2}\cdot\overrightarrow{OP_2},$$
where $O(0,0,0)$ is our origin.
Then
$$\overrightarrow{OM} = \tfrac{1}{2}\cdot[x_1, y_1, z_1] + \tfrac{1}{2}\cdot[x_2, y_2, z_2]$$
$$= [(x_1+x_2)/2, (y_1+y_2)/2), (z_1+z_2)/2].$$
Hence it follows our point
$$M((x_1+x_2)/2, (y_1+y_2)/2), (z_1+z_2)/2).$$
Thus we move halfway between points P_1 and P_2.

1.2 $\mathbb{R}^n$ and $\mathbb{C}^n$.

3. We prove (8) where $(cd)\vec{x} = c(d\vec{x})$ for all scalars c, d and vectors x.

Proof. Assume that $\vec{x} = [x_1, x_2, \ldots, x_n]$. Then by definition 1.2.1 we have the i th component of x to be $(cd)x_i = c(dx_i)$ via the associative law of our field $\mathbb{R}$. Thus it follows $(cd)\vec{x} = c(d\vec{x})$. ∎

4. We would have the following

(1) If $\vec{x}, \vec{y} \in F^1$ then $\vec{x} + \vec{y} = [x_1 + y_1] \in F^1$.

(2) If $c \in F$ and $\vec{x} \in F^1$ then $c\vec{x} = [cx_1] \in F^1$.

(3)
$$\begin{aligned}\vec{x}+\vec{y} &= [x_1]+[y_1] = [x_1+y_1] = [y_1+x_1]\\ &= [y_1]+[x_1] = \vec{y}+\vec{x} \quad \text{for all vectors } \vec{x},\vec{y} \in F^1.\end{aligned}$$

(4)
$$\begin{aligned}(\vec{x}+\vec{y})+\vec{z} &= ([x_1]+[y_1])+[z_1]\\ &= [x_1+y_1]+[z_1]\\ &= [(x_1+y_1)+z_1]\\ &= [x_1+(y_1+z_1)]\\ &= [x_1]+[y_1+z_1]\\ &= [x_1]+([y_1]+[z_1])\\ &= \vec{x}+(\vec{y}+\vec{z})\end{aligned}$$
for all vectors $\vec{x}, \vec{y}, \vec{z} \in F^1$.

(5)
$$\begin{aligned}\vec{x}+\vec{0} &= [x_1]+[0] = [x_1+0]\\ &= [x_1] = \vec{x} = [x_1]\\ &= [0+x_1] = [0]+[x_1]\\ &= \vec{0}+\vec{x} \quad \text{for all vectors } \vec{x} \in F.\end{aligned}$$

(6)
$$\begin{aligned}\vec{x}+(-\vec{x}) &= [x_1]+(-[x_1])\\ &= [x_1]+[-x_1]\\ &= [x_1+(-x_1)]\\ &= [0] = \vec{0}\\ &= [0] = [-x_1+x_1]\\ &= [-x_1]+[x_1]\\ &= (-[x_1])+[x_1]\\ &= -\vec{x}+\vec{x} \quad \text{for all vectors } \vec{x} \in F^1.\end{aligned}$$

(7)
$$\begin{aligned}c(\vec{x}+\vec{y}) &= c([x_1]+[y_1])\\ &= c([x_1+y_1])\\ &= [c(x_1+y_1)]\\ &= [cx_1+cy_1]\\ &= [cx_1]+[cy_1]\\ &= c[x_1]+c[y_1]\\ &= c\vec{x}+c\vec{y}\ ;\end{aligned}$$
$$\begin{aligned}(c+d)\vec{x} &= (c+d)[x_1]\\ &= [(c+d)x_1]\\ &= [cx_1+dx_1]\\ &= [cx_1]+[dx_1]\\ &= c[x_1]+d[x_1]\\ &= c\vec{x}+d\vec{x}\end{aligned}$$
for all vectors $\vec{x}, \vec{y} \in F^1$, and scalars $c, d \in F$.

(8)
$$\begin{aligned}(cd)\vec{x} &= (cd)[x_1]\\ &= [(cd)x_1]\\ &= [c(dx_1)]\\ &= c[dx_1]\\ &= c(d[x_1])\\ &= c(d\vec{x}) \quad \text{for all vectors } \vec{x} \in F^1.\end{aligned}$$

(9)
$$\begin{aligned}1\cdot\vec{x} &= 1\cdot[x_1]\\ &= [1\cdot x_1]\\ &= [x_1]\\ &= \vec{x} \quad \text{for all vectors } \vec{x} \in F^1.\end{aligned}$$

Therefore from (1)-(9) we see F^1 being identical to F.

Exercise.

We rewrite the symbols as

(1) $x+y \in V$ for all $x, y \in V$.
(2) for all $x \in V$, for all $c \in F$, $cx \in V$.
(3) $x+y = y+x$ for all $x, y \in V$.
(4) $(x+y)+z = x+(y+z)$ for all $x, y, z \in V$.
(5) there exits a $\vec{0} \in V$ such that $\vec{0}+x = x = x+\vec{0}$ for all $x \in V$.
(6) for all $x \in V$ there exits $-x \in V$ such that $x+(-x) = \vec{0} = -x+x$.
(7) $c(x+y) = cx+cy$, $(x+y)d = xd+yd$ for all $x, y \in V$ and for all $c, d \in F$.
(8) for all $c, d \in F$ and for all $x \in V$ $(cd)x = c(dx)$.
(9) $1x = x$ for all $x \in V$.

Exercise

Given $V = \mathcal{F}(T, F)$ we have as follow.

(3) If $x, y \in V$ then $(x+y)(t) = x(t)+y(t) \in F$ for all $t \in T$.

(4) If $x \in V$ and $c \in F$ then $(cx)(t) = cx(t) \in F$ for all $t \in T$.

(5) There exists $\vec{0} \in V$ such that for all $x \in V$ we have
$$\begin{aligned}(\vec{0}+x)(t) &= \vec{0}(t)+x(t)\\ &= 0+x(t)\\ &= x(t)\\ &= x(t)+0\\ &= x(t)+\vec{0}(t)\\ &= (x+\vec{0})(t) \quad \text{for all } t \in T.\end{aligned}$$

(6) For all $x \in V$ there exits $-x \in V$ such that
$$\begin{aligned}(x+(-x))(t) &= x(t)+(-x)(t)\\ &= 0\\ &= \vec{0}(t)\\ &= 0\\ &= -x(t)+x(t)\\ &= (-x+x)(t) \quad \text{for all } t \in T.\end{aligned}$$

(7) For all $x, y \in V$ and $c, d \in F$ we have
$$\begin{aligned}(c(x+y))(t) &= c(x+y)(t)\\ &= c(x(t)+y(t))\\ &= cx(t)+cy(t) \in F;\end{aligned}$$
$$\begin{aligned}((x+y)d)(t) &= (x+y)(t)d\\ &= (x(t)+y(t))d\\ &= x(t)d+y(t)d \in F,\end{aligned}$$
for all $t \in T$.

(8) For all $c, d \in F$ and $x \in V$ we have
$$\begin{aligned}((cd)x)(t) &= (cd)x(t)\\ &= c(dx(t))\\ &= (c(dx))(t) \in F,\end{aligned}$$
for all $t \in T$.

(9) For all $x \in V$ we have
$$\begin{aligned}(1\cdot x)(t) &= 1\cdot x(t)\\ &= x(t) \in F, \quad \text{for all } t \in T.\end{aligned}$$

Next for $T = \{1, 2, \ldots, n\}$ we would have $x(1), x(2), x(3), \ldots, x(n)$, so by giving an n-tuple we get $(x(1), x(2), \ldots, x(n))$ which is akin to F^n with our current $V = \mathcal{F}(T, F)$.