3.6 Example.

Proof. Given the set $\{x \in \emptyset \mid P(x)\}$. Then by Definition 3.5 for every $x \in \emptyset$ satifies property $P(x)$. Thus since $\emptyset$ is empty, by the Axiom of Extensionality $\{x \in \emptyset \mid P(x)\} = \emptyset$. ∎

Proof. Consider the set C' where $A \in C'$ and $B \in C'$. Then we choose element $x \in C$ where $x = A$ or $x = B$, so by the Axiom of Pair $x \in C'$. Thus by the Axiom of Extensionality we have $C = C'$. ∎

3.7

We prove U is unique.

Proof. Suppose we have an arbitrary set S. Next consider a set U' where $x \in U'$. Then by the Axiom of Union we have $x \in A$ for some $A \in S$. Then we have $x \in U$ as well. Thus by the Axiom of Extensionality $U = U'$. ∎

Exercises

3.1. *Proof.* Consider $P(x)$ to be "$x \notin B$". Then by the Axiom of Schema Comprehension $x \in A$ and $x \notin B$ implies $x \in \{y \in A \mid y \notin B\}$. Thus there is a set $\{y \in A \mid y \notin B\}$. $\blacksquare$

3.2. *Proof.* Suppose that the set A exits via our Weak Axiom of Existence. Let $P(x)$ to be "$x \neq x$". Then by the Axiom of Schema Comprehension we have $\{x \in A \mid x \neq x\}$. Thus since all elements of A are equal to itself, $\{x \in A \mid x \neq x\}$ has no elements, namely $\emptyset$. $\blacksquare$

3.3 (a) *Proof.* By contradiction let V be the set of all sets. Next let $P(x)$ be "$x \notin x$" and $x \in V$. Then by definition 3.5 we have the set $\{x \in V \mid x \notin x\}$. Thus if $V \in \{x \in V \mid x \notin x\}$ then by the axiom of schema comprehension $V \in V$ and $V \notin V$. Thus the set of all sets is an element to itself and not to itself, which contradicts. $\blacksquare$

(b) *Proof.* Let A be an arbitrary set. By contradiction suppose that every $x \in A$. Then A is the set of all sets, which contradicts due (a). $\blacksquare$

3.4. *Proof.* Consider two sets C and C'. Then for any $x \in C$ and $x \in C'$ we have $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$. Thus we have $C \subseteq C'$ and $C' \subseteq C$. Thus by the axiom of extensionality we have $C = C'$. For existence consider $x \in \{y \in A \mid y \notin B\}$ or $x \in \{y \in B \mid y \notin A\}$. Then by the axiom of union we have $x \in \{y \in A \mid y \notin B\} \cup \{y \in B \mid y \notin A\}$. Thus we have $x \in C$. $\blacksquare$

3.5. (a) *Proof.* Let $P = \{A, B\} \cup \{C\}$. $(\rightarrow)$ Suppose that $x \in P$. Then by the axiom of union we have $x \in \{A, B\}$ or $x \in \{C\}$, so $x = C$. Thus by the axiom of pair we have $x = A$ or $x = B$ or $x = C$. $(\leftarrow)$ Now suppose that $x = A$ or $x = B$ or $x = C$. Then by the axiom of pair we have $x \in \{A, B\}$ or $x = C$, so $x \in \{C\}$. Hence by the axiom of union we have $x \in \{A, B\} \cup \{C\}$, so $x \in P$. $\blacksquare$

(b) *Proof.* Let $P = \{A, B\} \cup \{C, D\}$. $(\rightarrow)$ Suppose that $x \in P$. Then by the axiom of union we have $x \in \{A, B\}$ or $x \in \{C, D\}$. Thus by the axiom of pair we have $x = A$ or $x = B$, or $x = C$ or $x = D$. $(\leftarrow)$ Now suppose that $x = A$ or $x = B$ or $x = C$ or $x = D$. Then the axiom of pair we have $x \in \{A, B\}$ or $x \in \{C, D\}$. Thus by the axiom of union $x \in \{A, B\} \cup \{C, D\}$, so $x \in P$. $\blacksquare$

3.6. *Proof.* By contradiction suppose that $\mathcal{P}(X) \subseteq X$ holds for any X. Let $Y = \{u \in X \mid u \notin u\}$ and $Y \in \mathcal{P}(X)$. Next we prove that $Y \notin X$. By contradiction suppose that $Y \in X$. Then we show $Y \in Y$ or $Y \notin Y$. If $Y \in Y$ then by the axiom of schema comprehension $Y \in X$ and $Y \notin Y$, which contradicts. Now if $Y \notin Y$ then we have $Y \notin X$ or $Y \in Y$. Hence if $Y \in Y$ then immediately that contradicts, and if $Y \notin X$ then immediatly that contradicts our hypothesis. Thus all cases show that $Y \notin X$. Hence since $Y \in \mathcal{P}(X)$ and $\mathcal{P}(X) \subseteq X$, we have $Y \in X$, which we contradicted earlier, so it follows $\mathcal{P}(X) \nsubseteq X$. $\blacksquare$

3.7 i) Weak axiom of pair.
Let A and B be arbitrary sets. By the weak axiom of pair we choose a set C such that $A \in C$ and $B \in C$. $(\rightarrow)$ Suppose that $x \in C$. Let $P(x)$ be "$x = A$ or $x = B$". Then by the axiom of schema comprehension we have $\{x \mid x \in A \text{ or } x = B\}$. Thus they have the same elements, so by the axiom of extensionality $C = \{x \mid x = A \text{ or } x = B\}$. Hence since $x \in C$, we have $x = A$ or $x = B$. $(\leftarrow)$ Suppose that $x = A$ or $x = B$. If $x = A$ then by the weak axiom of pair $x \in C$. Likewise if $x = B$ then $x \in C$. Thus both cases give $x \in C$. $\blacksquare$

ii) Let S be an arbitrary set. By the weak axiom of union we choose a set U where $X \in A$ and $A \in S$ implies $X \in U$. $(\rightarrow)$ Suppose that $x \in U$. Let $P(x)$ be "$x \in A$ for some $A \in S$." Then by the axiom of schema comprehension we have $\{x \mid x \in A \text{ for some } A \in S\}$. Hence there sets have the same elements, so by the axiom of extensionality we have $U = \{x \mid x \in A \text{ for some } A \in S\}$. Thus we have $x \in A$ for some $A \in S$. $(\leftarrow)$ Now suppose that $x \in A$ for some $A \in S$. Then $x \in A$ and $A \in S$, so by the weak axiom of union we have $x \in U$. $\blacksquare$

iii) Let S be an arbitrary set. By the weak axiom of power set we choose a set $\mathcal{P}$ where $X \subseteq S$ implies $X \in \mathcal{P}$. $(\rightarrow)$ Suppose that $X \in \mathcal{P}$. Let $P(X)$ be "$X \subseteq S$". Then by the axiom of schema comprehension we have $\{X \mid X \subseteq S\}$. Hence these sets have the same elements, so by the axiom of extensionality we have $\mathcal{P} = \{X \mid X \subseteq S\}$. Thus $X \in \mathcal{P}$. $(\leftarrow)$ Now suppose that $X \subseteq S$. Then by the weak axiom of power set we have $X \in \mathcal{P}$. $\blacksquare$

4. Proof. Let A be an arbitrary set. By contradiction suppose that the "complement" of A exits. Let A be an empty set. Then its "complement" is the set of all sets, which doesn't exits. Therefore its complement doesn't exits. $\blacksquare$

5. Let $S \neq \emptyset$ and A be an arbitrary set.

(a) Given $T_1 = \{Y \in \mathcal{P}(A) \mid Y = A \cap X \text{ for some } X \in S\}$.

Proof. ($\leftarrow$) Suppose that $a \in \bigcup T_1$. Then by the Axiom of Union we have $a \in Y$ for some $Y \in T_1$. Thus we have $Y = A \cap X$ for some $X \in S$. Hence we have $a \in A$ and $a \in X$ for some $X \in S$, so $a \in \bigcup S$. Thus $a \in A \cap \bigcup S$. ($\rightarrow$) Suppose that $b \in A \cap \bigcup S$ Then we have $b \in A$ and $b \in \bigcup S$. Thus by the Axiom of Union we have $b \in X$ for some $X \in S$. Hence we have $b \in A \cap X$ for some $X \in S$. Thus since $A \cap X \in T_1$ and $b \in A \cap X$, by the the Axiom of Union we have $b \in \bigcup T_1$. $\blacksquare$

(b) Given $T_2 = \{Y \in \mathcal{P}(A) \mid Y = A - X \text{ for some } X \in S\}$

i) Proof. ($\leftarrow$) Suppose that $a \in \bigcap T_2$. Then by the definition of intersection we have $a \in Y$ for all $Y \in T_2$. Hence we have $a \in A - X$ for some $X \in S$. Thus $a \in A$ and $a \notin X$ for some $X \in S$. Hence by the Axiom of Union we have $a \notin \bigcup S$, so we have $a \in A - \bigcup S$. ($\rightarrow$) Now suppose that $b \in A - \bigcup S$. Then we have $b \in A$ and $b \notin \bigcup S$. Hence by the Axiom of Union we have $b \notin X$ for all $X \in S$. Thus we have $b \in A - X$ for all $X \in S$. Thus we have every $Y = A - X \in T_2$, so we have $b \in Y$ for every $Y \in T_2$. Therefore by the definition of the intersection we have $b \in \bigcup T_2$. $\blacksquare$

ii) Proof. ($\leftarrow$) Suppose that $a \in \bigcup T_2$. Then by the Axiom of Union we have $a \in Y$ for some $Y \in T_2$. Hence we have $a \in A - X$ for some $X \in S$. Thus we have $a \in A$ and $a \notin X$ for some $X \in S$, so by the definition of intersection we have $a \notin \bigcap S$. Hence $a \in A - \bigcap S$. ($\rightarrow$) Now suppose that $b \in A - \bigcap S$. Then we have $b \in A$ and $b \notin \bigcap S$, so by the definition of intersection we have $b \notin X$ for some $X \in S$. Thus we have $b \in A - X$ for some $X \in S$. Hence we have $Y = A - X \in T_2$, so $b \in Y$. Therefore by the Axiom of Union we have $b \in \bigcup T_2$. $\blacksquare$

6. Proof. Consider an arbitrary element $A \in S$ and the property $P(x, S)$ to be "$x \in Q$ for all $Q \in S$". Then by the Axiom of Schema Comprehension we have $\{x \in A \mid x \in Q, \text{ for all } Q \in S\}$. Hence by the Axiom of Extensionality we have $\{x \in A \mid x \in Q, \text{ for all } Q \in S\} = \{x \in A \mid \text{for all } A \in S\} = \bigcap S$. $\blacksquare$

Hence due to $A \in S$ we can't have S to be empty.

1-5 , 1-4, 6, 7

1.1 Note that we have

$$\begin{aligned}\mathcal{P}(\mathcal{P}(\{a,b\})) &= \mathcal{P}(\{\emptyset, \{a\}, \{b\}, \{a,b\}\}) \\ &= \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \\ &\quad \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a,b\}\}, \\ &\quad \{\{a\}, \{b\}\}, \{\{a\}, \{a,b\}\}, \{\{b\}, \{a,b\}\}, \\ &\quad \{\emptyset, \{a\}, \{b\}\}, \{\emptyset, \{a\}, \{a,b\}\}, \\ &\quad \{\emptyset, \{b\}, \{a,b\}\}, \{\{a\}, \{b\}, \{a,b\}\}, \\ &\quad \{\emptyset, \{a\}, \{b\}, \{a,b\}\}\}.\end{aligned}$$

Thus we have an element where $\{\{a\}, \{a,b\}\} = (a,b)$, so $(a,b) \in \mathcal{P}(\mathcal{P}(\{a,b\}))$.

Next we consider $\bigcup(a,b) = \bigcup\{\{a\}, \{a,b\}\}$
$= \{a\} \cup \{a,b\} = \{a,b\}$,
so it follows $a, b \in \bigcup(a,b)$

1.2 Proof. (i) We first prove (a,b) exists. Consider the property $P(X, \{a\}, \{a,b\})$ to mean "$X = \{a\}$ or $X = \{a,b\}$". Then by the Axiom of Pair we have $\{X \mid X = \{a\} \text{ or } X = \{a,b\}\}$. Hence by the Axiom of Extensionality we have
$$\{X \mid X = \{a\} \text{ or } X = \{a,b\}\} = \{\{a\}, \{a,b\}\} = (a,b).$$

(ii) Next we prove (a,b,c) exists. Consider the property $P(X, \{(a,b)\}, \{(a,b),c\})$ to mean "$X = \{(a,b)\}$ or $X = \{(a,b),c\}$". Then by the Axiom of Pair we have
$$\{X \mid X = \{(a,b)\} \text{ or } X = \{(a,b),c\}\}.$$
Hence by the Axiom of Extensionality we have
$$\{X \mid X = \{(a,b)\} \text{ or } X = \{(a,b),c\}\} = \{\{(a,b)\}, \{(a,b),c\}\} = ((a,b),c) = (a,b,c).$$

(iii) Finally we prove (a,b,c,d) exists. Consider the property $P(X, \{(a,b,c)\}, \{(a,b,c),d\})$ to mean $X = \{(a,b,c)\}$ or $X = \{(a,b,c),d\}$. Then by the Axiom of Pair we have
$$\begin{aligned}&\{X \mid X = \{(a,b,c)\} \text{ or } X = \{(a,b,c),d\}\} \\ &= \{\{(a,b,c)\}, \{(a,b,c),d\}\} \quad \text{via Axiom of Extensionality} \\ &= ((a,b,c),d) = (a,b,c,d).\end{aligned}$$ ∎

1.3. Proof. Suppose that $(a,b) = (b,a)$. Then we have
$$\{\{a\}, \{a,b\}\} = \{\{b\}, \{a,b\}\}.$$
Thus we have $\{a\} = \{b\}$, so $a = b$. ∎

1.4. Proof. Suppose that $(a,b,c) = (a',b',c')$. Then we have
$$((a,b),c) = ((a',b'),c').$$
Thus by Theorem 1.2 we have $(a,b) = (a',b')$ and $c = c'$. Hence by Theorem 1.2 again we have $a = a'$, $b = b'$, and $c = c'$. Thus for the quaruples $(a,b,c,d) = (a',b',c',d')$ we have $((a,b,c),d) = ((a',b',c'),d')$, so by earlier we have $a = a'$, $b = b'$, $c = c'$, and $d = d'$. ∎

1.5. Proof. By contradiction suppose that $((a,b),c) = (a,(b,c))$. Then by Theorem 1.2 we have $(a,b) = a$ and $c = (b,c)$. Thus if we consider $a = c = \emptyset$ and $b \neq 0$ then we have $\emptyset = (\emptyset, b) = (b, \emptyset)$, which are both false. Hence $((a,b),c) \neq (a,(b,c))$.

Next we prove $((a,b,c),d) \neq (a,(b,c,d))$. Suppose otherwise. Then by Theorem 1.2 we have $a = (a,b,c)$ and $d = (b,c,d)$. Thus if $a = d = \emptyset$, $b \neq \emptyset$, $c \neq \emptyset$, and $b \neq c$ then $\emptyset = (\emptyset, b, c) = (b, c, \emptyset)$, which are both false. Hence we have
$$((a,b,c),d) \neq (a,(b,c,d)).$$ ∎

1.6. We first consider the analagous theorem
$$\langle a,b \rangle = \langle a',b' \rangle \text{ iff } a = a' \text{ and } b = b'.$$
Proof. ($\rightarrow$) Suppose that $\langle a,b \rangle = \langle a',b' \rangle$. Then we have
$$\{\{a, \square\}, \{b, \Delta\}\} = \{\{a', \square\}, \{b', \Delta\}\}.$$
Thus we have $\{a, \square\} = \{a', \square\}$ and $\{b, \Delta\} = \{b', \Delta\}$. Hence $a = a'$ and $b = b'$. ($\leftarrow$) Now suppose that $a = a'$ and $b = b'$. Then immediately $\langle a,b \rangle = \langle a',b' \rangle$. ∎

We define ordered triples as
$$\langle a,b,c \rangle = \{\{a, \square\}, \{b, \Delta\}, \{c, \bigcirc\}\},$$
and quadruples as
$$\langle a,b,c,d \rangle = \{\{a, \square\}, \{b, \Delta\}, \{c, \bigcirc\}, \{d, \cancel{\Delta}\}\}.$$

Exercise

2.1 (i) We first prove $x \in A$ and $y \in A$.
Proof Let $A = \bigcup(\bigcup R)$. Suppose that $(x,y) \in R$. Then by contradiction consider $x \notin A$ or $y \notin A$.
Case 1. $x \notin A$. Then by the Axiom of Union we have $x \notin X$ for all $X \in \bigcup R$. Hence since $X \in \bigcup R$, we have $X \in Y$ for some $Y \in R$. Thus since $(x,y) \in R$, we have $X \in (x,y)$, so $X = \{x\}$ or $X = \{x,y\}$. However, we assumed $x \notin X$, which contradicts both cases, so $x \in X$.
Case 2. $y \notin A$. Then similar to earlier we have $X' = \{x\}$ or $X' = \{x,y\}$ for all $X' \in \bigcup R$. However, this contradicts as well, so $y \in A$.
Therefore $x \in A$ and $y \in A$. $\blacksquare$

(ii) Next we show $\operatorname{dom} R$ and $\operatorname{ran} R$ exits.
Proof. Given by (i) we have $(x,y) \in R$ to imply $x \in A$, $y \in A$. Then consider the property $P(z)$ to mean "there is some z where $(x,y) \in R$." Thus since $x \in A$ and $P(x)$, by the Axiom of Schema Comprehension and Definition 2.3 (a) we have $x \in \operatorname{dom} R$. Hence in a similar manner we have $y \in \operatorname{ran} R$. $\blacksquare$

2.2 (a) We show that R^{-1} and $S \circ R$ exits.
(i) Proof. Suppose that $(x,y) \in R$. Then by exercise 2.1 we have $x \in \operatorname{dom} R$ and $y \in \operatorname{ran} R$. Now consider the property "there is some x, y where $(x,y) \in R$." Then by the Axiom of Schema Comprehension we have

$$(y,x) \in \{(y,x) \in \operatorname{ran} R \times \operatorname{dom} R \mid \exists x,y \text{ where } (x,y) \in R\}.$$

Hence by Definition 2.7 we have $(y,x) \in R^{-1}$, so $R^{-1} \subseteq \operatorname{ran} R \times \operatorname{dom} R$ exits. $\blacksquare$

(ii) Proof. Suppose that $(x,y') \in R$ and $(y'',z) \in S$. Then by Exercise 2.1 we have $x \in \operatorname{dom} R$ and $z \in \operatorname{ran} S$. Next consider the property "there is a y where xRy and ySz". Then by the Axiom of Schema Comprehension we have

$$(x,z) \in \{(x,z) \in \operatorname{dom} R \times \operatorname{ran} S \mid \exists y, xRy \text{ and } ySz\}.$$

Hence by Definition 2.10 we have $(x,z) \in S \circ R$, so $S \circ R \subseteq \operatorname{dom} R \times \operatorname{ran} S$ exits. $\blacksquare$

(b) Proof. Suppose that $a \in A$, $b \in B$, and $c \in C$. Then by the Axiom of the Power Set we have

$$A \times B = \{(a,b) \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \text{for some } a \in A \text{ and } b \in B\}.$$

Thus it follows

$$(A \times B) \times C = \{((a,b),c) \in \mathcal{P}(\mathcal{P}((A \times B) \cup C)) \mid \exists (a,b) \in A \times B, c \in C\}$$

to exits. Thus since $((a,b),c) = (a,b,c)$, we have $(a,b,c) \in (A \times B) \times C$ to imply $(a,b,c) \in A \times B \times C$, and vice versa. Thus $(A \times B) \times C = A \times B \times C$ exits. $\blacksquare$

2.3 (a) ($\rightarrow$) Suppose that $y \in R[A \cup B]$. Then we have $y \in \operatorname{ran} R$ such that we choose some $x \in A \cup B$ where xRy. Thus we have $x \in A$ or $x \in B$, so we choose two cases.
Case 1. $x \in A$. Then we have $y \in R[A]$, so $y \in R[A] \cup R[B]$.
Case 2. $x \in B$. Then we have $y \in R[B]$, so $y \in R[A] \cup R[B]$.
Hence we have $y \in R[A] \cup R[B]$. Thus $R[A \cup B] \subseteq R[A] \cup R[B]$.
($\leftarrow$) Suppose that $y' \in R[A] \cup R[B]$. Then we have $y' \in R[A]$ or $y' \in R[B]$, so two cases.
Case 1. $y' \in R[A]$. Then we have $y \in \operatorname{ran} R$ such that we choose some $x' \in A$ where $x'Ry'$. Hence since $x' \in A$, we have $x' \in A \cup B$. Thus we have $y' \in R[A \cup B]$.
Case 2. $y' \in R[B]$. Then in a similar way we have $y' \in R[A \cup B]$.
Hence we have $y' \in R[A \cup B]$. Therefore $R[A \cup B] = R[A] \cup R[B]$. $\blacksquare$

(b) Suppose that $y \in R[A \cap B]$. Then we have $y \in \operatorname{ran} R$ such that we choose some $x \in A \cap B$ where xRy. Hence since $A \cap B \subseteq A$, we have $x \in A$. Hence $y \in R[A]$. Moreover, since $A \cap B \subseteq B$, we have $x \in B$. Thus $y \in R[B]$. Therefore since $y \in R[A]$ and $y \in R[B]$, we have $y \in R[A] \cap R[B]$, so $R[A \cap B] \subseteq R[A] \cap R[B]$. $\blacksquare$

(c) Suppose that $y \in R[A] - R[B]$. Then $y \in R[A]$ and $y \notin R[B]$. Hence since $y \in R[A]$, we have $y \in \operatorname{ran} R$ such that we choose some $x \in A$ where xRy. Now by contradiction let $x \in B$. Then we have $x \in A \cap B$, so we have $y \in R[A \cap B]$. Hence by (b) we have $y \in R[A] \cap R[B]$, so $y \in R[A]$ and $y \in R[B]$. However, we have $y \notin R[B]$, so $x \notin B$. Therefore $y \in R[A-B]$. $\blacksquare$

(d) We show $R[A] \cap R[B] \nsubseteq R[A \cap B]$.
Consider $A = \{1,2\}$, $B = \{2,3\}$, and

$$R = \{(1,1),(1,3),(2,2),(3,3)\}.$$

Then we have

$$R[A] = \{1,2,3\}$$
$$R[B] = \{2,3\}$$
$$R[A \cap B] = \{2\}$$

Hence $R[A] \cap R[B] = \{2,3\}$, and $\{2,3\} \nsubseteq \{2\}$.

Next we show $R[A-B] \nsubseteq R[A] - R[B]$.
Consider $A = \{1,2,3\}$, $B = \{2,3\}$, and we have

$$R = \{(1,2),(2,2),(3,3),(2,1)\}.$$

Then we have

$$R[A] = \{2,3\},$$
$$R[B] = \{1,2,3\},$$
$$R[A-B] = \{2\}.$$

Hence we have $\{2\} \nsubseteq \emptyset$.

e) (a) We prove $R^{-1}[A] \cup R^{-1}[B] = R^{-1}[A \cup B]$.
Proof. Consider $x \in R^{-1}[A] \cup R^{-1}[B]$. Then $x \in R^{-1}[A]$ or $x \in R^{-1}[B]$, so we consider two cases.
Case 1. $x \in R^{-1}[A]$. Then $x \in \operatorname{dom} R$, there exits $y \in A$ where xRy. Hence we have $y \in A$ or $y \in B$, so $y \in A \cup B$. Thus $x \in R^{-1}[A \cup B]$, so $R^{-1}[A] \cup R^{-1}[B] \subseteq R^{-1}[A \cup B]$.
Case 2. $x \in R^{-1}[B]$. Then in a similarly $R^{-1}[A] \cup R^{-1}[B] \subseteq R^{-1}[A \cup B]$.
Now consider $x' \in R^{-1}[A \cup B]$. Then $x' \in \operatorname{dom} R$, there is a $y' \in A \cup B$ where xRy. Thus $y' \in A$ or $y' \in B$, so consider two cases.
Case 1. $y' \in A$. Then $x' \in R^{-1}[A]$
Case 2. $y' \in B$. Then $x' \in R^{-1}[B]$.
Therefore $R^{-1}[A \cap B] \subseteq R^{-1}[A] \cup R^{-1}[B]$, so they're equal. $\blacksquare$

e) (b) We prove $R^{-1}[A \cap B] \subseteq R^{-1}[A] \cap R^{-1}[B]$.
Proof. Consider $x \in R^{-1}[A \cap B]$. Then by the definition of R^{-1} we have $x \in \operatorname{dom} R$ such that there exits a $y \in A \cap B$. Thus we have $y \in A$ and $y \in B$. Hence we have $x \in R^{-1}[A]$ and $x \in R^{-1}[B]$, so $x \in R^{-1}[A] \cap R^{-1}[B]$. Therefore since x is arbitrary, $R^{-1}[A \cap B] \subseteq R^{-1}[A] \cap R^{-1}[B]$. $\blacksquare$

e) (c) We prove $R^{-1}[A-B] \supseteq R^{-1}[A] - R^{-1}[B]$.
Proof. Consider $x \in R^{-1}[A] - R^{-1}[B]$. Then we have $x \in R^{-1}[A]$ and $x \notin R^{-1}[B]$. Hence since $x \in R^{-1}[A]$, we have $x \in \operatorname{dom} R$ such that there is a $y \in A$ where xRy. Now by contradiction consider $y \in B$. Then we have $y \in A \cap B$, so it follows $x \in R^{-1}[A \cap B]$. Hence by (b) we have $x \in R^{-1}[A]$ and $x \in R^{-1}[B]$. However, this contradicts since $x \notin R^{-1}[B]$. Thus $y \notin B$, so $x \in R^{-1}[A-B]$. Therefore since x is arbitrary we have $R^{-1}[A-B] \supseteq R^{-1}[A] - R^{-1}[B]$. $\blacksquare$

e) (d)
(b) Consider $A = \{1,2\}$, $B = \{2,3\}$, and

$$R = \{(1,1),(3,1),(2,2),(3,3)\}.$$

Then we have $R^{-1} = \{(1,1),(1,3),(2,2),(3,3)\}$.
Thus

$$R^{-1}[A] = \{1,2\},\ R^{-1}[B] = \{1,2,3\}, \text{ and}$$
$$R^{-1}[A \cap B] = R^{-1}[\{2\}] = \{2\}.$$

However, $R^{-1}[A] \cap R^{-1}[B] \nsubseteq R^{-1}[A \cap B]$, since $\{1,2\} \cap \{1,2,3\} = \{2,3\}$, but $\{2,3\} \nsubseteq \{2\}$, so they can't be equal.

(c) Consider $A = \{1,2,3\}$, $B = \{2,3\}$, and

$$R = \{(1,2),(2,2),(3,3),(2,1)\}.$$

Then $R^{-1} = R$. Thus

$$R^{-1}[A-B] = R^{-1}[\{1\}] = \{2\},$$
$$R^{-1}[A] = \{1,2,3\}, \text{ and } R^{-1}[B] = \{1,2,3\}.$$

Thus $R^{-1}[A] - R^{-1}[B] = \emptyset$, but then $R^{-1}[A-B] \nsubseteq R^{-1}[A] - R^{-1}[B]$, since $\{2\} \nsubseteq \emptyset$. $\blacksquare$

(f) (i) We prove $R^{-1}[R[A]] \supseteq A \cap \operatorname{dom} R$.
Proof. Suppose that $x \in A \cap \operatorname{dom} R$. Then we have $x \in A$ and $x \in \operatorname{dom} R$. Hence by definition of $\operatorname{dom} R$ there exits a $y \in B$ such that xRy. Thus we have $y \in \operatorname{ran} R$, so $y \in R[A]$. Hence since $x \in \operatorname{dom} R$, $y \in R[A]$, and xRy, we have $x \in R^{-1}[R[A]]$. $\blacksquare$

(ii) $R[R^{-1}[B]] \supseteq B \cap \operatorname{ran} R$.
Proof. Suppose that $y \in B \cap \operatorname{ran} R$. Then we have $y \in B$ and $y \in \operatorname{ran} R$. Hence by definition of $\operatorname{ran} R$ there exits a $x \in A$ such that xRy. Thus we have $x \in \operatorname{dom} R$, so $x \in R^{-1}[B]$. Hence since $y \in \operatorname{ran} R$, $x \in R^{-1}[B]$, and xRy, we have $y \in R[R^{-1}[B]]$. $\blacksquare$

(iii) Counterexamples
(i) We show that $R^{-1}[R[A]] \nsubseteq A \cap \operatorname{dom} R$.
Consider intervals $A = (0,1)$ and $B = (-1,1)$, and let $R = \{(x,x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$.
Then we have the interval

$$R[A] = (0,1), \text{ so}$$
$$R^{-1}[(0,1)] = (-1,1),$$

so $(0,1) \cap \mathbb{R} = (0,1)$, but $(-1,1) \nsubseteq (0,1)$.

(ii) We show $R[R^{-1}[B]] \nsubseteq B \operatorname{ran} R$.
Consider intervals $A = (-1,1)$ and $B = (0,1)$, and let $R = \{(x,x^2) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$.
Then we have the interval

$$R[B] = (0,1), \text{ so}$$
$$R^{-1}[R[B]] = (-1,1),$$

so $B \cap \operatorname{ran} R = (0,1) \cap \mathbb{R}^+ = (0,1)$, but $(-1,1) \nsubseteq (0,1)$. $\blacksquare$

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2.4 Let $R \subseteq X \times Y$.

(a) *Proof.*

(i) We first prove $R[X] = \operatorname{ran} R$.
($\rightarrow$) Suppose that $a \in R[X]$. Then by Definition 2.5(a) we have $a \in \operatorname{ran} R$ where we choose some $x \in X$ for which xRa. Hence immediately, $R[X] \subseteq \operatorname{ran} R$.
($\leftarrow$) Now suppose that $a \in \operatorname{ran} R$. Then by Definition 2.3(b) we choose an x' such that $x'Ra$. Hence since $R \subseteq X \times Y$, we have $x' \in X$. Thus since $a \in \operatorname{ran} R$, some $x' \in X$, and $x'Ra$, by Definition 2.5(a) we have $a \in R[X]$. Therefore $\operatorname{ran} R \subseteq R[X]$, so $R[X] = \operatorname{ran} R$. ∎

(ii) Next we prove $R^{-1}[Y] = \operatorname{dom} R$.
($\rightarrow$) Suppose that $b \in R^{-1}[Y]$. Then by Definition 2.5(b) we have $b \in \operatorname{dom} R$ where we choose some $y \in Y$ for which bRy. Hence immediately, $R^{-1}[Y] \subseteq \operatorname{dom} R$.
($\leftarrow$) Now suppose that $b \in \operatorname{dom} R$. Then by Definition 2.3(a) we choose some y' such that bRy'. Hence since $R \subseteq X \times Y$, we have $y' \in Y$. Thus since $b \in \operatorname{dom} R$, some $y' \in Y$, and bRy', by Definition 2.5(b) we have $b \in R^{-1}[Y]$. Therefore $\operatorname{dom} R \subseteq R^{-1}[Y]$, so we have $R^{-1}[Y] = \operatorname{dom} R$. ∎

(b)(i) *Proof* by contrapositive. Consider $R[\{a\}] \neq \emptyset$. Then we have a nonempty element where we choose some $y \in R[\{a\}]$. Thus by Def. 2.5(a) we have $y \in \operatorname{ran} R$ such that we choose some $x \in \{a\}$ where xRy. Hence we have $x = a$, so by Def. 2.3(a) we have $x \in \operatorname{dom} R$, so $a \in \operatorname{dom} R$. Hence by contrapositive we have $a \notin \operatorname{dom} R$ implies $R[\{a\}] = \emptyset$. ∎

(ii) *Proof* by contrapositive. Consider $R^{-1}[\{a\}] \neq \emptyset$. Then it's nonempty, so we chose some $x \in R^{-1}[\{a\}]$. Hence by Def. 2.5(b) we have $x \in \operatorname{dom} R$ such that we choose some $y \in \{a\}$ where xRy. Thus we have $y = a$, so by Def 2.3(b) we have $y \in \operatorname{ran} R$, so $a \in \operatorname{ran} R$. Therefore by contrapositive we have $a \notin \operatorname{ran} R$ implies $R^{-1}[\{a\}] = \emptyset$. ∎

(c)(i) *Proof.* Consider $x \in \operatorname{dom} R$. Then $x \in X$ such that there is some $y \in Y$ where xRy. Thus we have $yR^{-1}x$, so $x \in \operatorname{ran} R^{-1}$. Hence $\operatorname{dom} R \subseteq \operatorname{ran} R^{-1}$. Next consider $y' \in \operatorname{ran} R^{-1}$. Then $y' \in Y$ such that we have some $x' \in X$ where $(x', y') \in R^{-1}$. Thus we have $(y', x') \in R$, so $y' \in \operatorname{dom} R$. Hence we have $\operatorname{ran} R^{-1} \subseteq \operatorname{dom} R$, so $\operatorname{ran} R^{-1} = \operatorname{dom} R$. ∎

(ii) *Proof.* Consider $y \in \operatorname{ran} R$. Then $y \in Y$ such that there is some $x \in X$ where xRy. Thus we have $yR^{-1}x$, so $y \in \operatorname{dom} R^{-1}$. Hence $\operatorname{ran} R \subseteq \operatorname{dom} R^{-1}$. Now assume $x' \in \operatorname{dom} R^{-1}$. Then $x' \in X$ such that there is some $y' \in Y$ where $(x', y') \in R^{-1}$. Thus we have $(y', x') \in R$. Hence $x' \in \operatorname{ran} R$. Therefore we have $\operatorname{dom} R^{-1} \subseteq \operatorname{ran} R$, so $\operatorname{ran} R = \operatorname{dom} R^{-1}$. ∎

(d) *Proof.* Suppose that $z \in R$. Then we have $(x, y) \in R$ for some $x \in X$ and $y \in Y$. Hence $(y, x) \in R^{-1}$, so $(x, y) \in (R^{-1})^{-1}$. Hence $R \subseteq (R^{-1})^{-1}$. Next consider that $z' \in (R^{-1})^{-1}$. Then we have $(x', y') \in (R^{-1})^{-1}$. Hence $(y', x') \in R^{-1}$, so $(x', y') \in R$. Thus we have $(R^{-1})^{-1} \subseteq R$. Thus $(R^{-1})^{-1} = R$. ∎

(e)

(i) We first prove $R^{-1} \circ R \supseteq I_{\operatorname{dom} R}$.
Proof. Suppose that $z \in I_{\operatorname{dom} R}$. Then we have $z = (a, b)$ for all $a, b \in \operatorname{dom} R$, and $a = b$. Hence since $a \in \operatorname{dom} R$, we have $a \in X$ where we choose some $c \in Y$ such that we have $(a, c) \in R$. Hence since $a = b$, $(b, c) \in R$. Thus we have $(a, c) \in R$ and $(c, b) \in R^{-1}$, so we have $(a, b) \in R^{-1} \circ R$, so $z \in R^{-1} \circ R$. ∎

(ii) Next we prove $R \circ R^{-1} \supseteq I_{\operatorname{ran} R}$.
Proof. Suppose that $z \in I_{\operatorname{ran} R}$. Then we have $z = (a, b)$ for all $a, b \in \operatorname{ran} R$ and $a = b$. Hence since $a, b \in \operatorname{ran} R$, we have $a, b \in Y$ where choose some $c \in X$ such that we have $(c, a) \in R$. Thus since $a = b$, $(c, b) \in R$, so we have $(a, c) \in R^{-1}$, hence $(a, b) \in R \circ R^{-1}$. Therefore $z \in R \circ R^{-1}$, so $I_{\operatorname{ran} R} \subseteq R \circ R^{-1}$. ∎

2.5

We have $\mathcal{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

(a) $\in_Y = \{(a, b) \mid a, b \in Y \text{ and } a \in b\}$
$= \{(\emptyset, \{\emptyset\}), (\emptyset, \{\emptyset, \{\emptyset\}\}),$
$(\{\emptyset\}, \{\{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\})\}$

(b) $I_Y = \{(a, b) \mid a, b \in Y \text{ and } a = b\}$
$= \{(\emptyset, \emptyset), (\{\emptyset\}, \{\emptyset\}),$
$(\{\{\emptyset\}\}, \{\{\emptyset\}\}),$
$(\{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}\})\}$.

2.6 We prove $T \circ (S \circ R) = (T \circ S) \circ R$.

Proof. Suppose that $z \in T \circ (S \circ R)$. Then we have