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COMPUTING METHODS

Linear Algebra Done Wrong

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目录

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第 1 章　Chapter one: Basic Notion

1.1 Vector spaces

1.1.1

1.1. Let $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$, $\mathbf{z} = (4, 2, 1)^T$. Compute $2\mathbf{x}$, $3\mathbf{y}$, $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Solution

$$2x = [2, 4, 6]^T = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \tag{1.1}$$

$$3y = [3y_1, 3y_2, 3y_3]^T = \begin{bmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{bmatrix} \tag{1.2}$$

$$x + 2y - 3z = [1 + 2y_1 - 12, 2 + 2y_2 - 6, 3 + 2y_3 - 3] = \begin{bmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{bmatrix} \tag{1.3}$$

1.1.2

1.2. Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces. Justify your answer.

a) The set of all continuous functions on the interval $[0, 1]$;
b) The set of all non-negative functions on the interval $[0, 1]$;
c) The set of all polynomials of degree *exactly* n;
d) The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^n$ such that $A^T = A$.

Solution

1.1.3

1.3. True or false:

a) Every vector space contains a zero vector;
b) A vector space can have more than one zero vector;
c) An $m \times n$ matrix has m rows and n columns;
d) If f and g are polynomials of degree n, then $f+g$ is also a polynomial of degree n;
e) If f and g are polynomials of degree at most n, then $f+g$ is also a polynomial of degree at most n

Solution

True; True; True; True; False

1.1.4

1.4. Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Solution

Assumption 1. *Exist 2 different zero vectors* $0_1, 0_2$.

For any $v \in V$, we have

$$v + 0_1 = v \tag{1.4}$$

$$v + 0_2 = v \tag{1.5}$$

So, we got

$$0_1 - 0_2 = 0, \text{then } 0_1 = 0_2, \text{The zero vector is unique.} \tag{1.6}$$

1.1.5

1.5. What matrix is the zero vector of the space $M_{2\times 3}$?

Solution

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.7}$$

1.1.6

1.6. Prove that the additive inverse, defined in Axiom 4 of a vector space is unique.

Solution

Assumption 2. *Exist 2 different additive inverses* w_1, w_2*.*

For any $v \in V$, we have

$$v + w_1 = 0 \tag{1.8}$$
$$v + w_2 = 0 \tag{1.9}$$

So, we got

$$w_1 - w_2 = 0, \text{then } w_1 = w_2, \text{The additive inverse is unique.} \tag{1.10}$$

1.1.7

1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Solution

1. We have $v \in V$
2. Acoording to **Multiplicative associativity**, we have $0v = (\beta 0)v = 0(\beta v)$
3. $0v - \beta(0v) = (1 - \beta)0v = 0$, so $0v = 0$.

1.1.8

1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Solution

1. According to **1.6**: additive inverse is unique.
2. $v - (-1)v = (1 - 1)v = 0$， so $(-1)v = -v$.

1.2 Linear combinations,base.

1.2.1

2.1. Find a basis in the space of 3×2 matrices $M_{3\times 2}$.

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.11}$$

1.2.2

2.2. True or false:

a) Any set containing a zero vector is linearly dependent

b) A basis must contain $\mathbf{0}$;

c) subsets of linearly dependent sets are linearly dependent;

d) subsets of linearly independent sets are linearly independent;

e) If $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n = \mathbf{0}$ then all scalars α_k are zero;

Solution

UNKNOWN; UNKNOWN; FALSE; TRUE; TRUE.

1.2.3

2.3. Recall, that a matrix is called *symmetric* if $A^T = A$. Write down a basis in the space of *symmetric* 2×2 matrices (there are many possible answers). How many elements are in the basis?

Solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.12}$$

1.2.4

2.4. Write down a basis for the space of

a) 3×3 symmetric matrices;

b) $n \times n$ symmetric matrices;

c) $n \times n$ *antisymmetric* ($A^T = -A$) matrices;

Solution

a)

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \tag{1.13}$$

b) And so on.

c)

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \tag{1.14}$$

1.2.5

2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent. **Hint:** Take for $\mathbf{v}_{r+1}$ any vector that cannot be represented as a linear combination $\sum_{k=1}^{r} \alpha_k \mathbf{v}_k$ and show that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

Solution

Let V be our overlying vector space. Consider $v_1, \ldots, v_n \in V$ such that they are linearly independent but $\text{span}(v_1, \ldots, v_n) \neq V$ (this is what it means by $v_1, \ldots, v_n$ does not "generate" the space, not every vector in V can be written in terms of our set. You may want to investigate the definition of span). Then there exists a vector $v_{n+1} \in V$ such that $v_{n+1} \notin \text{span}(v_1, \ldots, v_n)$ by our assumption that they are not generating. Now, for the sake of contradiction, assume $v_1, \ldots, v_{n+1}$ are linearly dependent. Then there exists scalars $b_1, \ldots, b_{n+1}$ not all 0 such that

$$b_1 v_1 + \cdots + b_{n+1} v_{n+1} = 0$$

We have that the only b that can be non-zero is b_{n+1} (why?). Then we can rearrange our equation and have that

$$v_{n+1} = -\frac{b_1}{b_{n+1}} v_1 - \cdots - \frac{b_n}{b_{n+1}} v_n$$

which is a contradiction to our assumption that $v_{n+1} \notin \text{span}(v_1, \ldots, v_n)$. So our set of $n+1$ vectors must be linearly independent.

1.2.6

2.6. Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*?

Solution

1. According to the statement, we can get

$$\begin{cases} \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \\ \beta_1 (v_1 + v_2) + \beta_2 (v_2 + v_3) + \beta_3 (v_1 + v_3) = 0 \end{cases} \rightarrow \begin{cases} \alpha_1 = \beta_1 + \beta_3 = 0 \\ \alpha_2 = \beta_2 + \beta_3 = 0 \\ \alpha_3 = \beta_1 + \beta_2 = 0 \end{cases} \tag{1.15}$$

2. We can know that $\beta_1, \beta_2, \beta_3$ can be arbitrary. So, it is impossible to be linearly independent.