Neoclassical Growth Transition Dynamics with One-Sided Commitment*

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Abstract

This paper characterizes the transition dynamics of a continuous-time neoclassical production economy with capital accumulation in which households face idiosyncratic income risk and cannot commit to repay their debt. Therefore, even though a full set of contingent claims that pay out conditional on the realization of idiosyncratic shocks is available, the equilibrium features imperfect insurance and a non-degenerate cross-sectional consumption distribution. When household labor productivity takes two values, one of which is zero, and the utility function is logarithmic, we characterize the entire transition dynamics induced by unexpected technology shocks, including the evolution of the consumption distribution, in closed form. We then use these analytical transition results to study the speed of convergence in income per capita of a poor (low TFP) to a rich (high TFP) economy and the evolution of consumption inequality over time in response to an increase in idiosyncratic income risk.

JEL Codes: E21, D11, D91, G22

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1 Introduction

Households face considerable idiosyncratic income and unemployment risk. Following the foundational work of Bewley (1986), Huggett (1993) and Aiyagari (1994) in quantitative macroeconomics, a large literature has arisen studying the macroeconomic consequences of this risk on the micro level, both theoretically as well as empirically.[1] The key assumption in much of this work (henceforth denoted as standard incomplete markets, SIM) is that the idiosyncratic risk is uninsurable, in the sense that explicit market or informal insurance arrangements are by assumption absent, and the best households can do is to engage in self-insurance through the accumulation of assets whose payoff is non-contingent on the realization of the idiosyncratic risk.

However, there is now considerable evidence that households seem able to insulate consumption better against idiosyncratic wage or income risks than what is implied by models of self-insurance. Blundell, Pistaferri and Preston (2008) develop a (by now standard) methodology to empirically measure the extent of consumption insurance against permanent and transitory income shocks, and Kaplan and Violante (2011) show that, quantitatively a standard life-cycle version of the SIM model implies too little insurance especially against permanent income shocks. A substantial follow-up literature, which includes Arellano, Blundell and Bonhomme (2017), Eika, Mogstad and Vestad (2020), Chatterjee, Morley and Singh (2021), Balke and Lamadon (2022), Commault (2022) and Braxton et al. (2023) has largely confirmed these findings. Thus, alternatives to the conventional self-insurance approach encoded in the SIM model are needed.

To make a contribution to this goal, in this paper we introduce limited commitment in the tradition of Kehoe and Levine (1993, 2001), Kocherlakota (1996), and Alvarez and Jermann (2000) into the same physical environment that Aiyagari (1994) studied with standard incomplete markets. Specifically, we develop and study a continuous time general equilibrium neoclassical production economy with idiosyncratic income risk and explicit insurance contracts against these risks. We assume that households cannot honor their debts, and therefore cannot sell these contracts short, limiting the extent of insurance households can achieve. Effectively, therefore, ours is a model with a full set of Arrow securities and tight short-sale constraints at zero.[2]

[1] See, e.g., Krueger, Mitman and Perri (2016) for an overview of this literature. It may be appropriate to also point to Imrohoroglu (1989) and the PhD thesis by Uhlig (1990), the latter of which also featured a choice of households between risky and riskless investments. The thesis is available here: https://voices.uchicago.edu/haralduhlig/thesis/.

[2] As Krueger and Uhlig (2006) show, this asset market structure is equivalent to a one-sided limited com-

The purpose of this paper is to understand the consequences of introducing this limited insurance as cleanly as possible, by examining the most tractable scenario in which closed-form solution of the entire macroeconomic dynamics can be given. The analysis here thus seeks to serve as an important stepping stone and complements to a more quantitative and empirical, but ultimately less tractable investigation. To this end, we assume that household labor productivity takes two values, one of which is zero, and that the period utility function is logarithmic. In Krueger and Uhlig (2025), we analytically characterized the steady state of this model for an arbitrary number of income states. In this paper, we show that the entire transition path of the economy induced by an MIT aggregate (transitory or permanent) productivity shock, including the dynamic evolution of the non-degenerate consumption and wealth distribution can be given in closed form as long as the aggregate shock is not too large.[3] This complete analytical tractability of the transition path not only sets our model apart from the standard Aiyagari (1994) SIM model, but also contrasts with the representative agent neoclassical growth model without any idiosyncratic income risk (or equivalently, with complete markets) for which no closed-form solution of its transitional dynamics is available.

This analytical tractability originates from the fact that under the assumptions made, the population endogenously separates into two groups: one group with only labor income but no capital income, and a second group with no labor income but heterogeneous asset holdings and thus asset incomes. Crucially, this latter group shares the same consumption growth rate and effective saving rate, which (given log utility) is a constant that does not depend on the current or future interest rates. This second group then aggregates exactly (both in steady state and along the transition), and the resulting macro economy is also characterized by a constant aggregate saving rate, as in the classic Solow model or as in the model of workers and entrepreneurs by Moll (2014), but unlike in the standard neoclassical growth model or the SIM model. As Sato (1963) and Jones (2000) show, the nonlinear ordinary differential equation characterizing the aggregate dynamics of an economy with constant saving rate is a Bernoulli differential equation with a closed-form solution – the same is then true in our economy. We wish to emphasize, though, that in contrast to the Solow model, the constant aggregate saving rate is a result rather than an assumption,

mitment model in which perfectly competitive and perfectly committed insurance companies offer long-term consumption insurance contracts to households that cannot commit to these long-term contracts and can switch to competing intermediaries without punishment. Without punishment, short-sale constraints at zero are then precisely Alvarez and Jermann's (2000) solvency constraints that are "not too tight."

[3]The transition path could also be induced by an initial capital stock that is not at its steady state value.

and that this rate depends on the structural parameters of the model, including the time preference rate as well as the parameters governing the idiosyncratic income process.

Given the dynamics of the aggregate capital stock, the speed of convergence to the new steady state and the entire transition path of the consumption distribution in response to the MIT shock can also be characterized in closed form. To demonstrate the potential usefulness of our tractable model for applied-quantitative work, we establish two results. First, we show that our model can slow down the speed of convergence of capital to its long-run steady state (relative to the standard neoclassical growth model), and therefore potentially contribute to a resolution of the puzzle originally identified by King and Rebelo (1993) that neoclassical convergence dynamics tends to be too fast, relative to what is observed in the data. Second, we show that in the model, consumption inequality is "procyclical:" it increases on impact in response to an (expansionary) positive productivity shock before converging back to its original level in the long run.

1.1 Related Literature

In this paper, we seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, as Bewley (1986), Imrohoroglu (1989), Uhlig (1990), Huggett (1993), and Aiyagari (1994). In a recent paper, Achdou et al. (2022) analyze a two-state continuous-time SIM model. As we do, they characterize the stationary equilibrium by two key differential equations: one governing the optimal solution of the consumption (self-)insurance problem, and one characterizing the associated stationary distribution. The papers complement each other by characterizing equilibria in the same physical environment, but with two different market structures. Furthermore, we achieve a full analytical characterization of the entire transition path of the economy, possibly opening a path for an analytical analysis of macroeconomic fluctuations.[4]

The second branch is the literature on recursive contracts and endogenously incomplete markets which permits explicit insurance, but whose scope is limited by contract enforcement frictions.[5] Specifically, we incorporate explicit insurance contracts offered by com-

[4]Heathcote, Storesletten and Violante (2014) construct an analytically tractable endowment economy with incomplete markets. Key to their results is that transitory shocks are assumed to be perfectly insurable whereas permanent shocks are not insurable at all.

[5]Recent work that builds on Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann

petitive financial intermediaries, as analyzed previously in partial equilibrium by Krueger and Uhlig (2006), into a neoclassical production economy. In doing so, we seek to provide the macroeconomic literature with a novel, fully micro-founded yet analytically tractable model of neoclassical investment, production, and the cross-sectional consumption distribution where the limits to cross-insurance are explicitly derived from contractual frictions. In Krueger and Uhlig (2025), we theoretically characterize steady states in a model where idiosyncratic risk follows an arbitrary N-state continuous-time Poisson process, and here we characterize the transitional dynamics of the model in closed form when labor productivity takes two values (because only then can the transition analysis be given in closed form) and apply it to the evolution of consumption inequality and income per capita convergence along the transition. Finally, in Ando, Krueger and Uhlig (2023) we study asset pricing in the model with aggregate risk cast in discrete time. While the approach and formulation in these papers are described from the perspective of a financial market, one can alternatively think of the insurance contracts offered by financial intermediaries as long-term wage contracts offered by firms that provide workers with partial insurance against productivity fluctuations to workers, in line with the formulation in Harris and Holmstrom (1982), Thomas and Worrall (1988), Guiso, Pistaferri and Schivardi (2005), Saporta-Eksten (2016), and Balke and Lamadon (2022).

Finally, our paper shares elements and insights with other work on dynamic macro models with limited commitment in endowment economies, such as Grochulski and Zhang (2011), Zhang (2013) and Miao and Zhang (2015), but models capital accumulation and production explicitly.[6] By casting our model in general equilibrium we follow Hellwig and Lorenzoni (2009), Gottardi and Kubler (2015) and Martins-da-Rocha and Santos (2019) but in contrast to these endowment economy papers emphasize the transitional dynamics induced by capital accumulation.[7]

In the next section, we describe the model and define the equilibrium. Section 3 characterizes the optimal household consumption-asset allocation for a given sequence of wages

(2000) includes Broer (2012), Kovrijnykh (2013), Abraham and Laczo (2018), and Sargent, Wang and Yang (2021). A common theme in this literature is the interaction between private and public insurance, see, e.g., Golosov and Tsyvinski (2007), Thomas and Worrall (2007), and Krueger and Perri (2011).

[6] An alternative literature introduces private information into economies with aggregate production and capital accumulation, see, e.g., Golosov and Tsyvinski (2007) and Khan and Ravikumar (2020).

[7] Furthermore, whereas Hellwig and Lorenzoni (2009) and Martins-da-Rocha and Santos (2019), motivated by the original Bulow and Rogoff (1989) paper, postulate that (sovereign) default can be punished at least with exclusion from future borrowing, we assume that there is no punishment for default at all. We discuss the implied differences for steady state interest rates in greater detail in Krueger and Uhlig (2025).

and interest rates. Section 4 aggregates these allocations to analytically characterize the equilibrium transition path, starting from an initial steady state with partial consumption insurance. It also provides sufficient conditions on the aggregate productivity and idiosyncratic risk processes for a partial insurance transition equilibrium to exist. Section 5 discusses two potential applications of our theoretical results: Section 5.1 contrasts the speed of convergence in our model to that of the representative agent neoclassical growth model, and Section 5.2 traces out the response of consumption inequality to an increase in idiosyncratic income risk along the transition. Section 6 concludes. All proofs and additional technical details and results are contained in the Appendix.

2 The Model

2.1 Endowments, Preferences, Technology and Financial Markets

Time is continuous. There is a unit mass continuum of infinitely lived agents who supply labor to the market, consume goods, and purchase financial assets.

2.1.1 Idiosyncratic Labor Productivity

Labor productivity z_{it} of an individual agent i at time t follows a two-state Poisson process that is independent across agents. More precisely, productivity can either be high ($z_{it} = \zeta_t > 0$) or zero ($z_{it} = 0$). Let $Z_t = \{0, \zeta_t\}$. The transition from high to low productivity occurs at rate $\xi_t > 0$, whereas the transition from low to high productivity occurs at rate $\nu > 0$. We permit the transition rate ξ_t to be time-varying to allow income risk at the household level and thus cross-sectional income inequality to be time-varying. Denote the associated share of low-productivity and high-productivity households by $(\Psi_{l,t}, \Psi_{h,t})$. If the transition rates are constant at (ξ, ν), then the stationary distribution is given by

$$(\Psi_l, \Psi_h) = \left(\frac{\xi}{\xi + \nu}, \frac{\nu}{\xi + \nu} \right). \tag{1}$$

We assume that the initial productivity distribution is given by (1). In addition, we normalize productivity such that aggregate labor efficiency units are equal to 1 for all t:

$$L_t = \zeta_t \Psi_{h,t} = 1. \tag{2}$$

Using equation (2), the cross-sectional variance of the productivity process is given by

$$Var_t(z_{it}) = (1-\Psi_{h,t})(0-1)^2 + \Psi_{h,t}(\zeta_t - 1)^2 = \zeta_t - 1 = (\Psi_{h,t})^{-1} - 1. \tag{3}$$

Thus, when the risk ξ_t of losing productivity rises and the share of high-productivity agents $\Psi_{h,t}$ falls, the cross-sectional variance of productivity rises.

2.1.2 Preferences

Agents have log utility $u(c) = \log(c)$ and discount the future at rate $\rho > 0$. Then the expected utility of an agent from time t onward is given by

$$U_t = E_t \left[\int_t^{+\infty} e^{-\rho(\tau - t)} \log(c)\, d\tau \right], \tag{4}$$

where the expectation depends on the current idiosyncratic state and risk of the agent.[8]

2.1.3 Technology

There is a competitive sector of production firms which uses labor and capital to produce the final output good according to the Cobb-Douglas production function $A_t F(K_t, L_t) = A_t K_t^{\theta} L_t^{1-\theta}$, where $\theta \in (0,1)$ denotes the capital share and $A_t > 0$ is a productivity parameter, evolving as an exogenous and non-stochastic function of time. Capital depreciates at a constant rate $\delta \geq 0$. Production firms seek to maximize profits, taking as given the market spot wage w_t per efficiency unit of labor and rental rate of capital (net of depreciation) r_t per unit of capital. We impose the following assumptions on the exogenous time-varying processes throughout the paper.

Assumption 0. *For all $t > 0$, both A_t and ξ_t are differentiable functions of time t and converge to finite and strictly positive limits,*

$$\tilde{A} = \lim_{t \to +\infty} A_t, \tilde{A} \in (0, +\infty), \tag{5}$$

$$\tilde{\xi} = \lim_{t \to +\infty} \xi_t, \tilde{\xi} \in (0, +\infty). \tag{6}$$

[8] We abstract from aggregate risk in this paper. A number of our results for the deterministic transition analysis generalize to CRRA utility, see Online Appendix H.

Furthermore, the process ξ_t (and the associated $\Psi_{h,t}$ and ζ_t) satisfies the normalization $L_t = 1$ in (2), for all t.

Note that this assumption permits productivity A_t and risk ξ_t to jump (or have a kink) at $t = 0$. It merely stipulates that after this initial MIT shock, productivity and risk evolve smoothly and converge to finite limits.

2.1.4 Financial Markets

As in Krueger and Uhlig (2006, 2022, 2024), agents seek to insure themselves against their productivity fluctuations. We assume that a full set of individual-specific insurance contracts is available, but individuals cannot commit to honor these contracts and there is no punishment from default. There are two ways to formulate the resulting consumption insurance contracts and the associated market structures, which turn out to be equivalent.

First, envision a market structure in which individuals buy long-term consumption insurance contracts from risk-neutral and perfectly competitive insurance companies. These financial intermediaries can fully commit to contracts, whereas individuals cannot commit and individuals can switch insurers without cost at any time, i.e., there is one-sided limited commitment. The intermediaries offer the utility-maximizing consumption allocation to individuals, subject to breaking even and subject to not losing an individual to the competition. This is the market structure that Krueger and Uhlig (2006) studied.

That paper, following the insights of Alvarez and Jermann (2000), also showed that this formulation is equivalent to an asset market-based structure in which individuals own assets (in the form of physical capital), either by themselves or through an account at a financial intermediary, and given this capital, maximize lifetime utility by buying idiosyncratic shock-contingent Arrow securities subject to state-contingent short-sale constraints. This is the formulation we pursue here. The key result in Krueger and Uhlig (2006), reminiscent of Bulow and Rogoff (1989), is that limited commitment by households and no punishment from default implies that individuals cannot borrow at all in this capital account. Note that the capital account is state-contingent, and its balance can jump when productivity changes and otherwise evolves due to new (possibly negative) investment x, given the current agent-specific state and calendar time t.

2.1.5 Household Optimization Problem

Denote by $U_t(k;z)$ the expected continuation lifetime utility of the agent, given the current capital account k, agent-specific productivity z and the aggregate state of the economy encapsulated by the time index t. This lifetime utility satisfies the Hamilton-Jacobi-Bellman (HJB) equation defining the optimal consumption-asset allocation.

Definition 1. *For $z \in Z$, wages w_t and interest rates r_t, let $\tilde{z}$ be the "other" state and $p_{z,t} \in \{\xi_t, \nu\}$ be the Poisson intensity for the transition from z to $\tilde{z}$. An optimal consumption allocation $\mathcal{C}_t = \left(U_t(k;z), c_t(k;z), x_t(k;z), \tilde{k}_t(k;z)\right)_{k\geq 0, z\in Z}$ is the solution to the program*

$$\begin{aligned} \rho U_t(k;z) &= \max_{c,\tilde{k}\geq 0,x} u(c) + \dot{U}_t(k;z) + U_t'(k;z)\,x + p_{z,t}\left(U_t\left(\tilde{k},\tilde{z}\right) - U_t(k;z)\right) && (7) \\ \textit{s.t.} \quad & c + x + p_{z,t}\left(\tilde{k} - k\right) = r_t k + w_t z && (8) \\ & x \geq 0 \quad \text{if} \quad k = 0. && (9) \end{aligned}$$

To build intuition for the HJB, consider the household problem in competitive equilibrium of the standard deterministic neoclassical growth model where the agent receives a wage w and owns capital k, earning interest r. The HJB equation in that model reads as

$$\begin{aligned} \rho U(k) &= \max_{c\geq 0,x}\{u(c) + U'(k)x\} \quad \text{s.t.} \quad c + x = rk + w \quad \text{or} \\ \rho U(k) &= \max_{c\geq 0}\{u(c) + U'(k)[rk + w - c]\}. \end{aligned}$$

The flow payoff $\rho U(k)$ of the value function $U(k)$ is the sum of the flow utility $u(c)$ from consuming c and the change in the value function $U'(k)\,x$ due to the investment $\dot{k}_t = x$. Investment and consumption have to respect the budget constraint $c + x = rk + w$.

The fact that wages and interest rates (w_t, r_t) are time-varying adds a time subscript to the value function $U_t(\cdot)$, and the payoff now also includes the time derivative in the value function $\dot{U}_t(\cdot)$ due to changing factor prices. In the presence of idiosyncratic labor productivity risk, the current state of the household includes both capital as well as current productivity (k, z), and the value function becomes a function of both these individual state variables. In addition, labor income is now $w_t z$.

For Definition 1, two further crucial features are added that embed the financial market structure with explicit insurance but limited commitment. First, the flow payoff $\rho U_t(k;z)$ also accounts for the expected instantaneous change in utility $p_{z,t}\left(U_t\left(\tilde{k};\tilde{z}\right) - U_t(k;z)\right)$

due to a possible change in productivity from z to $\tilde{z}$. Note the crucial feature that the capital stock upon a productivity change, $\tilde{k}$, is allowed to differ from the current one, k. This is the feature of explicit insurance against idiosyncratic agent-specific shocks, in contrast to the standard incomplete markets model which requires $\tilde{k} = k$. The change in the capital stock has to be paid for, though, which explains the actuarially fair "insurance premium" $p_{z,t}\left(\tilde{k} - k\right)$ in budget constraint[9] (8). Second, the lack of commitment is incorporated by the restriction that $\tilde{k} \geq 0$, as well as $x \geq 0$ when $k = 0$. Without punishment for walking away from a negative capital account (defaulting on an intermediary if the account is held with them), Krueger and Uhlig (2006) show that the state-contingent borrowing limits that are not too tight, in the sense of Alvarez and Jermann (2000), are exactly at zero.[10]

2.2 Equilibrium

In our model, agents hold capital to insure against a spell of low productivity. We will focus on equilibria in which agents never wish to purchase state-contingent capital for the high-productivity state – our definition of equilibrium below reflects that focus. For this to be optimal, the return on capital has to be sufficiently low and wage growth sufficiently high (in a way we make precise below). We will provide sufficient conditions on the parameters of the model such that this is indeed the case in equilibrium.[11]

The only reason for acquiring and subsequently holding capital is thus to finance the consumption stream of agents with zero productivity. High-productivity agents pay insurance premia to obtain a stock of capital should the transition to zero productivity occur, but hold no capital as long as they are productive. Thus, all these agents are identical and we do not need to keep track of their past productivity history. Low-productivity agents, in contrast, are distinguished by the length of time $\tau \geq 0$ elapsed since the transition from high to low productivity occurred. Let the density at time t be denoted by $\psi_{l,t}(\tau)$. In a stationary equilibrium where $\xi_t = \xi$ is constant, this density is given by $\psi_l(\tau) = \frac{\xi\nu}{\xi+\nu}e^{-\nu\tau}$, and in general $\psi_{l,t}(\tau)$ will depend on the time path of ξ_t. Note that the mass of high-

[9] We can think of insurance being offered by risk-neutral perfectly competitive intermediaries. In this economy with only idiosyncratic but no aggregate risk (in contrast to Alvarez and Jermann, 2000), a competitively behaving financial intermediary offering the insurance can always contract with a measure one of agents with current state z buying insurance for state $\tilde{z}$, and therefore faces the deterministic insurance payout $p_{z,t}(\tilde{k} - k)$. By perfect competition this then also is the price charged for this insurance in equilibrium.

[10] In Krueger and Uhlig's (2006) discrete-time environment, the interest rate is exogenous, but this does not change the fact that in the absence of any punishment, positive (possibly state-contingent) debt (a negative k) cannot be sustained, reminiscent of the classic Bulow and Rogoff (1989) result.

[11] Under these parameter restrictions, we also conjecture this is the *only* equilibrium.

productivity individuals is then given by $\Psi_{h,t} = 1 - \int_0^{+\infty} \psi_{l,t}(\tau)\, d\tau$, which integrates to $\Psi_h = 1 - \int_0^{+\infty} \psi_l(\tau)\, d\tau = \frac{\nu}{\xi+\nu}$ in steady state.

Low-productivity agents hold capital $k_{s,t}$ which depends on the date t and the time $s = t - \tau$ of the transition to low productivity. Thus, rather than keeping track of the joint state distribution across capital and productivity states $(k; z)$, it is more convenient to keep track of the capital holding $k_{s,t}$ as a function of the transition time s and the calendar time t. Similarly, we denote by $c_{s,t}$ the consumption of an agent at time t who made the transition to low productivity at time $s < t$. In what follows, time derivatives are always with respect to calendar time.

Definition 2. *An equilibrium consists of household allocations $\mathcal{C}_t$, equilibrium wages w_t, interest rates r_t, aggregate capital K_t, and capital holdings of low-productivity agents $(k_{s,t})_{s \leq t}$, as functions of time $t \in (-\infty, +\infty)$, such that:*

1. *Given (w_t, r_t), the household allocation $\mathcal{C}_t$ is optimal (in the sense of Definition 1).*

2. *The allocations $\mathcal{C}_t$ have the "only low-productivity agents hold capital" property that $\tilde{k}_t(k; 0) = 0$, for all $k = k_{s,t}$, for all $s \leq t$, as well as $x_t(0; \zeta) = 0$.*

3. *Capital holdings of low-productivity agents are consistent with allocations $\mathcal{C}_t$, i.e.,*

$$k_{t,t} = \tilde{k}_t(0; \zeta), \tag{10}$$

$$\dot{k}_{s,t} = x_t(k_{s,t}; 0), \tag{11}$$

where $\dot{k}_{s,t} = \frac{\partial k_{s,t}}{\partial t}$.

4. *The interest rates and wages (r_t, w_t) satisfy*

$$r_t = A_t F_K(K_t, 1) - \delta, \tag{12}$$

$$w_t = A_t F_L(K_t, 1). \tag{13}$$

5. *The goods and capital markets clear*

$$\int_0^{+\infty} c_t(k_{t-\tau,t}; 0)\, \psi_{l,t}(\tau) d\tau + c_t(0; \zeta)\, \Psi_{h,t} + \delta K_t = A_t F(K_t, 1), \tag{14}$$

$$\int_0^{+\infty} k_{t-\tau,t} \psi_{l,t}(\tau) d\tau = K_t. \tag{15}$$

In the market clearing condition (15), the supply of capital comes from agents with currently low productivity that were income rich τ periods ago, integrated over τ. Aggregate consumption in (15) is composed of the integral over the heterogeneous consumption levels of the income-poor and the uniform consumption of the income-rich times their mass.

The thought experiment envisions the economy initially (for $t < 0$) in a *stationary* equilibrium in which all entities in Definition 2 indexed by time t are constant. We establish that there is a unique stationary equilibrium in Section 4.1. Then, aggregate total factor productivity A and/or income risk ξ change at time $t = 0$ from the constant (A^*, ξ^*) towards time-varying paths (A_t, ξ_t) with $\lim_{t \to +\infty} (A_t, \xi_t) = (\tilde{A}, \tilde{\xi})$. Since this is a complete surprise to everyone (the typical "MIT shock"), the stationary equilibrium allocations chosen at $t < 0$ have not allowed for that contingency. We then characterize the dynamic transition path induced by this change in productivity. The aggregate capital stock and its distribution at $t = 0$ are predetermined by the steady state equilibrium associated with (A^*, ξ^*).

3 The Optimal Consumption Allocation

In this section, we characterize the optimal household consumption-asset allocation, given a path for wages and interest rates (r_t, w_t). We start with a graphical representation of the optimal allocation to provide intuition and to guide the ensuing theoretical analysis.

Figure 1 illustrates the consumption insurance allocation in the initial steady state (i.e., for $t < 0$) of an agent with productivity z_t and thus labor income $y_t = w^* z_t$ that switches idiosyncratic productivity at two Poisson dates from high to low and back to high productivity. In the high idiosyncratic income state of a simple equilibrium, the agent holds no capital, consumes less than his current income (see the upper panel) and uses the difference to make insurance payments against the possibility of a switch to low productivity. When the switch occurs, the agent receives a stock of capital as insurance payout (see the lower panel) and draws down this capital account to finance the consumption stream during the low-productivity (zero labor income) phase. Upon a transition back to high productivity, the capital account returns to its zero value and the allocation returns to the initial phase.[12]

[12]Figures 1 and 2 also clarify why a fully analytical characterization of the transition path with more than two states is infeasible: while the optimal consumption allocation has the same qualitative features with $N > 2$ states, as shown in Krueger and Uhlig (2025) for the steady state, it is now characterized by $N - 1$ consumption "plateaus" (and a constant drift between them). Consequently, the downward drift hits a consumption "plateau" in finite time, and this time depends on the interest rate. The associated consumption distribution is piece-wise Pareto, with interest rate-dependent bounds, and explicit aggregation giving rise to a tractable law of motion that we achieve in Section 4 is infeasible for a general N-state Poisson process.

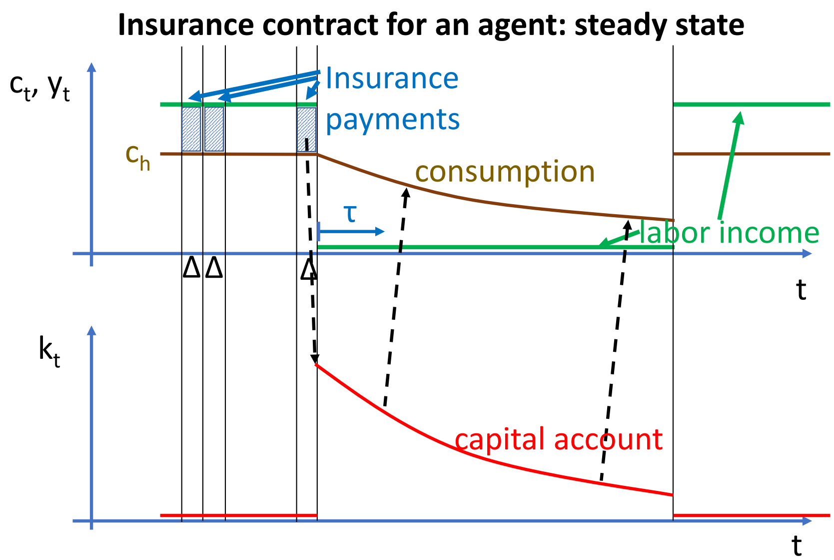


Figure 1: Consumption allocation in stationary equilibrium. In the high-productivity state, the agent holds no capital, consumes less than current income, and pays for insurance against a productivity change. When the productivity state changes to zero, the agent receives a stock of capital as insurance payout, running it down while productivity is zero. When productivity switches to the high state again, the capital account returns to zero.

Figure 2 depicts what happens to the consumption insurance allocation on the impact of the MIT shock at time $t = 0$. Whereas the capital held by low-productivity agent remains unchanged on impact (since capital is a state variable and the transition was completely unexpected), a different path for consumption emerges, due to changed aggregate dynamics in wages and interest rates (w_t, r_t) and the individual income process $y_t = w_t z_t$. In the high-productivity phase, consumption changes due to the changing wages along the transition, but as long as income growth is sufficiently fast relative to the interest rate (and thus savings incentives sufficiently weak, in a sense formalized below), the capital account will still be zero and a share of labor income will again be devoted to insurance payments against idiosyncratic productivity loss. During the low-productivity and thus zero labor income spell (assumed to encompass the instant $t = 0$ of the MIT shock), the aggregate shock will potentially induce an altered consumption path due to interest rate changes along the transition. Whereas consumption can in principle change discontinuously at $t = 0$ (as displayed in Figure 2), we will show that this is not the case when utility is logarithmic.[13]

We now proceed with the formal analysis. Given an allocation $\mathcal{C}_t$, we define the implied

[13] For a general CRRA utility function, a jump at $t = 0$ indeed occurs. See Online Appendix H.

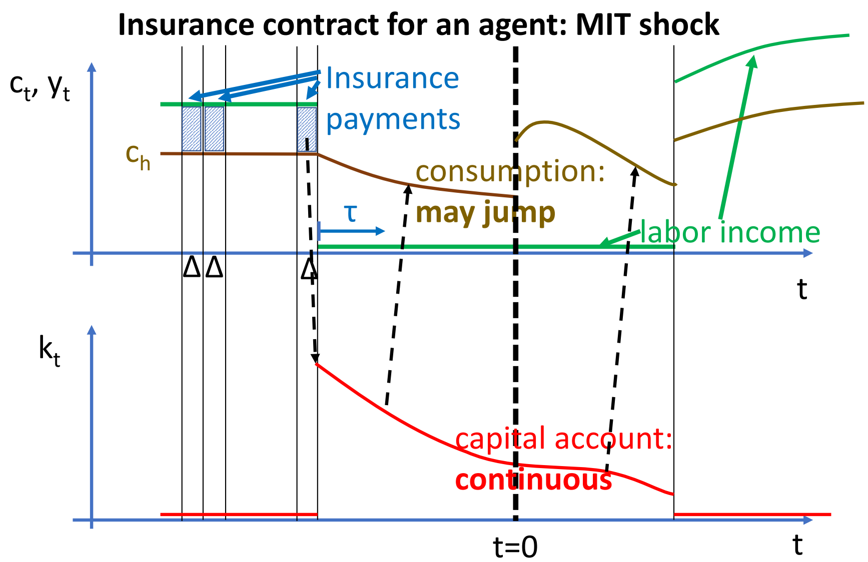


Figure 2: Consumption allocation around transition date $t = 0$. Low-productivity agents keep their capital. Since returns r_t, wages w_t and possibly the individual income process have changed, a different consumption path might now be optimal, given this initial capital.

time derivative of consumption (assuming no idiosyncratic z-state transition) as[14]

$$\dot{c}_t(k;z) \equiv \frac{\partial c_t(k;z)}{\partial t} + \frac{\partial c_t(k;z)}{\partial k} x_t(k;z). \tag{16}$$

We make the following assumption on the equilibrium interest rate path, which we will show in Section 4.2 to hold under suitable assumptions about the exogenous parameters.

Assumption 1. *For some $T \geq 0$, $r_t < \rho$ for all $t \geq T$.*

Lemma 1 (Optimal consumption allocation for $z = 0$ and $k > 0$)**.** *Let Assumption 1 be satisfied and let $k > 0$. Then the optimal allocation in definition 1 is characterized by:*

$$\frac{\dot{c}_t(k;z)}{c_t(k;z)} = r_t - \rho. \tag{17}$$

Furthermore, if $z_t = 0$ and if, for some $\bar{k} > 0$ we have $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$, then

[14]To provide some intuition for this definition of the time derivative, suppose that productivity remains constant at z for some time interval. In that case note that $\dot{k}_t = x_t(k;z)$ and that consumption evolves as $c(t) = c_t(k_t;z)$ as a function of time only. Taking the derivative with respect to time yields the expression.

there exists $\bar{\bar{k}}$ such that for all $k \leq \bar{\bar{k}}$,

$$c_t(k;0) = (\rho+\nu)k, \tag{18}$$
$$x_t(k;0) = (r_t-\rho)k. \tag{19}$$

The proof is in the Appendix. Note that Assumption 1 permits the possibility that $r_t > \rho$ for some time during the transition. When this happens, capital k and consumption c will be temporarily increasing. The optimality of the proposed allocation in the lemma is then threatened by low-productivity individuals having so much current wealth (capital) that they find it optimal to carry some of it into the high-productivity state. As long as the initial capital in the low-productivity state is less than $\bar{\bar{k}}$, then even when capital drifts up temporarily, it never exceeds the bound $\bar{k}$ that guarantees the optimality of $\tilde{k}_t(k;0) = 0$.

We now use this result to characterize the dynamics of consumption for individuals with currently high productivity. To do so, we make the following assumption concerning equilibrium variables, which in Section 4.2 again will be replaced by sufficient conditions exclusively on exogenous model parameters.

Assumption 2. *Suppose the aggregate wage and interest rate satisfy, for all $t \geq 0$,*

$$\frac{\dot{w}_t}{w_t} + \frac{\dot{\zeta}_t}{\zeta_t} + \rho > r_t, \tag{20}$$
$$\forall s < t, \quad \frac{\zeta_t w_t}{\zeta_s w_s} \geq \frac{\rho+\nu+\xi_t}{\rho+\nu+\xi_s} e^{\int_s^t (r_u-\rho)du}. \tag{21}$$

Generally, agents discount future payments at rate $\frac{\dot{c}_t}{c_t} + \rho$. Suppose that consumption is proportional to wages, as will be the case for individuals with high productivity. The assumption then ensures that this discount rate is higher than the interest rate r_t that can be earned on savings, i.e., high-productivity agents would rather not postpone consumption into the future by saving (and recall that they cannot borrow). The second part of the assumption ensures that low-productivity individuals do not want to save for the high productivity state during some period t along the transition.[15]

Now define the conjectured consumption rate (out of current labor income) of high-

[15] The assumption guarantees that for all agents with low-productivity, current consumption is weakly lower than that of high-productivity individuals. If this were not the case, these individuals could improve lifetime utility by (temporarily) saving for the contingency of a productivity reversal towards the high state.

productivity individuals as

$$\alpha_t \equiv \frac{\rho+\nu}{\rho+\nu+\xi_t}. \tag{22}$$

For ease of notation, let $c_{s,t} = c_t\left(k_{s,t};0\right)$ denote the time-t consumption of a low-productivity agent who switched from high to low productivity at time $s \leq t$ and thus holds capital $k_{s,t}$. This notation implies that $c_{s,s}$ and $k_{s,s}$ are the consumption and capital holdings of an agent whose productivity has turned to zero this very instant. Further, denote by $c_{h,t} = c_t\left(0;\zeta\right)$ the time-t consumption of a high-productivity agent with no assets.

Lemma 2 (The optimal consumption allocation)**.** *Let Assumption 2 be satisfied. Then the optimal allocation of Definition 1 is characterized by the following consumption process:*

$$\begin{aligned} c_{h,t} &= \alpha_t \zeta_t w_t, & (23)\\ c_{s,t} &= c_{h,s} e^{\int_s^t (r_u-\rho)du}, & (24) \end{aligned}$$

for all s,t, where the consumption rate α_t is defined in (22) *and where the instantaneous consumption growth rate of unconstrained agents is given by $r_t-\rho$, see equation* (17).

The proof is in the Appendix.[16] Equation (23) implies that high-productivity agents pay an insurance premium $(1-\alpha_t)\zeta_t w_t$ against their productivity falling. Since the insurance contracts are actuarially fair, this finances initial capital

$$k_{t,t} = \frac{1-\alpha_t}{\xi_t}\zeta_t w_t \tag{27}$$

after the switch to zero income. Equation (18) then implies

$$c_{t,t} = \frac{(\rho+\nu)\left(1-\alpha_t\right)}{\xi_t}\zeta_t w_t = \alpha_t\zeta_t w_t \tag{28}$$

[16] Denote by $U_t^L(k) = U_t(k;0)$ and by $U_t^H = U_t(0;\zeta_t)$ the lifetime utilities of agents with low productivity and capital k, and agents with high productivity and no capital, respectively. Conditional on the property that there is no capital accumulation for the high state (which the proof in the appendix establishes), the HJB equations in (7)-(9) can be simplified to

$$\begin{aligned} \rho U_t^L(k) &= \max_c \left\{\log(c) + \dot{U}_t^L(k) + U_t'^L(k)((r_t+\nu)k - c) + \nu\left(U_t^H - U_t^L(k)\right)\right\}, & (25)\\ \rho U_t^H &= \max_c \left\{\log(c) + \dot{U}_t^H + \xi_t\left(U_t^L\left(\frac{w_t\zeta_t}{\xi_t} - c\right) - U_t^H\right)\right\}. & (26) \end{aligned}$$

As an alternative to the proof in the appendix, the optimal consumption-savings allocation stated in Lemmas 1 and 2 in the main text can then also directly be obtained by taking first order conditions of the above (perhaps more familiar) HJB's, invoking the envelope theorem and combining them. See Online Appendix C for the detailed derivations.

for the consumption following the switch, which coincides with $c_{h,t}$ and with the expression in (24). Finally, solving for α_t in equation (28) delivers (22).

What is perhaps surprising is that, as equation (18) shows, the consumption rate $\alpha_t = c_{h,t}/\zeta_t w_t$ after switching to zero productivity takes the same form along the transition as in the initial steady state, despite the fact that the subsequent consumption path drifts down at a different (and time-varying) interest rate. However, since the rates at which future consumption is discounted also change, the present discounted value of this altered consumption stream remains the same with log-utility, and thus the wage-normalized entry level of consumption does not depend on the future interest rate path. This is a version of the well-known "income effect and substitution effect cancel" property of log preferences.[17]

4 Transition Dynamics

To compute the aggregate capital supply K_t at time t, we aggregate the capital holdings of low-productivity agents,

$$K_t = \int_{-\infty}^{t} k_{s,t}\psi_{l,t}(t-s)ds. \tag{29}$$

This allows us to characterize the evolution of the aggregate capital stock in closed form.

Lemma 3 (Dynamics of aggregate capital supply)**.** *Let the initial capital stock $K_0 = K^*$ be given by the capital associated with the steady state interest rate $r^* < \rho$ and let Assumptions 0-2 be satisfied.*[18] *Then the aggregate law of motion for capital is given by*

$$\dot{K}_t = \hat{s}_t A_t K_t^{\theta} - \hat{\delta} K_t, \tag{30}$$

$$\textit{where} \quad \hat{s}_t = 1-(1-\theta)\alpha_t \quad \textit{and} \quad \hat{\delta} = \delta + \rho + \nu. \tag{31}$$

The proof is in the Appendix. The aggregation result is intuitive: high-productivity agents consume the fraction α_t of all wages and save the rest $(1-\alpha_t)w_t$. Low-productivity agents own all capital, earn r_t on their capital accounts, and deplete it at rate $\rho + \nu$. Adding

[17] Recall that the MIT shock at $t = 0$ is completely unexpected, and the insurance contracts did not allow for this contingency. Nevertheless, since the insurance only stipulates delivery of capital and does not guarantee a specific return on that capital and the share of newly low-productivity individuals is continuous at $t = 0$, those providing the insurance will not make unexpected losses or profits induced by the MIT shock to either aggregate productivity or idiosyncratic risk.

[18] Assumption S1 in Section 4.1 below guarantees such a unique partial insurance steady state with $r < \rho$.

up yields

$$\dot{K}_t = (1-\alpha_t)\, w_t + (r_t - (\rho+\nu))\, K_t. \tag{32}$$

Using the expressions for the wage and interest rate (r_t, w_t) from (12) and (13) delivers the law of motion (30) with a possibly time-varying aggregate saving rate $\hat{s}_t$ in (31). This is a consequence of identical savings rates within each group of households, see equations (18) and (23): with log utility, low-productivity agents all consume the same share of their capital account and high-productivity agents consume the same share of their wage income. The law of motion is akin to that of the Solow model, but with a micro-founded saving rate $\hat{s}_t$ and depreciation rate $\hat{\delta}$ that are explicit functions of the underlying structural parameters of the model, including the potentially time-varying risk of losing productivity ξ_t.

As is well-known from the Solow model (or from Lemma H.5 of Online Appendix H.3), this differential equation has a closed-form solution given by:[19]

$$K_t = \left[e^{-(1-\theta)\hat{\delta}t}\,(K^*)^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)\hat{\delta}(t-\tau)}\hat{s}_\tau A_\tau d\tau\right]^{\frac{1}{1-\theta}}, \tag{33}$$

where $K_0 = K^* = \left(\frac{s^* A^*}{\hat{\delta}}\right)^{\frac{1}{1-\theta}}$ is the initial steady state capital stock associated with initial productivity and risk (A^*, ξ^*). The right-hand side of equation (33) is solely a function of parameters and the exogenous time paths $\{A_t, \xi_t\}$. The solution in (33), in principle, applies to any paths for $\{A_t, \xi_t\}$, but the requirement that Assumption 2 be satisfied imposes restrictions on these paths for which (33) is a valid characterization of the equilibrium law of motion for capital. We will characterize these restrictions in Section 4.2.

Defining the final steady state associated with $(\tilde{A}, \tilde{\xi})$ as $K_\infty = \tilde{K} = \left(\frac{\tilde{s}\tilde{A}}{\hat{\delta}}\right)^{\frac{1}{1-\theta}}$, the following corollary is a direct consequence of Assumption 0, Lemma 3 and equation (33).

Corollary 1. *Under the conditions of Lemma 3, we have* $K_t \to \tilde{K}$.

Note that the time paths of all other aggregate variables, such as the interest rate, the wage and aggregate consumption directly follow from the dynamics of the aggregate capital

[19]By using the transformation $X_t = K_t^{1-\theta}$, one can transform this so-called Bernoulli differential equation into a linear ODE, from which the solution readily follows.

stock, as in the standard neoclassical growth model and the Solow model, i.e.,

$$r_t = \theta A_t K_t^{\theta-1} - \delta, \tag{34}$$

$$w_t = (1-\theta) A_t K_t^{\theta}, \tag{35}$$

$$C_t = A_t K_t^{\theta} - \delta K_t - \dot{K}_t. \tag{36}$$

Although the dynamics of our model appears to be similar to that of the Solow model, there are fundamental differences. In the Solow model the aggregate saving rate s is not only constant, but an exogenous parameter, whereas in our model the saving rate $\hat{s}_t$ is not only potentially time varying, but (as is the effective depreciation rate $\hat{\delta}$) fully endogenous and micro-founded. Crucially, these rates change in response to underlying household-level risk. The applications of our model in Section 5 will leverage this observation.

4.1 Partial Insurance Steady State

For constant $(A_t, \xi_t) \equiv (A, \xi)$, the steady state capital stock K satisfying $\dot{K}_t = 0$ and associated interest rate r are given by

$$K = \left(\frac{\hat{s}A}{\hat{\delta}}\right)^{\frac{1}{1-\theta}} \quad \text{and} \quad r = \theta\frac{\hat{\delta}}{\hat{s}} - \delta, \tag{37}$$

where $(\hat{s}, \hat{\delta})$ are the steady state effective saving and depreciation rates defined in (31) for constant income risk $\xi_t = \xi$. Now define the constant

$$\chi(\xi) \equiv \frac{\xi}{\nu(\rho+\nu+\xi)} - \frac{\theta}{(1-\theta)(\rho+\delta)}. \tag{38}$$

Krueger and Uhlig (2025) show that χ is the difference between the steady-state wage-normalized capital supply and demand at $r = \rho$. We now impose the following assumption for the steady state (consequently labeled S1), which guarantees both a unique initial and a final stationary equilibrium with interest rates $r < \rho$.

Assumption S1. *Let the exogenous parameters of the model satisfy $\theta, \nu, \rho, \xi^*, \tilde{\xi}, > 0$ and*

$$\chi(\xi^*) > 0, \chi(\tilde{\xi}) > 0. \tag{39}$$

In Lemma B.4 of Online Appendix B.4, we show that the initial and final steady state interest rates satisfy $r(\xi^*) < \rho$ and $r(\tilde{\xi}) < \rho$ if and only if $\chi(\xi^*) > 0$ and $\chi(\tilde{\xi}) > 0$. The following proposition from Krueger and Uhlig (2025) summarizes our discussion above and fully characterizes the initial and final steady state of the model.[20]

Proposition 1 (Krueger and Uhlig, 2024)**.** *Let Assumption S1 be satisfied. Then there exists a unique initial and final stationary equilibrium. The unique equilibrium capital stock and interest rate (K, r) are given by equation (37), and r satisfies $r < \rho$ in the initial and final steady state. Both stationary equilibria feature partial insurance, i.e., consumption of the high-productivity agents is c_h and consumption of the low-productivity agents drifts downwards at rate $r - \rho < 0$. The steady state interest rate is independent of aggregate productivity A and strictly decreasing in income risk ξ and the steady state capital stock is strictly increasing in both aggregate productivity and income risk.*

The wage is given by $w = (1-\theta) A (K)^{\theta}$. The stationary consumption distribution has a mass point at $c_h = \alpha\zeta w$ for the mass $\frac{\nu}{\nu+\xi}$ of high-productivity agents, and

$$k_\tau = e^{-(\rho-r)\tau}\frac{c_h}{\nu+\rho}, \tag{40}$$

$$c_\tau = e^{-(\rho-r)\tau}c_h, \tag{41}$$

for the low-productivity agents as functions of τ (the time elapsed since their last transition to low productivity), where $k_\tau = k_{t-\tau,t}$ and $c_\tau = c(k_{t-\tau,t})$ are independent of t.

Note that k_τ is the net present value of the future zero-income consumption path $c_{\tau+s}$, taking into account the rate ν of switching out of the zero income state, i.e.,

$$k_\tau = \int_{s=0}^{+\infty} e^{-(\nu+r)s} c_{\tau+s} ds. \tag{42}$$

4.2 Sufficient Conditions for "No-Savings" Assumption 2

In this section, we derive sufficient conditions on the exogenous productivity and risk paths $\{A_t, \xi_t\}$ and the other model parameters under which Assumption 2 (which imposed conditions on endogenous entities) is indeed satisfied in equilibrium.[21]

In what follows, we impose Assumption S1 which guarantees the existence of a unique stationary equilibrium with partial insurance from which (by assumption) the transition

[20] See Lemma B.4 of Online Appendix B.4 for a reproduction of the argument here.

[21] Readers not interested in this technical discussion can skip to Section 5 without loss.

path starts and a unique stationary partial insurance equilibrium where it ends. This assumption also guarantees that the path for capital implied by (33) has an implied interest path that converges to $\tilde{r} < \rho$, and thus Assumption 1 is automatically satisfied.

Assumption 2 only involves the dynamics of wages and interest rates ensuring that incentives to save for the high states are sufficiently weak along the transition. Given that these wage and interest paths are functions of the dynamics of the capital stock only, a sharp characterization of it is paramount. To this end, first note that if productivity and risk change permanently at $t = 0$ from (A^*, ξ^*) to $(\tilde{A}, \tilde{\xi})$, then from equation (33) the aggregate capital stock evolves according to

$$K_t = \left[e^{-(1-\theta)\hat{\delta}t} \left(K^*\right)^{1-\theta} + \left(1 - e^{-(1-\theta)\hat{\delta}t}\right) \left(\tilde{K}\right)^{1-\theta} \right]^{\frac{1}{1-\theta}} \tag{43}$$

and thus the capital stock evolves monotonically from the old to the new steady state, at a speed determined by the constant $(1-\theta)(\delta + \rho + \nu)$.

4.2.1 Aggregate Productivity Shocks

Since a one-time positive aggregate productivity shock delivers the cleanest sufficient conditions under which Assumption 2 holds, and permits the clearest discussion why the assumption might be violated, we start with this case. We then turn to more general productivity paths in the remainder of Section 4.2.1. These paths are the key ingredients for our first application of the model to the cross-country convergence debate. We then consider risk shocks in Section 4.2.2, the main driving force of our second application – the study of income and consumption inequality over time.

Positive Aggregate Technology Shocks Consider first a permanent increase in productivity at time $t = 0$ from A^* to $\tilde{A} > A^*$ which implies $\tilde{K} > K^*$. From (43) it then follows that the capital stock is monotonically increasing over time. Recall that Assumption 2 bounds the saving incentives for the high idiosyncratic productivity state along the transition by requiring $\frac{\dot{w}_t}{w_t} > r_t - \rho$ for all t be satisfied.[22] Further, recall from equations (34) and (35) that wages and interest rates are given by $r_t = \theta A_t K_t^{\theta-1} - \delta$ and $w_t = (1-\theta) A_t K_t^{\theta}$, and thus wages and interest rates jump up at $t = 0$ with productivity. Wages then exhibit

[22] In the absence of risk shocks, ξ_t is constant over time and the second part of Assumption 2 then reads as

$$\frac{w_t}{w_s} \geq e^{\int_s^t (r_u - \rho) du} \tag{44}$$

further positive growth towards the new steady state whereas the interest rate declines towards the new (and old steady state), both induced by the dynamics of the capital stock. A sufficient condition for Assumption 2 to be satisfied then limits the size of the productivity-induced initial increase in marginal product of capital (and thus the interest rate) at $t = 0$ by imposing an upper bound on the rise in productivity. The following result formalizes this discussion.

Proposition 2 (Transition dynamics after a permanent productivity increase)**.** *Suppose Assumption S1 is satisfied and the economy is in the initial steady state with interest rate $r^* < \rho$. Now suppose at $t = 0$ productivity increases unexpectedly but permanently from A^* to $\tilde{A}$. Risk ξ and thus the saving rate $\hat{s}$ remains unchanged. Suppose that $\tilde{A} \leq \bar{A}$ where the upper bound $\bar{A}$ satisfies*

$$\bar{A} = \left(1 + \frac{\rho - r^*}{(r^* + \delta)(1 - \hat{s})}\right) > A^*. \tag{45}$$

Then equilibrium wage and interest rate processes jointly satisfy Assumption 2, the aggregate capital stock is given by equation (43), the capital stock and the wage are strictly increasing over time, and the equilibrium interest rate is strictly decreasing over time.

Since this result is a special case of Proposition 3 below, it will be proved there. Figure F.1 in Online Appendix F displays the content of Proposition 2 graphically.

The previous argument to limit the size of the temporary increase in the interest rate along the transition carries over to any weakly increasing (over time) productivity process, as the next proposition shows. To state it we make the following assumption imposing the upper bound $\bar{A}$ on the productivity process.[23]

and since $w_t = w_s e^{\int_s^t \frac{\dot{w}_u}{w_u} du}$, as long as the first part of the assumption, $\frac{\dot{w}_t}{w_t} > r_t - \rho$ for all t, is satisfied, so is the second part.

[23]The upper bound $\bar{A}$ follows from the following consideration, to be made precise in the proof of Proposition 3 in the Appendix. First, we can rewrite the first condition, equation (20) in Assumption 2 as

$$\frac{\dot{A}_t}{A_t} > (1 - \hat{s})(r_t - r^*) + r^* - \rho, \tag{46}$$

by exploiting $\dot{w}_t/w_t = \dot{A}_t/A_t + \theta \dot{K}_t/K_t$ and using the dynamics of aggregate capital in equation (30), substituting $\hat{s}\delta - \theta\hat{\delta}$ with $\hat{s}r^*$ and rearranging terms. Since A_t is increasing, K_t is increasing and $r_t < \bar{r} = \bar{A}\left(K_0^*\right)^{\theta-1} - \delta$, that is, the interest rate which would prevail at $t = 0$ if $A_0 = \bar{A}$. Note that $\bar{r} + \delta = \left(\bar{A}/A^*\right)(r^* + \delta)$. Since $\dot{A}_t/A_t \geq 0$, equation (46) is implied by $0 = (1 - \hat{s})\left(\left(\bar{A}/A^*\right) - 1\right)(r^* + \delta) + r^* - \rho$, and thus by equation (45).

Assumption 3 (Upper bound on the productivity increase)**.** *The productivity process* $\{A_t\}_{t\geq 0}$ *is weakly increasing and satisfies* $A_t < \bar{A}$ *for all* $t \geq 0$ *where the bound* $\bar{A}$ *is given by* (45)*.*

The following proposition, which is crucial for our first application on the speed of international convergence of income per capita levels, then generalizes Proposition 2.

Proposition 3 (Transition dynamics after general productivity increase)**.** *Impose Assumption S1. Assume that the productivity process* $\{A_t\}_{t\geq 0}$ *satisfies Assumptions 0 and 3 and income risk* ξ *remains unchanged. Then the dynamics of aggregate capital is given in* (33)*, the equilibrium wage and interest rate processes jointly satisfy Assumptions 1 and 2, and capital is weakly increasing along the transition.*

Negative Aggregate Technology Shocks When productivity declines, falling wages are the main threat to the no-savings condition $\dot{w}_t/w_t > r_t - \rho$ in Assumption 2. In fact, with a one-time permanent decline this condition cannot possibly be satisfied since at $t = 0$ the growth rate of wages is infinitely negative. Intuitively, under the proposed allocation the discontinuous drop in wages induced by the permanent productivity decline leads to an instant drop in consumption $c_{h,t}$ of the high-productivity wage earners, whereas consumption of the richest capital owners $c_{t,t}$ is continuous at $t = 0$ and therefore exceeds $c_{h,t}$ at (or close to) $t = 0$. A low productivity individual can then avoid the discontinuous (and thus suboptimal) decline in consumption upon a reversal to idiosyncratic productivity by saving into that state, invalidating the proposed allocation in Lemma 2 as optimal. Figure F.2 in Online Appendix F displays this argument graphically.

Although any discontinuous decline in productivity will necessarily violate Assumption 2, we now show that with a continuous decline in productivity the assumption is satisfied as long as the productivity *growth rate* (and thus the wage growth rate) is not too negative. This is the content of the next result, which requires the following assumption:

Assumption 4 (Lower bound on the growth rate of productivity)**.** *The growth rate* $\frac{\dot{A}_t}{A_t}$ *of the productivity process* $\{A_t\}_{t\geq 0}$ *satisfies* $\frac{\dot{A}_t}{A_t} > r^* - \rho$*.*

Proposition 4 (Transition dynamics after a continuous productivity decline)**.** *Impose Assumptions 0, S1 and 4, and let* $A_0 = A^*$ *and* $\dot{A}_t \leq 0$ *for all* $t > 0$*. Assume income risk* ξ *remains unchanged. Then the equilibrium aggregate capital stock is given in* (33)*, and the wage and interest rate processes jointly satisfy Assumptions 1 and 2. Furthermore, the* *following log-linearly decreasing productivity path* $\{A_t\}_{t\geq 0}$,

$$\log A_t = \log A^* + \frac{t + (T-t)\cdot \mathbf{1}\{t > T\}}{T}\left(\log \tilde{A} - \log A^*\right) \tag{47}$$

with $T > 0$ *satisfies Assumption 4 if the parameters* $\tilde{A}$ *and* T *satisfy*

$$\frac{\log A^* - \log \tilde{A}}{T} < \rho - r^*. \tag{48}$$

The proof is in the Appendix. Proposition 4 states that in order to discourage agents from accumulating capital for the high idiosyncratic productivity state, we need to bound the growth rate of the productivity decrease. This insures that wages do not fall too fast, otherwise even high-productivity agents would want to hold some capital to fund future consumption in light of low future wages. Note that this requirement contrasts with the case of a productivity increase for which we required a bound on the level of productivity in Assumption 3 to limit the increase in the interest rate. The remainder of the proposition gives a specific productivity path that declines over time and for which Assumption 2 is satisfied, in order to demonstrate that the first part of the proposition is not vacuous.

4.2.2 Idiosyncratic Risk Shocks

Aggregate productivity shocks only affect households through factor prices, but do not affect the decision rules given these prices. Therefore sufficient conditions for Assumption 2 to be satisfied are relatively easy to come by. In contrast, when idiosyncratic risk, as parameterized by ξ_t changes, under the conjectured allocation the share of income consumed by wage earners is altered as well, see equation (23) reproduced here

$$c_{h,t} = \frac{\nu + \rho}{\nu + \rho + \xi_t}\zeta_t w_t. \tag{49}$$

As before in the case of an aggregate productivity change, we need to rule out that individuals find it optimal to save for the high productivity state along the transition. If risk ξ_t increases immediately and permanently from ξ to its new long-run value $\tilde{\xi}$ at $t = 0$, then consumption of high-productivity individuals in (49) declines immediately and discontinuously after $t = 0$ whereas consumption of low-productivity individuals is continuous at (since their capital path is continuous). As in the case of an aggregate productivity decline this is suboptimal and can be avoided by saving into the good idiosyncratic productivity state, invalidating the proposed allocation as optimal.

Thus, to ensure the optimality of the consumption allocation we need to impose restrictions on the change in income risk. We first show in Proposition 5 that if income risk *declines* permanently at time zero, then the no-savings condition is satisfied as long as the decline is not too large (as made precise in Assumption 5). This is the equivalent of Proposition 2 for an aggregate productivity increase. The assumption bounds the decline in aggregate wages (relative to the increase in interest rates) to insure that savings incentives remain sufficiently weak. Second, and more relevantly for our second application, we show that if the increase in income risk ξ_t is gradual enough (as made precise in Assumption 6), so that consumption of high-productivity individuals does not fall below that of low-productivity individuals, then Assumption 2 remains satisfied and our equilibrium characterization remains valid (see Proposition 6).

A Permanent Decline in Income Risk ξ A permanent *decline* in income risk leads to an immediate increase in the share of wage income that is being consumed by high-productivity individuals, but lowers wage over time, potentially inducing currently high-wage earners to save for the future. The following assumption limits the magnitude of the latter effect.

Assumption 5 (Limit on the magnitude of decline in income risk)**.**

$$\rho - \tilde{r} > \xi^* - \tilde{\xi}. \tag{50}$$

Equipped with this assumption the following result now follows. Key to establishing it is to show that if the decline in productivity risk is sufficiently small, the resulting decline in labor income of high-productivity individuals and increase of the interest rate is so small that these individuals still do not want to save (conditional on making no productivity transition).

Proposition 5 (Transition dynamics after a permanent decline in income risk)**.** *Suppose Assumption S1 is satisfied and the economy is in the initial steady state with interest rate $r^* < \rho$. Now suppose at $t = 0$ income risk declines unexpectedly but permanently from ξ^* to $\tilde{\xi} < \xi^*$. Aggregate productivity A remains unchanged. Furthermore impose Assumption 5. Then equilibrium wage and interest rate processes jointly satisfy Assumptions 1 and 2, the aggregate capital stock is given by equation (43), the capital stock and the wage are strictly decreasing, and the equilibrium interest rate is strictly increasing over time.*

Gradual Increase in Income Risk ξ Although an immediate and permanent increase in income risk ξ will necessarily violate the no-savings condition in Assumption 2, as long as the increase is gradual (enough), Assumption 2 remains intact. To make this formal, we now state the following assumption.

Assumption 6. *Suppose that the process $\{\xi_t\}_{t\geq 0}$ satisfies Assumption 0, is weakly increasing over time and for all $t \geq 0$*

$$\rho - r^* > \tilde{\xi} - \xi + \frac{\dot{\xi}_t}{\rho + \nu + \xi_t},$$

where r^* is the interest rate in the initial steady state and $\tilde{\xi}$ is the income risk in the final steady state.

Note that this assumption limits the speed of the increase in income risk, but it defines a nonempty set of parameter values. For example, when the risk process satisfies

$$\xi_t = \tilde{\xi} + \left(\xi^* - \tilde{\xi}\right) e^{-\kappa t}, \tag{51}$$

then Assumption 6 imposes an upper bound on the parameter κ governing the speed of adjustment from the old to the new steady state value.

A gradual increase in income risk ξ_t introduces the complication that the saving rate $\hat{s}_t$ in the aggregate law of motion for capital (31) changes over time now, stemming from the fact that time-varying risk induces equally time-varying precautionary savings behavior by high wage earners, as evident from the consumption function (49). As a result, an explicit closed-form characterization of the capital stock and interest rate dynamics is now more difficult since (43) no longer applies and is replaced by (33). We can nevertheless establish the following proposition:

Proposition 6 (Transition dynamics after a continuous increase in income risk)**.** *Suppose Assumption S1 is satisfied and the economy is in the initial steady state with interest rate $r^* < \rho$. Now suppose at $t = 0$ income risk increases unexpectedly and gradually from ξ^* to $\tilde{\xi}$ and satisfies Assumption 6. Aggregate productivity A remains unchanged. Then equilibrium wage and interest rate processes jointly satisfy Assumptions 1 and 2, and the aggregate capital stock is given by equation* (33)*.*

In the next section we show how our tractable dynamic macro model with partially insured idiosyncratic income risk can fruitfully be applied to two substantive questions in macroeconomics. Here, we use our results to discuss the implications of the model for the speed of convergence in income per capita of poor countries to that of rich countries in Section 5.1. We then apply (in Section 5.2) our results on changes in income risk to the debate about the relative trends in income and consumption inequality.

5 Applications

5.1 The Speed of Convergence

So far we have characterized the transition path induced by a shock to productivity. This begs the question of how rapid the convergence to the new steady state is, given the exogenous shock. We can think of the initial steady state as describing an originally poor country that, all of a sudden, obtains access to frontier production technologies. The speed of convergence question then asks how quickly such a country will catch up with frontier economies. In an influential paper, King and Rebelo (1993) study the quantitative predictions of the standard representative agent neoclassical growth model for this question and conclude that outside parameter values they consider implausible, that model implies a speed of convergence that is too fast relative to the empirical evidence.[24]

In this section, we analyze the ability of our model to generate a slower speed of convergence, relative to the standard neoclassical growth model.[25] Barro and Sala-i-Martin (2004, p. 56) formally define the speed of convergence β_t as

$$\beta_t \equiv -\frac{\partial\left(\dot{K}_t/K_t\right)}{\partial \log\left(K_t\right)}. \tag{52}$$

For a permanent increase in TFP, the thought experiment considered here, this statistic measures how quickly the growth rate of capital (and thus income per capita) declines as the capital stock converges towards its new steady state. Since the partial derivative is negative, the minus sign in the definition turns the speed of convergence positive. The following proposition analytically characterizes this statistic in our model.

Proposition 7 (Speed of convergence)**.** *Suppose Assumptions 0, 1 and 2 hold. Then the speed of convergence in our model is characterized as follows.*

[24]See Barro and Sala-i-Martin (2004) for further discussion of the neoclassical growth model and a summary of the empirical evidence on this question.

[25]For the purpose of this subsection we assume that income risk ξ_t is constant over time.

1. *Along the transition*

$$\beta_t = \frac{(1-\theta)}{\theta}\hat{s}_t\,(r_t+\delta)\,. \tag{53}$$

2. *In the long run*[26]

$$\beta_t \to \beta = (1-\theta)\,\hat{\delta}. \tag{54}$$

Thus, the speed of convergence in the long run is independent of the path for productivity A_t or the risk of losing productivity ξ. Note that this proposition is general, in the sense that the required assumptions are always satisfied whenever the conditions of Proposition 3 or Proposition 4 hold. The proof of this proposition is in the appendix.

We now contrast the speed of convergence in our model to that of the standard neoclassical growth model. Since that model does not have a closed form solution despite the lack of household heterogeneity, for a theoretical comparison we log-linearize it around the steady state. No approximation is needed for our model since its speed of convergence around the steady state is available analytically, as shown in the previous proposition. The following proposition contains the result, whose proof is in the Appendix (and involves tedious calculations to conduct the log-linearization of the neoclassical growth model).

Proposition 8 (Comparison of speed of convergence)**.** *Suppose a permanent shock raises productivity from A^* to $\tilde{A}$. Furthermore impose Assumptions 0, S1 and 3. Around the new steady state, our model exhibits a slower speed of convergence than the neoclassical growth model if and only if*

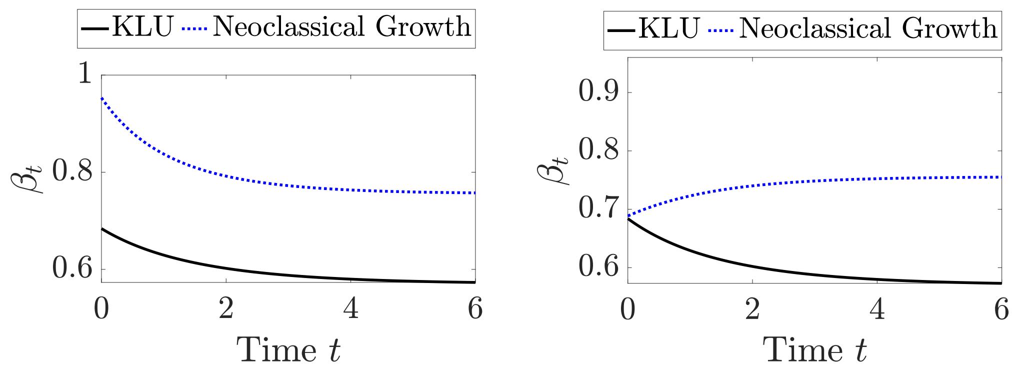
$$\theta\left(1+\frac{\nu}{\rho+\delta}\right)\left(1+\frac{\rho+\nu}{\frac{\rho}{1-\theta}+\delta}\right) < 1. \tag{55}$$

Figure 3 plots the speed of convergence, both to show Proposition 7 in action and to contrast it to the speed of convergence in the standard neoclassical growth model[27] characterized in Proposition 8. The parameter values satisfy equation (55) and thus in the long run our model displays slower convergence to the new steady state. Panels (a) and (b) differ in the initial levels of capital in the two models. In panel (a), each model starts from its own steady state capital stock which is lower in the neoclassical growth model since in that model the steady state interest rate is $r=\rho$, whereas our model features partial insurance

[26] The expression in equation (54) is also the speed of convergence along the transition if one log-linearizes the model around the new steady state.

[27] Online Appendix D.1.2 contains the computational details for the numerical solution of that model.

and thus $r < \rho$. In panel (b), we instead assume that both models start from the *same* capital stock, equal to the initial steady state capital stock in our model.



(a) Starting from steady state

(b) Starting from same capital

Figure 3: The figure compares the speed of convergence in our model and in the representative agent neoclassical growth model, following a permanent increase in productivity. In each panel, the solid black line represents our model and the dotted blue line is the neoclassical growth model. Productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Agents have log utility and the parameters are $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

The figure shows that our model displays slower convergence than the neoclassical growth model, both in the short run and the long run. As King and Rebelo's (1993) classic paper on this issue shows, the speed of convergence in the neoclassical growth model is fast initially if the capital stock is far below its new long-run steady state level, as is the case when there is a large permanent increase in productivity. Then the returns to capital are temporarily very high, and the representative agent endogenously chooses a high saving (and thus investment) rate early in the transition. In our model, akin to the classical Solow model, the saving rate of those making positive saving decisions is endogenously chosen to be constant. Thus convergence is slower in the short run, and, with the inequality in Proposition 8 satisfied, also in the long run. As King and Rebelo (1993) and Barro and Sala-i-Martin (2004) argue, the neoclassical growth model implies unreasonably fast convergence for commonly used calibrations. Thus our model can potentially alleviate this "fast convergence puzzle," as long as the parameter restrictions in Proposition 8 are satisfied.

5.2 Increased Idiosyncratic Income Risk and Consumption Inequality

Over the last 40 years income inequality has risen substantially (see, e.g., Heathcote et al., 2023), and a large literature has attributed a significant part of this rise to increased household income instability (see, e.g., Krueger and Perri (2006) and Gottschalk and Moffitt, 2009). In this section we use the model to study the relationship between income risk and consumption inequality along a transition induced by increased idiosyncratic risk, that is, by a gradual increase in the parameter ξ_t over time.

To this end, suppose the idiosyncratic productivity process is given by equation (51), that is, income risk as parameterized by ξ_t starts from the initial steady state value of ξ^* and increases over time to $\tilde{\xi} > \xi^*$ at a speed given by κ. To frame this discussion, note that the cross-sectional dispersion of idiosyncratic labor productivity, and thus labor income, as measured by the variance of z is given by[28]

$$\mathrm{Var}_{z,t} = (1 - \Psi_{h,t})(0 - 1)^2 + \Psi_{h,t}(\zeta_t - 1)^2 = \zeta_t - 1. \tag{56}$$

As income risk ξ_t gradually increases, the share of high-productivity individuals $\Psi_{h,t}$ declines[29] and their productivity ζ_t rises. Thus, cross-sectional income inequality $\mathrm{Var}_{z,t}$ increases as well. The thought experiment is then an (unexpected before time $t = 0$) mean-preserving spread in labor productivity whose size is increasing over time. We now study how consumption inequality evolves along the transition induced by this income risk shock.

It follows from Proposition 1 that the capital stock is higher and the interest rate is lower in the final than in the initial steady state. Along the transition, idiosyncratic risk and thus cross-sectional income inequality increases, precautionary savings incentives rise and the capital stock and aggregate wage gradually increase whereas the real interest rate falls.

In contrast to the two-point income distribution, the consumption distribution is com-

[28]Using the assumption that average labor productivity is equal to 1, that is, $\Psi_{h,t}\zeta_t = L_t = 1$.

[29]The share of high-productivity individuals shifts slowly from its initial steady state value towards its new steady state value and can be calculated to be

$$\Psi_{h,t} = 1 - e^{-\int_0^t (\xi_s + \nu) ds} \left(\int_0^t e^{\int_0^s (\xi_u + \nu) du} \xi_s ds + \frac{\xi^*}{\xi^* + \nu} \right).$$

If income risk changes immediately and permanently from ξ^* to $\tilde{\xi}$, then $\Psi_{h,t}$ is given by

$$\Psi_{h,t} = 1 - e^{-(\tilde{\xi}+\nu)t} \left(\tilde{\xi} \int_0^t e^{(\tilde{\xi}+\nu)s} ds + \frac{\xi^*}{\xi^* + \nu} \right) = \frac{\nu}{\tilde{\xi} + \nu} + \left(\frac{\tilde{\xi}}{\tilde{\xi} + \nu} - \frac{\xi^*}{\xi^* + \nu} \right) e^{-(\tilde{\xi}+\nu)t}.$$

posed of a mass point $\Psi_{h,t}$ at $c_{h,t}$ as well as a full consumption density below this mass point. A complete description of the consumption distribution is the Lorenz curve whose evolution we will plot below. One summary statistic that is especially tractable is the relative per-capita consumption of high labor-income households $c_{h,t} = C_{h,t}/\Psi_{h,t}$ to that of low income individuals $\bar{c}_{l,t} = C_{l,t}/\Psi_{l,t}$. We denote this statistic by $\varphi_{c,t}$ and it is given by[30]

$$\varphi_{c,t} = \frac{C_{h,t}/\Phi_{h,t}}{C_{l,t}/\Phi_{l,t}} = \frac{\frac{\rho+\nu}{\rho+\nu+\xi_t}(1-\theta)Y_t/\Phi_{h,t}}{(\rho+\nu)K_t/(1-\Phi_{h,t})} = \frac{(1-\theta)}{\theta}\frac{(r_t+\delta)}{(\rho+\nu+\xi_t)}(\zeta_t - 1). \tag{57}$$

In steady state, this statistic becomes

$$\varphi_c = \frac{(1-\theta)(\zeta-1)(r+\delta)}{\theta(\rho+\nu+\xi)} = \frac{(1-\theta)\frac{\xi}{\nu}\theta\frac{\hat{\delta}}{\hat{s}}}{\theta(\rho+\nu+\xi)} = \frac{(1-\theta)(\rho+\nu+\delta)}{\nu\left(\theta(\rho+\nu)/\xi+1\right)},$$

which is strictly increasing in ξ. Thus, in the long run, in response to an increase in income risk, consumption per capita of the income rich rises relative to the consumption of the income-poor. The share of the consumption-rich falls, and consumption among the income-poor becomes more dispersed, with a fatter left tail of the consumption distribution, since with a lower interest rate, consumption of low-productivity individuals drifts down more rapidly than in the old steady state.

Figure 4 plots relative consumption of the income-rich $\varphi_{c,t}$ along the transition, together with the dispersion of labor productivity (and thus income). It displays the increase in consumption inequality, following that of income inequality in the long run. However, it also shows that consumption inequality temporarily declines in the short run, as high-productivity, high-consumption agents increase their savings rate (and thus reduce their consumption rate $\alpha_t = \frac{\rho+\nu}{\rho+\nu+\xi_t}$) to engage in larger precautionary saving to hedge against the elevated income risk. This effect, in the short run, dominates the higher income stemming from gradually increasing individual productivity and aggregate wages.

These developments are reflected in the Lorenz curve plotted in Figure 5. Panel (a) shows the Lorenz curve in the initial and final steady state as well as in two different instances along the transition, and Panel (b) displays the deviation from the initial steady state Lorenz curve. The figure shows that in the long-run the Lorenz-curve unambiguously shifts further away from the 45-degree line, indicating an increase in consumption inequality. In the very short run the decline in relative consumption of high-productivity individuals mit-

[30]We have used Assumption 0 which implies that $\frac{1-\Phi_{h,t}}{\Phi_{h,t}} = 1/\Phi_{h,t} - 1 = \zeta_t - 1$ from the normalization of effective labor to $L_t = 1$.

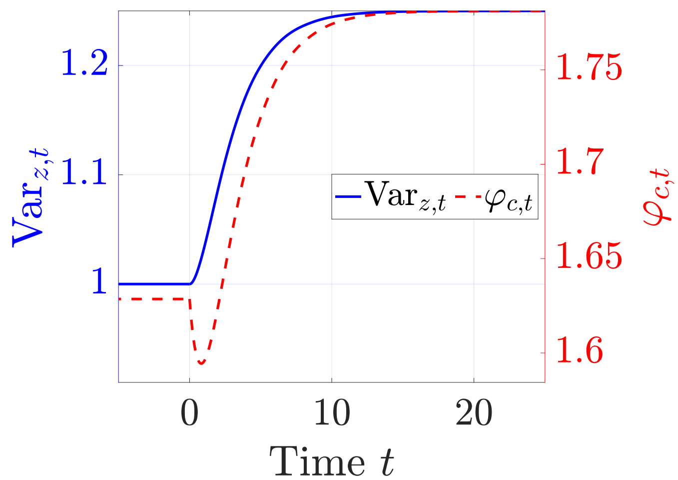
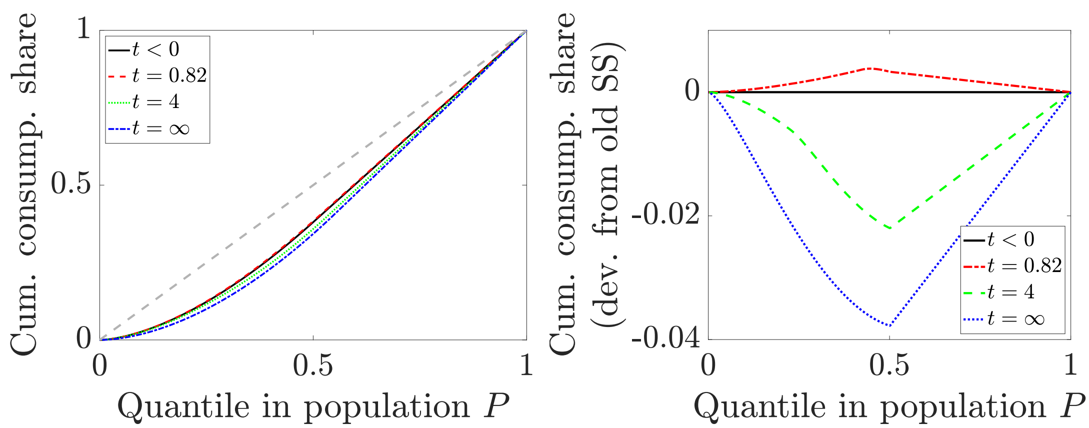


Figure 4: Income dispersion and consumption inequality (measured by $\varphi_{c,t}$) along the transition. Agents have log utility and the parameter values are $A = 1, \delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25$, $\xi = 0.2, \tilde{\xi} = 0.25, \kappa = 1$. The evolution of ξ_t is given by the process in (51).



(a) Lorenz Curve (b) Lorenz Curve: Deviation from Steady State

Figure 5: Lorenz curve for three periods along the transition induced by a change in income risk. Panel (a) shows the Lorenz curves themselves, and Panel (b) shows the deviation from the initial steady state. Parameter values are the same as in Figure 4.

igates this long-run trend. As a result, the Lorenz curve early in the transition[31] lies inside the one of the initial steady state, although quantitatively this effect is small.

[31]Specifically, $t = 0.82$ is the instant in which relative consumption $\varphi_{c,t}$ is smallest.

6 Conclusion

In this paper we have analytically characterized the transition dynamics in a neoclassical production economy with idiosyncratic income shocks and limited commitment. When income can only take two values one of which is zero (i.e., unemployment) and the utility function is logarithmic, the transition path induced by an unexpected productivity shock or a change in idiosyncratic income risk can be given in closed form, both for the macroeconomic variables as well as the consumption distribution which displays partial consumption insurance against the idiosyncratic income shocks. We have then traced out the implications of the model for two applied transition questions, the speed of convergence of an initially poor economy to a new, high TFP steady state, and the evolution of consumption inequality in response to an increase in idiosyncratic income risk.

Given the theoretical results in this paper, we think that two immediate next questions are worth studying. First, owing to the use continuous time, the characterization of the optimal consumption allocation is analytically tractable for a general N-state Poisson process (as we demonstrate in Krueger and Uhlig (2025) in steady state), and it would be useful to generalize our transition analysis to this more general case, even though the aggregate law of motion will not be available in closed form.

Second, this paper has focused on an environment with idiosyncratic risk, but no aggregate shocks, rendering the macroeconomic dynamics deterministic. Given the sharp analytical characterization of the equilibrium in the absence of aggregate shocks, the economy with aggregate shocks is at least partially analytically tractable as well, as we demonstrate in Ando, Krueger and Uhlig (2023) in a discrete-time version of this model.[32] Pursuing this agenda further seems an important topic for future research.

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[32]Related to this paper, Broer (2024) studies consumption insurance over the aggregate business cycle in this class of models, using both analytical and numerical techniques.

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APPENDIX

A Proofs of Propositions in the Main Text

Proof of Lemma 1. Let $\mu \geq 0$ denote the Lagrange multiplier (LM) on the budget constraint (8), λ the LM on the borrowing constraint (9), and $\omega \geq 0$ the LM on the constraint $\tilde{k} \geq 0$. Then the Lagrangian for the maximization problem in Definition 1 is

$$\begin{aligned} \mathcal{L} = & u(c) + \dot{U}_t(k;z) + U_t'(k;z)\,x + p_{z,t}\left(U_t\left(\tilde{k};\tilde{z}\right) - U_t(k;z)\right) \\ & - \mu\left(c + x + p_{z,t}\left(\tilde{k} - k\right) - r_t k - w_t z\right) + \lambda x + \omega \tilde{k}. \end{aligned} \tag{A.1}$$

The FOCs wrt to c, k and $\tilde{k}$ are

$$u'(c) = \mu, U_t'(k;z) = \mu - \lambda, U_t'\left(\tilde{k};\tilde{z}\right) = \mu - \frac{\omega}{p_{z,t}}. \tag{A.2}$$

Consider an agent with $k > 0$. In this case, condition (9) does not apply and $\lambda = 0$. Then the FOCs imply $u'(c) = U_t'(k;z)$. When productivity stays unchanged for a small interval of time, we differentiate both sides wrt time t and use $\dot{k}_t = x$ to obtain

$$u''(c)\,\dot{c} = \dot{U}_t'(k;z) + U_t''(k;z)\,x, \tag{A.3}$$

where $\dot{c}_t(k;z) \equiv \frac{\partial c_t(k;z)}{\partial t} + \frac{\partial c_t(k;z)}{\partial k} x_t(k;z)$, $U_t''(k;z) \equiv \frac{\partial^2 U_t(k;z)}{\partial k^2}$, and $\dot{U}_t'(k;z) \equiv \frac{\partial^2 U_t(k;z)}{\partial k \partial t}$. Differentiating the objective in equation (7) wrt the state k, we get the envelope condition

$$\rho U_t'(k;z) = \dot{U}_t'(k;z) + U_t''(k;z)\,x - p_{z,t} U_t'(k;z) + \mu\left(p_{z,t} + r_t\right). \tag{A.4}$$

Using the FOCs at $k > 0$ and equations (A.3) and (A.4), we have $\rho u'(c) = u''(c)\,\dot{c} + u'(c)\,r_t$. Hence, we get the consumption dynamics (17) for $k > 0$ under log utility.

When $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, the consumption dynamics (17) and the budget constraint (8) can be rewritten as the linear system of differential equations[33] $\dot{c}_t = (r_t - \rho)\,c_t$ and $\dot{k}_t = (r_t + \nu)\,k_t - c_t$ in the unknown functions c_t and k_t with the

[33]Note that equation (17) implies $c_s = e^{\int_t^s (r_u - \rho) du} c_t$. A less formal, but more meaningful argument in terms of economic theory is thus to recognize that the budget constraint and utility maximization implies that the current capital k is equal to the net present value of all future consumption, as long as the productivity state stays unchanged. This yields $k = \int_t^{+\infty} e^{-\int_t^s (r_u + \nu) du} c_s ds = c_t / (\rho + \nu)$.

boundary condition[34] $\lim_{t \to +\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t. Such a system of linear ODEs has a unique solution. With $x_t(k; 0) = \dot{k}_t$, it is easy to verify that the solutions are (18) and (19). The solutions are valid, as long as the implied path for k_s for $s \geq t$ does not cross the upper bound $\bar{k}$, since (19) is $\dot{k}_t = (r_t - \rho) k$ and since $r_t < \rho$ for $t \geq T$ per Assumption 1. This will be true for all $k_t \in \left(0, \bar{\bar{k}}\right)$ and some suitable $\bar{\bar{k}}$. □

Proof of Lemma 2. The lemma is a version of Section 3.3 in the Online Appendix of Krueger and Uhlig (2022), generalized to the case when wages and interest rates are functions of time. Rather than replicating the steps, here we provide the logic of the argument and point to the results and proofs in Krueger and Uhlig (2022) for the details. For a high-productivity agent, the Lagrangian, first-order and envelope conditions are as in the proof for Lemma 1 (see (A.1), (A.2) and (A.4)) but applied to $k = 0$, $z = \zeta_t$, and $p_{z,t} = \xi_t$.

1. We first consider the choice of $\tilde{k}$. The solution in Lemma 1 implies that

$$U_t'\left(\tilde{k}; 0\right) = u'\left((\rho + \nu)\tilde{k}\right) = \frac{1}{(\rho + \nu)\tilde{k}},$$

   and increases to infinity, as $\tilde{k} \to 0$. The third first-order condition (A.2) thus implies that $\tilde{k} > 0$ and thus $\omega = 0$. With the first and third first-order conditions in (A.2), we obtain consumption smoothing $u'\left((\rho + \nu)\tilde{k}\right) = u'(c)$ and thus $(\rho + \nu)\tilde{k} = c$. Therefore (23) follows from the budget constraint (8), provided that $x = 0$.

2. We then show that $x > 0$ is not optimal. Suppose otherwise, $x > 0$ were optimal, then $(\rho + \nu)\tilde{k} = c$ together with the budget constraint (8) implies that $c_t < \alpha_t \zeta_t w_t$, i.e., consumption is less than the right-hand side of equation (23). Furthermore, constraint (9) would not be binding, $\lambda = 0$, and consumption growth would satisfy equation (17). Let $[t, t + \Delta]$ be a "short" time interval of length $\Delta > 0$, during which this is the case and along a path where no productivity switch occurs. Assumption 2 then implies $c_s \leq \alpha_s \zeta_s w_s$ during the interval $s \in [t, t + \Delta]$, i.e., consumption is less than the consumption level proposed in equation (23) of Lemma 2 for that episode. The integral of utility during that time interval is then smaller than the utility of the solution proposed in equation (23) of Lemma 2. This loss in utility can only be justified by the additional utility gained from consuming the accumulated capital after a switch to lower productivity for $s > 0$, or alternatively, for $s > \Delta$ in case there is no

[34] The boundary condition ensures that no capital gets wasted and this follows from utility maximization.

switch to lower productivity. This amounts to postponing consumption compared to the solution proposed in Lemma 2. But this contradicts the impatience of the agent relative to wage growth, as expressed in Assumption 2. A precise formulation of that contradiction requires replicating the arguments in Section 3.3 of Krueger and Uhlig (2022), allowing for the additional time evolution of r_t and w_t.

For a low-productivity agent, given his consumption dynamics in equation (17) and the consumption of a high-productivity agent in equation (23), we have

$$c_{s,t} = e^{\int_s^t (r_u - \rho) du} c_{s,s} = c_{h,s} e^{-\int_s^t (\rho - r_u) du}.$$

Hence, the consumption of a low-productivity agent is given equation (24). $\square$

Proof of Lemma 3. We use the notation $\psi_l(t-s,t)$ instead of $\psi_{l,t}(t-s)$ in equation (29) to denote the density of the low-productivity agents with length of time $t-s \geq 0$ since the last transition from high to low productivity. Differentiate both sides of (29) wrt time t,

$$\dot{K}_t = k_{t,t}\psi_l(0,t) + \int_{-\infty}^{t} \Big(\dot{k}_{s,t}\psi_l(t-s,t) + k_{s,t}\left(\partial_\tau \psi_l(t-s,t) + \partial_t \psi_l(t-s,t)\right) \Big) ds. \tag{A.5}$$

Equations (27) and (B.3) (see Lemma B.2 of Online Appendix B.2) yield

$$k_{t,t}\psi_l(0,t) = \frac{1-\alpha_t}{\xi_t}\zeta_t w_t \xi_t \Psi_h(t) = (1-\alpha_t)\zeta_t w_t \Psi_h(t) = (1-\alpha_t) w_t. \tag{A.6}$$

The last equality uses the normalization in (2). Rewrite equation (19) with $\dot{k}_{s,t} = x_t$ as

$$\dot{k}_{s,t} = (r_t - \rho) k_{s,t}. \tag{A.7}$$

Substituting equations (B.2) (see Lemma B.2 of Online Appendix B.2), (A.6), and (A.7) into equation (A.5), we get

$$\begin{aligned}
\dot{K}_t &= (1-\alpha_t) w_t + \int_{-\infty}^{t} \left((r_t - \rho) k_{s,t}\psi_l(t-s,t) - \nu k_{s,t}\psi_l(t-s,t) \right) ds \\
&= (1-\alpha_t) w_t + (r_t - \rho - \nu) \int_{-\infty}^{t} k_{s,t}\psi_l(t-s,t)\, ds \\
&= (1-\alpha_t) w_t + (r_t - \rho - \nu) K_t, \\
&= (1-\alpha_t)(1-\theta) A_t K_t^{\theta} + \left(\theta A_t K_t^{\theta-1} - \delta - \rho - \nu\right) K_t = \hat{s}_t A_t K_t^{\theta} - \hat{\delta} K_t.
\end{aligned}$$

where the third line above uses the expression for aggregate capital in equation (29), the fourth line above uses the wage and interest rate in equations (34) and (35), and $\hat{s}_t$ and $\hat{\delta}$ are defined in equation (31). $\square$

Lemma A.1. *Fixing risk ξ, consider any productivity process $\{A_t\}_{t\geq 0}$ such that in equilibrium, the aggregate capital evolves according to equation (30).*

- *If $A_t > A^*$ and is weakly increasing for all $t > 0$, then $r_t > r^*$ and $\dot{K}_t > 0$.*
- *If $A_t < A^*$ and is weakly decreasing for all $t > 0$, then $r_t < r^*$ and $\dot{K}_t < 0$.*

A^* and r^* are the productivity and interest rate in the initial steady state, respectively.

Proof. It suffices to show this for the first case of an increasing A_t, since the proof for the second case of a decreasing A_t is entirely symmetric.

With constant ξ, we define the function $K(A_t) \equiv \left(\frac{\hat{s}A_t}{\hat{\delta}}\right)^{\frac{1}{1-\theta}}$, where $\hat{s} = \hat{s}_t$ and $\hat{\delta}$ are given in equation (31) of Lemma 3. Then $K(A)$ is the steady state capital under constant productivity A. From equation (37), the initial steady state interest rate is $r^* = \theta\frac{\hat{\delta}}{\hat{s}} - \delta$. Substituting equation (34) into equation (30), we get

$$\frac{\dot{K}_t}{K_t} = \hat{s}A_t K_t^{\theta-1} - \hat{\delta} = \frac{\hat{s}}{\theta}(r_t + \delta) - \hat{\delta}. \tag{A.8}$$

Then it follows that

$$r_t > r^* \iff \dot{K}_t > 0 \iff K_t < K(A_t). \tag{A.9}$$

For any $\tilde{t} > 0$, consider the solution $\tilde{K}_t$ to the ODE (A.8) starting at $\tilde{K}_0 = K^*$, but with $A_t \equiv A_{\tilde{t}}$ for $t \in [0, \tilde{t}]$ instead. It is clear that $\tilde{K}_t < K(A_{\tilde{t}})$: convergence to the new steady state $K(A_{\tilde{t}})$ for $A_t \equiv A_{\tilde{t}}$ does not happen in finite time and is strictly monotone, as one can see by examining the solution in (43). Given any $t \in [0, \tilde{t}]$ and any K_t, we have $\dot{K}_t \leq \dot{\tilde{K}}_t$, since the right-hand side of (A.8) weakly increases, when A_t is replaced by $A_{\tilde{t}}$. Since $K_0 = \tilde{K}_0 = K^*$, this implies that $K_t \leq \tilde{K}_t$ for $t \in [0, \tilde{t}]$. In particular now, $K_{\tilde{t}} \leq \tilde{K}_{\tilde{t}} < K(A_{\tilde{t}})$ and we must have $r_{\tilde{t}} > r^*$ as well as $\dot{K}_{\tilde{t}} > 0$ per equation (A.9).[35] Since $\tilde{t} > 0$ is arbitrary, this establishes the claim. $\square$

Proof of Proposition 2. The productivity process $\{A_t\}_{t\geq 0}$ satisfies Assumptions 0 and 3. Then following the proof of Proposition 3 below, the dynamics of aggregate capital is given

[35] An alternative way to see this is to examine the closed-form solution (33).

in equation (33), aggregate capital K_t is strictly increasing over time, and Assumption 2 holds in equilibrium. With a permanent increase in productivity, $\hat{s}_t = \hat{s} = 1 - \alpha + \alpha\theta$ with $\alpha = \frac{\rho+\nu}{\rho+\nu+\xi}$. Then equation (33) reduces to

$$\begin{aligned} K_t &= \left[e^{-(1-\theta)\hat{\delta}t} \left(K^*\right)^{1-\theta} + \frac{\hat{s}\tilde{A}}{\hat{\delta}} \int_0^t \left(1-\theta\right) \hat{\delta} e^{-(1-\theta)\hat{\delta}(t-\tau)} d\tau \right]^{\frac{1}{1-\theta}} \\ &= \left[e^{-(1-\theta)\hat{\delta}t} \left(K^*\right)^{1-\theta} + \left(1 - e^{-(1-\theta)\hat{\delta}t}\right) \left(\tilde{K}\right)^{1-\theta} \right]^{\frac{1}{1-\theta}}, \end{aligned}$$

where the last line above uses the expression for steady state aggregate capital in equation (37), and $\tilde{K}$ is the aggregate capital in the final steady state.

Since the aggregate capital K_t is strictly increasing over time, the interest rate r_t given in equation (34) is strictly decreasing over time and the wage w_t given in equation (35) is strictly increasing over time. □

Proof of Proposition 3. We proceed in two steps. We first conjecture Assumptions 1 and 2 hold and use the upper bound $\bar{A}$ in Assumption 3 to obtain an upper bound for the interest rates along the transition path. We then verify the conjecture by showing that the upper bound for interest rates is sufficiently low for Assumptions 1 and 2 to hold in equilibrium.

1. Conjecture that Assumptions 1 and 2 hold. Then Lemma 3 (together with Lemma H.5 of Online Appendix H.3) implies the aggregate capital dynamics in equation (33). Let $\bar{r}$ be the equilibrium interest rate that would prevail if capital was at its initial steady state value $K_t = K_0 = K^*$ and productivity was at its upper bound $\bar{A}$ in equation (45). Using the expression for interest rate in equation (34), we get

   $$\bar{r} = \theta \bar{A} \left(K^*\right)^{\theta-1} - \delta = \frac{\bar{A}}{A^*} \left(r^* + \delta\right) - \delta, \tag{A.10}$$

   where A^*, K^*, and r^* are the productivity, aggregate capital, and interest rate in the initial steady state, respectively.

   Lemma 3 implies (30). Lemma A.1 implies $r_t > r^*$ and $\dot{K}_t > 0$, $\forall t > 0$. Hence, K_t is increasing and $K_t > K_0 = K^*$ for $t > 0$. Comparing equations (34) and (A.10), we get $r_t = \theta A_t K_t^{\theta-1} - \delta \leq \bar{r}$, i.e., $\bar{r}$ is an upper bound for interest rates along the transition path.

2. We then verify the above conjecture. Assumption 0 and the steady state interest rate in equation (37) imply $\lim_{t\to+\infty} r_t = \theta\frac{\hat{\delta}}{\hat{s}} - \delta = r^*$. Assumption S1 and Lemma B.4 of Online Appendix B.4 implies $r^* < \rho$. Hence, Assumption 1 holds in equilibrium.

   Using the expression for wage in equation (35) and the aggregate capital dynamics in equation (30), we compute

$$\begin{aligned}
&\frac{\dot{w}_t}{w_t} + \frac{\dot{\zeta}_t}{\zeta_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} + \theta\frac{\dot{K}_t}{K_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} - (1-\hat{s})\, r_t + \hat{s}\delta - \theta\hat{\delta} + \rho \\
> \;& 0 - (1-\hat{s})\,\bar{r} - \hat{s}r^* + \rho \\
= \;& (1-\hat{s})\left(\delta - \left(1 + \frac{\rho - r^*}{(r^*+\delta)(1-\hat{s})}\right)(r^*+\delta)\right) + (1-\hat{s})\, r^* + \rho - r^* = 0,
\end{aligned}$$

   where the third line above uses equations (A.10) and (45). Hence, condition (20) of Assumption 2 holds. Next, we prove that condition (20) implies condition (21). With constant risk ξ and thus ζ, condition (21) becomes $\log(w_t) \geq \log(w_s) + \int_s^t (r_u - \rho)\, du$. Define the function $f(s,t) \equiv \log(w_s) + \int_s^t (r_u - \rho)\, du$. Then condition (20) implies

$$\frac{\partial f(s,t)}{\partial s} = \frac{\dot{w}_s}{w_s} + \rho - r_s > 0,$$

   and thus condition (21) of Assumption 2 holds. Taken together, Assumption 2 holds in equilibrium.

□

Proof of Proposition 4. WLOG, we focus on the case where A_t is strictly decreasing at $t = 0$ and weakly decreasing for any $t > 0$.[36] Conjecture that Assumptions 1 and 2 hold in equilibrium. Then Lemma 3 gives the dynamics of aggregate capital in equation (33) and implies Assumption 1. Combining the expression for wage in equation (35) with the dynamics of aggregate capital in equation (30), we calculate

$$\begin{aligned}
&\frac{\dot{w}_t}{w_t} + \frac{\dot{\zeta}_t}{\zeta_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} + \theta\frac{\dot{K}_t}{K_t} + \rho - r_t = \frac{\dot{A}_t}{A_t} - (1-\hat{s})\, r_t - \hat{s}\left(\theta\frac{\hat{\delta}}{\hat{s}} - \delta\right) + \rho \\
> \;& (r^* - \rho) - r^* + \rho = 0,
\end{aligned}$$

[36]For the case where A_t is constant for $t \in [0, \hat{t})$ and strictly decreasing at $t = \hat{t}$ with $\hat{t} > 0$, the economy will remain at its initial steady state for $t \in [0, \hat{t})$. Then we just need to show the transition dynamics for $t \geq \hat{t}$, the proof of which is the same as the case where A_t is strictly decreasing at $t = 0$.

where the last inequality above uses Assumption 4. Hence, condition (20) of Assumption 2 holds and the proof for condition (21) of Assumption 2 follows the proof of Proposition 3 above. $\square$

Proof of Proposition 5. Conjecture that Assumptions 1 and 2 hold. Then Lemma 3 gives the dynamics of aggregate capital dynamics in equation (33) and implies Assumption 1. With a permanent decrease in income risk, $\hat{s}_t = \tilde{\hat{s}} = 1 - \tilde{\alpha} + \tilde{\alpha}\theta$ with $\tilde{\alpha} = \frac{\rho+\nu}{\rho+\nu+\tilde{\xi}}$. Then equation (33) reduces to

$$\begin{aligned} K_t &= \left[e^{-(1-\theta)\hat{\delta}t}\left(K^*\right)^{1-\theta} + \frac{\tilde{\hat{s}}A}{\hat{\delta}}\int_0^t (1-\theta)\,\hat{\delta}e^{-(1-\theta)\hat{\delta}(t-\tau)}d\tau\right]^{\frac{1}{1-\theta}} \\ &= \left[e^{-(1-\theta)\hat{\delta}t}\left(K^*\right)^{1-\theta} + \left(1-e^{-(1-\theta)\hat{\delta}t}\right)\left(\tilde{K}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}, \end{aligned}$$

where the last line above uses the expression for steady state aggregate capital in equation (37), and $\tilde{K}$ is the aggregate capital in the final steady state. Since $\tilde{\hat{s}} < \hat{s} \equiv \frac{\rho+\nu}{\rho+\nu+\xi}$, the expression for steady state capital in equation (37) implies $\tilde{K} < K^*$. Then the closed form solution in equation (43) implies the aggregate capital K_t is strictly decreasing over time, and thus the interest rate r_t given in equation (34) is strictly increasing over time and the wage w_t given in equation (35) is strictly decreasing over time.

Next, we verify condition (20) of Assumption 2.

$$\begin{aligned} \frac{\dot{w}_t}{w_t} + \frac{\dot{\zeta}_t}{\zeta_t} + \rho - r_t &= -\frac{\dot{\Psi}_h(t)}{\Psi_h(t)} + \theta\frac{\dot{K}_t}{K_t} + \rho - r_t \\ &= -\frac{\dot{\Psi}_h(t)}{\Psi_h(t)} - \left(1-\tilde{\hat{s}}\right)r_t + \tilde{\hat{s}}\delta - \theta\hat{\delta} + \rho \\ &\geq \tilde{\xi} - \xi - \left(1-\tilde{\hat{s}}\right)\tilde{r} + \tilde{\hat{s}}\delta - \theta\hat{\delta} + \rho \\ &= \tilde{\xi} - \xi + \rho - \tilde{r} > 0, \end{aligned}$$

where the first line above uses the expression for wage in equation (35) and the normalization in equation (2), the second line above uses the expression for interest rate in equation (34) and the dynamics of aggregate capital in equation (30), the third line above uses equation (B.1) in Lemma B.1 of Online Appendix B.1 (which implies $-\frac{\dot{\Psi}_h(t)}{\Psi_h(t)} = \frac{\tilde{\xi}+\nu}{1+\frac{\xi+\nu}{\tilde{\xi}-\xi}e^{(\tilde{\xi}+\nu)t}} \in \left[\tilde{\xi}-\xi, 0\right)$) and the fact that r_t is strictly increasing over time, and the fourth line above uses the expression for steady state interest rate in equation (37) and

Assumption 5.

Finally, we prove that condition (20) implies condition (21). With a permanent decrease in income risk, $\xi_t \leq \xi_s$, $\forall s < t$. Then a sufficient condition for (21) is $\log(\zeta_t w_t) \geq \log(\zeta_s w_s) + \int_s^t (r_u - \rho)\, du$. Define the function $f(s,t) \equiv \log(\zeta_s w_s) + \int_s^t (r_u - \rho)\, du$. Then condition (20) implies

$$\frac{\partial f(s,t)}{\partial s} = \frac{\dot{w}_s}{w_s} + \frac{\dot{\zeta}_s}{\zeta_s} + \rho - r_s > 0,$$

and thus condition (21) of Assumption 2 holds. Taken together, Assumption 2 holds in equilibrium. □

Proof of Proposition 6. Conjecture that Assumptions 1 and 2 hold. Then Lemma 3 gives the dynamics of aggregate capital dynamics in equation (33) and implies Assumption 1. To verify Assumption 2, we prove the following sufficient condition for (21): $c_{s,t}$, as a function of s, is increasing in s. Define the following monotone transformation of $c_{s,t}$

$$f(s,t) \equiv \log(c_{s,t}) - \log(\rho+\nu) = \int_s^t (r_u - \rho)\, du + \log \zeta_s + \log w_s - \log(\rho + \nu + \xi_s).$$

Then

$$\begin{aligned}
\frac{\partial f(s,t)}{\partial s} &= \rho - r_s + \frac{\dot{\zeta}_s}{\zeta_s} + \frac{\dot{w}_s}{w_s} - \frac{\dot{\xi}_s}{\rho+\nu+\xi_s} \\
&= \rho - r_s + \xi_s + \nu - \frac{\nu}{\Psi_h(s)} + \theta\frac{\dot{K}_s}{K_s} - \frac{\dot{\xi}_s}{\rho+\nu+\xi_s} \\
&\geq \left(\theta\frac{\dot{K}_s}{K_s} + \rho - r_s\right) + \xi_s + \nu - \frac{\nu}{\Psi_h(\infty)} - \frac{\dot{\xi}_s}{\rho+\nu+\xi_s} \\
&\geq (\rho - r^*) - \left(\tilde{\xi} - \xi + \frac{\dot{\xi}_s}{\rho+\nu+\xi_s}\right) > 0,
\end{aligned}$$

where the second line above uses the expression for wage in equation (35), the normalization in equation (2), and the dynamics of the population share implied by equation (B.1) in Lemma B.1 of Online Appendix B.1, the third line above uses Assumption 6 (which implies $\Psi_h(t) \geq \Psi_h(\infty), \forall t \geq 0$), and the fourth line above use Lemma B.3 of Online Appendix B.3 and Assumption 6. Hence, $\frac{\partial f(s,t)}{\partial s} > 0$, $\forall s < t$, i.e., $c_{s,t}$, as a function of s, is increasing in s. This further implies condition (20) given Assumption 6. Taken together, Assumption 2 holds in equilibrium. $\square$

Proof of Proposition 7. From Lemma 3, we rewrite the dynamics of aggregate capital in equation (30) as

$$\frac{\dot{K}_t}{K_t} = \hat{s}_t A_t e^{(\theta-1)\log K_t} - \hat{\delta}. \tag{A.11}$$

Thus, the speed of convergence defined in equation (52) is

$$\beta_t = -\frac{\partial\left(\dot{K}_t/K_t\right)}{\partial \log\left(K_t\right)} = \left(1-\theta\right)\hat{s}_t A_t K_t^{\theta-1} = \frac{1-\theta}{\theta}\hat{s}_t\left(r_t+\delta\right),$$

where the last equality uses the interest rate in equation (34). Corollary 1 and the expression for steady state interest rate in equation (37) imply $r_t \to \tilde{r} = \theta\frac{\hat{\delta}}{\lim\limits_{t\to+\infty}\hat{s}_t} - \delta$ and thus (54) holds. $\square$

Proof of Proposition 8. Let β^{NG} and β denote the long-run speed of convergence in the neoclassical growth model and the Krueger-Li-Uhlig model, respectively. According to equation (D.1) in Online Appendix D.1.3 and Proposition 7,

$$\begin{aligned}
\beta^{NG} &= \frac{\left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta}-\delta\right)\left(1-\theta\right)\left(\rho+\delta\right)\right)^{\frac{1}{2}} - \rho}{2}, \\
\beta &= \left(1-\theta\right)\hat{\delta} = \left(1-\theta\right)\left(\delta+\rho+\nu\right).
\end{aligned}$$

We want to find the necessary and sufficient condition for $\beta < \beta^{NG}$. Note that both coefficients are positive. Then

$$\begin{aligned}
\beta < \beta^{NG} \quad &\Longleftrightarrow \quad \left(\rho + 2\left(1-\theta\right)\left(\delta+\rho+\nu\right)\right)^2 < \rho^2 + 4\left(\frac{\rho+\delta}{\theta}-\delta\right)\left(1-\theta\right)\left(\rho+\delta\right) \\
&\Longleftrightarrow \quad \rho\left(\delta+\rho+\nu\right) + \left(\delta+\rho+\nu\right)^2\left(1-\theta\right) < \left(\frac{\rho+\delta}{\theta}-\delta\right)\left(\rho+\delta\right) \\
&\Longleftrightarrow \quad \left(\rho+\delta+\nu\right)\left(\rho+\left(\rho+\delta+\nu\right)\left(1-\theta\right)\right) < \left(\rho+\delta\right)\frac{1}{\theta}\left(\rho+\delta\left(1-\theta\right)\right) \\
&\Longleftrightarrow \quad \theta\left(1+\frac{\nu}{\rho+\delta}\right)\left(1+\frac{\left(\rho+\nu\right)\left(1-\theta\right)}{\rho+\delta\left(1-\theta\right)}\right) < 1 \\
&\Longleftrightarrow \quad \theta\left(1+\frac{\nu}{\rho+\delta}\right)\left(1+\frac{\rho+\nu}{\frac{\rho}{1-\theta}+\delta}\right) < 1.
\end{aligned}$$

$\square$

ONLINE APPENDIX

B More Proofs

B.1 Lemma B.1 and Proof

Lemma B.1 (Population share of low-productivity agents). *Under Assumption 0, the population share of low-productivity agents at time t, denoted as $\Psi_l(t)$, follows*

$$\dot{\Psi}_l(t) + (\xi_t + \nu)\,\Psi_l(t) = \xi_t. \tag{B.1}$$

Proof. We provide a heuristic derivation. Over a "short" time period of length Δ, $\Psi_l(t)$ evolves according to

$$\begin{aligned} & \Psi_l(t+\Delta) = (1-\Delta\nu)\,\Psi_l(t) + \Delta\xi_t\,(1-\Psi_l(t)) \\ \Longrightarrow \quad & \frac{\Psi_l(t+\Delta) - \Psi_l(t)}{\Delta} = -\nu\Psi_l(t) + \xi_t\,(1-\Psi_l(t))\,. \end{aligned}$$

Taking the limit as $\Delta \to 0$, we get equation (B.1). $\square$

B.2 Lemma B.2 and Proof

Lemma B.2 (Density of low-productivity agents). *Let $\psi_l(\tau,t)$ denote the time-t density of the low-productivity agents whose last transition from high to low productivity occurred at time $t-\tau$. Under Assumption 0, $\psi_l(\tau,t)$ satisfies*

$$\partial_\tau\psi_l(\tau,t) + \partial_t\psi_l(\tau,t) = -\nu\psi_l(\tau,t)\,, \forall\tau \geq 0, \tag{B.2}$$

$$\psi_l(0,t) = \xi_t\Psi_h(t)\,, \tag{B.3}$$

where $\Psi_h(t) = 1 - \Psi_l(t)$[37] is the population share of high-productivity agents at time t.

Proof. We provide a heuristic derivation. Define

$$G_l(\tau,t) \equiv \int_0^\tau \psi_l(x,t)\,dx, \tau \in [0,t]\,, \tag{B.4}$$

[37] $\Psi_h(t)$ is also denoted as $\Psi_{h,t}$ in Section 2.1.

which is the time-t share of the low-productivity agents whose last transition from high to low productivity occurred between time $t-\tau$ and t.

We first derive the dynamics of $G_l(\tau,t)$. Over a "short" time period of length Δ, $G_l(\tau,t)$ evolves according to

$$G_l(\tau,t+\Delta) = (1-\Delta\nu)\, G_l(\tau-\Delta,t) + \Delta\xi_t\Psi_h(t)\,, \forall\tau\in[0,t]\,,$$

which implies

$$\frac{G_l(\tau,t+\Delta)-G_l(\tau,t)}{\Delta} = \frac{G_l(\tau-\Delta,t)-G_l(\tau,t)}{\Delta} - \nu G_l(\tau-\Delta,t) + \xi_t\Psi_h(t)\,.$$

Taking the limit as $\Delta\to 0$, we get

$$\partial_\tau G_l(\tau,t) + \partial_t G_l(\tau,t) + \nu G_l(\tau,t) = \xi_t\Psi_h(t)\,, \tag{B.5}$$

and thus, by the definition of $G_l(\tau,t)$ in equation (B.4), $\psi_l(\tau,t)$ satisfies

$$\partial_\tau\psi_l(\tau,t) + \partial_t\psi_l(\tau,t) = -\nu\psi_l(\tau,t)\,, \forall\tau\geq 0.$$

We then compute the density $\psi_l(\tau,t)$. Equation (B.5) is a linear first-order PDE, which can be solved as follows. Define $m(\tau,t)\equiv\tau$ and $n(\tau,t)\equiv t-\tau$, with $h(m(\tau,t),n(\tau,t)) = G_l(\tau,t)$. Using the chain rule, we have

$$\begin{aligned}\frac{\partial G_l(\tau,t)}{\partial\tau} &= \frac{\partial h}{\partial m}\frac{\partial m}{\partial\tau} + \frac{\partial h}{\partial n}\frac{\partial n}{\partial\tau} = \frac{\partial h}{\partial m} - \frac{\partial h}{\partial n},\\ \frac{\partial G_l(\tau,t)}{\partial t} &= \frac{\partial h}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial h}{\partial n}\frac{\partial n}{\partial t} = \frac{\partial h}{\partial n}.\end{aligned}$$

Substituting the above into equation (B.5), we get

$$
\begin{aligned}
& \frac{\partial h(m,n)}{\partial m} + \nu h(m,n) = \xi_{m+n}\Psi_h(m+n) \\
\implies \quad & h(m,n) = e^{-\nu m} h(0,n) + e^{-\nu m}\int_0^m e^{\nu u}\xi_{u+n}\Psi_h(u+n)\,du \\
\implies \quad & G_l(\tau,t) = e^{-\nu\tau}\int_0^\tau e^{\nu u}\xi_{u+t-\tau}\Psi_h(u+t-\tau)\,du, \forall\tau\in[0,t] \\
\implies \quad & \psi_l(\tau,t) = -\nu G_l(\tau,t) + \xi_t\Psi_h(t) \\
& + e^{-\nu\tau}\int_0^\tau e^{\nu u}\xi_{u+t-\tau}\Psi_h(u+t-\tau)\left(\frac{\dot{\xi}_{u+t-\tau}}{\xi_{u+t-\tau}} + \frac{\dot{\Psi}_h(u+t-\tau)}{\Psi_h(u+t-\tau)}\right)du, \\
& \forall\tau\in[0,t].
\end{aligned}
$$

Then for $\tau = 0$, $\psi_l(0,t) = \xi_t\Psi_h(t)$. Also note $\forall\tau\in(t,+\infty)$, the density is the stationary density in the initial steady state, i.e., $\psi_l(\tau,t) = \frac{\xi\nu}{\xi+\nu}e^{-\nu\tau}, \forall\tau > t$. $\square$

B.3 Lemma B.3 and Proof

Lemma B.3. *Let the initial capital stock $K_0 = K^*$ be given by the capital associated with the steady state interest rate $r^* < \rho$ and let Assumptions 0-2 and 6 be satisfied. Then the aggregate capital is increasing over time, i.e., $\dot{K}_t \geq 0$, $\forall t > 0$.*

Proof. We first prove that $\dot{K}_t > 0$ for $t\in(0,\varepsilon)$ with a sufficiently small $\varepsilon > 0$. Suppose otherwise, $\dot{K}_t < 0$ for $t\in(0,\hat{\varepsilon})$ for some small $\hat{\varepsilon} > 0$. Using the dynamics of aggregate capital in equation (30) of Lemma 3 and the expression for interest rate in equation (34), we get $\frac{\dot{K}_t}{K_t} = \hat{s}_t A K_t^{\theta-1} - \hat{\delta} = \frac{\hat{s}_t}{\theta}(r_t+\delta) - \hat{\delta}$. Since $\hat{s}_t$, defined in equation (31), is continuous and monotonically increasing over time under Assumption 6, we have $r_t < r^*$, $\forall t\in(0,\hat{\varepsilon})$. Then the expression for interest rate in equation (34) implies $K_t > K^*$, $\forall t\in(0,\hat{\varepsilon})$, a contradiction.

We then prove that $\dot{K}_t \geq 0, \forall t > 0$. Suppose otherwise, then $\exists\hat{t}\in(0,+\infty)$ such that $\dot{K}_t = 0$ at $t = \hat{t}$. Without loss of generality, let $\hat{t}$ denote the first time $\dot{K}_t = 0$, which means $\dot{K}_t > 0$, $\forall t\in\left(0,\hat{t}\right)$ and $\dot{K}_{\hat{t}+\Delta} < 0 \implies K_{\hat{t}+\Delta} < K_{\hat{t}}$ for a sufficiently small $\Delta > 0$. Then from $\frac{\dot{K}_{\hat{t}+\Delta}}{K_{\hat{t}+\Delta}} = s_{\hat{t}+\Delta}AK_{\hat{t}+\Delta}^{\theta-1} - \hat{\delta} < 0 = \frac{\dot{K}_{\hat{t}}}{K_{\hat{t}}} = s_{\hat{t}}AK_{\hat{t}}^{\theta-1} - \hat{\delta}$ and $s_{\hat{t}} < s_{\hat{t}+\Delta}$, we get $K_{\hat{t}+\Delta} > K_{\hat{t}}$, a contradiction. $\square$

B.4 Lemma B.4 and Proof

Lemma B.4. $r(\xi) < \rho \iff \chi(\xi) > 0$*, where* $r(\xi)$ *is the steady state interest rate defined in equation* (37) *and* $\chi(\xi)$ *is defined in equation* (38).

Proof. Using the definition of $\hat{s}(\xi)$ and $\hat{\delta}$ in equation (31) of Lemma 3, we can rewrite the steady state interest rate $r(\xi)$ in equation (37) as

$$r(\xi) = \theta \frac{\hat{\delta}}{\hat{s}(\xi)} - \delta = \frac{\theta(\rho+\nu)(\rho+\nu+\xi) - \delta\xi(1-\theta)}{\xi + \theta(\rho+\nu)}.$$

Then we have

$$\begin{aligned}
& \chi(\xi) > 0 \\
\iff \quad & \theta(\rho+\nu+\xi)\nu < \xi(1-\theta)(\rho+\delta) \\
\iff \quad & \theta(\rho+\nu+\xi)(\rho+\nu) < \xi(1-\theta)(\rho+\delta) + \theta(\rho+\nu+\xi)\rho \\
\iff \quad & \frac{\theta(\rho+\nu)(\rho+\nu+\xi) - \delta\xi(1-\theta)}{\xi+\theta(\rho+\nu)} < \rho \\
\iff \quad & r(\xi) < \rho.
\end{aligned}$$

□

C Derivations of the Optimal Consumption Allocations Using "Simplified" HJB Equations

In this section, we derive the optimal consumption allocations, assuming the high-productivity agents do not save.

For a high-productivity agent, his HJB equation is

$$\rho U_t^H = \max_{c_t} u(c_t) + \xi_t \left(U_t^L \left(\tilde{k}_t \right) - U_t^H \right) + \dot{U}_t^H.$$

His FOC (with log-utility) is thus

$$\frac{1}{c_t} = \left. \frac{\partial U_t^L(k)}{\partial k} \right|_{k=\tilde{k}_t},$$

where $\tilde{k}_t = \frac{w_t \zeta_t - c_t}{\xi_t}$ (implied by the budget constraint).

For a low-productivity agent who owns capital k_t at time t, his HJB equation is

$$\rho U_t^L(k_t) = \max_{c_t} u(c_t) + \nu\left(U_t^H - U_t^L(k_t)\right) + \left.\frac{\partial \dot{U}_t^L(k)}{\partial k}\right|_{k=k_t} x_t + \dot{U}_t^L(k_t).$$

His FOC (with log-utility) is thus

$$\frac{1}{c_t} = \left.\frac{\partial U_t^L(k)}{\partial k}\right|_{k=k_t}, \tag{C.1}$$

and the envelope condition is

$$\rho \left.\frac{\partial U_t^L(k)}{\partial k}\right|_{k=k_t} = -\nu \left.\frac{\partial U_t^L(k)}{\partial k}\right|_{k=k_t} + \left.\frac{\partial^2 \dot{U}_t^L(k)}{\partial k^2}\right|_{k=k_t} x_t + \left.\frac{\partial \dot{U}_t^L(k)}{\partial k}\right|_{k=k_t}. \tag{C.2}$$

Differentiating both sides of equation (C.1) wrt time t, we get

$$-\frac{\dot{c}_t}{c_t^2} = \left.\frac{\partial^2 \dot{U}_t^L(k)}{\partial k^2}\right|_{k=k_t} x_t + \left.\frac{\partial \dot{U}_t^L(k)}{\partial k}\right|_{k=k_t}. \tag{C.3}$$

Substituting equations (C.1) and (C.3) into the envelope condition (C.2), we get

$$\frac{\dot{c}_t}{c_t} = r_t - \rho.$$

Together with the budget constraint, we get the following linear system of differential equations

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= r_t - \rho, \\ \dot{k}_t &= (r_t + \nu)\, k_t - c_t, \end{aligned}$$

which has the unique solution[38]

$$c_t = (\rho + \nu)\, k_t, \tag{C.4}$$

$$\dot{k}_t = (r_t - \rho)\, k_t. \tag{C.5}$$

[38]It is straightforward to verify that equations (C.4) and (C.5) solve this system of equations. We show the uniqueness of the solution using the boundary condition $\lim_{t\to+\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t. See the proof of Lemma 1 for details.

This is exactly the optimal consumption allocation of the low-productivity agent in equations (18) and (19) of Lemma 1.

Finally, we derive the optimal consumption allocation of the high-productivity agent using the continuity of consumption. Equation (C.4) implies that a newly unproductive agent consumes $c_t = (\rho + \nu)\,\tilde{k}_t$, and recall $\tilde{k}_t = \frac{w_t \zeta_t - c_t}{\xi_t}$ from the continuity of consumption and the budget constraint of the high-productivity agent. Taken together, we get

$$c_t = (\rho + \nu)\,\frac{w_t \zeta_t - c_t}{\xi_t} \implies c_t = \alpha_t \zeta_t w_t,$$

where $\alpha_t \equiv \frac{\rho+\nu}{\rho+\nu+\xi_t}$. This is exactly the optimal consumption of the high-productivity agent in equation (23) of Lemma 2.

D Details of the Speed of Convergence Analysis

D.1 Neoclassical Growth Model

In this section, we consider the neoclassical growth model with a representative agent and complete market in continuous time. We first derive the speed of convergence along the transition using the definition in equation (52) and then compute its long-run value by log-linearizing the model.

D.1.1 Setup

There is a representative agent with utility function

$$\int_0^{+\infty} e^{-\rho t} u\left(C_t\right) dt,$$

where $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$. The production technology is

$$\begin{aligned}
Y_t &= A_t K_t^{\theta} L_t^{1-\theta}, \\
C_t + I_t &= Y_t, \\
\dot{K}_t &= I_t - \delta K_t, \\
C_t &\geq 0, K_t \geq 0,
\end{aligned}$$

where A_t, Y_t, K_t, C_t, I_t are the productivity, output, capital stock, consumption, and investment at time t, respectively. The economy is endowed with capital K_0 at time $t = 0$ and there is unit labor supply $L_t = 1$.

The first-order conditions give the interest rate and wage

$$\begin{aligned} r_t &= \theta A_t K_t^{\theta-1} - \delta, \\ w_t &= (1-\theta) A_t K_t^{\theta}. \end{aligned}$$

These yield the following consumption and capital dynamics

$$\begin{aligned} &\frac{\dot{C}_t}{C_t} = -\frac{\rho - r_t}{\sigma} = \frac{\theta A_t K_t^{\theta-1} - \rho - \delta}{\sigma}, \\ &\dot{K}_t = A_t K_t^{\theta} - \delta K_t - C_t. \end{aligned}$$

In the initial steady state with productivity A^*, aggregate capital and consumption are

$$\begin{aligned} &K^* = \left(\frac{\theta A^*}{\rho + \delta}\right)^{\frac{1}{1-\theta}}, \\ &C^* = A^* \left(K^*\right)^{\theta} - \delta K^*. \end{aligned}$$

D.1.2 Numerical Solution of the Speed of Convergence

We consider the transition dynamics under the productivity process $\{A_t\}_{t\geq 0}$. Let s_t denote the savings rate at time t. By definition,

$$C_t = (1 - s_t) Y_t = (1 - s_t) A_t K_t^{\theta}.$$

This implies

$$\begin{aligned} & \dot{K}_t = s_t A_t K_t^{\theta} - \delta K_t \\ \Longrightarrow \quad & \frac{\dot{K}_t}{K_t} = s_t A_t K_t^{\theta-1} - \delta = \frac{s_t (r_t + \delta)}{\theta} - \delta = s(\log K_t) A_t e^{(\theta-1)\log K_t} - \delta, \end{aligned}$$

where we view the savings rate as a function of log capital, i.e., $s_t = s(\log K_t)$.

Using the definition in equation (52), the speed of convergence can be expressed as

$$\beta_t = (1-\theta) s_t A_t K_t^{\theta-1} - \frac{\partial s(\log K_t)}{\partial \log K_t} A_t K_t^{\theta-1} = \frac{1-\theta}{\theta} s_t (r_t + \delta) - \frac{\partial s(\log K_t)}{\partial \log K_t} \frac{r_t + \delta}{\theta}.$$

In Figure 3 of the paper, we compute the above expression numerically.

D.1.3 Speed of Convergence in the Long Run

In this section, we derive the speed of convergence in the long run for a particular productivity process: at time 0, productivity increases unexpectedly but permanently from A^* to $\tilde{A}$, as in Proposition 2.

We first log-linearize the model. Note that

$$\begin{aligned}\frac{d\log K_t}{dt} &= \tilde{A}e^{-(1-\theta)\log K_t} - e^{\log\left(\frac{C_t}{K_t}\right)} - \delta,\\ \frac{d\log C_t}{dt} &= \frac{1}{\sigma}\left(\theta\tilde{A}e^{-(1-\theta)\log K_t} - \rho - \delta\right).\end{aligned}$$

In the new steady state, where $\frac{d\log K_t}{dt} = \frac{d\log C_t}{dt} = 0$, we have

$$\begin{aligned}\tilde{A}e^{-(1-\theta)\log K^{**}} - e^{\log\left(\frac{C^{**}}{K^{**}}\right)} &= \delta,\\ \theta\tilde{A}e^{-(1-\theta)\log K^{**}} &= \rho + \delta,\end{aligned}$$

where C^{**} and K^{**} denote aggregate consumption and capital in the new steady state (under productivity level $\tilde{A}$). Taking a first-order Taylor expansion, we get

$$\begin{bmatrix}\frac{d\log K_t}{dt}\\ \frac{d\log C_t}{dt}\end{bmatrix} = \begin{bmatrix}\rho & \delta - \frac{\rho+\delta}{\theta}\\ -(1-\theta)\frac{\rho+\delta}{\sigma} & 0\end{bmatrix}\begin{bmatrix}\log\left(\frac{K_t}{K^{**}}\right)\\ \log\left(\frac{C_t}{C^{**}}\right)\end{bmatrix}.$$

Let ϵ denote the eigenvalues of the first matrix on the right-hand side, i.e.,

$$\det\begin{bmatrix}\rho - \epsilon & \delta - \frac{\rho+\delta}{\theta}\\ -(1-\theta)\frac{\rho+\delta}{\sigma} & -\epsilon\end{bmatrix} = 0.$$

This implies

$$2\epsilon = \rho \pm \left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)(1-\theta)\frac{\rho+\delta}{\sigma}\right)^{\frac{1}{2}}.$$

Let ϵ_1 denote the positive root and ϵ_2 denote the negative root, then

$$\log K_t = \log K^{**} + \psi_1 e^{\epsilon_1 t} + \psi_2 e^{\epsilon_2 t}.$$

$\psi_1 = 0$ must hold and ψ_2 is determined from the initial condition

$$\psi_2 = \log\left(K^*\right) - \log\left(K^{**}\right).$$

Hence, in the log-linearized model, capital evolves according to

$$\log\left(K_t\right) = \left(1 - e^{-\beta t}\right)\log\left(K^{**}\right) + e^{-\beta t}\log\left(K_0\right),$$

where β is defined as

$$\beta \equiv -\epsilon_2 = -\frac{\rho - \left(\rho^2 + 4\left(\frac{\rho+\delta}{\theta} - \delta\right)\left(1-\theta\right)\frac{\rho+\delta}{\sigma}\right)^{\frac{1}{2}}}{2}. \tag{D.1}$$

The capital dynamics imply

$$\frac{\dot{K}_t}{K_t} = -\beta e^{-\beta t}\left(\log\left(K_0\right) - \log\left(K^{**}\right)\right) = -\beta\left(\log K_t - \log K^{**}\right).$$

Using the definition in equation (52), the speed of convergence in the log-linearized model is

$$-\frac{\partial\left(\frac{\dot{K}_t}{K_t}\right)}{\partial\left(\log K_t\right)} = -\frac{\frac{\dot{K}_t}{K_t}}{\log K_t - \log K^{**}} = \beta.$$

Hence, β, defined in equation (D.1), is the long-run speed of convergence in the neoclassical growth model.

E Details of the Consumption Inequality Analysis

In this section, we establish the mappings between the consumption inequality measures, the individual characteristic τ, the consumption ratio between a low-productivity agent and a high-productivity agent, and the quantiles in the population and consumption distribution. We consider a particular productivity process: at time 0, productivity increases unexpectedly but permanently from A^* to $\tilde{A}$, as in Proposition 2.

We introduce the following notations: let ι denote the consumption ratio of a low-productivity agent to a high-productivity agent, P denote the quantile in the population of agents ranked by consumption level, and G denote the quantile in the consumption distribution.

Symbol	Definition
τ	Length of time elapsed since the low-productivity agent's last transition to low productivity
ι	Consumption ratio of a low-productivity agent to a high-productivity agent
P	Quantile in the population ranked by consumption level
G	Quantile in the consumption distribution

E.1 Mapping between τ and P

We establish a one-to-one mapping between τ and the quantiles in the population ranked from the lowest to the highest consumption level. Using the optimal consumption allocation in equations (23) and (24) of Lemma 2, for all τ, τ' such that $0 < \tau' < \tau$, we have

$$\frac{c_{t-\tau,t}}{c_{t-\tau',t}} = \frac{w_{t-\tau}}{w_{t-\tau'}} e^{-\int_{t-\tau}^{t-\tau'} g_u du} < 1.$$

Hence, τ fully characterizes an agent's rank in the consumption distribution.

At any time t, we compute the fraction of agents who have lower consumption level than those that last transitioned from high to low productivity at time $t - \tau$. We use $P(\tau)$ to denote this metric. Then

$$P(\tau) = \int_{\tau}^{+\infty} \psi_l(x)\, dx = \int_{\tau}^{+\infty} \frac{\xi \nu}{\xi + \nu} e^{-\nu x} dx = \frac{\xi}{\xi + \nu} e^{-\nu \tau}.$$

Hence, $P(\tau)$ is the one-to-one mapping from τ to the quantiles in the population ranked by consumption level.

Inversely, we can also establish the mapping from the quantiles in the population to τ. Given a P, we find the τ such that P fraction of the agents have consumption level lower than those who last transitioned from high to low productivity at time $t - \tau$.

$$\tau(P) = \frac{1}{\nu} \log\left(\frac{\xi}{(\xi + \nu) P}\right).$$

E.2 Mapping from ι to P

Given an ι, we compute the fraction of agents whose consumption ratio is below ι.

For $\iota < 1$,

$$
\begin{aligned}
P(\iota, t) &= Pr\left(\frac{c_{t-\tau,t}}{c_{h,t}} \leq \iota\right) = Pr\left(-\log\left(\frac{c_{t-\tau,t}}{c_{h,t}}\right) \geq -\log(\iota)\right) \\
&= \int_{\underline{\tau}}^{+\infty} \psi_l(x)\, dx = \int_{\underline{\tau}}^{+\infty} \frac{\xi\nu}{\xi+\nu} e^{-\nu x} dx \\
&= \frac{\xi}{\xi+\nu} e^{-\nu \underline{\tau}(\iota,t)},
\end{aligned}
$$

where $\underline{\tau}(\iota, t)$ is the solution to the equation $\frac{c_{t-\underline{\tau},t}}{c_{h,t}} = \iota$.

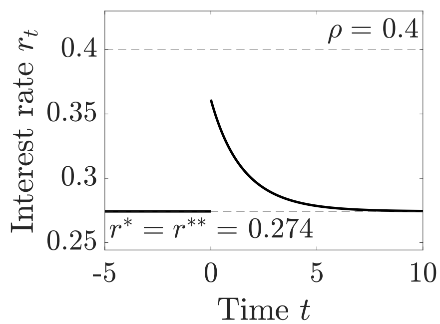
For $\iota = 1$, $P(1, t) = 1$.

E.3 Mapping from P to G

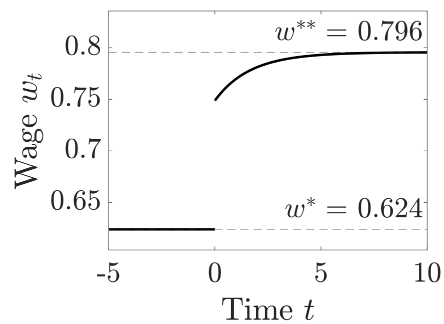
Given the quantile P for a group of agents, we compute their cumulative consumption share as follows,

$$
\begin{aligned}
G(P, t) &= \frac{1}{C_t} \int_{\tau(P)}^{+\infty} c_{t-x,t} \psi_l(x) dx \\
&= \frac{1}{C_t} \int_{\tau(P)}^{+\infty} c_{h,t-x} e^{-\int_{t-x}^{t} g_u du} \frac{\xi\nu}{\xi+\nu} e^{-\nu x} dx \\
&= \frac{1}{C_t} \frac{\xi\nu}{\xi+\nu} \alpha\zeta \int_{\tau(P)}^{+\infty} w_{t-x} e^{-\int_{t-x}^{t} g_u du} e^{-\nu x} dx \\
&= \alpha\xi \frac{w_t}{C_t} \int_{\tau(P)}^{+\infty} \frac{w_{t-x}}{w_t} e^{-\int_{t-x}^{t} g_u du} e^{-\nu x} dx.
\end{aligned}
$$

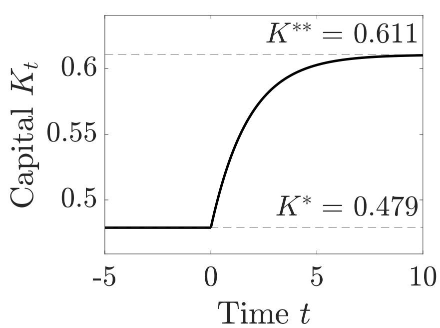
F Additional Figures



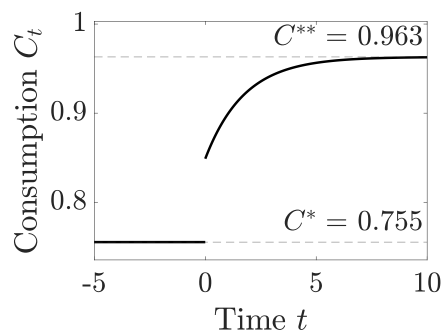
(a) Interest rate



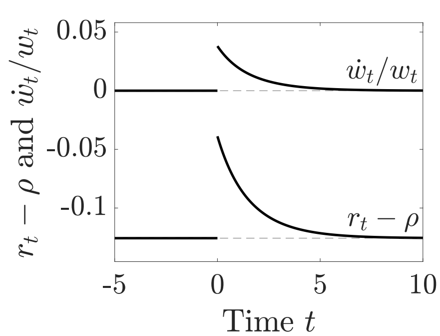
(b) Wage



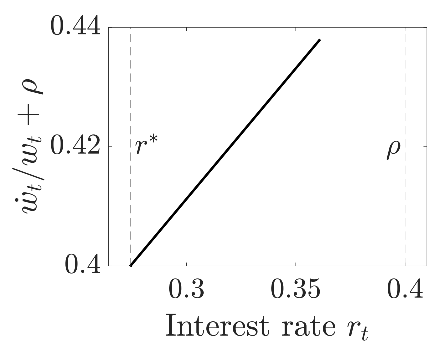
(c) Capital



(d) Consumption

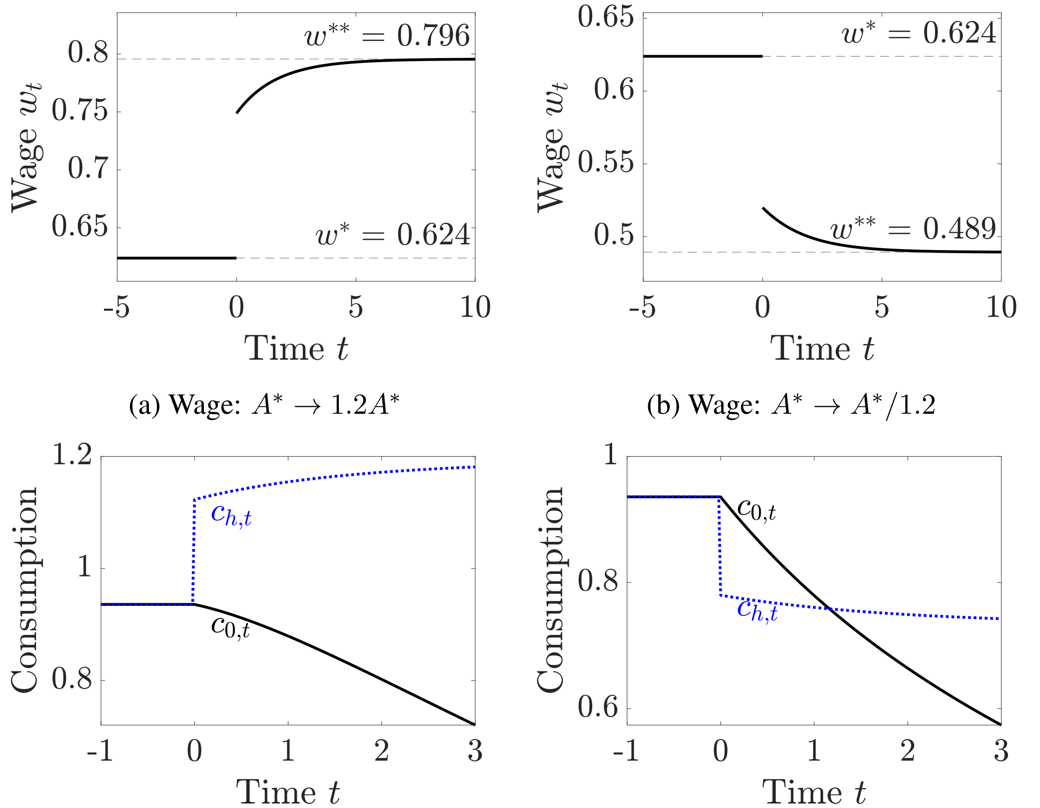


(e) Comparing $r_t - \rho$ with $\frac{\dot{w}_t}{w_t}$



(f) $\frac{\dot{w}_t}{w_t} + \rho$ as a function of r_t

Figure F.1: Transitional dynamics following a permanent increase in productivity. The figure plots the transition dynamics of aggregate variables when productivity permanently increases from $A^* = 1$ to $\widetilde{A} = 1.2$. Agents have log utility and the parameters are $\delta = 0.16$, $\nu = 0.2$, $\rho = 0.4$, $\theta = 0.25$, $\xi = 0.2$.



(c) Household consumption: $A^* \to 1.2A^*$ (d) Household consumption: $A^* \to A^*/1.2$

Figure F.2: Difference between permanent increase and decrease in productivity. We plot the dynamics of wages and individual agents' consumption, assuming that agents behave according to the consumption allocations in Lemmas 1 and 2. Panels a and c plot the dynamics when productivity permanently increases from $A^* = 1$ to $\tilde{A} = 1.2$. Panels b and d do the same when productivity permanently decreases from $A^* = 1$ to $\tilde{A} = 1/1.2$. In panels c and d, the solid black line plots consumption of a low-productivity agent who just transitioned from high productivity at time t, while the dotted blue line plots the consumption of a high-productivity agent at time t. Agents have log utility and the parameters are $\delta = 0.16, \nu = 0.2, \rho = 0.4, \theta = 0.25, \xi = 0.2$.

G Computational Details for Figures

We describe the features of all figures in Table G.1.

Table G.1: Description of Figures

Figure	Shocks A	Shocks ξ	x-axis	y-axis
3	Perm. $\uparrow$	–	Time	Capital & speed of convergence
4	–	Cont. $\uparrow$	Time	Income disp. & consump. ineq.
5	–	Cont. $\uparrow$	Population quantile	Cum. consump. share
F.1	Perm. $\uparrow$	–	Time	Aggregate variables
F.2	Perm. $\uparrow, \downarrow$	–	Time	Wage & individual consump.

We provide the computational details for the figures.

- Figure 3: This figure compares the speed of convergence to the new steady state in our model with that in the neoclassical growth model. The speed of convergence in our model is computed from equation (53) of Proposition 7, and that in the neoclassical growth model is computed according to Online Appendix D.1.2.

- Figure 4: This figure plots the transition dynamics of income dispersion and consumption, computed from equations (56) and (57), respectively.

- Figure 5: This figure plots the transition dynamics of the Lorenz curve. We construct panel a as follows: at each time t,

  1. A low-productivity household i is characterized by the time since he last transitioned from high to low productivity, which is denoted as τ_i. We create an equally spaced vector $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_n)$, which represents the cross section of low-productivity households at time t.

  2. For each household i, we compute the corresponding quantile in the population ranked by the consumption level, i.e., $P(\tau_i)$, using the definition of $P(\tau)$ in Section E.1.

  3. For each household i, we compute the cumulative consumption share (as a fraction of aggregate consumption) for those households with consumption level lower than i. The individual consumption level is computed from equations (23) and (24) of Lemma 2.

  4. Finally, we plot the population quantiles of these households (obtained from Step 2) on the x-axis and the cumulative consumption share from Step 3 on the y-axis.

In panel b, we plot the vertical distance between each line and the line for the initial steady state (i.e., $t < 0$) in panel a.

- Figure F.1: This figure plots the transition dynamics of the aggregate variables with a permanent increase in productivity. The steady state values and time paths of these variables are computed according to Sections 4.1 and 4.2.1.

- Figure F.2: This figure plots the transition dynamics of wage and individual households' consumption assuming that individual households consume according to the consumption allocations in equations (23) and (24) of Lemma 2. The time path of wage is computed according to Sections 4.1 and 4.2.1. Individual households' consumption in panels c and d are computed from equations (23) and (24).

H Transition Dynamics When Agents Have CRRA Utility

In this section, we show how to generalize our analysis, when agents have CRRA utility. Specifically, we assume agents have the period CRRA utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}$$

instead of log-utility $u(c) = \log(c)$, so that the time-t expected utility of an agent is

$$U_t = E_t \left[\int_t^{+\infty} e^{-\rho(\tau - t)} \frac{c_\tau^{1-\sigma} - 1}{1-\sigma} d\tau \right].$$

While much can still be calculated analytically, there is no longer a direct calculation of the equilibrium path. Rather, we obtain a fixed-point problem, which one would need to solve numerically.

More precisely, a transition equilibrium is characterized by the following fixed-point problem.

1. Conjecture a path for aggregate capital K_t along the transition, given the initial condition $K_0 = K^*$. Calculate interest rate r_t and wage w_t, using the first-order conditions of production firms, i.e., capital and labor demand. See Section H.1 below.

2. Characterize the optimal allocation $\mathcal{C}_t$, given the paths for r_t and w_t. See Section H.2 below.

3. Compute the path of aggregate capital supply K_t^S by aggregating the capital holdings across individual agents. See Section H.3 below.

4. Check whether the path of aggregate capital supply K_t^S matches the conjectured path of aggregate capital stock K_t in step 1.

For the remainder of Online Appendix H, we illustrate the above solution method in the case with aggregate productivity shocks, as discussed in Section 4.2.1.

H.1 Conjecture a Path for Aggregate Capital

We conjecture a path for aggregate capital $\{K_t\}_{t\geq 0}$ and then calculate the path for interest rate and wage using the first-order conditions (34) and (35) of production firms. Since the interest rate in stationary equilibrium does not depend on the value of aggregate productivity (exactly as in the standard neoclassical growth model), the interest rate will converge to the original steady state value r^*, i.e.,

$$\lim_{t\to+\infty} r_t = r^*.$$

This together with equation (34) implies that the aggregate capital K_t converges to

$$K_\infty = \left(\frac{\theta A_\infty}{r^* + \delta}\right)^{\frac{1}{1-\theta}}.$$

Hence, the conjectured capital path $\{K_t\}_{t\geq 0}$ is constrained by the two boundary conditions: $K_0 = K^*$ (i.e., the initial steady state as initial condition for $t = 0$) and $\lim_{t\to+\infty} K_t = K_\infty$.

H.2 Characterize the Optimal Allocation $\mathcal{C}_t$

The interest rate and wage are the only aggregate variables relevant for the dynamic insurance problem at the agent level. Equipped with a conjectured path of interest rate and wage from Section H.1, we can now characterize the optimal allocation by deriving conditions for the evolution of (individual) consumption and (individual) capital over time as functions of (individual) productivity. Define

$$g_t \equiv \frac{\rho - r_t}{\sigma}. \tag{H.1}$$

As might be expected from the standard consumption-savings problem with CRRA utility, g_t will turn out to be the negative of the consumption growth rate, if capital holdings are strictly positive. We define the discounting term

$$D_t \equiv \int_t^{+\infty} e^{-\int_t^s (r_u+\nu+g_u)du} ds. \tag{H.2}$$

This is the net present value of a consumption spell that starts at a level of 1, falls at a rate of $-g_t < 0$ defined above, and ends at the Poisson rate ν. This expression will be useful for calculating the cost of a consumption allocation for a newly unproductive agent.

Given the allocation $\mathcal{C}_t$, we define the implied time derivative of consumption as[39]

$$\dot{c}_t\left(k;z\right) \equiv \frac{\partial c_t\left(k;z\right)}{\partial t} + \frac{\partial c_t\left(k;z\right)}{\partial k} x_t\left(k;z\right).$$

The following result is the counterpart to Lemma 1 in the log-utility case.

Lemma H.1 (Optimal allocation for $z = 0$ and $k > 0$)**.** *Let Assumption 1 be satisfied. Then the optimal allocation in definition 1 is characterized by*

$$\frac{\dot{c}_t\left(k;z\right)}{c_t\left(k;z\right)} = -g_t, \tag{H.3}$$

where g_t is defined in equation (H.1).

Furthermore, if $z_t = 0$ and if, for some $\bar{k} > 0$ we have $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$, then there exists $\bar{\bar{k}}$ such that for all $k \leq \bar{\bar{k}}$,

$$c_t\left(k;0\right) = \frac{k}{D_t}, \tag{H.4}$$

$$x_t\left(k;0\right) = \left(r_t + \nu - \frac{1}{D_t}\right)k, \tag{H.5}$$

where D_t is defined in equation (H.2).

The proof is in Online Appendix H.4.1. We now use this result to characterize the consumption dynamics of high-productivity agents. We impose the following assumption.

[39] As a heuristic for this definition of the time derivative, suppose that productivity remains constant at z for some small time interval. In that case, note that $\dot{k}_t = x_t\left(k;z\right)$ and that consumption evolves as $c_t = c_t\left(k_t;z\right)$ which is a function of time only. Taking the derivative with respect to time yields the expression here.

Assumption H.1. *Suppose the aggregate wage and interest rate satisfy,* $\forall t \geq 0$*,*

$$g_t + \frac{\dot{w}_t}{w_t} - \frac{\xi \dot{D}_t}{\zeta w_t} > 0. \tag{H.6}$$

Lemma H.2 (Optimal allocation for $z = \zeta$ and $k = 0$)**.** *Let Assumption H.1 be satisfied. Then the optimal allocation in definition 1 implies the following consumption dynamics of low-productivity agents*

$$c_t(0;\zeta) = \frac{w_t \zeta}{1 + \xi D_t}, \tag{H.7}$$

$$x_t(0;\zeta) = 0, \tag{H.8}$$

where D_t *is defined in equation* (H.2)*.*

Furthermore,

$$\tilde{k}_t(0;\zeta) = \frac{w_t \zeta}{1 + \xi D_t} D_t. \tag{H.9}$$

The proof is in Online Appendix H.4.2. The term ξD_t in the denominator of the right-hand side of (H.7) is the insurance premium to obtain the capital stock $\tilde{k}_t(0;\zeta)$, in case of a transition to the low-productivity state and to assure desirable continuity of consumption via (H.4), if so. Indeed, the equality

$$\xi \tilde{k}_t(0;\zeta) = w_t \zeta - c_t(0;\zeta)$$

shows that this is an actuarially fair contract. Note that (H.7) implies

$$\frac{\dot{c}_t(0;\zeta)}{c_t(0;\zeta)} = \frac{\dot{w}_t}{w_t} - \frac{\xi \dot{D}_t}{\zeta w_t}. \tag{H.10}$$

Equation (H.10) rationalizes why we need Assumption H.1. If consumption could be chosen in an unconstrained fashion, then we would obtain (H.3). With (H.6), consumption would grow more slowly than the right-hand side of (H.10), but this can now only be accomplished per borrowing against future wages and choosing $x < 0$, subject to making the consumption-smoothing insurance payments against the transition to low productivity. But this is ruled out by the borrowing constraint (9). Put differently, Assumption H.1 assures that the high-productivity agent has no desire to accumulate capital.

For ease of notation, let $c_{s,t} = c_t(k_{s,t}; 0)$ denote the time-t consumption of a low-productivity agent who switched from high to low productivity at time $s \leq t$, and thus holds capital $k_{s,t}$. This notation implies that $c_{s,s}$ and $k_{s,s}$ are the consumption and capital holdings of an agent whose productivity has turned to zero this very instant. Furthermore, denote by $c_{h,t} = c_t(0; \zeta)$ the time-t consumption of a high-productivity agent with no assets.

From equation (H.4), capital holdings are proportional to consumption for low-productivity agents,

$$k_{s,t} = D_t c_{s,t}. \tag{H.11}$$

Equations (H.7) and (H.9) imply that

$$c_{s,s} = c_s(0; \zeta) = \frac{w_s \zeta}{1 + \xi D_s}, \tag{H.12}$$

$$k_{s,s} = \tilde{k}_s(0; \zeta) = \frac{w_s \zeta}{1 + \xi D_s} D_s. \tag{H.13}$$

Equation (H.12) is due to the fact that consumption is continuous and does not jump upon receiving a negative productivity shock (it in principle could, since it is a jump variable). For low-productivity agents, the consumption growth equation (H.3) or equivalently

$$\frac{\dot{c}_{s,t}}{c_{s,t}} = -\frac{\rho - r_t}{\sigma} = -g_t \tag{H.14}$$

holds except for the economy-wide "MIT-shock" at date $t = 0$ (on which the economy transitions to the productivity path $\{A_t\}_{t \geq 0}$). If an agent last switched from high to low productivity after that transition date, i.e., if $s > 0$, then equation (H.14) characterizes his consumption dynamics since that date. If the switch last happened at some date $s \leq 0$, this low-productivity agent will have started at some steady state capital k^*_{-s}, characterized by equation (40) in the log-utility case. More generally, using the results above applied to the steady state together with

$$g^* = \frac{\rho - r^*}{\sigma}, \tag{H.15}$$

$$D^* = \frac{1}{\nu + r^* + g^*}, \tag{H.16}$$

we have

$$
\begin{aligned}
c_h^* &= \frac{w^* \zeta}{1 + \xi D^*}, && \text{(H.17)}\\
c_\tau^* &= e^{-g^* \tau} c_h^*, && \text{(H.18)}\\
k_\tau^* &= D^* e^{-g^* \tau} c_h^*. && \text{(H.19)}
\end{aligned}
$$

The above two cases for s yield the consumption dynamics of low-productivity agents in Lemma H.3 below.

Lemma H.3 (Consumption dynamics of low-productivity agents)**.** *Let Assumption H.1 be satisfied. Consider the time-t consumption $c_{s,t}$ of a low-productivity agent who last switched from $z = \zeta$ to $z = 0$ at time $s \leq t$.*

1. *If $s > 0$, then*

$$
c_{s,t} = e^{-\int_s^t g_u du} \frac{w_s \zeta}{1 + \xi D_s}. \tag{H.20}
$$

2. *If $s \leq 0$, then*

$$
c_{s,t} = e^{-\int_0^t g_u du} \frac{k_{-s}^*}{D_0}. \tag{H.21}
$$

Equation (H.20) can be rewritten with equation (H.13) as

$$
c_{s,t} = e^{-\int_s^t g_u du} \frac{k_{s,s}}{D_s}, \tag{H.22}
$$

or more generally, as

$$
c_{s,t} = e^{-\int_q^t g_u du} \frac{k_{s,q}}{D_q}, \tag{H.23}
$$

for any $s \leq q \leq t$. Comparing equations (H.21) and (H.23), we see that the consumption of agents with $s < t$ will jump, if and only if $D_0 \neq D^*$: the change in the path of future interest rates may induce the agent to reduce or to increase current consumption, compared to the steady state and given the same budget or net present value at time $t = 0$.

H.3 Compute the Path of Aggregate Capital Supply

To compute the aggregate capital supply K_t at time t, we aggregate the capital holdings of low-productivity agents,

$$K_t = \int_{-\infty}^{t} k_{s,t}\psi_{l,t}\,(t-s)\,ds. \tag{H.24}$$

Lemma H.4 (Dynamics of aggregate capital supply)**.** *Let the initial capital stock $K_0 = K^*$ be given by the capital associated with the steady state interest rate $r^* < \rho$ and let Assumption H.1 be satisfied. Then the law of motion for aggregate capital is given by*

$$\dot{K}_t \;=\; \left(\frac{\xi D_t}{1+\xi D_t}(1-\theta)+\theta\right)A_t K_t^{\theta} - \left(\delta+\frac{1}{D_t}\right)K_t, \tag{H.25}$$

where D_t is defined in equation (H.2)*.*

The proof is in Online Appendix H.4.3. Given the initial capital stock $K_0 = K^*$, we can use the capital dynamics in equation (H.25) to solve for the path of aggregate capital supply. Lemma H.5 summarizes the results with proof in Online Appendix H.4.4.

Lemma H.5 (Aggregate capital supply)**.** *Suppose the aggregate capital evolves according to equation* (H.25) *for all $t \geq 0$, given the initial condition $K_0 = K^*$. Then the aggregate capital supply at any time $t \geq 0$ takes the following form*

$$K_t^S = \left(e^{-(1-\theta)\int_0^t b_u du}K_0^{1-\theta} + (1-\theta)\int_0^t e^{-(1-\theta)\int_s^t b_u du}a_s ds\right)^{\frac{1}{1-\theta}}, \tag{H.26}$$

where

$$a_t \;\equiv\; \left(\frac{\xi D_t}{1+\xi D_t}(1-\theta)+\theta\right)A_t, \tag{H.27}$$

$$b_t \;\equiv\; \delta+\frac{1}{D_t}, \tag{H.28}$$

and D_t is defined in equation (H.2)*.*

H.4 Proofs

H.4.1 Proof of Lemma H.1

Proof. The proof expands the proof of Lemma 1 in Appendix A. It is verbatim the same except that $\rho u'(c) = u''(c)\,\dot{c} + u'(c)\,r_t$ now implies

$$\frac{\dot{c}}{c} = \frac{r_t - \rho}{\sigma} = -g_t, \tag{H.29}$$

which is the consumption dynamics in equation (H.3).

When $z = 0$ and $\tilde{k}_t(k;0) = 0$ for all $k \leq \bar{k}$ and some $\bar{k}$, the consumption dynamics (H.3)[40] and the budget constraint (8) can be rewritten as the linear system of differential equations

$$\dot{c}_t = -g_t c_t, \tag{H.30}$$
$$\dot{k}_t = (r_t + \nu)\,k_t - c_t, \tag{H.31}$$

in the unknown functions c_t and k_t with the boundary condition[41] $\lim_{t\to+\infty} k_t = 0$, provided $k_t \leq \bar{k}$ for all t.[42] The solution is valid, as long as the implied path for k_s for $s \geq t$ does not cross the upper bound $\bar{k}$. This will be true for all $k_t \in \left(0, \bar{\bar{k}}\right)$ and some suitable $\bar{\bar{k}}$. □

H.4.2 Proof of Lemma H.2

Proof. The lemma is a version of Section D.3 (Lemma 7) in the Online Appendix of Krueger and Uhlig (2022), generalized to the case, where aggregate wages and interest rates are functions of time. Rather than replicating the steps of that section, we provide the

[40]Note that equation (H.3) implies $c_s = e^{-\int_t^s g_u du} c_t$. A less formal, but more meaningful argument in terms of economic theory is thus to recognize that the budget constraint and utility maximization imply that the current capital k is equal to the net present value of all future consumption, as long as the productivity state stays unchanged. This yields $k = \int_t^{+\infty} e^{-\int_t^s (r_u+\nu)du} c_s ds = D_t c_t$.

[41]The boundary condition ensures that no capital gets wasted and this follows from utility maximization.

[42]To derive equation (H.31), conjecture that the solution satisfies equation (H.4), which can be rewritten as

$$k_t = D_t c_t. \tag{H.32}$$

Note that

$$\dot{D}_t = -1 + (r_t + \nu + g_t) D_t. \tag{H.33}$$

Differentiate equation (H.32) w.r.t. time t and use equations (H.30) and (H.33) to derive equation (H.31).

key logic of the argument here and point to the results and proofs in Krueger and Uhlig (2022) mentioned above for a deeper foundation.

For a high-productivity agent, the Lagrangian, first-order conditions, and envelope condition are as in the proof for Lemma 1, see equations (A.1), (A.2), and (A.4), but applied to $k = 0$, $z = \zeta$, and $p_z = \xi$.

1. We first consider the choice of $\tilde{k}$. The solution in Lemma 1 implies that

$$U_t'\left(\tilde{k};0\right) = u'\left(\frac{\tilde{k}}{D_t}\right),$$

which increases to infinity, as $\tilde{k} \to 0$. The third first-order condition in (A.2) therefore implies that $\tilde{k} > 0$ and thus $\omega = 0$. With the first and third first-order conditions in (A.2), we obtain consumption smoothing

$$u'\left(\frac{\tilde{k}}{D_t}\right) = u'\left(c\right) \implies \tilde{k} = D_t c. \tag{H.34}$$

Hence, equation (H.7) follows from the budget constraint (8), provided that $x = 0$.

2. Then we need to show that $x > 0$ is not optimal. Suppose otherwise, $x > 0$ were optimal, then equation (H.34) together with the budget constraint (8) implies that $c_t < \frac{w_t\zeta}{1+\xi D_t}$, i.e., consumption is less than the right-hand side of equation (H.7). Furthermore, constraint (9) would not be binding, $\lambda = 0$, and consumption growth would satisfy equation (H.3). Let $[t, t+\Delta]$ be a time interval for some $\Delta > 0$, during which this is the case and along a path where no productivity switch occurs. Assumption H.1 then implies $c_s \leq \frac{w_s\zeta}{1+\xi D_s}$ during the interval $s \in [t, t+\Delta]$, i.e., consumption is less than the consumption level proposed in Lemma H.2 for that episode. The integral of utility during that time interval is then smaller than the utility of the solution proposed in Lemma H.2. This loss in utility can only be justified by the additional utility gained from consuming the accumulated capital after a switch to lower productivity for $s > 0$, or, alternatively, for $s > \Delta$ in case there is no switch to lower productivity. This amounts to postponing consumption compared to the solution proposed in Lemma H.2. But this can be seen to contradict the impatience of the agent relative to wage growth, as expressed in Assumption H.1. A precise formulation of that contradiction requires replicating the arguments in Section D.3

of Krueger and Uhlig (2022), allowing for the additional time evolution of r_t and w_t.

□

H.4.3 Proof of Lemma H.4

Proof. We use the notation $\psi_l\left(t-s,t\right)$ instead of $\psi_{l,t}\left(t-s\right)$ in equation (H.24) to denote the density of the low-productivity agents with length of time $t-s\geq 0$ elapsed since the last transition from high to low productivity.

We differentiate both sides of equation (H.24) wrt time t,

$$\begin{aligned}\dot{K}_t &= k_{t,t}\psi_l\left(0,t\right)\\ &\quad+\int_{-\infty}^{t}\Big(\dot{k}_{s,t}\psi_l\left(t-s,t\right)+k_{s,t}\left(\partial_\tau\psi_l\left(t-s,t\right)+\partial_t\psi_l\left(t-s,t\right)\right)\Big)\,ds.\end{aligned}\tag{H.35}$$

We compute $\frac{\dot{k}_{s,t}}{k_{s,t}}$ from equation (H.11),

$$\dot{k}_{s,t}=\dot{D}_t c_{s,t}+D_t\dot{c}_{s,t}\implies\frac{\dot{k}_{s,t}}{k_{s,t}}=\frac{\dot{D}_t}{D_t}+\frac{\dot{c}_{s,t}}{c_{s,t}}.\tag{H.36}$$

Equations (2), (B.3), and (H.13) yield

$$k_{t,t}\psi_l\left(0,t\right)=\frac{\xi w_t}{1+\xi D_t}D_t.\tag{H.37}$$

Using the definition of D_t in equation (H.2), we get

$$\frac{\dot{D}_t}{D_t}=r_t+\nu+g_t-\frac{1}{D_t}.\tag{H.38}$$

Substituting equation (H.14) for consumption growth and equation (H.38) into equation (H.36), we get

$$\frac{\dot{k}_{s,t}}{k_{s,t}}=r_t+\nu+g_t-\frac{1}{D_t}-g_t=r_t+\nu-\frac{1}{D_t}.\tag{H.39}$$

Substituting equations (B.2), (H.37), and (H.39) into equation (H.35),

$$\begin{aligned}\dot{K}_t &= \frac{\xi w_t}{1+\xi D_t} D_t + \int_{-\infty}^{t} \left(\left(r_t + \nu - \frac{1}{D_t} \right) k_{s,t} \psi_l \left(t-s,t \right) - \nu k_{s,t} \psi_l \left(t-s,t \right) \right) ds \\ &= \frac{\xi w_t}{1+\xi D_t} D_t + \left(r_t - \frac{1}{D_t} \right) K_t \\ &= \left(\frac{\xi D_t}{1+\xi D_t} \left(1-\theta \right) + \theta \right) A_t K_t^{\theta} - \left(\delta + \frac{1}{D_t} \right) K_t,\end{aligned}$$

where the third line above uses the expressions for interest rate and wage in equations (34) and (35). □

H.4.4 Proof of Lemma H.5

Proof. Equation (H.25) is a Bernoulli differential equation. Given an initial condition K_0, it can be solved as follows.

1. Rewrite equation (H.25) as a linear differential equation. Define $X_t \equiv K_t^{1-\theta}$, $a_t \equiv \left(\frac{\xi D_t}{1+\xi D_t} \left(1-\theta \right) + \theta \right) A_t$, and $b_t \equiv \delta + \frac{1}{D_t}$. We can rewrite equation (H.25) as

$$\dot{X}_t + \left(1-\theta \right) b_t X_t = \left(1-\theta \right) a_t. \tag{H.40}$$

2. Solve the linear differential equation (H.40). Multiply both sides of equation (H.40) by $e^{(1-\theta)\int_0^t b_u du}$,

$$\frac{d\left(e^{(1-\theta)\int_0^t b_u du} X_t \right)}{dt} = \left(1-\theta \right) e^{(1-\theta)\int_0^t b_u du} a_t.$$

Integrate both sides of the above equation from time 0 to t,

$$e^{(1-\theta)\int_0^t b_u du} X_t - X_0 = \int_0^t \left(1-\theta \right) e^{(1-\theta)\int_0^s b_u du} a_s ds,$$

which yields

$$X_t = e^{-(1-\theta)\int_0^t b_u du} X_0 + \left(1-\theta \right) \int_0^t e^{(1-\theta)\int_t^s b_u du} a_s ds. \tag{H.41}$$

3. Substitute the definition of X_t into equation (H.41),

$$\begin{aligned} K_t &= \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta) \int_0^t e^{(1-\theta)\int_t^s b_u du} a_s ds \right)^{\frac{1}{1-\theta}} \\ &= \left(e^{-(1-\theta)\int_0^t b_u du} K_0^{1-\theta} + (1-\theta) \int_0^t e^{-(1-\theta)\int_s^t b_u du} a_s ds \right)^{\frac{1}{1-\theta}} . \end{aligned}$$

□