Retail Trading and Asset Prices: The Role of Changing Social Dynamics*

Fulin Li†

November 1, 2022

Click here for the latest version

Abstract

Social-media-fueled retail trading poses new risks to institutional investors. This paper examines the origin and pricing of this new risk. I first present stylized facts on prices, quantities, and retail investors' beliefs, for a set of meme stocks. I establish that aggregate fluctuations in retail sentiment originate from a growing and concentrated social network. The retail sentiment fluctuations induce changes in investor composition. As sentiment increased throughout 2020 and 2021, retail investors built up long positions, while price-sensitive long-only institutions gradually exited the market since early 2020. Short interest stayed high in 2020, then dropped sharply following the price surge in January 2021, and remained low throughout 2021. I then develop a model that reconciles the price, quantity, and retail sentiment dynamics during this period. In the model, retail investors participate in a social network with concentrated linkages. The concentration of the network implies that idiosyncratic sentiment shocks do not "average out," which leads to aggregate fluctuations in retail sentiment. Aggregate retail sentiment shocks shift investor composition, which in turn determines the price of retail sentiment risk. In particular, following an increase in aggregate retail sentiment, price-elastic long-only institutions first hit their short-sale constraints. This leads to a decrease in aggregate demand elasticity in the market for an individual stock. Then a "small" retail sentiment shock can have a "large" price impact and even squeeze short sellers. Finally, I conduct counterfactuals, which show that social network dynamics shape the distribution of retail sentiment risk and have economically large impact on asset prices.

*I am grateful to my dissertation committee Lars Peter Hansen, Zhiguo He, Ralph Koijen (co-chair), Stefan Nagel (co-chair), and Harald Uhlig for their guidance and support. I also thank Francesca Bastianello, Filippo Cavaleri, Aditya Chaudhry, Manav Chaudhary, Eugene Fama, Niels Gormsen, John Heaton, Erica Xuewei Jiang, Jingoo Kwon, Dan Luo, Yueran Ma, Federico Mainardi, Sangmin S. Oh, Seongjin Park, Carolin Pflueger, Fabricio Previgliano, Yang Su, Quentin Vandeweyer, Anthony Lee Zhang, and seminar participants at Chicago Booth, Chicago Booth Asset Pricing Working Group, UChicago Economic Dynamics Working Group, and UChicago Joint Program and Friends Conference for helpful comments. This research was funded in part by the John and Serena Liew Fellowship Fund at the Fama-Miller Center for Research in Finance, University of Chicago Booth School of Business.

†The University of Chicago, fli3@chicagobooth.edu.

1 Introduction

Retail trading accounts for an increasing share of U.S. equity trading activity. Throughout 2020 and early 2021, retail investors were responsible for over 20% of the trading volume in the U.S. equity market.[1] New brokerage accounts opened by retail investors reached a record high in the first quarter of 2021.[2] This flood of new investors, many of which are young first-time traders, have transformed social media platforms (e.g., Reddit, Twitter, and TikTok) into virtual trading clubs, where they share investment ideas and encourage each other to pile into single stocks. Their sudden coordinated actions, fired by social media, present new risks to institutional investors in the market.

In this paper, I examine how social media has changed the nature of retail trading and the associated risks to institutional investors. I document that aggregate retail sentiment fluctuations originate from a growing and concentrated social network. Retail investors who communicate on the network cluster around a few "influencers." This concentration of influence implies that idiosyncratic shocks to retail investors' beliefs can lead to significant aggregate fluctuations in retail sentiment. The growth of the network would further amplify the fluctuations. Hence, social media has changed the nature of retail trading as a source of risk, where the risk distribution results from the interplay between idiosyncratic shocks and network effects.

For institutional short sellers, retail sentiment fluctuations create a second risk by shifting the investor composition in the market. I find that for a set of meme stocks, as retail sentiment increases, price-sensitive long-only institutions gradually exit the market. This can lead to a decrease in aggregate price elasticity in the market for an individual stock. Then a moderate retail sentiment shock can drive up the stock price and put institutional short sellers at risk. This compositional change determines the price of retail sentiment risk.

To analyze the retail sentiment dynamics, I obtain data from Reddit's WallStreetBets forum (hereafter referred to as WSB). This dataset allows me to recover the communication network of a representative group of retail investors and quantify the sentiment (or belief) of each individual investor. I combine this data with stock prices, short interest, and portfolio holdings of various classes of long investors. Using this comprehensive dataset, I present four facts on prices, quantities and beliefs, in the context of GameStop short squeeze.

Fact 1 establishes the relationship between GameStop's stock price and retail sentiment

[1]McCabe, C. (2021, June 18). It Isn't Just AMC. Retail Traders Increase Pull on the Stock Market. *WSJ*. https://www.wsj.com/articles/it-isnt-just-amc-retail-traders-increase-pull-on-the-stock-market-11624008602?mod=article_inline.

[2]Fitzgerald, M. (2021, April 15). Schwab adds 3.2 million new brokerage accounts in first quarter – more than all of 2020. *CNBC*. https://www.cnbc.com/2021/04/15/retail-trading-boom-schwab-first-quarter-2021-earnings.html.

from Reddit WSB forum. I document that the average retail sentiment of GameStop had been steadily increasing since the beginning of 2020, while the WSB discussion volume on GameStop spiked in January 2021. The spike in the discussion volume coincided with the price surge of GameStop.

The increase in average sentiment and discussion volume would contribute to an increase in aggregate retail sentiment. This effectively shifts the aggregate demand curve of retail investors, and its price impact crucially depends on the demand of investors who take the other side of the trade, which I explore next.

Fact 2 establishes that retail investors gradually built up their positions in GameStop throughout 2020 and early 2021, relative to long-only institutions. Retail investors' relative positions remained constant for the rest of 2021. This suggests that retail investors were relatively more optimistic than long-only institutions. Moreover, long hedge funds also built up their positions in 2020, but then liquidated almost all their long positions in 2021 Q1. This suggests that long hedge funds were initially riding the price increase. But their initial long strategies were no longer profitable after the price surge in January 2021, as they expected the price to quickly fall back to the pre-January level.

Fact 3 establishes that the short interest of GameStop increased from 60% to 80% from mid- to late 2020. But then it dropped sharply in January 2021 and stayed at below 20% throughout 2021. This is consistent with the narrative that short sellers got squeezed and were forced to cover their short positions.

Long-only institutions and short sellers are the two groups of investors who take the other side of the trade against retail investors. However, they are both constrained in terms of taking (large) short positions. Long-only institutions do not short for institutional reasons, while short sellers face margin constraints. If retail sentiment keeps rising and drives up the price, then both groups of investors will hit their portfolio constraints. In particular, when short sellers hit their margin constraints, they will be forced to cover their short positions, and stock price will rise even further.

Next, I examine the role of the WSB social network in driving retail sentiment fluctuations, in particular, how individual users' opinions factor into aggregate retail sentiment. To do so, I construct daily WSB communication networks from users' conversations and measure each user's influence based on their network connections.

Fact 4 establishes that the WSB communication network is highly concentrated, with a few influencers dominating the discussions. The influence distribution across users is right-skewed. This implies that the influencers' idiosyncratic sentiment shocks would propagate strongly through the social network, generating sizable aggregate fluctuations in retail sentiment.

Motivated by these observations, I develop a model to reconcile the price, quantity, and retail sentiment dynamics during this period. The model features three groups of investors: a large number of unconstrained retail investors, one long institution facing short-sale constraint, and one short institution facing margin constraint. The three groups of investors trade one risky asset, and they have heterogeneous beliefs (i.e., sentiment) about the asset's payoff.

The aggregate fluctuations in retail sentiment originate from a social network with highly concentrated linkages. A subset of retail investors participate in the social network. They draw idiosyncratic sentiment shocks and communicate according to their network connections. The concentration of the network implies that the influence distribution on the network is right-skewed. Influencers' views carry a disproportionately high weight in the aggregate view, and idiosyncratic sentiment shocks do not "average out" in the aggregate. This leads to aggregate fluctuations in retail sentiment.

The price of retail sentiment risk depends on the investor composition in the market. In particular, investors face heterogeneous financial constraints. As retail sentiment fluctuates over time, the constraints may bind for a sub-group of investors and effectively make them price-inelastic. This generates time-variation in aggregate price elasticity in the market for the risky asset, which determines the price impact of an aggregate retail sentiment shock.

Consider the case where retail investors (in aggregate) are relatively more optimistic than institutional investors. The two institutions have the same beliefs and only differ in their financial constraints – the long institution cannot short and thus faces a "tighter" constraint than the short institution. As retail sentiment increases and drives up the price, the long institution gradually reduces the long positions in the risky asset until he hits the short-sale constraint. Once the constraint binds, his demand does not respond to price changes. This translates into a decrease in the aggregate demand elasticity in the market for the risky asset. Now a "small" positive shock to retail sentiment can have a "large" price impact and even squeeze the short seller.

Retail sentiment fluctuations also redistribute wealth across investors with heterogeneous beliefs – those who happen to make the "right" bets gain wealth at the expense of others. The aggregate demand elasticity is a wealth-weighted average of individual investors' demand elasticities. Hence, wealth redistribution also generates time variation in aggregate demand elasticity and affects the price impact of retail sentiment shock. For example, as price increases, the short institution loses wealth and carries a smaller weight in the aggregate demand elasticity. In the extreme case where the short institution loses all the wealth, he exits the market, and only those investors who remain in the market determine the aggregate price elasticity. If these investors are sufficiently price-inelastic, then this also leads to a decrease in the aggregate demand elasticity in the market for the risky asset.

The model thus provides a unified explanation for the retail sentiment fluctuations originated from the social network, the price impact of the retail sentiment shocks, and the quantity dynamics induced by the retail sentiment fluctuations. I demonstrate (through a simple calibration) that the model can generate the price and quantity movements observed in the data.

Finally, I analyze two counterfactual scenarios through the lens of the model. First, I consider a scenario where the WSB discussion volume did not spike in January 2021. In the model, this corresponds to a smaller subset of retail investors participating in the social network, i.e., a smaller network "size." I show that the realized aggregate retail sentiment would be lower, and the short seller would not hit the margin constraint and would not get squeezed. Importantly, network concentration plays a fundamental role in driving the wedge between the sentiment realizations (under different network sizes). If the network linkages are not concentrated, then idiosyncratic sentiment shocks always "average out" regardless of the network size.

Second, I consider the case where short sellers updated their perceptions of retail sentiment risk after observing the influx of retail investors to WSB in January 2021. I demonstrate that the change in their risk perceptions can help explain GameStop's (and other meme stocks') price and short interest dynamics post the short squeeze episode.

The findings in this paper have broader implications on the changing market dynamics going forward, above and beyond a specific short squeeze episode or a specific set of meme stocks. Social media has fundamentally changed the nature of retail trading as a source of risk. In a world with financial constraints, even moderate fluctuations in retail sentiment can have significant consequences to institutional players in the market. The retail sentiment risk from social media and the investor composition change are two new risks for short sellers to adapt to.

My paper contributes to the empirical literature examining the impact of retail trading on asset prices. Recent work has focused on the the trading patterns of retail investors (Barber et al., 2021; Boehmer et al., 2021; Eaton et al., 2022) and how social media affects their trading behavior (Cookson and Niessner, 2020; Hu et al., 2021; Cookson et al., 2022). I present new facts on the interaction between retail investors and institutional investors. In particular, I show that retail trading can drive price-sensitive long-only institutions out of the market, causing a decrease in aggregate demand elasticity in the market for an individual stock. This mechanism rationalizes the price impact of retail trading observed from the data.

My paper demonstrates how social media has fundamentally changed the nature of retail trading as a source of risk. In particular, the dynamics of social connections shape the distribution of retail sentiment risk. This connects to the growing literature on social finance (Hirshleifer, 2020; Kuchler and Stroebel, 2021), which emphasizes the role of social interactions in shaping financial outcomes. For example, Bailey et al. (2018a) and Bailey et al. (2018b) examine the role of Facebook friendship network in driving economic decisions in the housing market and various other contexts. Compared with Facebook friendship network, the Reddit discussion network evolves much faster, since it is visible to all market participants including those who are not yet on the network. This fast-evolving nature of Reddit makes it harder to predict retail sentiment movement, and is crucial for understanding the distribution of retail sentiment risk.

A number of recent papers have explored various features of the Reddit community and its asset pricing implications. Bradley et al. (2021) focus on the due diligence reports on Reddit's WallStreetBets (WSB) forum. Hu et al. (2021) combine the information from Reddit with data on stock prices, shorting flows, and retail order flow to study the impact of social media activity on asset prices and retail trading, for a large sample of stocks. Allen et al. (2022) conduct a comprehensive analysis of the short squeeze episode in January 2021, using social media data from Reddit and data on stock prices, shorting activities, and retail trading of equities and options. My paper brings in additional institutional holdings data to study the interaction between retail investors and institutional investors. This is a unique setting where I observe prices, quantities, and retail investors' beliefs. I demonstrate that the information embedded in quantities (i.e., holdings by long-only institutions and short sellers) is important for understanding the asset pricing implications of social media activities. Moreover, I establish a direct mapping from network geometry to the asset price movements, both theoretically and empirically.

Retail investors are often thought of as noise traders (De Long et al., 1989; De Long et al., 1990), and their sentiment or beliefs is a "black box." My paper opens up the "black box" by empirically measuring the sentiment of individual investors and examining the changing social network structure that drives the day-to-day sentiment fluctuations. Through the lens of the model, I demonstrate that market participants can better predict retail sentiment movement by opening up this "black box."

I microfound the retail sentiment dynamics using a model of naive learning on social networks, which builds on the DeGroot-type models of social learning (DeGroot, 1974; DeMarzo et al., 2003; Golub and Jackson, 2010; Pedersen, 2022). My model extends the DeGroot-type framework to allow for interim sentiment shocks and time-varying network size. These extensions are essential for establishing retail sentiment as a source of risk.

I argue that the interim idiosyncratic sentiment shocks can lead to aggregate fluctuations in retail sentiment. This idea is borrowed from the literature that studies the effect of granularity (Gabaix, 2011) and network geometry (Acemoglu et al., 2012) on idiosyncratic shock aggregation. I apply this idea to the aggregation of idiosyncratic shocks to investor sentiment (or beliefs), and provide a statistic that captures the coordination among investors.

My model for pricing sentiment risk ties to the literature on disagreement and limits to arbitrage (Miller, 1977; Scheinkman and Xiong, 2003). In my model, there are two types of institutions: long institution facing short-sale constraint and short institution facing margin constraint (Brunnermeier and Pedersen, 2009; Gârleanu and Pedersen, 2011). The heterogeneity in financial constraints, together with the heterogeneity in beliefs, can reconcile the price, quantity and retail sentiment dynamics observed from the data.

This also fits into a broader theme on how heterogeneity matters for pricing (Caballero and Simsek, 2021; Gabaix and Koijen, 2022). The retail sentiment shocks in my context is a particular type of "flow" in Gabaix and Koijen (2022)'s definition. In my model, aggregate demand elasticity is one statistic that is closely related to the pricing of this "flow," which is consistent with Gabaix and Koijen (2022)'s argument. Moreover, my model microfounds the time variation in aggregate demand elasticity by introducing heterogeneous financial constraints and wealth effects. I illustrate that the time-varying aggregate demand elasticity is important for understanding the price of retail sentiment risk.

2 Data and methodology

2.1 Reddit data

2.1.1 Sample construction

I retrieve historical data on Reddit submissions and comments from the Pushshift API, using the Python Pushshift Multithread API Wrapper (PMAW). I restrict the data download to the subreddit r/wallstreetbets (WSB) and to the period from January 2020 to December 2021.

Occasionally, the Pushshift API does not return any submissions or comments for a given day, due to API outages. The missing data can be retrieved from the Pushshift dump files.[3] For any date that the Pushshift API returns zero submissions or comments, I pull data from these dump files.

In the raw data from Pushshift, submissions and comments are labeled with a UTC (Coordinated Universal Time) timestamp, which I convert to the New York time zone – a difference of 5 hours during Eastern Standard Time and 4 hours during Daylight Saving Time.

[3]See https://files.pushshift.io/reddit/.

Next, I construct a sample that includes submissions and comments about CRSP common stocks. To do so, I first obtain the list of tickers for CRSP common stocks, and then tag each submission with stock tickers through an iterative process of searching for tickers in the title and body text. If a submission is tagged with a ticker, then the associated comments are also tagged with the same ticker. Note that a submission or a comment can be associated with multiple stock tickers. Appendix A2.2 provides further details on the sample construction and the tagging algorithm.

2.1.2 Network construction

The WSB user network on day t can be represented by a directed graph $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$, where $\mathcal{V}_t = \{1, 2, \cdots, N_t\}$ is the set of users (or nodes, vertices) on the network, and $\mathcal{E}_t \subseteq \mathcal{V}_t \times \mathcal{V}_t \backslash \mathcal{D}_t$ is the set of directed edges between users, with $\mathcal{D}_t = \{(i, i) : i \in \mathcal{V}_t\}$ denoting self-loops.

To construct the node set $\mathcal{V}_t$ for day t, I select submissions and comments about CRSP common stocks,[4] made within the time window $[t - 30, t - 1]$. I define the node set $\mathcal{V}_t$ as the set of unique users who are authors of these selected submissions and comments. Hence, the nodes of the network are the users that have ever participated in the discussion of CRSP common stocks, during the 30-day window $[t - 30, t - 1]$.

To construct the edge set $\mathcal{E}_t$, I start by representing conversation threads as comment trees. A conversation thread consists of a particular submission and the associated comments. Figure 2 shows an example of a conversation thread. This thread consists of a submission made by the user Deep*******Value and the comments (on this submission) made by five other users. In particular, two of the users, YoloFDs4Tendies and FroazZ directly commented on the submission made by Deep*******Value. The other three users, smols1, GrowerNotAShower11, and DingLeiGorFei commented on FroazZ's comment. This thread is represented as a comment tree on the left side of Figure 3 panel (a). The comments made by YoloFDs4Tendies and FroazZ are called level-1 comments, since they were directly replying to the submission. The comments made by smols1, GrowerNotAShower11, and DingLeiGorFei are called level-2 comments, since they replied to a level-1 comment. The right side of Figure 3 panel (a) shows another tree, with quantkim being the author of the submission. The user FroazZ is the common user across the two trees.

I simplify each comment tree following Gianstefani et al. (2022). Specifically, I assume that any level-k comment is a direct reply to the submission, even if the comment was originally replying to some other comments. Figure 3 panel (b) shows the simplified trees

[4] The network constructed in this section is common to all stocks. Alternatively, one could also construct stock-specific networks, by selecting submissions and comments about a specific stock ticker and performing the rest of the construction in a similar way.

that correspond to the original ones in Figure 3 panel (a).

I construct one simplified tree for each selected submission within the $[t-30, t-1]$ time window. The nodes in each tree are the users who authored the submission or the associated comments. The set of directed edges are from users who commented on the submission to the user who authored the initial submission.

Finally, I define the edge set $\mathcal{E}_t$ as the union of the directed edges of all conversation trees. For example, in the two trees of Figure 3 panel (b), there is a common user FroazZ. When I take the union of the two trees, there are two edges that come out of FroazZ – one points to Deep*******Value (who is the author of the submission in the first conversation), and the other points to quantkim (who is the author of the submission in the second conversation). Figure 3 panel (c) shows the resulting network. Note that there are also cases where two distinct users i and j belong to multiple trees, and there is a directed edge from user i to user j in each tree. Then I only keep one edge from i to j in the edge set $\mathcal{E}_t$.[5] Furthermore, I drop self-loops, i.e., any edge from a user to himself.

To summarize, the user network on day t consists of node set $\mathcal{V}_t$ and edge set $\mathcal{E}_t$. The node set $\mathcal{V}_t$ is the set of unique users who are authors of the selected submissions and comments. The edge set $\mathcal{E}_t$ captures the connections between users. For any two distinct users $i, j \in \mathcal{V}_t$, if user j made a submission within the $[t-30, t-1]$ time window and user i commented on that submission, then there is a directed edge from i to j, i.e., $(i, j) \in \mathcal{E}_t$.

2.1.3 Influence measure

Based on the day-t network, I can measure the "influence" of each user on the network. In Section 3.4.1, I will explore the properties of the influence distribution in the cross section of users.

First define the adjacency matrix $\boldsymbol{A}_t = (a_{ij,t})$, which is an $N_t \times N_t$ square matrix with

$$a_{ij,t} \equiv \begin{cases} 1, & (i,j) \in \mathcal{E}_t \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

In other words, in the day-t network, there is a directed edge from user i to user j if and only if $a_{ij,t} = 1$. Hence, the adjacency matrix encodes the same information about user connections as the edge set $\mathcal{E}_t$. $a_{ij,t} = 1$ indicates that user i "listens to" or "attends to" user j, in the sense that i has commented on j's submission during the past 30 days.

[5] Alternatively, one could assign a positive weight to the edge from i to j, where the weight corresponds to the number of trees that have an edge from i to j, which is also the number of times user i commented on user j's submissions within the specific time window.

Then I normalize the rows of the adjacency matrix to be 1 to get the weighting matrix $\boldsymbol{W}_t = (\omega_{ij,t})$, where

$$\omega_{ij,t} \equiv \frac{a_{ij,t}}{\sum_{j=1}^{N_t} a_{ij,t}}. \tag{2}$$

I define the in-degree of user j on day t as

$$d_{j,t}^{in} \equiv \sum_{i=1}^{N_t} \omega_{ij,t}. \tag{3}$$

I call $d_{j,t}^{in}$ the "influence" of user j on day t. Intuitively, $\omega_{ij,t}$ captures the weight that user i assigns to user j, among all users that i listens to. Then $d_{j,t}^{in}$ sums up the weights that user j gets from all other users. A higher value of $d_{j,t}^{in}$ indicates that more users listen to or attend to j, and thus j is more influential.

2.1.4 Retail sentiment measures

For each submission (or comment), I conduct textual analysis on its augmented body text[6], using the Python sentiment analysis tool Valence Aware Dictionary and sEntiment Reasoner (VADER). VADER is a sentiment analysis tool attuned to social media text (Hutto and Gilbert, 2014). Its lexicon includes emojis and emoticons. Following Mancini et al. (2022), I further augment the VADER dictionary with the WSB-specific jargons listed in Table 4.

For a submission (or comment) l about stock n made by user j on day t, VADER returns a weighted composite sentiment score $Sent_l$ normalized to the range $[-1, 1]$.[7] A score in $[-1, -0.05]$ indicates that the submission has a negative tone, while a score in $[0.05, 1]$ indicates a positive tone. A score in $(-0.05, 0.05)$ indicates a neutral tone.

I aggregate the sentiment scores to stock-day level. I first compute an equal-weighted sentiment measure for stock n on day t, defined as

$$Sent_t^{EW}(n) \equiv \frac{1}{|\mathcal{L}_t(n)|} \sum_{l \in \mathcal{L}_t(n)} Sent_l, \tag{4}$$

where $\mathcal{L}_t(n)$ is the set of submissions and comments about stock n that came out within the window (4pm on day $t-1$, 4pm on day t], and $|\mathcal{L}_t(n)|$ is the number of submissions and comments in this set. For Monday sentiment, in addition to including the 4pm-midnight

[6] A submission has its title and body text. I obtain the augmented body text by appending the body text to the title, separated by a white space. A comment only has body text (without title).

[7] I use the compound score returned from VADER.

articles from Sunday, I also include articles from 4pm to midnight on the prior Friday.

Then I construct an influence-weighted sentiment measure, $Sent_t^{IW}(n)$, for stock n on day t. It is the average sentiment across users weighted by their influence, i.e.,

$$Sent_t^{IW}(n) \equiv \frac{1}{|\mathcal{J}_t(n)|} \sum_{j \in \mathcal{J}_t(n)} d_{j,t}^{in} \cdot Sent_{j,t}(n), \tag{5}$$

where $Sent_{j,t}(n) \equiv \frac{1}{|\mathcal{K}_{j,t}(n)|} \sum_{l \in \mathcal{K}_{j,t}(n)} Sent_l$ is the average sentiment of all submissions and comments about stock n made by user j on day t, $\mathcal{K}_{j,t}(n)$ is the set of submissions and comments about stock n made by user j on day t, $\mathcal{J}_t(n)$ is the set of users who made submissions or comments about stock n on day t, and $d_{j,t}^{in}$ is the influence of user j on day t defined in equation (3).

I use $Sent_t^{EW}(n)$ and $Sent_t^{IW}(n)$ as measures of retail investors' sentiment about stock n on day t. By construction, both measures are within the range $[-1, 1]$.

2.2 Financial data

I obtain data on stock prices and shares outstanding from CRSP, short interest data from IHS Markit and Compustat, holdings data of 13F institutions from FactSet, and retail order flow data from TAQ.

2.2.1 Short interest

I obtain the daily number of shares sold short from IHS Markit. I also obtain mid-month and end-month number of shares sold short from Compustat.

Short interest of stock n on day t, SI_t, is defined as the ratio of the number of shares sold short to the number of shares outstanding, i.e.,

$$SI_t(n) = \frac{S_t^{short}(n)}{S_t^{out}(n)}, \tag{6}$$

where $S_t^{short}(n)$ is the number of shares sold short, and $S_t^{out}(n)$ is the number of shares outstanding.[8]

2.2.2 Institutional and household holdings

I retrieve quarterly portfolio holdings of 13F institutions from FactSet. Following Gabaix and Koijen (2022) and Koijen et al. (2022), I classify 13F institutions into five groups: Hedge

[8]Figure A1 and A2 in the Internet Appendix compare the short interest measures constructed from IHS Markit data versus that from Compustat data.

Funds, Brokers, Private Banking, Investment Advisors, and Long-Term Investors. I then compute the total number of shares held by the institutions in each group. Appendix A3 includes further details on the data construction.

I back out household holdings from the market clearing condition. I assume that households do not short, and short sellers is a separate group of investors that are distinct from households and the long-only institutions in the 13F data. Then for stock n at the end of quarter t, the market clearing condition can be written as

$$Q_t^{\text{Households}}(n) + \sum_{g \in G} Q_t^g(n) \quad = \quad S_t^{out}(n) + S_t^{short}(n). \tag{7}$$

$Q_t^{\text{Households}}(n)$ is the number of shares held by households. $Q_t^g(n)$ is the total number of shares held by the 13F institutions in group $g \in G$, where $G = \{\text{Hedge Funds, Brokers, Private Banking, Investment Advisors, Long-Term Investors}\}$. $S_t^{out}(n)$ is the total number of shares outstanding, and $S_t^{short}(n)$ is the number of shares sold short from Compustat.

Equation (7) is an accounting identity. It says that the total number of shares held by long-only investors is equal to the number of shares outstanding plus the additional supply of shares from short selling. In the data, I observe the holdings of long-only institutions $\{Q_t^g(n)\}_{g \in G}$, the number of shares outstanding $S_t^{out}(n)$, and the number of shares sold short $S_t^{short}(n)$. Hence, I can back out the number of shares held by households from equation (7), i.e.,

$$Q_t^{\text{Households}}(n) \quad = \quad S_t^{out}(n) + S_t^{short}(n) - \sum_{g \in G} Q_t^g(n). \tag{8}$$

For each investor group $k \in G \cup \{\text{Households}\}$, I construct two measures of its percentage holdings.

- Shares held by investor group k as a percentage of the number of shares outstanding:

$$q_t^k(n) \quad \equiv \quad \frac{Q_t^k(n)}{S_t^{out}(n)}. \tag{9}$$

- Shares held by investor group k as a percentage of the sum of the number of shares outstanding and the number of shares sold short:

$$\hat{q}_t^k(n) \quad \equiv \quad \frac{Q_t^k(n)}{S_t^{out}(n) + S_t^{short}(n)}. \tag{10}$$

Note that $\sum_k \hat{q}_t^k(n) = 1$.

For the rest of the paper, I treat households and retail investors as the same group of investors, and I use household holdings as a measure of retail investors' positions.

Figure A3 and A4 in the Internet Appendix plot the total institutional holdings versus the sum of the number of shares outstanding and the number of shares shorted, for GameStop and AMC. After correcting for the additional supply from short selling (equation (7)), the total institutional holdings do not exceed the total supply.

2.2.3 Retail order flow

Section 2.2.2 constructs an indirect measure of retail investors' positions. In this Section, I present a direct yet noisy measure based on retail order flow. It serves as a cross-check of the indirect measure.

Boehmer et al. (2021), referred to as BJZZ hereafter, proposed an algorithm to identify off-exchange trades made by retail investors, based on sub-penny price improvement. Importantly, they assumed that the bid-ask spread is equal to one cent, and thus the price improvement has to be a fraction of one cent. If a trade was executed at less (more) than 0.4 (0.6) of a cent, then they labeled it as a retail sell (buy) trade. Barber et al. (2022) modified the BJZZ algorithm to take into account cases where the bid-ask spread is much larger than one cent. I use the modified BJZZ algorithm in Barber et al. (2022) to identify retail buy trades and sell trades. Appendix A4 includes details of the algorithm.

For stock n on day t, I first compute the total volume of retail buy orders $Mrbvol_t\left(n\right)$ and the total volume of retail sell orders $Mrsvol_t\left(n\right)$. Then I define cumulative net retail buy volume of stock n on day t as the cumulative difference between the two, i.e.,

$$Cum\ Net\ Retail\ Buy\ Vol_t\left(n\right) \equiv \sum_{s=0}^{t} Mrbvol_s\left(n\right) - Mrsvol_s\left(n\right). \tag{11}$$

Finally, I define cumulative net retail flow of stock n on day t as the ratio of the cumulative net retail buy volume to the sum of the number of shares outstanding and the number of shares shorted, i.e.,

$$Cum\ Net\ Retail\ Flow_t\left(n\right) \equiv \frac{Cum\ Net\ Retail\ Buy\ Vol_t\left(n\right)}{S_t^{out}\left(n\right) + S_t^{short}\left(n\right)}. \tag{12}$$

3 Stylized facts: prices, quantities, and beliefs

3.1 Price and aggregate retail sentiment

On January 28, 2021, GameStop hit an intra-day high price of \$483, compared to a price of less than \$20 throughout 2020. This price surge was believed to be driven by retail investors who communicated on WSB. So I begin by analyzing the relationship between GameStop's stock price and the aggregate retail sentiment from WSB.

Figure 4 plots the daily close price of GameStop (solid blue line), together with the equal-weighted retail sentiment from WSB (dotted red line).[9] The equal-weighted sentiment started at close to 0 in 2020 Q2, steadily increased to 0.2 till 2021 Q1, and remained stable for the rest of 2021. Recall from Section 2.1.4 that a sentiment score in $[0.05, 1]$ indicates an optimistic tone. Then the average sentiment level of 0.2 in 2021 suggests that retail investors were indeed optimistic, but far from being extremely optimistic.

More importantly, at different points in time, the same change in average retail sentiment had dramatically different price impact. For example, the equal-weighted sentiment increased by 15% from mid- to late December 2020, and also from early to late January 2021. Yet the price of GameStop increased by 1700% in the latter period, compared to 36% in the former. Moreover, the average retail sentiment of GameStop was stable in the latter half of 2021, but despite that, the price of GameStop still exhibited substantial volatility.

The price impact of average retail sentiment shocks not only had significant time variation, but also differed across stocks. Figure 5 panel (a) compares the equal-weighted sentiment of GameStop with that of two tech stocks – Amazon and Microsoft.[10] From late 2020 to early 2021, retail sentiment of Amazon and Microsoft had a similar increase as that of GameStop. However, Figure A5 and A6 in the Internet Appendix show that the prices of the two stocks did not soar as GameStop did in January 2021.

Aggregate retail sentiment is a combination of the average sentiment across users and the number of users who participate in the discussions on the social network. Figure 6 shows that, despite the moderate increase in average retail sentiment, the discussion volume about GameStop spiked in January 2021. Hence, the aggregate retail sentiment increased more than the average retail sentiment.

The change in aggregate retail sentiment effectively shifted the aggregate demand curve of retail investors, and its price impact crucially depends on the demand of investors who took the other side of the trade. In the extreme case where other investors (who traded GameStop) are perfectly price-elastic, they would willingly take the other side and prices would be unaffected. And thus retail sentiment change would have zero price impact. On the hand, a lack of price-elastic investors in this market could help explain the price surge

[9] In Figure 4, I plot 30-day moving averages of the daily sentiment series.

[10] In Figure 5, I plot 30-day moving averages of the daily sentiment series.

of GameStop in late January of 2021. In Section 3.2 and 3.3, I present facts on who took the other side of the trade and how their positions changed over time.

As a robustness check, I plot the price and sentiment of AMC in Figure A7 of the Internet Appendix. The price of AMC had a similar spike in late January of 2021, and its equal-weighted sentiment had a similar steady increasing trend.

I summarize the findings of this section into the following fact.

Fact 1: In the time series, the average retail sentiment of GameStop has been steadily increasing since the beginning of 2020, while the discussion volume on WSB about GameStop spiked in January 2021. The spike in discussion volume coincided with the price surge of GameStop. In the cross section, there are tech stocks that had similar trends in the average sentiment but did not have a price surge as GameStop did.

3.2 Positions of long investors

Figure 7 plots the quarterly holdings of households and long-only institutions of GameStop, as a fraction of the number of shares outstanding plus the number of shares sold short (equation (10)). From 2020 Q1 to 2021 Q1, households (blue shaded area) gradually built up their positions in GameStop, relative to long-only institutions. Households' relative positions remained constant for the rest of 2021. This suggests that households (or retail investors) were relatively more optimistic than long-only institutions, and the dynamics of household holdings is consistent with the dynamics of retail sentiment documented in Section 3.1.

Interestingly, long hedge funds (red shaded area) also built up their positions in 2020, but then liquidated almost all their long positions in 2021 Q1. One story is that long hedge funds were initially riding the price increase in 2020.[11] But after the price surge in January 2021, their initial long strategies were no longer profitable, as they expected the price to quickly fall back to the pre-January level.

Figure 8 panel (d), (e), and (f) plot the holdings of households, investment advisors, and hedge funds, using the number of shares outstanding as the denominator (equation (9)). These figures show the "absolute" holdings of each group of investors, which had similar patterns as the relative holdings in Figure 7 and Figure 8 panel (a)-(c). For AMC, Figure A8 and A9 in the Internet Appendix show similar patterns in the holdings of households versus long-only institutions.

In Figure 9 (and Figure A10 for AMC), I compare the quarterly household holdings measure in equation (10) with the daily cumulative net retail flow measure in equation (12). Both measures exhibit an increasing trend, though the latter has a temporary drop in late

[11] Brunnermeier and Nagel (2004) document similar behaviors of hedge funds during the technology bubble.

January of 2021, and the change in the latter from early 2020 to late 2021 is only half of the change in the former.

I summarize the key results in the following fact.

Fact 2: Households built up their positions in GameStop from 2020-2021, while long-only institutions reduced their positions. As a notable exception, long hedge funds initially built up their positions throughout 2020, then liquidated almost all their positions after 2021 Q1.

3.3 Positions of short sellers

Section 3.2 documents that the long-only institutions reduced their positions in GameStop, possibly because they thought the price was "too high" in January 2021, and it would quickly drop to the pre-January level. If short sellers (e.g., short hedge funds) held the same belief, then they would short more of GameStop in January 2021, hoping to profit from the subsequent price drop.

However, the data suggest the opposite. Figure 10 plots the daily short interest of GameStop (dotted red line) together with the price of GameStop (solid blue line). Short interest started out high at 80% of the outstanding shares till the end of 2020. But surprisingly, it dropped sharply in January 2021 and stayed at below 20% throughout 2021.[12] Given the high price of GameStop in 2021, it would be profitable for short sellers to take even larger short positions. But instead, they seem to have dropped out the market since January 2021.

Anecdotally, some short sellers were squeezed and lost capital. For example, Melvin Capital was forced to cover its short positions in GameStop and lost 53% on its investments in January 2021.[13] If these short sellers account for a large fraction of the short positions opened prior to January, then the sharp drop in short interest is consistent with the fact that they lost capital and exited the market.

The short squeeze might have been triggered by the 15% retail sentiment increase from early to late January 2021 (see Section 3.1). Consider a short seller who already had a large short position in GameStop prior to January and who faced a margin constraint. A 15%

[12] A short interest of 20% of outstanding shares is still considered high relative to an average stock. So the puzzle here is not the absolute level of the short interest in January 2021, but the time series patterns of the short interest.

[13] Adinarayan, T. (2021, January 28). Explainer: How retail traders squeezed Wall Street for bets against GameStop. *Reuters*. `https://www.reuters.com/business/retail-consumer/how-retail-traders-squeezed-wall-street-bets-against-gamestop-2021-01-27/`; Chung, J. (2021, February 1). Melvin Capital Lost 53% in January, Hurt by GameStop and Other Bets. *WSJ*. `https://www.wsj.com/articles/melvin-capital-lost-53-in-january-hurt-by-gamestop-and-other-bets-11612103117`.

increase in the average retail sentiment could make the margin constraint bind, and force the short seller to close part of the short position.

However, the remaining question is why "sophisticated" short sellers failed to anticipate the increase in retail sentiment and still maintained a large short position till January 2021. In Section 3.4, I explore the changing social dynamics on WSB, which likely led to an "unexpected" retail sentiment increase from the short sellers' perspective.

I sum up the findings of this section into the following fact.

Fact 3: The short interest of GameStop started out high at 80% of the outstanding shares until the end of 2020. But then it dropped sharply in January 2021, and stayed at below 20% throughout 2021.

Long-only institutions and short sellers are the two groups of investors who can take the other side of the trade against retail investors. However, they are both constrained in terms of taking (large) short positions. Long-only institutions like Fidelity do not short for institutional reasons, while short sellers like Melvin Capital face margin constraints. If retail sentiment keeps rising and drives up the price, then both group of investors will hit their constraints at some point. Once short sellers hit their margin constraints, they will be forced to cover their short positions, and price could rise even further. In Section 4, I present a model to formalize this idea.

3.4 Changing social dynamics on Reddit's WallStreetBets forum

In this section, I document the changing dynamics of WSB community leading up to January 2021. If short sellers failed to anticipate these changes, then they would likely make "mistakes" in opening or covering their short positions, or even get squeezed.

I first examine the aggregate dynamics of the WSB community. Figure 11 presents some descriptive statistics of daily submissions, comments, and user activity on WSB. Panel (a) shows that the number of subscribers to WSB (solid blue line) grew exponentially in late January of 2021, and then the growth rate reverted back to its pre-January level. Consistent with the growth of subscribers, there was a concurrent surge in the daily number of new submissions (panel (b) solid blue line), the daily number of new comments (panel (b) dotted red line), and the daily number of users who participated[14] in the discussions of CRSP stocks (panel (c)), in late January of 2021. Moving to the subjects of the discussions, panel (d) shows that the number of stock tickers mentioned (on a given day) also spiked in late January – over 700 tickers were mentioned on a given day, compared to less than 200 tickers before January.

[14] I define "participation" as follows: A user participated in the discussions about CRSP stocks on a given day, if and only if he made a new submission or a new comment about CRSP stock(s) on that day.

These facts suggest that WSB users became more engaged in the discussions in January 2021, and the engagement coincided with the price surge of GameStop. But how exactly did individual users' engagement translate into "collective actions" that could squeeze short sellers? And how is it related to the 15% sentiment increase from early to late January of 2021?

To answer these questions, I inspect the day-to-day activities of WSB users, and in particular, how influential users manage to spur others. Figure 12 shows the user communications on January 14, 2021.[15] Panel (a) plots user activities from 6-9am, right before the market opened. Each node represents a unique user who made a new submission or comment within this 3-hour window. For any two users i and j in this figure, if i commented on j's submission (within the 3-hour window), then I draw a directed edge from i to j. For example, the largest red dot represents the AutoModerator, and the dots clustered around it represent the users who commented on AutoModerator's submission.

The AutoModerator created "Daily Discussion Thread for January 14, 2021" at 06:00:18 on January 14, 2021. This thread quickly became the center of WSB discussions, as it received 46,228 comments, which is 94.26% of the comments received by new threads that came out between 6-9am. A similar discussion "hub" emerged right after market closed: At 16:00:16 on the same day, the AutoModerator started another thread titled "What Are Your Moves Tomorrow, January 15, 2021." Just like the morning discussion thread, this afternoon thread was the dominant thread on WSB between 4-7pm (Figure 12 panel (b)), which received 80.28% of the comments. These two types of threads are routine discussions on WSB. On each weekday, the AutoModerator will publish a new "Daily Discussion Thread" before market opens, and a new "What Are Your Moves Tomorrow" thread after market closes. Users typically discuss the market conditions and their trading strategies under these threads (Boylston et al., 2021; Mancini et al., 2022).

"Daily Discussion Thread" and "What Are Your Moves Tomorrow" are two prominent examples of "megathreads" on WSB, which are user-initiated discussions designated for a specific topic or issue. There are other megathreads for discussing individual stocks, e.g., GME megathreads. Figure 13 plots the WSB discussions between 6-8am on January 21, 2021. At 07:49:03, user grebfar created a thread titled "GME Megathread - Lemon Party 2: Electric Boogaloo." It received 67.84% of the comments, which is twice as many as the comments received by the daily discussion thread.

Figure 14 shows further evidence on the relative influence of GME megathreads versus the daily discussion threads, and how the relative influence evolves over time. The y-axis is the fraction of comments (on each day) received by a particular type of thread. The solid

[15] This figure is inspired by Mancini et al. (2022).

black line represents "GME Megathread," the dotted red line represents "Daily Discussion Thread" at market open, and the dash-dotted blue line represents "What Are Your Moves Tomorrow" at market close.[16] On January 20, 2021, the first GME megathread appeared and garnered as many comments as the daily discussion threads. It continued to be as influential as the daily discussion threads until mid-April, after which no new GME megathreads were created.

Megathreads could facilitate "collective actions" in the following sense: They make users' views visible to each other at a designated place. A particular user is able to gain influence within a short period of time and his view can suddenly dominate the community, which then leads to the kind of "collective actions" that short sellers fail to anticipate. In Section 3.4.1 and 3.4.2, I explore the dynamics of the influence distribution among users and the dynamics of influencers' views.

3.4.1 Dynamics of the influence distribution across users

Figure 15 plots the user network for GameStop discussion on January 14, 2021.[17] The red dots represent the top five most influential users. For each of these influencers, the percentage in parentheses is the fraction of users (on this network) that had commented on his posts within the past 30 days. Deep*******Value turns out to be the most influential user for GME discussion, and he attracted 20% of the users to comment on his posts.

Figure 15 also reveals that the influence distribution is highly skewed, with a few influencers receiving a lot of attention. This is a common feature of many empirical social networks, and the heavy right tail of the influence distribution can be approximated by a power-law distribution (Newman, 2005; Rantala, 2019). If user influence $d_{j,t}^{in}$ (defined in equation (3)) is drawn from a power-law distribution, then it has PDF

$$f_{d_{j,t}^{in}}(x) = \frac{\xi - 1}{d_{\min}} \left(\frac{x}{d_{\min}} \right)^{-\xi}, \xi > 1 \tag{13}$$

with support $[d_{\min}, +\infty)$. The exponent ξ captures the skewness of the influence distribution. Lower values of ξ correspond to heavier right tails and more right-skewed influence distribution. $d_{\min}$ is the lowest value at which the power law is obeyed (Newman, 2005).

[16] To identify GME megathreads, I search for the keyword "GME Megathread" (in a case-insensitive way) in the title of the threads. I identify "Daily Discussion Thread" and "What Are Your Moves Tomorrow" in a similar way. On a given day, there could be multiple threads of the same type, for example, multiple threads with "GME Megathread" in their titles. In that case, I take the total number of comments received by each type of thread, and then compute the fraction of comments each type received, which is what I plot on the y-axis of Figure 14.

[17] Here I only use submissions and comments about GameStop to construct the network, and the rest of the construction follows Section 2.1.2.

The power-law relationship implies that the log of influence $d_{j,t}^{in}$ and the log of the corresponding empirical frequencies (in the cross section of users) have a linear relationship. Figure 16 plots this relationship for January 14, 2021. The x-axis is the log of user influence (or in-degree), and the y-axis is the log empirical frequency. The relationship is approximately linear, which is consistent with the power-law distribution.

I then fit the power-law distribution to the vector of user influence on each day. Following Rantala (2019), I estimate the exponent $\hat{\xi}_t$ and the cutoff value $\hat{d}_{\min,t}$ for each day t using the maximum likelihood method, and I compute the confidence bands using bootstrap methods. Appendix A5 includes the computational details.

Figure 17 plots the time series of the $\hat{\xi}_t$ estimates with the bootstrapped confidence intervals. $\hat{\xi}_t$ is below 3 throughout the sample, which means the influence distribution is highly skewed. From the beginning to the end of January 2021, $\hat{\xi}_t$ dropped by 10%, from 2.1 to 1.9. This suggests that the influence distribution became increasingly skewed, which would allow influencers to spur more people.

Figure 18 plots the time series of the cutoff value $\hat{d}_{\min,t}$, which remains relatively stable within the range $[5, 15]$. Furthermore, Figure A12 in the Internet Appendix plots the p-value of the Kolmogorov-Smirnov test. A small p-value (less than 0.05) indicates that the test rejects the hypothesis that the original data could have been drawn from the fitted power-law distribution. For most of the dates, the test cannot reject the hypothesis that the original data was drawn from a power-law distribution.

Taken together, the influence distribution on WSB is right-skewed. This implies that influencers' views would quickly become dominant. If they happen to be optimistic, then the WSB community would quickly become optimistic as well. This could help explain the 15% increase in average retail sentiment from early to late January, 2021. In the next section, I document that influencers were indeed optimistic about GameStop.

3.4.2 Dynamics of influencers' views

In Section 3.4.1, I document that Deep*******Value was the most influential user in mid-January 2021. Figure 19 plots some examples of his posts. The titles of his posts always started with "GME YOLO." "YOLO" is a jargon on WSB and is considered a positive word – it means "You Only Live Once." Hence, the influencer Deep*******Value was indeed optimistic about GameStop, and his influence would allow him to spur a large number of users in the community.

Figure 4 shows the time variation of influencers' views. The dash-dotted green line is the influence-weighted sentiment for GameStop defined in equation (5), while the dotted red line is the equal-weighted sentiment in equation (4). From July to November 2020, the influence-weighted sentiment led the equal-weighted sentiment, which suggests that influencers happened to be optimistic and they spurred other users on the network.

I collect the results from this section in the following fact.

Fact 4: The distribution of user influence on WSB follows the power law with a heavy right tail, i.e., the influence distribution is right-skewed. Moreover, the influencers on WSB happened to be optimistic leading up to January 2021.

3.5 Proposed mechanism

Section 3.1-3.4 present a complete picture of the price, quantity and retail sentiment movements pre and post the GameStop frenzy. In this section, I propose a mechanism that reconciles these facts. In Section 4, I will formalize the idea within a model.

At the beginning of 2020, short sellers like Melvin Capital were pessimistic about GameStop's future prospects and believed that GameStop was "over-valued." Hence, they maintained large short positions, hoping to profit from a future price drop.

In mid-2020, influencers on WSB like Deep*******Value started to express their optimistic views about GameStop. Other users on WSB adopted the optimistic views and started to take long positions in GameStop. This resulted in a moderate price increase, which "drove out" price-elastic long-only institutions and attracted (more) short sellers to further increase their short positions, as they all thought the price was too high.

In January 2021, WSB went through a structural change – more users joined the network and the influence distribution remained highly skewed. This allowed influencers like Deep*******Value to be more influential and spur more people. Aggregate retail sentiment further increased, driving up the price and pushing short sellers towards their margin constraints. Short sellers did not expect this further sentiment increase, i.e., they were "surprised."

In late January of 2021, short sellers had to cover their short positions and suffered losses. Due to the short covering, price increased even further, and short sellers suffered from more significant losses. This ultimately led to the price surge on January 28, 2021. Some short sellers lost a large fraction of their capital and exited the market.

For the rest of 2021, retail investors and price-inelastic institutions like index funds remained in the market. Retail investors continued to be optimistic throughout 2021. Price-elastic long-only institutions and short sellers both dropped out of the market, and they no longer took the other side of the trade against the optimistic retail investors. Then a "small" retail sentiment shock would have a "large" price impact, due to a lack of price-elastic investors in this market.

Short sellers also changed their perceptions of retail sentiment risk, after observing a large influx of retail investors to the WSB forum in January 2021. They traded less aggressively in the latter half of 2021, being aware that the social network structure could change dramatically within a short period of time – this is a new risk for them to adapt to.

4 The pricing of retail sentiment risk

In this section, I present a model to reconcile the price, quantity and retail sentiment dynamics documented in Section 3. In particular, I show that a moderate increase in aggregate retail sentiment can have a large price impact, if it drives out price-elastic long-only institutions and squeezes short sellers. The price of retail sentiment risk depends on this shift in investor composition.

4.1 Setup

Time is discrete and is indexed by $t \in \{-1, 0, 1, 2\}$. There are $\bar{N} + 2$ investors who are divided into three groups: $\bar{N}$ retail investors indexed by j, a long institution (IL), and a short institution (IS). Investors trade a risky asset and a risk-free asset. And they differ in their beliefs about the risky asset's payoff, their risk aversion, and the portfolio constraints they face.

Assets Assets are traded at time $t \in \{0, 1\}$. The risk-free asset is in zero net supply, and has raw return $R_{f,t} = 1$ which is exogenously given.

The risky asset has a constant supply of $\bar{S}$ shares, and pays a one-time dividend $\tilde{D}$ at time 2. Let $\tilde{d} \equiv \log \tilde{D}$ denote its log payoff. The dividend payment is unobserved at time $t \in \{-1, 0, 1\}$. And the time-t conditional distribution of $\tilde{d}$ is truncated normal on the interval $\left[\underline{d}, \bar{d}\right]$, with post-truncation mean μ_d and variance σ_d^2. Let P_t and $p_t \equiv \log P_t$ denote the price and log price of the risky asset at time t, and let $\log X_t$ denote its log payoff at time t. Then

$$\log X_0 = p_0, \log X_1 = p_1, \log X_2 = p_2 = \tilde{d}.$$

Further define $\mathbb{E}_t \left[\log X_{t+1}\right]$ and $\sigma_t^2 \equiv \mathrm{Var}_t \left(\log X_{t+1}\right)$ as the time-t conditional mean and variance of next period's log payoff, respectively. And note that $\sigma_1^2 = \sigma_d^2$.

Then the risky asset has one-period raw return $R_{t+1} \equiv \frac{X_{t+1}}{P_t}$ from time t to $t+1$. Define $r_{t+1} \equiv \log R_{t+1}$ as the one-period log return of the risky asset, $r_{f,t} \equiv \log R_{f,t} = 0$ as the one-period log return of the risk-free asset.

Investors' subjective beliefs Investors have subjective beliefs about the risky asset's payoff. Specifically, at time $t \in \{0, 1\}$, investor i believes that the log payoff of the risky asset at time $t+1$ has mean $\mathbb{E}_t^i [\log X_{t+1}]$ and variance $\text{Var}_t^i (\log X_{t+1})$. I assume that investors know the true variance of the log payoff, i.e.,

$$\text{Var}_t^i (\log X_{t+1}) = \sigma_t^2, \forall i. \tag{14}$$

Investors disagree about the mean of the log payoff. First consider the institutional investors. At time 0, the two institutions IL and IS have subjective beliefs (about the mean)

$$\mathbb{E}_0^{IL} [\log X_1] = \mathbb{E}_0 [p_1] + \delta_0^{IL}, \tag{15}$$

$$\mathbb{E}_0^{IS} [\log X_1] = \mathbb{E}_0 [p_1] + \delta_0^{IS}, \tag{16}$$

where $\mathbb{E}_0 [p_1]$ is the conditional mean of time-1 log price, which is an equilibrium outcome. δ_0^{IL} and δ_0^{IS} capture the wedges between the subjective beliefs and the objective beliefs, and are given exogenously. At time 1, the two institutions have subjective beliefs that are consistent with the objective mean, i.e.,

$$\mathbb{E}_1^{IL} [\log X_2] = \mathbb{E}_1^{IS} [\log X_2] = \mathbb{E}_1 [p_2] = \mu_d. \tag{17}$$

Hence, at time 0, institutions disagree about the mean, while at time 1 they know the "true" mean.

There are two types of retail investors: At time t, the first N_t retail investors (labeled as "type 1") have subjective beliefs that deviate from the objective beliefs, while the rest $\bar{N} - N_t$ retail investors (labeled as "type 2") have subjective beliefs that conform with the objective ones. In particular, at time $t \in \{0, 1\}$, the subjective belief of type-1 retail investor $j \in \{1, 2, \cdots, N_t\}$ is

$$\mathbb{E}_t^j [\log X_{t+1}] = \mathbb{E}_t [p_{t+1}] + y_t^j, \tag{18}$$

where y_t^j is the deviation of j's belief from the objective expectation. I call y_t^j the "sentiment" of retail investor j.

Type-1 retail investors communicate on a social network, and form their subjective beliefs (and thus sentiment) by "listening to" other people on the network. In Section 5, I microfound their sentiment dynamics using a model of naive learning on networks. The model yields the conditional distribution of retail investor sentiment $\left\{y_t^j\right\}_{j=1}^{N_t}$.

Note that the number of type-1 retail investors, N_t, is time-varying. Assume that $0 \leq N_t \leq \bar{N}$, and define the fraction of type-1 retail investors at time t as

$$\theta(N_t) \equiv \frac{N_t}{\bar{N}} \in [0,1]. \tag{19}$$

Investors' preferences, budget constraint, and wealth share dynamics Investor i solves the following myopic portfolio choice problem

$$\max_{w_t^i} w_t^i \left(\mathbb{E}_t^i[r_{t+1}] - r_{f,t}\right) + \frac{1}{2} w_t^i \left(1 - w_t^i\right) \mathrm{Var}_t^i(r_{t+1}) + \frac{1}{2}\left(1 - \gamma^i\right)\left(w_t^i\right)^2 \mathrm{Var}_t^i(r_{t+1}), \tag{20}$$

where γ^i is his constant relative risk aversion, and w_t^i is the fraction of end-of-period wealth invested in the risky asset, i.e., the portfolio weight on the risky asset. Define risk tolerance $\tau^i \equiv \frac{1}{\gamma^i}$. I assume that institutional investors (IL and IS) have the same relative risk tolerance $\tau^I = \frac{1}{\gamma^I}$. The $\bar{N}$ retail investors have the same risk tolerance $\tau^R = \frac{1}{\gamma^R}$.

The budget constraint for investor i is

$$A_{t+1}^i = A_t^i \left(w_t^i \exp(r_{t+1}) + \left(1 - w_t^i\right)\exp(r_{f,t})\right), \tag{21}$$

where A_t^i is the investor's wealth entering period t.

Since the risk-free asset is in zero net supply, the aggregate wealth is equal to the market value of the risky asset. Hence, the time-1 wealth share of investor i is

$$\alpha_t^i \equiv \frac{A_t^i}{P_t \bar{S}}. \tag{22}$$

Appendix A1.1 shows that the budget constraint (21) implies the following wealth share dynamics

$$\alpha_{t+1}^i = \alpha_t^i \left(\left(1 - w_t^i\right)\exp(p_t - p_{t+1}) + w_t^i\right). \tag{23}$$

Non-negative wealth constraint All investors are subject to the non-negative wealth constraint

$$A_t^i \geq 0, \forall t.$$

If an investor loses all his wealth, then he cannot invest and has to exit the market.

Portfolio constraints Institutional investors face portfolio constraints. The long institution IL faces short-sale constraint of the following form

$$w_t^{IL} \geq 0. \tag{24}$$

The short institution IS faces margin constraint on short selling. Following Gârleanu and Pedersen (2011), I assume the margin constraint limits the leverage short sellers can take, i.e.,

$$w_t^{IS} \geq -\frac{1}{m}, \tag{25}$$

where $m \in (0, 1)$.

Market clearing Following Caballero and Simsek (2021), I show in Appendix A1.2 that the market clearing conditions for the risky asset and the risk-free asset are equivalent to the set of conditions

$$\sum_i A_t^i = \sum_i w_t^i A_t^i = P_t \bar{S}. \tag{26}$$

Equation (26) says that aggregate wealth is equal to the market value of the risky asset, both before and after investors make portfolio decisions. The conditions in equation (26) are also equivalent to

$$\sum_i \alpha_t^i w_t^i = 1, \tag{27}$$

where the wealth share α_t^i is defined in equation (22). This condition says that the wealth-share-weighted sum of portfolio weights is equal to 1.

Endowment and implicit price at time -1 At time -1, investor i is endowed with wealth share α_{-1}^i and portfolio weight w_{-1}^i. I assume that at time -1, investors do not anticipate future sentiment shocks. They all believe that the prices at time 0 and 1 will reflect the present value of the final dividend payment. In Appendix A1.10, I derive the implicit price p_{-1} that is consistent with this belief. Under this price, investors do not want to trade at time -1 and they enter time 0 with their initial endowment.

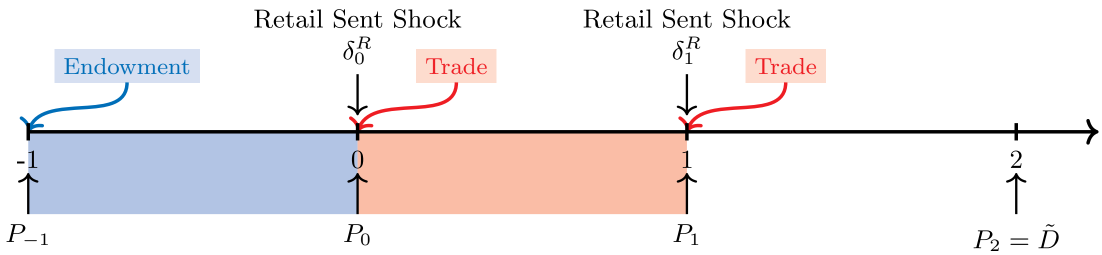


Figure 1. Timeline of the model.

Timeline Figure 1 shows the timeline of the model. At time -1, investors receive their endowment. At time 0 and 1, investors form subjective beliefs about next period's asset payoff, and then trade according to their beliefs. At time 2, the risky asset pays dividend.

In addition, I impose the following assumption.

Assumption 1. *At time $t \in \{0, 1\}$ before trading, retail investors first split their time $t-1$ end-of-period wealth equally among themselves. In particular, they split their aggregate stock position as well as aggregate bond position equally. Then they make portfolio choices based on their wealth after the splitting.*

Assumption 1 says that retail investors split their wealth equally before trading. This assumption together with linear demand implies that there exists an aggregate retail investor whose sentiment matters for asset prices. Lemma 1 in Section 4.2 formalizes this argument.

4.2 Investor demand

In this section, I first derive the asset demand of individual investors. Then I show that there exists an aggregate retail investor whose sentiment matters for asset prices.

Retail investors Type-1 retail investor j solves the portfolio problem in (20). His subjective expectation deviates from the objective expectation by y_t^j. Appendix A1.3.1 shows that his optimal portfolio weights on the risky asset are

$$w_0^j = \tau^R \left(\frac{\mathbb{E}_0 [p_1] + y_0^j - p_0}{\sigma_0^2} + \frac{1}{2} \right), \quad (28)$$

$$w_1^j = \tau^R \left(\frac{\mu_d + y_1^j - p_1}{\sigma_d^2} + \frac{1}{2} \right). \quad (29)$$

Type-2 retail investors' subjective beliefs conform with the objective beliefs. Hence, a type-2 retail investor j' chooses portfolio weights

$$w_0^{j'} = \tau^R \left(\frac{\mathbb{E}_0 [p_1] - p_0}{\sigma_0^2} + \frac{1}{2} \right), \tag{30}$$

$$w_1^{j'} = \tau^R \left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2} \right). \tag{31}$$

Long institution The long institution solves the portfolio problem in (20), subject to the short-sale constraint in (24). Appendix A1.3.2 shows that his optimal portfolio weights on the risky asset are

$$w_0^{IL} = \max \left\{ 0, \tau^I \left(\frac{\mathbb{E}_0 [p_1] + \delta_0^{IL} - p_0}{\sigma_0^2} + \frac{1}{2} \right) \right\}, \tag{32}$$

$$w_1^{IL} = \max \left\{ 0, \tau^I \left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2} \right) \right\}. \tag{33}$$

Short institution The short institution solves the portfolio problem in (20), subject to the margin constraint in (25). Appendix A1.3.3 shows that his optimal portfolio weights on the risky asset are

$$w_0^{IS} = \max \left\{ -\frac{1}{m}, \tau^I \left(\frac{\mathbb{E}_0 [p_1] + \delta_0^{IS} - p_0}{\sigma_0^2} + \frac{1}{2} \right) \right\}, \tag{34}$$

$$w_1^{IS} = \max \left\{ -\frac{1}{m}, \tau^I \left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2} \right) \right\}. \tag{35}$$

For the rest of the paper, I focus on scenarios where in equilibrium, the portfolio constraints for institutions do not bind at time 0, while they may bind at time 1 depending on the retail sentiment realization $\left\{y_1^j\right\}_{j=1}^N$.

Before characterizing the equilibrium, I first show that there exists an aggregate retail investor, whose sentiment drives asset prices.

Lemma 1 (Existence of an aggregate retail investor). *Under Assumption 1, the aggregate demand of the $\bar{N}$ retail investors is equal to the demand of an aggregate retail investor (R).*

- *The aggregate retail investor has subjective beliefs*

$$\mathbb{E}_0^R [p_1] = \mathbb{E}_0 [p_1] + \delta_0^R, \mathrm{Var}_0^R (p_1) = \sigma_0^2,$$
$$\mathbb{E}_1^R \left[\tilde{d}\right] = \mu_d + \delta_1^R, \mathrm{Var}_1^R \left(\tilde{d}\right) = \sigma_d^2.$$

His time-t sentiment δ_t^R ($t \in \{0,1\}$) aggregates individual retail investors' sentiment in the following way

$$\delta_t^R = \theta(N_t)\, y_t^R, \tag{36}$$

$$y_t^R \equiv \frac{1}{N_t}\sum_{j=1}^{N_t} y_t^j, \tag{37}$$

where N_t is the number of type-1 retail investors at time t, and $\theta(N_t)$ is the fraction of type-1 retail investors defined in equation (19).

- *The aggregate retail investor's demand for the risky asset (in terms of portfolio weights) takes the form*

$$w_0^R = \tau^R\left(\frac{\mathbb{E}_0[p_1] + \delta_0^R - p_0}{\sigma_0^2} + \frac{1}{2}\right), \tag{38}$$

$$w_1^R = \tau^R\left(\frac{\mu_d + \delta_1^R - p_1}{\sigma_d^2} + \frac{1}{2}\right). \tag{39}$$

- *The aggregate retail investor's time-t wealth aggregates individual retail investors' wealth*

$$A_t^R = \sum_{j=1}^{\bar{N}} A_t^j, \alpha_t^R = \sum_{j=1}^{\bar{N}} \alpha_t^j,$$

where A_t^R and α_t^R are his dollar wealth and wealth share, respectively. And his wealth share evolves according to

$$\alpha_{t+1}^R = \alpha_t^R\left(\left(1 - w_t^R\right)\exp\left(p_t - p_{t+1}\right) + w_t^R\right). \tag{40}$$

- *The time-t equilibrium price of the risky asset is determined by the market clearing condition*

$$\alpha_t^R w_t^R + \alpha_t^{IL} w_t^{IL} + \alpha_t^{IS} w_t^{IS} = 1. \tag{41}$$

Proof. See Appendix A1.4. □

This existence result comes from Assumption 1 and the linearity of investors' demand. From equations (28) and (29), an individual investor's demand is linear in his own sentiment. After retail investors split their wealth equally, their aggregate demand will be linear in the aggregate retail sentiment δ_t^R.

Lemma 1 allows me to study the pricing of aggregate retail sentiment risk in Section 4.3

and Section 4.4, for a given distribution of sentiment risk.

The aggregate retail sentiment δ_t^R depends on the fraction of type-1 investors in the retail investor population ($\theta(N_t)$), and also the average sentiment among the type-1 investors (y_t^R). In Section 5, I will show that the average sentiment y_t^R depends on the network geometry, in particular, the skewness of influence distribution on the network.

4.3 Equilibrium at time 1

At time 1, the aggregate retail sentiment δ_1^R drives the price of the risky asset. And the time-1 equilibrium price $p_1\left(\delta_1^R\right)$ is a function of retail sentiment. I assume that the time-0 conditional distribution of δ_1^R is truncated normal on the interval $\left[\underline{\delta}_1, \bar{\delta}_1\right]$, with CDF $\Psi(\cdot)$.

Under certain realizations of the retail sentiment, the portfolio constraints will be binding for institutions, and there will be multiple equilibria. I focus on the class of monotone equilibria defined below.

Definition 1 (Monotone equilibrium at time 1). *A monotone equilibrium at time 1 is an equilibrium where the price of the risky asset is strictly increasing in the retail sentiment realization, i.e., $p_1\left(\delta_1^R\right)$ is strictly increasing in δ_1^R.*

To characterize the time-1 equilibrium, I first derive two cutoff prices p_1^m and p_1^h such that: if $p_1 < p_1^m$, then none of the investors are constrained; if $p_1 \in \left[p_1^m, p_1^h\right)$, then the long institution is constrained by the short sale constraint, while the short institution is unconstrained: if $p_1 \geq p_1^h$, then both the long institution and the short institution are constrained. Since p_1^m is the cutoff price at which the short-sale constraint exactly binds for the long institution, we can calculate p_1^m by setting IL's unconstrained demand to 0, which yields

$$p_1^m \equiv \mu_d + \frac{1}{2}\sigma_d^2. \tag{42}$$

Similarly, p_1^h is the cutoff price at which the margin constraint exactly binds for the short institution, which yields

$$p_1^h \equiv \mu_d + \left(\frac{1}{2} + \frac{1}{m\tau^I}\right)\sigma_d^2. \tag{43}$$

Importantly, $p_1^m < p_1^h$, which means in the type of monotone equilibrium of Definition 1, these corresponds to two cutoff sentiment shocks $\delta_1^m = (p_1)^{-1}(p_1^m)$ and $\delta_1^h = (p_1)^{-1}\left(p_1^h\right)$[18],

[18] $(p_1)^{-1}(\cdot)$ denotes the inverse function of $p_1(\cdot)$.

with $\delta_1^m < \delta_1^h$. Impose market clearing condition (27) to derive these cutoffs

$$\delta_1^m \equiv \frac{\sigma_d^2}{\alpha_1^R(p_1^m)\tau^R}, \tag{44}$$

$$\delta_1^h \equiv \frac{\frac{1}{m\tau^I}\hat{\tau}_1\left(p_1^h\right)+1}{\alpha_1^R\left(p_1^h\right)\tau^R}\sigma_d^2, \tag{45}$$

where $\hat{\tau}_1\left(p_1^h\right) \equiv \alpha_1^R\left(p_1^h\right)\tau^R + \alpha_1^{IS}\left(p_1^h\right)\tau^I$.

For low retail sentiment shock realization, $\delta_1^R < \delta_1^m$, none of the investors are constrained. For intermediate shock realization $\delta_1^R \in \left[\delta_1^m, \delta_1^h\right)$, the long institution is constrained while the short institution is unconstrained. And for $\delta_1^R > \delta_1^h$, both the long institution and the short institution are constrained. If $\underline{\delta}_1 < \delta_1^m$ and $\delta_1^h < \bar{\delta}_1$, then as sentiment increases from $\underline{\delta}_1$ to $\bar{\delta}_1$, the long institution first hits the short-sale constraint, and then the short institution hits the margin constraint. Table 1 below summarizes the features of each sentiment region.

Table 1
Sentiment Regions and Binding Constraints

Sentiment	Shock	Constrained		
region	realization	Rep. Retail	Long Inst.	Short Inst.
Low	$\delta_1^R \in [\underline{\delta}_1, \delta_1^m)$	No	No	No
Medium	$\delta_1^R \in \left[\delta_1^m, \delta_1^h\right)$	No	Yes	No
High	$\delta_1^R \in \left[\delta_1^h, \bar{\delta}_1\right]$	No	Yes	Yes

For the rest of the paper, I focus on equilibria where the three sentiment regions are non-empty, i.e., $\underline{\delta}_1 < \delta_1^m < \delta_1^h < \bar{\delta}_1$.

Proposition 1 (Time-1 price). *Suppose a monotone equilibrium of Definition 1 exists at time 1, and $\underline{\delta}_1 < \delta_1^m < \delta_1^h < \bar{\delta}_1$. Take time-0 portfolios $\{w_0^i\}$ and wealth shares $\{\alpha_0^i\}$ as given, the time-1 equilibrium price function $p_1\left(\delta_1^R\right)$ is determined as follows.*

- *For $\delta_1^R \in [\underline{\delta}_1, \delta_1^m)$, the equilibrium features a price $p_1 < p_1^m$ that solves*

$$J\left(p_1, \delta_1^R\right) \equiv \mu_d + \left(\frac{1}{2}\sigma_d^2 + \frac{\alpha_1^R(p_1)\tau^R\delta_1^R - \sigma_d^2}{\tau_1(p_1)}\right) - p_1 = 0, \tag{46}$$

where $\tau_1(p_1)$ is the aggregate risk tolerance of unconstrained investors, which is defined as

$$\tau_1(p_1) \equiv \alpha_1^R(p_1)\tau^R + \left(1-\alpha_1^R(p_1)\right)\tau^I. \tag{47}$$

- *For* $\delta_1^R \in \left[\delta_1^m, \delta_1^h\right)$, *the equilibrium features a price* $p_1 \in \left[p_1^m, p_1^h\right)$ *that solves*

$$H\left(p_1, \delta_1^R\right) \equiv \mu_d + \left(\frac{1}{2}\sigma_d^2 + \frac{\alpha_1^R\left(p_1\right)\tau^R\delta_1^R - \sigma_d^2}{\hat{\tau}_1\left(p_1\right)}\right) - p_1 = 0, \tag{48}$$

where $\hat{\tau}_1\left(p_1\right)$ *is the aggregate risk tolerance of unconstrained investors, which is defined as*

$$\hat{\tau}_1\left(p_1\right) \equiv \alpha_1^R\left(p_1\right)\tau^R + \alpha_1^{IS}\left(p_1\right)\tau^I. \tag{49}$$

- *For* $\delta_1^R \in \left[\delta_1^h, \bar{\delta}_1\right]$, *the equilibrium features a price* $p_1 > p_1^h$ *that solves*

$$G\left(p_1, \delta_1^R\right) \equiv \mu_d + \delta_1^R + \left(\frac{1}{2} - \frac{1 + \alpha_1^{IS}\left(p_1\right)\frac{1}{m}}{\alpha_1^R\left(p_1\right)\tau^R}\right)\sigma_d^2 - p_1 = 0. \tag{50}$$

The cutoff prices p_1^m *and* p_1^h *are defined in equations* (42) *and* (43), *and the cutoff sentiment shocks* δ_1^m *and* δ_1^h *are defined in equations* (44) *and* (45).

Proof. See Appendix A1.5. □

Proposition 1 shows that in each of the sentiment region, the equilibrium price solves an implicit function. This is because the equilibrium price not only enters investors' demand, but also determines their wealth shares. These implicit functions may have multiple solutions, which means there could be multiple equilibria. As retail sentiment realization δ_1^R increases, certain class of equilibria may disappear, this gives rise to endogenous discontinuity in equilibrium price. Proposition 2 below presents the formal argument.

Proposition 2 (Endogenous discontinuity in time-1 price). *Consider an equilibrium with the following properties:*

- *Investors' time-0 optimal portfolios satisfy:* $w_0^R > 1$, $w_0^{IS} < 0 < w_0^{IL} < w_0^R$.
- *For any sentiment shock realization* $\delta_1^R \in \left(\underline{\delta}_1, \bar{\delta}_1\right)$, *the equilibrium price* $p_1\left(\delta_1^R\right)$ *is such that all investors have strictly positive wealth at time 1.*
- *The time-1 equilibrium is a monotone equilibrium of Definition 1.*

If $p_1\left(\delta_1^R\right)$ *is continuous on* $\left[\underline{\delta}_1, \delta_1^h\right)$ *and* $\left.\frac{\partial G\left(p_1, \delta_1^h\right)}{\partial p_1}\right|_{p_1 = p_1^h} > 0$, *then* $p_1\left(\delta_1^R\right)$ *jumps discontinuously at* $\delta_1^R = \delta_1^h$, *i.e.,*

$$\lim_{\delta_1^R \to \left(\delta_1^h\right)^-} p_1\left(\delta_1^R\right) < \lim_{\delta_1^R \to \left(\delta_1^h\right)^+} p_1\left(\delta_1^R\right).$$

Proof. See Appendix A1.7. □

To understand the endogenous jump, I provide a numerical example and Section 5.3 explains the parameter choices. Figure 20 plots the time-1 equilibrium price $p_1\left(\delta_1^R\right)$ as a function of the sentiment shock δ_1^R. There is an endogenous jump at the cutoff δ_1^h, at which the margin constraint exactly binds for the short institution. Figure 23 plots all the time-1 equilibria in this numerical example. Generically, for a given sentiment shock realization δ_1^R, there are one or three equilibria. And in the knife edge cases, there are two equilibria. In particular, there are two equilibrium prices at $\delta_1^R = \delta_1^h$, with p_1^h being the lower price. As sentiment increases further above δ_1^h, the low-price equilibrium disappears and the high-price equilibrium becomes the unique equilibrium, and this gives rise to the endogenous jump. Moreover, under this set of parameter values, we cannot find a price path $p_1\left(\delta_1^R\right)$ that is continuous in the sentiment shock δ_1^R. Hence, we can pick any other class of equilibrium (i.e., not necessarily the low-price equilibrium), and there will still be a price jump at certain sentiment shock realization.

Hence, the endogenous jump in price is a result of multiple equilibria. Next, I show that the margin constraint and wealth effect generate multiple equilibria. I first analyze demand and supply around the cutoff sentiment δ_1^h, from the short institution's perspective. The demand curve of the short institution can be written as

$$\frac{Q_1}{\bar{S}} = \begin{cases} \alpha_1^{IS}\left(p_1\right)\tau^I\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right), & p_1 \in \left[p_1^m, p_1^h\right] \\ -\frac{1}{m}\alpha_1^{IS}\left(p_1\right), & p_1 > p_1^h \end{cases}. \tag{51}$$

Around the cutoff δ_1^h, long institution demands zero shares due to the binding short-sale constraint (recall from Table 1). Hence, the "residual supply curve" faced by short institution is 1 minus the demand of the retail investor, i.e.,

$$\frac{Q_1}{\bar{S}} = 1 - \alpha_1^R\left(p_1\right)\tau^R\left(\frac{\mu_d + \delta_1^R - p_1}{\sigma_d^2} + \frac{1}{2}\right). \tag{52}$$

Figure 24 panel (a) plots the inverse demand curve (solid black line), and the inverse supply curves (blue lines) under different sentiment shock realizations. The demand curve is downward sloping for $p_1 \leq p_1^h$, but is upward sloping for $p_1 < p_1^h$. For a price higher than p_1^h, the margin constraint binds for the short institution and he can only allocate a constant fraction $-\frac{1}{m}$ of his wealth to the risky asset. As price increases, he loses wealth on the short position. This wealth effect together with the margin constraint limits the number of shares he can short, and make the demand curve upward sloping. The supply curves are upward sloping for $p_1 > p_1^h$, but is downward sloping for $p_1 < p_1^h$ due to the wealth effect. In this numerical example, the retail investor has a levered position in the risky asset. As price decreases below p_1^h, he loses wealth and demands less shares. This effectively "increases" the number of shares supplied to the short institution.

The yellow dots represent the three equilibria under a sentiment shock that is slightly below δ_1^h. As sentiment increases to δ_1^h, the lower and middle equilibria collapse into one, so there are two equilibria represented by the two green dots. As sentiment increases further above δ_1^h, the low-price equilibrium disappears, and price jumps discontinuously to the red dot (high-price equilibrium).

Intuitively, when sentiment increases further above δ_1^h, an unconstrained short seller would increase his short position and there will still be a low-price equilibrium. With the margin constraint, short seller would short less than in the unconstrained case, and the low-price equilibrium no longer clears the market and price has to rise further. As price rises further, the short seller loses wealth and has to short even less, this again drives up price. This feedback loop implies that market only clears at a very high price, which is the high-price equilibrium.

This phenomenon has a tight connection to Gennotte and Leland (1990), who analyze an endogenous price drop due to multiple equilibria. To see this, I define the short institution's "excess demand" as his demand minus "supply" from the retail investor, i.e.,

$$\begin{aligned}&\frac{Q_1^{IS}}{\bar{S}} + \frac{Q_1^R}{\bar{S}} \\ &= \begin{cases} \left(\alpha_1^{IS}\left(p_1\right)\tau^I + \alpha_1^R\left(p_1\right)\tau^R\right)\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) + \alpha_1^R\left(p_1\right)\tau^R\frac{\delta_1^R}{\sigma_d^2}, & p_1 \in \left[p_1^m, p_1^h\right] \\ -\frac{1}{m}\alpha_1^{IS}\left(p_1\right) + \alpha_1^R\left(p_1\right)\tau^R\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) + \alpha_1^R\left(p_1\right)\tau^R\frac{\delta_1^R}{\sigma_d^2}, & p_1 > p_1^h \end{cases}\end{aligned} \tag{53}$$

Then market clearing implies that the "excess supply" is equal to 1. Figure 24 panel (b) plots the "excess demand" and "excess supply," which is a mirror image of the scenario in Gennotte and Leland (1990).

Proposition 2 shows that price can jump discontinuously at certain sentiment cutoff. And the jump is one reason why moderate sentiment shock can have large price impact. Proposition 3 then characterizes the price impact within each sentiment region.

Proposition 3 (Price impact of time-1 aggregate retail sentiment shock). *Consider an equilibrium where $p_1\left(\delta_1^R\right)$ is continuous and differentiable in the interior of the three sentiment regions. The price impact of aggregate retail sentiment shock, $\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R}$, can be decomposed into two components – the direct effect and the redistribution effect.*

- *Low sentiment region* $\delta_1 \in [\underline{\delta}_1, \delta_1^m)$:

$$\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R} = \underbrace{\frac{\alpha_1^R\left(p_1\right)\tau^R}{\tau_1\left(p_1\right)}}_{\text{direct effect}} \cdot \underbrace{\frac{1}{1-\frac{1}{\tau_1(p_1)}\left(\frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\delta_1^R+\frac{d\tau_1(p_1)}{dp_1}\left(\mu_d+\frac{1}{2}\sigma_d^2-p_1\right)\right)}}_{\text{redistribution effect}}.$$

- *Medium sentiment region* $\delta_1 \in \left(\delta_1^h, \delta_1^h\right)$:

$$\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R} = \underbrace{\frac{\alpha_1^R\left(p_1\right)\tau^R}{\hat{\tau}_1\left(p_1\right)}}_{\text{direct effect}} \cdot \underbrace{\frac{1}{1-\frac{1}{\hat{\tau}_1(p_1)}\left(\frac{d\left(\alpha_1^R(p_1)\right)}{dp_1}\tau^R\delta_1^R+\frac{d\hat{\tau}_1(p_1)}{dp_1}\left(\mu_d+\frac{1}{2}\sigma_d^2-p_1\right)\right)}}_{\text{redistribution effect}}.$$

- *High sentiment region* $\delta_1 \in \left(\delta_1^h, \bar{\delta}_1\right]$:

$$\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R} = \underbrace{1}_{\text{direct effect}} \cdot \underbrace{\frac{1}{1-\frac{1}{\alpha_1^R(p_1)\tau^R}\left(\frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\delta_1^R+\frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\left(\mu_d+\frac{1}{2}\sigma_d^2-p_1\right)-\frac{d\alpha_1^{IS}(p_1)}{dp_1}\frac{1}{m}\sigma_d^2\right)}}_{\text{redistribution effect}}.$$

Proof. See Appendix A1.8. □

Within each sentiment region, the price impact can be decomposed into the direct effect and the redistribution effect. The direct effect says that the effect of retail sentiment shock depends on the contribution of retail investor's risk tolerance to aggregate risk tolerance. In the high sentiment region, both institutions are constrained, and thus retail investor's risk aversion is the aggregate risk aversion, and thus the direct effect is large. The redistribution effect captures the wealth redistribution triggered by the retail sentiment shock.

4.4 Equilibrium at time 0

Proposition 4 characterizes the time-0 equilibrium.

Proposition 4 (Equilibrium at time 0). *Consider an equilibrium where the short-sale constraint for the long institution and the margin constraint for the short institution are not binding at time 0 (under the equilibrium price p_0), and the time-1 equilibrium is a monotone equilibrium of Definition 1. Then the time-0 price is determined as follows.*

1. *Investors' time-*0 *beliefs about time-*1 *price distribution is consistent with the time-*1 *pricing function* $p_1\left(\delta_1^R\right)$ *and the shock distribution* $\Psi\left(\delta_1^R\right)$*, i.e.,*

$$\mathbb{E}_0^i\left[p_1\left(\delta_1^R\right)\right] = \mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right] + \delta_0^i = \int_{\underline{\delta}_1}^{\bar{\delta}_1} p_1\left(\delta_1^R\right) d\Psi\left(\delta_1^R\right) + \delta_0^i,$$

$$\mathrm{Var}_0^i\left(p_1\left(\delta_1^R\right)\right) = \sigma_0^2 = \int_{\underline{\delta}_1}^{\bar{\delta}_1} \left(p_1\left(\delta_1\right) - \mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right]\right)^2 d\Psi\left(\delta_1^R\right).$$

2. *Given the time-*1 *pricing function* $p_1\left(\delta_1^R\right)$*, time-*0 *equilibrium price* p_0 *clears the market at* $t = 0$*:*

$$p_0 \quad = \quad \mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right] + \left(\frac{1}{2}\sigma_0^2 + \frac{\sum_i \alpha_0^i\left(p_0\right)\tau^i\delta_0^i - \sigma_0^2}{\tau_0\left(p_0\right)}\right)$$

where $\tau_0\left(p_0\right)$ *is the aggregate risk tolerance at time* 0*, defined as*

$$\tau_0\left(p_0\right) \quad \equiv \quad \alpha_0^R\left(p_0\right)\tau^R + \left(1 - \alpha_0^R\left(p_0\right)\right)\tau^I$$

Hence, the equilibrium is a fixed problem. The time-0 price depends on the shape of the time-1 pricing function $p_1\left(\delta_1^R\right)$ through investors' beliefs, while the time-1 pricing function depends on p_0 through the wealth shares.

5 The network origins of aggregate retail sentiment fluctuations

Section 4 shows how investor composition matters for the pricing of retail sentiment risk. In this section, I microfound the sentiment risk distribution. I assume that type-1 retail investors communicate on a social network and update their beliefs by "listening to" other investors on the network. The influence distribution on the network is highly skewed, which means influencers' views will carry a disproportionately high weight in the aggregate view of retail investors. Then idiosyncratic sentiment shocks to retail investors would not cancel out, and would instead translate into an aggregate retail sentiment shock.

This microfoundation allows me to study two counterfactual scenarios in Section 6. These two counterfactuals shed light on why short sellers got squeeze in January 2021, and why they exited the market thereafter.

5.1 Naive learning on a growing random network

At time $t = 1$, type-1 retail investor j draws a noisy signal

$$x_t^j = \rho y_{t-1}^R + \varepsilon_t^j,$$

where y_{t-1}^R is the average retail sentiment at time $t-1$, ε_t^j is an error term that is i.i.d. across investors and time. I assume that ε_t^j follows a truncated normal distribution on $[-\bar{\varepsilon}, \bar{\varepsilon}]$, with post-truncation mean 0 and variance σ_ε^2. $\rho \in (0, 1]$ is a parameter that captures the persistence of sentiment.

Type-1 retail investors communicate on a social network, and reveal their signals to others. Then each investor on the network updates his belief by "listening to" other people on the network. I use the adjacency matrix $\boldsymbol{A}_t = (a_{jk,t})$ to capture the relationship between pairs of investors. If investor j "listens to" (or "attends to") investor k at time t, then $a_{jk,t} = 1$, otherwise $a_{jk,t} = 0$. Investor j assigns weight $\omega_{jk,t}$ to investor k's signal, and $\omega_{jk,t}$ is defined as

$$\omega_{jk,t} \equiv \frac{a_{jk,t}}{\sum_{k=1}^{N_t} a_{jk,t}}.$$

Hence, each investor on the network assigns equal weights to people he listens to. Also note that $\sum_{k=1}^{N_t} \omega_{jk,t} = 1$.

After the updating, investor j's view becomes

$$y_t^j = \sum_{k=1}^{N_t} \omega_{jk,t} x_t^k = \sum_{k=1}^{N_t} \omega_{jk,t} \left(\rho y_{t-1}^R + \varepsilon_t^k \right) = \rho y_{t-1}^R + \sum_{j=1}^{N_t} \omega_{jk,t} \varepsilon_t^k.$$

y_t^j is the sentiment of investor j in equation (18).

Dynamics of aggregate retail sentiment Using the definition in equation (36), time-t aggregate retail sentiment is

$$\delta_t^R \equiv \theta(N_t) \frac{1}{N_t} \sum_{j=1}^{N_t} y_t^j = \frac{\theta(N_t)}{\theta(N_{t-1})} \rho \delta_{t-1}^R + \theta(N_t) \frac{1}{N_t} \sum_{j=1}^{N_t} d_{j,t}^{in} \varepsilon_t^j, \tag{54}$$

where $d_{j,t}^{in}$ is the time-t "influence" (or in-degree) of retail investor j, defined as

$$d_{j,t}^{in} \equiv \sum_{i=1}^{N_t} \omega_{ij,t}.$$

This is the same definition of influence as in equation (3). δ_t^R has support $\left[\underline{\delta}_t, \bar{\delta}_t\right]$, where

$$\begin{aligned} \underline{\delta}_t &= \frac{\theta\left(N_t\right)}{\theta\left(N_{t-1}\right)} \rho \delta_{t-1}^R - \theta\left(N_t\right) \bar{\varepsilon}, \\ \bar{\delta}_t &= \frac{\theta\left(N_t\right)}{\theta\left(N_{t-1}\right)} \rho \delta_{t-1}^R + \theta\left(N_t\right) \bar{\varepsilon}. \end{aligned}$$

Motivated by the findings in Section 3.4, I assume that $d_{j,t}^{in}$ is drawn from a power-law distribution, and is i.i.d. in the cross section of the N_t retail investors on the social network. The PDF of $d_{j,t}^{in}$ is

$$f_{d_{j,t}^{in}}(x) = \frac{\xi - 1}{d_{\min}} \left(\frac{x}{d_{\min}}\right)^{-\xi}, \xi > 1, \tag{55}$$

with support $[d_{\min}, d_{\max}(N_t)]$. The exponent ξ captures the skewness of the influence distribution. Lower values of ξ correspond to heavier tails and more right-skewed influence distribution. The upper bound $d_{\max}(N_t) = d_{\min} \cdot N_t^{\frac{1}{\xi-1}}$.[19] Lemma 2 computes the moments of the influence distribution.

Lemma 2 (Moments of the influence distribution). *In the cross section of N_t retail investors (type-1), the m-th moment of influence $d_{j,t}^{in}$ is*

$$\mathbb{E}^{CS}\left[\left(d_{j,t}^{in}\right)^m\right] = \frac{\xi - 1}{\xi - m - 1} \frac{1}{d_{\min}^{1-\xi}} \left(d_{\min}^{m+1-\xi} - \left(d_{\max}\left(N_t\right)\right)^{m+1-\xi}\right).$$

The cross-sectional variance of $d_{j,t}^{in}$ is

$$\begin{aligned} \mathrm{Var}^{CS}\left(d_{j,t}^{in}\right) = & \frac{\xi - 1}{3 - \xi} \frac{1}{d_{\min}^{1-\xi}} \left(\left(d_{\max}\left(N_t\right)\right)^{3-\xi} - d_{\min}^{3-\xi}\right) \\ & - \left(\frac{\xi - 1}{\xi - 2}\right)^2 \frac{1}{d_{\min}^{2-2\xi}} \left(d_{\min}^{2-\xi} - \left(d_{\max}\left(N_t\right)\right)^{2-\xi}\right)^2. \end{aligned} \tag{56}$$

[19] Following Newman (2005) and Acemoglu et al. (2012), the upper bound can be computed in a heuristic way.

$$\Pr\left(d_j^{in} > x\right) = \int_x^{+\infty} \frac{\xi - 1}{d_{\min}} \left(\frac{y}{d_{\min}}\right)^{-\xi} dy = -\int_x^{+\infty} d\left(\frac{y}{d_{\min}}\right)^{1-\xi} = \left(\frac{x}{d_{\min}}\right)^{1-\xi}$$

$d_{\max}(N)$ is computed from

$$\Pr\left(d_j^{in} > d_{\max}(N)\right) = \frac{1}{N} \implies d_{\max}(N) = d_{\min} \cdot N^{\frac{1}{\xi-1}}$$

And $\mathrm{Var}^{CS}\left(d_{j,t}^{in}\right) = O\left(N_t^{\frac{3-\xi}{\xi-1}}\right)$ *for* $\xi > 1$*,*

Proof. See Appendix A1.11. $\square$

5.2 Aggregate fluctuations in retail sentiment

Proposition 5 below relates the volatility of the aggregate sentiment shock to the volatility of idiosyncratic shock σ_ε and the network parameters. This is a direct application of Acemoglu et al. (2012) Theorem 2 and Corollary 1.

Proposition 5 (Moments of aggregate retail sentiment). *Suppose the network size* N_t *evolves deterministically over time. Then at time* $t-1$*, the conditional mean and variance of aggregate retail sentiment* δ_t^R *are*

$$\mathbb{E}_{t-1}\left[\delta_t^R\right] = \frac{\theta\left(N_t\right)}{\theta\left(N_{t-1}\right)}\rho\delta_{t-1}^R, \tag{57}$$

$$\mathrm{Var}_{t-1}\left(\delta_t^R\right) = \left(\theta\left(N_t\right)\right)^2 \frac{2d_{\min}^{\xi-1}}{N_t}\frac{1}{3-\xi}\left(\left(d_{\max}\left(N_t\right)\right)^{3-\xi} - d_{\min}^{3-\xi}\right)\sigma_\varepsilon^2. \tag{58}$$

And the conditional volatility satisfies

$$\sqrt{\mathrm{Var}_{t-1}\left(\delta_t^R\right)} = O\left(N_t^{\frac{2-\xi}{\xi-1}}\right).$$

Proof. See Appendix A1.12. $\square$

Proposition 5 shows that the volatility of aggregate retail sentiment shock decreases with ξ. Intuitively, a smaller ξ corresponds to a more skewed influence distribution. Then idiosyncratic shocks to influencers will carry a higher weight in the aggregate sentiment, which leads to more aggregate fluctuations.

$\xi = 3$ corresponds to the standard Central Limit Theorem, which says that the aggregate volatility decreases at a rate of $\sqrt{N_t}$. Section 3.4.1 shows that for the Reddit WSB social network, $\xi < 3$. Hence, volatility decreases at a much lower rate. Even with a large number of users on the network, idiosyncratic sentiment shocks may still lead to large aggregate sentiment fluctuations. The 15% increase in average sentiment in January 2021 is thus a result of influencers' idiosyncratic sentiment shocks and a small $\hat{\xi}_t$.

5.3 Numerical example

Table 2
Model Parameters

Description	Parameter	Value
Risky Asset		
Mean of log dividend	μ_d	4
Volatility of log dividend	σ_d^2	0.1
Lower bound of log dividend	$\underline{d}$	-2.5
Upper bound of log dividend	$\bar{d}$	10.5
Supply of shares	$\bar{S}$	100
Endowment		
Retail investors	α_{-1}^{R}	0.3
	w_{-1}^{R}	1.194
Long institution	α_{-1}^{IL}	0.14
	w_{-1}^{IL}	4.800
Short institution	α_{-1}^{IS}	0.56
	w_{-1}^{IS}	-0.054
Risk Aversion		
Retail investors	γ^R	2
Institutions	γ^I	1

Description	Parameter	Value
Constraints		
Margin constraint	m	0.5
Sentiment Shocks		
Retail investors	δ_0^R	1.028
	$\bar{\varepsilon}$	2.872
	σ_{ε}^2	0.081
Long institution	δ_0^{IL}	0.256
Short institution	δ_0^{IS}	-0.505
Network		
Population of type-1 retail investors	N_L	80000
	N_H	140000
Population of retail investors	$\bar{N}$	200000
Exponent of power-law distribution	ξ	2.1
Cutoff value of power-law distribution	$d_{\min}$	10
Persistence of agg. retail sent shock	ρ	1

I present a numerical example that matches the price and quantity patterns observed in the data. Table 2 shows the parameters.

I assume that the network size remains constant over time, with $N_0 = N_1 = N_L$. When investors form their subjective expectations, they also perceive the network size as constant. When drawing time-1 sentiment shocks, I assume that the aggregate sentiment shock δ_1^R follows a truncated normal distribution with post-truncation mean and variance given by equations (57) and (58), and support $\left[\underline{\delta}_1, \bar{\delta}_1\right]$. Appendix A1.13 shows that the true distribution of δ_1^R (by aggregating the y_1^j's) can be approximated by this truncated normal distribution, if the influence distribution is skewed.

Figure 20 panel (a) plots the time-1 price as a function of the aggregate sentiment shock realization. And Figure 20 panel (b) plots the pricing function together with the PDF of the aggregate retail sentiment shock. As shown in Section 4.3, the price impact within each sentiment region is determined by the direct effect and wealth redistribution. At the cutoff sentiment δ_1^h, there is an endogenous jump in price, due to margin constraint and wealth effect.

In this example, investors' time-0 portfolio weights are $w_0^R = 1.90$, $w_0^{IL} = 1.76$, and $w_0^{IS} = -0.25$. Both the aggregate retail investor and the long institution take a levered position in the risky asset. Hence, as retail sentiment drives up price, wealth redistributes from the short institution to retail investors and the long institution (Figure 22 panel (c)).

Figure 21 shows the time series predictions from the model. The time-1 values correspond to an aggregate sentiment shock $\delta_1^R = 2.18$. The model can match the price and quantity patterns documented in Section 3.1-3.3. In particular, panel (a) shows that short sellers increase their short positions following the first retail sentiment shock δ_0^R, while significantly reduce their short positions after the second sentiment shock.

6 Counterfactuals

I conduct two counterfactuals, which shed light on the reasons why short sellers got squeezed in January 2021, and why they stayed out of the market thereafter.

6.1 Why did short sellers get squeezed in January 2021?

In Section 3.1, I document that average retail sentiment on GME had been steadily increasing from mid-2020 to Jan 2021, while the discussion volume on GME spiked in January 2021. Both forces would contribute to a large positive realization of aggregate retail sentiment, as is shown in equation (36). This realized retail sentiment shock not only drove out price-sensitive long investors, but also got short sellers squeezed.

I formalize this idea through the lens of the model, using the parameters for the numerical example in Section 5.3. In particular, an increase in discussion volume in the data corresponds to an unexpected increase in network size in the model, i.e., an "MIT shock" to network size. Given the skewness of the influence distribution and how optimistic influencers are, the growth of the network translates into a large sentiment realization, which exceeds the short squeeze cutoff δ_1^h in equation (43), i.e., the long institution liquidates his position and the short institution gets squeezed under this sentiment shock realization. Next, I consider a counterfactual scenario where the discussion volume does not spike in January, i.e., the network size does not change in the model. In this case, the average sentiment still remains positive, but the aggregate sentiment is lower than the realized sentiment and short sellers would not get squeezed.

I begin by analyzing the factors that contribute to the large positive realization of aggregate retail sentiment: network size, network geometry (or influence distribution), and optimism of individual retail investors on the network. I assume that the network size grows from time 0 to time 1, with $N_0 = N_L < N_H = N_1$, and the values of N_L and N_H are given in Table 2. Substitute into equation (54) to get the realized aggregate retail sentiment

$$\delta_1^R = \underbrace{\frac{\theta(N_H)}{\theta(N_L)}\rho\delta_0^R}_{\text{persistence}} + \underbrace{\theta(N_H)\frac{1}{N_H}\sum_{j=1}^{N_H} d_{j,1}^{in}\varepsilon_1^j}_{\text{aggregation of idio. shocks}}. \tag{59}$$

The first component captures the persistence of aggregate retail sentiment ($\rho\delta_0^R$) and the amplification effect through a growing network ($\frac{\theta(N_H)}{\theta(N_L)}$). $\frac{\theta(N_H)}{\theta(N_L)} = \frac{N_H}{N_L} > 1$ captures the growth of the social network from time 0 to time 1. $\rho > 0$ captures the persistence of aggregate retail sentiment. Suppose $\delta_0^R > 0$, i.e., at time 0, retail investors are optimistic in aggregate. Then retail investors who newly join the network will adopt the optimistic views from existing investors, and the average optimism of existing investors will get amplified and be reflected in the aggregate retail sentiment.

The second component ($\frac{1}{N_H}\sum_{j=1}^{N_H} d_{j,1}^{in}\varepsilon_1^j$) captures the aggregation of idiosyncratic sentiment shocks to investors on the network. Since the influence distribution is skewed, idiosyncratic sentiment shocks do not average out across investors, and influencers' sentiment shocks will carry a higher weight in the aggregate sentiment, leading to fluctuations in aggregate sentiment. If influencers happen to draw a positive sentiment shock, then aggregate sentiment will also be positive. And importantly, on the intensive margin, the aggregate optimism will depend on the skewness of the influence distribution and the network size. To see this, if we have a large number of retail investors on the network, i.e., $N_H \to +\infty$, then first apply the Law of Large Numbers in the cross section of retail investors,

$$\frac{1}{N_H}\sum_{j=1}^{N_H} d_{j,1}^{in}\varepsilon_1^j \xrightarrow{p} \mathbb{E}\left[d_{j,1}^{in}\varepsilon_1^j\right] = \text{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)\sqrt{\text{Var}\left(d_{j,1}^{in}; N_H\right)}\sigma_\varepsilon. \tag{60}$$

$\text{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)$ is the cross-sectional correlation between users' influence and the idiosyncratic shocks they draw. If $\text{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right) > 0$, then it means that influencers are optimistic. And how much influencers' views carry in the aggregate view depends on the cross-sectional dispersion in user influence, which is captured by $\sqrt{\text{Var}\left(d_{j,1}^{in}; N_H\right)}$. In Section 3.4.1, I estimated that the power-law exponent $\hat{\xi}_t \in (1,3)$. Then it immediately follows from Lemma 2 that, as the network grows, the influence distribution is more dispersed in the cross section of retail investors, and influencers' views will get amplified more and carry a higher weight.

Next, I consider a counterfactual scenario where the network size remains constant from time 0 to time 1, i.e., $N_0 = N_1 = N_L$. Using (60) to approximate the aggregation of idiosyncratic sentiment shocks, the realized aggregate sentiment in (59) can be approximated by

$$\delta_1^R \approx \frac{\theta(N_H)}{\theta(N_L)}\rho\delta_0^R + \theta(N_H)\,\mathrm{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)\sqrt{\mathrm{Var}\left(d_{j,1}^{in}; N_H, \xi\right)}\sigma_\varepsilon. \tag{61}$$

The counterfactual aggregate retail sentiment is

$$\hat{\delta}_1^R \approx \rho\delta_0^R + \theta(N_L)\,\mathrm{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)\sqrt{\mathrm{Var}\left(d_{j,1}^{in}; N_L, \xi\right)}\sigma_\varepsilon \tag{62}$$

In this counterfactual scenario, influencers remain as optimistic as they are in the realized scenario, i.e., $\mathrm{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)$ remains the same. But due to a smaller network size, the counterfactual aggregate retail sentiment shock is smaller, i.e., $\hat{\delta}_1^R < \delta_1^R$.

The model in Section 4 allows me to quantify the pricing impact of the counterfactual sentiment shock. From the pricing function $P_1\left(\delta_1^R\right)$ in Figure 20 and the price of GameStop observed from the data ($P_1 = 349.73$ in January 2021), I can back out the realized aggregate retail sentiment $\delta_1^R = 2.18$. Given the network parameters $\left(N_L, N_H, \bar{N}, d_{\min}, \xi\right)$ in Table 2, and using equation (61), I then back out how optimistic influencers are, which is $\mathrm{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right) = 0.00135$. Now fix the optimism of influencers, I can calculate the counterfactual retail sentiment from equation (62), which yields $\hat{\delta}_1^R = 1.20$. Finally, using the pricing function $P_1\left(\delta_1^R\right)$ in Figure 20, the counterfactual price is thus $P_1\left(\hat{\delta}_1^R\right) = 65.63$.

Figure 25 panel (a) plots the equilibrium price under the realized sentiment shock $\delta_1^R = 2.18$ versus that under the counterfactual sentiment shock $\hat{\delta}_1^R = 1.20$. In the latter case, short sellers do not get squeezed, since the counterfactual sentiment shock is smaller than the short squeeze cutoff δ_1^h.

As is discussed above, we can decompose the gap between the realized sentiment and counterfactual sentiment into two parts: one captures the persistence of aggregate retail sentiment and the amplification through a growing network, while the other captures the aggregation of idiosyncratic shocks on a network with skewed influence distribution. Formally, compare equation (61) with equation (62), and compute the difference

$$\begin{aligned}\delta_1^R - \hat{\delta}_1^R = &\underbrace{\left(\frac{\theta(N_H)}{\theta(N_L)} - 1\right)\rho\delta_0^R}_{\Delta_1} \\ &+ \underbrace{\mathrm{Corr}\left(d_{j,1}^{in}, \varepsilon_1^j\right)\left(\theta(N_H)\sqrt{\mathrm{Var}\left(d_{j,1}^{in}; N_H, \xi\right)} - \theta(N_L)\sqrt{\mathrm{Var}\left(d_{j,1}^{in}; N_L, \xi\right)}\right)\sigma_\varepsilon}_{\Delta_2}\end{aligned} \tag{63}$$

$\Delta_1 = 0.771$ is the component due to persistence of aggregate retail sentiment, and $\Delta_2 = 0.206$ is the component due to aggregation of idiosyncratic sentiment shocks. Figure 25 panel (b) plots the two components. The second component alone would be sufficient to squeeze short sellers, which suggests that the skewed influence distribution on social network has an economically large impact on asset price.

6.2 Why did short sellers exit the market after January 2021?

In the model, there are three mechanisms that can help explain why short sellers stayed out of the market and price remained high after January 2021: (1) Short sellers updated their perceptions about retail sentiment risk post the GameStop frenzy; (2) The market for GameStop became price inelastic due to financial constraints and wealth redistribution; (3) Short sellers lost wealth and were forced to exit the market.

Change in short sellers' risk perceptions After observing a large influx of retail investors to WSB in January 2021, short sellers may have updated their perceptions about the sentiment risk distribution, and thus would trade less aggressively. This can explain why price stayed high and short interest stayed low after January 2021.

In the model, the short institution's risk perception depends on his perception about the growth of the social network. In the numerical example of Section 5.3, short sellers believe that the network size remains constant from time 0 to time 1. Let $\tilde{N}_1$ denote short sellers' time-0 perception about the network size at time 1, then $\tilde{N}_1 = N_L$. Their perception of the risk distribution corresponding to $\tilde{N}_1$ is plotted in the solid blue line of Figure 26.

Now consider a counterfactual scenario where short sellers perfectly anticipate the growth of the network from time 0 to time 1, i.e., their perception about time-1 network size is $\tilde{N}_1 = N_H$. This corresponds to a different perception of the risk distribution, which is plotted in the dashed red line of Figure 26. I solve the time-0 equilibrium under this counterfactual risk perception. Table 3 compares the time-0 price under these two different risk perceptions. The time-0 price under risk perception $\tilde{N}_1 = N_H$ (column 4) is higher than that under $\tilde{N}_1 = N_L$ (column 3). This is primarily because the expected payoff of the risky asset is higher under the new risk perception. Table A2 in the Internet Appendix compares other equilibrium outcomes at time 0.

Table 3
Time-0 Equilibrium Price under Different Risk Perceptions

This table compares the time-0 equilibrium prices when changing investors' time-0 perceptions of risk. Column 3 shows the equilibrium outcomes when all investors believe that the size of the network at time 1 will remain the same as that at time 0, i.e., $\tilde{N}_1 = N_L = N_0$. Column 4 shows the equilibrium outcomes when all investors believe that the the size of the network will grow (deterministically) from time 0 to time 1, i.e., $\tilde{N}_1 = N_H > N_L = N_0$. The time-0 equilibrium price in each scenario is decomposed into three components: the expected log payoff after risk adjustment, the price of time-0 realized retail sentiment, and the price of time-1 retail sentiment risk. The parameter values are given in Table 2.

		Value	
Description	Notation	$\tilde{N}_1 = N_L$	$\tilde{N}_1 = N_H$
(1)	(2)	(3)	(4)
Expected log payoff (risk-adjusted)	$\mathbb{E}_0[p_1] + \frac{1}{2}\sigma_0^2$	4.658	5.664
Price of time-0 realized retail sentiment	$\frac{\sum_i \alpha_0^i(p_0)\tau^i\delta_0^i}{\tau_0(p_0)}$	0.039	0.163
Price of time-1 risk	$-\frac{1}{\tau_0(p_0)}\sigma_0^2$	-0.449	-1.215
Sum	p_0	4.249	4.612

Change in aggregate demand elasticity in the market for GameStop After the January 2021 short squeeze episode, the market for GameStop may have become price-inelastic for two reasons. First, price-elastic institutions hit their constraints and effectively became price-inelastic. Second, retail investors' wealth share increases and they are less price-elastic than (unconstrained) institutions. Since aggregate price elasticity is a weighted average of individual elasticity. This implies that the market for GameStop may have become price-inelastic after January 2021, and a moderate retail sentiment shock can have a large price impact.

Capital loss Short sellers like Melvin Capital lost a large fraction of wealth and exited the market.

7 Conclusion

This paper demonstrates how social media has fundamentally changed the nature of retail trading. The growth and concentration of social network can lead to extreme realizations of retail sentiment and amplify the fluctuations in asset prices. Moreover, after a "disaster" realization, short sellers may update their perceptions of retail sentiment risk and be more conservative in taking large short positions. Social-media-fueled retail trading becomes a new risk to institutional investors, and social network dynamics shape the risk distribution.

This paper also argues that retail trading can induce a shift in investor composition, which determines the price of this new risk. In particular, positive retail sentiment can drive out price-sensitive long-only institutions, causing a decline in aggregate demand elasticity in the market for an individual stock. Then a moderate retail sentiment shock can drive up the price and put short sellers at risk. From short sellers' perspective, price-sensitive long-only institutions act like a "buffer" against retail sentiment fluctuations. However, over the past two decades, this "buffer" has been shrinking due to the rise of passive investing. This implies that short sellers are now more "vulnerable" to retail sentiment risk. Hence, this change in investor composition is also a new risk for short sellers to watch out for.

References

Acemoglu, D., Carvalho, V. M., Ozdaglar, A., and Tahbaz-Salehi, A. (2012). The network origins of aggregate fluctuations. *Econometrica*, 80(5):1977–2016.

Allen, F., Haas, M., Nowak, E., Pirovano, M., and Tengulov, A. (2022). Squeezing shorts through social media platforms. *SSRN Electronic Journal*.

Bailey, M., Cao, R., Kuchler, T., and Stroebel, J. (2018a). The economic effects of social networks: Evidence from the housing market. *Journal of Political Economy*, 126(6):2224–2276.

Bailey, M., Cao, R., Kuchler, T., Stroebel, J., and Wong, A. (2018b). Social connectedness: Measurement, determinants, and effects. *Journal of Economic Perspectives*, 32(3):259–80.

Barber, B. M., Huang, X., Jorion, P., Odean, T., and Schwarz, C. (2022). A (sub) penny for your thoughts: Tracking retail investor activity in taq. *Available at SSRN 4202874*.

Barber, B. M., Huang, X., Odean, T., and Schwarz, C. (2021). Attention induced trading and returns: Evidence from robinhood users. *Journal of Finance, forthcoming*.

Boehmer, E., Jones, C. M., Zhang, X., and Zhang, X. (2021). Tracking retail investor activity. *Journal of Finance*, 76(5):2249–2305.

Boylston, C., Palacios, B., Tassev, P., and Bruckman, A. (2021). Wallstreetbets: positions or ban. *arXiv preprint arXiv:2101.12110*.

Bradley, D., Jr., J. H., Jame, R., and Xiao, Z. (2021). Place your bets? the market consequences of investment research on reddit's wallstreetbets. *SSRN Electronic Journal*.

Brunnermeier, M. K. and Nagel, S. (2004). Hedge funds and the technology bubble. *Journal of Finance*, 59(5):2013–2040.

Brunnermeier, M. K. and Pedersen, L. H. (2009). Market liquidity and funding liquidity. *Review of Financial Studies*, 22(6):2201–2238.

Caballero, R. J. and Simsek, A. (2021). A model of endogenous risk intolerance and lsaps: asset prices and aggregate demand in a "covid-19" shock. *Review of Financial Studies*, 34(11):5522–5580.

Cookson, J. A., Engelberg, J., and Mullins, W. (2022). Echo chambers. *Available at SSRN 3603107*.

Cookson, J. A. and Niessner, M. (2020). Why don't we agree? evidence from a social network of investors. *Journal of Finance*, 75(1):173–228.

De Long, J. B., Shleifer, A., Summers, L. H., and Waldmann, R. J. (1989). The size and incidence of the losses from noise trading. *Journal of Finance*, 44(3):681–696.

De Long, J. B., Shleifer, A., Summers, L. H., and Waldmann, R. J. (1990). Noise trader risk in financial markets. *Journal of Political Economy*, 98(4):703–738.

DeGroot, M. H. (1974). Reaching a consensus. *Journal of the American Statistical association*, 69(345):118–121.

DeMarzo, P. M., Vayanos, D., and Zwiebel, J. (2003). Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics*, 118(3):909–968.

Eaton, G. W., Green, T. C., Roseman, B. S., and Wu, Y. (2022). Retail trader sophistication and stock market quality: Evidence from brokerage outages. *Journal of Financial Economics*, 146(2):502–528.

Gabaix, X. (1999). Zipf's law for cities: an explanation. *Quarterly Journal of Economics*, 114(3):739–767.

Gabaix, X. (2011). The granular origins of aggregate fluctuations. *Econometrica*, 79(3):733–772.

Gabaix, X. and Koijen, R. S. J. (2022). In search of the origins of financial fluctuations: The inelastic markets hypothesis. *SSRN Electronic Journal*.

Gârleanu, N. and Pedersen, L. H. (2011). Margin-based asset pricing and deviations from the law of one price. *Review of Financial Studies*, 24(6):1980–2022.

Gennotte, G. and Leland, H. (1990). Market liquidity, hedging, and crashes. *American Economic Review*, pages 999–1021.

Gianstefani, I., Longo, L., and Riccaboni, M. (2022). The echo chamber effect resounds on financial markets: a social media alert system for meme stocks. *arXiv preprint arXiv:2203.13790*.

Golub, B. and Jackson, M. O. (2010). Naive learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–49.

Hirshleifer, D. (2020). Presidential address: Social transmission bias in economics and finance. *Journal of Finance*, 75(4):1779–1831.

Hu, D., Jones, C. M., Zhang, V., and Zhang, X. (2021). The rise of reddit: How social media affects retail investors and short-sellers' roles in price discovery. *Available at SSRN 3807655*.

Hutto, C. and Gilbert, E. (2014). Vader: A parsimonious rule-based model for sentiment analysis of social media text. In *Proceedings of the international AAAI conference on web and social media*, volume 8, pages 216–225.

Koijen, R. S. J., Richmond, R., and Yogo, M. (2022). Which investors matter for equity valuations and expected returns? *SSRN Electronic Journal*.

Kuchler, T. and Stroebel, J. (2021). Social finance. *Annual Review of Financial Economics*, 13:37–55.

Mancini, A., Desiderio, A., Di Clemente, R., and Cimini, G. (2022). Self-induced consensus of reddit users to characterise the gamestop short squeeze. *Scientific reports*, 12(1):1–11.

Miller, E. M. (1977). Risk, uncertainty, and divergence of opinion. *Journal of Finance*, 32(4):1151–1168.

Newman, M. E. (2005). Power laws, pareto distributions and zipf's law. *Contemporary Physics*, 46(5):323–351.

Pedersen, L. H. (2022). Game on: Social networks and markets. *Journal of Financial Economics*.

Rantala, V. (2019). How do investment ideas spread through social interaction? evidence from a ponzi scheme. *Journal of Finance*, 74(5):2349–2389.

Scheinkman, J. A. and Xiong, W. (2003). Overconfidence and speculative bubbles. *Journal of Political Economy*, 111(6):1183–1220.

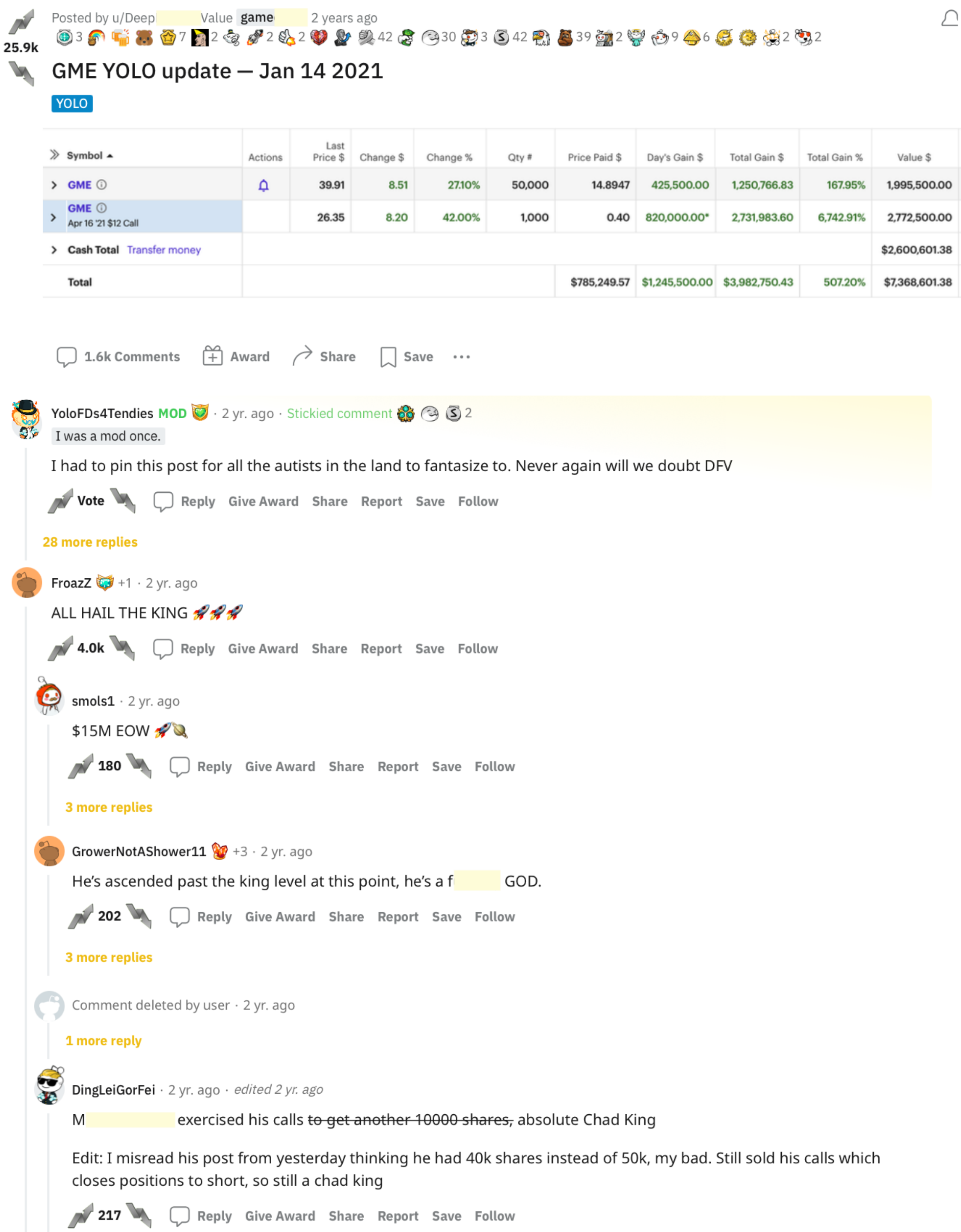
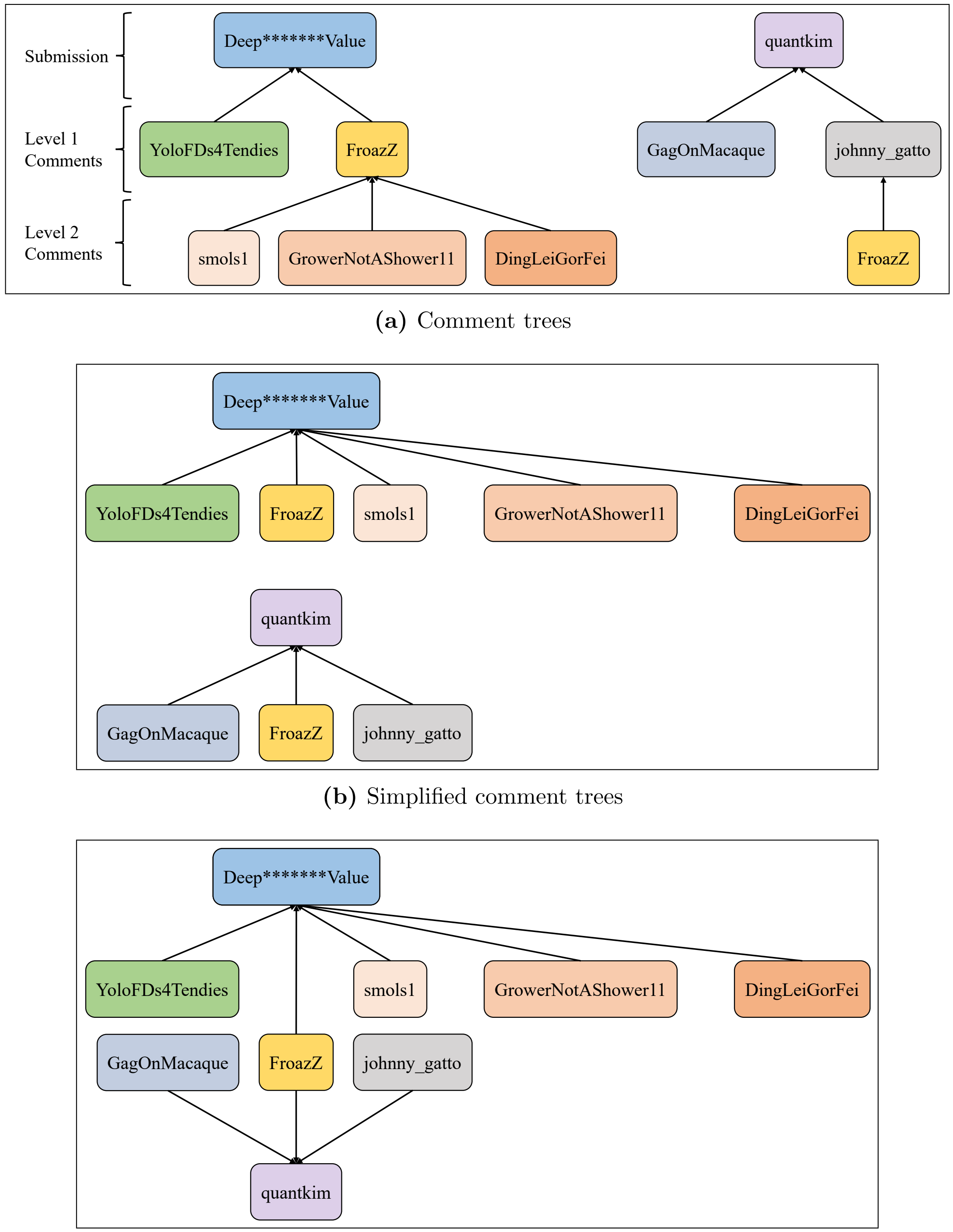


Figure 2. Example of a conversation tree. This figure shows an example of a conversation tree on Reddit's WallStreetBets (WSB) forum. The conversation is retrieved from https://www.reddit.com/r/wallstreetbets/comments/kxeq23/gme_yolo_update_jan_14_2021/.



(a) Comment trees

(b) Simplified comment trees

(c) User network

Figure 3. Generic representations of comment trees and user network. This figure shows an example of two comment trees from WSB and the corresponding user network. Panel (a) plots two trees, and the left one corresponds to the conversation shown in Figure 2. Panel (b) plots the simplified trees that correspond to the original ones in panel (a). Panel (c) plots the user network constructed from these two simplified trees.

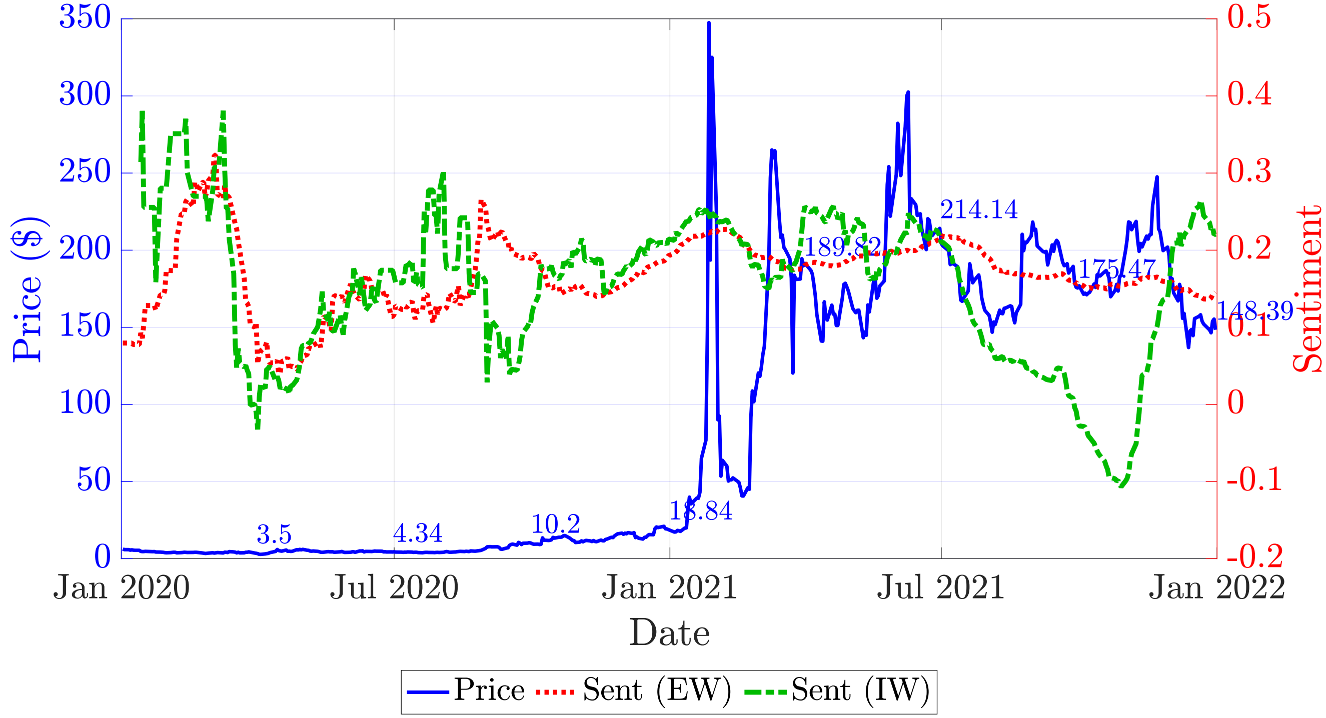
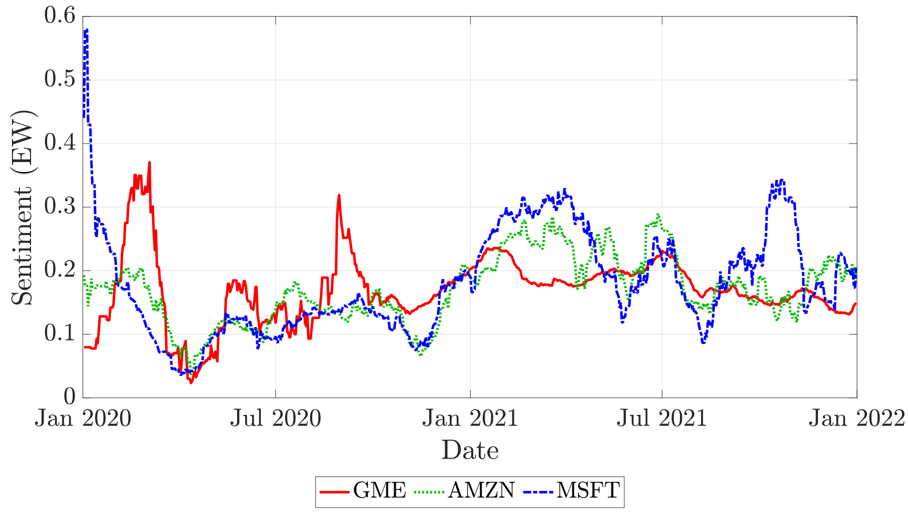
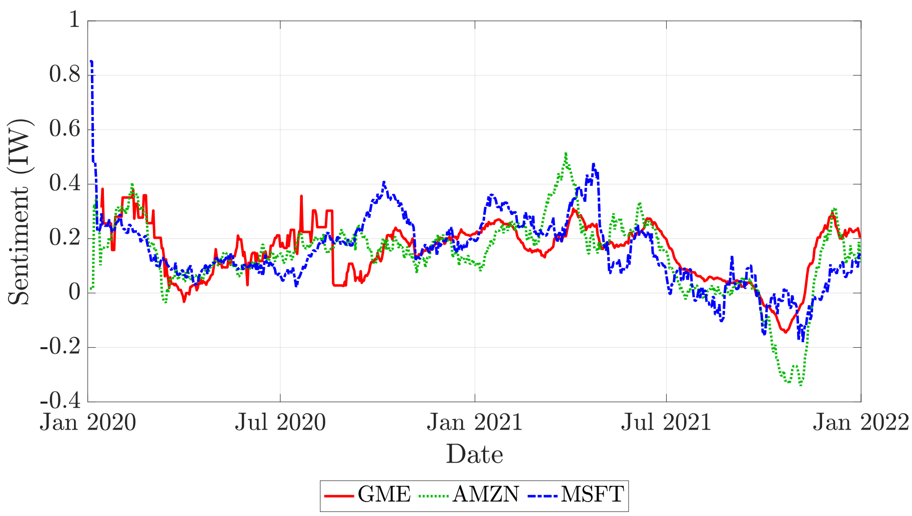


Figure 4. Price and sentiment of GameStop. This figure shows the daily close price (left y-axis) and the daily WSB sentiment measures (right y-axis) of GameStop, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price, the dotted red line plots the equal-weighted sentiment defined in equation (4), and the dash-dotted green line plots the influence-weighted sentiment defined in equation (5). The sentiment series are 30-day moving averages.

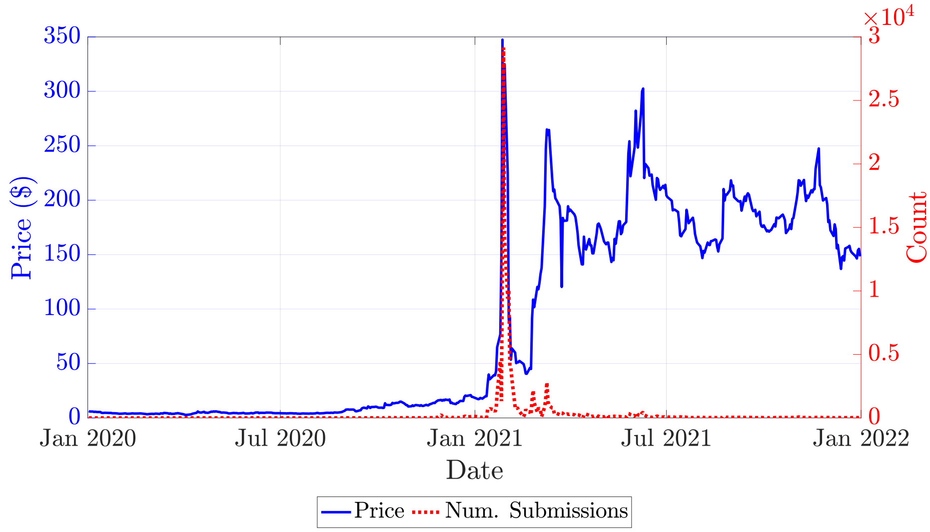


(a) Equal-weighted sentiment

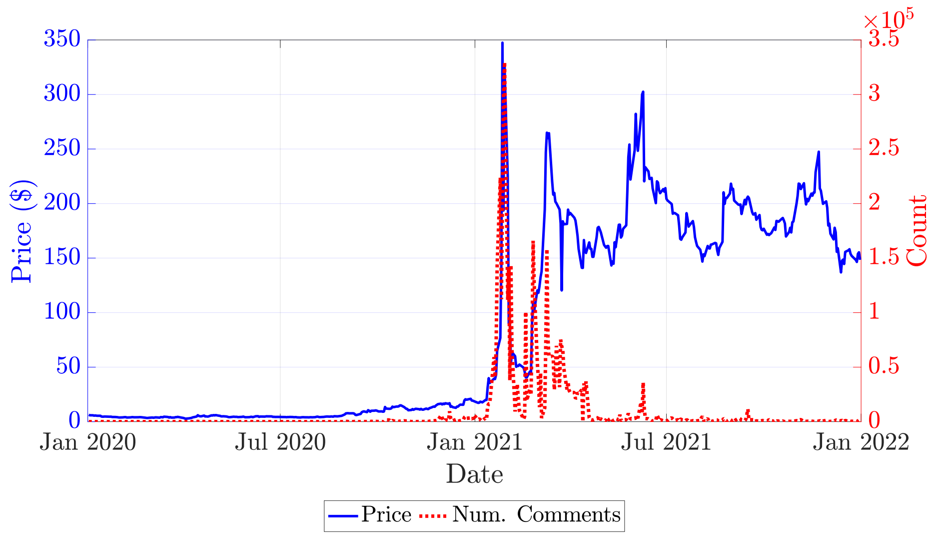


(b) Influence-weighted sentiment

Figure 5. Sentiment of GameStop versus tech stocks. This figure plots the daily WSB sentiment of GameStop versus two tech stocks, Amazon and Microsoft, for the period from January 1, 2020 to December 31, 2021. Panel (a) plots the equal-weighted sentiment defined in equation (4). Panel (b) plots the influence-weighted sentiment defined in equation (5). In each panel, the solid red line represents GameStop, the dotted green line represents Amazon, and the dash-dotted blue line represents Microsoft. The sentiment series are 30-day moving averages.



(a) Number of submissions



(b) Number of comments

Figure 6. Price and discussion volume of GameStop. This figure shows the daily close price (left y-axis) and the daily WSB discussion volume (right y-axis) of GameStop, for the period from January 1, 2020 to December 31, 2021. Panel (a) plots the close price of GameStop (solid blue line) and the daily number of new submissions about GameStop on WSB (dotted red line). Panel (b) plots the close price of GameStop (solid blue line) and the daily number of new comments about GameStop on WSB (dotted red line).

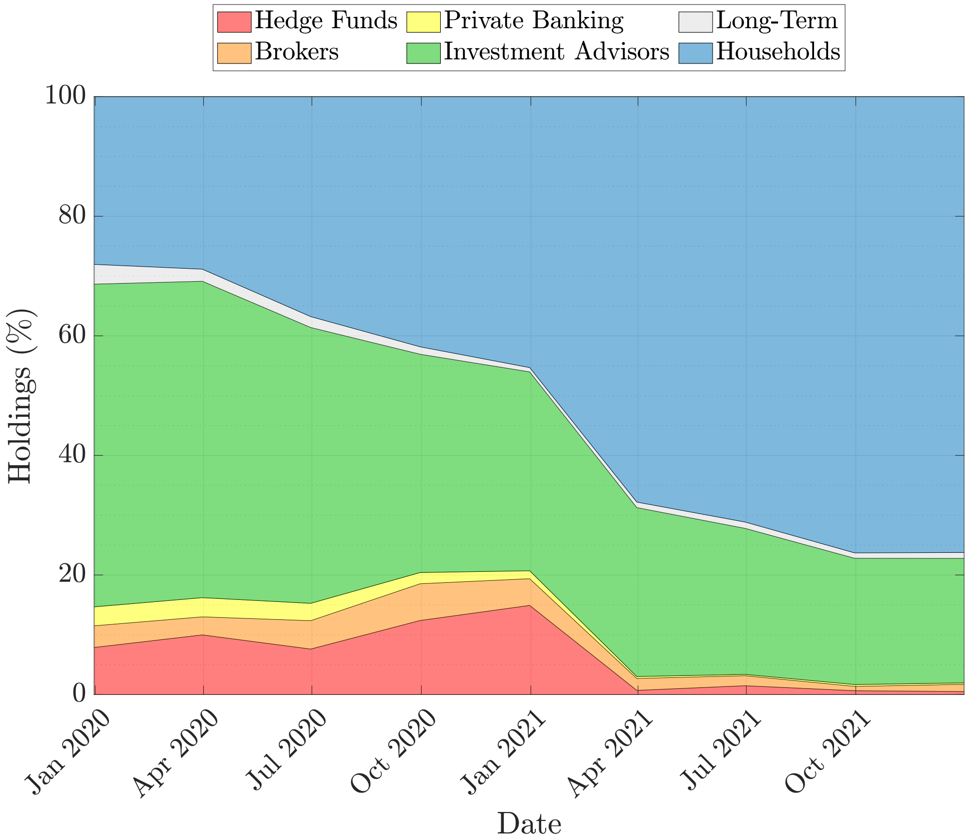
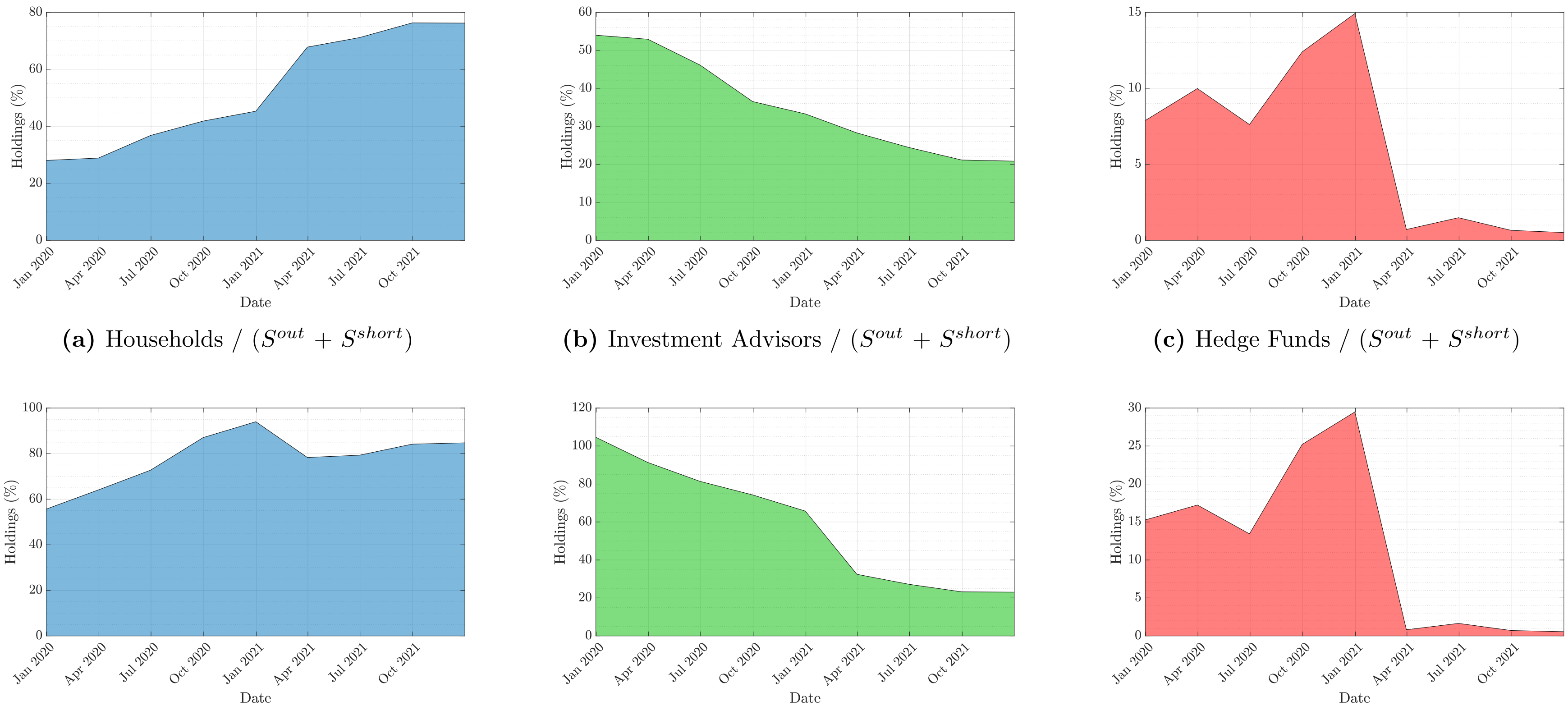


Figure 7. Ownership of GameStop by investor type. This figure plots the end-of-quarter holdings of GameStop by 13F institutions and households, for the period from 2019 Q4 to 2021 Q4. 13F holdings data are from FactSet. I aggregate 13F institutional holdings to investor-type level, using the method in Appendix A3. And the five institutional investor types are: Hedge Funds (red area), Brokers (orange aread), Private Banking (yellow area), Investment Advisors (green area), and Long-Term Investors (gray area). I calculate household holdings from equation (8), using data on the number of shares sold short from Compustat. And the blue area represents households. The y-axis is the percentage holdings defined in equation (10), which is the number of shares held by each type of investor divided by the sum of the number of shares outstanding and the number of shares sold shorted.



(a) Households / (S^{out} + S^{short}) **(b)** Investment Advisors / (S^{out} + S^{short}) **(c)** Hedge Funds / (S^{out} + S^{short})

(d) Households / S^{out} **(e)** Investment Advisors / S^{out} **(f)** Hedge Funds / S^{out}

Figure 8. Ownership of GameStop by Households, Investment Advisors, and Hedge Funds. This figure plots the end-of-quarter holdings of GameStop by Households (panel (a) and (d)), Investment Advisors (panel (b) and (e)), and Hedge Funds (panel (c) and (f)), for the period from 2019 Q4 to 2021 Q4. 13F institutional investors are classified into Investment Advisors and Hedge Funds according to Appendix A3, and the 13F holdings data are from FactSet. Household holdings are calculated from equation (8). In panel (a), (b), and (c), the y-axis is the number of shares held by the investor group, divided by the sum of the number of shares outstanding and the number of shares sold short (equation (10)). Data on the number of shares sold short is from Compustat. In panel (d), (e), and (f), the y-axis is the number of shares held by the investor group, divided by the number of shares outstanding (equation (9)).

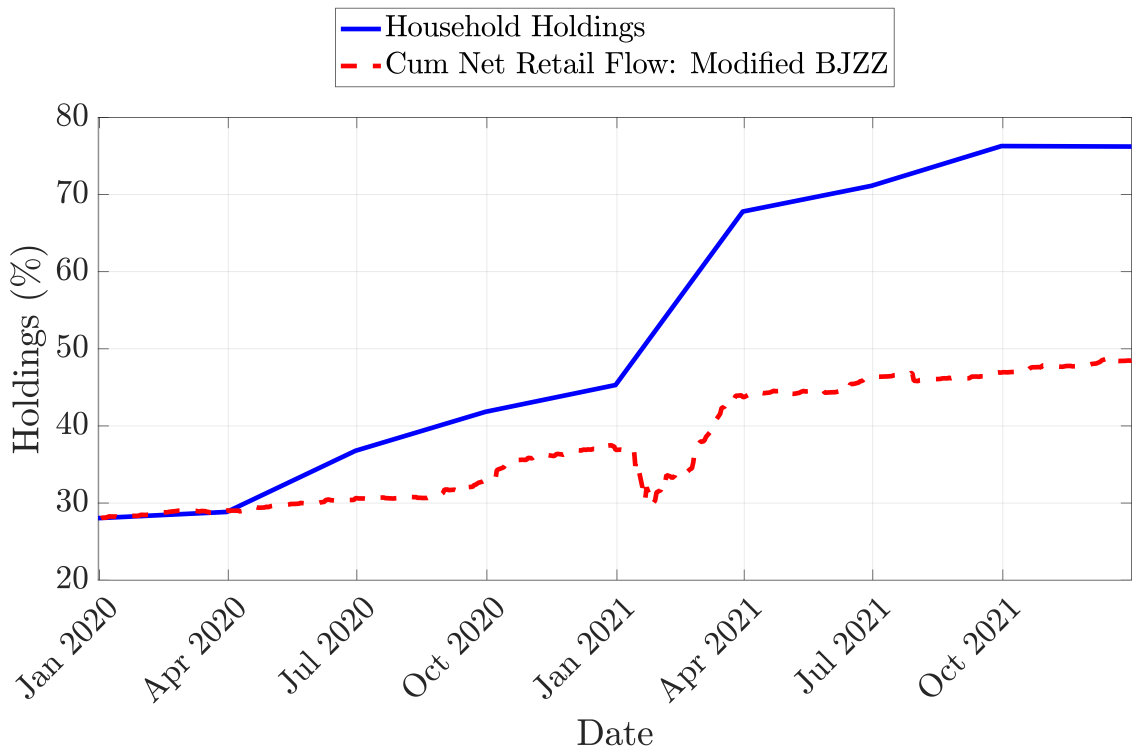


Figure 9. Ownership by households versus cumulative net retail buy volume of GameStop. This figure plots the end-of-quarter holdings of GameStop by Households (solid blue line), and the daily cumulative net retail buy volume (dashed red line), for the period from January 1, 2020 to December 31, 2021. Percentage holdings by households is defined in equation (10), which is the number of shares held by households (equation (8)) divided by the sum of the number of shares outstanding and the number of shares sold shorted. Cumulative net retail flow is defined in equation (12), which is the cumulative net retail buy volume (equation (11)) divided by the sum of the number of shares outstanding and the number of shares sold short. Data on the number of shares sold short is from Compustat. The initial value of the cumulative net retail flow (on Dec 31, 2019) is set to be the percentage holdings by households at the end of 2019 Q4. I apply the modified BJZZ algorithm in Appendix A4 to identify retail trades from the TAQ data.

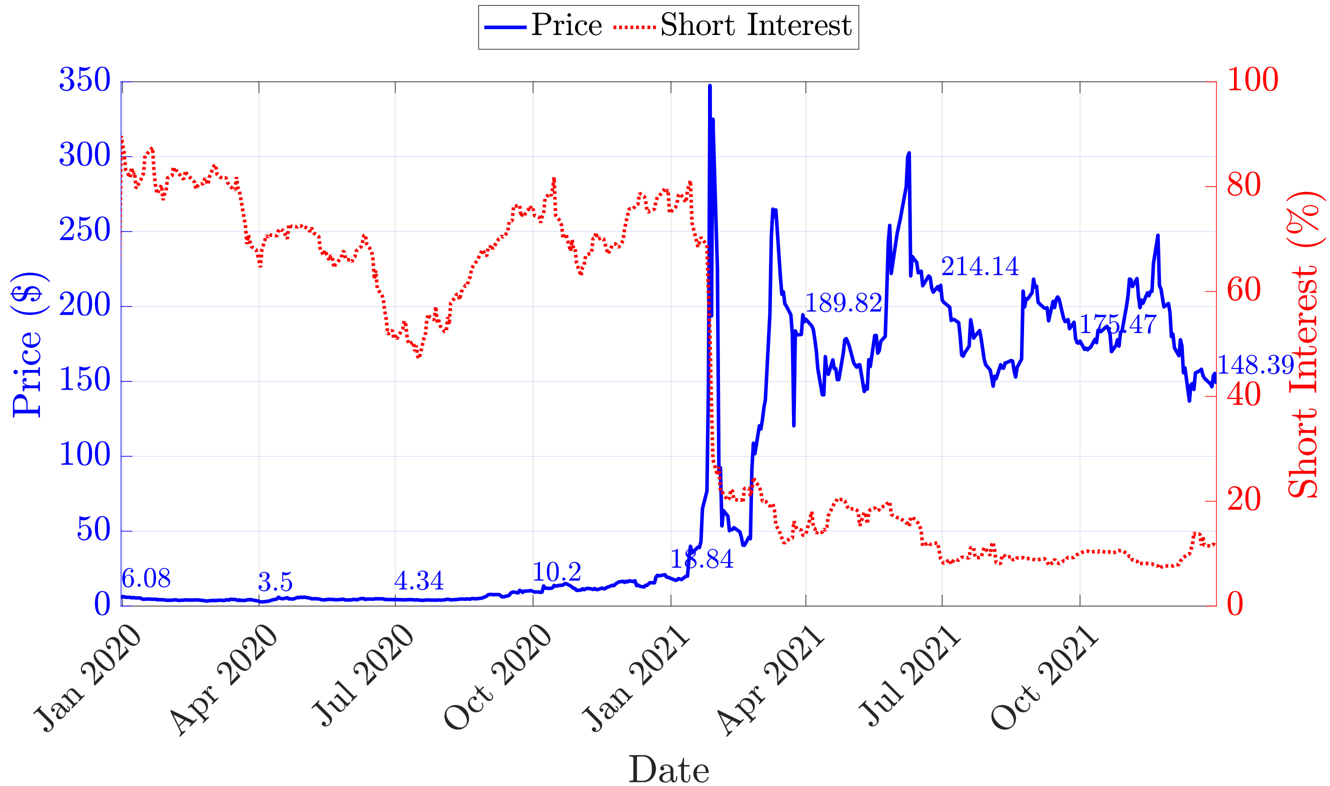
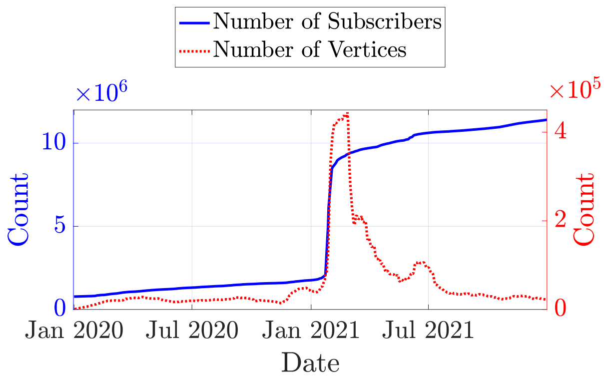
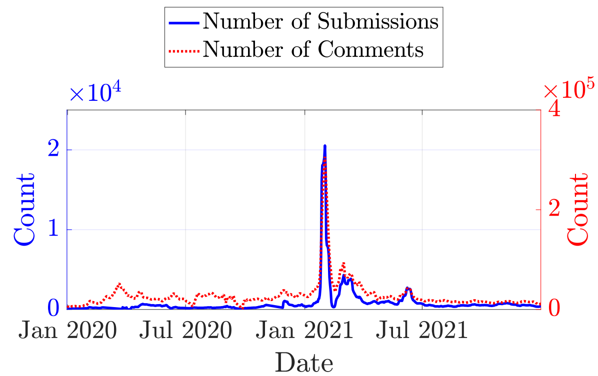


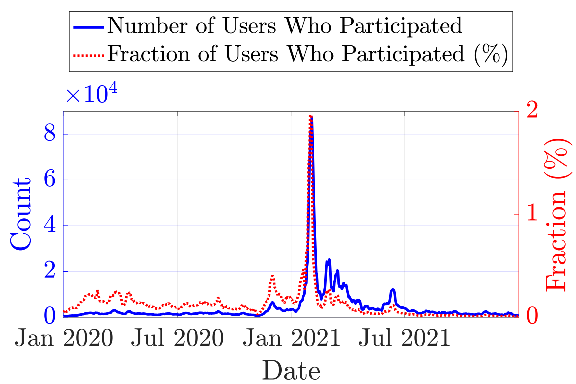
Figure 10. Price and short interest of GameStop. This figure shows the daily close price (left y-axis) and the daily short interest (right y-axis) of GameStop, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price. The dotted red line plots the short interest, which is defined as the ratio of the number of shares sold short to the number of shares outstanding (equation (6)). Data on the number of shares sold short is from IHS Markit.



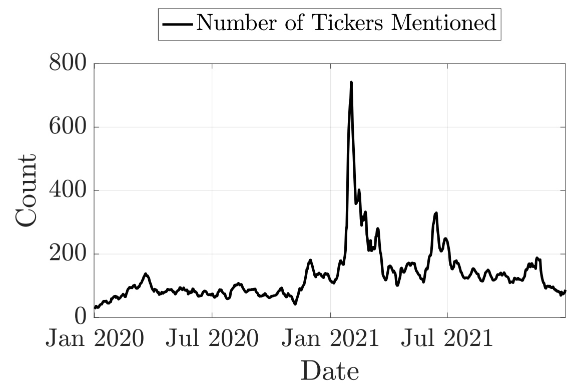
(a) Number of users



(b) Number of submissions and comments

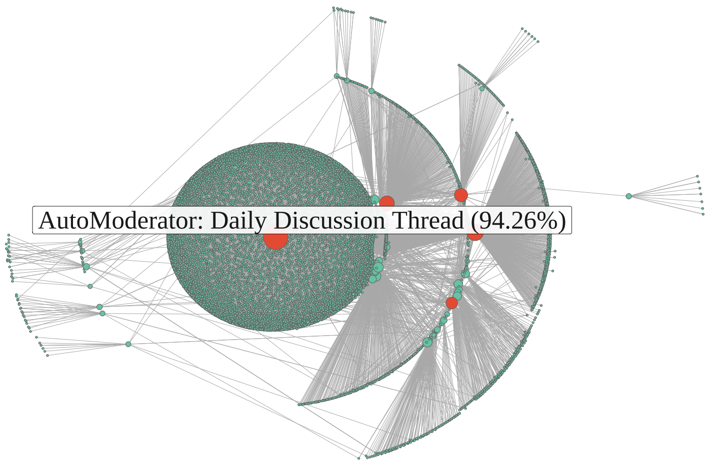


(c) Number of users who participated

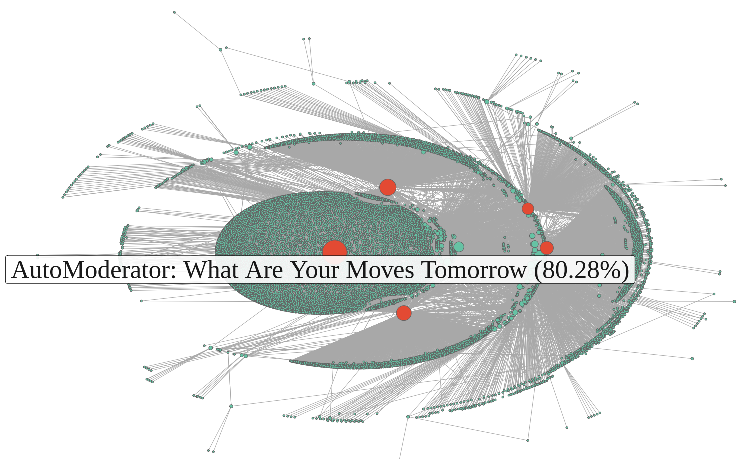


(d) Number of tickers mentioned

Figure 11. WSB statistics. This figure shows the time variation in WSB statistics, during the period from January 1, 2020 to December 31, 2021. Each line is a daily time series. Panel (a) plots the total number of subscribers to WSB (solid blue line) and the number of vertices (i.e., nodes) of the constructed network (dotted red line), on each day. When calculating the number of vertices, I use the network constructed from the sample of submissions and comments about CRSP common stocks, over a 30-day rolling window (see Section 2.1.2 for details). Panel (b) plots the number of new submissions (solid blue line) and the number of new comments (dotted red line) on WSB forum on each day. Panel (c) plots the number of users who participated in the discussion of CRSP common stocks (solid blue line) and the fraction of WSB subscribers who participated in these discussions (dotted red line), on each day. Panel (d) plots the number of stock tickers mentioned on WSB on each day. The series in panel (b)-(d) are 7-day moving averages.



(a) 6-9am



(b) 4-7pm

Figure 12. WSB user communications on January 14, 2022. This figure shows WSB user communications on January 14, 2022. Panel (a) plots the user communications from 6-9am, and panel (b) plots the user communications from 4-7pm. Each dot represents a unique user who made a new submission or new comment within this 3-hour window. For any two users i and j in this plot, if i commented on j's submission within the 3-hour window, then I draw a directed edge from i to j. For example, the largest red dot represents the AutoModerator, and the dots clustered around it represent the users who commented on AutoModerator's submission. The number in the parentheses is the number of comments received by the corresponding user, as a fraction of the total number comments received by new submissions that came out within the 3-hour window. The five red dots represent the top five users by the fraction of comments they received.

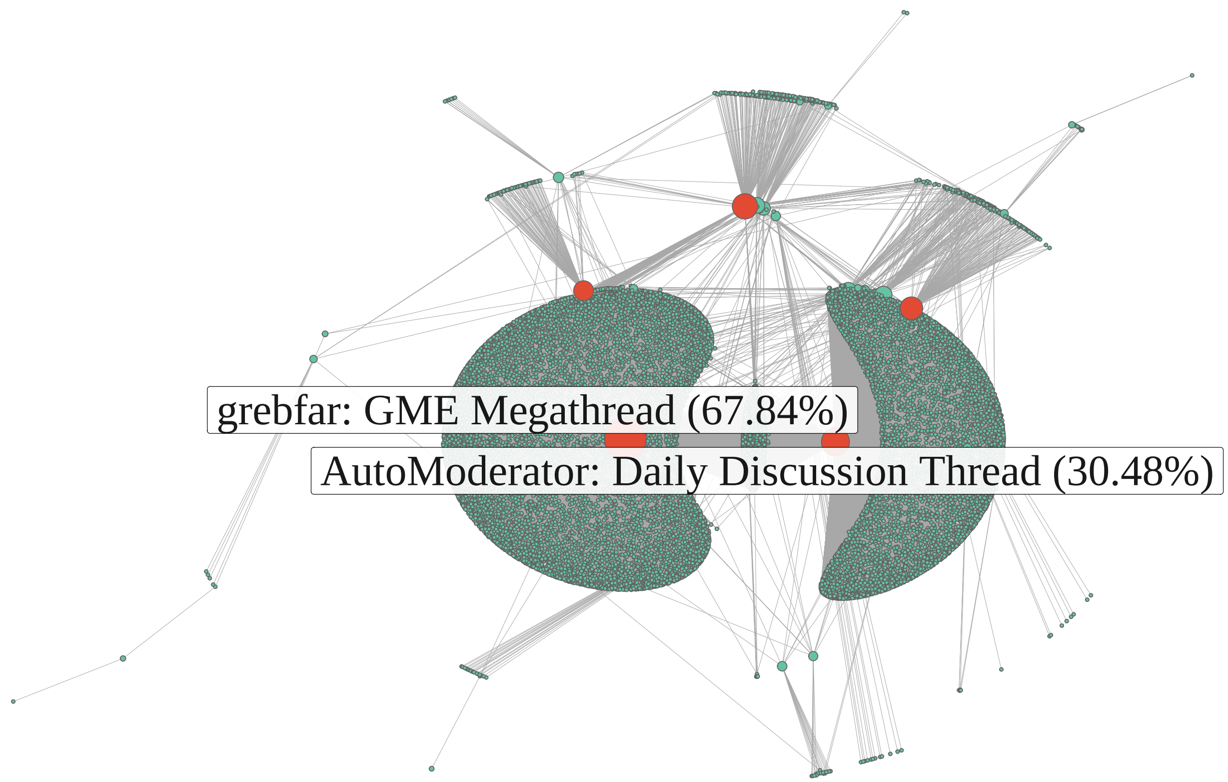


Figure 13. WSB user communications on January 21, 2021. This figure shows WSB user communications from 6-8am on January 21, 2021. Each dot represents a unique user who made a new submission or new comment within this 2-hour window. For any two users i and j in this plot, if i commented on j's submission within the 2-hour window, then I draw a directed edge from i to j. For example, the largest red dot represents user grebfar, and the dots clustered around it represent the users who commented on grebfar's submission titled "GME Megathread." The number in the parentheses is the number of comments received by the corresponding user, as a fraction of the total number comments received by new submissions that came out within the 2-hour window. The five red dots represent the top five users by the fraction of comments they received.

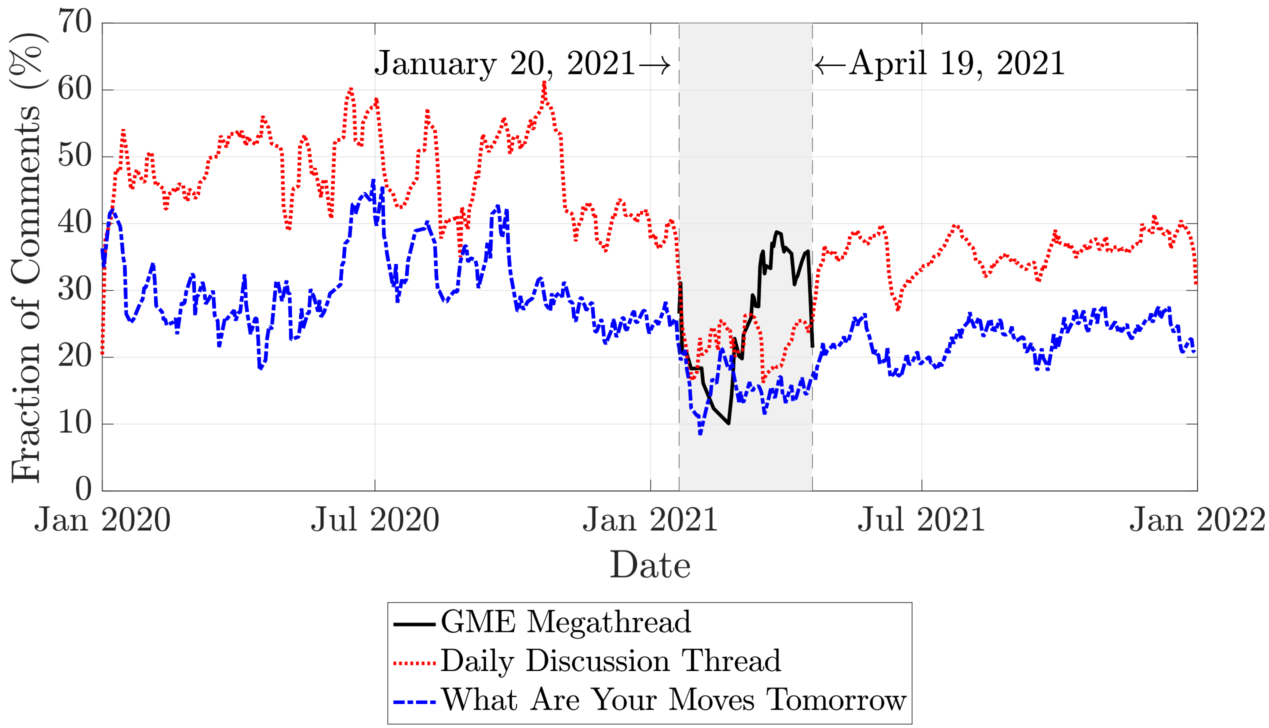


Figure 14. Fraction of comments received by different types of megathreads. This figure plots the fraction of comments received by three types of megathreads: GME Megathreads (solid black line), Daily Discussion Threads (dotted red line), and What Are Your Moves Tomorrow (dash-dotted blue line). On a given day, there could be multiple threads of the same type, e.g., multiple threads with "GME Megathread" in their titles. In that case, the fraction of comments received by each type of thread is the total number of comments received by all threads of the type divided by the total number of new comments that came out on that day. In this figure, each line is a daily time series, and I plot the 7-day moving average of each daily series. The sample period is from January 1, 2020 to December 31, 2021.

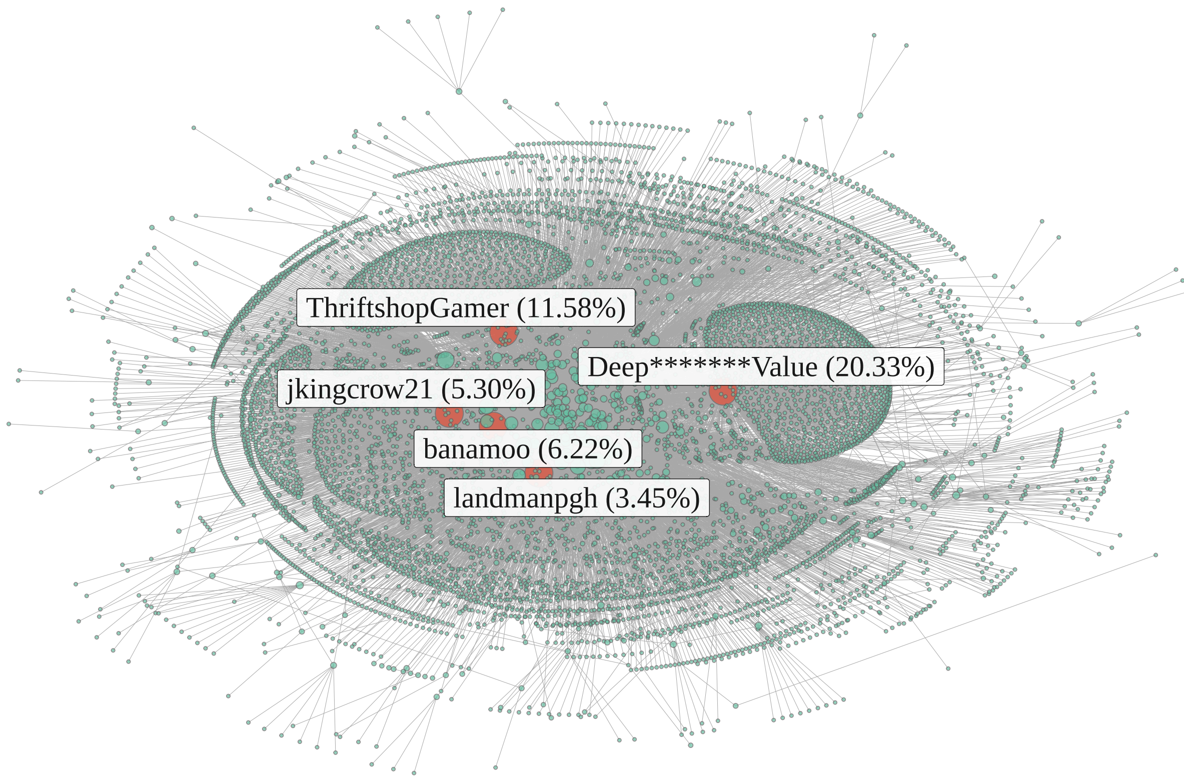


Figure 15. WSB user network on January 14, 2021 constructed from GameStop discussions. This figure shows the WSB user network on January 14, 2021, constructed from submissions and comments about GameStop during the 30-day window from December 15, 2020 to January 13, 2021. Each dot represents a unique user who authored at least one of the submissions or comments. For any two users i and j in this plot, if i commented on j's submission, then i "listened to" j, and I draw a directed edge from i to j. The five red dots represent the top five users by the fraction of users (on the network) who "listened to" them, and the numbers in the parentheses correspond to this fraction.

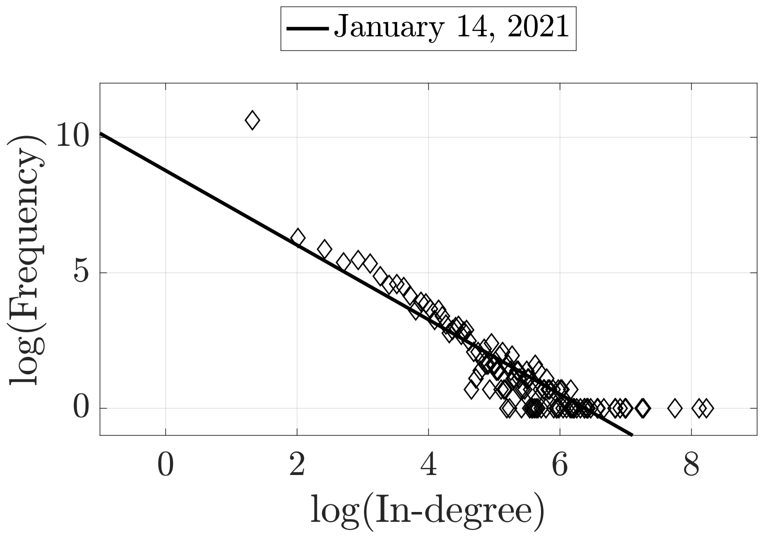


Figure 16. Log-log plot of influence distribution on January 14, 2021. This figure plots the cross-sectional distribution of user influence for the user network on January 14, 2021. The network is constructed according to Section 2.1.2. User influence (or in-degree) is defined in equation (3). The x-axis is the log of in-degree, and the y-axis is the log empirical frequency. The solid black line is a fitted linear regression line.

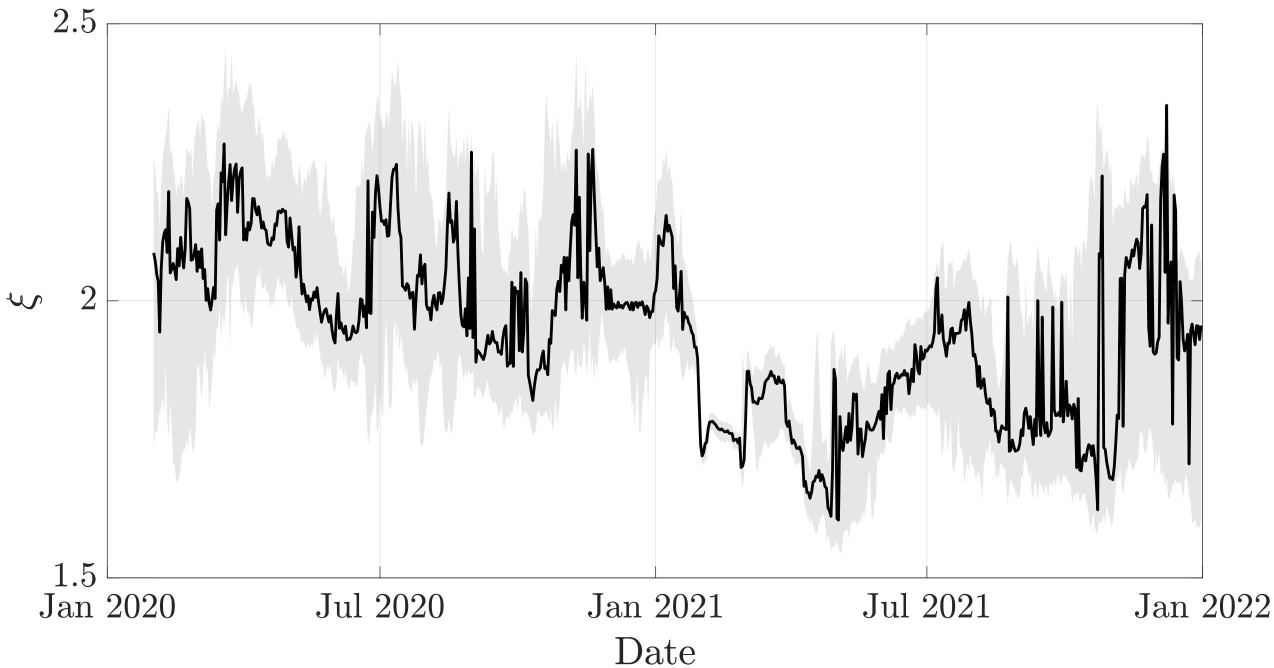


Figure 17. Estimates of the power-law exponent $\hat{\xi}_t$. This figure plots the daily estimate of the power-law exponent $\hat{\xi}_t$, for the period from January 1, 2020 to December 31, 2021. On each day t, I fit a power-law distribution to the vector of user influence (defined in equation (3)) and estimate the exponent ξ in equation (13). The solid black line plots the $\hat{\xi}_t$ estimates from the maximum likelihood method as in Rantala (2019). The gray area shows the 95% confidence interval for the estimates, computed from the bootstrap method in Appendix A5.

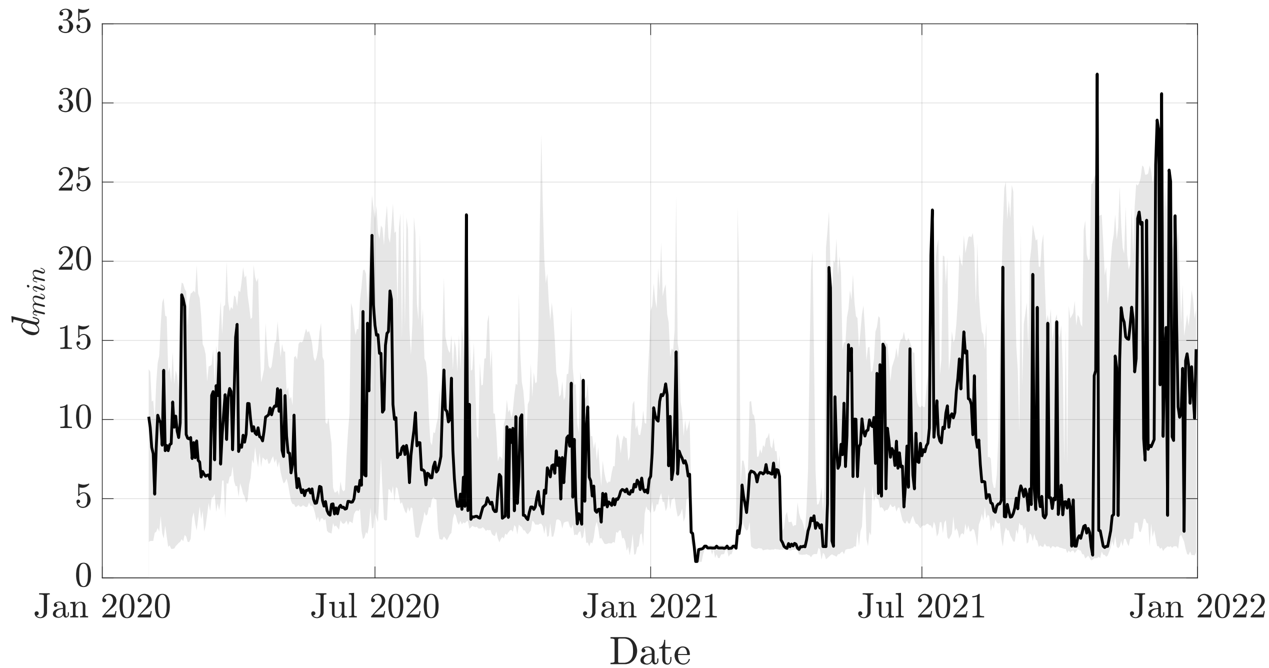


Figure 18. Estimates of the power-law cutoff $\hat{d}_{\min,t}$. This figure plots the daily estimate of the power-law cutoff $\hat{d}_{\min,t}$, for the period from January 1, 2020 to December 31, 2021. On each day t, I fit a power-law distribution to the vector of user influence (defined in equation (3)) and estimate the cutoff value $d_{\min}$ in equation (13). The solid black line plots the $\hat{d}_{\min,t}$ estimates from the maximum likelihood method as in Rantala (2019). The gray area shows the 95% confidence interval for the estimates, computed from the bootstrap method in Appendix A5.

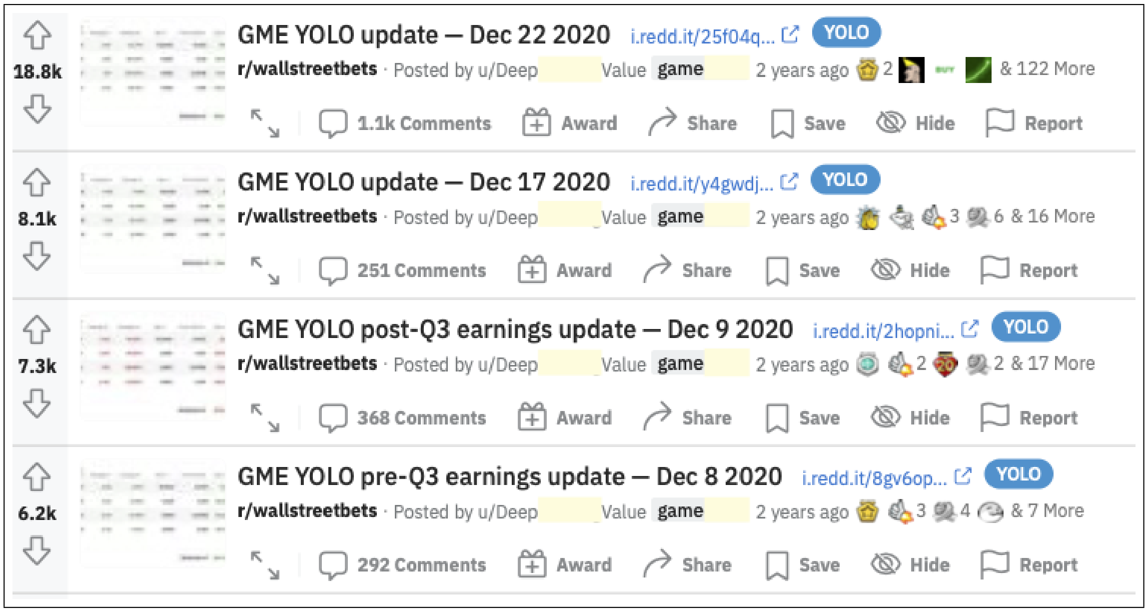
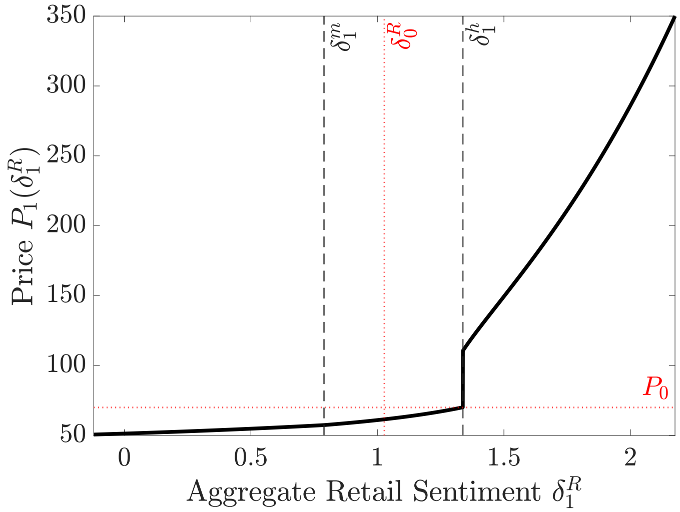
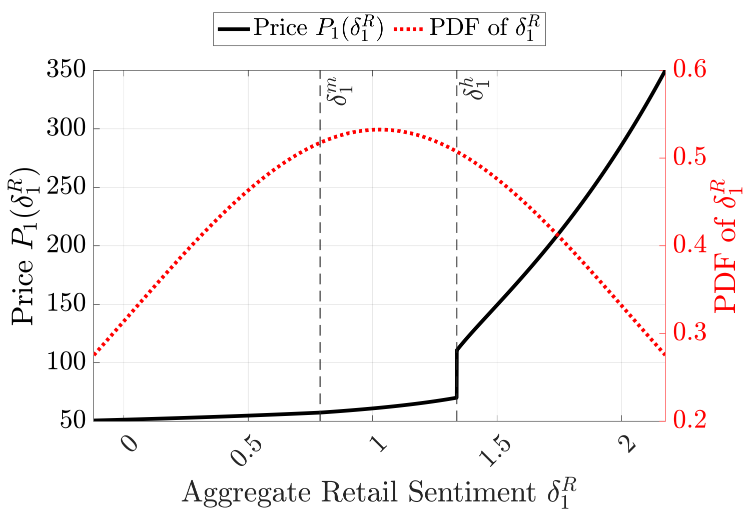


Figure 19. Examples of Deep*****Value's submissions.** This figure shows examples of WSB submissions made by user Deep*******Value in December 2020.

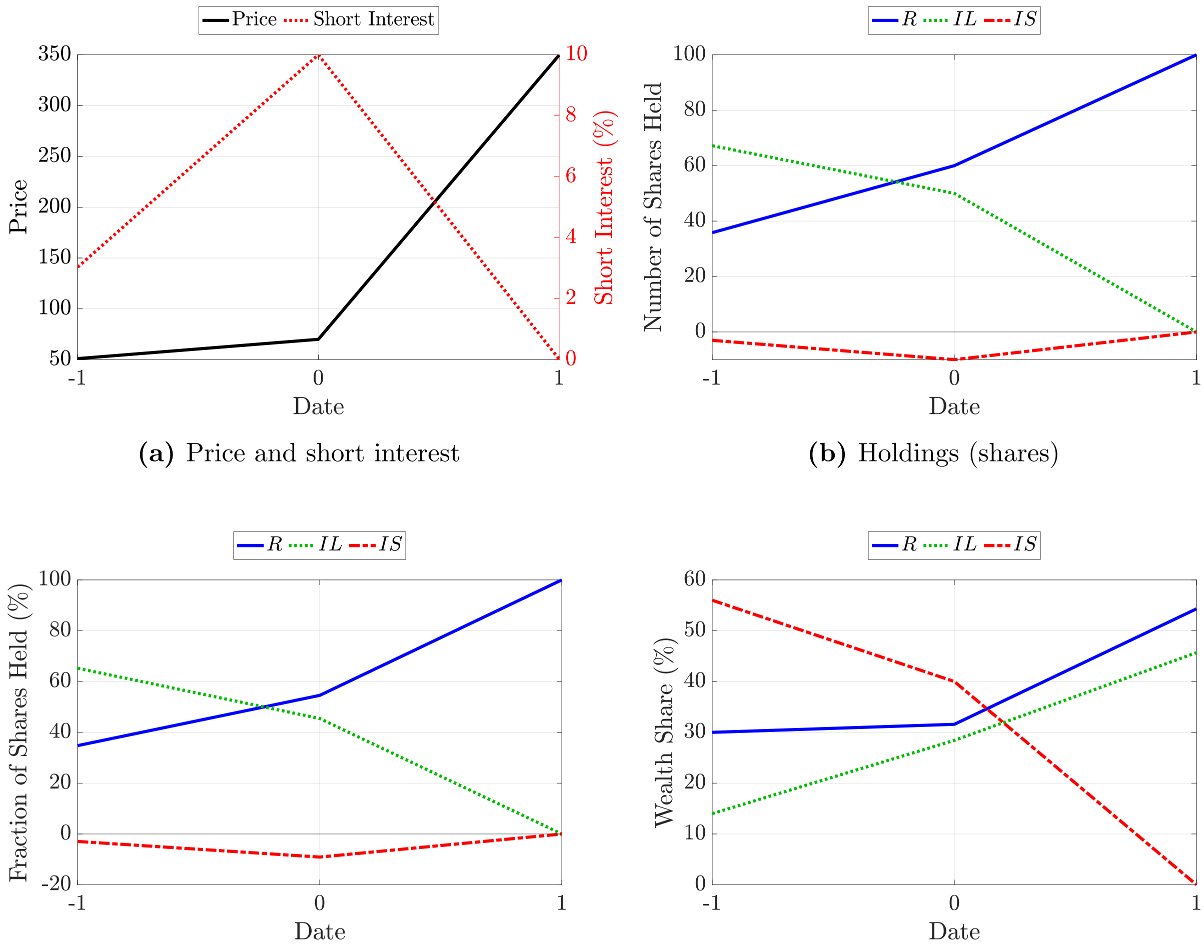


(a) Price function



(b) Price function and PDF of aggregate retail sentiment

Figure 20. Price impact of aggregate retail sentiment at time 1. This figure shows the time-1 equilibrium price $P_1\left(\delta_1^R\right)$ as a function of the aggregate retail sentiment realization δ_1^R. In panel (a), the solid black line is the price function $P_1\left(\delta_1^R\right)$. The two vertical dashed lines represent the two sentiment cutoffs δ_1^m and δ_1^h defined in equation (44) and (45), respectively. The vertical dotted line represents the time-0 aggregate retail sentiment, and the horizontal dotted line corresponds to the time-0 equilibrium price. In panel (b), the solid black line is the price function $P_1\left(\delta_1^R\right)$, and the dotted red line is the PDF of the aggregate retail sentiment δ_1^R perceived by investors. In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The parameter values are given in Table 2.



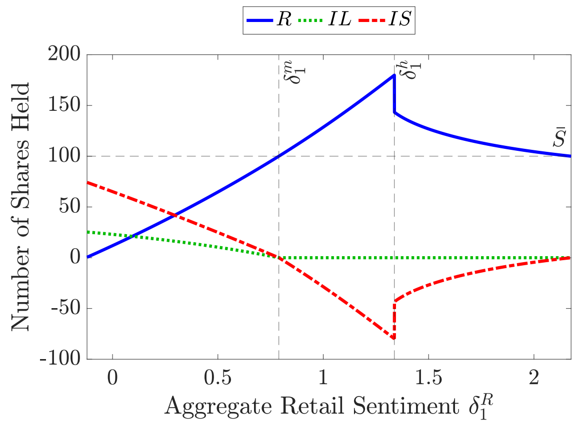
(a) Price and short interest

(b) Holdings (shares)

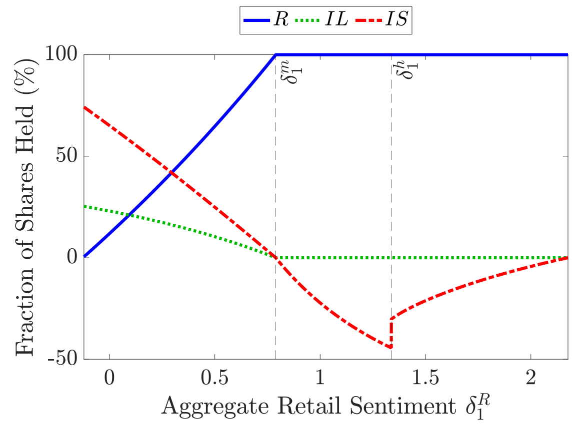
(c) Holdings (%)

(d) Wealth shares

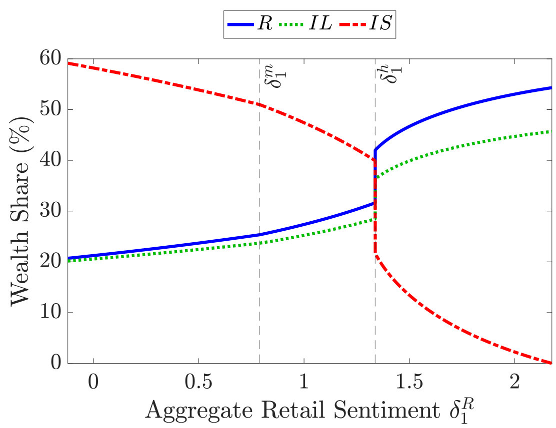
Figure 21. Time series predictions from the model. Panel (a) plots the equilibrium price (solid black line) and short interest (dotted red line). Short interest is defined as the number of shares shorted divided by the number of shares outstanding. Panel (b) plots the number of shares held by the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). Panel (c) plots the percentage holdings by the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). Percentage holdings is defined as the number of shares held divided by the sum of the number of shares outstanding and the number of shares sold short. Panel (d) plots the wealth shares of the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The time-1 equilibrium outcomes correspond to an aggregate retail sentiment realization $\delta_1^R = 2.18$. The parameter values are given in Table 2.



(a) Holdings (shares)



(b) Holdings (%)



(c) Wealth shares

Figure 22. Holdings and wealth shares at time 1. This figure shows the time-1 holdings and wealth shares of different investors, as functions of the aggregate retail sentiment realization δ_1^R. Panel (a) plots the number of shares held by the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). The horizontal dashed black line represents the number of shares outstanding. Panel (b) plots the percentage holdings by the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). Percentage holdings is defined as the number of shares held divided by the sum of the number of shares outstanding and the number of shares sold short. Panel (c) plots the wealth shares of the aggregate retail investor (solid blue line), the long institution (dotted green line), and the short institution (dash-dotted red line). In each panel, the two vertical dashed black lines represent the two sentiment cutoffs δ_1^m and δ_1^h defined in equation (44) and (45). In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The parameter values are given in Table 2.

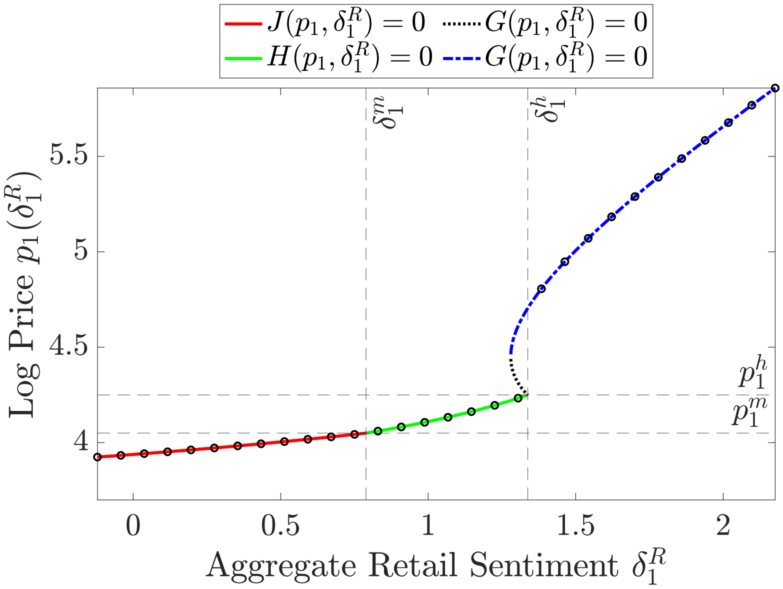
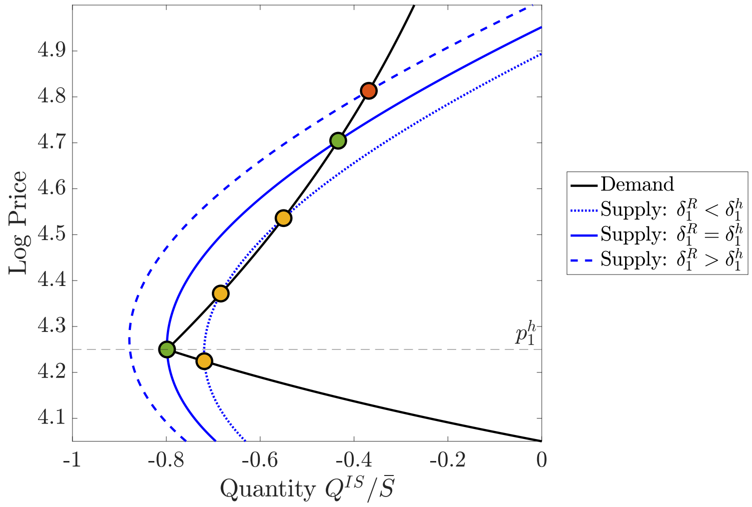
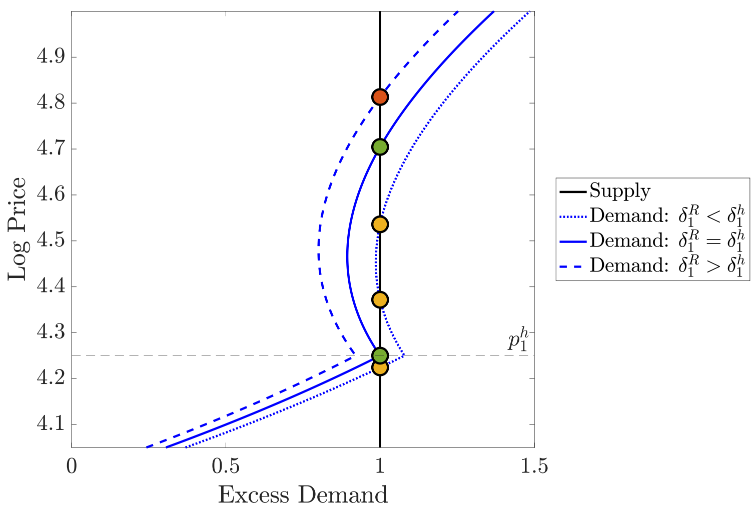


Figure 23. Multiple equilibria. This figure shows all the equilibria at time 1. The x-axis is the aggregate retail sentiment at time 1, and the y-axis is the log price at time 1. There are three classes of equilibria: the low-price equilibria (solid red line and solid green line), the medium-price equilibria (dotted black line), and the high-price equilibria (dash-dotted blue line). In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The parameter values are given in Table 2.

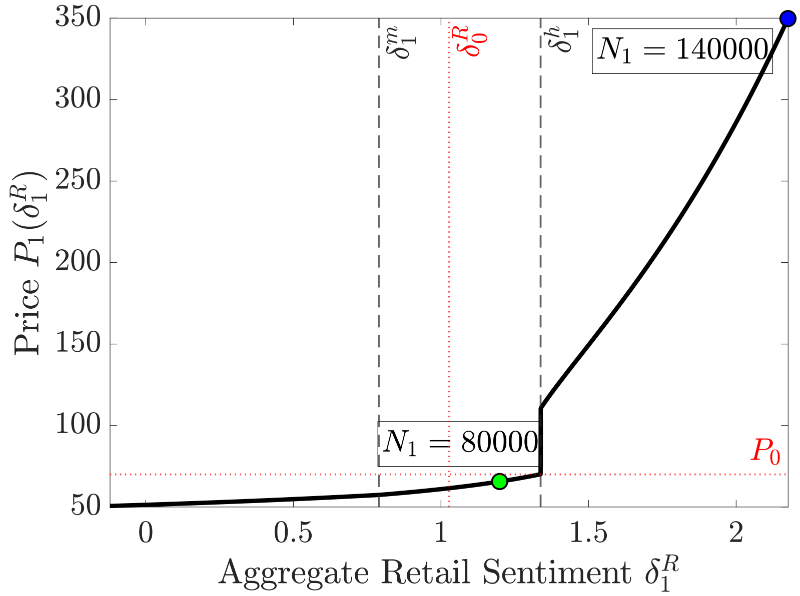


(a) Demand and supply

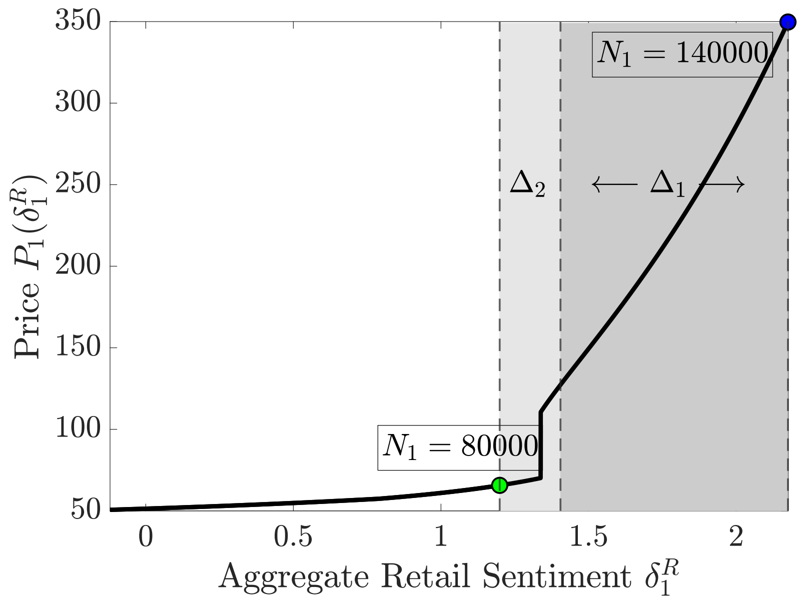


(b) Excess demand and supply

Figure 24. Demand and supply at time 1 from the short institution's perspective. This figure shows the demand and supply curves from the short institution's perspective. Panel (a) plots the demand curve of the short institution (solid black line) defined in equation (51), together with three supply curves (blue lines) defined in equation (52) which correspond to different aggregate retail sentiment shock realizations. Panel (b) plots three excess demand curves (blue lines) that correspond to different aggregate retail sentiment shock realizations, together with the excess supply (solid black line). Excess demand is defined in equation (53). In each panel, the horizontal dashed line represents the cutoff price p_1^h defined in equation (43), and each dot represents an equilibrium. In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The parameter values are given in Table 2.



(a) Price under different network size



(b) Decomposing the price difference

Figure 25. Time-1 aggregate retail sentiment realization under different network size. This figure shows the time-1 aggregate retail sentiment realization under different network size. In each panel, the blue dot represents the realized aggregate retail sentiment under network size $N_1 = N_H = 140000$, while the green dot represents the realized aggregate retail sentiment under $N_1 = N_L = 80000$. Panel (b) decomposes the difference between the two sentiment realizations according to equation (63). In this numerical example, investors believe that the size of the network will remain constant from time 0 to time 1 with $\tilde{N}_1 = N_0 = N_L$. The parameter values are given in Table 2.

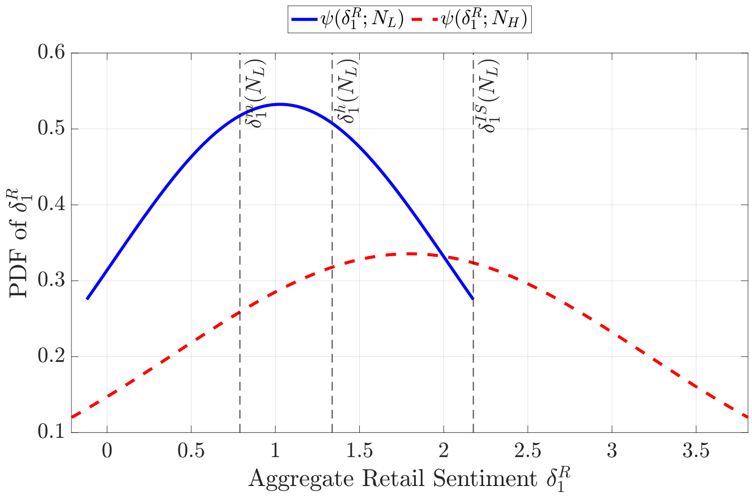
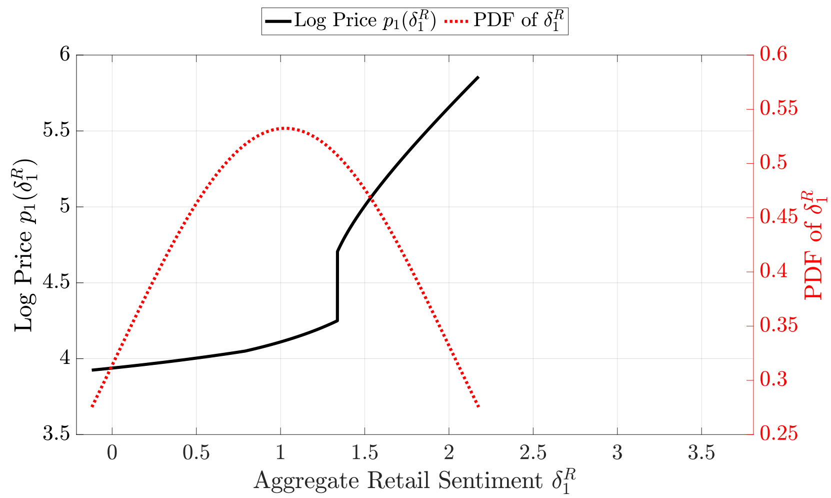
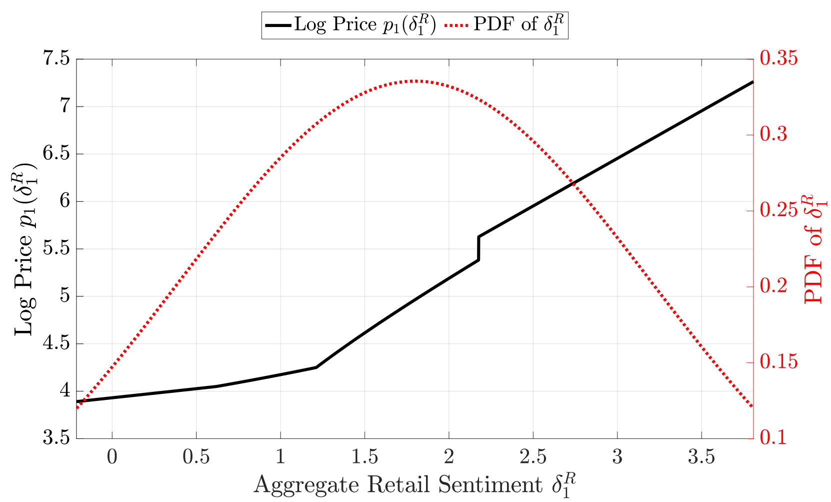


Figure 26. Time-1 retail sentiment distribution under different network size. This figure plots the distribution of time-1 aggregate retail sentiment (δ_1^R) under different network size. The solid blue line is the PDF of δ_1^R under $N_1 = N_L$. The dashed red line is the PDF of δ_1^R under $N_1 = N_H$. The parameter values are given in Table 2.



(a) Risk perception $\tilde{N}_1 = N_L$



(b) Risk perception $\tilde{N}_1 = N_H$

Figure 27. Time-1 equilibrium price under different risk perceptions. The figure shows the time-1 price function when changing investors' time-0 perceptions of risk. In panel (a), investors believe that the size of the network at time 1 will remain the same as that at time 0, i.e., $\tilde{N}_1 = N_L = N_0$. In panel (b), investors believe that the the size of the network will grow (deterministically) from time 0 to time 1, i.e., $\tilde{N}_1 = N_H > N_L = N_0$. The parameter values are given in Table 2.

Table 4
Modified VADER Lexicon

This table shows the modification to the VADER lexicon.

Positive			Negative		
Word	Emoji	Score	Word	Emoji	Score
rocket	🚀	4.0	bear(s)		-2.0
moon(ing)		4.0	paper		-4.0
diamond		4.0			
gem (stone)	💎	4.0			
hold(ing)		4.0			
tendie(s)		4.0			
yolo		4.0			
retard(s-ed)		2.0			
autist(s)		2.0			
degenerate(s)		2.0			
ape(s)		2.0			
gorilla(s)	🦍	2.0			

Table 5
Top 13F Institutions by Long Positions in GameStop in 2020 Q4

This table shows the top two 13F institutions within each institution type, ranked by their long positions in GameStop in 2020 Q4. 13F holdings data are from FactSet. I classify 13F institutions into five types using the method in Appendix A3. The five investor types are: Hedge Funds, Brokers, Private Banking, Investment Advisors, and Long-Term Investors.

Hedge Funds	Maverick Capital Ltd.
	Senvest Management LLC
Brokers	Goldman Sachs & Co. LLC
	Morgan Stanley & Co. LLC
Private Banking	Aperio Group LLC
	Permit Capital LLC (Private Equity)
Investment Advisors	Fidelity Management & Research Co. LLC
	BlackRock Fund Advisors
Long-Term Investors	The Public Sector Pension Investment Board
	The California Public Employees Retirement System

Internet Appendix for "Retail Trading and Asset Prices: The Role of Changing Social Dynamics"

Fulin Li*

November 1, 2022

Contents

*The University of Chicago, fli3@chicagobooth.edu.

A1 Omitted derivations and proofs

A1.1 Dynamics of wealth shares

Since the risk-free asset is in zero net supply, the time-t aggregate wealth is equal to the market value of the risky asset, $P_t\bar{S}$.

Investor i's wealth share at time $t+1$ is thus

$$\begin{aligned}
\alpha_{t+1}^i &\equiv \frac{A_{t+1}^i}{P_{t+1}\bar{S}} \\
&= \frac{A_t^i\left(w_t^i\frac{P_{t+1}}{P_t}+1-w_t^i\right)}{P_{t+1}\bar{S}} \\
&= \frac{\alpha_t^i P_t\bar{S}\left(w_t^i\frac{P_{t+1}}{P_t}+1-w_t^i\right)}{P_{t+1}\bar{S}} \\
&= \alpha_t^i\left(\left(1-w_t^i\right)\exp\left(p_t-p_{t+1}\right)+w_t^i\right),
\end{aligned}$$

where the second line uses the budget constraint (21) together with the assumption of constant risk-free rate $R_{f,t}=1$.

A1.2 Market clearing

Market clearing for the risk-free asset holds if and only if the aggregate wealth is equal to the market value of the risky asset, i.e.,

$$\sum_i A_t^i = P_t\bar{S}.$$

Market clearing condition for the risky asset is

$$\sum_i Q_t^i = \bar{S} \iff \sum_i \frac{w_t^i A_t^i}{P_t} = \bar{S} \iff \sum_i w_t^i A_t^i = P_t \bar{S}.$$

Hence, the two market clearing conditions reduce to

$$\sum_i A_t^i = \sum_i w_t^i A_t^i = P_t \bar{S}.$$

This is equivalent to the following condition

$$\sum_i \alpha_t^i w_t^i = 1, \alpha_t^i = \frac{A_t^i}{P_t \bar{S}}. \tag{A1}$$

From equation (A1), we can solve for the equilibrium price.

A1.3 Optimal portfolio choice

A1.3.1 Retail investors

Retail investor j solves the following problem

$$\begin{aligned} U_t^j\left(A_t^j\right) &= \max_{w_t^j} w_t^j \left(\mathbb{E}_t^j\left[r_{t+1}\right] - r_{f,t}\right) + \frac{1}{2} w_t^j \left(1 - w_t^j\right) \mathrm{Var}_t^j\left(r_{t+1}\right) \\ &\quad + \frac{1}{2}\left(1 - \gamma^R\right)\left(w_t^j\right)^2 \mathrm{Var}_t^i\left(r_{t+1}\right). \end{aligned}$$

The F.O.C. is

$$\begin{aligned} & \mathbb{E}_t^j\left[r_{t+1}\right] - r_{f,t} + \frac{1}{2}\mathrm{Var}_t^j\left(r_{t+1}\right) - \gamma^R w_t^j \mathrm{Var}_t^j\left(r_{t+1}\right) = 0 \\ \Longrightarrow \quad & w_t^j = \frac{\mathbb{E}_t^j\left[r_{t+1}\right] - r_{f,t} + \frac{1}{2}\mathrm{Var}_t^j\left(r_{t+1}\right)}{\gamma^R \mathrm{Var}_t^j\left(r_{t+1}\right)} = \tau^R \frac{\mathbb{E}_t^j\left[r_{t+1}\right] - r_{f,t} + \frac{1}{2}\mathrm{Var}_t^j\left(r_{t+1}\right)}{\mathrm{Var}_t^j\left(r_{t+1}\right)}. \end{aligned}$$

Substitute retail investors' subjective beliefs into the above expression, we get their demands for the risky asset.

- For a type-1 retail investor j, his time-0 and time-1 demands for the risky asset are

$$w_0^j = \tau^R \left(\frac{\mathbb{E}_0 [p_1] + y_0^j - p_0}{\sigma_0^2} + \frac{1}{2} \right), \tag{A2}$$

$$w_1^j = \tau^R \left(\frac{\mu_d + y_1^j - p_1}{\sigma_d^2} + \frac{1}{2} \right). \tag{A3}$$

- For a type-2 retail investor j', his time-0 and time-1 demands for the risky asset are

$$w_0^{j'} = \tau^R \left(\frac{\mathbb{E}_0 [p_1] - p_0}{\sigma_0^2} + \frac{1}{2} \right), \tag{A4}$$

$$w_1^{j'} = \tau^R \left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2} \right). \tag{A5}$$

A1.3.2 Long institution

The long institution IL solves the following problem

$$\begin{aligned} U_t^{IL}\left(A_t^{IL}\right) &= \max_{w_t^{IL}} w_t^{IL}\left(\mathbb{E}_t^{IL}[r_{t+1}] - r_{f,t}\right) + \frac{1}{2} w_t^{IL}\left(1 - w_t^{IL}\right) \mathrm{Var}_t^{IL}(r_{t+1}) \\ &\quad + \frac{1}{2}\left(1 - \gamma^I\right)\left(w_t^{IL}\right)^2 \mathrm{Var}_t^{IL}(r_{t+1}) \\ &\text{s.t.} \quad w_t^{IL} \geq 0. \end{aligned}$$

Since the objective function is quadratic in portfolio weight w_t^{IL} and has a global maximum, the solution to this constrained problem is

$$w_t^{IL} = \max\left\{0, \tau^I \frac{\mathbb{E}_t^{IL}[r_{t+1}] - r_{f,t} + \frac{1}{2}\mathrm{Var}_t^{IL}(r_{t+1})}{\mathrm{Var}_t^{IL}(r_{t+1})}\right\}.$$

Substitute IL's beliefs (equations (14), (15), and (17)) into the above expression, we get his time-0 and time-1 demands for the risky asset

$$w_0^{IL} = \max\left\{0, \tau^I \left(\frac{\mathbb{E}_0 [p_1] + \delta_0^{IL} - p_0}{\sigma_0^2} + \frac{1}{2} \right)\right\},$$

$$w_1^{IL} = \max\left\{0, \tau^I \left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2} \right)\right\}.$$

A1.3.3 Short institution

The short institution IS solves the following problem

$$\begin{aligned} U_t^{IS}\left(A_t^{IS}\right) &= \max_{w_t^{IS}} w_t^{IS}\left(\mathbb{E}_t^{IS}\left[r_{t+1}\right]-r_{f,t}\right)+\frac{1}{2}w_t^{IS}\left(1-w_t^{IS}\right)\mathrm{Var}_t^{IS}\left(r_{t+1}\right) \\ & +\frac{1}{2}\left(1-\gamma^I\right)\left(w_t^{IS}\right)^2\mathrm{Var}_t^{IS}\left(r_{t+1}\right) \\ \text{s.t.} \quad & w_t^{IS} \geq -\frac{1}{m}. \end{aligned}$$

The solution is

$$w_t^{IS} = \max\left\{-\frac{1}{m}, \tau^I\frac{\mathbb{E}_t^{IS}\left[r_{t+1}\right]-r_{f,t}+\frac{1}{2}\mathrm{Var}_t^{IS}\left(r_{t+1}\right)}{\mathrm{Var}_t^{IS}\left(r_{t+1}\right)}\right\}.$$

Substitute IS's beliefs (equations (14), (16), and (17)) into the above expression, we get his time-0 and time-1 demands for the risky asset

$$\begin{aligned} w_0^{IS} &= \max\left\{-\frac{1}{m}, \tau^I\left(\frac{\mathbb{E}_0\left[p_1\right]+\delta_0^{IS}-p_0}{\sigma_0^2}+\frac{1}{2}\right)\right\}, \\ w_1^{IS} &= \max\left\{-\frac{1}{m}, \tau^I\left(\frac{\mu_d-p_1}{\sigma_d^2}+\frac{1}{2}\right)\right\}. \end{aligned}$$

A1.4 Proof of Lemma 1

Proof. I prove the existence result in two steps. First, I show that the aggregate demand of the $\bar{N}$ retail investors is equal to the demand of the aggregate retail investor specified in equations (38) and (39), and thus the equilibrium price can be solved from the market clearing condition (41). Then I derive the wealth share dynamics of the aggregate retail investor in equation (40).

I begin by restating the timeline and the wealth share dynamics of individual retail investors. At time $t-1$ after trading, retail investor j has dollar wealth A_t^j and wealth share α_t^j. At time t before trading, the $\bar{N}$ retail investors first split their aggregate wealth $\sum_{k=1}^{\bar{N}} A_t^k$ equally. In particular, they split their aggregate stock position and aggregate bond position equally. After that, retail investor j has wealth $\hat{A}_t^j = \frac{1}{\bar{N}}\sum_{k=1}^{\bar{N}} A_t^k$ and wealth share

$$\hat{\alpha}_t^j \equiv \frac{\hat{A}_t^j}{A_t} = \frac{\frac{1}{\bar{N}}\sum_{k=1}^{\bar{N}} A_t^k}{A_t} = \frac{1}{\bar{N}}\sum_{k=1}^{\bar{N}}\alpha_t^k. \tag{A6}$$

Retail investors then trade with each other. Specifically, retail investor j allocates his wealth

$\hat{A}_t^j$ into the risky asset and the risk-free asset. His demand for the risky asset (in terms of the number of shares) is $Q_t^j = \frac{w_t^j \hat{A}_t^j}{P_t}$, where w_t^j is his optimal portfolio weight. After trading, his end-of-period wealth share becomes

$$\alpha_{t+1}^j \quad = \quad \hat{\alpha}_t^j \left(\left(1 - w_t^j\right) \exp\left(p_t - p_{t+1}\right) + w_t^j \right). \tag{A7}$$

Next, I show that the equilibrium price of the risky asset is the same as that in an economy with three investors – an aggregate retail investor, the long institution, and the short institution. And the demand of the aggregate retail investor is the sum of the demand of the $\bar{N}$ retail investors.

At time t, market clearing for the risky asset implies that

$$\begin{aligned}
& \sum_{j=1}^{\bar{N}} Q_t^j + Q_t^{IL} + Q_t^{IS} = \bar{S} \\
\Longrightarrow \quad & \sum_{j=1}^{\bar{N}} \frac{w_t^j \hat{A}_t^j}{P_t} + \frac{w_t^{IL} A_t^{IL}}{P_t} + \frac{w_t^{IS} A_t^{IS}}{P_t} = \bar{S} \\
\Longrightarrow \quad & \sum_{j=1}^{\bar{N}} w_t^j \hat{\alpha}_t^j + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1 \\
\Longrightarrow \quad & \sum_{j=1}^{\bar{N}} w_t^j \left(\frac{1}{\bar{N}} \sum_{k=1}^{\bar{N}} \alpha_t^k \right) + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1 \\
\Longrightarrow \quad & \left(\sum_{k=1}^{\bar{N}} \alpha_t^k \right) \frac{1}{\bar{N}} \sum_{j=1}^{N_t} \tau^R \left(\frac{\mathbb{E}_t\left[p_{t+1}\right] + y_t^j - p_t}{\sigma_t^2} + \frac{1}{2} \right) \\
& + \left(\sum_{k=1}^{\bar{N}} \alpha_t^k \right) \frac{1}{\bar{N}} \sum_{j=N_t+1}^{\bar{N}} \tau^R \left(\frac{\mathbb{E}_t\left[p_{t+1}\right] - p_t}{\sigma_t^2} + \frac{1}{2} \right) \\
& + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1 \\
\Longrightarrow \quad & \left(\sum_{k=1}^{\bar{N}} \alpha_t^k \right) \tau^R \left(\frac{\mathbb{E}_t\left[p_{t+1}\right] - p_t}{\sigma_t^2} + \frac{1}{2} \right) + \left(\sum_{k=1}^{\bar{N}} \alpha_t^k \right) \theta\left(N_t\right) \frac{1}{N_t} \sum_{j=1}^{N_t} \tau^R \frac{y_t^j}{\sigma_t^2} \\
& + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1 \\
\Longrightarrow \quad & \left(\sum_{k=1}^{\bar{N}} \alpha_t^k \right) \tau^R \left(\frac{\mathbb{E}_t\left[p_{t+1}\right] + \theta\left(N_t\right) \frac{1}{N_t} \sum_{j=1}^{N_t} y_t^j - p_t}{\sigma_t^2} + \frac{1}{2} \right) + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1,
\end{aligned}$$

where the fourth line uses the definition of $\hat{\alpha}_t^j$ in equation (A6), and the fifth line uses the optimal portfolio weights of retail investors in equations (28)-(31).

Define

$$A_t^R \equiv \sum_{j=1}^{\bar{N}} A_t^j, \alpha_t^R \equiv \sum_{j=1}^{\bar{N}} \alpha_t^j, \tag{A8}$$

$$\delta_t^R \equiv \theta\left(N_t\right) \frac{1}{N_t} \sum_{j=1}^{N_t} y_t^j, \tag{A9}$$

$$w_t^R \equiv \tau^R \left(\frac{\mathbb{E}_t\left[p_{t+1}\right] + \delta_t^R - p_t}{\sigma_t^2} + \frac{1}{2} \right) = \frac{1}{\bar{N}} \sum_{j=1}^{\bar{N}} w_t^j. \tag{A10}$$

Then the market clearing condition can be written as

$$w_t^R \alpha_t^R + w_t^{IL} \alpha_t^{IL} + w_t^{IS} \alpha_t^{IS} = 1,$$

with $\alpha_t^R + \alpha_t^{IL} + \alpha_t^{IS} = \sum_{j=1}^{\bar{N}} \alpha_t^j + \alpha_t^{IL} + \alpha_t^{IS} = 1$.

Hence, the equilibrium price of the risky asset is the same as that in an economy with three investors – an aggregate retail investor R, the long institution IL, and the short institution IS, where the three investors have demand $\left(w_t^R, w_t^{IL}, w_t^{IS}\right)$, and wealth shares $\left(\alpha_t^R, \alpha_t^{IL}, \alpha_t^{IS}\right)$. In other words, there exists an aggregate retail investor whose demand for the risky asset is given by equation (A10). The aggregate retail investor has constant relative risk tolerance $\tau^R = \frac{1}{\gamma^R}$ and subjective beliefs

$$\begin{aligned} \mathbb{E}_0^R\left[p_1\right] &= \mathbb{E}_0\left[p_1\right] + \delta_0^R, \mathrm{Var}_0^R\left(p_1\right) = \sigma_0^2, \\ \mathbb{E}_1^R\left[\tilde{d}\right] &= \mu_d + \delta_1^R, \mathrm{Var}_1^R\left(\tilde{d}\right) = \sigma_d^2. \end{aligned}$$

Finally, I derive the wealth share dynamics of the aggregate retail investor. From the definition of α_{t+1}^R in (A8),

$$\begin{aligned}
\alpha_{t+1}^R &\equiv \sum_{j=1}^{\bar{N}} \alpha_{t+1}^j \\
&= \sum_{j=1}^{\bar{N}} \hat{\alpha}_t^j \left(\left(1-w_t^j\right) \exp\left(p_t - p_{t+1}\right) + w_t^j\right) \\
&= \left(\frac{1}{\bar{N}} \sum_{k=1}^{\bar{N}} \alpha_t^k\right) \sum_{j=1}^{\bar{N}} \left(\left(1-w_t^j\right) \exp\left(p_t - p_{t+1}\right) + w_t^j\right) \\
&= \alpha_t^R \left(\left(1 - \frac{1}{\bar{N}} \sum_{j=1}^{\bar{N}} w_t^j\right) \exp\left(p_t - p_{t+1}\right) + \frac{1}{\bar{N}} \sum_{j=1}^{\bar{N}} w_t^j\right) \\
&= \alpha_t^R \left(\left(1-w_t^R\right) \exp\left(p_t - p_{t+1}\right) + w_t^R\right),
\end{aligned}$$

where the second line uses investor j's wealth share dynamics in equation (A7), and the last line uses the aggregate retail investor's demand in equation (A10). □

A1.5 Proof of Proposition 1

Proof. I focus on monotone equilibrium of Definition 1, with sentiment cutoffs δ_1^m, δ_1^h satisfying $\underline{\delta}_1 < \delta_1^m < \delta_1^h < \bar{\delta}_1$. Hence, $\forall \delta_1^R \in [\underline{\delta}_1, \delta_1^m)$, the equilibrium price $p_1\left(\delta_1^R\right) < p_1^m$. Similarly, $\forall \delta_1^R \in \left[\delta_1^m, \delta_1^h\right)$, the price $p_1\left(\delta_1^R\right) \in \left[p_1^m, p_1^h\right)$. And $\forall \delta_1^R \in \left[\delta_1^h, \bar{\delta}_1\right]$, the price $p_1\left(\delta_1^R\right) \geq p_1^h$.

Next, I solve the equilibrium price from the market clearing condition in equation (41).

- For $\delta_1^R \in [\underline{\delta}_1, \delta_1^m)$, I look for an equilibrium price $p_1 < p_1^m$. Substitute the optimal portfolio choices of the three investors, (39), (33), and (35) into the market clearing condition (41), I get

$$\begin{aligned}
&\frac{\alpha_1^R\left(p_1\right)\tau^R}{\sigma_d^2}\delta_1^R + \sum_i \alpha_1^i\left(p_1\right)\tau^i\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) = 1 \\
\Longrightarrow \quad & p_1 = \mu_d + \left(\frac{\frac{\alpha_1^R(p_1)\tau^R}{\sigma_d^2}\delta_1^R - 1}{\sum_i \alpha_1^i\left(p_1\right)\tau^i} + \frac{1}{2}\right)\sigma_d^2 \\
\Longrightarrow \quad & p_1 = \mu_d + \left(\frac{1}{2} + \frac{\frac{\alpha_1^R(p_1)\tau^R}{\sigma_d^2}\delta_1^R - 1}{\tau_1\left(p_1\right)}\right)\sigma_d^2
\end{aligned}$$

where

$$\tau_1(p_1) \equiv \sum_i \alpha_1^i(p_1)\tau^i = \alpha_1^R(p_1)\tau^R + \left(1-\alpha_1^R(p_1)\right)\tau^I$$

Define the function

$$J\left(p_1, \delta_1^R\right) \equiv \mu_d + \left(\frac{1}{2}\sigma_d^2 + \frac{\alpha_1^R(p_1)\tau^R\delta_1^R - \sigma_d^2}{\tau_1(p_1)}\right) - p_1$$

Then the equilibrium price p_1 solves $J\left(p_1, \delta_1^R\right) = 0$.

The cutoff sentiment shock δ_1^m solves $J\left(p_1^m, \delta_1^m\right) = 0$, which yields

$$\delta_1^m = \frac{\left(p_1^m - \mu_d - \frac{1}{2}\sigma_d^2\right)\tau_1(p_1^m) + \sigma_d^2}{\alpha_1^R(p_1^m)\tau^R} = \frac{\sigma_d^2}{\alpha_1^R(p_1^m)\tau^R}$$

- For $\delta_1^R \in \left[\delta_1^m, \delta_1^h\right)$, I look for an equilibrium price $p_1 \in \left[p_1^m, p_1^h\right)$. Substitute the optimal portfolio choices of the three investors, (39), (33), and (35) into the market clearing condition (41), I get

$$\begin{aligned}
& \alpha_1^R(p_1)\tau^R\left(\frac{\mu_d+\delta_1^R-p_1}{\sigma_d^2}+\frac{1}{2}\right) + \alpha_1^{IS}(p_1)\tau^I\left(\frac{\mu_d-p_1}{\sigma_d^2}+\frac{1}{2}\right) = 1 \\
\Longrightarrow\quad & \frac{\alpha_1^R(p_1)\tau^R}{\sigma_d^2}\delta_1^R + \left(\alpha_1^R(p_1)\tau^R + \alpha_1^{IS}(p_1)\tau^I\right)\left(\frac{\mu_d-p_1}{\sigma_d^2}+\frac{1}{2}\right) = 1 \\
\Longrightarrow\quad & p_1 = \mu_d + \left(\frac{1}{2} + \frac{\frac{1}{\sigma_d^2}\alpha_1^R(p_1)\tau^R\delta_1^R - 1}{\alpha_1^R(p_1)\tau^R + \alpha_1^{IS}(p_1)\tau^I}\right)\sigma_d^2 \\
\Longrightarrow\quad & p_1 = \mu_d + \left(\frac{1}{2} + \frac{\frac{1}{\sigma_d^2}\alpha_1^R(p_1)\tau^R\delta_1^R - 1}{\hat{\tau}_1(p_1)}\right)\sigma_d^2
\end{aligned}$$

where

$$\hat{\tau}_1(p_1) \equiv \alpha_1^R(p_1)\tau^R + \alpha_1^{IS}(p_1)\tau^I$$

Define the function

$$H\left(p_1, \delta_1^R\right) \equiv \mu_d + \left(\frac{1}{2}\sigma_d^2 + \frac{\alpha_1^R(p_1)\tau^R\delta_1^R - \sigma_d^2}{\hat{\tau}_1(p_1)}\right) - p_1$$

Then the equilibrium price p_1 solves $H\left(p_1, \delta_1^R\right) = 0$.

The cutoff sentiment shock δ_1^h solves $H\left(p_1^h, \delta_1^h\right) = 0$, which yields

$$\delta_1^h = \frac{\left(p_1^h - \mu_d - \frac{1}{2}\sigma_d^2\right)\hat{\tau}_1\left(p_1^h\right) + \sigma_d^2}{\alpha_1^R\left(p_1^h\right)\tau^R} = \frac{\frac{1}{m\tau^I}\hat{\tau}_1\left(p_1^h\right) + 1}{\alpha_1^R\left(p_1^h\right)\tau^R}\sigma_d^2$$

- For $\delta_1^R \in \left[\delta_1^h, \bar{\delta}_1\right]$, I look for an equilibrium price $p_1 \geq p_1^h$. Substitute the optimal portfolio choices of the three investors, (39), (33), and (35) into the market clearing condition (41), I get

$$\alpha_1^R\left(p_1\right)\tau^R\left(\frac{\mu_d + \delta_1^R - p_1}{\sigma_d^2} + \frac{1}{2}\right) - \alpha_1^{IS}\left(p_1\right)\frac{1}{m} = 1$$
$$\Longrightarrow \quad p_1 = \mu_d + \delta_1^R + \left(\frac{1}{2} - \frac{1 + \alpha_1^{IS}\left(p_1\right)\frac{1}{m}}{\alpha_1^R\left(p_1\right)\tau^R}\right)\sigma_d^2$$

Define the function

$$G\left(p_1, \delta_1^R\right) \;=\; \mu_d + \delta_1^R + \left(\frac{1}{2} - \frac{1 + \alpha_1^{IS}\left(p_1\right)\frac{1}{m}}{\alpha_1^R\left(p_1\right)\tau^R}\right)\sigma_d^2 - p_1$$

Then the equilibrium price p_1 solves $G\left(p_1, \delta_1^R\right) = 0$.

$\square$

A1.6 Lemma A1 and proof

Lemma A1 (Properties of the implicit function $G\left(p_1, \delta_1^R\right)$). *Consider a monotone equilibrium of Definition 1, where the time-0 portfolios satisfy $w_0^R > 1$, $w_0^{IS} < 0$, $w_0^R > w_0^{IL} > w_0^{IS}$, and investors always have strictly positive wealth $\forall\delta_1 \in \left(\underline{\delta}_1, \bar{\delta}_1\right)$. Let p_1^R denote the price at which the retail investor's time-1 wealth is zero,*

$$p_1^R \;\equiv\; p_0 + \log\left(1 - \frac{1}{w_0^R}\right)$$

Then the implicit function $G\left(p_1, \delta_1^R\right)$ has the following properties on $p_1 \in \left(p_1^R, +\infty\right)$:

1. *$G\left(p_1, \delta_1^R\right)$ is continuous and strictly increasing in δ_1^R: $\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial\delta_1} = 1 > 0$.*
2. *$G\left(p_1, \delta_1^R\right)$ is continuous and strictly concave in p_1: $\frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1^2} < 0$.*
3. *$\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}$ does not depend on δ_1^R: $\frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1 \partial\delta_1^R} = 0$.*
4. *$G\left(p_1, \delta_1^R\right)$, as a function of p_1, has at most two distinct roots on $p_1 \in \left(p_1^R, +\infty\right)$.*

Proof. First, I derive p_1^R from

$$\begin{aligned} & \alpha_1^R\left(p_1^R\right)=0 \\ \Longrightarrow\quad & 0=\alpha_0^R\left(\left(1-w_0^R\right)\exp\left(p_0-p_1^R\right)+w_0^R\right) \\ \Longrightarrow\quad & p_1^R=p_0+\log\left(1-\frac{1}{w_0^R}\right) \end{aligned}$$

Then $\forall p_1>p_1^R$, $\alpha_1\left(p_1\right)>0$. And thus $G\left(p_1,\delta_1^R\right)$ is continuous and twice differentiable, $\forall p_1>p_1^R$, $\forall\delta_1^R$.

To show Properties 1-3, compute the following derivatives

$$\begin{aligned} \frac{\partial G\left(p_1,\delta_1^R\right)}{\partial\delta_1^R} &= 1 \\ \frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1} &= -\left(\alpha_1^R\left(p_1\right)\tau^R\right)^{-2} \\ &\quad\cdot\left(\frac{d\alpha_1^{IS}\left(p_1\right)}{dp_1}\frac{1}{m}\alpha_1^R\left(p_1\right)\tau^R-\frac{d\alpha_1^R\left(p_1\right)}{dp_1}\tau^R\left(1+\alpha_1^{IS}\left(p_1\right)\frac{1}{m}\right)\right)\sigma_d^2-1 \\ &= \left(\alpha_1^R\left(p_1\right)\tau^R\right)^{-2}\exp\left(p_0-p_1\right) \\ &\quad\cdot\tau^R\left(\alpha_0^{IS}\left(1-w_0^{IS}\right)\frac{1}{m}\alpha_1^R\left(p_1\right)-\alpha_0^R\left(1-w_0^R\right)\left(1+\alpha_1^{IS}\left(p_1\right)\frac{1}{m}\right)\right)\sigma_d^2 \\ &\quad-1 \\ &= \left(\alpha_1^R\left(p_1\right)\tau^R\right)^{-2}\exp\left(p_0-p_1\right) \\ &\quad\cdot\alpha_0^R\tau^R\left(w_0^R-1+\frac{1}{m}\alpha_0^{IS}\left(w_0^R-w_0^{IS}\right)\right)\sigma_d^2-1 \\ \frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1\partial\delta_1^R} &= 0 \\ \frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1^2} &= -\left(\alpha_1^R\left(p_1\right)\tau^R\right)^{-2}\sigma_d^2 \\ &\quad\cdot\left(\frac{d\alpha_1^R\left(p_1\right)}{dp_1}\tau^R\left(1+\alpha_1^{IS}\left(p_1\right)\frac{1}{m}\right)-\frac{d\alpha_1^{IS}\left(p_1\right)}{dp_1}\frac{1}{m}\alpha_1^R\left(p_1\right)\tau^R\right) \\ &\quad\cdot\left(1+\frac{2}{\alpha_1^R\left(p_1\right)}\frac{d\alpha_1^R\left(p_1\right)}{dp_1}\right) \end{aligned}$$

From the wealth share dynamics, we get

$$\begin{aligned} & \alpha_{t+1}^i\left(p_{t+1}\right)=\alpha_t^i\left(\left(1-w_t^i\right)\left(p_t-p_{t+1}\right)+w_t^i\right) \\ \Longrightarrow\quad & \frac{d\alpha_{t+1}^i\left(p_{t+1}\right)}{dp_{t+1}}=-\alpha_t^i\left(1-w_t^i\right)\exp\left(p_t-p_{t+1}\right) \end{aligned}$$

Since $w_0^R > 1$ and $w_0^{IS} < 0$, we have

$$\frac{d\alpha_1^R(p_1)}{dp_1} > 0, \frac{d\alpha_1^{IS}(p_1)}{dp_1} < 0$$

Hence, $\frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1^2} < 0$, i.e. $G\left(p_1, \delta_1^R\right)$ is strictly concave in p_1, $\forall p_1 \in \left(\delta_1^R, +\infty\right)$.

Next, I show property 4. For a given δ_1^R, suppose $G\left(p_1, \delta_1^R\right)$ has more than two roots. Let x_1, x_2, x_3 denote three of the roots, with $x_1 < x_2 < x_3$. Then $\exists \lambda \in (0,1)$, such that $x_2 = \lambda x_1 + (1-\lambda) x_3$. Since $G\left(p_1, \delta_1^R\right)$ is continuous and strictly concave in p_1,

$$0 = \lambda G\left(x_1, \delta_1^R\right) + (1-\lambda) G\left(x_3, \delta_1^R\right) = G\left(\lambda x_1 + (1-\lambda) x_3, \delta_1^R\right) < G\left(x_2, \delta_1^R\right) = 0$$

A contradiction. Hence, $\forall p_1 \in \left(p_1^R, +\infty\right)$, $G\left(p_1, \delta_1^R\right)$ (as a function of p_1) has at most two distinct roots. □

A1.7 Proof of Proposition 2

Proof. I first show that $\forall \delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$, $G\left(p_1, \delta_1^R\right) = 0$ has exactly one root that satisfies $p_1 > p_1^h$. Suppose otherwise, then from Lemma A1, there are two roots x_1 and x_2 which satisfy $p_1^h < x_1 < x_2$, and $G\left(x_1, \delta_1^R\right) = G\left(x_2, \delta_1^R\right) = 0$. Since $G\left(p_1^h, \delta_1^h\right) = 0$ and $\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial \delta_1^R} = 1 > 0$, then $G\left(p_1^h, \delta_1^R\right) > G\left(p_1^h, \delta_1^h\right) = 0$, $\forall \delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$. $p_1^h < x_1 < x_2 \rightarrow \exists \lambda \in (0,1)$ such that $x_1 = \lambda p_1^h + (1-\lambda) x_2$. And since $G\left(p_1, \delta_1^R\right)$ is strictly concave in p_1, we have

$$0 < \lambda G\left(p_1^h, \delta_1^R\right) + (1-\lambda) G\left(x_2, \delta_1^R\right) < G\left(\lambda p_1^h + (1-\lambda) x_2, \delta_1^R\right) = G\left(x_1, \delta_1^R\right) = 0$$

A contradiction. Hence, $\forall \delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$, $G\left(p_1, \delta_1^R\right)$ has exactly one root that satisfies $p_1 > p_1^h$. In a monotone equilibrium of Definition 1, this is the unique equilibrium price in the high sentiment region $\delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$.

Next, I derive conditions for discontinuity in price. Consider the following two cases:

- Case 1: $\left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1 = p_1^h} \leq 0$.

  From the strict concavity of $G\left(p_1, \delta_1^R\right)$ in Lemma A1, $\forall p_1 > p_1^h$, $\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1} < \left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1 = p_1^h} \leq 0$. This implies that $G\left(p_1, \delta_1^h\right) < G\left(p_1^h, \delta_1^h\right) = 0, \forall p_1 > p_1^h$. Hence, p_1^h is the largest root of $G\left(p_1, \delta_1^h\right) = 0$.

From Lemma A1, $\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1 \partial \delta_1^R}=0$ and $\frac{\partial^2 G\left(p_1,\delta_1^R\right)}{\partial p_1^2}<0$. Then

$$
\begin{aligned}
& \left.\frac{\partial G\left(p_1, \delta_1^h\right)}{\partial p_1}\right|_{p_1=p_1^h} \leq 0 \\
\Longrightarrow \quad & \left.\frac{\partial G\left(p_1, \delta_1^R\right)}{\partial p_1}\right|_{p_1=p_1^h} \leq 0, \forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right] \\
\Longrightarrow \quad & \frac{\partial G\left(p_1, \delta_1^R\right)}{\partial p_1}<0, \forall p_1>p_1^h, \forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]
\end{aligned}
$$

Moreover, if $\left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1=p_1^h}=0$, then $\left.\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\right|_{p_1=p_1^h}=0$, $\forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]$. Otherwise, $\left.\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\right|_{p_1=p_1^h}<0$, $\forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]$.

Using the implicit function theorem, $\forall p_1>p_1^h$, $\forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]$,

$$
\begin{aligned}
& \frac{\partial G\left(p_1, \delta_1^R\right)}{\partial p_1} \frac{d p_1\left(\delta_1^R\right)}{d \delta_1^R}+\frac{\partial G\left(p_1, \delta_1^R\right)}{\partial \delta_1^R}=0 \\
\Longrightarrow \quad & \frac{\partial G\left(p_1, \delta_1^R\right)}{\partial p_1} \frac{d p_1\left(\delta_1^R\right)}{d \delta_1^R}+1=0 \\
\Longrightarrow \quad & \frac{d p_1\left(\delta_1^R\right)}{d \delta_1^R}=-\frac{1}{\frac{\partial G\left(p_1, \delta_1^R\right)}{\partial p_1}}>0
\end{aligned}
$$

Hence, $\forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]$, the equilibrium price $p_1\left(\delta_1^R\right)$ is strictly increasing in δ_1^R. Furthermore, $p_1\left(\delta_1^R\right)$ is continuous in δ_1^R on $\delta_1^R \in\left(\delta_1^h, \bar{\delta}_1\right]$, and is right-continuous at $\delta_1^R=\delta_1^h$.

- Case 2: $\left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1=p_1^h}>0$.

  First, I prove that $\forall \delta_1^R \in\left[\delta_1^h, \bar{\delta}_1\right]$, $G\left(p_1, \delta_1^R\right)=0$ has two distinct roots, denoted as $x_1\left(\delta_1^R\right)$ and $x_2\left(\delta_1^R\right)$, with $x_1\left(\delta_1^R\right) \leq p_1^h<x_2\left(\delta_1^R\right)$. And $x_1\left(\delta_1^R\right)=p_1^h$ if and only if $\delta_1^R=\delta_1^h$.

  - $\forall \delta_1^R \in\left(\delta_1^h, \bar{\delta}_1\right]$, we have $G\left(p_1^h, \delta_1^R\right)>G\left(p_1^h, \delta_1^h\right)=0$, and $G\left(+\infty, \delta_1^R\right)=-\infty$. Let p_1^R denote the price at which the retail investor's time-1 wealth share is exactly zero, then p_1^R satisfies

$$\begin{aligned}
&\alpha_1^R\left(p_1^R\right)=0\\
\Longrightarrow\quad & 0=\alpha_0^R\left(\left(1-w_0^R\right)\exp\left(p_0-p_1^R\right)+w_0^R\right)\\
\Longrightarrow\quad & p_1^R=p_0+\log\left(1-\frac{1}{w_0^R}\right)
\end{aligned}$$

And we have $G\left(p_1^R,\delta_1^R\right)=-\infty$. Then $G\left(p_1^R,\delta_1^R\right)=G\left(+\infty,\delta_1^R\right)=-\infty<0<G\left(p_1^h,\delta_1^R\right)$. By the intermediate value theorem, $G\left(p_1,\delta_1^R\right)=0$ has two distinct roots $x_1\left(\delta_1^R\right),x_2\left(\delta_1^R\right)$ such that $p_1^R<x_1\left(\delta_1^R\right)<p_1^h<x_2\left(\delta_1^R\right)$, $\forall\delta_1^R\in\left(\delta_1^h,\bar{\delta}_1\right]$. In a monotone equilibrium of Definition 1, $x_2\left(\delta_1^R\right)$ is the unique equilibrium price.

Next, I show that $\forall\delta_1^R\in\left(\delta_1^h,\bar{\delta}_1\right]$, $\left.\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\right|_{p_1=x_2\left(\delta_1^R\right)}<0$. Suppose otherwise, then $\left.\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\right|_{p_1=x_2\left(\delta_1^R\right)}\geq 0\implies\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}>0,\forall p_1<x_2\left(\delta_1^R\right)$. This implies $0=G\left(p_1^h,\delta_1^h\right)<G\left(p_1^h,\delta_1^R\right)<G\left(x_2\left(\delta_1^R\right),\delta_1^R\right)=0$, a contradiction.

- At the cutoff $\delta_1^R=\delta_1^h$, $\left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1=p_1^h}>0$ implies that, $\exists\varepsilon>0$ and small, $G\left(p_1^h+\varepsilon,\delta_1^h\right)>G\left(p_1^h,\delta_1^h\right)=0$. Together with $G\left(+\infty,\delta_1^h\right)=-\infty<0$, this implies that $G\left(p_1,\delta_1^h\right)$ has two distinct roots $x_1\left(\delta_1^h\right),x_2\left(\delta_1^h\right)$ such that $x_1\left(\delta_1^h\right)=p_1^h<x_2\left(\delta_1^h\right)$.

  Next, I show that $\left.\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}\right|_{p_1=x_2\left(\delta_1^h\right)}<0$. Suppose otherwise, then $\left.\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\right|_{p_1=x_2\left(\delta_1^h\right)}\geq 0\implies\frac{\partial G\left(p_1,\delta_1^h\right)}{\partial p_1}>0,\forall p_1<x_2\left(\delta_1^h\right)$. This implies $0=G\left(p_1^h,\delta_1^h\right)<G\left(x_2\left(\delta_1^h\right),\delta_1^h\right)=0$, a contradiction.

In a monotone equilibrium of Definition 1, $\forall\delta_1^R\in\left(\delta_1^h,\bar{\delta}_1\right]$, the equilibrium price has to be greater than p_1^h. Hence, $x_2\left(\delta_1^R\right)$ is the unique equilibrium price on $\delta_1^R\in\left(\delta_1^h,\bar{\delta}_1\right]$. And since $p_1^h<x_2\left(\delta_1^h\right)$, the pricing function $p_1\left(\delta_1^R\right)$ is discontinuous at $\delta_1^R=\delta_1^h$.

Using the implicit function theorem, $\forall p_1>x_2\left(\delta_1^h\right)$, $\forall\delta_1^R\in\left[\delta_1^h,\bar{\delta}_1\right]$,

$$\begin{aligned}
&\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R}+\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial\delta_1^R}=0\\
\Longrightarrow\quad&\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R}+1=0\\
\Longrightarrow\quad&\frac{dp_1\left(\delta_1^R\right)}{d\delta_1^R}=-\frac{1}{\frac{\partial G\left(p_1,\delta_1^R\right)}{\partial p_1}}>0
\end{aligned}$$

Hence, $\forall \delta_1^R \in \left[\delta_1^h, \bar{\delta}_1\right]$, the equilibrium price $p_1\left(\delta_1^R\right)$ is strictly increasing in δ_1^R. Furthermore, $p_1\left(\delta_1^R\right)$ is continuous in δ_1^R on $\delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$, and is discontinuous at $\delta_1^R = \delta_1^h$.

□

A1.8 Proof of Proposition 3

Proof.

- Low sentiment $\delta_1^R \in [\underline{\delta}_1, \delta_1^m)$: from the optimal portfolio choices of the three investors, (39), (33), (35), and the market clearing condition (41), we get

$$
\begin{aligned}
& \frac{\alpha_1^R\left(p_1\right)\tau^R}{\sigma_d^2}\delta_1^R + \sum_i \alpha_1^i\left(p_1\right)\tau^i\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) = 1 \\
\Longrightarrow \quad & \frac{\alpha_1^R\left(p_1\right)\tau^R}{\sigma_d^2}\delta_1^R + \tau_1\left(p_1\right)\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) = 1 \\
\Longrightarrow \quad & \alpha_1^R\left(p_1\right)\tau^R\delta_1^R + \tau_1\left(p_1\right)\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) = \sigma_d^2
\end{aligned}
$$

  Using the implicit function theorem,

$$
\alpha_1^R\left(p_1\right)\tau^R + \frac{d\left(\alpha_1^R\left(p_1\right)\tau^R\delta_1^R\right)}{dp_1}\frac{dp_1}{d\delta_1^R} + \frac{d\tau_1\left(p_1\right)}{dp_1}\frac{dp_1}{d\delta_1^R}\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) - \tau_1\left(p_1\right)\frac{dp_1}{d\delta_1^R} = 0
$$

$$
\Longrightarrow \quad \frac{dp_1}{d\delta_1^R} = \frac{\frac{\alpha_1^R(p_1)\tau^R}{\tau_1(p_1)}}{1 - \frac{1}{\tau_1(p_1)}\left(\frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\delta_1^R + \frac{d\tau_1(p_1)}{dp_1}\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right)\right)}
$$

- Medium sentiment $\delta_1^R \in \left(\delta_1^m, \delta_1^h\right)$: from the optimal portfolio choices of the three investors, (39), (33), (35), and the market clearing condition (41), we get

$$
\begin{aligned}
& \frac{\alpha_1^R\left(p_1\right)\tau^R}{\sigma_d^2}\delta_1^R + \left(\alpha_1^R\left(p_1\right)\tau^R + \alpha_1^{IS}\left(p_1\right)\tau^I\right)\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) = 1 \\
\Longrightarrow \quad & \frac{\alpha_1^R\left(p_1\right)\tau^R}{\sigma_d^2}\delta_1^R + \hat{\tau}_1\left(p_1\right)\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) = 1 \\
\Longrightarrow \quad & \alpha_1^R\left(p_1\right)\tau^R\delta_1^R + \hat{\tau}_1\left(p_1\right)\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) = \sigma_d^2
\end{aligned}
$$

Using the implicit function theorem,

$$\alpha_1^R(p_1)\tau^R + \frac{d\left(\alpha_1^R(p_1)\tau^R\delta_1^R\right)}{dp_1}\frac{dp_1}{d\delta_1^R} + \frac{d\hat{\tau}_1(p_1)}{dp_1}\frac{dp_1}{d\delta_1^R}\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) - \hat{\tau}_1(p_1)\frac{dp_1}{d\delta_1^R} = 0$$

$$\implies \frac{dp_1}{d\delta_1^R} = \frac{\frac{\alpha_1^R(p_1)\tau^R}{\hat{\tau}_1(p_1)}}{1 - \frac{1}{\hat{\tau}_1(p_1)}\left(\frac{d\left(\alpha_1^R(p_1)\right)}{dp_1}\tau^R\delta_1^R + \frac{d\hat{\tau}_1(p_1)}{dp_1}\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right)\right)}$$

- High sentiment $\delta_1^R \in \left(\delta_1^h, \bar{\delta}_1\right]$: from the optimal portfolio choices of the three investors, (39), (33), (35), and the market clearing condition (41), we get

$$\begin{aligned}
&\alpha_1^R(p_1)\tau^R\left(\frac{\mu_d + \delta_1^R - p_1}{\sigma_d^2} + \frac{1}{2}\right) - \alpha_1^{IS}(p_1)\frac{1}{m} = 1 \\
\implies\ &\frac{\alpha_1^R(p_1)\tau^R}{\sigma_d^2}\delta_1^R + \alpha_1^R(p_1)\tau^R\left(\frac{\mu_d - p_1}{\sigma_d^2} + \frac{1}{2}\right) - \alpha_1^{IS}(p_1)\frac{1}{m} = 1 \\
\implies\ &\alpha_1^R(p_1)\tau^R\delta_1^R + \alpha_1^R(p_1)\tau^R\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) - \alpha_1^{IS}(p_1)\frac{1}{m}\sigma_d^2 = \sigma_d^2
\end{aligned}$$

Using the implicit function theorem,

$$\begin{aligned}
&\alpha_1^R(p_1)\tau^R + \frac{d\left(\alpha_1^R(p_1)\tau^R\delta_1^R\right)}{dp_1}\frac{dp_1}{d\delta_1^R} + \frac{d\alpha_1^R(p_1)}{dp_1}\frac{dp_1}{d\delta_1^R}\tau^R\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) - \alpha_1^R(p_1)\tau^R\frac{dp_1}{d\delta_1^R} \\
&-\frac{d\alpha_1^{IS}(p_1)}{dp_1}\frac{dp_1}{d\delta_1^R}\frac{1}{m}\sigma_d^2 = 0 \\
\implies\ &\frac{dp_1}{d\delta_1^R} = \frac{1}{1 - \frac{1}{\alpha_1^R(p_1)\tau^R}\left(\frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\delta_1^R + \frac{d\alpha_1^R(p_1)}{dp_1}\tau^R\left(\mu_d + \frac{1}{2}\sigma_d^2 - p_1\right) - \frac{d\alpha_1^{IS}(p_1)}{dp_1}\frac{1}{m}\sigma_d^2\right)}
\end{aligned}$$

□

A1.9 Proof of Proposition 4

Proof. To derive the time-0 equilibrium price, substitute the optimal portfolio choices of the three investors, (38), (32), and (34) into the market clearing condition (41),

$$\begin{aligned}
&\left(\alpha_0^R(p_0)\tau^R + \left(1 - \alpha_0^R(p_0)\right)\tau^I\right)\left(\frac{\mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right] - p_0}{\sigma_0^2} + \frac{1}{2}\right) + \sum_i \frac{\alpha_0^i(p_0)\tau^i\delta_0^i}{\sigma_0^2} = 1 \\
\implies\ &\tau_0(p_0)\left(\mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right] - p_0 + \frac{1}{2}\sigma_0^2\right) + \sum_i \alpha_0^i(p_0)\tau^i\delta_0^i = \sigma_0^2 \\
\implies\ &p_0 = \mathbb{E}_0\left[p_1\left(\delta_1^R\right)\right] + \left(\frac{1}{2}\sigma_0^2 + \frac{\sum_i \alpha_0^i(p_0)\tau^i\delta_0^i - \sigma_0^2}{\tau_0(p_0)}\right)
\end{aligned}$$

where

$$\tau_0(p_0) \equiv \sum_i \alpha_0^i(p_0)\,\tau^i = \alpha_0^R(p_0)\,\tau^R + \left(1-\alpha_0^R(p_0)\right)\tau^I$$

The rest of the proof follows Proposition 1. $\square$

A1.10 Implicit price at time −1

I assume that, at time -1, investors do not anticipate future sentiment shocks. They believe that the prices at time 0 and 1 will reflect the present value of the terminal dividend, and the prices are deterministic. Hence, from time -1 to 0 and from time 0 to 1, the risky asset should have the same one-period return as the risky-free asset. This implies that $p_{-1} = \tilde{p}_0 = \tilde{p}_1$, where $\tilde{p}_0$ and $\tilde{p}_1$ denote investors' beliefs about time-0 and time-1 prices, respectively.

The implicit price p_{-1} is such that investors do not want to trade at time -1. Since $p_{-1} = \tilde{p}_0 = \tilde{p}_1$, investors believe that they will not have incentives to trade at time 0 and 1, and thus they believe their asset positions and wealth shares remain constant from time -1 to time 1. In this case, the aggregate risk tolerance remains constant from time -1 to time 1, and is equal to

$$\tau_{-1} = \alpha_{-1}^R\tau^R + \left(1-\alpha_{-1}^R\right)\tau^I.$$

Impose the market clearing condition in equation (41), we can solve for the implicit price

$$p_{-1} = \tilde{p}_0 = \tilde{p}_1 = \mu_d + \left(\frac{1}{2} - \frac{1}{\tau_{-1}}\right)\sigma_d^2.$$

Note that at time -1, investors do not want to trade, because they believe that the risky asset has the same return as the risk-free asset.

A1.11 Proof of Lemma 2

Proof. I first compute the m-th moment of $d_{j,t}^{in}$ in the cross section of retail investors, using the PDF specified in equation (55) with support $[d_{\min}, d_{\max}(N_t)]$.

$$\begin{aligned}
\mathbb{E}^{CS}\left[\left(d_{j,t}^{in}\right)^m\right] &= \int_{d_{\min}}^{d_{\max}(N_t)} x^m \frac{\xi-1}{d_{\min}}\left(\frac{x}{d_{\min}}\right)^{-\xi} dx \\
&= \frac{\xi-1}{d_{\min}^{1-\xi}} \int_{d_{\min}}^{d_{\max}(N_t)} x^{m-\xi} dx \\
&= \frac{\xi-1}{d_{\min}^{1-\xi}} \frac{1}{m+1-\xi} x^{m+1-\xi}\Big|_{d_{\min}}^{d_{\max}(N_t)} \\
&= \frac{\xi-1}{\xi-m-1} \frac{1}{d_{\min}^{1-\xi}} \left(d_{\min}^{m+1-\xi} - \left(d_{\max}(N_t)\right)^{m+1-\xi}\right).
\end{aligned}$$

The cross-sectional variance of $d_{j,t}^{in}$ is thus

$$\begin{aligned}
\mathrm{Var}^{CS}\left(d_{j,t}^{in}\right) &= \mathbb{E}\left[\left(d_{j,t}^{in}\right)^2\right] - \left(\mathbb{E}\left[d_{j,t}^{in}\right]\right)^2 \\
&= \frac{\xi-1}{3-\xi} \frac{1}{d_{\min}^{1-\xi}} \left(\left(d_{\max}(N_t)\right)^{3-\xi} - d_{\min}^{3-\xi}\right) - \left(\frac{\xi-1}{\xi-2}\right)^2 \frac{1}{d_{\min}^{2-2\xi}} \left(d_{\min}^{2-\xi} - \left(d_{\max}(N_t)\right)^{2-\xi}\right)^2.
\end{aligned}$$

□

A1.12 Proof of Proposition 5

Proof. The proof follows from Acemoglu et al. (2012).

Using the PDF of $d_{j,t}^{in}$ in equation (55), I first derive the counter-CDF

$$P_{N_t}(x) \equiv \Pr\left(d_{j,t}^{in} > x\right) = \int_x^{+\infty} \frac{\xi-1}{d_{\min}}\left(\frac{y}{d_{\min}}\right)^{-\xi} dy = \left(\frac{x}{d_{\min}}\right)^{1-\xi} \tag{A11}$$

Define the empirical counterpart as

$$\hat{P}_{N_t}(x) = \frac{1}{N_t}\left|\left\{j \in \mathcal{I}_{N_t} : d_{j,t}^{in} > x\right\}\right| = \frac{1}{N_t}\sum_{j=1}^{N_t} \mathbf{1}\left\{d_{j,t}^{in} > x\right\}$$

Let $\boldsymbol{B}_t = \{b_{1,t}, b_{2,t}, \cdots, b_{m_t,t}\}$ denote the set of values $d_{j,t}^{in}$ takes, with $b_{1,t} < b_{2,t} < \cdots < b_{m_t,t}$, and the convention that $b_{0,t} = 0$. Then

$$
\begin{aligned}
&\sum_{j=1}^{N_t} \left(d_{j,t}^{in}\right)^2 = N_t \sum_{k=1}^{m_t} \left(b_{k,t}\right)^2 \left(\hat{P}_{N_t}\left(b_{k-1,t}\right) - \hat{P}_{N_t}\left(b_{k,t}\right)\right) \\
= \;& N_t \left(b_{1,t}^2 \left(\hat{P}_{N_t}\left(b_{0,t}\right) - \hat{P}_{N_t}\left(b_{1,t}\right)\right) + \cdots + b_{m_t}^2 \left(\hat{P}_{N_t}\left(b_{m_t-1,t}\right) - \hat{P}_{N_t}\left(b_{m_t,t}\right)\right)\right) \\
= \;& N_t \left(\left(b_{1,t}^2 - b_{0,t}^2\right) \hat{P}_{N_t}\left(b_{0,t}\right) + \cdots + \left(b_{m_t,t}^2 - b_{m_t-1,t}^2\right) \hat{P}_{N_t}\left(b_{m_t-1,t}\right) - b_{m_t,t}^2 \hat{P}_{N_t}\left(b_{m_t,t}\right)\right) \\
= \;& N_t \sum_{k=0}^{m_t-1} \left(b_{k+1,t}^2 - b_{k,t}^2\right) \hat{P}_{N_t}\left(b_{k,t}\right) \\
= \;& N_t \sum_{k=0}^{m_t-1} \left(b_{k+1,t} + b_{k,t}\right)\left(b_{k+1,t} - b_{k,t}\right) \hat{P}_{N_t}\left(b_{k,t}\right) \\
= \;& 2N_t \sum_{k=0}^{m_t-1} \left(\frac{b_{k,t} + b_{k+1,t}}{2}\right)\left(b_{k+1,t} - b_{k,t}\right) \hat{P}_{N_t}\left(b_{k,t}\right)
\end{aligned}
$$

Replace the empirical counter-CDF $\hat{P}_{N_t}\left(b_{k,t}\right)$ with the continuous function in (A11).

$$
\begin{aligned}
\sum_{j=1}^{N_t} \left(d_{j,t}^{in}\right)^2 &= 2N_t \int_{d_{\min}}^{d_{\max}(N_t)} x \left(\frac{x}{d_{\min}}\right)^{1-\xi} dx \\
&= 2N_t \int_{d_{\min}}^{d_{\max}(N_t)} x \frac{d_{\min}}{2-\xi} d\left(\frac{x}{d_{\min}}\right)^{2-\xi} \\
&= 2N_t \frac{d_{\min}}{2-\xi}\left(x\left(\frac{x}{d_{\min}}\right)^{2-\xi}\Bigg|_{d_{\min}}^{d_{\max}(N_t)} - \int_{d_{\min}}^{d_{\max}(N_t)} \left(\frac{x}{d_{\min}}\right)^{2-\xi} dx\right) \\
&= 2N_t \frac{d_{\min}}{2-\xi}\left(x\left(\frac{x}{d_{\min}}\right)^{2-\xi}\Bigg|_{d_{\min}}^{d_{\max}(N_t)} - \frac{d_{\min}}{3-\xi}\left(\frac{x}{d_{\min}}\right)^{3-\xi}\Bigg|_{d_{\min}}^{d_{\max}(N_t)}\right) \\
&= 2N_t \frac{d_{\min}}{2-\xi}\left(d_{\max}\left(N_t\right)\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{2-\xi} - d_{\min} - \frac{d_{\min}}{3-\xi}\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{3-\xi} + \frac{d_{\min}}{3-\xi}\right) \\
&= 2N_t \frac{d_{\min}}{2-\xi}\left(\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{2-\xi}\left(d_{\max}\left(N_t\right) - \frac{1}{3-\xi} d_{\max}\left(N_t\right)\right) - \left(d_{\min} - \frac{d_{\min}}{3-\xi}\right)\right) \\
&= 2N_t \frac{d_{\min}}{2-\xi}\left(\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{2-\xi} \frac{2-\xi}{3-\xi} d_{\max}\left(N_t\right) - \frac{2-\xi}{3-\xi} d_{\min}\right)
\end{aligned}
$$

Using the dynamics of aggregate retail sentiment δ_t^R in equation (54), we can compute the conditional mean of δ_t^R

$$
\mathbb{E}_{t-1}\left[\delta_t^R\right] = \frac{\theta\left(N_t\right)}{\theta\left(N_{t-1}\right)} \rho \delta_{t-1}^R,
$$

and the conditional variance

$$
\begin{aligned}
\mathrm{Var}_{t-1}\left(\delta_t^R\right) &= \left(\theta\left(N_t\right)\right)^2 \frac{1}{N_t^2}\sum_{j=1}^{N_t}\left(d_{j,t}^{in}\right)^2 \sigma_\varepsilon^2 \\
&= \left(\theta\left(N_t\right)\right)^2 \frac{1}{N_t}\frac{2d_{\min}}{2-\xi}\sigma_\varepsilon^2\left(\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{2-\xi}\frac{2-\xi}{3-\xi}d_{\max}\left(N_t\right)-\frac{2-\xi}{3-\xi}d_{\min}\right) \\
&= \left(\theta\left(N_t\right)\right)^2 \frac{2d_{\min}}{N_t}\frac{1}{3-\xi}\left(\left(\frac{d_{\max}\left(N_t\right)}{d_{\min}}\right)^{2-\xi}d_{\max}\left(N_t\right)-d_{\min}\right)\sigma_\varepsilon^2 \\
&= \left(\theta\left(N_t\right)\right)^2 \frac{2d_{\min}^{\xi-1}}{N_t}\frac{1}{3-\xi}\left(\left(d_{\max}\left(N_t\right)\right)^{3-\xi}-d_{\min}^{3-\xi}\right)\sigma_\varepsilon^2 \\
&= O\left(N_t^{\frac{4-2\xi}{\xi-1}}\right).
\end{aligned}
$$

The last equality uses $d_{\max}\left(N_t\right) = O\left(N_t^{\frac{1}{\xi-1}}\right)$. Hence, the conditional volatility of aggregate retail sentiment is

$$\sqrt{\mathrm{Var}_{t-1}\left(\delta_t^R\right)} = O\left(N_t^{\frac{2-\xi}{\xi-1}}\right).$$

□

A1.13 Distribution of time-1 aggregate retail sentiment shock

Define $c_j \equiv \frac{1}{N}d_j^{in}$, and the random variable $X_j = \mu + \varepsilon_1^j$, $\mu = \delta_0^R$. Let σ^2 denote the pre-truncation variance of ε_1^j, then X_j follows a truncated normal distribution on $[-\bar{\varepsilon}, \bar{\varepsilon}]$ with pre-truncation mean μ and variance σ^2, and X_j is i.i.d. in the cross section. Further define $\rho \equiv \frac{\bar{\varepsilon}}{\sigma}$, $a = \mu - \rho\sigma$, $b = \mu + \rho\sigma$. Then the PDF of X_j is

$$f_{X_j}\left(x\right) = \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} = \frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{2\Phi\left(\rho\right)-1}$$

where $\phi\left(\cdot\right)$ and $\Phi\left(\cdot\right)$ are the PDF and CDF of a standard normal random variable, respectively.

The time-1 aggregate retail sentiment shock δ_1^R can be written as

$$\delta_1^R = \sum_{j=1}^{N} c_j X_j$$

Hence, the characteristic function of δ_1^R is

$$\begin{aligned}
\varphi_{\delta_1^R}(t) &= \varphi_{X_1}(c_1 t)\,\varphi_{X_2}(c_2 t)\cdots\varphi_{X_N}(c_N t) \\
&= \prod_{j=1}^{N}\varphi_{X_j}(c_j t) = \prod_{j=1}^{N}\mathbb{E}\left[e^{itc_jX_j}\right] \\
&= \prod_{j=1}^{N}\left[\int_a^b e^{itc_jx}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{2\Phi(\rho)-1}dx\right]
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_a^b e^{itc_jx}\frac{1}{\sigma_j}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{2\Phi(\rho)-1}dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\int_a^b\frac{1}{\sqrt{2\pi}\sigma}\exp\left(itc_jx-\frac{(x-\mu)^2}{2\sigma^2}\right)dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\int_a^b\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2-2\mu x+\mu^2-2itc_jx\sigma^2}{2\sigma^2}\right)dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\exp\left(\frac{\left(\mu+itc_j\sigma^2\right)^2-\mu^2}{2\sigma^2}\right)\int_a^b\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(x-\left(\mu+itc_j\sigma^2\right)\right)^2}{2\sigma^2}\right)dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\int_a^b\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(x-\left(\mu+c_j\sigma^2it\right)\right)^2}{2\sigma^2}\right)dx
\end{aligned}$$

Define $y\equiv\frac{x-\left(\mu+c_j\sigma^2it\right)}{\sigma}\implies x=\sigma y+\left(\mu+c_j\sigma^2it\right)\implies dx=\sigma dy$. And note that $\frac{a-\left(\mu+c_j\sigma^2it\right)}{\sigma}=-\rho-c_j\sigma it$, $\frac{b-\left(\mu+c_j\sigma^2it\right)}{\sigma}=\rho-c_j\sigma it$. Then

$$\begin{aligned}
&\int_a^b e^{itc_jx}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{2\Phi(\rho)-1}dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\int_a^b\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\left(x-\left(\mu+c_j\sigma^2it\right)\right)^2}{2\sigma^2}\right)dx \\
=\;& \frac{1}{2\Phi(\rho)-1}\exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\int_{\frac{a-\left(\mu+c_j\sigma^2it\right)}{\sigma}}^{\frac{b-\left(\mu+c_j\sigma_j^2it\right)}{\sigma}}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\
=\;& \frac{1}{2\Phi(\rho)-1}\exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\int_{-\rho-c_j\sigma it}^{\rho-c_j\sigma it}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\
=\;& \exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\frac{\Phi(\rho-c_j\sigma it)-\Phi(-\rho-c_j\sigma it)}{2\Phi(\rho)-1} \\
=\;& \exp\left(c_j\mu it-\frac{1}{2}c_j^2\sigma^2t^2\right)\frac{\Phi(\rho-c_j\sigma it)+\Phi(\rho+c_j\sigma it)-1}{2\Phi(\rho)-1}
\end{aligned}$$

Hence,

$$\begin{aligned}\varphi_{S_n}(t) &= \prod_{j=1}^{n}\left[\int_a^b e^{itc_jx}\frac{1}{\sigma}\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{2\Phi(\rho)-1}dx\right] \\ &= \exp\left(\left(\sum_{j=1}^{n}c_j\mu\right)it-\frac{1}{2}\left(\sum_{j=1}^{n}c_j^2\sigma^2\right)t^2\right)\prod_{j=1}^{n}\frac{\Phi(\rho-c_j\sigma it)+\Phi(\rho+c_j\sigma_j it)-1}{2\Phi(\rho)-1}\end{aligned}$$

The characteristic function of δ_1^R is

$$\varphi_{S_n}(t)=\exp\left(\left(\sum_{j=1}^{N}c_j\mu\right)it-\frac{1}{2}\left(\sum_{j=1}^{N}c_j^2\sigma^2\right)t^2\right)\prod_{j=1}^{N}\frac{\Phi(\rho-c_j\sigma it)+\Phi(\rho+c_j\sigma it)-1}{2\Phi(\rho)-1}$$

$$\Longrightarrow\quad \varphi_{\delta_1}(t)=\exp\left(\left(\sum_{j=1}^{N}c_j\right)\mu it-\frac{1}{2}\left(\sum_{j=1}^{N}c_j^2\right)\sigma_\varepsilon^2t^2\right)\prod_{j=1}^{N}\frac{\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}-c_j\sigma_\varepsilon it\right)+\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}+c_j\sigma_\varepsilon it\right)-1}{2\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}\right)-1}$$

$$\Longrightarrow\quad \varphi_{\delta_1}(t)=\exp\left(\mu it-\frac{1}{2}\left(\sum_{j=1}^{N}c_j^2\right)\sigma_\varepsilon^2t^2\right)\prod_{j=1}^{N}\frac{\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}-c_j\sigma_\varepsilon it\right)+\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}+c_j\sigma_\varepsilon it\right)-1}{2\Phi\left(\frac{\bar{\varepsilon}}{\sigma_\varepsilon}\right)-1}$$

Compare the characteristic function of δ_1^R with another random variable $\tilde{\delta}_1$, which follows a truncated normal distribution on $[\mu-\bar{\varepsilon},\mu+\bar{\varepsilon}]$, with mean $\sum_{j=1}^N c_j\mu=\mu$ and variance $\sum_{j=1}^N c_j^2\sigma_\varepsilon^2$.

$$\begin{aligned}\varphi_{\tilde{\delta}_1}(t) &= \exp\left(\mu it-\frac{1}{2}\left(\sum_{j=1}^{N}c_j^2\right)\sigma_\varepsilon^2t^2\right) \\ &\cdot\frac{\Phi\left(\frac{\bar{\varepsilon}}{\sqrt{\sum_{j=1}^N c_j^2}\sigma_\varepsilon}-\sqrt{\sum_{j=1}^N c_j^2}\sigma_\varepsilon it\right)+\Phi\left(\frac{\bar{\varepsilon}}{\sqrt{\sum_{j=1}^N c_j^2}\sigma_\varepsilon}+\sqrt{\sum_{j=1}^N c_j^2}\sigma_\varepsilon it\right)-1}{2\Phi\left(\frac{\bar{\varepsilon}}{\sqrt{\sum_{j=1}^N c_j^2}\sigma_\varepsilon}\right)-1}\end{aligned}$$

Hence, the distribution of δ_1^R can be approximated by a truncated normal distribution, if the cross sectional distribution of c_j is skewed.

A2 Reddit data

A2.1 Variable definitions

I construct two data frames following the steps in Section 2.1.1 – one includes all the submissions, and the other includes all the comments.

In the data frame of submissions, each row is a unique submission. And it has the following fields:

- `id`: the unique id of the submission, e.g., "eifjq5". I add the prefix "t3_" to the submission `id` to facilitate the mapping between the submission and its associated comments.
- `author`: the name of the author of the submission, e.g., "Ituglobal".
- `author_fullname`: the unique user id of the author of the submission, prefixed by "t2_", e.g., "t2_6rjw5".
- `created_utc`: the UTC date and time at which the submission was created.
- `title`: the textual content of the title of the submission.
- `selftext`: the textual content of the body text of the submission.

In the data frame of comments, each row is a unique comment. And it has the following fields:

- `id`: the unique id of the comment, e.g., "fctzgly". I add the prefix "t1_" to the `id` to facilitate the mapping between the comment in question and its parent comment.
- `link_id`: the unique id of the submission that the comment in question replies to, e.g., "t3_eiwx9h".
- `parent_id`: the unique id of the parent comment (or submission) of the comment in question. If the comment is a reply to another comment, then the is prefixed by "t1_". Otherwise, it is a reply to a submission, and it's prefixed by "t3_".
- `created_utc`: the UTC date and time at which the comment was created.
- `author`: the name of the author of the comment, e.g., "urfriendosvendo".
- `author_fullname`: the unique user id of the author of the comment, prefixed by "t2_", e.g., "t2_12ol3k".
- `body`: the textual content of the comment.

A2.2 Constructing the sample of submissions and comments

I first run the following algorithm to tag submissions and comments with stock tickers, and then select samples of submissions and comments.

1. Retrieve the list of tickers of CRSP common stocks.

2. Search for stock tickers in the text of the submission.[1]

   (a) First pass search: search for CRSP stock tickers in the augmented body text[2].

      i. Preprocess the augmented body text in the following order:

         - Replace ‘’ / - with space.
         - Replace & with space if it appears between words.
         - Replace . with space.
         - Remove all other punctuation marks.
         - Tokenize augmented body text and only keep non-empty tokens.

      ii. Search for CRSP stock tickers in the augmented body text in a case-insensitive way. A submission is tagged with a ticker if the ticker can be found in the list of tokens.

   (b) Manually go over the matched tickers, add $ sign in front of those tickers that are common words, and use this updated list of tickers in the second pass search.

   (c) Second pass search: repeat the procedures in the first pass search, but using the updated list of tickers from the previous step.

3. Drop submissions where `author_fullname` is empty, or “[deleted]”, or “[removed]”. I also drop those where `id` is empty, or “[deleted]”, or “[removed]”.

4. Drop submissions where `author` is one of the bots in Table A1.

5. Only keep submissions tagged with at least one CRSP common stock ticker, and only keep the comments associated with these selected submissions (see Appendix A2.3 below for the procedure of matching submissions with comments).

   If a submission is tagged with a ticker, then the associated comments are also tagged with the same ticker. A submission or comment can be tagged with multiple stock tickers.

[1] For GameStop, I search for both its ticker “GME” and the company name “GameStop”.

[2] A submission has its title and body text. I obtain the augmented body text by appending the body text to the title, separated by a white space.

Finally, I construct the following two samples of submissions and comments:

- Sample of submissions and comments for CRSP common stocks, by performing steps 1-5 above.
- Sample of all submissions and comments, by performing steps 1-4 above.

For each of the sample, I keep one data frame for submissions and another data frame for comments, with the structure described in Appendix A2.1. And I construct the network using these two data frames.

A2.3 Constructing the network

As is described in Appendix A2.1, the submission data frame and the comment data frame has a common field – the field `id` in the submission data frame corresponds to the `link_id` in the comment data frame. This allows me to recover the comment tree described in.

For each of the sample described in Appendix A2.2, I merge the submission data frame and comment data frame by the common field described above, and only keep submissions with at least one comment. In the merged dataset, each row corresponds to a comment, with information on the author of the comment, and the author of the submission that the comment replies to. This allows me to construct the network of users from the commenting relationship.

A3 FactSet data

I following the procedure in Gabaix and Koijen (2022) and Koijen et al. (2022):

1. Merge the holdings data (`[own_v5].[own_inst_eq_v5].[own_inst_13f_detail]`) with the entity sub type data (`[own_v5].[own_hub_ent_v5].[own_ent_institutions]`), by `factset_entity_id`.

   Each record in this merged dataset corresponds to a filer entity (with unique id `factset_entity_id`).

2. For those filer entities with missing entity sub type (from the previous step), find the corresponding roll-up entity (from `[own_v5].[own_hub_ent_v5].[own_ent_13f_combined_inst]`), and assign the sub type of the roll-up entity to the filer entity.

- To identify the sub type of the roll-up entity: merge the roll-up entity data ([own_v5].[own_hub_ent_v5].[own_ent_13f_combined_inst]) with the entity sub type data ([own_v5].[own_hub_ent_v5].[own_ent_institutions]), by factset_rollup_entity_id in the former (factset_entity_id in the latter).
$\Longrightarrow$ 12,276 out of the 12,295 roll-up entities have non-missing entity sub type.

3. Classify institutions into five types using entity_sub_type:

   - Hedge Funds: entity_sub_type = "AR", "FH", "FF", "FU", "FS", "HF".
   - Brokers: entity_sub_type = "BM", "IB", "ST", "MM", "BR".
   - Private Banking: entity_sub_type = "CP", "FY", "VC", "PB".
   - Investment Advisors: entity_sub_type = "IC", "RE", "PP", "SB", "MF", "IA".
   - Long-Term Investors: entity_sub_type = "FO", "SV", "IN", "PF".

A4 Modified BJZZ algorithm to identify retail trades

1. Start with any trade with price not at the midpoint of bid and ask.
2. Match the NBBO to the timestamp of the trade, and then compute bid-ask spread quoted before the trade.
3. If the spread quoted before the trade is one cent, use the original BJZZ algorithm to sign the trade.
4. If the trade price is outside the bid-ask spread, use the original BJZZ algorithm to sign the trade.
5. Otherwise, if the trade is below the midpoint, label the trade as a sell. If the trade is above the midpoint, label the trade as a buy.

   I also implement the $[0.4, 0.6]$ "donut" in this step, as in the original BJZZ algorithm.

A5 Fitting power-law distribution

For each calendar day t, I fit a power-law distribution to the vector of user influence, $\left(d^{in}_{1,t}, d^{in}_{2,t}, \cdots, d^{in}_{N_t,t}\right)^{\intercal}$ computed in Section 2.1.3, and estimate the exponent $\hat{\xi}_t$ and the threshold value $\hat{d}^{in}_{\min,t}$. Following Rantala (2019), I use maximum likelihood method to estimate these parameters. Specifically, I use the power.law.fit function of the igraph package in R, with the "plfit" implementation.

I use bootstrap methods to compute the confidence intervals. The steps are:

1. Generate a bootstrap sample $\left\{d_{k,t}^{in}\left(b\right)\right\}_{k=1}^{N_t}$ by sampling the original data $\left(d_{1,t}^{in}, d_{2,t}^{in}, \cdots, d_{N_t,t}^{in}\right)^{\intercal}$ randomly with replacement.

2. Estimate the parameters $\xi_t\left(b\right)$ and $d_{\min,t}\left(b\right)$ for this bootstrapped sample, using the maximum likelihood method described above.

3. Repeat steps 1 and 2 for $B = 5000$ times, and obtain the vector of estimates $\left\{\xi_t\left(b\right)\right\}_{b=1}^{B}$, $\left\{d_{\min,t}\left(b\right)\right\}_{b=1}^{B}$.

4. For the $\hat{\xi}_t$ estimate, the lower (upper) bound of the 95% confidence interval is the 2.5th (97.5th) percentile of the empirical distribution $\left\{\xi_t\left(b\right)\right\}_{b=1}^{B}$. Similarly, for the $\hat{d}_{min,t}$ estimate, the lower (upper) bound of the 95% confidence interval is the 2.5th (97.5th) percentile of the empirical distribution $\left\{d_{\min,t}\left(b\right)\right\}_{b=1}^{B}$.

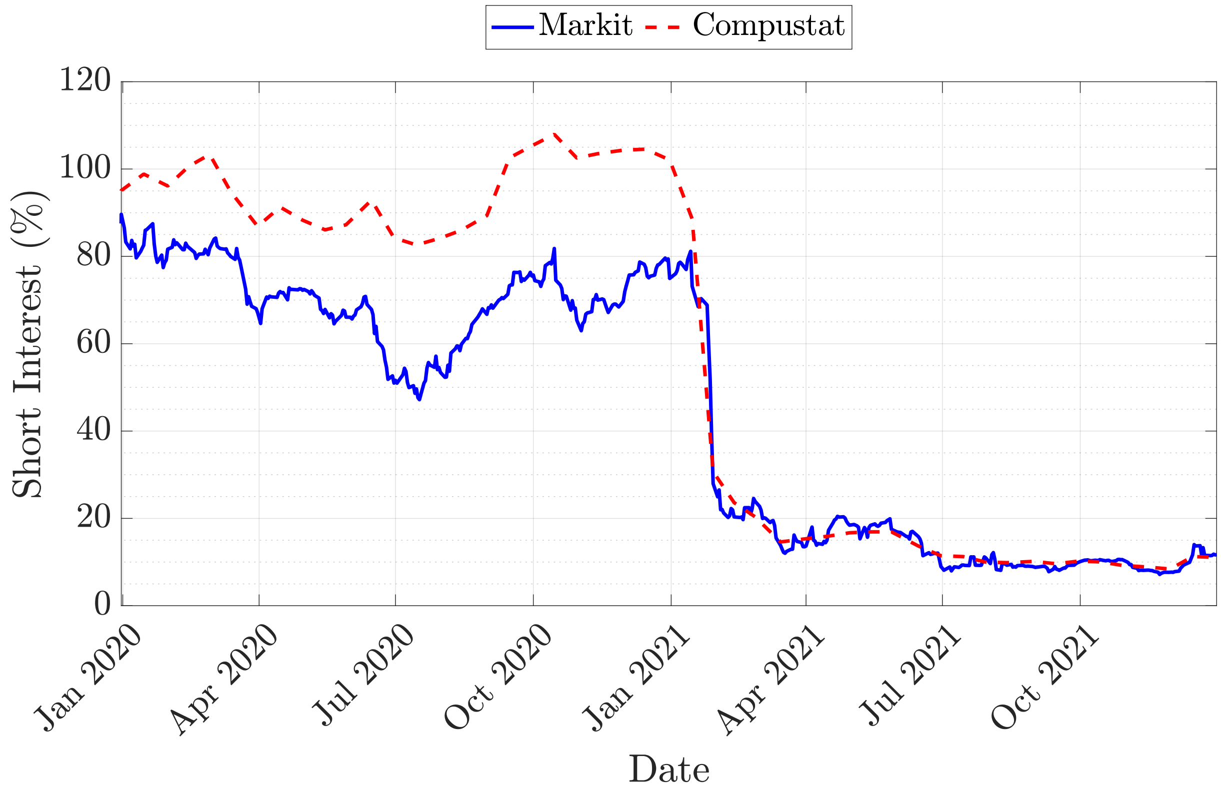


Figure A1. Short interest of GameStop from IHS Markit versus Compustat. This figure compares the short interest of GameStop computed using IHS Markit data versus that using Compustat data, for the period from January 1, 2020 to December 31, 2021. Short interest is defined as the ratio of the number of shares sold short to the number of shares outstanding (equation (6)). The solid blue line is the short interest computed using daily data on number of shares sold short from IHS Markit. The dashed red line is the short interest computed using mid-month and month-end number of shares sold short from Compustat.

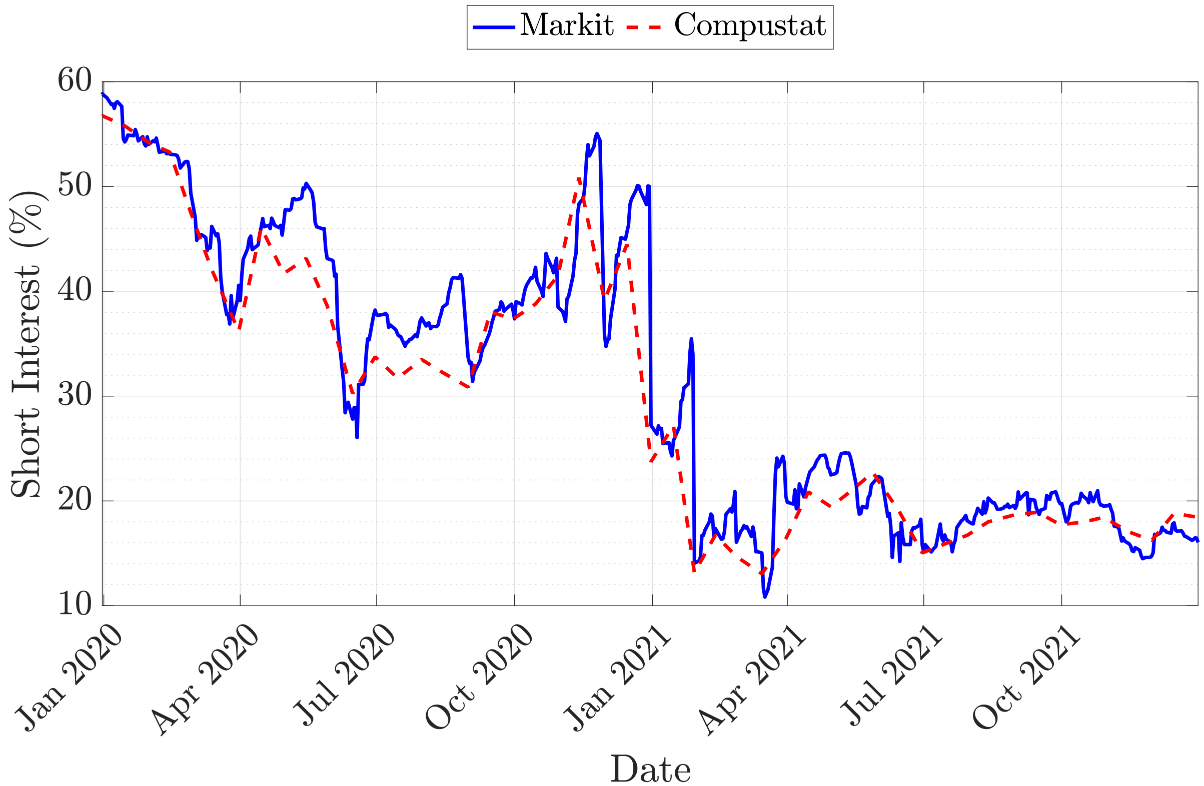


Figure A2. Short interest of AMC from IHS Markit versus Compustat. This figure compares the short interest of AMC computed using IHS Markit data versus that using Compustat data, for the period from January 1, 2020 to December 31, 2021. Short interest is defined as the ratio of the number of shares sold short to the number of shares outstanding (equation (6)). The solid blue line is the short interest computed using daily data on number of shares sold short from IHS Markit. The dashed red line is the short interest computed using mid-month and month-end number of shares sold short from Compustat.

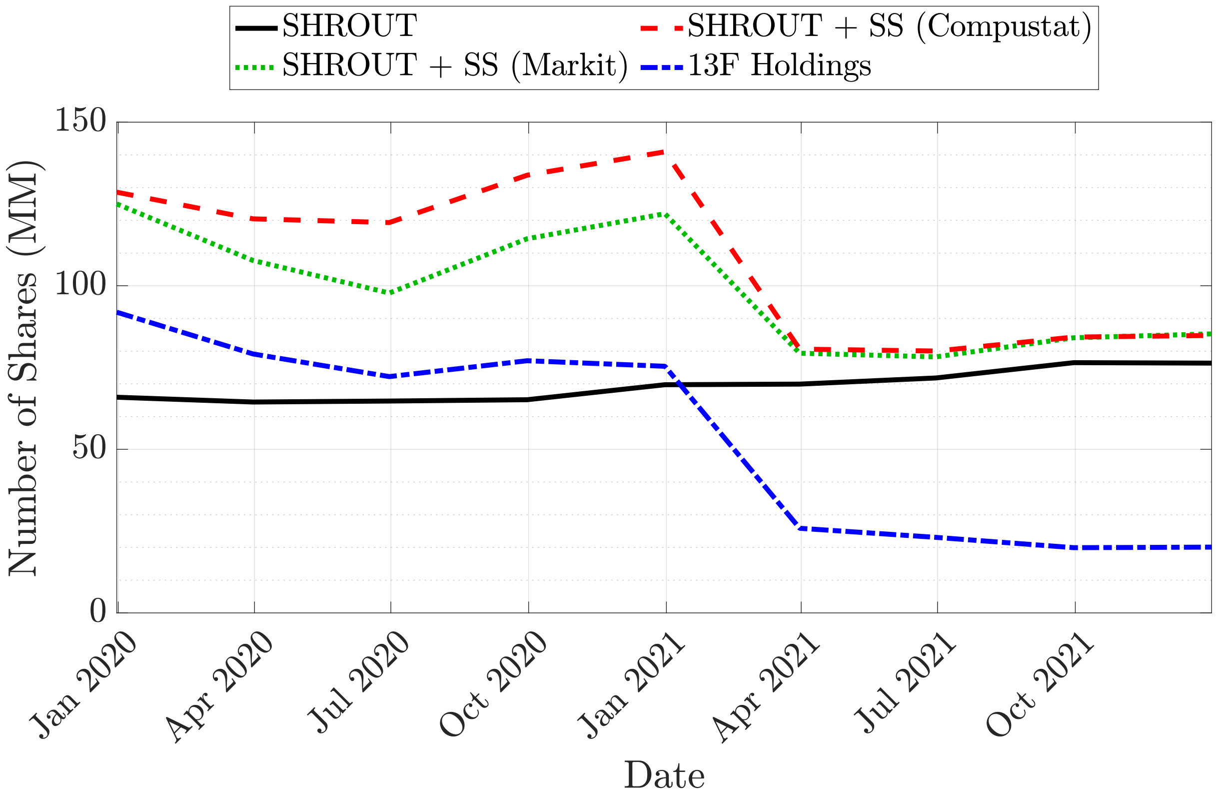


Figure A3. Shares outstanding and 13F institutional ownership of GameStop. This figure compares the number of shares outstanding with 13F institutional ownership of GameStop, for the period from January 1, 2020 to December 31, 2021. The solid black line is the number of shares outstanding. The dashed red line is the number of shares outstanding plus the number of shares sold short from Compustat. The dotted green line is the number of shares outstanding plus the number of shares sold short from IHS Markit. The dash-dotted blue line is the number of shares held by 13F institutions.

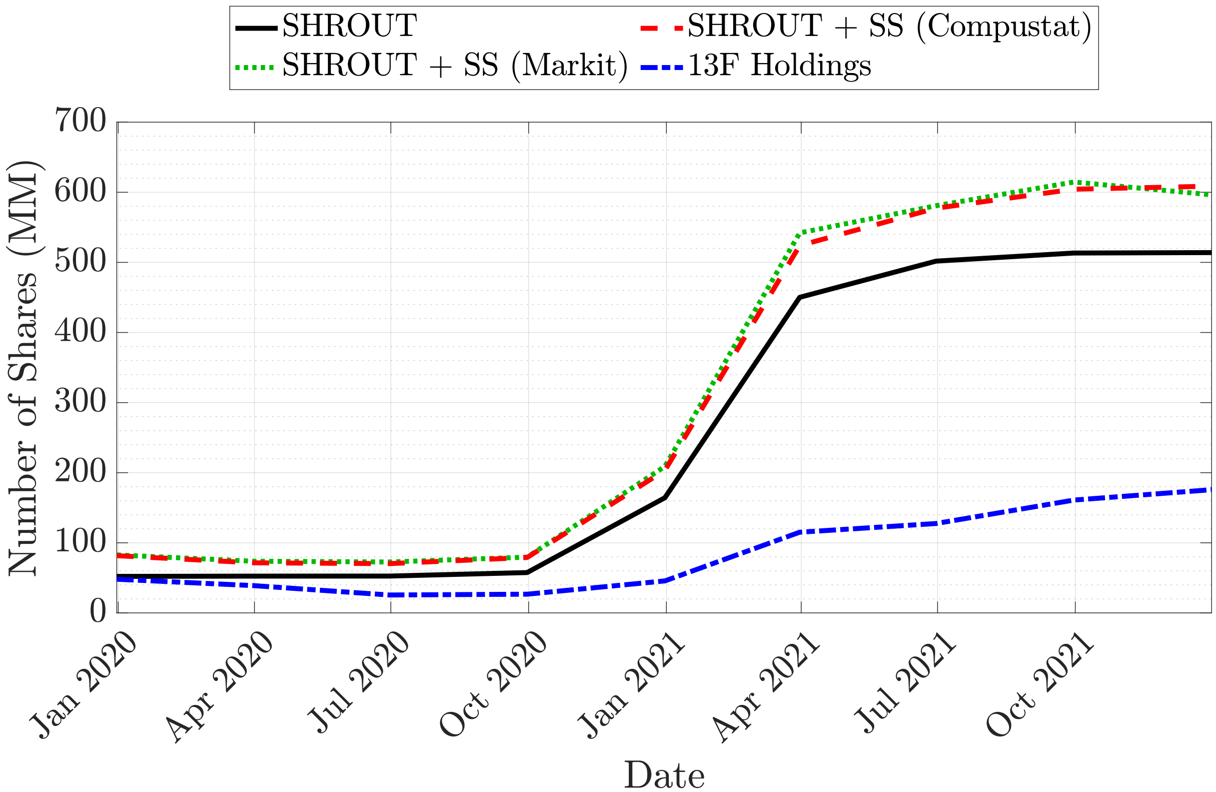


Figure A4. Shares outstanding and 13F institutional ownership of AMC. This figure compares the number of shares outstanding with 13F institutional ownership of AMC, for the period from January 1, 2020 to December 31, 2021. The solid black line is the number of shares outstanding. The dashed red line is the number of shares outstanding plus the number of shares sold short from Compustat. The dotted green line is the number of shares outstanding plus the number of shares sold short from IHS Markit. The dash-dotted blue line is the number of shares held by 13F institutions.

Figure A5. Price and sentiment of Amazon. This figure shows the daily close price (left y-axis) and the daily WSB sentiment measures (right y-axis) of Amazon, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price, the dotted red line plots the equal-weighted sentiment defined in equation (4), and the dash-dotted green line plots the influence-weighted sentiment defined in equation (5). The sentiment series are 30-day moving averages.

Figure A6. Price and sentiment of Microsoft. This figure shows the daily close price (left y-axis) and the daily WSB sentiment measures (right y-axis) of Microsoft, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price, the dotted red line plots the equal-weighted sentiment defined in equation (4), and the dash-dotted green line plots the influence-weighted sentiment defined in equation (5). The sentiment series are 30-day moving averages.

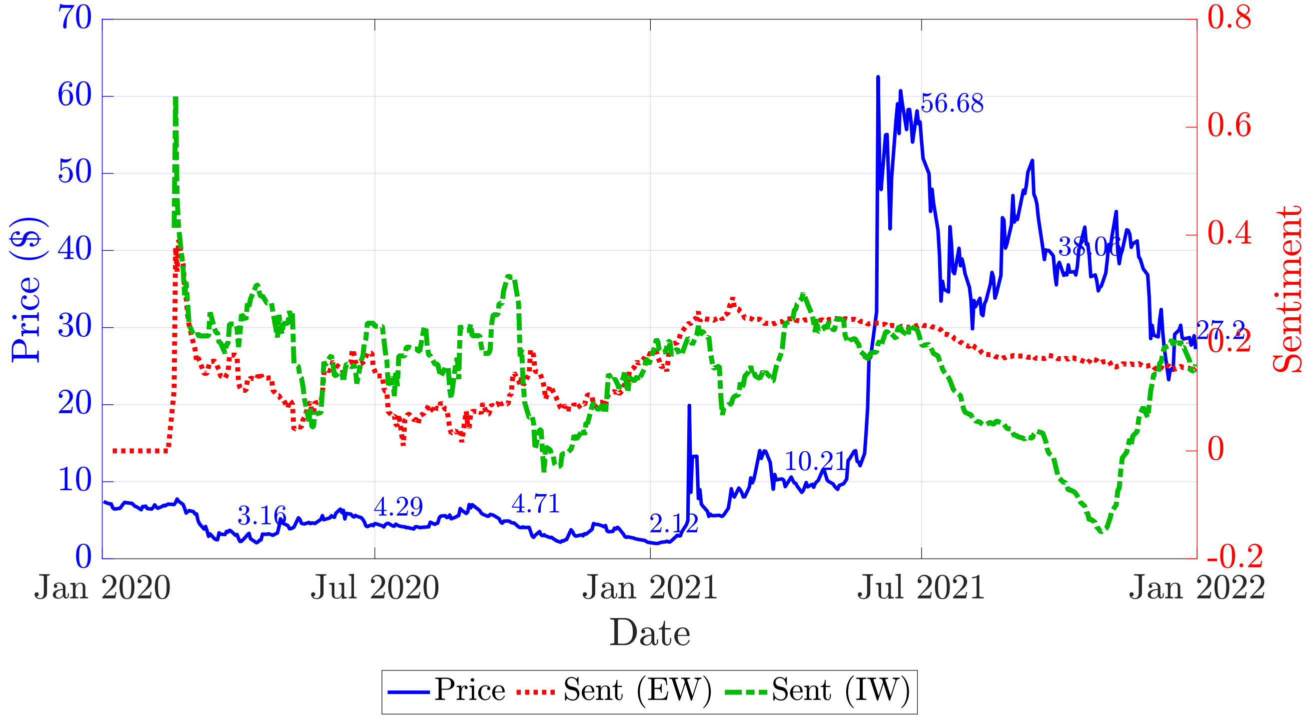


Figure A7. Price and sentiment of AMC. This figure shows the daily close price (left y-axis) and the daily WSB sentiment measures (right y-axis) of AMC, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price, the dotted red line plots the equal-weighted sentiment defined in equation (4), and the dash-dotted green line plots the influence-weighted sentiment defined in equation (5). The sentiment series are 30-day moving averages.

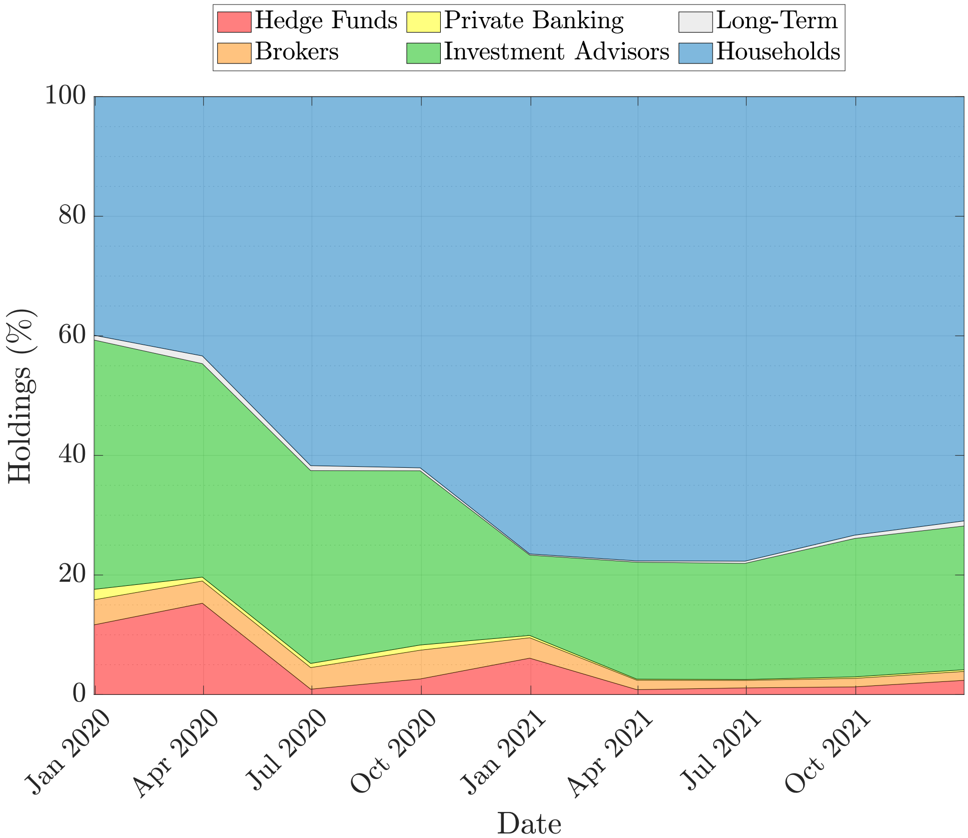
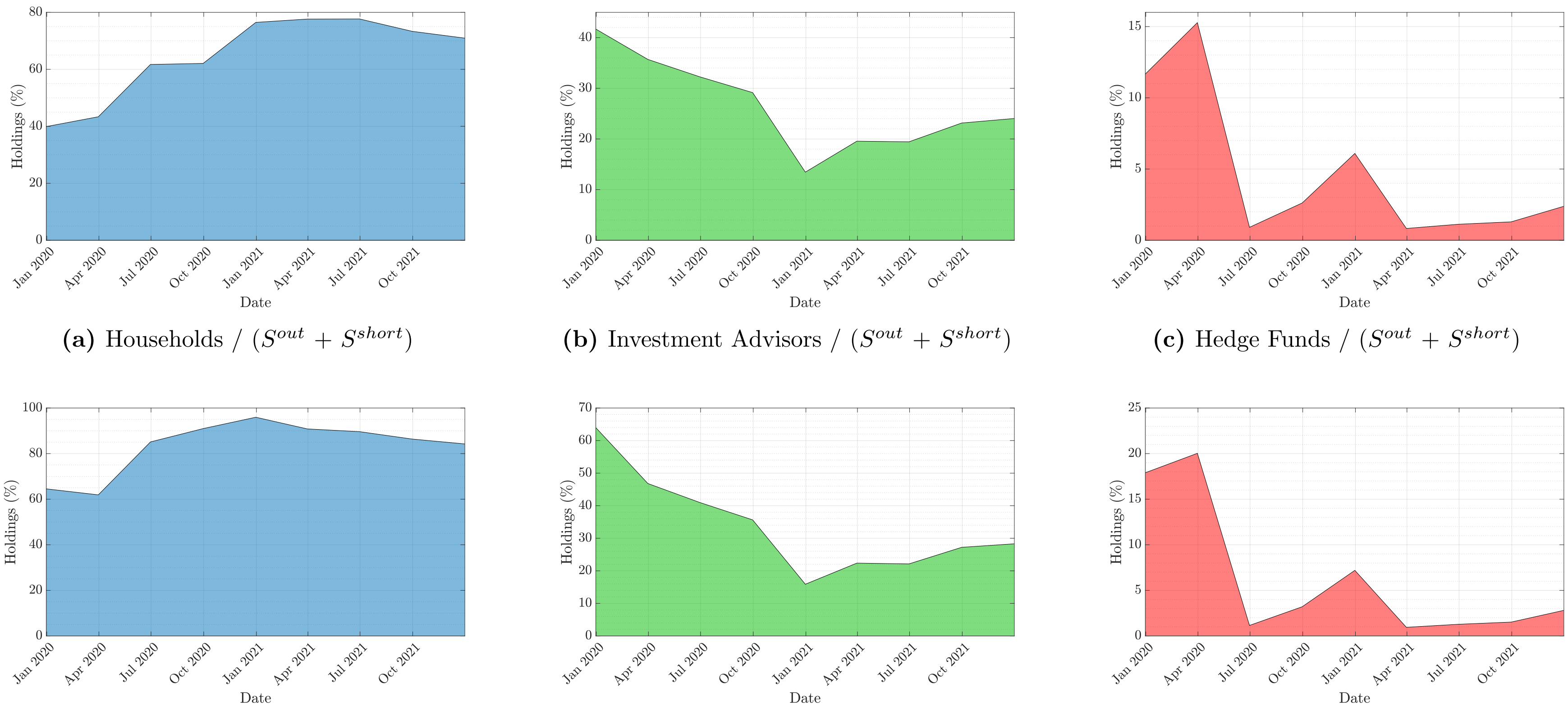


Figure A8. Ownership of AMC by investor type. This figure plots the end-of-quarter holdings of AMC by 13F institutions and households, for the period from 2019 Q4 to 2021 Q4. 13F holdings data are from FactSet. I aggregate 13F institutional holdings to investor-type level, using the method in Appendix A3. And the five institutional investor types are: Hedge Funds (red area), Brokers (orange aread), Private Banking (yellow area), Investment Advisors (green area), and Long-Term Investors (gray area). I calculate household holdings from equation (8), using data on the number of shares sold short from Compustat. And the blue area represents households. The y-axis is the percentage holdings defined in equation (10), which is the number of shares held by each type of investor divided by the sum of the number of shares outstanding and the number of shares sold shorted.



(a) Households / (S^{out} + S^{short}) **(b)** Investment Advisors / (S^{out} + S^{short}) **(c)** Hedge Funds / (S^{out} + S^{short})

(d) Households / S^{out} **(e)** Investment Advisors / S^{out} **(f)** Hedge Funds / S^{out}

Figure A9. Ownership of AMC by Households, Investment Advisors, and Hedge Funds. This figure plots the end-of-quarter holdings of AMC by Households (panel (a) and (d)), Investment Advisors (panel (b) and (e)), and Hedge Funds (panel (c) and (f)), for the period from 2019 Q4 to 2021 Q4. 13F institutional investors are classified into Investment Advisors and Hedge Funds according to Appendix A3, and the 13F holdings data are from FactSet. Household holdings are calculated from equation (8). In panel (a), (b), and (c), the y-axis is the number of shares held by the investor group, divided by the sum of the number of shares outstanding and the number of shares sold short (equation (10)). Data on the number of shares sold short is from Compustat. In panel (d), (e), and (f), the y-axis is the number of shares held by the investor group, divided by the number of shares outstanding (equation (9)).

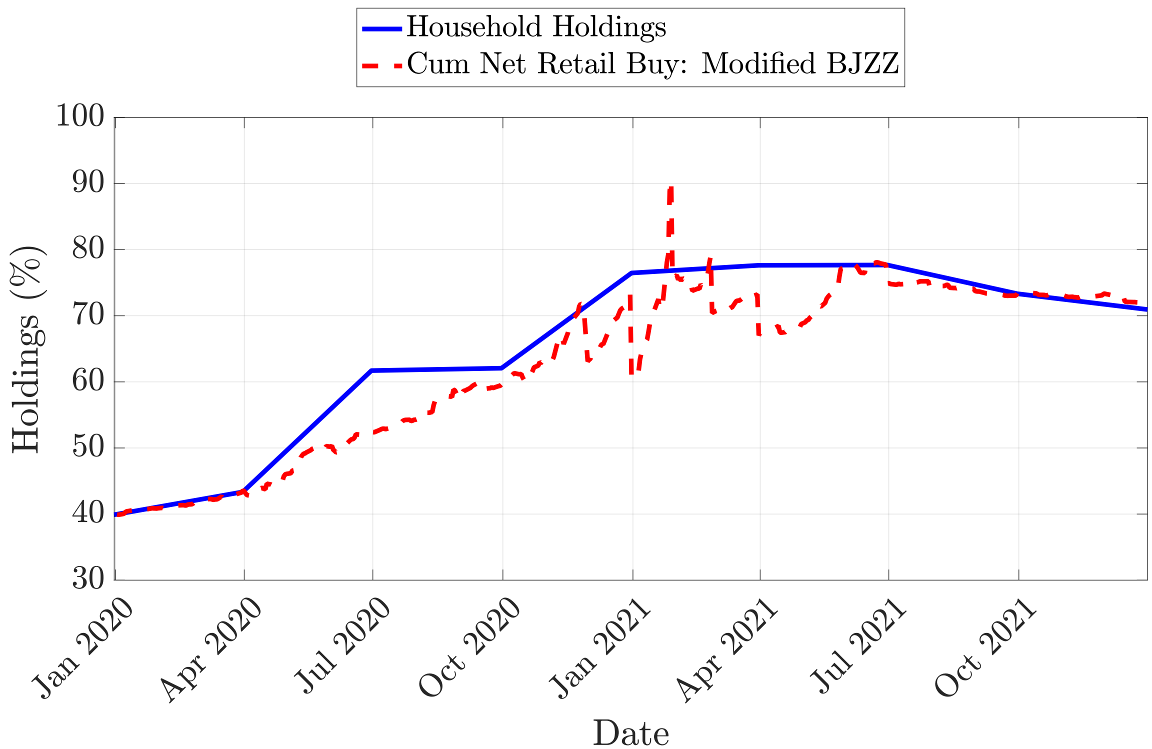


Figure A10. Ownership by households versus cumulative net retail buy volume of AMC. This figure plots the end-of-quarter holdings of AMC by Households (solid blue line), and the daily cumulative net retail buy volume (dashed red line), for the period from January 1, 2020 to December 31, 2021. Percentage holdings by households is defined in equation (10), which is the number of shares held by households (equation (8)) divided by the sum of the number of shares outstanding and the number of shares sold shorted. Cumulative net retail flow is defined in equation (12), which is the cumulative net retail buy volume (equation (11)) divided by the sum of the number of shares outstanding and the number of shares sold short. Data on the number of shares sold short is from Compustat. The initial value of the cumulative net retail flow (on Dec 31, 2019) is set to be the percentage holdings by households at the end of 2019 Q4. I apply the modified BJZZ algorithm in Appendix A4 to identify retail trades from the TAQ data.

Figure A11. Price and short interest of AMC. This figure shows the daily close price (left y-axis) and the daily short interest (right y-axis) of AMC, for the period from January 1, 2020 to December 31, 2021. The solid blue line plots the close price. The dotted red line plots the short interest, which is defined as the ratio of the number of shares sold short to the number of shares outstanding (equation (6)). Data on the number of shares sold short is from IHS Markit.

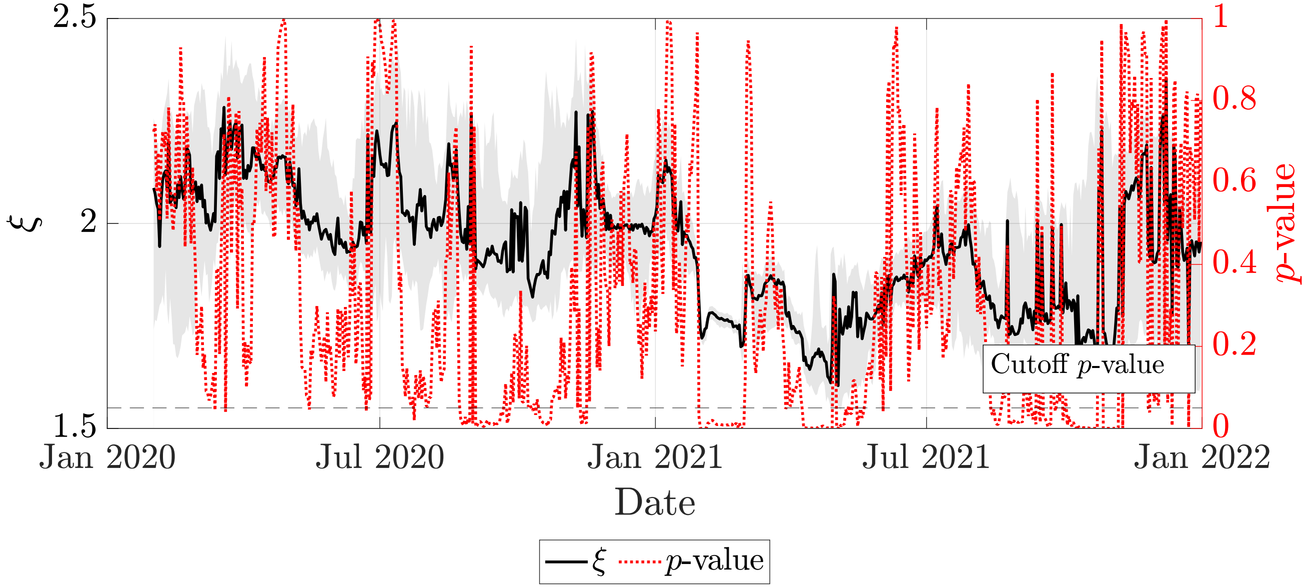


Figure A12. ***p*-value for fitting the power-law distribution.** This figure plots the daily estimate of the power-law exponent $\hat{\xi}_t$ and the p-value of the Kolmogorov-Smirnov test, for the period from January 1, 2020 to December 31, 2021. On each day t, I fit a power-law distribution to the vector of user influence (defined in equation (3)) and estimate the exponent ξ in equation (13). The solid black line plots the $\hat{\xi}_t$ estimates from the maximum likelihood method as in Rantala (2019). The gray area shows the 95% confidence interval for the estimates, computed from the bootstrap method in Appendix A5. The dotted red line plots the p-value of the Kolmogorov-Smirnov test. The cutoff p-value is 0.05 (dashed horizontal line). Small p-values (less than 0.05) indicate that the test rejected the hypothesis that the original data could have been drawn from the fitted power-law distribution.

Table A1
Reddit Bots Removed from the Sample

This table shows the Reddit bots whose submissions are removed from the sample.

Bot Name
WSBVoteBot
RemindMeBot
Generic_Reddit_Bot
ReverseCaptioningBot
LimbRetrieval-Bot
NoGoogleAMPBot
RepostSleuthBot
GetVideoBot
CouldWouldShouldBot

Table A2
Time-0 Equilibrium Outcomes under Different Risk Perceptions

This table compares the time-0 equilibrium outcomes when changing investors' time-0 perceptions of risk. Column 3 shows the equilibrium outcomes when all investors believe that the size of the network at time 1 will remain the same as that at time 0, i.e., $\tilde{N}_1 = N_L = N_0$. Column 4 shows the equilibrium outcomes when all investors believe that the the size of the network will grow (deterministically) from time 0 to time 1, i.e., $\tilde{N}_1 = N_H > N_L = N_0$. The parameter values are given in Table 2.

		Value	
Description	Notation	$\tilde{N}_1 = N_L$	$\tilde{N}_1 = N_H$
(1)	(2)	(3)	(4)
Log price	p_0	4.249	4.612
	w_0^R	1.900	1.024
Portfolio weights	w_0^{IL}	1.759	1.288
	w_0^{IS}	−0.250	0.539
	Q_0^R	60	34
Num. shares held	Q_0^{IL}	50	52
	Q_0^{IS}	−10	14
	α_0^R	0.316	0.329
Wealth shares	α_0^{IL}	0.284	0.403
	α_0^{IS}	0.400	0.269
Expected log payoff	$\mathbb{E}_0[p_1]$	4.469	5.157
Variance of log return	σ_0^2	0.378	1.015