

# Chapter 8

## NP and Computational Intractability



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# Algorithm Design Patterns and Anti-Patterns

## Algorithm design patterns.

Greed.

Divide-and-conquer.

Dynamic programming.

Max-flow min-cut.

Reductions.

Ex.

$O(n \log n)$  interval scheduling.

$O(n \log n)$  FFT.

$O(n^2)$  edit distance.

$O(n^3)$  bipartite matching.

## Algorithm design anti-patterns.

NP-completeness.

PSPACE-completeness.

Undecidability.

$O(n^k)$  algorithm unlikely.

$O(n^k)$  certification algorithm unlikely.

No algorithm possible.

## 8.1 Polynomial-Time Reductions

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# Classify Problems According to Computational Requirements

Q. Which problems will we be able to solve in practice?

*A working definition.* [Cobham 1964, Edmonds 1965, Rabin 1966]

Those with polynomial-time algorithms. (In practice, poly-time algorithms scale to huge problems.)

Yes	Probably no
Shortest path	Longest path
Matching	3D-matching
Min cut	Max cut
2-SAT	3-SAT
Planar 4-color	Planar 3-color
Bipartite vertex cover	Vertex cover
Primality testing (2002)	Factoring

# Classify Problems

**Desiderata.** Classify problems according to those that can be solved in polynomial-time and those that cannot.

**Provably requires exponential-time.**

Given a program of size  $\log k$ , does it halt in at most  $k$  steps?

Given a board position in an  $n$ -by- $n$  generalization of checker, can black guarantee a win?

**Frustrating news.** Huge number of fundamental problems have defied classification for decades.

**This chapter.** Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one **really hard** problem.

# Polynomial-Time Reduction

**Desiderata'**. Suppose we could solve X in polynomial-time. What else could we solve in polynomial time?

don't confuse with reduces from

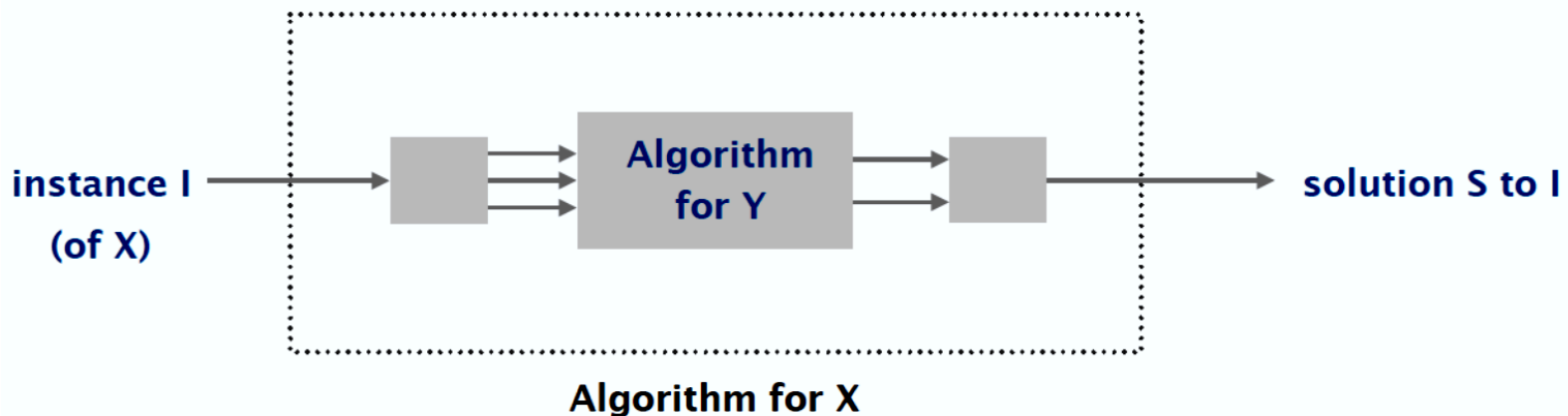
**Reduction.** Problem X **polynomial-time reduces to** problem Y if arbitrary instances of problem X can be solved using:

Polynomial number of standard computational steps, plus

Polynomial number of calls to oracle that solves problem Y.



A black box that solves instances of Y in a single step



# Polynomial-Time Reduction

**Desiderata'**. Suppose we could solve  $X$  in polynomial-time. What else could we solve in polynomial time?

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**Reduction.** Problem  $X$  **polynomial-time reduces to** problem  $Y$  if arbitrary instances of problem  $X$  can be solved using:

Polynomial number of standard computational steps, plus

Polynomial number of calls to oracle that solves problem  $Y$ .



A black box that solves instances of  $Y$  in a single step

**Notation.**  $X \leq_p Y$ .

**Remarks.**

We pay for time to write down instances sent to black box  $\Rightarrow$  instances of  $Y$  must be of polynomial size.

# Polynomial-Time Reduction

**Purpose.** Classify problems according to **relative** difficulty.

**Design algorithms.** If  $X \leq_p Y$  and  $Y$  can be solved in polynomial-time, then  $X$  **can** also be solved in polynomial time.

**Establish intractability.** If  $X \leq_p Y$  and  $X$  cannot be solved in polynomial-time, then  $Y$  **cannot** be solved in polynomial time.

**Establish equivalence.** If  $X \leq_p Y$  and  $Y \leq_p X$ , we use notation  $X \equiv_p Y$ . In this case,  $X$  can be solved in polynomial time iff  $Y$  can be.



# Reduction By Simple Equivalence

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Basic reduction strategies.

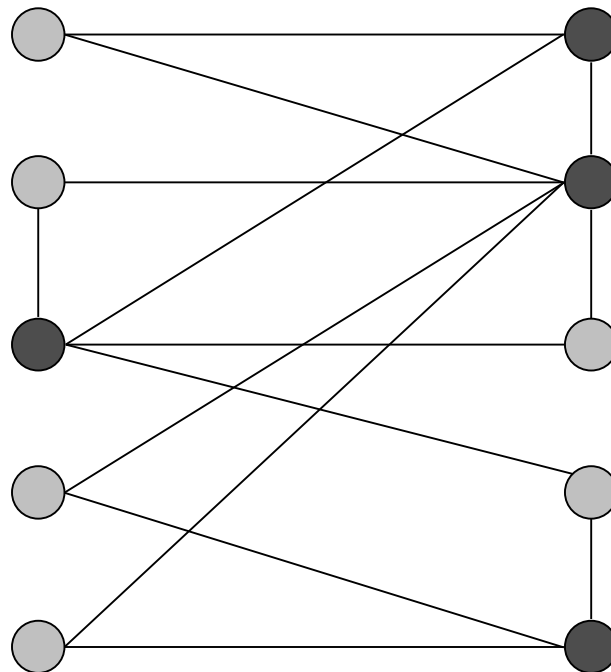
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

# Independent Set

**INDEPENDENT SET:** Given a graph  $G = (V, E)$  and an integer  $k$ , is there a subset of vertices  $S \subseteq V$  such that  $|S| \geq k$ , and for each edge at most one of its endpoints is in  $S$ ?

**Ex.** Is there an independent set of size  $\geq 6$ ? Yes.

**Ex.** Is there an independent set of size  $\geq 7$ ? No.



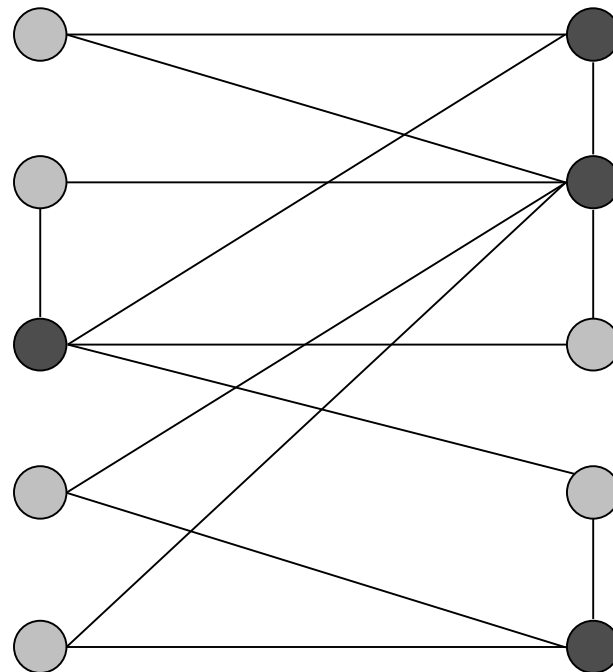
● independent set

# Vertex Cover

**VERTEX COVER:** Given a graph  $G = (V, E)$  and an integer  $k$ , is there a subset of vertices  $S \subseteq V$  such that  $|S| \leq k$ , and for each edge, at least one of its endpoints is in  $S$ ?

**Ex.** Is there a vertex cover of size  $\leq 4$ ? Yes.

**Ex.** Is there a vertex cover of size  $\leq 3$ ? No.

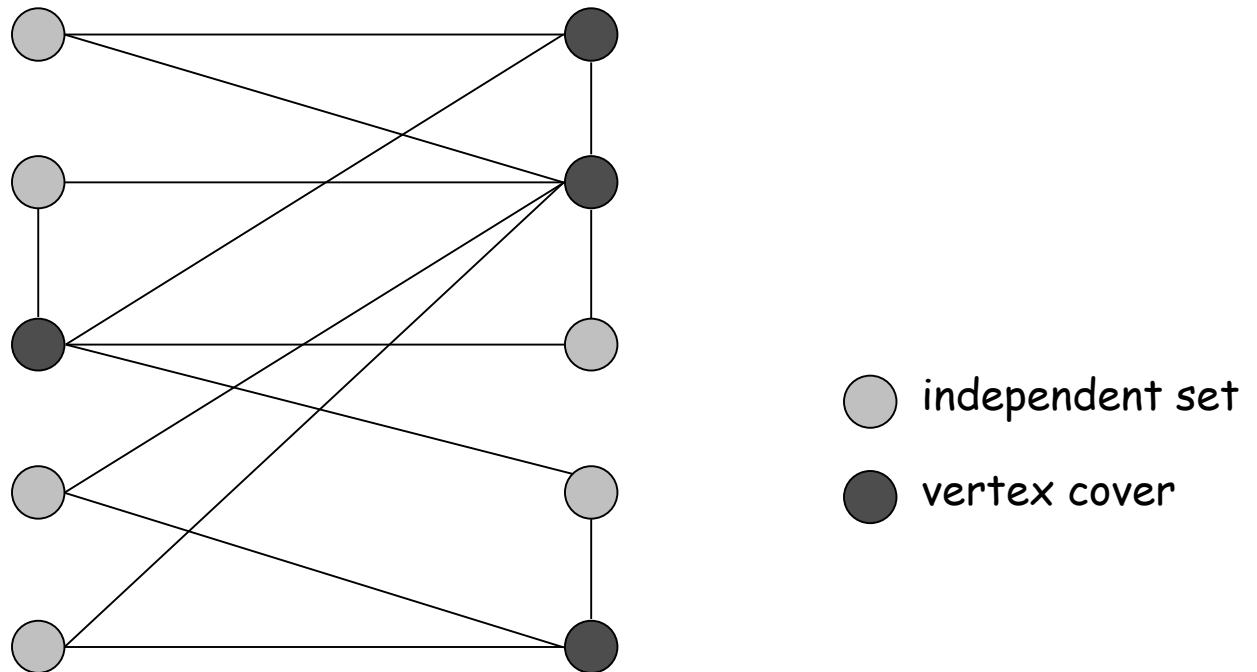


● vertex cover

# Vertex Cover and Independent Set

**Claim.** VERTEX-COVER  $\equiv_p$  INDEPENDENT-SET.

**Pf.** We show  $S$  is an independent set iff  $V - S$  is a vertex cover.



## Vertex Cover and Independent Set

**Claim.** VERTEX-COVER  $\equiv_p$  INDEPENDENT-SET.

**Pf.** We show  $S$  is an independent set iff  $V - S$  is a vertex cover.

$\Rightarrow$

Let  $S$  be any independent set.

Consider an arbitrary edge  $(u, v)$ .

$S$  independent  $\Rightarrow u \notin S$  or  $v \notin S \Rightarrow u \in V - S$  or  $v \in V - S$ .

Thus,  $V - S$  covers  $(u, v)$ .

$\Leftarrow$

Let  $V - S$  be any vertex cover.

Consider two nodes  $u \in S$  and  $v \in S$ .

Observe that  $(u, v) \notin E$  since  $V - S$  is a vertex cover.

Thus, no two nodes in  $S$  are joined by an edge  $\Rightarrow S$  independent set. ■

# Reduction from Special Case to General Case

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Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

# Set Cover

**SET COVER:** Given a set  $U$  of elements, a collection  $S_1, S_2, \dots, S_m$  of subsets of  $U$ , and an integer  $k$ , does there exist a collection of  $\leq k$  of these sets whose union is equal to  $U$ ?

## Sample application.

$m$  available pieces of software.

Set  $U$  of  $n$  capabilities that we would like our system to have.

The  $i$ -th piece of software provides the set  $S_i \subseteq U$  of capabilities.

Goal: achieve all  $n$  capabilities using fewest pieces of software.

Ex:

$$U = \{1, 2, 3, 4, 5, 6, 7\}$$

$$k = 2$$

$$S_1 = \{3, 7\}$$

$$S_4 = \{2, 4\}$$

$$S_2 = \{3, 4, 5, 6\}$$

$$S_5 = \{5\}$$

$$S_3 = \{1\}$$

$$S_6 = \{1, 2, 6, 7\}$$

# Vertex Cover Reduces to Set Cover

**Claim.** VERTEX-COVER  $\leq_p$  SET-COVER.

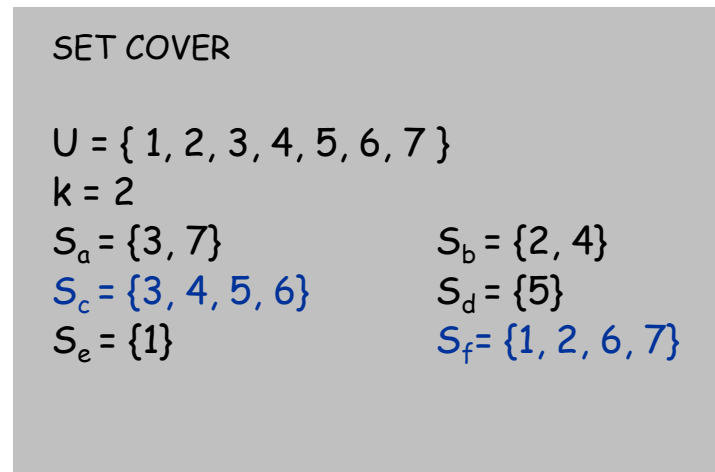
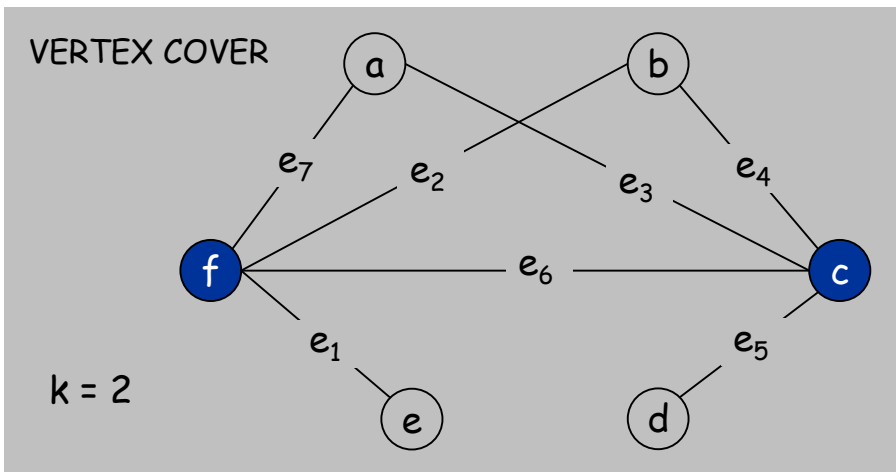
**Pf.** Given a VERTEX-COVER instance  $G = (V, E)$ ,  $k$ , we construct a set cover instance  $(U, S)$  whose size equals the size of the vertex cover instance.

## Construction.

Create SET-COVER instance:

-  $k = k$ ,  $U = E$ ,  $S_v = \{e \in E : e \text{ incident to } v\}$

Set-cover of size  $\leq k$  iff vertex cover of size  $\leq k$ . ■





## 8.2 Reductions via "Gadgets"

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Basic reduction strategies.

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction via "gadgets."

# Satisfiability

**Literal:** A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

**Clause:** A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

**Conjunctive normal form:** A propositional formula  $\Phi$  that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

**SAT:** Given CNF formula  $\Phi$ , does it have a satisfying truth assignment?

**3-SAT:** SAT where each clause contains exactly 3 literals.

↑  
each corresponds to a different variable

**Ex:**  $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

**Yes:**  $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}, x_4 = \text{false}.$

### 3 Satisfiability Reduces to Independent Set

**Claim.**  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$ .

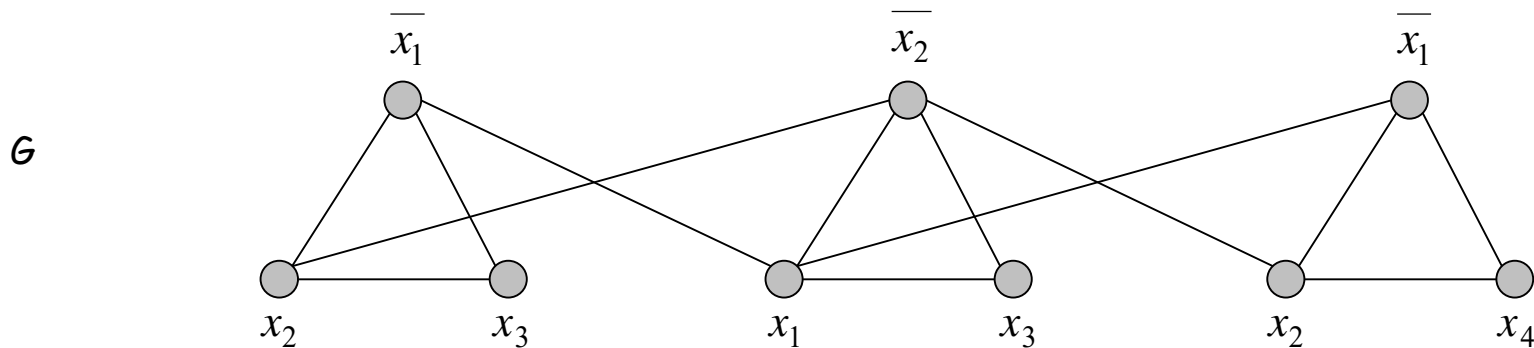
**Pf.** Given an instance  $\Phi$  of 3-SAT, we construct an instance  $(G, k)$  of INDEPENDENT-SET that has an independent set of size  $k$  iff  $\Phi$  is satisfiable.

**Construction.**

$G$  contains 3 vertices for each clause, one for each literal.

Connect 3 literals in a clause in a triangle.

Connect literal to each of its negations.



$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

### 3 Satisfiability Reduces to Independent Set

**Claim.**  $G$  contains independent set of size  $k = |\Phi|$  iff  $\Phi$  is satisfiable.

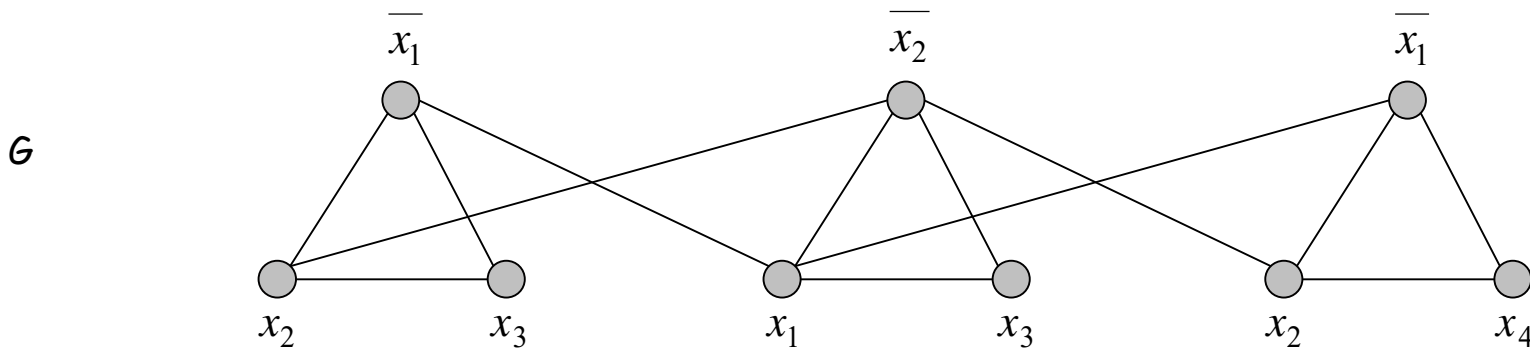
**Pf.**  $\Rightarrow$  Let  $S$  be independent set of size  $k$ .

$S$  must contain exactly one vertex in each triangle.

Set these literals to true.  $\leftarrow$  and any other variables in a consistent way

Truth assignment is consistent and all clauses are satisfied.

**Pf**  $\Leftarrow$  Given satisfying assignment, select one true literal from each triangle. This is an independent set of size  $k$ .  $\blacksquare$



$k = 3$

$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

# Review

## Basic reduction strategies.

Simple equivalence:  $\text{INDEPENDENT-SET} \equiv_p \text{VERTEX-COVER}$ .

Special case to general case:  $\text{VERTEX-COVER} \leq_p \text{SET-COVER}$ .

Encoding with gadgets:  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET}$ .

**Transitivity.** If  $X \leq_p Y$  and  $Y \leq_p Z$ , then  $X \leq_p Z$ .

**Pf idea.** Compose the two algorithms.

**Ex:**  $3\text{-SAT} \leq_p \text{INDEPENDENT-SET} \leq_p \text{VERTEX-COVER} \leq_p \text{SET-COVER}$ .

# Self-Reducibility

**Decision problem.** Does there **exist** a vertex cover of size  $\leq k$ ?

**Optimization problem.** **Find** vertex cover of minimum cardinality.

**Self-reducibility.** Optimization problem  $\leq_p$  decision version.

Applies to all (NP-complete) problems in this chapter.

Justifies our focus on decision problems.

**Ex:** to find min cardinality vertex cover.

(Binary) search for cardinality  $k^*$  of min vertex cover.

Find a vertex  $v$  such that  $G - \{v\}$  has a vertex cover of size  $\leq k^* - 1$ .

- $v$  must be in a vertex cover of size  $\leq k^*$
- Prove by construction: a vertex cover of  $G - \{v\}$  plus  $v$  must be a vertex cover of  $G$

Include  $v$  in the vertex cover.

Recursively find a min vertex cover in  $G - \{v\}$ .

↑  
delete  $v$  and all incident edges

## 8.3 Definition of NP

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# Decision Problems

Decision problem.

$X$  is a set of strings.

Instance: string  $s$ .

Algorithm  $A$  solves problem  $X$ :  $A(s) = \text{yes}$  iff  $s \in X$ .

**Polynomial time.** Algorithm  $A$  runs in polynomial time if for every string  $s$ ,  $A(s)$  terminates in at most  $p(|s|)$  "steps", where  $p(\cdot)$  is some polynomial.

↑  
length of  $s$

**PRIMES:**  $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, \dots \}$

**Instance:**  $s = 592335744548702854681$

**Algorithm:** [Agrawal-Kayal-Saxena, 2002]  $p(|s|) = |s|^8$ .



# Definition of P

P. Decision problems for which there is a poly-time algorithm.

Problem	Description	Algorithm	Yes	No
MULTIPLE	Is $x$ a multiple of $y$ ?	Grade school division	51, 17	51, 16
RELPRIME	Are $x$ and $y$ relatively prime?	Euclid (300 BCE)	34, 39	34, 51
PRIMES	Is $x$ prime?	AKS (2002)	53	51
EDIT-DISTANCE	Is the edit distance between $x$ and $y$ less than 5?	Dynamic programming	niether neither	acgggt ttttta
LSOLVE	Is there a vector $x$ that satisfies $Ax = b$ ?	Gauss-Edmonds elimination	$\left[ \begin{array}{ccc c} 0 & 1 & 1 & 4 \\ 2 & 4 & -2 & 2 \\ 0 & 3 & 15 & 36 \end{array} \right]$	$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right]$

# NP

## Certification algorithm intuition.

Certifier views things from "managerial" viewpoint.

Certifier doesn't determine whether  $s \in X$  on its own;  
rather, it checks a proposed proof  $t$  that  $s \in X$ .

Def. Algorithm  $C(s, t)$  is a **certifier** for problem  $X$  if for every string  $s$ ,  
 $s \in X$  iff there exists a string  $t$  such that  $C(s, t) = \text{yes}$ .

↖  
"certificate" or "witness"

**NP.** Decision problems for which there exists a **poly-time** certifier.

$C(s, t)$  is a poly-time algorithm

$|t| \leq p(|s|)$  for some polynomial  $p(\cdot)$ .

**Remark.** NP stands for **nondeterministic** polynomial-time.

# Certifiers and Certificates: Composite

**COMPOSITES.** Given an integer  $s$ , is  $s$  composite?

**Certificate.** A nontrivial factor  $t$  of  $s$ . Note that such a certificate exists iff  $s$  is composite. Moreover  $|t| \leq |s|$ .

**Certifier.**

```
boolean C(s, t) {  
    if (t ≤ 1 or t ≥ s)  
        return false  
    else if (s is a multiple of t)  
        return true  
    else  
        return false  
}
```

**Instance.**  $s = 437,669$ .

**Certificate.**  $t = 541$  or  $809$ .  $\longleftarrow 437,669 = 541 \times 809$

**Conclusion.** COMPOSITES is in NP.

## Certifiers and Certificates: 3-Satisfiability

**SAT.** Given a CNF formula  $\Phi$ , is there a satisfying assignment?

**Certificate.** An assignment of truth values to the  $n$  boolean variables.

**Certifier.** Check that each clause in  $\Phi$  has at least one true literal.

**Ex.**

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee x_4) \wedge (\overline{x_1} \vee \overline{x_3} \vee \overline{x_4})$$

instance  $s$

$$x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1$$

certificate  $t$

**Conclusion.** SAT is in NP.

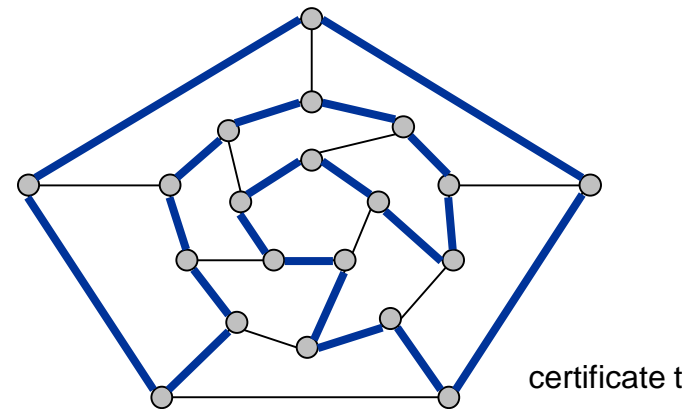
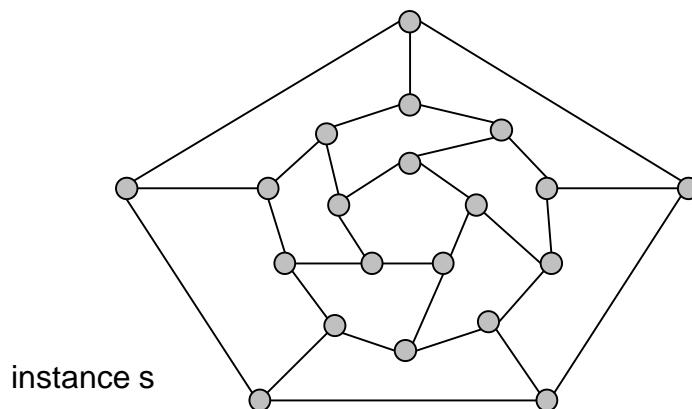
# Certifiers and Certificates: Hamiltonian Cycle

**HAM-CYCLE.** Given an undirected graph  $G = (V, E)$ , does there exist a simple cycle  $C$  that visits every node?

**Certificate.** A permutation of the  $n$  nodes.



**Certifier.** Check that the permutation contains each node in  $V$  exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

**Conclusion.** HAM-CYCLE is in NP.



# Definition of NP

**NP.** Decision problems for which there exists a poly-time certifier.

Problem	Description	Algorithm	Yes	No
LSOLVE	Is there a vector $x$ that satisfies $Ax = b$ ?	Gauss-Edmonds elimination	$\left[ \begin{array}{ccc c} 0 & 1 & 1 & 4 \\ 2 & 4 & -2 & 2 \\ 0 & 3 & 15 & 36 \end{array} \right]$	$\left[ \begin{array}{ccc c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array} \right]$
PRIMES	Is $x$ prime?	AKS (2002)	53	51
FACTOR	Does $x$ have a nontrivial factor less than $y$ ?	?	(56159, 50)	(55687, 50)
SAT	Is there a truth assignment that satisfies the formula?	?	$\neg x_1 \vee x_2$ $x_1 \vee x_2$	$\neg x_2$ $\neg x_1 \vee x_2$ $x_1 \vee x_2$
HAM-CYCLE	Does there exist a simple cycle $C$ that visits every node?	?		

## P, NP, EXP

**P.** Decision problems for which there is a **poly-time algorithm**.

**EXP.** Decision problems for which there is an **exponential-time algorithm**.

**NP.** Decision problems for which there is a **poly-time certifier**.

**Claim.**  $P \subseteq NP$ .

**Pf.** Consider any problem  $X$  in  $P$ .

By definition, there exists a poly-time algorithm  $A(s)$  that solves  $X$ .

Certificate:  $t = \varepsilon$ , certifier  $C(s, t) = A(s)$ . ▀

**Claim.**  $NP \subseteq EXP$ .

**Pf.** Consider any problem  $X$  in  $NP$ .

By definition, there exists a poly-time certifier  $C(s, t)$  for  $X$ .

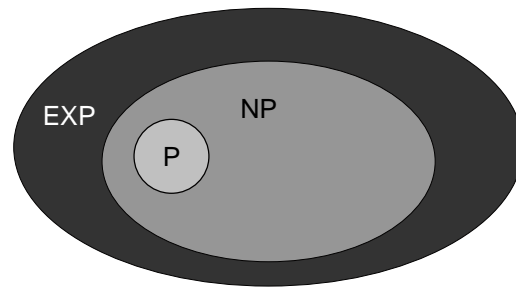
To solve input  $s$ , run  $C(s, t)$  on all strings  $t$  with  $|t| \leq p(|s|)$ .

Return **yes**, if  $C(s, t)$  returns **yes** for any of these. ▀

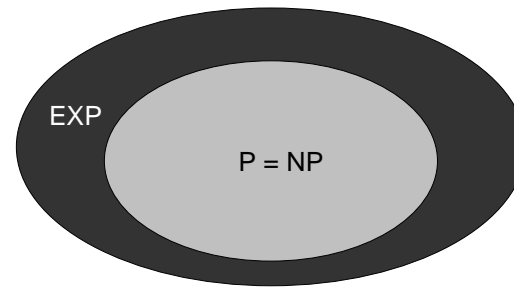
# The Main Question: P Versus NP

Does  $P = NP$ ? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

Is the decision problem as easy as the certification problem?



If  $P \neq NP$



If  $P = NP$

would break RSA cryptography  
(and potentially collapse economy)



**If yes:** Efficient algorithms for 3-COLOR, TSP, SAT, FACTOR, ...

**If no:** No efficient algorithms possible for 3-COLOR, TSP, SAT, ...



# Consensus

Consensus opinion on  $P = NP$ ? Probably no.

2002 poll of 100 researchers

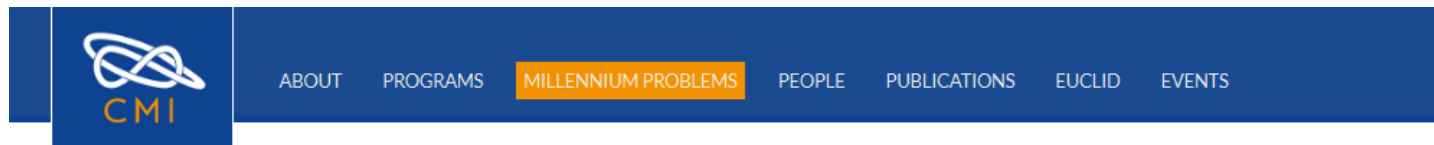
- 61: no
- 9: yes
- 22: unsure
- 4: the question may be independent of the currently accepted axioms and therefore is impossible to prove or disprove

2012 poll of 151 researchers

- 126: no
- 12: yes
- 8: don't know / don't care
- 5: the question may be independent of the currently accepted axioms and therefore is impossible to prove or disprove

# Millennium prize

One of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a **US\$1,000,000** prize for the first correct solution



## Millennium Problems

### Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

### Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part  $1/2$ .

### P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given  $N$  cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

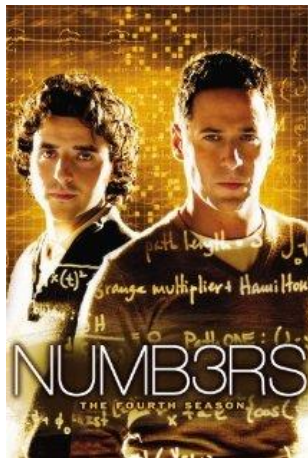
# In popular culture



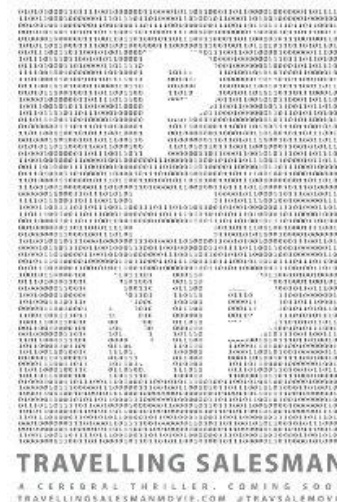
*The Simpsons* (season 7 episode 6)



*Futurama* (season 2 episode 10)



*Numb3rs*  
(season 1  
episode 2)



*Travelling Salesman*  
(2012 film)

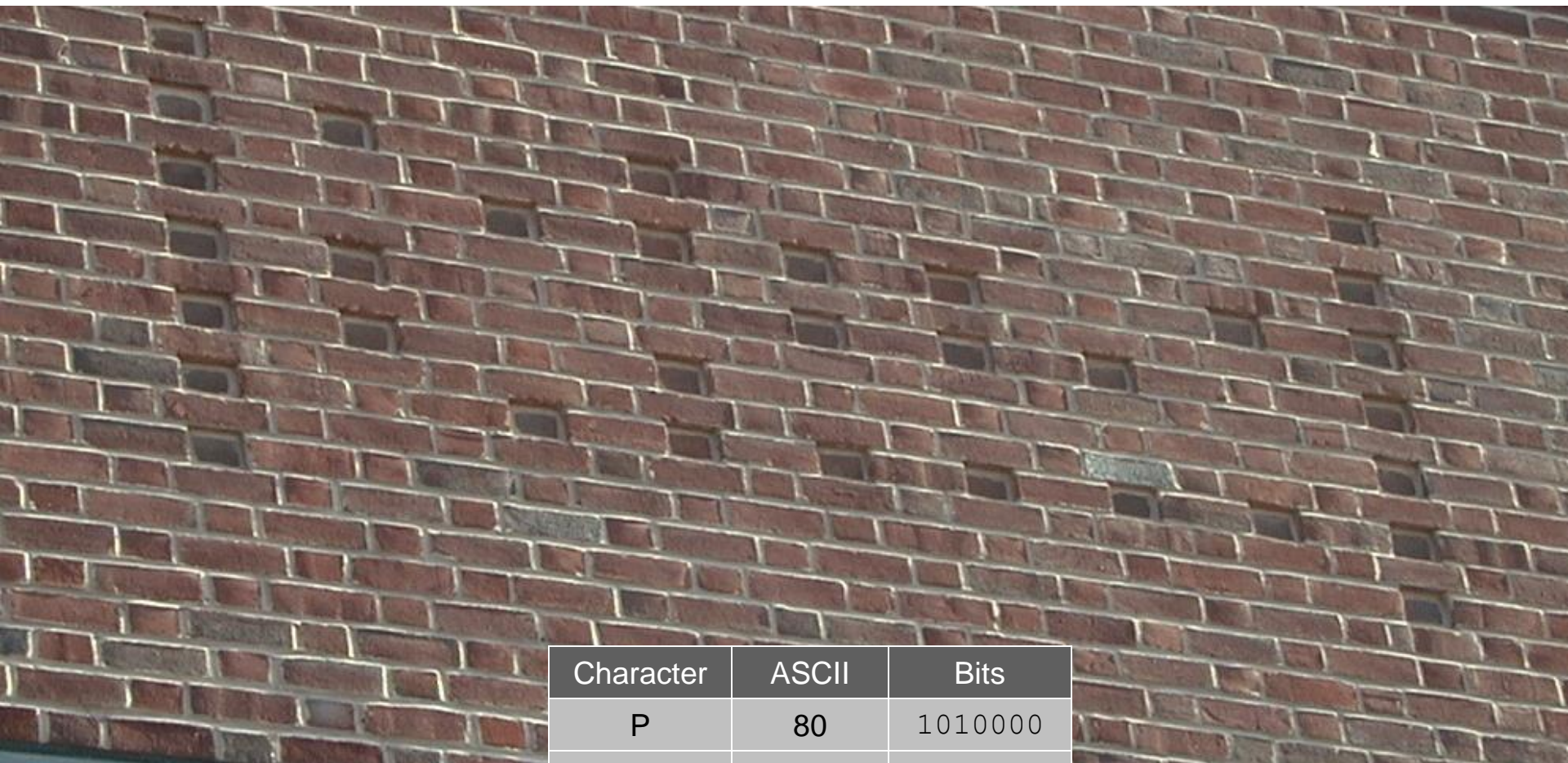
## Princeton CS Building, West Wall





## Princeton CS Building, West Wall





Character	ASCII	Bits
P	80	1010000
=	61	0111101
N	78	1001110
P	80	1010000
?	63	0111111

## 8.4 NP-Completeness

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# Polynomial Transformation

**Def.** Problem X **polynomial reduces** (Cook) to problem Y if arbitrary instances of problem X can be solved using:

Polynomial number of standard computational steps, plus

Polynomial number of calls to oracle that solves problem Y.

**Def.** Problem X **polynomial transforms** (Karp) to problem Y if given any input  $x$  to X, we can construct an input  $y$  in poly-time such that  $x$  is a  $y_{\text{yes}}$  instance of X iff  $y$  is a  $y_{\text{yes}}$  instance of Y.

↑  
we require  $|y|$  to be of size polynomial in  $|x|$

**Note.** Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

**Open question.** Are these two concepts the same?

↑  
we abuse notation  $\leq_p$  and blur distinction



# NP-Complete

**NP-complete.** A problem  $Y$  in NP with the property that for every problem  $X$  in NP,  $X \leq_p Y$ .

**Theorem.** Suppose  $Y$  is an NP-complete problem. Then  $Y$  is solvable in poly-time iff  $P = NP$ .

**Pf.**  $\Leftarrow$  If  $P = NP$  then  $Y$  can be solved in poly-time since  $Y$  is in NP.

**Pf.**  $\Rightarrow$  Suppose  $Y$  can be solved in poly-time.

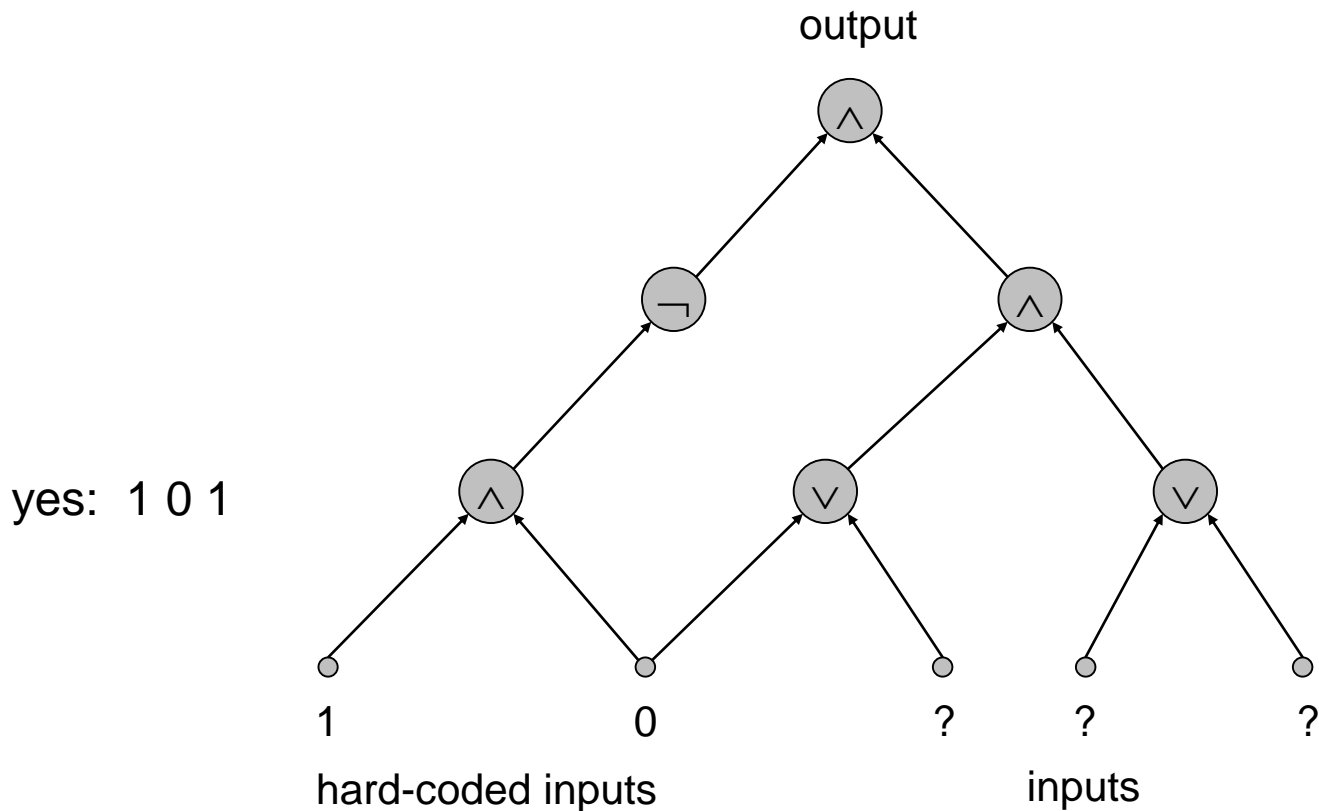
Let  $X$  be any problem in NP. Since  $X \leq_p Y$ , we can solve  $X$  in poly-time. This implies  $NP \subseteq P$ .

We already know  $P \subseteq NP$ . Thus  $P = NP$ . ■

**Fundamental question.** Do there exist "natural" NP-complete problems?

# Circuit Satisfiability

**CIRCUIT-SAT.** Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?



# The "First" NP-Complete Problem

**Theorem.** CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

**Pf.** (sketch)

Clearly, CIRCUIT-SAT is in NP

Any algorithm that takes a fixed number of bits  $n$  as input and produces a yes/no answer can be represented by such a circuit.

Moreover, if algorithm takes poly-time, then circuit is of poly-size.

sketchy part of proof; fixing the number of bits is important,  
and reflects basic distinction between algorithms and circuits

Consider some problem  $X$  in NP. It has a poly-time certifier  $C(s, t)$ .

To determine whether  $s$  is in  $X$ , need to know if there exists a certificate  $t$  of length  $p(|s|)$  such that  $C(s, t) = \text{yes}$ .

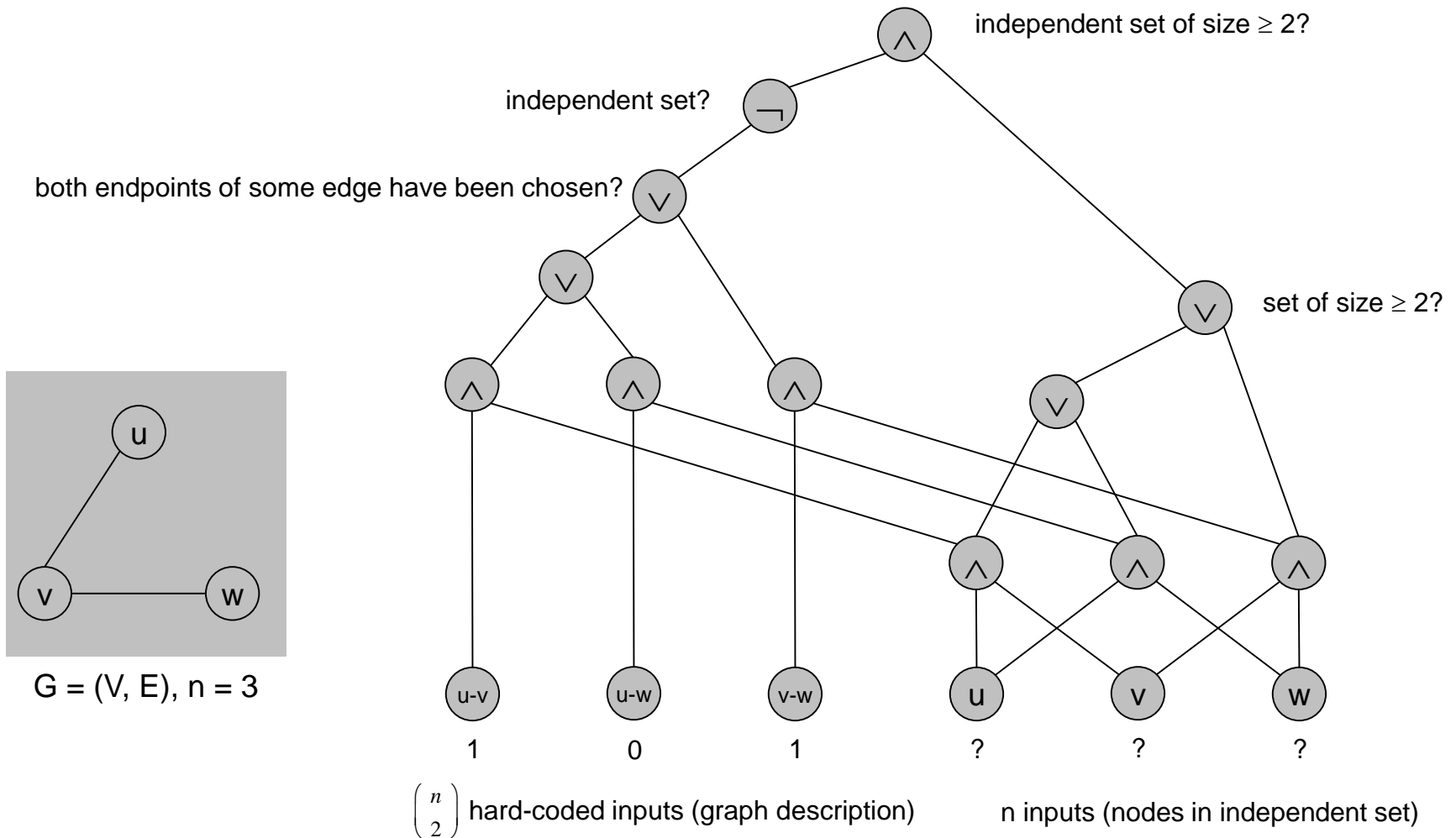
View  $C(s, t)$  as an algorithm on  $|s| + p(|s|)$  bits (input  $s$ , certificate  $t$ ) and convert it into a poly-size circuit  $K$ .

- first  $|s|$  bits are hard-coded with  $s$
- remaining  $p(|s|)$  bits represent (unknown) bits of  $t$

Circuit  $K$  is satisfiable iff  $\exists t, C(s, t) = \text{yes}$ .

# Example

Ex. Construction below creates a circuit  $K$  whose inputs can be set so that  $K$  outputs true iff graph  $G$  has an independent set of size  $\geq 2$ .



# Establishing NP-Completeness

**Remark.** Once we establish first "natural" NP-complete problem, others fall like dominoes.

**Recipe to establish NP-completeness of problem  $Y$ .**

Step 1. Show that  $Y$  is in NP.

Step 2. Choose an NP-complete problem  $X$ .

Step 3. Prove that  $X \leq_p Y$ .

**Theorem.** If  $X$  is an NP-complete problem and  $Y$  is a problem in NP with the property that  $X \leq_p Y$ , then  $Y$  is NP-complete.

**Pf.** Let  $W$  be any problem in NP. Then  $W \leq_p X \leq_p Y$ .

By transitivity,  $W \leq_p Y$ .

Hence  $Y$  is NP-complete. ■

↑  
by definition of  
NP-complete

↑  
by assumption

# 3-SAT is NP-Complete

## Review of 3-SAT:

**Literal:** A Boolean variable or its negation.

$$x_i \text{ or } \overline{x_i}$$

**Clause:** A disjunction of literals.

$$C_j = x_1 \vee \overline{x_2} \vee x_3$$

**Conjunctive normal form:** A propositional formula  $\Phi$  that is the conjunction of clauses.

$$\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$$

**SAT:** Given CNF formula  $\Phi$ , does it have a satisfying truth assignment?

**3-SAT:** SAT where each clause contains exactly 3 literals.

**Ex:**  $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

**Yes:**  $x_1 = \text{true}, x_2 = \text{true}, x_3 = \text{false}, x_4 = \text{false}.$

# 3-SAT is NP-Complete

**Theorem.** 3-SAT is NP-complete.

**Pf.** Suffices to show that  $\text{CIRCUIT-SAT} \leq_p \text{3-SAT}$  since 3-SAT is in NP.

Let  $K$  be any circuit. We can construct an instance of 3-SAT  $\Phi$ :

Step 1: Create a 3-SAT variable  $x_i$  for each circuit element  $i$ .

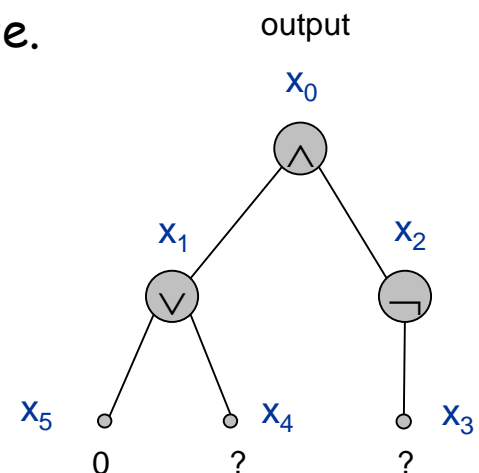
Step 2: Make circuit compute correct values at each node:

- $x_2 = \neg x_3 \Rightarrow$  add 2 clauses:  $x_2 \vee x_3, \overline{x_2} \vee \overline{x_3}$
- $x_1 = x_4 \vee x_5 \Rightarrow$  add 3 clauses:  $x_1 \vee \overline{x_4}, x_1 \vee \overline{x_5}, \overline{x_1} \vee x_4 \vee x_5$
- $x_0 = x_1 \wedge x_2 \Rightarrow$  add 3 clauses:  $\overline{x_0} \vee x_1, \overline{x_0} \vee x_2, x_0 \vee \overline{x_1} \vee \overline{x_2}$

Step 3: Hard-coded input values and output value.

- $x_5 = 0 \Rightarrow$  add 1 clause:  $\overline{x_5}$
- $x_0 = 1 \Rightarrow$  add 1 clause:  $x_0$

Final step: turn clauses of length  $< 3$  into clauses of length exactly 3 (see book p472) ■



# 3-SAT is NP-Complete

**Lemma.**  $\Phi$  is satisfiable iff. the inputs of  $K$  can be set s.t. it outputs 1.

**Pf.**  $\Leftarrow$

Suppose there are inputs of  $K$  s.t. it outputs 1.

Compute the values of all the nodes in  $K$ .

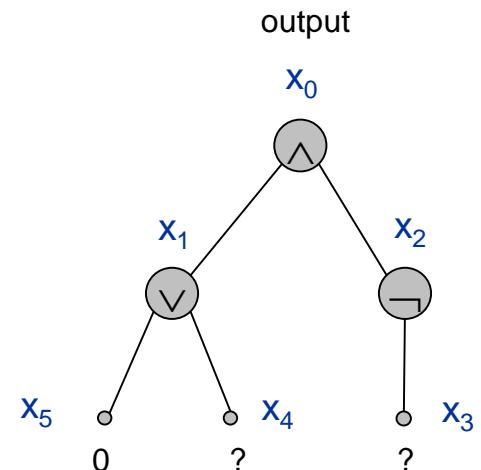
According to our construction, this set of values satisfies  $\Phi$ .

$\Rightarrow$

Suppose  $\Phi$  is satisfiable.

The variable assignment of  $\Phi$  specifies the values of all the nodes in  $K$ .

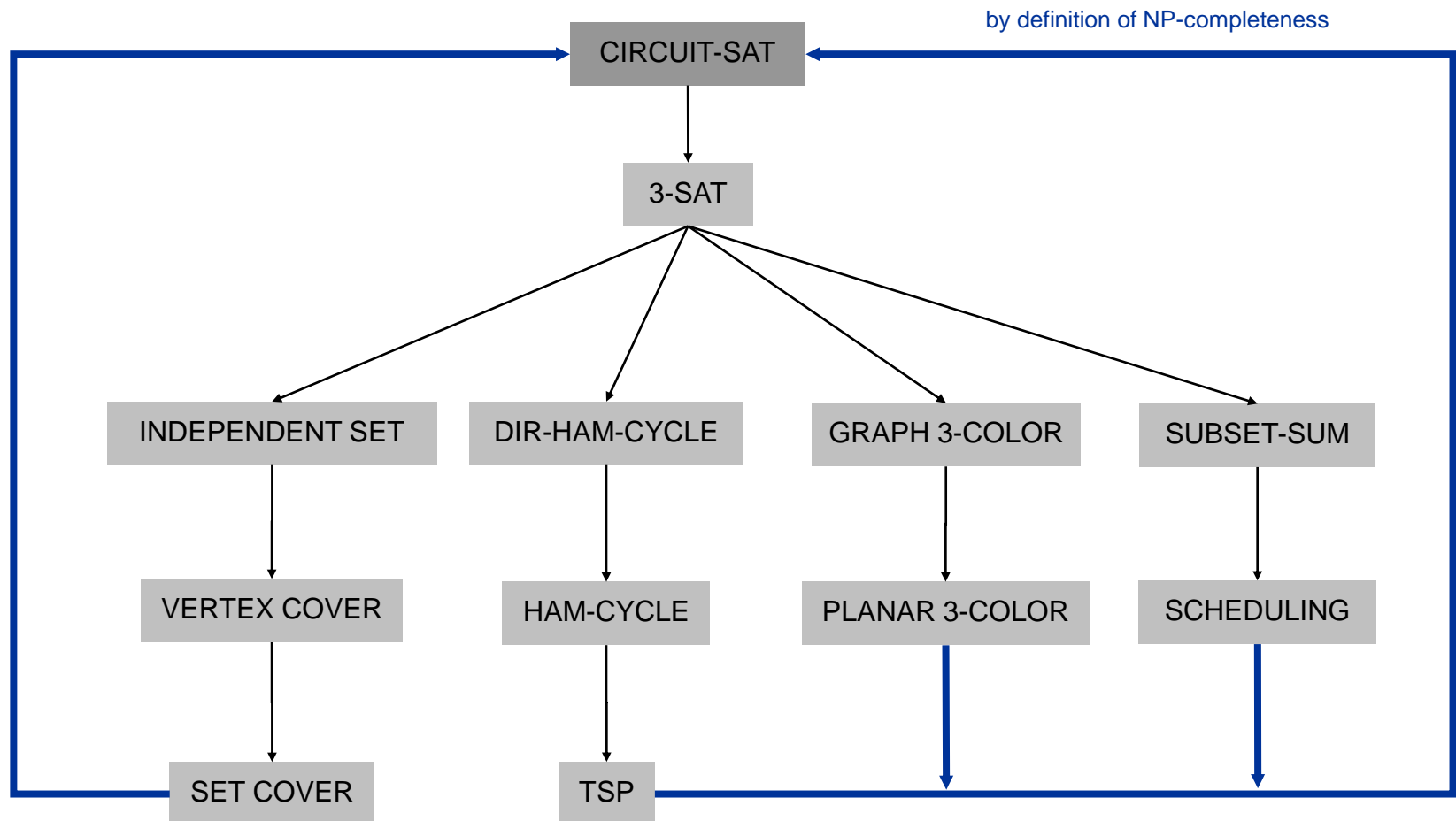
Our construction ensures that all the nodes in  $K$  are correctly computed and the output is 1.





# NP-Completeness

**Observation.** All problems below are NP-complete and polynomial reduce to one another!



# Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

Packing problems: SET-PACKING, INDEPENDENT SET.

Covering problems: SET-COVER, VERTEX-COVER.

Constraint satisfaction problems: SAT, 3-SAT.

Sequencing problems: HAMILTONIAN-CYCLE, TSP.

Partitioning problems: 3D-MATCHING, 3-COLOR.

Numerical problems: SUBSET-SUM, KNAPSACK.

**Practice.** Most NP problems are known to be either in P or NP-complete.

**Notable exceptions.** Factoring, graph isomorphism, Nash equilibrium.

**Theorem.** [Ladner 1975] Unless  $P = NP$ , there exist problems in NP that are neither in P nor NP-complete.

# Extent and Impact of NP-Completeness

Extent of NP-completeness. [Papadimitriou 1995]

6,000 citations per year (title, abstract, keywords).

- more than "compiler", "operating system", "database"

Prime intellectual export of CS to other disciplines.

Broad applicability and classification power.

"Captures vast domains of computational, scientific, mathematical endeavors, and seems to roughly delimit what mathematicians and scientists had been aspiring to compute feasibly."

NP-completeness can guide scientific inquiry.

1926: Ising introduces simple model for phase transitions.

1944: Onsager finds closed-form solution to 2D-Ising.

19xx: Feynman and other top minds seek 3D solution.

2000: Istrail proves 3D-Ising is NP-complete; thus search for closed-form solution to 3D-Ising appears doomed.

# More Hard Computational Problems

- Aerospace engineering:** optimal mesh partitioning for finite elements.
- Biology:** protein folding.
- Chemical engineering:** heat exchanger network synthesis.
- Civil engineering:** equilibrium of urban traffic flow.
- Economics:** computation of arbitrage in financial markets with friction.
- Electrical engineering:** VLSI layout.
- Environmental engineering:** optimal placement of contaminant sensors.
- Financial engineering:** find minimum risk portfolio of given return.
- Game theory:** find Nash equilibrium that maximizes social welfare.
- Genomics:** phylogeny reconstruction.
- Mechanical engineering:** structure of turbulence in sheared flows.
- Medicine:** reconstructing 3-D shape from biplane angiocardiogram.
- Operations research:** optimal resource allocation.
- Physics:** partition function of 3-D Ising model in statistical mechanics.
- Politics:** Shapley-Shubik voting power.
- Pop culture:** Minesweeper consistency.
- Statistics:** optimal experimental design.

# Terminology

**NP-complete.** A problem in NP such that every problem in NP polynomial reduces to it.

**NP-hard.** [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]  
A problem such that every problem in NP reduces to it.



Stephen Cook

## Stephen Cook

1971 paper "The Complexity of Theorem Proving Procedures"

- formalized the notions of polynomial-time reduction and NP-completeness
- proved the existence of an NP-complete problem (SAT)

### Bio

- Joined UC Berkeley in 1966 but was denied tenure in 1970
- Joined U of Toronto in 1970
- 1982 Turing award



Richard Karp

"It is to our everlasting shame that we were unable to persuade the math department to give him tenure." -- Richard Karp (1985 Turing award)

## 8.9 co-NP and the Asymmetry of NP

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# Asymmetry of NP

NP. Decision problems for which there is a poly-time certifier.

Ex. SAT, HAM-CYCLE, COMPOSITES.

Asymmetry of NP. We only need to have short proofs of *yes* instances.

It may not be easy to show proofs of *no* instances.

Ex 1. SAT vs. TAUTOLOGY.

Can prove a CNF formula is satisfiable by giving such an assignment.

How could we prove that a formula is **not** satisfiable?

Ex 2. HAM-CYCLE vs. NO-HAM-CYCLE.

Can prove a graph is Hamiltonian by giving such a Hamiltonian cycle.

How could we prove that a graph is **not** Hamiltonian?

Q. how do we classify TAUTOLOGY?

SAT is NP-complete and  $SAT \equiv_p TAUTOLOGY$

But TAUTOLOGY is not even known to be in NP



## NP and co-NP

**NP.** Decision problems for which there is a poly-time certifier.

**Ex.** SAT, HAM-CYCLE, COMPOSITES.

**Def.** Given a decision problem  $X$ , its **complement**  $\overline{X}$  is the same problem with the <sub>yes</sub> and <sub>no</sub> answers reverse.

**Ex.**  $\overline{X} = \{ 0, 1, 4, 6, 8, 9, 10, 12, 14, 15, \dots \}$   
 $X = \{ 2, 3, 5, 7, 11, 13, 17, 23, 29, \dots \}$

**co-NP.** Complements of decision problems in NP.

- for <sub>no</sub> instance, there is a succinct disqualifier

**Ex.** TAUTOLOGY, NO-HAM-CYCLE, PRIMES.

# NP = co-NP ?

**Fundamental question.** Does  $NP = co-NP$ ?

Do <sub>yes</sub> instances have succinct certificates iff <sub>no</sub> instances do?

Consensus opinion: no.

**Theorem.** If  $NP \neq co-NP$ , then  $P \neq NP$ .

**Pf idea.**

$P$  is closed under complementation.

If  $P = NP$ , then  $NP$  is closed under complementation.

In other words,  $NP = co-NP$ .

This is the contrapositive of the theorem.

# Good Characterizations

Good characterization. [Edmonds 1965]  $NP \cap co-NP$ .

If problem X is in both NP and co-NP, then:

- for <sub>yes</sub> instance, there is a succinct certificate
- for <sub>no</sub> instance, there is a succinct disqualifier

Provides conceptual leverage for reasoning about a problem.

Ex. Given a bipartite graph, is there a perfect matching.

If yes, can exhibit a perfect matching.

If no, can exhibit a set of nodes S such that  $|N(S)| < |S|$ .

# Good Characterizations

Observation.  $P \subseteq NP \cap \text{co-NP}$ .

Fundamental open question. Does  $P = NP \cap \text{co-NP}$ ?

Mixed opinions.

Many examples where problem found to have a non-trivial good characterization, but only years later discovered to be in P.

- linear programming [Khachiyan, 1979]
- primality testing [Agrawal-Kayal-Saxena, 2002]

Fact. Factoring is in  $NP \cap \text{co-NP}$ , but not known to be in P.



if poly-time algorithm for factoring,  
can break RSA cryptosystem

## FACTOR is in $NP \cap co-NP$

**FACTORIZE.** Given an integer  $x$ , find its prime factorization.

**FACTOR.** Given two integers  $x$  and  $y$ , does  $x$  have a nontrivial factor less than  $y$ ?

**Theorem.**  $FACTOR \equiv_p FACTORIZE$ .

**Pf.** Binary search to find a factor, divide the factor and repeat

**Theorem.**  $FACTOR$  is in  $NP \cap co-NP$ .

**Pf.**

Certificate: a factor  $p$  of  $x$  that is less than  $y$ .

Disqualifier: the prime factorization of  $x$  (where each prime factor is  $\geq y$ ), along with a Pratt's certificate that each factor is prime.

# Primality Testing and Factoring

Q: Is PRIMES in P?

Yes! (Agrawal-Kayal-Saxena, 2002)

Q: Is FACTOR in P?

Consensus: No.

RSA cryptosystem.

Based on dichotomy between complexity of two problems.

To use RSA, must generate large primes efficiently.

To break RSA, must find efficient factoring algorithm.

## 8.5 Sequencing Problems

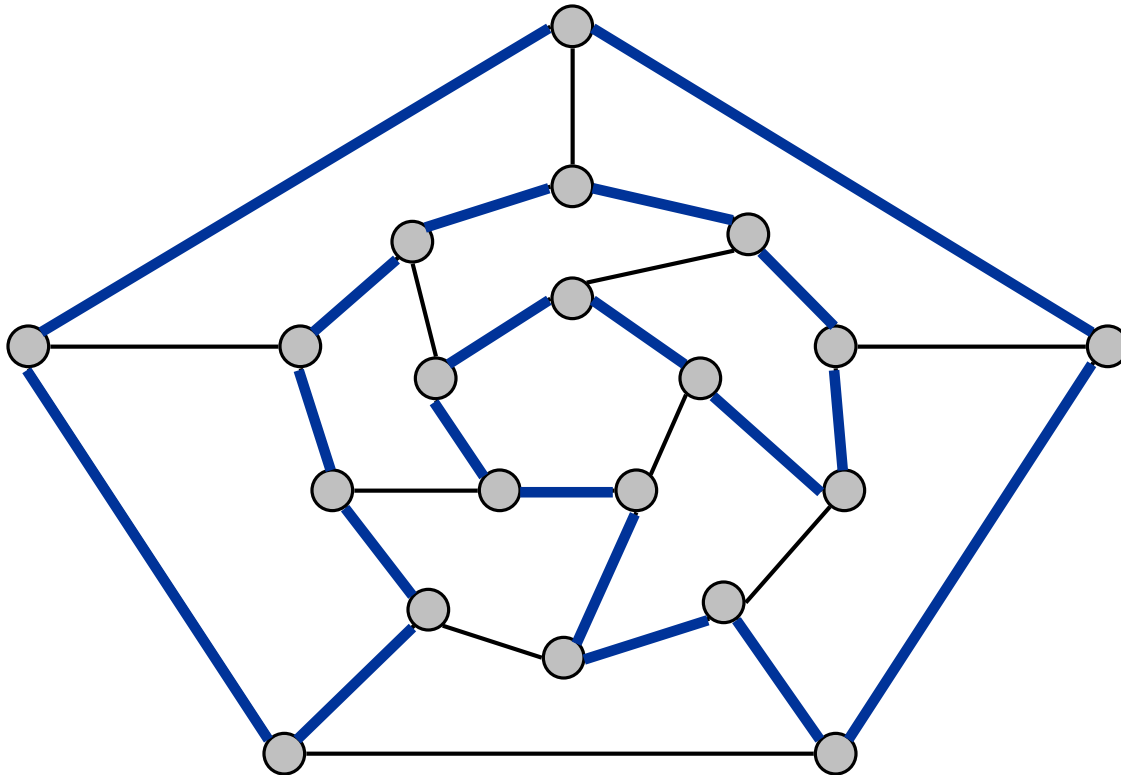
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Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- **Sequencing problems:** HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

# Hamiltonian Cycle

**HAM-CYCLE:** given an undirected graph  $G = (V, E)$ , does there exist a simple cycle  $\Gamma$  that contains every node in  $V$ .

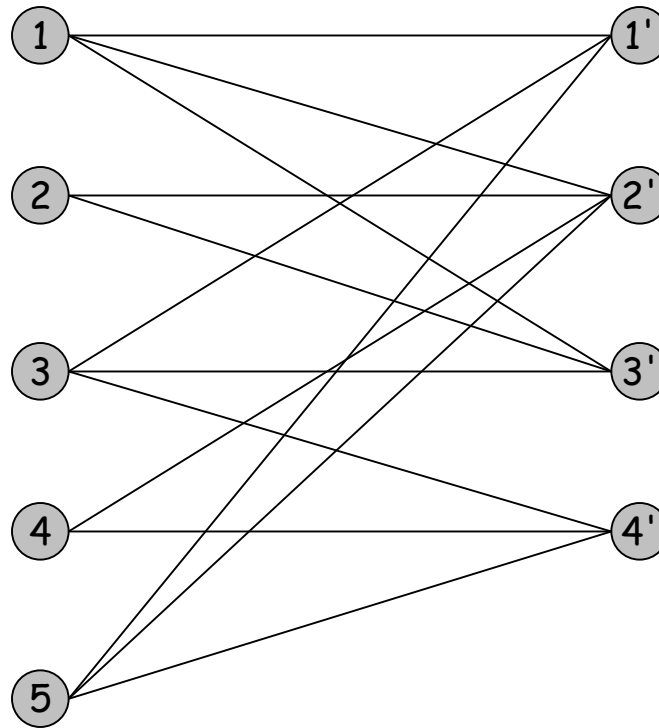


YES: vertices and faces of a dodecahedron.



# Hamiltonian Cycle

**HAM-CYCLE:** given an undirected graph  $G = (V, E)$ , does there exist a simple cycle  $\Gamma$  that contains every node in  $V$ .



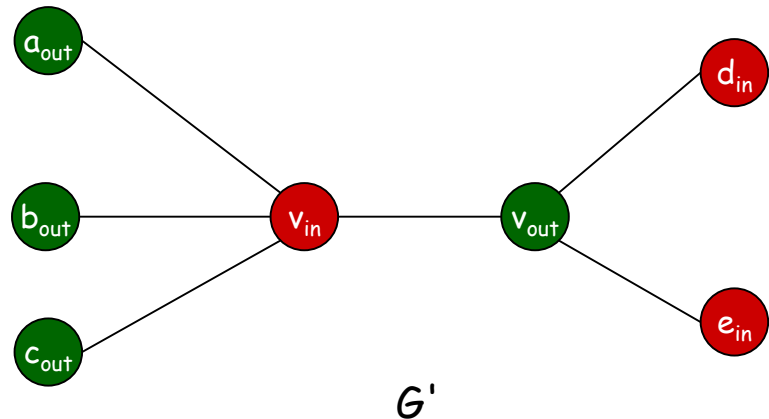
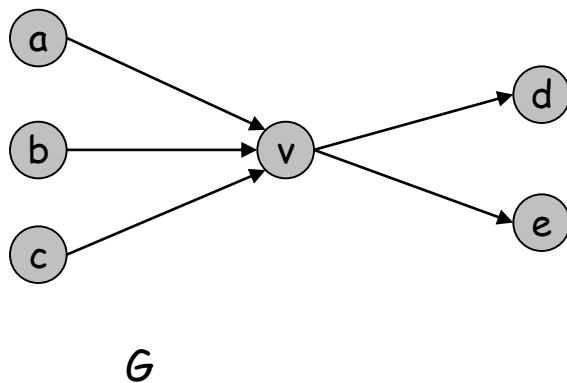
NO: bipartite graph with odd number of nodes.

# Directed Hamiltonian Cycle

**DIR-HAM-CYCLE:** given a **digraph**  $G = (V, E)$ , does there exist a simple directed cycle  $\Gamma$  that contains every node in  $V$ ?

**Claim.**  $\text{DIR-HAM-CYCLE} \leq_p \text{HAM-CYCLE}$ .

**Pf.** Given a directed graph  $G = (V, E)$ , construct an undirected graph  $G'$

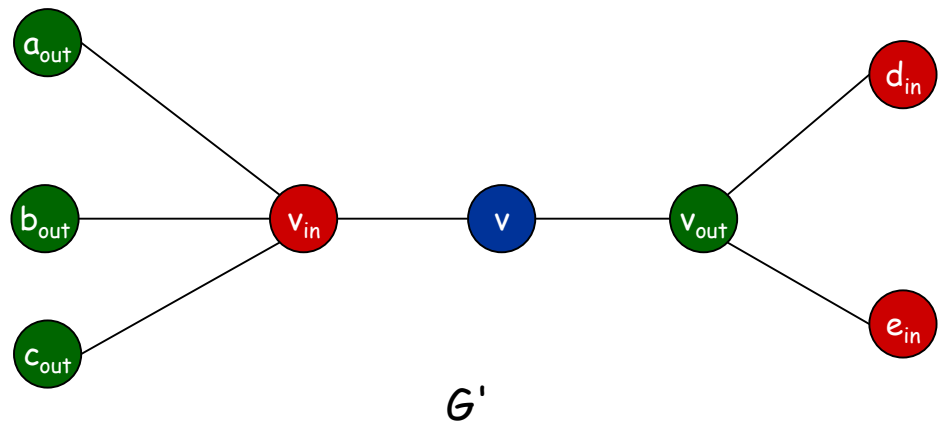
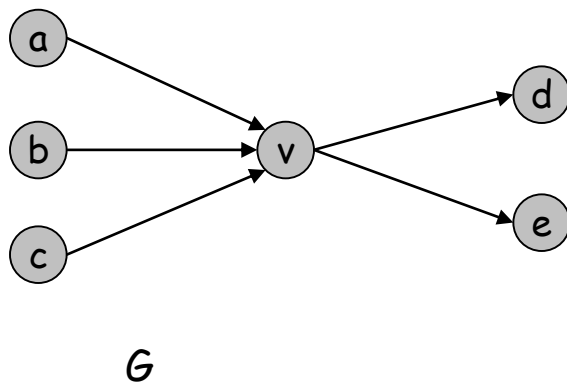


# Directed Hamiltonian Cycle

**DIR-HAM-CYCLE:** given a **digraph**  $G = (V, E)$ , does there exist a simple directed cycle  $\Gamma$  that contains every node in  $V$ ?

**Claim.**  $\text{DIR-HAM-CYCLE} \leq_p \text{HAM-CYCLE}$ .

**Pf.** Given a directed graph  $G = (V, E)$ , construct an undirected graph  $G'$  with  $3n$  nodes.



# Directed Hamiltonian Cycle

**Claim.**  $G$  has a Hamiltonian cycle iff  $G'$  does.

**Pf.**  $\Rightarrow$

Suppose  $G$  has a directed Hamiltonian cycle  $\Gamma$ .

Then  $G'$  has an undirected Hamiltonian cycle (same order).

**Pf.**  $\Leftarrow$

Suppose  $G'$  has an undirected Hamiltonian cycle  $\Gamma'$ .

$\Gamma'$  must visit nodes in  $G'$  using one of following two orders:

..., B, G, R, B, G, R, B, G, R, B, ...

..., B, R, G, B, R, G, B, R, G, B, ...

Blue nodes in  $\Gamma'$  make up directed Hamiltonian cycle  $\Gamma$  in  $G$ , or reverse of one. ■

## 3-SAT Reduces to Directed Hamiltonian Cycle

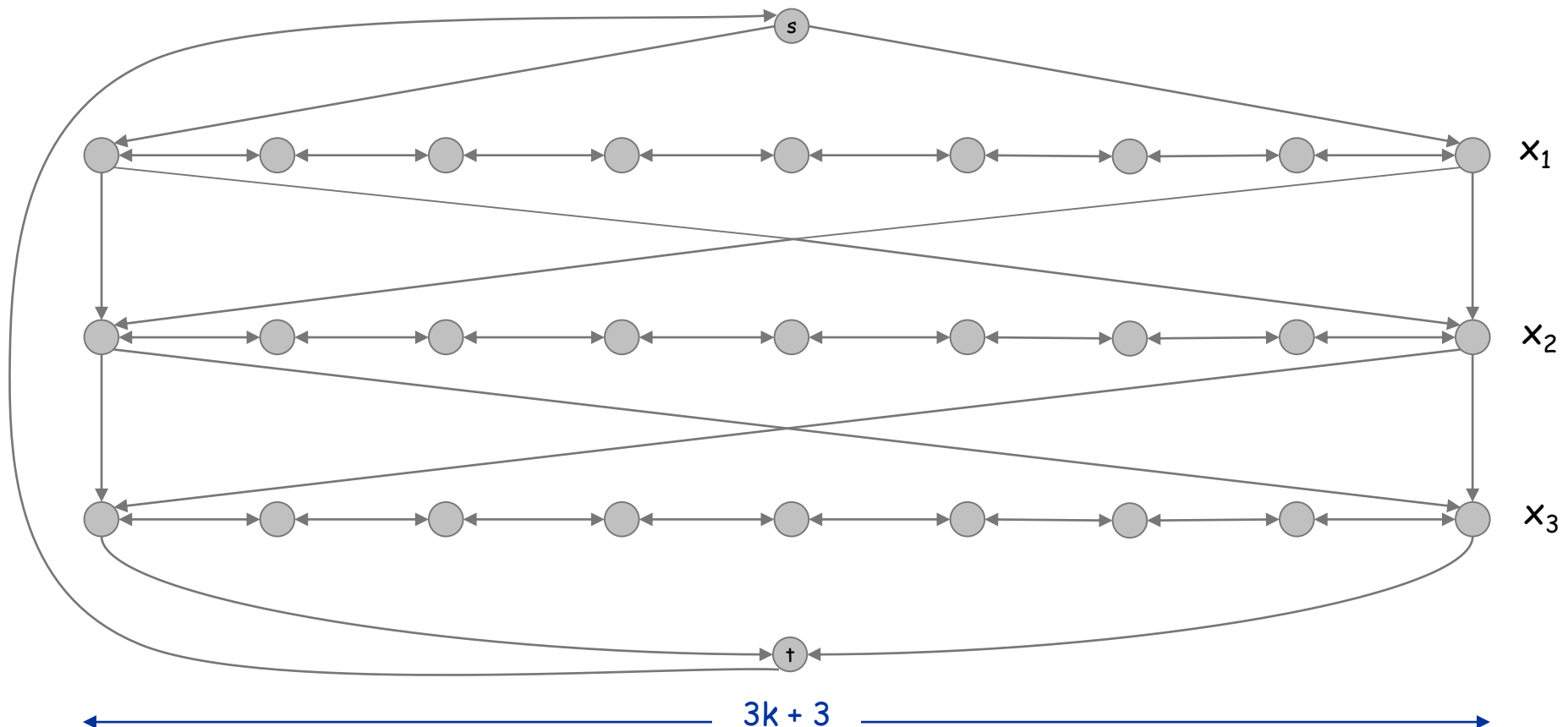
**Claim.**  $3\text{-SAT} \leq_p \text{DIR-HAM-CYCLE}$ .

**Pf.** Given an instance  $\Phi$  of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff  $\Phi$  is satisfiable.

# 3-SAT Reduces to Directed Hamiltonian Cycle

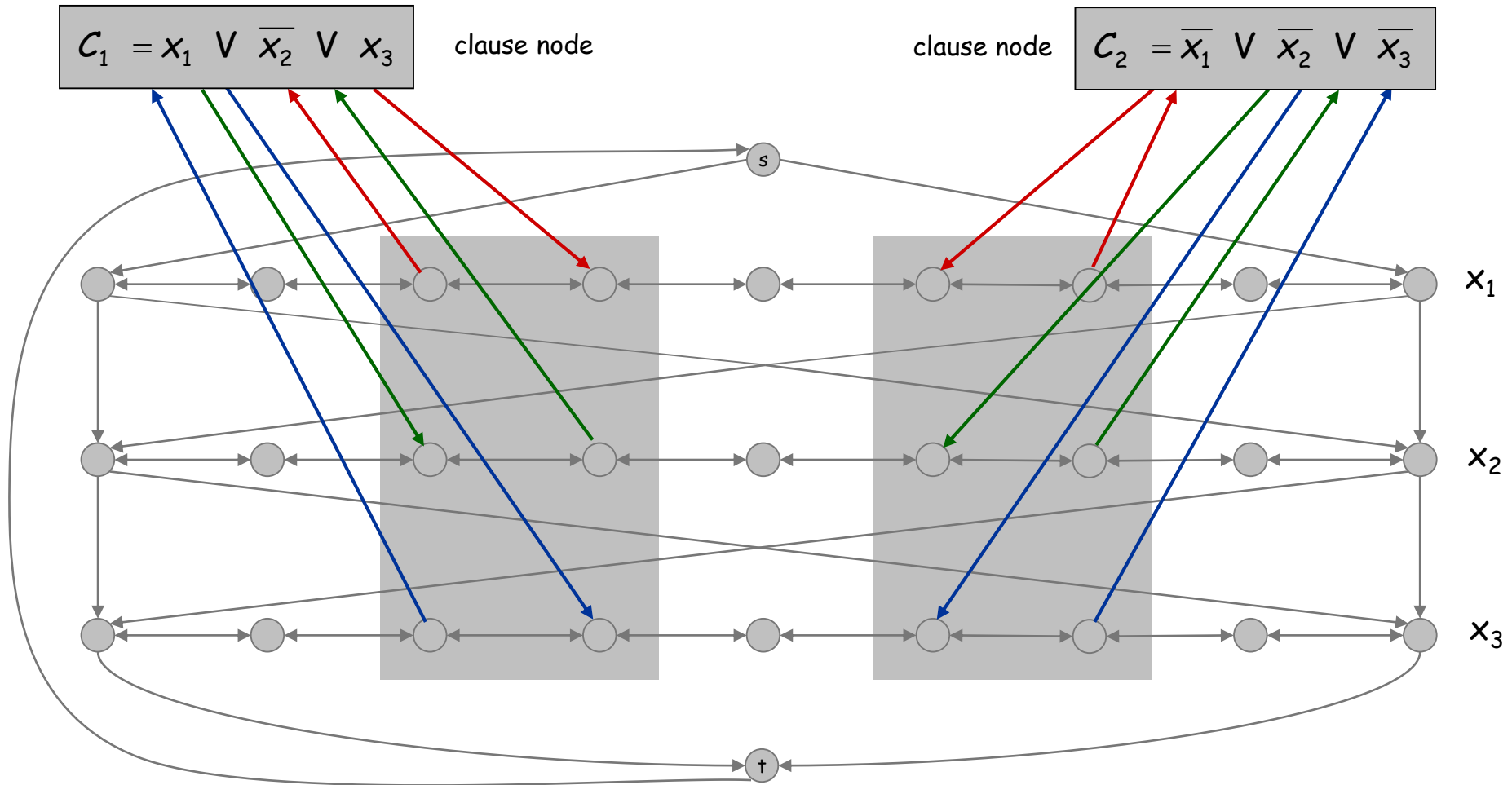
**Construction.** Given 3-SAT instance  $\Phi$  with  $n$  variables  $x_i$  and  $k$  clauses. Construct  $G$  to have  $2^n$  Hamiltonian cycles.

Intuition: traverse path  $i$  from left to right  $\Leftrightarrow$  set variable  $x_i = 1$ .



# 3-SAT Reduces to Directed Hamiltonian Cycle

**Construction.** Given 3-SAT instance  $\Phi$  with  $n$  variables  $x_i$  and  $k$  clauses. For each clause: add a node and 6 edges.



## 3-SAT Reduces to Directed Hamiltonian Cycle

**Claim.**  $\Phi$  is satisfiable iff  $G$  has a Hamiltonian cycle.

**Pf.**  $\Rightarrow$

Suppose 3-SAT instance has satisfying assignment  $x^*$ .

Then, define Hamiltonian cycle in  $G$  as follows:

- if  $x_i^* = 1$ , traverse row  $i$  from left to right
- if  $x_i^* = 0$ , traverse row  $i$  from right to left
- for each clause  $C_j$ , there will be at least one row  $i$  in which we are going in "correct" direction to splice node  $C_j$  into tour



## 3-SAT Reduces to Directed Hamiltonian Cycle

**Claim.**  $\Phi$  is satisfiable iff  $G$  has a Hamiltonian cycle.

**Pf.**  $\Leftarrow$

Suppose  $G$  has a Hamiltonian cycle  $\Gamma$ .

If  $\Gamma$  enters clause node  $C_j$ , it must depart on mate edge.

- thus, nodes immediately before and after  $C_j$  are connected by an edge  $e$  in  $G$
- removing  $C_j$  from cycle, and replacing it with edge  $e$  yields Hamiltonian cycle on  $G - \{C_j\}$

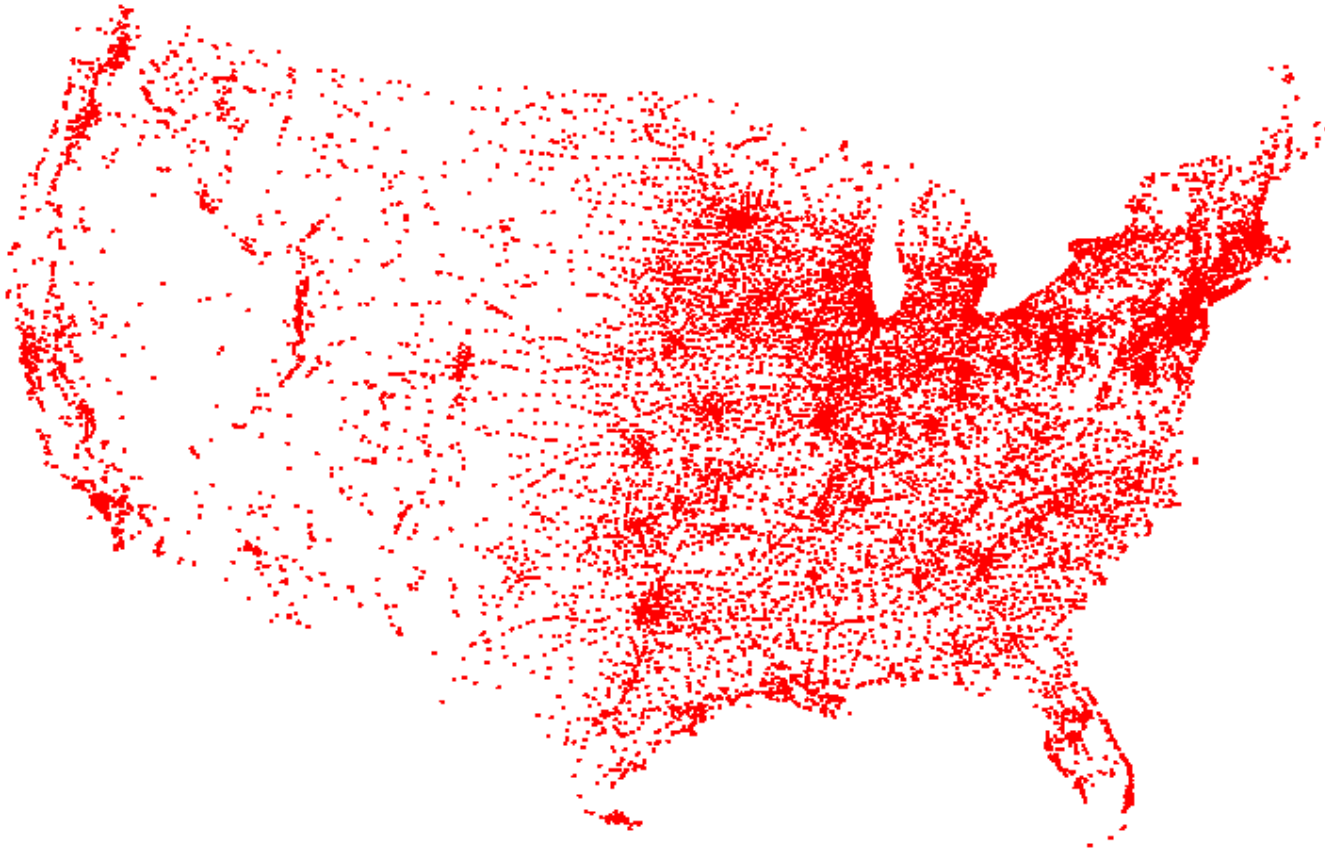
Continuing in this way, we are left with Hamiltonian cycle  $\Gamma'$  in  $G - \{C_1, C_2, \dots, C_k\}$ .

Set  $x_i^* = 1$  iff  $\Gamma'$  traverses row  $i$  left to right.

Since  $\Gamma$  visits each clause node  $C_j$ , at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ■

# Traveling Salesperson Problem

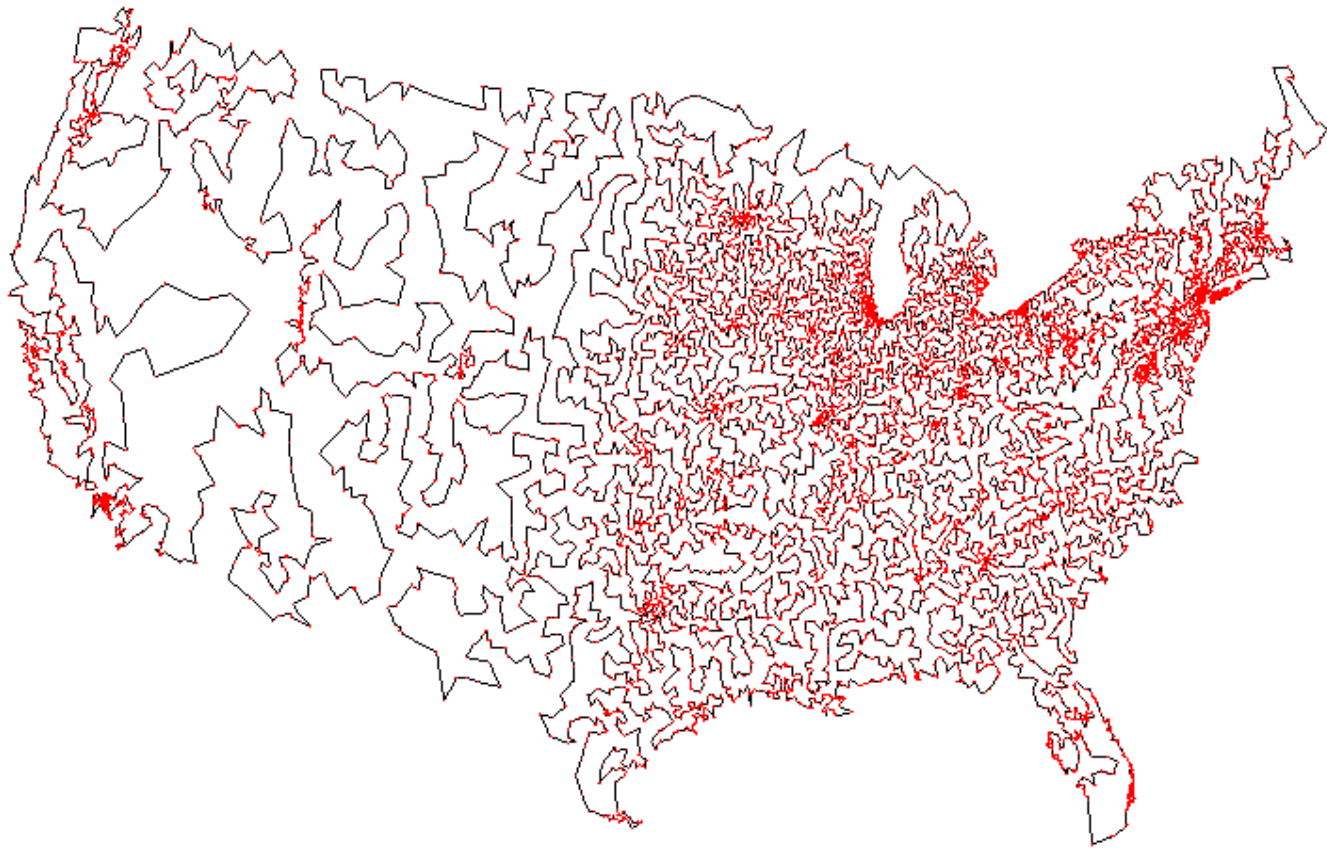
**TSP.** Given a set of  $n$  cities and a pairwise distance function  $d(u, v)$ , is there a tour of length  $\leq D$ ?



All 13,509 cities in US with a population of at least 500  
Reference: <http://www.tsp.gatech.edu>

# Traveling Salesperson Problem

**TSP.** Given a set of  $n$  cities and a pairwise distance function  $d(u, v)$ , is there a tour of length  $\leq D$ ?



Optimal TSP tour  
Reference: <http://www.tsp.gatech.edu>

# Traveling Salesperson Problem

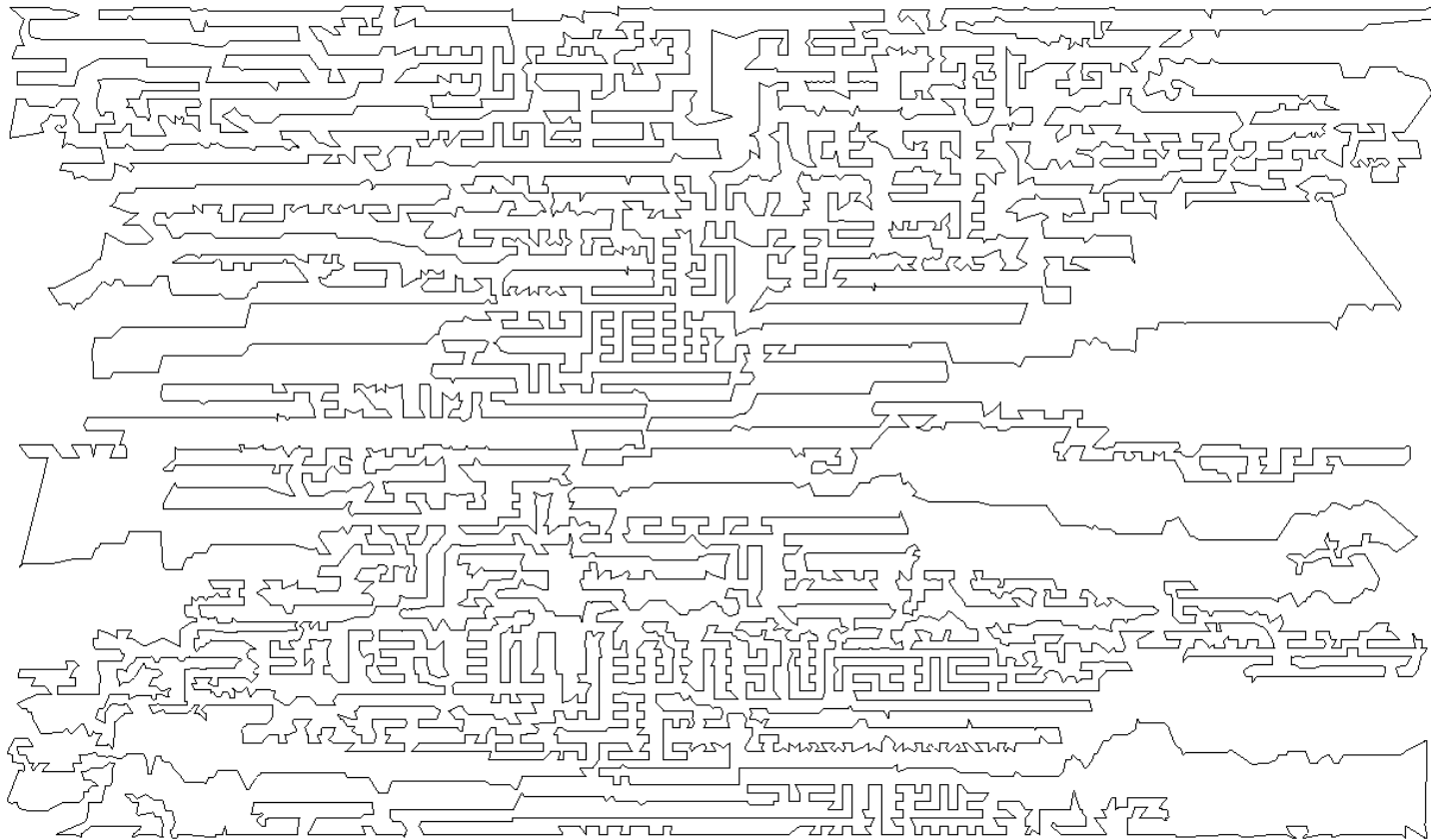
**TSP.** Given a set of  $n$  cities and a pairwise distance function  $d(u, v)$ , is there a tour of length  $\leq D$ ?



11,849 holes to drill in a programmed logic array  
Reference: <http://www.tsp.gatech.edu>

# Traveling Salesperson Problem

**TSP.** Given a set of  $n$  cities and a pairwise distance function  $d(u, v)$ , is there a tour of length  $\leq D$ ?



Optimal TSP tour  
Reference: <http://www.tsp.gatech.edu>

# Traveling Salesperson Problem

**TSP.** Given a set of  $n$  cities and a pairwise distance function  $d(u, v)$ , is there a tour of length  $\leq D$ ?

**HAM-CYCLE:** given a graph  $G = (V, E)$ , does there exist a simple cycle that contains every node in  $V$ ?

**Claim.**  $\text{HAM-CYCLE} \leq_p \text{TSP}$ .

**Pf.**

Given instance  $G = (V, E)$  of HAM-CYCLE, create  $n$  cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

TSP instance has tour of length  $\leq n$  iff  $G$  is Hamiltonian. ■

# Longest Path

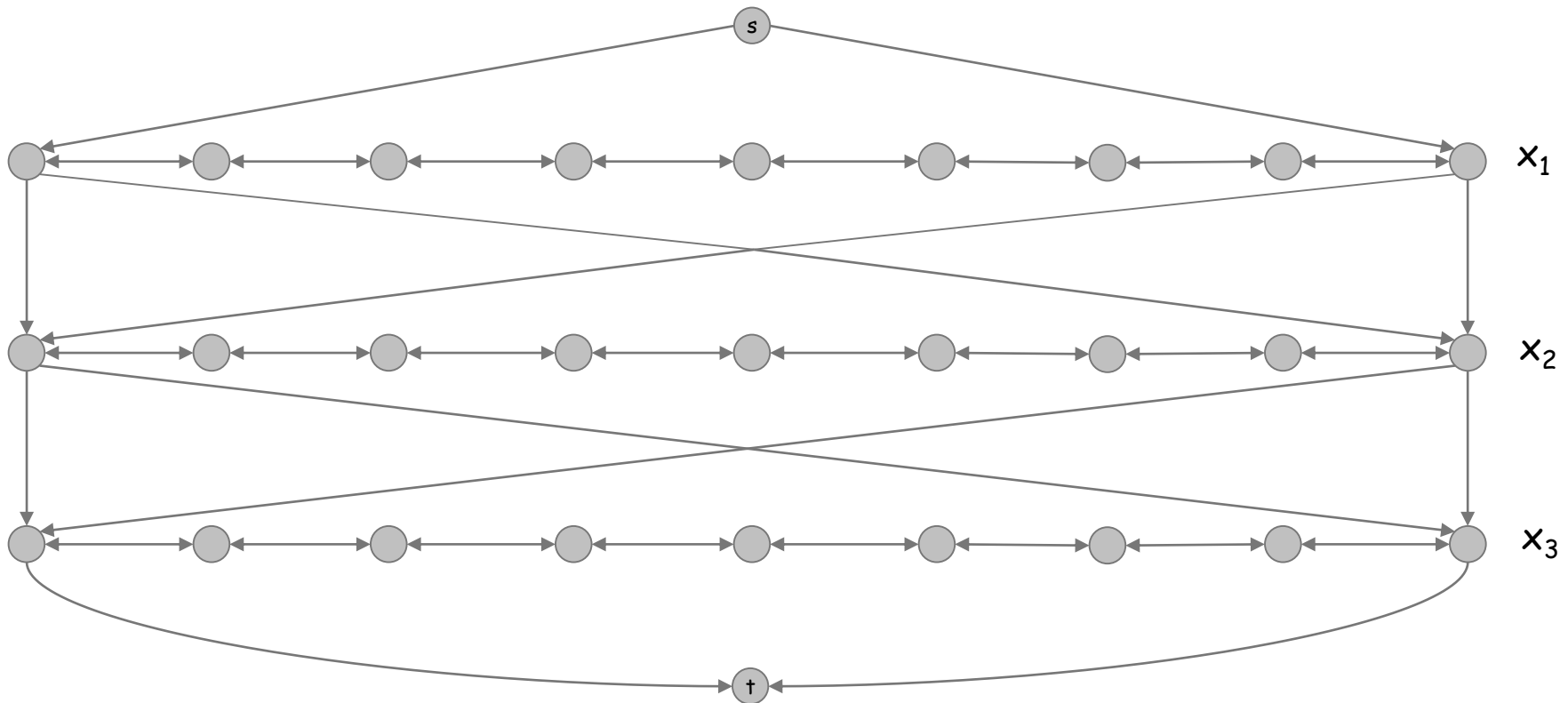
**SHORTEST-PATH.** Given a digraph  $G = (V, E)$ , does there exist a simple path of length **at most**  $k$  edges?

**LONGEST-PATH.** Given a digraph  $G = (V, E)$ , does there exist a simple path of length **at least**  $k$  edges?

# Longest Path

**Claim.**  $3\text{-SAT} \leq_p \text{LONGEST-PATH}$ .

**Pf 1.** Redo proof for DIR-HAM-CYCLE, ignoring back-edge from  $t$  to  $s$ .



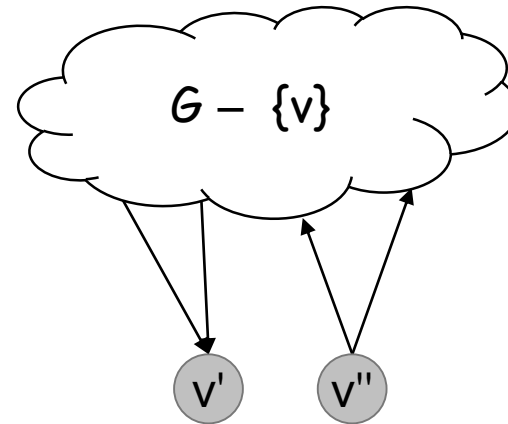
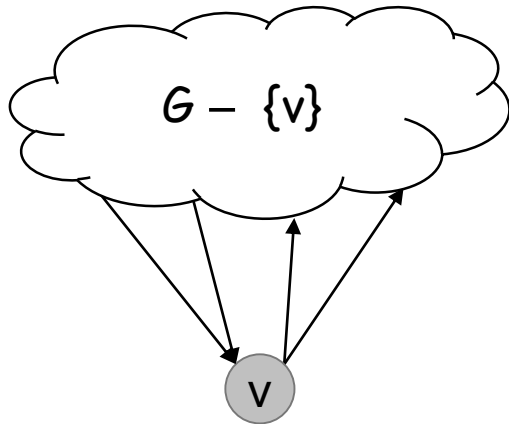


# Longest Path

**Claim.**  $3\text{-SAT} \leq_p \text{LONGEST-PATH}$ .

**Pf 2.** Show  $\text{DIR-HAM-CYCLE} \leq_p \text{LONGEST-PATH}$ .

Given an instance of DIR-HAM-CYCLE, split an arbitrary node  $v$  of the digraph into two nodes  $v'$  and  $v''$ ; connect all the incoming edges to  $v'$  and all the outgoing edges to  $v''$ .



# The Longest Path †

**Lyrics.** Copyright © 1988 by Daniel J. Barrett.

**Music.** Sung to the tune of *The Longest Time* by Billy Joel.



Woh-oh-oh-oh, find the longest path!  
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,  
There would still be papers left to write,  
I have a weakness,  
I'm addicted to completeness,  
And I keep searching for the longest  
path.

The algorithm I would like to see  
Is of polynomial degree,  
But it's elusive:  
Nobody has found conclusive  
Evidence that we can find a longest path.

I have been hard working for so long.  
I swear it's right, and he marks it wrong.  
Some how I'll feel sorry when it's done:  
GPA 2.1  
Is more than I hope for.

Garey, Johnson, Karp and other men (and  
women)  
Tried to make it order  $N \log N$ .  
Am I a mad fool  
If I spend my life in grad school,  
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!  
Woh-oh-oh-oh, find the longest path!  
Woh-oh-oh-oh, find the longest path.

† Recorded by Dan Barrett while a grad student at Johns Hopkins during a difficult algorithms final.

## 8.6 Partitioning Problems

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Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- **Partitioning problems:** 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

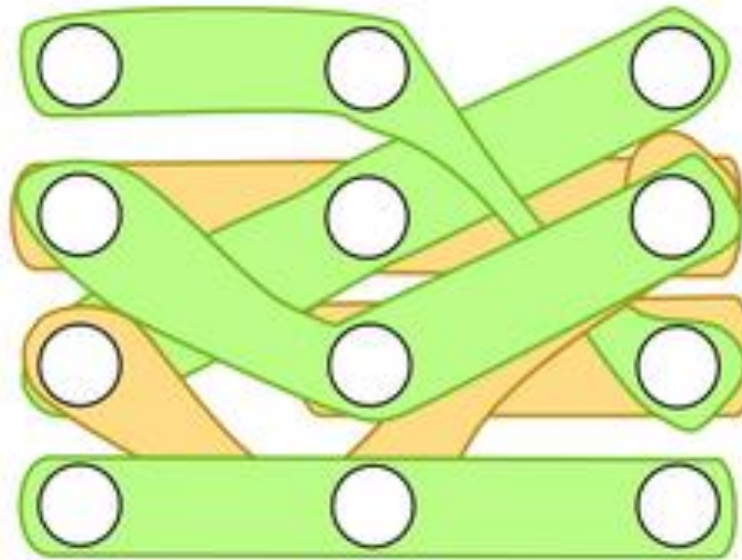
## 3-Dimensional Matching

**3D-MATCHING.** Given  $n$  instructors,  $n$  courses, and  $n$  times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

Instructor	Course	Time
Wayne	COS 423	MW 11-12:20
Wayne	COS 423	TTh 11-12:20
Wayne	COS 226	TTh 11-12:20
Tardos	COS 523	TTh 3-4:20
Tardos	COS 423	TTh 11-12:20
Tardos	COS 423	TTh 3-4:20
Kleinberg	COS 226	TTh 3-4:20
Kleinberg	COS 226	MW 11-12:20
Kleinberg	COS 423	MW 11-12:20

## 3-Dimensional Matching

**3D-MATCHING.** Given disjoint sets  $X$ ,  $Y$ , and  $Z$ , each of size  $n$  and a set  $T \subseteq X \times Y \times Z$  of triples, does there exist a set of  $n$  triples in  $T$  such that each element of  $X \cup Y \cup Z$  is in exactly one of these triples?



## 3-Dimensional Matching

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**Claim.**  $3\text{-SAT} \leq_p 3\text{D-MATCHING}$ .

**Pf.** Given an instance  $\Phi$  of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff  $\Phi$  is satisfiable.

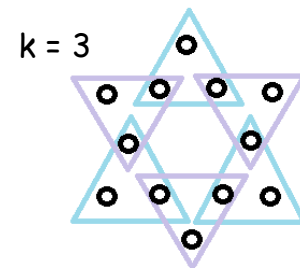
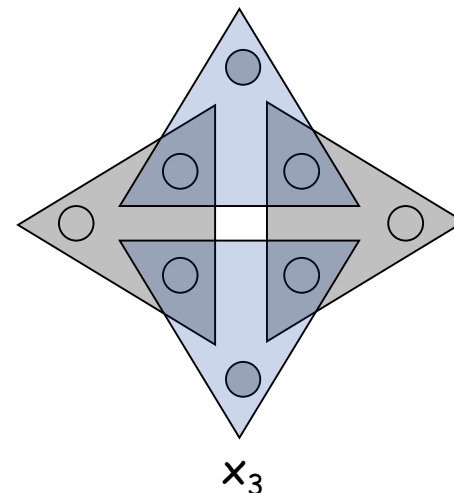
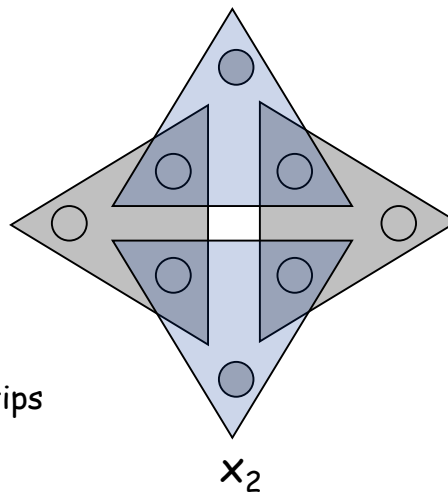
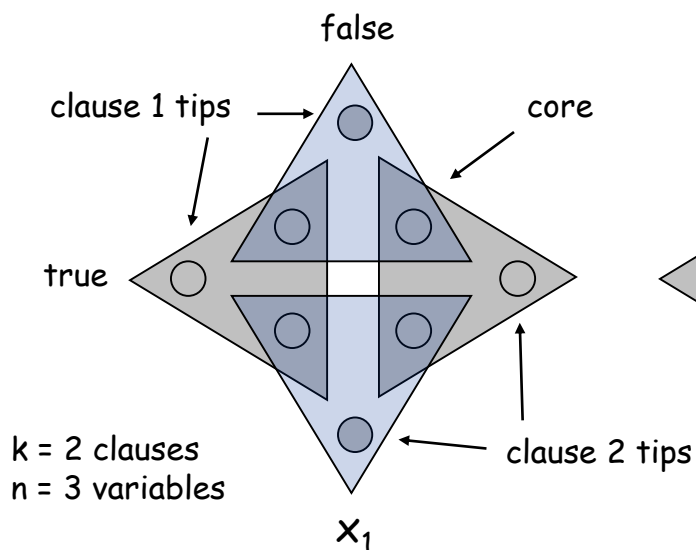
# 3-Dimensional Matching

## Construction. (part 1)

Create gadget for each variable  $x_i$  with  $2k$  core elements and  $2k$  tip elements.

No other triples will use core elements.

In gadget  $i$ , 3D-matching must use either all the grey triples (corresponding to  $x_i = \text{true}$ ) or all the blue ones (corresponding to  $x_i = \text{false}$ ).



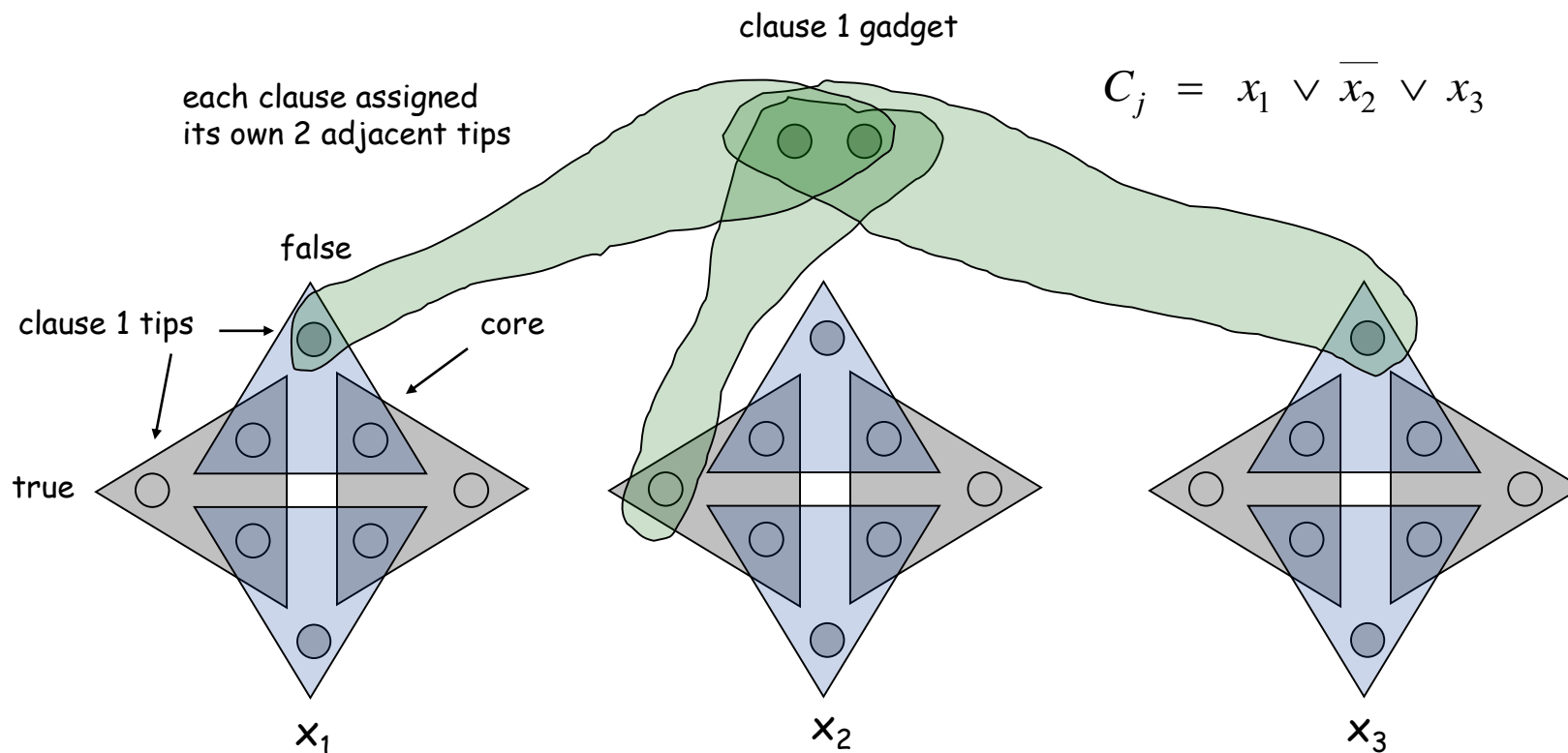
## 3-Dimensional Matching

## Construction. (part 2)

For each clause  $C_i$  create gadget with two elements and three triples.

Exactly one of these triples will be used in any 3D-matching.

Ensures any 3D-matching uses either (i) grey core of  $x_1$  or (ii) blue core of  $x_2$  or (iii) grey core of  $x_3$ .



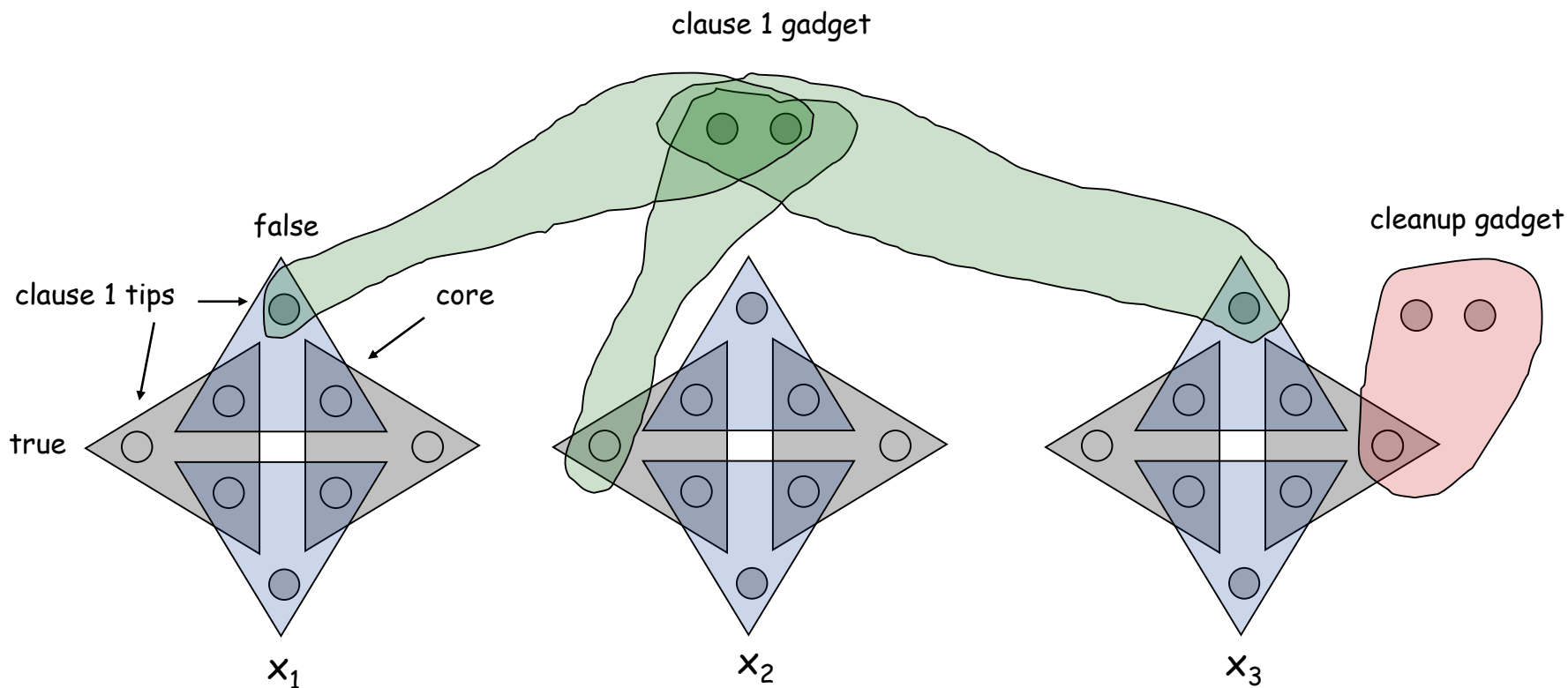


# 3-Dimensional Matching

## Construction. (part 3)

There are  $2nk$  tips.  $nk$  covered by blue/gray triples,  $k$  covered by clause triples.

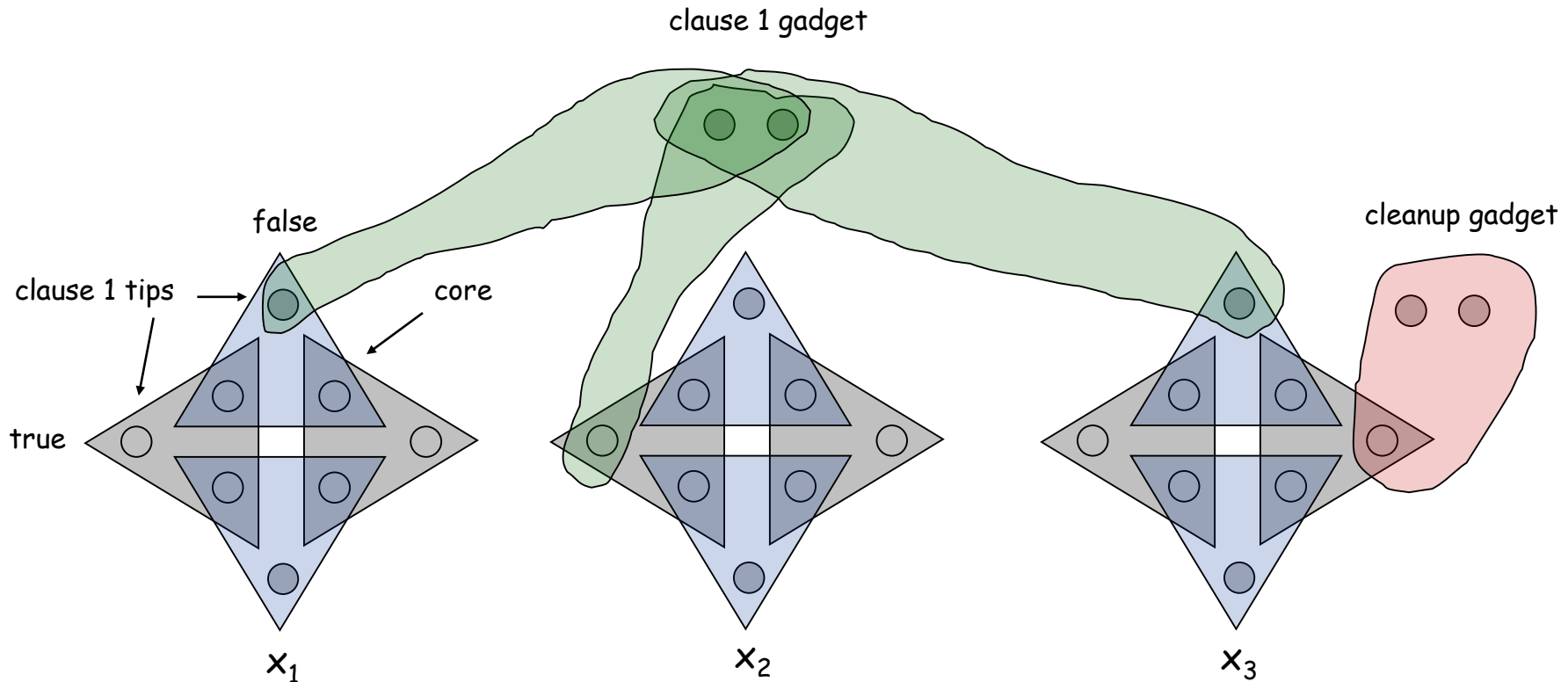
To cover the remaining  $(n-1)k$  tips, create  $(n-1)k$  cleanup gadgets, each connected to all the  $2nk$  tips.



# 3-Dimensional Matching

**Claim.** Instance has a 3D-matching iff  $\Phi$  is satisfiable.

**Q.** What are  $X$ ,  $Y$ , and  $Z$ ? Does each triple contain one element from each of  $X$ ,  $Y$ ,  $Z$ ?

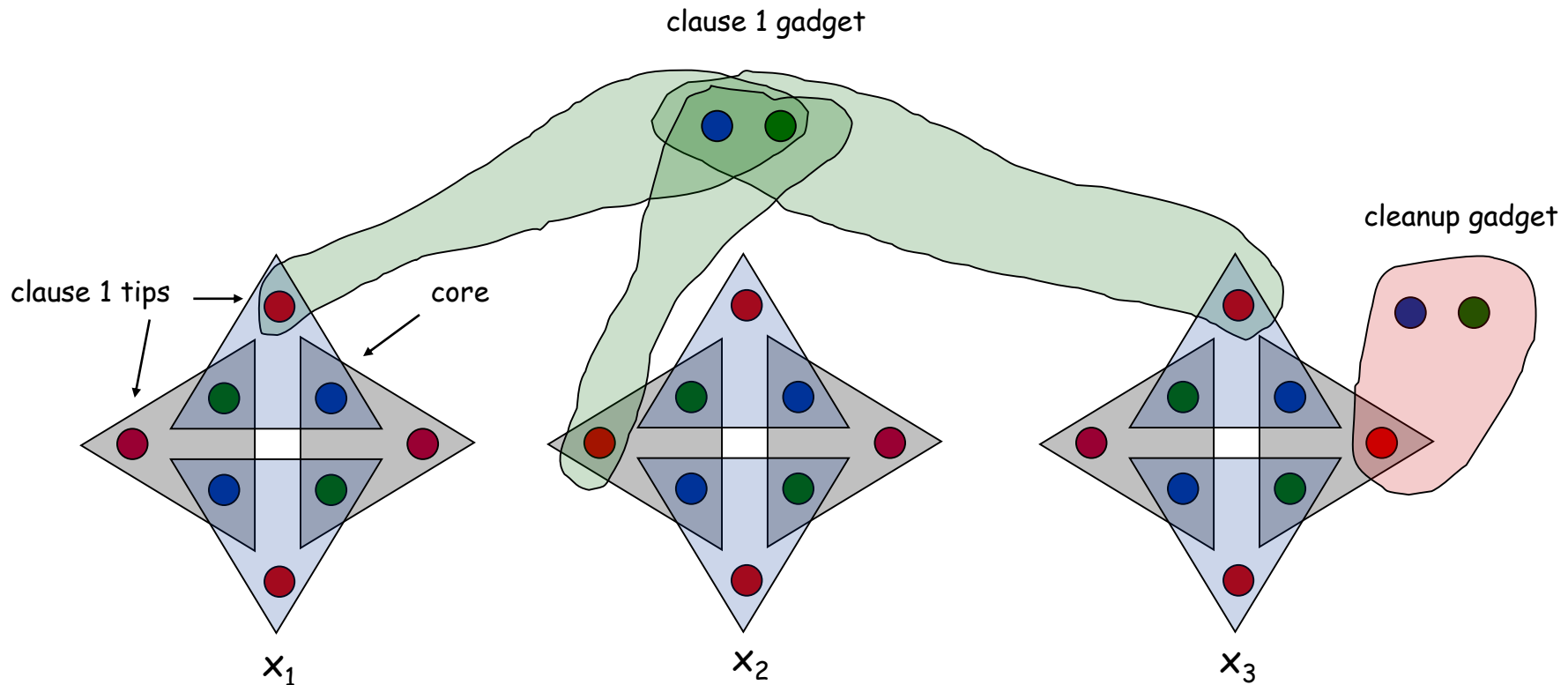


# 3-Dimensional Matching

**Claim.** Instance has a 3D-matching iff  $\Phi$  is satisfiable.

**Q.** What are  $X$ ,  $Y$ , and  $Z$ ? Does each triple contain one element from each of  $X$ ,  $Y$ ,  $Z$ ?

**A.**  $X$  = red,  $Y$  = green,  $Z$  = blue

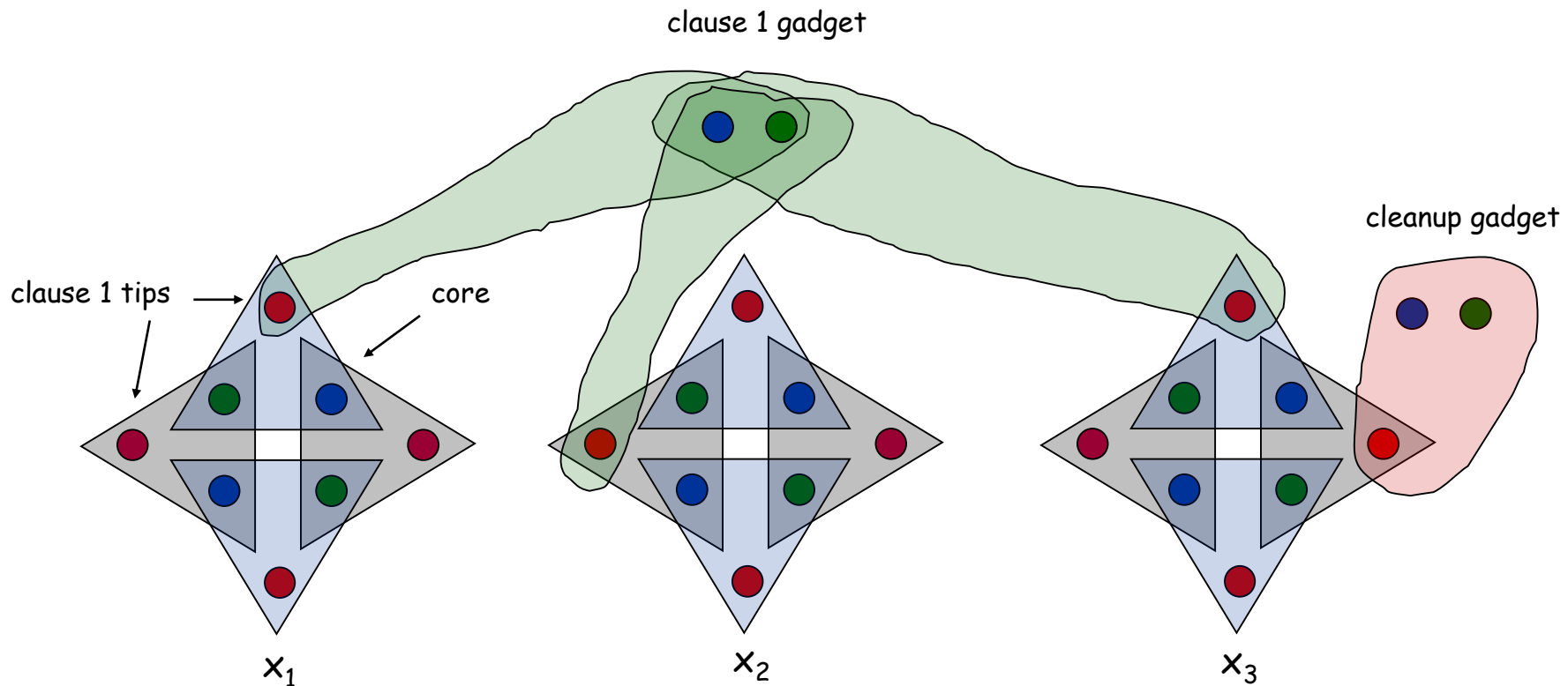


# 3-Dimensional Matching

**Claim.** Instance has a 3D-matching iff  $\Phi$  is satisfiable.

**Pf.  $\Rightarrow$**  If 3D-matching, then assign  $x_i$  according to the color of the selected triples in gadget  $x_i$

**$\Leftarrow$**  If  $\Phi$  is satisfiable, then use assignment of  $x_i$  to select gadget  $x_i$  triples and use true literal in  $C_j$  to select gadget  $C_j$  triple



## 8.7 Graph Coloring

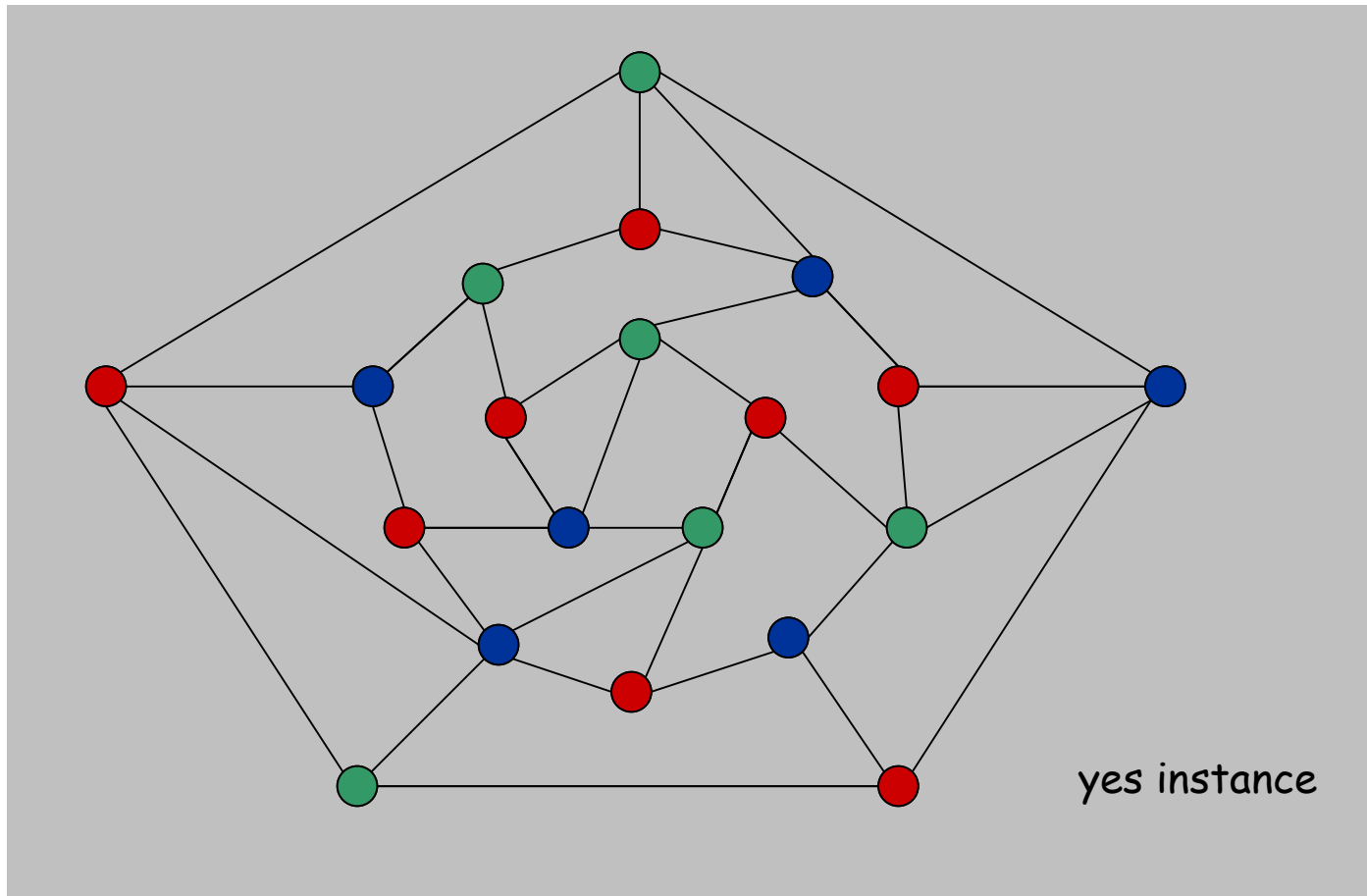
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### Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- **Partitioning problems:** 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

## 3-Colorability

**3-COLOR:** Given an undirected graph  $G$  does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



## Application: Register Allocation

**Register allocation.** Assign program variables to machine register so that no more than  $k$  registers are used and no two program variables that are needed at the same time are assigned to the same register.

**Interference graph.** Nodes are program variables names, edge between  $u$  and  $v$  if there exists an operation where both  $u$  and  $v$  are "live" at the same time.

**Observation.** [Chaitin 1982] Can solve register allocation problem iff interference graph is  $k$ -colorable.

**Fact.**  $3\text{-COLOR} \leq_p k\text{-REGISTER-ALLOCATION}$  for any constant  $k \geq 3$ .

## 3-Colorability

**Claim.**  $3\text{-SAT} \leq_p 3\text{-COLOR}$ .

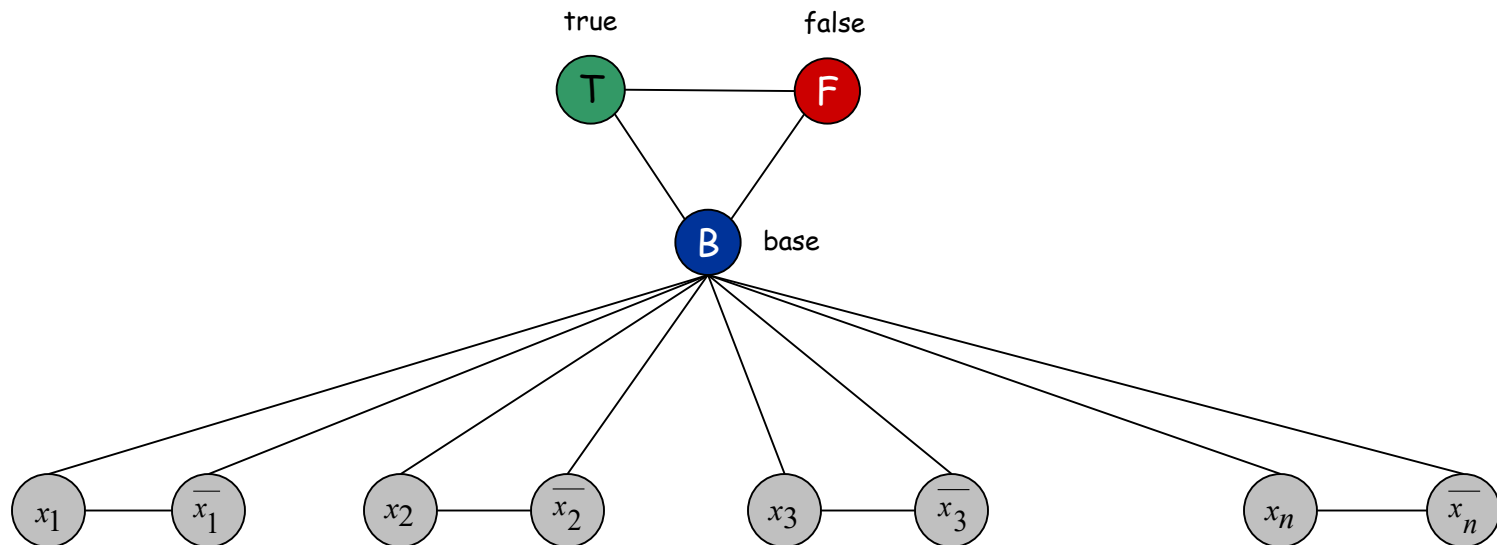
**Pf.** Given 3-SAT instance  $\Phi$ , we construct an instance of 3-COLOR that is 3-colorable iff  $\Phi$  is satisfiable.



# 3-Colorability

## Construction.

- i. For each literal, create a node.
- ii. Connect each literal to its negation.
- iii. Create 3 new nodes T, F, B; connect them in a triangle
- iv. Connect each literal to B.
- v. For each clause, add gadget of 6 nodes and 13 edges (to be described later)



# 3-Colorability

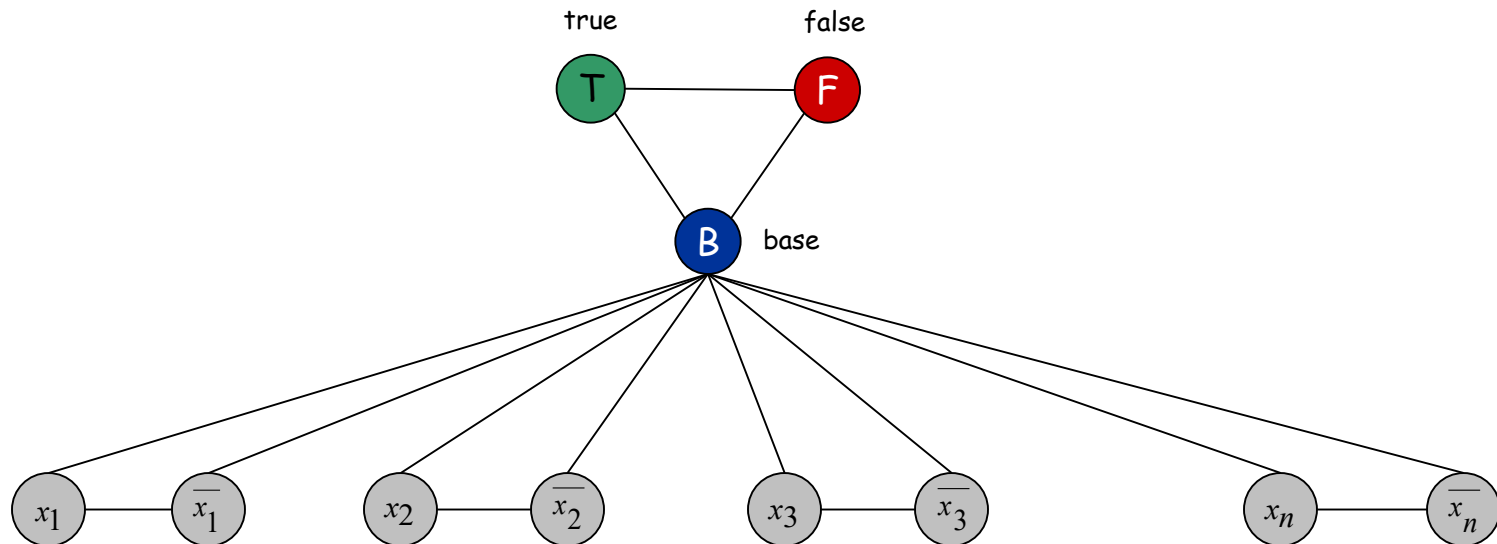
**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

**Pf.**  $\Rightarrow$  Suppose graph is 3-colorable.

Set all the literals with T color to true.

(iv) ensures each literal is T or F.

(ii) ensures a literal and its negation are opposites.



# 3-Colorability

**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

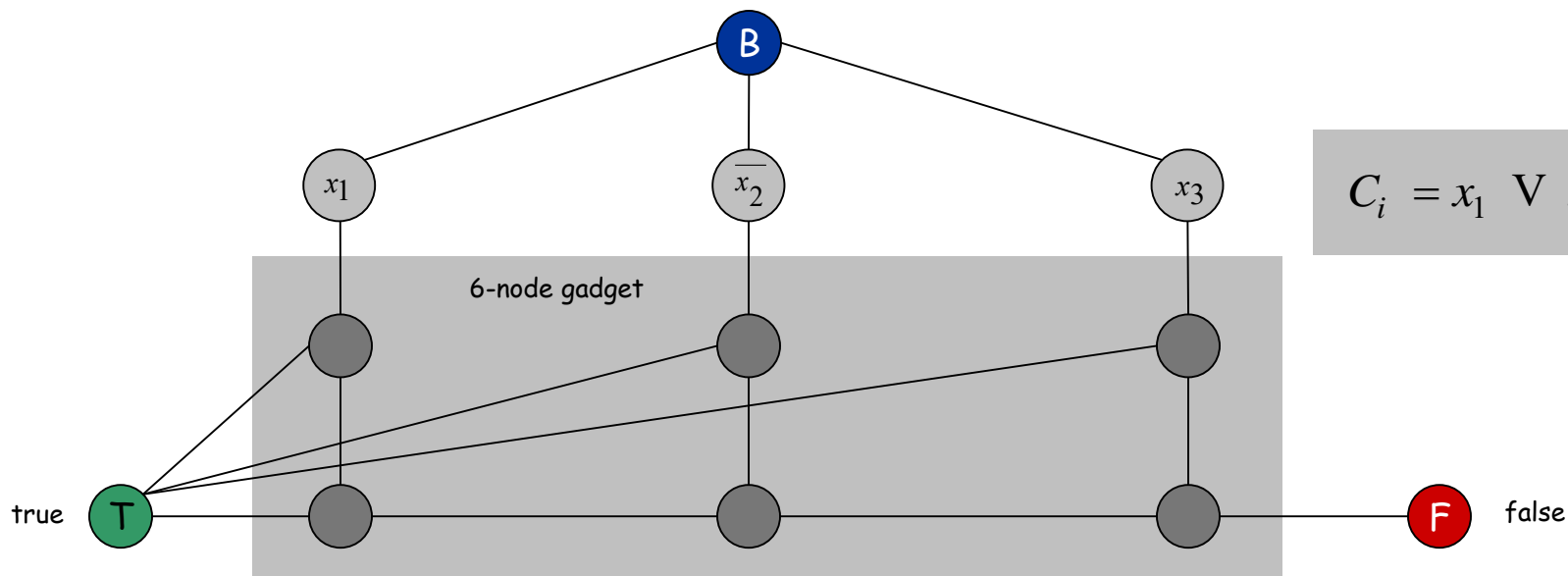
**Pf.**  $\Rightarrow$  Suppose graph is 3-colorable.

Set all the literals with T color to true.

(iv) ensures each literal is T or F.

(ii) ensures a literal and its negation are opposites.

(v) ensures at least one literal in each clause is T.



# 3-Colorability

**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

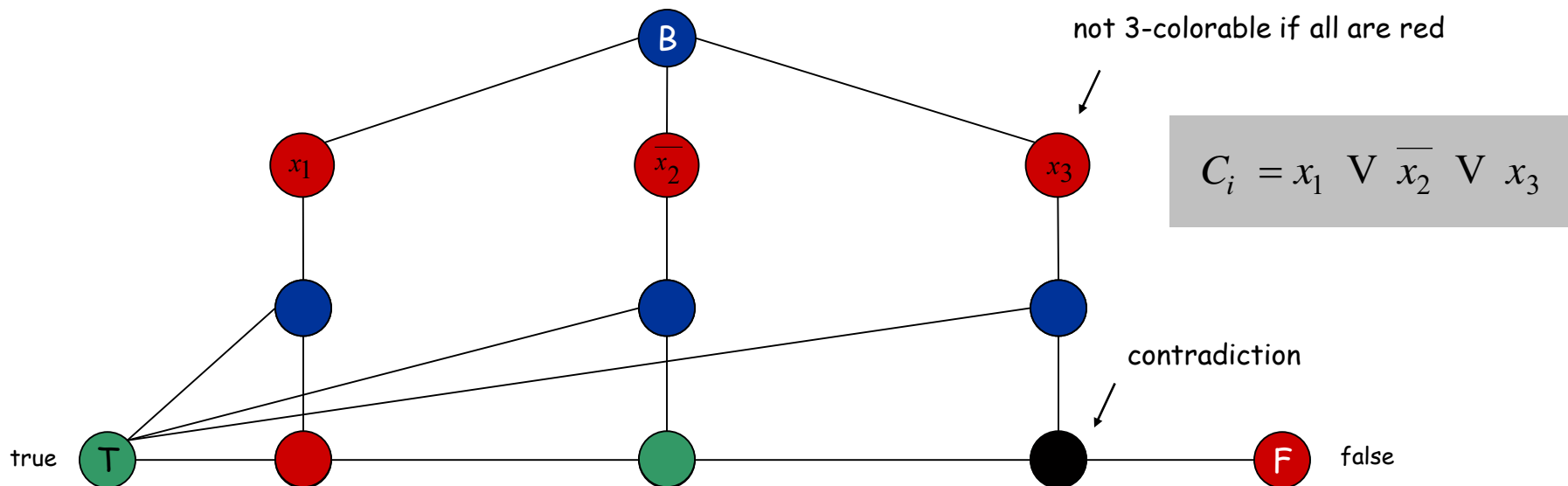
**Pf.**  $\Rightarrow$  Suppose graph is 3-colorable.

Set all the literals with T color to true.

(iv) ensures each literal is T or F.

(ii) ensures a literal and its negation are opposites.

(v) ensures at least one literal in each clause is T.



# 3-Colorability

**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

**Pf.**  $\Leftarrow$  Suppose 3-SAT formula  $\Phi$  is satisfiable.

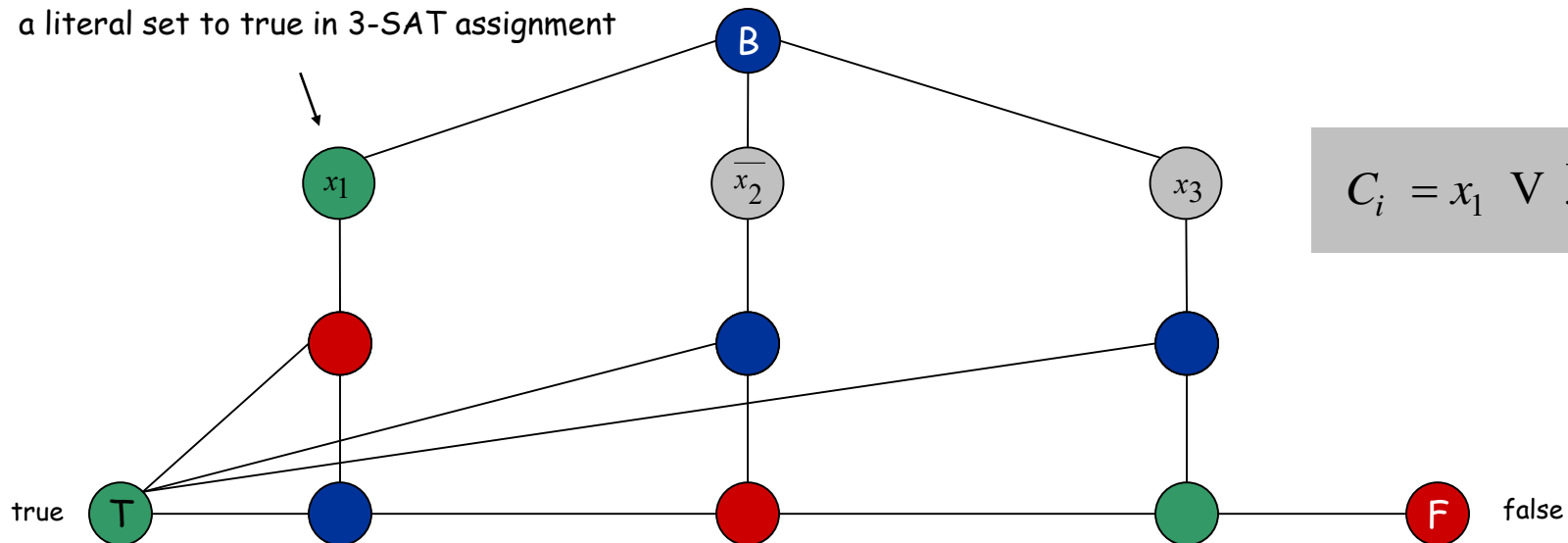
Color all true literals T.

Color node below green node F, and node below that B.

Color remaining middle row nodes B.

Color remaining bottom nodes T or F as forced. ■

a literal set to true in 3-SAT assignment



# 3-Colorability

**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

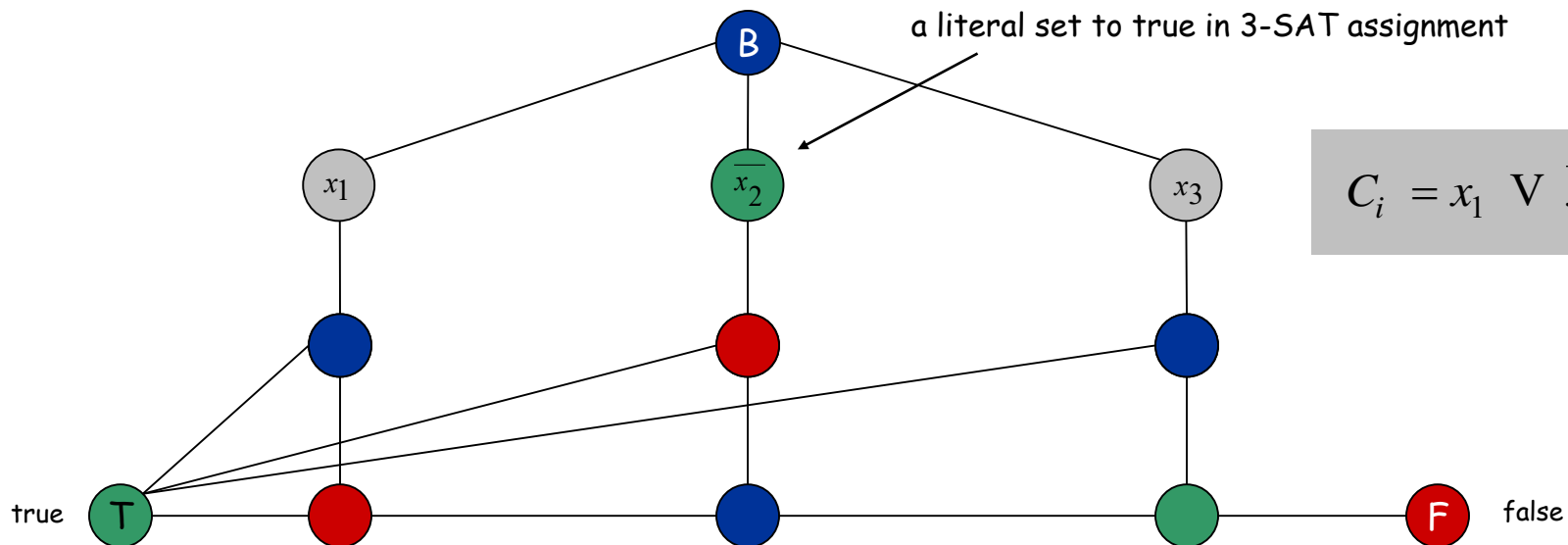
**Pf.**  $\Leftarrow$  Suppose 3-SAT formula  $\Phi$  is satisfiable.

Color all true literals T.

Color node below green node F, and node below that B.

Color remaining middle row nodes B.

Color remaining bottom nodes T or F as forced. ■



# 3-Colorability

**Claim.** Graph is 3-colorable iff  $\Phi$  is satisfiable.

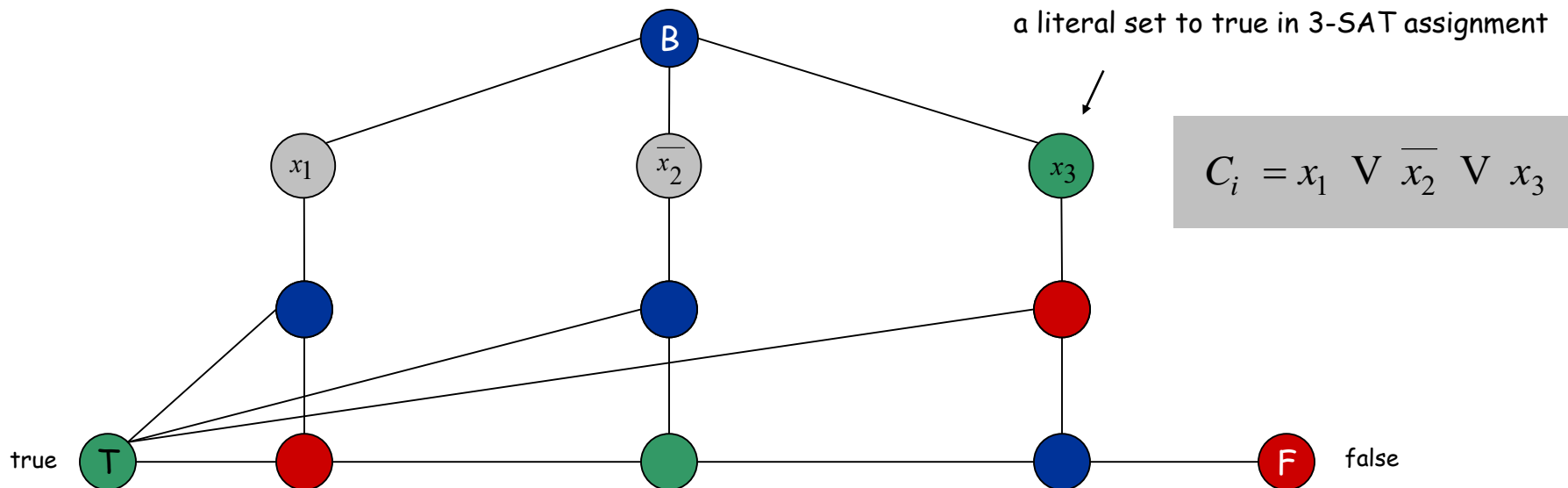
**Pf.**  $\Leftarrow$  Suppose 3-SAT formula  $\Phi$  is satisfiable.

Color all true literals T.

Color node below green node F, and node below that B.

Color remaining middle row nodes B.

Color remaining bottom nodes T or F as forced. ■



## 8.8 Numerical Problems

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### Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.



## Subset Sum

**SUBSET-SUM.** Given natural numbers  $w_1, \dots, w_n$  and an integer  $W$ , is there a subset that adds up to exactly  $W$ ?

**Ex:**  $\{ 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 \}$ ,  $W = 3754$ .

**Yes.**  $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$ .

**Remark.** With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in **binary** encoding.

## 3-SAT reduces to SUBSET-SUM

**Claim.**  $3\text{-SAT} \leq_p \text{SUBSET-SUM}$ .

**Pf.** Given an instance  $\Phi$  of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff  $\Phi$  is satisfiable.

# 3-SAT reduces to SUBSET-SUM

**Construction.** Given 3-SAT instance  $\Phi$  with  $n$  variables and  $k$  clauses, form  $2n + 2k$  decimal integers, each of  $n+k$  digits, as illustrated below.

One digit for each variable and for each clause

Two numbers for each variable

Two numbers for each clause

Sum of each variable digit is 1;

sum of each clause digit is 4.

$$\begin{aligned} C_1 &= \bar{x} \vee y \vee z \\ C_2 &= x \vee \bar{y} \vee z \\ C_3 &= \bar{x} \vee \bar{y} \vee \bar{z} \end{aligned}$$

dummies to get  
clause columns  
to sum to 4

	x	y	z	$C_1$	$C_2$	$C_3$	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

# 3-SAT reduces to SUBSET-SUM

**Claim.**  $\Phi$  is satisfiable iff there exists a subset that sums to  $W$ .

**Pf.  $\Rightarrow$**  Suppose  $\Phi$  is satisfiable

Choose numbers corresponding to each true literal

Since  $\Phi$  is satisfiable, each clause digit sums to at least 1 from the chosen numbers

Choose dummy numbers to make each clause digit sum to 4.

$$C_1 = \bar{x} \vee y \vee z$$

$$C_2 = x \vee \bar{y} \vee z$$

$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

dummies to get  
clause columns  
to sum to 4

	x	y	z	$C_1$	$C_2$	$C_3$	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

# 3-SAT reduces to SUBSET-SUM

**Claim.**  $\Phi$  is satisfiable iff there exists a subset that sums to  $W$ .

**Pf.**  $\Leftarrow$  Suppose there is a subset that sums to  $W$

Each variable digit forces the subset to choose exactly one from  $x$  and  $\neg x$

Each clause digit forces the subset to choose at least one literal in the clause

Assign each variable according to the chosen literal

$$\begin{aligned} C_1 &= \bar{x} \vee y \vee z \\ C_2 &= x \vee \bar{y} \vee z \\ C_3 &= \bar{x} \vee \bar{y} \vee \bar{z} \end{aligned}$$

dummies to get  
clause columns  
to sum to 4

	x	y	z	$C_1$	$C_2$	$C_3$	
x	1	0	0	0	1	0	100,010
$\neg x$	1	0	0	1	0	1	100,101
y	0	1	0	1	0	0	10,100
$\neg y$	0	1	0	0	1	1	10,011
z	0	0	1	1	1	0	1,110
$\neg z$	0	0	1	0	0	1	1,001
}	0	0	0	1	0	0	100
	0	0	0	2	0	0	200
	0	0	0	0	1	0	10
	0	0	0	0	2	0	20
	0	0	0	0	0	1	1
	0	0	0	0	0	2	2
W	1	1	1	4	4	4	111,444

## From book: 3D-MATCHING reduces to SUBSET-SUM

**Construction.** Let  $X \cup Y \cup Z$  be an instance of 3D-MATCHING with triplet set  $T$ . Let  $n = |X| = |Y| = |Z|$  and  $m = |T|$ .

Let  $X = \{x_1, x_2, x_3, x_4\}$ ,  $Y = \{y_1, y_2, y_3, y_4\}$ ,  $Z = \{z_1, z_2, z_3, z_4\}$

For each triplet  $t = (x_i, y_j, z_k) \in T$ , create an integer  $w_t$  in base  $m+1$  with  $3n$  digits that has a 1 in positions  $i$ ,  $n+j$ , and  $2n+k$ .

**Claim.** 3D-matching iff some subset sums to  $W = 111, \dots, 111$ .

Triplet $t_i$			$x_1$	$x_2$	$x_3$	$x_4$	$y_1$	$y_2$	$y_3$	$y_4$	$z_1$	$z_2$	$z_3$	$z_4$	$w_i$
$x_1$	$y_2$	$z_3$	1	0	0	0	0	1	0	0	0	0	1	0	100,001,000,010
$x_2$	$y_4$	$z_2$	0	1	0	0	0	0	0	1	0	1	0	0	10,000,010,100
$x_1$	$y_1$	$z_1$	1	0	0	0	1	0	0	0	1	0	0	0	100,010,001,000
$x_2$	$y_2$	$z_4$	0	1	0	0	0	1	0	0	0	0	0	1	10,001,000,001
$x_4$	$y_3$	$z_4$	0	0	0	1	0	0	1	0	0	0	0	1	100,100,001
$x_3$	$y_1$	$z_2$	0	0	1	0	1	0	0	0	0	1	0	0	1,010,000,100
$x_3$	$y_1$	$z_3$	0	0	1	0	1	0	0	0	0	0	1	0	1,010,000,010
$x_3$	$y_1$	$z_1$	0	0	1	0	1	0	0	0	1	0	0	0	1,010,001,000
$x_4$	$y_4$	$z_4$	0	0	0	1	0	0	0	1	0	0	0	1	100,010,001
															111,111,111,111

## Scheduling With Release Times

**SCHEDULE-RELEASE-TIMES.** Given a set of  $n$  jobs with processing time  $t_i$ , release time  $r_i$ , and deadline  $d_i$ , is it possible to schedule all jobs on a single machine such that job  $i$  is processed with a contiguous slot of  $t_i$  time units in the interval  $[r_i, d_i]$ ?

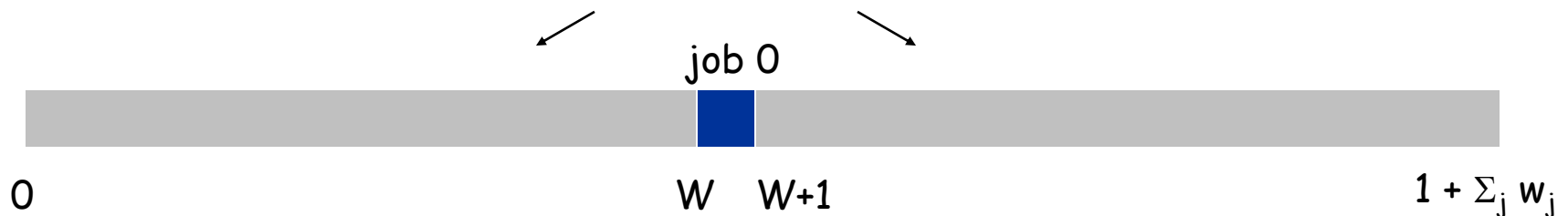
**Claim.**  $\text{SUBSET-SUM} \leq_p \text{SCHEDULE-RELEASE-TIMES}$ .

**Pf.** Given an instance of SUBSET-SUM  $w_1, \dots, w_n$ , and target  $W$ ,

Create  $n$  jobs with processing time  $t_i = w_i$ , release time  $r_i = 0$ , and deadline  $d_i = 1 + \sum_j w_j$ .

Create job 0 with  $t_0 = 1$ , release time  $r_0 = W$ , and deadline  $d_0 = W+1$ .

Can schedule jobs 1 to  $n$  anywhere but  $[W, W+1]$



## 8.10 A Partial Taxonomy of Hard Problems

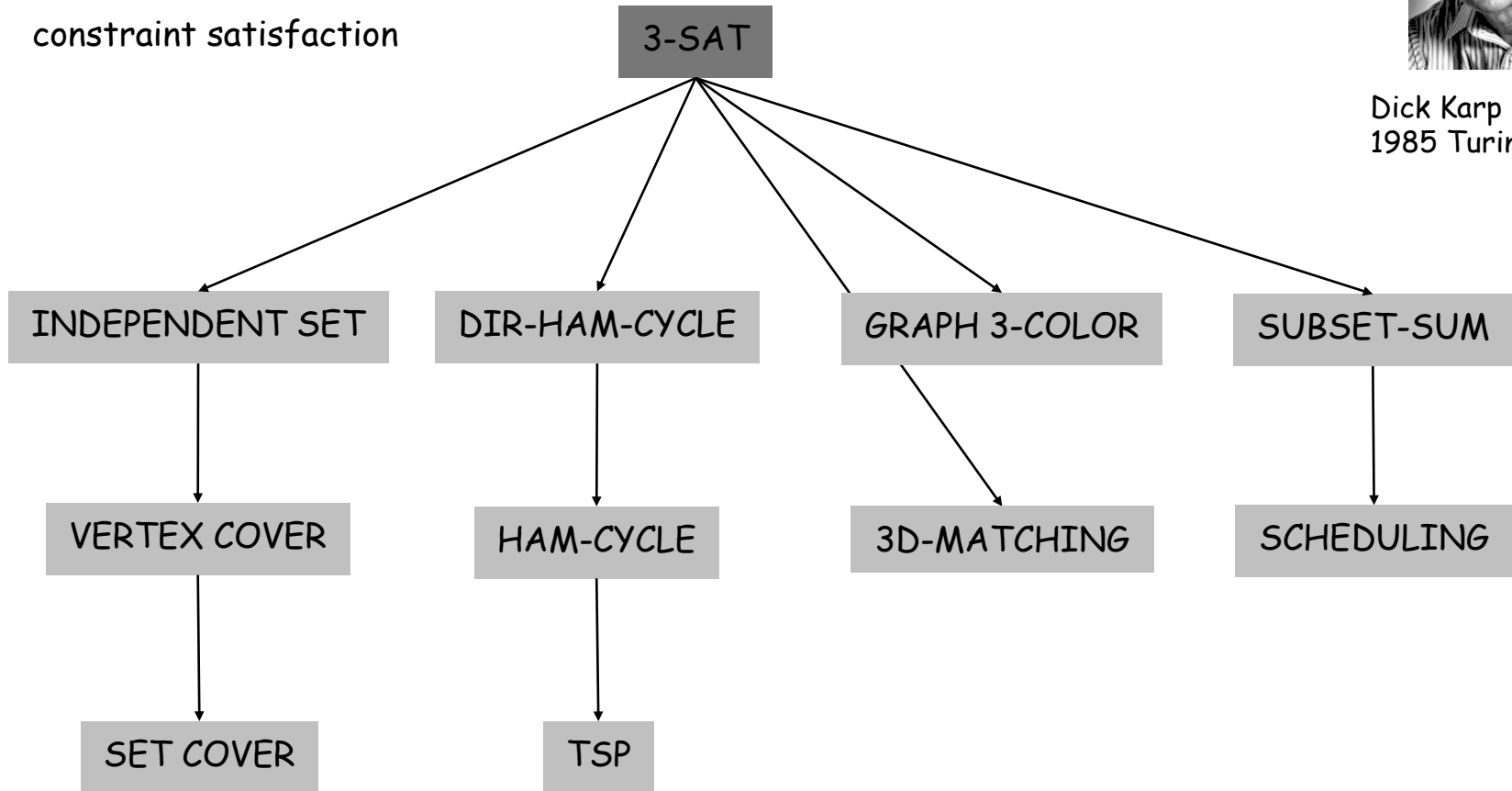
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# Polynomial-Time Reductions



Dick Karp (1972)  
1985 Turing Award



packing and covering

sequencing

partitioning

numerical

# Chapter Summary

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# Key Concepts

## Decision problem

Answer yes/no

**P.** Decision problems for which there is a **poly-time** algorithm.

**NP.** Decision problems for which there exists a **poly-time** certifier.

Algorithm  $C(s, t)$  is a **certifier** for problem  $X$  if for every string  $s$ ,  $s \in X$  iff there exists a string  $t$  such that  $C(s, t) = \text{yes}$ .

**co-NP.** Complements of decision problems in NP.

**EXP.** Decision problems for which there is an **exponential-time** algorithm.

**Claim.**  $P \subseteq NP$ ,  $\text{co-NP} \subseteq \text{EXP}$

# Polynomial-Time Reduction

**Reduction.** Problem X **polynomial-time reduces to** problem Y if arbitrary instances of problem X can be solved using:

Polynomial number of standard computational steps, plus

Polynomial number of calls to oracle that solves problem Y.

**Notation.**  $X \leq_p Y$ .

**Common approach:** **polynomial transformation**

Given any input  $x$  to  $X$ , **construct** an input  $y$  in poly-time such that  $x$  is a yes instance of  $X$  **iff**  $y$  is a yes instance of  $Y$ .

# NP-Completeness

**NP-complete.** A problem  $Y$  in NP with the property that for every problem  $X$  in NP,  $X \leq_p Y$ .

**Recipe to establish NP-completeness of problem  $Y$ .**

Step 1. Show that  $Y$  is in NP.

Step 2. Choose an NP-complete problem  $X$ .

Step 3. Prove that  $X \leq_p Y$ .

MY HOBBY:  
EMBEDDING NP-COMPLETE PROBLEMS IN RESTAURANT ORDERS

CHOTCHKIES RESTAURANT	
~ APPETIZERS ~	
MIXED FRUIT	2.15
FRENCH FRIES	2.75
SIDE SALAD	3.35
HOT WINGS	3.55
MOZZARELLA STICKS	4.20
SAMPLER PLATE	5.80
~ SANDWICHES ~	
BARBECUE	6.55

