



10. EXTENDING TRACTABILITY

- ▶ *finding small vertex covers*
- ▶ *solving NP-hard problems on trees*
- ▶ *circular arc coverings*
- ▶ *vertex cover in bipartite graphs*

Lecture slides by Kevin Wayne

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Coping with NP-completeness

Q. Suppose I need to solve an **NP**-complete problem. What should I do?

A. Theory says you're unlikely to find poly-time algorithm.

Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve **arbitrary instances** of the problem.

This lecture. Solve some special cases of **NP**-complete problems.

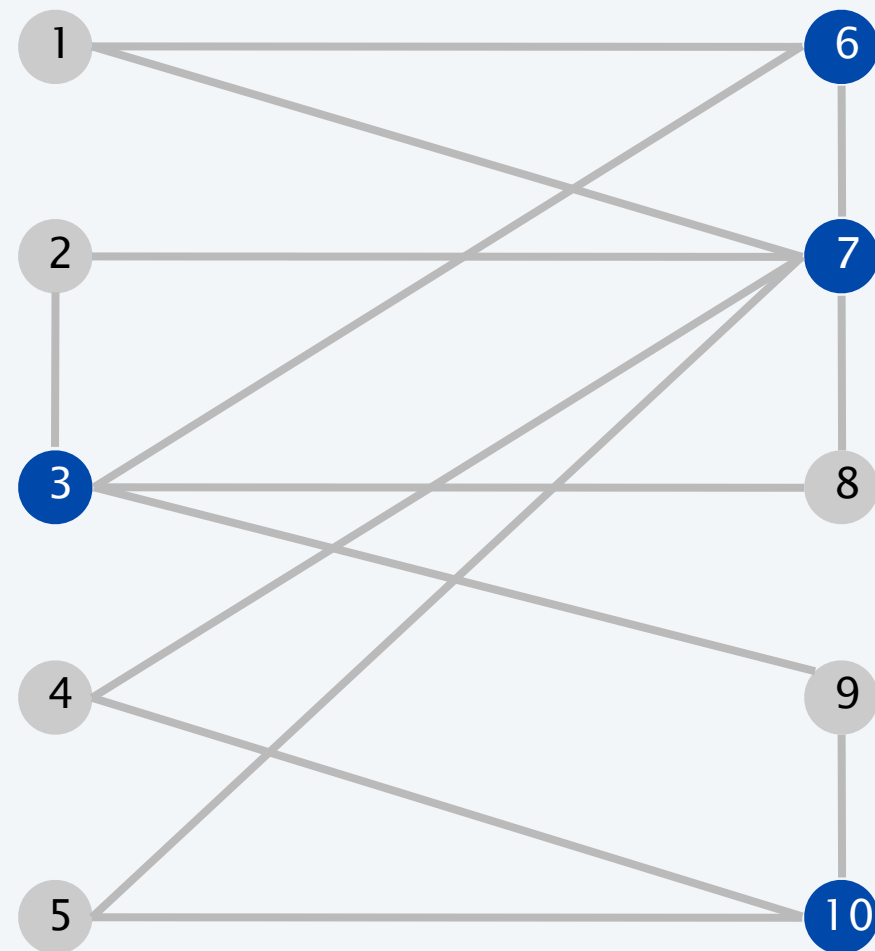


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Vertex cover

Given a graph $G = (V, E)$ and an integer k , is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge (u, v) either $u \in S$ or $v \in S$ or both?



$S = \{ 3, 6, 7, 10 \}$ is a vertex cover of size $k = 4$

Finding small vertex covers

Q. VERTEX-COVER is **NP**-complete. But what if k is small?

Brute force. $O(k n^{k+1})$.

- Try all $C(n, k) = O(n^k)$ subsets of size k .
- Takes $O(k n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on k , say to $O(2^k k n)$.

Ex. $n = 1,000, k = 10$.

Brute. $k n^{k+1} = 10^{34} \Rightarrow$ infeasible.

Better. $2^k k n = 10^7 \Rightarrow$ feasible.

Remark. If k is a constant, then the algorithm is poly-time;
if k is a small constant, then it's also practical.

Finding small vertex covers

Claim. Let (u, v) be an edge of G . G has a vertex cover of size $\leq k$ iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\leq k - 1$.

 delete v and all incident edges

Pf. \Rightarrow

- Suppose G has a vertex cover S of size $\leq k$.
- S contains either u or v (or both). Assume it contains u .
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

Pf. \Leftarrow

- Suppose S is a vertex cover of $G - \{u\}$ of size $\leq k - 1$.
- Then $S \cup \{u\}$ is a vertex cover of G . ■

Claim. If G has a vertex cover of size k , it has $\leq k(n - 1)$ edges.

Pf. Each vertex covers at most $n - 1$ edges. ■

Finding small vertex covers: algorithm

Claim. The following algorithm determines if G has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

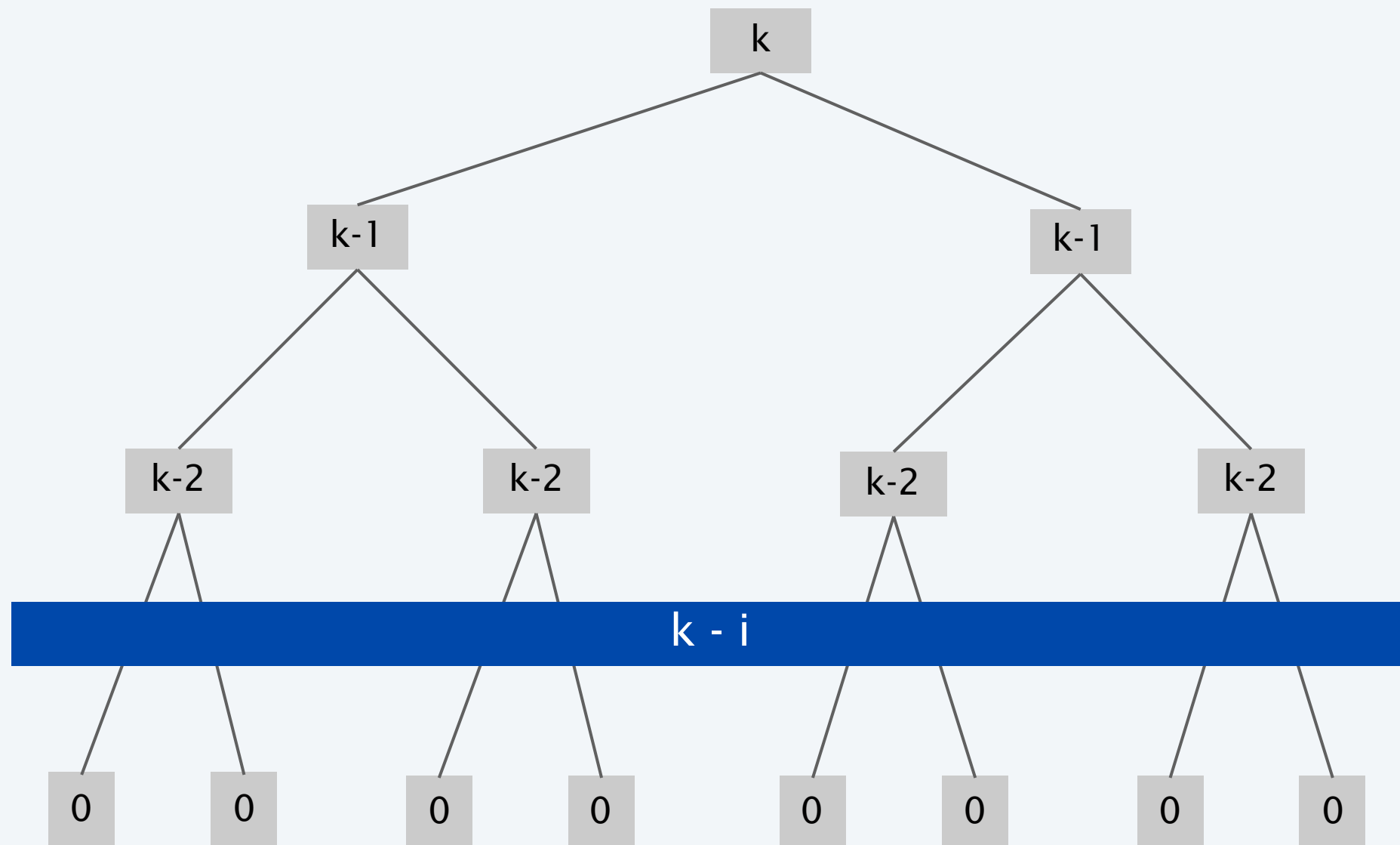
```
Vertex-Cover( $G, k$ ) {  
    if ( $G$  contains no edges)    return true  
    if ( $G$  contains  $\geq kn$  edges) return false  
  
    let  $(u, v)$  be any edge of  $G$   
     $a = \text{Vertex-Cover}(G - \{u\}, k-1)$   
     $b = \text{Vertex-Cover}(G - \{v\}, k-1)$   
    return  $a$  or  $b$   
}
```

Pf.

- Correctness follows from previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. ■

Finding small vertex covers: recursion tree

$$T(n, k) \leq \begin{cases} c & \text{if } k = 0 \\ cn & \text{if } k = 1 \\ 2T(n, k-1) + ckn & \text{if } k > 1 \end{cases} \Rightarrow T(n, k) \leq 2^k c k n$$





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Independent set on trees

Independent set on trees. Given a **tree**, find a maximum cardinality subset of nodes such that no two share an edge.

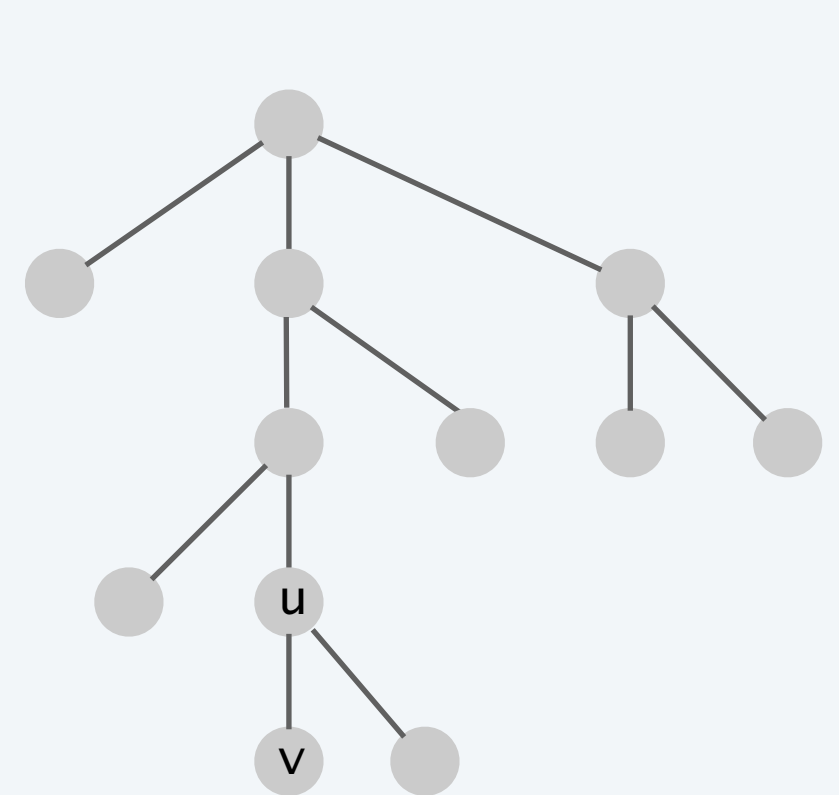
Fact. A tree on at least two nodes has at least two leaf nodes.

degree = 1

Key observation. If v is a leaf, there exists a maximum size independent set containing v .

Pf. (exchange argument)

- Consider a max cardinality independent set S .
- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{v\}$ is independent $\Rightarrow S$ not maximum.
- If $u \in S$ and $v \notin S$, then $S \cup \{v\} - \{u\}$ is independent. ■



Independent set on trees: greedy algorithm

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
Independent-Set-In-A-Forest(F) {  
  S ←  $\phi$   
  while (F has at least one edge) {  
    Let  $e = (u, v)$  be an edge such that  $v$  is a leaf  
    Add  $v$  to  $S$   
    Delete from  $F$  nodes  $u$  and  $v$ , and all edges  
    incident to them.  
  }  
  return  $S$   
}
```

Pf. Correctness follows from the previous key observation. ■

Remark. Can implement in $O(n)$ time by considering nodes in postorder.

Weighted independent set on trees

Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\sum_{v \in S} w_v$.

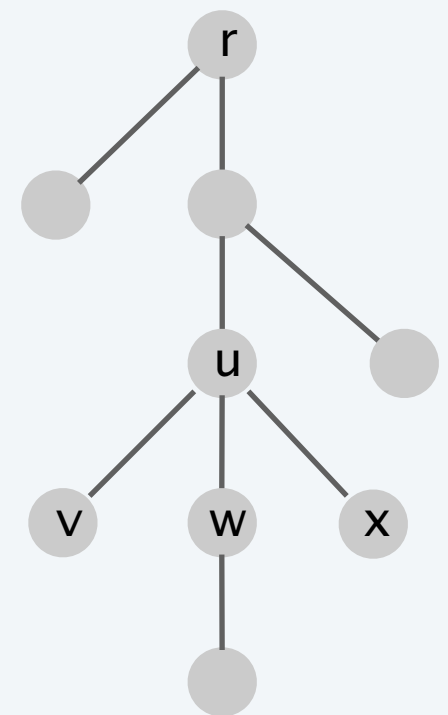
Observation. If (u, v) is an edge such that v is a leaf node, then either OPT includes u or OPT includes all leaf nodes incident to u .

Dynamic programming solution. Root tree at some node, say r .

- $OPT_{in}(u)$ = max weight independent set of subtree rooted at u , containing u .
- $OPT_{out}(u)$ = max weight independent set of subtree rooted at u , not containing u .

$$OPT_{in}(u) = w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) = \sum_{v \in \text{children}(u)} \max \{OPT_{in}(v), OPT_{out}(v)\}$$



children(u) = { v, w, x }

Weighted independent set on trees: dynamic programming algorithm

Theorem. The dynamic programming algorithm finds a maximum weighted independent set in a tree in $O(n)$ time.

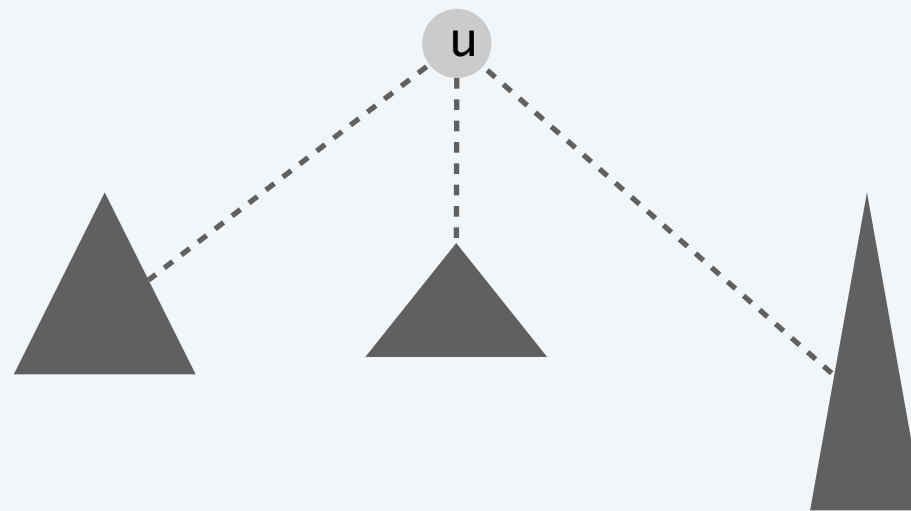
can also find
independent set itself
(not just value)

```
Weighted-Independent-Set-In-A-Tree(T) {  
  Root the tree at a node r  
  foreach (node u of T in postorder) {  
    if (u is a leaf) {  
       $M_{in}[u] = w_u$   
       $M_{out}[u] = 0$   
    }  
    else {  
       $M_{in}[u] = w_u + \sum_{v \in \text{children}(u)} M_{out}[v]$   
       $M_{out}[u] = \sum_{v \in \text{children}(u)} \max(M_{in}[v], M_{out}[v])$   
    }  
  }  
  return  $\max(M_{in}[r], M_{out}[r])$   
}
```

ensures a node is visited
after all its children

Context

Independent set on trees. This structured special case is tractable because we can find a node that **breaks the communication** among the subproblems in different subtrees.



see Chapter 10.4
(but proceed with caution)

Graphs of bounded tree width. Elegant generalization of trees that:

- Captures a rich class of graphs that arise in practice.
- Enables decomposition into independent pieces.



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Wavelength-division multiplexing

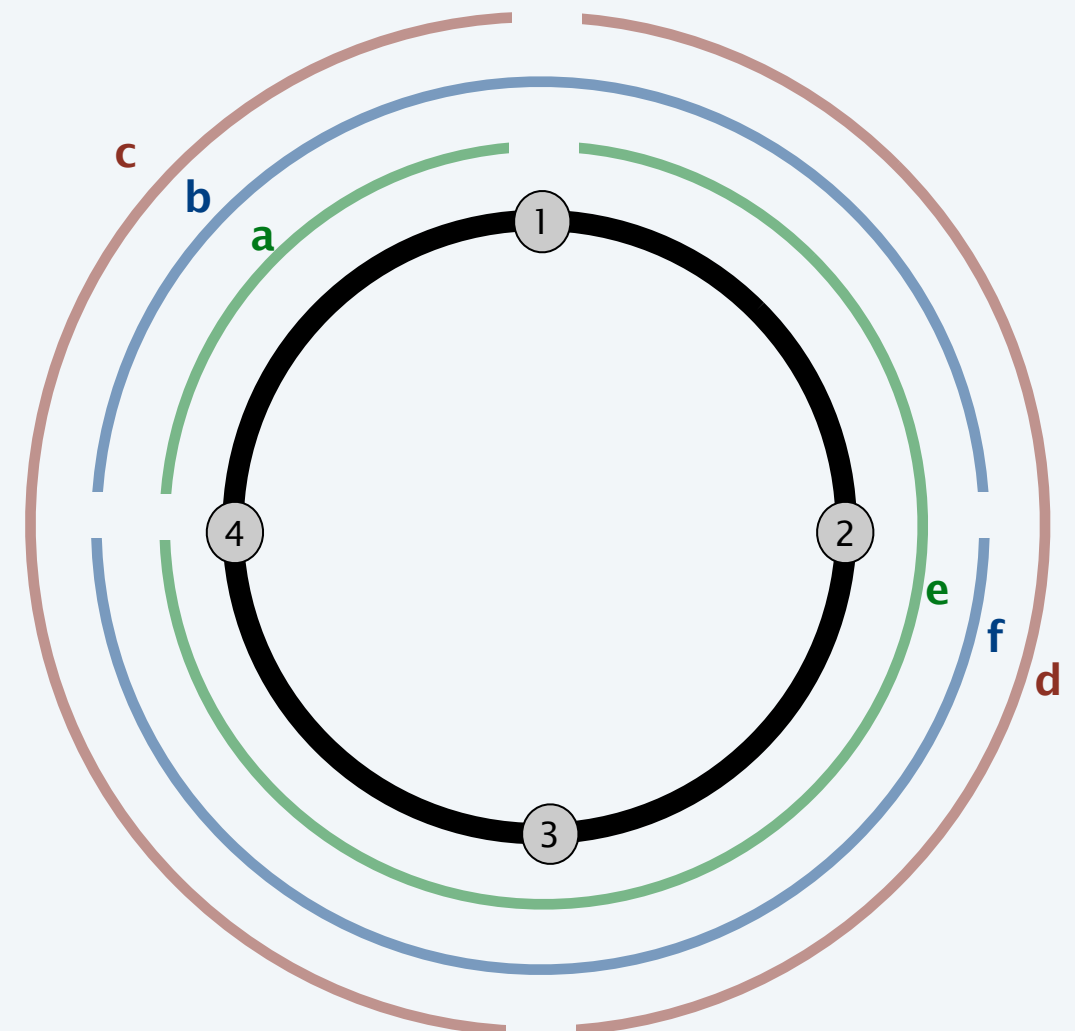
Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a **cycle** on n nodes.

Bad news. **NP**-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k -colorings.

Goal. $O(f(k)) \cdot \text{poly}(m, n)$ on rings.



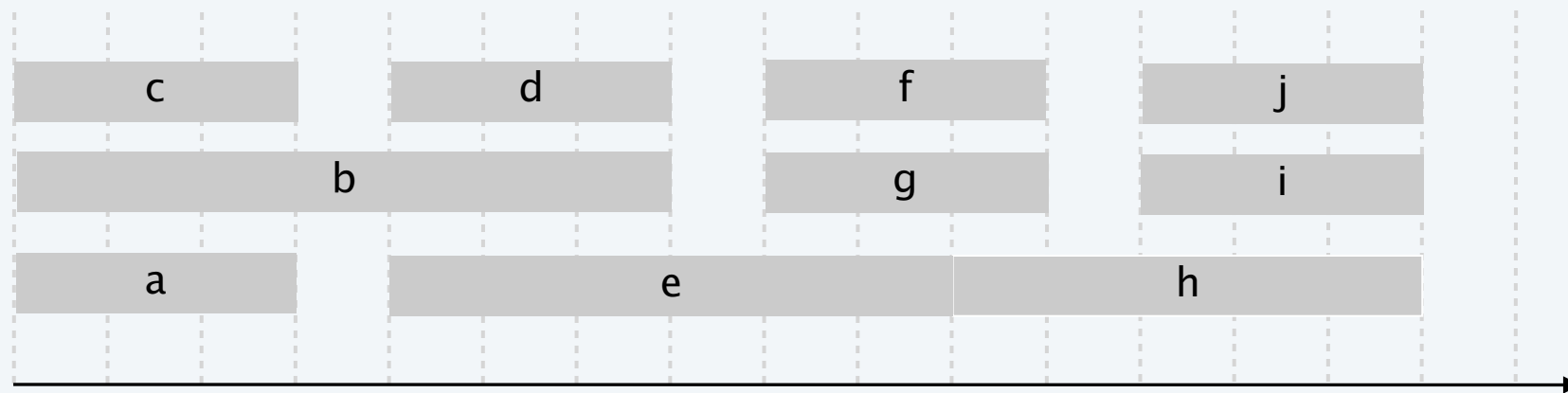
$n = 4, m = 6$

$\{c, d\}, \{b, f\}, \{a, e\}$

Review: interval coloring

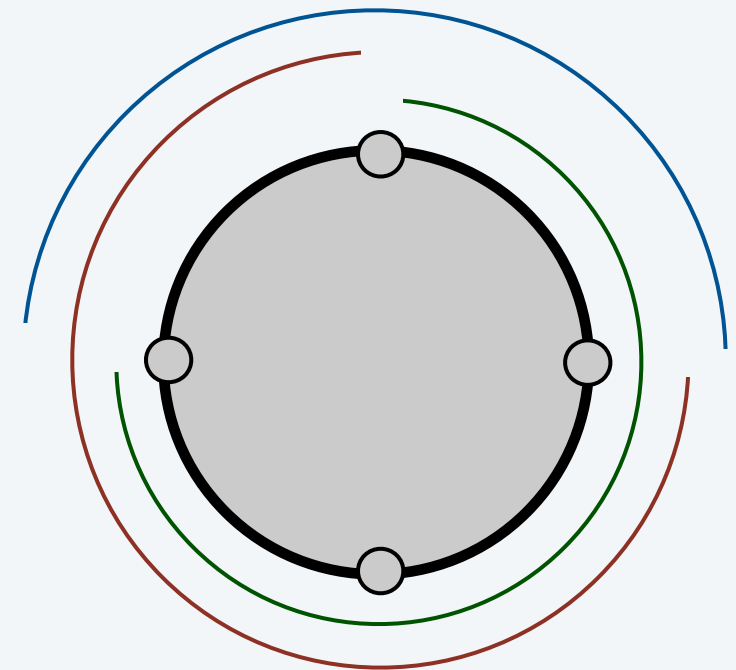
Interval coloring. Greedy algorithm finds coloring such that number of colors equals depth of schedule.

↖ maximum number of streams at one location



Circular arc coloring.

- Weak duality: number of colors \geq depth.
- Strong duality does not hold.

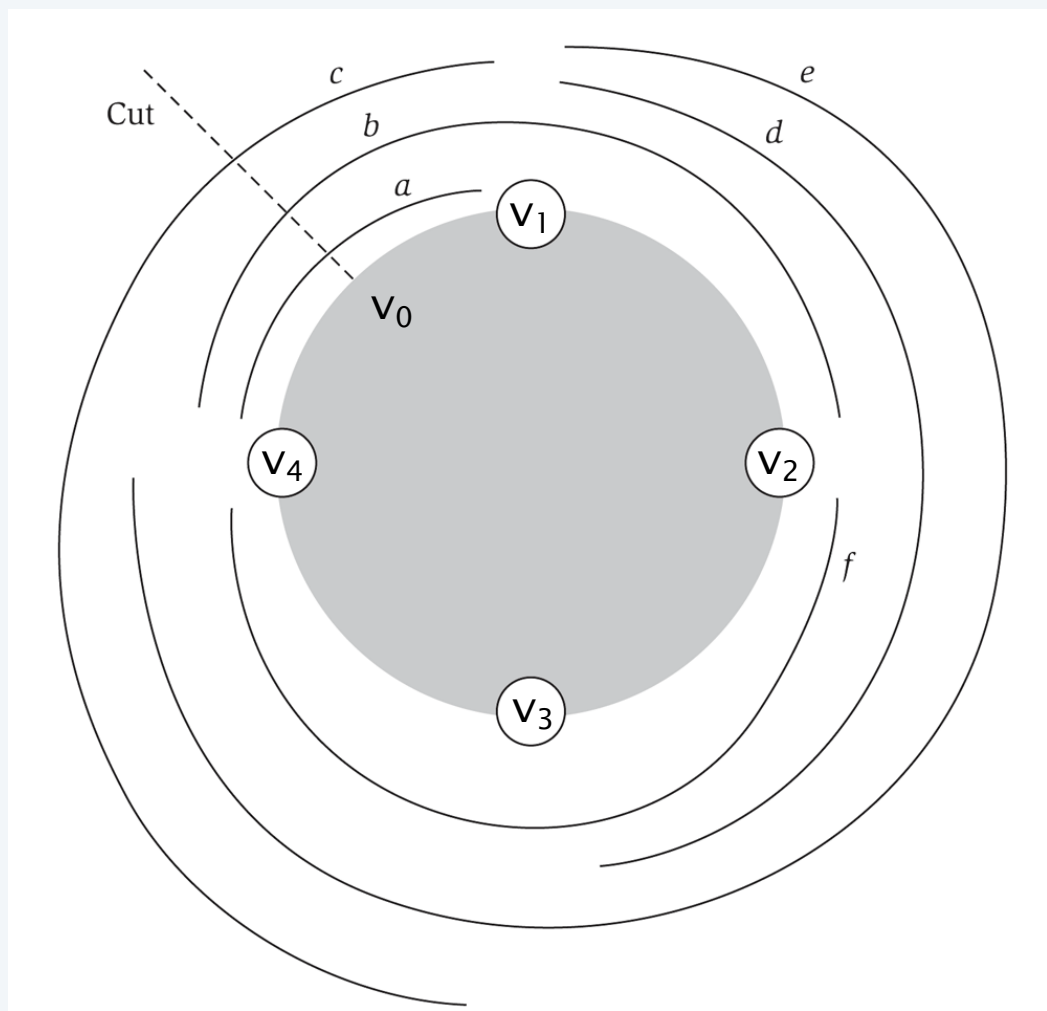


max depth = 2
min colors = 3

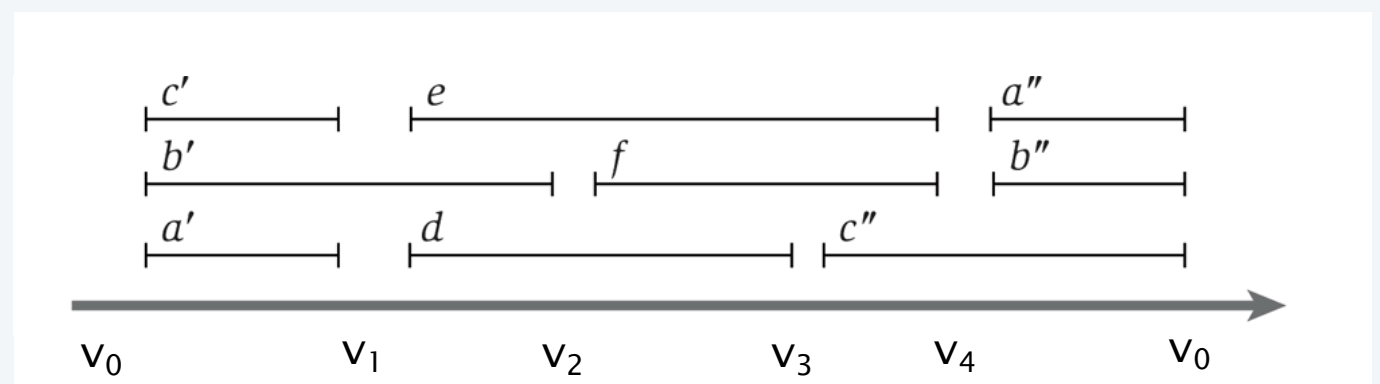
(Almost) transforming circular arc coloring to interval coloring

Circular arc coloring. Given a set of n arcs with depth $d \leq k$, can the arcs be colored with k colors?

Equivalent problem. Cut the network between nodes v_1 and v_n . The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that “sliced” arcs have the same color.



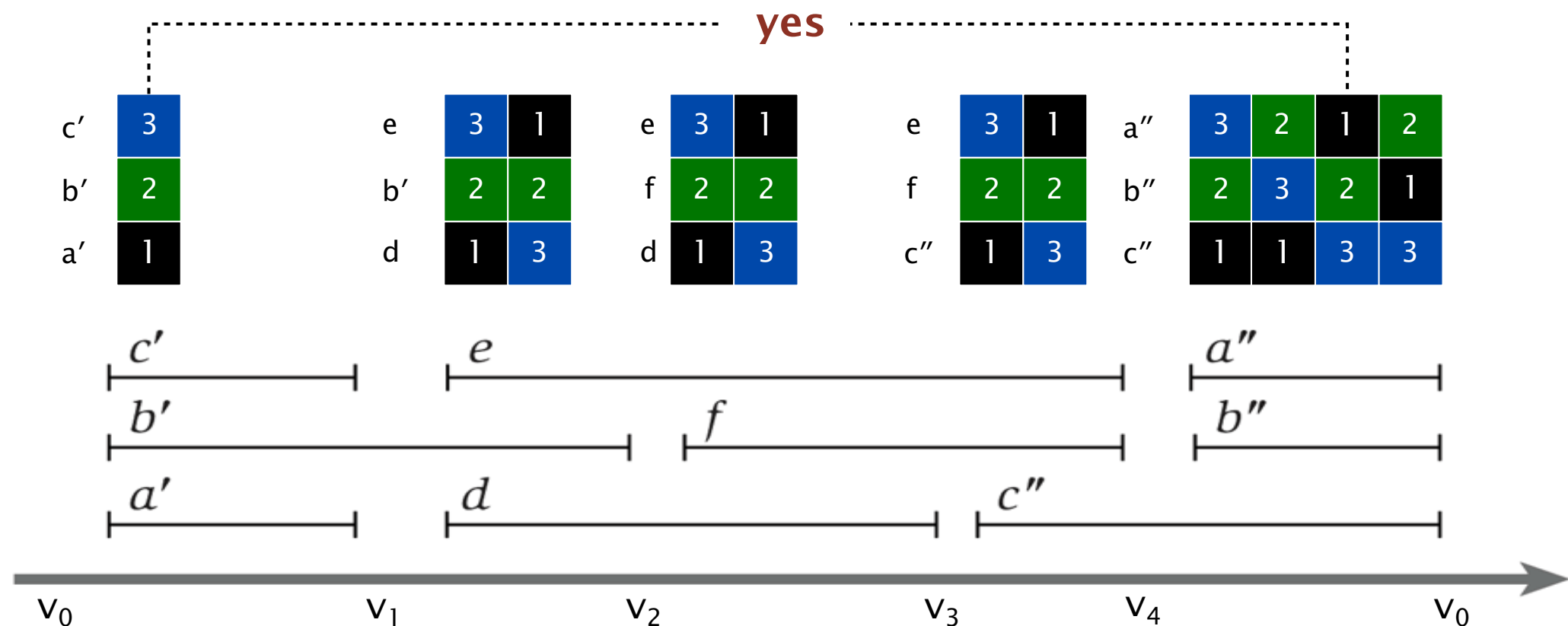
colors of a' , b' , and c' must correspond to colors of a'' , b'' , and c''



Circular arc coloring: dynamic programming algorithm

Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node v_0 .
- At each node v_i , some intervals may finish, and others may begin.
- Enumerate all k -colorings of the intervals through v_i that are consistent with the colorings of the intervals through v_{i-1} .
- The arcs are k -colorable iff some coloring of intervals ending at cut node v_0 is consistent with original coloring of the same intervals.



Circular arc coloring: running time

Running time. $O(k! \cdot n)$.

- The algorithm has n phases.
- Bottleneck in each phase is enumerating all consistent colorings.
- There are at most k intervals through v_i , so there are at most $k!$ colorings to consider.

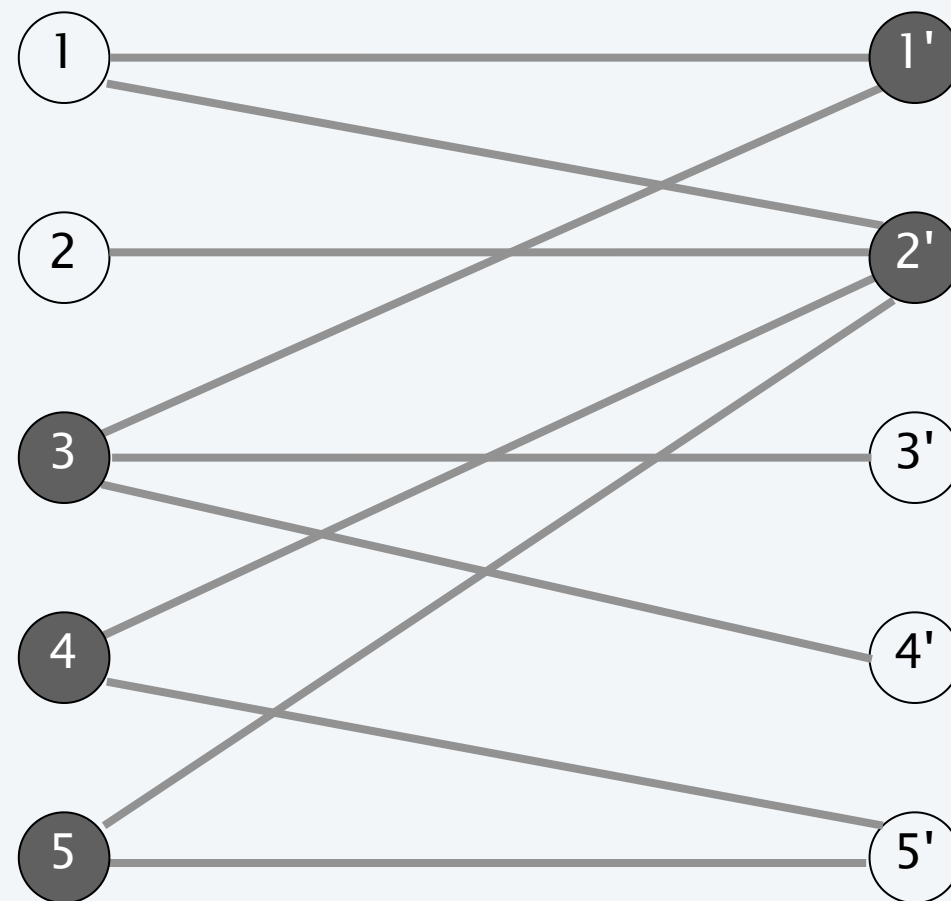
Remark. This algorithm is practical for small values of k (say $k = 10$) even if the number of nodes n (or paths) is large.

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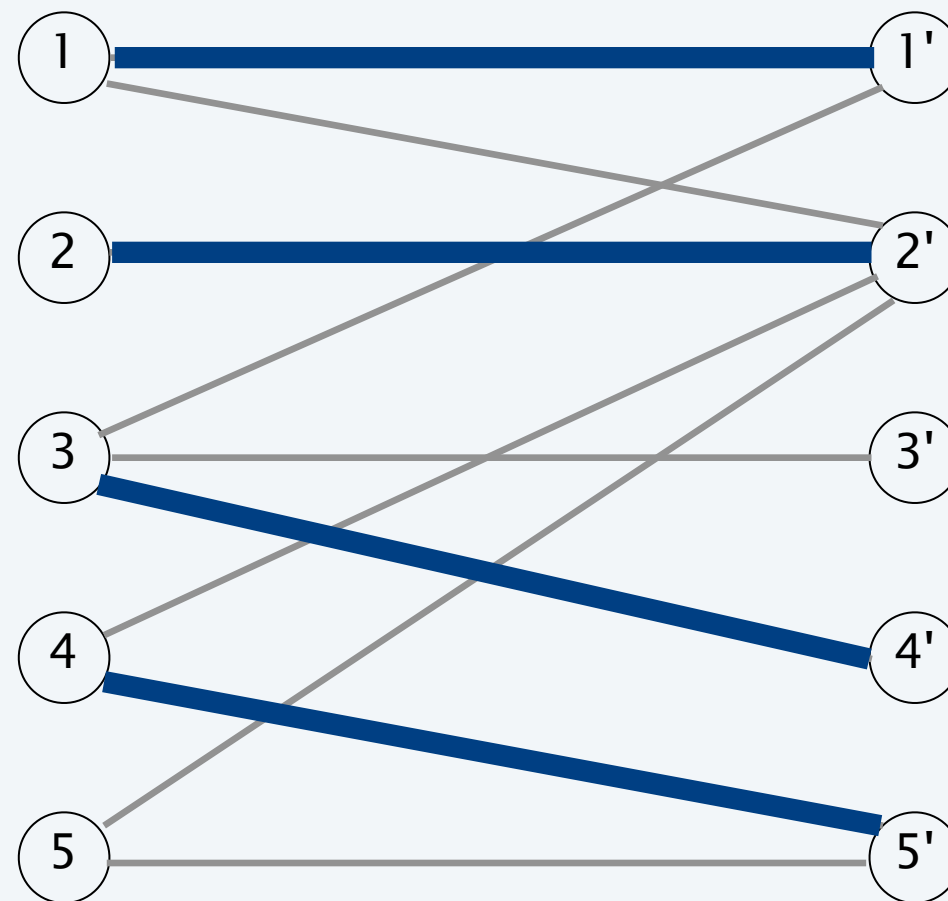


vertex cover $S = \{ 3, 4, 5, 1', 2' \}$

Vertex cover and matching

Weak duality. Let M be a matching, and let S be a vertex cover. Then, $|M| \leq |S|$.

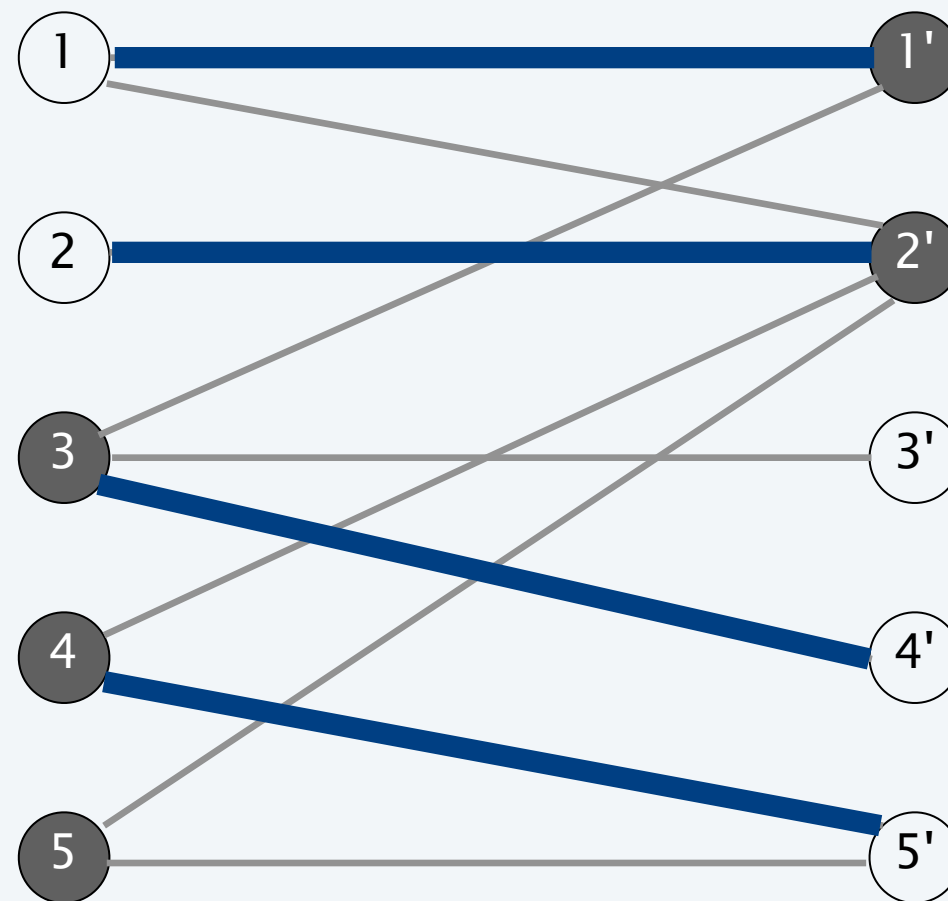
Pf. Each vertex can cover at most one edge in any matching.



matching M: 1-1', 2-2', 3-4', 4-5'

Vertex cover in bipartite graphs: König-Egerváry Theorem

Theorem. [König-Egerváry] In a **bipartite** graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.



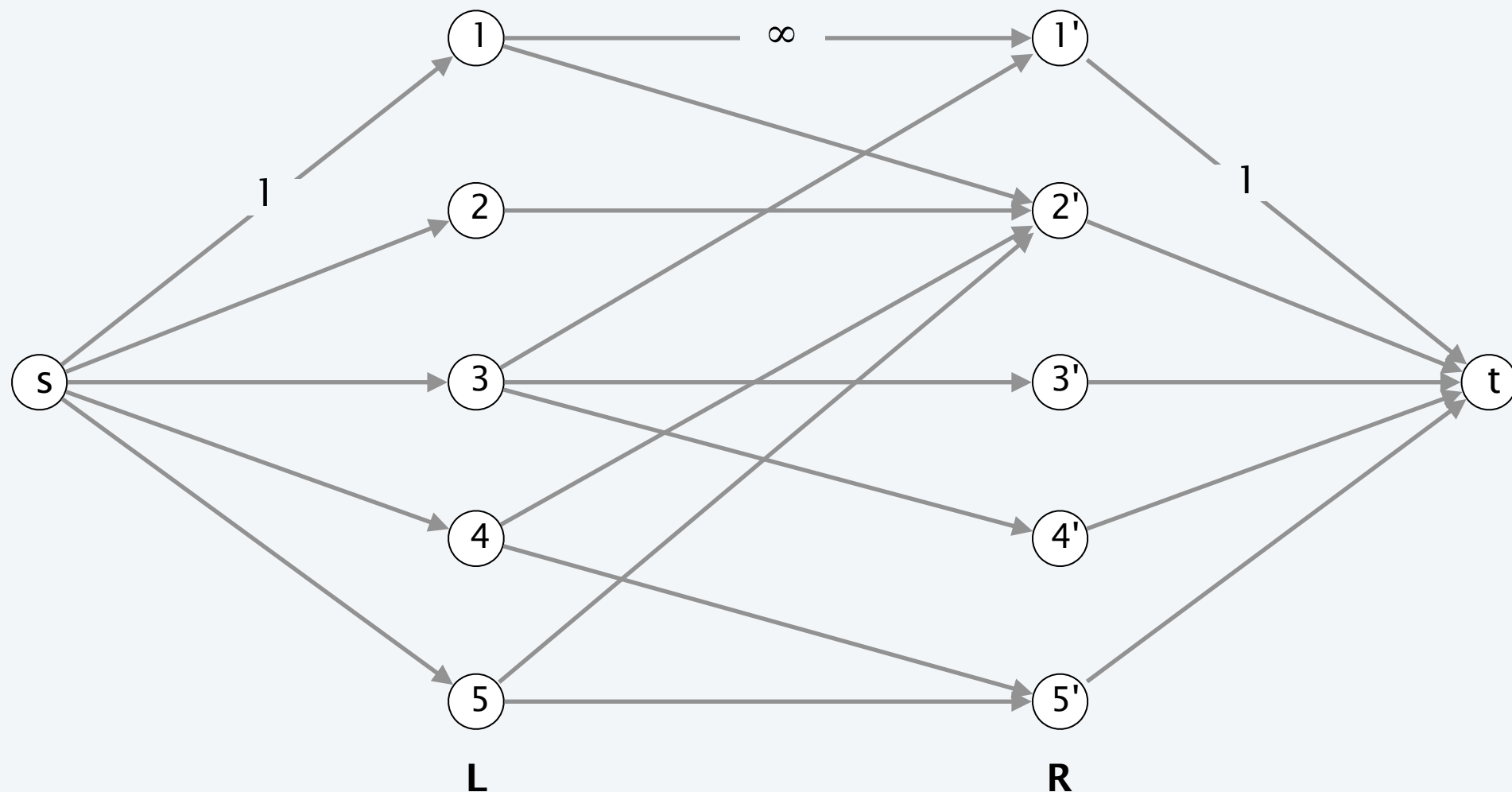
matching M: 1-1', 2-2', 3-4', 4-5'

vertex cover $S = \{ 3, 4, 5, 1', 2' \}$

Proof of König-Egerváry theorem

Theorem. [König-Egerváry] In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that $|M| = |S|$.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.



Proof of König-Egerváry theorem

Theorem. [König-Egerváry] In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching M and cover S such that $|M| = |S|$.
- Formulate max flow problem as for bipartite matching.
- Let M be max cardinality matching and let (A, B) be min cut.
- Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$, $R_B = R \cap B$.

- Claim 1. $S = L_B \cup R_A$ is a vertex cover.

- consider $(u, v) \in E$
- $u \in L_A, v \in R_B$ impossible since infinite capacity
- thus, either $u \in L_B$ or $v \in R_A$ or both

- Claim 2. $|M| = |S|$.

- max-flow min-cut theorem $\Rightarrow |M| = \text{cap}(A, B)$
- only edges of form (s, u) or (v, t) contribute to $\text{cap}(A, B)$
- $|M| = \text{cap}(A, B) = |L_B| + |R_A| = |S|$. ■