Homework 1 : Solutions

Problem 1 : (10pts)

$$R_H(t,t';f,f') = \mathbb{E}\left[L_H(t,f)\,L_H^*(t',f')\right] = \mathbb{E}\left[\int_\tau h(t,\tau)e^{-j2\pi f\tau}d\tau \int_{\tau'} h^*(t',\tau')e^{j2\pi f'\tau'}d\tau'\right]$$

$$= \iint_{\tau\tau'} \mathbb{E}\left[h(t,\tau)h^*(t',\tau')\right] e^{-j2\pi f\tau} e^{j2\pi f'\tau'} d\tau d\tau'$$

WSSUS

$$= \iint_{\tau\tau'} R_h(t-t';\tau)\,\delta(\tau-\tau')\,e^{-j2\pi f\tau} e^{j2\pi f'\tau'} d\tau d\tau'$$

$$= \int_\tau R_h(\underbrace{t-t'}_{\Delta t};\tau)\,e^{-j2\pi\tau\underbrace{(f-f')}_{\Delta f}} d\tau$$

$$= R_H(\Delta t;\Delta f) \quad \blacksquare$$

Problem 2 : (30pts)

- part 1: There is no delay in the channel:

  10pts

$$\boxed{h(t,\tau) = m(t)\,\delta(\tau)} \leftarrow \text{guess}$$

check : $r_b(t) = \int_\tau h(t,\tau)s_b(t-\tau)\,d\tau = m(t)\int_\tau \delta(\tau)s_b(t-\tau)\,d\tau = \underline{m(t)\,s_b(t).}$

- part 2:

  5pts

$$S_H(\tau,\nu) = \int_{t\in\mathbb{R}} h(t,\tau)e^{-j2\pi\nu t}dt = \delta(\tau)\underbrace{\int_t m(t)e^{-j2\pi\nu t}dt}_{M(\nu)} = \underline{\delta(\tau)\,M(\nu)}$$

Fourier transform

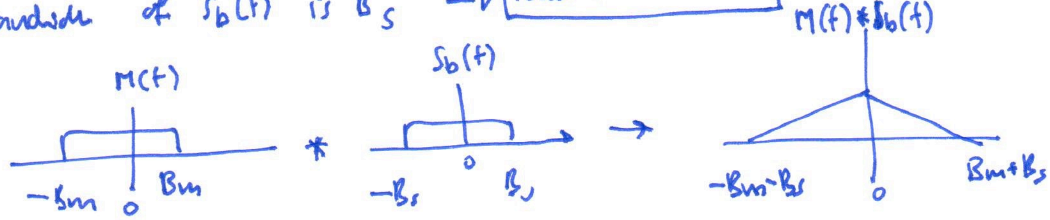
$$L_H(t,f) = \int_\tau h(t,\tau)e^{-j2\pi f\tau}d\tau = m(t)\int_\tau \delta(\tau)e^{-j2\pi f\tau}d\tau = \underline{m(t)}$$

does not introduce any time shifts!

• part 3 : bandwidth of $m(t)$ is B_m

5 pts

bandwidth of $S_b(t)$ is B_S $\Rightarrow$ $\boxed{\text{total BW is } B_S + B_m}$



$\Rightarrow$ multiplication in time is convolution in frequency !!!

• part 4 : $m(t)$ is stationary Gaussian with $\mathbb{E}[m(t)] = 0$

and $R_m(\tau) = \mathbb{E}[m(t+\tau)\, m^*(t)]$.

5 pts $\quad R_H(\Delta t;\Delta f) = \mathbb{E}[L_H(t+\Delta t; f+\Delta f)\, L_H^*(t;f)] =$

$\Delta t = t' - t \quad \Delta f = f' - f$

$$\underset{\substack{\text{from part 2} \\ L_H(t,f) = m(t)}}{=} \mathbb{E}[m(t+\Delta t)\, m^*(t)] = \underline{R_m(\Delta t)} = (*)$$

5 pts $\quad C_H(\tau,\nu) = \int_{\Delta t}\int_{\Delta f} R_H(\Delta t,\Delta f)\, e^{-j2\pi\nu\Delta t}\, e^{j2\pi\tau\Delta f}\, d\Delta t\, d\Delta f =$

$$\overset{(*)}{=} \underbrace{\int_{\Delta f} e^{j2\pi\tau\Delta f}\, d\Delta f}_{\delta(\tau)} \underbrace{\int_{\Delta t} R_m(\Delta t)\, e^{-j2\pi\nu\Delta t}\, d\Delta t}_{S_m(\nu)} = \underline{\delta(\tau)\, S_m(\tau)}$$

power spectral density of $R_m(\Delta t)$

Problem ~~4~~ 3: 20

Part 1: $h(\tau) = \alpha_0 \delta(\tau) + \alpha_1 \delta(t-\tau_0)$

15 pts

FT in τ

5 pts $L_H(t,f) = H(f) = \alpha_0 + \alpha_1 e^{-j2\pi f \tau_0}$

$$R_H(\Delta t; \Delta f) = \mathbb{E}\left[L_H(t,f) L_H^*(t',f') \right] =$$

$$= \mathbb{E}\left[\underbrace{|\alpha_0|^2}_{} + \underbrace{\alpha_0 \alpha_1^*}_{0} e^{j2\pi f \tau} + \underbrace{\alpha_0^* \alpha_1}_{0} e^{-j2\pi f' \tau} + |\alpha_1|^2 e^{-j2\pi \tau_0 (f-f')} \right]$$

5 pts
$$= \sigma_0^2 + \sigma_1^2 e^{-j2\pi \tau_0 \Delta f}$$

cross terms disappear because α_0 and α_1 are uncorrelated and zero mean!

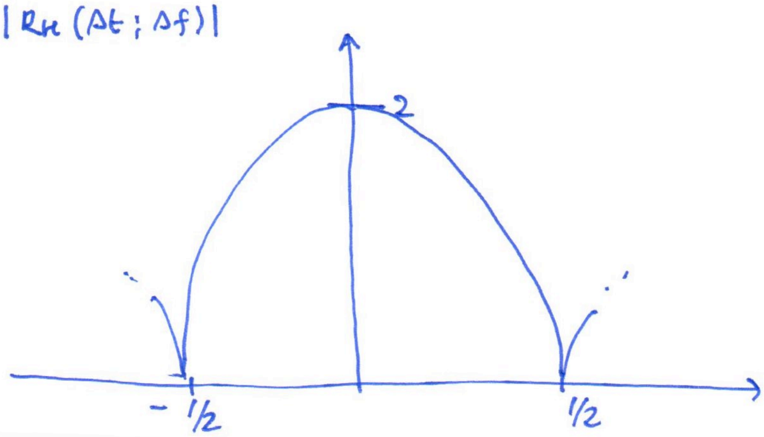
$$C_H(\tau;\nu) = \int_{\Delta t} \int_{\Delta f} R_H(\Delta t; \Delta f) e^{-j2\pi \nu \Delta t} e^{j2\pi \tau \Delta f} \, d\Delta t \, d\Delta f =$$

$$= \underbrace{\int_{\Delta t} e^{-j2\pi \nu \Delta t} \, d\Delta t}_{\delta(\nu)} \underbrace{\int_{\Delta f} \left(\sigma_0^2 + \sigma_1^2 e^{-j2\pi \tau_0 \Delta f} \right) e^{j2\pi \tau \Delta f} \, d\Delta f}_{\text{inverse FT}}$$

5 pts
$$\underline{= \delta(\nu) \cdot \left(\sigma_0^2 \delta(\tau) + \sigma_1^2 \delta(\tau - \tau_0) \right)}$$

Part 2: $|R_H(\Delta t; \Delta f)|$

5 pts



plot with matlab

$\tau_0 = 1$, $\sigma_0 = \sigma_1 = 1$

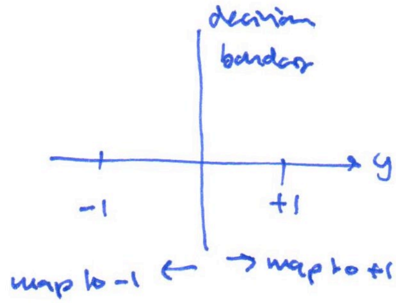
Problem 4: 40 pts

Part 1: 5 pts

optimal detector : MAP

$$f_{y|x}(y|x=+1) \underset{\hat{x}=-1}{\overset{\hat{x}=+1}{\gtrless}} f_{y|x}(y|x=-1)$$

$$y \underset{-1}{\overset{+1}{\gtrless}} 0$$



Part 2: 5 pts

$$P_{err} = \frac{1}{2}P[\hat{x}=1|x=-1] + \frac{1}{2}P[\hat{x}=-1|x=1]$$

$$= \frac{1}{2}P[y \geq 0|x=-1] + \frac{1}{2}P[y<0|x=1]$$

$$\overset{\text{symmetry}}{=} P[y \geq 0|x=-1] = P[-1+n \geq 0] = P[n \geq 1]$$

$$\Rightarrow n \sim N(0,\sigma^2) \qquad \frac{1}{\sigma}n \sim N(0,1)$$

$$= P\left[\frac{1}{\sigma}n \geq \frac{1}{\sigma}\right] = \underline{Q\left(\frac{1}{\sigma}\right)} = Q\left(\sqrt{SNR}\right)$$

Part 3: see printout & plot 20 pts

Part 4: see plot 10 pts

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% ------------------------------------------------------------------------
% Homework 1 : AWGN simulation
% (c) 2015 studer@cornell.edu
% ------------------------------------------------------------------------

% set up SNR range
SNRdB_list = [-10:20];

% number of Monte-Carlo trials
T = 10000;

% set random seed (allows to repeat simulation)
rng(0);

% convert SNR to noise variance (signal power is Es=1)
sigma2 = 10.^(-SNRdB_list/10);

% predefine SER vector
SER = zeros(length(SNRdB_list),1);

% generate zero-mean additive Gaussian noise with variance 1
n = randn(T,1);

% generate BPSK symbols {-1,+1}
x = sign(randn(T,1));

% main SNR simulation loop
for kk=1:length(SNRdB_list)

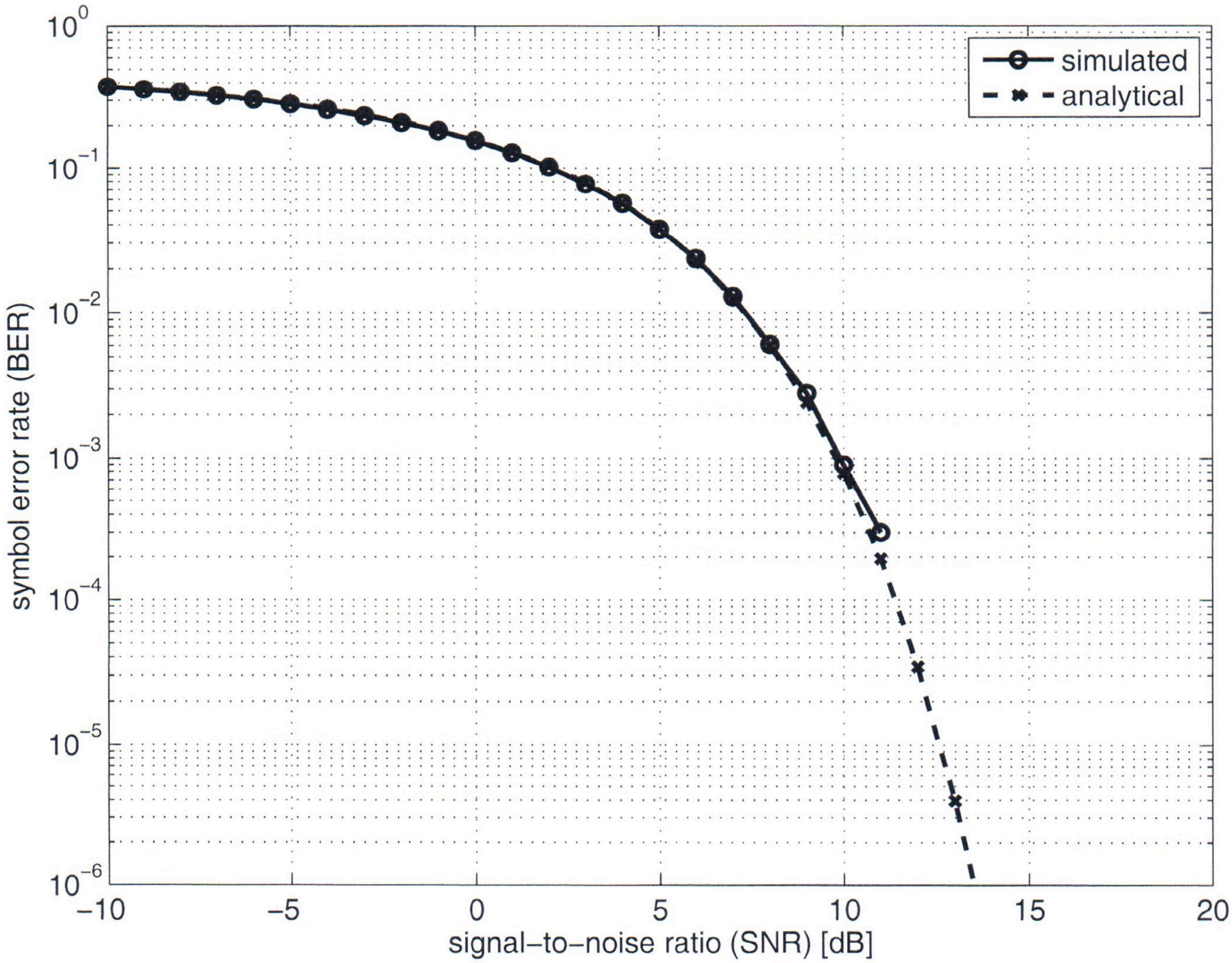
  % transmit over AWGN channel
  y = x+sqrt(sigma2(kk))*n;

  % ML detector (MAP is equivalent)
  xhat = sign(y);

  % compute bit error rate (BER)
  SER(kk,1) = sum(x~=xhat)/T;

end

% plot results
figure(1)
% simulated SER
semilogy(SNRdB_list,SER,'bo-','LineWidth',2)
hold on
% analytical SER
semilogy(SNRdB_list,qfunc(1./sqrt(sigma2)),'rx--','LineWidth',2)
hold off
grid on
axis([-10 20 1e-6 1])
xlabel('signal-to-noise ratio (SNR) [dB]','FontSize',12)
ylabel('symbol error rate (BER)','FontSize',12)
legend('simulated','analytical')
set(gca,'FontSize',12)
```


simulated
analytical
symbol error rate (BER)
signal–to–noise ratio (SNR) [dB]
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
−10
−5
0
5
10
15
20