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Homework-3

P.1.1 SIMO :

$$\underline{y} = \underline{h}\, x + \underline{n}$$

With coherent detection, doing matched filtering

$$\tilde{y} = \frac{\underline{h}}{\|h\|} \cdot \underline{y} = \|h\| x + \frac{\underline{h}}{\|h\|} \cdot \underline{n}$$

$\uparrow$ Scalar sufficient statistic

$\because x \in (+a, -a)$ i.e. Real only

Taking Real sufficient statistic

$$\underbrace{\tilde{\gamma} = \text{Real}(\tilde{y})}_{\sim N(\|h\|x,\ N_0/2)} = \|h\| x + \underbrace{\text{Re}\left\{\frac{\underline{h}}{\|h\|} \cdot \underline{n}\right\}}_{\sim N(0,\ N_0/2)}$$

$\because p(+a) = p(-a) = 0.5$

M.L. detector gives optimal detection Rule

$$P(\tilde{\gamma} \mid x = +a) \underset{-a}{\overset{+a}{\gtrless}} P(\tilde{\gamma} \mid x = -a)$$

$$\frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(\tilde{\gamma} - \|h\|a)^2}{2N_0/2}\right) \underset{-a}{\overset{+a}{\gtrless}} \frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(\tilde{\gamma} + \|h\|a)^2}{2N_0/2}\right)$$

$$\Rightarrow (\tilde{\gamma} - \|h\|a)^2 \underset{-a}{\overset{+a}{\lessgtr}} (\tilde{\gamma} + \|h\|a)^2$$

$\uparrow$

Optimal detector, for $x \in \{+a, -a\}$

P1.2 Prob. of error

$$P_e(h) = P\left((\tilde{r} - \|h\|a)^2 > (\tilde{r} + \|h\|a)^2 \mid x = +a \right)$$

$$= P\left(\left(Re\left\{\frac{h}{\|h\|} n\right\}\right)^2 > \left(\|h\|a + Re\left\{\frac{h}{\|h\|} n\right\} + \|h\|a \right)^2 \right)$$

$$= P\left(0 > (2\|h\|a)^2 + 2 \cdot 2\|h\|a \cdot Re\left\{\frac{h}{\|h\|} n\right\} \right)$$

$$= P\left((2\|h\|a)^2 < -4\|h\|a \; Re\left\{\frac{h}{\|h\|} n\right\} \right)$$

$$= P\left(\|h\|a < \underbrace{-Re\left\{\frac{h}{\|h\|} n\right\}}_{\sim N(0,\, N_0/2)} \right)$$

$$= P\left(\frac{\|h\|a}{\sqrt{N_0/2}} < \underbrace{-Re\left\{\frac{h}{\|h\|} n\right\}\frac{1}{\sqrt{N_0/2}}}_{\sim N(0,1)} \right)$$

$$P_e(h) = Q\left(\sqrt{\frac{2\|h\|^2 a^2}{N_0}} \right) = Q\left(\sqrt{2\|h\|^2 SNR} \right); \quad SNR = \frac{a^2}{N_0}$$

$$\therefore P_e = E_{\|h\|}\left[P_e(h) \right] = E\left[Q\left(\sqrt{2\|h\|^2 SNR} \right) \right]$$

$\|h\|^2 \sim \chi^2_{2L}$ - chi-squared with $2L$ degrees of freedom

$$f_{\chi^2_{2L}}(x) = \frac{1}{(L-1)!\, 2^L} x^{L-1} e^{-x} \sigma(x)$$

Using this

$$P_e = \left(\frac{1}{2}(1-\mu) \right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2} \right)^k; \quad \mu = \sqrt{\frac{SNR}{1+SNR}}$$

$$P_e \xrightarrow{\text{large SNR}} (4\,SNR)^{-L} \binom{2L-1}{L}$$

End of Ques 1

P.2.1 SIMO: Selection Combining

$$P(\text{"deep fade"}) = P\left(|h_{max}|^2 SNR < 1\right)$$
$$= P\left(|h_{max}|^2 < \frac{1}{SNR}\right) \quad -(1)$$

where $|h_{max}|^2 = \max\{|h_1|^2, |h_2|^2 \cdots |h_L|^2\}$

Now, we are given

$$z = \max\{|x_1|^2, \cdots, |x_L|^2\}$$

$$\& \underset{\text{CDF}}{F(z)} = P\left(\max\{|x_1|^2, \cdots |x_L|^2\} < z\right)$$

$$= P\left(|x_1|^2 < z, |x_2|^2 < z \cdots\right)$$

$$\stackrel{\text{iid}}{=} \underbrace{P(|x_1|^2 < z)}_{\text{CDF of } |x_i|^2} \cdot P(|x_2|^2 < z) \cdots$$

$$= (1 - \exp(-z)) \cdot (1 - \exp(-z)) \cdots$$

$$F(z) = (1 - \exp(-z))^L \cdot \sigma(z) \quad -(2)$$

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P. 2.2 From 1 & 2

$$P\left(|h_{max}|^2 < \frac{1}{SNR}\right) = F\left(\frac{1}{SNR}\right)$$

$$= \left(1 - \exp\left(-\frac{1}{SNR}\right)\right)^L \quad \cancel{\leq \left(\frac{1}{SNR}\right)}^{1}$$

$\because 1/_{SNR} > 0$

$e^x \approx 1 + x$ for small x

$$P(\text{"Deep Fade"}) = \left(1 - \left(1 - \frac{1}{SNR}\right)\right)^L = \left(\frac{1}{SNR}\right)^L = (SNR)^{-L}$$

$\therefore$ It can be seen that we still achieve the diversity order of L. which is what we achieved doing MRC in Ques 1

End of Ques 2

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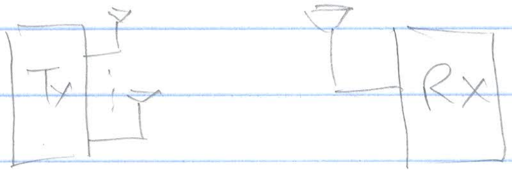
P-3.1



MISO.

$$y = \sum_{l=1}^{L} h_l x + n$$

$h_l \sim CN(0,1)$
$x \in \{-1,+1\}$
$n \sim CN(0, N_0)$

$$SNR = \frac{\mathbb{E}\left[\left|\sum_{l=1}^{L} h_l x\right|^2\right]/1}{\mathbb{E}\left[|n|^2\right]/1}$$

$\mathbb{E}\left[|n|^2\right] = N_0$

$$\mathbb{E}\left[\left|\sum_{l=1}^{L} h_l x\right|^2\right] = \mathbb{E}\left[\left|\sum_{l=1}^{L} h_l\right|^2 \cancel{|x|^2}\right]$$

$$= \mathbb{E}\left[\sum_{l=1}^{L} |h_l|^2\right] \quad \because \text{all } h_l \text{ are iid.}$$

$$= \sum_{l=1}^{L} \mathbb{E}\left[|h_l|^2\right] = L$$

$$\therefore SNR = \frac{L}{N_0}$$

Optimal detector says

$$P(y \mid x=+1) \underset{-1}{\overset{+1}{\gtrless}} P(y \mid x=-1)$$

when $P(x=+1) = P(x=-1) = 1/2$

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$$y = \underbrace{\sum_{l=1}^{L} h_l}\, x + n$$

$$\sum_{l=1}^{L} h_l \sim CN(0, L) \quad \& \quad n \sim CN(0, N_0)$$

$$\sum h_l \cdot x \sim CN(0, L \cdot 1^2) \quad \because \quad x \in (+1, -1)$$

Doing Coherent detection

$$\tilde{y} \rightarrow \frac{\sum h_l}{|\sum h_l|} \cdot y = |\sum h_l| x + \underbrace{\frac{\sum h_l}{|\sum h_l|} \cdot n}_{\sim CN(0, N_0)}$$

$\because$ x is Real, taking Real sufficient statistic

$$\underbrace{Re(\tilde{y})}_{\sim N(|\sum h_l| x,\ N_0/2)} = |\sum h_l| x + Re\left(\frac{\sum h_l}{|\sum h_l|} \cdot n\right)$$

$$\therefore P(\tilde{y} \mid x = +1) \underset{-1}{\overset{+1}{\gtrless}} P(\tilde{y} \mid x = -1)$$

$$\frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(\tilde{y} - |\sum h_l| \cdot (+a))^2}{N_0/2}\right) \underset{-a}{\overset{+a}{\gtrless}} \frac{1}{\sqrt{2\pi N_0/2}} \left(\exp\left(-\frac{(\tilde{y} - |\sum h_l| (-a))^2}{N_0/2}\right)\right.$$

$$(\tilde{y} - |\sum h_l| \cdot a)^2 \underset{-a}{\overset{+a}{\lessgtr}} (\tilde{y} + |\sum h_l| \cdot a)^2$$

Optimal detector

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Prob. of error

$$\therefore P\left(\text{``}(\tilde{y} - \Sigma|h_l|\cdot a)^2 > (\tilde{y} + \Sigma|h_l|a)^2\text{''} \,\middle|\, x=+a\right)$$

$$\tilde{y} = |\Sigma h_l|\cdot a + \frac{\Sigma h_l}{|\Sigma h_l|}\cdot n$$

$$P\left(\left(\frac{\Sigma h_l}{|\Sigma h_l|}\cdot n\right)^2 > \left(2|\Sigma h_l|\cdot a + \frac{\Sigma h_l}{|\Sigma h_l|}\cdot n\right)^2\right)$$

$$\Sigma h_l = \gamma$$

$$P\left(\cancel{\left(\frac{\gamma}{|\gamma|}\cdot n\right)^2} > \left[(2|\gamma|\cdot a)^2 + \cancel{\left(\frac{\gamma}{|\gamma|}\cdot n\right)^2} + 4|\gamma|\cdot a\cdot\frac{\gamma}{|\gamma|}\cdot n\right]\right)$$

$$P\left((2|\gamma|\cdot a)^2 < 4\cdot a\cdot|\gamma|\cdot\frac{\gamma}{|\gamma|}\cdot n\right)$$

$$P\left(|\gamma|\cdot a < \underbrace{\frac{\gamma}{|\gamma|}\cdot n}_{\sim N(0,\, N_0/2)}\right)$$

$$P\left(\frac{1}{\sqrt{N_0/2}}|\gamma|\cdot a < \underbrace{\frac{1}{\sqrt{N_0/2}}\cdot\frac{\gamma}{|\gamma|}\cdot n}_{\sim N(0,1)}\right)$$

$$= Q\left(\frac{|\Sigma h_l|\cdot a}{\sqrt{N_0/2}}\right)$$

$$= Q\left(\sqrt{\frac{2\,|\Sigma h_\ell|^2 a^2}{N_0}}\right)$$

$$\therefore = Q\left(\sqrt{\frac{2|\Sigma h_\ell|^2\, SNR}{L}}\right) = P_e(\Sigma h_\ell)$$

$$\because SNR = \frac{L.a^2}{N_0}$$

$$\therefore P_e(\Sigma h_\ell) = \mathbb{E}_{\Sigma h_\ell}\left[Q\left(\sqrt{\frac{2|\Sigma h_\ell|^2 SNR}{L}}\right)\right)$$

$|\Sigma h_\ell|^2 \rightarrow$ exponentially distributed

$\because \Sigma h_\ell \sim CN(0,L)$

For large SNR

$$P_e = \left(\frac{1}{4SNR}\right)$$

Diversity order:

$$P(\text{"Deep fade"}) = P\left(|\Sigma h_\ell|^2 \frac{SNR}{L} < 1\right)$$

$$= P\Big(\underbrace{|\Sigma h_\ell|^2}_{} < \frac{L}{SNR}\Big)$$

exponential distribution with mean L since $\Sigma h_\ell \sim CN(0, L)$

$$= \int_0^{L/SNR} \frac{1}{L} \exp\left(\frac{-x}{L}\right) dx \quad ; x = \Sigma h_\ell$$

Let $r = x/L \quad \therefore \frac{dr}{dx} = \frac{1}{L}$ $\qquad e^{-r} \qquad r = \frac{x}{L}$

$dx = L.dr$ $\qquad \frac{dr}{dx} = \frac{1}{L}$

$$= \int_0^{1/SNR} \frac{1}{\cancel{L}} \exp(-r) \cdot \cancel{L}.dr \qquad dx \quad dr.L$$

$$= \Big| -\exp(-r) \Big|_0^{1/SNR} = -\exp(-1/SNR) - (-\exp(0))$$

$$= 1 - \exp(-1/SNR)$$

$\xrightarrow[SNR]{large}$ $1/SNR$ $\therefore$ Diversity Order is 1

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P.4 $y = \underline{h}^T \underline{s} + n$

$\underline{s} = x\sqrt{E_s} \cdot \underline{a} \quad ; \underline{a} = [a_1 \cdots a_L]$

$\therefore \underline{s} = [x\sqrt{E_s} \cdot a_1, x\sqrt{E_s} \cdot a_2, \cdots]$

$\|\underline{s}\|_2^2 = (x\sqrt{E_s}a_1)^2 + (x\sqrt{E_s}a_2)^2 + \cdots$
$= (x\sqrt{E_s})^2(a_1^2 + a_2^2 \cdots)$

$\|\underline{s}\|^2 = (x\sqrt{E_s})^2\|\underline{a}\|^2 = x^2 E_s \quad \because \|\underline{a}\|^2 = 1$ (Given)

$\therefore \quad \therefore \mathbb{E}[\|s\|^2] = \mathbb{E}[x^2 E_s] = E_s \quad \because \mathbb{E}[x^2] = 1$

$$SNR = \frac{\mathbb{E}[|\underline{h}^T \underline{s}|^2]/1}{\mathbb{E}[|n|^2]/1} = \frac{\mathbb{E}[|\underline{h}^T \underline{s}|^2]}{N_0}$$

$\mathbb{E}[|\underline{h}^T\underline{s}|^2] = \mathbb{E}[|\sum_{l=1}^{L} h_l s_l|^2] = \mathbb{E}[(\sum_l h_l s_l) \cdot (\sum h_{l'} s_{l'})^*]$

$= \mathbb{E}[(\sum h_l s_l) \cdot (\sum h_{l'}^* s_{l'}^*)]$

$= \mathbb{E}[\sum_{l=1}^{L} \underbrace{|h_l s_l|^2}_{l = l'} + \sum_{l \neq l'} (h_l s_l \cdot h_{l'}^* s_{l'}^*)]$

$= \mathbb{E}[\sum_{l=1}^{L} |h_l s_l|^2] = \sum[\mathbb{E}|h_l s_l|^2]$ $*$

$|h_l s_l|^2 = (h_l s_l) \cdot (h_l^* s_l^*)$
$= h_l \cdot h_l^* \; s_l \cdot s_l^*$
$= |h_l|^2 |s_l|^2$

$\therefore \mathbb{E}|h_l s_l|^2 = \underbrace{\cancel{\mathbb{E}|h_l|^2}}_{1} \cdot \mathbb{E}|s_l|^2 = \mathbb{E}|s_l|^2$

$\therefore * \sum[\mathbb{E}|h_l s_l|^2] = \sum(\mathbb{E}|s_l|^2) = \mathbb{E}[\sum|s_l|^2]$
$= \mathbb{E}[\|s\|^2] = E_s$

$$\therefore SNR = \frac{E_s}{N_0}$$

Part 1 MAP receiver

$$\because \quad x \in \{+1, -1\} \quad \& \quad p(+1) = p(-1) = 0.5$$

$$y = \underline{h}^T \underline{s} + n \quad ; \quad \underline{s} = x\sqrt{E_s} \cdot \underline{a}$$

$$y = \underline{h}^T (x\sqrt{E_s}\, \underline{a}) + n$$

$$= \underbrace{(\underline{h}^T \underline{a})}_{\text{Effective channel}} \cdot x\sqrt{E_s} + n \quad ; \text{ Assuming } E_s = 1$$

$$\therefore \text{SNR} = \frac{1}{N_0}$$

Taking Real sufficient statistic, since x is real

$$\tilde{y} = \text{Real}\left(\frac{(h^T a)^*}{|h^T a|} \times y\right) = |h^T a| \cdot x + \text{Real}\left(\frac{(h^T a)^*}{|h^T a|} \times n\right)$$

$$\therefore \tilde{y} = |h^T a| \cdot x + \underbrace{\text{Real}\left(\frac{(h^T a)^*}{|h^T a|} \cdot n\right)}_{w \sim N(0, N_0/2)}$$

Detection Rule:

$$P(\tilde{y} \mid x = +1) \underset{-1}{\overset{+1}{\gtrless}} P(\tilde{y} \mid x = -1)$$

$$\tilde{y} \sim N(|h^T a| \cdot x, N_0/2)$$

$$\therefore \frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(\tilde{y} - |h^T a| x_a)}{2 \cdot N_0/2}\right) \underset{x_b}{\overset{x_a}{\gtrless}} \frac{1}{\sqrt{2\pi N_0/2}} \exp\left(-\frac{(\tilde{y} - |h^T a| \cdot x_b}{2 \cdot N_0/2}\right)$$

$$(\tilde{y} - |\underline{h}^T \underline{a}|\, x_a)^2 \underset{x_b}{\overset{x_a}{\lessgtr}} (\tilde{y} - |\underline{h}^T \underline{a}|\, x_b)^2$$

$x_a = +1 \; ; \; x_b = -1$

$$(\tilde{y} - |\underline{h}^T \underline{a}|)^2 \underset{-1}{\overset{+1}{\lessgtr}} (\tilde{y} + |\underline{h}^T \underline{a}|)^2$$

Part 2 Prob. of error

$$P_e(|h^Ta|) = P\left((\tilde{y} - |h^Ta|x_a)^2 > (\tilde{y} - |h^Ta|x_b)^2 \mid x = x_a\right)$$

$$= P\left((w)^2 > (|h^Ta|x_a + w - |h^Ta|x_b)^2\right)$$

$$= P\left(\cancel{(w)^2} > (|h^Ta|(x_a - x_b))^2 + \cancel{(w)^2} + 2 \cdot |h^Ta|(x_a - x_b) \cdot w\right)$$

$$= P\left((|h^Ta|(x_a - x_b))^2 < 2 \cdot |h^Ta|(x_b - x_a) \cdot w\right)$$

$$= P\left(\frac{|h^Ta|\,|x_a - x_b|}{2} < \underbrace{w}_{\sim N(0, N_0/2)}\right)$$

$$= P\left(\frac{|h^Ta|\,|x_a - x_b|}{\sqrt{N_0/2}\;2} < \underbrace{\frac{w}{\sqrt{N_0/2}}}_{\sim N(0,1)}\right)$$

$$= Q\left(\sqrt{\frac{|h^Ta|^2\,|x_a - x_b|^2}{2N_0}}\right) \qquad \because x_a = +1,\; x_b = -1$$

$$= Q\left(\sqrt{\frac{2\,|h^Ta|^2}{N_0}}\right) = Q\sqrt{2\,|h^Ta|^2\,SNR}$$

$$\because SNR = \frac{1}{N_0} \text{ as } E_s = 1 \text{ was assumed}$$

If $\underline{a}$ does not have any relation with $\underline{h}$

$$h^Ta = z = h_1 a_1 + h_2 a_2 + \cdots$$

$$h_i \sim CN(\mu_i, \sigma_i^2)\ ; \text{ where } \mu_i = 0,\ \sigma_i^2 = 1$$

$$\therefore z \sim CN\left(\Sigma a_i \mu_i,\ \Sigma a_i^2 \sigma_i^2\right)$$

$$z \sim CN(0, \|a\|^2)$$

$$\underline{h^Ta} = z \sim CN(0, 1) \qquad \because \|a\|^2 = 1$$

$$\therefore P_e = \mathbb{E}\left(Q\sqrt{2|h^Ta|^2 SNR}\right) \underset{\text{SNR}}{\overset{\text{large}}{\approx}} (4\,SNR)^{-1}$$

$\because h^Ta$ is exponentially distributed

Part3. Now, if Tx has knowledge about the channel, then it should try to maximise

$$|h^T a|^2$$

$$= |h_1 a_1 + h_2 a_2 + \cdots|^2$$

$$= |h_1 a_1|^2 + |h_2 a_2|^2 + \cdots \quad \because h_i \text{ are iid.}$$

To maximise, each

$|h_i a_i|^2$ should be maximised

$$\therefore a_i = \frac{h_i^*}{|h_i|}$$, which cancels the phase of channel

$$\therefore \underline{a} = \left[\frac{h_1^*}{|h_1|}, \frac{h_2^*}{|h_2|}, \cdots \frac{h_L^*}{|h_L|} \right]^T$$

Now since $\|\underline{a}\|$ to be 1

$$\therefore \underline{a} \Leftarrow \frac{1}{\sqrt{L}} \underline{a}$$

$$\therefore \text{final } \underline{a} = \frac{1}{\sqrt{L}} \left[\frac{h_1^*}{|h_1|}, \frac{h_2^*}{|h_2|}, \cdots \frac{h_L^*}{|h_L|} \right]^T$$

Part4 $$P_e(h) = Q\left(\sqrt{2|h^T a|^2 \, SNR}\right)$$

$$h^T a = \frac{1}{\sqrt{L}} \left[\frac{h_1 h_1^*}{|h_1|} + \frac{h_2 h_2^*}{|h_2|} + \cdots \right]$$

$$= \frac{1}{\sqrt{L}} \left[|h_1| + |h_2| + |h_3| + \cdots \right]$$

$$\therefore |h^T a|^2 = \frac{1}{L} (|h_1| + |h_2| + \cdots)^2$$

Diversity Order is L

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P.5 Part (1) & Part (2) in matlab (HW3_Q5.m & HW3_Q5_2.m

Part (3):

- Yes, the slopes are correct. For $L=1$, Slope is $\sim(-\frac{1}{10})$
For $L=2$, Slope is $\sim(-\frac{1}{5})$
$L=3$, Slope is $\sim(-\frac{1}{3})$
$L=4$, Slope is $\sim(-\frac{1}{2.5})$

- Yes, selection combining is also giving the same slope as MRC, but the performance is 2-4 dB degraded. i.e. to achieve the same Pe, around 2-4dB higher SNR is required
This is expected because in Selection Combining we are only using the best antenna & ignoring other antenna signal while in MRC we use the best channel as well as the other channels. Therefore the amount of Energy we pick is more which is providing the extra gain in MRC.

- for $P_e = 1\% = 10^{-2}$, with $L=4$, $SNR_{MRC} = 0\,dB$
$SNR_{Selection} = 4\,dB$

P. 6. Part (1) & (2) in matlab (HW3_Q6.m & HW3_Q6_2.m

Part (3): HW3_Q6_3.m

- Yes, curves look OK

- With Beamforming we get better performance as it give us diversity gain.

- For $P_e = 1\% = 10^{-2}$, with $L=4$, $SNR_{Normal} \approx 14\,dB$
$SNR_{BF} \sim 1\,dB$