# HW 1: Some Basic Properties and a Simple Simulation

ECE 5680 – Wireless Communication

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- **Due date:** Tuesday, September 8, 2015 at 11:59pm
- **Submission instructions:** Upload your solutions (solutions and Matlab code) to Cornell Blackboard as a *single* zip-file; name the file ⟨netid⟩-hw⟨number⟩.zip
- **Total points:** 100pts

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### Problem 1: Time-frequency correlation function (10pts)

In class, we used the fact that the WSSUS (short for wide-sense stationary, uncorrelated scattering) assumption leads to the following time-frequency correlation function:

$$R_{\mathbb{H}}(t,t';f,f') \triangleq \mathsf{E}[L_{\mathbb{H}}(t,f)L^*_{\mathbb{H}}(t',f')] = R_{\mathbb{H}}(t-t',f-f'),$$

where $L_{\mathbf{H}}(t,f)$ is the time-varying transfer function of the channel $\mathbb{H}$. Show that this relation holds true. *Hint: Use the fact that $L_{\mathbb{H}}(t,f) = \int_\tau h(t,\tau)e^{-j2\pi f\tau}d\tau$, and exploit the WSS and US properties of the channel $\mathbb{H}$.*

### Problem 2: Linear frequency-invariant system (30pts)

A linear frequency-invariant (LFI) system is a special case of a linear time-varying system. Its input-output relation can be written as $r(t) = m(t)s(t)$, where $s(t)$ is the time-domain input signal, $m(t)$ characterizes the effect of the LFI system, and $r(t)$ is the output signal.

**Part 1** Compute the time-varying impulse response $h(t,\tau)$ as a function of $m(t)$.

**Part 2** Start from the result of Part 1 to compute the delay-Doppler spreading function $S_{\mathbb{H}}(\tau,\nu)$ and the time-varying transfer function $L_{\mathbb{H}}(t,f)$ as a function of $m(t)$. Explain briefly how the frequency invariance manifests itself in each of these system functions.

**Part 3** Assume that the bandwidth of $m(t)$ is $B_m$, i.e.,

$$M(f) = \int_{\mathbb{R}} m(t)e^{-j2\pi ft}\mathrm{d}t \quad \text{for} \quad |f| > B_m.$$

Compute the bandwidth of $R(f) = \int_{\mathbb{R}} r(t)e^{-j2\pi ft}\mathrm{d}t$ as a function of $B_m$ and the bandwidth $B_s$ of the input signal $s(t)$. Give a brief physical interpretation of the result. *Hint: Remember the convolution operator.*

**Part 4** Assume that $m(t)$ is a stationary Gaussian random process with mean zero and the following correlation function:

$$R_m(\tau) = \mathsf{E}[m(t+\tau)m^*(t)].$$

Provide expressions for the time-frequency correlation function $R_{\mathbb{H}}(\Delta t, \Delta f)$ and for the scattering function $C_{\mathbb{H}}(\tau, \nu)$ as a function of the correlation function $R_m(\tau)$. *Hint: Note that $\Delta t = t - t'$ and $\Delta f = f - f'$.*

## Problem 3: A simple channel with two taps (20pts)

Consider a two-tap channel with the following impulse response:

$$h(\tau) = \alpha_0 \delta(\tau) + \alpha_1 \delta(\tau - \tau_0),$$

where $\alpha_0$ and $\alpha_1$ are independent complex circularly-symmetric Gaussian random variables with variances $\sigma_0^2 = \mathsf{E}[|\alpha_0|^2]$ and $\sigma_1^2 = \mathsf{E}[|\alpha_1|^2]$, respectively; the delay $\tau_0$ is known and deterministic.

**Part 1** Compute the scattering function $C_{\mathbb{H}}(\tau, \nu)$ and the time-frequency correlation function $R_{\mathbb{H}}(\Delta t, \Delta f)$, and discuss the impact of $\sigma_0^2$, $\sigma_1^2$, and $\tau_0$ on these functions.

**Part 2** Plot the function $|R_{\mathbb{H}}(\Delta t, \Delta f)|$ for $\sigma_0^2 = \sigma_1^2$ using MATLAB.

## Problem 4: Error-rate performance of BPSK signaling in an AWGN channel (40pts)

The goal of this problem is to characterize the symbol error rate (SER) of binary phase shift keying (BPSK) in an additive white Gaussian noise (AWGN) channel. Consider the following input-output relation:

$$y = x + n, \tag{1}$$

where $y \in \mathbb{R}$ is the received signal, $x \in \{-1, +1\}$ is the transmit signal (assume that both symbols are transmitted with probability 0.5), and $n$ is zero-mean Gaussian noise with variance $\sigma^2$.

**Part 1** Design the optimal detector for (1), i.e., come up with a decision rule which uses the received signal $y$ to generate an estimate $\hat{x} \in \{-1, +1\}$ for which the symbol error rate $\Pr\{x \neq \hat{x}\}$ is minimized.

**Part 2** Derive an analytical expression for the symbol error rate $\Pr\{x \neq \hat{x}\}$. Your expression should depend on the signal-to-noise ratio (SNR), which we define as follows:

$$SNR = \frac{\mathsf{E}[|x|^2]}{\mathsf{E}[|n|^2]} = \frac{1}{\sigma^2}.$$

*Hint: Assume that the symbol $+1$ was transmitted but the detector designed in Part 1 generates $\hat{x} = -1$. Then, compute the probability of this event.*

**Part 3** Compute the SER using Monte-Carlo simulations in MATLAB. To this end, remember the fact that $\Pr\{x \neq \hat{x}\} = \mathsf{E}[I(x \neq \hat{x})]$, where $I(x \neq \hat{x})$ is the indicator function that generates a 1 if a detection error occurs, i.e., if $x \neq \hat{x}$, and 0 if the detector generates the correct transmit signal, i.e., if $x = \hat{x}$. One can now numerically approximate the SER by replacing the expectation operator with the sample mean

$$\Pr\{x \neq \hat{x}\} = \mathsf{E}[I(x \neq \hat{x})] \approx \frac{1}{T}\sum_{t=1}^{T} I(x_t \neq \hat{x}_t), \tag{2}$$

and by performing the following steps for every Monte-Carlo trial $t = 1, \ldots, T$:

1. Generate a transmit symbol $x_t$ from the BPSK set $\{-1, +1\}$ with equal probability
2. Generate a zero-mean Gaussian noise sample $n_t$ with variance $\sigma^2$
3. Compute the received signal $y_t = x_t + n_t$ as in the model (1)
4. Use the detector from Part 1 to generate an estimate $\hat{x}_t$ from $y_t$
5. Evaluate $I(x_t \neq \hat{x}_t)$

After repeating these steps $T$ times, you can approximate $\Pr\{x \neq \hat{x}\}$ using (2). The more Monte-Carlo trials you perform, the more accurate your SER result will be.

Write a MATLAB script that performs the above Monte-Carlo simulation. Simulate the SER for different SNR values, i.e., for $SNR_{\text{dB}} \in \{-10, -9, \ldots, 19, 20\}$ decibel; remember that $SNR_{\text{dB}} = 10\log_{10}(SNR)$. For each value of $SNR_{\text{dB}}$, perform $T = 10{,}000$ Monte-Carlo trials. Plot the resulting SER function $SER(SNR_{\text{dB}})$ in MATLAB as a doubly-logarithmic plot (the SNR values are already in decibels) using the following MATLAB commands:

```
figure(1)
semilogy(SNR_dB,SER,'bo-')
grid on
axis([-10 20 1e-6 1])
xlabel('signal-to-noise ratio (SNR) [dB]')
ylabel('symbol error rate (SER)')
legend('simulated SER')
```

where `SNR_dB` is a vector of SNR values in decibels and `SER` a vector containing the corresponding simulated SER values; the option `'bo-'` generates a blue curve with o-markers and a solid line.

**Part 4** Use your analytical expression of the SER from Part 2 and include it in the MATLAB plot from Part 3. Compare both results. *Hint: Use MATLAB's '`hold on`' and '`hold off`' commands to superimpose multiple curves in one plot; see the '`help`' command.*

***Remark: For many practical wired and wireless channels, modulation schemes, and detectors, the SER cannot be evaluated analytically (or it would simply be too complicated). Hence, one often resorts to Monte-Carlo simulations to obtain SER values.***