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P.1.1 $x \in \mathbb{C}^N$

$\hookrightarrow x_R + i x_I$

$\& \quad z = \begin{bmatrix} x_R \\ x_I \end{bmatrix}_{2n \times 1}$

$K_z = \mathbb{E}[z z^T]$, will be a $2n \times 2n$ matrix

$= \mathbb{E}\left[\begin{bmatrix} x_R \\ x_I \end{bmatrix} \begin{bmatrix} x_R & x_I \end{bmatrix} \right]$, where x_R has n elements, x_I has n elements.

$$= \mathbb{E}\begin{array}{c|cc} & \text{Real} & \text{Imag} \\ \hline \text{Real} & [x_R \cdot x_R^T]_{n \times n} & [x_R \cdot x_I^T]_{n \times n} \\ \text{Imag} & [x_I \cdot x_R^T]_{n \times n} & [x_I \cdot x_I^T]_{n \times n} \end{array}_{2n \times 2n}$$

$x = x_R + i x_I$

$\therefore K_x = \mathbb{E}[x x^H]$

$\therefore K_x(i,j) = \mathbb{E}[(x_{R_i} + i x_{I_i})(x_{R_j} - i x_{I_j})]$

$\uparrow$ the i th row & j th column

$= \mathbb{E}[x_{R_i} x_{R_j} + x_{I_i} x_{I_j} + i(-x_{R_i} x_{I_j} + x_{I_i} x_{R_j})]$

$\& \quad J_x = \mathbb{E}[x x^T]$

$J_x(i,j) = \mathbb{E}[(x_{R_i} + i x_{I_i})(x_{R_j} + x_{I_j})]$

$= \mathbb{E}[x_{R_i} x_{R_j} - x_{I_i} x_{I_j} + i(x_{R_i} x_{I_j} + x_{I_i} x_{R_j})]$ — 1

Now, $[x_R . x_R^T](i,j) = x_{Ri} . x_{Rj}$

$$\therefore \ [x_R . x_R^T] = \frac{1}{2}\{R(K_x) + R(J_x)\}$$

Similarly

$$[x_I . x_I^T] = \frac{1}{2}\{R(K_x) - R(J_x)\}$$

$$[x_R . x_I^T] = \frac{1}{2}\{-I(K_x) + I(J_x)\}$$

$$[x_I . x_R^T] = \frac{1}{2}\{I(K_x) + I(J_x)\}$$

$$\therefore \ K_Z = \frac{1}{2}\left[\begin{array}{c|c} R(K_x) & -I(K_x) \\ \hline I(K_x) & R(K_x) \end{array}\right]_{2n\times 2n} + \frac{1}{2}\begin{bmatrix} R(J_x) & I(J_x) \\ I(J_x) & -R(J_x) \end{bmatrix}$$

P.1.2. Since x is circularly symmetric

Therefore x & $e^{j\theta}x$ should have same distribution

$$\therefore \ K_x = K_{e^{j\theta}x} \quad \& \quad J_x = J_{xe^{j\theta}}$$

$$\therefore \ J_x \triangleq \mathbb{E}[x x^T] = J_{xe^{j\theta}} \triangleq \mathbb{E}[x e^{j\theta}(x e^{j\theta})^T]$$

$$\mathbb{E}[x x^T] = \mathbb{E}[x e^{j\theta} . e^{j\theta} x^T] = e^{2j\theta} . \mathbb{E}(x x^T)$$

$\therefore$ To satisfy above

$\mathbb{E}(x x^T) = 0$ as $e^{2j\theta}$ is non-zero

$\therefore \ J_x = 0$

$\therefore$ for circular symmtric case

$$K_z = \frac{1}{2}\begin{bmatrix} R(K_x) & -I(K_x) \\ I(K_x) & R(K_x) \end{bmatrix}$$

P.1.3 From eq① in P 1.1

$$J_x(i,j) = E\left(x_{Ri}\, x_{Rj} - x_{Ii}\, x_{Ij} + i\left(x_{Ri}\, x_{Ij} + x_{Ij}\, x_{Rj}\right)\right)$$

& from P 1.2

$$J_x B = 0$$

$$\therefore E(x_{Ri}\, x_{Rj}) - E(x_{Ii}\, x_{Ij}) = 0$$

$\therefore K_{x_R} = K_{x_I}$ (covariance of x_R & x_I are equal

But more Importantly

$$E(x_{Ri}\, x_{Ij} + x_{Ii}\, x_{Rj}) = 0$$

$\therefore$ $E(x_{Ri}\, x_{Ij}) = 0$ for all i, j

$\therefore$ which is covariance between x_R & x_I

& since covariance is zero x_R & x_I are indpendent

P 1.4 $\quad x = x_R + i\,x_I$

$\quad\quad\quad$ & $x_R, x_I \sim N(0, N_0/2)$

Now, each element of K_x is

$$K_x(i,j) = x_{Ri}\,x_{Rj} + x_{Ii}\,x_{Ij} + i\big(\underbrace{-x_{Ri}\,x_{Ij} + x_{Ii}\,x_{Rj}}_{0}\big)$$

as they are independent

$$\therefore K_x = \underbrace{\begin{bmatrix} N_0/2 & 0 & \cdots \\ 0 & N_0/2 & \\ \vdots & & \\ \vdots & & \end{bmatrix}_{n\times n}}_{\text{covariance matrix of } x_R \text{ vector}} + \underbrace{\begin{bmatrix} N_0/2 & \cdots & 0 \\ \vdots & N_0/2 & \\ 0 & & N_0/2 \end{bmatrix}_{n\times n}}_{\text{covariance matrix of } x_I \text{ vector}}$$

$$K_x = \begin{bmatrix} N_0 & 0 & 0 \\ 0 & N_0 & \\ 0 & & \ddots \end{bmatrix}_{n\times n}$$

= Diagonal matrix with each element of diagonal as N_0

Similarly

$$J_x = x_{Ri}\,x_{Rj} - x_{Ii}\,x_{Ij} + i\big(\overbrace{x_{Ri}\,x_{Ij} + x_{Ii}\,x_{Rj}}^{0}\big)$$

$$J_x = \underset{\text{of } x_R}{\text{Covariance matrix}} - \underset{\text{of } x_I}{\text{Covariance matrix}}$$

$$= [\,0\,]$$

$\therefore$ Its a circular symmetric vector

P2.1 802.11 b WLAN in office.

Doppler spread :

$$D_s = \frac{v f_c}{c}$$

Now, since we are in office

Assumption 1 : Nodes are moving at low speed, say ~~10~~ 2 km/hr

$$\therefore v = \frac{\cancel{10}\ 2 \times 1609 \text{ meter}}{3600\ s} \approx \frac{4}{5} m/s \sim 0.8 m/s$$

$\therefore$ ~~D_s~~ f_c for WLAN (802.11 b) $\approx$ 2500 MHz

$$\therefore D_s = \frac{(4/5) \times 2500 \times 10^6}{3 \times 10^8} = \frac{33.33}{5} Hz = 6.6 Hz$$

Now, $T_c \propto \frac{1}{D_s}$, for simplicity we take

$$T_c = 1/D_s$$

$$\therefore T_c = \frac{5}{33.33} \approx \cancel{30 ms}\ 150 ms$$

Now, from table 2

Max. Frame / Slot for 802.11 b = 18.96 ms

$$\therefore \frac{\cancel{30}\ 150}{18.96} \approx 8$$ frames experience same channel

$\therefore$ Channel is slow/moderate fading as it changes over 7 symbols ~~for about~~ length. ~~every frame~~.

Now in office at $f_c = 2400$ MHz, max Delay spread is

$\tau = 0.074\ \mu s$

$\therefore$ min $B_c = \frac{1}{0.074} = 13.5$ MHz

& the bandwidth for 802.11b is $= 22$ MHz

$\therefore$ Channel is freq-selective

$\therefore$ 802.11b channel is ~~fast~~ slow/moderate fading & freq. selective.

P.22. UMTS terminal on high-speed moving train
$V = 155$ mph & rural environment

f_c for UMTS terminal receive $= 2200$ MHz (Downlink)

$\therefore$ ~~D_s~~ $V = 155$ mph $= \frac{155 \times 1609}{3600} = 69$ m/s

$\therefore D_s = \frac{69 \times 2200 \times 10^6}{3 \times 10^8} \approx 506$ Hz

$\therefore T_c = 1/D_s = \frac{1}{506} = \frac{10^3}{506}$ ms ~ 2 ms

Frame per slot for UMTS $= 10$ ms

$\therefore$ ~~App~~ Channel is changing within one frame
$\therefore$ its time-selective

Now, in Rural env. at $f_c = 2200$ MHz

Worst case $\tau = 14.4\ \mu s$

$\therefore$ Min $B_c = \frac{1}{14.4} = 69$ KHz

Bandwidth of UMTS = 5 MHz

$\therefore$ Channel is not freq. selective.

$\therefore$ UMTS Channel is time selective / freq. Dispersive.

* End of Ques 2

P.3.1 $x \in \mathcal{X} = \{+1+j,\ 1-j,\ -1+j,\ -1-j\}$

Let this be represented as

$$\mathcal{X} = \{X_a,\ X_b,\ X_c,\ X_d\}$$

We receive, $y = x + n$, $y \in \mathbb{C}$

Lets represent a complex variable by 2x1 Real Vector

$$\therefore Y = \begin{bmatrix} y_R \\ y_I \end{bmatrix},\quad X = \begin{bmatrix} x_R \\ x_I \end{bmatrix},\quad N = \begin{bmatrix} n_R \\ n_I \end{bmatrix}$$

$\because$ n is $\sim \mathcal{CN}(0, N_0)$

$\therefore$ n_R & n_I are iid with $\mathcal{N}(0, N_0/2)$

& $N \sim \mathcal{N}(0, I\, N_0/2)$

$$\mathcal{X} = \{X_a,\ X_b,\ X_c,\ X_d\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

Therefore my received signal can be written as.

$$Y = X + N \quad \text{, all } Y, X, N \text{ are } 2\times1 \text{ vectors}$$

Now

$$\hat{X}^{MAP} = \underset{X \in \mathcal{X}}{\arg\max}\ f(Y \mid X)\, p(X)$$

$$p(X_a) = p(X_b) = p(X_c) = p(X_d) = 0.25 \text{ (given)}$$

$\therefore$ its constant w.r.t. X

$$\therefore \hat{X}^{MAP} = \underset{X \in \mathcal{X}}{\arg\max}\ f(Y \mid X)$$

Now $f(y|x)$

$$= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \exp\left(\frac{-\|y - x\|_2^2}{2\sigma^2}\right)$$

for our case $n = 2$

$$\therefore = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-\|y - x\|_2^2}{2\sigma^2}\right)$$

$$\|y - x\|_2^2 = \left\| \begin{bmatrix} y_R \\ y_I \end{bmatrix} - \begin{bmatrix} x_R \\ x_I \end{bmatrix} \right\|_2^2 = \left(\sqrt{(y_R - x_R)^2 + (y_I - x_I)^2}\right)^2$$

$$= (y_R - x_R)^2 + (y_I - x_I)^2$$

$$\therefore \hat{x}^{MAP} = \underset{x \in \mathcal{X}}{\operatorname{argmax}} \left(\underbrace{\frac{1}{2\pi\sigma^2}}_{\substack{\text{Same} \\ \text{for all } x\text{, so removing}}} \exp\left(\frac{-\|y - x\|_2^2}{2\sigma^2}\right)\right)$$

& then taking log

$$\therefore = \underset{x \in \mathcal{X}}{\operatorname{argmax}} \left[-\frac{\|y - x\|_2^2}{\underbrace{2\sigma^2}} \right]$$

constant, can be removed.

$$= \underset{x \in \mathcal{X}}{\operatorname{argmin}} \left[+\|y - x\|_2^2 \right]$$

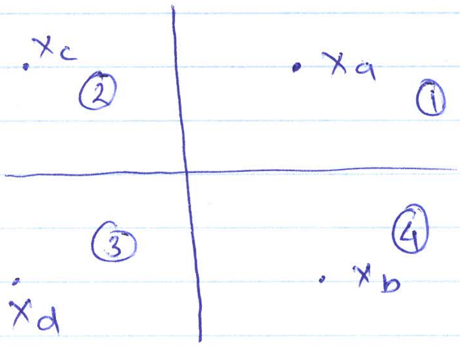
$$= \min\left[\underbrace{\|y - x_a\|_2^2}_{x_a}, \underbrace{\|y - x_b\|_2^2}_{x_b}, \underbrace{\|y - x_c\|_2^2}_{x_c}, \underbrace{\|y - x_d\|_2^2}_{x_d} \right]$$

$\hat{x} = x_a$

↑

Nearest - neighbour rule

$\therefore$ Detection Rule can be shown as



If y in Quad (1) $\rightarrow \hat{x} = x_a$
y in Quad 2 $\rightarrow \hat{x} = x_c$
y in " 3 $\hat{x} = x_d$
y in " 4 $\hat{x} = x_b$

Prob. of error

$P_e = Pr[y \text{ in Quad (2) or } y \text{ in Quad (3) or } y \text{ in Quad (4)} \mid x = x_a]$

$= 1 - P_c$

$P_c \longrightarrow Pr[y \text{ in Quad (1)} \mid x = x_a]$

$\downarrow$

$Pr[y_R > 0,\ y_I > 0 \mid x = x_a]$

$= p(y_R > 0 \mid x = x_a) \cdot p(y_I > 0 \mid x = x_a)$ as they are i.i.d.

$\therefore P_c = p(x_{aR} + w_R > 0) \cdot p(x_{aI} + w_I > 0)$

$\because x_{aR} = x_{aI} = +k$

$\therefore P_c = P(k > \underbrace{-w_R}_{\mathcal{N}(0, N_0/2)}) \cdot P(k > -w_I)$

$= P\left(\frac{k}{\sqrt{N_0/2}} > \underbrace{-\frac{w_R}{\sqrt{N_0/2}}}_{\mathcal{N}(0,1)}\right) \cdot P\left(\frac{k}{\sqrt{N_0/2}} > -\frac{w_I}{\sqrt{N_0/2}}\right)$

$= Q\left(-\frac{k}{\sqrt{N_0/2}}\right) \cdot Q\left(\frac{-k}{\sqrt{N_0/2}}\right)$

$= \left(1 - Q\left(\frac{k}{\sqrt{N_0/2}}\right)\right)^2$ as $Q(-x) = 1 - Q(x)$

$= 1 + \left(Q\left(\frac{k}{\sqrt{N_0/2}}\right)\right)^2 - 2\,Q\left(\frac{k}{\sqrt{N_0/2}}\right)$

$$\therefore \quad P_e = 1 - P_c = 2Q\left(\frac{k}{\sqrt{N_0/2}}\right) - \left[Q\left(\frac{k}{\sqrt{N_0/2}}\right)\right]^2$$

$$SNR = \frac{\mathbb{E}(|x|^2)}{\mathbb{E}(|n|^2)} = \frac{(\sqrt{2}\,k)^2}{N_0} \quad \text{as} \quad x_R = x_I = k$$

$$= \frac{2k^2}{N_0}$$

$$\therefore \quad \sqrt{SNR} = \sqrt{\frac{2k^2}{N_0}}$$

$$\therefore \quad P_e = 2Q\left(\sqrt{SNR}\right) - \left[Q\left(\sqrt{SNR}\right)\right]^2$$

P3.2 In matlab file HW2_Q3.m

P3.3 Difference between SER of QPSK & BPSK is that the QPSK ~~SER~~ SER is almost twice that of BPSK case as can be seen from Proof in 3.1

$$P_e^{QPSK} \sim 2Q\left(\sqrt{SNR}\right)$$

$$P_e^{BPSK} \sim Q\left(\sqrt{SNR}\right)$$

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P4.1 $\mathcal{X} = \{1+j, 1-j, -1+j, -1-j\}$

$$y[k] = h[k]\,x[k] + n[k]$$

$n[k] \sim \mathcal{CN}(0, N_0)$

$$\text{Average SNR} = \frac{\mathbb{E}[|hx|^2]}{\mathbb{E}[|n|^2]}$$

$$\mathbb{E}[|n|^2] = N_0$$

$h = h_r + i h_I$ & $\because h \sim \mathcal{CN}(0,1)$

$\therefore h_r, h_I \sim N(0, 1/2)$

$$\therefore \mathbb{E}[|hx|^2] = \mathbb{E}[|(h_r + i h_I)(1+j)|^2]$$

$$= \mathbb{E}[|(h_r - h_I) + i(h_r + h_I)|^2]$$

$$= \mathbb{E}[(h_r - h_I)^2 + (h_r + h_I)^2]$$

$$= \mathbb{E}[h_r^2 + h_I^2 - 2h_r h_I + h_r^2 + h_I^2 + 2 h_r h_I]$$

$$= \mathbb{E}[2h_r^2 + 2h_I^2]$$

$\because \mathbb{E}[h_r^2] = 1/2 = \mathbb{E}[h_I^2]$

$$= 2 \times \frac{1}{2} + 2 \times \frac{1}{2} =$$

$$= 2$$

$$\therefore \text{SNR} = \frac{2}{N_0}$$

Detection Rule for QPSK in fading channel

Receive ~~vector~~ symbol is

$$y = hx + n$$

Since, we assume coherent detection, $\therefore$ we know h

$\therefore$ Doing Dot Product of the receive signal with $\frac{h}{|h|}$, gives

$$\frac{h}{|h|} \cdot y = \frac{h}{|h|} \cdot h\, x + \frac{h}{|h|} \cdot n$$

Basically projection of h over y

$$\underbrace{\frac{h}{|h|}\, y}_{y'} = |h|\, x + z$$

we put our decision rule on this quantity

$$\hat{x}^{MAP} = \underset{x \in \mathcal{X}}{\arg\max} \left\{ -\| y' - |h| x \|^2 \right\}$$

$$= \min \left\{ \underbrace{\| y' - |h| x_a \|_2^2}_{x_a}, \overbrace{\| y' - |h| x_b \|_2^2}^{x_b}, \| y' - |h| x_c \|^2, \underbrace{\| y' - |h| x_d \|^2}_{x_d} \right\}$$

x_c

P.4.2 In Matlab

p.4.3 Since the channel is fading, its gain keeps changing over time,