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P.1 To show: $R_H(t,t';f,f') \triangleq R_H(t-t', f-f')$

Now, $R_H(t,t';f,f') = \mathbb{E}[L_H(t,f)\, L_H^*(t',f')]$ — ①

& $L_H(t,f) = \int_{\tau} h(t,\tau)\, e^{-j2\pi f\tau}\, d\tau$

Putting this in eqn' ①

$$LHS. = \mathbb{E}\left[\int_{\tau} h(t,\tau) e^{-j2\pi f\tau} d\tau \int_{\tau'} h^*(t',\tau') e^{+j2\pi f'\tau'} d\tau'\right]$$

$$= \mathbb{E}\left[\int_{\tau}\int_{\tau'} h(t,\tau)\,.\,h^*(t',\tau')\right].e^{-j2\pi f\tau}.e^{+j2\pi f'\tau'} d\tau d\tau'\Big]$$

$$= \int_{\tau}\int_{\tau'} \underbrace{\mathbb{E}[h(t,\tau), h^*(t',\tau')]}\,.\,e^{-j2\pi f\tau}.e^{+j2\pi f'\tau'} d\tau d\tau' \;*$$

$R_h(t,t',\tau,\tau') \Rightarrow$ By WSSUS assumption becomes

$$R_h(t-t';\tau)\,\delta(\tau-\tau')$$
$$= R_h(\Delta t,\tau)\,\delta(\tau-\tau')$$

$$\therefore LHS = \int_{\tau}\int_{\tau'} R_h(\Delta t,\tau)\,\delta(\tau-\tau').e^{-j2\pi f\tau}.e^{+j2\pi f'\tau'} d\tau d\tau'$$

Now multiply RHS by $e^{-j2\pi f'\tau}.e^{j2\pi f'\tau} = 1$

$$\therefore LHS = \int_{\tau}\int_{\tau'} R_h(\Delta t,\tau)\,\delta(\tau-\tau').\underbrace{e^{-j2\pi f\tau}.e^{+j2\pi f'\tau}}_{\text{Combine}}\, e^{+j2\pi f'\tau'}.e^{-j2\pi f'\tau} d\tau d\tau'$$

$$= \underbrace{\int_{\tau} R_h(\Delta t,\tau)\, e^{-j2\pi(f-f')\tau} d\tau}_{F.T.\ \tau\to\Delta f} \int_{\tau'} \underbrace{\delta(\tau-\tau')\, e^{-j2\pi f'(\tau-\tau')} d\tau'}_{F.T.\ of\ Impulse. = 1}$$

$$= R_H(\Delta t, \Delta f) \cdot 1$$

$$= R_H(\Delta t, \Delta f)$$

P.2.1 Given : LFI system
& $r(t) = m(t) \cdot s(t)$ —①

To find $h(t, \tau)$

Now, since the system is LFI, it does not have time dispersion, which means that the time delay is constant & let that constant be τ_0

$$\therefore \quad r(t) = \int_\tau h(t,\tau)\, s(t-\tau)\, d\tau$$

$$\Downarrow \tau = \tau_0$$

$$= h(t,\tau_0) \int_\tau \underbrace{s(t-\tau)\, \delta(\tau-\tau_0)\, d\tau}_{\text{only present for } \tau=\tau_0}$$

$$= h(t,\tau_0) \times s(t-\tau_0)$$

$$\therefore \quad h(t,\tau_0) = \frac{r(t)}{s(t-\tau_0)}$$

$$= \frac{m(t) \cdot s(t)}{s(t) \cdot e^{-j2\pi f \tau_0}}$$

$$h(t,\tau_0) = m(t) \cdot e^{j2\pi f \tau_0}$$

& can also be written as

$$h(t,\tau) = \underbrace{m(t) \cdot e^{j2\pi f \tau_0} \cdot \delta(\tau-\tau_0)}_{\text{only present for } \tau=\tau_0}$$

For case where $\tau_0 = 0$

we get $h(t,\tau) = m(t).\delta(\tau)$

will use further for simplicity.

P.2 To compute: $S_H(\tau,\nu)$ & $L_H(t,f)$

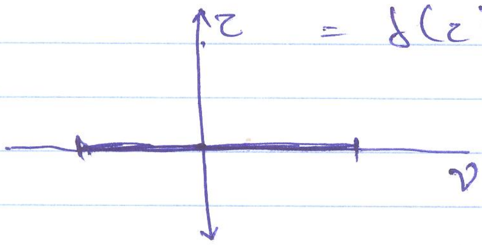
Now

$$S_H(\tau,\nu) = \int_t h(t,\tau)\, e^{-j2\pi\nu t}\, dt$$

$$= \int_t m(t).\delta(\tau)\, e^{-j2\pi\nu t}\, dt$$

$$= \delta(\tau) \int_t \underbrace{m(t).e^{-j2\pi\nu t}}_{F.T\ t \to \nu}\, dt$$

$$= \delta(\tau).M(\nu)$$



$$L_H(t,f) = \int_\tau h(t,\tau)e^{-j2\pi f\tau}\, d\tau$$

$$= \int_\tau m(t)\,\delta(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= m(t) \underbrace{\int_\tau \delta(\tau)e^{-j2\pi f\tau}\, d\tau}_{F.T.\ of\ impulse\ = 1}$$

$$= m(t)$$

∴ Transfer function is just the function of time & does not ~~depend~~ change with frequency ∴ it is not frequency selective.

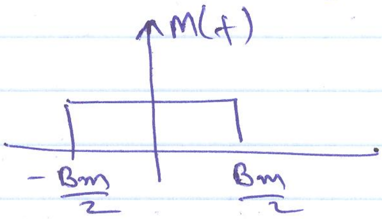
P.3 Given B.W. of $m(t)$ is B_m
& B.W. of $s(t) = B_s$

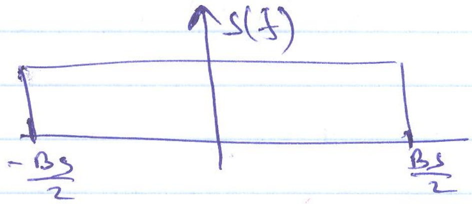
To find Bandwidth of $r(t)$

$\because$ $r(t) = m(t) \cdot s(t)$

$\therefore$ $R(f) = M(f) * S(f)$
$\uparrow$
Convolution

Assume $\rightarrow$



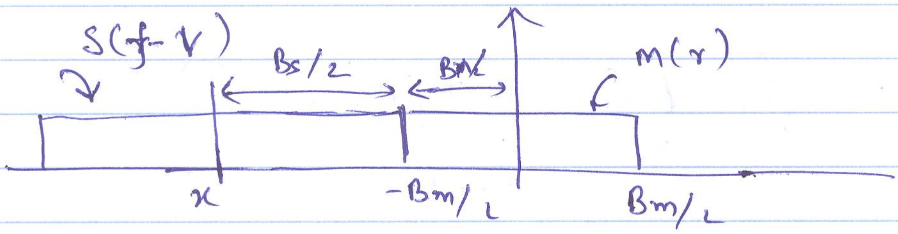


$$R(f) = \int_{-\infty}^{\infty} M(\gamma)\, S(f-\gamma)\, d\gamma$$

Physically it means to reverse one function
here $S(\gamma) \xrightarrow{to} S(-\gamma)$ & then shift
with value of f $\quad S(\gamma) \rightarrow S(-\gamma) \rightarrow S(-(\gamma - f))$
$= S(f-\gamma)$

$\therefore$ $R(f)$ becomes non-zero

from the instance as shown



$\therefore$ x marked above is $-\dfrac{(B_m + B_s)}{2}$

& it remains non-zero till $+\dfrac{(B_m + B_s)}{2}$

$$\therefore \boxed{B.W. = B_m + B_s}$$

P.4. Given: $R_m(\tau) = \mathbb{E}[m(\underbrace{t+\tau}_{t'})\, m^*(t)]$

so here $\underline{\tau = \Delta t}$

Now

$$R_H(\Delta t, \Delta t) = \mathbb{E}[L_H(t,t)\, L^*_H(t', t')]$$

$$= \mathbb{E}\underbrace{[m(t) \cdot m^*(t')]}_{\Downarrow\ R_m(\tau) \text{ as given.}}$$

$$C_H(\tau, \nu) = \mathbb{E}[S_H(\tau, \nu)\, S_H^*(\tau', \nu')]$$

$$= \mathbb{E}[\delta(\tau) M(\nu) \cdot \delta(\tau') M^*(\nu')]$$

$$= \mathbb{E}[M(\nu) \cdot M^*(\nu')]\, \delta(\tau)$$

$$= \mathbb{E}\left[\int_t m(t) e^{-j2\pi\nu t} dt \cdot \int_{t'} m^*(t') e^{j2\pi\nu' t'} dt' \right] \delta(\tau)$$

$$= \iint_{t\,t'} \underbrace{E[m(t)\, m^*(t')]}_{R_m(\Delta t)} e^{-j2\pi(\nu t - \nu' t')} dt\, dt'$$

$$= \iint_{t\,t'} \underbrace{R_m(\Delta t) e^{-j2\pi\nu(t-t')} d\Delta t} \cdot e^{-j2\pi(\nu-\nu')t'} dt'$$

$$= R_m(\nu) \int_{t'} e^{-j2\pi(\nu-\nu')t'} dt'$$

$$C_H(\tau,\nu) = R_m(\nu) \cdot \delta(\nu-\nu')$$

P3.1 Given

$$h(\tau) = \alpha_0 \delta(\tau) + \alpha_1 \delta(\tau-\tau_0)$$

$\because$ h does not change w.r.t t, its time invariant system.
i.e. symbol $T \ll T_c$

$$\therefore S_H(\tau,\nu) = h(\tau)$$

as $h(t,\tau) \xrightarrow{\text{F.T. over } t\to\nu} S_H(\tau,\nu)$

But its not a function of t.

$$\therefore S_H(\tau) = h(\tau)$$

$$\therefore C_H(\tau) = \mathbb{E}[S_H(\tau)\, S_H^*(\tau')]$$

$$= \mathbb{E}\left[(\alpha_0 \delta(\tau) + \alpha_1 \delta(\tau-\tau_0)) \times [\alpha_0^* \delta(\tau') + \alpha_1^* \delta(\tau'-\tau_0)] \right]$$

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as they are independent

$$= \mathbb{E}\Big[\underbrace{\alpha_0 \alpha_0^* \delta(\tau)\,\delta(\tau')}_{\text{Present for } \tau=\tau'=0} + \cancel{\alpha_0 \alpha_1^*}\,\delta(\tau)\,\delta(\tau'-\tau_0) + \cancel{\alpha_1 \alpha_0^*}\,\delta(\tau-\tau_0)\,\delta(\tau') + \underbrace{\alpha_1 \alpha_1^* \delta(\tau-\tau_0)\,\delta(\tau'-\tau_0)}_{\text{Present for } \tau=\tau'=\tau_0}\Big]$$

~~only~~ present ~~rooted~~ for ~~$\tau=\tau_0=\tau'$~~

$$= \mathbb{E}\left[|\alpha_0|^2 + |\alpha_1|^2\right] \delta(\tau-\tau_0)$$

$$= \sigma_0^2\,\delta(\tau) + \sigma_1^2\,\delta(\tau-\tau_0)$$

Impact of σ_0^2 & σ_1^2 is that, for path with $\tau=0$ the scattering is of order σ_0^2
& for path with $\tau=\tau_0$, the scattering is of σ_1^2

τ_0 is just the delay in 2nd path

$$R_H(\Delta t, \Delta f) = \mathbb{E}\left[L_H(f) \cdot L_H^*(f)\right]$$

$$L_H(f) = \int_\tau h(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int \alpha_0\, \delta(\tau)\, e^{-j2\pi f\tau}\, d\tau + \int \alpha_1\, \delta(\tau-\tau_0)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \alpha_0 \cdot 1 + \alpha_1 \int_\gamma \delta(\gamma)\, e^{-j2\pi f\gamma} \cdot e^{-j2\pi f\tau_0}\, d\gamma$$

where $\gamma = \tau - \tau_0$

$$= \alpha_0 \cdot 1 + \alpha_1 \cdot e^{-j2\pi f\tau_0} \cdot 1$$

$$= \alpha_0 + \alpha_1\, e^{-j2\pi f\tau_0}$$

$$\therefore R_H(\cdot, \Delta f) = \mathbb{E}\left[(\alpha_0 + \alpha_1 e^{-j2\pi f\tau_0})(\alpha_0^* + \alpha_1^* e^{j2\pi f'\tau_0})\right]$$

$$= \mathbb{E}\left[\alpha_0\alpha_0^* + \alpha_1\alpha_1^*\, e^{-j2\pi(f-f')\tau_0}\right] \quad \because \alpha_0\alpha_1^* = 0 \;\&\; \alpha_1\alpha_0^* = 0$$

$$R_H(\cdot, \Delta f) = \sigma_0^2 + \sigma_1^2\, e^{-j2\pi\Delta f\tau_0}$$

Impact on $R_H(\Delta t, \Delta f)$

Higher the value of σ_0^2, σ_1^2 higher the amplitude
Higher the value of τ_0, higher the fluctuation/variation in amplitude of ~~in~~ ~~a~~ time-freq. correlation function is increasing.

P 3.2 In Matlab.

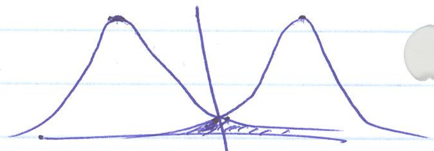
P. 4.1 Given: $y = x + n$; $x \in \{-1, 1\}$

$P_r(x=-1) = P_r(x=1) = 0.5$

~~n is 0~~

$n \sim N(0, \sigma^2)$

Optimal detector would be one which minimizes the following Prob.



$$P_{error} = P_r(x=1)\cdot P_r(\hat{x}=-1 \mid x=1) + P_r(x=-1)\cdot P_r(\hat{x}=1 \mid x=-1)$$

$$P_{error} = f(\cancel{x}y) = 0.5\left[\int_{-\infty}^{d} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_1)^2}{2\sigma^2}} + \int_{d}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_{-1})^2}{2\sigma^2}}\right]$$

∴ Upon doing $f'(\cancel{x}y) = 0$ & solving
we get d to be $\frac{\mu_1 + \mu_{-1}}{2}$

In our ~~so~~ case $\mu_1 = 1$ (mean) when $x = 1$
$\mu_{-1} = -1$ (mean) when $x = -1$

∴ $d = 0$

∴ $\hat{x} = \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases}$

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P.4.2 Given $SNR = \frac{E[|x|^2]}{E[|n|^2]}$

$$P_{error} = 0.5\left[\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_1)^2}{2\sigma^2}} dy + \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_{-1})^2}{2\sigma^2}} dy\right]$$

$\Downarrow$

~~let $\frac{y-\mu_1}{2\sigma^2} = z$~~ let $\frac{y-\mu_1}{\sqrt{2\sigma^2}} = z$

~~$\therefore \frac{\partial x 1}{2\sigma^2} = \frac{dz}{dx}$~~ $\therefore \frac{dz}{dy} = \frac{1}{\sqrt{2\sigma^2}}$

~~$\therefore dy = 2\sigma^2 dz$~~ $\therefore dy = \sqrt{2}\,\sigma\, dz$

$$0.5\frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{\frac{-\mu_1}{\sqrt{2\sigma^2}}} e^{-z^2} dz + \int_{\frac{-\mu_{-1}}{\sqrt{2\sigma^2}}}^{\infty} e^{-z^2} dz\right]$$

Now $\mu_1 = 1$ & $\mu_{-1} = -1$

$$\therefore \frac{0.5}{\sqrt{\pi}}\left[\int_{-\infty}^{\frac{-1}{\sqrt{2\sigma^2}}} e^{-z^2} dz + \int_{1/\sqrt{2\sigma^2}}^{\infty} e^{-z^2} dz\right]$$

$\Downarrow$

Same as

$\int_{1/\sqrt{2\sigma^2}}^{\infty} e^{-z^2} dz$ as e^{-z^2} is symmtric arrowd y axis in y, z plane.

$$\therefore \frac{0.5}{\sqrt{\pi}} \times 2 \underbrace{\int_{1/\sqrt{2\sigma^2}}^{\infty} e^{-y^2} dz}_{\frac{\sqrt{\pi}}{2}\, erfc(1/\sqrt{2\sigma^2})}$$

$$\cancel{0.25}\ \frac{1}{\cancel{\sqrt{\pi}}} = \frac{1}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2}\, erfc(1/\sqrt{2\sigma^2})$$

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$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{\sqrt{2\sigma^2}}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{SNR}{2}}\right)$$

$\because SNR = \frac{1}{\sigma^2}$

P4.3 matlab

P4.4 In matlab.