Homework 2: Solutions

Problem 1 : (20pts)

- part 1 : (5pts) $K_z = \mathbb{E}\left\{\begin{bmatrix} x_R \\ x_I \end{bmatrix}\begin{bmatrix} x_R^T & x_I^T \end{bmatrix}\right\} = \begin{bmatrix} K_{x_R} & K_{x_R x_I} \\ K_{x_I x_R} & K_{x_I} \end{bmatrix}$

$$K_{x_R} = \mathbb{E}\left[x_R x_R^T\right]$$

$$K_{x_R x_I} = \mathbb{E}\left[x_R x_I^T\right]$$

$$K_{x_I x_R} = \mathbb{E}\left[x_I x_R^T\right]$$

$$K_{x_I} = \mathbb{E}\left[x_I x_I^T\right]$$

now,

$$K_x = \mathbb{E}\left[x x^H\right] = \mathbb{E}\left[(x_R + j x_I)(x_R - j x_I)^T\right]$$

$$= K_{x_R} + K_{x_I} + j\left(K_{x_I x_R} - K_{x_R x_I}\right) \quad (a)$$

$$J_x = \mathbb{E}\left[x x^T\right] = \mathbb{E}\left[(x_R + j x_I)(x_R + j x_I)^T\right]$$

$$= K_{x_R} - K_{x_I} + j\left(K_{x_I x_R} + K_{x_R x_I}\right) \quad (b)$$

summing (a)+(b) yields

$$K_x + J_x = 2\left(K_{x_R} + j K_{x_I x_R}\right)$$

which implies that

$$K_{x_R} = \frac{1}{2} \mathrm{Re}\left\{K_x + J_x\right\} \qquad K_{x_I x_R} = \frac{1}{2} \mathrm{Im}\left\{K_x + J_x\right\}$$

similarly: $K_x - J_x = 2\left(K_{x_I} + j K_{x_R x_I}\right)$

which implies

$$K_{x_I} = \frac{1}{2} \mathrm{Re}\left\{K_x - J_x\right\} \qquad K_{x_R x_I} = \frac{1}{2} \mathrm{Im}\left\{J_x - K_x\right\}$$

this means that K_x and J_x are sufficient to describe the 2nd order statistics of $z = \begin{pmatrix} x_R \\ x_I \end{pmatrix}$

- part 2: (5pb)

$$\mathbb{E}[xx^T] = \mathbb{E}[xe^{i\varphi}(xe^{i\varphi})^T] = e^{i2\varphi}\,\mathbb{E}[xx^T]$$

with $\varphi = \pi/2$ we get $J_x = \mathbb{E}[xx^T] = 0$

we can now use $J_x = 0$ in the above results:

$$K_{x_R} = K_{x_I} = \frac{1}{2}\,\mathrm{Re}\{K_x\}$$

$$K_{x_I x_R} = -K_{x_R x_I} = \frac{1}{2}\,\mathrm{Im}\{K_x\} \overset{\text{also}}{=\!=} -K^T_{x_I x_R}$$

$\Rightarrow K_z$ is skew-symmetric! $\quad A = -A^T$ ☺

- part 3: (5pb)

K_z is skew symmetric $\rightarrow$ diagonal of K_z <u>must</u> be zero.
this means that the real and imaginary part of each entry x_n are uncorrelated. But $J_x = 0$ does <u>not</u> imply that x_n and x_m are uncorrelated for $n \neq m$!

- part 4: (5pb)

$$K_x = K_{x_R} + K_{x_I} + j(K_{x_I x_R} - K_{x_R x_I}) =$$
$$= I\frac{N_0}{2} + I\frac{N_0}{2} + j(0 \quad - 0) = \underline{N_0 \cdot I}$$

$$J_x = K_{x_R} - K_{x_I} + j(K_{x_I x_R} + K_{x_R x_I}) =$$

$$= I\frac{N_0}{2} - I\frac{N_0}{2} + j(0 \; - 0) = \underline{0}$$

$$\mathbb{E}[e^{i\varphi}x(xe^{i\varphi})^H] = e^{i\varphi}\,\mathbb{E}[xx^H]\,e^{-i\varphi} = K_x \rightarrow \text{remains}$$

$\rightarrow$ circular symmetric!

Problem 2: (20 pts)

• part 1: (10 pts)

office environment. motion at most 5 km/h (≈ 1.4 m/s)

thus, maximum Doppler spread at $f_c = 2.4$ GHz is

$$D_s = \frac{1.4 \cdot 2.4 \cdot 10^9}{3 \cdot 10^8} \approx \underline{11.2\ Hz}$$

hence $T_c = 1/D_s = \underline{90\ ms}$

Table 2 shows IEEE 802.11b, frame duration is $\underline{\leq 18.96\ ms} \ll T_c$

$\Rightarrow$ channel is <u>slow fading</u>

Table 1 shows that delay spread in office: $0{,}025 - 0.074\ \mu s = T_d$

$B_c = 1/T_d \rightarrow \quad 13.5\ MHz < B_c < 40\ MHz$

Table 2 shows that BW is 20 MHz, thus $\quad 13.5\ MHz < 20\ MHz$

channel is (mildly) <u>frequency selective</u>!

• part 2: (10 pts)

speed $v = 250$ km/h ≈ 70 m/s
$f_c = 2$ GHz

$\Big\}$ Doppler: $D_s = \frac{v \cdot f_c}{c} \cong \underline{466\ Hz}$

coherence time: $T_c = 1/D_s \approx \underline{2\ ms}$

Table 2 shows slot length of 10 ms which exceeds coherence time of 2 ms

$\Rightarrow$ fast fading!

Delay spread: rural ($f_c = 1718$ MHz in Norway)

$$B_c \leq \frac{1}{2.8\ \mu s} \approx \underline{357\ kHz} \quad \text{coherence BW}$$

UMTS BW is 5 MHz $\rightarrow$ <u>frequency selective</u>

Problem 3 (30 pts)

- part 1: optimal decision rule $\quad y = x + n$
(10 pts)

$$\hat{x} = \arg\max_{x \in \mathcal{X}} p(x|y) \underset{\text{Bayes' rule}}{=} \arg\max_{x \in \mathcal{X}} p(y|x)p(x) \underset{\text{iid symbols}}{=} \arg\max_{x \in \mathcal{X}} p(y|x) = *$$

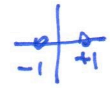
$$p(y|x) = \frac{1}{\pi N_0} \exp\left(-\frac{|y-x|^2}{N_0}\right)$$

$$(*) = \arg\max_{x \in \mathcal{X}} \exp\left(-\frac{|y-x|^2}{N_0}\right) \underset{\text{log is monotone}}{=} \arg\max_{x \in \mathcal{X}} -\frac{|y-x|^2}{N_0} \underset{\text{constants don't matter}}{=} \underline{\arg\min_{x \in \mathcal{X}} |y-x|^2}$$

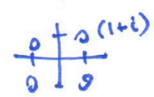
$$\hat{x}^{MAP} = \arg\min_{x \in \mathcal{X}} |y-x|^2 \quad \leftarrow \text{nearest neighbor rule!}$$

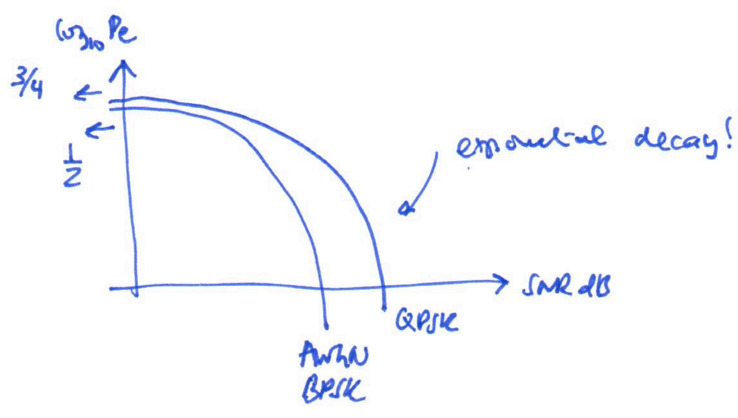
- part 2 (10 pts): see code and plot. careful with $SNR = E_s/N_0$
- part 3: see plot
(10 pts)

  - SNR for BPSK: $SNR = 1/N_0$

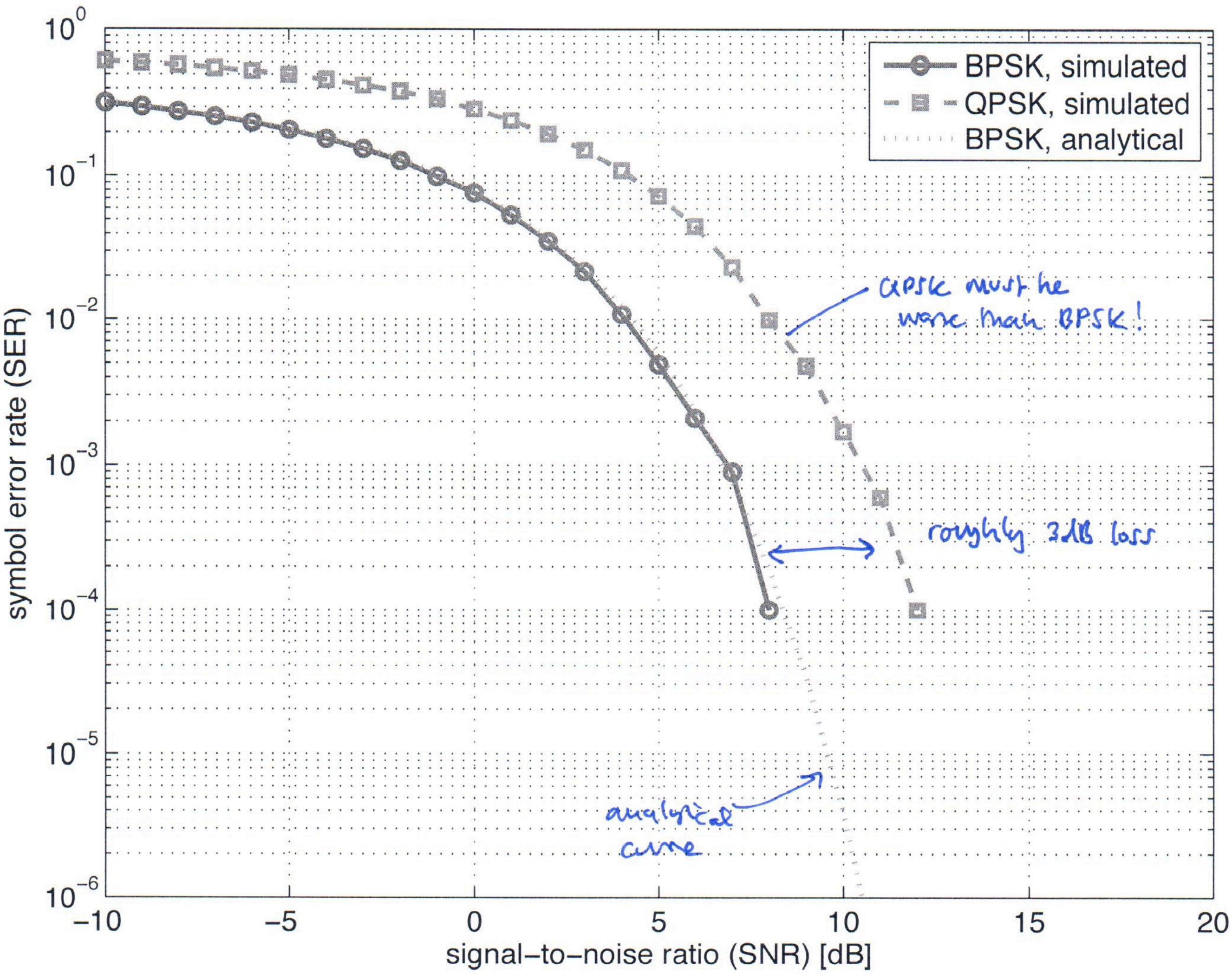


  - SNR for QPSK: $SNR = 2/N_0$





part 2 and 3:



Reasons: ① QPSK have higher energy $E_s=2$ than BPSK: $E_s=1$

equivalent

$SNR = \frac{E_s}{N_0}$ hence noise is larger for a given SNR!

② normalize QPSK constellation: then constellation points are closer than for BPSK!

$\frac{1-i}{\sqrt{2}}$ $\frac{1+i}{\sqrt{2}}$ $\frac{-1-i}{\sqrt{2}}$ $\frac{-1+i}{\sqrt{2}}$ $\rightarrow E_s=1$

Problem 4: (30 pts)

- part 1: (10 pts)

$$SNR = E_{h,x}[|hx|^2] = E_{h,x}[hxx^*h^*] = E_s \cdot \underbrace{E[|h|^2]}_{=1} = E_s$$

for QPSK (1+i) $\rightarrow E_s = 2$

$$\boxed{SNR = 2/N_0}$$

coherent MAP decision rule:

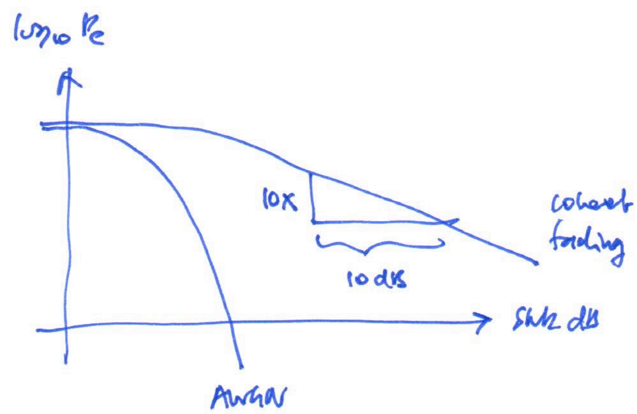
$y = hx + n$ $\quad$ h: known

$$\hat{x} = \underset{x\in\mathcal{X}}{\operatorname{argmax}}\ p(x|y,h) = \underset{x\in\mathcal{X}}{\operatorname{argmax}}\ p(y|hx) = \underset{x\in\mathcal{X}}{\operatorname{argmin}}\ |y - hx|^2$$

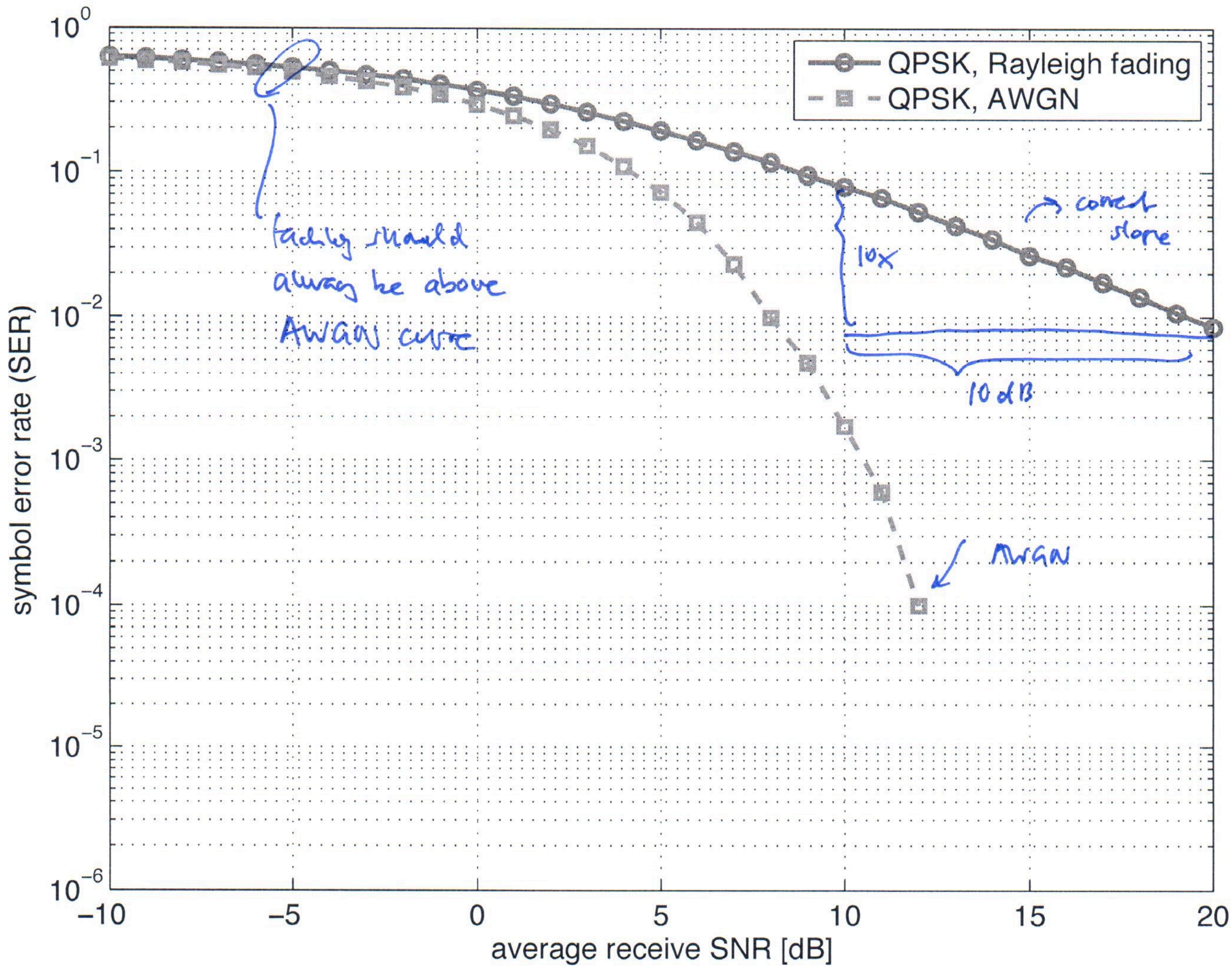
equivalent: $\hat{x} = \underset{x\in\mathcal{X}}{\operatorname{argmin}}\ |\underbrace{y/h}_{\tilde{y}} - x|^2$

- part 2: (10 pts) see code and plot
- part 3: (10 pts) see plot

explanation: fading $\quad P(\text{'deep fade'}) \approx \frac{1}{SNR}$ for fading which causes poor slope.



7/7



QPSK, Rayleigh fading
QPSK, AWGN
symbol error rate (SER)
average receive SNR [dB]
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
−10
−5
0
5
10
15
20
fading should always be above AWGN curve
10x
10 dB
→ correct slope
AWGN

```
% Homework 2 : Problem 3
% (c) 2015 studer@cornell.edu

runid = 0; % controls random seed

% -- set up simulation parameters
par.SNRdB_list = [-10:20];      % SNR (dB) range
par.trials = 10000;             % simulation trials
par.channel = 'AWGN';           % 'AWGN' or 'Rayleigh'
par.mod = 'BPSK';               % modulation scheme (constellation)

% -- run simulator
[BPSK.res,par] = hw2_simulator(runid,par);

% -- set up simulation parameters
par.mod = 'QPSK';               % modulation scheme (constellation)

% -- run simulator
[QPSK.res] = hw2_simulator(runid,par);

% -- plot simulation results
figure(3)
semilogy(par.SNRdB_list,BPSK.res.SER,'bo-','LineWidth',2)
hold on
semilogy(par.SNRdB_list,QPSK.res.SER,'rs--','LineWidth',2)
% analytical BER of BPSK transmission in complex AWGN channel
semilogy(par.SNRdB_list,0.5*(1+erf((-1)./sqrt(par.N0))),'m:','LineWidth',2)
hold off
grid on
axis([-10 20 1e-6 1])
xlabel('signal-to-noise ratio (SNR) [dB]','FontSize',12)
ylabel('symbol error rate (SER)','FontSize',12)
legend('BPSK, simulated','QPSK, simulated','BPSK, analytical')
set(gca,'FontSize',12)

% export plot as eps file
print -depsc 'hw2_problem3.eps'
```

```
% Homework 2 : Problem 4
% (c) 2015 studer@cornell.edu

runid = 0; % controls random seed

% -- set up simulation parameters
par.SNRdB_list = [-10:20];      % SNR (dB) range
par.trials = 10000;             % simulation trials
par.channel = 'Rayleigh';       % 'AWGN' or 'Rayleigh'
par.mod = 'QPSK';               % modulation scheme (constellation)

% -- run simulator
[QPSK.res,par] = hw2_simulator(runid,par);

% -- set up simulation parameters
par.channel = 'AWGN';           % 'AWGN' or 'Rayleigh'
par.mod = 'QPSK';               % modulation scheme (constellation)

% -- run simulator
[AWGN.res] = hw2_simulator(runid,par);

% -- plot simulation results
figure(4)
semilogy(par.SNRdB_list,QPSK.res.SER,'bo-','LineWidth',2)
hold on
semilogy(par.SNRdB_list,AWGN.res.SER,'rs--','LineWidth',2)
hold off
grid on
axis([-10 20 1e-6 1])
xlabel('average receive SNR [dB]','FontSize',12)
ylabel('symbol error rate (SER)','FontSize',12)
legend('QPSK, Rayleigh fading','QPSK, AWGN')
set(gca,'FontSize',12)

% export plot as eps file
print -depsc 'hw2_problem4.eps'
```

```
% Homework 2 : AWGM/Rayleigh fading simulator
% Complex-valued; supports BPSK and QPSK
% (c) 2015 studer@cornell.edu

function [res,par] = hw2_simulator(runid,par)

    % == initialize simulator

    % set random seed
    rng(runid);

    % select constellation alphabet
    switch (par.mod)
        case 'BPSK',
            par.symbols = [ -1 1 ];
        case 'QPSK',
            par.symbols = [ -1-1i,-1+1i,+1-1i,+1+1i ];
        otherwise,
            error('par.mod not defined')
    end

    % compute average symbol energy
    par.Es = mean(abs(par.symbols).^2);

    % compute number of bits per constellation
    par.Q = log2(length(par.symbols));

    % convert SNR to noise variance (signal power is par.Es)
    par.N0 = par.Es*10.^(-par.SNRdB_list/10);

    % pre-allocate SER vector
    res.SER = zeros(length(par.SNRdB_list),1);

    % == start simulation

    tic; % count seconds

    % generate Gaussian noise ~CN(0,1)
    n = sqrt(0.5)*(randn(1,par.trials)+1i*randn(1,par.trials));

    % generate channel coefficients
    switch (par.channel)
        case 'AWGN',
            h = ones(1,par.trials);
        case 'Rayleigh', % Ralyeigh fading coefficients ~CN(0,1)
            h = sqrt(0.5)*(randn(1,par.trials)+1i*randn(1,par.trials));
        otherwise,
            error('par.channel not defined')
    end

    % generate iid constellation symbols
    idx = randi([1 2^par.Q],1,par.trials);
    x = par.symbols(idx);

    % main SNR simulation loop
    for kk=1:length(par.SNRdB_list)
        % transmit over AWGN channel
        y = h.*x+sqrt(par.N0(kk))*n;
```

```
        % coherent MAP detector (nearest neighbor rule)
        [~,idxhat] = min(abs(ones(2^par.Q,1)*y-par.symbols.'*h),[],1);

        % compute symbol error rate (SER)
        res.SER(kk,1) = sum(idx~=idxhat)/par.trials;
    end

    toc % end of counting seconds

end
```