HW 3: Wireless Systems with Spatial Diversity

ECE 5680 – Wireless Communication

- **Due date:** Friday, October 9, 2015 at 11:59pm
- **Submission instructions:** Upload your solutions (solutions and Matlab code) to Cornell Blackboard as a *single* zip-file; name the file ⟨netid⟩-hw⟨number⟩.zip. If you want to get detailed comments on your homework olutions, create a *single* pdf file that contains all answers. I strongly suggest the use of LaTeX to typeset your responses, but a scanned version of your handwritten answers is also OK.
- **Total points:** 100pts

Problem 1: Single-input multiple-output (SIMO) wireless system (20pts)

In class, you have seen how we can improve the link reliability of systems operating in fading channels by means of diversity techniques, i.e., by coding and interleaving information over independent channel fades in time. If it is possible to transmit several independently faded replicas of the transmit signal, then the probability that all independent diversity links are at the same time in a "deep fade" can be reduced. This is the key of improving the link reliability of fading wireless systems. The same idea can also be used over *space*. In fact, if the receiver has multiple receive antennas, and each antenna element is placed sufficiently separated at the receiver, the channel gains from the transmitter to each receive antenna fade more or less independently. In practice, separating the receive antennas by about half carrier-frequency wavelength is often sufficient to obtain independently fading links. In this problem, we derive the optimal detector and analyze its error probability of such a single-input multiple-output (MISO) wireless system.

Assume that we want to transmit equally likely BPSK signals $x \in \{+a, -a\}$ from a single transmit antenna to L receive antennas. Furthermore, assume that we have a flat Rayleigh fading channel and that the receiver knows the channel perfectly, i.e., we consider coherent transmission. The input-output relation of such a MISO system can be modeled as $\mathbf{y} = \mathbf{h}x + \mathbf{n}$. Here, $\mathbf{y} \in \mathbb{C}^L$ is the L-dimensional receive vector (each entry corresponds to the signal obtained at each receive antenna), $\mathbf{h} \in \mathbb{C}^L$ models the individual paths from the transmit antenna to the L receive antennas, and $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, N_0\mathbf{I})$ models additive Gaussian receive noise. Assume that the receive antennas are spaced sufficiently far apart. This assumption together with the flat Rayleigh fading assumption implies that the channel gains are distributed as $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$.

Part 1 Derive the optimal detector.

Part 2 Compute the error probability. *Hint: use the same approach as in class for the time-diversity case.*

Remark: As you will see, having multiple antennas at the receiver significantly improves the error-rate performance compared to single antenna systems. We shall see next, there are simpler schemes that also achieve the same diversity order (but slightly worse error-rate performance).

Problem 2: Antenna selection in a SIMO system (10pts)

The optimal receiver for the SIMO system in Problem 1 uses maximum ratio combining (MRC), where each diversity branch (or receive antenna) is weighted by the corresponding complex conjugated channel coefficient, and all branches are added *coherently*. Hardware implementations of such a receiver are often expensive as every antenna requires its own radio-frequency (RF) chain (consisting of amplifiers, analog-to-digital-converters, mixers, etc.). In practice, it is therefore often desirable to use a less complex receiver structure. Instead of coherently combining all signals from all antennas, the receiver complexity can be reduced significantly if only the signal from the strongest diversity branch (i.e., the one that offers the highest per-branch SNR) is used. This can be realized by selecting the antenna corresponding to the best receive SNR. Such an antenna-selection scheme is often called *selection combining* and commonly found in low-cost products (e.g., Wi-Fi base stations with two antennas). A receiver that performs selection combining evaluates the channel gains at each receive antenna and only uses the antenna signal with the strongest channel gain (in terms of magnitude). In this problem, we characterize the diversity order of selection combining in a SIMO system with the same parameters as that of Problem 1.

Note that selection combining is an ad-hoc concept. We therefore do not start from the derivation of the optimal detector, but use the approximate typical error event approach explained in class. For the SIMO case of Problem 1, the typical error event occurs with probability

$$P(\text{“deep fade”}) = P(\|\mathbf{h}\|^2 SNR < 1).$$

For a system performing selection combining, a deep fade occurs with probability

$$P(\text{“deep fade”}) = P(|h_{\max}|^2 SNR < 1), \tag{1}$$

where $|h_{\max}|^2 = \max\{|h_1|^2, \ldots, |h_L|^2\}$ is just the channel gain with largest magnitude. Since $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$, we only need to know the distribution of $|h_{\max}|^2$ before we can evaluate (1).

Part 1 Let the random variables X_i, $i = 1, \ldots, L$, be i.i.d. with $X_i \sim \mathcal{CN}(0, 1)$. We learned in class that the random variables $|X_i|^2$ are exponentially distributed with rate parameter 1. Hence, the cumulative distribution function (CDF) of $|X_i|^2$ is $F(x) = (1 - \exp(-x))\sigma(x)$, where $\sigma(x)$ is the Heaviside step function. To obtain the distribution of $|h_{\max}|^2$, we consider the CDF of the following random variable:

$$Z = \max\{|X_1|^2, \ldots, |X_L|^2\},$$

which is defined as

$$F(z) = P(\max\{|X_1|^2, \ldots, |X_L|^2\} < z).$$

Compute the CDF $F(z)$. To this end, use (i) the fact that

$$P(\max\{|X_1|^2, \ldots, |X_L|^2\} < z) = P(|X_1|^2 < z \wedge \cdots \wedge |X_L|^2 < z)$$

and (ii) the assumption that the variables X_i, $i = 1, \ldots, L$, are i.i.d. with exponential distribution.

Part 2 In Part 1, you computed the CDF of the random variable $|h_{\max}|^2$. It is now straightforward to derive the probability of a deep fade (1). Show that selection combining achieves in fact the same diversity order as that of the (much more expensive) MRC receiver considered in Problem 1.

Problem 3: Multiple-input single-output (MISO) wireless system (10pts)

Consider a wireless system where there are L transmit antennas and only one receive antenna. This configuration is known as multiple-input single-output (MISO) and common in the downlink of cellular systems, as it is often cheaper (and feasible because of limited power consumption at the mobile terminals) to have multiple antennas at the base station than at the mobile terminals (aka. cell phones).

Part 1 What is the diversity order of a scheme that transmits the same symbol over all antennas in each time slot? The input-output relation of such a MISO system corresponds to $y = \sum_{\ell=1}^{L} h_\ell x + n$, where $y \in \mathbb{C}$ is the received symbol, the channel gains h_ℓ, $\ell = 1, \dots, L$ are i.i.d. with $h_\ell \sim \mathcal{CN}(0,1)$, $x \in \{-1, +1\}$, and $n \sim \mathcal{CN}(0, N_0)$. Derive the optimal detector and the error probability of this scheme.

Problem 4: MISO wireless system with full diversity (20pts)

The goal is now to design an improved transmission scheme for MISO systems that realizes L-th order diversity. Assume that the transmitter still has M antennas and a fixed transmit energy of E_s per time slot. Furthermore, assume that, in each time slot, equiprobable BPSK messages $x \in \{-1, +1\}$ are sent. Let $\mathbf{s} = [s_1, \dots, s_L]^T$ be the vector of symbols transmitted from each antenna. Assume $\mathbf{s} = x\sqrt{E_s}\mathbf{a}$, where $\mathbf{a} = [a_1 \cdots a_L]$ is some complex row vector that is cleverly chosen by the transmitter and $\|\mathbf{a}\|_2 = 1$. This implies that the signal transmitted from the ℓ-th antenna is a weighted version of x, weighted by some complex coefficient a_ℓ. The scalar input-output relation of this MISO transmission scheme is

$$y = \mathbf{h}^T\mathbf{s} + n,$$

where $y \in \mathbb{C}$ is the received symbol (per time slot), $\mathbf{h} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ with h_ℓ denoting the channel gain from transmit antenna ℓ to the receiver, and $n \sim \mathcal{CN}(0, N_0)$ models additive Gaussian receive noise. *Hint: Your answers to Part 1 and Part 2 should depend on the vector* $\mathbf{a}$.

Part 1 Derive the maximum a-posteriori (MAP) receiver.

Part 2 Derive the average error probability for the detection rule of Part 1.

Part 3 Now, assume that the vector $\mathbf{a}$ can be chosen by the transmitter. In practice, it is often possible that the transmitter is able to acquire channel-state information (CSI) through feedback. Hence, assume that the transmitter knows the channel vector $\mathbf{h}$ and that the vector $\mathbf{a}$ is a function of $\mathbf{h}$. What is the optimal choice for $\mathbf{a}$ that minimizes the error-rate probability? *Hint: Look at your error-rate expression from Part 2 and find a vector* $\mathbf{a}$ *that depends on* $\mathbf{h}$ *with* $\|\mathbf{a}\|_2 = 1$ *that maximizes the expression.*

Part 4 What is the diversity order of the scheme derived in Part 3? Use math to answer your response and compare your result to the one derived in Problem 3.

Remark: The MISO transmission technique derived above, which uses the optimal vector a is called transmit maximum ratio combining (MRC) or *beamforming*, as the multi-antenna transmitter focuses the signal energy in the direction of the single-antenna receiver.

Problem 5: Error-rate simulation of SIMO in a fading channel (20pts)

Consider the SIMO wireless system of Problem 1 with BPSK transmission over an L-antenna SIMO flat Rayleigh fading system with a coherent receiver. As done in previous homework assignments, write a MATLAB script that performs Monte-Carlo simulations to compute the average symbol error rate (SER). Perform $T = 10,000$ trials, sweep the SNR between 0 dB and 30 dB, and plot the resulting average SER vs. SNR performance in a doubly-logarithmic plot. Also, generate SER results for $L \in \{1, 2, 3, 4\}$ receive antennas. *Hint: Be extremely careful with the SNR definition and the normalization of your vectors.*

Part 1 Simulate the SER performance of the optimal detector for the SIMO system of Problem 1.

Part 2 Simulate the SER performance of selection combining as in Problem 2. To this end, implement a receiver that checks which of the L channel gains h_ℓ has the largest magnitude for every time slot and then, only take this received signal $y_\ell = h_\ell x + n_\ell$ to generate an estimate for $x \in \{-1, +1\}$.

Part 3 Generate a plot that includes all your simulated SER results (including different antenna configurations) from Part 1 and Part 2. Answer the following questions:

- Are the slopes of all simulated SER curves correct?
- Do you expect selection combining to perform as it should (compared to MRC of Part 1)?
- Imagine you have to design a system with $L = 4$ receive antennas. What is the minimum SNR your system requires in order to achieve a symbol error rate of 1%?

Problem 6: Error-rate simulation of MISO in a fading channel (20pts)

Consider the MISO wireless system of Problems 3 and 4 with BPSK transmission over a MISO flat Rayleigh fading channel with L transmit antennas and a single-antenna coherent receiver. As in Problem 5, write a MATLAB script that performs Monte-Carlo simulations to compute the average symbol error rate (SER). Perform $T = 10,000$ trials, sweep the SNR between 0 dB and 30 dB, and plot the resulting average SER vs. SNR performance in a doubly-logarithmic plot. Generate SER results for $L \in \{1, 2, 3, 4\}$ transmit antennas. *Hint: Be extremely careful with the SNR definition and the normalization of your vectors.*

Part 1 Simulate the SER performance of the transmission scheme of Problem 3.

Part 2 Simulate the SER performance of beamforming as in Problem 4.

Part 3 Generate a plot that includes all simulated SER results (including different antenna configurations) from Part 1 and Part 2. Answer the following questions:

- Are the slopes of all simulated SER curves correct?
- Do you expect that beamforming performs as it should (compared to the scheme used in Part 1)?
- Imagine you have to design a system with $L = 4$ transmit antennas. What is the minimum SNR your system requires in order to achieve a symbol error rate of 1%?