Université de Lomé

ECOLE POLYTECHNIQUE DE LOME (EPL)

Applications of Linear Algebra

By

Pr, PhD Adekunlé Akim SALAMI

Electrical Engineer

Course Overview

Combined with calculus, linear algebra facilitates the solution of linear systems of differential equations. Techniques from linear algebra are also used in analytic geometry, engineering, physics, natural sciences, computer science, computer animation, and the social sciences (particularly in economics).

The course likewise gives hands-on preparing to assist you with composing and test your using linear algebra aptitude to solve many problems in Engineering.

Course objectives?

Before the finish of this Applications of Linear Algebra course, you will be conversant in using sympy module in python, and furthermore how to use Linear algebra technics to solve real problem in engineering. This course will assist you with achieving the accompanying:

Gain information on principal ideas of sympy module in python
Learn about constructing curves and surfaces passing through Specified points, Least Square approximation Functions, traffic Flow, Electrical Circuits
Graph Theory, ..., and Markov Chain by using Linear Algebra tools.

Course Outline

A. Constructing Curves and surfaces passing through Specified points
B. Least Square approximation Functions
C. Traffic Flow
D. Electrical Circuits
E. Determinant
F. Graph Theory
G. Genetics
H. Cryptography
I. Markov Chain
J. Leonteif Economic Model

B. Least Square approximation Functions

We learned how to construct a curve through specified points. For example how to construct a line passing through two given points, or a parabola passing through three points. In general you can construct a polynomial of degree n that passes through n specified points. Such a polynomial is called Interpolating polynomial. What happens if you have for example more than two points and you want to represent your data with a straight line? In cases like this we face an inconsistent system of linear equations Ax=b. Instead of solving Ax=b we try to find an x such that Ax is good approximation of b.

B. Least Square approximation Functions

Inconsistent systems arise often in applications. Scientists try to find a functional relationship between variables. They collect data, which usually involve experimental error and by studying this data or from other findings, they suggest a mathematical model for it, ``functional relationship between variables." Because of the measurement error, usually a polynomial that passes through all The given points is not the a true representation of the relationship between variable. A lower degree polynomial may represent the relationship better. To find coefficients of such polynomial the method introduced in A is useful. But most of the times this method gives us an inconsistent system, which requires an approximation. In this laboratory we discuss such approximation, `` Least Square Solution ".

B. Least Square approximation Functions

This is an overdetermined system. That is it has more equation than unknowns. Over determined systems usually are inconsistent.

Writing the matrix equation for this linear system we get

$$\begin{bmatrix} \frac{5}{2} & 1 \\ 3 & 1 \\ \frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{9}{2} \\ 2 \\ 1 \end{bmatrix}$$

Find the reduced row echelon form of the augmented matrix.

$$\begin{bmatrix} \frac{5}{2} & 1 \\ 3 & 1 \\ \frac{3}{2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{9}{2} \\ 2 \\ 1 \end{bmatrix}$$

As you see the rank of rref of the augmented matrix is 3 while the rank of the coefficient matrix is 2. Therefore the system is inconsistent. So there is no x such that Ax=b or Ax - b =0. Now the question is : Since Ax - b is non-zero we try to make it as small as possible. So the goal is to find an approximation, Ax, for b or to minimize Ax-b?

B. Least Square approximation Functions

Example 1

Suppose that you know two variables x and y have linear relationship that is y= mx +c . Suppose in an experiment you obtained the following data:

$$\begin{aligned} x_1 &= \tfrac{5}{2} \\ x_2 &= 3 \\ x_3 &= \tfrac{3}{2} \\ x_4 &= 1 \end{aligned} \qquad \begin{aligned} y_1 &= 2 \\ y_2 &= \tfrac{9}{2} \\ y_3 &= 2 \\ y_4 &= 1 \end{aligned}$$

If you want to fit a line with equation y= mx +b through these points. As you learned in Laboratary A, you need to form the following system of linear equations.

$$\begin{array}{rcccc} \frac{5}{2}m & + & 1 & = & 2 \\ 3m & + & 1 & = & \frac{9}{2} \\ \frac{3}{2}m & + & 1 & = & 2 \\ 1m & + & 1 & = & 1 \end{array}$$

This is an overdetermined system. That is it has more equation than unknowns. Over determined systems usually are inconsistent.

Writing the matrix equation for this linear system we get

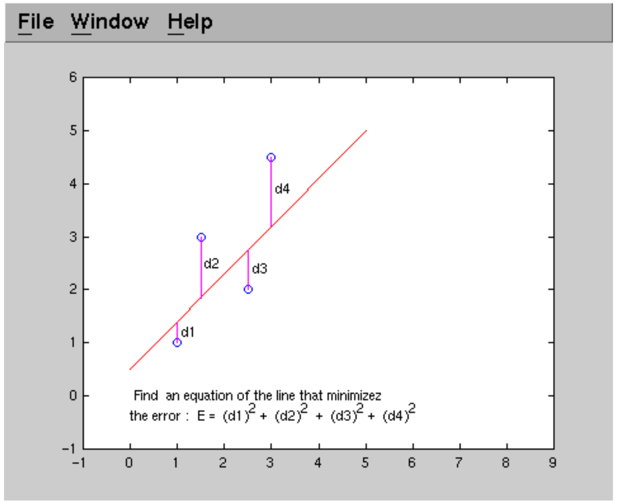
B. Least Square approximation Functions

Restating the question, how could we find an x such that Ax is as close as possible to b?

Minimizing Ax-b

Let A be an mxn Matrix. Suppose that Ax=b is an inconsistent system, we are interested in finding an x such that Ax is as close as possible to b.

Lets first look at the exercise 1. There are several ways to make your line ``close'' to given points, depending how we define ``closeness". Usual way is to add the square of d1, d2 d3 , ... dn, then minimize the sum of squares. See figure bellow.



This method is called `` least square Approximation ".

B. Least Square approximation Functions

We may also think that Ax in a $m \times 1$ vector and b is another $m \times 1$ vector. We want to minimize Ax-b. Assume that our vector space is an Inner Product Space with the usual Euclidean inner product, minimizing (ax-b) translates to minimizing the distance $\| Ax - b \|$ between the two vectors Ax and b . Note that $\| ax-b \|$ is the length of the vector Ax-b.

One thing which helps understanding the procedure of minimizing the distance between Ax and b is the fact that Ax is a vector in column space of A. (Why?)
Since Ax=b is not consistent, b is not in the column space of A. So we are looking for a vector, Ax, in the column space of A which is closest to the vector b. It can be proved that such a vector is the orthogonal projection of b onto the column space of A. Now if Ax is the orthogonal projection of b onto $col(A)$ then Ax-b is orthogonal to $clo(A)$. (why?) That is Ax-b is in $null(A^t)$. So

$$A^t A x = A^t b \qquad A^t A = I \qquad x = A^t b \qquad p = AA^t b$$

$$x = (A^t A)^{-1} A^t b$$

B. Least Square approximation Functions

Assume that y is a linear function of x that is $y=mx+b$ and experimental data Set is given.

solve the linear system .

$$A^t A x = A^t b \qquad\qquad A^t(b - Ax) = 0$$

We want to find the parameters m and b such that the line y=mx+b be as `` close '' as possible to the given points.

$$x = \begin{bmatrix} m \\ b \end{bmatrix} \qquad A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$x = (A^t A)^{-1} A^t b$$

B. Least Square approximation Functions

find equation of a polynomial of degree 3, $y = a_3x^3 + a_2x^2 + a_1x + a_0$ that best fits the following data.

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \\ x_4 &= -2 \\ x_5 &= 5 \\ x_6 &= -1 \end{aligned} \qquad \begin{aligned} y_1 &= 7 \\ y_2 &= -2 \\ y_3 &= -7 \\ y_4 &= -27 \\ y_4 &= 110 \\ y_4 &= -20 \end{aligned}$$

Therefore the linear system Ax=b would be

$$\begin{bmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \\ x_5^3 & x_5^2 & x_5 & 1 \\ x_6^3 & x_6^2 & x_6 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

B. Least Square approximation Functions

Pratice

A new type of thermocouple is being investigated by your company’s process control group.
These devices produce an almost linear voltage (millivolt) response at different temperatures.
In practice though it is used the other way around: use the millivolt reading to predict the temperature.
The process of fitting this linear model is called calibration.

Temperature [K]	273	293	313	333	353	373	393	413	433	453
Reading [mV]	0.01	0.12	0.24	0.38	0.51	0.67	0.84	1.01	1.15	1.31