Statistical Computing - CWB - 2019

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```
# Sources, libraries, seed
source('CWB2019code.R')
library(tidyverse)
set.seed(10)
```

Question 1

Task 1

The function `negloglike` shown below takes as input the values N, θ, y_1 and y_2 and outputs $l(N, \theta)$.

```
negloglike <- function(param, Y) {
  if (param[1] < max(Y)) { # If N >= max(y1, y2) then return +Infinity
    return(+Inf)
  } else { # Otherwise we calculate the negated log-likelihood
    return(sum(
      lgamma(Y+1),
      lgamma(param[1]-Y+1),
      -2*lgamma(param[1]+1),
      2*param[1]*log(1+exp(param[2])),
      -param[2]*sum(Y)
    ))
  }
}
```

Task 2

We seek to use the `optim` function with `negloglike` to find a maximum likelihood estimate of N and θ (and in turn ϕ). Since `optim` is a numerical optimiser it is only guaranteed to find a local minima. It therefore makes sense to try `optim` at different sensible starting values to try and find the best MLE we can in a grid search. The following parameter values were tried as starting points:

- N: We know we must have $N > max(y_1, y_2)$ so we try both $N = max(y_1, y_2)+1$ and $N = 2max(y_1, y_2)$.
- θ: Since this is derived from the actual probability ϕ we choose sensible values of ϕ and convert them to a value of θ using the `logit` function provided. Since ϕ is a probability is makes sense to try starting at the values of 0.01, 0.5 and 0.99.

```
Y <- c(256, 237) # Given data
bestopt <- list(value=+Inf) # Initialise our optimisation
```

```r
# Perform the grid search
for (N_start in list(max(Y)+1, 2*max(Y))) {
    for (theta_start in lapply(list(0.01, 0.5, 0.99), logit)) {
        # Use optim with the current starting values
        opt <- optim(par = c(N_start, theta_start), fn = negloglike, Y = Y)
        if (opt$value < bestopt$value) { # Update if we found a better minima
            bestopt <- opt
        }
    }
}

# Record MLEs of N and theta
N_hat <- bestopt$par[1]
theta_hat <- bestopt$par[2]
# Obtain MLE of phi
phi_hat <- ilogit(theta_hat)
```

Table 1: MLEs for $\hat{N}$, $\hat{\theta}$ and $\hat{\phi}$

$\hat{N}$	$\hat{\theta}$	$\hat{\phi}$
388.13	0.55	0.64

In Table 1 we see the maximum likelihood estimates for N, θ and consequently ϕ. This means that in order to maximise the likelihood $p(y|N, \phi)$ we would require there to be around 388 people buried at the site, with a probability 0.64 of finding a femur.

Task 3

Question 2

Task 1

The negated log-likelihood $l(N, \theta)$ is given in the Introduction.

The first derivative of $\Gamma(x)$ is $\Psi(x) = \Gamma'(x)$ and the second derivative of $\Gamma(x)$ is $\Psi'(x) = \Gamma''(x)$. The derivatives $\Psi(x)$ and $\Psi'(x)$ are given respectively in R by the functions `digamma(x)` and `trigamma(x)`.

The first partial derivative of $l(N, \theta)$ with respect to N is given by

$$\frac{\partial l(N,\theta)}{\partial N} = \Psi(N - y_1 + 1) + \Psi(N - y_2 + 1) - 2\Psi(N + 1) + 2\log(1 + e^{\theta}),$$

and the first partial derivative of $l(N, \theta)$ with respect to θ is given by

$$\frac{\partial l(N,\theta)}{\partial \theta} = \frac{2Ne^{\theta}}{1 + e^{\theta}} - (y_1 + y_2).$$

The second derivatives of $l(N, \theta)$ are given by

$$\begin{aligned}
\frac{\partial^2 l(N,\theta)}{\partial N^2} &= \Psi'(N - y_1 + 1) + \Psi'(N - y_2 + 1) - 2\Psi'(N + 1), \\
\frac{\partial^2 l(N,\theta)}{\partial \theta^2} &= \frac{2Ne^\theta}{(1 + e^\theta)^2}, \text{ and,} \\
\frac{\partial^2 l(N,\theta)}{\partial N \partial \theta} &= \frac{\partial}{\partial N} \frac{\partial l(N,\theta)}{\partial \theta} = \frac{2e^\theta}{1 + e^\theta}.
\end{aligned}$$

Task 2

Task 3

Task 4

Question 3

Task 1

Task 2

Task 3

Question 4

Task 1

Task 2

Task 3

Task 4

Appendix