A FIELD WITH ONE ELEMENT

LIGHT EDIAND

Abstract: This paper defines the field $\mathbb{F}_{un}$, in which division by 0 is defined, all elements have multiplicative inverses, and the underlying set of the field forms groups with respect to addition and multiplication.

1. Introduction

A group can be thought of as a set with 1 operation, whilst a field can be thought of as a set with 2 operations. Suppose we have the field $\mathbb{C}$. We may wish to break this apart to become $(\mathbb{C}, +)$ and $(\mathbb{C}, \cdot)$. However, we do not have 2 groups here. There exists an element in $\mathbb{C}$ that does not have a multiplicative inverse, and so it fails the relevant axiom, thereby depriving $(\mathbb{C}, \cdot)$ of group status. The field axioms contain a caveat for multiplicative inverses, so this dear element does not additionally prevent $\mathbb{C}$ from becoming a field. All things considered, that is quite a convenient caveat, as the field of complex numbers is a useful field to have. Regardless, this bothered me. In fact, it infuriated me! Despite the described *completeness* and *closure* contained within the complex numbers, there was still a number that lacked what I deem to be the essential and basic property of having a multiplicative inverse.

Furthermore, this element in $\mathbb{C}$ is one that we cannot even divide by. Division is simply left undefined because there is no way we can define it that makes sense. To me, $\mathbb{C}$ is incomplete, lacking, devoid of the warm embrace of the number that can wrap its arms around 0 and say, “You can be my divisor! You don’t have to be alone anymore!”

Perhaps you will suggest, “If you want to divide by 0, you could use an extension of the complex numbers such as the Riemann Sphere.” This is no good, as it results in the underlying set now having terms that lack multiplicative *and* additive inverses, unable to preserve any field or group structure. Moreover, this solution is incomplete as $\frac{0}{0}$ is still left undefined [1]. Essentially, any extension we try to devise for the complex field to define division by 0 will ultimately cause us to lose the field itself. Therefore, we must accept that fields only have multiplicative inverses for their non-zero elements, and the underlying set of a field can only form a group with respect to multiplication if we exclude 0.

The only problem with that is: I’m me. I’m a big fan of *including* 0, everywhere, and especially in the set of natural numbers. Why, if we do that, then $\mathbb{N}$ forms a monoid with respect to both addition and multiplication, subsequently forming a semiring! That doesn’t happen if you start $\mathbb{N}$ at 1. I digress, but the point is, I seek a field where we can a divide by 0, 0 has a multiplicative inverse, and where the underlying set forms a group with respect to addition and multiplication. I am delighted to inform you that such a field exists, and such a field is beautiful. Our solution exists not by extending the complex numbers, but by restricting them. Immensely.

2. A field of nothing

There is an object called ‘a field with one element’ commonly referred to as $\mathbb{F}_1$ which is neither a field, nor does it have one element [2]. Mathematicians are not very good at naming things. However, when I am discussing with you a field with one element, I am actually referring to a literal field with one element that I have willed into existence. The field is constructed from the underlying set $\{0\}$. We use the alternative notation of $\mathbb{F}_{un}$ to denote our field with one element in this paper. In this section, we verify all the field axioms hold, at least if we apply them generously. Let a, b and c be arbitrary elements of the set $\{0\}$. Noting that since there is only one element they can all be, rather elegantly we have

$$(a+b)+c=(0+0)+0=0+0=0.$$

Furthermore,

$$a+(b+c)=0+(0+0)=0+0=0.$$

Hence addition is associative. With the same elements we can verify the commutative property holds as follows.

$$a+b=0+0=0, \quad b+a=0+0=0.$$

Now suppose e is the additive identity, then $0+e=0$. Rearranging for e gives $e=0$, and indeed this belongs to our set as required. Finally, we seek our additive inverses. Let a again be an arbitrary element of the set $\{0\}$. Noting our additive identity is 0, the inverse of a is the element $-a$ such that

$$a+(-a)=0.$$

Since a is of course 0, we have $0+(-a)=0$, which shows our inverse is $-a=0$. Once again we see this is an element of the set $\{0\}$, and every element in our set has an additive inverse. In conclusion, $\mathbb{F}_{un}$ satisfies all the addition axioms as required. We have also proven $(\{0\},+)$ to be an abelian group in the process. We now verify the distributive property holds. Given any three elements from our set that will all be 0, we have

$$0(0+0)=0+0=0,$$
$$(0\cdot 0)+(0\cdot 0)=0+0=0.$$

Thus the distributive property is satisfied, as required and as expected. Similar to addition, given three 0s from our set we have the following.

$$0\cdot(0\cdot 0)=0\cdot 0=0,$$
$$(0\cdot 0)\cdot 0=0\cdot 0=0.$$

This shows multiplication is associative. Considering now two zeroes from our set we see that $0\cdot 0=0$, regardless of the order we put our 0s in. Hence multiplication commutes. Now suppose e is our multiplicative identity, then $0\cdot e=0$, and rearranging for e gives $e=0$. Once again, this belongs to our set as required.

A multiplicative inverse for an element a is defined as a^{-1} such that

$$a\cdot a^{-1}=e,$$

where e is the multiplicative identity. In the complex field where the multiplicative identity is 1, 0 cannot have a multiplicative inverse, since there is no $z\in\mathbb{C}$ such that $0z=1$.

However, since our multiplicative identity is 0, our inverse elements will be defined as

$$a\cdot a^{-1}=0.$$

As a is a member of the set $\{0\}$, $a=0$, and it follows from the above equation to be true that $a^{-1}=0$ too, meaning that it is a self inverse.

Now this means $(\{0\},\cdot)$ forms an abelian group, and hence all the field axioms are satisfied so $\mathbb{F}_{un}$ is a field.

3. A fun fact about $\mathbb{F}_{un}$

For all elements in $z\in\mathbb{F}_{un}$, we can define division by 0 as equal to 0.

$$z\div 0=0.$$

This is fine as rearranging gives $z=0\cdot 0$, and since $z\in\mathbb{F}_{un}$, it follows that $z=0$ which does not lead us to contradiction. Hurray!

4. Concluding remarks

It is now with a heavy heart that I inform you this paper is not to be taken seriously. Unlike constructing a subring of $\mathbb{C}$ to divide the prime numbers [3], constructing a field in which to divide 0 creates quite a lot of problems. The object $\mathbb{F}_{un}$ as I have described it is nothing more than the trivial ring. Finite fields must have p^n elements where p is prime and n is a positive integer, and 1 is generally not considered prime. Additionally, there is an issue with stating 0 is the multiplicative identity since the field axioms explicitly state the additive and multiplicative identities must be unique and thus a field should have at least two elements [4]. Further still, despite $\{0\}$ adhering to the definition of an integral domain, it is convention to exclude it due to convenience, and if a finite set does not form an integral domain then it cannot be a finite field [5].

There are likely even more issues than I am aware of about the consequences of a field having only one element. Though I feel most of the issues could be resolved if we simply went back to saying 1 is prime, like the good old days. The Fundamental Theorem of Arithmetic doesn't sound like anything important anyways so I'm sure it's a perfectly reasonable readjustment to the definition, much how including 0 in the natural numbers is also a perfectly reasonable readjustment.

Alas, 0 does not form a field in and of itself. Maybe we will never have a field where we can divide by 0, but it sure is nice to imagine a field with nothing in it. However, this adventure was not in vain! Though beyond the scope of this paper (and my current expertise), the actual field with one element, that's not a field, and doesn't have one element, is an active area of research. So even though we can't divide by 0, the rabbit hole I fell down to seek an answer led me to discover something really interesting that I wish to know more about! I hope to write to you about it some day too.

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References

[1] L.V. Ahlfors, *Complex Analysis*, Third Edition, McGraw-Hill, (1979), page 18
[2] O. Lorscheid, $\mathbb{F}_1$ *for everyone*, arXiv:1801.05337v1, (2018)
[3] L. Ediand, *An Adventurer's Guide to Dividing the Primes*, ediand.xyz, (2022)
[4] G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, First Edition, MacMillan Publishing Company, (1977), pages 36 & 445
[5] R.B.J.T. Allenby, *Rings, Fields and Groups*, Second Edition, Butterworth-Heinemann, (1991), page 94

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