COHOMOLOGY OF LIE ALGEBRAS

3(b) Cohomology of Lie algebras

Let G be any connected semisimple Lie group, $K \subset G(\mathbb{R})$ a maximal compact subgroup, (ρ, V) a finite-dimensional complex representation of G, $\Gamma \subset G(\mathbb{R})$ a torsion-free discrete arithmetic subgroup. We let $X = G(\mathbb{R})/K$ be the symmetric space attached to G and define the local system

$$\tilde{V} = \Gamma\backslash(X \times V) \to X_\Gamma = \Gamma\backslash X.$$

Here Γ acts diagonally on $X \times V$. Note that $\Gamma = \pi_1(X_\Gamma)$ and $\tilde{V}$ is a local system. Define

$$C^\infty(\tilde{V}) = \{f \in C^\infty(G(\mathbb{R}), V) \mid f(\gamma g) = \rho(\gamma) f(g), \gamma \in \Gamma,\ g \in G(\mathbb{R})\} = Ind_\Gamma^G(V).$$

(We write G for $G(\mathbb{R})$ in much of this section.) This is a representation space for G and $\mathfrak{g}$, under the right regular representation R. There is an isomorphism of G-modules

$$C^\infty(\tilde{V}) \xrightarrow{\sim} C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes V;\ f \mapsto F(g) = \rho(g)^{-1} f(g).$$

Indeed, if $h \in G$ then $R_h(F)(g) = \rho(gh)^{-1} f(gh)$, hence under this isomorphism, $R_h(f)$ maps to

$$[g \mapsto \rho(g^{-1}) R_h(f)(g) = \rho(h) \cdot \rho((gh)^{-1}) R_h(f)(g) = \rho(h) \cdot R_h F(g) = (R \otimes \rho(h) F)(g)].$$

We want to compute the cohomology groups $H^*(\Gamma\backslash G, \tilde{V})$ and especially $H^*(\Gamma\backslash X, \tilde{V})$. Now $\tilde{V}$ is locally constant (in the euclidean topology) and so its cohomology is computed by the complex of global sections of the (twisted) de Rham complex:

$$0 \to \tilde{V} \to \mathcal{A}^0(\tilde{V}) \xrightarrow{d} \mathcal{A}^1(\tilde{V}) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{A}^N(\tilde{V}) \to 0.$$

Here $\mathcal{A}^i(\tilde{V})$ is the locally constant sheaf that on a small ball U is just the C^∞ differential i-forms on U with coefficients in V. Then

$$A^i(\tilde{V}) := \Gamma(\Gamma\backslash G, \tilde{V}) = A^i(G, V)^\Gamma.$$

Now $A^i(G, V)$ is the space of smooth functions on G that at the point g takes $\bigwedge^i T_{G,g}$ to V. But $\bigwedge^i T_{G,g} = \bigwedge^i \mathfrak{g}$. So

$$A^i(G, V) = C^\infty(G, Hom(\bigwedge^i \mathfrak{g}, V)) = Hom(\bigwedge^i \mathfrak{g}, C^\infty(G, V)).$$

We will compute exterior differentiation and then look at what this does to the Γ-invariant subspaces.

Differential forms and differentials.

The exterior derivative of a differential form is calculated by linearity using the formula

$$d(f\omega_1 \wedge \omega_2 \wedge \dots \omega_q) = df \wedge \omega_1 \wedge \omega_2 \wedge \dots \omega_q) + \sum_{i=1}^{q} (-1)^{i+1} f\omega_1 \wedge \cdots \wedge d\omega_i \wedge \dots \omega_q.$$

This simplifies if $\omega_i = dx_i$ for a coordinate system $x_1, \dots, x_n$, whose derivatives commute, but we have identified differentials with linear maps from the Lie algebra to functions, and elements of the Lie algebra certainly don't commute. Let $X_1, \dots, X_N$ be a basis for $\mathfrak{g}$, $\omega_1, \dots, \omega_N$ the dual basis that parallelizes the cotangent space. Note in any case that if $f \in C^\infty(G)$ then $df = \sum X_j(f)\omega_j$. On the other hand, $d^2\ f = 0$, and this allows us to compute $d\omega_i$ for each i, as follows:

$$\begin{aligned}
0 = d^2\ f = d(\sum X_j(f)\omega_j) &= \sum_j d(X_j(f) \wedge \omega_j) + \sum_k X_k(f) d\omega_k \\
&= \sum_{i,j} X_i \circ X_j(f)\omega_i \wedge \omega_j + \sum_k X_k(f) d\omega_k \\
&= \sum_{i<j} (X_i \circ X_j - X_j \circ X_i)(f)\omega_i \wedge \omega_j + \sum_k X_k(f) d\omega_k \\
&= \sum_{i<j} [X_i, X_j](f)\omega_i \wedge \omega_j + \sum_k X_k(f) d\omega_k.
\end{aligned}$$

Now we write $[X_i, X_j] = \sum_k c_{ij}^{j} X_k$ and thus

$$0 = \sum_k X_k(f)[\sum_{i<j} c_{ij}^k \omega_i \wedge \omega_j + d\omega_k].$$

But this is true for any f, and by letting f vary, we see that the term in brackets vanishes, in other words

$$d\omega_k = -\sum_{i<j} c_{ij}^k \omega_i \wedge \omega_j.$$

Continuing the manipulations, we eventually find that if

$$\omega \in A^q(G, V) = Hom(\bigwedge^q \mathfrak{g}, C^\infty(G, V))$$

$$\begin{aligned}
d\omega(Y_0 \wedge \cdots \wedge Y_q) = &\sum_{j=0}^{q} (-1)^j Y_j(\omega(Y_0 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q)) \\
&+ \sum_{r<s} \omega([Y_r, Y_s] \wedge Y_0 \wedge \cdots \wedge \hat{Y}_r \wedge \cdots \wedge \hat{Y}_s \wedge \cdots \wedge Y_q)
\end{aligned}$$

The action $Y_j(\omega(\bullet))$ is right-differentiation of functions. The complete calculation can be found in Knapp, *Lie Groups, Lie Algebras, and Cohomology*, pp. 155-160.

The Lie algebra complex.

If (π, W) is any (complex) $U(\mathfrak{g})$-module, we can define a complex $C^\bullet(\mathfrak{g}, W) = Hom(\bigwedge^\bullet \mathfrak{g}, W)$ with differential given by the same formula

$$df(Y_0, \ldots, Y_q) = \sum (-1)^j \pi(Y_i) f(Y_0, \ldots, \hat{Y}_j, \ldots, Y_q) + \sum_{r<s} f([Y_r, Y_s], Y_0, \ldots, \hat{Y}_r, \ldots, \hat{Y}_s, \ldots, Y_q).$$

Let $H^q(\mathfrak{g}, W) = \ker(d_q)/im(d_{q-1})$. Then

$$H^0(\mathfrak{g}, W) = \{f \in Hom(\mathbb{C}, W) \mid df(X) = 0, \forall X \in \mathfrak{g}\} = \{f \in Hom(\mathbb{C}, W) \mid \pi(X) = 0, \forall X \in \mathfrak{g}\}.$$

In other words

$$H^0(\mathfrak{g}, W) = W^{\mathfrak{g}} := Hom_{U(\mathfrak{g})}(\mathbb{C}, W).$$

Theorem. *The functor $W \mapsto W^{\mathfrak{g}}$ is left-exact and $W \mapsto H^q(\mathfrak{g}, W)$ are its right-derived functors.*

Proof. Since $W^{\mathfrak{g}} := Hom_{U(\mathfrak{g})}(\mathbb{C}, W)$, we know that the functor is left-exact and its right-derived functors are given by $Ext^q_{U(\mathfrak{g})}(\mathbb{C}, W)$. So we need to identify $C^\bullet(\mathfrak{g}, W)$ with $Hom_{U(\mathfrak{g})}(C^\bullet, W)$ where $C^\bullet$ is an acyclic resolution of $\mathbb{C}$ in the category of $U(\mathfrak{g})$-modules. This will be sketched later in the setting of $(\mathfrak{g}, K)$-modules.

We certainly don't want to compute the cohomology of $C^\infty(G, V)$; but we have seen that

$$A^i(\tilde{V}) = A^i(G, V)^\Gamma = Hom(\bigwedge^i \mathfrak{g}, C^\infty(\Gamma\backslash G) \otimes V)$$

So

$$H^q(\Gamma\backslash G, \tilde{V}) = H^q(\mathfrak{g}, C^\infty(\Gamma\backslash G) \otimes V).$$

This is still not what we want to compute, which is $H^\bullet(X_\Gamma, \tilde{V}) = H^\bullet(\Gamma\backslash G/K, \tilde{V})$. We can compute this by a complex of differential forms $A^\bullet(X_\Gamma, \tilde{V}) = A^\bullet(X, V)^\Gamma$ and there is a functorial embedding

$$i_K : A^q(X, V)^\Gamma \hookrightarrow A^q(G, V)^\Gamma.$$

given by $G \mapsto X; g \mapsto gK$ (pullback of differentials). The image consists of $f : \bigwedge^q \mathfrak{g} \to C^\infty(\Gamma\backslash G) \otimes V$ such that

(1) $f(Y_0, \ldots, Y_{q-1}) = 0$ if one of the $Y_i \in \mathfrak{k} = Lie(K)$;
(2) $f \in (A^q(G, V)^\Gamma)^K$ (right-invariant under K).

Here (1) says that the pullback of differentials from X to G vanish on vectors tangent to K, and (2) says that the coefficients of the differential are functions on $X = G/K$. More precisely – ignore Γ for the moment, since the right and left actions don't interfere: Say $f(g) \in Hom(\bigwedge^q T_{G,g}, V)$;

$$f(gk) \in Hom(\bigwedge^q T_{G,gk}, V) \xrightarrow[R(k^{-1})]{\sim} Hom(\bigwedge^q T_{G,g}, V)$$

and the condition that f be K-invariant is that $R(k^{-1}) f(gk) = f(g)$. But $R(k^{-1})$ acting on left-invariant vector fields is just $Ad(k)$ acting on $\mathfrak{g}$. So to conclude

Lemma. *The image of* $i_K(A^q(X,V)^\Gamma)$ *is*

$$Hom_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G, V)) = Hom_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)\otimes V).$$

Define $C^\infty(\Gamma\backslash G)_0 \subset C^\infty(\Gamma\backslash G)$ to be the subspace of K*-finite* vectors: a vector whose translates under K generate a finite-dimensional subspace. Then it is clear that for any q,

$$Hom_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)\otimes V) = Hom_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)_0\otimes V).$$

because both V and $\bigwedge^q(\mathfrak{g}/\mathfrak{k})$ are finite-dimensional.

Thus for any representation (π, W) of $U(\mathfrak{g})$ on which the action of $\mathfrak{k}$ integrates to a consistent K-finite action of K, we define

$$C^q(\mathfrak{g}, K; W) = Hom_K(\bigwedge^q(\mathfrak{g}/\mathfrak{k}), W).$$

This is a subspace of $C^q(\mathfrak{g}, W)$ and it is easy to see that it is preserved by the differential, so that we have a complex and can define the *relative Lie algebra cohomology* $H^q(\mathfrak{g}, K; W)$.

Proposition. $H^\bullet(X_\Gamma, \tilde{V}) \xrightarrow{\sim} H^\bullet(\mathfrak{g}, K; C^\infty(\Gamma\backslash G)_0\otimes V)$.

Again $H^q(\mathfrak{g}, K; \bullet)$ is the right-derived functor of a left-exact functor on the category of $(\mathfrak{g}, K)$-modules (see below). In what follows, G is a connected reductive Lie group, $K \subset G$ is a maximal compact subgroup (modulo the center of G) and $\mathfrak{g} = Lie(G), \mathfrak{k} = Lie(K)$.

Definition. *1. A* $(\mathfrak{g}, \mathfrak{k})$*-module is a* $U(\mathfrak{g})$*-module whose restriction to* $U(\mathfrak{k})$ *is semi-simple and a sum of finite-dimensional* $\mathfrak{k}$*-modules.*

2. A $(\mathfrak{g}, K)$*-module is a* $(\mathfrak{g}, \mathfrak{k})$*-module* (π, V) *whose* $\mathfrak{k}$ *action integrates to an action of* K *(note:* K *may be disconnected) in such a way that, for* $k \in K$ *and* $X \in \mathfrak{g}$*, we have* $\pi(k)\pi(X)\pi(k^{-1})v = \pi(ad(k)X)v$ *for all* $v \in V$*.*

3. A $(\mathfrak{g}, K)$*-module* V *is admissible if for any irreducible representation* τ *of* K*,* $Hom_K(\tau, V)$ *is finite-dimensional.*

One also calls $(\mathfrak{g}, K)$-modules *Harish-Chandra modules.* They form an abelian category (a full subcategory of the category of $U(\mathfrak{g})$-modules) with enough projectives and injectives. If V, W are $(\mathfrak{g}, K)$-modules, one can compute $Ext^\bullet(V, W)$ – extensions in the category of $(\mathfrak{g}, K)$-modules – by using a projective resolution of V:

$$\ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to V;\ C^i = Hom(P_i, W);\ Ext^q(V, W) = H^q(C^\bullet).$$

Let $P_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bigwedge^i(\mathfrak{g}/\mathfrak{k})$ and define $\partial_q : P_q \to P_{q-1}$ by the expected formula

$$\begin{aligned}\partial_q(r\otimes x_1\wedge\cdots\wedge x_q) &= \sum(-1)^{i-1}x_i\cdot r\otimes x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_q)\\ &\quad+\sum_{i<i}(-1)^{i+j}r\otimes([x_i, Y_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_q)\end{aligned}$$

Let $\varepsilon : P_0 = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbb{C} \to \mathbb{C}$ be the augmentation.

Theorem. *The P_j are projective, the maps ∂_q and ε are well-defined, and the sequence $\ldots \to P_q \to P_{q-1} \to \ldots \to \to P_0 \to \mathbb{C}$ is exact.*

Proof. The detailed proofs are contained in §VII.8 of Knapp's yellow book. There are two points:

(1) tensor product $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \bullet$ takes projectives to projectives and everything in the category is $U(\mathfrak{k})$-projective;
(2) The sequence is exact.

Exactness is a long calculation that reduces ultimately using filtrations to the exactness of the Koszul complex. The first point can be easily explained. Let V be any $(\mathfrak{g}, K)$-module and let U be a K-module. Let $I(U) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} U$. We check that this is a $(\mathfrak{g}, \mathfrak{k})$-module (that it is a $(\mathfrak{g}, K)$-module follows easily). Let $W \subset U(\mathfrak{g})$ be a finite-dimensional subspace invariant under $ad(\mathfrak{k})$, $X \in W$, $Y \in U(\mathfrak{k})$; then

$$YX \otimes u = [Y, X] \otimes u + X \otimes Yu \subset W \otimes U.$$

So the action is semisimple and $I(U)$ is a $(\mathfrak{g}, \mathfrak{k})$-module.

Suppose $f : B \to A$ is a surjective morphism of $(\mathfrak{g}, K)$-modules. Now Frobenius reciprocity is a canonical isomorphism

$$Hom_{(\mathfrak{g},K)}(I(U), \bullet) = Hom_K(U, \bullet).$$

So the map $Hom_{(\mathfrak{g},K)}(I(U), B) \to Hom_{(\mathfrak{g},K)}(I(U), A)$ is surjective if and only if $Hom_K(U, B) \to Hom_K(U, A)$ is surjective; but this is clear because B and A are semisimple as K-modules. Thus $I(U)$ is projective for any U, and in particular all the P_j are projective.

Corollary. *The functors $H^\bullet(\mathfrak{g}, K; \bullet)$ are the right-derived functors of*

$$W \mapsto Hom_{\mathfrak{g},K}(\mathbb{C}, W).$$

In particular short exact sequences give rise to long exact sequences in the usual way.

Definition. *Let U be a $(\mathfrak{g}, K)$-module. The* contragredient *of U is the subspace $\tilde{U} \subset Hom(U, \mathbb{C})$ of vectors on which K acts finitely.*

Suppose U is *admissible*, so that as K-module, $U = \oplus_{\sigma \in \hat{K}} U(\sigma)$ with $U(\sigma) = Hom_K(\sigma, U) \otimes \sigma$ finite-dimensional. Then $\tilde{U} = \oplus_{\sigma \in \hat{K}} U(\sigma)^*$ where $*$ is the usual contragredient for finite-dimensional representations. In particular, $\tilde{U}$ is again admissible.

Definition. *Let $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ be an algebra homomorphism. The $(\mathfrak{g}, K)$-module U has* infinitesimal character *χ if $zu = \chi(z)u$ for all $z \in Z(\mathfrak{g})$, $u \in U$.*

Corollary. *Suppose U and V are $(\mathfrak{g}, K)$-modules with infinitesimal characters χ_U and χ_V, respectively. Suppose V is finite-dimensional and $\chi_U \neq \chi_{\tilde{V}}$. Then $H^q(\mathfrak{g}, K; U \otimes V) = 0$ for all q.*

Proof. The natural equivalence of (bi)-functors:

$$Hom_{(\mathfrak{g},K)}(\mathbb{C}, U \otimes V) \xrightarrow{\sim} Hom_{(\mathfrak{g},K)}(\tilde{V}, U)$$

gives rise to isomorphisms of derived functors

$$H^q(\mathfrak{g}, K; U \otimes V) \xrightarrow{\sim} Ext^q(\tilde{V}, U).$$

So it suffices to show that $Ext^q(\tilde{V}, U) = 0$ for all q. Now by hypothesis, there is $z \in Z(\mathfrak{g})$ that acts as 1 on U and as 0 on $\tilde{V}$. If $q = 0$ and $h : \tilde{V} \to U$ is a $(\mathfrak{g}, K)$-morphism then $h(v) = zh(v) = h(zv) = 0$. If $q > 0$, let $S \in Ext^q(\tilde{V}, U)$ correspond to a Yoneda extension, i.e. a long exact sequence of $(\mathfrak{g}, K)$-modules

$$0 \to U \to E_{q-1} \to \ldots \to E_0 \to \tilde{V} \to 0.$$

Then z acts consistently on this sequence and as the identity on U and as 0 on $\tilde{V}$, and defines an equivalence with the trivial Yoneda extension.

Alternatively, multiplication by z defines two natural transformations of bifunctors $(A, B) \mapsto Ext^q_{(\mathfrak{g},K)}(A, B)$, say z_1 and z_2, by acting in the first and second variable, respectively. For $q = 0$ we have $z_1 = z_2$. Suppose they are equal up to $q-1$. Let $B \to C \to B'$ be an exact sequence of $(\mathfrak{g}, K)$-modules with C injective. Then the exact cohomology sequence breaks up as $Ext^{j-1}(U, V') \to Ext^j(U, V)$ which is surjective for $j = 1$ and an isomorphism for $j \geq 2$, and which commutes with z_1 and z_2. This reduces the case of q to that of $q - 1$. In our setting, $z_1 = 0$ and $z_2 = 1$, so we are done.

Compact quotients.

Suppose $\Gamma\backslash G$ is *compact*. Then $C^\infty(\Gamma\backslash G)_0 \subset L_2(\Gamma\backslash G)$ where the measure defining L_2 is any right-invariant Haar measure dg on $G = G(\mathbb{R})$. It is not hard to show that if G is reductive, then any right-invariant Haar measure is also left-invariant. Now $L_2(\Gamma\backslash G)$ is a Hilbert space on which G acts by a unitary representation – unitary because

$$< r(h)f, r(h)f' >_{L_2} = \int_{\Gamma\backslash G} f(gh)f'(gh)dg = \int_{\Gamma\backslash G} f(g)f'(g)dg = < f, f' >_{L_2},$$

where the second equality follows from invariance of dg.

The following theorem is due to Gelfand and Piatetski-Shapiro:

Theorem. *Assume $\Gamma\backslash G$ is compact. Then $L_2(\Gamma\backslash G)$ decomposes as G-module as a countable Hilbert space direct sum:*

$$L_2(\Gamma\backslash G) = \widehat{\bigoplus_\pi} m(\pi, \Gamma)\pi$$

where π runs over irreducible unitary representations of G and $m(\pi, \Gamma) \in \mathbb{N}$. In particular, the multiplicity of π is always finite, and is zero except for a countable set of π.

The classification of irreducible unitary representations is incomplete, but the following theorem is fundamental. Choose a maximal compact (or compact connected) subgroup K of G. We define a K-finite vector in a Hilbert space representation $(\pi, \mathcal{V})$ as above; a *smooth vector* $v \in \mathcal{V}$ is one for which, for any element $X \in Lie(G)$ and any $w \in \mathcal{V}$ the function $t \mapsto < \pi(exp(tX)v, w >$ from $\mathbb{R}$ to $\mathcal{V}$ is infinitely differentiable.

Theorem(Harish-Chandra). *Let π be an irreducible unitary Hilbert space representation of the reductive Lie group G and let $\pi_0 \subset \pi$ denote the subspace of K-finite smooth vectors. Then $U(\mathfrak{g})$ acts naturally on π_0 by $\pi(X)v = \frac{d}{dt}\pi(exp(tX)v$ and makes it an irreducible $(\mathfrak{g}, K)$ module.*

This is proved in several stages, the most important being that any irreducible representation of K occurs with finite multiplicity in π_0 – in other words, that π_0 is an *admissible* $(\mathfrak{g}, K)$ module. This implies that every K-finite vector is automatically smooth. The proofs can be found in the beginning of Chapter VIII of Knapp's book *Representation Theory of Semisimple Groups.*

We see that $L^2(\Gamma\backslash G)_0 = C^\infty(\Gamma\backslash G)_0$ and that this in turn is $\widehat{\bigoplus}_\pi m(\pi,\Gamma)\pi_0$. Thus

Theorem. *Assume $\Gamma\backslash G$ is compact. Then for any finite-dimensional representation V of G,*

$$H^\bullet(X_\Gamma, \tilde{V}) \xrightarrow{\sim} \widehat{\bigoplus_\pi} m(\pi,\Gamma) H^\bullet(\mathfrak{g}, K; \pi_0 \otimes V).$$

Thus the calculation of the cohomology of X_Γ divides into two parts: a global part, which is the determination of $m(\pi,\Gamma)$, and a purely Lie-theoretic part, which is the calculation of $H^\bullet(\mathfrak{g}, K; \pi_0 \otimes V)$. The first part is very hard, the second part was solved some time ago. First I switch to the adelic setting.

Theorem (Borel-Harish-Chandra). *Let G be a reductive group over $\mathbb{Q}$ with center Z, and for any open compact subgroup $K \subset G(\mathbf{A}_f)$, let ${}_KS(G) = G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty Z(\mathbb{R}) \times K$. Then ${}_KS(G)$ is compact for one K if and only it it is compact for all K if and only if G/Z has $\mathbb{Q}$-rank 0; in other words, if G/Z does not contain a subgroup isomorphic to $GL(1)$.*

One says that G/Z is anisotropic if it is of $\mathbb{Q}$-rank 0. The adelic version of the Gelfand-Piatetski-Shapiro theorem is the following:

Theorem. *Let G be a (connected) reductive group over $\mathbb{Q}$ with center Z, and assume G/Z is anisotropic. Then $L_2(G(\mathbb{Q})\backslash G(\mathbf{A}))$ decomposes as $G(\mathbf{A})$-module as a countable Hilbert space direct sum:*

$$L_2(G(\mathbb{Q})\backslash G(\mathbf{A})) = \widehat{\bigoplus_\pi} m(\pi)\pi$$

where π runs over irreducible unitary representations of $G(\mathbf{A})$.

We fix a maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. With π as in the theorem, a vector $v \in \mathcal{V}_\pi$ is *smooth* if it is C^∞ with regard to the action of $G(\mathbb{R})$ and if there is an open compact subgroup $K_f \subset G(\mathbf{A}_f)$ that fixes v. Write $\pi_0 \subset \pi$ for the space of K_∞-finite smooth vectors. Then

(1) π_0 determines π (and vice versa); in particular, π_0 is irreducible as a representation of $(U(\mathfrak{g}), K_\infty) \times G(\mathbf{A}_f)$;
(2) π_0 admits a unique factorization (up to scalar multiples) $\pi_0 \xrightarrow{\sim} \pi_\infty \otimes \otimes'_p \pi_p$ where the product is taken over all prime numbers, π_∞ is an irreducible $(\mathfrak{g}, K_\infty)$-module, and each π_p is an irreducible (smooth) representation of $G(\mathbb{Q}_p)$.

If V is a representation of G, we can then define $H^\bullet(\mathfrak{g}, K_\infty; \pi \otimes V) := H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty \otimes V) \otimes \pi_f$ where $\pi_f = \otimes'_p \pi_p$ is an irreducible smooth representation of $G(\mathbf{A}_f)$. If $K_f \subset G(\mathbf{A}_f)$ is open compact, we can similarly define

$$H^\bullet(\mathfrak{g}, K_\infty; \pi^{K_f} \otimes V) := H^\bullet(\mathfrak{g}, K_\infty; \pi_\infty \otimes V) \otimes \pi_f^{K_f}.$$

Working through the comparison of ${}_K S(G)$ with a union of spaces of the form $\Gamma_i \backslash G(\mathbb{R})/K_\infty Z(\mathbb{R})$, we find

Proposition. *For any representation V of G, there is a canonical isomorphism*

$$H^\bullet({}_K S(G), \tilde{V}) \xrightarrow{\sim} \bigoplus_\pi H^\bullet(\mathfrak{g}, K; \pi_\infty \otimes V) \otimes \pi_f^{K_f};$$

$$H^\bullet(S(G), \tilde{V}) \xrightarrow{\sim} \bigoplus_\pi H^\bullet(\mathfrak{g}, K; \pi_\infty \otimes V) \otimes \pi_f,$$

where the latter isomorphism commutes with the action of $G(\mathbf{A}_f)$ on both sides.

Automorphic vector bundles. Now assume (G, X) a Shimura datum, so that X is of Hermitian type. Fix $h \in X$ and let $[W]$ be the automorphic vector bundle on $S(G, X)$ attached to a finite-dimensional representation of K_h. We can replace the de Rham complex by the Dolbeault complex in the discussion above. Instead of

$$A^i(G, V) = C^\infty(G, Hom(\bigwedge^i \mathfrak{g}, V)) = Hom(\bigwedge^i \mathfrak{g}, C^\infty(G, V)).$$

we have

$$\begin{aligned} A^{0,q}([W]) &= [C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}), Hom(\bigwedge^i(\mathfrak{p}^-), W))]^{K_h} \\ &= Hom_{K_h}(\bigwedge^i(\mathfrak{p}^-), C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes W) \\ &= [\bigwedge^i(\mathfrak{p}^-)^* \otimes C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes W]^{K_h}. \end{aligned}$$

Now $\mathfrak{k}_h \oplus \mathfrak{p}^-$ is the Lie algebra of a subgroup $P_h \subset G_\mathbb{C}$ (not defined over $\mathbb{R}$ and the definition of relative Lie algebra cohomology in this case identifies $A^{0,q}([W])$ with $C^q(Lie(P_h), K_h; C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes W)$. Thus Dolbeault's theorem gives us an isomorphism

$$H^q(S(G, X), [W] \xrightarrow{\sim} H^q(Lie(P_h), K_h; C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \otimes W).$$

Again, we can replace $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ by $\bigoplus_\pi m(\pi)\pi_0$ and then we find an isomorphism of $G(\mathbf{A}_f)$-spaces

$$H^q(S(G, X), [W]) \xrightarrow{\sim} \bigoplus_\pi m(\pi) H^q(Lie(P_h), K_h; \pi_\infty \otimes W) \otimes \pi_f.$$

Hodge theory for $(\mathfrak{g}, K)$-modules.

The analytic arguments used to prove the Hodge theorem in complex geometry becomes completely algebraic in the setting of relative Lie algebra cohomology. Change notation: let π be a $(\mathfrak{g}, K)$-module that comes from a *unitary* representation of G. We want to compute $H^q(\mathfrak{g}, K; \pi \otimes V)$ for any irreducible finite-dimensional representation (ρ, V). We already know that this vanishes unless π and V have opposite infinitesimal characters. To apply Hodge theory, we endow W with an *admissible scalar product*: one that is invariant under K and with respect to which the action of $\mathfrak{p}$ is self-adjoint. The existence of such a scalar product is easy to show: V is a representation of G, therefore of the compact real form G_u, and therefore possesses a G_u-invariant (hermitian) scalar product (unique up to positive scalar multiples), say $(,)_V$. This means that $Lie(G_u) = \mathfrak{k} \oplus i\mathfrak{p}$ acts by skew-adjoint operators: if $X \in Lie(G_u)$ then $(Xv, v') + (v, Xv') = 0$. For $X = iY$ with $Y \in \mathfrak{p}$ this means

$$0 = i(Yv, v') + \bar{i}(v, Yv') = i[(Yv, v') - (v, Yv')]$$

which means that $\mathfrak{p}$ is self-adjoint.

Let $D^q(\pi \otimes V) = \bigwedge^q \mathfrak{p}^* \otimes \pi \otimes V$ with scalar product given by the tensor product of the hermitian Killing form on $\mathfrak{p}_{\mathbb{C}}$:

$$(x, y) = B(x, \bar{y})$$

(dualized and taken to the q-th power) with the scalar products already defined on the other two factors. All of these scalar products are invariant under $\mathfrak{k}$. Write $\tau = \pi \otimes \rho$, so that $\tau(x) = \pi(x) \otimes 1 + 1 \otimes \rho(x)$ for $x \in \mathfrak{g}$. The adjoint with respect to the scalar product on τ is given by

$$\tau^*(x) = -\tau(x), x \in \mathfrak{k};\ \tau^*(x) = -\pi(x) + \rho(x), x \in \mathfrak{p}.$$

In what follows, we choose a (real) basis $x_1, \ldots, x_D$ of $\mathfrak{p}$ (and don't confuse the dimension D with the differential d. If $\eta \in D^q(\pi \otimes V) = Hom(\bigwedge^q \mathfrak{p}, \pi \otimes V)$ and $J \subset \{1, \ldots, D\}$, $|J| = q$, we write

$$\eta_J = \eta(x_{j_1}, \ldots, x_{j_q}).$$

Proposition. *Define $d^* : D^q(\pi \otimes V) \to D^{q-1}(\pi \otimes V)$ by the formula*

$$(d^*\eta)_J = \sum_{j=1}^{D} \tau(x_j)^* \eta_{\{j\} \cup J}.$$

Then d^* commutes with K, maps the K-invariant subspace $C^q(\pi \otimes V)$ into the K-invariant subspace $C^{q-1}(\pi \otimes V)$, and is formally adjoint to d:

$$(d^*\eta, f) = (\eta, df)$$

for $\eta \in D^q$, $f \in D^{q-1}$.

Proof. We recall the formula for df (with a shift of indices):

$$\begin{aligned} &df(Y_{j_1}, \ldots, Y_{q+1}) \\ &= \sum_j (-1)^{j-1} \tau(Y_j) f(Y_1, \ldots, \hat{Y}_j, \ldots, Y_{q+1}) + \sum_{r<s} f([Y_r, Y_s], Y_1, \ldots, \hat{Y}_r, \ldots, \hat{Y}_s, \ldots, Y_{q+1}) \\ &= \sum_j (-1)^{j-1} \tau(Y_j) f(Y_1, \ldots, \hat{Y}_j, \ldots, Y_{q+1}) \end{aligned}$$

because $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ so the bracket terms vanish. Thus (replacing the Y_u's by x_{j_u}

$$(\eta, df) = \sum_{|I|=q} (\eta_I, (df)_I)_\tau = \sum_{|I|=q} (\eta_I, \sum_u (-1)^{u-1} \tau(x_{j_u}) f_{I(u)})_\tau$$

where $I = \{j_1, \ldots, j_q\}$ and $I(u)$ is I with the j_u term removed. And thus by adjunction

$$(\eta, df) = \sum_{I,u} ((-1)^{u-1} \tau^*(x_{j_u}) \eta_I, f_{I(u)}.$$

Using the anticommutation formula

$$\eta_I = (-1)^{u-1} \eta_{\{j_u\} \cup I(u)}$$

this shows after reviewing the notation that $(\eta, df) = (d^*\eta, f)$.

Now if $x \in k$ we find

$$\begin{aligned} -(\tau(x) d^*\eta, f) &= (\tau(x)^* d^*\eta, f) = (d^*\eta, \tau(x) f) = (\eta, d\tau(x) f) \\ &= (\eta, \tau(x) df) = (-\tau(x)\eta, df) = -(d^*\tau(x)\eta, f). \end{aligned}$$

Thus d^* commutes with K and in particular takes the K-invariants to K-invariants.

Let $\Delta = dd^* + d^*d$. For each q, Δ is an endomorphism of $D^q(\pi \otimes V)$ and of $C^q(\pi \otimes V)$. Moreover, for $\eta \in D^q(\pi \otimes V)$

$$(\Delta\eta, \eta) = (d\eta, d\eta) + (d^*\eta, d^*\eta)$$

and since the scalar product is positive non-degenerate,

$$\Delta\eta = 0 \Leftrightarrow d\eta = 0 \text{ and } d^*\eta = 0 \Leftrightarrow (\Delta\eta, \eta) = 0.$$

In this case we say η is *harmonic*; and we let $\mathcal{H}^q(\pi \otimes V) \subset C^q(\pi \otimes V)$ denote the subspace of harmonic forms.

Proposition. *For every q, the map $\mathcal{H}^q(\pi \otimes V) \to H^q(\mathfrak{g}, K; \pi \otimes V)$ is an isomorphism.*

Proof. This is equivalent to the orthogonal decomposition

$$C^q(\pi \otimes V) = \mathcal{H}^q(\pi \otimes V) \oplus Im(d) \oplus Im(d^*).$$

Note for example that for $\eta \in C^q$,

$$d\eta = 0 \Leftrightarrow (d\eta, f) = 0 \ \forall f \in C^{q+1} \Leftrightarrow \eta \perp Im(d^*).$$

So $C^q = \ker(d) \oplus Im(d^*)$ is an orthogonal decomposition. Similarly, $Z^q := \ker d = \mathcal{H}^q \oplus Im(d)$ is an orthogonal decomposition because $\mathcal{H}^q = Z^q \cap \ker d^*$.

Kuga's formula calculates the action of Δ in terms of a specific element, the Casimir element, in $Z(\mathfrak{g})$.