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C^*-ALGEBRAIC INTERTWINERS FOR PRINCIPAL SERIES: CASE OF $\mathrm{SL}_2(\mathbb{R})$

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ABSTRACT. We construct and normalise intertwining operators at the level of Hilbert modules describing the principal series of $\mathrm{SL}_2(\mathbb{R})$. Normalisation is achieved through the use of a Fourier transform defined on some homogenous space and twisted by a Weyl element. Normalising factors are also explicitely obtained. In the appendix we relate reducibility points to a certain distribution arising from the non-normalised intertwiners.

1. INTRODUCTION: A C^*-ALGEBRAIC POINT OF VIEW ON PRINCIPAL SERIES

The study of the tempered dual $\widehat{G}_r$ of a semisimple Lie group G in relation to the Plancherel formula is essentially the work of Harish-Chandra [7], who described the various series of representations carrying the Plancherel measure. Because of its measure-theoretic nature, this approach does not require *all* tempered representations to be taken into account. However, in order to understand $\widehat{G}_r$ as a topological space, a full description of the intertwining relations among the principal series is necessary, for which A. W. Knapp and E. M. Stein developped the main tools in [9, 10]. Facts and references about these matters may be found in [8] and [14].

If G admits more than one conjugacy class of Cartan subgroups, $\widehat{G}_r$ is not Hausdorff when equipped with the Fell-Jacobson topology. According to the general philosophy of Non-commutative Geometry, the relevant algebra related to $\widehat{G}_r$ is the reduced C^*-algebra of the group, denoted by $\mathrm{C}^*_r(G)$. A classical reference on the relations between the properties of $\mathrm{C}^*_r(G)$ and the representation theory of G is [6]; other aspects are discussed in the more recent book [1].

In order to analyse $\mathrm{C}^*_r(G)$, it is natural to seek a formulation of the basic objects and results of semisimple theory in terms of operator algebras. Elements of such a C^*-algebraic description of the principal series were obtained in [3] and [4]. The purpose of the present article is now to discuss the analogue of Knapp-Stein theory of intertwining operators in this framework for the group $\mathrm{SL}_2(\mathbb{R})$.

Outline. In the remainder of the introduction, we recall classical facts about the principal series and the Hilbert modules general setting in which the main result will be stated. The problem of normalising C^*-algebraic intertwiners is precisely formulated in Paragraph 1.4. Section 2 is devoted to the description of the module $\mathcal{E}$ associated to the principal series of $\mathrm{SL}_2(\mathbb{R})$. In particular, Proposition 1 gives a characterisation of functions in $\mathcal{E}$ by a norm estimate. Finally the normalisation theorem is established in Section 3. In the appendix, we relate the reducibility

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points in the principal series to a certain distribution responsible for the unboundedness of the generalised intertwining operators before normalisation.

Notation. The carrying space of a representation π will always be denoted by $\mathcal{H}_\pi$.

1.1. **Structure and general notations.** Let G be a connected semisimple Lie group with finite center. An Iwasawa decomposition $G = KAN$ being fixed, let M' and M respectively denote the normaliser and the centraliser of A in K. In addition, we will assume that G has real rank 1, that is $\dim A = 1$. Then, the Weyl group W defined as the quotient M'/M contains exactly one non-trivial element w. Let $P = MAN$ be the standard Borel subgroup of G, and $\bar{N}$ denote the image of N under the Cartan involution determined by the choice of K. Then $\bar{N} = w^{-1}Nw$ and the Bruhat decomposition implies that $G \setminus \bar{N}MAN$ has Haar measure 0. Almost every element $g \in G$ thus admits a unique decomposition

$$g = \bar{\boldsymbol{n}}(g)\boldsymbol{m}(g)\boldsymbol{a}(g)n_g = \bar{\boldsymbol{n}}(g)\boldsymbol{l}(g)n_g$$

according to $\bar{N}MAN$. See [9, p.513] for general properties of the projections $\bar{\boldsymbol{n}}$, $\boldsymbol{a}$ and $\boldsymbol{l}$ and Paragraph 2.1 for concrete formulas in the case of $\mathrm{SL}_2(\mathbb{R})$.

Finally, if dg is a fixed left Haar measure on G, it decomposes according to KAN as

$$dg = e^{2\rho_N \log a} dk\, da\, dn, \tag{1.1}$$

where ρ_N is the half sum of positive roots of the Lie algebra of A with respect to N.

1.2. **Principal series.** If ν is a purely imaginary complex number, we still denote by ν the character of A defined by $a \longmapsto e^{\nu \log a}$. The *principal series* associated to the parabolic subgroup P consists in the representations of the form

$$\pi_P^{\sigma,\nu} = \mathrm{Ind}_P^G\, \sigma \otimes \nu \otimes 1_N.$$

The Weyl group acts on $\widehat{M} \times \widehat{A}$ *via* $w.(\sigma,\nu) = \big(\sigma(w^{-1}\cdot w), -\nu\big)$. Results of F. Bruhat [2] show that parameters (σ,ν) in the same orbit under W induce equivalent representations and that $\pi_P^{\sigma,\nu}$ is irreducible if (σ,ν) is not a fixed point under W: the principal series representations are said to be *generically irreducible*.

Remark 1. Another classical result, valid in higher rank, is the fact that if P_1 and P_2 are associate parabolic subgroups of G, that it $P_i = L_iN_i$ such that L_1 and L_2 are conjugate, then the representations in the principal series $\pi_{P_1}^{\widehat{L}_1}$ are equivalent to the ones in $\pi_{P_2}^{\widehat{L}_2}$.

The problem of determining if reducibility indeed occurs at the Weyl-fixed points was dealt with by Knapp and Stein in [9]. Natural candidates for intertwiners appear as integral operators formally satisfying the intertwining relations. However, these operators are given by non-locally integrable kernels. The central object in this theory is the *standard intertwining integral* associated to the element w, formally defined by the formula

$$I_w^{\sigma,\nu} f(g) = \int_{\bar{N}} f(xw\bar{n})\, d\bar{n} \tag{1.2}$$

where f is a function in a dense subspace of $\mathcal{H}_{\pi_P^{\sigma,\nu}}$. Knapp and Stein then proceed in two steps to construct intertwiners. The first consists in making sense of (1.2) by letting ν take non-purely imaginary values. The operators $I_w^{\sigma,\nu}$ are then defined as meromorphic functions in the complex variable ν. The second step is called *normalisation* and yields unitary self intertwiners at the reducibility parameters. More precisely, Knapp and Stein construct complex-valued meromorphic functions γ_σ such that the operator defined by

$$\widetilde{I}_w^{\sigma,\nu} = \frac{1}{\gamma_\sigma(\nu)} I_w^{\sigma,\nu} \tag{1.3}$$

is unitary for $\nu \in i\mathbb{R}$ provided that ν is not a pole. When defined, the operators $\widetilde{I}_w^{\sigma,0}$, where $w.\sigma = \sigma$, yield the splitting of $\pi_P^{\sigma,0}$ and the coefficients $\gamma_\sigma(\nu)$ are related to the densities in Harish-Chandra's Plancherel formula.

1.3. **Induction Hilbert modules.** The theory of induced representations of C*-algebras originated in M.A. Rieffel's seminal work [17]. It relies on the use of bimodules over the C*-algebras of the ambient and inducing group and contains Mackey's theory of induced representations of locally compact groups, seen at the level of group C*-algebras, as a special case. One advantage of using Hilbert modules to describe the induction process is for instance the neat expression of Mackey's Imprimitivity Theorem in terms of strong Morita equivalence and crossed product.

However, a direct application of Rieffel's theory in the situation depicted in the previous paragraph fails to enclose all of its specifities. More precisely, it yields a $\mathrm{C}^*(P)$-module, whereas the results of Bruhat discussed above and Remark 1 indicate that the relevant parameter space for the principal series is the dual of the Θ-stable Levi component L of P, modulo the action of W. It suggests that principal series should be induced by a Hilbert module over $\mathrm{C}^*(L)$. Following this idea, a slight generalisation of Rieffel's construction was obtained in [3, 4], which leads in particular to a $\mathrm{C}^*(L)$-module $\mathcal{E}(G/N)$ providing an accurate description of the principal series induced from P. For the convenience of the reader, we recall here some of the properties of this object which will be of use in what follows. Proofs and details can be found in [4].

The Hilbert module $\mathcal{E}(G/N)$ is obtained as the completion of $C_c(G/N)$ with respect to certain inner product taking values in $C_c(L) \subset \mathrm{C}^*(L)$. Explicit formulas will be given in the case of $\mathrm{SL}_2(\mathbb{R})$ in Paragraph 2.3. Beside the right $\mathrm{C}^*(L)$-module structure, $\mathcal{E}(G/N)$ also carries an action by linear bounded (in fact compact) operators of $\mathrm{C}^*(G)$, that factorises through $\mathrm{C}^*_r(G)$. Eventually, a goal is to characterize the image of the morphism

$$\mathrm{C}^*_r(G) \longrightarrow \mathcal{K}_{\mathrm{C}^*(L)}\left(\mathcal{E}(G/N)\right),$$

for it is expected to be complemented and help understand the structure of $\mathrm{C}^*_r(G)$ in relation with the representation theory of G, as advocated in [4].

1.3.1. *Induction and generic irreducibility.* The first feature of Rieffel's modules consists in the way they implement the induction functor by taking tensor products. This still holds with $\mathcal{E}(G/N)$: for any $\sigma, \nu \in \widehat{L}$, there is a unitary map

$$\mathcal{E}(G/N) \otimes_{\mathrm{C}^*(L)} \mathcal{H}_{\sigma\otimes\nu} \xrightarrow{\sim} \mathcal{H}_{\pi_P^{\sigma,\nu}}$$

which intertwines the left actions of $\mathrm{C}^*_r(G)$ on both Hilbert spaces. It is in this sense that $\mathcal{E}(G/N)$ will be considered to globally enclose the principal series representations.

As an example of the way in which classical properties of principal series reflect at the level of Hilbert modules, it was proved in [4] that the commutant of $\mathrm{C}^*_r(G)$ in

$\mathcal{L}_{\mathrm{C}^*(L)}\left(\mathcal{E}(G/N)\right)$ reduces to the center of the multiplier algebra of $\mathrm{C}^*(L)$, acting by right multiplications. These operators correspond to the homotheties of the module $\mathcal{E}(G/N)$, so that the result may be interpreted as a generic irreducibility statement in this global context.

1.3.2. *Open picture.* The existence of an open Bruhat cell manifests at the Hilbert module level through an isometry

$$\mathcal{E}(G/N) \simeq L^2(\bar{N}) \otimes \mathrm{C}^*(L) \tag{1.4}$$

of $\mathrm{C}^*(L)$-modules. This isomorphism allows to transport the left $\mathrm{C}^*(G)$-structure on $\mathcal{E}(G/N)$ naturally coming from the action $G \curvearrowright G/N$, to $L^2(\bar{N})\otimes\mathrm{C}^*(L)$ although G does not act on $\bar{N}MA$. A practical consequence of (1.4) is that $C_c(\bar{N}) \otimes C_c(L)$ may be used as a dense subset of $\mathcal{E}(G/N)$. The correspondence between this isomorphism and the classical non-compact picture of the principal series is discussed in [4].

1.3.3. *Intertwining integrals.* Another property of the generalised induction modules describing the principal series is the possibility to define intertwining integrals rather easily on a dense subset. More precisely, the convergence of

$$\mathcal{I}_w\, f(g) = \int_{\bar{N}} f(xw\bar{n})\, d\bar{n} \tag{1.5}$$

is established in [4] for any compactly supported continuous function f on G/N. This integral is the analogue of the standard operator $I_w^{\sigma,\nu}$ in Knapp-Stein theory. However, although $\mathcal{I}_w$ defines a map $C_c(G/N) \longrightarrow C(G/N)$, it does not extend to $\mathcal{E}(G/N)$, the obstruction being concentrated in a certain distribution T_w, discussed in the appendix.

1.4. **Normalisation: principle.** Let us discuss the linearity properties of the operator

$$\mathcal{I}_w : C_c(G/N) \longrightarrow C(G/N).$$

Both the source and the target of this map are equipped with the same right module structure over $C_c(L)$, obtained by integrating the action of L on G/N. Besides, the Weyl element w acts by conjugation on $C_c(L)$: for $\varphi \in C_c(L)$, we denote $\varphi^w : l \mapsto \varphi(w^{-1}lw)$, and an easy computation shows that

$$\mathcal{I}_w(f.\varphi) = \mathcal{I}_w(f).\varphi^w. \tag{1.6}$$

The action of w extends to an involutive automorphism $a \mapsto a^w$ of $\mathrm{C}^*(L)$, which defines another $\mathrm{C}^*(L)$-Hilbert module structure on $\mathcal{E}(G/N)$. More precisely for $\xi, \eta \in \mathcal{E}(G/N)$ and $a \in \mathrm{C}^*(L)$, we define

$$\xi ._w a = \xi . a^w \tag{1.7a}$$

$$\langle \xi, \eta\rangle_w = \left(\langle \xi, \eta\rangle\right)^w. \tag{1.7b}$$

Notation. The module thus obtained will be denoted by $\mathcal{E}(G/N)^w$ and the subscript w will usually be omitted when letting elements of $\mathrm{C}^*(L)$ act from the right.

We are now ready to state the properties that an operator should satisfy in order to be considered as a normalisation of $\mathcal{I}_w$, by analogy with (1.3):

Definition 1. An operator $\mathcal{U}_w$ shall be said to *normalise* $\mathcal{I}_w$ if:

(i) $\mathcal{U}_w$ is a unitary operator between $\mathcal{E}(G/N)^w$ and $\mathcal{E}(G/N)$,

(ii) there exists a continuous function $\gamma : L \longrightarrow \mathbb{C}$ such that the composition $\mathcal{I}_w \circ \mathcal{U}_w$ acts on a dense subspace of functions in $\mathcal{E}(G/N)^w$ by right convolution with γ over L.

The central result in this paper is an explicit construction in the case of $\mathrm{SL}_2(\mathbb{R})$ of an operator $\mathcal{U}_w$ together with a function γ satisfying the properties of the above definition.

1.5. **General notations.** The proof of our main result essentially relies on the application of a certain Fourier transform on an appropriate subspace of functions in $\mathcal{E}(G/N)^w$. Following classical texts, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions over $\mathbb{R}^n$. We also consider the space $\mathcal{S}_0(\mathbb{R}^n)$ of Schwartz functions all of whose derivatives vanish at 0. Finally, δ denotes the usual Dirac distribution supported at 0.

We shall use the following normalisation of the Fourier transform on $\mathbb{R}^n$:

$$\mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi\langle x,\xi\rangle}\, dx,$$

for any $f \in \mathcal{S}(\mathbb{R}^n)$. The action of real numbers by dilation will be denoted in the following way: if f is a Schwartz function, then for any real number $\alpha \neq 0$, we define f^α by

$$f^\alpha(x) = f(\alpha x)$$

for $x \in \mathbb{R}^n$. Then, the well-known behaviour of the Fourier transform under dilations may be expressed as

$$\mathcal{F}(f^\alpha) = |\alpha|^{-n} (\mathcal{F} f)^{\frac{1}{\alpha}}. \tag{1.8}$$

2. Case of $\mathrm{SL}_2(\mathbb{R})$

From now on, G shall denote the group $\mathrm{SL}_2(\mathbb{R})$ of real matrices of size 2, with determinant 1:

$$G = \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ,\ ad - bc = 1 \right\}.$$

2.1. **Structure and identifications.** Let Θ be the Cartan involution defined on G by $\Theta(g) = {}^t g^{-1}$. The corresponding maximal compact subgroup is $K = \mathrm{SO}(2)$ and G admits an Iwasawa decomposition KAN with

- K identified to $\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}^1$ by $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \longmapsto \theta$,
- A identified to $\mathbb{R}_+^\times$ by $a_\rho = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix} \longmapsto \rho$,
- N identified to $\mathbb{R}$ by $n_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \longmapsto s$.

Denoting by 1 the identity matrix, and by w the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the normaliser and the centraliser of A in K respectively appear as $M' = \{\pm 1, \pm w\}$ and $M = \{\pm 1\}$. The standard Borel subgroup P of upper triangular matrices then admits the Langlands decomposition $P = MAN$ and its Θ-stable Levi component $L = MA$ identifies to $\mathbb{R}^\times$. It also follows that w is a representative of the non-trivial generator of the Weyl group $W = M'/M$. The subgroup $\bar{N} = \Theta(N)$ also identifies to $\mathbb{R}$ *via*

$$\bar{n}_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \longmapsto t.$$

Under these identifications, one has for $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a \neq 0$,

$$\begin{aligned} \bar{\boldsymbol{n}}(g) &= \frac{c}{a} \\ \boldsymbol{m}(g) &= \frac{a}{|a|} \\ \boldsymbol{a}(g) &= |a| \\ n_g &= \frac{b}{a} \\ \boldsymbol{l}(g) &= a. \end{aligned}$$

Finally, the element ρ_N acts on the Lie algebra of A by $\rho_N(\log(a_\rho)) = \rho$.

2.2. **Topology and measure on G/N.** The Iwasawa decomposition implies that G/N topologically identifies to $KA \simeq \mathbb{S}^1 \times \mathbb{R}_+^\times \simeq \mathbb{R}^2 \setminus \{0\}$. More precisely, if x is the column matrix associated to a non-zero vector in the euclidean plane, with polar coordinates (ρ, θ), then

$$R_\theta . a_\rho = \left[\begin{array}{c|c} x & w.\dfrac{x}{\|x\|^2} \end{array} \right],$$

so that x may be seen as a representative of $R_\theta . a_\rho$ in G/N. Under this identification, the left action of an element $g \in G$ on G/N is the linear action on the plane, denoted by $x \mapsto g.x$.

Moreover, using the decomposition (1.1) of the measure, it appears that the G-invariant measure $e^{2\rho_N \log a} dk\, da$ on G/N expresses as $\rho^2\, d\theta\, \frac{d\rho}{\rho} = \rho\, d\rho\, d\theta$, that is the restriction to $\mathbb{R}^2 \setminus \{0\}$ of the Lebesgue measure on the euclidean plane.

2.3. **The Hilbert modules $\mathcal{E}$ and $\mathcal{E}^w$.** Under the above identifications, the modules $\mathcal{E} = \mathcal{E}(G/N)$ and $\mathcal{E}^w = \mathcal{E}(G/N)^w$ of Paragraph 1.3, initially defined in [4] as completions of $C_c(G/N)$, can be recovered from spaces of functions over the euclidean plane. For instance, it will appear that functions in $\mathcal{S}_0(\mathbb{R}^2)$ seen as functions on $\mathbb{R}^2 \setminus \{0\}$ define elements in $\mathcal{E}$ and $\mathcal{E}^w$.

Let f, g be functions on $\mathbb{R}^2$ and α a real number. We define

$$\langle f, g \rangle(\alpha) = |\alpha| \int_{\mathbb{R}^2} \overline{f(x)} g(\alpha x)\, dx \tag{2.1}$$

whenever it makes sense. Since the measure on $G/N \simeq \mathbb{R}^2 \setminus \{0\}$ is the restriction of the Lebesgue measure on the plane, for which singletons have measure 0, it is easily seen that formula (2.1) coincides with the definition of the inner product defined in [4] in the case of $\mathrm{SL}_2(\mathbb{R})$ for $\alpha \in \mathbb{R}^\times$ when f, g have compact supports away from 0.

In the same way, the formulas defining the respectively left and right actions of G and $L = \mathbb{R}^\times$ on $\mathcal{E}$ can be defined for a wide class of functions: for $f \in \mathcal{S}(\mathbb{R}^2)$,

$g \in \mathrm{SL}_2(\mathbb{R})$ and $l \in \mathbb{R}^\times$, we let

$$g.f = f\left(g^{-1}\cdot\right) \tag{2.2}$$

$$f.l = |l|^{-1} f^{\frac{1}{l}}. \tag{2.3}$$

Again, when applied to compactly supported functions, these formulas are the same that define the pre-Hilbert bimodule structure on $C_c(G/N)$.

The corresponding formulas for $\mathcal{E}^w$ are, according to (1.7):

$$\langle f, g\rangle_w(l) = \langle f, g\rangle(l^{-1}) = |l|^{-1}\int_{\mathbb{R}^2} \overline{f(x)} g(l^{-1}x)\, dx \tag{2.4a}$$

$$f\cdot_w l = |l| f^l, \tag{2.4b}$$

while the left action of G in the same as (2.2).

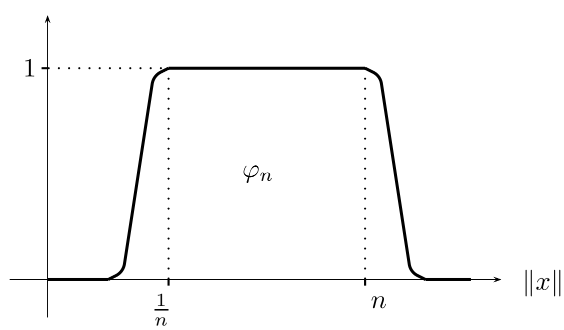
Definition 2. The Hilbert module $\mathcal{E}$ (resp. $\mathcal{E}^w$) is obtained from $C_c(\mathbb{R}^2 \setminus \{0\})$ by integrating and extending the action (2.3) (resp. (2.4b)) to $\mathrm{C}^*(L)$, then completing with respect to the norm induced by (2.1) (resp. (2.4a)). The action of $\mathrm{C}^*_r(G)$ is obtained from (2.2) by integration over G.

This definition is a special case of the construction in [4]. Further, we will need the following characterisation of elements in $\mathcal{E}$ and $\mathcal{E}^w$ by an estimate of the $\mathrm{C}^*(L)$-valued norm.

Proposition 1. *Let $f \in \mathcal{S}\left(\mathbb{R}^2\right)$. If $\langle f, f\rangle \in \mathcal{S}_0(\mathbb{R})$, then f defines elements in $\mathcal{E}$ and $\mathcal{E}^w$.*

Proof. Let $f \in \mathcal{S}\left(\mathbb{R}^2\right)$ be such that $\langle f, f\rangle \in \mathcal{S}_0(\mathbb{R})$ and let $\{f_n\}_{n\geq 1}$ be the family of truncations defined by $f_n = f.\chi_n$, where for any integer n, the function χ_n is chosen on $\mathbb{R}^2 \setminus \{0\}$

- smooth,
- compactly supported,
- vanishing on a neighbourhood of the origin,
- constantly equal to 1 on the annulus $\left\{\frac{1}{n} \leq \|x\| \leq n\right\}$.



For any n, the truncated function f_n is compactly supported on $\mathbb{R}^2 \setminus \{0\}$. We will prove that $\langle f - f_n, f - f_n\rangle$ converges to 0 with respect to the norm of $\mathrm{C}^*(L)$, so that $\{f_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{E}$, the limit of which can be identified with f.

It follows from the definition (2.1) that the support of $\langle f, g\rangle$ is a compact of $L \simeq \mathbb{R}^\times$ whenever f or g is compactly supported. Then, by sesquilinearity of the inner product, $\langle f - f_n, f - f_n \rangle$ belongs to $\mathcal{S}_0(\mathbb{R})$, hence can be seen as an element of $\mathrm{C}^*(L)$, denoted by φ_n. In order to prove that f_n converges to f in $\mathcal{E}$, it suffices to prove that $\{\varphi_n\}_{n\geq 1}$ converges to 0 in $L^1(\mathbb{R}^\times)$. By definition, for $l \in \mathbb{R}^\times$, one has

$$\varphi_n(l) = |l| \int_{\{\|x\|<\frac{1}{n}\}\cup\{\|x\|>n\}} \overline{(f-f_n)(x)}\,(f-f_n)(lx)\,dx.$$

It follows that

$$\begin{aligned} |\varphi_n(l)| &\leq |l|\,.\,\|f\|_\infty \left(\int_{\{\|x\|<\frac{1}{n}\}} |f(lx)|\,dx + \int_{\{\|x\|>n\}} |f(lx)|\,dx \right) \\ &\leq \frac{\|f\|_\infty}{|l|} \left(\int_{\{\|x\|<\frac{|l|}{n}\}} |f(x)|\,dx + \int_{\{\|x\|>|l|n\}} |f(x)|\,dx \right) \end{aligned}$$

Since l is fixed, the limit as $n \to \infty$ of the first summand is 0 by continuity of f. The same holds for the second summand because f is rapidly decreasing, hence integrable. It follows that $\{\varphi_n\}_{n\geq 1}$ converges pointwise to 0.

Let $\varphi(l) = |\langle f, f\rangle(l)|$. Then φ_n satisfies $|\varphi_n| \leq \varphi$ for any $n \geq 1$. Moreover, for $l \neq 0$,

$$\begin{aligned} \left|\frac{\varphi(l)}{l}\right| &\leq \int_{\mathbb{R}^2} \left|\overline{f(x)} f(lx)\right|\,dx \\ &\leq \|f\|_\infty \int_{\mathbb{R}^2} |f(lx)|\,dx \leq \frac{\|f\|_\infty^2}{|l|^2}. \end{aligned}$$

so that φ is integrable as $|l| \to \infty$ with respect to the Haar measure $\frac{dl}{|l|}$ of $\mathbb{R}^\times$. By invariance under inversion $l \mapsto l^{-1}$, the same holds as $|l| \to 0$ so $\varphi \in L^1(\mathbb{R}^\times)$. Then Lebesgue's Dominated Convergence Theorem implies that $\varphi_n \xrightarrow{L^1} 0$, which concludes the proof. The statement relative to $\mathcal{E}^w$ is proved along the exact same lines using formulas (2.4a) and (2.4b). $\square$

2.4. **Intertwining integrals.** Let us now turn to the operator $\mathcal{I}_w$. With the notations of Paragraph 2.2, for $x \in \mathbb{R}^2 \setminus \{0\}$ and $t \in \mathbb{R}$,

$$\left[\begin{array}{c|c} x & w.\dfrac{x}{\|x\|^2} \end{array}\right] w\bar{n}_t = \left[\begin{array}{c|c} -tx + w.\frac{x}{\|x\|^2} & -x \end{array}\right]$$

so that (1.5) becomes

$$\mathcal{I}_w f(x) = \int_{-\infty}^{+\infty} f\left(tx + w.\frac{x}{\|x\|^2}\right) dt, \tag{2.5}$$

and $\mathcal{I}_w$ can be seen to be defined by the kernel

$$K_{\mathcal{I}}(x,z) = \int_{\mathbb{R}} \delta_0 \left(z - tx - w.\frac{x}{\|x\|^2}\right) \tag{2.6}$$

3. Normalisation

Let $\mathcal{F}_w$ be the operator defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}_w\, f(x) = \int_{\mathbb{R}^2} e^{-2i\pi\langle w.y,x\rangle} f(y)\, dy = \mathcal{F}\, f(w^{-1}.x).$$

We denote by K_w the kernel defining this operator, that is

$$K_w(z,y) = e^{-2i\pi\langle w.y,z\rangle}. \tag{3.1}$$

Theorem 1. *The operator $\mathcal{F}_w$ extends to an operator $\mathcal{U}_w$ on $\mathcal{E}^w$ such that:*
(1) $\mathcal{U}_w$ normalises $\mathcal{I}_w$,
(2) the corresponding function γ is given on $L \simeq \mathbb{R}^\times$ by

$$\gamma(l) = e^{-\frac{2i\pi}{l}}.$$

3.1. **Proof of Theorem 1.** In view of Definition 2, linearity with respect to $\mathrm{C}^*(L)$ reduces to equivariance with respect to $L = \mathbb{R}^\times$, which in turn follows directly from the definitions (2.3) and (2.4b) together with the property (1.8) of the Fourier transform. Let $f \in \mathcal{S}_0\left(\mathbb{R}^2\right)$. Then, given a fixed $l \in \mathbb{R}^\times$,

$$\begin{aligned}
\langle \mathcal{F}_w\, f, \mathcal{F}_w\, f\rangle(l) &= |l|\langle \mathcal{F}\, f, (\mathcal{F}\, f)^l\rangle_{L^2} = |l|^{-1}\langle \mathcal{F}\, f, \mathcal{F}\left(f^l\right)\rangle_{L^2} && \text{by (1.8)}\\
&= |l|^{-1}\langle f, f^l\rangle_{L^2} && \text{by the Plancherel formula on } \mathbb{R}^2\\
&= \langle f, f\rangle_w(l)
\end{aligned}$$

Starting with a function f satisfying the hypotheses in Proposition 1, the above computation proves that $\mathcal{F}_w$ maps $\mathcal{E}^w$ to $\mathcal{E}$. Formulas (2.6) and (3.1) imply that the composition $\mathcal{I}_w \circ \mathcal{F}_w$ is defined by the kernel K:

$$\mathcal{I}_w \circ \mathcal{F}_w\, f(x) = \int_{\mathbb{R}^2} K(x,y) f(y)\, dy$$

where

$$K(x,y) = \int_{\mathbb{R}^2} K_{\mathcal{I}}(x,z) K_w(z,y)\, dz,$$

that is

$$K(x,y) = \int_{\mathbb{R}^2}\int_{\mathbb{R}} e^{-2i\pi\langle w.y,z\rangle}\delta_0\left(z - tx - w.\frac{x}{\|x\|^2}\right) dt\, dz.$$

For any vector $x \neq 0$ of the plane, we denote by $\widetilde{x}$ its projection on the euclidean sphere, so that $x = \|x\|\widetilde{x}$. Then,

$$\begin{aligned}
K(x,y) &= \int_{\mathbb{R}} e^{-2i\pi\langle w.y, tx + w.\frac{x}{\|x\|^2}\rangle}\, dt\\
&= e^{-2i\pi\langle w.y, w.x\rangle\|x\|^{-2}}\int_{\mathbb{R}} e^{-2i\pi t\langle w.y,x\rangle}\, dt\\
&= \|x\|^{-1}\|y\|^{-1} e^{-2i\pi\langle y,x\rangle\|x\|^{-2}}\,\widehat{\mathbf{1}_{\mathbb{R}}}\left(\langle w.\widetilde{y}, \widetilde{x}\rangle\right)
\end{aligned}$$

so that

$$K(x,y) = \frac{e^{-2i\pi\langle y,x\rangle\|x\|^{-2}}}{\|x\|\|y\|}\delta\left(\langle w.\widetilde{y}, \widetilde{x}\rangle\right).$$

A vector x being fixed, T_x denotes the distribution $\delta(\langle \cdot, \widetilde{x}\rangle)$, that is if φ is a Schwartz function, $T_x(\varphi)$ is the integral of φ along the orthogonal of x, denoted by $x^\perp$:

$$T_x(\varphi) = \int_{x^\perp} \varphi.$$

It follows that

$$\begin{aligned}
\mathcal{I}_w \circ \mathcal{F}_w\, f(x) &= \|x\|^{-1} \int_{\mathbb{R}x} e^{-2i\pi\langle y,x\rangle \|x\|^{-2}} f(y) \frac{dy}{\|y\|} \\
&= \|x\|^{-1} \int_{\mathbb{R}} e^{-2i\pi\langle \lambda\widetilde{x},x\rangle \|x\|^{-2}} f(\lambda\widetilde{x}) \frac{d\lambda}{|\lambda|} \\
&= \|x\|^{-1} \int_{\mathbb{R}} e^{-2i\pi\lambda \|x\|^{-1}} f(\lambda\widetilde{x}) \frac{d\lambda}{|\lambda|} \\
&= \int_{\mathbb{R}^\times} e^{-2i\pi\lambda} f(\lambda x)\, d^\times \lambda,
\end{aligned}$$

where $d^\times\lambda$ denotes the classical Haar measure on $\mathbb{R}^\times$, which makes the composition a convolution of the form $f *_{\mathbb{R}^\times} \gamma$, where γ is the expected function.

3.2. **Concluding remark.** A normalisation result has already been obtained for $\mathrm{SL}_2(\mathbb{R})$ in the author's thesis [3] by means of functionnal calculus and differential operators. It seems to us that the point of view adopted in this article is more natural, for it relies on a more geometric interpretation of the space G/N. The approach to Knapp-Stein operators as geometric integral transforms seems to originate in the work of A. Unterberger (see [18, 16]) in a context of geometrical analysis. It has been promoted in various recent works in representation theory, especially in the case of degenerate principal series. For instance, they were obtained *via* Radon transforms on light cones in [11], understood in terms of $\cos^\lambda$ transformations in [15] and normalised by symplectic Fourier transforms in [12] and [5].

4. Appendix: residual intertwining distribution and reducibility

The purpose of this rather independant part is to suggest that the information about reducibility points in the principal series contained in the normalising factors of the classical Knapp-Stein operators can be extracted from the non-normalised operators $\mathcal{I}_w$. The algebraic similarities make it natural to deal with $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ at the same time. In particular, the formulas and identifications of Paragraph 2.1 hold for complex matrices. The main difference in the complex case is that the group M identifies to $\mathrm{U}(1)$. The analysis of $\mathcal{I}_w$ is done in the open picture of Paragraph 1.3.2.

4.1. **Open picture and truncation.** Whether in the classical context or in the C*-algebraic framework, one of the advantages of working in the open picture is that it essentially reduces to some analysis on $\bar{N}$, where the action of A by dilations can be quantitatively measured. More precisely, there exists a *norm function* $|\cdot|$ on $\bar{N} \setminus \{1\}$ defined by

$$|\bar{n}| = e^{\rho_N \log \boldsymbol{a}(w^{-1}\bar{n})}.$$

This norm function satisfies homogeneity properties under dilations by the elements of A (see [9, p.513]).

An expression for $\mathcal{I}_w$ in the open picture was obtained in [3]. For an elementary tensor $f \otimes \varphi$ and an element $x_0 = \bar{n}_0 l_0 = \bar{n}_0 m_0 a_0 \in \bar{N}L$, it is given by

$$\mathcal{I}_w(f\otimes\varphi)(x_0)=\int_{\overline{N}} f(\bar n_0\bar n)\left[U_{\boldsymbol{l}(w^{-1}\bar n)}\varphi\right]^w(l_0)\,\frac{d\bar n}{|\bar n|}$$

where U_γ denotes the multiplier associated to an element γ in a group Γ acting on $\mathrm{C}^*(\Gamma)$ by the formula $U_\gamma.\varphi=\varphi(\gamma^{-1}\cdot)$ for $\varphi\in C_c(\Gamma)$, and $\varphi^w=\varphi\circ c_w$ where c_w is the conjugation by w.

Let us consider the following decomposition of the integral:

$$\mathcal{I}_w=\mathcal{I}_w^0+\mathcal{I}_w^\infty+\mathcal{R}_w$$

with

$$\mathcal{I}_w^\infty(f\otimes\varphi)(x_0)=\int_{|\bar n|>1} f(\bar n_0\bar n)\left[U_{\boldsymbol{l}(w^{-1}\bar n)}\varphi\right]^w(l_0)\,\frac{d\bar n}{|\bar n|}$$

$$\mathcal{I}_w^0(f\otimes\varphi)(x_0)=\int_{|\bar n|<1}\left(f(\bar n_0\bar n)-f(\bar n_0)\right)\left[U_{\boldsymbol{l}(w^{-1}\bar n)}\varphi\right]^w(l_0)\,\frac{d\bar n}{|\bar n|}$$

$$(\dagger)\qquad \mathcal{R}_w(f\otimes\varphi)(x_0)=f(\bar n_0)\int_{|\bar n|<1}\left[U_{\boldsymbol{l}(w^{-1}\bar n)}\varphi\right]^w(l_0)\,\frac{d\bar n}{|\bar n|}$$

This truncation will allow to extract what prevents $\mathcal{I}_w$ to define a Hilbert module operator, precisely enough to recover the information that is encoded in the singularities of the Knapp-Stein operators, as the main result of this note shows.

It is proved in [3] that $\mathcal{I}_w^\infty$ and $\mathcal{I}_w^0$ give densely defined operators between the Hilbert modules $\mathcal{E}$ and $\mathcal{E}^w$. The remaining term $\mathcal{R}_w$ hence concentrates the singularity of the global intertwining integral.

4.2. **Residual distribution.** The first remark about $\mathcal{R}_w$ is that it acts non-trivially only on the right factor $\mathrm{C}^*(L)$ in the open picture of $\mathcal{E}$. In other terms, it is supported in L seen in $G/N\doteq\bar NL$. The following results show that $\mathcal{R}_w$ is characterised by a certain distribution on L.

Let us first recall the following lemma, proved in [3, p.97]. We sketch the proof below for the reader's convenience.

Lemma 1. *Let $T_{w,\bar n}$ be the distribution defined on L by*

$$T_{w,\bar n}(\lambda)=\delta_1(c_{w^{-1}}\left(\boldsymbol{l}(w^{-1}\bar n)^{-1}\right)\lambda).$$

The integral

$$T_w=\int_{|\bar n|<1} T_{w,\bar n}\,\frac{d\bar n}{|\bar n|}$$

then defines a Radon measure on L.

Proof. Consider $\varphi=\varphi_M\otimes\varphi_A\in C_c^\infty(M)\otimes C_c^\infty(A)$. By definition of $T_{w,\bar n}$,

$$\begin{aligned}T_{w,\bar n}(\varphi)&=\varphi_M\left(c_w(\boldsymbol{m}(w^{-1}\bar n))\right)\varphi_A\left(c_w(\boldsymbol{a}(w^{-1}\bar n))\right)\\&=\varphi_M\left(c_w(\boldsymbol{m}(w^{-1}\bar n))\right)\varphi_A\left(\boldsymbol{a}(w^{-1}\bar n)^{-1}\right)\end{aligned}$$

Elementary properties of the projection $\boldsymbol{m}$ imply that the first factor is homogeneous of degree 0 under dilations, while the second one only depends on $|\bar n|$. More precisely, $\varphi_A\left(\boldsymbol{a}(w^{-1}\bar n)^{-1}\right)=\varphi_A(|\bar n|^c)$ for some positive constant c depending on the identification between the Lie algebra of A and $\mathbb{R}$. One can then apply the integration formula of [9, Proposition 3, p.496], which gives

$$T_w(\varphi) = C.M_w(\varphi_M)\int_0^{+\infty} \varphi_A(r)\,\frac{dr}{r}$$

where C is a constant and $M_w(\varphi_M)$ denotes the mean value over the unit sphere defined relative to $|\bar{n}|$ of the function $\bar{n} \mapsto \varphi_M\left(c_w(\boldsymbol{m}(w^{-1}\bar{n}))\right)$. A straightforward estimate relying on the same integral formula proves that this mean value defines a Radon measure on M. The distribution T acts on φ_A like the Haar measure, hence the result. □

Plugging the formula defining T_w into the expression (†) of the residual part of the global intertwining integral, one immediately obtains the following characterisation of $\mathcal{R}_w$ as a convolution operator by this distribution:

Proposition 2. *For any $f \otimes \varphi \in L^2(\bar{N}) \otimes C_c(L)$,*

$$\mathcal{R}_w(f \otimes \varphi) = f \otimes (T_w * \varphi^w)\,.$$

From now on, T_w will be called *the residual distribution* associated to $\mathcal{I}_w$.

4.3. **Principal series of $\mathrm{SL}_2(k)$.** Let k be $\mathbb{R}$ or $\mathbb{C}$. In the real case, $\widehat{M} = \{\mathbf{1}, \varepsilon\}$ where $\mathbf{1}$ is the one-dimensional trivial representation and $\varepsilon(\pm I_2) = \pm 1$. In the complex case, $\widehat{M} = \{\varphi_n, n \in \mathbb{Z}\} \simeq \mathbb{Z}$ where $\varphi_n(z) = z^n$. In both cases, A identifies to $\mathbb{R}_+^\times$, hence $\widehat{A} = \{\nu_t, t \in \mathbb{R}\} \simeq \mathbb{R}$ where $\nu_t(a) = a^{2i\pi t}$.

The action of w by conjugation is given as follows:

$$\begin{aligned} w.\varepsilon &= \varepsilon \\ w.\mathbf{1} &= \mathbf{1} \\ w.\varphi_n &= \varphi_{-n} \\ w.\nu_t &= \nu_{-t}. \end{aligned}$$

It follows that the Weyl-fixed points in $\widehat{L}$ are

- $(\mathbf{1}, \varphi_0) = (\mathbf{1}, \mathbf{1})$ and $(\varepsilon, \varphi_0) = (\varepsilon, \mathbf{1})$ for $\mathrm{SL}_2(\mathbb{R})$,
- $(\mathbf{1}, \varphi_0) = (\mathbf{1}, \mathbf{1})$ for $\mathrm{SL}_2(\mathbb{C})$.

Applying the theory of Knapp and Stein (see [8]), one obtains that

- $\pi^{\mathbf{1},\mathbf{1}}$ is irreducible on $\mathrm{SL}_2(\mathbb{R})$,
- $\pi^{\varepsilon,\mathbf{1}}$ is reducible,
- $\pi^{\mathbf{1},\mathbf{1}}$ is irreducible on $\mathrm{SL}_2(\mathbb{C})$.

The first and third points are in fact examples of the more general phenomenon discovered by B. Kostant [13] that principal series of the form $\pi^{\mathbf{1},\nu}$ are always irreducible. On the other hand, the second point can be made more precise: $\pi^{\varepsilon,\mathbf{1}}$ splits into the direct sum of the so-called *limits of the discrete series*, which is a special case of the Schmid equality (see [8]). Finally, the third point should also be seen as a manifestation of a structural fact about $\mathrm{SL}_2(\mathbb{C})$: N. Wallach proved in [19] that complex groups, or more generally groups having exactly one conjugacy class of Cartan subgroups, only admit irreducible principal series representations. This situation corresponds to the case of a group G having a Hausdorff tempered dual $\widehat{G}_r$, in which case the reduced C*-algebra is stably equivalent to $C_0(\widehat{G}_r)$.

4.4. **Detection of the reducibility parameters.** Let us now state the main result of this appendix. We denote by $\mathcal{F}_H$ the Fourier transform defined on characters of an abelian group H. The other notations are the same as above.

Theorem 2. *Let $(\sigma, \nu) \in \widehat{L}$ be Weyl-fixed and T_w the distribution on L associated to the residual part of the global interwining integral. The principal series representation $\pi^{\sigma,\nu}$ is reducible if and only if $\mathcal{F}_L\, T_w(\sigma, \nu) = 0$.*

The remainding paragraphs consist in a verification of this statement by determining the Fourier transform of the residual distribution associated to $\mathcal{I}_w$ in the cases of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$.

4.4.1. *Determination of T_w.* Using the notations and identifications relative to the projections with respect to $\bar{N}$, M and A of Paragraph 2.1, one observes for $\bar{n}_t \in \bar{N}$ that $w^{-1}\bar{n}_t = \begin{bmatrix} -t & -1 \\ 1 & 0 \end{bmatrix}$, hence if $t \neq 0$,

$$\boldsymbol{l}(w^{-1}\bar{n}_t) = -t$$
$$\boldsymbol{a}(w^{-1}\bar{n}_t) = |t|$$

and the norm function on $\bar{N}$ coincides with the absolute value on k. Moreover, for $\lambda \in k^\times$,

$$c_{w^{-1}}\left(\boldsymbol{l}(w^{-1}\bar{n}_t)^{-1}\right)\lambda = \lambda t,$$

so that if φ is a test function on L, then

$$T_w\varphi = \int_{|t|<1} \varphi\left(-\frac{1}{t}\right)\frac{dt}{|t|} = \int_{|t|>1} \varphi(t)\,\frac{dt}{|t|}.$$

The common expression for the residual distribution in both the real and the complex case hence is

$$T_w = 1_{\{|x|>1\}}$$

where $|\cdot|$ denotes the usual absolute value on $\mathbb{R}$ or $\mathbb{C}$ and 1_X the characteristic function of the set X.

4.4.2. *Fourier transform of T_w.* Let us recall the expressions of Fourier transforms on the multiplicative abelian groups $\mathbb{R}^\times \simeq \{\pm 1\} \times \mathbb{R}_+^*$ and $\mathbb{C}^\times \simeq \mathbb{S}^1 \times \mathbb{R}_+^*$. With the notations introduced above for characters, one has, for a function S,

$$\mathcal{F}_{\mathbb{R}^\times}\, S(\mathbf{1}, \nu_t) = \int_{-\infty}^{+\infty} S(x)\overline{\mathbf{1}\left(\frac{x}{|x|}\right)|x|^{-2i\pi t}}\,\frac{dx}{|x|} = \int_{-\infty}^{+\infty} S(x)|x|^{-2i\pi t}\,\frac{dx}{|x|}$$
$$\mathcal{F}_{\mathbb{R}^\times}\, S(\varepsilon, \nu_t) = \int_{-\infty}^{+\infty} S(x)\overline{\varepsilon\left(\frac{x}{|x|}\right)|x|^{-2i\pi t}}\,\frac{dx}{|x|} = \int_{-\infty}^{+\infty} S(x)\mathrm{sign}(x)|x|^{-2i\pi t}\,\frac{dx}{|x|}$$
$$\mathcal{F}_{\mathbb{C}^\times}\, S(\varphi_n, \nu_t) = \int_{\mathbb{S}^1}\int_0^{+\infty} S(\theta, |x|)e^{-in\theta}|x|^{-2i\pi t}\,\frac{dx}{|x|}\,d\theta$$

and those formulas extend to distributions by duality. It follows, in the case of $\mathrm{SL}_2(\mathbb{R})$, that

- $\mathcal{F}_{\mathbb{R}^\times}\, T_w(\mathbf{1}, \nu_t) = 2.\int_1^{+\infty} |x^{-2i\pi t}|\,\dfrac{dx}{|x|} = 2.\mathcal{F}_{\mathbb{R}}\left(1_{\mathbb{R}_+}\right)(t) = \dfrac{1}{i\pi t} + \delta_0(t)$
- $\mathcal{F}_{\mathbb{R}^\times}\, T_w(\varepsilon, \nu_t) = \int_1^{+\infty} |x^{-2i\pi t}|\,\dfrac{dx}{|x|} - \int_{-\infty}^{-1} |x^{-2i\pi t}|\,\dfrac{dx}{|x|} = 0$

while in the case of $\mathrm{SL}_2(\mathbb{C})$, the same computation yields

- $\mathcal{F}_{\mathbb{C}^\times} T_w(\mathbf{1}, \nu_t) = \dfrac{1}{i\pi t} + \delta_0(t)$.

4.4.3. *Conclusion.* In view of the above discussion, it is immediate to check that the Fourier transform of the residual distribution vanishes exactly in the parameters for which the corresponding principal series splits into irreducibles. This observation suggests that this distribution really plays the same in the Hilbert module picture of principal series as the poles in the classical Knapp-Stein theory. In particular, the globality of the C*-algebraic point of view does not lead to the loss of information about individual representations. It seems plausible that Theorem 2 holds in general. If so, it would be interesting to find a direct proof, that is not involving the knowledge of the principal series resulting from the application of Knapp and Stein's theory.

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