On Some Formulas for the Characteristic Classes of Group-actions

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1. Introduction. This is an account only of the material of my last two lectures of the Rio conference, as the earlier lectures dealt with well known matters.

My emphasis here then is the study of our foliation invariants in the context of group actions on a manifold, and I will start by showing you the naturality of our basic construction now pays off by extending directly to the equivariant situation.

Recall then the main conclusions of our earlier discussions. Essentially they amounted to this (see also [1],[2] for details):

If we let JM denote the space of jets - based at $0 \in \mathbb{R}^n$ - of diffeomorphisms of $\mathbb{R}^n$ to M, then JM carries an algebra of natural forms which is isomorphic to the cochain algebra $C^*(\mathfrak{a}_n)$ where $\mathfrak{a}_n$ is the Lie algebra of formal vector-fields on $\mathbb{R}^n$.

In short there is a natural arrow:

$$C^*(\mathfrak{a}_n) \longrightarrow \operatorname*{Inv}_{\text{Diff } M} \Omega^* JM \tag{1.1}$$

of $C^*(\mathfrak{a}_n)$ onto the algebra of $\text{Diff}(M)$ - invariant forms on JM. Further we saw that although (1.1) induces the 0 - homomorphism in cohomology, it induces an interesting one once we divide JM by the natural action of O_n = the orthogonal group of $\mathbb{R}^n$.

Indeed then (1.1) induces a "basic" homomorphism:

$$C^*(\mathfrak{a}_n;O_n) \longrightarrow \operatorname*{Inv}_{\text{Diff } M} \Omega^*(JM/O_n) \tag{1.2}$$

and this map certainly recaptures all the usual characteristic classes of M.

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More precisely we saw that:

$$H^*(\mathfrak{a}_n, O_n) \simeq H^*(WO_n) \tag{1.3}$$

where WO_n is the differential algebra given by:

$$WO_n = \underline{\mathbb{R}}[c_1, \cdots, c_n] \otimes E(h_1, h_3, \cdots h_k) \quad k \text{ odd and } = n \text{ or } n-1 \tag{1.4}$$

with differential:

$$d\, c_i \otimes 1 = 0 \quad , \quad d(1 \otimes h_i) = c_i$$

and $\underline{\mathbb{R}}$ denoting the quotient of the indicated polynomial ring by the elements of $\dim > 2n$, the c_i having dimension $2i$.

Further we saw - and this is of course quite standard - that

$$H^*(\Omega^*(JM/O_n)) \simeq H^*(M)$$

so that (1.2) induces a natural map

$$H^*(\mathfrak{a}_n, O_n) \longrightarrow H^*(M) \quad ,$$

and finally we identified the image of c_{2i} under this arrow with the Pontryagin classes of M.

The main virtue of this point of view is then, that we see that a manifold determines its own Pontryagin forms naturally on the space

$$\widetilde{M} = JM/O_n \quad ,$$

which I think of as a naturally thickened version of M, and that furthermore these forms are invariant under any diffeomorphism of M.

Now let us put this construction to use when an abstract (i. e., discrete) group Γ, acts on M via diffeomorphism. These data naturally define two semisimplicial manifolds

(1.5) $$M\Gamma : M \leftleftarrows \Gamma \times M \Lleftarrow \Gamma \times \Gamma \times M$$

and

(1.6) $$B\Gamma : * \leftleftarrows \Gamma \Lleftarrow \Gamma \times \Gamma$$

whose geometric realizations then respectively correspond to, the M - bundle, $|M\Gamma|$, induced by the Γ action over the classifying space $|B\Gamma|$ of Γ, and $|B\Gamma|$ itself.

Now if

(1.7) $$X : X_0 \leftleftarrows X_1 \Lleftarrow X_2 \cdots$$

is any semisimplicial manifold the cohomology of its geometric realization $|X|$ can be computed in various ways. First of all the double complex $\Omega^{**}X$

(1.8) $$\Omega^{**}X : \Omega^*(X_0) \xrightarrow{\delta} \Omega^*(X_1) \xrightarrow{\delta} \cdots$$

obtained from (1.7) by applying the de Rham functor Ω^* to each (X_i) and then giving $\Omega^{**}X$ the sum of the de Rham differential and the differential operator δ derived from the simplicial structure, computes $H^*(|X|)$ thus:

(1.9) $$H^*(|X|) = H\{\Omega^{**}(X)\}$$

On the other hand one also has the complex of "compatible forms" on the realization of X:

(1.10) $$|X| = X_0 \cup X_1 \times \Delta_1 \cup X_2 \times \Delta_2 \cup \cdots$$

in the sense of Whitney-Thom-Sullivan. This complex is denoted by $\Omega^*|X|$, and once again one has:

(1.11) $$H^*(|X|) = H^*(\Omega^*|X|) \ .$$

For a fine account of all this I refer you to Dupont, [5].
The formula (1.9) is also to be found in [4].

In particular then, we can compute the cohomology of $M\Gamma$ from the double complex

(1.12) $$\Omega^{**}M\Gamma \ : \ \Omega^*(M) \xrightarrow{\delta} \Omega^*(M \times \Gamma) \longrightarrow$$

Furthermore as we will see later, on the first constitutent δ takes the form

(1.13) $$\delta\omega \mid M \times j = \omega \otimes 1 - j^*\omega \otimes 1 \ .$$

Hence the natural linear map

(1.14) $$\Omega^* M \longrightarrow \Omega^{**} M\Gamma$$

<u>becomes a cochain map only on the Γ - invariant forms on M</u>.

In view of this state of affairs it suggests itself to replace $M\Gamma$ by $\widetilde{M\Gamma}$.

Then the arrows (1.2) and (1.14) combine to yield a chain map

(1.15) $$C^*(a_n, O_n) \longrightarrow \Omega^*(\widetilde{M}) \longrightarrow \Omega^{**}\widetilde{M\Gamma} \ .$$

On the other hand the homotopy equivalence of M with $\widetilde{M}$ easily implies that of $M\Gamma$ and $\widetilde{M\Gamma}$. Hence in homology (1.15) induces a homomorphism

(1.16) $$H^*(a_n, O_n) \longrightarrow H^*(|M\Gamma|) \ .$$

Thus we see that the invariance of our construction immediately yields "equivariant characteristic classes" for a Γ - manifold in $|M\Gamma|$. And here of course, as the cohomology of $M\Gamma$ has no a priori bound, all the classes of WO_n potentially come into play.

Finally if M is a compact Γ - orientable manifold we can define the equivariant characteristic numbers of M by following (1.16) with integration over the fiber in the fibering

$$\begin{array}{c} M\Gamma \\ \big\downarrow \pi \\ B\Gamma \end{array} .$$

There results an additive homomorphism

$$H^*(\mathfrak{a}_n; O_n) \longrightarrow H^*(B\Gamma) \quad , \tag{1.17}$$

and my aim in the next sections will be to derive some explicit recipes for (1.17), and to review some of Thurston's examples in this framework.

2. Formulas for the Godbillon-Vey Class in $H^*(B\Gamma)$. Note that (1.16) can also be thought of in this manner.

The discreteness of Γ naturally defines a folitation $\mathfrak{F}_\Gamma$ on $M\Gamma$ which is transversal to the fibers in the projection $M\Gamma \to B\Gamma$, and of codimension n. The characteristic classes of $\mathfrak{F}_\Gamma$ integrated over the fiber induce (1.16). With this interpretation it suggests itself that the constructions which are known to represent the characteristic classes of folitations should extend to the semisimplicial situation provided only that one has a suitable de Rham theory at hand. Now the double complex $\Omega^{**}X$ does not fit the bill because its multiplication is not anticommutative. On the other hand the compatible complex of Dupont is perfectly suitable and therefore yields explicit recipes quite readily.

Let me start with the Godbillon-Vey class ω, itself. This invariant - corresponding to a generator of $H^3(\mathfrak{a}_1, O_1)$ - is defined on oriented foliations $\mathfrak{F}$ of codimension one, and can be computed according to the algorithm:

Let $\mathfrak{F}$ be described as the kernel of a non-degenerate 1 - form φ. Then integrability implies that there exists a 1 - form η with

$$d\varphi = \eta \wedge \varphi \quad .$$

(2.1) Now set

$$\omega = \eta \wedge d\eta \quad .$$

Then $d\omega = 0$. Further the cohomology class of ω is independent of the choices involved, and represents $\omega(\mathfrak{F})$.

The extension of (2.1) to the semisimplicial case is now immediate.

Given a s. s. manifold

$$X : X_0 \leftleftarrows X_1 \; \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \; X_2 \cdots$$

a foliation $\mathfrak{F}$ on X is simply a foliation $\mathfrak{F}_k$ on each X_k, such that, all the structure maps:

$$X(\alpha) \; : \; X_k \longrightarrow X_{k'}$$

are transversal to $\mathfrak{F}_{k'}$ and induce isomorphisms

$$\mathfrak{F}(\alpha) \; : \; \mathfrak{F}_k \longrightarrow X(\alpha)^{-1} \mathfrak{F}_{k'} \quad .$$

Such data then define a foliation $|\mathfrak{F}|$ on the geometric realization $|X|$ of X, in the following manner:

On $X_k \times \Delta^k$, the natural projection

$$X_k \xleftarrow{\pi_L} X_k \times \Delta^k \tag{2.2}$$

induces the foliation $\pi_L^{-1} \circ \mathfrak{F}_k$, and these foliations are compatible under the identifications which assemble the $X_k \times \Delta^k$ to form $|X|$. Precisely we have in mind here the so-called "Fat realization" of $|X|$, which therefore only identifies by the boundary maps. Thus if α is such a boundary map, and

$$\Delta(\alpha) : \Delta^{k-1} \longrightarrow \Delta^k$$

the corresponding map of the $k - 1$ simplex Δ^{k-1} onto a face of Δ^k, then the identification corresponding to α is described by the diagram:

$$\begin{tikzcd} X_k \times \Delta^{k-1} \arrow[r, "1\times\Delta(\alpha)"] \arrow[d, "X(\alpha)\times 1"'] & X_k \times \Delta^k \\ X_{k-1} \times \Delta^{k-1} \end{tikzcd} \tag{2.3}$$

That is, the two images of a point $(p,q) \in X_k \times \Delta^{k-1}$ under the horizontal and vertical arrows are to be identified in $|X|$.

It is clear then that the $\pi_L^{-1} \circ \mathfrak{F}_k$ do define a compatible collection of foliations on $|X|$, and that is precisely what one means by a foliation on $|X|$.

Similarly, one defines the de Rham complex $\Omega^*|X|$ of "compatible forms" on $|X|$ in terms of the diagram (2.3): Thus:

(2.4) A q - form φ on $|X|$ is a collection $\{\varphi_k\}$ of q - forms on $X_k \times \Delta^k$ such that

$$\{1 \times \Delta(\alpha)\}^* \varphi_k = \{X(\alpha) \times 1\}^* \varphi_{k-1} \quad .$$

Finally one has the Dupont extension of the Whitney-Thom-Sullivan Theorem to the effect that

$$H^*(\Omega X) \simeq H(\Omega^{**}X)$$

the isomorphism being induced by "integration over the simplexes"

$$\varphi \mapsto \sum_k \pi_*^L \varphi_k \tag{2.5}$$

where π_L is the projection (2.2) and π_*^L denotes integration over the fiber of π_L. Note that the sum has only q nonzero terms, when φ is of dimension q.

Now then, with all this understood our first remark is:

PROPOSITION 2.6. <u>Let $\mathfrak{F}$ be an oriented codimension 1 foliation on the paracompact simplicial manifold</u> X. <u>Then</u> $\omega(\mathfrak{F}) \in H^*(\Omega|X|)$ <u>can be computed by the Godbillon-Vey algorithm</u> (2.1) <u>provided only we interpret "form" to mean "compatible form" on</u> $|X|$.

The proof of this is quite straightforward, so let me only start the argument and in the process derive an explicit algorithm for $\omega(\mathfrak{F})$ in terms of the structure maps of X.

Consider then an $\mathfrak{F}$ as envisaged in the proposition, and let $\mathfrak{F}_0$ on X_0 be represented as kernel of φ with

$$d\varphi = \eta\varphi \quad .$$

We next try to extend φ and η to $X_1 \times \Delta^1$ in a compatible manner. To fix the notation let 0 and 1 be the vertices of Δ^1 and let x^0 and x^1 be the corresponding barycentric coordinates on Δ^1. Also let

$$\sigma^0 : \Delta^0 \longrightarrow \Delta^1 , \quad \text{and} \quad \sigma^1 : \Delta^0 \longrightarrow \Delta^1$$

be the inclusions sending Δ^0 to 0 and 1 respectively, and let σ_0 and σ_1 be the corresponding maps:

$$X_0 \underset{\sigma_1}{\overset{\sigma_0}{\leftleftarrows}} X_1 \quad .$$

Now by our hypothesis on $\mathfrak{F}$ the forms

$$\sigma_0^* \varphi \quad \text{and} \quad \sigma_1^* \varphi$$

both represent $\mathfrak{F}_1$ in the same orientation. Hence there exists a smooth positive function μ_1 on X_1 such that

(2.6) $$\sigma_1^* \varphi = \mu_1 \, \sigma_0^* \varphi \quad .$$

Using μ_1 we now construct the form

(2.7) $$\varphi_1 = \sigma_0^* \varphi \cdot \mu_1^{x^1} \qquad \text{on} \quad X_1 \times \Delta^1 \quad .$$

Here we have identified the forms on X_1 with their pullback to $X_1 \times \Delta^1$ under π_L and $\mu_1^{x^1}$ is defined by

$$\mu^{x^1}(p, a) = \mu_1(p)^{x^1(a)} \quad .$$

Hence φ_1 restricts to $\sigma_i^*\varphi$ on $\partial(X_1 \times \Delta^1)$ and is compatible. Furthermore the kernel of φ_1 clearly represents $\pi_L^{-1}\mathfrak{F}_1$. Indeed any nonzero multiple of $\sigma_0^*\varphi$ would.

Next we wish to extend η compatibly to $X_1 \times \Delta^1$. For this purpose differentiate (2.7). One obtains

(2.8)
$$d\varphi_1 = (\log \mu_1 \cdot dx^1 + x^1 \, d \log \mu_1 + \sigma_0^*\eta) \wedge \varphi_1 \quad .$$

Now the term in the bracket is not compatible with η as it stands. However, it can be modified to become so in the following manner:

By differentiating (2.6) we obtain

$$d\,\sigma_1^*\varphi = \sigma_1^*\eta \cdot \sigma_1^*\varphi = \{d \log \mu_1 + \sigma_0^*\eta\}\,\sigma_1^*\varphi$$

whence

(2.9)
$$\sigma_1^*\eta \equiv d \log \mu_1 + \sigma_0^*\eta \mod (\sigma_1^*\varphi) \quad .$$

Hence we may replace $d \log \mu_1$ in (2.8) by $\sigma_1^*\eta - \sigma_0^*\eta$ to obtain:

$$d\varphi_1 = \{\log \mu_1 \, dx^1 + x^1 \, \sigma_1^*\eta_1 + x^0 \, \sigma_0^*\eta\} \wedge \varphi_1$$

and this time the term

(2.10)
$$\eta_1 = \{\log \mu_1 \, dx^1 + x^1 \, \sigma_1^*\eta + x^0 \, \sigma_0^*\eta\}$$

is clearly compatible with η.

This construction now extends to all of $|X|$ to yield the following compatible collection φ_k and η_k on $X_k \times \Delta^k$.

For each k, we let $\sigma_0, \sigma_1, \cdots, \sigma_k$ be the maps of $X_k \to X_0$ corresponding to the inclusion of Δ^0 into Δ^k as the $k+1$ vertexes.

Then we define μ_i, $i = 1, \cdots, k$, by

$$\sigma_i^* \varphi = \mu_i \sigma_0^* \varphi$$

and finally set

$$\varphi_k = \sigma_0^* \varphi \left(\prod_{i=0}^{k} \mu_i^{x^i} \right) \tag{2.11}$$

and correspondingly set

$$\eta_k = \sum_{i=0}^{k} \{ \sigma_i^* \eta \cdot x^i + \log \mu_i \cdot dx^i \} \quad . \tag{2.12}$$

<u>The forms</u> $\eta_k \wedge d\eta_k = \omega_k(\mathfrak{F})$ <u>are therefore again compatible, closed, and give an algorithm for computing</u> $\omega(\mathfrak{F})$ <u>in</u> $H^*(|X|)$.

At this stage one may of course return to the more economical complex $\Omega^{**}(X)$ by integrating over the simplices. Then

$$\pi_*^L \omega(\mathfrak{F}) = \omega^{3,0} + \omega^{2,1} + \omega^{1,2} + \omega^{0,3} \tag{2.13}$$

with $\omega^{i,j} \in \Omega^i(X_j)$ and using (2.13) one obtains explicit formulae. For example:

$$\omega^{3,0} = \eta \wedge \varphi , \tag{2.14}$$

while $\omega^{2,1}$ is obtained by integrating $\eta_1 \wedge d\eta_1$ over the 1 - simplex Δ^1. Thus this component is given by:

$$\int_{\Delta^1} (x^0 \sigma_0^*\eta + x^1 \sigma_1^*\eta + \log \mu_1 \, dx^1) \wedge (dx^1(\sigma_1^*\eta - \sigma_0^*\eta) + d \log \mu_1 \, dx^1 + x^0 \, d\sigma_0^* + x^1 \, d\sigma_1^*\eta) \, .$$

Only the terms involving a dx survive and hence with a little algebra one arrives at

$$\omega^{2,1} = \sigma_1^*\eta \wedge \sigma_0^*\eta - d\{\log \mu_1 (\frac{\sigma_1^*\eta + \sigma_0^*\eta}{2})\} \tag{2.15}$$

To obtain the next term we proceed similarly with $\eta_2 \wedge d\eta_2$. The result is

$$\omega^{1,2} = \sum_{i,j=0}^{2} \int_{\Delta^2} \log \mu_i \, (d \log \mu_j - \sigma_j^*\eta) \, dx^i \, dx^j \qquad 0 \leq i \, , \, j \leq 2$$

(2.16)

$$= \sum_{0 \leq i < j \leq 2} (-1)^{i-j+1} \log u_i (d \log \mu_j - \sigma_j^*\eta)$$

Finally $\omega^{0,3}$ vanishes because there is no term involving three dx's in $\eta_3 \wedge d\eta_3$.

So far I have discussed only the original Godbillon-Vey class ω for foliations of codimension 1. If $\mathfrak{F}$ is orientable but has higher codimension say q - their class ω is generalized to $\mathfrak{F}$, by again describing $\mathfrak{F}$ as the kernel of a <u>decomposable form</u> φ, this time of dimension q. Then integrability again implies that

$$d\varphi = \eta \wedge \varphi$$

but now it is the form

$$\omega(\mathfrak{F}) = \eta \wedge (d\eta)^q$$

which is an invariant of $\mathfrak{F}$ and thus gives the extended Godbillon-Vey class in $H^{2q+1}(M)$. In WO_q this class then represents $h_1 c_1^q$.

In the semisimplicial situation this extension works equally well and (2. 12) again yields an explicit algorithm for the computations of $\omega(\mathfrak{F})$. Note by the way that in codimension q, only <u>components of the type</u>:

$$\pi_*^L \, \omega(\mathfrak{F}) = \omega^{2q+1,0} + \cdots + \omega^{q,q+1} \tag{2.17}$$

survive because $\eta_k \wedge d\eta_k{}^q$ can contain no more than $(q+1)$ dx's for any k. Furthermore the extremal terms are easy to compute from (2. 12):

$$\begin{aligned} \omega^{2q+1,0} &= \eta \wedge (d\eta)^q \\ \omega^{q,q+1} &= \int_{\Delta^q} (\log \mu_i \, dx^i) \{(d \log \mu_j - \sigma_j^* \eta) \, dx^j)^q \} \end{aligned} \tag{2.18}$$

There are corresponding formulae for the other classes in $H^*(WO_n)$ but they are harder to write down explicitly, because, as in the case of just a single foliation $\mathfrak{F}$ on M, one has to bring in a comparison of torsion-free and Riemannian connections.

However the compatible complex again enables one to extend the single-space recipes to the semisimplicial case in a more or less straightforward manner, but rather than discussing these let me now specialize our formulae to the case of a group action. We therefore have to recall briefly how $M\Gamma$ and $B\Gamma$ are constructed and why $M\Gamma$ carries a natural foliation.

Recall first, that if $\mathcal{C}$ is a a category, then it defines a natural semisimplicial object

$$B\mathcal{C} : O(\mathcal{C}) \leftleftarrows \mathfrak{m}_1(\mathcal{C}) \; \substack{\leftarrow \\ \leftarrow \\ \leftarrow} \; \mathfrak{m}_2(\mathcal{C}) \tag{2.19}$$

starting with the objects $O(\mathcal{C})$ of $\mathcal{C}$, and $\mathfrak{m}_i(\mathcal{C})$ being the i-tuples of composable morphisms in $\mathcal{C}$. Thus a typical element in $\mathfrak{m}_k(\mathcal{C})$ is given by the diagram:

(2.20)
$$\cdot \xleftarrow{\gamma_1} \cdot \xleftarrow{\gamma_2} \cdot \ \cdots \ \cdot \xleftarrow{\gamma_k} \cdot$$

the γ's being arrows in $\mathcal{C}$. The boundary maps in $B\mathcal{C}$ are then given by sending such a diagram into one with either an end arrow deleted, or two consecutive one composed. For example

(2.21)
$$\begin{aligned} \partial_0(\gamma_1,\gamma_2) &= \gamma_2 \\ \partial_1(\gamma_1,\gamma_2) &= \gamma_1 \circ \gamma_2 \\ \partial_2(\gamma_1,\gamma_2) &= \gamma_1 \end{aligned}$$

This construction applied to the category determined by the group Γ yields $B\Gamma$. When applied to the category Γ_M determined by the action of Γ on M, it yields $M\Gamma$: Thus

$$M\Gamma \equiv B\Gamma_M$$

where the category Γ_M has as objects the manifold M, and as morphisms pairs (γ, m) with $\gamma \in \Gamma$, $m \in M$. Finally the composition

(2.22)
$$(\gamma_1, m_1) \circ (\gamma_2, m_2) = (\gamma_1 \circ \gamma_2, m_2)$$

exists only if $\gamma_2(m_2) = m_1$ and is then given by the RHS of (2.22).

Pictorially these are then best described by arrows of the form

(2.23)
$$\gamma(m) \xleftarrow{\gamma} m$$

In any case, all of this granted it is easy to see that the following proposition holds.

PROPOSITION 2.24. *Let $\mathfrak{F}$ be a foliation on M, such that Γ acts transversally to $\mathfrak{F}$ on M. Then $\mathfrak{F}$ extends to a foliation $\mathfrak{F}\Gamma$ on $M\Gamma$.*

In particular the foliation of M by points induces a natural foliation $\mathfrak{F}_M$ on $M\Gamma$.

Note of course that $\mathfrak{F}_M$ then corresponds to the foliation by points of $M\Gamma$ considered as a simplicial manifold in the obvious sense.

When we specialize our formulae to $\mathfrak{F}_M$, matters simplify to some extent as now η can clearly be taken to be zero. If we furthermore follow our recipe, in the case when M is oriented and compact, and integrate $\omega(\mathfrak{F}_M)$ over the fiber in the projection

$$M\Gamma \xrightarrow{\pi} B\Gamma$$

then - M being q - dimensional - only $\omega^{q,q+1}$ enters into the computation. In this manner we obtain the following algorithm for

$$\pi_* \, \omega(\mathfrak{F}_M) \quad .$$

PROPOSITION 2.25. *The characteristic "number" $\pi_* \, \omega(\mathfrak{F}_M) \in H^{q+1}(B\Gamma)$ is represented by the cocycle:*

$$\omega(\gamma_1, \cdots, \gamma_{q+1}) = \int_{\Delta^{q+1} \times M} (\log \mu_i \, dx^i)(d \log \mu_j \, dx^j)^q \tag{2.26}$$

where the μ_i are functions on M, determined in the following manner:

Choose a volume φ on M, and then define μ_i by:

$$(\gamma_i \gamma_{i+1} \cdots \gamma_{q+1})^* \varphi = \mu_i \varphi \quad . \tag{2.27}$$

Remark. (1) Thurston has been aware of this formula all along although his conventions and his setting might have differed somewhat. For those of us less geometrically inspired it was however not so obvious. On the other hand - and this is the main point of my remarks - in the context of the compatible complex the formula does become easily accessible.

(2) The following geometrical interpretation of 2.25 was pointed out to me by Thurston quite recently. If $y_1, \cdots, y_{q+1}$ are coordinates in $\mathbb{R}^{q+1}$ then the form

$$\Psi = \Sigma(-1)^i y_i \, dy_1 \wedge \cdots \widehat{dy_i} \wedge \cdots dy_n \tag{2.28}$$

clearly satisfies the identity

$$d\Psi = n \text{ volume in } \mathbb{R}^n .$$

On the other hand, up to a universal constant (2.26) clearly gives the integral of Ψ pulled back to M under the map

$$f_\Gamma : M \longrightarrow \mathbb{R}^{q+1}$$

defined by

$$y^i \circ f_\Gamma = \log(\mu_i) \tag{2.29}$$

Thus our formula for the Eilenberg MacLane cocycle $\omega(\mathfrak{F})$ can be written;

$$\omega(\mathfrak{F})(\gamma_1, \cdots, \gamma_{q+1}) = \text{volume enclosed by } f_\Gamma M \text{ in } \mathbb{R}^{q+1} . \tag{2.30}$$

3. The Infinitesimal Case. When the action of Γ on a foliation $\mathfrak{F}$ over M comes by restriction of an action of a connected Lie group G containing Γ, the characteristic classes of the Γ - action can be described in Lie-algebra terms. Such a description is often more easy to handle, and our aim in this section is therefore to describe an infinitesimal analogue of (2.18) in this framework. The technique involves the lifting of $\mathfrak{F}$ to to $(G/K) \times M$, with K a maximal compact subgroup of G, and is thus typical of the Van-Est theorems and theorems on continuous cohomology, as well as of the whole approach of Kamber and Tondeur to foliated bundles. See [7].

However I will not pursue these aspects, but rather head as quickly as possible towards the infinitesimal formula for $\omega(\mathfrak{F})$.

Recall then, first of all, that G/K is contractible:

$$G/K \simeq pt \quad . \tag{3.1}$$

Hence if we consider the natural Γ action on G/K then (3.1) extends to a homology equivalence of the quotient of G/K by the action of Γ (denoted $[G/K]\Gamma$) with $B\Gamma$,"

$$[G/K]\Gamma \simeq B\Gamma \tag{3.2}$$

Thus $[G/K]\Gamma$ is a thickened version of $B\Gamma$. Now consider the edge homomorphism in $[G/K]\Gamma$,

$$\operatorname*{inv}_{\Gamma} \Omega^*(G/K) \to \Omega^{**}([G/K]\Gamma) \quad . \tag{3.3}$$

Because $\Gamma \subset G$, the left hand side certainly contains the subcomplex of G - invariant forms on G/K, which is finally identified with the relative Lie algebra complex $\Omega^*(\underline{g}; K)$. Thus we have

(3.4) $$\Omega^*(\underline{g};K) \cong \operatorname*{Inv}_G \Omega^*(G/K) \subset \operatorname*{Inv}_\Gamma \Omega^*(G/K)$$

combining (3.4) with (3.3) and (3.2) gives rise to an arrow †

(3.5) $$\iota_* : H^*(\underline{g};K) \to H^*(B\Gamma) \quad ,$$

and our first aim is then to show that:

PROPOSITION 3.6. <u>In the situation envisaged the characteristic classes of the</u> Γ <u>action lift naturally to</u> $H^*(\underline{g};K)$.

To see this result one first has to understand the following diagram of G spaces:

(3.7) $$G \underset{K}{\times} M \xrightarrow[\approx]{t} (G/K) \times M \xrightarrow{\pi} M \quad ,$$

where π is the natural product projection, and the action of G on $(G/K) \times M$ is the product action. On the other hand G <u>acts on</u> $G \underset{K}{\times} M$ <u>purely on the left, and</u> $G \underset{K}{\times} M$ <u>is</u> defined as the quotient

(3.8) $$G \underset{K}{\times} M = (G \times M)/K \quad ,$$

with K acting by

(3.9) $$(g, m) \cdot k = (gk, k^{-1}m) \quad .$$

Finally the twist map t is induced by

(3.10) $$\tilde{t} : G \times M \to G \times M$$

sending

(3.11) $$(g, m) \quad \text{to} \quad (g, g \cdot m) \quad .$$

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† In [5] and also independently in a recent paper of Shulman and Tischler, this arrow is explicitly described on the chain level.

It is then clear that $\tilde{t}$ is a diffeomorphism which sends the K action (3.9) into the K action $(g, m) \to (gk, m)$ and so induces the equivalence t of (3.7). The virtue of this twisting map t, is of course that the G - action on $G \times_K M$ is given by the left translation of G. It follows that the G - invariant forms on $G \times_K M$ can be identified with the forms on $G \times M$, which are

(1) invariant under left translation of G by G, and

(2) K - basic under the action (3.8) of K on $G \times M$. Thus:

$$\operatorname{Inv}_G \Omega^*(G \times_K M) \simeq K \text{ basic forms in } \Omega^*(\underline{g}) \otimes \Omega^* M \quad . \tag{3.12}$$

In any case, the plan of procedure is now suggested by the diagram:

$$\begin{array}{ccccc} \operatorname{Inv}_G \Omega^* (G \times_K M) & \to & \Omega^{**}(G \times_K M\Gamma) & \approx & \Omega^{**}(M\Gamma) \\ \downarrow \pi_* & & & & \downarrow \pi_* \\ \operatorname{Inv}_G \Omega^*(G/K) & \to & \Omega^* (G/K\Gamma) & \approx & \Omega^{**}(B\Gamma) \end{array} \tag{3.13}$$

where the lower line induces (3.5), and π_* denotes integration over the fiber. The homotopy equivalence in the upper line is of course again a consequence of the contractibility of G/K. In view of (3.13) our lifting problem clearly amounts to realizing the characteristic classes of $\mathfrak{F}$ lifted to $G \times M$, by G - invariant forms. Thus we need the following.

PROPOSITION 3.14. The characteristic classes of $\mathfrak{F}$ admit a natural lifting to the complex (3.12).

Let me carry out the proof, but again only for our Godbillon-Vey class $\omega(\mathfrak{F})$. Consider then the pull-back $\tilde{\mathfrak{F}}$ of $\mathfrak{F}$ to $G \times M$, under the map

$$G \times M \xrightarrow{\pi \circ \tilde{t}} M \quad . \tag{3.15}$$

To describe $\tilde{\mathfrak{F}}$ let us identify the tangent space of $G \times M$, at (g, m) with $\underline{g} \oplus T_m M$, using the left invariant vector-fields on G to identify G_g with $\underline{g}$, and using the projections for the direct sum decomposition. Also if $x \in \underline{g}$ is an element of the Lie algebra of G, let $\dot{x} \in \Gamma(TM)$ be the corresponding infinitesimal motion on M, induced by the action of G on M.

Precisely if e^{tx} is the one-parameter subgroup generated by $x \in \underline{g}$, then

$$\dot{x}_m = \text{tangent of } e^{tx}m \text{ at } m \quad . \tag{3.16}$$

With this understood, the kernel of $\tilde{\mathfrak{F}}$ is described by:

$x + y \in T_{(g,m)}(G \times M)$ <u>is in</u> $\tilde{\mathfrak{F}}$ <u>if and only if</u>

$$\dot{x}_m + y_m \in \mathfrak{F} \quad . \tag{3.17}$$

Indeed the curve (ge^{tx}, m) goes over into $ge^{tx}m$ under our map, and hence its tangent at $t = 0$ goes to $g_* \dot{x}_m$. On the other hand y goes to $g_* y_m$. Hence $g_*(\dot{x}_m + y_m) \in \mathfrak{F}$. But G preserves the foliation so that (3. 17) follows.

An immediate corollary of (3. 17) is the following:

PROPOSITION 3. 18. <u>Let the foliation</u> $\mathfrak{F}$ <u>be described as the kernel of the decomposable</u> q <u>form</u> φ. <u>Then</u> $\tilde{\mathfrak{F}}$ <u>admits a natural representation as the kernel of a decomposable form</u> $\tilde{\varphi} \in \Omega^*(\underline{g}) \otimes \Omega^*(M)$. <u>Furthermore, in the natural double grading of this complex</u> $\tilde{\varphi}$ <u>has components:</u>

$$\tilde{\varphi} = \tilde{\varphi}^{q,0} + \cdots + \tilde{\varphi}^{0,q}$$

with:

$$\tilde{\varphi}^{0,q} = \varphi \quad .$$

(3.19) $$\widetilde{\varphi}^{1,q-1} = \sum_{\alpha} x'_{\alpha} \wedge \iota(\dot{x}_{\alpha})\, \varphi$$

where x_{α} runs over a basis of g, and x'_{α} over a dual base in g^*.

Proof. This is purely a linear algebra matter. If θ is a 1 - form with $\mathfrak{F}$ in its kernel, then

(3.20) $$\widetilde{\theta} = \theta + \sum_{\alpha} x'_{\alpha}\, \iota(\dot{x}_{\alpha})\, \theta$$

will have $\widetilde{\mathfrak{F}}$ in its kernel. Indeed for any $x + y$ subject to (3.17) we then have $\iota(x + y)\widetilde{\theta} = \theta(y) + \theta(\dot{x}) = 0$.

Hence if $\theta^1 \wedge \cdots \wedge \theta^q = \varphi$ locally, then $\widetilde{\theta}^1 \wedge \cdots \wedge \widetilde{\theta}^q$ describes $\widetilde{\mathfrak{F}}$ locally, and expanding this product clearly yields (3.19).

To proceed further we need to compute $d\widetilde{\varphi}$ and express it as $\widetilde{\eta} \wedge \widetilde{\varphi}$ in the double complex $\Omega^*(g) \otimes \Omega^*(M)$.

For this purpose, let us set

$$\iota(\dot{x}_{\alpha})\varphi = \varphi_{\alpha}$$

and use the double index summation convention, so that

(3.21) $$\widetilde{\varphi} = \varphi + x'_{\alpha} \wedge \varphi_{\alpha} + \cdots$$

describes the "beginning" of $\widetilde{\varphi}$.

Also, let us assume that on M the integrability of $\mathfrak{F}$ is expressed by

(3.22) $$d\varphi = \eta \wedge \varphi$$

where η is a global 1 - form . Then I claim that:

PROPOSITION 3.23. *The* η *of* (3.22) *lifts naturally to one* $\tilde{\eta}$ *in* $\Omega^*\underline{g} \otimes \Omega^*M$ *such that*

$$d\tilde{\varphi} = \tilde{\eta} \wedge \tilde{\varphi} \quad .$$

Further the $\tilde{\eta}$ *is given by*

$$\tilde{\eta} = \eta - x'_\alpha \{\mu(x_\alpha) - \eta(\dot{x}_\alpha)\} \text{ where } \mu(x) \text{ is defined by (3.27).} \tag{3.24}$$

Proof. Differentiating (3.21) yields

$$d\tilde{\varphi} = d\varphi - x'_\alpha \wedge d\varphi_\alpha + \cdots \quad . \tag{3.25}$$

Further

$$d\varphi_\alpha = d\iota(\dot{x}_\alpha)\varphi = \mathcal{L}(\dot{x}_\alpha)\varphi - \iota(\dot{x}_\alpha)d\varphi. \tag{3.26}$$

where $\mathcal{L}(\dot{x}_\alpha)$ is the Lie derivative in the direction $\dot{x}$. Because G preserves $\mathfrak{F}$, $\mathcal{L}(\dot{x})$, $x \in \underline{g}$ must preserve φ up to multiples, whence

$$\mathcal{L}(\dot{x})\varphi = \mu(x)\varphi \ , \ x \in \underline{g} \ , \ \text{with} \ \mu(x) \in \Omega^0(M) \quad . \tag{3.27}$$

This $\mu(x)$ is, of course, the infinitesimal analogue of the $\mu(\sigma)$ in Section 2.

In any case, combining (3.25), (3.26), (3.27) with (3.22) one obtains the formula

$$\begin{aligned} d\tilde{\varphi} &= \eta \wedge \varphi - \mu(x_\alpha)x'_\alpha \wedge \varphi \\ &\quad + x'_\alpha \wedge \{\eta(\dot{x}_\alpha)\varphi - \eta \wedge \varphi_\alpha\} + \cdots \end{aligned}$$

which, up to terms of order ≥ 2 in the $\underline{g}$ direction, is given by

$$d\tilde{\varphi} = \{\eta - x'_\alpha(\mu(x_\alpha) - \eta(\dot{x}_\alpha)\} \wedge \tilde{\varphi} \quad . \tag{3.28}$$

But as $\tilde{\eta}$ exists and clearly is the sum of forms of type $(1,0)$ and $(0,1)$ this equation fixes $\tilde{\eta}$. Q. E. D.

To assemble the pieces, we shall have to determine whether $\tilde{\varphi}$ and $\tilde{\eta}$ are K basic in our complex.

In general this will, of course, not be the case. However, by averaging over K, we can arrange it that both the φ and the η of our discussion are invariant under the action of K, i. e., that infinitesimally

(3. 29) $$\mathcal{L}(\dot{x})\varphi = 0 \quad ; \quad \mathcal{L}(\dot{x})\eta = 0 \quad \text{for} \quad x \in \underline{k}$$

and under this hypothesis we have the following.

PROPOSITION 3. 30. The condition (3. 29) implies that $\tilde{\varphi}$ and $\tilde{\eta}$ are K basic† relative to the action of K on $G \times M$.

The proof of this fact is a straightforward check (though not quite trivial) which I will take up in greater generality at another time.

At this stage, we are ready to give an infinitesimal recipe for $\omega(\mathfrak{F})$.

Indeed, expanding $\tilde{\eta}(d\tilde{\eta})^q$, will have all possible type of components:

$$\omega(\mathfrak{F}) = \omega^{2q+1,0} + \cdots + \omega^{0,2q+1} \quad ,$$

of which the simplest are given by:

(3. 31) $$\omega^{0,2q+1} = \eta \cdot d\eta^q \quad ,$$

(3. 32) $$\omega^{q+1,q} = - x'_{\alpha_1} \wedge \cdots x'_{\alpha_{q+1}} \nu_{\alpha_1} d\nu_{\alpha_2} \wedge \cdots d\nu_{\alpha_{q+1}}$$

where we have now set

† The K basic forms are those annihilated by the vector fields along the orbit of the K action and invariant under that action.

(3.33) $$\nu(x) = \mu(x) - \eta(\dot{x}) \qquad x \in \underline{g}$$

and have abbreviated $\nu(x_\alpha)$ to ν_α.

If we think of $\Omega^*(\underline{g}) \otimes \Omega^*(M)$ as the complex $\Omega^*(\underline{g}; \Omega^*(M))$ of forms on $\underline{g}$ with values in Ω^*M, then (3.33) can be thought of as a 1 - form on $\underline{g}$ with values in $\Omega^o(M)$:

$$\nu \in \Omega^1(\underline{g}; \Omega^o(M)) \quad ,$$

and there (3.32) takes the form

(3.34) $$\omega^{q+1,q}(x_1, \cdots, x_{q+1}) = \frac{1}{(q+1)} \Sigma (-1)^i \nu(x_i)\, d\nu(x_1) \cdots d\nu(\hat{x}_i) \cdots d\nu(x_{q+1}) \quad .$$

Hence we get the following corollary, which is an infinitesimal analogue of 3.25.

PROPOSITION 3.35. Suppose $Z \subset M$ is a cycle of dimension q on M. Then

(3.36) $$\int_Z \omega(\mathfrak{F}) \in H^{q+1}(\underline{g}; K)$$

is represented by the cocycle

(3.37) $$\omega_Z(\mathfrak{F})(x_1, \cdots, x_q) = \int_Z \nu(x_1) d\nu(x_2) \cdots d\nu(x_{q+1})$$

where $\nu(x)$ is defined by

(3.38) $$\{\mathcal{L}(x) - \eta(\dot{x})\}\varphi = \nu(x)\varphi \quad .$$

4. On the Examples of Thurston and Heitsch. The history of examples of foliations with varying classes $\omega(\mathfrak{F})$ is roughly as follows.

In the complex analytic case, I had observed already before 1970 that the foliation:

$$\mathcal{F}_\lambda = \{\lambda_1 z_1 \frac{\partial}{\partial z_1} + \lambda_2 z_2 \frac{\partial}{\partial z_2}\} \tag{4.1}$$

on $\mathbb{C}^2 - 0$, had for its g.v. invariant:

$$\int_{S^3} \omega(\mathcal{F}_\lambda) = c\,\{\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} - 2\} \quad ,$$

and thus varied continuously with λ.

At that time, I thought that the corresponding real invariant would always vanish. However, soon thereafter in 1971, the paper of Godbillon-Vey appeared with the Roussarie example of the foliation $\mathcal{F}$ on $\Gamma\backslash SL(2;\mathbb{R})$ induced by the Lie algebra of triangular matrices in $SL(2;\mathbb{R})$, Γ being a discrete subgroup with compact quotient space.

Thereafter, Thurston produced his examples. In particular he constructed examples of a family of actions of a group Γ acting on S^1, such that the corresponding g.v. number $\in H^2(B\Gamma)$ varied continuously. In an appendix, Robert Brooks has written up the details treating this example with the formula (2.25) much like Dupont in [5] treated the Euler class of flat bundles. Here let me just outline and comment on the plan of this very ingenious example.

We start by observing that $SL(2;\mathbb{R})$ acts on S^1 in the classical manner

$$z \longrightarrow \frac{az+b}{cz+d} \quad , \quad |z| = 1, \begin{pmatrix} a b \\ c d \end{pmatrix} \in SL(2,\mathbb{R}) \quad . \tag{4.1}$$

Hence for any $\Gamma \subset SL(2,\mathbb{R})$ there is a natural action of Γ on S^1.
Where $\Gamma\backslash SL(2,\mathbb{R})$ is compact, $H^2(\Gamma;\mathbb{R}) \neq 0$ and the g.v. class of the action will also be nonzero. On the other hand, as we let Γ vary in a continuous family of such subgroups,

the corresponding characteristic class does not vary. Thus, the moduli of Riemann surfaces do not furnish varying examples.

Thurston therefore twisted these homogeneous actions on S^1 in the following manner.

Consider the double cover

$$S^1 \xrightarrow{\pi} S^1$$

given by sending z to z^2; $|z| = 1$. Then every diffeomorphism f of S^1, admits precisely two liftings $\widetilde{f}$ relative to π. Thus,

$$\widetilde{f} : S^1 \longrightarrow S^1$$

is a diffeomorphism of S^1, with

$$\pi \circ \widetilde{f} = f \circ \pi \quad .$$

On the double cover $\mathrm{Diff}^{(2)} S^1$ of $\mathrm{Diff}\, S^1$ the function $f \longrightarrow \widetilde{f}$ now becomes single valued and defines a homomorphism:

$$\mathrm{Diff}^{(2)} S^1 \xrightarrow{\Psi_2} \mathrm{Diff}\,(S^1) \tag{4.2}$$

Note that if f is lifted to an "equivariant map" : $\underline{f}(x + 2\pi) = \underline{f}(x) + 2\pi$,

$$\underline{f} : \mathbb{R} \longrightarrow \mathbb{R}$$

then $\widetilde{f}$ is represented by:

$$\widetilde{\underline{f}}(x) = 1/2\, \underline{f}\,(2x) \quad \text{or} \quad 1/2\, \{\underline{f}(2x) + \pi\} \quad .$$

Hence, in particular, if f is a rotation by α and hence represented by

$$x \longrightarrow x + \alpha$$

then $\tilde{f}$ is represented by:

$$x \longrightarrow x + \alpha/2 \,, \quad \text{and} \quad x \longrightarrow x + \frac{\alpha}{2} + \pi \,. \tag{4.3}$$

It follows that if the map

$$SL(2,\mathbb{R}) \longrightarrow PSL(2,\mathbb{R}) \longrightarrow \text{Diff } S^1$$

given by (4.1), is lifted to a map of $SL(2,\mathbb{R})$ to $\text{Diff}^{(2)}(S^1)$ and then followed by Ψ_2, there results a homomorphism of

$$SL(2,\mathbb{R}) \underset{SO(2)}{*} SL(2,\mathbb{R}) \xrightarrow{1 \times \Psi_2} \text{Diff }(S^1) \tag{4.4}$$

where on the left we have in mind the free product amalgamated along the rotations according to (4.3), and it is this action which gives rise to Thurston's example. More precisely, he takes the Γ represented by generators

$$\{X, Y, Z, W\}$$

and with the relation

$$[X,Y] = [Z,W] \,, \tag{4.5}$$

chooses $X, Y \in SL(2;\mathbb{R})$ so that $[X,Y]$ is a rotation by $0 < \alpha < \pi$, (see Appendix for details), and also $Z', W' \in SL(2,\mathbb{R})$ so that $[Z', W']$ is a rotation by 2α. Then lifting Z' and W' and applying Ψ_2, one obtains Z, W in $\text{Diff } S^1$, which clearly satisfy the relation (4.5).

Now varying α, Thurston obtains his example. Note that as this example is obtained by amalgamation of two infinitesimal situations, it cannot be directly treated by our infinitesimal method, although, as Bob Brooks point out our global and infinitesimal cocycles agree where they should.

More recently Thurston has found examples of varying the higher g. v. classes by actions of certain Γ 's on the spheres. Thus, his examples actually vary the characteristic numbers of certain Γ actions.

During the Rio conference, James Heitsch suggested a different approach to these examples, which quite recently has enabled him to vary a large number of characteristic classes independently (see [6]).

In our terminology, Heitsch passes from the sphere, where Thurston worked, to a foliation $\mathfrak{F}_\lambda$ on $\mathbb{R}^n - 0$ of the type I used in the complex case, but here he starts out with a sufficiently special λ so that $\mathfrak{F}_\lambda$ admits a large group of automorphism. Let me conclude by taking up the first instance of his construction to vary $h_1 c_1^3 \in H^4(B\Gamma)$.

We will use our infinitesimal recipe for this purpose, so recall first of all that the homomorphism

$$H^*(\underline{g}; K) \longrightarrow H^*(B\Gamma) \tag{4.6}$$

is injective for any semi-simple Lie group and any discrete subgroup Γ with $\Gamma\backslash G$ compact. Indeed Γ will then have a subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' acts freely on G/K. The natural map

$$H^*(\underline{g}; K) \longrightarrow H^*(\Gamma' \backslash G/K) \tag{4.7}$$

is then injective in the top dimension and both sides satisfy Poincaré duality. Hence (4. 7) is injective, but $\Gamma'\backslash G/K \simeq B\Gamma'$, and $B\Gamma'$ and $B\Gamma$ have equal rational cohomology. Q. E. D.

Finally recall that by a theorem of Borel's any semi-simple G admits a Γ with $\Gamma\backslash G$ compact. Thus for our purposes, varying the infinitesimal class with a semi-simple group G also varies the class in some $B\Gamma$.

With these remarks we are ready to take up the Heitsch example.

Let then $\mathfrak{F}_\lambda$ be generated by X_λ in $\mathbb{R}^4 - 0$ where

$$X_\lambda = \sum_{i=1}^{2} \lambda_i \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) . \tag{4.8}$$

Then the natural action of $SL(2;\mathbb{R})$ on the (x_i, y_i) space defines an action of

$$G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$$

on $\mathbb{R}^4$, which obviously preserves X_λ and hence $\mathfrak{F}_\lambda$.

The infinitesimal action of $\underline{g}$ on $\mathbb{R}^4$ is therefore generated by

$$u_i = x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i} \quad ; \quad v_i = x_i \frac{\partial}{\partial y_i} + y_i \frac{\partial}{\partial x_i} \tag{4.9}$$

and

$$h_i = x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \quad .$$

Thus the h_i generate the action of K , and a class in $H^4(\underline{g}; K)$ is determined by its value on $u_1 \wedge v_1 \wedge u_2 \wedge v_2$.

Let us now apply the infinitesimal recipe to $\mathfrak{F}_\lambda$. For φ we may choose

$$\varphi = \iota(X_\lambda)v \quad , \quad v = dx_1\, dy_1\, dx_2\, dy_2 \quad . \tag{4.10}$$

Then

$$d\varphi = \mathcal{L}(X_\lambda)v = 2(\lambda_1 + \lambda_2)v \quad .$$

Hence an admissible η is given by

$$\eta = (\lambda_1 + \lambda_2) \cdot \frac{\Sigma \lambda_i^2 \, dr_i^2}{\Sigma \lambda_i^2 \, r_i^2} \quad , \quad i = 1,2 \tag{4.11}$$

where we have set

$$r_i^2 = x_i^2 + y_i^2 \quad . \tag{4.12}$$

Indeed η is clearly invariant under K, and satisfies the relation

$$d\varphi = \eta \wedge \varphi \quad .$$

Further note that φ is invariant under all of G, so that the ν of (3.33) is entirely given by the formula

$$\nu(x) = \eta(x) \quad .$$

Now by direct computation

$$\begin{aligned} u_i \cdot r_i^2 &= 2(x_i^2 - y_i^2) \\ v_i \cdot r_i^2 &= 4\, x_i\, y_i \end{aligned} \tag{4.13}$$

hence

$$\nu(u_i) = -\eta(u_i) = \frac{\left(\lambda_1 + \lambda_2\right) 2\lambda_i \left(x_i^2 - y_i^2\right)}{\Sigma\, \lambda_i^2\, r_i^2}$$

(4.14)

$$\nu(v_i) = -\eta(v_i) = \frac{\left(\lambda_1 + \lambda_2\right) 4\lambda_i\, x_i\, y_i}{\Sigma\, \lambda_i^2\, r_i^2}$$

Now (3.37) together with (2.28) yield the formula:

$\omega(\mathfrak{F}_\lambda)\,(u_1, v_1, u_2, v_2)$ = <u>volume enclosed by the map</u> $S^3 \longrightarrow R^4$ <u>given by the four functions</u> $\nu(u_1), \nu(v_1), \nu(u_2), \nu(v_2)$ <u>on</u> $S^3 \subset R^4$.

Clearly this map is homogeneous of degree zero, hence we may change coordinates from x_i to x_i/λ_i, and setting

(4.15) $$z_i = x_i + \sqrt{-1}\, y_i$$

we see that the volume enclosed by the map in question will be proportional to $(\lambda_1 + \lambda_2)^4/(\lambda_1\lambda_2)^2$ times the volume enclosed by the unit sphere under the map,

(4.16) $$\{z_1, z_2\} \longrightarrow \{z_1^2, z_2^2\}/|z_1|^2 + |z_2|^2 \quad ,$$

which is easily seen to be nonzero. <u>Hence</u>

$$\int \omega(\mathfrak{F}_\lambda)\,(u_1, v_1, u_2, v_2) = \text{const.}\ \frac{\left(\lambda_1 + \lambda_2\right)^4}{\left(\lambda_1\lambda_2\right)^2}$$

<u>and therefore varies with</u> λ.

In higher even dimensions this method of Heitsch's works equally well and leads to an independent variation of all the $h_1 c^{\alpha}$ classes. In odd dimensions the argument is a little more subtle. However, he can also treat this case by construction which - on the sphere - goes back to Thurston.

All in all then Thurston and Heitsch have set us well on the way of showing that all the potential classes of $H^*(a_n; o_n)$ are non-trivial, independent and, in the appropriate cases, variable. For classes involving many h's, corresponding independence theorems were first obtained by Kamber and Tondeur [7].

Appendix

by

Robert Brooks

In this appendix, we will evaluate the Godbillon-Vey class in the case of some specific actions of groups on the circle. We will then show how these calculations lead to an example due to Thurston showing how one can vary the Godbillon-Vey "number."

Given an action of G on S^1, recall that on BG, ω is given by the 2 - cocycle

$$\omega(g,f) = \int_{S^1} \log(\mu_f)\, d\log(\mu_{gf}) \quad .$$

If G is a discrete subgroup of $PSL(2,\mathbb{R})$, then we have a natural action of G on S^1, which we view as the boundary of the upper half plane, given by the linear fractional transformations -

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}$$

For ease in computation, we can conjugate this action by the linear fractional transformation $z \to \frac{z-i}{z+i}$ taking the upper half plane into the disk $|z| \leq 1$ - we will want to pass back and forth freely between the two pictures.

Viewing f and g as acting on $|z| = 1$, we may now write

$$f(z) = \theta_1\left(\frac{z-\eta_1}{\overline{\eta}_1 z - 1}\right); \quad g \circ f(z) = \theta_2\left(\frac{z-\eta_2}{\overline{\eta}_2 z - 1}\right)$$

with $|\theta_1| = |\theta_2| = 1$, $|\eta_1|$, $|\eta_2| < 1$

and so viewing S^1 as $\mathbb{R}/\mathbb{Z}$, then f and g become

$$f(x) = \frac{1}{2\pi i}\log(\theta_1(\frac{e^{2\pi i x} - \eta_1}{\overline{\eta}_1 e^{2\pi i x} - 1}))\ ;\quad g\circ f(x) = \frac{1}{2\pi i}\log(\theta_2(\frac{e^{2\pi i x} - \eta_2}{\overline{\eta}_2 e^{2\pi i x} - 1}))\quad .$$

Choosing the standard volume on S^1, $\mu_f = f'$ and $\mu_g = g'$, we can now easily compute by residues to get

$$\begin{aligned}
\omega(f, g) &= \int_{S^1} \log(\mu_f)\, d\log(\mu_{gf}) \\
&= \int_0^1 \log(1 - e^{-2\pi i x}\eta_1)(1 - e^{2\pi i x}\overline{\eta}_1)[\,2\pi i\,]\left(\frac{e^{-2\pi i x}\eta_2}{1 - e^{-2\pi i x}\eta_2} - \frac{\overline{\eta}_2}{1 - e^{2\pi i x}\overline{\eta}_2}\right) dx \\
&= 2\pi i \log\left(\frac{1 - \eta_2\overline{\eta}_1}{1 - \overline{\eta}_2\eta_1}\right) \quad .
\end{aligned}$$

We can give this number a somewhat more geometric flavor by now passing to the upper half plane - writing $\eta_1 = \frac{a - i}{a + i}$, $\eta_2 = \frac{b - i}{b + i}$, we have

$$\omega(f, g) = 2\pi i \log\left(\frac{a + i}{\overline{a} - i}\right)\left(\frac{\overline{b} - i}{b + i}\right)\left(\frac{\overline{a} - b}{\overline{b} - a}\right) \quad .$$

Now in the upper half plane, a geodesic is of the form $re^{i\theta} + k$, where r, k and θ are real. So given $z_1 = re^{i\theta_1} + k$, $z_2 = re^{i\theta_2} + k$, the expression $\log(\frac{z_1 - \overline{z}_2}{z_2 - \overline{z}_1}) = i(\theta_1 - \theta_2)$.

Hence well-known formulas in non-Euclidean geometry give us

$$(*)\qquad \omega(g, g) = (-4\pi^2)\ \text{area}\ (\Delta)\ ,$$

where Δ is the geodesic triangle with vertices i, a, and b. Here, of course, a and b satisfy $f(a) = i$, $(g \circ f)(b) = i$.

The formula $(*)$ relates well to the formulas of [5] in the following manner: If a Lie group G operates on a manifold M of dimension q, then we get a map from $\mathfrak{g} = \text{Lie}(G)$ to the Lie algebra of vector fields on M in the following way: given $X \in \mathfrak{g}$, then it defines a one-parameter group f_t of diffeomorphisms of M, and the infinitesimal generator of this flow is a vector field on M.

Let K be a maximal compact subgroup of G, with Lie algebra $\mathfrak{k}$, and choose a K-invariant volume θ on M. Then $\mathcal{L}_X(\theta) = \mu_X \cdot \theta$ defines a function μ_X on M, which we may think of as the "divergence" of X. One may define an "infinitesimal" Godbillon-Vey class by the formula

$$\omega(X_0, X_1, \cdots, X_q) = \int \mu_{X_0} \, d\mu_{X_1} \wedge \cdots \wedge d\mu_{X_q}$$

and one sees easily that this defines an element of $H^{q+1}(\mathfrak{g}/\mathfrak{k})$, independent of θ.

In [5] is constructed, given a discrete subgroup Γ of G, a map $H^*(\mathfrak{g}/\mathfrak{k}) \to H^*(B\Gamma)$. In the specific case of our standard $PSL(2, \mathbb{R})$ action on S^1, one can check easily that this "infinitesimal" Godbillon-Vey class agrees with our "global" Godbillon-Vey class.

Using the formula $(*)$, we can construct foliated circle bundles having arbitrary Godbillon-Vey number, according to the following scheme due to Thurston. Let a and b be any two points in the upper half plane. Then there is a unique parallelogram (in the sense that opposite sides are of equal length) having as three vertices i, a, and b. Labelling the fourth vertex c, then there are unique elements f and g satisfying

$$f(c) = a \qquad f(b) = i$$

$$g(c) = b \qquad g(a) = i$$

Then $fgf^{-1}g^{-1}(i) = i$, and so $fgf^{-1}g^{-1}$ is a rotation. A little non-Euclidean geometry shows us that the rotation is the non-Euclidean area of the parallelogram. Since the parallelogram may have an arbitrary area, we have shown

LEMMA. For any $0 < \alpha < 2\pi$, there are $f, g \in PSL(2, \mathbb{R})$ whose commutator is a rotation through angle α.

Now let G be the fundamental group of the two-holed torus. G is given as $\{X, Y, Z, W : XYX^{-1}Y^{-1} = ZWZ^{-1}W^{-1}\}$, and BG has a fundamental 2-cycle

$$[BG] = (X, Y) - (XYX^{-1}, X) - (XYX^{-1}Y^{-1}, Y) - (Z, W) + (ZWZ^{-1}, Z) + (ZWZ^{-1}W^{-1}, W)$$

and this is the object on which, for careful choices of X, Y, Z, and W, we want to evaluate our cocycle ω.

Fixing a positive integer n, and some $\alpha < \frac{2\pi}{n}$, choose f and g with commutator α, and f' and g' with commutator $n\alpha$.

Set $X = g$, and $Y = f$. We set $\widetilde{f} = (x) = \frac{1}{n} f'(nx)$, and similarly for $\widetilde{g}$ - we may think of $\widetilde{f}$ and $\widetilde{g}$ as liftings of f' and g' to an n-fold covering of the circle. Of course it is clear that the commutator of $\widetilde{f}$ and $\widetilde{g}$ is now α.

Now $\omega(g, f) - \omega(gfg^{-1}, g)$ is simply $(-4\pi^2) \cdot \alpha$, as can be seen by applying $(*)$ to the parallelogram with vertices i, a, b, and c. $\omega(gfg^{-1}f^{-1}, g) = 0$, since $gfg^{-1}f^{-1}$ is a rotation. Similarly, $\omega(g', g') - \omega(g'f'g'^{-1}, g') = (-4\pi^2) \cdot n\alpha$, and it is easy to check by the original integral formula that passing to an n-fold covering simply multiplies the result by n. Hence choosing $X = g$, $Y = f$, $Z = \widetilde{g}$, $W = \widetilde{f}$, we get $\omega([BG]) = 4\pi^2(n^2 - 1)\alpha$.

Varying α between 0 and $\frac{2\pi}{n}$, and taking n arbitrarily large, we see that ω may take any real value.

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