

Classifying Space for Proper Actions and K-Theory of Group C*-algebras

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ABSTRACT. We announce a reformulation of the conjecture in [8,9,10]. The advantage of the new version is that it is simpler and applies more generally than the earlier statement. A key point is to use the universal example for proper actions introduced in [10]. There, the universal example seemed somewhat peripheral to the main issue. Here, however, it will play a central role.

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0. Introduction

Let G be a group which is locally compact, Hausdorff and second countable. Denote by $C_r^*(G)$ the reduced C^* -algebra of G . It is the completion in the operator norm of the convolution algebra $L^1(G)$, viewed as an algebra of operators on $L^2(G)$.

The purpose of this article is to describe in some detail a conjecture concerning the C^* -algebra K -theory groups

$$K_j(C_r^*(G)) \quad (j = 0, 1).$$

We shall propose a means of calculating these groups which blends group homology with the representation theory of compact subgroups of G . The conjecture, if true, would have a number of implications in geometry and topology, as well as C^* -algebra theory. In addition there are close connections with the tempered representation theory of Lie groups and p -adic groups, for example with the Selberg principle and the problem of finding explicit realizations of discrete series representations.

To state the conjecture in full generality we need the KK -theory of G. Kasparov [34,35]. But for various classes of groups the conjecture assumes a quite simple and concrete form which does not involve Kasparov's theory. We hope to make this point clear in the later sections of the paper, where we shall focus in turn on Lie groups, p -adic groups and discrete groups.

We begin with a few remarks of a general character. If G is a locally compact *abelian* group then the structure of $C_r^*(G)$ is readily described by the Fourier transform, which provides an isomorphism between $C_r^*(G)$ and the algebra $C_0(\hat{G})$ of continuous functions on the Pontrjagin dual of G which vanish at infinity. So, repeating a well known slogan, we can say that studying the algebra $C_r^*(G)$ amounts to the same thing as studying the topological space \hat{G} . The K -theory groups $K_j(C_r^*(G))$ identify with the Atiyah-Hirzebruch K -theory groups of the locally compact space \hat{G} ,

$$(0.1) \quad K_j(C_r^*(G)) \cong K^j(\hat{G}).$$

(In (0.1) we are only using the structure of \hat{G} as a topological space.) Thus our conjecture (which has been verified for abelian groups) concerns the algebraic-topological invariants of the space \hat{G} .

The K -theory of \hat{G} , as opposed to say its ordinary cohomology, has particular relevance to us for two reasons. First of all, the K -theory functors extend readily to the non-abelian situations which are our main interest. Secondly there is a very direct link between the K -theory of \hat{G} and the index of elliptic operators. Suppose that M is a smooth closed manifold with abelian fundamental group G , and let D be an elliptic partial differential operator on M . It has an integer valued index, namely

$$\text{Index}(D) = \dim_{\mathbb{C}}(\text{kernel}(D)) - \dim_{\mathbb{C}}(\text{cokernel}(D)).$$

A more refined index, lying in $K^0(\hat{G})$, can be defined as follows. Each element of \hat{G} (a character of G) determines a line bundle L_α on M . We “twist” the operator D by L_α , so as to obtain an operator D_α acting on sections of L_α , and form the families of vector spaces

$$(0.2) \quad \{\ker(D_\alpha)\}_{\alpha \in \hat{G}} \quad \text{and} \quad \{\text{cokernel}(D_\alpha)\}_{\alpha \in \hat{G}}$$

In favorable circumstances these constitute two vector bundles on \hat{G} , and we define the quantity

$$\text{Index}_G(D) \in K^0(\hat{G})$$

by taking the difference of the two K -theory classes represented by these vector bundles. In general the families (0.2) may be perturbed so as to become vector bundles, and we define $\text{Index}_G(D)$ to be the difference of the resulting K -theory classes (it does not depend on the choice of perturbation).

This construction was introduced by G. Lusztig [42] in connection with Novikov’s conjecture on the homotopy invariance of higher signatures. A. Mischenko and Kasparov subsequently generalized the construction quite significantly, and their work is our starting point.

Suppose that X is a smooth manifold on which a locally compact group G acts properly, with the quotient space $G \backslash X$ compact. Let D be a G -equivariant elliptic operator on X . In [38] Kasparov has defined an index for D lying in the group $K_0(C_r^*(G))$. His definition generalizes Lusztig’s (which can be put into the present context by choosing for X the universal covering space of the closed manifold M).

Following Atiyah [3], Kasparov formalizes a notion of abstract elliptic operator D on a locally compact space X . If X admits a proper G -action, with $G \backslash X$ compact, and if D is G -equivariant, then it has an index, lying in the group $K_0(C_r^*(G))$. Roughly speaking, our conjecture is that $K_0(C_r^*(G))$ is generated by these indices, and furthermore that the only relations among them are those imposed from simple geometric considerations.

To formulate the conjecture precisely we associate to each locally compact group G a space $\underline{E}G$ which plays roughly the same role in the theory of proper G -actions as the space EG (familiar from topology) plays in the theory of principal G -actions. Like EG , the space $\underline{E}G$ is only defined up to equivariant homotopy, but an important feature of the new notion is that in many cases there is a natural model which is of special geometric interest. For instance, if G is a reductive Lie group then $\underline{E}G$ is the associated symmetric space. Similarly, if G is a reductive group over a p -adic field then $\underline{E}G$ is the affine Bruhat-Tits building. These and other examples are discussed in Section 2.

Using Kasparov’s KK -theory we form the equivariant K -homology groups $K_j^G(\underline{E}G)$, with G -compact supports. A class in $K_j^G(\underline{E}G)$ is represented by an abstract G -equivariant elliptic operator on $\underline{E}G$ which is supported on a G -invariant subset $X \subset \underline{E}G$ with $G \backslash X$ compact. Following Kasparov we define a homomor-

phism of abelian groups

$$(0.3) \quad \mu : K_j^G(\underline{EG}) \longrightarrow K_j(C_r^*G) \quad (j = 0, 1),$$

by assigning to each elliptic operator its index.¹ Precise definitions are given in Section 3.

Our conjecture is that (0.3) is an isomorphism of abelian groups.

In many cases the K -homology groups $K_j^G(\underline{EG})$ may be explicitly identified (at least after tensoring with \mathbb{C}). If G is torsion-free and discrete then they are the K -homology groups of the classifying space BG :

$$K_j^G(\underline{EG}) \cong K_j(BG), \quad \text{for } G \text{ discrete and torsion-free.}$$

If G is a connected Lie group then $K_j^G(\underline{EG})$ can be calculated very easily from the representation theory of the maximal compact subgroup. If G is a reductive p -adic group then our K -homology groups identify, after tensoring by \mathbb{C} , with certain very interesting homology groups associated to the Bruhat-Tits building of G .

These identifications lead to various views on the conjecture (0.3) in various special cases:

If G is discrete then the groups $K_j(C_r^*(G))$ are best viewed as analytic counterparts to the Witt groups and L -groups studied in surgery theory, and our conjecture may be thought of as an analytic counterpart to the Borel conjecture (see [71]). The injectivity of (0.3) for discrete groups implies the Novikov higher signature conjecture.

If G is a reductive Lie or p -adic group then our conjecture provides, more or less, a description of the tempered dual of G , at the level of cohomology. This is because the identification (0.1) is close to correct for reductive groups, as long as \hat{G} is replaced by the tempered dual of G (= the support of the Plancherel measure).² If G is a semisimple Lie group with discrete series representations then the description of the tempered dual implied by (0.3) is closely related to the realization of the discrete series as solution spaces of twisted Dirac equations [6, 49]. If G is a reductive p -adic group the conjecture is closely related to the problem of realizing supercuspidal representations as induced from compact open subgroups [41].

These issues are explored in Sections 4–8.

A more general version of our conjecture may be formulated, involving “coefficients.” Let A be a C^* -algebra equipped with an action of G as C^* -algebra automorphisms. Form the reduced crossed-product C^* -algebra $C_r^*(G, A)$ and

¹Classes in the group $K_1^G(\underline{EG})$ are represented by *self-adjoint* operators. Kasparov’s construction associates to each such operator an index in $K_1(C_r^*(G))$.

²The identification is precise in certain instances where the tempered dual is a Hausdorff space—if G is complex semisimple, for example—but in general the groups $K_j(C_r^*(G))$ depend on more detailed representation theory, such as the reducibility of principal series representations and the theory of intertwining operators.

denote its K theory by $K_*(C_r^*(G, A))$. We define a homomorphism of abelian groups

$$(0.4) \quad \mu : K_j^G(\underline{E}G; A) \longrightarrow K_j(C_r^*(G, A)) \quad (j = 0, 1).$$

Again, our conjecture is that μ is always an isomorphism. In (0.4), $K_*^G(\underline{E}G; A)$ is the equivariant K -homology of $\underline{E}G$ with coefficients A , and with G -compact supports. See Section 9. The validity of (0.4) for a given group G implies the validity of (0.3) for G and all its closed subgroups.

There is by now considerable evidence in support of the conjecture (0.3) in cases where the group G admits a model for $\underline{E}G$ possessing geometric properties similar to those of a complete, simply connected, non-positively curved Riemannian manifold. This includes Lie groups, p -adic groups, and their closed subgroups. However, it must be said that for arbitrary groups the evidence in support of the conjecture is fragmentary. In this generality the conjecture may still not be in its final form.

The first steps towards the conjecture of this note were taken in [8, 59]. Hints and clues were provided by the conjecture of Connes-Kasparov [23, 36] and the Baum-Douglas isomorphism of analytic and topological K -homology [11]. Conjectures analogous to ours have been stated in other contexts, for example L -theory and algebraic K -theory [27] and the coarse geometry of metric spaces [57].

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1. Proper Actions

Let G be a second countable, locally compact and Hausdorff topological group. A G -space is a topological space X with a given continuous action of G on X ,

$$(1.1) \quad G \times X \longrightarrow X.$$

In order to avoid any extraneous issues in general topology we shall make the following assumptions:

$$(1.2) \quad \text{The spaces } X \text{ and } G \backslash X \text{ are metrizable.}$$

If X, Y are two G -spaces, a G -map from X to Y is a continuous G -equivariant map $f : X \longrightarrow Y$.

(1.3) DEFINITION. The action (1.1) of G on X is *proper* if for every $p \in X$ there exists a triple (U, H, ρ) such that:

- (i) U is an open neighborhood of p in X , with $gu \in U$ for all $(g, u) \in G \times U$,
- (ii) H is a compact subgroup of G ,
- (iii) $\rho : U \longrightarrow G/H$ is a G -map from U to the homogeneous space G/H .

In most geometric situations our notion of proper action coincides with the other available definitions (compare [48]).

(1.4) **EXAMPLE.** Let H be a compact subgroup of G and let S be an H -space. Then the space $G \times_H S$, formed by dividing $G \times S$ by the equivalence relation $(gh, s) \sim (g, hs)$, is a proper G -space (with the evident action of G).

Suppose that the group G has the following property:

(1.5) Every compact subgroup H of G is contained within a compact subgroup H' such that the projection $G \rightarrow G/H'$ admits continuous local sections. (In other words, G is a locally trivial principal H' -bundle over G/H' .)

This holds whenever G is finite dimensional [47]. It follows from (1.5) that every proper G -space X is covered by a family of G -invariant open sets, each of which is equivariantly homeomorphic to a proper G -space of the special form $G \times_H S$. Thus the action of G on X is “locally induced from an action of a compact subgroup.”

Two G -maps $f_0, f_1 : X \rightarrow Y$ are G -homotopic if they are homotopic through G -maps, that is, if there exists a homotopy $\{f_t\}$ $0 \leq t \leq 1$ with each f_t a G -map.

(1.6) **DEFINITION.** A *universal example* for proper actions of G , denoted \underline{EG} , is a proper G -space with the following property: *If X is any proper G -space, then there exists a G -map $f : X \rightarrow \underline{EG}$, and any two G -maps from X to \underline{EG} are G -homotopic.*

It is immediate from the definition that if \underline{EG} and $(\underline{EG})'$ are two universal examples then there exist G -maps

$$\begin{aligned} f : \underline{EG} &\rightarrow (\underline{EG})' \\ f' : (\underline{EG})' &\rightarrow \underline{EG} \end{aligned}$$

such that the compositions $f' \circ f$ and $f \circ f'$ are G -homotopic to the identity maps on \underline{EG} and $(\underline{EG})'$ respectively. Furthermore f and f' are unique up to G -homotopy.

(1.7) **PROPOSITION.** *There exists a universal example for proper actions of G .*

PROOF. Let W be the disjoint union, over all compact subgroups of G , of the homogeneous spaces G/H . Following Milnor [45], form the infinite join

$$\underline{EG} = W * W * W * \dots$$

It is a universal example. Details of the argument are given in Appendix 1. \square

As we shall see in the next section, it is usually possible to provide a much simpler and more interesting model for \underline{EG} . The following result is useful for this purpose.

(1.8) PROPOSITION. *Let Y be a proper G -space. Then Y is universal if and only if the following axioms hold:*

Axiom 1. *If H is any compact subgroup of G , then there exists $p \in Y$ with $hp = p$ for all $h \in H$.*

Axiom 2. *View $Y \times Y$ as a G -space with the usual diagonal action $g(y_0, y_1) = (gy_0, gy_1)$. Denote by $\rho_0, \rho_1: Y \times Y \rightarrow Y$ the two projections*

$$\rho_0(y_0, y_1) = y_0 \quad \text{and} \quad \rho_1(y_0, y_1) = y_1.$$

Then $\rho_0, \rho_1: Y \times Y \rightarrow Y$ are G -homotopic.

PROOF. See Appendix 2 below. \square

(1.9) COROLLARY. *Let G' be a closed subgroup of G . If Y is a universal example for proper G -actions, and if Y is proper as a G' -space then (upon restricting the action to G') it is also a universal example for proper G' -actions.*

PROOF. Viewed as a G' -space, Y is proper and satisfies the two axioms. \square

We remark that if G is a Lie group (not necessarily connected), and if G' is any closed subgroup, then every proper G -space is proper as a G' -space [48].

2. Examples of $\underline{E}G$

Suppose that Y is a simply-connected, complete Riemannian manifold with non-positive sectional curvatures. A well known theorem of E. Cartan asserts that every action of a compact group on Y by isometries has a fixed point. In addition, for any two points y_0 and y_1 in Y there is a unique geodesic $\gamma(t)$ such that $\gamma(0) = y_0$ and $\gamma(1) = y_1$. The prescription

$$\rho_t(y_0, y_1) = \gamma(t)$$

gives a homotopy between the two projection maps $\rho_0, \rho_1: Y \times Y \rightarrow Y$. So it follows from Proposition 1.8 that if any G acts properly on Y by isometries then Y is a universal example for proper G -actions. For example, if G is a semisimple Lie group then the action of G on the associated symmetric space $Y = G/K$ fits into this framework.

With this in mind, Proposition 1.8 presents us with an alternative view of $\underline{E}G$: it plays the role, at least at the crude level of homotopy, of a symmetric space for G . This point is amplified in some of the examples below.

NOTE. $\underline{E}G$ must be viewed as a G -space. In asserting that an example X is $\underline{E}G$ the action of G on X must be taken into account.

Compact Groups. If G is compact then every G -space is proper. So the trivial G -space consisting of a single point is universal.

Groups with no Compact Subgroups. If G has no compact subgroups other than the one-element subgroup then every proper G -space X is a locally trivial principal G -bundle over $G \backslash X$. Thus $\underline{E}G$ coincides with the universal principal G -space EG , familiar from topology.

Lie Groups. Let G be any Lie group with finitely many connected components. Up to conjugacy G has a unique maximal compact subgroup K , and $\underline{E}G = G/K$. For semisimple groups this follows from the discussion at the end of the previous section. Compare [1,17] for the general case.

It follows from Corollary 1.9 that if Γ is any discrete subgroup of G then $\underline{E}\Gamma = G/K$. The quotient space $\underline{B}\Gamma = \Gamma \backslash \underline{E}\Gamma$ is $\Gamma \backslash G/K$. Note that if Γ is torsion-free then $\Gamma \backslash G/K$ is a manifold. Generally it is an orbifold.

Almost-Connected Groups. It follows from the solution of Hilbert's fifth problem that if G is any locally compact group with a compact component group then there is a unique (up to conjugacy) maximal compact subgroup K . Once again, $\underline{E}G = G/K$.

p-adic Groups. Let G be a reductive p -adic algebraic group and let βG be its affine Bruhat-Tits building [66,67]. Then G acts properly on βG , and $\beta G = \underline{E}G$. As noted in [66], the building is geometrically similar to a symmetric space: there are unique geodesics and Cartan's fixed point theorem holds. So the remarks at the end of Section 1 apply.

Adelic Groups. Let \mathbb{A} be the ring of adeles, the restricted direct product of \mathbb{R} and the \mathbb{Q}_p (over all primes p). Let $\beta_{\mathbb{R}}$ be the symmetric space for $GL(n, \mathbb{R})$. Let $\beta_{\mathbb{Q}_p}$ be the affine building for $GL(n, \mathbb{Q}_p)$ and let v_p be the unique vertex in $\beta_{\mathbb{Q}_p}$ fixed by the compact subgroup $GL(n, \mathcal{O}_p)$ (where \mathcal{O}_p denotes the p -adic integers). Then the universal example for proper actions of the locally compact group $GL(n, \mathbb{A})$ is the restricted product

$$\beta_{\mathbb{A}} = \beta_{\mathbb{R}} \times \prod'_p \beta_{\mathbb{Q}_p},$$

consisting of sequences $(x_0, x_2, x_3, x_5, \dots)$ in the usual direct product such that $x_p = v_p$ for almost all p .

Discrete Groups. If Γ is a discrete group then there is a model for $\underline{E}\Gamma$ which is somewhat simpler than the infinite join construction of Proposition 1.7. Let

$$X_{\Gamma} = \{ f: \Gamma \rightarrow [0, 1] \mid f \text{ has finite support and } \sum_{\gamma \in \Gamma} f(\gamma) = 1 \},$$

equipped with the evident action of Γ by translation and the topology determined by the metric

$$d(f_1, f_2) = \sup_{\gamma \in \Gamma} |f_1(\gamma) - f_2(\gamma)|.$$

This is a proper Γ -space, and the axioms in Proposition 1.8 hold, so it is universal.

Note that as a set, X_{Γ} is simply the geometric realization of the simplicial complex whose p -simplices are all the $(p+1)$ -element subsets of Γ .

Hyperbolic Groups. Suppose that Γ is a hyperbolic group in the sense of Gromov [29]. Endow Γ with a word-length metric and for $R > 0$ let $X_{\Gamma}(R)$ be the geometric realization of the simplicial complex whose p -simplices are all the

$(p + 1)$ -element subsets of Γ of diameter less than R (this is the *Rips complex*). Of course $X_\Gamma(R)$ is a subspace of X_Γ above. With the inherited G -action and topology it is a universal example for proper Γ -actions, as long as R is sufficiently large.

Groups Acting on Trees. Let T be a tree on which G acts by a simplicial (and continuous) action, such that the stabilizer group of each vertex is compact. Then $T = \underline{EG}$. An example of this is $G = SL(2, \mathbb{Q}_p)$ acting on its tree [19, 63] (the tree for $SL(2, \mathbb{Q}_p)$ is its affine Bruhat-Tits building). Another example is the tree T which Serre [63] assigns to a free product with amalgamation $G = \Gamma_1 *_H \Gamma_2$, where Γ_1, Γ_2 and H are *finite* groups. Thus T is the double mapping cylinder of $G/\Gamma_1 \leftarrow G/H \rightarrow G/\Gamma_2$.

As is well known, $SL(2, \mathbb{Z})$ is the free product with amalgamation

$$(2.1) \quad SL(2, \mathbb{Z}) = (\mathbb{Z}/4\mathbb{Z}) \underset{\mathbb{Z}/2\mathbb{Z}}{*} (\mathbb{Z}/6\mathbb{Z})$$

The corresponding tree has alternately two and three edges emanating from each vertex. On the other hand $SL(2, \mathbb{Z})$ is a discrete subgroup of $SL(2, \mathbb{R})$, and so the symmetric space for $SL(2, \mathbb{R})$ —which is the Poincaré disc—is also a universal example for proper actions of $SL(2, \mathbb{Z})$. The relation between these two models for $\underline{ESL}(2, \mathbb{Z})$ can be seen by dividing the Poincaré disc into fundamental domains for $SL(2, \mathbb{Z})$ as shown in the illustration.

FIGURE 1. The Poincaré Disc and the Tree for $SL(2, \mathbb{Z})$

Each fundamental domain is a geodesic triangle with one vertex at infinity (the boundary of the disc) and two vertices in the interior. Thus each fundamental domain has precisely one edge both of whose vertices are in the interior of the Poincaré disc (drawn with bold lines in the figure). Let T be the union of all these interior edges, viewed as an $SL(2, \mathbb{Z})$ -space. It is the tree which Serre assigns to the free product with amalgamation (2.1).

According to the definition of universal example there should be an $SL(2, \mathbb{Z})$ -equivariant map from the Poincaré disc to T . In fact one can construct a deformation retraction by contracting any point p in the disc to T along the geodesic passing through p from the vertex at infinity of the triangle (= fundamental domain) containing p . This verifies that T is $SL(2, \mathbb{Z})$ -homotopy equivalent to the Poincaré disc.

\mathbb{R} -Trees. Let Γ be a discrete group acting by a proper isometric action on an \mathbb{R} -tree T [64]. Then T is $\underline{E}\Gamma$.

3. Equivariant K-Homology

In this section we use Kasparov's KK -theory to precisely formulate our conjecture. It should be emphasized that the equivariant K -homology given below differs from the Borel construction familiar to topologists.

Let A be a C^* -algebra. A *pre-Hilbert A -module* is a left³ A -module equipped with an A -valued inner product $\langle \cdot, \cdot \rangle$ (which satisfies the usual axioms for a Hilbert space inner product, but with the scalars \mathbb{C} replaced by A). A *Hilbert A -module* is a pre-Hilbert A -module which is complete with respect to the norm

$$(3.1) \quad \|v\| = \|\langle v, v \rangle\|^{1/2}.$$

Thus a Hilbert \mathbb{C} -module is simply a Hilbert space.

In what follows we restrict our attention to *countably generated* Hilbert A -modules (those having a countable subset with dense A -linear span).

An *operator* between two Hilbert A -modules is an A -linear map T which possesses an adjoint T^* with respect to the given inner products:

$$\langle Tv, w \rangle = \langle v, T^*w \rangle.$$

It is automatically bounded with respect to the norm (3.1).

An operator is *compact* if it is a norm limit of operators of the form

$$v \mapsto \sum_{i=1}^n \langle v, v_i \rangle w_i.$$

An operator is *Fredholm* if it is invertible modulo compact operators. See [14] for details. If $A = \mathbb{C}$ then these definitions give the usual notions in Hilbert space.

Kasparov shows [34] that each Fredholm operator \mathcal{F} between Hilbert A -modules has an index

$$(3.2) \quad \text{Index}(\mathcal{F}) \in K_0(A).$$

³Kasparov works with right modules, but for our purposes left modules seem more convenient since this choice agrees with the usual conventions regarding Hilbert spaces.

If A is unital, and if the kernel and cokernel of F are finitely generated projective A -modules, then

$$(3.3) \quad \text{Index}(\mathcal{F}) = [\text{kernel}(\mathcal{F})] - [\text{cokernel}(\mathcal{F})] \in K_0(A).$$

In general neither the kernel nor the cokernel of \mathcal{F} is a finitely generated projective module, in which case a somewhat more complicated construction of the index must be used. There is a parallel index theory for *self-adjoint* Fredholm operators, with

$$(3.4) \quad \text{Index}(\mathcal{F}) \in K_1(A), \quad \text{if } \mathcal{F} \text{ is self-adjoint.}$$

See [34].⁴

(3.5) DEFINITION. A proper G -space X is *G -compact* if the quotient space $G \backslash X$ is compact. Note that since G is locally compact, every G -compact, proper G -space is locally compact.

(3.6) DEFINITION. Let X be a G -compact, proper G -space. A *G -equivariant abstract elliptic operator* on X is a triple (H_+, H_-, F) , where:

- (1) H_+ and H_- are Hilbert spaces equipped with unitary G -representations and G -covariant representations π_{\pm} of the C^* -algebra $C_0(X)$.
- (2) F is a bounded, G -equivariant Hilbert space operator from H_+ to H_- .
- (3) The operators

$$\pi_-(\varphi)F - F\pi_+(\varphi)$$

are compact, for every $\varphi \in C_0(X)$.

- (4) There exists a bounded, G -equivariant operator $Q: H_- \rightarrow H_+$ such that the operators

$$\pi_-(\varphi)(FQ - I) \quad \text{and} \quad \pi_+(\varphi)(QF - I)$$

are compact, for every $\varphi \in C_0(X)$.

For simplicity we shall work with operators F which are *properly supported*. This means that for each compactly supported function $\varphi \in C_c(X)$ there is another compactly supported function $\varphi' \in C_c(X)$ such that

$$\pi_-(\varphi')F\pi_+(\varphi) = F\pi_+(\varphi).$$

Every G -equivariant abstract elliptic operator may be perturbed to one which is properly supported.

⁴In fact it is possible to *define* the K -theory groups as universal receivers for the indices of elliptic operators, subject to a few conditions such as the index of an invertible operator being zero, and so on. This is more or less how Kasparov proceeds.

(3.7) **EXAMPLE.** Let X be a smooth manifold, equipped with a smooth, proper action of a Lie group G with $G \backslash X$ compact. Let E_{\pm} be G -equivariant, smooth hermitian vector bundles on X , and let

$$H_{\pm} = L^2(X; E_{\pm})$$

be the Hilbert spaces of square integrable sections of E_{\pm} , equipped with their natural representations of G and $C_0(X)$. Let F be a properly supported, G -equivariant, order zero elliptic pseudodifferential operator mapping sections of E_+ to sections of E_- . Then the triple (H_+, H_-, F) satisfies the conditions of Definition 3.6.

(3.8) **DEFINITION.** Following Kasparov, we associate to each G -equivariant abstract elliptic operator a G -index

$$(3.9) \quad \text{Index}_G(F) \in K_0(C_r^*(G)).$$

Form the complex vector spaces

$$\mathcal{H}_{\pm}^0 = \pi_{\pm}(C_c(X))H_{\pm}.$$

They are modules for the convolution algebra $C_c(G) \subseteq C_r^*(G)$ and carry the following $C_c(G)$ -valued inner product:

$$\langle v_1, v_2 \rangle(g) = (v_1, U_{\pm}(g)v_2).$$

Here $(\ , \)$ denotes the Hilbert space inner product in H_{\pm} and $U_{\pm}(g)$ is the unitary operator on H_{\pm} implementing the action of $g \in G$. We define

$$\mathcal{H}_{\pm} = \text{Completion of } \mathcal{H}_{\pm}^0 \text{ in the norm (3.1).}$$

These are Hilbert $C_r^*(G)$ -modules, and the operator $F: H_+ \rightarrow H_-$ passes to an operator

$$\mathcal{F}: \mathcal{H}_+ \rightarrow \mathcal{H}_-.$$

Thanks to the axioms for an abstract elliptic operator (and the fact that X is G -compact), \mathcal{F} is a Fredholm operator. We define

$$\text{Index}_G(F) = \text{Index}(\mathcal{F}) \in K_0(C_r^*(G)).$$

If F is self-adjoint then (3.4) gives a G -index

$$(3.10) \quad \text{Index}_G(F) \in K_1(C_r^*(G)), \quad \text{for } F \text{ self-adjoint.}$$

(3.11) **EXAMPLE.** Let M be a smooth, closed manifold with *abelian* fundamental group. Denote by X the universal covering space of M ,

$$X = \widetilde{M},$$

equipped with the natural action of $G = \pi_1(M)$ as deck transformations. It is a G -compact, proper G -space. Let D be an order zero, elliptic pseudodifferential operator on M (for simplicity we shall think of it as acting on scalar functions).

Assume that the distribution kernel for D is supported sufficiently near the diagonal in $M \times M$ that D lifts to an operator

$$F: L^2(X) \rightarrow L^2(X).$$

As in example (3.7), F is a G -equivariant abstract elliptic operator on X .

To identify the G -index of F we use the Fourier transform isomorphism

$$C_r^*(G) \cong C(\hat{G}),$$

where \hat{G} is the Pontrjagin dual, viewed as a compact space (the group structure of \hat{G} is ignored). Every Hilbert $C(\hat{G})$ -module identifies with the space of continuous sections of some continuous field of Hilbert spaces over \hat{G} . The module \mathcal{H} formed from the Hilbert space $L^2(X)$ corresponds to the field $\{L^2(M; L_\alpha)\}_{\alpha \in \hat{G}}$, where L_α is the flat complex line bundle over M with holonomy α . The operator $\mathcal{F}: \mathcal{H} \rightarrow \mathcal{H}$ formed from F identifies with the field of operators $\{D_\alpha\}_{\alpha \in \hat{G}}$ obtained by lifting D so as to act on sections of the bundles L_α .

Assume, as we did in the introduction, that the families of vector spaces $\{\ker(D_\alpha)\}_{\alpha \in \hat{G}}$ and $\{\text{cokernel}(D_\alpha)\}_{\alpha \in \hat{G}}$ are vector bundles on \hat{G} (this will be so if $\dim(\ker(D_\alpha))$ is a locally constant function of α). Then, in view of (3.3), $\text{Index}(\mathcal{F})$ identifies with the element

$$[\{\ker(D_\alpha)\}_{\alpha \in \hat{G}}] - [\{\text{cokernel}(D_\alpha)\}_{\alpha \in \hat{G}}]$$

of the group $K^0(\hat{G}) \cong K_0(C(\hat{G}))$. So in this special case the G -index of Definition 3.8 is the same as the index discussed in the introduction.

We return now to general G -compact, proper G -spaces X . Kasparov defines

$$K_0^G(X) = \left\{ \begin{array}{l} \text{homotopy classes of } G\text{-equivariant} \\ \text{abstract elliptic operators on } X \end{array} \right\}$$

(his notation for $K_0^G(X)$ is $KK_G^0(C_0(X), \mathbb{C})$). This is an abelian group, and there is a companion group

$$K_1^G(X) = \left\{ \begin{array}{l} \text{homotopy classes of } \textit{self-adjoint } G\text{-equivariant} \\ \text{variant abstract elliptic operators on } X \end{array} \right\}.$$

Kasparov shows that the two combine to form a periodic homology theory for G -compact proper G -spaces. See [34, 35] for precise definitions and further details.

(3.12) NOTES.

(i) As we have already mentioned, this is not the same as the “Borel construction” of equivariant K -homology.

(ii) If G is a discrete group and if the action of G on X is free, as well as proper, then $K_j^G(X)$ identifies with the K -homology (as in topology—the homology theory associated to the Bott spectrum) of the quotient space $G \backslash X$.

(iii) In particular, if G is the trivial one-element group then $K_j^G(X)$ is the K -homology of X .

(iv) If G is *not* discrete then the analogue of item (ii) above does not hold.

We extend Kasparov's definition to arbitrary (as opposed to G -compact) proper G -spaces in the following way.

(3.13) DEFINITION. Let Z be any proper G -space. The equivariant K -homology of Z with G -compact supports, denoted $K_j^G(Z)$, is

$$K_j^G(Z) = \varinjlim_{\substack{X \subset Z \\ X \text{ } G\text{-compact}}} K_j^G(X) \quad (j = 0, 1),$$

where the direct limit is over the directed system of all G -invariant, G -compact subsets of Z .

If X is a G -compact, proper G -space then associating to each abstract elliptic operator its G -index we obtain a map

$$\begin{aligned} \mu_X : K_j^G(X) &\rightarrow K_j(C_r^*(G)) & (j = 0, 1) \\ \mu_X(H_+, H_-, F) &= \text{Index}_G(F). \end{aligned}$$

If Z is any proper G -space then the maps μ_X , for X a G -invariant G -compact subset of Z , are compatible with the direct limit in (3.13), and yield a homomorphism

$$(3.14) \quad \mu : K_j^G(Z) \longrightarrow K_j(C_r^*(G)) \quad (j = 0, 1).$$

We can now precisely formulate our conjecture.

(3.15) CONJECTURE. *Let G be a locally compact, Hausdorff, second countable, topological group, and let $\underline{E}G$ be a universal example for proper actions of G . Then*

$$\mu : K_j^G(\underline{E}G) \longrightarrow K_j(C_r^*(G)) \quad (j = 0, 1),$$

is an isomorphism.

4. Lie Groups

The Reduced C^* -Algebra of a Lie Group. The purpose of this subsection is to give the reader some insight into the structure of $C_r^*(G)$ and its K -theory by means of several examples.⁵

(4.1) EXAMPLE. Let $G = \mathbb{R}^2$. As noted already, the Fourier transform gives an isomorphism

$$C_r^*(\mathbb{R}^2) \cong C_0(\mathbb{R}^2).$$

The K -theory of $C_r^*(\mathbb{R}^2)$ identifies with the Atiyah-Hirzebruch K -theory of the locally compact space \mathbb{R}^2 . So by the Bott Periodicity Theorem,

$$K_j(C_r^*(\mathbb{R}^2)) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1. \end{cases}$$

⁵If G is a compact Lie group (or indeed any compact group at all) then our conjecture is readily verified from the Peter-Weyl Theorem. For this reason we shall concentrate on non-compact groups.

(4.2) **EXAMPLE.** Let $G = SL(2, \mathbb{C})$. Denote by M the diagonal unitary matrices, A the diagonal matrices with positive real entries, and N the unipotent upper triangular matrices, so that $P = MAN$ is the “Borel subgroup” of G comprised of the upper triangular matrices. The diagonal group MA is isomorphic to $S^1 \times \mathbb{R}$ via the correspondence

$$(e^{i\theta}, t) \leftrightarrow \begin{pmatrix} e^{t+i\theta} & 0 \\ 0 & e^{-(t+i\theta)} \end{pmatrix}.$$

Each character

$$(n, \lambda): (e^{i\theta}, t) \mapsto e^{in\theta + i\lambda t}$$

in $\widehat{MA} \cong \mathbb{Z} \times \mathbb{R}$ determines a unitary principal series representation $\pi_{(n, \lambda)}$ by first extending the character to MAN then inducing unitarily to G [40]. These representations are all irreducible and distinct, except for the relation that

$$(4.3) \quad \pi_{(n, \lambda)} \cong \pi_{(-n, -\lambda)}.$$

The space $\widehat{MA}/\mathbb{Z}_2$ obtained by making the identification in (4.3) comprises the tempered dual of G . The C^* -algebra $C_r^*SL(2, \mathbb{C})$ may be described as the algebra of continuous compact operator-valued functions, vanishing at infinity, on the locally compact space $\widehat{MA}/\mathbb{Z}_2$:

$$(4.4) \quad C_r^*SL(2, \mathbb{C}) \cong C_0(\widehat{MA}/\mathbb{Z}_2, \mathcal{K}).$$

The isomorphism is implemented by mapping an L^1 -function f on G to the operator valued function $(n, \lambda) \mapsto \pi_{(n, \lambda)}(f)$.

To explicitly calculate the K -theory we note that $\widehat{MA}/\mathbb{Z}_2$ identifies with the following subspace of \widehat{MA} :

$$\widehat{MA}/\mathbb{Z}_2 \cong \{0\} \times [0, \infty) \cup \{1\} \times \mathbb{R} \cup \{2\} \times \mathbb{R} \cup \{3\} \times \mathbb{R} \cup \dots.$$

So we see that

$$(4.5) \quad K_j(C_r^*(SL(2, \mathbb{C}))) = \bigoplus_{n=1}^{\infty} K^j(\{n\} \times \mathbb{R}) = \begin{cases} \bigoplus_{n=1}^{\infty} \mathbb{Z} & \text{if } j = 1 \\ 0 & \text{if } j = 0. \end{cases}$$

Similar remarks apply to any complex semisimple group G . Using Harish-Chandra’s Plancherel Theorem, M. Penington and R. Plymen [52] obtain a Morita equivalence⁶

$$(4.6) \quad C_r^*(G) \underset{\text{Morita}}{\sim} C_0(\widehat{MA}/W),$$

where $P = MAN$ is a minimal parabolic subgroup of G and W is the Weyl group. By analyzing the action of W on \widehat{MA} one can show that

$$(4.7) \quad K_j(C_r^*(G)) = \begin{cases} \bigoplus_{[\sigma] \text{ regular}} \mathbb{Z} & \text{if } j \equiv \dim(A) \pmod{2} \\ 0 & \text{if } j \not\equiv \dim(A) \pmod{2}, \end{cases}$$

⁶In other words an isomorphism after tensoring with the compact operators—an operation which does not alter K -theory groups.

where the sum is over the W -orbits of weights $\sigma \in \hat{M}$ which are fixed by no non-trivial element of W .

(4.8) **EXAMPLE.** Let $G = SL(2, \mathbb{R})$. Its reduced C^* -algebra may be determined in much the same way as Example 4.2 above, but there are one or two noteworthy differences between the real and complex groups. The principal series representations of $SL(2, \mathbb{R})$ are parametrized by characters

$$(\sigma, \lambda) \in \widehat{MA} \cong \{\pm 1\} \times \mathbb{R},$$

modulo the action of the Weyl group \mathbb{Z}_2 . But unlike the complex case not all the principal series representations are irreducible (the representation $\pi_{(-1,0)}$ decomposes as a sum of two “limit of discrete series” representations), a fact which must be taken into account when calculating $C_r^*SL(2, \mathbb{R})$ and its K -theory. Whereas

$$\begin{aligned} \widehat{MA}/\mathbb{Z}_2 &\cong \{+1\} \times [0, \infty) \cup \{-1\} \times [0, \infty) \\ &\cong \{+1\} \times \mathbb{R}/\mathbb{Z}_2 \cup \{-1\} \times \mathbb{R}/\mathbb{Z}_2, \end{aligned}$$

the principal series contribute summands to $C_r^*SL(2, \mathbb{R})$ of the form

$$C_0(\mathbb{R}/\mathbb{Z}_2) \quad \text{and} \quad C_0(\mathbb{R}) \rtimes \mathbb{Z}_2,$$

up to Morita equivalence (the second term is a C^* -algebra crossed product [50]). In addition $SL(2, \mathbb{R})$ has discrete series representations each of which contributes a summand of \mathbb{C} to $C_r^*SL(2, \mathbb{R})$, up to Morita equivalence. We obtain:

$$C_r^*SL(2, \mathbb{R}) \underset{\text{Morita}}{\sim} C_0(\mathbb{R}/\mathbb{Z}_2) \oplus C_0(\mathbb{R}) \rtimes \mathbb{Z}_2 \oplus \bigoplus_{n \in \mathbb{Z} \setminus \{0\}} \mathbb{C},$$

where the last sum has one term for each discrete series representation (the labels are the Harish-Chandra parameters). The middle summand contributes one copy of \mathbb{Z} to K_0 , and if we label it by $n = 0$ (the Harish-Chandra parameter for the limit of discrete series representations) we get

$$(4.9) \quad K_j(C_r^*SL(2, \mathbb{R})) = \begin{cases} \bigoplus_{n=-\infty}^{\infty} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1. \end{cases}$$

An analysis of real rank one semisimple Lie groups along these lines has been carried out by A. Valette [68]. A. Wassermann [70] has analyzed the structure of $C_r^*(G)$ for any connected, linear reductive group. His final result is a Morita equivalence

$$(4.10) \quad C_r^*(G) \underset{\text{Morita}}{\sim} \bigoplus_P \bigoplus_{[\sigma] \in \widehat{M}/W} C_0(\widehat{A}/W'_\sigma) \rtimes R_\sigma,$$

where: the first sum is over conjugacy classes of cuspidal parabolic subgroups $P = MAN$; the second is over representatives of the Weyl-group orbits of discrete series representations $\sigma \in \widehat{M}$; W_σ denotes the stabilizer of σ ; and $W_\sigma = W'_\sigma \rtimes R_\sigma$

is the R -group decomposition of W_σ [40]. Using equivariant K -theory for locally compact spaces he obtains:

$$K_j(C_r^*(G)) = \oplus_P \oplus_{[\sigma] \in \widehat{M}/W} K_{R_\sigma}^j(\widehat{A}/W'_\sigma),$$

and goes on to show that

$$K_{R_\sigma}^j(\widehat{A}/W'_\sigma) = \begin{cases} \mathbb{Z} & \text{if } W'_\sigma = 1 \text{ and } j \equiv \dim G/K \\ 0 & \text{otherwise} \end{cases}$$

(K denotes a maximal compact subgroup). Hence

$$(4.11) \quad K_j(C_r^*(G)) = \begin{cases} \oplus_P \oplus_{\{[\sigma]: W'_\sigma=1\}} \mathbb{Z} & \text{if } j \equiv \dim(G/K) \\ 0 & \text{if } j \not\equiv \dim(G/K). \end{cases}$$

We refer the reader to [70] for a reparametrization of (4.11) in terms of limits of discrete series and, ultimately, in terms of weights of the maximal compact subgroup K . See also [51].

(4.12) EXAMPLE. Let G be the “ $ax + b$ group”

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+ \text{ and } b \in \mathbb{R} \right\}.$$

From the point of view of abstract C^* -algebra theory, $C_r^*(G)$ is a somewhat more complicated object than in the previous examples. To describe its structure we must proceed as follows. If $f(a, b)$ is a smooth and compactly supported function on G then denote by $\hat{f}(a, \xi)$ the function obtained by taking Fourier transform in the b -variable. Let

$$\mathcal{K}_\pm^0 = \{ f(a, b) : \hat{f}(a, \xi) = 0 \text{ if } \pm \xi > 0 \},$$

and let \mathcal{K}_\pm be the completions of \mathcal{K}_\pm^0 in $C_r^*(G)$. They are orthogonal, two sided ideals in $C_r^*(G)$, each abstractly isomorphic to the C^* -algebra of compact operators on a separable Hilbert space. Taking the quotient of $C_r^*(G)$ by the ideal $\mathcal{K}_+ \oplus \mathcal{K}_-$ we obtain a short exact sequence of C^* -algebras

$$(4.13) \quad 0 \rightarrow \mathcal{K}_+ \oplus \mathcal{K}_- \rightarrow C_r^*(G) \xrightarrow{\pi} C_r^*(\mathbb{R}^+) \rightarrow 0,$$

where π maps $f(a, b)$ to $\hat{f}(a, 0)$. Of course $C_r^*(\mathbb{R}^+)$ is isomorphic to $C_0(\mathbb{R})$. Now it is a simple special case of the theory developed by L. Brown, R. Douglas and P. Fillmore [18] that two C^* -algebra extensions of $C_r^*(\mathbb{R}^+)$ by $\mathcal{K}_+ \oplus \mathcal{K}_-$, like (4.13), which give rise to the same connecting homomorphism

$$(4.14) \quad K_1(C_r^*(\mathbb{R}^+)) \xrightarrow{\partial} K_0(\mathcal{K}_+) \oplus K_0(\mathcal{K}_-)$$

in K -theory, are in fact unitarily equivalent. So a determination of the K -theory of $C_r^*(G)$, which is more or less the same thing as calculating (4.14), allows

one to characterize $C_r^*(G)$ as a C^* -algebra. Compare [73] for the K -theory calculation, the result of which is that

$$K_j(C_r^*(G)) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ 0 & \text{if } j = 1. \end{cases}$$

The C^* -algebras of certain other solvable groups may be analyzed in a similar way. Although the conclusions are not typically as strong as those above (K -theory calculations rarely characterize $C_r^*(G)$ up to isomorphism) they do provide insight into the structure of the C^* -algebra.

Reformulation of the Conjecture. Let G be a connected Lie group and let K be a maximal compact subgroup of G . We shall reformulate our conjecture (3.15) for G in the language of Dirac operators.⁷

Denote by \mathfrak{p} a vector space complement of the Lie algebra of K inside the Lie algebra of G ,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

which is invariant under the adjoint action of K . Provide it with a K -invariant inner product.

We review a few facts about spinors. A *Clifford algebra representation* of the vector space \mathfrak{p} is an \mathbb{R} -linear map

$$c: \mathfrak{p} \rightarrow \text{End}(E)$$

from \mathfrak{p} into the endomorphisms of a finite dimensional complex inner product space such that

$$(4.15) \quad c(P)^* = -c(P), \quad \text{and} \quad c(P)^2 = -\|P\|^2 I,$$

for all $P \in \mathfrak{p}$. Every Clifford algebra representation decomposes into a direct sum of irreducibles. If \mathfrak{p} is even dimensional then there is a unique irreducible representation S , up to unitary equivalence, while if \mathfrak{p} is odd dimensional there are two: choose one of them and denote it by S . In either case there is a natural Lie algebra representation

$$\mathfrak{so}(\mathfrak{p}) \xrightarrow{\sigma} \text{End}(S).$$

It is compatible with the Clifford algebra representation on S , in the sense that

$$(4.16) \quad c(\text{Ad}(k)P) = \sigma(k) \cdot c(P),$$

⁷This reformulation of the conjecture seems most appropriate for *unimodular* groups.

for all $P \in \mathfrak{p}$ and $k \in \mathfrak{so}(\mathfrak{p})$. For the rest of this section we assume the following condition holds:⁸

The composition

$$\mathfrak{k} \xrightarrow{\text{ad}} \mathfrak{so}(\mathfrak{p}) \xrightarrow{\sigma} \text{End}(S)$$

(4.17) *exponentiates to a representation*

$$\chi: K \rightarrow \text{Aut}(S).$$

Let V be an irreducible representation of K and form the tensor product $S \otimes V$. Extend the Clifford structure on S to one on $S \otimes V$ by taking the tensor products $c(P) \otimes 1$. Denote by $C_c^\infty(G/K; S \otimes V)$ the vector space of smooth, compactly supported functions $\zeta: G \rightarrow V$ which transform according to the law

$$\zeta(gk) = \pi(k^{-1})\zeta(g) \quad (g \in G \text{ and } k \in K).$$

The group G acts on $C_c^\infty(G/K; S \otimes V)$ by left translation. Define an operator by the formula

$$(4.18) \quad D_V \zeta = \sum_i c(P_i) P_i(\zeta),$$

where $\{P_1, \dots, P_k\}$ is any orthonormal basis for \mathfrak{p} and we view P_i as a left invariant vector field on G . D_V does not depend on the choice of orthonormal basis. It follows from (4.16) that D_V is an operator on $C_c^\infty(G/K; V)$; clearly it is G -equivariant.

(4.19) REMARK. The space $C_c^\infty(G/K; S \otimes V)$ identifies with the smooth, compactly supported sections of the G -equivariant vector bundle on G/K induced from the representation $S \otimes V$. In this way the operator D_V identifies with the elliptic partial differential operator of order one on G/K obtained from the Dirac operator on G/K by “twisting” with the vector bundle induced from V .

The vector space $C_c^\infty(G/K; S \otimes V)$ completes to a Hilbert $C_r^*(G)$ -module $C_r^*(G/K, S \otimes V)$ as indicated in Section 3, and D_V determines a Fredholm operator on this module.⁹ If G/K is even dimensional then the spin representation S splits, as a representation of K , into a direct sum $S = S_+ \oplus S_-$, and we view D_V as an operator

$$D: C_r^*(G/K; S_+ \otimes V) \rightarrow C_c^\infty(G/K; S_- \otimes V).$$

If G/K is odd dimensional then no such reduction is possible, and we view D_V as a self-adjoint operator on $C_r^*(G/K; S \otimes V)$. In either case we can form the

⁸In the case where the Lie algebra representation does not exponentiate one considers a suitable double cover G_1 of G where (4.17) is valid. Analogues for G of the various assertions made below are then easily worked out using G_1 .

⁹It is an unbounded operator, although it is not difficult to manufacture from D_V a bounded operator, and so remain within the scope of the theory outlined in Section 3.

quantity

$$\text{Index}(D_V) \in K_j(C_r^*(G)) \quad (j \equiv \dim(G/K) \pmod{2}).$$

(4.20) CONNES-KASPAROV CONJECTURE. *Let $j \equiv \dim(G/K) \pmod{2}$. Define*

$$(4.21) \quad \tilde{\mu}: R(K) \rightarrow K_j(C_r^*(G))$$

by associating to each representation $[V] \in R(K)$ the G -index of the twisted Dirac operator D_V . Then $\tilde{\mu}$ is an isomorphism of abelian groups. In addition,

$$K_{j+1}(C_r^*(G)) = 0.$$

The following result of Kasparov shows that this is equivalent to the conjecture (3.15):

(4.22) PROPOSITION. *Let $j \equiv \dim(G/K) \pmod{2}$. The map*

$$R(K) \rightarrow K_j^G(G/K)$$

which associates to $[V] \in R(K)$ the K -homology class of the operator D_V is an isomorphism of abelian groups. In addition

$$K_{j+1}^G(G/K) = 0.$$

(4.23) EXAMPLE. Let $G = \mathbb{R}^n$. Of course the maximal compact subgroup of \mathbb{R}^n is the trivial group, and so the left hand side of (4.22) is simply the abelian group \mathbb{Z} . The Dirac operator defined in (4.18) is the usual (Euclidean) Dirac operator on \mathbb{R}^n . To calculate its index we use the Fourier transform, under which D corresponds to the matrix valued function

$$\sqrt{-1} \sum_{i=1}^n x_i c(X_i)$$

on \mathbb{R}^n . The corresponding K -theory class

$$\text{Index}_{\mathbb{R}^n}(D) \in K_j(C_r^*(\mathbb{R}^n)) \cong K_j(C_0(\mathbb{R}^n)) \cong K^j(\mathbb{R}^n)$$

(where $j \equiv n \pmod{2}$) is the “Bott generator,” and the conjecture (4.20) follows from the Bott Periodicity Theorem.

(4.24) EXAMPLE. Penington and Plymen [52] have verified (4.20) for all connected complex semisimple groups. We give a brief account of their argument, specialized to the group $G = SL(2, \mathbb{C})$.

The irreducible representations of $K = SU(2)$ are parametrized by their “highest weights” $k = 0, 1, 2, \dots$, so that

$$R(SU(2)) \cong \bigoplus_{k=0}^{\infty} \mathbb{Z}.$$

We shall denote by D_k the Dirac operator on G/K twisted by the irreducible representation with highest weight k . To calculate the index of D_k it is convenient to use the Morita equivalence

$$C_r^*SL(2, \mathbb{C}) \underset{\text{Morita}}{\sim} C_0(\widehat{MA}/\mathbb{Z}_2)$$

which gives an isomorphism

$$K_1(C_r^*SL(2, \mathbb{C})) \cong K^1(\widehat{MA}/\mathbb{Z}_2) \cong \bigoplus_{n=1}^{\infty} K^1(\{n\} \times \mathbb{R}).$$

The module on which the Dirac operator D_k acts corresponds to the vector bundle E over $\widehat{MA}/\mathbb{Z}_2$ whose fiber at (n, λ) is

$$E_{(n, \lambda)} = (H_{\pi_{(n, \lambda)}} \otimes S \otimes V_k)^{SU(2)}.$$

Using Frobenius reciprocity one calculates that

$$(4.25) \quad \dim_{\mathbb{C}}((H_{\pi_{(n, \lambda)}} \otimes S \otimes V_k)^{SU(2)}) = \begin{cases} 1 & \text{if } n = k + 1 \\ 0 & \text{if } k + 1 < n \end{cases}$$

(the dimension for $n < k + 1$ does not concern us), and on the basis of this calculation we see that the index of D_k lies in the components of K^1 labelled by $n \geq k + 1$.

The Dirac operator¹⁰ acts as an endomorphism of the vector bundle E . A modest amount of calculation¹¹ reveals that D_k^2 acts as multiplication by the scalar

$$(4.26) \quad D_k^2 = (k + 1)^2 - n^2 + \lambda^2$$

in the fiber over (n, λ) . So D_k is invertible over the components of $\widehat{MA}/\mathbb{Z}_2$ where $n < k + 1$ and on the basis of this we see that the index of D_k lies in the components of K^1 where $n \leq k + 1$.

It follows that the index of D_k lies in the component of K^1 where $n = k + 1$. But when $n = k + 1$ another representation theory calculation reveals that the function mapping λ to the endomorphism $D_k: E_{(n, \lambda)} \rightarrow E_{(n, \lambda)}$ is linear (note that the vector spaces $E_{(n, \lambda)}$ are canonically isomorphic for different λ). Putting this together (4.26) we see that D_k gives in effect a Clifford algebra representation for the vector space $\hat{A} \cong \mathbb{R}$ (made into an inner product space using the Killing form). By (4.25) it is an irreducible representation, and so by the Bott Periodicity Theorem the index of D_k is a generator for the K -theory group $K^1(\{n\} \times \mathbb{R})$.

¹⁰To form the Dirac operator we choose \mathfrak{p} be the orthogonal complement of $\mathfrak{su}(2)$ with respect to the Killing form—the self-adjoint, trace zero matrices. The restriction of the Killing form to \mathfrak{p} is an invariant, \mathbb{R} -valued inner product; to simplify formulas we scale it by $1/8$.

¹¹We have compressed two calculations into one: the first expresses D_k^2 as a scalar translate of the Casimir operator and the second calculates the action of the Casimir operator in a principal series representation.

(4.25) EXAMPLE. Let $G = SL(2, \mathbb{R})$. Then of course $K = SO(2)$ and

$$R(K) \cong \bigoplus_{k \in \mathbb{Z}} \mathbb{Z}.$$

Denote by D_k the Dirac operator twisted by the representation of $SO(2)$ of weight k .

As far as the principal series representations are concerned, the calculation of the map μ proceeds in much the same fashion as for $SL(2, \mathbb{C})$. One finds that the index on the Dirac operator D_1 is the K -theory generator for the principal series and that for $k \neq 0$ the principal series component of the index of D_k is zero.

Denote by H_n the discrete series representation with parameter $n > 0$ (the representations with parameter $n < 0$ are treated similarly). When restricted to $SO(2)$, it decomposes into a direct sum of irreducible representations with weights $n, n+2, n+4, \dots$. It follows that the tensor product representation $H_n \otimes S_{\pm} \otimes V_k$ has weights

$$n \pm 1 + k, n \pm 1 + k + 2, n \pm 1 + k + 4, \dots,$$

and therefore

$$\dim((H_n \otimes S_+ \otimes V_k)^{SO(2)}) - \dim((H_n \otimes S_- \otimes V_k)^{SO(2)}) = \begin{cases} -1 & \text{if } n = 1 - k \\ 0 & \text{if } n \neq 1 - k \end{cases}.$$

This calculation implies that the discrete series part of $\text{Index}_G(D_k)$ is (minus) the K -theory generator corresponding to H_{1-k} .

We find that the map μ is the isomorphism which corresponds (up to sign) the generators $k \leftrightarrow n = 1 - k$ in (4.9).

The above examples give a good indication of the many ingredients in Wassermann's proof of the conjecture (4.20) for general linear reductive Lie groups: the precise calculation of $C_r^*(G)$ using the Plancherel Theorem and the theory of intertwining operators; a calculation of the Dirac operator in the principal series; a description of the K -types in discrete series¹² representations (= Blattner's Conjecture); and finally, the Bott Periodicity Theorem in equivariant K -theory.

Discrete Series Representations. According to Harish-Chandra's classification, a semisimple group G (connected, with finite center) possesses discrete series representations if and only if it has a compact Cartan subgroup T (which we can take to be a maximal torus in K). Supposing this to be the case, the discrete series representations of G are in one to one correspondence with the *regular* characters of T , modulo the action of the Weyl group of K on \hat{T} . As shown in [6, 49], they may be realized “geometrically” as follows.

¹²And limit of discrete series.

(4.26) **THEOREM.** *Let G be a connected semisimple Lie group with finite center. Suppose that $\text{rank}(G) = \text{rank}(K)$, and let $T \subseteq K$ be a maximal torus. Let π_η be the discrete series representation of G corresponding to the regular character $\eta \in \hat{T}_{\text{reg}}$. Denote by $\rho_K \in \hat{T}$ the half-sum of the positive weights of K and denote by $D_{\eta-\rho_K}$ the Dirac operator twisted by the irreducible representation of K with highest weight $\eta - \rho_K$. Then*

$$\ker(D_{\eta-\rho_K}^+) \cong \pi_\eta \quad \text{and} \quad \ker(D_{\eta-\rho_K}^-) = 0$$

(for a suitable orientation of G/K).

As (4.11) indicates, each discrete series representation contributes a direct factor of \mathcal{K} (the compact operators) to $C_r^*(G)$, and so contributes a summand \mathbb{Z} to $K_0(C_r^*G)$. So there is obviously a very close correspondence between (4.26) and the conjecture (4.20).

Kasparov has made the following conjecture [38]:

(4.27) **CONJECTURE.** *Let π be an irreducible, square integrable representation of a unimodular Lie group G . There is a unique Clifford module V such that $0 \in \mathbb{R}$ is an isolated point in the spectrum of the Dirac operator D_V , and*

$$\ker(D_V^+) \cong \pi \quad \text{and} \quad \ker(D_V^-) = 0.$$

Deformations. We mention an interesting reformulation of the Connes-Kasparov conjecture. Let G be a connected Lie group and let K be a maximal compact subgroup of G . Denote by V the quotient of the Lie algebras of G and K . There is a natural adjoint action of K on V , and we form the semidirect product

$$G_0 = K \ltimes V.$$

For $t > 0$ we let $G_t = G$, and then form the disjoint union

$$\mathbf{G} = \cup_{t \in [0,1]} G_t.$$

It may be given the structure of a connected smooth manifold with boundary in such a way that the group operations—defined fiberwise—are smooth maps, presenting us with a “smooth deformation” of Lie groups.

The deformation $\{G_t\}$ has been studied in mathematical physics (in the literature it is called a *contraction* of G). It has been observed that the representation theories of G_0 and G_t are in close correspondence with one another—see for example Mackey’s article [43]. The language of C^* -algebra K -theory, particularly the theory introduced in [24,25], allows us to make this precise, at least at the level of cohomology.

The deformation \mathbf{G} gives rise to a continuous field of C^* -algebras $\{C_r^*(G_t)\}$. We observe that for $t > 0$ the family is constant, and so $\{C_r^*(G_t)\}$ is a deformation of the sort considered in [24]. As explained there, it induces a K -theory map

$$K_j(C_r^*(G_0)) \rightarrow K_j(C_r^*(G)).$$

CONJECTURE. *The above map is an isomorphism of abelian groups.*

Now it follows from the Bott Periodicity Theorem [2,62] that, assuming the orientation condition (4.17) holds,

$$K_j(C_r^*(G_0)) \cong \begin{cases} R(K) & \text{if } j \equiv \dim(G/K) \pmod{2} \\ 0 & \text{if } j \not\equiv \dim(G/K) \pmod{2}. \end{cases}$$

This identifies $K_*(C_r^*(G_0))$ with the right hand side of (4.21), and in fact one can show that in this way the index map $\tilde{\mu}$ identifies with the K -theory maps obtained from the deformation $\{C_r^*(G_t)\}$.

Dual Dirac Construction. There exists a natural candidate ν for the inverse of the map $\mu: K_j(C_r^*G) \rightarrow K_j^G(G/K)$. The definition of this *Dual Dirac* map (invented by Kasparov [35]) involves ideas which are very closely related to the Bott periodicity theorem, as proved in [2]. Naturally, one would like to establish the conjecture for Lie groups by proving that both of the compositions $\mu \circ \nu$ and $\nu \circ \mu$ are the identity, using purely K -theoretic arguments. Kasparov shows in [35] that $\nu \circ \mu$ is the identity on $K_j^G(G/K)$. The other composition is more difficult, but by this method Connes [23] and Kasparov [35] have proved (4.4) for simply connected solvable groups, and Kasparov has proved (4.4) for amenable groups.

5. Cosheaf Homology

We introduce some homology groups relevant to our conjecture in the cases where G is a discrete or totally disconnected group.

We shall use the term *simplicial complex* to refer to a topological space with a given triangulation, that is, with a given decomposition into simplices satisfying the usual rules.

A *polysimplicial complex* is a finite product

$$X = X_1 \times X_2 \times \cdots \times X_l$$

where each X_j is a simplicial complex.

A *polysimplex* of X is a subset of the form

$$\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_l,$$

where σ_j is a (closed) simplex of X_j .

An *orientation* of a polysimplex $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_l$ is an equivalence class of orientations of each σ_j , where two sets of orientations of the σ_j are equivalent if they differ on an even number of the σ_j . If σ is oriented and $\eta \subseteq \sigma$ is a sub-polysimplex (a *face*) of codimension 1 then η inherits an orientation from σ . Indeed, η is a product $\eta_1 \times \cdots \times \eta_l$, where each η_j is either equal to the corresponding σ_j or is a codimension 1 face. In either case, η_j inherits an orientation from σ_j .

If η and σ are oriented polysimplices and if η is a codimension 1 face of σ then we define an *incidence number*

$$[\eta : \sigma] = \begin{cases} +1 & \text{if the orientation on } \eta \text{ is the inherited one from } \sigma \\ -1 & \text{otherwise.} \end{cases}$$

By an *orientation* of a polysimplicial complex X we mean the assignment of an orientation to each polysimplex. This may be done in a completely arbitrary fashion, with no regard to the inherited orientations on faces.

Let X be a polysimplicial complex. A *cosheaf* \mathcal{A} on X consists of the following data:

(5.1) For each polysimplex σ of X an abelian group A_σ .

(5.2) For each inclusion of polysimplices $\eta \subseteq \sigma$ a homomorphism of abelian groups $\varphi_\eta^\sigma : A_\sigma \rightarrow A_\eta$ with $\varphi_\tau^\sigma = \varphi_\tau^\eta \varphi_\eta^\sigma$ whenever $\tau \subseteq \eta \subseteq \sigma$, and with $\varphi_\sigma^\sigma = \text{id}$ for each σ .

Let \mathcal{A} be a cosheaf on a polysimplicial complex X . We define some homology groups as follows.

Denote by $C_n(X; \mathcal{A})$ the abelian group whose elements are all finite formal sums

$$\sum_{\dim \sigma = n} a_\sigma [\sigma],$$

where σ ranges over all polysimplices of dimension n and $a_\sigma \in A_\sigma$. Define homomorphisms of abelian groups

$$\partial : C_{n+1}(X; \mathcal{A}) \rightarrow C_n(X; \mathcal{A})$$

by first orienting X in any fashion and then using the formula

$$\partial(a_\sigma[\sigma]) = \sum_{\substack{\eta \subseteq \sigma \\ \dim(\eta) = \dim(\sigma) - 1}} [\eta : \sigma] \varphi_\eta^\sigma(a_\sigma) [\eta].$$

The *homology groups of X with coefficients in the cosheaf \mathcal{A}* are the homology groups of the complex

$$(5.3) \quad 0 \leftarrow C_0(X; \mathcal{A}) \xleftarrow{\partial} C_1(X; \mathcal{A}) \xleftarrow{\partial} C_2(X; \mathcal{A}) \xleftarrow{\partial} \dots$$

They do not depend on the choice of orientation of X .

We shall be interested in situations where a group G acts on the space X in such a way that G maps polysimplices to polysimplices. Suppose for simplicity that X is oriented and the action of G is orientation preserving.

By an action of G on a cosheaf \mathcal{A} over X we mean a family of maps

$$\Phi_g : A_\sigma \rightarrow A_{g\sigma} \quad (\text{where } g \in G \text{ and } \sigma \text{ is a polysimplex in } X),$$

which is compatible, in the natural sense, with composition in G and with the maps φ_η^σ in (5.2).

If G acts on the cosheaf \mathcal{A} then it acts on the complex (5.3). The *equivariant homology groups of X with coefficients in \mathcal{A}* are the homology groups of the complex obtained by dividing each $C_n(X; \mathcal{A})$ by the subgroup generated by elements of the form $a_\sigma[\sigma] - \Phi_g(a_\sigma)[g\sigma]$. In other words the complex (5.3) is replaced by the associated complex of *coinvariants*.

6. p -adic Groups

In this section we report on joint work with Roger Plymen.

Let F be a non-archimedean local field and let G be the F -rational points of a reductive algebraic group defined over F .¹³ Examples are $SL(n, F)$ and $GL(n, F)$. We write

$$\beta G = \text{the affine Bruhat-Tits building for } G$$

(see [66, 67]). It is a proper G -space and Proposition 1.8 proves that $\beta G = \underline{E}G$. So our conjecture (3.15) becomes:

(6.1) CONJECTURE. *Let G be a reductive group over a non-archimedean local field and let βG be its affine Bruhat-Tits building. Then*

$$\mu: K_j^G(\beta G) \rightarrow K_j(C_r^*(G)) \quad (j = 0, 1)$$

is an isomorphism of abelian groups.

The Reduced C^* -Algebra and the Plancherel Theorem. The tempered dual of G (= support of the Plancherel measure) is comprised of discrete series representations and one or more families of principal series representations, induced from parabolic subgroups of G . The Plancherel Theorem of Harish-Chandra, together with the theory of intertwining operators, leads to an isomorphism

$$(6.2) \quad C_r^*(G) \cong \oplus_M C_0(E_2M, \mathcal{K})^{W_M},$$

where: the sum is over conjugacy classes of Levi subgroups of G ; E_2M denotes the discrete series representations of M ; and the group W_M is the Weyl group associated to M . The algebra $C_0(E_2M, \mathcal{K})$ is the compact operator-valued endomorphisms of the bundle of principal series representations associated to E_2M , on which W_M acts in a manner prescribed by the theory of intertwining operators (and $C_0(E_2M, \mathcal{K})^{W_M}$ denotes the endomorphisms fixed by the action of the Weyl group). See [56].

¹³We shall confine our attention to *reductive* groups in order make use of the geometry of the Bruhat-Tits building. See Rosenberg's article [58] for an analysis of the K -theory associated to a certain solvable p -adic group.

(6.3) **EXAMPLE.** Let $G = GL(n, F)$. All the (unitary) principal series representations of G are irreducible, and for this reason (6.2) simplifies somewhat to a Morita equivalence

$$(6.4) \quad C_r^*(G) \underset{\text{Morita}}{\sim} \oplus_M C_0(E_2M/W_M).$$

The space E_2M is a disjoint union of tori, on which W_M (a symmetric group) acts by permuting the tori and the coordinates within individual tori. The quotient space E_2M/W_M has the homotopy type of a disjoint union of tori. See [55].

(6.5) **EXAMPLE.** Let $G = SL(2, F)$. Here the tempered dual is not a Hausdorff space, and the Morita equivalence (6.4) is not valid. There are a number of “double points,” corresponding one-to-one with the quadratic extensions of F (for example, if F has odd residual characteristic then there are three pairs of double points corresponding to the three distinct quadratic extensions of F). This is a simple example of the way in which the arithmetic of the field F enters into the representation theory of the group. A complete description of the reduced C^* -algebra of G is given in [54].

Thus far we have proceeded in very close analogy with the theory for reductive Lie groups. One can see from (6.2) that the reduced C^* -algebra for G and its K -theory depend more or less on the topology of the tempered dual of G . However a detailed description of $K(C_r^*(G))$ for a general reductive group G , comparable to Lie groups case, is not yet available. (We remark that even for $SL(2, F)$ it is not a simple matter to explicitly parametrize the generators for the K -theory of the C^* -algebra.)

Equivariant Homology for the Building. As it naturally comes to us (that is, with its standard decomposition into chambers) βG is a polysimplicial complex, and the action of G on βG maps polysimplices to polysimplices.

Let σ be a polysimplex in βG and set

$$G_\sigma = \{g \in G \mid gp = p \text{ for all } p \in \sigma\}.$$

This is a compact open subgroup of G . Let

$$\text{Cl}(G_\sigma) = \{\text{locally constant class functions on } G_\sigma\}$$

(recall that a *class function* on a group is a function which is invariant under inner automorphisms). We remark that since G_σ is compact and totally disconnected $\text{Cl}(G_\sigma)$ is the complexification of the representation ring of G_σ :

$$\text{Cl}(G_\sigma) = R(G_\sigma) \otimes_{\mathbb{Z}} \mathbb{C}.$$

If η is a face of σ then G_σ is a finite index subgroup of G_η . Hence induction gives a linear map

$$\begin{aligned} \text{Ind}_{G_\sigma}^{G_\eta} : \text{Cl}(G_\sigma) &\rightarrow \text{Cl}(G_\eta), \\ \text{Ind}_{G_\sigma}^{G_\eta} \varphi(g) &= \sum_{h \in G_\eta/G_\sigma} \varphi(h^{-1}gh). \end{aligned}$$

(In the above formula ϕ —a class function on G_σ —is extended by zero to a function on G_η , and then summed over coset representatives so as to obtain a class function on G_η .) Consider the cosheaf \mathcal{C} on βG determined by

$$\begin{aligned}\mathcal{C}_\sigma &= \text{Cl}(G_\sigma) \\ \varphi_\eta^\sigma &= \text{Ind}_{G_\sigma}^{G_\eta}.\end{aligned}$$

As in Section 5 we form the complex

$$(6.6) \quad 0 \leftarrow C_0(\beta G; \mathcal{C}) \leftarrow C_1(\beta G; \mathcal{C}) \leftarrow C_2(\beta G; \mathcal{C}) \leftarrow \dots$$

which computes the cosheaf homology groups of \mathcal{C} on βG . The group G acts on the cosheaf \mathcal{C} , and hence on the complex (6.6), in the following way. If $g \in G$ then conjugation by g^{-1} determines an isomorphism $G_{g\sigma} \xrightarrow{\cong} G_\sigma$, and hence a linear isomorphism

$$\Phi_g = \text{Ad}(g^{-1})^*: \text{Cl}(G_\sigma) \xrightarrow{\cong} \text{Cl}(G_{g\sigma}).$$

These maps are compatible with induction, in the obvious sense. We note that :

$$(6.7) \quad \text{If } g\sigma = \sigma \text{ then } \Phi_g: \text{Cl}(G_\sigma) \rightarrow \text{Cl}(G_\sigma) \text{ is the identity map.}$$

This is because an inner automorphism of G_σ acts trivially on $\text{Cl}(G_\sigma)$.

Following the procedure outlined at the end of Section 5, we divide (6.6) by the action of G . In other words we form the complex of co-invariants

$$(6.8) \quad 0 \leftarrow C_0(\beta G; \mathcal{C})_G \leftarrow C_1(\beta G; \mathcal{C})_G \leftarrow C_2(\beta G; \mathcal{C})_G \leftarrow \dots$$

(6.9) DEFINITION. The homology groups of the complex (6.8) shall be denoted by $H_j^G(\beta G; \mathbb{C})$.

The relevance of $H_*^G(\beta G; \mathbb{C})$ to our conjecture is made precise by the following result.

(6.10) THEOREM. (See [7]). *There is a Chern character homomorphism*

$$\text{ch}: K_j^G(\beta G) \rightarrow \bigoplus_n H_{2n+j}^G(\beta G; \mathbb{C})$$

which becomes an isomorphism upon tensoring with \mathbb{C} .

Using this Chern isomorphism we obtain from the map μ in (6.1) a homomorphism

$$(6.11) \quad \mu_{\mathbb{C}}: \bigoplus_n H_{2n+j}^G(\beta G; \mathbb{C}) \rightarrow K_j(C_r^* G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Modulo torsion, (6.1) is equivalent to the assertion that (6.11) is an isomorphism.

(6.12) **EXAMPLE.** To illustrate the definition we consider the groups $G = SL(n, F)$. The affine building for G is a simplicial complex (see [19] for an elementary description of it). The quotient of βG by the action of G identifies with a single chamber Δ in βG . In other words, every simplex in βG can be mapped by some element of G to a unique face of Δ . Using this fact, along with (6.7), the complex of coinvariants (6.8) identifies with the complex

$$0 \leftarrow C_0(\Delta; \mathcal{C}|_\Delta) \leftarrow C_1(\Delta; \mathcal{C}|_\Delta) \leftarrow C_2(\Delta; \mathcal{C}|_\Delta) \leftarrow \dots$$

obtained by restricting the cosheaf \mathcal{C} to Δ . This we can write down quite explicitly. Denote by \mathcal{O} the ring of integers in our field F and by $\varpi \in \mathcal{O}$ a generator for the maximal ideal of \mathcal{O} . Form the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \varpi & 0 & 0 & \dots & 0 \end{pmatrix}$$

(it is an element of $GL(n, F)$, not $SL(n, F)$), and denote by J_0, \dots, J_n the compact subgroups of G obtained by conjugating $SL(n, \mathcal{O})$ by powers of the matrix A :

$$J_i = A^i SL(n, \mathcal{O}) A^{-i}.$$

These are the stabilizers of the vertices of a fundamental chamber Δ in βG . For each index set $\alpha = \{\alpha_0 < \dots < \alpha_p\}$ of length $|\alpha| = p + 1$ let

$$J_\alpha = J_{\alpha_0} \cap \dots \cap J_{\alpha_p}.$$

Then

$$C_p(\Delta; \mathcal{C}) = \bigoplus_{|\alpha|=p+1} \text{Cl}(J_\alpha),$$

so that the homology groups $H_j^G(\beta G; \mathcal{C})$ are computed from a complex

$$(6.13) \quad 0 \leftarrow \bigoplus_{|\alpha|=1} \text{Cl}(J_\alpha) \xleftarrow{\partial} \bigoplus_{|\alpha|=2} \text{Cl}(J_\alpha) \xleftarrow{\partial} \bigoplus_{|\alpha|=3} \text{Cl}(J_\alpha) \xleftarrow{\partial} \dots$$

The boundary map ∂ is given by the formula

$$\partial \phi = \bigoplus_{t=0}^p (-1)^p \text{Ind}_{J_\alpha}^{J_{\alpha(t)}} \phi,$$

where $\phi \in \text{Cl}(J_\alpha)$, $\alpha = \{\alpha_0 < \dots < \alpha_p\}$, and $\alpha(t)$ denotes the index set obtained from α by omitting α_t . Suppose for instance we consider the special case $G = SL(2, F)$. Then

$$J_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2, F) \quad \text{and} \quad J_1 = \begin{pmatrix} \mathcal{O} & \varpi^{-1}\mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2),$$

while their intersection is the Iwahori subgroup

$$I = J_0 \cap J_1 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi \mathcal{O} & \mathcal{O} \end{pmatrix} \cap SL(2).$$

The complex which computes $H_j^G(\beta G; \mathcal{C})$ in this case is

$$0 \leftarrow \text{Cl}(J_0) \oplus \text{Cl}(J_1) \xleftarrow{-\text{Ind}_I^{J_0} + \text{Ind}_I^{J_0}} \text{Cl}(I) \leftarrow 0 \leftarrow 0 \leftarrow \dots$$

A detailed examination of the corresponding homology groups is carried out in [13].

(6.14) EXAMPLE. If G is any absolutely almost simple group (such as the group $SL(n, F)$ just considered) then βG , with its standard decomposition into chambers, is a simplicial complex and the quotient $G \backslash \beta G$ identifies with a single chamber in βG . The chain complex (6.8) identifies with the complex

$$0 \leftarrow C_1(\Delta; \mathcal{C}) \leftarrow C_1(\Delta; \mathcal{C}) \leftarrow C_2(\Delta; \mathcal{C}) \leftarrow \dots,$$

just as in the case of $SL(n, F)$.

(6.15) EXAMPLE. If $G = GL(n, F)$ then the building βG is the direct product

$$\beta G = \beta SL(n, F) \times \mathbb{R}$$

of the building for $SL(n, F)$ (the action of $SL(n, F)$ extends to an action of G and the real line \mathbb{R} (on which G acts by translation through the homomorphism $GL(n) \rightarrow \mathbb{Z}$ mapping an invertible matrix to the valuation of its determinant). The complex computing $H_*^G(\beta G; \mathbb{C})$ is a little more difficult to describe than for $SL(n)$. This reflects the fact that the quotient space $G \backslash \beta G$ is topologically more complicated: for example, in the case $n = 2$ it is a Möbius band. Denote by ${}^\circ G$ the subgroup of $G = GL(n)$ consisting of matrices whose determinant is a unit in the ring of integers \mathcal{O} . Form the complex

$$(6.16) \quad 0 \leftarrow \tilde{C}_0 \xleftarrow{\partial} \tilde{C}_1 \xleftarrow{\partial} \tilde{C}_2 \xleftarrow{\partial} \dots$$

This is the same as the complex (6.13) used to compute equivariant homology for $SL(n, F)$, except that the groups J_i are replaced by

$$\tilde{J}_i = A^i \cdot GL(n, \mathcal{O}) \cdot A^{-i}.$$

Conjugation with the matrix A permutes the groups $\tilde{J}_\alpha = \tilde{J}_{\alpha_0} \cap \dots \cap \tilde{J}_{\alpha_p}$ among themselves, and yields an automorphism A_* of the complex (6.16). The complex computing $H^G(\beta G; \mathbb{C})$ for $G = GL(n, F)$ is obtained by totalizing the double complex

$$\begin{array}{ccccccc} 0 & \longleftarrow & \tilde{C}_0 & \xleftarrow{\partial} & \tilde{C}_1 & \xleftarrow{\partial} & \tilde{C}_2 & \xleftarrow{\partial} & \dots \\ & & \downarrow 1-A_* & & \downarrow 1-A_* & & \downarrow 1-A_* & & \\ 0 & \longleftarrow & \tilde{C}_0 & \xleftarrow{\partial} & \tilde{C}_1 & \xleftarrow{\partial} & \tilde{C}_2 & \xleftarrow{\partial} & \dots \end{array}.$$

Supercuspidal Representations. In this subsection we assume for simplicity that G is a semisimple p -adic group.

A representation of G on a complex vector space V is *admissible* if the isotropy subgroup of each vector $v \in V$ is an open subgroup of G , and if the fixed point set of each compact open subgroup of G is a finite dimensional subspace of V .

An admissible representation is *supercuspidal* if its matrix coefficients are compactly supported functions on G . See [41].

Each irreducible supercuspidal representation of G may be completed to an integrable unitary representation of G . It follows that each irreducible supercuspidal representation contributes a summand of \mathbb{Z} to the group $K_0(C_r^*G)$.

A long standing conjecture in the representation theory of p -adic groups asserts that:

(6.7) CONJECTURE. (See [41].) *Every supercuspidal representation of G is obtained by induction¹⁴ from a representation of a compact open subgroup of G .*

Let ρ be a representation of a compact open subgroup H of G . The group H lies within the stabilizer $G_{\{p\}}$ of some vertex $p \in \beta G$, and by inducing the representation ρ to $G_{\{p\}}$ we obtain a representation of $G_{\{p\}}$. The character of this induced representation is a class function on $G_{\{p\}}$. It determines an element

$$(6.18) \quad [\rho] \in H_0^G(\beta G).$$

By definition of $H_0^G(\beta G)$, $[\rho]$ does not depend on the choice of group $G_{\{p\}}$ (in the event that H fixes more than one vertex in βG). In addition the classes $[\rho]$ span the space $H_0^G(\beta G)$.

Now let e_ρ be a normalized matrix coefficient for ρ :

$$e_\rho(h) = \frac{\dim(\rho)}{\text{vol}(H)\|v\|^2} \langle \rho(h^{-1})v, v \rangle$$

Extended by zero to a function on G , the matrix coefficient e_ρ is a projection in the C^* -algebra $C_r^*(G)$ and so determines a K -theory class

$$[e_\rho] \in K_0(C_r^*G).$$

If ρ induces to an irreducible supercuspidal representation π of G then e_ρ is a normalized matrix coefficient for π , and so $[e_\rho]$ is the K -theory generator associated to π .

Using the map $\mu_{\mathbb{C}}$ of (6.11) we have

$$\mu_{\mathbb{C}}([\rho]) = [e_\rho].$$

A slight refinement of the conjecture (6.1) asserts that:

¹⁴The method of induction used is one appropriate to the theory of admissible representations.

(6.19) CONJECTURE. *The K -theory classes associated to the discrete series representations of G (in particular, the K -theory classes associated to the supercuspidal representations) lie in the image of the map*

$$\mu_{\mathbb{C}}: H_0^G(\beta G) \rightarrow K_0(C_r^*G) \otimes \mathbb{C}.$$

Conjecture (6.19) stands in relation (6.18) more or less as the Atiyah-Schmid Theorem (4.26) stands in relation to the assertion that each discrete series representation of a semisimple Lie group is the index *in the sense of K -theory* of some twisted Dirac operator on the symmetric space G/K .

We remark that the Steinberg representation [16] of a p -adic semisimple group is not induced from a compact open subgroup, so a stronger version of (6.19), asserting that if π is any discrete series representation then $\mu_{\mathbb{C}}([\rho]) = [\pi]$ for some $[\rho]$ is as in (6.18), does not hold. However the K -theory class of the Steinberg representation does lie in the image of $\mu_{\mathbb{C}}$.

The Chern character isomorphism (6.9) suggests a uniqueness counterpart to (6.17): if two representations ρ and ρ' both induce to a given supercuspidal then up to conjugacy both ρ and ρ' are induced from a common subgroup. The injectivity of (6.11) (which is known, thanks to work of Kasparov and Skandalis [39]) implies a weaker result, at the level of homology and K -theory. We leave it to the reader to formulate this.

Finally, we remark that the calculations for $H_0^G(\beta G)$ in [13] extend in a straightforward way to general semisimple groups, and provide a left inverse to $\mu_{\mathbb{C}}$ in the language of orbital integrals on G .

Deformations and Affine Hecke Algebras. Let G be a reductive p -adic group, as above, and let I be an Iwahori subgroup of G [67]. Normalize Haar measure on G so that I has volume 1.

Denote by $C_c^\infty(G)$ the convolution algebra of locally constant, compactly supported functions on G and let $e \in C_c^\infty(G)$ be the characteristic function of I . Then e is an idempotent in $C_c^\infty(G)$ and we define

$$H(G//I) = eC_c^\infty(G)e.$$

Thus $H(G//I)$ consists of those functions in $C_c^\infty(G)$ which are constant on the double cosets IgI ($g \in G$).

Suppose now that G is split over F , and assume for simplicity that G is semisimple and simply connected. Denote by \widetilde{W} the affine Weyl group associated to G [31]. It is provided with a set of generators \widetilde{S} , and the pair $(\widetilde{W}, \widetilde{S})$ is a Coxeter group. There is a natural isomorphism of sets

$$I \backslash G / I \cong \widetilde{W}.$$

Associating to each $w \in \widetilde{W}$ the characteristic function T_w of the double coset labelled by w we obtain a vector space basis

$$\{ T_w \mid w \in \widetilde{W} \}$$

for $H(G//I)$. Iwahori and Matsumoto [31] have given the following presentation of $H(G//I)$ in terms of this basis:

$$(6.20) \quad \begin{aligned} T_{w_1} T_{w_2} &= T_{w_1 w_2} & \text{if } l(w_1) + l(w_2) &= l(w_1 w_2) \\ (T_s + 1)(T_s - q) &= 0 & \text{for all } s &\in \tilde{S} \end{aligned}$$

Here q denotes the cardinality of the residue field $\mathcal{O}/\varpi\mathcal{O}$, and $l(w)$ denotes the word length in \widetilde{W} .

For any value of q the relations (6.20) define an algebra structure on the vector space with basis $\{T_w\}$. Denote the algebra so obtained by $H(\widetilde{W}, q)$.

From now on let $q \geq 1$. The adjoint operation

$$T_w^* = T_{w^{-1}}$$

makes $H(\widetilde{W}, q)$ into a $*$ -algebra. The functional

$$\begin{aligned} \tau: H(\widetilde{W}, q) &\rightarrow \mathbb{C} \\ \tau(\sum \alpha_w T_w) &= \alpha_e \end{aligned}$$

is a positive definite trace on $H(\widetilde{W}, q)$. Denote by $H_r^*(\widetilde{W}, q)$ the C^* -algebra completion of $H(\widetilde{W}, q)$ in the representation associated to τ by the GNS construction.

The $*$ -representations of $H_r^*(\widetilde{W}, q)$ correspond in a natural way to irreducible tempered representations of G which contain a vector fixed by the Iwahori subgroup I [16]. The K -theory of $H_r^*(\widetilde{W}, q)$ (when q is the cardinality of the residue field of F) lies as a component in the K -theory of $C_r^*(G)$.

(6.21) CONJECTURE. *For $q \geq 1$ the field of C^* -algebras $H_r^*(\widetilde{W}, q)$ has constant K -theory.*

This is in many respects analogous to the reformulation of the Connes-Kasparov conjecture in terms of deformations. It may be recast in several ways, and at the present time constitutes the most geometric component of our conjecture for p -adic groups. In addition, as the next subsection indicates, a detailed study of Hecke algebras will probably be key to a treatment of the full conjecture (6.1).

Theory of Types. A very promising analysis of the smooth representation theory of $GL(n, F)$ is being carried out by C. Bushnell and P. Kutzko [21].

They explicitly construct a distinguished family of representations of compact open subgroups (in the Bushnell-Kutzko terminology, *simple types*). Given a simple type (H, ρ) the function

$$E_\rho(g) = \begin{cases} \dim(\rho) \operatorname{vol}(H)^{-1} \operatorname{trace}(\rho(g^{-1})) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

is an idempotent in $C_c^\infty(G)$, the convolution algebra of locally constant, compactly supported functions on G . Restricted to $C^\infty(H)$, E_ρ is the standard central idempotent associated to the irreducible representation ρ . Bushnell and

Kutzko establish a Morita equivalence between $E_\rho C_c^\infty(G)E_\rho$ and an affine Hecke algebra. This is quite promising for our conjecture since it suggests that a more complete analysis along these lines will decompose $K_*(C_r^*(G))$ into manageable summands of finite rank.

Status of the Conjecture for p -adic Groups. Thus far the conjecture for p -adic groups has been verified (using Pimsner's theorem) only for reductive p -adic groups which are locally products of split-rank one groups. Examples are $SL(2, F)$ and $GL(2, F)$.

The homology calculations in [13], combined with the explicit description of the tempered dual, lead to a direct representation theoretic proof for $SL(2, F)$.

For general reductive p -adic groups a KK -theoretic argument of Kasparov and Skandalis shows that the map μ is split injective.

Finally, aspects of the Selberg principle are closely related to our conjecture (6.1). The reader is referred to the papers [15, 20, 69] which look at the Selberg principle from the point of view of Hochschild and cyclic homology.

7. Discrete Groups

Reduced C^* -Algebra of a Discrete Group. If Γ is a discrete group then the C^* -algebra $C_r^*\Gamma$ may be viewed as a completion of the complex group algebra $\mathbb{C}[\Gamma]$. Each element of $C_r^*\Gamma$ is an infinite formal sum

$$(7.1) \quad x = \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma]$$

with complex coefficients. The coefficients satisfy $\sum |\lambda_\gamma|^2 < \infty$, but not every square-summable formal sum corresponds to an element of $C_r^*\Gamma$ (unless Γ happens to be finite).

It is generally impossible to describe the C^* -algebra $C_r^*\Gamma$ in any detail. In the special case where Γ is abelian we have already mentioned that the Fourier isomorphism $l^2\Gamma \cong L^2(\hat{\Gamma})$ identifies $C_r^*\Gamma$ with the C^* -algebra of continuous complex valued functions on the dual $\hat{\Gamma}$. But for a general discrete group the reduced dual $\hat{\Gamma}_r$ is usually very poorly behaved as a topological space (for instance there may be no non-trivial open sets), so it is unrealistic to expect a simple description of $C_r^*\Gamma$ as an algebra of functions on it.

For this reason it is difficult to view the K -theory groups $K_j(C_r^*\Gamma)$ from a geometric perspective. But they nevertheless play an important role as receivers for geometrically defined indices and signatures.

Equivariant Homology and Chern Character. Let Γ be a finitely generated discrete group and let X be a simplicial complex on which Γ acts properly.

Form a cosheaf on X by associating to each simplex σ in X the space of class functions on the isotropy subgroup of σ :

$$\sigma \mapsto \text{Cl}(\Gamma_\sigma) \cong R(\Gamma_\sigma) \otimes_{\mathbb{Z}} \mathbb{C}.$$

We associate to each inclusion $\sigma \subseteq \eta$ the induction map

$$\Phi_\eta^\sigma = \text{Ind}_{\Gamma_\eta}^{\Gamma_\sigma}: \text{Cl}(\Gamma_\sigma) \rightarrow \text{Cl}(\Gamma_\sigma)$$

(compare Section 6). The group Γ acts on this cosheaf.

(7.2) **DEFINITION.** Denote by $H_j^\Gamma(X; \mathbb{C})$ the equivariant homology groups of the above cosheaf (see Section 5) In other words $H_j^\Gamma(X; \mathbb{C})$ is the homology of the complex of coinvariants of the complex (5.3).

(7.3) **THEOREM.** (See [10].) *There is a functorial Chern character*

$$\begin{aligned} \text{ch}^\Gamma : K_0^\Gamma(X) &\longrightarrow \bigoplus_j H_{2j}^\Gamma(X; \mathbb{C}) \\ \text{ch}^\Gamma : K_1^\Gamma(X) &\longrightarrow \bigoplus_j H_{2j+1}^\Gamma(X; \mathbb{C}) \end{aligned}$$

which becomes an isomorphism after tensoring with \mathbb{C} .

If the action of Γ on X is free, in addition to being proper, then the Chern character (7.3) is compatible with the ordinary Chern character, in the sense that there is a natural commutative diagram

$$\begin{array}{ccc} K_j^\Gamma(X) & \xrightarrow{\text{ch}^\Gamma} & \bigoplus_n H_{2n+j}^\Gamma(X; \mathbb{C}) \\ \cong \downarrow & & \downarrow \cong \\ K_j(X/\Gamma) & \xrightarrow[\text{ch}]{} & \bigoplus_n H_{2n+j}(X/\Gamma; \mathbb{C}). \end{array}$$

The vertical isomorphisms come from our definition of equivariant homology and K -homology; the bottom arrow is the ordinary Chern character.

Equivariant Homology of $\underline{E}\Gamma$. Using the model for $\underline{E}\Gamma$ given in Section 2 we can interpret $H_*^\Gamma(\underline{E}\Gamma; \mathbb{C})$ in terms of group homology. Denote by $S\Gamma$ the set of all elements in Γ which are of finite order ($S\Gamma$ is not empty—it contains at least the identity element). The group Γ acts on $S\Gamma$ by conjugation and we let $F\Gamma$ be the associated permutation module with coefficients the complex numbers \mathbb{C} : an element of $F\Gamma$ is a finite formal linear combination

$$\sum_{\alpha \in S\Gamma} \lambda_\alpha [\alpha]$$

with complex coefficients. As usual, denote by $H_j(\Gamma; F\Gamma)$ the j -th homology group of Γ with coefficients in the Γ -module $F\Gamma$.

(7.4) **PROPOSITION.** *We have an isomorphism*

$$H_j(\Gamma; F\Gamma) \cong H_j^\Gamma(\underline{E}\Gamma; \mathbb{C}) \quad (j = 0, 1, \dots).$$

In view of (7.3) and (7.4) our conjecture (3.15) can be reformulated, modulo torsion, as follows

(7.5) CONJECTURE. Let $\rho: K_j(C_r^*\Gamma) \rightarrow K_j(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$ be the natural map and let $\mu_{\mathbb{C}}$ be the unique map which makes the diagram

$$\begin{array}{ccc} K_j^{\Gamma}(\underline{E}\Gamma) & \xrightarrow{\mu} & K_j(C_r^*\Gamma) \\ \text{ch}^{\Gamma} \downarrow & & \downarrow \rho \\ \bigoplus_r H_{j+2r}(\Gamma; F\Gamma) & \xrightarrow{\mu_{\mathbb{C}}} & K_j(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

commute. Then for any discrete countable group Γ , the map $\mu_{\mathbb{C}}$ ($j = 0, 1$) is an isomorphism.

Let $\gamma_1, \gamma_2, \dots$ be a list representatives of the conjugacy classes in Γ consisting of finite order elements (called the *elliptic* conjugacy classes). Note that the identity element of Γ is contained in the list. Let $F[\gamma_n]$ be the Γ -submodule of $F\Gamma$ generated by $[\gamma_n]$. Then

$$F\Gamma = \bigoplus_n F[\gamma_n],$$

Denote by $Z(\gamma_n)$ the centralizer of γ_n in Γ . It is a simple consequence of Shapiro's Lemma that

$$(7.6) \quad H_j(\Gamma; F\Gamma) \cong \bigoplus_n H_j(\Gamma; F[\gamma_n]) \cong \bigoplus_n H_j(Z(\gamma_n); \mathbb{C})$$

(in the last term we have group homology with trivial complex coefficients).

Strong Novikov Conjecture. Let $E\Gamma$ be the universal principal Γ -space. According to the universal property enjoyed by $\underline{E}\Gamma$ there is a Γ -map (unique up to homotopy)

$$\sigma: E\Gamma \rightarrow \underline{E}\Gamma.$$

Now the equivariant homology of $E\Gamma$ identifies with the group homology $H_j(\Gamma, \mathbb{C})$, and the map σ induces a map on homology which identifies with the inclusion of the summand in (7.6) labelled by the conjugacy class of the identity element:

$$\begin{array}{ccc} H_j^{\Gamma}(E\Gamma; \mathbb{C}) & \xrightarrow{\sigma_*} & H_j^{\Gamma}(\underline{E}\Gamma; \mathbb{C}) \\ \cong \downarrow & & \downarrow \cong \\ H_j(\Gamma; \mathbb{C}) & \xrightarrow{\text{inclusion}} & \bigoplus_n H_j(Z(\gamma_n); \mathbb{C}) \end{array}$$

(SNC) STRONG NOVIKOV CONJECTURE. The composite map

$$K_j(B\Gamma) \cong K_j^{\Gamma}(E\Gamma) \xrightarrow{\sigma_*} K_j^{\Gamma}(\underline{E}\Gamma) \xrightarrow{\mu} K_j(C_r^*\Gamma) \quad (j = 0, 1)$$

is rationally injective. Equivalently, the maps

$$\begin{aligned} \mu_{\mathbb{C}}: \bigoplus_j H_{2j}(\Gamma; \mathbb{C}) &\longrightarrow K_0(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \\ \mu_{\mathbb{C}}: \bigoplus_j H_{2j+1}(\Gamma; \mathbb{C}) &\longrightarrow K_1(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \end{aligned}$$

obtained from (7.5) by restricting to the summand $H_n(\Gamma; \mathbb{C})$ in (7.6), are injective.

Higher Signatures and Higher \hat{A} -Genera. Let M be a C^∞ manifold. We recall that associated to each power series $f(x^2)$ with rational coefficients there is a cohomology class lying in $H^*(M, \mathbb{Q})$. It is defined by expressing the product $\prod_i f(x_i^2)$ as a series in the elementary symmetric polynomials of x_1^2, x_2^2, \dots , and then substituting the j -th Pontrjagin class $p_j(M)$ for the j -th elementary symmetric polynomial. See [46].

The L -polynomial and \hat{A} -polynomial of M are the cohomology classes LM and $\hat{A}M$ associated to the power series $x/\tanh(x)$ and $(x/2)/\sinh(x/2)$, respectively:

$$\begin{aligned} x/\tanh(x) &\leftrightarrow LM \in \oplus_k H^{4k}(M; \mathbb{Q}) \\ (x/2)/\sinh(x/2) &\leftrightarrow \hat{A}M \in \oplus_k H^{4k}(M; \mathbb{Q}). \end{aligned}$$

Suppose now that M is connected, closed and oriented. Let Γ be the fundamental group of M and let

$$\tau: M \rightarrow B\Gamma$$

be the classifying map for the universal covering space \widetilde{M} of M . Associated to any cohomology class $u \in H^*(B\Gamma; \mathbb{Q})$ are rational numbers

$$\text{Sgn}_u(M) = (LM \cup \tau^*u)[M]$$

$$\hat{A}_u(M) = (\hat{A}M \cup \tau^*u)[M],$$

where, as usual, $[M]$ denotes the homology class determined by the orientation of M , and we use the map τ to pull back the cohomology class u on $B\Gamma$ to one on M .

(NC) **NOVIKOV CONJECTURE.** *Let M and N be closed, connected and oriented C^∞ manifolds. Let $f: M \rightarrow N$ be an orientation preserving homotopy equivalence, which we use to identify the fundamental groups of M and N . Let $\Gamma = \pi_1(M) = \pi_1(N)$. Then*

$$\text{Sgn}_u(M) = \text{Sgn}_u(N),$$

for all $u \in H^*(B\Gamma; \mathbb{Q})$.

(GLRC) **GROMOV-LAWSON-ROSENBERG CONJECTURE.** *Let M be a closed, connected C^∞ spin manifold with fundamental group Γ . Assume that there exists a Riemannian metric on M with positive scalar curvature. Then*

$$\hat{A}_u(M) = 0,$$

for all $u \in H^*(B\Gamma; \mathbb{Q})$.

Signature Operator. Let M be a closed connected oriented C^∞ manifold. Choose a Riemannian metric for M and using it construct the signature operator ∂ . If M is even dimensional then ∂ is the usual Hirzebruch signature operator. If M is odd-dimensional then ∂ is the Atiyah-Patodi-Singer operator [5]. Since ∂ is elliptic, it determines an element $[\partial]$ in the K -homology of M ,

$$[\partial] \in K_j(M) \quad j = \begin{cases} 0 & \text{if } \dim(M) \text{ is even} \\ 1 & \text{if } \dim(M) \text{ is odd.} \end{cases}$$

It follows from the Atiyah-Singer Index Theorem that the Chern character of $[\partial]$ is

$$(7.7) \quad \text{ch}[\partial] = 2^l \mathcal{L}(M) \cap [M], \quad l = \begin{cases} \dim(M)/2 & \text{if } \dim(M) \text{ is even} \\ (\dim(M) - 1)/2 & \text{if } \dim(M) \text{ is odd.} \end{cases}$$

Here $\mathcal{L}(M)$ is the cohomology class—a variant of $\mathbf{L}M$ —obtained from the power series $(x_i/2)/\tanh(x_i/2)$.

Let \widetilde{M} be the universal covering space of M . In an evident fashion the signature operator ∂ lifts to a Γ -equivariant elliptic operator $\widetilde{\partial}$ on \widetilde{M} . The Γ -index of $\widetilde{\partial}$ is an element of the group $K_j(C_r^*\Gamma)$, where j is the dimension of M , modulo 2:

$$\text{Index}_\Gamma(\widetilde{\partial}) \in K_j(C_r^*\Gamma) \quad j = \begin{cases} 0 & \text{if } \dim(M) \text{ is even} \\ 1 & \text{if } \dim(M) \text{ is odd.} \end{cases}$$

It does not depend on the choice of Riemannian metric on M .

(7.8) **PROPOSITION.** *Let M and N be two closed, connected and oriented C^∞ manifolds. Let $f: M \rightarrow N$ be an orientation preserving homotopy equivalence. Let $\Gamma = \pi_1(M) = \pi_1(N)$. Denote the signature operators of M and N by ∂_M and ∂_N . Then in $K_j(C_r^*\Gamma)$ we have*

$$\text{Index}_\Gamma(\widetilde{\partial}_M) = \text{Index}_\Gamma(\widetilde{\partial}_N).$$

See [33] for a proof of this important result of Kasparov (in the case where Γ is abelian the result is due to Lusztig [42]). It is essential here that we work with the C^* -algebra $C_r^*\Gamma$ and not merely some Banach algebra completion of the group algebra $\mathbb{C}[\Gamma]$.

Dirac Operator. Let M be a closed oriented C^∞ spin manifold and denote by D by the Dirac operator on M . Since D is elliptic it determines a K -homology class

$$[D] \in K_j(M) \quad j = \begin{cases} 0 & \text{if } \dim(M) \text{ is even} \\ 1 & \text{if } \dim(M) \text{ is odd.} \end{cases}$$

It follows from the Atiyah-Singer Index Theorem that the Chern character of $[D]$ is the Poincaré dual of $\hat{A}M$,

$$(7.9) \quad \text{ch}[D] = \hat{A}M \cap [M].$$

As with the signature operator, the Dirac operator D lifts to a Γ -equivariant elliptic operator \tilde{D} on \tilde{M} and we may form the Γ -index

$$\text{Index}_\Gamma(\tilde{D}_M) \in K_j(C_r^*\Gamma).$$

(7.10) PROPOSITION. *Let M be a closed connected C^∞ spin manifold. Assume there exists a Riemannian metric for M with positive scalar curvature. Then in $K_j(C_r^*\Gamma)$*

$$\text{Index}_\Gamma(\tilde{D}) = 0.$$

The proof is a consequence of the Lichnerowicz formula, expressing D^*D and DD^* as

$$\nabla^*\nabla + \frac{1}{4}(\text{scalar curvature}),$$

where ∇ denotes the connection operator on positive and negative spinors, respectively. If the scalar curvature is positive then D and its adjoint are bounded below, and the same holds for \tilde{D} . For the ordinary Fredholm index, this is enough to imply that $\text{Index}(D) = 0$, and one can show that the same is true of the Γ -index. As with (7.8), it is of crucial importance here that $C_r^*\Gamma$ is a C^* -algebra. For more details see [60].

Implications of the Strong Novikov Conjecture.

(7.11) THEOREM. *Let Γ be a countable discrete group. The following implications hold*

$$\begin{array}{ccccccc} (3.15) & \Rightarrow & (7.4) & \Rightarrow & (\text{SNC}) & \Rightarrow & (\text{NC}) \\ & & & & \Downarrow & & \\ & & & & (\text{GLRC}) & & \end{array}$$

To prove $(\text{SNC}) \Rightarrow (\text{NC})$ let M be a closed connected oriented C^∞ manifold and let $\tau: M \rightarrow B\Gamma$ be the map which classifies the universal cover of M . We first observe that the Novikov conjecture amounts to the assertion that the homology class

$$(7.12) \quad \tau_*(LM \cap [M]) \in H_*(B\Gamma; \mathbb{Q})$$

is an oriented homotopy invariant. But consider now the diagram

$$\begin{array}{ccccc} H_*(M; \mathbb{Q}) & \xleftarrow{\text{ch}} & K_*(M) & \xrightarrow{\text{Index}_\Gamma} & K_*(C_r^*\Gamma) \\ \downarrow & & \downarrow & & \downarrow \\ H_*(B\Gamma; \mathbb{Q}) & \xleftarrow{\text{ch}} & K_*(B\Gamma) & \xrightarrow{\text{Index}_\Gamma} & K_*(C_r^*\Gamma) \end{array}$$

According to (7.8) the image of $[\partial] \in K_*(M)$ in the bottom right group is an oriented homotopy invariant. Since (SNC) implies that the bottom right map is rationally injective it implies that the image of $[\partial]$ in $K_*(B\Gamma)$ is an oriented homotopy invariant, modulo torsion. Hence its Chern character, which by (7.7) is $2^l \tau_*(\mathcal{L}(M) \cap [M])$, is an oriented homotopy invariant. The homotopy invariance of (7.12) follows easily from this.

To prove (SNC) \Rightarrow (GLRC) let D be the Dirac operator on a closed C^∞ spin manifold M . By (7.10) the image of $[D] \in K_*(M)$ in the bottom right group in the above diagram is zero (assuming the existence of a metric of positive scalar curvature). So (SNC) implies that the image of $[D]$ in $H_*(B\Gamma; \mathbb{Q})$ is zero too. By (7.9) this is $\tau_*(\hat{A}M \cap [M])$, and (GLRC) follows easily.

Relative η -invariants. The geometric applications examined above depend on the validity of Strong Novikov Conjecture, and not the full conjecture (7.5). We mention here very briefly a geometric topic where the full conjecture (surjectivity of $\mu_{\mathbb{C}}$ as well as injectivity) appears to be needed.

Let M be a closed, connected and oriented odd-dimensional Riemannian manifold. Let

$$\eta(M) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \text{trace}(\partial \exp(-\partial^2 t)) dt$$

be the η -invariant of the signature operator on M [5]. Let \tilde{M} be the universal covering space of M , equipped with the action of $\Gamma = \pi_1(M)$ by deck transformations. J. Cheeger and M. Gromov [22] have defined an eta-type invariant

$$\eta_\Gamma(M) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \text{tr}_{\Gamma}(\tilde{\partial} \exp(-\tilde{\partial}^2 t)) dt$$

using the Γ -trace on $L^2(\tilde{M})$. Let

$$\rho_\Gamma(M) = \eta_\Gamma(M) - \eta(M).$$

This may be shown to be a differential invariant. Following V. Mathai [44] and S. Weinberger [72] we have:

(7.13) **PROPOSITION.** *If $\Gamma = \pi_1(M)$ is torsion free and K -amenable, and if (3.15) is valid for Γ , then the quantity $\rho_\Gamma(M)$ is an invariant of oriented homotopy type.*

Torsion Free Groups. If Γ is a torsion free discrete group then $\underline{E}\Gamma = E\Gamma$ and so

$$K_j^\Gamma(\underline{E}\Gamma) \cong K_j(B\Gamma) \quad (j = 0, 1),$$

where $K_j(B\Gamma)$ denotes the K -homology of $B\Gamma$ (with compact supports). Thus for a torsion free discrete group (3.15) conjectures an isomorphism

$$(7.14) \quad \mu: K_j(B\Gamma) \rightarrow K_j(C_r^*\Gamma) \quad (j = 0, 1).$$

Range of the Trace. There is a natural trace on the C^* -algebra $C_r^*\Gamma$ defined by

$$\text{tr}\left(\sum_{\gamma \in \Gamma} \lambda_\gamma[\gamma]\right) = \lambda_e.$$

In other words the trace associates to each formal sum the coefficient of the identity element of Γ . We note that with x as in (7.1),

$$\text{tr}(x^*x) = \sum_{\gamma \in \Gamma} |\lambda_\gamma|^2,$$

so that $\text{tr}(x^*x) = 0$ if and only if $x = 0$. In other words the trace is *faithful*.

Passing to K -theory, the trace gives a homomorphism of abelian groups

$$\text{tr}: K_0(C_r^*\Gamma) \rightarrow \mathbb{R}$$

(see [14]).

(7.15) **PROPOSITION.** *Let Γ be a torsion-free discrete group and assume that the map $\mu: K_0(B\Gamma) \rightarrow K_0(C_r^*\Gamma)$ is surjective. Then the range of the trace map on $K_0(C_r^*\Gamma)$ is the integers \mathbb{Z} .*

This follows from the fact that the composite map

$$K_0(B\Gamma) \xrightarrow{\mu} K_0(C_r^*\Gamma) \xrightarrow{\text{tr}} \mathbb{R}$$

may be shown to associate to each elliptic operator its ordinary Fredholm index (this is an abstract version of an index theorem of Atiyah [4] on covering spaces).

Non-Existence of Idempotents. R. Kadison and I. Kaplansky have conjectured that if Γ is a torsion-free discrete group then the C^* -algebra $C_r^*\Gamma$ contains no projections other than 0 and 1.¹⁵

(7.16) **PROPOSITION.** *Let Γ be a torsion-free discrete group and assume that the map $\mu: K_0(B\Gamma) \rightarrow K_0(C_r^*\Gamma)$ is surjective. Then the C^* -algebra $C_r^*\Gamma$ contains no projections other than 0 and 1.*

PROOF. Let p be a projection (= self-adjoint idempotent) in $C_r^*\Gamma$. Then $1 - p$ is also a projection. In the equation

$$\text{tr}(1) = \text{tr}(p) + \text{tr}(1 - p)$$

the terms are non-negative, and all are integers by 7.15. Since $\text{tr}(1) = 1$, one of $\text{tr}(p)$ or $\text{tr}(1 - p)$ must be zero, and so by the faithfulness of the trace, one of p or $1 - p$ must be zero. \square

Status of the Conjecture for Discrete Groups. If Γ is any discrete subgroup of a Lie group G with $\pi_0 G$ finite, then $\underline{E}\Gamma = G/K$, where K is the maximal compact subgroup of G . The Dirac-Dual Dirac method of Kasparov [35] proves that $\mu: K_j^\Gamma(G/K) \rightarrow K_j(C_r^*\Gamma)$ is split injective. If $G = SO(n, 1)$ or $SU(n, 1)$ then Kasparov's approach proves that μ is an isomorphism [37, 28, 32]. But for more general groups Skandalis has pointed out [65] that there is a basic difficulty with the Dirac-Dual Dirac approach, and as yet the conjecture has not been verified for any infinite, discrete property T group.

If Γ acts on a tree, without inversion, and if our conjecture (3.15) is valid for the isotropy subgroup of each vertex and edge (for example, this is so if the action is proper), then (3.15) holds for Γ . This follows from a theorem of Pimsner [53].

¹⁵A well known conjecture asserts that if Γ is torsion-free then the group algebra $\mathbb{C}[\Gamma]$ contains no zero divisors. The relation between this and our conjecture is unclear.

If Γ is any discrete subgroup of a reductive p -adic algebraic group G then $\underline{E}\Gamma$ is equal to βG , the affine Bruhat-Tits building of G . Results of Kasparov and Skandalis [39] imply that $\mu : K_j^\Gamma(\beta G) \rightarrow K_j(C_r^*\Gamma)$ is split injective.

Finally, Connes and Moscovici [26] prove that (SNC) is valid for all hyperbolic groups. One can strengthen this result to show that (3.15) is split injective for hyperbolic groups.

8. An Equivariant Novikov Conjecture

Let G be any Lie group. We allow countably many connected components; in particular G may be any countable discrete group. Let M be a smooth, oriented manifold (with no boundary), equipped with a smooth, proper, orientation-preserving action of G . Assume that the quotient space $G \backslash M$ is compact.

Under our assumptions there exist Riemannian metrics for M which are G -invariant. We shall fix one such metric (the choice is unimportant).

The signature operator is an equivariant elliptic operator on M , and it determines a class

$$[\partial_M] \in KK_G^j(C_0(M), \mathbb{C}),$$

where

$$j \equiv \dim(M) \pmod{2}.$$

By the universal property (1.2) of $\underline{E}G$, there is a G -map

$$\epsilon : M \longrightarrow \underline{E}G$$

which is unique up to G -homotopy. Since $G \backslash M$ is compact the map ϵ yields a homomorphism of abelian groups

$$\epsilon_* : KK_G^j(C_0(M), \mathbb{C}) \longrightarrow K_j^G(\underline{E}G).$$

DEFINITION. The G -signature of M , denoted $G\text{-Sgn}(M)$, is the quantity

$$G\text{-Sgn}(M) = \epsilon_*[\partial_M]$$

in $K_j^G(\underline{E}G)$, where $j \equiv \dim(M) \pmod{2}$.

(8.1) **EQUIVARIANT NOVIKOV CONJECTURE.** *Let M, N be two oriented C^∞ manifolds, each with a given proper action of G by orientation preserving diffeomorphisms. Assume that the quotient spaces $G \backslash M$ and $G \backslash N$ are both compact. Suppose that there exists an orientation preserving G -map $f : M \rightarrow N$ which induces an isomorphism on ordinary homology with rational coefficients. Then in $K_j^G(\underline{E}G)$,*

$$G\text{-Sgn}(M) = G\text{-Sgn}(N).$$

Suppose for instance that $X \rightarrow Y$ is a homotopy equivalence of oriented, smooth closed manifolds. It lifts to a Γ -map of universal covers $\tilde{X} \rightarrow \tilde{Y}$, where Γ is the fundamental group of X and Y , and all the hypotheses of the conjecture are satisfied. In this case the signature of \tilde{X} lies in the image of the map

$$(8.2) \quad K_j(B\Gamma) \cong K_j^\Gamma(E\Gamma) \rightarrow K_j^\Gamma(\underline{E}\Gamma).$$

It is the image of the class $\epsilon_*([\partial_X]) \in K_j(B\Gamma)$ discussed in Section 7. Since (8.2) is rationally injective the conjecture (8.1) implies the usual Novikov higher signature conjecture. Note that (8.1) is a little more precise in that it asserts the homotopy invariance of an integral, rather than rational, K -homology class.

Interesting examples for conjecture (8.1) may be constructed as follows. Begin with a smooth, orientation-preserving action of a compact Lie group K on a connected, oriented, smooth, closed manifold V . For $k \in K$ denote by $T(k)$ the corresponding diffeomorphism of V . Form the group G of all diffeomorphisms $T: \tilde{V} \rightarrow \tilde{V}$ of the universal cover which lift some $T(k): V \rightarrow V$. This means that the diagram

$$\begin{array}{ccc} \tilde{V} & \xrightarrow{T} & \tilde{V} \\ \pi \downarrow & & \downarrow \pi \\ V & \xrightarrow{T(k)} & V \end{array}$$

commutes. The group G fits into an exact sequence of groups

$$1 \rightarrow \pi_1(V) \rightarrow G \rightarrow K \rightarrow 1,$$

and the action of G on $M = \tilde{V}$ satisfies our hypotheses. Bearing this in mind, (8.1) should be compared to the conjecture of J. Rosenberg and S. Weinberger in [61].

If our main conjecture (3.15) is valid for G , then so is (8.1). The reason is that one may show quite directly that the *analytic signature* $\mu(G\text{-Sgn}(M))$ is invariant under equivalences $M \rightarrow N$ as in (8.1).

If G is a discrete group, then the Chern character of Section 7, together with an index theorem for proper actions of discrete groups, can be used to give a more explicit version of (8.1). Details will be given elsewhere [12].

9. The Conjecture with Coefficients

In this section we shall formulate a more general version of our conjecture, involving a coefficient C^* -algebra. For brevity we shall use various features of Kasparov's KK -theory without comment.

Let G be as in Section 1. Let A be a C^* -algebra with a given continuous action of G as C^* -algebra automorphisms. Form the reduced crossed-product C^* -algebra $C_r^*(G, A)$ [50] and consider its K -theory $K_j(C_r^*(G, A))$.

(9.1) DEFINITION. Let Z be any proper G -space. The equivariant K -homology of Z with G -compact supports and coefficients A , denoted $K_j^G(Z; A)$, is

$$K_j^G(Z; A) = \varinjlim_{\substack{X \subset Z \\ X \text{ } G\text{-compact}}} KK_G^j(C_0(X), A)$$

where the direct limit is taken over the directed system of all G -compact subsets of Z .

There is a homomorphism of abelian groups

$$(9.2) \quad \mu : K_j^G(Z; A) \longrightarrow K_j(C_r^*(G, A)) \quad (j = 0, 1).$$

It is defined in a similar manner to the map μ in Section 3, but we shall take advantage of the Kasparov product to present the definition in a slightly different way.

For clarity, write the crossed product of $C_0(X)$ by G as $C_r^*(G, X)$.

We first define, for any G -invariant, G -compact subset X of Z , a homomorphism of abelian groups

$$(9.3) \quad \mu_X : KK_G^j(C_0(X), A) \longrightarrow K_j(C_r^*(G, A)).$$

It is the composition of the maps

$$(9.4) \quad KK_G^j(C_0(X), A) \longrightarrow KK^j(C_r^*(G, X), C_r^*(G, A))$$

and

$$(9.5) \quad KK^j(C_r^*(G, X), C_r^*(G, A)) \longrightarrow KK^j(\mathbb{C}, C_r^*(G, A)),$$

where (9.4) is Kasparov's map j_r^G from [35] and (9.5) is given by Kasparov product with the class in $KK^0(\mathbb{C}, C_r^*(G, X))$ determined by the trivial line bundle on X . The maps μ_X are compatible with the direct limit in (9.1), and so yield a map (9.2).

Take Z to be the universal example \underline{EG} .

(9.6) CONJECTURE. *The map*

$$\mu : K_j^G(\underline{EG}; A) \rightarrow K_j(C_r^*(G, A)) \quad (j = 0, 1)$$

is always an isomorphism.

This is a good deal stronger than (3.15). For example, if (9.6) is true for G then (3.15) is true for any closed subgroup of G .¹⁶

If G is compact or abelian then (9.6) is valid for G .

If \underline{EG} is a tree, then Pimsner's theorem [53] proves that (9.6) is valid for G . More generally, suppose that G acts on a tree by a simplicial (and continuous) action such that (9.6) is valid for the stabilizer group of each vertex and each edge. Then [53] applies to prove that (9.6) is valid for G .

The Dirac-Dual Dirac method of Kasparov proves that (9.6) is valid for $SO(n, 1)$ [37]. Recent work of Julg and Kasparov [32] shows that it is also valid for $SU(n, 1)$.

Injectivity of $\mu : K_j^G(\underline{EG}; A) \rightarrow K_j(C_r^*(G, A))$ is known for a much larger class of groups: for example almost-connected Lie groups [35], reductive p -adic algebraic groups [39], fundamental groups of complete, non-positively curved manifolds [35].

¹⁶This is true modulo the same considerations as in (1.9).

Appendix 1: Infinite Join Construction of \underline{EG}

Let W be the disjoint union of all the homogeneous spaces G/H for H a compact subgroup of G .

Let CW be the cone on W , formed from $[0, 1] \times W$ by making the identifications

$$(0, w) = (0, w') \quad \text{for all } w, w' \in W.$$

We write a typical point in CW as a formal product tw , where $(t, w) \in [0, 1] \times W$.

The *infinite join*

$$\underline{EG} = W * W * W * \dots$$

is the set of sequences (t_1w_1, t_2w_2, \dots) in CW such that $t_j = 0$ for almost all j and $\sum t_j = 1$. It is given the weakest topology such that the maps

$$\begin{aligned} p_i &: (t_1w_1, t_2w_2, \dots) \mapsto t_i \\ q_i &: (t_1w_1, t_2w_2, \dots) \mapsto w_i \end{aligned}$$

are continuous (the map q_i is defined on the open set where p_i is non-zero). The group G acts on \underline{EG} by

$$g(t_1w_1, t_2w_2, \dots) = (t_1gw_1, t_2gw_2, \dots).$$

The infinite join is metrizable, as is its quotient by G .

To see that \underline{EG} is a proper G -space, note that the open sets $U_i = p_i^{-1}(0, 1]$ constitute an open cover for \underline{EG} . Each set U_i maps to W via q_i , and since W is a disjoint union of homogeneous spaces G/H , with H compact, each U_i is a disjoint union of open sets satisfying the conditions of Definition 1.3.

LEMMA. *Let X be a proper G -space. There exists a countable partition of unity $\alpha_1, \alpha_2, \dots$, consisting of G -invariant functions, such that each of the spaces $\alpha_i^{-1}(0, 1]$ admits a G -map to W .*

PROOF. The proof is the same as the proof of Proposition 3.12.1 of [30]. The key point is that the disjoint union of any collection of G -spaces which admit G -maps to W itself admits a G -map to W . \square

THEOREM. *\underline{EG} is a universal example for proper actions of G*

PROOF. Let X be a proper G -space and let $\alpha_1, \alpha_2, \dots$ be a partition of unity as in the Lemma. Fix G -maps $\psi_i: \alpha_i^{-1}(0, 1] \rightarrow W$. Define a G -map $\psi: X \rightarrow \underline{EG}$ by

$$\psi(x) = (\alpha_1(x)\psi_1(x), \alpha_2(x)\psi_2(x), \dots).$$

The proof that ψ is unique up to G -homotopy is the same as the proof of Theorem 3.12.4 in [30]. \square

Appendix 2: Axioms for \underline{EG}

In this appendix we shall prove Proposition 1.8.

Let Y be a proper G -space which satisfies the two axioms in Proposition 1.8. Using Axiom 2 fix a G -map

$$\Phi : Y \times Y \times [0, 1] \rightarrow Y$$

such that

$$\Phi(y_0, y_1, 0) = y_0 \quad \text{and} \quad \Phi(y_0, y_1, 1) = y_1.$$

Let X be a proper G -space and let $\alpha_1, \alpha_2, \dots$ be a partition of unity as in the lemma of Appendix 1. We shall construct a sequence of G -maps

$$\Psi_N : \cup_1^N \alpha_i^{-1}(0, 1] \rightarrow Y$$

such that for every $x \in X$ there is a neighbourhood of x with $\Phi_N = \Phi_{N+1}$ in that neighbourhood, for large enough N .

By Axiom 1 of (1.8), for every i there is a G -map $\psi_i : \alpha_i^{-1}(0, 1] \rightarrow Y$. We define

$$\Psi_1 = \psi_1 : \alpha_1^{-1}(0, 1] \rightarrow Y.$$

Suppose that Ψ_{N-1} has been defined. Define a partition of unity $\{\beta_0, \beta_1\}$ on $\cup_1^N \alpha_i^{-1}(0, 1]$ as follows. Let

$$\begin{aligned} \beta'_0(x) &= \max\{0, \alpha_1(x) + \dots + \alpha_{N-1}(x) - 1/2\alpha_N(x)\}, \\ \beta'_1(x) &= \max\{0, \alpha_N(x) - \alpha_1(x) - \dots - \alpha_{N-1}(x)\}, \end{aligned}$$

and form β_0 and β_1 by dividing β'_0 and β'_1 by $\beta'_0 + \beta'_1$. We set

$$\Psi_N(x) = \begin{cases} \Psi_{N-1}(x) & \text{if } x \notin \alpha_N^{-1}(0, 1] \\ \psi_N(x) & \text{if } x \notin \cup_1^{N-1} \alpha_i^{-1}(0, 1] \\ \Phi(\Psi_{N-1}(x), \psi_N(x), \beta_1(x)) & \text{if } x \in \alpha_N^{-1}(0, 1] \cap \cup_1^{N-1} \alpha_i^{-1}(0, 1]. \end{cases}$$

It is easily verified that Ψ_N is a G -map and that $\Psi_N = \Psi_{N-1}$ on the set where $\alpha_N = 0$.

We define

$$\Psi : X \rightarrow Y,$$

$$\Psi(x) = \Psi_N(x) \quad \text{for } N \text{ large enough}$$

(observe that since the partition of unity $\alpha_1, \alpha_2, \dots$ is locally finite, $\Psi_N = \Psi_{N+1}$ near any point of X , for N large enough). Ψ is a G -map.

If Ψ, Ψ' are two G -maps from X to Y then

$$\begin{aligned} X \times [0, 1] &\rightarrow Y \\ (x, t) &\mapsto \Phi(\Psi x, \Psi' x, t) \end{aligned}$$

is a G -homotopy from Ψ to Ψ' . This completes the proof that every proper G -space satisfying the Axioms in (1.8) is universal. The converse is straightforward, and is left to the reader.

Appendix 3: What does \underline{BG} classify?

Denote by \underline{BG} the quotient space $G \backslash \underline{EG}$.

Let X be any metrizable space and denote by $[X, \underline{BG}]$ the set of homotopy classes of maps from X into \underline{BG} . Our objective is to give a description of $[X, \underline{BG}]$ analogous to the well known description of $[X, BG]$ as isomorphism classes of principal G -bundles over X .

A *proper G -space over X* is a pair (Z, π) where Z is a proper G -space and $\pi : Z \rightarrow X$ is a continuous map such that

- (i) For all $(g, z) \in G \times Z$, $\pi(gz) = \pi(z)$.
- (ii) The map $G \backslash Z \rightarrow X$ determined by π is a homeomorphism of $G \backslash Z$ onto X .

Two proper G -spaces (Z, π) , (Z', π') over X are *isomorphic* if there exists a G -map $f : Z \rightarrow Z'$ with:

- (i) f is a homeomorphism of Z onto Z' .
- (ii) $\pi = \pi' \circ f$.

Suppose that X' is another metrizable space and that (Z, π) is a proper G -space over X' . Let

$$\psi : X \longrightarrow X'$$

be a continuous map and form the space

$$X \times_{X'} Z = \{(w, x) \in X \times Z \mid \psi(x) = \pi(z)\}$$

on which G acts by $g(x, z) = (x, gz)$. This is a proper action and the evident projection

$$\rho : X \times_{X'} Z \rightarrow X$$

identifies the quotient space $G \backslash (X \times_{X'} Z)$ with X . Hence $(X \times_{X'} Z, \rho)$ is a proper G -space over X , called the *pull-back* of (Z, π) along ψ . We shall use the notation

$$\psi^*(Z, \pi) = (X \times_{X'} Z, \rho).$$

Two proper G -spaces (Z_0, π_0) , (Z_1, π_1) over X are *homotopic* if there exists a proper G -space (Z, π) over $X \times [0, 1]$ such that $i_0^*(Z, \pi)$ is isomorphic to (Z_0, π_0) and $i_1^*(Z, \pi)$ is isomorphic to (Z_1, π_1) , where $i_0, i_1 : X \rightarrow X \times [0, 1]$ are the inclusions

$$\begin{aligned} i_0(x) &= (x, 0), \\ i_1(x) &= (x, 1). \end{aligned}$$

Denote by $P(G, X)$ the set of homotopy classes of proper G -spaces over X .

LEMMA. *Let (Z, π) be a proper G -space over X' . If $\psi_0, \psi_1 : X \rightarrow X'$ are homotopic maps then the proper G -spaces $\psi_0^*(Z, \pi)$ and $\psi_1^*(Z, \pi)$ are homotopic.*

□

PROPOSITION. *The function $[X, \underline{BG}] \rightarrow P(G, X)$, which assigns to the homotopy class of a map $\psi: X \rightarrow \underline{BG}$ the pull-back along ψ of the proper G -space \underline{EG} over \underline{BG} , is a bijection of sets.*

PROOF. An inverse map

$$P(G, X) \rightarrow [X, \underline{BG}]$$

is defined as follows. Given a proper G -space (Z, π) over X , there is a G -map $\alpha: Z \rightarrow \underline{EG}$, unique up to G -homotopy. Define $\psi: X \rightarrow \underline{BG}$ so that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & \underline{EG} \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{\psi} & \underline{BG} \end{array}$$

commutes, and let

$$(Z, \pi) \mapsto [\psi].$$

It is immediate that the composition

$$[X, \underline{BG}] \rightarrow P(G, X) \rightarrow [X, \underline{BG}]$$

is the identity. The proof is completed by showing that the other composition is the identity. This follows easily from the following Lemma.

LEMMA. *Let (Z_0, π_0) and (Z_1, π_1) be proper G -spaces over X . Assume that there exists a G -map $f: Z_0 \rightarrow Z_1$ with $\pi_0 = \pi_1 \circ f$. Then (Z_0, π_0) and (Z_1, π_1) are homotopic.*

PROOF. Let M_f be the mapping cylinder of $f: Z_0 \rightarrow Z_1$. Thus a point in M_f is a pair (z, t) with

$$\begin{cases} z \in Z_0 & \text{if } 0 \leq t < 1 \\ z \in Z_1 & \text{if } t = 1. \end{cases}$$

Equipped with the action of G on the z -coordinate it is a proper G -space. Define $\rho: M_f \rightarrow X \times [0, 1]$ by

$$\begin{aligned} \rho(z, t) &= (\pi_0 z, t) & 0 \leq t < 1 \\ \rho(z, 1) &= (\pi_1 z, 1). \end{aligned}$$

Then (M_f, ρ) is a proper G -space over $X \times [0, 1]$. It is a homotopy between (Z_0, π_0) and (Z_1, π_1) . \square

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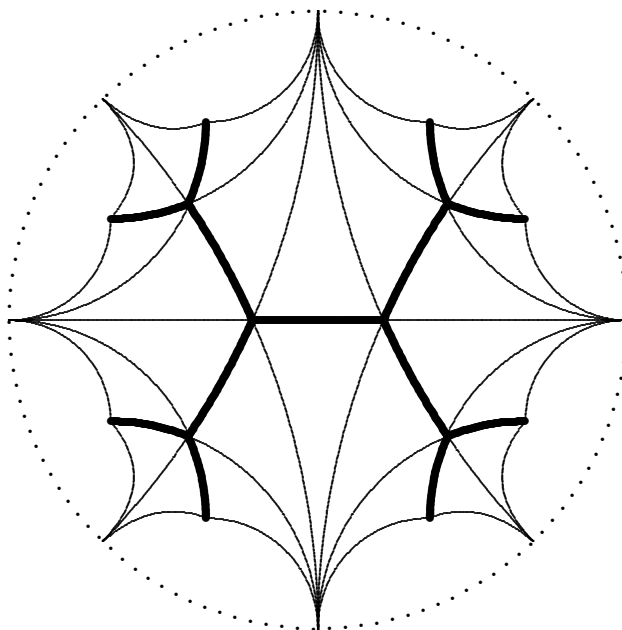


Figure 1.