TORUS EMBEDDINGS

6(a) Torus embeddings

The constructions of this section are valid over any field, but in most cases the ground field, occasionally denoted k, will be $\mathbb{C}$ or $\mathbb{Q}$, and in any case will be a field of characteristic zero. For the purposes of this section, a *torus* is a connected affine algebraic group T of multiplicative type, isomorphic to $\mathbb{G}_m^N$ for some integer N. In particular, T is split. There is an anti-equivalence of categories between (split) tori T over k and free $\mathbb{Z}$-modules X of finite rank:

$$T \mapsto X^*(T) = Hom(T, \mathbb{G}_m);\ \ X \mapsto Hom(X, \mathbb{G}_m)$$

where the latter means the representable functor from (affine) k-schemes $S = Spec(R)$ to abelian groups that takes S to $Hom(X, \mathbb{G}_m(S)) = Hom(X, R^\times)$. We also let $X_*(T) = Hom(\mathbb{G}_m, T)$ so that there is an obvious non-degenerate pairing

$$X_*(T) \times X^*(T) \to \mathbb{Z} = Hom(\mathbb{G}_m, \mathbb{G}_m).$$

Definition. *An equivariant affine embedding (resp. equivariant embedding) of T is an affine variety V (resp. variety V) given with an open immersion $\phi : T \hookrightarrow V$ and an action*

$$T \times V \to V$$

that extends the natural action of T on itself as an open subset. A morphism between equivariant embeddings is a map $V \to V'$ that is the identity on T (and that is then necessarily compatible with the T action).

The morphism ϕ is dual to a map of affine algebras $k[V] \hookrightarrow k[T]$ that extends to an isomorphism $k(V) \xrightarrow{\sim} k(T)$ of function fields.

We fix the notation T, $X^* = X^*(T)$, $X_* = X_*(T)$.

Example: Let $T = \mathbb{G}_m^N$, $V = \mathbb{A}^N$ (affine N-space), and the embedding ϕ is the morphism dual to the inclusion of affine algebras $k[X_1, \dots, X_N] \hookrightarrow k[X_1, X_1^{-1}, \dots, X_N, X_N^{-1}]$. Note that every function in the affine algebra $k[T]$ is a k-linear combination of elements of X^* – this is always true – and that we have chosen a basis $X_1, \dots, X_N$. Thus every function in $k[T]$ is a linear combination $\sum_{a=(a_1,\dots,a_N)\in\mathbb{Z}^N} X^a$, where we write $X^a = \prod X_i^{a_i}$. The inclusion $k[V] \hookrightarrow k[T]$ corresponds to the subring where all $a \in \mathbb{N}$. This is the general situation.

Example: Continuing the previous example, we can embed $\mathbb{A}^N \hookrightarrow \mathbb{P}^N$ as the complement of a hyperplane at infinity; then the action of T extends canonically and $T \hookrightarrow \mathbb{P}^N$ is a (projective) equivariant embedding, obtained as usual by patching $N+1$ affine embeddings.

As the first example makes clear, an affine embedding $T \hookrightarrow V$ is determined by the inclusion $k[V] \hookrightarrow k[T]$. Such an embedding is equivariant provided $k[V]$ carries a grading indexed by X^*:

$$k[V] = \oplus_{r \in X^*} k[V]_r; \; k[V]_r = \{f \in k[V] \mid f(tv) = r(t)f(v), t \in T, v \in V\}$$

where by $t \in T$ we mean t is an R-valued point for some k-algebra R. Note that

$$k[V]_r \times k[V]_{r'} \subset k[V]_{r+r'}$$

so that $S(V) := \{r \in X^* \mid k[V]_r \neq 0\}$ is a sub-semigroup of X^* (the constants k form $k[V]_0$).

Say the semigroup $S \subset X^*$ is *saturated* if, whenever $r \in X^*$ and $n \geq 1$ is an integer such that $r^n \in S$, then $r \in S$.

Theorem. *(a) The correspondence $S \mapsto Speck[S]$ defines a bijection between the set of finitely generated semigroups $S \subset X^*$ that generate X^* as a group and isomorphism classes of equivariant affine embeddings, and morphisms of equivariant affine embeddings correspond (dually) to inclusions of semi-groups.*

(b) The equivariant T-variety $Speck[S]$ is normal if and only if S is saturated.

Part (a) is obvious; part (b) is included only for completeness and will not be used. Of course, if $r^n \in S$ then $Y^n - r^n$ is an integral equation over $k[S]$, hence has a root in $k[S]$ provided $k[S]$ is normal, which implies $r \in k[S] = k[V]$. The other direction uses the fact that we can test normality on homogeneous elements of $k[T]$.

Saturated semigroups arise geometrically as follows. Write $X^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$, $X_{*,\mathbb{R}}$ similarly. The following lemma is obvious

Lemma. *For a subset $\sigma \in X_{*,\mathbb{R}}$, the following conditions are equivalent:*

(i) σ is the intersection of a finite set of rational half-spaces, i.e. there exists a finite subset $\lambda_i, i = 1, \ldots, m$ of $X^* \otimes \mathbb{Q}$ such that

$$\sigma = \{x \mid \lambda_i(x) \geq 0, i = 1, \ldots, m\};$$

(ii) σ is generated by a finite set $x_1, \ldots, x_n \in X_* \otimes \mathbb{Q}$, i.e.

$$\sigma = \{\sum_{i=1}^{n} \alpha_i x_i, \alpha_i \in \mathbb{R}_+\}.$$

Note that $X_{*,\mathbb{R}}$ satisfies these conditions: in this case $m = 0$ and n is twice the rank (so if $X_* = \mathbb{Z}$ then $x_1 = +1, x_2 = -1$). A subset σ satisfying these conditions is called a *(convex) rational polyhedral cone*, or just *polyhedral cone*. We let

$$\check{\sigma} = \{\lambda \in X^*_{\mathbb{R}} \mid \; < \lambda, x > \geq 0 \forall x \in \sigma\}.$$

Then the lemma above is equivalent to the duality $\check{\check{\sigma}} = \sigma$ and $\check{\sigma}$ is also a polyhedral cone.

Moreover, σ contains no linear subspace if and only if $\check{\sigma}$ is contained in no hyperplane.

If σ is a polyhedral cone, then a *face* of σ is a subset $\sigma' \subset \sigma$ of the form $\{x \in \sigma \mid \lambda(x) = 0\}$ for some $\lambda \in \check{\sigma}$. The intersection of two faces is a face (take $\lambda_1 + \lambda_2$; since both $\lambda_i(x) \geq 0$ the condition $\lambda_1(x) + \lambda_2(x) = 0$ implies $\lambda_1(x) = \lambda_2(x) = 0$).

Theorem. *The correspondence $\sigma \mapsto Speck[\check{\sigma} \cap X^*] = V_\sigma$ defines a bijection between the set of polyhedral cones in $X_{*,\mathbb{R}}$ that contain no linear subspace and the set of equivariant affine normal embeddings of T. Moreover, if $\mu \in X_*$ then $\mu \in \sigma$ if and only if $\lim_{t\to 0} \mu(t)$ exists in V_σ, in which case it is denoted $\mu(0)$.*

Here $\mu \in Hom(\mathbb{G}_m, T)$ so $\mu(t) \in T \subset V_\sigma$. The proof is easy once we know "Gordan's Lemma" which is that the semi-group of integral solutions to a finite set of homogeneous linear integral inequalities is finitely generated.

Letting T vary in the category of tori, we can reformulate the theorem as an equivalence of categories between cones in (rational) vector spaces and equivariant affine embeddings, if interpreted correctly. Now I will describe the T orbits in $V = V_\sigma$. The set of orbits is finite and in $1-1$ correspondence with the faces of σ, as follows. For any face $\sigma' \subset \sigma$, the *interior* of σ' is the complement of the union of all smaller faces.

Theorem. *Let $\mu, \mu' \in \sigma$. Then $\mu(0) = \mu'(0)$ if and only if μ and μ' are in the interior of the same face σ', and we call this point $v_{\sigma'}$. The T-orbit $O_{\sigma'}$ of $v_{\sigma'}$ is equivariantly isomorphic to the quotient of T by the subtorus generated by the images of μ in the interior of σ', and in particular $\dim O_{\sigma'} + \dim \sigma' = \dim T$. The open orbit corresponds to the vertex of σ. Finally, $\sigma_1 \subset \sigma_2$ if and only if O_{σ_2} is contained in the closure of O_{σ_1}.*

Here is how it works for the case $\mathbb{G}_m^N \subset \mathbb{A}^N$, which corresponds to the positive sector $\sigma = \mathbb{R}^N_{\geq 0} \subset \mathbb{R}^N$. Each face $\sigma' = \sigma_J$ corresponds to a set $J \subset \{1, \ldots, N\}$ such that $z_j \equiv 0$ on σ' if and only if $j \notin J$, so that $\dim \sigma_J = |J|$. It follows that O_{σ_J} is of dimension $N - |J|$ and it is naturally the open $\mathbb{G}_m^{N-|J|}$ corresponding to the coordinates not in J. So if $N = 1$, $\sigma_\emptyset$ is the vertex which corresponds to the open orbit, σ_1 is the open half-line which corresponds to the origin in $\mathbb{A}^1$. By a change of basis, every polyhedral cone can be identified with a face of this σ, and in this way it is easy to see that

Theorem. *If $\sigma_1 \subset \sigma_2$ then there is a morphism $V_{\sigma_1} \to V_{\sigma_2}$, and this morphism is an open immersion if and only if σ_1 is a face of σ_2.*

In the model above, σ_J is a face of $\sigma_{J'}$ if and only if $J \subset J'$. Then V_{σ_J} is obtained from $V_{\sigma_{J'}}$ by removing the orbits $O_{\sigma'}$ where σ' is a face of $\sigma_{J'}$ not contained in σ_J, in other words by removing the O_{σ_I} where $I \subset J'$ is not entirely contained in J. So if $J = \{1, 2\}$ and $J' = \{1, 2, 3\}$ then we remove any O_{σ_I} when $3 \in I$. It is an easy combinatorial exercise to show that this is an open immersion.

Theorem. *The equivariant affine embedding V_σ is smooth if and only if $\sigma \cap X_*$ can be generated by a subset of a basis of X_* over $\mathbb{Z}$.*

This condition is combinatorially equivalent to the condition that V_σ is a product of copies of $\mathbb{G}_m$ by $\mathbb{A}^1$.

Patching.

Now that we have an equivalence of categories between face inclusions of polyhedral cones and open immersions, we can patch.

Definition. *A fan in X_* is a collection F of (rational convex) polyhedral cones $\{\sigma\} \subset X_{*,\mathbb{R}}$ such that (i) every face of an element of F belongs to F and (ii) if $\sigma, \sigma' \in F$ then $\sigma \cap \sigma'$ is a face of σ and of σ'.*

Each $\sigma \in F$ defines an affine embedding V_σ. We patch V_σ and $V_{\sigma'}$ along their common open subset $V_{\sigma\cap\sigma'}$. This gives us a well-defined k-variety V_F which is an equivariant embedding, but not affine.

Theorem. *The correspondence $F \mapsto V_F$ is a bijection between fans and isomorphism classes of equivariant (normal) T-embeddings. The equivariant open affine subsets of V_F are in $1-1$ correspondence with $\sigma \in F$.*

Definition. *A refinement of the fan F is a fan F' such that (i) every $\sigma' \in F'$ is contained in a $\sigma \in F$ and (ii) every $\sigma \in F$ is a finite union of $\sigma' \in F'$.*

Theorem. *Let F' and F are fans in X_*. If they satisfy condition (i) above, then there is a canonical equivariant morphism $X_{F'} \to X_F$. This morphism is proper if and only if F' is a refinement of F. In particular, X_F is complete if and only if F is a refinement of the trivial fan with sole cone $X_{*,\mathbb{R}}$.*

The first part follows from what we did before, the second from the valuative criterion of properness. The final statement seems to be in contradiction with what we did before, since the "trivial fan" obviously contains a rational linear subspace; but any cone containing a linear subspace W should be replaced by its quotient by W in $X_*(T')$ where T' is the quotient of T by the subtorus generated by the cocharacters in W. The first part of the theorem remains valid in this generality. In the case of the last statement, T' is the 0-dimensional torus and the morphism to $X_{trivial}$ is the map to the point.

Corollary. *Any F has a refinement F' such that $V_{F'}$ is a smooth resolution of V_F.*

One needs to use the criterion above for smoothness. If the dimension is 1 it is already smooth. We induct upwards on dimension and can assume that F consists of a single simplex, each of whose maximal faces σ' satisfies the criterion that its linear generators generate the corresponding lattice $N_{\sigma'} = X_* \cap \mathbb{R}\sigma'$ and such that $N_{\sigma'}$ is a direct summand of X_*. Then we just need to find a vector in X_* in the interior of σ that together with the linear generators of each maximal face σ' generates X_* over $\mathbb{Z}$.

Theorem. *Any complete F has a refinement F' such that $V_{F'}$ is a smooth projective toric variety.*

This is based on an ampleness criterion for line bundles.

Cohomology.

The cohomology of sheaves on torus embeddings can be calculated purely topologically. Let T be a torus, F a fan, $T \hookrightarrow V = V_F$ the equivariant embedding. Let $|F|$ denote the topological realization of F. For simplicity, we assume every $\sigma \in F$ is a simplex, i.e. the number of 1-dimensional faces equals its dimension.

We want to calculate $H^i(V, \mathcal{O}_V)$ topologically. Since $\mathcal{O}_V$ is T-equivariant, there is an action of T on $H^i(V, \mathcal{O}_V)$ that decomposes it as

$$H^i(V, \mathcal{O}_V) = \oplus_{\chi \in X^*(T)} H^i(V, \mathcal{O}_V)[\chi].$$

For $\chi \in X^*(T)$, let $F(\chi) = |V(\chi)|$ (topological realization), where $V(\chi) \subset V$ is the largest T-invariant open subset U such that $\chi \in \Gamma(U, \mathcal{O}_U)$; equivalently, $F(\chi) \subset |F|$ is the subset of x where $< \chi, x > \geq 0$.

Theorem. *For any $\chi \in X^*(T)$,*

$$H^i(V, \mathcal{O}_V)[\chi] = H^i_{F(\chi)}(|F|, k).$$

The long exact sequence for cohomology with support is the following:

$$0 \to H^0_{F(\chi)}(|F|, k) \to H^0(|F|, k) \to H^0(|F| \setminus F(\chi), k) \to H^1_{F(\chi)}(|F|, k) \to$$

Corollary. *Let F' be a refinement of F. Then the natural map*

$$H^i(V_F, \mathcal{O}_{V_F}) \to H^i(V_{F'}, \mathcal{O}_{V_{F'}})$$

is an isomorphism for all i and if $\phi : V_{F'} \to V_F$ is the natural map, then $R^i\phi_*(\mathcal{O}_{V_{F'}}) = 0$, $i > 0$, and there is a natural isomorphism $\phi_*(\mathcal{O}_{V_{F'}}) = \mathcal{O}_{V_F}$.

The first part of the corollary follows from the Theorem because $|F'|$ and $|F|$ are homeomorphic (the former is a triangulation of the latter) and the homeomorphism identifies $F'(\chi)$ with $F(\chi)$. The second part of the corollary is a formal consequence: it suffices to compute $R^i\phi_*(\mathcal{O}_{V_{F'}}) = 0$ on an affine open of V_F, so we may assume V_F is affine and F is a single (closed) simplex. Then $R^i\phi_*(\mathcal{O}_{V_{F'}})$ is the coherent sheaf attached to the $\mathcal{O}_{V_F}$ module $H^i(V_{F'}, \mathcal{O}_{V_{F'}})$, but by part 1 this is just $H^i(V_F, \mathcal{O}_{V_F}) = 0$ if $i > 0$ because V_F is affine. For $i = 0$, the first part shows that $H^0(V_{F'}, \mathcal{O}_{V_{F'}}) = \Gamma(V_F, \mathcal{O}_{V_F})$ and likewise for any T-equivariant affine open, which implies the final statement.

We now prove the Theorem. First assume F is a single simplex, so V is affine. Then we have to show that $H^i_{F(\chi)}(|F|, k) = 0$ for all χ if $i > 0$ and that $H^0_{F(\chi)}(|F|, k) = k$ if and only if $\chi \in \Gamma(V_F, \mathcal{O}_{V_F})$ and is 0 otherwise.

Now $|F|$ is a cone and by definition, $|F| \setminus F(\chi)$ is the intersection of a cone with the open halfspace where $< \chi, x >< 0$. So both $|F|$ and $|F| \setminus F(\chi)$ are convex, hence contractible, and the long exact sequence comes down to

$$0 \to H^0_{F(\chi)}(|F|, k) \to H^0(|F|, k) \to H^0(|F| \setminus F(\chi), k) \to H^1_{F(\chi)}(|F|, k) \to 0$$

with all the other terms equal to 0. By definition, $\chi \in \Gamma(V_F, \mathcal{O}_{V_F})$ if and only if $< \chi, x > \geq 0$ for all $x \in |F|$ which means that

$$\chi \in \Gamma(V_F, \mathcal{O}_{V_F} \Leftrightarrow |F| \setminus F(\chi) = \emptyset \Rightarrow H^0_{F(\chi)}(|F|, k) = H^0(|F|, k), H^1_{F(\chi)}(|F|, k) = 0$$

by the long exact sequence. On the other hand, if $|F|\backslash F(\chi) \neq \emptyset$ then $H^0_{F(\chi)}(|F|, k) = 0$ and the map $H^0(|F|, k) \to H^0(|F| \setminus F(\chi), k)$ is an isomorphism of 1-dimensional spaces, so we still have $H^1_{F(\chi)}(|F|, k) = 0$. This completes the proof of the affine case.

One also checks that the isomorphisms in the affine case are compatible with inclusions of affine torus embeddings. To prove the general case, we use a Cech covering by open affines to calculate the cohomology $H^i(V, \mathcal{O}_V)[\chi]$ and a Cech covering by closed simplices to compute $H^i_{F(\chi)}(|F|, k)$. This gives an isomorphism of complexes computing cohomology and therefore an isomorphism of cohomology spaces.