[F3] H. G. FEICHTINGER, Minimal Banach spaces and atomic decompositions, *Publ. Math. Debrecen* **33** (1986), 167–168 and **34** (1987), 231–240.

[F4] H. G. FEICHTINGER, Atomic characterizations of modulation spaces, *in* "Proceedings, Conference on Constructive Function Theory, Edmonton, July 1986." *Rocky Mountain J.*, to appear.

[FGr] H. G. FEICHTINGER AND P. GRÖBNER, Banach spaces of distributions defined by decomposition methods, I, *Math. Nachr.* **123** (1985), 97–120.

[FG] H. G. FEICHTINGER AND K. GRÖCHENIG, A unified approach to atomic decompositions via integrable group representations, *in* "Proceedings, Conference on Functions, Spaces and Applications, Lund, 1986," Lecture Notes in Mathematics, Springer-Verlag, **1302** (1988), 52–73, New York/Berlin, 1987.

[FJ1] M. FRAZIER AND B. JAWERTH, Decomposition of Besov spaces, *Indiana Univ. Math. J.* **34** (1985), 777–799.

[FJ2] M. FRAZIER AND B. JAWERTH, The φ-transform and decompositions of distribution spaces, *in* Proceedings, Conference on Functions, Spaces and Applications, Lund, 1986," Lecture Notes in Mathematics, Springer-Verlag, **1302** (1988), Heidelberg, 1987.

[GMP1] A. GROSSMANN, J. MORLET, AND T. PAUL, Transforms associated to square integrable group representations, I, *J. Math. Phys.* **26**, No. 10 (1985), 2473–2479.

[GMP2] A. GROSSMANN, J. MORLET, AND T. PAUL, Transforms associated to square integrable group representations, II, Examples, *Ann. Inst. H. Poincaré* **45** (1986), 293–309.

[KS] J. KLAUDER AND B. SKAGERSTAM (Eds.), Coherent States—Applications in Physics and Mathematical Physics," World Sci. Publ., Singapore, 1985.

[LT] J. LINDENSTRAUSS AND L. TZAFRIRI, "Classical Banach Spaces," Springer-Verlag, Berlin/Heidelberg/New York, 1977.

[P1] J. PEETRE, Paracommutators and minimal spaces, *in* "Operators and Function Theory" (S. C. Powers, Ed.), NATO ASI Series, Reidel, Dordrecht, 1985.

[P2] J. PEETRE, Some calculations related to Fock space and the Shale–Weil representation, preprint, 1987.

[RT] F. RICCI AND M. TAIBLESON, Boundary values of harmonic functions in mixed norm spaces and their atomic structure, *Ann. Scuola Norm. Sup. Pisa* (4) **10** (1983), 1–54.

[R] R. ROCHBERG, Decomposition theorems for Bergman spaces and their applications, *in* "Operators and Function Theory" (S. C. Powers, Eds.), NATO ASI Series, Reidel, Dordrecht, 1985.

[Z] A. ZAANEN, "Integration," 2nd ed., North-Holland, Amsterdam, 1967.

JOURNAL OF FUNCTIONAL ANALYSIS **86**, 341–359 (1989)

On Lie Algebras of Operators

OLA BRATTELI

Institute of Mathematics, University of Trondheim, N-7034 Trondheim-NTH, Norway

GEORGE A. ELLIOTT

Mathematics Institute, University of Copenhagen, DK-2100 Copenhagen, Denmark, and Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

AND

F. M. GOODMAN AND PALLE E. T. JORGENSEN

Department of Mathematics, University of Iowa, Iowa City, Iowa 52242

Communicated by the Editors

Received March 18, 1988

We consider the integrability problem for Lie algebras of (generally unbounded) operators in Banach space $\mathscr{X}$. In addition, a Lie group G is given acting strongly continuously on $\mathscr{X}$. Smoothness is defined as a *relative* notion with respect to the "*basepoint action.*" We consider a class of smooth perturbations of Lie algebras and establish integrability for the perturbed operator Lie algebra. We also have a structure theoretic result for the components of the Levi decomposition of the perturbed Lie algebra. We give applications to automorphic Lie actions on C^*-algebras, and to Lie algebras of derivations. A sequel paper restricts the setting further to the case of the irrational rotation C^*-algebras. There a classification of smooth actions is given using the general results of the present paper.

1. INTRODUCTION

In this paper we consider some questions regarding the integrability and structure of Lie algebras of operators which were motivated by our interest in smooth actions of Lie groups and Lie algebras on C^* algebras. First of all, we introduce a notion of smoothness for actions of Lie groups and Lie

0022-1236/89 $3.00

580/86/2-9

algebras which is an appropriate generalization of the usual notions of smoothness for actions on manifolds.

We then consider finite dimensional Lie algebras which are perturbations of a given integrable Lie algebra in the following sense. Let α be a strongly continuous action of a Lie group G on a Banach space $\mathscr{X}$. A perturbation class $\mathfrak{P}$ for $d\alpha(\mathfrak{G})$ is a Lie algebra of bounded operators leaving $\mathscr{X}^\infty(\alpha)$ invariant, such that $[d\alpha(\mathfrak{G}), \mathfrak{P}] \subset \mathfrak{P}$. We show that finite dimensional Lie subalgebras of $d\alpha(\mathfrak{G}) + \mathfrak{P}$ exponentiate to smooth Lie group actions. Our main result, Theorem 3.4, is a structure theorem for perturbations of *abelian* operator Lie algebras (under an additional assumption which ensures that the Lie algebras exponentiate to uniformly bounded Lie group representations). We show that in a Levi decomposition $\mathfrak{S} + \mathfrak{R}$ of such an operator Lie algebra, $\mathfrak{S}$ and $\mathfrak{R}$ are commuting ideals, $\mathfrak{S}$ is compact, and consists of bounded operators, and $\mathfrak{R}$ is a two step solvable algebra of a special type.

In the final Section 4, we describe a class of examples which motivated the investigation. Certain C^* dynamical systems $(\mathfrak{A}, G, \alpha)$ have the property that every derivation in $\mathrm{Der}(\mathfrak{A}^\infty(\alpha))$ has a unique decomposition $\delta = \delta_0 + \delta_1$, where $\delta_0 \in d\alpha(\mathfrak{G})$, and δ_1 is bounded. We show that if G is abelian, then the class $\mathfrak{P}$ of bounded smooth derivations is a perturbation class for $d\alpha(\mathfrak{G})$. It follows that every finite dimensional Lie subalgebra of $\mathrm{Der}(\mathfrak{A}^\infty(\alpha))$ exponentiates to a smooth Lie group action, and that the structure theorem 3.4 applies to every such finite dimensional Lie algebra of smooth derivations. In a companion paper [BEGJ], we use these ideas to give a nearly complete analysis of smooth Lie algebra actions on the irrational rotation C^* algebras.

2. Smooth Actions of Lie Groups and Lie Algebras

Let $\mathscr{X}$ be a Banach space, G a Lie group, and $\alpha: G \to \mathscr{B}(\mathscr{X})$ a strongly continuous representation of G. Let $\mathscr{X}^\infty = \mathscr{X}^\infty(\alpha)$ denote the space of C^∞-elements for this action and $d\alpha$ the derived action of the Lie algebra $\mathfrak{G}$ of G on $\mathscr{X}^\infty$:

$$d\alpha(X)(a) = \frac{d}{dt}\bigg|_{t=0} \alpha(\exp(tX))(a) \qquad (X \in \mathfrak{G}, a \in \mathscr{X}^\infty).$$

If H is another Lie group and $\rho: H \to \mathscr{B}(\mathscr{X})$ a strongly continuous representation, we shall say that ρ is smooth (with respect to α) if for all $a \in \mathscr{X}^\infty(\alpha)$, the map $(g, h) \mapsto \alpha_g \rho_h a$ is smooth from $G \times H$ to $(\mathscr{X}, \|\cdot\|)$. In particular, this implies that

(1) ρ_h maps $\mathscr{X}^\infty(\alpha)$ into $\mathscr{X}^\infty(\alpha)$ for all $h \in H$, and

(2) $\mathscr{X}^\infty(\alpha) \subseteq \mathscr{X}^\infty(\rho)$.

We shall call elements of $\mathrm{End}(\mathscr{X}^\infty(\alpha))$ smooth operators, and call a representation of a Lie algebra $\mathfrak{h}$ in $\mathrm{End}(\mathscr{X}^\infty(\alpha))$ a smooth representation of $\mathfrak{h}$ (with respect to α).

Let us now fix the system $(\mathscr{X}, G, \alpha)$, so that smoothness will always refer to α. Our first observation is that differentiation of a smooth Lie group representation yields a smooth Lie algebra representation.

Lemma 2.1. *Let $\rho: H \to \mathscr{B}(\mathscr{X})$ be a smooth representation, and let $\mathfrak{h}$ denote the Lie algebra of H. Then*

(a) *$\mathscr{X}^\infty(\alpha)$ is invariant under the operators $d\rho(Y)$ $(Y \in \mathfrak{h})$, and*

(b) *$\mathscr{X}^\infty(\alpha)$ is a core for $d\rho(Y)$ $(Y \in \mathfrak{h})$.*

Proof. By hypothesis, for any $a \in \mathscr{X}^\infty(\alpha)$ the function $f_a(g, h) = \alpha_g \rho_h a$ is smooth on $G \times H$. Hence for each $Y \in \mathfrak{h}$, the derivative

$$(Yf_a)(g, e) = \frac{d}{dt}\bigg|_{t=0} \alpha_g \rho_{\exp(tY)} a = \alpha_g\, d\rho(Y) a$$

is a smooth function of $g \in G$. But this means that $d\rho(Y) a \in \mathscr{X}^\infty(\alpha)$. Finally, since $\mathscr{X}^\infty(\alpha)$ is invariant under ρ_h $(h \in H)$, the core theorem yields conclusion (b); see, for example, [Po, Theorem 1.3]. ∎

Next we show that if both ρ and $d\rho$ leave $\mathscr{X}^\infty(\alpha)$ invariant, then ρ is smooth.

Proposition 2.2. *Suppose that $\rho: L \to B(\mathscr{X})$ is a strongly continuous representation of a Lie group L on $\mathscr{X}$ such that*

(1) *$\mathscr{X}^\infty(\rho) \supseteq \mathscr{X}^\infty(\alpha)$,*

(2) *$d\rho(Y): \mathscr{X}^\infty(\alpha) \to \mathscr{X}^\infty(\alpha)$ for all Y in the Lie algebra $\mathfrak{L}$ of L, and*

(3) *$\rho_h: \mathscr{X}^\infty(\alpha) \to \mathscr{X}^\infty(\alpha)$ for all $h \in L$.*

Then ρ is smooth with respect to α.

Proof. Denote by τ^∞ the Fréchet topology on $\mathscr{X}^\infty(\alpha)$ generated by the semi-norms $a \mapsto \|d\alpha(X)a\|$, X in the enveloping algebra $\mathscr{E}(\mathfrak{G})$ of the Lie algebra $\mathfrak{G}$ of G. It follows from the closed graph theorem that $\rho_h: \mathscr{X}^\infty(\alpha) \mapsto \mathscr{X}^\infty(\alpha)$ is τ^∞–τ^∞ continuous for all $h \in L$.

We assert that ρ is a strongly continuous representation on $(\mathscr{X}^\infty(\alpha), \tau^\infty)$, i.e., that for each $a \in \mathscr{X}^\infty(\alpha)$ the function $\Phi: h \mapsto \rho_h a$ is continuous from L

to $(\mathscr{X}^\infty(\alpha), \tau^\infty)$. Let $(\varphi_n)_{n \geq 1}$ be an approximate identity in $L^1(G)$ consisting of non-negative C^∞ functions of compact support. The operators

$$\alpha(\varphi_n)x = \int_G \varphi_n(g)\alpha_g(x)\, dg$$

are continuous from $(\mathscr{X}, \|\cdot\|)$ to $(\mathscr{X}^\infty(\alpha), \tau^\infty)$, and $\lim_{n\to\infty} \|\alpha(\varphi_n)x - x\| = 0$ for all x. Therefore the functions $\Phi_n: h \mapsto \alpha(\varphi_n)\rho_h a$ are continuous from L to $(\mathscr{X}^\infty(\alpha), \tau^\infty)$ and converge pointwise to Φ. Since L is a Baire space, it follows that Φ has at least one point of continuity h_0 [Bour, Chap. IX, Exercise 22]. But since ρ is a group homomorphism, and each ρ_h is τ^∞–τ^∞ continuous, this implies that Φ is continuous at every point.

For $a \in \mathscr{X}^\infty(\alpha)$, write $\alpha_g\rho_h(a) = f_a(g, h)$. Let us prove that for $X \in \mathscr{E}(\mathfrak{G})$, the derivative

$$(Xf_a)(g, h) = \alpha_g\, d\alpha(X)\rho_h a$$

is continuous from $G \times L$ to $(\mathscr{X}, \|\cdot\|)$. For fixed $g \in G$, $h \mapsto (Xf_a)(g, h)$ is continuous, since $h \mapsto \rho_h a$ is continuous from L to $(\mathscr{X}^\infty(\alpha), \tau^\infty)$. Joint continuity follows because $\{\alpha_g\}$ is uniformly bounded in a compact neighborhood of any point $g_0 \in G$, by the uniform boundedness principle.

For $Y \in \mathscr{E}(\mathfrak{L})$, we have $(Yf_a)(g, h) = \alpha_g\rho_h\, d\rho(Y)a = f_{d\rho(Y)a}(g, h)$. Hence for $X \in \mathscr{E}(\mathfrak{G})$,

$$(XYf_a)(g, h) = (Xf_{d\rho(Y)a})(g, h),$$

and the right-hand side is continuous on $G \times L$ by the previous observation. It remains to be shown that $(YXf_a)(g, h) = (XYf_a)(g, h)$ for all $X \in \mathscr{E}(\mathfrak{G})$ and $Y \in \mathscr{E}(\mathfrak{L})$.

For $Y \in \mathfrak{L}$ and $a \in \mathscr{X}^\infty(\alpha)$, the map $t \mapsto \rho_{\exp(tY)}a$ is continuous from $\mathbf{R}$ to $(\mathscr{X}^\infty(\alpha), \tau^\infty)$. (See above and [MZ].) We assert that this function is also differentiable at $t = 0$ with respect to τ^∞, with derivative $d\rho(Y)a$. For all t the integral

$$\int_0^t \rho_{\exp(sY)}\, d\rho(Y)a\, ds,$$

viewed simply as a limit of Riemann sums, converges with respect to $\|\cdot\|$ and with respect to τ^∞ to the same value. Since $(d/ds)\rho_{\exp(sY)}a = \rho_{\exp(sY)}\, d\rho(Y)a$ (with respect to $\|\cdot\|$), an application of the fundamental theorem of calculus gives

$$\rho_{\exp(tY)}a - a = \int_0^t \rho_{\exp(sY)}\, d\rho(Y)a\, ds.$$

Since the integrand is τ^∞-continuous, a second application of the fundamental theorem gives the assertion. It follows that for $h \in L$, $g \in G$, and $X \in \mathscr{E}(\mathfrak{G})$,

$$\begin{aligned}(YXf_a)(g, h) &= \frac{d}{dt}\bigg|_{t=0} \alpha_g\, d\alpha(X)\rho_h\rho_{\exp(tY)}a \\ &= \alpha_g\, d\alpha(X)\rho_h\, d\rho(Y)a = Xf_{d\rho(Y)a}(g, h) = (XYf_a)(g, h).\end{aligned}$$

Hence, $YXf_a = XYf_a$ for all $X \in \mathscr{E}(\mathfrak{G})$ and $Y \in \mathscr{E}(\mathfrak{L})$. ∎

3. Perturbations of Abelian Operator Lie Algebras

Recall that a Lie algebra $\mathfrak{L}$ of operators on a domain D in $\mathscr{X}$ is said to be exponentiable if there is a strongly continuous representation $\rho: L \to \mathscr{B}(\mathscr{X})$, where L is the simply connected Lie group with Lie algebra isomorphic to $\mathfrak{L}$, such that

(1) $D \subseteq \mathscr{X}^\infty(\rho)$,
(2) $d\rho(X)|_D = X|_D$ $(X \in \mathfrak{L})$, and
(3) D is a core for $d\rho(X)$ $(X \in \mathfrak{L})$.

In general, the correspondence between representations of Lie groups on $\mathscr{X}$ and representations of Lie algebras is inexact, because a representation of a Lie algebra may fail to exponentiate. However, we shall now single out a class of Lie algebras of smooth operators which do exponentiate to smooth representations of the corresponding simply connected Lie groups.

A Lie algebra $\mathfrak{P}$ of bounded operators on $\mathscr{X}$ is said to be a perturbation class for $\mathfrak{L}_0 = d\alpha(\mathfrak{G})$ if

(1) $\mathscr{X}^\infty(\alpha)$ is invariant under the operators in $\mathfrak{P}$, and
(2) $[\mathfrak{L}_0, \mathfrak{P}] \subseteq \mathfrak{P}$.

$\mathfrak{P}$ is permitted to be infinite dimensional. For example, let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system and set

$$\mathfrak{P} = \{\mathrm{ad}(h): h \in \mathfrak{A}^\infty(\alpha), h \text{ skew adjoint}\}.$$

For $\delta \in \mathfrak{L}_0 = d\alpha(\mathfrak{G})$ and $\mathrm{ad}(h) \in \mathfrak{P}$, $[\delta, \mathrm{ad}(h)] = \mathrm{ad}(\delta(h))$, so $\mathfrak{P}$ is a perturbation class for $\mathfrak{L}_0$.

Note that $\mathfrak{L}_0 + \mathfrak{P}$ is a Lie subalgebra of $\mathrm{End}(\mathscr{X}^\infty)$. We will consider finite dimensional Lie subalgebras of $\mathfrak{L}_0 + \mathfrak{P}$; such Lie algebras were called "semi-direct product perturbations of $\mathfrak{L}_0$" in [JM, Chap. 9].

PROPOSITION 3.1. *Let $\mathfrak{P}$ be a perturbation class for $\mathfrak{L}_0 = d\alpha(\mathfrak{G})$, and let $\mathfrak{L}$ be a finite dimensional Lie subalgebra of $\mathfrak{L}_0 + \mathfrak{P}$. Then $\mathfrak{L}$ exponentiates to a smooth representation of the simply connected Lie group L with algebra isomorphic to $\mathfrak{L}$.*

Proof. That $\mathfrak{L}$ exponentiates to a continuous representation ρ follows at once from [JM, Theorem 9.9]. Because $d\rho(X)$ extends X $(X \in \mathfrak{L})$, $\mathscr{X}^\infty(\alpha)$ is invariant under $d\rho(X)$ and hence $\mathscr{X}^\infty(\alpha) \subseteq \mathscr{X}^\infty(\rho)$.

We next show that $\mathscr{X}^\infty(\alpha)$ is invariant under ρ_h $(h \in L)$. For this, it is enough to show that $\mathscr{X}^\infty(\alpha)$ is invariant under $\exp(X+P)$ for $X \in \mathfrak{L}_0$ and $P \in \mathfrak{P}$.

We assert that for each $n \in \mathbf{N}$, P maps $\mathscr{X}^n(\alpha)$, the space of C^n-elements for the action α, into itself, that P is bounded with respect to the norm $\| \ \|_n$ of $\mathscr{X}^n(\alpha)$, and, finally, that $\exp(X+P)$ leaves $\mathscr{X}^n(\alpha)$ invariant. Let $Y_1, ..., Y_d$ be a basis of $\mathfrak{L}_0$. Then $Y_i P = PY_i + [Y_i, P]$, as operators on $\mathscr{X}^\infty(\alpha)$, and $[Y_i, P] \in \mathfrak{P} \subseteq B(\mathscr{X})$. Hence

$$\|Y_i Pa\| \leqslant \|P\| \ \|Y_i a\| + \|[Y_i, P]\| \ \|a\| \qquad (a \in \mathscr{X}^\infty(\alpha)).$$

This shows that P is bounded with respect to the norm

$$\|a\|_1 = \max\{\|Y_i a\| + \|a\| : 1 \leqslant i \leqslant d\},$$

and it follows also that for $a \in \mathscr{X}^1(\alpha)$, $Pa \in \bigcap_{1 \leqslant i \leqslant d} D(Y_i) = \mathscr{X}^1(\alpha)$.

The one-parameter group of operators, $t \mapsto \exp tX$, restricts to a strongly continuous group on $\mathscr{X}^1(\alpha)$. Indeed, if $a \in \mathscr{X}^1(\alpha)$ and $Y \in \mathfrak{L}_0$, then

$$\lim_{t \to 0} \|d\alpha(Y)(\alpha(\exp(tX))a - a)\|$$
$$= \lim_{t \to 0} \|\alpha(\exp(tX))\, d\alpha(\mathrm{Ad}(\exp(-tX))(Y))a - d\alpha(Y)a\|,$$

which is zero due to the uniform boundedness of $\{\alpha(\exp(tX)) : |t| \leqslant 1\}$ and the continuity of the adjoint representation of G on the finite dimensional space $\mathfrak{L}_0$. Denote the infinitesimal generator of this restricted group by X_1; if a is in the domain of X_1 then also $a \in D(X)$ and $X_1(a) = X(a)$, since

$$\|t^{-1}(e^{tX}a - a) - X_1 a\| \leqslant \|t^{-1}(e^{tX}a - a) - X_1 a\|_1.$$

Since $P_1 := P|_{\mathscr{X}^1(\alpha)}$ is a bounded operator in the Banach space $\mathscr{X}^1(\alpha)$, it follows from Phillips' perturbation theorem that $X_1 + P_1$ is also the infinitesimal generator of a strongly continuous one-parameter group on

$\mathscr{X}^1(\alpha)$, and it is easy to see that this group agrees with $\exp t(X+P)$ on $\mathscr{X}^1(\alpha)$, because for $a \in D(X_1)$

$$\frac{d}{dt} e^{-t(X+P)} e^{t(X_1+P_1)} a = 0.$$

The invariance of $\mathscr{X}^1(\alpha)$ under $\exp t(X+P)$ is immediate from this.

This establishes the case $n = 1$ of our assertion. The general case $(n > 1)$ follows at once by induction, as the space $\mathscr{X}^{n+1}(\alpha)$ is the space of C^1-vectors for the action of G on the Banach space $\mathscr{X}^n(\alpha)$.

Since $\mathscr{X}^n(\alpha)$ is invariant under $\exp(X+P)$ for all n, so also is $\mathscr{X}^\infty(\alpha)$. Hence ρ is smooth by Proposition 2.2. ∎

The following theorem, a restatement of results from [JM, Appendix G], is a useful tool for analyzing Lie algebras of operators which exponentiate to uniformly bounded Lie group representations, i.e., representations whose image in $\mathscr{B}(\mathscr{X})$ is norm bounded. The theorem generalizes a result of Singer [Si] for unitary representations; see also [KS, NS, Sa].

Given a real Lie algebra $\mathfrak{L}$ of operators in $\mathscr{X}$, we let $\mathfrak{L}_b$ denote the Lie algebra of bounded elements in $\mathfrak{L}$, and $\mathfrak{L}^{\mathbf{C}}$ the complexification of $\mathfrak{L}$, which may be identified with the complex span of $\mathfrak{L}$ in the operators on X.

THEOREM 3.2 ([JM]; The Generalized Singer Theorem). *Let $\mathfrak{L}$ be a Lie algebra of operators in a Banach space which exponentiates to a uniformly bounded Lie group representation. Then*

(a) *$\mathfrak{L}_b$ is an ideal in $\mathfrak{L}$.*

(b) *For all $\xi \in \mathfrak{L}$, $ad\,\xi|_{\mathfrak{L}_b^{\mathbf{C}}}$ is diagonalizable, with purely imaginary eigenvalues.*

(c) *Let $\mathfrak{L} = \mathfrak{S} \dotplus \mathfrak{R}$ be a Levi decomposition of $\mathfrak{L}$ into the solvable radical ideal $\mathfrak{R}$ and a semisimple subalgebra $\mathfrak{S}$. Then $\mathfrak{G}_b$ and $\mathfrak{R}_b$ commute, and $\mathfrak{L}_b = \mathfrak{S}_b + \mathfrak{R}_b$. In other words, $\mathfrak{L}_b$ is the direct sum of the commuting ideals $\mathfrak{S}_b$ and $\mathfrak{R}_b$. Furthermore $\mathfrak{S}_b$ is compact and $\mathfrak{R}_b$ is abelian.* ∎

To be able to use Theorem 3.2, we must make sure that our exponentiable Lie algebras generate uniformly bounded Lie group representations. There is no problem with this in the case of Lie algebras of *-derivations in C^*-algebras, the application of primary interest to us, since these generate representations by *-automorphisms, which are isometric. The following observations will suffice for the purpose of the present exposition.

An operator P in a Banach space $\mathscr{X}$ is called conservative if $\pm P$ are dissipative, i.e., if for each $a \in D(P)$ and $\omega \in \mathscr{X}^*$ such that $\omega(a) = \|\omega\| \ \|a\|$, we have $\mathrm{Re}\,\omega(P(a)) = 0$. As is well known, a strongly continuous one-parameter group of operators is isometric if, and only if, its infinitesimal generator is conservative.

LEMMA 3.3. *Let $\alpha: G \to \mathscr{B}(\mathscr{X})$ be a strongly continuous representation by isometries, and let $\mathfrak{P}$ be a perturbation class for $\mathfrak{L}_0 = d\alpha(\mathfrak{G})$ consisting of conservative operators. If $\mathfrak{L}$ is a finite dimensional Lie subalgebra of $\mathfrak{L}_0 + \mathfrak{P}$ then $\exp(\mathfrak{L})$ is a representation by isometries.*

Proof. It suffices to prove that $\exp(t(X+P))$ is a group of isometries for each $X \in \mathfrak{L}_0$ and $P \in \mathfrak{P}$, and this is evident since the sum of conservative operators is conservative. ∎

THEOREM 3.4. *Suppose that G is an abelian Lie group, $\alpha: G \to \mathscr{B}(\mathscr{X})$ is a strongly continuous representation by isometries of the Banach space $\mathscr{X}$, and $\mathfrak{P}$ is a perturbation class for $\mathfrak{L}_0 = d\alpha(\mathfrak{G})$ consisting of conservative operators.*
Let $\mathfrak{L}$ be a finite dimensional Lie subalgebra of $\mathfrak{L}_0 + \mathfrak{P}$. Let $\mathfrak{L} = \mathfrak{S} \dotplus \mathfrak{R}$ be a Levi decomposition, where $\mathfrak{R}$ denotes the solvable radical of $\mathfrak{L}$, and $\mathfrak{S}$ is semisimple. Then

(a) *$\mathfrak{S}$ is compact and consists of bounded operators.*

(b) *$\mathfrak{S}$ and $\mathfrak{R}$ are commuting ideals of $\mathfrak{L}$.*

(c) *$\mathfrak{R}_b$ is abelian and $\mathfrak{R}_b \supseteq [\mathfrak{R}, \mathfrak{R}]$. Thus the derived series of $\mathfrak{R}$ has only two steps: $\mathfrak{R} \supseteq [\mathfrak{R}, \mathfrak{R}] \supseteq (0)$.*

(d) *$\mathfrak{R}_b = \mathfrak{R}_b(0) \oplus [\mathfrak{R}, \mathfrak{R}_b]$, where $\mathfrak{R}_b(0)$ denotes the centralizer of $\mathfrak{R}$ in $\mathfrak{R}_b$. Moreover, $[\mathfrak{R}, \mathfrak{R}_b]$ has even dimension, and is the direct sum of two-dimensional minimal ideals.*

(e) *The adjoint representation of $\mathfrak{R}$ on $\mathfrak{R}_b^{\mathbf{C}}$ is diagonalizable, and the weights (eigenvalues) are purely imaginary.*

(f) *Suppose in addition that $(\mathfrak{L}_0)_b = (0)$. Then $\mathfrak{R}/\mathfrak{R}_b$ is canonically isomorphic to a Lie subalgebra of $\mathfrak{L}_0$.*

Proof. $\mathfrak{L}$ is exponentiable and $\exp(\mathfrak{L})$ is an isometric representaton by 3.1 and 3.3, so Theorem 3.2 applies. Since $\mathfrak{L}_0$ is abelian we have $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathfrak{L} \cap \mathfrak{P} \subseteq \mathfrak{L}_b$. Because $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{S}$, it follows that $\mathfrak{S} = \mathfrak{S}_b$, and therefore $\mathfrak{S}$ is compact by Theorem 3.2. Likewise $[\mathfrak{R}, \mathfrak{R}] \subseteq \mathfrak{R}_b$, and $\mathfrak{R}_b$ is abelian by the same theorem. This takes care of (a) and (c).

To prove (b), we have to show that $[\mathfrak{S}, \mathfrak{R}] = 0$. But $\mathfrak{S}$ and $\mathfrak{R}_b$ commute, again by Theorem 3.2, and $[\mathfrak{S}, \mathfrak{R}] \subseteq \mathfrak{R}_b$. So each $\xi \in \mathfrak{S}$ satisfies $(\operatorname{ad} \xi)^2|_{\mathfrak{R}} = 0$. On the other hand, the adjoint representation of $\mathfrak{S}$ on $\mathfrak{R}^{\mathbf{C}}$ exponentiates to a representation of a compact group. $\mathfrak{R}^{\mathbf{C}}$ has an inner product with respect to which this representation is unitary, and the operators $\operatorname{ad} \xi|_{\mathfrak{R}^{\mathbf{C}}}$ are skew adjoint. Hence $\operatorname{ad} \xi|_{\mathfrak{R}^{\mathbf{C}}}$ is diagonalizable with

purely imaginary eigenvalues. Then since $(\operatorname{ad} \xi)^2|_{\mathfrak{R}^{\mathbf{C}}} = 0$, it follows that $\operatorname{ad} \xi|_{\mathfrak{R}^{\mathbf{C}}} = 0$. This proves (b).

We prove (d) and (e) together. Consider the adjoint representation of $\mathfrak{R}$ on $\mathfrak{R}_b^{\mathbf{C}}$. By Theorem 3.2, the operators $\operatorname{ad} \xi|_{\mathfrak{R}_b^{\mathbf{C}}}$ $(\xi \in \mathfrak{R})$ are each diagonalizable with purely imaginary eigenvalues. Note that $\operatorname{ad} \mathfrak{R}|_{\mathfrak{R}_b^{\mathbf{C}}}$ is abelian, since $[\operatorname{ad} \xi_1, \operatorname{ad} \xi_2]|_{\mathfrak{R}_b^{\mathbf{C}}} = \operatorname{ad}([\xi_1, \xi_2])|_{\mathfrak{R}_b^{\mathbf{C}}}$, but $[\xi_1, \xi_2] \in \mathfrak{R}_b$, and $\mathfrak{R}_b$ is abelian. Hence the operators $\operatorname{ad} \xi|_{\mathfrak{R}_b^{\mathbf{C}}}$ are simultaneously diagonalizable. It follows that there are distinct nonzeo real linear functionals $\varphi_1, ..., \varphi_k$ on $\mathfrak{R}$ such that

$$\mathfrak{R}_b^{\mathbf{C}} = W_b(0) \oplus \sum_j{}^{\oplus} W_b(\varphi_j),$$

where

$$W_b(\varphi) = \{\eta \in \mathfrak{R}_b^{\mathbf{C}} : \forall \delta \in \mathfrak{R},\ [\delta, \eta] = i\varphi(\delta)\eta\}$$

and

$$W_b(0) = \{\eta \in \mathfrak{R}_b^{\mathbf{C}} : \forall \delta \in \mathfrak{R},\ [\delta, \eta] = 0\}.$$

(The functionals φ_j are the nonzero weights of the adjoint representation of $\mathfrak{R}$ on $\mathfrak{R}_b^{\mathbf{C}}$. Note that if φ is a weight, then $-\varphi$ is also, and

$$W_b(-\varphi) = \{\eta^* : \eta \in W_b(\varphi)\},$$

where * denotes the involution $\xi_1 + i\xi_2 \mapsto \xi_1 - i\xi_2$ on $\mathfrak{R}^{\mathbf{C}}$.) Set

$$\mathfrak{R}_b(\varphi) = \left\{\operatorname{Re} \eta = \frac{\eta + \eta^*}{2} : \eta \in W_b(\varphi)\right\}.$$

Note that

$$\mathfrak{R}_b(\varphi) = \mathfrak{R}_b(-\varphi),$$

and also

$$\mathfrak{R}_b(\varphi) = \left\{\operatorname{Im} \eta = \frac{\eta - \eta^*}{2i} : \eta \in W_b(\varphi)\right\}.$$

Furthermore, $[\delta, \eta] = i\varphi(\delta)\eta$ implies

$$[\delta, \operatorname{Re} \eta] = -\varphi(\delta) \operatorname{Im} \eta,$$
$$[\delta, \operatorname{Im} \eta] = \varphi(\delta) \operatorname{Re} \eta,$$

and

$$[\delta, [\delta, \operatorname{Re} \eta]] = -\varphi(\delta)^2 \operatorname{Re} \eta.$$

It follows that:

(1) For $\varphi \neq 0$, $\dim_{\mathbf{R}} \mathfrak{R}_b(\varphi) = 2 \dim_{\mathbf{C}} W_b(\varphi)$.

(2) If φ_1, φ_2 are weights with $\varphi_1 \neq \pm\varphi_2$, then $\mathfrak{R}_b(\varphi_1) \cap \mathfrak{R}_b(\varphi_2) = (0)$.

(3) $\sum_{\varphi \neq 0}^{\oplus} \mathfrak{R}_b(\varphi) = [\mathfrak{R}, \mathfrak{R}_b]$; of course, for each pair $(\varphi, -\varphi)$, only one copy of $\mathfrak{R}_b(\varphi)$ is taken in the direct sum.

This proves (d) and (e).

Finally, if $(\mathfrak{L}_0)_b = (0)$, then each $\delta \in \mathfrak{R}$ has a unique decomposition $\delta = \delta_0 + P$ with $\delta_0 \in \mathfrak{L}_0$ and $P \in \mathfrak{P}$, and the map $\delta \mapsto \delta_0$ is a Lie algebra homomorphism of $\mathfrak{R}$ with kernel $\mathfrak{R}_b = \mathfrak{R} \cap \mathfrak{P}$. This proves (f). ∎

The remainder of this section concerns the structure of solvable subalgebras of $\mathfrak{L}_0 + \mathfrak{P}$. This material is used in an essential way in the analysis of smooth Lie algebra actions on non-commutative tori in [BEGJ].

THEOREM 3.5. *Let* $(\mathcal{X}, G, \alpha)$ *and* $\mathfrak{P}$ *verify the hypotheses of Theorem* 3.4, *and let* $\mathfrak{R}$ *be a solvable finite dimensional Lie subalgebra of* $\mathfrak{L}_0 + \mathfrak{P}$. *Then either*

(a) $\mathfrak{R}$ *contains a Lie subalgebra isomorphic to the three dimensional Heisenberg Lie algebra, or*

(b) $\mathfrak{R}$ *contains an abelian Lie subalgebra* $\mathfrak{T}$ *such that* $\mathfrak{R} = \mathfrak{T} \dotplus \mathfrak{R}_b$, *where the symbol* $\dotplus$ *denotes semidirect sum of Lie algebras.*

Possibilities (a) *and* (b) *are mutually exclusive.*

Proof. Suppose that $\mathfrak{R}$ contains no Lie subalgebra isomorphic to the three dimensional Heisenberg Lie algebra. Let $\xi_1, ..., \xi_r$ be elements of $\mathfrak{R}$ which are linearly independent modulo $\mathfrak{R}_b$ and mutually commuting ($[\xi_i, \xi_j] = 0$ for all i, j). We will show that if $r < d = \dim_{\mathbf{R}}(\mathfrak{R}/\mathfrak{R}_b)$, then there exist elements $\xi'_1, ..., \xi'_r, \xi'_{r+1}$ that are linearly independent modulo $\mathfrak{R}_b$ and mutually commuting.

Set $\mathfrak{T}_0 = \operatorname{span}_{\mathbf{R}}\{\xi_1, ..., \xi_r\}$, and consider the adjoint representation of $\mathfrak{T}_0$ on $\mathfrak{R}^{\mathbf{C}}$. Since $\mathfrak{T}_0$ is abelian, there exists a family of distinct nonzero linear functionals (weights) ψ on $\mathfrak{T}_0$ such that

$$\mathfrak{R}_{\mathbf{C}} = M(0) \oplus \sum_{\psi}^{\oplus} M(\psi),$$

where

$$M(0) = \{\eta \in \mathfrak{R}^{\mathbf{C}} : \forall \xi \in \mathfrak{T}_0 \ \exists k \ (\operatorname{ad} \xi)^k(\eta) = 0\}$$

and

$$M(\psi) = \{\eta \in \mathfrak{R}^{\mathbf{C}} : \forall \xi \in \mathfrak{T}_0 \ \exists k \ ((\operatorname{ad} \xi) - \psi(\xi))^k(\eta) = 0\}.$$

If ψ is one of the nonzero weights, then there is a $\xi \in \mathfrak{T}_0$ such that $\psi(\xi) \neq 0$. If $\eta \in M(\psi)$, there is a k such that

$$\begin{aligned} 0 &= ((\operatorname{ad} \xi) - \psi(\xi))^k(\eta) \\ &= \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix} (-\psi(\xi))^l (\operatorname{ad} \xi)^{k-l}(\eta) \\ &= (-\psi(\xi))^k \eta + \{k(-\psi(\xi))^{k-1}(\operatorname{ad} \xi)(\eta) + \cdots\}. \end{aligned}$$

This shows that $\eta \in [\mathfrak{R}, \mathfrak{R}^{\mathbf{C}}] \subseteq \mathfrak{R}_b^{\mathbf{C}}$, since the terms inside the braces on the last expression are in $[\mathfrak{R}, \mathfrak{R}^{\mathbf{C}}]$. Thus $M(\psi) \subseteq \mathfrak{R}_b^{\mathbf{C}}$. Hence $\dim_{\mathbf{C}}(M(0)/(M(0) \cap \mathfrak{R}_b^{\mathbf{C}})) = d$, and it follows, since $r < d$, that $M(0)$ contains an element whose real part ξ' is not in $\mathfrak{T}_0 + \mathfrak{R}_b$. Since $M(0) = M(0)^*$, ξ' is in $M(0)$ as well. We note that for all $\xi \in \mathfrak{T}_0$, $(\operatorname{ad} \xi)^2(\xi') = 0$. In fact for each ξ there is a k such that $(\operatorname{ad} \xi)^k(\xi') = 0$. But $(\operatorname{ad} \xi)(\xi') \in \mathfrak{R}_b$ and $(\operatorname{ad} \xi)^{k-1}(\operatorname{ad} \xi)(\xi') = 0$; hence by Theorem 3.2, $(\operatorname{ad} \xi)((\operatorname{ad} \xi)(\xi')) = 0$.

Recall (from the proof of Theorem 3.4) the weight space decomposition of $\mathfrak{R}_b^{\mathbf{C}}$, with respect to action of $\mathfrak{R}$ on $\mathfrak{R}_b^{\mathbf{C}}$:

$$\mathfrak{R}_b^{\mathbf{C}} = W_b(0) \oplus \sum_{\varphi}^{\oplus} W_b(\varphi).$$

Decompose each element $[\xi_i, \xi']$ accordingly:

$$[\xi_i, \xi'] = \eta_0^i + \sum_{\varphi} \eta_{\varphi}^i,$$

where $\eta_0^i \in W_b(0)$ and $\eta_{\varphi}^i \in W_b(\varphi)$. Then

$$0 = [\xi_i, [\xi_i, \xi']] = \sum_{\varphi} \sqrt{-1}\, \varphi(\xi_i)\, \eta_{\varphi}^i.$$

Since the $W_b(\varphi)$ are linearly independent subspaces,

$$\varphi(\xi_i)\eta_{\varphi}^i = 0 \qquad \text{for all } i \text{ and } \varphi. \tag{3.5.1}$$

Next, since $[\xi_i, \xi_j]=0$ for all i, j,

$$0=[[\xi_i,\xi_j],\xi']=[[\xi_i,\xi'],\xi_j]+[\xi_i,[\xi_j,\xi']]$$
$$=\sum_{\varphi}\sqrt{-1}\,(\varphi(\xi_i)\eta^j_\varphi-\varphi(\xi_j)\eta^i_\varphi).$$

Therefore

$$\varphi(\xi_i)\eta^j_\varphi-\varphi(\xi_j)\eta^i_\varphi=0 \qquad \text{for all } i, j, \varphi. \tag{3.5.2}$$

Set $A=\{\varphi\colon \varphi(\xi')\neq 0\}$, $A'=\{\varphi\colon \varphi(\xi')=0\}$, and

$$\xi'_i=\xi_i+\sum_{\varphi\in A}\frac{1}{\sqrt{-1}\,\varphi(\xi')}\eta^i_\varphi.$$

One computes that

$$[\xi'_i,\xi'_j]=\sum_{\varphi\in A}\frac{1}{\sqrt{-1}\,\varphi(\xi')}(\varphi(\xi_i)\eta^j_\varphi-\varphi(\xi_j)\eta^i_\varphi),$$

which is zero by (3.5.2).

Furthermore, a short computation gives

$$[\xi'_i,\xi']=\eta^i_0+\sum_{\varphi\in A'}\eta^i_\varphi.$$

Set $[\xi'_i,\xi']=\zeta^i$. We note that $[\xi',\zeta^i]=[\xi'_i,\zeta^i]=0$. In fact,

$$[\xi',\zeta^i]=\sum_{\varphi\in A'}\sqrt{-1}\,\varphi(\xi')\eta^i_\varphi,$$

which is zero by definition of A', and

$$[\xi'_i,\zeta^i]=[\xi_i,\zeta^i]=\sum_{\varphi\in A'}\varphi(\xi_i)\eta^i_\varphi,$$

which is zero by (3.5.1). If $\zeta^i\neq 0$ for some i, then $\{\xi'_i,\xi',\zeta^i\}$ spans a Heisenberg subalgebra of $\mathfrak{N}$, which is against our assumption. Thus we have $[\xi'_i,\xi']=0$ for all i. Set $\xi'_{r+1}=\xi'$; then $\{\xi'_1,\ldots,\xi'_r,\xi'_{r+1}\}$ satisfies our requirements.

It follows by induction that, if $\mathfrak{N}$ contains no Lie subalgebra isomorphic to $\mathfrak{h}_3$, then $\mathfrak{N}_b$ has an abelian complement $\mathfrak{T}$.

Finally we show that the two possibilities (a) and (b) are mutually exclusive. Suppose that $\mathfrak{N}$ has the form $\mathfrak{N}=\mathfrak{T}\dotplus\mathfrak{N}_b$, and that $\{\xi_1,\xi_2,\zeta\}$ verify the ralations

$$[\xi_1,\xi_2]=\zeta, \qquad [\xi_i,\zeta]=0 \qquad (i=1,2).$$

We show that $\zeta=0$. Decompose ξ_i (in $\mathfrak{N}^{\mathbb{C}}$) as

$$\xi_i=\delta_i+\eta^i_0+\sum_{\varphi}\eta^i_\varphi \qquad (i=1,2),$$

where $\eta^i_0\in W_b(0)$, $\eta^i_\varphi\in W_0(\varphi)$, and $\delta_i\in\mathfrak{T}$. Then

$$\zeta=\sum_{\varphi}\sqrt{-1}\,(\varphi(\delta_1)\eta^2_\varphi-\varphi(\delta_2)\eta^1_\varphi),$$

and

$$0=[\xi_i,\zeta]=\sum_{\varphi}(-1)\varphi(\delta_i)(\varphi(\delta_1)\eta^2_\varphi-\varphi(\delta_2)\eta^1_\varphi) \qquad (i=1,2).$$

It follows that for all φ, $\varphi(\delta_1)\eta^2_\varphi-\varphi(\delta_2)\eta^1_\varphi=0$. Hence $\zeta=0$. ∎

LEMMA 3.6. *Suppose that $(\mathscr{X}, G, \alpha)$ and $\mathfrak{P}$ verify the hypotheses of Theorem 3.4, and that $\mathfrak{N}$ is a solvable finite dimensional Lie subalgebra of $\mathfrak{L}_0+\mathfrak{P}$. If $\{\xi_1,\xi_2,\zeta\}$ is a basis of a Heisenberg Lie subalgebra of $\mathfrak{N}$, such that*

$$[\xi_1,\xi_2]=\zeta \quad \textit{and} \quad [\xi_i,\zeta]=0 \qquad (i=1,2),$$

then ξ_1 and ξ_2 are linearly independent modulo $\mathfrak{N}_b$ and ζ lies in the center of $\mathfrak{N}$.

Proof. First we note that ξ_1 and ξ_2 are unbounded. If ξ_2 is bounded, it has a decomposition in $\mathfrak{N}^{\mathbb{C}}_b$,

$$\xi_2=\eta_0+\sum_{\varphi}\eta_\varphi,$$

where $\eta_0\in W_b(0)$ and $\eta_\varphi\in W_b(\varphi)$. Then

$$\zeta=[\xi_1,\xi_2]=\sum_{\varphi}\sqrt{-1}\,\varphi(\xi_1)\eta_\varphi,$$

and

$$0=[\xi_1,\zeta]=\sum_{\varphi}(-1)\,\varphi(\xi_1)^2\eta_\varphi.$$

It follows that $\zeta=0$, a contradiction.

If ξ_1 and ξ_2 are linearly dependent modulo $\mathfrak{N}_b$, then there is a $t \in \mathbf{R}$ such that $\xi_2 - t\xi_1$ is bounded. But then

$$\{\xi_1, \xi_2 - t\xi_1, \zeta\}.$$

is a Heisenberg system whose second element is bounded.

Next we show that ζ is in the center of $\mathfrak{N}$. Fix $\xi \in \mathfrak{N}$ and consider the decompositions

$$\zeta = \zeta_0 + \sum_{\varphi} \zeta_\varphi,$$

$$[\xi, \xi_i] = w_0^i + \sum_{\varphi} w_\varphi^i \qquad (i = 1, 2),$$

where $\zeta_0, w_0^i \in W_b(0)$, and $\zeta_\varphi, w_\varphi^i \in W_b(\varphi)$. Since $[\xi_i, \zeta] = 0$, we have

$$\varphi(\xi_i)\zeta_\varphi = 0 \qquad \text{for all } \varphi \text{ and } i = 1, 2. \tag{3.6.1}$$

On the other hand,

$$\begin{aligned}\sum_{\varphi} \sqrt{-1}\, \varphi(\xi)\zeta_\varphi &= [\xi, \zeta] \\ &= [\xi, [\xi_1, \xi_2]] = [[\xi, \xi_1], \xi_2] + [\xi_1, [\xi, \xi_2]] \\ &= \sqrt{-1} \sum_{\varphi} (\varphi(\xi_1) w_\varphi^2 - \varphi(\xi_2) w_\varphi^1).\end{aligned}$$

Thus for all φ,

$$\varphi(\xi)\zeta_\varphi = \varphi(\xi_1) w_\varphi^2 - \varphi(\xi_2) w_\varphi^1. \tag{3.6.2}$$

If $\zeta_\varphi \neq 0$ for some φ, then $\varphi(\xi_1) = \varphi(\xi_2) = 0$ by (3.6.1), and therefore $\varphi(\xi)\zeta_\varphi = 0$ by (3.6.2). Hence, $[\xi, \zeta] = 0$. (It follows that $\zeta_\varphi = 0$ for all nonzero φ.) ∎

4. C^*-Dynamical Systems with the Decomposition Property

In this section we describe a class of examples to which the results of Section 3 apply.

A C^*-dynamical system $(\mathfrak{A}, G, \alpha)$, with G a Lie group, is said to have the decomposition property for smooth derivations if each $\delta \in \mathrm{Der}(\mathfrak{A}^\infty(\alpha))$ has a unique decomposition $\delta = \delta_0 + \tilde{\delta}$, where $\delta_0 \in \mathfrak{L}_0 = d\alpha(\mathfrak{G})$, and $\tilde{\delta}$ is a bounded derivation. A systematic study of this and related decompositions is made in [Bra]; here we only mention some particular cases:

Example 4.1 (Noncommutative Tori). If G is the d-torus $\mathbf{T}^d$, and α is an ergodic action of G on a simple C^*-algebra, then $(\mathfrak{A}, G, \alpha)$ has the following structure (see, e.g., [OPT]): For each $\mathbf{n}$ in $\mathbf{Z}^d = (\mathbf{T}^d)\hat{}$, the spectral subspace $\mathfrak{A}^\alpha(\mathbf{n})$ is one-dimensional, and is spanned by a unitary element $U(\mathbf{n})$. These unitaries satisfy the relations

$$U(\mathbf{n}) U(\mathbf{m}) = \chi(\mathbf{n}, \mathbf{m}) U(\mathbf{m}) U(\mathbf{n}), \tag{4.1.1}$$

where $\chi: \mathbf{Z}^d \times \mathbf{Z}^d \to \mathbf{T}$ is a nondegenerate antisymmetric bicharacter. Conversely, given a nondegenerate antisymmetric bicharacter χ on $\mathbf{Z}^d$, there is a unique simple C^*-algebra $\mathfrak{A}_\chi$ with an ergodic action $\alpha: \mathbf{T}^d \to \mathrm{Aut}(\mathfrak{A}_\chi)$ such that the unitary eigenelements $U(\mathbf{n})$ $(\mathbf{n} \in \mathbf{Z}^d)$ satisfy (4.1.1). If in addition χ has generic Diophatine properties, i.e., if $|\chi(\mathbf{n}, \mathbf{m}) - 1|^{-1}$ grows polynomially in $\|\mathbf{m}\|$ for fixed $\mathbf{n} \neq 0$ in $\mathbf{Z}^d$, then $(\mathfrak{A}_\chi, \mathbf{T}^d, \alpha)$ has a strong form of the decompostion property for smooth derivations: Every $\delta \in \mathrm{Der}(\mathfrak{A}_\chi^\infty(\alpha))$ has a unique decomposition $\delta = \delta_0 + \mathrm{ad}(h)$, where $\delta_0 \in d\alpha(\mathbf{R}^d)$ and h is a skew-adjoint element of $\mathfrak{A}_\chi^\infty(\alpha)$ [BEJ; Co, Proposition 49].

Example 4.2. If G is a compact abelian group, and α is an action of G on a C^*-algebra $\mathfrak{A}$ such that $\Gamma(\alpha) = \hat{G}$ and $\mathfrak{A}$ admits an α-invariant pure state ω such that the associated representation π_ω is faithful, then $(\mathfrak{A}, G, \alpha)$ has the decomposition property. This is an immediate consequence of the main theorems in [Kis, KR]; see [Bra, Theorems 2.9.10 and 2.6.6]. The decomposition is even valid assuming only that δ maps the algebra $\mathfrak{A}_F^\alpha$ of G-finite elements into $\mathfrak{A}$. If $\mathfrak{A}$ is separable, the dynamical assumption above can be stated in several other equivalent ways, e.g.,

$$\text{If } x, y \in \mathfrak{A} \setminus \{0\}, \text{ then } x\mathfrak{A}^\alpha y \neq \{0\}. \tag{4.2.1}$$

There exists an irreducible representation π of $\mathfrak{A}$ such that $\pi(\mathfrak{A}^\alpha)'' = \pi(\mathfrak{A})$. (4.2.2)

There exists an α-covariant representation π of $\mathfrak{A}$ such that $\pi(\mathfrak{A}^\alpha)' \cap \pi(\mathfrak{A})'' = \mathbf{C}\mathbf{1}$, or (4.2.3)

$\mathfrak{A}$ and $\mathfrak{A}^\alpha$ are prime, and α_g is properly outer for each $g \neq 0$; see [BEEK]. (4.2.4)

Example 4.3. Let G be a second-countable compact group and α a faithful action of G on a simple separable unital C^*-algebra $\mathfrak{A}$. Assume that there exists a sequence $\tau_n \in \mathrm{Aut}(\mathfrak{A})$ such that $[\tau_n, \alpha] = 0$ and

$$\lim_{n \to \infty} \|[\tau_n(x), y]\| = 0 \tag{4.3.1}$$

for all $x, y \in \mathfrak{A}$. Then $(\mathfrak{A}, G, \alpha)$ has the decomposition property, and even all derivations from $\mathfrak{A}_F^\alpha$ into $\mathfrak{A}$ decompose; see [BK, Theorem 1.1] or [Bra, Theorem 2.9.31].

EXAMPLE 4.4. Finally, if G is a compact group, α is a faithful action of G on a C^*-algebra $\mathfrak{A}$, and there exists a faithful G-covariant representation π with $\pi(\mathfrak{A}^\alpha)' \cap \pi(\mathfrak{A})'' = \mathbf{C}1$, then any derivation $\delta\colon \mathfrak{A}_F^\alpha \to \mathfrak{A}$ has a decomposition $\delta = \delta_0 + \tilde{\delta}$ where δ_0 is the generator of a one-parameter subgroup of α, and $\tilde{\delta}$ is bounded [BG, Theorem 2.5; Bra, Theorem 2.9.22; L, Cor. 4.3].

So, for example, if $\mathfrak{A}$ is the UHF C^*-algebra of type n^∞ and G is a closed subgroup of $U(n)$ acting on $\mathfrak{A}$ via the canonical product action, then $(\mathfrak{A}, G)$ has the decomposition property for smooth derivations.

Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system with the decomposition property; then $\mathrm{Der}(\mathfrak{A}^\infty(\alpha)) = \mathfrak{L}_0 + \mathfrak{P}$, where $\mathfrak{P}$ is the space of smooth bounded derivations of $\mathfrak{A}$, and the sum is direct as a sum of linear spaces. To be able to use the results of Section 2, we need to know that $\mathfrak{P}$ is a perturbation class for $\mathfrak{L}_0$. We prove this in the case that G is abelian.

PROPOSITION 4.5. *If G is an abelian Lie group, and the C^*-dynamical system $(\mathfrak{A}, G, \alpha)$ has the decomposition property, then the space $\mathfrak{P}$ of bounded smooth *-derivations of $\mathfrak{A}$ satisfies*

$$[\mathfrak{L}_0, \mathfrak{P}] \subseteq \mathfrak{P}.$$

Proof. Let $\delta_0 \in \mathfrak{L}_0$ and $\tilde{\delta} \in \mathfrak{P}$; we have to show that $[\delta_0, \tilde{\delta}] \in \mathfrak{P}$. By the decomposition property, $[\delta_0, \tilde{\delta}]$ has a unique decomposition

$$[\delta_0, \tilde{\delta}] = \xi_0 + \tilde{\xi}, \tag{4.5.1}$$

where $\xi_0 \in \mathfrak{L}_0$ and $\tilde{\xi} \in \mathfrak{P}$. We have to show that $\xi_0 = 0$.

First we consider the case that G is compact. For $a \in \mathfrak{A}^\infty(\alpha)$ and $\mu \in \mathfrak{P}$ the function $g \mapsto \alpha_g \mu \alpha_g^{-1} a$ is continuous from G to $\mathfrak{A}^\infty(\alpha)$, endowed with the Fréchet topology τ_∞; this follows from Banach–Steinhaus, which yields the τ_∞–τ_∞ equicontinuity of $\{\alpha_g : g \in G\}$, together with the closed graph theorem, which gives the τ_∞–τ_∞ continuity of μ. Therefore the integral $\mu_{\mathrm{inv}} a = \int_G dg\, \alpha_g \mu \alpha_g^{-1} a$ converges with respect to τ_∞ and defines an element μ_{inv} in $\mathfrak{P}$. By the τ_∞–τ_∞ continuity of δ_0, we have

$$[\delta_0, \mu_{\mathrm{inv}}] a = \int_G dg [\delta_0, \alpha_g \mu \alpha_g^{-1}] a. \tag{4.5.2}$$

Furthermore, by the G-invariance of μ_{inv}, we have

$$[\delta_0, \mu_{\mathrm{inv}}] a = \frac{d}{dt}\bigg|_{t=0} e^{t\delta_0} \mu_{\mathrm{inv}} e^{-t\delta_0} a = 0. \tag{4.5.3}$$

As G is abelian, (4.5.1) yields

$$[\delta_0, \alpha_g \mu \alpha_g^{-1}] = \alpha_g [\delta_0, \mu] \alpha_g^{-1} = \xi_0 + \alpha_g \tilde{\xi} \alpha_g^{-1}.$$

Combining this with (4.5.2) and (4.5.3), we obtain

$$0 = \xi_0 + \int_G \alpha_g \tilde{\xi} \alpha_g^{-1} = \xi_0 + \tilde{\xi}_{\mathrm{inv}}.$$

It follows that $\xi_0 \in \mathfrak{L}_0 \cap \mathfrak{P} = (0)$.

If G is noncompact (but still abelian) we can modify this argument as follows. Realize $\mathfrak{A}$ in a faithful G-covariant representation; then α extends to a point $-\sigma(\mathcal{M}, \mathcal{M}_*)$ continuous representation $\bar{\alpha}$ in $\mathcal{M} = \mathfrak{A}''$. If $\delta_0 = d\alpha(X)$, then $d\bar{\alpha}(X) = \bar{\delta}_0$ is a $\sigma(\mathcal{M}, \mathcal{M}_*)$-closed extension of δ_0. For $a \in \mathfrak{A}^\infty(\alpha)$, the functions

$$g \mapsto \alpha_g \tilde{\delta} \alpha_g^{-1} a,$$
$$g \mapsto \alpha_g \tilde{\delta} \alpha_g^{-1} \delta_0 a,$$
$$g \mapsto [\delta_0, \alpha_g \tilde{\delta} \alpha_g^{-1}] a = \xi_0 a + \alpha_g \tilde{\xi} \alpha_g^{-1} a$$

are continuous and bounded from G to $(\mathfrak{A}, \|\cdot\|)$. Hence so is the function

$$g \mapsto \delta_0 \alpha_g \tilde{\delta} \alpha_g^{-1} a.$$

We now apply an invariant mean over G to each of these functions; the value of the mean is in $\mathcal{M}$. For $\mu \in \mathfrak{P}$ we denote the mean of $g \mapsto \alpha_g \mu \alpha_g^{-1} a$ by $\mu_{\mathrm{inv}} a$. By the $\sigma(\mathcal{M}, \mathcal{M}_*)$-closedness of $\bar{\delta}_0$ we have

$$\operatorname*{mean}_g(\delta_0 \alpha_g \tilde{\delta} \alpha_g^{-1} a) = \bar{\delta}_0(\operatorname*{mean}_g \alpha_g \tilde{\delta} \alpha_g^{-1} a) = \bar{\delta}_0 \tilde{\delta}_{\mathrm{inv}} a.$$

Hence, using (4.5.1), we obtain

$$\xi_0 a + \tilde{\xi}_{\mathrm{inv}} a = \operatorname*{mean}_g([\delta_0, \alpha_g \tilde{\delta} \alpha_g^{-1}] a) = [\bar{\delta}_0, \tilde{\delta}_{\mathrm{inv}}] a.$$

Finally (using the fact that $\tilde{\delta}_{\text{inv}} a \in D(\delta_0)$) we have

$$[\delta_0, \tilde{\delta}_{\text{inv}}] a = \frac{d}{dt}\bigg|_{t=0} e^{t\delta_0} \tilde{\delta}_{\text{inv}} e^{-t\delta_0} a = 0.$$

It follows that ξ_0 is bounded, hence in $\mathfrak{L}_0 \cap \mathfrak{P} = (0)$. ∎

COROLLARY. *Let G be an abelian Lie group and let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system with the decomposition property for smooth derivations. Then any finite dimensional Lie subalgebra of* $\mathrm{Der}(\mathfrak{A}^\infty(\alpha))$ *is exponentiable and satisfies the conclusions of Theorem* 3.4.

ACKNOWLEDGMENTS

F. G. and P. J. were partially supported by the NSF, USA. Part of this work was done during a visit of O.B., F.G., and P.J. at the University of Toronto, with support from the NSERC of Canada.

REFERENCES

[Bra] O. BRATTELI, Derivations, dissipations and group actions on C^*-algebras, *in* "Lecture Notes in Mathematics," Vol. 1229, Springer-Verlag, Berlin/Heidelberg/New York, 1986.

[BEEK] O. BRATTELI, G. A. ELLIOTT, D. E. EVANS AND A. KISHIMOTO, Quasi-product actions of a compact abelian group on a C^*-algebra, preprint.

[BEGJ] O. BRATTELI, G. A. ELLIOTT, F. M. GOODMAN, AND P.E.T. JORGENSEN, Smooth Lie group actions on noncommutative tori, *Nonlinearity* **2** (1989), 271–286.

[BEJ] O. BRATTELI, G. A. ELLIOTT, AND P. E. T. JORGENSEN, Decomposition of unbounded derivations into invariant and approximately inner parts, *J. Reine Angew. Math.* **346** (1984), 166–193.

[BG] O. BRATTELI AND F. GOODMAN, Derivations tangential to compact group actions: Spectral conditions in the weak closure, *Canad. J. Math.* **37** (1985), 160–192.

[BK] O. BRATTELI AND A. KISHIMOTO, Derivations and free group actions on C^*-algebras, *J. Operator Theory* **15** (1986), 377–410.

[Bour] N. BOURBAKI, "General Topology," Part 2, Hermann/Addison-Wesley, Paris/Reading, MA, 1966.

[Co] A. CONNES, Non-commutative differential geometry, *Inst. Hautes Études Sci. Publ. Math.* **62** (1985), 257–360.

[JM] P. E. T. JORGENSEN AND R. T. MOORE, "Operator Communication Relations," Reidel, Dordrecht/Boston, 1984.

[Kis] A. KISHIMOTO, Derivations with a domain condition, *Yokohama Math. J.* **32** (1982), 215–333.

[KR] A. KISHIMOTO AND D. W. ROBINSON, Derivations, dynamical systems and spectral restrictions, *Math. Scand.* **56** (1985), 83–95.

[KS] R. V. KADISON AND I. M. SINGER, Some remarks on representations of connected groups, *Proc. Natl. Acad. Sci. U.S.A.* **38** (1952), 419–423.

[L] R. LONGO, Restricting a compact action to an injective subfactor, *Ergod. Th. and Dynam. Sys.* **9** (1989), 127–135.

[MZ] D. MONTGOMERY AND L. ZIPPIN, "Topological Transformation Groups," Interscience, New York, 1955.

[NS] J. VON NEUMANN AND I. E. SEGAL, A theorem on unitary representations of semisimple Lie groups, *Ann. of Math.* **52** (1950), 509–517.

[OPT] D. OLESEN, G. K. PEDERSEN, AND M. TAKESAKI, Ergodic actions of compact abelian groups, *J. Operator Theory* **3** (1980), 237–269.

[Po] N. S. POULSEN, On C^∞-vectors and intertwining bilinear forms for representations of Lie groups, *J. Funct. Anal.* **9** (1972), 87–120.

[Sa] S. SAKAI, Derivations and dynamical systems in C^*-algebras, lecture notes for a college on representation theory of lie groups, November, 1985, International Centre for Theoretical Physics, Trieste, Italy; excerpted from a forthcoming book: "Operator Algebras in Dynamical Systems,"

[Si] I. M. SINGER, Uniformly continuous representations of Lie groups, *Ann. of Math.* **56** (1952), 242–247.