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OPERATOR K-THEORY FOR GROUPS WHICH ACT PROPERLY AND ISOMETRICALLY ON HILBERT SPACE

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Abstract. Let G be a countable discrete group which acts isometrically and metrically properly on an infinite-dimensional Euclidean space. We calculate the K-theory groups of the C^*-algebras $C^*_{\max}(G)$ and $C^*_{\mathrm{red}}(G)$. Our result is in accordance with the Baum-Connes conjecture.

1. Introduction

Let G be a countable discrete group and denote by $C^*_{\max}(G)$ and $C^*_{\mathrm{red}}(G)$ its full and reduced group C^*-algebras. This note is concerned with the computation of the K-theory of these C^*-algebras, for groups G which admit the following sort of action on a Euclidean space:

1.1. Definition. Let V be a real inner product vector space which is possibly infinite-dimensional (hereafter we shall call V a *Euclidean space*). An affine, isometric action of a discrete group G on V is *metrically proper* if $\lim_{g\to\infty} \|g\cdot v\| = \infty$, for every $v \in V$.

Gromov [10] calls groups which admit such an action *a-T-menable.* The terminology is justified by the well-known theorem that every affine, isometric action of a property T group on a Euclidean space has bounded orbits [7], and by the recent observation [3] that every countable amenable discrete group admits a metrically proper, affine, isometric action on a Euclidean space. Other examples of a-T-menable groups are proper groups of isometries of real or complex hyperbolic space, finitely generated Coxeter groups, and groups which act properly on locally finite trees. The reader is referred to the short article [3] and the monograph [12] for further information.

If G is any discrete group then denote by $\mathcal{E}G$ the universal proper G-space, as described in [2]. It is a paracompact and Hausdorff G-space which has a paracompact and Hausdorff quotient by G, along with the following additional properties:

(i) it is covered by open G-sets, each of which admits a continuous G-map to some coset space G/H, where H is finite; and
(ii) if X is any other G-space satisfying (i) then there is a unique-up-to-G-homotopy G-map from X into $\mathcal{E}G$.

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Clearly $\mathcal{E}G$ is unique up to G-homotopy. If G is torsion-free then $\mathcal{E}G$ is the universal principal space EG, whose quotient by G is the classifying space BG.

The *Baum-Connes assembly maps* for the C^*-algebras $C^*_{\max}(G)$ and $C^*_{\mathrm{red}}(G)$ are homomorphisms of abelian groups

$$\mu_{\max}\colon K^G_*(\mathcal{E}G) \to K_*(C^*_{\max}(G)),$$
$$\mu_{\mathrm{red}}\colon K^G_*(\mathcal{E}G) \to K_*(C^*_{\mathrm{red}}(G)).$$

On the left hand side is the equivariant K-homology of $\mathcal{E}G$, defined using KK-theory [2], [16]. If G is torsion-free then $K^G_*(\mathcal{E}G)$ is isomorphic to $K_*(BG)$, the K-homology (with compact supports) of the classifying space for G. See [2] and Section 3 below for a further description of both equivariant K-homology and the assembly map.

We are going to outline a proof of the following result:

1.2. Theorem. *If G is a discrete group which admits an affine, isometric and metrically proper action on a Euclidean space* (*possibly infinite-dimensional*) *then the Baum-Connes assembly maps $\mu_{\max}$ and μ_{red} are isomorphisms.*

The assertion that for *any* G the assembly map μ_{red} is an isomorphism is known as the *Baum-Connes conjecture* [2]. By an index theory argument, the injectivity of either $\mu_{\max}$ or μ_{red}, for a given G, implies the Novikov higher signature conjecture [8] for manifolds with fundamental group G. Thus our theorem has the following consequence:

1.3. Corollary. *If G is a discrete group which admits an affine, isometric and metrically proper action on a Euclidean space* (*possibly infinite-dimensional*) *then the Baum-Connes and Novikov conjectures hold for G. In particular, these conjectures hold for any countable amenable group.*

To prove the theorem we shall use a variant of KK-theory, the E-theory of Connes and Higson [6], which appears to be more appropriate in this case. As in other instances where the K-theory of group C^*-algebras may be computed, or partially computed, a key role is played by a Bott periodicity argument—in this case periodicity for infinite-dimensional Euclidean space [14]. Given periodicity, the proof that $\mu_{\max}$ is an isomorphism is a relatively simple consequence of the formal machinery of E-theory. Thanks to a peculiarity of E-theory, the proof for μ_{red} is rather more complicated, unless G has the additional functional-analytic property that $C^*_{\mathrm{red}}(G)$ is exact [22]. Conjecturally all discrete groups have this property, but in any case we shall end this note with an *ad hoc* proof that μ_{red} is an isomorphism for the groups under consideration here.

2. E-theory

We begin by reviewing some of the foundational results in E-theory proved in [6] and [11]. Let A and B be C^*-algebras. An *asymptotic morphism* from A to B is a family of functions

$$\{\phi_t\}_{t\in[1,\infty)}\colon A \to B$$

such that $t \mapsto \phi_t(a)$ is bounded and norm-continuous, for every $a \in A$, and

$$\lim_{t\to\infty} \left\{ \begin{array}{c} \phi_t(a_1a_2) - \phi_t(a_1)\phi_t(a_2) \\ \phi_t(a_1 + a_2) - \phi_t(a_1) - \phi_t(a_2) \\ \phi_t(\alpha a_1) - \alpha\phi_t(a_1) \\ \phi_t(a_1^*) - \phi_t(a_1)^* \end{array} \right\} = 0,$$

for every $a_1, a_2 \in A$ and $\alpha \in \mathbb{C}$. This is the definition used in [6]; it agrees with the definition in [11] if we identify those asymptotic morphisms ϕ', ϕ'' which are *asymptotically equivalent,* in the sense that $\phi'_t(a) - \phi''_t(a) \to 0$ as $t \to \infty$ (we shall write $\phi'_t(a) \sim \phi''_t(a)$). If A and B are $\mathbb{Z}/2$-graded C^*-algebras then we shall consider only asymptotic morphisms which map homogeneous elements to homogeneous elements and preserve the grading degree. If A and B are equipped with actions of a countable discrete group G, then by an *equivariant* asymptotic morphism from A to B we shall mean an asymptotic morphism $\{\phi_t\}_{t\in[1,\infty)}$ such that $\phi_t(g(a)) \sim g(\phi_t(a))$, for every $a \in A$ and every $g \in G$.

Two asymptotic morphisms from A to B are *homotopic* if there is an asymptotic morphism from A to $B[0, 1]$ (the C^*-algebra of continuous functions mapping the unit interval into B) from which the two may be recovered by evaluation at 0 and 1. Homotopy is an equivalence relation and we denote by $[A, B]$ the set of homotopy classes of asymptotic morphisms from A to B. The same definition may be made in the graded and equivariant contexts.

Let us denote an asymptotic morphism from A to B by a broken arrow:

$$\phi\colon A \dashrightarrow B.$$

It is clearly possible to compose an asymptotic morphism with a $*$-homomorphism, on either side, so as to obtain asymptotic morphisms

$$A \longrightarrow B \dashrightarrow C$$

and

$$A \dashrightarrow B \longrightarrow C.$$

Now every $*$-homomorphism from A to B determines a *constant* asymptotic morphism from A to B, and the starting point for the theory of asymptotic morphisms is the following result [6]:

2.1. Theorem. *On the class of separable, $\mathbb{Z}/2$-graded G-C^*-algebras there is an associative composition law*

$$[A, B] \times [B, C] \to [A, C]$$

which is compatible with the above compositions of asymptotic morphisms and $*$-homomorphisms.

(The proof in [6] is written for C^*-algebras with no grading or G-action, but the argument extends to the present context with no essential change.) We shall use the following features of the homotopy category of asymptotic morphisms. Denote by $A_1 \hat{\otimes} A_2$ the *maximal* graded tensor product of A_1 and A_2 [4, Section 14]. There is then a tensor product functor on the homotopy category of asymptotic morphisms which associates to the asymptotic morphisms $\phi_1\colon A_1 \dashrightarrow B_1$ and $\phi_2\colon A_2 \dashrightarrow B_2$ a tensor product

$$\phi_1\hat{\otimes}\phi_2\colon A_1\hat{\otimes}A_2 \dashrightarrow B_1\hat{\otimes}B_2,$$

where $\phi_1\hat{\otimes}\phi_2(a_1\hat{\otimes}a_2) \sim \phi_1(a_1)\hat{\otimes}\phi_2(a_2)$. If $C^*_{\max}(G,B)$ denotes the *maximal,* or *full* crossed product C^*-algebra [19, Section 7.6] then there is a *descent functor* on the homotopy category of equivariant asymptotic morphisms which associates to an equivariant asymptotic morphism $\phi\colon A \dashrightarrow B$ an asymptotic morphism

$$\phi\colon C^*_{\max}(G,A) \dashrightarrow C^*_{\max}(G,B),$$

where $\phi_t(\sum a_g[g]) \sim \sum \phi_t(a_g)[g]$.

Let $\mathcal{S} = C_0(\mathbb{R})$, graded according to even and odd functions and equipped with the trivial action of G. There are $*$-homomorphisms

$$\Delta\colon \mathcal{S} \to \mathcal{S}\hat{\otimes}\mathcal{S}, \qquad \epsilon\colon \mathcal{S} \to \mathbb{C},$$
$$\Delta\colon f(X) \mapsto f(X\hat{\otimes}1 + 1\hat{\otimes}X), \qquad \epsilon\colon f(X) \mapsto f(0).$$

Using them, we construct the *amplified category of* $\mathbb{Z}/2$*-graded* C^**-algebras,* in which the morphisms from A to B are graded $*$-homomorphisms (equivariant, in the presence of a group G) from $\mathcal{S}A = \mathcal{S}\hat{\otimes}A$ to B. Similarly we can form the amplification of the homotopy category of asymptotic morphisms.

2.2. Definition. Let $\mathcal{K}(\mathcal{H}_G)$ be the C^*-algebra of compact operators on the $\mathbb{Z}/2$-graded G-Hilbert space

$$\mathcal{H}_G = \ell^2(G) \oplus \ell^2(G) \oplus \cdots$$

(the summands are graded alternately even and odd). If A and B are separable, $\mathbb{Z}/2$-graded G-C^*-algebras then denote by $E_G(A,B)$ the homotopy classes of equivariant asymptotic morphisms from $\mathcal{S}A\hat{\otimes}\mathcal{K}(\mathcal{H}_G)$ into $B\hat{\otimes}\mathcal{K}(\mathcal{H}_G)$:

$$E_G(A,B) = [\mathcal{S}A\hat{\otimes}\mathcal{K}(\mathcal{H}_G), B\hat{\otimes}\mathcal{K}(\mathcal{H}_G)].$$

When G is trivial we drop the subscript and write $E(A,B)$.

The sets $E_G(A,B)$ are in fact abelian groups. There is an associative law of composition

$$E_G(A,B) \otimes E_G(B,C) \to E_G(A,C),$$

derived from the composition law in the amplified homotopy category of asymptotic morphisms. There are tensor product and descent functors on the E-theory category,

$$E_G(A_1,B_1) \otimes E_G(A_2,B_2) \to E_G(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2)$$

and

$$E_G(A,B) \to E(C^*_{\max}(G,A), C^*_{\max}(G,B)).$$

If G is trivial then the E-theory groups specialize in one variable to K-theory of graded C^*-algebras:

$$E(\mathbb{C},B) \cong K_0(B).$$

For ungraded C^*-algebras these facts are proved in [11]. The graded case will be considered elsewhere.

If $\mathcal{H}$ is a separable, $\mathbb{Z}/2$-graded G-Hilbert space then any equivariant asymptotic morphism from A to $B\hat{\otimes}\mathcal{K}(\mathcal{H})$, or from $\mathcal{S}A$ to $B\hat{\otimes}\mathcal{K}(\mathcal{H})$, determines—after tensoring with $\mathcal{K}(\mathcal{H}_G)$—an element of $E_G(A,B)$. This is because $\mathcal{H}\hat{\otimes}\mathcal{H}_G \cong \mathcal{H}_G$ (cf. [18]).

Suppose that there is a continuous, one-parameter family of actions of G on a Hilbert space $\mathcal{H}$, and suppose that an asymptotic morphism

$$\phi\colon A \dashrightarrow B\hat{\otimes}\mathcal{K}(\mathcal{H})$$

is equivariant with respect to this family, in the sense that $\phi_t(g(a)) \sim g_t(\phi_t(a))$. Then $\{\phi_t\}_{t\in[1,\infty)}$ determines a class in $E_G(A,B)$. The reason is that after tensoring with $\mathcal{H}_G$ the one parameter family of actions on $\mathcal{H}$ is conjugate to a constant action (cf. [18] again). We shall use this observation, and a small extension of it, in Section 4.

3. The assembly map

Let Y be a proper G-space and let B be a G-C^*-algebra. We define

$$E_G(Y,B) = \varinjlim_{X\subset Y} E_G(C_0(X),B),$$

where the direct limit is over all G-compact, locally compact and second countable subsets $X \subset Y$. For each such X there is a class $\phi_X \in E(\mathbb{C}, C^*_{\max}(G,C_0(X)))$, determined by a basic projection in the crossed product algebra, and we define the *maximal Baum-Connes assembly map* to be the composition

$$E_G(C_0(X),B) \xrightarrow{\text{descent}} E_G(C^*_{\max}(G,C_0(X)), C^*_{\max}(G,B)) \xrightarrow[\phi_X\in E_G(\mathbb{C},C^*_{\max}(G,C_0(X)))]{\text{composition with}} E_G(\mathbb{C}, C^*_{\max}(G,B)).$$

See [11, Chapter 10]. If Y is a general proper G-space, not necessarily G-compact, then the maximal Baum-Connes assembly map for Y is the homomorphism

$$\mu_{\max}\colon E_G(Y,B) \to E(\mathbb{C}, C^*_{\max}(G,B))$$

obtained as the direct limit of the assembly maps for the G-compact subsets of Y. The *reduced* Baum-Connes assembly map

$$\mu_{\mathrm{red}}\colon E_G(Y,B) \to E(\mathbb{C}, C^*_{\mathrm{red}}(G,B))$$

is defined by composing $\mu_{\max}$ with the E-theory map induced from the regular representation $\rho\colon C^*_{\max}(G,B) \to C^*_{\mathrm{red}}(G,B)$. The assembly maps are usually defined using KK-theory, as in [2], but our definitions are equivalent to these [13].

Let us say that a G-C^*-algebra D is *proper* if there exists a second countable, locally compact, proper G-space Z, and an equivariant $*$-homomorphism from $C_0(Z)$ into the center of the multiplier algebra of D, such that $C_0(Z)D$ is dense in D (cf. [16, Definition 1.5]). The main theorem in [11], and a key tool for us here, is the following result:

3.1. Theorem ([11, Theorem 14.1])**.** *Let G be a countable discrete group and let D be a proper G-C^*-algebra. The assembly map*

$$\mu_{\max}\colon E_G(\mathcal{E}G,D) \to E(\mathbb{C}, C^*(G,D))$$

is an isomorphism.

We will deduce our main result on the Baum-Connes conjecture, Theorem 1.2, from Theorem 3.1 (in our first proof of Theorem 1.2 we used instead of Theorem 3.1 a longer argument based on Poincaré duality for manifolds modelled on Hilbert space). As explained in [11], Theorem 3.1 may be viewed as a generalization of a theorem of Green [9] and Julg [15] which identifies equivariant K-theory (in the sense of Atiyah and Segal [20]) with the C^*-algebra K-theory of crossed product algebras. An immediate consequence is the following result:

3.2. Theorem (see [11, Theorem 15.1] and also [21, Theorem 2.1])**.** *Let G be a discrete group and suppose that there is a proper G-C^*-algebra D, and elements $\beta \in E_G(\mathbb{C}, D)$ and $\alpha \in E_G(D, \mathbb{C})$, whose composition is $\alpha \circ \beta = 1 \in E_G(\mathbb{C}, \mathbb{C})$. Then for any G-C^*-algebra B the assembly map*

$$\mu_{\max} \colon E_G(\mathcal{E}G, B) \to E(\mathbb{C}, C^*(G, B))$$

is an isomorphism.

Proof. The hypotheses imply that the assembly map for $B = B \otimes \mathbb{C}$ may be viewed as a direct summand of the assembly map for the proper C^*-algebra $B \otimes D$, and by Theorem 3.1 the latter is an isomorphism. □

4. The C^*-algebra of a Euclidean space

Let G be a discrete group which admits an affine, isometric and metrically proper action on a Euclidean space. Then G is countable and it admits such an action on a *countably infinite-dimensional* Euclidean space. With this in mind, let us fix such a Euclidean space V.

The following definitions are adapted from [14, Section 3]. Denote by V_a, V_b, and so on, the finite-dimensional, affine subspaces of V. In addition, denote by V_a^0 the finite-dimensional linear subspace of V comprised of differences of elements in V_a. Let $\mathcal{C}(V_a)$ be the $\mathbb{Z}/2$-graded C^*-algebra of continuous functions from V into the complexified Clifford algebra of V_a^0 which vanish at infinity (the grading on $\mathcal{C}(V_a)$ is inherited from the grading on the Clifford algebra). Let $\mathcal{A}(V_a)$ be the graded tensor product of $\mathcal{S} = C_0(\mathbb{R})$ and $\mathcal{C}(V_a)$.

If $V_a \subset V_b$ then there is a canonical decomposition $V_b = V_{ba}^0 + V_a$, where V_{ba}^0 is the orthogonal complement of V_a^0 in V_b^0. We shall write elements of V_b as $v_b = v_{ba} + v_a$ in accordance with this decomposition. Every function h on V_a may be extended to a function $\tilde{h}$ on V_b by the formula $\tilde{h}(v_b) = h(v_a)$.

4.1. Definition. If $V_a \subset V_b$ then denote by C_{ba} the Clifford algebra-valued function on V_b which maps v_b to $v_{ba} \in V_{ba}^0 \subset \mathrm{Cliff}(V_b^0)$. Define a $*$-homomorphism

$$\beta_{ba} \colon \mathcal{A}(V_a) \to \mathcal{A}(V_b)$$

by the formula

$$\beta_{ba}(f \hat{\otimes} h) = f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba})(1 \hat{\otimes} \tilde{h}).$$

Remark. The element $X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba}$ is regarded as an *unbounded multiplier* of $\mathcal{A}(V_b)$; see for example [5, Section 6.5]. The elements $f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_{ba})$ and $1 \hat{\otimes} \tilde{h}$ are then bounded multipliers of $\mathcal{A}(V_b)$, and their product lies in $\mathcal{A}(V_b)$.

If $V_a \subset V_b \subset V_c$ then $\beta_{cb} \circ \beta_{ba} = \beta_{ca}$. In view of this we can construct the C^*-algebra

$$\mathcal{A}(V) = \varinjlim \mathcal{A}(V_a),$$

where the limit is over the directed set of all $V_a \subset V$. If G acts on V by affine isometries then there is an induced action of G on $\mathcal{A}(V)$ by $*$-automorphisms.

4.2. Proposition. *If G acts metrically properly on V then $\mathcal{A}(V)$ is a proper G-C^*-algebra.*

Proof. The following elegant argument, which much improves our original proof, is due to G. Skandalis. The center $\mathcal{Z}(V_a)$ of the C^*-algebra $\mathcal{A}(V_a)$ contains the algebra of continuous functions, vanishing at infinity, on the locally compact space $[0,\infty)\times V_a$ (if V_a is even-dimensional and nonzero this is the full center). The linking map β_{ba} takes $\mathcal{Z}(V_a)$ into $\mathcal{Z}(V_b)$, and so we can form the direct limit $\mathcal{Z}(V)$. It has the property that $\mathcal{Z}(V)\cdot\mathcal{A}(V)$ is dense in $\mathcal{A}(V)$. Its Gelfand spectrum is the locally compact space $Z = [0,\infty)\times\overline{V}$, where $\overline{V}$ is the Hilbert space completion of V and Z is given the weakest topology for which the projection to $\overline{V}$ is weakly continuous and the function $t^2 + \|v\|^2$ is continuous, and if G acts metrically properly on V then the induced action on the locally compact space Z is proper, in the ordinary sense of the term. □

In view of Proposition 4.2 and Theorem 3.2, to prove Theorem 1.2 it suffices to exhibit classes $\alpha \in E_G(\mathcal{A}(V),\mathbb{C})$ and $\beta \in E_G(\mathbb{C},\mathcal{A}(V))$ such that $\alpha\circ\beta = 1 \in E_G(\mathbb{C},\mathbb{C})$. This is what we shall now do. We shall follow the article [14], which is in turn an adaptation of Atiyah's elliptic operator proof [1] of the Bott periodicity theorem. The construction of β is quite simple:

4.3. Definition. Denote by $\beta\colon \mathcal{S} = \mathcal{A}(0)\to\mathcal{A}(V)$ the $*$-homomorphism associated to the inclusion of the zero-dimensional linear space 0 into V. Denote by $\beta_t\colon \mathcal{S}\to\mathcal{A}(V)$ the $*$-homomorphism $\beta_t(f) = \beta(f_t)$, where $f_t(x) = f(t^{-1}x)$. The family of $*$-homomorphisms $\{\beta_t\}_{t\in[1,\infty)}$ is asymptotically equivariant and so is an equivariant asymptotic morphism from $\mathcal{S}$ to $\mathcal{A}(V)$. Denote by $\beta \in E_G(\mathbb{C},\mathcal{A}(V))$ its E-theory class.

As in Atiyah's paper [1], to construct $\alpha \in E_G(\mathcal{A}(V),\mathbb{C})$ we shall need some elliptic operator theory. In the present context the operators must be defined on the infinite-dimensional space V. Denote by $\mathcal{H}_a$ the Hilbert space of square integrable functions from V_a into $\mathrm{Cliff}(V_a^0)$. If $V_a\subset V_b$ then there is a canonical isomorphism

$$\mathcal{H}_b \cong \mathcal{H}_{ba}\hat{\otimes}\mathcal{H}_a,$$

where $\mathcal{H}_{ba}$ denotes the Hilbert space associated to V_{ba}^0. We define a unit vector $\xi_0\in\mathcal{H}_{ba}$ by

$$\xi_0(v_{ba}) = \pi^{-n_{ba}/4}\exp(-\tfrac{1}{2}\|v_{ba}\|^2),$$

where $n_{ba} = \dim(V_{ba}^0)$, and we regard $\mathcal{H}_a$ as included in $\mathcal{H}_b$ via the isometry $\xi\mapsto\xi_0\hat{\otimes}\xi$. We define

$$\mathcal{H} = \varinjlim \mathcal{H}_a$$

and denote by $\mathfrak{s} = \varinjlim \mathfrak{s}_a$ the direct limit of the Schwartz subspaces $\mathfrak{s}_a\subset\mathcal{H}_a$. If $V_a\subset V$ is a finite-dimensional affine subspace then the *Dirac operator* D_a, an unbounded operator on $\mathcal{H}$ with domain $\mathfrak{s}$, is defined by

$$D_a\xi = \sum_{i=1}^{n}(-1)^{\deg(\xi)}\frac{\partial\xi}{\partial x_i}v_i,$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V_a^0, and $\{x_1, \ldots, x_n\}$ are the dual coordinates to $\{v_1, \ldots, v_n\}$. If V_a is a *linear* subspace then we also define the *Clifford operator* by

$$C_a\xi = \sum_{i=1}^{n} x_i v_i \xi$$

(cf. Definition 4.1).

4.4. Definition. Fix an algebraic, direct sum decomposition

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots ,$$

where each V_j is a finite-dimensional linear subspace of V. For each n, define an unbounded operator $B_{n,t}$ on $\mathcal{H}$, by the formula

$$\begin{aligned} B_{n,t} = t_0 D_0 + t_1 D_1 + \cdots + t_{n-1} D_{n-1} \\ + t_n(D_n + C_n) + t_{n+1}(D_{n+1} + C_{n+1}) + \cdots , \end{aligned}$$

where $t_j = 1 + t^{-1} j$.

The infinite sum is well defined because when $B_{n,t}$ is applied to any vector in the Schwartz space $\mathfrak{s}$ the resulting infinite series has only finitely many nonzero terms. This is because the function $\exp(-\frac{1}{2}\|v_j\|^2) \in s_j$, used in the definition of the direct limit $\mathfrak{s}$, lies in the kernel of $D_j + C_j$.

If $h \in \mathcal{C}(V_0 \oplus \cdots \oplus V_n)$ then the function $h_t(v) = h(t^{-1}v)$ acts as a bounded operator $\pi(h_t)$ on $\mathcal{H}$ by pointwise multiplication. With this notation in hand, the main technical results concerning the operators $B_{n,t}$ are as follows.

4.5. Proposition (cf. [14, Lemma 2.9 and Proposition 4.2])**.** *The operators $B_{n,t}$ are essentially selfadjoint. The formula*

$$\alpha_t^n \colon f \hat{\otimes} h \mapsto f_t(B_t)\pi(h_t) \qquad (f \in \mathcal{S},\ h \in \mathcal{C}(V_0 \oplus \cdots \oplus V_n))$$

defines an asymptotic morphism from $\mathcal{A}(V_0 \oplus \cdots \oplus V_n)$ into $\mathcal{K}(\mathcal{H})$, and the diagram

$$\begin{array}{ccc} \mathcal{A}(V_0 \oplus \cdots \oplus V_n) & \overset{\alpha^n}{\dashrightarrow} & \mathcal{K}(\mathcal{H}) \\ \downarrow{\scriptstyle \beta_{n+1}} & & \downarrow{\scriptstyle =} \\ \mathcal{A}(V_0 \oplus \cdots \oplus V_{n+1}) & \underset{\alpha^{n+1}}{\dashrightarrow} & \mathcal{K}(\mathcal{H}) \end{array}$$

is asymptotically commutative.

It follows from the second part of the proposition that the asymptotic morphisms α^n combine to form a single asymptotic morphism

$$\alpha \colon \mathcal{A}(V) \dashrightarrow \mathcal{K}(\mathcal{H}).$$

Now let G act on the Euclidean space V according to the affine, isometric action

$$g \cdot v = \pi(g)v + \kappa(g),$$

where π is a linear, isometric representation of G on V. For $s \geq 0$ form the scaled G-action

$$g_s \cdot v = \pi(g)v + s\kappa(g),$$

and denote the induced actions on $\mathcal{A}(V)$ and $\mathcal{K}(\mathcal{H})$ in the same way.

4.6. Proposition (cf. [14, Lemma 5.15])**.** *Suppose that the direct sum decomposition $V = \bigoplus_0^\infty V_j$ is chosen in such a way that for every $g \in G$ there is an $N \in \mathbb{N}$ such that if $n > N$ then $g[\bigoplus_0^n V_j] \subset \bigoplus_0^{n+1} V_j$. Then for every $a \in \mathcal{A}(V)$ and every $g \in G$, $\alpha_t(g(a)) \sim g_t(\alpha_t(a))$. Consequently the asymptotic morphism $\{\alpha_t\}_{t\in[1,\infty)}$ determines a class $\alpha \in E_G(\mathcal{A}(V), \mathbb{C})$.*

We are ready to prove the following result:

4.7. Theorem. *The composition $\alpha \circ \beta \in E_G(\mathbb{C}, \mathbb{C})$ is the identity.*

Proof. Let $s \in [0,1]$ and denote by $\mathcal{A}_s$ the C^*-algebra $\mathcal{A} = \mathcal{A}(V)$, but with the scaled G-action $(g,a) \mapsto g_s(a)$. The algebras $\mathcal{A}_s$ form a continuous field of G-C^*-algebras over the unit interval, and we shall denote by $\mathcal{A}[0,1]$ the G-C^*-algebra of continuous sections. In a similar way, form a continuous field of G-C^*-algebras $\mathcal{K}_s = \mathcal{K}(\mathcal{H})$ and denote by $\mathcal{K}[0,1]$ the G-C^*-algebra of continuous sections.

The asymptotic morphism $\alpha\colon \mathcal{A} \dashrightarrow \mathcal{K}$ induces an asymptotic morphism

$$\bar\alpha\colon \mathcal{A}[0,1] \dashrightarrow \mathcal{K}[0,1]\ ,$$

and similarly the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}$ determines an asymptotic morphism

$$\bar\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}[0,1]$$

by forming the tensor product $\mathcal{S}[0,1] \dashrightarrow \mathcal{A}[0,1]$ and composing with the inclusion $\mathcal{S} \subset \mathcal{S}[0,1]$ as constant functions. Consider now the diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[r, dashed, "\bar\beta"] \arrow[d, "="'] & \mathcal{A}[0,1] \arrow[r, dashed, "\bar\alpha"] \arrow[d, "\epsilon_s"] & \mathcal{K}[0,1] \arrow[d, "\epsilon_s"] \\
\mathcal{S} \arrow[r, dashed, "\beta"'] & \mathcal{A}_s \arrow[r, dashed, "\alpha"'] & \mathcal{K}_s
\end{tikzcd}$$

where ϵ_s denotes evaluation at $s \in [0,1]$. All the asymptotic morphisms determine classes in E_G-theory, and the $*$-homomorphism $\epsilon_s\colon \mathcal{K}[0,1] \to \mathcal{K}_t$ defines an isomorphism, so to prove the theorem it suffices to show that when $s = 0$ the bottom composition

$$\mathcal{S} \overset{\beta}{\dashrightarrow} \mathcal{A}_0 \overset{\alpha}{\dashrightarrow} \mathcal{K}_0$$

determines the identity element of $E_G(\mathbb{C}, \mathbb{C})$. When $s = 0$ the action of G on V is linear and the asymptotic morphism $\beta\colon \mathcal{S} \dashrightarrow \mathcal{A}_0$ is homotopic to the equivariant $*$-homomorphism $\beta\colon \mathcal{S} \to \mathcal{A}_0$ of Definition 4.3. The composition of the $*$-homomorphism β with the asymptotic morphism α is the asymptotic morphism

$$\gamma\colon \mathcal{S} \dashrightarrow \mathcal{K},$$
$$\gamma_t\colon f \mapsto f_t(B_{0,t}).$$

Replacing $B_{0,t}$ with $s^{-1}B_{0,t}$, for $0 < s \leq 1$ we obtain a homotopy to the $*$-homomorphism $\gamma_0\colon f \mapsto f(0)P$, where P is the projection onto the kernel of $B_{0,t}$. Since the kernel is one-dimensional and G-invariant, γ_0 determines the identity in $E_G(\mathbb{C}, \mathbb{C})$. $\square$

5. Reduced group C^*-algebras

If $C^*_{\mathrm{red}}(G)$ is an exact C^*-algebra [22] then there is a *reduced descent functor* on the homotopy category of asymptotic morphisms which associates to an equivariant asymptotic morphism $\phi\colon A \dashrightarrow B$ an asymptotic morphism

$$\phi\colon C^*_{\mathrm{red}}(G,A) \dashrightarrow C^*_{\mathrm{red}}(G,B).$$

We can therefore define a descent functor in E-theory and the arguments of the previous sections then apply equally well to the reduced Baum-Connes assembly map. In the absence of exactness the following *ad hoc* argument proves that μ_{red} is an isomorphism for the groups we are considering.

5.1. Theorem. *If G is a discrete group which admits an affine, isometric and metrically proper action on a Euclidean space then for every G-C^*-algebra B the regular representation*

$$\rho\colon C^*_{\max}(G,B) \to C^*_{\mathrm{red}}(G,B)$$

determines an invertible morphism $\rho \in E(C^*_{\max}(G,B), C^*_{\mathrm{red}}(G,B))$.

Proof. Let us consider the case where $B = \mathbb{C}$. We shall define a class $\sigma \in E(C^*_{\mathrm{red}}(G), C^*_{\max}(G))$ which is inverse to $\rho \in E(C^*_{\max}(G), C^*_{\mathrm{red}}(G))$. Suppose given an equivariant asymptotic morphism $\phi\colon \mathcal{S} \dashrightarrow A$, where A is any G-C^*-algebra. Both A and $C^*_{\mathrm{red}}(G)$ embed in the multiplier algebra of $C^*_{\mathrm{red}}(G,A)$, and $C^*_{\mathrm{red}}(G)$ asymptotically commutes with the image of the asymptotic morphism ϕ (this is because ϕ is asymptotically equivariant). It follows that ϕ induces an asymptotic morphism

$$\phi_{\mathrm{red}}\colon \mathcal{S}\hat{\otimes} C^*_{\mathrm{red}}(G) \dashrightarrow C^*_{\mathrm{red}}(G,A).$$

In particular, β induces an asymptotic morphism

$$\beta_{\mathrm{red}}\colon \mathcal{S}\hat{\otimes} C^*_{\mathrm{red}}(G) \dashrightarrow C^*_{\mathrm{red}}(G,\mathcal{A}).$$

Now since $\mathcal{A}$ is a proper C^*-algebra the regular representation $C^*_{\max}(G,\mathcal{A}) \to C^*_{\mathrm{red}}(G,\mathcal{A})$ is an isomorphism. We define $\sigma \in E(C^*_{\mathrm{red}}(G), C^*_{\max}(G))$ to be the E-theory class of the composition

$$\begin{tikzcd}
\mathcal{S}\hat{\otimes} C^*_{\mathrm{red}}(G) \arrow[r, dashed, "\beta_{\mathrm{red}}"] & C^*_{\mathrm{red}}(G,\mathcal{A}) & & \\
& C^*_{\max}(G,\mathcal{A}) \arrow[u, "\cong"', "\rho"] \arrow[r, dashed, "\alpha"] & C^*_{\max}(G,\mathcal{K}) \arrow[r, "\cong"'] & C^*_{\max}(G)\hat{\otimes}\mathcal{K}.
\end{tikzcd}$$

It follows from Theorem 4.7 that $\sigma \circ \rho = 1 \in E_G(C^*_{\max}(G), C^*_{\max}(G))$. To calculate the reverse composition, we note that since $C^*_{\max}(G,\mathcal{A}) \cong C^*_{\mathrm{red}}(G,\mathcal{A})$ the asymptotic morphism $\alpha\colon \mathcal{A} \dashrightarrow \mathcal{K}$ induces an asymptotic morphism

$$\alpha_{\mathrm{red}}\colon C^*_{\mathrm{red}}(G,\mathcal{A}) \dashrightarrow C^*_{\mathrm{red}}(G,\mathcal{K}).$$

The composition $\rho \circ \sigma \in E(C^*_{\mathrm{red}}(G), C^*_{\mathrm{red}}(G))$ is given by the composition

$$\mathcal{S}\hat{\otimes} C^*_{\mathrm{red}}(G) \overset{\beta_{\mathrm{red}}}{\dashrightarrow} C^*_{\mathrm{red}}(G,\mathcal{A}) \overset{\alpha_{\mathrm{red}}}{\dashrightarrow} C^*_{\mathrm{red}}(G,\mathcal{K}).$$

The composition $\alpha_{\text{red}} \circ \beta_{\text{red}}$ is the asymptotic morphism

$$\gamma_{\text{red}} \colon \mathcal{S}\hat{\otimes} C^*_{\text{red}}(G) \dashrightarrow C^*_{\text{red}}(G, \mathcal{K})$$

induced, according to the remark made above, by the equivariant asymptotic morphism $\gamma \colon \mathcal{S} \to \mathcal{K}$ which is the composition of β and α and is defined by the formula: $\gamma_t(f) = f_t(B_{0,t})$. (This follows from a similar fact for full crossed products.) It is now possible to deform the action of G on V to a linear action, as in the proof of Theorem 4.7, and verify that $\gamma_{\text{red}} = 1 \in E(C^*_{\text{red}}(G), C^*_{\text{red}}(G))$. □

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