5

Categorical Galois theorem and factorization systems

This chapter is the core of the book. It could even be given a longer title, namely *Categorical Galois theorem, "non-Grothendieck" examples and factorization systems.*

We show first how the situation for commutative rings, studied in chapter 4, generalizes to develop Galois theory with respect to an axiomatic categorical setting. This setting consists basically in an adjunction with "well-behaved" properties, which mimic the situation of the *Pierce spectrum* functor and its adjoint, as in section 4.3. This categorical setting will contain the situations of the previous chapters: in particular the cases of fields and commutative rings. But our categorical Galois theorem will also apply to many other contexts.

First we apply the categorical Galois theorem to the study of central extensions of groups. This topic is generally not considered as part of Galois theory, but nevertheless the central extensions of groups turn out to be precisely the objects split over extension in a special case of categorical Galois theory.

This chapter also provides a good help for understanding the relationship between the Galois theory and factorization systems, which began in [18]. We focus in particular on the case of semi-left-exact reflections and apply it to the monotone–light factorization of continuous maps between compact Hausdorff spaces. It should be noticed that the situation of chapter 4, the categorical Galois theory of rings, is also a special case of a semi-left-exact reflection.

It might look strange for a topologist to describe light maps of compact Hausdorf spaces as the actions of a compact totally disconnected equivalence relation considered as a topological groupoid. However if we replace the category of compact Hausdorff spaces by the dual category of commutative C^*-algebras, which is equivalent to it, the surprise disappears since general profinite topological groupoids are already used in Galois theory of commutative rings.

5.1 The abstract categorical Galois theorem

This section presents the Galois theorem of Janelidze in its general form. The Galois theory for rings, as developed in section 4.7, is in fact a special case of this more general theory.

Definition 5.1.1 Let $\mathcal{C}$ be a category. A class $\overline{\mathcal{C}}$ of arrows in $\mathcal{C}$ is admissible when

(i) every isomorphism is in $\overline{\mathcal{C}}$,
(ii) $\overline{\mathcal{C}}$ is closed under composition,
(iii) in the pullback

$$\begin{tikzcd} \bullet \arrow[r, "c"] \arrow[d, "d"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "a"'] & \bullet \end{tikzcd}$$

if $a, b \in \overline{\mathcal{C}}$, then $c, d \in \overline{\mathcal{C}}$.

Definition 5.1.2 Let $\overline{\mathcal{C}}$ be an admissible class of morphisms in a category $\mathcal{C}$. For an object $C \in \mathcal{C}$, we write $\overline{\mathcal{C}}/C$ for the following category:

(i) the objects are the pairs (X, f) where $f\colon X \longrightarrow C$ is in $\overline{\mathcal{C}}$;
(ii) the arrows $h\colon (X, f) \longrightarrow (X', f')$ are all arrows $h\colon X \longrightarrow X'$ in $\mathcal{C}$ such that $f' \circ h = f$.

$\overline{\mathcal{C}}/C$ is thus a full subcategory of $\mathcal{C}/C$.

Definition 5.1.3 A relatively admissible adjunction consists in

(i) an adjunction $\mathcal{A} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathcal{P}$; $\mathcal{S} \dashv \mathcal{C}$,
(ii) two admissible classes $\overline{\mathcal{A}} \subseteq \mathcal{A}$, $\overline{\mathcal{P}} \subseteq \mathcal{P}$ of arrows,

such that

(i) $\forall f \in \mathcal{A} \quad f \in \overline{\mathcal{A}} \Rightarrow \mathcal{S}(f) \in \overline{\mathcal{P}}$,
(ii) $\forall g \in \mathcal{P} \quad g \in \overline{\mathcal{P}} \Rightarrow \mathcal{C}(g) \in \overline{\mathcal{A}}$,
(iii) $\forall A \in \mathcal{A} \quad \eta_A\colon A \longrightarrow \mathcal{CS}(A)$, the unit of the adjunction $\mathcal{S} \dashv \mathcal{C}$, is an arrow of $\overline{\mathcal{A}}$,

(iv) $\forall X \in \mathcal{P}\ \ \varepsilon_X \colon \mathcal{SC}(X) \longrightarrow X$, the counit of the adjunction $\mathcal{S} \dashv \mathcal{C}$, is an arrow of $\overline{\mathcal{P}}$.

The various conditions in definitions 5.1.1 and 5.1.3 obviously imply the following two lemmas.

Lemma 5.1.4 *Let $\mathcal{A}$ be a category with pullbacks. In the conditions of definition 5.1.3, lemma 4.3.4 yields an adjunction between $\mathcal{A}/A$ and $\mathcal{P}/S(A)$, which restricts to an adjunction*

$$\overline{\mathcal{A}}/A \underset{\mathcal{S}_A}{\overset{\mathcal{C}_A}{\rightleftarrows}} \overline{\mathcal{P}}/\mathcal{S}(A), \quad \mathcal{S}_A \dashv \mathcal{C}_A. \qquad \square$$

Lemma 5.1.5 *Let $\overline{\mathcal{C}}$ be an admissible class of arrows in a category with pullbacks. For every morphism $\sigma \colon S \longrightarrow R$ in $\overline{\mathcal{C}}$, the adjunction $\Sigma_\sigma \dashv \sigma^*$ of section 4.4 restricts to an adjunction*

$$\overline{\mathcal{C}}/R \underset{\sigma^*}{\overset{\Sigma_\sigma}{\rightleftarrows}} \overline{\mathcal{C}}/S, \quad \Sigma_\sigma \dashv \sigma^*. \qquad \square$$

Definition 5.1.6 Let $\overline{\mathcal{C}}$ be an admissible class of arrows in a category $\mathcal{C}$ with pullbacks. An arrow $\sigma \colon S \longrightarrow R$ of $\mathcal{C}$ is an effective descent morphism relatively to $\overline{\mathcal{C}}$ when

(i) $\sigma \in \overline{\mathcal{C}}$,
(ii) the functor $\sigma^* \colon \overline{\mathcal{C}}/R \longrightarrow \overline{\mathcal{C}}/S$ is monadic.

Definition 5.1.7 With the notation of definition 5.1.3, consider a relatively admissible adjunction

$$\mathcal{S} \dashv \mathcal{C} \colon (\mathcal{A}, \overline{\mathcal{A}}) \rightleftarrows (\mathcal{P}, \overline{\mathcal{P}})$$

where $\mathcal{A}$ is a category with pullbacks. An object $(A, a) \in \overline{\mathcal{A}}/R$ is split by a morphism $\sigma \colon S \longrightarrow R$ of $\overline{\mathcal{A}}$ when the unit

$$\eta^S_{\sigma^*(A,a)} \colon \sigma^*(A, a) \longrightarrow \mathcal{C}_S \mathcal{S}_S \sigma^*(A, a)$$

of the adjunction $\mathcal{S}_S \dashv \mathcal{C}_S$ is an isomorphism at the object $\sigma^*(A, a)$.

Definition 5.1.8 With the notation of definition 5.1.3, consider a relatively admissible adjunction

$$\mathcal{S} \dashv \mathcal{C} \colon (\mathcal{A}, \overline{\mathcal{A}}) \rightleftarrows (\mathcal{P}, \overline{\mathcal{P}})$$

where $\mathcal{A}$ and $\mathcal{P}$ are categories with pullbacks. A morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{A}$ is of relative Galois descent with respect to these data when

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \mathcal{C}(X) \arrow[d, "\mathcal{C}(f)"] \\
S \arrow[r, "\eta_S"] \arrow[d, "\sigma"'] & \mathcal{CS}(S) \\
R &
\end{tikzcd}$$

Diagram 5.1

(i) σ is an effective descent morphism relatively to $\overline{\mathcal{A}}$,

(ii) the counit of the adjunction $\mathcal{S}_S \dashv \mathcal{C}_S$

$$\mathcal{S}_S \circ \mathcal{C}_S \Longrightarrow \mathsf{id}_{\overline{\mathcal{P}}/\mathcal{S}(S)}$$

is an isomorphism,

(iii) for every object $(X, f) \in \overline{\mathcal{P}}/\mathcal{S}(S)$, the object $(\Sigma_\sigma \circ \mathcal{C}_S)(X, f) \in \overline{\mathcal{A}}/R$ is split by σ.

Let us recall that the object indicated in condition (iii) is the left vertical composite in diagram 5.1, where the square is a pullback and η is the unit of the adjunction $\mathcal{S} \dashv \mathcal{C}$.

For clarity, let us put on a single diagram the various functors involved in definition 5.1.8.

$$\begin{tikzcd}
\overline{\mathcal{A}}/S \arrow[r, shift right, "\mathcal{S}_S"'] \arrow[d, shift right, "\Sigma_\sigma"'] & \mathcal{P}/\mathcal{S}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \arrow[d, shift right, "\Sigma_{\mathcal{S}(\sigma)}"'] \\
\overline{\mathcal{A}}/R \arrow[r, shift right, "\mathcal{S}_R"'] \arrow[u, shift right, "\sigma^*"'] & \mathcal{P}/\mathcal{S}(R) \arrow[l, shift right, "\mathcal{C}_R"'] \arrow[u, shift right, "\mathcal{S}(\sigma)^*"']
\end{tikzcd}$$

$$\Sigma_\sigma \dashv \sigma^*, \quad \mathcal{S}_R \dashv \mathcal{C}_R, \quad \Sigma_{\mathcal{S}(\sigma)} \dashv \mathcal{S}(\sigma)^*, \quad \mathcal{S}_S \dashv \mathcal{C}_S.$$

Lemma 5.1.9 *In the conditions of definition 5.1.8,*

$$\Sigma_{\mathcal{S}(\sigma)} \circ \mathcal{S}_S \cong \mathcal{S}_R \circ \Sigma_\sigma, \quad \mathcal{C}_S \circ \mathcal{S}(\sigma)^* = \sigma^* \circ \mathcal{C}_R.$$

Proof The first relation is obvious and the second follows by adjunction. □

Before going on, it is useful to specify in what sense the situation for rings is a special case of the present situation. In the case of rings,

- $\mathcal{A}$ is the dual of the category of rings,
- $\overline{\mathcal{A}}$ is the class of all morphisms of $\mathcal{A}$, thus $\overline{\mathcal{A}}/S = \mathcal{A}/S$,
- $\mathcal{P}$ is the category of profinite spaces,
- $\overline{\mathcal{P}}$ is the class of all morphisms of $\mathcal{P}$, thus $\overline{\mathcal{P}}/\mathcal{S}(S) = \mathcal{P}/\mathcal{S}(S)$,
- $\mathcal{S}$ is the 'Pierce spectrum' functor,
- $\mathcal{C}$ is the functor $\mathcal{C}(-,\mathbb{Z})$.

Indeed, a ring is exactly a $\mathbb{Z}$-algebra and given a ring R, $\mathsf{Ring}^{\mathsf{op}}/R$ is the dual category of R-algebras (see 4.3.1).

The first two of the following results are then obvious, by definition 5.1.7.

Lemma 5.1.10 *In the conditions of definition 5.1.8, the following conditions are equivalent for an object $(A,a) \in \overline{\mathcal{A}}/S$:*

(i) *(A,a) is split by $\mathsf{id}_S \colon S =\!= S$;*
(ii) *the morphism $\eta^S_{(A,a)} \colon (A,a) \longrightarrow \mathcal{C}_S\mathcal{S}_S(A,a)$ is an isomorphism.* □

Corollary 5.1.11 *In the conditions of definition 5.1.8, the following conditions are equivalent for an object $(A,a) \in \overline{\mathcal{A}}/R$:*

(i) *(A,a) is split by $\sigma \colon S \longrightarrow R$;*
(ii) *$\sigma^*(A,a)$ is split by $\mathsf{id}_S \colon S \longrightarrow S$.* □

Lemma 5.1.12 *In the conditions of definition 5.1.8, the following conditions are equivalent for an object $(A,a) \in \overline{\mathcal{A}}/S$:*

(i) *(A,a) is split by $\mathsf{id}_S \colon S =\!= S$;*
(ii) *$(A,a) \cong \mathcal{C}_S(X,\varphi)$ for some $(X,\varphi) \in \overline{\mathcal{P}}/\mathcal{S}(S)$.*

Proof Choosing $(X,\varphi) = \mathcal{S}_S(A,a)$ yields (i) $\Rightarrow$ (ii) by lemma 5.1.10. Conversely, by definition 5.1.8,

$$(A,a) \cong \mathcal{C}_S(X,\varphi) \cong \mathcal{C}_S\mathcal{S}_S\mathcal{C}_S(X,\varphi) \cong \mathcal{C}_S\mathcal{S}_S(A,a)$$

from which we obtain the conclusion by lemma 5.1.10. □

Corollary 5.1.13 *In the conditions of definition 5.1.8, the following conditions are equivalent for an object $(A,a) \in \overline{\mathcal{A}}/R$:*

(i) *(A,a) is split by $\sigma \colon S \longrightarrow R$;*
(ii) *$\sigma^*(A,a) \cong \mathcal{C}_S(X,\varphi)$ for some $(X,\varphi) \in \overline{\mathcal{P}}/\mathcal{S}(S)$.* □

Lemma 5.1.14 *We assume the conditions of definition 5.1.8 and write $\mathsf{Split}_S(S)$ for the full subcategory $\mathsf{Split}_S(S) \subseteq \overline{\mathcal{A}}/S$ of those objects which are split by $\mathsf{id}_S \colon S = S$. The functors*

$$\mathsf{Split}_S(S) \underset{\mathcal{S}_S}{\overset{\mathcal{C}_S}{\rightleftarrows}} \overline{\mathcal{P}}/\mathcal{S}(S)$$

constitute an equivalence of categories.

Proof By lemma 5.1.12, $\mathcal{C}_S$ takes values in $\mathsf{Split}_S(S)$. One has $\mathcal{S}_S \circ \mathcal{C}_S \cong \mathsf{id}$ by definition 5.1.8 and $\mathsf{id} \cong \mathcal{C}_S \circ \mathcal{S}_S$ by lemma 5.1.10. □

Lemma 5.1.15 *In the conditions of definition 5.1.8, if an object $(A,a) \in \overline{\mathcal{A}}/S$ is split by $\mathsf{id}_S \colon S = S$, the same property holds for the object $(\sigma^* \circ \Sigma_\sigma)(A,a)$.*

Proof By lemma 5.1.12, $(A,a) \cong \mathcal{C}_S(X,\varphi)$ for some $(X,\varphi) \in \overline{\mathcal{P}}/\mathcal{S}(S)$. By definition 5.1.8, $\Sigma_\sigma \mathcal{C}_S(X,\varphi)$ is split by σ. By corollary 5.1.11, this implies that $\sigma^* \Sigma_\sigma \mathcal{C}_S(X,\varphi) \cong \sigma^* \Sigma_\sigma(A,a)$ is split by id_S. □

Lemma 5.1.16 *In the conditions of definition 5.1.8, $(S, \mathsf{id}_S) \in \overline{\mathcal{A}}/S$ is split by $\mathsf{id}_S \colon S = S$.*

Proof The lemma immediately follows from the previous equality $\mathcal{C}_S\big(\mathcal{S}_S(S), \mathsf{id}_{\mathcal{S}(S)}\big) = (S, \mathsf{id}_S)$ and lemma 5.1.12. □

Corollary 5.1.17 *In the conditions and with the notation of definition 5.1.8, the object $(R, \mathsf{id}_R) \in \overline{\mathcal{A}}/R$ is split by $\sigma \colon S \longrightarrow R$.* □

Lemma 5.1.18 *In the conditions of definition 5.1.8, for all integers $n \in \mathbb{N}$ and $1 \le i \le n$, the objects $\big(\prod_{i=1}^n (S,\sigma), p_i\big) \in \overline{\mathcal{A}}/S$ are split by $\mathsf{id}_S \colon S = S$.*

Proof By iterated application of lemma 5.1.15, starting with $(A,a) = (S, \mathsf{id}_S)$ via lemma 5.1.16. □

Lemma 5.1.19 *In the conditions of definition 5.1.8, if (A, a) and (B, b) are objects of $\overline{\mathcal{A}}/S$ split by id_S, then $(A, a) \times (B, b) \in \overline{\mathcal{A}}/S$ is split by id_S as well.*

Proof By lemma 5.1.12, we can write $(A, a) \cong \mathcal{C}_S(X, \varphi)$ and $(B, b) \cong \mathcal{C}_S(Y, \psi)$. We deduce

$$(A, a) \times (B, b) \cong \mathcal{C}_S(X, \varphi) \times \mathcal{C}_S(Y, \psi) \cong \mathcal{C}_S\big((X, \varphi) \times (Y, \psi)\big)$$

because $\mathcal{C}_S$, as a right adjoint, preserves products. One concludes the proof by lemma 5.1.12 again. □

Lemma 5.1.20 *Assume the conditions of definition 5.1.8 and write $\mathsf{Split}_R(\sigma)$ for the full subcategory $\mathsf{Split}_R(\sigma) \subseteq \overline{\mathcal{A}}/R$ of those objects which are split by $\sigma\colon S \longrightarrow R$. The functor*

$$\sigma^*\colon \mathsf{Split}_R(\sigma) \longrightarrow \mathsf{Split}_S(S)$$

is monadic.

Proof By corollary 5.1.11, this functor is correctly defined. Observe that when $(A, a) \in \mathsf{Split}_S(S)$, $\sigma^*\Sigma_\sigma(A, a) \in \mathsf{Split}_S(S)$ by lemma 5.1.15, and thus $\Sigma_\sigma(A, a) \in \mathsf{Split}_R(\sigma)$ by corollary 5.1.11 (which in fact is shown directly in the proof of lemma 5.1.15). This proves that the adjunction

$$\overline{\mathcal{A}}/R \underset{\sigma^*}{\overset{\Sigma_\sigma}{\rightleftarrows}} \overline{\mathcal{A}}/S, \quad \Sigma_\sigma \dashv \sigma^*$$

of lemma 5.1.5 restricts to an adjunction

$$\mathsf{Split}_R(\sigma) \underset{\sigma^*}{\overset{\Sigma_\sigma}{\rightleftarrows}} \mathsf{Split}_S(S), \quad \Sigma_\sigma \dashv \sigma^*.$$

This is one of the conditions of the Beck criterion for monadicity (see [66]).

Since σ is an effective descent morphism relatively to $\overline{\mathcal{A}}$, the functor

$$\sigma^*\colon \overline{\mathcal{A}}/R \longrightarrow \overline{\mathcal{A}}/S$$

is monadic, thus reflects isomorphisms. The same thus holds for its restriction to the full subcategories of split objects.

Finally consider two morphisms $u, v\colon (B, b) \rightrightarrows (A, a)$ in $\mathsf{Split}_R(\sigma)$ such that the following diagram is a split coequalizer of $\sigma^*(u)$, $\sigma^*(v)$ in $\mathsf{Split}_S(S)$.

$$\begin{tikzcd} \sigma^*(B,b) \arrow[r, shift left, "\sigma^*(u)"] \arrow[r, shift right, "\sigma^*(v)"'] & \sigma^*(A,a) \arrow[l, bend right, "t"'] \arrow[r, "q"'] & (Q,x) \arrow[l, bend right, "r"'] \end{tikzcd}$$

This split coequalizer is preserved by every functor, thus in particular by the inclusion $\mathsf{Split}_S(S) \subseteq \overline{\mathcal{A}}/S$. The Beck criterion in the case of the functor

$$\sigma^* \colon \overline{\mathcal{A}}/R \longrightarrow \overline{\mathcal{A}}/S$$

thus implies the existence of a coequalizer in $\overline{\mathcal{A}}/R$

$$\begin{tikzcd} (B,b) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & (A,a) \arrow[r, two heads, "p"] & (C,c) \end{tikzcd}$$

which is preserved by σ^*. To conclude the proof, it remains to check that $(C,c) \in \mathsf{Split}_R(\sigma)$. By corollary 5.1.11, this reduces to $\sigma^*(C,c) \in \mathsf{Split}_S(S)$. The preservation of the coequalizer $p = \mathsf{Coker}\,(u,v)$ by σ^* implies precisely $\sigma^*(C,c) \cong (Q,x) \in \mathsf{Split}_S(S)$. $\square$

Corollary 5.1.21 *In the conditions of definition 5.1.8, the functor*

$$\mathsf{Split}_R(\sigma) \longrightarrow \overline{\mathcal{P}}/\mathcal{S}(S), \qquad (A,a) \mapsto (\mathcal{S}_S \circ \sigma^*)(A,a)$$

is monadic.

Proof This functor is the composite of those in lemmas 5.1.20 and 5.1.14. $\square$

For clarity, let us consider on a single diagram the various functors involved in our discussion.

$$\begin{tikzcd} \mathsf{Split}_R(\sigma) \arrow[d, tail] \arrow[r, shift right, "\sigma^*"'] & \mathsf{Split}_S(S) \arrow[l, shift right, "\Sigma_\sigma"'] \arrow[d, tail] \arrow[r, shift right, "\mathcal{S}_S"'] & \overline{\mathcal{P}}/\mathcal{S}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \arrow[d, equal] \\ \overline{\mathcal{A}}/R \arrow[r, shift right, "\sigma^*"'] & \overline{\mathcal{A}}/S \arrow[l, shift right, "\Sigma_\sigma"'] \arrow[r, shift right, "\mathcal{S}_S"'] & \overline{\mathcal{P}}/\mathcal{S}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \end{tikzcd}$$

Lemma 5.1.22 *In the conditions and with the notation of definition 5.1.8, the functor $\mathcal{S} \colon \mathcal{A} \longrightarrow \mathcal{P}$ transforms the kernel pair of $\sigma \colon S \longrightarrow R$, seen as a groupoid in $\mathcal{A}$ (see example 4.6.2), into a groupoid in $\mathcal{P}$.*

Proof The kernel pair of σ, viewed as a groupoid in $\mathcal{A}$,

$$\begin{tikzcd}[column sep=large]
(S \times_R S) \times_S (S \times_R S) \arrow[r, "(p_1, p_4)"] & S \times_R S \arrow[loop below, "\tau"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & S, \arrow[l, "\Delta"']
\end{tikzcd}$$

can also be seen as the "object part" of the following groupoid in $\mathcal{A}/R$:

$$\begin{tikzcd}[column sep=large]
\big((S,\sigma) \times (S,\sigma)\big) \underset{(S,\sigma)}{\times} \big((S,\sigma) \times (S,\sigma)\big) \arrow[r, "(p_1, p_4)"] & (S,\sigma) \times (S,\sigma) \arrow[loop below, "\tau"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & (S,\sigma). \arrow[l, "\Delta"']
\end{tikzcd}$$

The pullback defining the left hand object can even be seen as the image by Σ_σ of the pullback

$$\begin{tikzcd}
\big((S \times_R S) \times_S (S \times_R S), p\big) \arrow[r] \arrow[d] & (S \times_R S, p_1) \arrow[d, "p_1"] \\
(S \times_R S, p_2) \arrow[r, "p_2"'] & (S, \mathsf{id}_S)
\end{tikzcd}$$

in $\overline{\mathcal{A}}/S$. Also observe that

$$\big((S \times_R S) \times_S (S \times_R S), p\big) \cong \big(S \times_R S \times_R S, p_2\big).$$

By lemma 5.1.18, the last pullback in $\overline{\mathcal{A}}/S$ lies in fact in $\mathsf{Split}_S(S)$. By lemma 5.1.14, we get therefore a pullback in $\overline{\mathcal{P}}/\mathcal{S}(S)$

$$\begin{tikzcd}
\mathcal{S}_S\big((S \times_R S) \times_S (S \times_R S), p\big) \arrow[r] \arrow[d] & \mathcal{S}_S(S \times_R S, p_1) \arrow[d, "\mathcal{S}_S(p_1)"] \\
\mathcal{S}_S(S \times_R S, p_2) \arrow[r, "\mathcal{S}_S(p_2)"'] & \mathcal{S}_S(S, \mathsf{id}_S)
\end{tikzcd}$$

Since pullbacks in $\overline{\mathcal{P}}/\mathcal{S}(S)$ are computed as in $\mathcal{P}$, we get the pullback in $\mathcal{P}$

$$\begin{tikzcd}
\mathcal{S}\big((S \times_R S) \times_S (S \times_R S)\big) \arrow[r] \arrow[d] & \mathcal{S}(S \times_R S) \arrow[d, "\mathcal{S}(p_1)"] \\
\mathcal{S}(S \times_R S) \arrow[r, "\mathcal{S}(p_2)"'] & \mathcal{S}(S)
\end{tikzcd}$$

which indicates that applying the functor $\mathcal{S}$ to the kernel pair of σ yields the following situation in $\mathcal{P}$:

$$\begin{tikzcd}[column sep=large] \mathcal{S}(S\times_R S)\times_{\mathcal{S}(S)}\mathcal{S}(S\times_R S) \arrow[r, "{(\mathcal{S}(p_1),\mathcal{S}(p_4))}"] & \mathcal{S}(S\times_R S) \arrow[loop below, "\mathcal{S}(\tau)"] \arrow[r, shift left=2, "\mathcal{S}(p_1)"] \arrow[r, shift right=2, "\mathcal{S}(p_2)"'] & \mathcal{S}(S). \arrow[l, "\mathcal{S}(\Delta)"' description] \end{tikzcd}$$

A perfectly analogous argument applies to prove that

$$\begin{aligned} &\mathcal{S}\big((S\times_R S)\times_S (S\times_R S)\times_S (S\times_R S)\big) \\ &\qquad\cong \mathcal{S}(S\times_R S)\times_{\mathcal{S}(S)}\mathcal{S}(S\times_R S)\times_{\mathcal{S}(S)}\mathcal{S}(S\times_R S). \end{aligned}$$

This object is useful to describe the associativity of the composition in the expected groupoid.

The axioms for an internal groupoid are expressed by the commutativity of some diagrams involving the previous pullbacks. We have just seen that these pullbacks are the images under $\mathcal{S}$ of the corresponding pullbacks defining the kernel pair of σ, seen as a groupoid. Since $\mathcal{S}$, like every functor, preserves the commutativity of diagrams, the proof is done. □

Definition 5.1.23 Let $\mathcal{S}\dashv\mathcal{C}\colon(\mathcal{A},\overline{\mathcal{A}})\overset{\longleftarrow}{\longrightarrow}(\mathcal{P},\overline{\mathcal{P}})$ be a relatively admissible adjunction (notation of definition 5.1.3) and $\sigma\colon S\longrightarrow R$ a morphism of $\mathcal{A}$ which is of relative Galois descent with respect to those data. The Galois groupoid $\mathsf{Gal}\,[\sigma]$ of σ is the internal groupoid in $\mathcal{P}$

$$\begin{tikzcd}[column sep=large] \mathcal{S}(S\times_R S)\times_{\mathcal{S}(S)}\mathcal{S}(S\times_R S) \arrow[r, "{(\mathcal{S}(p_1),\mathcal{S}(p_4))}"] & \mathcal{S}(S\times_R S) \arrow[loop below, "\mathcal{S}(\tau)"] \arrow[r, shift left=2, "\mathcal{S}(p_1)"] \arrow[r, shift right=2, "\mathcal{S}(p_2)"'] & \mathcal{S}(S) \arrow[l, "\mathcal{S}(\Delta)"' description] \end{tikzcd}$$

given by lemma 5.1.22.

Theorem 5.1.24 (Galois theorem) *Let $\mathcal{S}\dashv\mathcal{C}\colon(\mathcal{A},\overline{\mathcal{A}})\overset{\longleftarrow}{\longrightarrow}(\mathcal{P},\overline{\mathcal{P}})$ be a relatively admissible adjunction (notation of definition 5.1.3) and $\sigma\colon S\longrightarrow R$ a morphism of $\mathcal{A}$ which is of relative Galois descent with respect to these data. In these conditions, there exists an equivalence of categories*

$$\mathsf{Split}_R(\sigma)\approx\overline{\mathcal{P}}^{\mathsf{Gal}\,[\sigma]}$$

between the category of those objects $(A,a)\in\overline{\mathcal{A}}/R$ which are split by σ and the category of internal covariant presheaves (P,p,π) on the internal groupoid $\mathsf{Gal}\,[\sigma]$ in $\mathcal{P}$, in which $p\in\overline{\mathcal{P}}$.

$$\begin{tikzcd}
(S \times_R S) \times_S A \arrow[r, "a_2"] \arrow[d, "a_1"'] & A \arrow[d, "a"] \\
S \times_R S \arrow[r, "p_1"] \arrow[d, "p_2"'] & S \arrow[d, "\sigma"] \\
S \arrow[r, "\sigma"'] & R
\end{tikzcd}$$

Diagram 5.2

Proof The category $\mathcal{P}^{\mathsf{Gal}\,[\sigma]}$ is the category of algebras for the monad on $\overline{\mathcal{P}}/\mathcal{S}(S)$ described in proposition 4.6.1. We shall prove that $\mathsf{Split}_R(\sigma)$ is also monadic on $\overline{\mathcal{P}}/\mathcal{S}(S)$, for a monad which is isomorphic to that given by proposition 4.6.1. This will yield the expected result.

By corollary 5.1.21, the functor

$$\mathsf{Split}_R(\sigma) \longrightarrow \overline{\mathcal{P}}/\mathcal{S}(S), \quad (A,a) \mapsto \mathcal{S}_S\sigma^*(A,a)$$

is monadic; its left adjoint is $\Sigma_\sigma \circ \mathcal{C}_S$. Observe that the functorial part of the corresponding monad is thus

$$T\colon \overline{\mathcal{P}}/\mathcal{S}(S) \longrightarrow \overline{\mathcal{P}}/\mathcal{S}(S), \quad (X,\varphi) \mapsto \mathcal{S}_S\sigma^*\Sigma_\sigma\mathcal{C}_S(X,\varphi).$$

Let us show that this monad coincides with that given by proposition 4.6.1.

Putting $(A,a) = \mathcal{C}_S(X,\varphi)$, we must first consider pullbacks in $\mathcal{A}$ as in diagram 5.2, which show that

$$\big((S \times_R S) \times_S A, p_2 \circ a_1\big) = \sigma^*\Sigma_\sigma(A,a).$$

The objects $(S \times_R S, p_1)$ and (A,a) of $\overline{\mathcal{A}}/S$ are split by id_S, by lemmas 5.1.18 and 5.1.12 respectively. Thus by lemma 5.1.19, their product

$$(S \times_R S, p_1) \times (A,a) = \big((S \times_R S) \times_S A, p_1 \circ a\big)$$

in $\overline{\mathcal{A}}/S$ is split by id_S as well. By lemma 5.1.14, this product is transformed by $\mathcal{S}_S$ into a product in $\overline{\mathcal{P}}/\mathcal{S}(S)$. Since this product in $\overline{\mathcal{P}}/\mathcal{S}(S)$ corresponds to a pullback in $\overline{\mathcal{P}}$, we get diagram 5.3, where the square is a pullback in $\overline{\mathcal{P}}$. Notice that indeed $\mathcal{S}(A) \cong X$ because $\mathcal{S}_S(A,a) = \mathcal{S}_S\mathcal{C}_S(X,\varphi) \cong (X,\varphi)$. This diagram shows that

$$\begin{tikzcd}
\mathcal{S}\big((S \times_R S) \times_S A\big) \arrow[r] \arrow[d, "\mathcal{S}(a_1)"'] & \mathcal{S}(A) \cong X \arrow[d, "\mathcal{S}(a) = \varphi"] \\
\mathcal{S}(S \times_R S) \arrow[r, "\mathcal{S}(p_1)"'] \arrow[d, "\mathcal{S}(p_2)"'] & \mathcal{S}(S) \\
\mathcal{S}(S) &
\end{tikzcd}$$

Diagram 5.3

$$\begin{aligned}
\mathcal{S}_S\sigma^*\Sigma_\sigma\mathcal{C}_S(X,\varphi) &\cong \mathcal{S}_S\big((S \times_R S) \times_S A, p_2 \circ a_1\big) \\
&\cong \Big(\mathcal{S}\big((S \times_R S) \times_S A\big), \mathcal{S}(p_2) \circ \mathcal{S}(a_1)\Big) \\
&\cong \big(\Sigma_{\mathcal{S}(p_2)} \circ \mathcal{S}(p_1)^*\big)\,(X,\varphi).
\end{aligned}$$

This shows that the functor T of the present proof coincides on the objects with the functorial part of the monad given by proposition 4.6.1.

This is the difficult part of the proof. Completing the proof that the two monads we have indicated are isomorphic is now routine left to the reader. □

5.2 Central extensions of groups

This section is devoted to applying the general Galois theory of Janelidze (see section 5.1) to recapture various results on central extensions of groups. All groups are written multiplicatively.

We shall apply the results of section 5.1 to the following adjunction, which will be shown to be relatively admissible:

$$(\mathsf{Gr}, \overline{\mathsf{Gr}}) \underset{\mathsf{ab}}{\overset{i}{\leftrightarrows}} (\mathsf{Ab}, \overline{\mathsf{Ab}}), \quad \mathsf{ab} \dashv i$$

where

- Gr is the category of groups,

- Ab is the category of abelian groups,
- i is the canonical inclusion,
- ab is the abelianization functor,
- $\overline{\mathsf{Gr}} \subseteq \mathsf{Gr}$ is the class of surjective homomorphisms,
- $\overline{\mathsf{Ab}} \subseteq \mathsf{Ab}$ is the class of surjective homomorphisms.

We first recall some elementary results on the abelianization of a group.

Definition 5.2.1 Let $H \subseteq G$ be a subgroup of a group G. The group of H-commutators of G is the subgroup of G generated by the elements of the form

$$xyx^{-1}y^{-1} \quad \text{with} \quad x \in H, \quad y \in G,$$

which are called the "elementary commutators" This group of commutators is denoted by $[H, G]$.

Lemma 5.2.2 *When $H \subseteq G$ is a normal subgroup, the subgroup of commutators $[H, G] \subseteq G$ is normal as well. Moreover, in the quotient $G/[H, G]$,*

$$x \in H \quad \textit{and} \quad y \in G \Rightarrow [x][y] = [y][x].$$

Proof An element of $[H, G]$ is a finite composite of elements of one of the two forms,

$$xyx^{-1}y^{-1} \quad \text{with} \quad x \in H, \quad y \in G,$$

or the inverse

$$xyx^{-1}y^{-1} \quad \text{with} \quad x \in G, \quad y \in H$$

of such an element. Observe that for all elements $x, y, z \in G$,

$$z(xyx^{-1}y^{-1})z^{-1} = (zxz^{-1})(zyz^{-1})(zxz^{-1})^{-1}(zyz^{-1})^{-1},$$

which suffices to prove that $[H, G]$ is stable for the operation $u \mapsto zuz^{-1}$. Thus $[H, G]$ is a normal subgroup.

Finally given $x \in H$ and $y \in G$, one has $[x][y][x]^{-1}[y]^{-1} = [1]$ in $G/[H, G]$, from which we obtain the last part of the statement. □

Proposition 5.2.3 *The left adjoint functor of the inclusion functor $i \colon \mathsf{Ab} \longrightarrow \mathsf{Gr}$ is given by*

$$\mathsf{Ab} \colon \mathsf{Gr} \longrightarrow \mathsf{Ab}, \quad G \mapsto G/[G, G].$$

Proof Every group G is normal in itself, from which $G/[G,G]$ is an abelian group by lemma 5.2.2. This yields a surjective group homomorphism

$$\eta_G \colon G \longrightarrow G/[G,G], \quad x \mapsto [x].$$

If $f \colon G \longrightarrow A$ is a group homomorphism, with A an abelian group, for all elements $x, y \in G$

$$f\big(xyx^{-1}y^{-1}\big) = f(x)f(y)f(x)^{-1}f(y)^{-1} = 1$$

since A is abelian. Thus f factors through the quotient $G/[G,G]$ and this factorization is unique, since η_G is surjective. □

Proposition 5.2.4 *The adjunction* $\mathsf{ab} \dashv i \colon \big(\mathsf{Gr}, \overline{\mathsf{Gr}}\big) \underset{\longrightarrow}{\overset{\longleftarrow}{}} \big(\mathsf{Ab}, \overline{\mathsf{Ab}}\big)$, *with* $\overline{\mathsf{Gr}}$ *and* $\overline{\mathsf{Ab}}$ *the classes of surjective homomorphisms, is relatively admissible in the sense of definition 5.1.3.*

Proof In Ab or Gr, a composite or a pullback of surjections is again a surjection. Obviously, the canonical inclusion $i \colon \mathsf{Ab} \longrightarrow \mathsf{Gr}$ preserves surjections and the unit of the adjunction $\eta_G \colon G \longrightarrow G/[G,G]$ is surjective at each object $G \in \mathsf{Gr}$.

Since i is full and faithful, the counit of the adjunction is an isomorphism, thus certainly a surjection at each object $A \in \mathsf{Ab}$. More directly, when A is abelian, $[A,A] = \{1\}$ and thus $A/[A,A] \cong A$.

It remains to prove that the functor ab respects surjections. Given a group homomorphism $f \colon G_1 \longrightarrow G_2$, the commutative diagram

$$\begin{tikzcd}
G_1 \arrow[r, two heads, "\eta_{G_1}"] \arrow[d, "f"'] & G_1/[G_1,G_1] \arrow[d, "\mathsf{ab}(f)"] \\
G_2 \arrow[r, two heads, "\eta_{G_2}"'] & G_2/[G_2,G_2]
\end{tikzcd}$$

shows at once that when f is surjective, $\mathsf{ab}(f)$ is surjective as well. □

Proposition 5.2.5 *For every group G, the counit of the adjunction*

$$\overline{\mathsf{Gr}}/G \underset{\mathsf{ab}_G}{\overset{i_G}{\underset{\longrightarrow}{\longleftarrow}}} \overline{\mathsf{Ab}}/\mathsf{ab}(G)$$

is an isomorphism.

Proof Let us consider a surjection $\varphi\colon A \longrightarrow \mathsf{ab}(G)$ in the category of abelian groups and the corresponding pullback

$$\begin{tikzcd} A' \arrow[r, two heads, "\varphi''"] \arrow[d, two heads, "\varphi'"'] & A \arrow[d, two heads, "\varphi"] \\ G \arrow[r, two heads, "\eta_G"'] & \mathsf{ab}(G) \end{tikzcd}$$

We thus have $i_G(A, \varphi) = (A', \varphi')$ and it remains to prove that

$$\Big(\mathsf{ab}(A') \xrightarrow{\mathsf{ab}(\varphi')} \mathsf{ab}(G)\Big) \cong \Big(A \xrightarrow{\ \varphi\ } \mathsf{ab}(G)\Big).$$

Observe that since η_G and φ are surjective, so are φ'' and φ'. Considering the factorization φ''' obtained from proposition 5.2.4

$$\begin{tikzcd} A' \arrow[r, two heads, "\eta_{A'}"] \arrow[dr, two heads, "\varphi''"'] & A'/[A', A'] \arrow[d, dotted, "\varphi'''"] \\ & A \end{tikzcd}$$

we deduce that φ''' is surjective as well.

In fact, φ''' is also injective. To prove this, consider $a' \in A'$ and the corresponding element $[a'] \in A'/[A', A']$; assume that $\varphi'''\big([a']\big) = 1$. The element $a' \in A'$ is thus a pair $a' = (x, a)$ with $x \in G$, $a \in A$ and $\varphi(a) = [x] \in G/[G, G]$. This yields

$$a = \varphi''(x, a) = (\varphi''' \circ \eta_{A'})(x, a) = \varphi'''\big([a']\big) = 1.$$

We must prove that $a' = (x, a) = (x, 1) \in [A', A']$. Since

$$\eta_G(x) = (\eta_G \circ \varphi')(x, a) = (\varphi \circ \varphi'')(x, a) = \varphi(1) = 1,$$

one has $x \in [G, G]$. By definition 5.2.1, this implies

$$x = x_1 y_1 x_1^{-1} y_1^{-1} \dots x_n y_n x_n^{-1} y_n^{-1}$$

with $x_i, y_i \in G$. Since φ is surjective, for each index $i = 1, \dots, n$ let us choose

$$a_i \in A \text{ with } \varphi(a_i) = [x_i], \quad b_i \in A \text{ with } \varphi(b_i) = [y_i]$$

so that all (x_i, a_i) and (y_i, b_i) are in A'. Since the group A is abelian

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1.$$

It follows that

$$(x,1) = (x_1,a_1)(y_1,b_1)(x_1^{-1},a_1^{-1})(y_1^{-1},b_1^{-1}) \cdots \\ \cdots (x_n,a_n)(y_n,b_n)(x_n^{-1},a_n^{-1})(y_n^{-1},b_n^{-1})$$

and this last composite is in $[A', A']$ by definition 5.2.1. Thus $a' = (x, 1) \in [A', A']$. This proves that φ''' is injective, and thus an isomorphism.

It remains to notice that $\varphi \circ \varphi''' = \mathsf{ab}(\varphi')$. But

$$\varphi \circ \varphi''' \circ \eta_{A'} = \varphi \circ \varphi'' = \eta_G \circ \varphi' = \mathsf{ab}(\varphi') \circ \eta_{A'}$$

by naturality of η. Thus $\varphi \circ \varphi''' = \mathsf{ab}(\varphi')$ since $\eta_{A'}$ is surjective. □

Proposition 5.2.6 *Every surjection in* Gr *is an effective descent morphism relatively to the class of surjective homomorphisms.*

Proof Let $\sigma \colon S \longrightarrow R$ be a surjective homomorphism of groups. We shall apply the Beck criterion (see [66]).

The functors σ^* and Σ_σ restrict to functors

$$\overline{\mathsf{Gr}} \underset{\sigma^*}{\overset{\Sigma_\sigma}{\leftrightarrows}} \overline{\mathsf{Gr}}/S$$

from which σ^* has a left adjoint.

This functor σ^* reflects isomorphisms since, by lemma 4.4.6, this is the case for the functor

$$\sigma^* \colon \mathsf{Gr}/R \longrightarrow \mathsf{Gr}/S.$$

Finally the categories $\overline{\mathsf{Gr}}/R$ and $\overline{\mathsf{Gr}}/S$ have coequalizers computed as in Gr/R and Gr/S, that is, finally, computed as in Gr. Thus the Beck condition on split coequalizers is valid for $\sigma^* \colon \overline{\mathsf{Gr}}/R \longrightarrow \overline{\mathsf{Gr}}/S$ since it is valid for $\sigma^* \colon \mathsf{Gr}/R \longrightarrow \mathsf{Gr}/S$. □

Lemma 5.2.7 *Let* $\sigma \colon S \longrightarrow R$ *be a surjective homomorphism of groups. Consider the relatively admissible adjunction*

$$(\mathsf{Gr}, \overline{\mathsf{Gr}}) \underset{\mathsf{ab}}{\overset{i}{\leftrightarrows}} (\mathsf{Ab}, \overline{\mathsf{Ab}}).$$

Given $(H, h) \in \overline{\mathsf{Gr}}/R$*, we consider the following pullback in* Gr*:*

$$\begin{tikzcd} S \times_R H \arrow[r, two heads, "p_2"] \arrow[d, two heads, "p_1"'] & H \arrow[d, two heads, "h"] \\ S \arrow[r, two heads, "\sigma"'] & R \end{tikzcd} \qquad \sigma^*(H,h) = (S \times_R H, p_1)$$

The following conditions are then equivalent:

(i) $(H,h) \in \overline{\mathsf{Gr}}/R$ *is split by* σ*;*

(ii) *the following diagram is a pullback –*

$$\begin{tikzcd} S \times_R H \arrow[r, "\eta_{S \times_R H}"] \arrow[d, "p_1"'] & \mathsf{ab}(S \times_R H) \arrow[d, "\mathsf{ab}(p_1)"] \\ S \arrow[r, "\eta_S"'] & \mathsf{ab}(S) \end{tikzcd}$$

(iii) $p_1 \colon [S \times_R H, S \times_R H] \longrightarrow [S,S]$ *is an isomorphism;*

(iv) $p_1 \colon [S \times_R H, S \times_R H] \longrightarrow [S,S]$ *is injective.*

Proof Let us first compute explicitly condition (i). One considers the pullback

$$\begin{tikzcd} H' \arrow[r] \arrow[d, "h'"'] & \mathsf{ab}(S \times_R H) \arrow[d, "\mathsf{ab}(p_1)"] \\ S \arrow[r, two heads, "\eta_S"'] & \mathsf{ab}(S) \end{tikzcd} \qquad (H',h') = i_S \mathsf{ab}_S \sigma^*(H,h)$$

Condition (i) means $(H',h') \cong (S \times_R H, p_1)$, that is condition (ii).

Let us next prove that the map

$$p_1 \colon [S \times_R H, S \times_R H] \longrightarrow [S,S]$$

is always surjective, which will prove (iii) $\Leftrightarrow$ (iv). Notice that obviously, the image of an element of the form $xyx^{-1}y^{-1}$ is again an element of that form, proving that p_1 indeed restricts to the groups of commutators. Let us now choose $s,t \in S$ and prove that $sts^{-1}t^{-1}$ has the form $p_1(u)$ for some commutator u; by definition 5.2.1, this is enough to prove the required surjectivity. By surjectivity of h, we choose $x,y \in H$ such that

$h(x) = \sigma(s)$, $h(y) = \sigma(t)$; this yields $(s,x) \in S\times_R H$ and $(t,y) \in S\times_R H$. Thus

$$(s,x)(t,y)(s^{-1},x^{-1})(t^{-1},y^{-1}) \in [S \times_R H, S \times_R H]$$

that is

$$(sts^{-1}t^{-1}, xyx^{-1}y^{-1}) \in [S \times_R H, S \times_R H].$$

This element is of course mapped by p_1 onto $sts^{-1}t^{-1}$.

It remains to prove (ii) $\Leftrightarrow$ (iv). For this consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & {[S \times_R H, S \times_R H]} \arrow[r, tail] \arrow[d, two heads, "p_1"] & S \times_R H \arrow[r, two heads, "\eta_{S\times_R H}"] \arrow[d, two heads, "p_1"] & \mathsf{ab}(S \times_R H) \arrow[r] \arrow[d, two heads, "\mathsf{ab}(p_1)"] & 0 \\
0 \arrow[r] & {[S,S]} \arrow[r, tail] & S \arrow[r, two heads, "\eta_S"'] & \mathsf{ab}(S) \arrow[r] & 0
\end{tikzcd}$$

Since h is surjective, p_1 is surjective as well. Since $\mathsf{ab} \dashv i$, ab preserves epimorphisms and $\mathsf{ab}(p_1)$ is also surjective. The three vertical morphisms in the above diagram are thus surjections.

Suppose first condition (ii), that is, the right hand square is a pullback. We must prove that

$$\forall u \in [S \times_R H, S \times_R H] \quad p_1(u) = 1 \Rightarrow u = 1.$$

We have

$$u = (s,x), \quad s \in S, \quad x \in H, \quad \sigma(s) = h(x).$$

From $p_1(u) = 1$ we get $s = 1$ and from $u \in [S \times_R H, S \times_R H]$ we deduce $\eta_{S\times_R H}(u) = 1$. Since the right hand square is a pullback, we have thus $u = (1,1)$, which proves the injectivity of p_1 on the groups of commutators.

Conversely assume (iii), or equivalently (iv). Consider diagram 5.4, which is commutative and where the square is a pullback. We must prove that the factorization θ is an isomorphism.

To prove the injectivity of θ, consider $(s,x) \in S \times_R H$, thus $s \in S$, $x \in H$ with $\sigma(s) = h(x)$. Suppose $\theta(s,x) = 1$: we must deduce $(s,x) = (1,1)$. One immediately gets

$$s = p_1(s,x) = (\pi_1 \circ \theta)(s,x) = \pi_1(1) = 1,$$

$$\eta_{S\times_R H}(s,x) = (\pi_2 \circ \theta)(s,x) = \pi_2(1) = 1.$$

$$\begin{tikzcd}
S \times_R H \arrow[dr, dotted, "\theta"] \arrow[drr, "\eta_{S \times_R H}"] \arrow[ddr, "p_1"'] & & \\
& P \arrow[r, two heads, "\pi_2"'] \arrow[d, "\pi_1"] & \mathsf{ab}(S \times_R H) \arrow[d, "\mathsf{ab}(p_1)"] \\
& S \arrow[r, two heads, "\eta_S"'] & \mathsf{ab}(S)
\end{tikzcd}$$

Diagram 5.4

This proves $(s,x) \in [S \times_R H, S \times_R H]$ with $p_1(s,x) = 1$. Thus $(s,x) = (1,1)$ by condition (iv).

To prove the surjectivity of θ, choose $\big(s,[t,x]\big) \in P$, thus $s \in S$, $t \in S$, $x \in H$ with $\sigma(t) = h(x)$ and $[s] = [t]$. This yields $[t^{-1}s] = [1]$, that is $t^{-1}s \in [S,S]$. By condition (iii), there exists a unique $y \in H$ such that $(t^{-1}s, y) \in [S \times_R H, S \times_R H]$. In particular, $\sigma(t^{-1}s) = h(y)$ and $[t^{-1}s, y] = [1,1]$. Therefore

$$[s,xy] = [tt^{-1}s, xy] = [t,x][t^{-1}s,y] = [t,x][1,1] = [t,x].$$

Finally,

$$\pi_1\theta(s,xy) = p_1(s,xy) = s, \quad \pi_2\theta(s,xy) = [s,xy] = [t,x],$$

that is $\theta(s,xy) = \big(s,[t,x]\big)$ and θ is surjective. □

Definition 5.2.8

(i) An extension of a group R is an exact sequence of groups

$$0 \longrightarrow K \overset{k}{\rightarrowtail} S \overset{\sigma}{\twoheadrightarrow} R \longrightarrow 0.$$

(ii) This extension of the group R is central when K is contained in the centre $Z(S)$ of S, where as usual

$$Z(S) = \{x \in S \mid \forall y \in S \;\; xy = ys\}.$$

(iii) This central extension of the group R is weakly universal when, for every other central extension (k', σ') of R, there exists a factorization φ making the following diagram commutative:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, dotted, "\varphi"'] & S \arrow[r, two heads, "\sigma"] \arrow[d, dotted, "\varphi"'] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K' \arrow[r, tail, "k'"'] & S' \arrow[r, two heads, "\sigma'"'] & R \arrow[r] & 0
\end{tikzcd}
$$

(iv) The central extension of the group R is universal when it is weakly universal and, in the previous condition, the factorization φ is unique.

Proposition 5.2.9 *Consider a weakly universal central extension of the group R;*

$$0 \longrightarrow K \overset{k}{\rightarrowtail} S \overset{\sigma}{\twoheadrightarrow} R \longrightarrow 0.$$

For a surjective homomorphism $h\colon H \twoheadrightarrow R$ of groups, the following conditions are equivalent:

(i) *the object $(H, h) \in \overline{\mathsf{Gr}}/R$ is split by σ;*
(ii) *$(\mathsf{Ker}\, h, h)$ is a central extension of R.*

Proof (i) $\Rightarrow$ (ii) is a more general fact, which does not require any centrality assumption on σ. We must prove that $\mathsf{Ker}\, h \subseteq Z(H)$, that is

$$\forall x \in H \quad \forall y \in H \qquad h(x) = 1 \Rightarrow xy = yx.$$

Since σ is surjective, choose $s \in S$ such that $\sigma(s) = h(y)$, from which $(s, y) \in S \times_R H$. One also has $h(x) = 1 = \sigma(1)$, thus $(1, x) \in S \times_R H$. Therefore

$$(s, y)(1, x)(s^{-1}, y^{-1})(1, x^{-1}) = (1, yxy^{-1}x^{-1}) \in [S \times_R H, S \times_R H].$$

Condition (iv) of lemma 5.2.7 yields $yxy^{-1}x^{-1} = 1$, that is $yx = xy$.

Let us prove now (ii) $\Rightarrow$ (i). We use condition (iv) of lemma 5.2.7. Since the central extension (k, σ) is weakly universal, we get a factorization φ making commutative the following diagram, where i denotes the canonical inclusion:

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "\varphi"'] & S \arrow[r, two heads, "\sigma"] \arrow[d, "\varphi"'] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Ker}\, h \arrow[r, tail, "i"'] & H \arrow[r, two heads, "h"'] & R \arrow[r] & 0
\end{tikzcd}
$$

Given $s \in S$, one has $h\varphi(s) = \sigma(s)$, thus $\bigl(s, \varphi(s)\bigr) \in S \times_R H$. This yields a group homomorphism $S \longrightarrow S \times_R H$, which thus restricts through the corresponding groups of commutators

$$\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} : [S, S] \longrightarrow [S \times_R H, S \times_R H], \quad s \mapsto \bigl(s, \varphi(s)\bigr).$$

We consider now the composite

$$[S \times_R H, S \times_R H] \xrightarrow{p_1} [S, S] \xrightarrow{\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix}} [S \times_R H, S \times_R H];$$

proving it is the identity will in particular imply that p_1 is injective as required. It clearly suffices to prove that the composite is the identity on each elementary commutator. Consider thus (s, x) and (t, y) in $S \times_R H$:

$$s, t \in S, \quad x, y \in H, \quad \sigma(s) = h(x), \quad \sigma(t) = h(y).$$

One gets

$$\begin{aligned}
&\left(\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} \circ p_1\right) \bigl((s,x)(t,y)(s^{-1},x^{-1})(t^{-1},y^{-1})\bigr) \\
&\quad = \left(\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} \circ p_1\right) (sts^{-1}t^{-1}, xyx^{-1}y^{-1}) \\
&\quad = \begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} (sts^{-1}t^{-1}) \\
&\quad = \bigl(sts^{-1}t^{-1}, \varphi(sts^{-1}t^{-1})\bigr) \\
&\quad = \bigl(sts^{-1}t^{-1}, \varphi(s)\varphi(t)\varphi(s)^{-1}\varphi(t)^{-1}\bigr).
\end{aligned}$$

It suffices now to prove that

$$\varphi(s)\varphi(t)\varphi(s)^{-1}\varphi(t)^{-1} = xyx^{-1}y^{-1}.$$

The equalities $h(x) = \sigma(s) = (h \circ \varphi)(s)$ imply $h\bigl(\varphi(s) \cdot x^{-1}\bigr) = 1$, thus $\varphi(s) \cdot x^{-1} \in \mathsf{Ker}\, h$. Putting $u = \varphi(s) \cdot x^{-1} \in \mathsf{Ker}\, h$, we have thus $\varphi(s) = ux$ with $u \in \mathsf{Ker}\, h$. In the same way, $\varphi(t) = vy$ with $v \in \mathsf{Ker}\, h$. By assumption, $\mathsf{Ker}\, h \subseteq Z(H)$; therefore

$$\varphi(s)\varphi(t)\varphi(s)^{-1}\varphi(t)^{-1} = ux \cdot uy \cdot x^{-1}u^{-1} \cdot y^{-1}v^{-1} = xyx^{-1}y^{-1}.$$

This completes the proof that $\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} \circ p_1$ maps every elementary commutator onto itself. $\square$

$$\begin{tikzcd}
A' \arrow[r, two heads, "f''"] \arrow[d, two heads, "f'"'] & A \arrow[d, two heads, "f"] \\
S \arrow[r, two heads, "\eta_S"'] \arrow[d, two heads, "\sigma"'] & S/[S,S] \\
R &
\end{tikzcd}$$

Diagram 5.5

Proposition 5.2.10 *With the notation of 5.2.8, if* (k, σ) *is a weakly universal central extension of groups, then the morphism* $\sigma\colon S \twoheadrightarrow R$ *is of Galois descent with respect to the relatively admissible adjunction*

$$(\mathsf{Gr}, \overline{\mathsf{Gr}}) \underset{\mathsf{ab}}{\overset{i}{\rightleftarrows}} (\mathsf{Ab}, \overline{\mathsf{Ab}}).$$

Proof By propositions 5.2.4, 5.2.5, 5.2.6, it remains to check condition (iii) of definition 5.1.8. For this consider a surjective homomorphism $f\colon A \twoheadrightarrow S/[S,S]$ in the category of abelian groups. The object $(\Sigma_\sigma \circ i_S)(A, f)$ is the left vertical composite in diagram 5.5, where the square is a pullback: We must prove that this composite is split by σ, that is, by proposition 5.2.9, that $\big(\sigma \circ f', \mathsf{Ker}\,(\sigma \circ f')\big)$ is a central extension. Observe that the surjectivitiy of f implies that of f'. It remains thus to check that the kernel of $\sigma \circ f'$ is contained in the centre of A'.

Let $a \in A'$ such that $(\sigma \circ f')(a) = 1$. Thus

$$a' = (s, a), \quad s \in S, \quad a \in A, \quad [s] = f(a), \quad \sigma(s) = 1.$$

Consider an arbitrary element $a'' \in A'$, that is

$$a'' = (t, b), \quad t \in S, \quad b \in A, \quad [t] = f(b).$$

We must prove $a'a'' = a''a'$, that is $(st, ab) = (ts, ba)$. Since A is abelian, this reduces to proving $st = ts$. But $s \in \mathsf{Ker}\,\sigma$ by choice of a' and $\mathsf{Ker}\,\sigma \subseteq Z(S)$ by centrality; thus $st = ts$. □

Theorem 5.2.11 *Given a weakly universal central extension of groups*

$$0 \longrightarrow K \overset{k}{\rightarrowtail} S \overset{\sigma}{\twoheadrightarrow} R \longrightarrow 0$$

(i) *the kernel pair of σ, seen as an internal groupoid in the category of groups, is transformed by the functor* $\mathsf{Ab}\colon \mathsf{Gr} \longrightarrow \mathsf{Ab}$ *into an internal groupoid* $\mathsf{Gal}\,[\sigma]$ *in the category of abelian groups,*

(ii) *the category of central extensions of the group R is equivalent to the category of internal covariant presheaves (P, p, π), with p surjective, on the internal groupoid* $\mathsf{Gal}\,[\sigma]$*, in the category of abelian groups.*

Proof The category of extensions of the group R has for morphisms the morphisms φ as in condition (iii) of definition 5.2.8. With the notation of definition 5.2.8, k is entirely determined by σ, since $k = \mathsf{Ker}\,\sigma$. Thus the category of extensions of R is equivalent to the category $\overline{\mathsf{Gr}}/R$. The result then follows at once from theorem 5.1.24 and proposition 5.2.9. □

Definition 5.2.12 A group G is perfect when $G = [G, G]$.

A group is thus perfect when it is "perfectly non-abelian": indeed, the abelian group $G/[G, G]$ associated with it is the trivial group.

Let us mention now a standard property of groups, expressed here in more categorical terms. We recall that in a category with binary products and terminal object 1, an internal group consists in giving an object G and three morphisms

$$\mu\colon G \times G \longrightarrow G,$$
$$\upsilon\colon 1 \longrightarrow G,$$
$$\iota\colon G \longrightarrow G$$

called the "multiplication" μ, the "unit" υ and the "inverse" ι, which satisfy diagrammatically the usual group axioms (see [8], volume 2). In other words, an internal group is an internal groupoid whose object of objects is the terminal object.

Lemma 5.2.13 *Consider a group G, written multiplicatively. Suppose G is provided with the structure of an internal group in the category of groups:*

$$\mu\colon G \times G \longrightarrow G,$$

$$\upsilon \colon 1 \longrightarrow G,$$
$$\iota \colon G \longrightarrow G$$

with thus μ the multiplication, υ the unit and ι the inverse. In these conditions, the two group structures on G coincide and are commutative.

Proof Since ι is a group homomorphism for the original group structure on G, the two units coincide and are thus written as 1. Since μ is a group homomorphism for the original group structure on G, for all elements $a, b, c, d \in G$ one has

$$\mu(a \cdot c, b \cdot d) = \mu\big((a, b) \cdot (c, d)\big) = \mu(a, b) \cdot \mu(c, d).$$

Putting $b = 1 = c$ yields then $\mu(a, d) = a \cdot d$ while putting $a = 1 = d$ yields $\mu(c, b) = b \cdot c$. □

Proposition 5.2.14 *Consider a weakly universal central extension of groups*

$$0 \longrightarrow K \rightarrowtail^{k} S \xrightarrow{\sigma} \!\!\!\!\twoheadrightarrow R \longrightarrow 0.$$

When S is a perfect group

(i) *R is a perfect group,*

(ii) *the Galois groupoid $\mathsf{Gal}\,[\sigma]$ of theorem 5.2.11 is an abelian group isomorphic to K,*

(iii) *the central extension $(\mathsf{Ker}\,\sigma, \sigma)$ is universal.*

Proof Since σ is surjective, every element $r \in R$ can be written $r = \sigma(s)$ with $s \in S$. Since S is perfect, s can be written as a product of elements of the form $xyx^{-1}y^{-1}$. Thus r is a product of elements of the form $\sigma(x)\sigma(y)\sigma(x^{-1})\sigma(y^{-1})$, proving $r \in [R, R]$. Therefore $R = [R, R]$ and R is perfect.

Again since S is perfect, the "object of objects" of the groupoid $\mathsf{Gal}\,[\sigma]$ is the abelian group $\mathsf{ab}(S) = S/[S, S] \cong \{1\}$. Therefore the groupoid $\mathsf{Gal}\,[\sigma]$ is an internal group in Ab, thus by lemma 5.2.13, is an abelian group whose structure coincides with the internal group structure of $\mathsf{Gal}\,[\sigma]$.

By the condition (ii) of lemma 5.2.7, the diagram

$$\begin{tikzcd} S \times_R S \arrow[r, "\eta_{S\times_R S}"] \arrow[d, "p_1"'] & \mathsf{ab}(S \times_R S) = \mathsf{Gal}\,[\sigma] \arrow[d, "\mathsf{ab}(p_1)"] \\ S \arrow[r, "\eta_S"'] & \mathsf{ab}(S) = \{1\} \end{tikzcd}$$

is a pullback, and so $\mathsf{Gal}\,[\sigma] \cong \mathsf{Ker}\, p_1 \cong \mathsf{Ker}\, \sigma$.

It remains to prove the last assertion of the proposition. Consider for this the situation of condition (iii) in definition 5.2.8; we must prove the uniqueness of φ. Let ψ be another factorization. By assumption, $S = [S, S]$ and, via the proofs of proposition 5.2.9 and lemma 5.2.7, we have reciprocal isomorphisms

$$[S \times_R S, S \times_R S] \underset{p_1}{\overset{\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix}}{\leftrightarrows}} [S, S] = S.$$

The same observation applies to ψ, yielding finally

$$\begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} = p_1^{-1} = \begin{pmatrix} \mathsf{id} \\ \psi \end{pmatrix} : S = [S, S] \longrightarrow [S \times_R S, S \times_R S] \subseteq S \times_R S.$$

Composing with the second projection of the pullback yields

$$\varphi = p_2 \circ \begin{pmatrix} \mathsf{id} \\ \varphi \end{pmatrix} = p_2 \circ \begin{pmatrix} \mathsf{id} \\ \psi \end{pmatrix} = \psi. \qquad \square$$

We now need an existence theorem.

Proposition 5.2.15 *Every group has a weakly universal central extension.*

Proof Every group R is a quotient of a free group F (see [61]), which yields an exact sequence

$$0 \longrightarrow K \rightarrowtail^{k} F \overset{p}{\twoheadrightarrow} R \longrightarrow 0.$$

The group K itself is free, as subgroup of a free group (see [61] again). This last fact will not be needed in the proof. A situation as just exhibited is called a “free resolution of the group” R.

The subgroup K, as a kernel, is a normal subgroup of F. By lemma 5.2.2, $[K, F]$ too is a normal subgroup of F. For every $u \in K = \mathsf{Ker}\, p$

and $v \in F$,

$$p(uvu^{-1}v^{-1}) = p(u)p(v)p(u)^{-1}p(v)^{-1} = 1p(v)1p(v)^{-1} = 1,$$

proving that $[K, F] \subseteq \mathsf{Ker}\, p$. Therefore we get a factorization σ

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, tail, "k"] & F \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] & R \arrow[r] & 0 \\ & & S = F/[K,F] \arrow[ur, dotted, "\sigma"'] & & \end{tikzcd}$$

which is still surjective.

Let us prove now that the extension

$$0 \longrightarrow \mathsf{Ker}\,\sigma \rightarrowtail S \xrightarrow{\sigma} R \longrightarrow 0$$

is central. Choose for this $u, v \in F$ and suppose $\sigma\big([u]\big) = 1$, that is, $p(u) = 1$. We must prove

$$[u][v] = [v][u] \in F/[K, F].$$

But

$$\begin{aligned} u \in \mathsf{Ker}\, p = K &\Rightarrow uvu^{-1}v^{-1} \in [K, F] \\ &\Rightarrow [u][v][u]^{-1}[v]^{-1} = 1 \\ &\Rightarrow [u][v] = [v][u], \end{aligned}$$

which proves the centrality of the extension.

Let us see next that this central extension is weakly universal. Let us choose another central extension

$$0 \longrightarrow \mathsf{Ker}\,h \rightarrowtail H \xrightarrow{h} R \longrightarrow 0$$

of R. We consider diagram 5.6, where $\sigma \circ q \circ k = p \circ k = 0$ induces the existence of the factorization q'. Since F is a free group, it is projective and p factors as $p = h \circ \psi$ through the epimorphism h. Thus $h \circ \psi \circ k = p \circ k = 0$ and we get a factorization ψ' of $\psi \circ k$ through the kernel of h. Given an elementary commutator $uvu^{-1}v^{-1} \in [K, F]$,

$$\psi(uvu^{-1}v^{-1}) = \psi(u)\psi(v)\psi(u^{-1})\psi(v^{-1}) = \psi(u)\psi(u^{-1})\psi(v)\psi(v^{-1}) = 1$$

since $v \in K$ implies $\psi(v) = \psi'(v) \in \mathsf{Ker}\, h \subseteq Z(H)$, by centrality of the bottom extension. This implies that ψ factors as $\psi = \varphi \circ q$ through the quotient q and, since q is surjective, this yields $h \circ \varphi = \sigma$. Consequently we get the further factorization φ'. $\square$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, tail, "k"] \arrow[d, "q'"] \arrow[dd, bend right, "\psi'"'] & F \arrow[r, two heads, "p"] \arrow[d, two heads, "q"] \arrow[dd, bend right, "\psi"'] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Ker}\,\sigma \arrow[r, tail] \arrow[d, dotted, "\varphi'"] & S = F/[K,F] \arrow[r, two heads, "\sigma"] \arrow[d, "\varphi"] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{Ker}\,h \arrow[r, tail] & H \arrow[r, two heads, "h"] & R \arrow[r] & 0
\end{tikzcd}$$

Diagram 5.6

Proposition 5.2.16 *Every perfect group has a universal central extension.*

Proof Let us use freely the notation of the proof of proposition 5.2.15. The surjection σ restricts at once to a surjection $\sigma\colon [S,S] \twoheadrightarrow [R,R] = R$, where $[R,R] = R$ since R is perfect. This yields the diagram of groups

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, tail, "l"] \arrow[d, tail] & {[S,S]} \arrow[r, two heads, "\sigma"] \arrow[d, tail] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & S \arrow[r, two heads, "\sigma"'] & R \arrow[r] & 0
\end{tikzcd}$$

where both lines are exact sequences. The top line is obviously a weakly universal central extension, since so is the bottom line (see definition 5.2.8). To conclude the proof by proposition 5.2.14, it remains to prove that $[S,S]$ is perfect.

For this we observe first that, trivially by definition 5.2.1,

$$[S,S] = \left[\frac{F}{[K,F]}, \frac{F}{[K,F]}\right] = \frac{[F,F]}{[K,F]}.$$

Next, since R is perfect, F is "perfect up to K", i.e. every element $x \in F$ can be written as $x = ck$ with $c \in [F,F]$ and $k \in K$. Indeed, since R is perfect, write

$$[x] = [x_1][y_1][x_1]^{-1}[y_1]^{-1} \cdots [x_n][y_n][x_n]^{-1}[y_n]^{-1}$$

in $F/K \cong R$. This implies

$$x = x_1 y_1 x_1^{-1} y_1^{-1} \cdots x_n y_n x_n^{-1} y_n^{-1} k$$

for some $k \in K$.

Now any $t \in T = [S, S] = [F, F]/[K, F] \subseteq F/[K, F]$ can be written as a product of elements of the form $[xyx^{-1}y^{-1}] = [x][y][x^{-1}][y^{-1}]$ in $F/[K, F]$. Applying the previous observation to each of these elements, we write $x = ck$ and $y = dl$ for some $c, d \in [F, F]$ and $k, l \in K$. Since for every $z \in F$ and every $m \in K$, we have $[z][m] = [m][z]$ in $F/[K, F]$,

$$\begin{aligned} [xyx^{-1}y^{-1}] &= [c][k][d][l][k]^{-1}[c]^{-1}[l]^{-1}[d]^{-1} \\ &= [c][d][c]^{-1}[d]^{-1}[k][l][k]^{-1}[l]^{-1} \\ &= [cdc^{-1}d^{-1}][klk^{-1}l^{-1}] \\ &= [cdc^{-1}d^{-1}] \\ &\in [T, T] \end{aligned}$$

because $[klk^{-1}l^{-1}] = 1$ in $F/[K, F]$. This holds for each pair x, y, thus finally $t \in [T, T]$, proving that $T = [S, S]$ is perfect. □

Let us conclude this section with a lemma providing a link with a famous Hopf formula.

Lemma 5.2.17 *In the conditions and with the notation of proposition 5.2.16, the following isomorphism holds:*

$$\mathsf{Ker}\, \sigma \cong \frac{K \cap [F, F]}{[K, F]}.$$

Proof To construct an isomorphism

$$\beta \colon \frac{K \cap [F, F]}{[K, F]} \longrightarrow \mathsf{Ker}\, \sigma,$$

we consider diagram 5.7. The morphisms i, u, k, t, v, w are canonical inclusions, thus square (1) is commutative. The commutativity of square (2) holds by definition of σ in the proof of 5.2.16. Moreover the bottom sequence is exact, thus in particular $p \circ k = 1$. Therefore $\sigma q u = p k t = 1$, which implies that qu factors through the kernel w of σ; this yields α such that $w\alpha = qu$. But then $w\alpha v = quv = 1$, proving $\alpha v = 1$ since w is injective. This yields the expected factorization β through the cokernel s of v, completing at the same time the commutative diagram 5.7.

To prove the injectivity of β, choose $a \in K \cap [F, F]$ such that $\beta([a]) =$

$$
\begin{tikzcd}
 & \frac{K\cap[F,F]}{[K,F]} \arrow[rr, dotted, "\beta"] & & \operatorname{Ker}\sigma \arrow[d, tail, "w"] & \\
{[K,F]} \arrow[r, tail, "v"] & K\cap[F,F] \arrow[u, two heads, "s"] \arrow[urr, "\alpha"] \arrow[r, tail, "u"] \arrow[d, tail, "t"'] \arrow[dr, phantom, "(1)"] & {[F,F]} \arrow[r, two heads, "q"] \arrow[d, tail, "i"] \arrow[dr, phantom, "(2)"] & \frac{[F,F]}{[K,F]} \arrow[r, equal] \arrow[d, two heads, "\sigma"] & S \\
0 \arrow[r] & K \arrow[r, tail, "k"'] & F \arrow[r, two heads, "p"'] & R \arrow[r] & 0
\end{tikzcd}
$$

Diagram 5.7

1. This yields

$$qu(a) = w\alpha(a) = w\beta s(a) = w(1) = 1$$

thus $a = u(a) \in [K, F]$ and $[a] = 1$.

To prove the surjectivity of β, let $b \in \operatorname{Ker}\sigma$. There is thus an element $a \in [F, F]$ such that $q(a) = b$ and $p(a) = \sigma(b) = 1$. Thus $a \in \operatorname{Ker} p = K$ and finally $a \in K \cap [F, F]$. Therefore we can write $b = \alpha(a) = \beta\big(s(a)\big)$. $\square$

The formula of lemma 5.2.17 is precisely the Hopf formula defining $H_2(R, \mathbb{Z})$, starting from a free resolution of R (see [69]). By proposition 5.2.14, in the case where S is a perfect group, the group $H_2(R, \mathbb{Z})$ is thus exactly the Galois group $\mathsf{Gal}\,[\sigma]$.

5.3 Factorization systems

We first review some classical facts about factorization systems; for a more detailed treatment, see [20] or [8], volume 1.

Definition 5.3.1 A factorization system on a category $\mathcal{C}$ consists in two classes $\mathcal{E}$, $\mathcal{M}$ of arrows of $\mathcal{C}$, such that

(i) every isomorphism is in both $\mathcal{E}$ and $\mathcal{M}$,
(ii) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition,
(iii) every morphism f of $\mathcal{C}$ factors as $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$,

(iv) in a commutative square where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal arrow making both triangles commutative:

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d] & \bullet \arrow[d] \arrow[dl, dotted] \\ \bullet \arrow[r, "m"'] & \bullet \end{tikzcd}$$

It is important to emphasize the fact that in definition 5.3.1, it is not required at all that the morphisms in $\mathcal{E}$ be epimorphisms or the morphisms in $\mathcal{M}$, monomorphisms.

It is also useful to observe that definition 5.3.1 is selfdual.

Example 5.3.2 In a regular category $\mathcal{C}$, one gets a factorization system $(\mathcal{E}, \mathcal{M})$ by choosing

$$\begin{array}{lll} e \in \mathcal{E} & \text{iff} & e \text{ is a regular epimorphism,} \\ m \in \mathcal{M} & \text{iff} & m \text{ is a monomorphism.} \end{array}$$

Proof Let us recall that a regular category has enough finite limits (let us say, for short, pullbacks) and coequalizers of kernel pairs; moreover pulling back a regular epimorphism along an arbitrary morphism again yields a regular epimorphism. In such a category, it is well known that conditions (i), (ii), (iii) of 5.3.1 are satisfied by regular epimorphisms and arbitrary monomorphisms (see [4] or [8], volume 2).

To check condition (iv) of 5.3.1, consider diagram 5.8 where e is a regular epimorphism, m is a monomorphism and $m \circ a = b \circ e$. Considering the factorizations $m' \circ e'$ and $m'' \circ e''$ of a and b as a regular epimorphism followed by a monomorphism, we get two factorizations of $m \circ a = b \circ e$ as a regular epimorphism followed by a monomorphism

$$(m \circ m') \circ e' = m \circ a = b \circ e = m'' \circ (e'' \circ e).$$

By uniqueness of such a factorization, we get an isomorphism s making the diagram commutative, and therefrom the expected factorization $m' \circ s^{-1} \circ e''$. □

Proposition 5.3.3 *Let $(\mathcal{E}, \mathcal{M})$ be a factorization system on a category $\mathcal{C}$. The following properties hold:*

(i) *$f \in \mathcal{E} \cap \mathcal{M} \Rightarrow f$ is an isomorphism;*

$$\begin{tikzcd}
A \arrow[rrr, two heads, "e"] \arrow[dr, two heads, "e'"] \arrow[dd, "a"'] & & & B \arrow[dl, two heads, "e''"'] \arrow[dd, "b"] \\
& I \arrow[r, "s", "\cong"'] \arrow[dl, tail, "m'"] & J \arrow[dr, tail, "m''"'] & \\
C \arrow[rrr, tail, "m"'] & & & D
\end{tikzcd}$$

Diagram 5.8

(ii) *the factorization* $f = m \circ e$, $m \in \mathcal{M}$, $e \in \mathcal{E}$, *of a morphism of* $\mathcal{C}$ *is unique up to an isomorphism;*

(iii) $f \in \mathcal{M}$ *iff for every commutative square*

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "a"'] & \bullet \arrow[d, "b"] \arrow[dl, dotted] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

with $e \in \mathcal{E}$, *there exists a unique diagonal morphism making both triangles commutative;*

(iv) $f \circ g \in \mathcal{M}$ *and* $f \in \mathcal{M} \Rightarrow g \in \mathcal{M}$;

(v) *in a pullback*

$$\begin{tikzcd}
\bullet \arrow[r, "n"] \arrow[d, "a"'] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "m"'] & \bullet
\end{tikzcd}$$

if $m \in \mathcal{M}$, *then* $n \in \mathcal{M}$.

The dual properties of (iii), (iv), (v) *are valid for the arrows in* $\mathcal{E}$.

Proof (i) It suffices to consider the square

$$\begin{tikzcd}
A \arrow[r, "f \in \mathcal{E}"] \arrow[d, equal] & B \arrow[d, equal] \arrow[dl, dotted, "g"'] \\
A \arrow[r, "f \in \mathcal{M}"'] & B
\end{tikzcd}$$

to get g such that $g \circ f = \mathsf{id}_A$, $f \circ g = \mathsf{id}_B$. The converse holds by 5.3.1.

(ii) If $f = m' \circ e'$, with $m' \in \mathcal{M}$, $e' \in \mathcal{E}$, is another factorization, considering both diagrams

$$\begin{tikzcd}
A \arrow[r, "e \in \mathcal{E}"] \arrow[d, "e'"'] & I \arrow[d, "m"] \arrow[dl, dotted, "\alpha"'] \\
I' \arrow[r, "m' \in \mathcal{M}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "e' \in \mathcal{E}"] \arrow[d, "e"'] & I' \arrow[d, "m'"] \arrow[dl, dotted, "\beta"'] \\
I \arrow[r, "m \in \mathcal{M}"'] & B
\end{tikzcd}$$

yields the factorizations α and β. Considering both diagrams

$$\begin{tikzcd}
A \arrow[r, "e \in \mathcal{E}"] \arrow[d, "e"'] & I \arrow[d, "m"] \arrow[dl, shift right, "\beta \circ \alpha"'] \arrow[dl, shift left, "\mathsf{id}_I"] \\
I \arrow[r, "m \in \mathcal{M}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "e' \in \mathcal{E}"] \arrow[d, "e'"'] & I' \arrow[d, "m'"] \arrow[dl, shift right, "\alpha \circ \beta"'] \arrow[dl, shift left, "\mathsf{id}_{I'}"] \\
I' \arrow[r, "m' \in \mathcal{M}"'] & B
\end{tikzcd}$$

and the uniqueness of the diagonal factorization yields $\beta \circ \alpha = \mathsf{id}_I$ and $\alpha \circ \beta = \mathsf{id}_{I'}$.

(iii) By 5.3.1(iv), the arrows $f \in \mathcal{M}$ have the expected property. Conversely, consider f with property as in condition (iii) of 5.3.1 and write $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. Considering the square

$$\begin{tikzcd}
A \arrow[r, "e \in \mathcal{E}"] \arrow[d, equal] & I \arrow[d, "m"] \arrow[dl, dotted, "g"'] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

yields a factorization g, by assumption on f. The following diagram is then commutative:

$$\begin{tikzcd}
A \arrow[r, "e \in \mathcal{E}"] \arrow[d, "e"'] & I \arrow[d, "m"] \arrow[dl, "e \circ g"'] \\
I \arrow[r, "m \in \mathcal{M}"'] & B
\end{tikzcd}$$

This proves $e \circ g = \mathsf{id}_I$ by the uniqueness condition in 5.3.1(iv). Since on the other hand we have already $g \circ e = \mathsf{id}_A$, g is an isomorphism. Therefore $f = m \circ g^{-1} \in \mathcal{M}$ by conditions (i) and (ii) in 5.3.1.

(iv) Consider a diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "a"'] & Y \arrow[d, "b"] \arrow[dl, dotted, "\alpha"'] & \\
A \arrow[r, "g"'] & B \arrow[r, "f"'] & C
\end{tikzcd}$$

with $f \circ g \in \mathcal{M}$, $f \in \mathcal{M}$, $g \circ a = b \circ e$ and $e \in \mathcal{E}$. Considering the square

$$\begin{tikzcd}
X \arrow[r, "e \in \mathcal{E}"] \arrow[d, "a"'] & Y \arrow[d, "f \circ b"] \arrow[dl, dotted, "\alpha"'] \\
A \arrow[r, "f \circ g \in \mathcal{M}"'] & C
\end{tikzcd}$$

yields a unique factorization α. Considering now the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "e \in \mathcal{E}"] \arrow[d, "g \circ a"'] & Y \arrow[d, "f \circ b"] \arrow[dl, shift right, "b"'] \arrow[dl, shift left, "g \circ \alpha"] \\
B \arrow[r, "f \in \mathcal{M}"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, "e \in \mathcal{E}"] \arrow[d, "c"'] & \bullet \arrow[d, "d"] \arrow[dl, dotted, "\beta"'] \arrow[ddl, dotted, "\alpha"] \\
\bullet \arrow[r, "n"'] \arrow[d, "a"'] & \bullet \arrow[d, "b"] \\
\bullet \arrow[r, "m \in \mathcal{M}"'] & \bullet
\end{tikzcd}$$

Diagram 5.9

yields $b = g \circ \alpha$, by uniqueness of the factorization. Thus α is already a morphism such that $\alpha \circ e = a$ and $g \circ \alpha = b$. If α' is another such morphism, then $\alpha' \circ e = a$ and $(f \circ g) \circ \alpha' = f \circ b$, thus $\alpha = \alpha'$ by the uniqueness condition in 5.3.1(iv). One concludes the proof by 5.3.3(iii).

(v) Consider the given pullback and a commutative square $n \circ c = d \circ e$, with $e \in \mathcal{E}$, as in diagram 5.9. Since $e \in \mathcal{E}$ and $m \in \mathcal{M}$, we first get a factorization α such that $\alpha \circ e = a \circ c$ and $m \circ \alpha = b \circ d$. This last equality yields a factorization β through the bottom pullback, such that $a \circ \beta = \alpha$ and $n \circ \beta = d$. Therefore

$$a \circ \beta \circ e = \alpha \circ e = a \circ c, \quad n \circ \beta \circ e = d \circ e = n \circ c,$$

from which $\beta \circ e = c$ since the bottom square is a pullback. If β' is another factorization in the upper square,

$$(a \circ \beta') \circ e = a \circ c, \quad m \circ (a \circ \beta') = b \circ n \circ \beta' = b \circ d,$$

from which $a \circ \beta' = \alpha$, by 5.3.1(iv). But then $a \circ \beta' = a \circ \beta$ and $n \circ \beta' = n \circ \beta$, from which $\beta = \beta'$ since the bottom square is a pullback. One concludes the proof again by 5.3.3(iii)). □

5.4 Reflective factorization systems

We now investigate the relation between factorization systems and reflective subcategories.

Definition 5.4.1 A category $\mathcal{C}$ is finitely well-complete when

(i) $\mathcal{C}$ is finitely complete,
(ii) $\mathcal{C}$ admits arbitrarily large intersections of subobjects.

Most categories we consider in general are finitely well-complete, for one of the following reasons:

(i) the category is complete and each object has only a set of subobjects, like the categories of sets, groups, topological spaces, and so on;
(ii) the category is finitely complete and each object has only a finite set of subobjects, like the categories of finite sets, finite groups, finite topological spaces, and so on.

Theorem 5.4.2 *Let $\mathcal{C}$ be a finitely well-complete category. There exists a bijection between*

(i) *the replete reflective full subcategories of $\mathcal{C}$,*
(ii) *the factorization systems $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ which satisfy the additional condition*

$$f \circ g \in \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathcal{E}.$$

Moreover, all the morphisms of a given reflective full subcategory belong to the corresponding class $\mathcal{M}$.

We recall that a full subcategory $\mathcal{R} \subseteq \mathcal{C}$ is replete when every object $C \in \mathcal{C}$ which is isomorphic to an object $R \in \mathcal{R}$ is itself in $\mathcal{R}$. This is just a canonical way to choose one specific subcategory in each class of equivalent full subcategories. This allows us also, without any loss of generality, to assume freely that the counit of a replete reflective subcategory is the identity.

Also observe that in view of axiom 5.3.1(ii) and the dual of property 5.3.3(iv), the additional requirement in 5.4.2(ii) can be stated as

If two sides of a commutative triangle are in $\mathcal{E}$,
so is the third side.

Obviously, example 5.3.2 is not of this type.

Proof of theorem Let us begin with a replete reflective full subcategory

$$r \dashv i \colon \mathcal{R} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{C}.$$

Let us put

$$\begin{array}{lll} f \in \mathcal{E} & \text{iff} & r(f) \text{ is an isomorphism,} \\ f \in \mathcal{M} & \text{iff} & f \text{ satisfies condition 5.3.3(iii).} \end{array}$$

$$\begin{tikzcd}
A \arrow[rr, "e"] \arrow[dr, "\eta_A"] \arrow[dd, "\alpha"'] & & B \arrow[dr, "\eta_B"] \arrow[dd, "\beta"'] & \\
& ir(A) \arrow[rr, "ir(e)"] \arrow[dl, dotted, "\alpha'"] & & ir(B) \arrow[dl, dotted, "\beta'"] \\
X \arrow[rr, "f"'] & & Y &
\end{tikzcd}$$

Diagram 5.10

It follows at once that both classes $\mathcal{E}$ and $\mathcal{M}$ contain isomorphisms, that the class $\mathcal{E}$ is closed under composition and that the class $\mathcal{M}$ satisfies condition 5.3.1(iv).

Next, if $m_1, m_2 \in \mathcal{M}$ and the rectangle below is commutative, with $e \in \mathcal{E}$,

$$\begin{tikzcd}
\bullet \arrow[rr, "e \in \mathcal{E}"] \arrow[d] & & \bullet \arrow[d] \arrow[dl, dotted, "\alpha"'] \arrow[dll, dotted, "\beta"'] \\
\bullet \arrow[r, "m_1 \in \mathcal{M}"'] & \bullet \arrow[r, "m_2 \in \mathcal{M}"'] & \bullet
\end{tikzcd}$$

we get at once the unique factorizations α, since $e \in \mathcal{E}$, $m_2 \in \mathcal{M}$, and β, since $e \in \mathcal{E}$, $m_1 \in \mathcal{M}$. By 5.3.3(iii), it follows that $m_2 \circ m_1 \in \mathcal{M}$.

To get a factorization system $(\mathcal{E}, \mathcal{M})$, it remains to prove that every morphism f factors as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. To achieve this, we prove first the last assertion, namely that every arrow of $\mathcal{R}$ is in $\mathcal{M}$.

Given $f\colon X \longrightarrow Y$ in $\mathcal{R}$ and and $e\colon A \longrightarrow B$ such that $r(e)$ is an isomorphism, consider diagram 5.10 where $f \circ \alpha = \beta \circ e$ and $\eta\colon \mathsf{id} \Rightarrow ir$ is the unit of the adjunction $r \dashv i$. The factorizations α' and β' exist by the adjunction property, since X and Y are in $\mathcal{R}$. This yields the "diagonal" morphism

$$B \xrightarrow{\quad \eta_B \quad} ir(B) \xrightarrow{\ (ir(e))^{-1}\ } ir(A) \xrightarrow{\quad \alpha' \quad} X$$

$$
\begin{tikzcd}
A \arrow[dr, dotted, "\gamma"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\
& I \arrow[r, "\beta"] \arrow[d, "\alpha"'] & ir(A) \arrow[d, "ir(f)"] \\
& B \arrow[r, "\eta_B"'] & ir(B)
\end{tikzcd}
$$

Diagram 5.11

which makes the diagram commutative. This factorization is unique because given $g\colon B \longrightarrow X$ with $g \circ e = \alpha$, $f \circ g = \beta$, we get at once

$$g = \eta_X \circ g = ir(g) \circ \eta_B = ir(\alpha) \circ \big(ir(e)\big)^{-1} \circ \eta_B = \alpha' \circ \big(ir(e)\big)^{-1} \circ \eta_B$$

because $X \in \mathcal{R}$, thus η_X can be chosen to be id_X, as already observed. This concludes the proof that $f \in \mathcal{M}$.

Next observe that

$$f \circ g \in \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathcal{E}$$

since when two sides of the triangle $r(f \circ g)$, $r(f)$, $r(g)$ are isomorphisms, so certainly is the third side.

It remains to prove that a morphism $f \in \mathcal{C}$ factors as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$. To do this, consider diagram 5.11, where the square is a pullback and γ is the factorization through it of the external commutative quadrilateral. Since $ir(f) \in \mathcal{R}$, we know already that $ir(f) \in \mathcal{M}$; it is easy to see that the proof of 5.3.3(v) applies to show that $\alpha \in \mathcal{M}$. But in general, γ has no reason to be a morphism in $\mathcal{E}$. Thus we must factor γ itself as a morphism in $\mathcal{E}$ followed by a morphism in $\mathcal{M}$.

Let us consider all the possible factorizations

$$
\begin{tikzcd}
A \arrow[rr, "\gamma"] \arrow[dr, "\gamma_i"'] & & I \\
& S_i \arrow[ur, tail, "s_i \in \mathcal{M}"'] &
\end{tikzcd}
$$

$$\begin{tikzcd}
X \arrow[rr, "e \in \mathcal{E}"] \arrow[d, "a"'] & & Y \arrow[d, "b"] \arrow[dll, dotted, "t"'] \arrow[dl, dotted, "t_i"'] \\
S \arrow[r, tail, "s'_i"] \arrow[rr, bend right, "s"'] & S_i \arrow[r, tail, "s_i"] & I
\end{tikzcd}$$

Diagram 5.12

where s_i is a monomorphism belonging to $\mathcal{M}$; there is of course at least one such factorization, namely, $\gamma = \mathrm{id}_I \circ \gamma$. Since $\mathcal{C}$ is finitely well-complete, we can consider $S = \bigcap S_i$ and we get a factorization

$$\begin{tikzcd}
A \arrow[rr, "\gamma"] \arrow[dr, "\gamma'"'] & & I \\
& S \arrow[ur, tail, "s"'] &
\end{tikzcd}$$

We shall prove that $s \in \mathcal{M}$ and $\gamma' \in \mathcal{E}$. This will yield the factorization $f = (\alpha \circ s) \circ \gamma'$ with $\gamma' \in \mathcal{E}$ and $\alpha \circ s \in \mathcal{M}$, since $\alpha \in \mathcal{M}$ and $s \in \mathcal{M}$.

It is easy to observe that $s \in \mathcal{M}$. Consider diagram 5.12, where the outer part is commutative and $e \in \mathcal{E}$. The monomorphism s factors through each s_i via a monomorphism s'_i. Since $s_i \in \mathcal{M}$ and $e \in \mathcal{E}$, we get a unique factorization t_i. Since $s_i \circ t_i = b = s_j \circ t_j$ for all indices i, j, this yields a further unique factorization t through the intersection S. This proves precisely $s \in \mathcal{M}$.

To prove that $\gamma' \in \mathcal{E}$, that is, $r(\gamma')$ is an isomorphism, we consider diagram 5.13 where the left hand square is a pullback and w is the corresponding factorization arising from $\eta_S \circ \gamma' = ir(\gamma') \circ \eta_A$. The right hand square is commutative since

$$ir(\beta) \circ ir(s) \circ ir(\gamma') = ir(\beta \circ s \circ \gamma') = ir(\beta \circ \gamma) = ir(\eta_A) = \mathrm{id}_{ir(A)}.$$

This last equality implies already that $ir(\gamma')$ is a monomorphism; on the other hand $ir(\gamma') \in \mathcal{R}$, thus $ir(\gamma') \in \mathcal{M}$ by a previous part of the proof. By pulling back, we get that u as well is a monomorphism belonging to $\mathcal{M}$. But the definition of S as intersection of the subobjects of I

$$\begin{tikzcd}
A \arrow[dr, dotted, "w"] \arrow[drr, "\eta_A"] \arrow[ddr, "\gamma'"'] & & & \\
& P \arrow[r, "v"'] \arrow[d, "u"] & ir(A) \arrow[r, equal] \arrow[d, "ir(\gamma')"] & irir(A) \\
& S \arrow[r, "\eta_S"'] & ir(S) \arrow[r, "ir(s)"'] & ir(I) \arrow[u, "ir(\beta)"']
\end{tikzcd}$$

Diagram 5.13

in $\mathcal{M}$ then forces u to be an isomorphism. This allows considering the following situation:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "v \circ u^{-1}"'] & ir(S) \arrow[d, dotted, "i(z)"] \\
& ir(A)
\end{tikzcd}$$

where the unique morphism $z\colon r(S) \longrightarrow r(A)$ making the diagram commutative exists by the adjointness property. This implies

$$i\big(r(\gamma') \circ z\big) \circ \eta_S = ir(\gamma') \circ i(z) \circ \eta_S = ir(\gamma') \circ v \circ u^{-1} = \eta_S \circ u \circ u^{-1} = \eta_S$$

from which $r(\gamma') \circ z = \mathrm{id}_S$ by uniqueness of the factorization through the unit η_S of the adjunction. Therefore $ir(\gamma') \circ i(z) = \mathrm{id}_{i(S)}$ and $ir(\gamma')$ is a retraction in $\mathcal{C}$; since we know already it is also a monomorphism, it is an isomorphism. Thus $r(\gamma')$ is an isomorphism as well, because $\mathcal{R}$ is full in $\mathcal{C}$. This means $\gamma' \in \mathcal{E}$. This ends the proof that $(\mathcal{E}, \mathcal{M})$ is a factorization system satisfying all the requirements of the statement.

Let us now start with a factorization system $(\mathcal{E}, \mathcal{M})$ satisfying the additional condition

$$f \circ g \in \mathcal{E} \text{ and } f \in \mathcal{E} \Rightarrow g \in \mathcal{E}.$$

Let us define $\mathcal{R}$ as the full subcategory of $\mathcal{C}$ generated by those objects $C \in \mathcal{C}$ such that the unique morphism $\xi_C\colon C \longrightarrow \mathbf{1}$ to the terminal

$$
\begin{tikzcd}
r(C) \arrow[d, "t", "\cong"'] \\
I \arrow[d, "m"] \\
X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & \mathbf{1} \\
C \arrow[urr, "\xi_C"] \arrow[r, "e_C"] \arrow[dr, "e"] \arrow[ddr, "f"'] & r(C) \arrow[ur, "m_C"'] \arrow[d, "t", "\cong"'] & \\
 & I \arrow[d, "m"] & \\
 & X \arrow[uuur, "\xi_X \in \mathcal{M}"'] &
\end{tikzcd}
$$

Diagram 5.14

object is in $\mathcal{M}$. Let us prove that the inclusion $i\colon \mathcal{R} \hookrightarrow \mathcal{C}$ has a left adjoint functor r. For this, we consider diagram 5.14 in the category $\mathcal{C}$, with the left hand part of this diagram in the subcategory $\mathcal{R}$.

Given $C \in \mathcal{C}$, we consider the unique morphism ξ_C to the terminal object and we factor it as $\xi_C = m_C \circ e_C$, with $m_C \in \mathcal{M}$ and $e_C \in \mathcal{E}$. By definition of $\mathcal{R}$, the corresponding object $r(C)$ belongs to $\mathcal{R}$. To prove the adjointness property, choose $X \in \mathcal{R}$ and $f\colon C \longrightarrow X$. By definition of $\mathcal{R}$, $\xi_X \in \mathcal{M}$. Factorizing f as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, we get two $(\mathcal{E}, \mathcal{M})$-factorizations of ξ_C, namely, $m_C \circ e_C$ and $(\xi_X \circ m) \circ e$. The uniqueness of such a factorization implies the existence of an isomorphism $t\colon r(C) \xrightarrow{\cong} I$ making diagram 5.14 commutative. Since the full subcategory $\mathcal{R}$ is replete, $I \in \mathcal{R}$ and thus $t \in \mathcal{R}$ as well. This yields the expected factorization $m \circ t$. If $t'\colon r(C) \longrightarrow X$ is another morphism such that $t' \circ e_C = f$, the trivial equality $\xi_X \circ t' = m_C$ implies $t' \in \mathcal{M}$ by 5.3.3(iv). Thus $f = t' \circ e_C$ is an $(\mathcal{E}, \mathcal{M})$-factorization of f and, by uniqueness, $t' = m \circ t$. This concludes the proof that the inclusion functor i admits a left adjoint functor taking values $r(C)$ on the objects.

It remains to check that both constructions that we have just described define mutually inverse bijections. First, in the construction of the reflection $\mathcal{R}$, starting from a factorization system $(\mathcal{E}, \mathcal{M})$, we must prove that the morphisms inverted by the reflexion r are exactly those of $\mathcal{E}$.

Let $g\colon C \longrightarrow D$ be a morphism of $\mathcal{C}$; $r(g)\colon r(C) \longrightarrow r(D)$ is the unique morphism of $\mathcal{R}$ making commutative the following diagram:

$$\begin{tikzcd} C \arrow[r, "e_C"] \arrow[d, "g"'] & r(C) \arrow[r, "m_C"] \arrow[d, "r(g)"'] & \mathbf{1} \arrow[d, equal] \\ D \arrow[r, "e_D"'] & r(D) \arrow[r, "m_D"'] & \mathbf{1} \end{tikzcd}$$

where $e_C, e_D \in \mathcal{E}$ are the units of the adjunction and $m_C, m_D \in \mathcal{M}$. If $r(g)$ is an isomorphism, then $r(g) \circ e_C$ and e_D are in $\mathcal{E}$, thus $g \in \mathcal{E}$. Conversely if $g \in \mathcal{E}$, then $m_C \circ e_C$ and $m_D \circ (e_D \circ g)$ are two $(\mathcal{E}, \mathcal{M})$-factorizations of the same arrow, thus $r(g)$ is an isomorphism.

On the other hand starting from a reflective subcategory $\mathcal{R}$ and constructing the corresponding factorization system $(\mathcal{E}, \mathcal{M})$ as at the beginning of the proof, we must check that for every object $C \in \mathcal{C}$, the composite

$$C \xrightarrow{\eta_C} ir(C) \xrightarrow{\xi_{ir(C)}} \mathbf{1}$$

is such that $\eta_C \in \mathcal{E}$ and $\xi_{ir(C)} \in \mathcal{M}$.

Indeed, $r(\eta_C) = \mathrm{id}_{r(C)}$ by adjunction, thus $\eta_C \in \mathcal{E}$. Moreover $ir(C) \in \mathcal{R}$ and $\mathbf{1} \in \mathcal{R}$, thus $\xi_{ir(C)} \in \mathcal{R}$ since $\mathcal{R}$ is full in $\mathcal{C}$; we know already that this implies $\xi_{ir(C)} \in \mathcal{M}$. □

5.5 Semi-exact reflections

First we recall without proof a result of Gabriel and Ulmer (see [32] or [8], volume 2). This result will not be used in this book, but will at least justify the terminology "semi-left-exact" used in this section. The notion of "semi-left-exact reflection" appears also under the name "fibred reflection" in [11].

Proposition 5.5.1 *Let $\mathcal{C}$ be a category with finite limits. For a full reflective subcategory $r \dashv i\colon \mathcal{R} \rightleftarrows \mathcal{C}$, the following conditions are equivalent:*

(i) *the reflection $r\colon \mathcal{C} \longrightarrow \mathcal{R}$ preserves finite limits, that is, is left exact;*

(ii) *the class of those arrows which are inverted by the reflection r is stable under change of base; this means, given a pullback*

$$\begin{tikzcd} \bullet \arrow[r, "c"] \arrow[d, "a"'] & \bullet \arrow[d, "b"] \\ \bullet \arrow[r, "d"'] & \bullet \end{tikzcd}$$

if $r(d)$ is an isomorphism, then $r(c)$ is an isomorphism as well. □

In other words, the reflection is left exact iff the class $\mathcal{E}$ of the corresponding factorization system (see theorem 5.4.2) is stable under change of base. The next proposition presents a weakening of this situation.

Proposition 5.5.2 *Let $\mathcal{C}$ be a finitely well-complete category. Let $r \dashv i\colon \mathcal{R} \rightleftarrows \mathcal{C}$ be a full reflective subcategory of $\mathcal{C}$ and $(\mathcal{E}, \mathcal{M})$ the corresponding factorization system, as in theorem 5.4.2. The following conditions are equivalent:*

(i) *in the pullback in $\mathcal{C}$*

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "b"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "e"'] & \bullet \end{tikzcd}$$

if $e \in \mathcal{E}$ and $m \in \mathcal{M}$, then $a \in \mathcal{E}$ (and, of course, $b \in \mathcal{M}$);

(ii) *in the pullback in $\mathcal{C}$*

$$\begin{tikzcd} \bullet \arrow[r, "b"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ A \arrow[r, "\eta_A"'] & ir(A) \end{tikzcd}$$

where $A \in \mathcal{C}$, η_A is the unit of the adjunction $r \dashv i$ and $X \in \mathcal{R}$, one has $b \in \mathcal{E}$;

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$$\begin{tikzcd}
A \arrow[dr, "\gamma"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& I \arrow[r, "\beta"] \arrow[d, "\alpha"'] & ir(A) \arrow[d, "ir(f)"] \\
& B \arrow[r, "\eta_B"'] & ir(B)
\end{tikzcd}$$

Diagram 5.15

(iii) *for every object $C \in \mathcal{C}$, the adjunction*

$$\mathcal{R}/r(C) \underset{i_C}{\overset{r_C}{\leftrightarrows}} \mathcal{C}/C, \quad r_C \dashv i_C$$

given by lemma 4.3.4 still presents $\mathcal{R}/r(C)$ as a full reflective subcategory of $\mathcal{C}/C$.

Moreover, in these conditions,

- *a morphism f is in the class $\mathcal{M}$ iff the following square is a pullback –*

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ir(A) \arrow[d, "ir(f)"] \\
B \arrow[r, "\eta_B"'] & ir(B)
\end{tikzcd}$$

- *the reflection $r\colon \mathcal{C} \longrightarrow \mathcal{R}$ preserves those pullbacks mentioned in condition* (i).

Proof Condition (ii) is just a special case of condition (i) since $\eta_A \in \mathcal{E}$ and $f \in \mathcal{R} \subseteq \mathcal{M}$ (see theorem 5.4.2). Moreover in condition (i), the statement $b \in \mathcal{M}$ holds by 5.3.3(v). To prove (ii) $\Rightarrow$ (i), we show first that condition (ii) implies the characterization of the arrows $f \in \mathcal{M}$ as at the end of the statement.

If the square

$$\begin{tikzcd}
P \arrow[r, "u"] \arrow[dd, "v"'] & ir(C) \arrow[r, equal] \arrow[d, "ir(m)"'] & ir(C) \arrow[dd, "ir(m)"] \\
 & ir(D) \arrow[d, "(ir(e))^{-1}"', "\cong"] & \\
B \arrow[r, "\eta_B"'] & ir(B) \arrow[r, "\cong", "ir(e)"'] & ir(D)
\end{tikzcd}$$

Diagram 5.16

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ir(A) \arrow[d, "ir(f)"] \\
B \arrow[r, "\eta_B"'] & ir(B)
\end{tikzcd}$$

is a pullback, then $ir(f) \in \mathcal{R} \subseteq \mathcal{M}$ (see 5.4.2) gives $f \in \mathcal{M}$ (see 5.3.3(v)). Conversely if $f \in \mathcal{M}$, consider diagram 5.15, already used in the proof of 5.4.2 and where the square is a pullback. By assumption (ii), $\beta \in \mathcal{E}$; but since $\eta_A \in \mathcal{E}$, it follows from 5.4.2(ii) that $\gamma \in \mathcal{E}$. On the other hand $ir(f) \in \mathcal{R} \subseteq \mathcal{M}$ and thus $\alpha \in \mathcal{M}$ by 5.3.3(v). This yields $f = \alpha \circ \gamma$ with $\alpha \in \mathcal{M}$ and $\gamma \in \mathcal{E}$. But when $f \in \mathcal{M}$, the uniqueness of the $(\mathcal{E}, \mathcal{M})$-factorization implies that γ is an isomorphism. Thus the outer part of the diagram is a pullback, since the inner square is.

We are now able to prove (ii) $\Rightarrow$ (i). Let us consider the situation

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "b"'] & C \arrow[r, "\eta_C"] \arrow[d, "m"] & ir(C) \arrow[d, "ir(m)"] \\
B \arrow[r, "e"'] & D \arrow[r, "\eta_D"'] & ir(D)
\end{tikzcd}$$

where the left hand square is a pullback as in condition (i), thus with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. From $m \in \mathcal{M}$ and the characterization of arrows

$$
\begin{tikzcd}
A \arrow[rr, "a"] \arrow[dr, "\eta_A"] \arrow[dd, "b"'] & & C \arrow[dr, "\eta_C"] \arrow[dd, "m"' near start] & \\
& ir(A) \arrow[rr, "ir(a)" near start] \arrow[dd, "ir(b)" near end] & & ir(C) \arrow[dd, "ir(m)"] \\
B \arrow[rr, "e" near start] \arrow[dr, "\eta_B"'] & & D \arrow[dr, "\eta_D"'] & \\
& ir(B) \arrow[rr, "\cong", "ir(e)"'] & & ir(D)
\end{tikzcd}
$$

Diagram 5.17

in $\mathcal{M}$ we have just exhibited, we deduce that the right hand square is a pullback as well. Let us compare this with diagram 5.16, where the right hand rectangle is obviously a pullback and the left hand rectangle is a pullback by definition. Notice that $ir(e)$ is indeed an isomorphism since $e \in \mathcal{E}$. Since $ir(C) \in \mathcal{R}$, by assumption (ii), $u \in \mathcal{E}$. By naturality, $\eta_D \circ e = ir(e) \circ \eta_B$, proving that the two diagrams have the same bottom arrow. Therefore $\eta_C \circ a \colon A \longrightarrow ir(C)$ coincides with $u \colon P \longrightarrow ir(C)$ up to an isomorphism. Since $\eta_C \in \mathcal{E}$ and $u \in \mathcal{E}$, it follows that $a \in \mathcal{E}$ by 5.4.2(ii). This concludes the proof of the equivalence (i) $\Leftrightarrow$ (ii).

Let us prove now (i) $\Rightarrow$ (iii). It is useful to go back to lemma 4.3.4 to observe how pullbacks occur at once in the definition of the functor i_C.

Thus choose $f \colon X \longrightarrow r(C)$ in $\mathcal{R}$ and consider the following pullback in $\mathcal{C}$:

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"'] & i(X) \arrow[d, "i(f)"] \\
C \arrow[r, "\eta_C"'] & ir(C)
\end{tikzcd}
$$

We must prove that the counit of the adjunction $r_C \dashv i_C$ is an isomorphism (see [8], volume 1), that is the existence of an isomorphism in $\mathcal{R}/r(C)$

$$\begin{tikzcd} r(X') \arrow[rr, "\cong"] \arrow[dr, "r(f')"'] & & X \arrow[dl, "f"] \\ & r(C) & \end{tikzcd}$$

Observe first that given the pullback of condition (i), one gets the "cubical" diagram 5.17, about which we use freely a "three-dimensional" terminology. Since $e \in \mathcal{E}$ implies that $ir(e)$ is an isomorphism, for the face containing $ir(a)$, $ir(e)$ to be a pullback it is necessary and sufficient that $ir(a)$ be an isomorphism as well. This is further equivalent to $a \in \mathcal{E}$, which is precisely condition (i). In particular, condition (i) implies that the pullbacks mentioned in this condition are preserved by ir, thus by r since pullbacks in $\mathcal{R}$ are computed as in $\mathcal{C}$.

Observe next that the pullback defining f' is such that $\eta_C \in \mathcal{E}$ and $i(f) \in \mathcal{R} \subseteq \mathcal{M}$, thus it is preserved by the reflection r, as we have already seen. This yields the following pullback in $\mathcal{R}$, where the bottom line is an identity:

$$\begin{tikzcd} r(X') \arrow[r, "s"] \arrow[d, "r(f')"'] & ri(X) = X \arrow[d, "f"] \\ r(C) \arrow[r, equal] & r(C) \end{tikzcd}$$

The arrow s is thus the expected isomorphism.

Finally we prove (iii) $\Rightarrow$ (ii). Thus assume condition (iii) and consider the situation of condition (ii):

$$\begin{tikzcd} X' \arrow[r, "f''"] \arrow[d, "f'"'] & i(X) \arrow[d, "i(f)"] & X \arrow[d, "f"'] & X \in \mathcal{R} \\ C \arrow[r, "\eta_C"'] & ir(C) & r(C) & \end{tikzcd}$$

We must prove that $f'' \in \mathcal{E}$, that is, $r(f'')$ is an isomorphism. But $(X', f') = i_C(X, f)$ and, by assumption, the counit of the adjunction

$$r_C i_C(X, f) = \big(r(X'), r(f')\big) \longrightarrow (X, f) \cong \big(ri(X), f\big)$$

is an isomorphism. This counit is precisely $r(f'')$. □

Definition 5.5.3 Let $\mathcal{C}$ be a category with pullbacks and $r \dashv i \colon \mathcal{R} \rightleftarrows \mathcal{C}$ a reflective full subcategory of $\mathcal{C}$. The reflection is semi-left-exact when, for every object $C \in \mathcal{C}$, the adjunction

$$\mathcal{R}/r(C) \underset{i_C}{\overset{r_C}{\rightleftarrows}} \mathcal{C}/C, \quad r_C \dashv i_C$$

given by lemma 4.3.4 still presents $\mathcal{R}/r(C)$ as a reflective full subcategory of $\mathcal{C}/C$.

Definition 5.5.4 Let us consider the following data:

- a category $\mathcal{C}$ with pullbacks;
- a semi-left-exact reflection $r \dashv i \colon \mathcal{R} \rightleftarrows \mathcal{C}$;
- a morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{C}$.

We particularize definitions 5.1.7 and 5.1.8 by saying that

(i) a pair $(A, f) \in \mathcal{C}/R$ is split by σ with respect to the semi-left-exact reflection $r \dashv i$, when it is so in the sense of definition 5.1.7, relatively to the classes $\overline{\mathcal{R}}$, $\overline{\mathcal{C}}$ of all arrows of $\mathcal{R}$ and $\mathcal{C}$,

(ii) σ is a morphism of Galois descent with respect to the semi-left-exact reflection $r \dashv i$, when it is so in the sense of definition 5.1.8, relativeley to the classes $\overline{\mathcal{R}}$, $\overline{\mathcal{C}}$ of all arrows of $\mathcal{R}$ and $\mathcal{C}$.

In the special case of semi-left-exact reflections, we shall now investigate simpler descriptions of split objects and Galois descent morphisms. First, we observe that restricting one's attention to semi-left-exact reflections allows working with finitely complete categories instead of finitely well-complete ones (see definition 5.4.1).

Proposition 5.5.5 *Let $\mathcal{C}$ be a category with finite limits. There exists a bijection between*

(i) *the semi-left-exact reflective replete full subcategories of $\mathcal{C}$,*

(ii) *the factorization systems $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ which satisfy the two aditional conditions*

 (a) $f \circ g \in \mathcal{E}$ *and* $f \in \mathcal{E} \Rightarrow g \in \mathcal{E}$,

 (b) *given a pullback*

$$\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, "b"'] & \bullet \arrow[d, "m"] \\ \bullet \arrow[r, "e"'] & \bullet \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, one has $a \in \mathcal{E}$ (and, of course, $b \in \mathcal{M}$).

In these conditions, an arrow $f \in \mathcal{C}$ is in $\mathcal{M}$ precisely when the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & ir(A) \arrow[d, "ir(f)"] \\ B \arrow[r, "\eta_B"'] & ir(B) \end{tikzcd}$$

is a pullback, with η the unit of the adjunction $r \dashv i$. Moreover the reflection $r\colon \mathcal{C} \longrightarrow \mathcal{R}$ preserves the pullbacks mentioned in condition (ii).(b) *and each arrow of $\mathcal{R}$ is in $\mathcal{M}$.*

Proof In the case of a finitely well-complete category $\mathcal{C}$, this is an immediate consequence of 5.4.2 and 5.5.2. So it suffices to observe that when restricting one's attention to semi-left-exact reflections, one can get rid of the existence of arbitrary intersections. This assumption has been used at a single place, in the proof of 5.4.2, to prove the existence of the $(\mathcal{E}, \mathcal{M})$-factorization of a morphism f, starting with a given reflection.

Observe first that the class $\mathcal{M}$ constructed in 5.4.2 from an arbitrary reflection is stable under change of base. The proof of this fact is exactly the same as that of point (v) in proposition 5.3.3.

Observe next that the following facts remain valid here, since they have been proved without any reference to the existence of arbitrary intersections in $\mathcal{C}$:

(i) the inclusion $\mathcal{R} \subseteq \mathcal{M}$, in the proof of 5.4.2;
(ii) the implication $f \circ g \in \mathcal{E}$ and $f \in \mathcal{E} \Rightarrow g \in \mathcal{E}$, in the proof of 5.4.2;
(iii) the implication (iii) $\Rightarrow$ (ii), in the proof of 5.5.2.

When the reflection $r \dashv i\colon \mathcal{R} \leftrightarrows \mathcal{C}$ is semi-left-exact and $\mathcal{E}$, $\mathcal{M}$ are defined as in the proof of 5.4.2, the $(\mathcal{E}, \mathcal{M})$-factorization of an arbitrary

$$\begin{tikzcd}
A \arrow[dr, "\gamma"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& I \arrow[r, "\beta"] \arrow[d, "\alpha"'] & ir(A) \arrow[d, "ir(f)"] \\
& B \arrow[r, "\eta_B"'] & ir(B)
\end{tikzcd}$$

Diagram 5.18

morphism $f\colon A \longrightarrow B$ can be constructed at once, without any intersection. Indeed, as in the proof of 5.5.2, we consider diagram 5.18 where the square is a pullback and γ is the factorization through it of the external quadrilateral. The previous observations show that $ir(f) \in \mathcal{R} \subseteq \mathcal{M}$ thus, by pulling back, $\alpha \in \mathcal{M}$. On the other hand, $\beta \in \mathcal{E}$ and $\eta_A \in \mathcal{E}$ imply $\gamma \in \mathcal{E}$. So $f = \alpha \circ \gamma$ is the expected $(\mathcal{E}, \mathcal{M})$-factorization. $\square$

Avoiding the existence of arbitrary intersections in 5.5.5 is useful if one intends to apply the result to, for example, elementary toposes.

We now investigate the form of split pairs.

Proposition 5.5.6 *Let us consider the following data:*

- *a finitely complete category $\mathcal{C}$;*
- *a semi-left-exact reflection $r \dashv i\colon \mathcal{R} \underset{\longrightarrow}{\overset{\longleftarrow}{}} \mathcal{C}$;*
- *the corresponding factorization system $(\mathcal{E}, \mathcal{M})$;*
- *a morphism $\sigma\colon S \longrightarrow R$ in $\mathcal{C}$.*

For an object $(A, f) \in \mathcal{C}/R$, the following conditions are equivalent:

(i) *the pair (A, f) is split by σ with respect to the semi-left-exact reflection $r \dashv i$;*

(ii) *the canonical morphism $\sigma^*(A, f) \longrightarrow i_S r_S \sigma^*(A, f)$ is an isomorphism;*

(iii) *in the pullback square*

$$\begin{tikzcd}
A' \arrow[dr, dotted, "\alpha"] \arrow[drr, bend left, "\eta_{A'}"] \arrow[ddr, bend right, "f'"'] & & \\
& \overline{A} \arrow[r, "\widetilde{f}"] \arrow[d, "\overline{f}"'] & ir(A) \arrow[d, "ir(f')"] \\
& S \arrow[r, "\eta_S"'] & ir(S)
\end{tikzcd}$$

Diagram 5.19

$$\begin{tikzcd}
A' \arrow[r, "f''"] \arrow[d, "f'"'] & A \arrow[d, "f"] \\
S \arrow[r, "\sigma"'] & R
\end{tikzcd}$$

one has $f' \in \mathcal{M}$.

Proof With the notation of the statement, one has $(A', f') = \sigma^*(A, f)$. Next computing the pullback

$$\begin{tikzcd}
\overline{A} \arrow[r, "\widetilde{f}"] \arrow[d, "\overline{f}"'] & ir(A') \arrow[d, "ir(f')"] \\
S \arrow[r, "\eta_S"'] & ir(S)
\end{tikzcd}$$

we find $i_S r_S(A', f') = (\overline{A}, \overline{f})$. The canonical morphism involved in the statement is then the factorization α in diagram 5.19. This factorization is an isomorphism precisely when the outer part of the diagram is a pullback, that is, when $f' \in \mathcal{M}$ (see 5.5.5). But α being an isomorphism is also the definition of (A, f) being split by σ relatively to $r \dashv i$. $\square$

Next we investigate, in the present context, condition (iii) in definition 5.1.8.

$$\begin{tikzcd} X' \arrow[r, "x''"] \arrow[d, "x'"'] & i(X) \arrow[d, "i(x)"] \\ S \arrow[r, "\eta_S"'] \arrow[d, "\sigma"'] & ir(S) \\ R & \end{tikzcd}$$

Diagram 5.20

Lemma 5.5.7 *Let us consider the following data:*

- *a finitely complete category $\mathcal{C}$;*
- *a semi-left-exact reflection $r \dashv i \colon \mathcal{R} \rightleftarrows \mathcal{C}$;*
- *the corresponding factorization system $(\mathcal{E}, \mathcal{M})$;*
- *a morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{C}$.*

The following conditions are equivalent:

(i) *for every object $(X, x) \in \mathcal{R}/r(S)$, the object $(\Sigma_\sigma \circ i_S)(X, x) \in \mathcal{C}/R$ is split by σ with respect to $r \dashv i$;*

(ii) *in the pullback*

$$\begin{tikzcd} S \times_R S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "\sigma"] \\ S \arrow[r, "\sigma"'] & R \end{tikzcd}$$

one has $p_1 \in \mathcal{M}$.

Proof Considering diagram 5.20, where the square is a pullback, we get $(X', \sigma \circ x') = (\Sigma_\sigma \circ i_S)(X, x)$. We consider next diagram 5.21, where both squares are pullbacks: this yields $(\overline{X}, p_1 \circ \overline{x}) = \sigma^*(X', \sigma \circ x')$. By proposition 5.5.6, $(\Sigma_\sigma \circ i_S)(X, x)$ is split by σ when $p_1 \circ \overline{x} \in \mathcal{M}$. But $i(x) \in \mathcal{R} \subseteq \mathcal{M}$ (see 5.5.5), which implies $x' \in \mathcal{M}$ and $\overline{x} \in \mathcal{M}$ (see 5.3.3(v)). Thus when $p_1 \in \mathcal{M}$, one has $p_1 \circ \overline{x} \in \mathcal{M}$ (see definition 5.3.1(ii)). Conversely if condition (i) of the statement holds, putting

$$\begin{tikzcd}
\overline{X} \arrow[r, "\overline{x}"] \arrow[d, "\overline{x}"'] & X' \arrow[d, "x'"] \\
S \times_R S \arrow[r, "p_2"] \arrow[d, "p_1"'] & S \arrow[d, "\sigma"] \\
S \arrow[r, "\sigma"'] & R
\end{tikzcd}$$

Diagram 5.21

$(X, x) = \big(r(S), \mathsf{id}_{r(S)}\big)$ yields $x' = \mathsf{id}_S$ and $\overline{x} = \mathsf{id}_{S \times_R S}$; therefore $p_1 \circ \overline{x} \in \mathcal{M}$ implies $p_1 \in \mathcal{M}$. □

Observe that in condition 5.5.7(ii), we can equivalently write $p_1 \in \mathcal{M}$, or $p_2 \in \mathcal{M}$, or $p_1, p_2 \in \mathcal{M}$ since the twisting isomorphism

$$\tau \colon S \times_R S \longrightarrow S \times_R S,$$

transforms p_1 into p_2. In other words $p_1 \in \mathcal{M}$ implies $p_2 = p_1 \circ \tau \in \mathcal{M}$ by definition 5.3.1.

Finally we are able to characterize morphisms of Galois descent in the present situation:

Corollary 5.5.8 *Let us consider the following data:*

- *a finitely complete category $\mathcal{C}$;*
- *a semi-left-exact reflection $r \dashv i \colon \mathcal{R} \overset{\longleftarrow}{\longrightarrow} \mathcal{C}$;*
- *the corresponding factorization system $(\mathcal{E}, \mathcal{M})$;*
- *a morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{C}$.*

The following conditions are equivalent for an effective descent morphism σ:

(i) *σ is a morphism of Galois descent with respect to the semi-left-exact reflection $r \dashv i$;*

(ii) *$p_1 \colon S \times_R S \longrightarrow S$ is in the class $\mathcal{M}$.*

These conditions are in particular satisfied when $\sigma \in \mathcal{M}$.

Proof The equivalence (i) ⇔ (ii) follows from lemma 5.5.7 and condition (v) of proposition 5.3.3. □

5.6 Connected components of a space

First, we recall some basic facts about connected components in an arbitrary topological space.

Definition 5.6.1 A topological space X is connected when it is non-empty and cannot be written as the union of two disjoint non-trivial open subsets.
A subset $A \subseteq X$ of a topological space is connected when, provided with the induced topology, it is a connected space.

Lemma 5.6.2 *Let X be a topological space. Let $(A_i)_{i\in I}$ be a family of connected subsets of X. When the intersection $\bigcap_{i\in I} A_i$ is non-empty, the union $\bigcup_{i\in I} A_i$ is connected.*

Proof Let $x \in \bigcap_{i\in I} A_i$. We prove the statement by a reduction *ad absurdum.* Assume $\bigcup_{i\in I} A_i = M \cup N$ where M and N are non trivial disjoint open subsets of $\bigcup_{i\in I} A_i$. For each A_i we have

$$A_i = (A_i \cap M) \cup (A_i \cap N)$$

with $A_i \cap M$ and $A_i \cap N$ open in A_i. Since A_i is connected, this implies $A_i \cap M = A_i$ or $A_i \cap N = A_i$, that is, $A_i \subseteq M$ or $A_i \subseteq N$. On the other hand, $x \in M \cup N$, thus $x \in M$ or $x \in N$. Let us handle the case $x \in M$. If $x \in M$, since $x \in A_i$ for all $i \in I$, then $A_i \subseteq M$ for all $i \in I$. This proves $\bigcup_{i\in I} A_i = M$, which is a contradiction. □

Definition 5.6.3 The connected component of a point x in a topological space X is the union of all connected subsets containing x, that is, the largest connected subset containing x. We shall write Γ_x for the connected component of the point x.

By lemma 5.6.2 and the fact that the singleton $\{x\}$ is obviously connected, definition 5.6.3 indeed makes sense.

Lemma 5.6.4 *In a topological space, the closure of a connected subset is again connected.*

Proof Let $A \subseteq X$ be a connected subset of the space X. Assume $\overline{A} = M \cup N$ where M, N are non trivial disjoint open subsets of $\overline{A}$. One has in particular

$$A = (A \cap M) \cup (A \cap N)$$

with $A \cap M$ and $A \cap N$ open subsets of A. This forces $A \cap M = \emptyset$ or $A \cap N = \emptyset$. Let us handle the case $A \cap N = \emptyset$. Since N is non-empty, choose $x \in N$. Since $x \in \overline{A}$ and N is open in $\overline{A}$, one has $A \cap N \neq \emptyset$, which is a contradiction. □

Corollary 5.6.5 *In a topological space, the connected component of a point is closed.*

Proof The closure of the connected component of a point x is still connected by lemma 5.6.4, thus is equal to Γ_x by maximality of Γ_x (see definition 5.6.3). □

Lemma 5.6.6 *The image of a connected subset under a continuous map is still connected.*

Proof Let $f\colon X \longrightarrow Y$ be a continuous map and $A \subseteq X$ a connected subset. If $f(A) = M \cup N$ with M, N disjoint non-empty open subsets in $f(A)$, then

$$A \subseteq f^{-1}f(A) = f^{-1}(M) \cup f^{-1}(N)$$

with $f^{-1}(M)$ and $f^{-1}(N)$ disjoint non-empty open subsets of $f^{-1}f(A)$. Thus

$$A = \left(A \cap f^{-1}(M)\right) \cup \left(A \cap f^{-1}(N)\right)$$

and since A is connected, one of those open subsets of A is empty while the other one equals A. Let us handle the case $A = A \cap f^{-1}(M)$. In this case $A \subseteq f^{-1}(M)$ and thus $f(A) \subseteq ff^{-1}(M) \subseteq M$. Since $M \subseteq f(A)$, we get $M = f(A)$, which is a contradiction. □

Corollary 5.6.7 *If $f\colon X \longrightarrow Y$ is a continuous map and $x \in X$, then $f(\Gamma_x) \subseteq \Gamma_{f(x)}$.* □

Lemma 5.6.8 *If x is a point of a topological space X, then*

$$\Gamma_x \subseteq \bigcap \{U \,|\, U \text{ is a clopen of } X, \quad x \in U\}.$$

Proof If U is a clopen containing x, then

$$\Gamma_x = (\Gamma_x \cap U) \cup (\Gamma_x \cap \complement U)$$

and by connectedness of Γ_x, one of those two terms of the union must be empty. Since $x \in \Gamma_x \cap U$, this forces $\Gamma_x \cap \complement U = \emptyset$, that is $\Gamma_x \subseteq U$. □

5.7 Connected components of a compact Hausdorff space

We shall provide the set of connected components of a compact Hausdorff space with a profinite topology. To achieve this, we shall first show that in a compact Hausdorff space, the inclusion of lemma 5.6.8 is in fact an equality. The proof uses the uniform structure of a compact space.

Let us recall that a relation R on a set X is a subset $R \subseteq X \times X$. Given relations R, S on X we use the classical notation

$$\begin{aligned} R^{-1} &= \{(x,y) \big| (y,x) \in R\}, \\ R \circ S &= \{(x,z) \big| \exists y \in X \quad (x,y) \in R, \quad (y,z) \in S\}, \\ \Delta_X &= \{(x,x) \big| x \in X\}. \end{aligned}$$

When no confusion can occur, we shall just write Δ instead of Δ_X for the diagonal of X. With the previous notation, it follows at once that

$$\begin{aligned} R \text{ is reflexive} \quad &\text{iff} \quad \Delta \subseteq R, \\ R \text{ is symmetric} \quad &\text{iff} \quad R = R^{-1}, \\ R \text{ is transitive} \quad &\text{iff} \quad R \circ R \subseteq R. \end{aligned}$$

The notion of neighbourhood of a point x tries to recapture the idea of "points which are sufficiently close to the fixed point x"; for example in a metric space X, given $\varepsilon > 0$,

$$V = \{y \in X \mid d(x,y) < \varepsilon\}.$$

The notion of entourage tries to recapture an analogous idea, but "uniformly", for all the points of the space at the same time; for example on the metric space X:

$$V = \{(y,z) \big| y \in X, \quad z \in X, \quad d(y,z) < \varepsilon\}.$$

An "entourage" in a set X is thus a relation V on X expressing the "proximity" of the points $(y,z) \in V$.

There is an abstract notion of *uniform space*, which is a set X provided with a family of subsets $V \subseteq X \times X$, called "entourages" and satisfying precisely conditions (i) to (iii) of 5.7.2 below . The reader interested in these questions will find an explicit treatment of them in [10]. In this book, we shall only be interested in the uniform structure which is naturally present on a compact Hausdorff space, not in the abstract theory of uniform spaces.

Definition 5.7.1 In a compact Hausdorff space X, a neighbourhood of the diagonal $\Delta \subset X \times X$ is called an entourage of X.

A neighbourhood of the diagonal is of course a subset containing an open subset containing the diagonal.

Proposition 5.7.2 *Let X be a compact Hausdorff space and $\mathcal{V}$ the set of its entourages. The following properties hold:*

(i) $\forall V \in \mathcal{V} \quad \Delta \subseteq V$;
(ii) $\forall V \in \mathcal{V} \quad V^{-1} \in \mathcal{V}$;
(iii) $\forall V \in \mathcal{V} \quad \exists W \in \mathcal{V}$ *with* $W \circ W \subseteq V$.

Proof Property (i) holds by definition of an entourage.

To prove (ii), it suffices to consider the twisting morphism

$$\tau \colon X \times X \longrightarrow X \times X, \quad (x,y) \mapsto (y,x)$$

and observe that given $V \in \mathcal{V}$, one has $V^{-1} = \tau^{-1}(V)$. Since τ is continuous, V^{-1} is again a neighbourhood of Δ.

We prove (iii) by reduction *ad absurdum*. Assume thus the existence of $V \in \mathcal{V}$ such that for every $W \in \mathcal{V}$, one has $W \circ W \not\subseteq V$, that is, $(W \circ W) \cap \complement V \neq \emptyset$. Trivially, for all $W_1, W_2 \in \mathcal{V}$, one has

$$(W_1 \cap W_2) \circ (W_1 \cap W_2) \subseteq (W_1 \circ W_1) \cap (W_2 \circ W_2)$$

which proves that the subsets $(W \circ W) \cap \complement V$, with $W \in \mathcal{V}$, constitute a filter base in $X \times X$. Since X and thus $X \times X$ are compact Hausdorff, there exists a point (x,y) in the adherence of this filter base, that is, every neighbourhood of (x,y) meets every subset in the filter base (see [58]). Since V is in particular a neighbourhood of (x,x) and does not meet any $(W \circ W) \cap \complement V$, it follows that $(x,x) \neq (x,y)$, thus $x \neq y$. By normality of the compact Hausdorff space X (see [58]), we can find

$$x \in A' \subseteq A, \quad y \in B' \subseteq B$$

where A, B are open and disjoint and A', B' are, respectively, closed neighbourhoods of x and y. We consider $C = \complement(A' \cup B')$ which is thus an open subset of X and finally

$$W_0 = (A \times A) \cup (B \times B) \cup (C \times C)$$

which is open in $X \times X$ and contains the diagonal, since $A \cup B \cup C = X$. Thus $W_0 \in \mathcal{V}$. Since $A' \times B'$ is a neighbourhood of (x,y), we have thus

$$(A' \times B') \cap (W_0 \circ W_0) \cap \complement V \neq \emptyset;$$

we choose (u,v) in this intersection. Since $(u,v) \in W_0 \circ W_0$, there exists $w \in X$ such that $(u,w) \in W_0$ and $(w,v) \in W_0$. But $u \in A'$ implies

$w \in A$ by definition of W_0, just because A' is disjoint from B and C. Analogously, $v \in B'$ implies $w \in B$. But then $w \in A \cap B$, which is a contradiction. □

Proposition 5.7.3 *In a compact Hausdorff space, every entourage contains an open symmetric entourage.*

Proof With the notation of 5.7.2, $V \in \mathcal{V}$ contains an open entourage $W \subseteq V$ and $W^{-1} = \tau^{-1}(W)$ is thus another open entourage. It follows that $W \cap W^{-1} \subseteq W \subseteq V$ is still an open neighbourhood of the diagonal, obviously symmetric. □

In view of proposition 5.7.3, we shall concentrate our attention on open symmetric entourages, since these "generate" the uniform structure of X.

Lemma 5.7.4 *Consider a subset $B \subset X$ of a compact Hausdorff space X. For every open symmetric entourage V, we define*

$$V(B) = \{x \in X \,|\, \exists b \in B \quad (x,b) \in V\}.$$

This subset $V(B)$ is open in X.

Proof Let us consider, for each element $y \in X$, the continuous map

$$(-,y)\colon X \longrightarrow X \times X, \quad x \mapsto (x,y).$$

One gets at once

$$\begin{aligned} V(B) &= \{x \in X \,|\, \exists b \in B \quad (x,b) \in V\} \\ &= \bigcup_{b \in B} \{x \in X \,|\, (x,b) \in V\} \\ &= \bigcup_{b \in B} (-,b)^{-1}(V) \end{aligned}$$

which proves that $V(B)$ is a union of open subsets. □

Lemma 5.7.5 *Let B, C be disjoint closed subspaces of a compact Hausdorff space X. There exists an open symmetric entourage V such that*

$$\big(V(B) \times V(C)\big) \cap V = \emptyset.$$

In particular, $V(B) \cap V(C) = \emptyset$.

Proof In the space $X \times X$, the subset $B \times C$ is closed and disjoint from the diagonal Δ, which is closed as well, by Hausdorffness of X (see [58]). By normality of $X \times X$, we choose open subsets M, N of $X \times X$ such that

$$B \times C \subseteq M, \quad \Delta \subseteq N, \quad M \cap N = \emptyset.$$

By 5.7.2(iii) and 5.7.3, we choose an open symmetric entourage V such that $V \circ V \circ V \subseteq N$. We proceed by a reduction *ad absurdum*. If there exists a point

$$(x, y) \in \big(V(B) \times V(C)\big) \cap V,$$

one gets

$$\begin{aligned} x \in V(B) \quad &\Rightarrow \quad \exists b \in B \;\; (x, b) \in V, \\ y \in V(C) \quad &\Rightarrow \quad \exists c \in C \;\; (y, c) \in V. \end{aligned}$$

Together with $(x, y) \in V$, the symmetry of V and the inclusion $V \circ V \circ V \subseteq N$ then imply $(b, c) \in N$. This is a contradiction since $(b, c) \in B \times C \subseteq M$ and M is disjoint from N. This concludes the proof of

$$\big(V(B) \times V(C)\big) \cap V = \emptyset$$

which implies a fortiori

$$\big(V(B) \times V(C)\big) \cap \Delta = \emptyset.$$

This last equality means precisely $V(B) \cap V(C) = \emptyset$. □

We shall now focus on the nearness relation associated with an entourage.

Definition 5.7.6 Let X be a compact Hausdorff space.

(i) For an entourage V, the relation of V-nearness is the equivalence relation on X generated by the pairs $(x, y) \in V$.
(ii) The nearness relation on the space X is the intersection of all the V-nearness relations, for all entourages V.

A pair (x, y) is thus in the relation of V-nearness when there exists a finite sequence of elements of X

$$x = z_1, \ldots, z_i, \ldots, z_n = y$$

with each pair (z_i, z_{i+1}) in V or V^{-1}. When V is symmetric, this reduces to each pair (z_i, z_{i+1}) being in V.

Observe that in the example of a metric space already mentioned, if

$(z_i, z_{i+1}) \in V$ means $d(z_i, z_{i+1}) < \varepsilon$, two points x, y are in the relation of V-nearness when "one can travel from x to y by steps of length less than ε". The case of the nearness relation can be interpreted analogously, but using "arbitrarily small steps". Thus the V-nearness relations do not tell anything about the distance between points, but about the "width of the cracks" which can prevent you from travelling from one point to another one. Theorem 5.7.9 will emphasize the mathematical relevance of this intuition.

Lemma 5.7.7 *The nearness relation on a compact Hausdorff space X is the equivalence relation obtained as intersection of all the V-nearness relations, for all the open symmetric entourages V.*

Proof By proposition 5.7.3 and the fact that an intersection of equivalence relations is still an equivalence relation. □

Lemma 5.7.8 *Let X be a compact Hausdorff space. For every open symmetric entourage V, the equivalence classes in X for the relation of V-nearness are clopens of X.*

Proof Refering to lemma 5.7.4, let us consider

$$V(z) = V\big(\{z\}\big) = \big\{x \in X \big| (x, z) \in V\big\}$$

which is thus open in X. Writing further $[z]_V$ for the equivalence class of z for the relation of V-nearness, we have trivially $z \in V(z) \subseteq [z]_V$. Since $V(z)$ is open, $[z]_V$ is a neighbourhood of each of its points, thus is open. But since distinct equivalence classes are disjoint, an equivalence class is the complement of the union of the other classes, which are open; thus it is closed as well. □

Theorem 5.7.9 *Let $x \in X$ be a point of a compact Hausdordff space. Write Γ_x for the connected component of x and $[x]_\sim$ for the equivalence class of x for the nearness relation. The following equalities hold:*

$$\Gamma_x = [x]_\sim = \bigcap \{U \subseteq X \mid U \text{ clopen, } x \in U\}.$$

Proof By lemma 5.7.7, one has

$$[x]_\sim = \bigcap \big\{[x]_V \big| V \text{ open symmetric entourage}\big\}.$$

By lemma 5.7.8, $[x]_\sim$ is thus an intersection of clopens containing x, which proves already, together with lemma 5.6.8, that

$$\Gamma_x \subseteq \bigcap\{U \subseteq X \mid U \text{ clopen}, \ x \in U\} \subseteq [x]_\sim.$$

To conclude the proof, it remains to prove that $[x]_\sim$ is connected, since this will imply $[x]_\sim \subseteq \Gamma_x$. We do this by a reduction *ad absurdum*.

If $[x]_\sim$ is not connected, let us write $[x]_\sim = B \cup C$ where B, C are non trivial disjoint open subsets of $[x]_\sim$. Notice that since B, C are mutual complements in $[x]_\sim$, they are also closed in $[x]_\sim$. But $[x]_\sim$ itself is closed as intersection of the clopens $[x]_V$, as in lemma 5.7.8. Thus B and C are also closed in X. Applying lemma 5.7.5, let us choose an open symmetric entourage V such that

$$\big(V(B) \times V(C)\big) \cap V = \emptyset, \quad V(B) \cap V(C) = \emptyset.$$

Further, applying propositions 5.7.2 and 5.7.3, we choose an open symmetric entourage W such that $W \circ W \subseteq V$. Clearly $W = W \circ \Delta \subseteq W \circ W \subseteq V$, thus $W(B) \subseteq V(B)$ and $W(C) \subseteq V(C)$. In particular,

$$\big(W(B) \times W(C)\big) \cap W = \emptyset, \quad W(B) \cap W(C) = \emptyset.$$

Moreover, by lemma 5.7.4, $W(B)$ and $W(C)$ are open. We shall consider $H = \complement\big(W(B) \cup W(C)\big)$ which is thus closed in X.

Since $x \in B \cup C$, one has $x \in B$ or $x \in C$; let us handle the case $x \in B$. Since C is non-empty, choose $y \in C$. Since $x, y \in [x]_\sim$, the pair (x, y) is in the E-nearness relation for every open symmetric entourage $E \subseteq W$; thus there exists a chain

$$x = z_1, z_2, \ldots, z_i, \ldots, z_{n-1}, z_n = y$$

with each pair (z_i, z_{i+1}) in the E-nearness relation. Let us prove that at least one z_i must be in H. Otherwise one would have

$$\forall i = 1, \ldots, n \quad z_i \in \complement H = W(B) \cup W(C) \subseteq V(B) \cup V(C)$$

where $V(B) \cap V(C) = \emptyset$ by lemma 5.7.5. Notice that

$$\begin{aligned}
z_i \in W(B) \quad &\Rightarrow \quad \exists b_i \in B \quad (z_i, b_i) \in W \\
&\Rightarrow \quad \exists b_i \in B \quad (z_i, b_i) \in W \text{ and } (z_i, z_{i+1}) \in E \subseteq W \\
&\Rightarrow \quad \exists b_i \in B \quad (z_{i+1}, b_i) \in W \circ W \subseteq V \\
&\Rightarrow \quad z_{i+1} \in V(B) \\
&\Rightarrow \quad z_{i+1} \in W(B)
\end{aligned}$$

where the last implication holds because $z_{i+1} \in W(B) \cup W(C)$ and $W(C)$ is disjoint from $V(B)$. Starting from $x = z_1 \in B \subseteq W(B)$ and iterating the previous argument would yield $z_n = y \in W(B)$. But this is impossible because $y \in W(C)$ and $W(C)$ is disjoint from $W(B)$. Thus indeed, at least one of the elements z_i belongs to H. On the other hand, by definition of the E-nearness relation, all elements z_i are in $[x]_E$. This proves that for each open symmetric entourage $E \subseteq W$, one has $[x]_E \cap H \neq \emptyset$.

Now when E runs through all the open symmetric subentourages of W, the corresponding subsets $[x]_E \cap H$ constitute a filter basis constituted of closed subsets of X. Since X is a compact Hausdorff space, there exists a point x_0 in the intersection of the filter basis, that is a point $x_0 \in [x]_\sim \cap H$. This is a contradiction since

$$[x]_\sim = B \cup C \subseteq W(B) \cup W(C) = \complement H. \qquad \square$$

Corollary 5.7.10 *For a compact Hausdorff space X, the following conditions are equivalent:*

(i) *X is totally disconnected;*
(ii) *the connected component of any point $x \in X$ is reduced to that point.*

Proof (i) $\Rightarrow$ (ii) If X is totally disconnected and $x \in X$, for every $y \in X$, $y \neq x$, choose a clopen U_y such that $x \in U_y$ and $y \notin U_y$. This implies $\{x\} = \bigcap_{y \neq x} U_y$ and one concludes the proof by theorem 5.7.9.

Conversely, if $x \neq y$, then $y \notin \{x\} = \Gamma_x$ and by theorem 5.7.9, there exists a clopen U such that $x \in U$ and $y \notin U$. $\square$

Corollary 5.7.11 *Let X be a compact Hausdorff space. The quotient of X by the nearness relation is a profinite space. The equivalence classes for this quotient are exactly the connected components of X.*

Proof By theorem 5.7.9, the equivalence classes for the nearness relation are exactly the connected components. The quotient is obviously compact, as continuous image of a compact. It remains to prove that it is totally disconnected (see theorem 3.4.7).

Let $\Gamma_x \neq \Gamma_y$ be distinct connected components. One has

$$\Gamma_x = \bigcap \{U \mid U \text{ clopen}, \quad x \in U\}$$

by theorem 5.7.9. Since $y \notin \Gamma_x$, there exists a clopen U such that $x \in U$ and $y \notin U$. Observe that this clopen U is saturated for the nearness relation, that is, by theorem 5.7.9, the relation "having the same connected component": indeed, if $z \in U$, then $\Gamma_z \subseteq U$ by theorem

5.7.9. Since U is a saturated clopen in X which separates x and y, the image of U in the quotient is still a clopen which saturates the classes of x and y. □

The next result makes more precise a part of the statement of 3.4.6.

Proposition 5.7.12 *The category of profinite spaces is a reflective full subcategory of the category of compact Hausdorff spaces. The reflection of a compact Hausdorff space is its space of connected components.*

Proof Write $\Gamma(X)$ for the space of connected components of the compact Hausdorff space X and $\gamma_X : X \twoheadrightarrow \Gamma(X)$ for the canonical quotient, sending a point $x \in X$ onto its connected component Γ_x. By corollary 5.7.11, the space $\Gamma(X)$ is profinite and γ_X is continuous.

Consider now a profinite space Y together with a continuous map $f: X \longrightarrow Y$. We must prove the existence of a continous factorization g,

$$\begin{tikzcd} X \arrow[r, two heads, "\gamma_X"] \arrow[dr, "f"'] & \Gamma(X) \arrow[d, "g"] \\ & Y \end{tikzcd}$$

necessarily unique since γ_X is surjective.

Given $x, y \in X$ such that $\Gamma_x = \Gamma_y$, let us prove that $f(x) = f(y)$, from which the definition $g(\Gamma_x) = f(x)$ will make sense. Indeed if $f(x) \neq f(y)$, there is a clopen $U \subseteq Y$ such that $f(x) \in U$ and $f(y) \notin U$. This yields a clopen $f^{-1}(U) \subseteq X$ such that $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. But then $\Gamma_x \neq \Gamma_y$ by theorem 5.7.9, which contradicts our assumption. This allows us thus to define $g(\Gamma_x) = f(x)$, yielding a map g such that $g \circ \gamma_X = f$. Since f is continuous and $\Gamma(X)$ is provided with the quotient topology, g is continuous as well. □

5.8 The monotone–light factorization

We shall now describe categorically, in the category of compact Hausdorff spaces, the *monotone* and the *light* maps introduced by Eilenberg (in the metric case) and Whyburn (see [28] and [75]). Let us mention – we shall not prove it and shall not need it – that every continuous map between compact Hausdorff spaces factors as a monotone map followed by a light one.

Definition 5.8.1 Let $f\colon X \longrightarrow Y$ be a continuous map between compact Hausdorff spaces.

(i) The map f is monotone when the inverse image of each point is connected.

(ii) The map f is light when the inverse image of each point is totally disconnected.

The "opposition" between both notions of monotone and light map is emphasized by corollary 5.7.10. In the situation of definition 5.8.1, for each point $y \in Y$, $f^{-1}(y)$ has exactly one connected component in the monotone case, while in the light case, it has as many connected components as points.

Lemma 5.8.2 *Every monotone map between compact Hausdorff spaces is surjective.* $\square$

Proposition 5.8.3 *Consider the reflection* $r \dashv i\colon \mathsf{Prof} \rightleftarrows \mathsf{Comp}$ *of the category of profinite spaces in that of compact Hausdorff spaces (see 5.7.12) and the corresponding factorization system* $(\mathcal{E}, \mathcal{M})$ *(see theorem 5.4.2). For a continuous map* $f\colon X \longrightarrow Y$ *between compact Hausdorff spaces, the following conditions are equivalent:*

(i) *f is monotone;*

(ii) *for every pullback*

$$\begin{tikzcd}
\bullet \arrow[r, "f'"] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

in Comp, *one has* $f' \in \mathcal{E}$, *that is,* $r(f')$ *is an isomorphism.*

Proof In the presence of condition (ii), for every element $y \in Y$, the pullback

$$\begin{tikzcd}
f^{-1}(y) \arrow[r] \arrow[d, tail] & \{*\} \arrow[d, tail, "\ulcorner y \urcorner"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

shows that $f^{-1}(y)$ and $\{*\}$ are identified by r, that is, in view of proposition 5.7.12, $f^{-1}(y)$ has exactly one connected component. Thus $f^{-1}(y)$ is connected and non-empty and f is monotone.

Conversely, let f be monotone and thus surjective (lemma 5.8.2). Consider the pullback

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "k"'] & Z \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

By construction of a pullback in Comp, if $z \in Z$,

$$h^{-1}(z) = \{(z, x) \,|\, g(z) = f(x)\} \cong f^{-1}\big(g(z)\big).$$

Thus $h^{-1}(z)$ is connected and non-empty since, by assumption, this is the case for $f^{-1}\big(g(z)\big)$. This proves already that h is monotone. It remains to prove that $r(h)$ is an isomorphism.

Since $h: P \longrightarrow Z$ is monotone, it is surjective. With the notation of the proof of proposition 5.7.12, in the commutative square

$$\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, two heads, "\gamma_P"'] & Z \arrow[d, two heads, "\gamma_Z"] \\
r(P) \arrow[r, "r(h)"'] & r(Z)
\end{tikzcd}$$

the arrows h, γ_P and γ_Z are surjective, thus $r(h)$ is surjective as well. It remains to prove that $r(h)$ is injective, since a continuous bijection between profinite, thus compact Hausdorff, spaces is a homeomorphism.

Consider thus $a, b \in P$ such that $\Gamma_{h(a)} = \Gamma_{h(b)}$; we must prove that $\Gamma_a = \Gamma_b$. For this observe first that $h^{-1}(\Gamma_{h(a)})$ is connected. Indeed, otherwise one would have $h^{-1}(\Gamma_{h(a)}) = F_1 \cup F_2$ where F_1, F_2 are disjoint non-empty clopens of $h^{-1}(\Gamma_{h(a)})$. Since $h^{-1}(\Gamma_{h(a)})$ is itself closed in P by corollary 5.6.5, F_1 and F_2 are in fact closed in P as well. Let us prove that these closed subsets are saturated for the equivalence relation $\sim$ generated by h, i.e. defined by $c \sim d$ when $h(c) = h(d)$. Indeed given $c \in F_1$, one has $h(c) \in \Gamma_{h(a)}$. We know that $h^{-1}\big(h(c)\big)$ is connected, because h is monotone. Since F_1, F_2 are clopens in $h^{-1}(\Gamma_{h(a)})$ and $h^{-1}\big(h(c)\big) \subseteq h^{-1}(\Gamma_{h(a)})$, the connectedness of $h^{-1}\big(h(c)\big)$ forces $h^{-1}\big(h(c)\big) \subseteq F_1$ or $h^{-1}\big(h(c)\big) \subseteq F_2$. We thus have $h^{-1}\big(h(c)\big) \subseteq F_1$ because $c \in h^{-1}\big(h(c)\big) \cap F_1$. Thus F_1 is saturated for the equivalence relation induced by h, and analogously for F_2. But the continuous surjection $h\colon P \twoheadrightarrow Z$ between compact Hausdorff spaces necessarily has the quotient topology; indeed, writing $h(P)$ for the image of h provided with the quotient topology, $h(P)$ is compact as continuous image of a compact and thus the identity $h(P) = Z$ is a continuous bijection between compact Hausdorff spaces, thus a homeomorphism. Therefore $h(F_1)$ and $h(F_2)$ remain disjoint non-empty closed subsets of Z and, by surjectivity of h, they cover $hh^{-1}(\Gamma_{h(a)}) = \Gamma_{h(a)}$. But this is a contradiction, because $\Gamma_{h(a)}$ is connected. This completes the proof that $h^{-1}(\Gamma_{h(a)})$ is connected. Analogously, $h^{-1}(\Gamma_{h(b)})$ is connected.

Since $a \in h^{-1}(\Gamma_{h(a)})$ and $b \in h^{-1}(\Gamma_{h(b)})$ and these subsets are connected, we have $h^{-1}(\Gamma_{h(a)}) \subseteq \Gamma_a$ and $h^{-1}(\Gamma_{h(b)}) \subseteq \Gamma_b$. On the other hand, by corollary 5.6.7, $h(\Gamma_a) \subseteq \Gamma_{h(a)}$ and $h(\Gamma_b) \subseteq \Gamma_{h(b)}$. This implies

$$\Gamma_a \subseteq h^{-1}h(\Gamma_a) \subseteq h^{-1}(\Gamma_{h(a)}), \quad \Gamma_b \subseteq h^{-1}h(\Gamma_b) \subseteq h^{-1}(\Gamma_{h(b)})$$

and thus finally, since by assumption $\Gamma_{h(a)} = \Gamma_{h(b)}$,

$$\Gamma_a = h^{-1}(\Gamma_{h(a)}) = h^{-1}(\Gamma_{h(b)}) = \Gamma_b. \qquad \square$$

Corollary 5.8.4 *The reflection $r \dashv i\colon \mathsf{Prof} \overset{\longleftarrow}{\longrightarrow} \mathsf{Comp}$ of profinite spaces in compact Hausdorff spaces is semi-left-exact.*

Proof If X is a compact Hausdorff space, by definition, the unit of the adjunction $\gamma_X\colon X \longrightarrow \Gamma_X$ is such that the inverse image of a point is a connected component of X. Thus γ_X is monotone. Condition (ii) of proposition 5.8.3 implies at once condition (ii) of proposition 5.5.2. $\square$

We recall now a very classical result; we refer to [58] for a detailed treatment of the Stone–Čech compactifications.

Theorem 5.8.5 (Stone–Čech compactification) *The forgetful functor $|-|\colon \mathsf{Comp} \longrightarrow \mathsf{Set}$ of the category of compact Hausdorff spaces to* *that of sets has a left adjoint which takes values in the category of profinite spaces.*

Proof It is well known that $|-|$ has a left adjoint functor β, namely, the Stone–Čech compactification functor. We recall the construction of β. Given a set X, we consider the map

$$\eta_X \colon X \longrightarrow 2^X, \quad x \mapsto \{x\}$$

where 2^X is identified with the power set of X, that is, $\{x\}$ is the family $(\varepsilon_y)_{y \in X}$ with $\varepsilon_x = 1$ and $\varepsilon_y = 0$ for $y \neq x$. Viewing 2 as a discrete space, we put on 2^X the product topology and define $\beta(X)$ as the closure of the image of η_X in this product space. Since 2 is finite and discrete, 2^X is profinite by corollary 3.4.8. Thus $\beta(X)$ is closed in a profinite, that is compact, totally disconnected space (see theorem 3.4.7); thus it is still compact and totally disconnected, that is, profinite. □

Lemma 5.8.6 *The unit η of the Stone–Čech adjunction $\beta \dashv |-|$ (see theorem 5.8.5) is injective in each component. The counit σ of the same adjunction is surjective in each component.*

Proof With the notation of theorem 5.8.5, $|\beta(X)|$ is a subset of 2^X in which η_X takes values. The unit of the adjunction is given, for each set X, by the corestriction

$$\eta_X \colon X \longrightarrow |\beta(X)|, \quad x \mapsto \{x\}$$

of this map η_X; it is indeed injective.

On the other hand, given a compact Hausdorff space Y and writing σ for the counit of the adjunction $\beta \dashv |-|$, the following diagram is one of the triangular identities of this adjunction:

$$\begin{tikzcd} |Y| \arrow[r, "\eta_{|Y|}"] \arrow[dr, equal] & |\beta|Y|| \arrow[d, "|\sigma_Y|"] \\ & |Y| \end{tikzcd}$$

This proves the surjectivity of the map $|\sigma_Y|$, that is, of the continuous map σ_Y. □

Proposition 5.8.7 *The forgetful functor* $|-|\colon \mathsf{Comp} \longrightarrow \mathsf{Set}$ *from the category of compact Hausdorff spaces to that of sets is monadic.*

Proof We know already that this functor has a left adjoint (see 5.8.5). It reflects isomorphisms, since a continuous bijection between compact Hausdorff spaces is necessarily a homeomorphism (see [58]). Since the category of compact Hausdorff spaces has coequalizers, it remains, by the Beck criterion (see [8], volume 2), to prove that the functor $|-|$ preserves the coequalizer of those pairs (u, v) such that $\big(|u|, |v|\big)$ has a split coequalizer in the category of sets.

Thus we consider two continuous maps $u, v\colon X \rightrightarrows Y$, with X, Y compact Hausdorff spaces. We suppose they have a split coequalizer in the category of sets; this means the existence of a set Q and maps q, r, s such that

$$\begin{tikzcd} |X| \arrow[r, shift left, "|u|"] \arrow[r, shift right, "|v|"'] & |Y| \arrow[r, "q"] \arrow[l, bend right, "r"'] & Q. \arrow[l, bend right, "s"'] \end{tikzcd}$$

$$q \circ s = 1_Q, \quad |u| \circ r = 1_Y, \quad q \circ |u| = q \circ |v|, \quad |v| \circ r = s \circ q.$$

Let us provide Q with the quotient topology, induced by the topology of Y. If we prove that Q is Hausdorff, Q will be compact Hausdorff as image of the compact Hausdorff space Y by the continuous surjection q. It will follow at once that q is the coequalizer of (u, v) in the category of compact Hausdorff spaces, since it is in the category of sets and Q is provided with the quotient topology. But proving the Hausdorffness of Q reduces to proving that its kernel pair is a closed equivalence relation in $Y \times Y$ (see [58]).

First we consider the relation

$$R = \Big\{ \big(u(x), v(x)\big) \Big| x \in X \Big\}$$

which is a closed subspace of $Y \times Y$. Indeed, R is the image of the compact Hausdorff space X by the continuous map

$$X \longrightarrow Y \times Y, \quad x \mapsto \big(u(x), v(x)\big);$$

thus it is a compact, and therefore closed, subset of the compact Hausdorff space $Y \times Y$. The kernel pair of q is the equivalence relation R^+ generated by R, and we must prove it remains closed.

Let us prove that the fact of having a split coequalizer in the category of sets forces R^+ to be $R^{-1} \circ R$, the composite of the relation R with its opposite relation R^{-1}. If $y \in Y$, one has immediately

$$\big(y, sq(y)\big) = \big(ur(y), vr(y)\big) \in R.$$

Therefore, if $y, y' \in Y$ with $q(y) = q(y')$

$$\big(y, sq(y)\big) \in R \text{ and } \big(sq(y'), y'\big) \in R^{-1} \Rightarrow (y', y) \in R^{-1} \circ R.$$

Thus the kernel pair of q is contained in $R^{-1} \circ R$, while the converse inclusion is obvious.

Now R^{-1} is the image of R by the twisting homeomorphism $\tau\colon Y \times Y \longrightarrow Y \times Y$, thus it is compact and therefore closed. Next $R^{-1} \circ R$ is the image of the pullback $R^{-1} \times_Y R$ by the projection $p_{1,3}\colon Y \times Y \times Y \longrightarrow Y \times Y$, thus it is again compact and therefore closed. □

Proposition 5.8.8 *Consider the reflection $r \dashv i\colon \mathsf{Prof} \rightleftarrows \mathsf{Comp}$ of profinite spaces in compact Hausdorff spaces and the corresponding factorization system $(\mathcal{E}, \mathcal{M})$. For a continuous map $f\colon X \longrightarrow Y$ between compact Hausdorff spaces, the following conditions are equivalent:*

(i) *f is light;*

(ii) *in the following pullback, where $|Y|$ denotes the underlying set of Y and σ is the counit of the Stone–Čech adjunction (see theorem 5.8.5) the space P is profinite:*

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ \beta|Y| \arrow[r, two heads, "\sigma_Y"'] & Y \end{tikzcd}$$

(iii) *in the above pullback, $h \in \mathcal{M}$;*

(iv) *$(X, f) \in \mathsf{Comp}/Y$ is split by σ_Y.*

Proof (i) $\Rightarrow$ (ii) By corollary 5.7.10, it suffices to prove that given $(z, x) \in P$, its connected component $\Gamma_{(z,x)}$ is reduced to the point (z, x). By corollary 5.6.7, $h(\Gamma_{(z,x)})$ is connected, thus

$$h(\Gamma_{(z,x)}) \subseteq \Gamma_{h(z,x)} = \Gamma_z = \{z\},$$

since $\beta|Y|$ is profinite (see theorem 5.8.5 and corollary 5.7.10). It follows that $\Gamma_{(z,x)} = \{z\} \times g(\Gamma_{(z,x)})$. By lemma 5.6.6, $g(\Gamma_{(z,x)})$ is connected and contained in $f^{-1}\big(\sigma_Y(z)\big)$. But since f is light, $f^{-1}\big(\sigma_Y(z)\big)$ is totally disconnected (see definition 5.8.1), from which $g(\Gamma_{(z,x)})$ has only one element. This proves that $\Gamma_{(z,x)} = \{(z,x)\}$ and P is totally disconnected by corollary 5.7.10.

(ii) $\Leftrightarrow$ (iii) Consider the following commutative diagram:

$$\begin{tikzcd}
P \arrow[r, "\gamma_P"] \arrow[d, "h"'] & \Gamma(P) \arrow[d, "r(h)"] \\
\beta|Y| \arrow[r, "\gamma_{\beta|Y|}"'] & \Gamma(\beta|Y|)
\end{tikzcd}$$

By theorem 5.8.5, $\beta|Y|$ is profinite; via corollary 5.8.4 and proposition 5.5.2, we get

$$\begin{aligned}
h \in \mathcal{M} \quad &\Leftrightarrow \quad \text{the square above is a pullback} \\
&\Leftrightarrow \quad \gamma_P \text{ is an isomorphism} \\
&\Leftrightarrow \quad P \text{ is totally disconnected.}
\end{aligned}$$

Indeed, since $\beta|Y|$ is totally disconnected, $\gamma_{\beta|Y|}$ is an isomorphism by corollary 5.7.10.

(iii) $\Leftrightarrow$ (iv) This holds by definition 5.5.4 and proposition 5.5.6.

(ii) $\Rightarrow$ (i) For each element $y \in Y$, applying lemma 5.8.6, let us choose $z \in \beta|Y|$ such that $\sigma_Y(z) = y$. This yields

$$h^{-1}(z) = \{(z,x) | \sigma_Y(z) = f(x)\} \cong f^{-1}(\sigma_Y(z)) = f^{-1}(y).$$

But $h^{-1}(z)$ is totally disconnected as a subspace of the totally disconnected space P. This proves that $f^{-1}(y)$ is totally disconnected and thus f is light. □

Theorem 5.8.9 *For every compact Hausdorff space Y,*

(i) *the morphism $\sigma_Y \colon \beta|Y| \longrightarrow Y$, which is the counit of the Stone–Čech adjunction (see theorem 5.8.5), is a morphism of Galois descent with respect to the adjunction $r \dashv i \colon \mathsf{Prof} \overset{\longleftarrow}{\longrightarrow} \mathsf{Comp}$ between compact Hausdorff spaces and profinite spaces,*

(ii) *the objects $(X, f) \in \mathsf{Comp}/Y$ which are split by σ_Y are precisely those for which $f \colon X \longrightarrow Y$ is light.*

Proof The arrow $\sigma_Y \colon \beta|Y| \twoheadrightarrow Y$ is a surjection between compact Hausdorff spaces, thus is a topological quotient. By proposition 5.8.7

and lemma 4.4.6, σ_Y is thus an effective descent morphism. We also know, by corollary 5.8.4, that the reflection $r \dashv i$ is semi-left-exact. Referring to corollary 5.5.8, it remains to verify that in the pullback

$$\begin{tikzcd}
\beta|Y| \times \beta|Y| \arrow[r, "p_2"] \arrow[d, "p_1"'] & \beta|Y| \arrow[d, two heads, "\sigma_Y"] \\
\beta|Y| \arrow[r, "\sigma_Y"'] & Y
\end{tikzcd}$$

one has $p_1 \in \mathcal{M}$. By proposition 5.8.8, this is equivalent to proving that $\sigma_Y \colon \beta|Y| \longrightarrow Y$ is light, which is indeed the case since each $\sigma_Y^{-1}(y)$ is totally disconnected as a subspace of the totally disconnected space $\beta|Y|$ (see theorem 5.8.5). An alternative argument consists in observing that p_1 is in Prof, and applying the last statement of theorem 5.4.2. This proves that σ_Y is a morphism of Galois descent.

The second assertion follows from proposition 5.8.8. □

To conclude this chapter, let us recall the Gelfand duality theorem between the category of commutative $\mathbb{C}^*$-algebras and that of compact Hausdorff spaces. With every commutative $\mathbb{C}^*$-algebra, one associates the spectrum of its locale of closed ideals, which is a compact Hausdorff space. And with every compact Hausdorff space X, one associates the $\mathbb{C}^*$-algebra $\mathcal{C}(X, \mathbb{C})$ of continuous maps. This yields a contravariant equivalence of categories.

Theorem 5.8.9, via the Gelfand duality, can thus also be seen as a Galois theorem for commutative $\mathbb{C}^*$-algebras, the corresponding Galois equivalence being this time contravariant, as in the case of rings.