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A coarse Mayer–Vietoris principle

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Introduction

In [**1**], [**4**], and [**6**] the authors have studied index problems associated with the 'coarse geometry' of a metric space, which typically might be a complete noncompact Riemannian manifold or a group equipped with a word metric. The second author has introduced a cohomology theory, coarse cohomology, which is functorial on the category of metric spaces and coarse maps, and which can be computed in many examples. Associated to such a metric space there is also a C^*-algebra generated by locally compact operators with finite propagation. In this note we will show that for suitable decompositions of a metric space there are Mayer–Vietoris sequences both in coarse cohomology and in the K-theory of the C^*-algebra. As an application we shall calculate the K-theory of the C^*-algebra associated to a metric cone. The result is consistent with the calculation of the coarse cohomology of the cone, and with a 'coarse' version of the Baum–Connes conjecture.

1. *Mayer–Vietoris sequence in coarse cohomology*

In [**4**], the second author remarked that there is not in general a Mayer–Vietoris sequence for coarse cohomology. In other words, if M is a proper metric space ('proper' means that closed and bounded sets are compact), and if A and B are closed subspaces with $M = A \cup B$, then it is not in general true that there is a long exact sequence

$$\ldots \to HX^q(M) \to HX^q(A) \oplus HX^q(B) \to HX^q(A \cap B) \to HX^{q+1}(M) \to \ldots.$$

One can see this simply by taking M to be a two point space, and A and B disjoint one point subspaces.

Even in ordinary cohomology, though, one does not expect to have a Mayer–Vietoris sequence for every decomposition of a space; some kind of excisiveness property is needed, for instance that $A^\circ \cup B^\circ = M$ (compare section 4·6 of [**5**]). Since in coarse theory definitions involving small open sets get replaced by definitions involving large bounded neighbourhoods, the following is perhaps not entirely unexpected.

Definition 1. Let M be a proper metric space, and let A and B be closed subspaces with $M = A \cup B$. We say that (A,B) is an ω-excisive couple, or that $X = A \cup B$ is an ω-excisive decomposition, if for each $R > 0$ there is some $S > 0$ such that

$$\mathrm{Pen}(A;R) \cap \mathrm{Pen}(B;R) \subseteq \mathrm{Pen}(A \cap B;S).$$

(As in [**4**], $\mathrm{Pen}(A;R)$ denotes the set of points in M of distance at most R from A.)

Example 1. Let $M = \mathbb{R}$, with $A = \{x \in \mathbb{R} : x \geqslant 0\}$ and $B = \{x \in \mathbb{R} : x \leqslant 0\}$. Then (A,B) is an ω-excisive couple. More generally let N be a compact path metric space and let $\Phi : [0,\infty) \to [0,\infty)$ be a weight function, tending to infinity, describing a metric on the cone CN (see paragraph 3·46 in [**4**]). If $N = N_1 \cup N_2$ is a decomposition into closed subspaces, the corresponding decomposition $C_\Phi N = C_\Phi N_1 \cup C_\Phi N_2$ is ω-excisive.

Example 2. Let M be the space of Remark 2·70 in [**4**], that is,

$$M = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y \in \{0,1\}, \text{ or } x = 0 \text{ and } 0 \leqslant y \leqslant 1\},$$

equipped with the metric induced from $\mathbb{R}^2$. Let

$$A = \{(x,y) \in M : y \leqslant \tfrac{1}{2}\} \quad \text{and} \quad B = \{(x,y) \in M : y \geqslant \tfrac{1}{2}\}.$$

Then $A \cap B$ contains just one point, but $\mathrm{Pen}(A;1) \cap \mathrm{Pen}(B;1) = M$, so that this decomposition is not ω-excisive.

LEMMA 1. *The decomposition $M = A \cup B$ is ω-excisive if and only if for each $R > 0$, the natural map*

$$A \cap B \to \mathrm{Pen}(A;R) \cap \mathrm{Pen}(B;R)$$

is a bornotopy-equivalence.

We remind the reader that two coarse maps $F_1, F_2 : M \to M'$ are bornotopic if there is a constant $R > 0$ such that $d(F_1(m), F_2(m)) \leqslant R$, for all $m \in M$ (the definition of coarse map is given in Section 4). This notion of bornotopy leads to a notion of bornotopy equivalence, just as from homotopy we derive the notion of homotopy equivalence.

Proof. If (A,B) is ω-excisive there is an $S > 0$ such that

$$\mathrm{Pen}(A;R) \cap \mathrm{Pen}(B;R) \subseteq \mathrm{Pen}(A \cap B;S).$$

Therefore, $A \cap B$ is ω-dense in $\mathrm{Pen}(A;R) \cap \mathrm{Pen}(B;R)$, and by Proposition 2·6 of [**4**] the inclusion is a bornotopy equivalence. Conversely, if the natural map is a bornotopy equivalence then the existence of a bornotopy inverse implies the existence of a suitable $S > 0$ as in the definition above. ∎

The main result of this section is as follows.

THEOREM 1. *Suppose that $M = A \cup B$ is an ω-excisive decomposition. Then there is an exact Mayer–Vietoris sequence in coarse cohomology, of the form*

$$\ldots \to HX^q(M) \to HX^q(A) \oplus HX^q(B) \to HX^q(A \cap B) \to HX^{q+1}(M) \to \ldots.$$

The proof of this requires a couple of lemmas. We begin by considering certain inverse limit complexes. For $n = 0, 1, 2, \ldots$ let

$$C_n^* = CX^*(\mathrm{Pen}(A;n)) \oplus CX^*(\mathrm{Pen}(B;n)).$$

The complexes C_n^* form an inverse sequence under the obvious surjective restriction maps, and we define

$$C^* = \varprojlim C_n^*.$$

We may define C^* concretely as follows: an element of C^q is a pair (ϕ_A, ϕ_B) of ω-bounded Borel functions on M^{q+1}, such that the restriction of ϕ_A to any penumbra $\mathrm{Pen}(A;n)$ is a coarse co-chain, and similarly for ϕ_B. We also let

$$D_n^* = CX^*(\mathrm{Pen}(A;n) \cap \mathrm{Pen}(B;n)),$$

and let

$$D^* = \varprojlim D_n^*.$$

It has a similar explicit description.

LEMMA 2. *If C^* and D^* are the complexes defined above for an ω-excisive decomposition $X = A \cup B$, then the natural restriction maps induce isomorphisms*

$$H^q(C) \cong HX^q(A) \oplus HX^q(B)$$

and

$$H^q(D) \cong HX^q(A \cap B).$$

Proof. By standard results on cohomology and inverse limits [2], there is a short exact sequence

$$0 \to \varprojlim{}^1 H^{q-1}(C_n) \to H^q(C) \to \varprojlim H^q(C) \to 0.$$

But since the inclusions $A \to \mathrm{Pen}(A;n)$ and $B \to \mathrm{Pen}(B;n)$ are bornotopy equivalences, it follows from Proposition 2·6 of [4] that the cohomology groups $H^q(C_n)$ are all isomorphic by restriction to $HX^q(A) \oplus HX^q(B)$. The result for the complex C^* follows. The proof for D^* is similar, making use of Lemma 1. ∎

Consider the sequences of complexes

$$0 \to CX^*(M) \xrightarrow{i_n} C_n^* \xrightarrow{j_n} D_n^* \to 0,$$

where the maps are the usual ones of the Mayer–Vietoris sequence, that is, i_n is a sum of two restriction maps and j_n is a difference of two restriction maps. These sequences are not exact in general. However, by proceeding to the inverse limit we obtain a sequence

$$(*) \qquad 0 \to CX^*(M) \xrightarrow{i} C^* \xrightarrow{j} D^* \to 0,$$

and we have:

LEMMA 3. *The sequence $(*)$ is exact (whether or not (A, B) is ω-excisive).*

Proof. We will make use of the explicit descriptions of the inverse limit complexes C^* and D^* given above. It is clear that i is injective, so that the sequence is exact at CX^*. An element of $\mathrm{Ker}(j)$ can be described as a function $\phi: M^{q+1} \to \mathbb{R}$ such that the restriction of ϕ to each of the sets $\mathrm{Pen}(A;n)$ and $\mathrm{Pen}(B;n)$ is a coarse co-chain there. Let ϕ be such a function. Suppose that

$$(x_0, \ldots, x_q) \in \mathrm{Supp}(\phi) \cap \mathrm{Pen}(\Delta; R).$$

Then $d(x_0, x_k) \leqslant 2R$ for $k = 0, \ldots, q$, and so if n is the least integer greater than $2R$, then either all the x_k belong to $\mathrm{Pen}(A; n)$ or else all the x_k belong to $\mathrm{Pen}(B; n)$. Since ϕ restricts a coarse cocycle on each of these two sets, we find that $\mathrm{Supp}(\phi) \cap \mathrm{Pen}(\Delta; R)$ is compact. In other words, $\phi \in \mathrm{Image}(i)$. This shows that the sequence is exact at C^*.

Finally we must prove the exactness at D^*. An element of D^q is a function $\psi: M^{q+1} \to \mathbb{R}$ whose restriction to each $\mathrm{Pen}(A; n) \cap \mathrm{Pen}(B; n)$ is a coarse co-chain. Choose a bounded, continuous bump function β on M with $\mathrm{Supp}(\beta) \subseteq \mathrm{Pen}(\mathrm{A}; 1)$ and $\mathrm{Supp}(1-\beta) \subseteq \mathrm{Pen}(B; 1)$, and define functions ϕ_A and ϕ_B on M^{q+1} by

$$\phi_A(x_0, \ldots, x_q) = (1-\beta(x_0))\, \psi(x_0, \ldots, x_q),$$

$$\phi_B(x_0, \ldots, x_q) = \beta(x_0)\, \psi(x_0, \ldots, x_q).$$

Then $\psi = \phi_A + \phi_B$, and we claim that $(\phi_A, -\phi_B) \in C^*$; this will then show that j is surjective. It is enough to show that ϕ_B restricts to a coarse co-chain on each $\mathrm{Pen}(B; n)$, the proof for ϕ_A being analogous. Suppose then that

$$(x_0, \ldots, x_q) \in \mathrm{Supp}(\phi_B) \cap \mathrm{Pen}(\Delta; R),$$

with each $x_k \in \mathrm{Pen}(B; n)$. Necessarily, $x_0 \in \mathrm{Pen}(A; 1)$, and so each $x_k \in \mathrm{Pen}(A; m)$, where m is the least integer greater than $2R+1$. Thus $(x_0, \ldots, x_q)$ belongs to the support of the restriction of ψ to $\mathrm{Pen}(A; m) \cap \mathrm{Pen}(B; n)$, which is, by hypothesis, a compact set. ∎

We can now prove Theorem 1. By Lemma 3, the sequence $(*)$ is a short exact sequence of complexes. By standard homological algebra, there is associated to it a long exact sequence of cohomology groups. Lemma 2 identifies the cohomology groups of the complexes C^* and D^*, and thereby shows that this long exact sequence is the Mayer–Vietoris sequence we require.

2. *Decompositions of the coarse compactification*

The following ideas were introduced in [1] and [**4**].

Definition 1. Let M be a proper metric space. A bounded continuous function f on M has *vanishing variation at infinity* if for every $R > 0$ the function

$$V_R f(x) = \max\{|f(x) - f(y)| : d(x, y) \leqslant R\}$$

converges to zero at infinity. Denote by $C_h(M)$ the C^*-algebra of all bounded continuous functions on M with vanishing variation at infinity.

Definition 2. A coarse compactification of M is a compactification $\bar{M}$ (that is, a compact Hausdorff space which contains M as a dense open subset) with the property that every continuous function on $\bar{M}$ restricts to a bounded continuous function on M with vanishing variation at infinity.

There is a universal coarse compactification, characterized by the property that every bounded continuous function on M with vanishing variation at infinity extends to a continuous function on $\bar{M}$. See [1, **4**]. Thus $C_h(M) \cong C(\bar{M})$ if $\bar{M}$ is universal.

In this section we shall prove the following result.

PROPOSITION 1. *Let M be a proper metric space, and let A and B be closed subspaces whose union is M. The decomposition (A, B) is ω-excisive if and only if*

$$\bar{A} \cap \bar{B} = \overline{A \cap B},$$

where the bar denotes the closure inside the universal compactification.

For F a closed subset of M denote by $\mathscr{I}(F)$ the ideal in $C_h(M)$ consisting of functions which vanish on F. In view of the Gelfand–Neumark correspondence between compact spaces and commutative C^*-algebras, Proposition 1 is easily seen to be equivalent to the following assertion about $C_h(M)$.

PROPOSITION 2. *The decomposition $M = A \cup B$ is ω-excisive if and only if*

$$\mathscr{I}(A)+\mathscr{I}(B) = \mathscr{I}(A \cap B).$$

Proof. Let $f \in \mathscr{I}(A \cap B)$, and choose a continuous partition of unity $\{i_A, i_B\}$ with i_A and i_B supported within distance 1 of A and B respectively. Then

$$f = i_A f + i_B f,$$

and the functions $i_A f$ and $i_B f$ are continuous and vanish on $B \backslash \mathrm{Pen}(A;1)$ and $A \backslash \mathrm{Pen}(B;1)$ respectively. Suppose now that (A,B) is ω-excisive. Given $R > 1$, choose $S > R$ such that

$$\mathrm{Pen}(A;2R) \cap \mathrm{Pen}(B;2R) \subseteq \mathrm{Pen}(A \cap B;S).$$

The set $M \backslash \mathrm{Pen}(A \cap B;S)$ falls into two pieces, one contained in A and one in B, with a distance of more than R separating the two. On the first we have $i_A f = f$; on the second we have $i_A f = 0$; and on $\mathrm{Pen}(A \cap B;S)$ we have $f \to 0$ at infinity, since $f \in \mathscr{I}(A \cap B)$. Considering $\mathrm{Pen}(A \cap B;S)$ and these two pieces separately it follows easily that the variation $V_R(i_A f)$ vanishes at infinity on M, so that $i_A f, i_B f \in C_h(M)$. This shows that if (A,B) is ω-excisive then $\mathscr{I}(A)+\mathscr{I}(B) = \mathscr{I}(A \cap B)$. Suppose, on the other hand, that (A,B) is not ω-excisive. Then for some $R > 0$ there is a sequence of points $x_n \in M$ such that

$$d(x_n, A) \leqslant R \quad \text{and} \quad d(x_n, B) \leqslant R, \quad \text{but} \quad d(x_n, A \cap B) \geqslant 2^n.$$

We may also arrange that $d(x_n, x_k) \geqslant 2^n$, for $k < n$, and then it is a simple matter to build a bounded continuous function f on M, as a sum of smoother and smoother bump functions centred at the points x_n, for which $V_R f(x) \to 0$, as $x \to \infty$, and $f = 0$ on $A \cap B$, but $f(x_n) = 1$ for all n. Note that if $g \in \mathscr{I}(A)+\mathscr{I}(B)$ then $g(x_n) \to 0$. So our function $f \in \mathscr{I}(A \cap B)$ does not lie in $\mathscr{I}(A)+\mathscr{I}(B)$. ∎

3. *Some K-theory preliminaries*

We gather together a few facts from K-theory (none of them are new) which we shall need in the remaining sections of the paper.

LEMMA 1. *Let $\mathscr{A}$ and $\mathscr{B}$ be closed, two-sided ideals in a C^*-algebra $\mathscr{M}$. Assume that $\mathscr{A}+\mathscr{B}$ is dense in $\mathscr{M}$. Then $\mathscr{A}+\mathscr{B} = \mathscr{M}$, and the map $a \oplus b \mapsto a+b$ produces an isomorphism of C^*-algebras*

$$\mathscr{A}/(\mathscr{A} \cap \mathscr{B}) \oplus \mathscr{B}/(\mathscr{A} \cap \mathscr{B}) \cong \mathscr{M}/(\mathscr{A} \cap \mathscr{B}).$$

Proof. Since $\mathscr{A}\mathscr{B} \subseteq \mathscr{A} \cap \mathscr{B}$ the map $a \oplus b \mapsto a+b$ passes to an injective $*$-homomorphism

$$\mathscr{A}/(\mathscr{A} \cap \mathscr{B}) \oplus \mathscr{B}/(\mathscr{A} \cap \mathscr{B}) \to \mathscr{M}/(\mathscr{A} \cap \mathscr{B}).$$

By basic C^*-algebra theory the range is closed, while by hypothesis the range is dense. Consequently our map is an isomorphism. The fact that $\mathscr{A}+\mathscr{B} = \mathscr{M}$ follows immediately from this. ∎

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{M}$ be C^*-algebras, as in Lemma 1. There is a Mayer–Vietoris sequence in K-theory:

$$\ldots \to K_j(\mathscr{A} \cap \mathscr{B}) \xrightarrow{j} K_j(\mathscr{A}) \oplus K_j(\mathscr{B}) \xrightarrow{i} K_j(\mathscr{M}) \xrightarrow{\partial} K_{j-1}(\mathscr{A} \cap \mathscr{B}) \to \ldots.$$

One way to define this is to form the C^*-algebra

$$\mathscr{C} = \{f \in C([0,1], \mathscr{M}) : f(0) \in \mathscr{A}, f(1) \in \mathscr{B}\},$$

and analyse the exact sequence in K-theory arising from the ideal

$$\mathscr{T} = \{f \in C([0,1], \mathscr{M}) : f(0) = f(1) = 0\}.$$

Since $\mathscr{T}$ is just the suspension of $\mathscr{M}$, we have that $K_*(\mathscr{T}) \cong K_{*+1}(\mathscr{M})$. The quotient $\mathscr{C}/\mathscr{T}$ is isomorphic to $\mathscr{A} \oplus \mathscr{B}$. The inclusion into $\mathscr{C}$ of the algebra of continuous $\mathscr{A} \cap \mathscr{B}$-valued functions on $[0,1]$ is easily seen to induce an isomorphism on K-theory. So the exact K-theory sequence associated to $\mathscr{C}$ and $\mathscr{T}$ gives a Mayer–Vietoris sequence as claimed. It is functorial, in the sense that if $\mathscr{M}', \mathscr{A}', \mathscr{B}'$ is another system of C^*-algebras, as in Lemma 1, and if $\Phi: \mathscr{M}' \to \mathscr{M}$ maps $\mathscr{A}'$ into $\mathscr{A}$, and $\mathscr{B}'$ into $\mathscr{B}$, then the obvious diagram relating Mayer–Vietoris sequences commutes.

At several points we shall need the following observation.

LEMMA 2. *Let $\Phi: \mathscr{A} \to \mathscr{B}$ be a homomorphism of C^*-algebras and let W be a partial isometry in the multiplier algebra of $\mathscr{B}$ such that $\Phi(a)W^*W = \Phi(a)$, for all $a \in \mathscr{A}$. Then* $\mathrm{Ad}(W) \circ \Phi(a) = W\Phi(a)W^*$ *is a $*$-homomorphism from $\mathscr{A}$ to $\mathscr{B}$ and passing to the induced maps on K-theory we have*

$$(\mathrm{Ad}(W) \circ \Phi)_* = \Phi_* : K_*(\mathscr{A}) \to K_*(\mathscr{B}).$$

Proof. Embedding $\mathscr{B}$ into $\mathrm{Mat}_2(\mathscr{B})$ in the 'top left corner' (which gives an isomorphism on K-theory), and replacing W by

$$\begin{pmatrix} W & 1-WW^* \\ 1-W^*W & W^* \end{pmatrix},$$

we reduce to the case where W is a unitary, which is well known. ∎

Finally, we shall need

LEMMA 3. *Let $\Phi, \Psi: \mathscr{A} \to \mathscr{B}$ be orthogonal homomorphisms of C^*-algebras, meaning that $\Phi[\mathscr{A}]\Psi[\mathscr{A}] = 0$. Suppose that there is an isometry V in the multiplier algebra of $\mathscr{B}$ such that*

$$V(\Phi(a) + \Psi(a))V^* = \Psi(a), \quad \textit{for all } a \in \mathscr{A}.$$

Then the induced map

$$\Phi_* : K_*(\mathscr{A}) \to K_*(\mathscr{B})$$

is the zero map.

Proof. We note that under the hypothesis of orthogonality the map $\Psi + \Phi$ is a $*$-homomorphism. By hypothesis, $\mathrm{Ad}(V) \circ (\Phi + \Psi) = \Psi$. Passing to the induced maps on K-theory and using Lemma 2, we get

$$\Psi_* = (\Phi + \Psi)_*.$$

But it is easily shown that

$$(\Phi + \Psi)_* = \Phi_* + \Psi_*,$$

and so subtracting Ψ_* from everything we get $\Phi_* = 0$. ∎

4. *The algebra* $C^*(M)$

Let M be a proper metric space. Recall from [**4**] that a standard M-module is a separable Hilbert space equipped with a faithful and non-degenerate representation of $C_0(M)$ whose range contains no non-zero compact operator.

Definition 1. Let H_M and $H_{M'}$ be standard M and M'-modules, respectively. The support of a bounded linear operator $T: H_M \to H_{M'}$ is the complement of the set of points $(m, m') \in M \times M'$ for which there exist functions $f \in C_0(M)$ and $f' \in C_0(M')$ such that

$$f'Tf = 0, \quad f(m) \neq 0, \quad \text{and} \quad f'(m') \neq 0.$$

We shall say that T is properly supported if the projection from $\operatorname{Supp}(T)$ to M and M' are proper maps.

Definition 2. A bounded linear operator $T: H_M \to H_{M'}$ is *locally compact* if the operators $f'T$ and Tf are compact, for every $f \in C_0(M)$ and $f' \in C_0(M')$.

LEMMA 1. (*a*) *If* $T: H_M \to H_{M'}$ *and* $T': H_{M'} \to H_{M''}$ *are bounded operators then*

$$\operatorname{Supp}(T'T) \subseteq \left\{ (m, m'') \in M \times M'' : \begin{array}{l} \exists m' \in M' : (m, m') \in \operatorname{Supp}(T) \\ \textit{and } (m', m'') \in \operatorname{Supp}(T') \end{array} \right\}.$$

(*b*) *If T is properly supported and S is locally compact then* (*assuming the compositions make sense*) *the operators ST and TS are locally compact.*

Proof. Straightforward. ∎

Definition 3. An operator $T: H_M \to H_M$ has finite propagation if

$$\sup \{ d(m_1, m_2) : (m_1, m_2) \in \operatorname{Supp}(T) \} < \infty.$$

It follows from part (*a*) of Lemma 1 that the set of finite propagation operators on H_M is a $*$-subalgebra of the algebra of all bounded operators on H_M.

Definition 4. Denote by $C^*(M, H_M)$ the norm-closure of the $*$-algebra of all locally compact, finite propagation operators on H_M.

It is easy to prove that $C^*(M, H_M)$ is the same as the C^*-algebra $\overline{\mathscr{B}_{H_M}}$ of [**4**]. It follows from Lemma 1 that any finite propagation operator is a multiplier of $C^*(M, H_M)$; this fact will be useful later.

We are interested in investigating the functoriality of $C^*(M, H_M)$ within the context of coarse geometry.

Definition 5. A coarse map from M to M' is a proper† Borel map $F: M \to M'$ such that for every $R > 0$ there exists $S > 0$ with

$$d(m_1, m_2) \leqslant R \Rightarrow d(F(m_1), F(m_2)) \leqslant S.$$

The composition of coarse maps is a coarse map, and we obtain the coarse category of proper metric spaces, denoted UBB in [**4**].

† We say that a Borel map between proper metric spaces is a proper map if the inverse image of any bounded set is bounded.

LEMMA 2. *Let H_M and $H_{M'}$ be standard M and M'-modules and let $F: M \to M'$ be a coarse map. There exists an isometry $V: H_M \to H_{M'}$ such that for some $R > 0$*

$$\operatorname{Supp}(V) \subseteq \{(m, m') \in M \times M' : d(F(m), m') \leqslant R\}.$$

Proof. By spectral theory we can extend the representations of $C_0(M)$ and $C_0(M')$ on H_M and $H_{M'}$ to representations of the algebras of bounded Borel functions. Partition M' into Borel components M'_j, each with non-empty interior and uniformly bounded diameter. Denote by μ_j and μ'_j the characteristic functions of $F^{-1}[M'_j]$ and M'_j. Define an isometry V by taking an arbitrary direct sum of isometries $V_i: \mu_j H_M \to \mu'_j H'_M$. If we choose $S > 0$ so that

$$d(m_1, m_2) \leqslant 1 \Rightarrow d(F(m_1), F(m_2)) \leqslant S$$

then our isometry V satisfies the required support condition with

$$R = S + \sup \operatorname{diam}(M'_j) + 1. \quad ∎$$

With V as in the Lemma, it follows from Lemma 1 that the homomorphism $\operatorname{Ad}(V)$ maps $C^*(M, H_M)$ into $C^*(M', H_{M'})$.

LEMMA 3. *Let $F: M \to M'$ be a morphism and let $V_1, V_2: H_M \to H_{M'}$ be isometries satisfying the support condition in Lemma 2. The induced maps on K-theory are equal:*

$$\operatorname{Ad}(V_1)_* = \operatorname{Ad}(V_2)_* : K_*(C^*(M, H_M)) \to K_*(C^*(M', H_{M'})).$$

Proof. It follows from Lemma 1 that the partial isometry $V_2 V_1^*$ is a multiplier of $C^*(M', H_{M'})$. So the result follows from Lemma 2 of the previous section. ∎

The correspondence $M \mapsto K_*(C^*(M, H_M))$ becomes a functor on the category whose objects are pairs (M, H_M) and whose morphisms are coarse maps $F: M \to M'$. But it follows from functoriality that if H_M and H'_M are two standard M-modules then the map $\operatorname{Id}_*: K_*(C^*(M, H_M)) \to K_*(C^*(M, H'_M))$ is an isomorphism, so up to canonical isomorphism the group $K_*(C^*(M, H_M))$ does not depend on the choice of module.‡ So we might as well view $K^*(C^*(M, H_M))$ as a functor on the coarse category of proper metric spaces.

We note that our functor is 'bornotopy invariant', in the sense that bornotopic morphisms give rise to the same map in K-theory. This is because if F_1 and F_2 are bornotopic then the same isometry V will satisfy the support condition in Lemma 2 for both F_1 and F_2.

5. *Mayer–Vietoris sequence for $K_*(C^*(M))$*

In this section we shall drop the module H_M from our notation and write $C^*(M)$ in place of $C^*(M, H_M)$.

Definition 1. Let A be a closed subspace of a proper metric space M and let H_M be a standard M-module. Denote by $C^*(A, M)$ the operator-norm closure of the set of all locally compact, finite propagation operators T on H_M whose support is contained in $\operatorname{Pen}(A; R) \times \operatorname{Pen}(A; R)$, for some $R > 0$ (depending on T).

† It is easy to check that up to non-canonical isomorphism the C^*-algebra $C^*(M, H_M)$ itself does not depend on H_M.

We note that $C^*(A,M)$ is a closed two sided ideal in $C^*(M)$. If $V\colon H_A \to H_M$ is an isometry associated to the inclusion morphism $A \to M$ (as in Lemma 2 of the previous section) then the range of the map $\mathrm{Ad}(V)\colon C^*(A) \to C^*(M)$ lies within $C^*(A,M)$.

LEMMA 1. *The induced map*

$$\mathrm{Ad}(V)\colon K_*(C^*(A)) \to K_*(C^*(A,M))$$

is an isomorphism.

Proof. The C^*-algebra $C^*(A,M)$ is an inductive limit

$$C^*(A,M) = \varinjlim C^*(\mathrm{Pen}(A;n)) = \overline{\bigcup_{n=1}^{\infty} C^*(\mathrm{Pen}(A;n))},$$

where $C^*(\mathrm{Pen}(A,n))$ is viewed as acting on the standard module $\overline{C_0(\mathrm{Pen}(A;n))H_M}$. Consequently

$$K_*(C^*(A,M)) = \varinjlim K_*(C^*(\mathrm{Pen}(A,n))).$$

Since the inclusions $A \subset \mathrm{Pen}(A;n)$ and $\mathrm{Pen}(A;n) \subset \mathrm{Pen}(A;n+1)$ are bornotopy equivalences the induced maps on K-theory are all isomorphisms. ∎

LEMMA 2. *Let (A,B) be a decomposition of M. Then*

$$C^*(A,M) + C^*(B,M) = C^*(M).$$

If (A,B) is ω-excisive then

$$C^*(A,M) \cap C^*(B,M) = C^*(A \cap B,M).$$

Proof. Let T be a locally compact, finite propagation operator on H_M. Extend the representation of $C_0(M)$ on H_M to a representation of the bounded Borel functions, and let $P\colon H_M \to H_M$ be the projection operator corresponding to the characteristic function of A. Then $T = PT + (I-P)T$ is a decomposition of T into a sum of an operator in $C^*(A,M)$ and an operator in $C^*(B,N)$. This shows that $C^*(A,M) + C^*(B,M)$ is dense in $C^*(M)$, and we can apply Lemma 1 of Section 3 to complete the first part of the proof.

For the second part, note that $C^*(A \cap B,M) \subseteq C^*(A,M) \cap C^*(B,M)$, whether or not the decomposition is ω-excisive. For the converse, recall that by basic C^*-algebra theory the intersection of the ideals $C^*(A,M)$ and $C^*(B,M)$ is equal to their product. So it suffices to show that if (A,B) is ω-excisive, and if

$$\mathrm{Supp}(T_A) \subseteq \mathrm{Pen}(A;R') \times \mathrm{Pen}(A;R')$$

and

$$\mathrm{Supp}(T_B) \subseteq \mathrm{Pen}(B;R'') \times \mathrm{Pen}(B;R''),$$

then

$$\mathrm{Supp}(T_A T_B) \subseteq \mathrm{Pen}(A \cap B;S) \times \mathrm{Pen}(A \cap B;S),$$

for some $S > 0$. But this follows immediately from Lemma 1 of Section 4, together with the definition of ω-excisiveness. ∎

Combining Lemmas 1 and 2 with the discussion in Section 3 we obtain the following Mayer–Vietoris sequence for an ω-excisive decomposition of M:

$$\ldots \to K_j(C^*(A \cap B)) \to K_j(C^*(A)) \oplus K_j(C^*(B)) \to K_j(C^*(M)) \to K_{j-1}(C^*(A \cap B)) \to \ldots.$$

6. *Relation with K-homology*

Definition 1. Let X be a compact metric space and let Y be a closed subset of X. Let H be a Hilbert space equipped with a faithful non-degenerate representation of $C(X)$ whose range contains no non-zero compact operator. Denote by $D^*(X, Y)$ the C^*-algebra of bounded operators T on H such that

(1) if $f \in C(X)$ then $fT - Tf$ is a compact operator; and

(2) if $f \in C(X)$ and $f = 0$ on Y then Tf and fT are compact operators.

This definition is taken from [1], where the notation

$$D^*(X, Y) = \bar{D}(C(X), C_0(X \backslash Y))$$

is used. The following result is proved in [1].

THEOREM 1. *Suppose that X and Y are as above, with Y non-empty. Denote by $\tilde{K}_*(Y)$ the reduced Steenrod K-homology of Y. There is a natural isomorphism*

$$K_j(D^*(X, Y)) \cong \tilde{K}_{j-1}(Y). \quad \blacksquare$$

Of course if Y is empty (but X is not) then $D^*(X, Y)$ is just the algebra of compact operators, so that $K_0(D^*(X, Y)) \cong \mathbb{Z}$ and $K_1(D^*(X, Y)) \cong 0$.

The term 'natural' in the statement of the theorem is explained by the following result.

PROPOSITION 1. *If $F: (X, Y) \to (X', Y')$ is a continuous map of compact metric space pairs then there is an isometry*

$$V: H \to H'$$

with the property that $V(f \circ F) - fV$ is a compact operator, for every $f \in C(X')$. The homomorphism $\mathrm{Ad}(V)$ *maps $D^*(X, Y)$ into $D^*(X', Y')$, and the induced map on K-theory is independent of the choice of V.*

Proof. See [1]. $\blacksquare$

It follows that up to canonical isomorphism, $K_*(D^*(X, Y))$ does not depend on the choice of Hilbert space H,† and we obtain a functor $(X, Y) \mapsto K_*(D^*(X, Y))$ on the category of compact metric space pairs. Of course, in view of Theorem 1 this functor factors through the functor $(X, Y) \mapsto Y$.

Suppose now that $X_M = \bar{M}$ is a metrizable coarse compactification of M. Let H_M be a standard M-module. As we have pointed out earlier, the representation of $C_0(M)$ on H_M extends to a representation of the bounded Borel functions; in particular it extends to a representation of $C(X_M) = C(\bar{M})$. Let

$$Y_M = \bar{M} \backslash M$$

be the 'corona' of M in X_M and form the algebra of operators $D^*(X_M, Y_M)$ on H_M.

LEMMA 1.

(*a*) $C^*(M) \subseteq D^*(X_M, Y_M)$.

(*b*) *Let A be a closed subset of M, and let $Y_A = Y_M \cap \bar{A}$ (the bar denotes closure in X_M). Then $C^*(A; M) \subseteq D^*(X_M, Y_A)$.*

† In fact different choices of H lead to isomorphic C^*-algebras $D^*(X, Y)$, but the isomorphism is not canonical.

Proof. See Proposition 5·18 in [4]. ∎

Definition 2. Let M be a proper metric space and let X_M be a metrizable coarse compactification of M with corona Y_M. We define a homomorphism

$$\beta(M, Y_M): K_*(C^*(M)) \to \tilde{K}_{*-1}(Y_M)$$

by composing the K-theory map $K_*(C^*(M)) \to K_*(D(X_M, Y_M))$ induced by the inclusion in Lemma 1 (*a*) with the isomorphism $K(D^*(X_M, Y_M)) \cong \tilde{K}_{*-1}(Y_M)$ given by Theorem 1.

The main result of this section is as follows. Let $M = A \cap B$ be an ω-excisive decomposition of a proper metric space. Let X_M and Y_M be as above and let

$$Y_A = Y_M \cap \bar{A} \quad \text{and} \quad Y_B = Y_M \cap \bar{B}.$$

Assume that
$$Y_A \cap Y_B = Y_M \cap \overline{A \cap B},$$

so that Y_A, Y_B, and $Y_A \cap Y_B$ may be regarded as coronas of A, B and $A \cap B$, respectively. Notice that Proposition 1 of Section 3 states that this assumption always holds for the *universal* compactification; however, since the universal compactification is not metrizable, it does not seem possible to use it directly in this context.

THEOREM 2. *If the maps $\beta(A, Y_A)$, $\beta(B, Y_A)$ and $\beta(A \cap B, Y_A \cap Y_B)$ are isomorphisms then so is $\beta(M, N_M)$.*

The key to the proof is the following observation. Fix a Hilbert space H, equipped with a faithful non-degenerate representation of $C(X)$ whose range contains no non-zero compact operator, and view all the C^*-algebras below as subalgebras of $B(H)$.

LEMMA 2. *Let $Y = Y_1 \cup Y_2$ be any decomposition of Y into closed subsets, and form the C^*-subalgebras $D^*(X, Y_1 \cap Y_2)$, $D^*(X, Y_1)$, $D^*(X, Y_2)$, and $D^*(X, Y)$ of $B(H)$. Then*
(*a*) *$D^*(X, Y_1)$ and $D^*(X, Y_2)$ are ideals in $D^*(X, Y)$.*
(*b*) *$D^*(X, Y_1) + D^*(X, Y_2) = D^*(X, Y)$.*
(*c*) *$D^*(X, Y_1) \cap D^*(X, Y_2) = D^*(X, Y_1 \cap Y_2)$.*

Proof. A simple partition of unity argument. ∎

Proof of Theorem 2. The inclusion maps provided by Lemma 2 give rise to a commutative diagram of Mayer–Vietoris sequences

$$\begin{tikzcd}
K_j(C^*(A \cap B; M)) \arrow[r] \arrow[d, "i_{A \cap B}"'] & K_j(C^*(A; M)) \oplus K_j(C^*(B; M)) \arrow[r] \arrow[d, "i_A \oplus i_B"'] & {} \\
K_j(D^*(X_M, Y_A \cap Y_B)) \arrow[r] & K_j(D^*(X_M, Y_A)) \oplus K_j(D^*(X_M, Y_B)) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
K_j(C^*(M)) \arrow[r] \arrow[d, "i_M"] & K_{j-1}(C^*(A \cap B; M)) \arrow[r] \arrow[d, "i_{A \cap B}"] & {} \\
K_j(D^*(X_M, Y_M)) \arrow[r] & K_{j-1}(D^*(X_M, Y_A \cap Y_B)) \arrow[r] & {}
\end{tikzcd}$$

It follows from the hypotheses, together with Lemma 1 in Section 5 and Theorem 1 above, that the maps i_A, i_B and $i_{A\cap B}$ are isomorphisms. So it follows from the Five Lemma that i_M is an isomorphism. ∎

Remark. There is a similar result in coarse cohomology, based on the commutativity of the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q-1}(Y_M) \arrow[r] \arrow[d, "T"'] & H^{q-1}(Y_A) \oplus H^{q-1}(Y_B) \arrow[r] \arrow[d, "T"'] & {} \\
\cdots \arrow[r] & HX^q(M) \arrow[r] & HX^q(A) \oplus HX^q(B) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
H^{q-1}(Y_A \cap Y_B) \arrow[r] \arrow[d, "T"] & H^q(Y_M) \arrow[r] \arrow[d, "T"] & \cdots \\
HX^q(A \cap B) \arrow[r] & HX^{q+1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

in which the top row is the Mayer–Vietoris sequence of ordinary cohomology, the bottom row is the Mayer–Vietoris sequence in coarse cohomology, and the vertical maps are the transgressions of [**4**] (which exist as long as the spaces Y are sufficiently well behaved, e.g. finite polyhedra).

7. *Cones*

In this section we will use the Mayer–Vietoris sequence to calculate the K-theory of C^*-algebra for a space M which is a Euclidean cone CN, where N is a finite simplicial complex.

The metric space CN may be defined as follows. Embed N piecewise linearly (or piecewise smoothly) into a sphere centred at the origin in a Euclidean space. Then CN is the union of all half lines beginning at the origin and passing through a point in N. We give CN the metric it inherits as a subspace of Euclidean space. Up to bornotopy equivalence the space CN is independent of the embedding of N used. We note that CN has an obvious coarse compactification, for which the corona is N.

PROPOSITION 1. *Let $C\Delta$ be the Euclidean cone on a single simplex Δ. Then*

$$K_*(C^*(C\Delta)) = 0.$$

Proof. The cone on an n-simplex is (bornotopy equivalent to) the octant

$$M = \{(x_0, \ldots, x_n) \in \mathbb{R}^{N+1} : x_i \geqslant 0\}$$

in Euclidean space. For our standard module we take $L^2(M)$ (with respect to Lebesgue measure). Let

$$L^2(M)_\infty = L^2(M) \oplus L^2(M) \oplus \ldots,$$

and consider the inclusion

$$\Phi: \quad T \mapsto T \oplus 0 \oplus 0 \oplus \ldots$$

of $C^*(M, L^2(M))$ into $C^*(M, L^2(M)_\infty)$. By Lemma 3 of Section 4, and the remarks following it, the induced map

$$\Phi_*: K_*(C^*(M, L^2(M))) \to K_*(C^*(M, L^2(M)_\infty))$$

is an isomorphism, so it suffices to show that $\Phi_* = 0$. Define an isometry W on $L^2(M)$ by

$$W^*\phi(x_0, x_1, \ldots, x_n) = \phi(x_0 + 1, x_1, \ldots, x_n),$$

and define an isometry V on $L^2(M)_\infty$ by

$$V(\phi_1 \oplus \phi_2 \oplus \ldots) = (0 \oplus W\phi_1 \oplus W\phi_2 \oplus \ldots).$$

It has finite propagation, and is consequently a multiplier of $C^*(M, L^2(M)_\infty)$. Define a $*$-homomorphism

$$\Psi: C^*(M, L^2(M)) \to C^*(M, L^2(M)_\infty)$$

by the formula

$$\Psi(T) = 0 \oplus WTW^* \oplus W^2TW^{*2} \oplus W^3TW^{*3} \oplus \ldots.$$

Note that despite the fact that the direct sum defining $\Psi(T)$ is infinite the resulting operator is still locally compact and finite propagation. To complete the proof we note that the homomorphisms Φ and Ψ are orthogonal, and that $\mathrm{Ad}(V) \circ (\Phi + \Psi) = \Psi$, so that by Lemma 3 of Section 4, $\Phi_* = 0$. ∎

PROPOSITION 2. *Let N be a finite simplicial complex. Then the map*

$$\beta: K_*(C^*(C(N))) \to \tilde{K}_{*-1}(N)$$

is an isomorphism.

Proof. If N is empty, let us define $\tilde{K}_{*-1}(N)$ to be $K_*(D^*(CN, N))$; CN is a single point, and $D^*(CN, N)$ is the algebra of compact operators. Since $C^*(CN)$ is also the algebra of compact operators, the result is true for N empty. If N consists of a single simplex, the result is true by Proposition 1. The general result now follows by induction on the number of simplices, using Theorem 2 of Section 6. ∎

This result is a C^*-analogue of a purely algebraic theorem of Pedersen and Weibel [3]. As suggested in [4], the result can also be considered to be a verification of the Baum–Connes conjecture in the context of coarse geometry.

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REFERENCES

[1] N. HIGSON. On the relative K-homology theory of Baum and Douglas. *J. Functional Anal.*, to appear.

[2] J. MILNOR. On axiomatic homology theory. *Pacific J. Math.* **12** (1962), 337–341.

[3] E. K. PEDERSEN and C. A. WEIBEL. A nonconnective delooping of algebraic K-theory. *Springer Lecture Notes in Mathematics* **1126** (1985), 306–320.

[4] J. ROE. Coarse cohomology theory and index theory on complete Riemannian manifolds. *Memoirs Amer. Math. Soc.*, to appear.

[5] E. SPANIER. *Algebraic topology* (McGraw-Hill, 1966).

[6] G. YU. K-theoretic indices of Dirac type operators on complete manifolds and the Roe algebra. PhD Thesis, SUNY at Stony Brook, 1991.