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The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space*

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1 Introduction

Let Γ be a metric space; let H be a separable and infinite-dimensional Hilbert space. A map $f : \Gamma \to H$ is said to be a uniform embedding [10] if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}$ such that

(1) $\rho_1(d(x, y)) \leq \| f(x) - f(y)\| \leq \rho_2(d(x, y))$ for all $x, y \in \Gamma$;
(2) $\lim_{r\to+\infty} \rho_i(r) = +\infty$ for $i = 1, 2$.

The main purpose of this paper is to prove the following result:

Theorem 1.1. *Let Γ be a discrete metric space with bounded geometry. If Γ admits a uniform embedding into Hilbert space, then the coarse Baum–Connes conjecture holds for Γ.*

Recall that a discrete metric space Γ is said to have bounded geometry if $\forall r > 0$, $\exists N(r) > 0$ such that the number of elements in $B(x, r)$ is at most $N(r)$ for all $x \in \Gamma$, where $B(x, r) = \{y \in \Gamma : d(y, x) \leq r\}$. Every finitely generated group, as a metric space with a word-length metric, has bounded geometry.

Corollary 1.2. *Let Γ be a finitely generated group. If Γ, as a metric space with a word-length metric, admits a uniform embedding into Hilbert space, and its classifying space $B\Gamma$ has the homotopy type of a finite CW complex, then the strong Novikov conjecture holds for Γ, i.e. the index map from $K_*(B\Gamma)$ to $K_*(C_r^*(\Gamma))$ is injective.*

Corollary 1.2 follows from Theorem 1.1 and the descent principle [23]. By index theory, the strong Novikov conjecture implies the Novikov conjecture on the homotopy invariance of higher signatures (cf. [8] for an excellent

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survey of the Novikov conjecture). The class of finitely generated groups which admit a uniform embedding into Hilbert space contains a subclass of groups, groups with property A, which includes Gromov's word hyperbolic groups, discrete subgroups of connected Lie groups and amenable groups, and is closed under semi-direct product (cf. Theorem 2.2 and Proposition 2.6). In general, it is an open question if every finitely generated group (or every discrete metric space with bounded geometry) admits a uniform embedding into Hilbert space ([10], page 218, [11], page 67).

The following result follows from Theorem 1.1, Proposition 4.33 in [22], and the Lichnerowicz argument.

Corollary 1.3. *Let M be a complete Riemannian manifold with bounded geometry. If M is uniformly contractible and admits a uniform embedding into Hilbert space, then M can not have uniformly positive scalar curvature.*

In general, Gromov conjectures that a uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature [12]. Recall that a Riemannian manifold is said to have bounded geometry if it has bounded sectional curvature and positive injectivity radius, and a Riemannian manifold is said to be uniformly contractible if for every $r > 0$, there exists $R \geq r$ such that every ball with radius r can be contracted to a point in a ball with radius R. Riemannian manifolds with property A admit a uniform embedding into Hilbert space (cf. Theorem 2.7).

This paper is organized as follows. In Sect. 2, we introduce a geometric property (called property A) which guarantees the existence of a uniform embedding into Hilbert space. We also show that the class of groups with property A is closed under semi-direct product. In Sect. 3, we briefly recall the coarse Baum–Connes conjecture. In Sect. 4, we recall the localization algebra and discuss its application to the coarse Baum–Connes conjecture. The coarse Baum–Connes conjecture holds if and only if the evaluation map from the K-theory of localization algebra to the K-theory of Roe algebra is an isomorphism. In Sect. 5, we introduce certain twisted Roe algebras and twisted localization algebras for bounded geometry spaces which admit a uniform embedding into Hilbert space. In Sect. 6, we prove that the evaluation map from the K-theory of twisted localization algebra to the K-theory of twisted Roe algebra is an isomorphism for bounded geometry spaces which admit a uniform embedding into Hilbert space. This establishes "the twisted coarse Baum–Connes conjecture" for bounded geometry spaces which admit a uniform embedding into Hilbert space. In Sect. 7, we establish a geometric analogue of Higson–Kasparov–Trout's infinite-dimensional Bott periodicity for bounded geometry spaces which admit a uniform embedding into Hilbert space. This geometric analogue of infinite-dimensional Bott periodicity is then used to reduce the coarse Baum–Connes conjecture to the twisted coarse Baum–Connes conjecture for bounded geometry spaces which admit a uniform embedding into Hilbert space.

This work is inspired by Gromov's deep questions concerning uniform embedding into Hilbert space ([10], [11]) and by the remarkable work of Higson and Kasparov on the Baum–Connes conjecture [14].

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2 Uniform embeddings into Hilbert space and property A

In this section, we shall introduce the concept of property A for metric spaces. We prove that metric spaces with property A admit a uniform embedding into Hilbert space. This result is inspired by the Bekka–Cherix–Valette theorem which states that every amenable group admits a proper and isometric action on Hilbert space [2]. The class of finitely generated groups with property A, as metric spaces with word-length metrics, includes word hyperbolic groups, discrete subgroups of connected Lie groups and amenable groups, and is closed under semi-direct product.

Definition 2.1. A discrete metric space Γ is said to have property A if for any $r > 0$, $\varepsilon > 0$, there exist a family of finite subsets $\{A_\gamma\}_{\gamma\in\Gamma}$ of $\Gamma \times \mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers) such that

(1) $(\gamma, 1) \in A_\gamma$ for all $\gamma \in \Gamma$;

(2) $\frac{\#(A_\gamma - A_{\gamma'}) + \#(A_{\gamma'} - A_\gamma)}{\#(A_\gamma \cap A_{\gamma'})} < \varepsilon$ for all γ and $\gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq r$, where, for each finite set A, $\#A$ is the number of elements in A;

(3) $\exists R > 0$ such that if $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$ for some $\gamma \in \Gamma$, then $d(x, y) \leq R$.

Notice that property A is invariant under quasi-isometry. In the case of a finitely generated group, property A does not depend on the choice of the word-length metric.

The following result is inspired by the Bekka–Cherix–Valette theorem which states that every amenable group admits a proper and isometric action on Hilbert space [2].

Theorem 2.2. *If a discrete metric space Γ has property A, then Γ admits a uniform embedding into Hilbert space.*

Proof. Let

$$H = \bigoplus_{k=1}^{\infty} \ell^2(\Gamma \times \mathbb{N}).$$

By the definition of property A, there exist a family of finite subsets $\{A_\gamma^{(k)}\}_{\gamma\in\Gamma}$ such that

(1) $(\gamma, 1) \in A_\gamma^{(k)}$ for all $\gamma \in \Gamma$;

(2) $\exists R_k > 0$ such that if $(x, m) \in A_\gamma^{(k)}$, $(y, n) \in A_\gamma^{(k)}$ for some k and $\gamma \in \Gamma$, then $d(x, y) \leq R_k$;

(3)

$$\left\| \frac{\chi_{A_\gamma^{(k)}}}{\left(\#A_\gamma^{(k)}\right)^{1/2}} - \frac{\chi_{A_{\gamma'}^{(k)}}}{\left(\#A_{\gamma'}^{(k)}\right)^{1/2}} \right\|_{l^2(\Gamma\times\mathbb{N})} < \frac{1}{2^k}$$

for all $\gamma, \gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq k$ and $k \in \mathbb{N}$, where $\chi_{A_\gamma^{(k)}}$ is the characteristic function of $A_\gamma^{(k)}$.

Fix $\gamma_0 \in \Gamma$. Define $f : \Gamma \to H$ by:

$$f(\gamma) = \bigoplus_{k=1}^{\infty} \left(\frac{\chi_{A_\gamma^{(k)}}}{\left(\#A_\gamma^{(k)}\right)^{1/2}} - \frac{\chi_{A_{\gamma_0}^{(k)}}}{\left(\#A_{\gamma_0}^{(k)}\right)^{1/2}} \right).$$

One can easily check that f is a uniform embedding. □

Example 2.3. Let Γ be a finitely generated amenable group. $\forall\ r > 0$, $0.5 > \varepsilon > 0$, there exists a finite subset F of Γ such that

$$\frac{\#(Fg - F) + \#(F - Fg)}{\#F} < \varepsilon/10 \quad \text{if} \quad d(g, 1) \leq r,$$

where d is the word-length metric. Set $A_\gamma = \{(x, 1) \in \Gamma \times \mathbb{N} : x \in \gamma F\}$. It is easy to see that $\{A_\gamma\}_{\gamma\in\Gamma}$ satisfies the conditions of property A.

Hence amenable groups admit a uniform embedding into Hilbert space. This is also a consequence of the Bekka–Cherix–Valette Theorem mentioned before (cf. [2]) since groups acting properly and isometrically on Hilbert space admit a uniform embedding into Hilbert space. However, by Example 2.5 and Proposition 2.6 below, the class of groups admitting a uniform embedding into Hilbert space is much larger than the class of groups acting properly and isometrically on Hilbert space since infinite property T groups can not act properly and isometrically on Hilbert space.

Example 2.4. Let F_2 be the free group of two generators. Let T be the tree associated to F_2, where the set of all vertices of T is F_2. Endow T with the simplicial metric. Fix a geodesic ray γ_0 on T. For each $g \in F_2$, there exists a unique geodesic ray γ_g on T starting from g such that $\gamma_g \cap \gamma_0$ is a non-empty geodesic ray. For each natural number N, define

$$A_g = \{(x, 1) \in F_2 \times \mathbb{N}, x \in \gamma_g, d(x, g) \leq N\}$$

for all $g \in F_2$.

Given $r > 0$, $\varepsilon > 0$, it is not difficult to see that there exists a large N such that $\{A_g\}_{g \in F_2}$ satisfies the conditions of property A.

Example 2.5. Let M be a compact Riemannian manifold with negative sectional curvature, let $\Gamma = \pi_1(M)$, the fundamental group of M. Fix a point $x_0 \in \widetilde{M}$, the universal cover of M, and a geodesic ray (with unit speed) γ_0 on $\widetilde{M}$ starting from x_0. For each $g \in \Gamma$, let γ_g be the geodesic ray (with unit speed) on $\widetilde{M}$ starting from gx_0 such that γ_g is asymptotic to γ_0 (cf. Proposition 1.7.3 in "Geometry of Nonpositively Curved Manifolds" by Patrick B. Eberlein, Chicago Lecture Notes in Mathematics, 1996). For each natural number N, we define

$$B_g = \{x \in \widetilde{M}, d(x, \gamma_g(t)) < 1 \quad \text{for some } t \in [0, N]\}.$$

Let F be a fundamental domain in $\widetilde{M}$. For each $\delta_0 > 0$, $\delta_1 > 0$, there exists a natural number k such that if volume $(g'F \cap B_g) \geq \delta_0$ for some $g, g' \in \Gamma$, then there exists an integer $\ell_{g',g}$ satisfying

$$\left| \text{volume } (g'F \cap B_g) - \frac{\ell_{g',g}}{k} \right| < \delta_1.$$

We define

$$A_g = \{(g', n) : \text{volume } (g'F \cap B_g) \geq \delta_0, \;\; 1 \leq n \leq \ell_{g',g}\} \subseteq \Gamma \times \mathbb{N}.$$

Using comparison theorems in Riemannian geometry and the negative curvature property, it is not difficult to verify that if N is large enough, δ_0 and δ_1 are small enough, then $\{A_g\}_{g \in \Gamma}$ satisfies the conditions of property A.

More generally, one can show that word hyperbolic groups have property A by similar argument using constructions from [25].

Proposition 2.6. *Let Γ_1 and Γ_2 be two finitely generated groups with property A (as metric spaces with word-length metrics). If Γ_1 acts on Γ_2 by automorphisms, then the semi-direct product $\Gamma_2 \rtimes \Gamma_1$ has property A.*

Proof. Given $r > 0$, $\varepsilon > 0$. For each $r_1 > 0$, $\varepsilon_1 > 0$, $r_2 > 0$, $\varepsilon_2 > 0$, let $\{A^{(1)}_\gamma\}_{\gamma \in \Gamma_1}$ and $\{A^{(2)}_\gamma\}_{\gamma \in \Gamma_2}$ be respectively as in the definition of property A for Γ_1 with respect to r_1 and ε_1, and for Γ_2 with respect to r_2 and ε_2. Set

$$f(x \cdot y, (\gamma_1 \cdot \gamma_2, (m, n))) = \chi_{A^{(1)}_x}(\gamma_1, m)\chi_{A^{(2)}_{\gamma_1^{-1}xyx^{-1}\gamma_1}}(\gamma_2, n),$$

where $m, n \in \mathbb{N}$, $x \in \Gamma_1$, $y \in \Gamma_2$, $x \cdot y \in \Gamma_2 \rtimes \Gamma_1$, $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$, $\gamma_1 \cdot \gamma_2 \in \Gamma_2 \rtimes \Gamma_1$, and, for each set A, χ_A is the characteristic function of A.

Let h be a bijective map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ such that $h(1) = (1, 1)$. For each $\gamma = x \cdot y \in \Gamma_2 \rtimes \Gamma_1$, we define

$$A_\gamma = \{(\gamma_1 \cdot \gamma_2, n) \in (\Gamma_2 \rtimes \Gamma_1) \times \mathbb{N} : \; f(x \cdot y, (\gamma_1 \cdot \gamma_2, h(n))) \neq 0\}.$$

Now it is straightforward to verify that if r_i and ε_i $(i = 1, 2)$ are chosen appropriately, then $\{A_\gamma\}_{\gamma \in \Gamma_2 \rtimes \Gamma_1}$ satisfies the conditions of property A. □

In a recent paper [17], Higson and Roe proved that a finitely generated group with a word-length metric has property A if and only if the group acts amenably on some compact Hausdorff space. It follows that discrete subgroups of connected Lie groups have property A [17].

A general metric space X is said to have property A if there exists a discrete subspace Γ of X such that (1) there exists $c > 0$ for which $d(x, \Gamma) \leq c$ for all $x \in X$; (2) Γ has property A.

The following result follows from Theorem 2.2.

Theorem 2.7. *If a metric space X has property A, then X admits a uniform embedding into Hilbert space.*

In general, it is an open question if every discrete metric space with bounded geometry admits a uniform embedding into Hilbert space [10]. However it is not difficult to show that every separable metric space admits a uniform embedding into a separable Banach space.

3 The coarse Baum–Connes conjecture

In this section, we shall briefly recall the coarse Baum–Connes conjecture and its applications.

Let M be a proper metric space (a metric space is called proper if every closed ball is compact). Let H_M be a separable Hilbert space equipped with a faithful and non-degenerate $*$-representation of $C_0(M)$ whose range contains no nonzero compact operator, where $C_0(M)$ is the algebra of all complex-valued continuous functions on M which vanish at infinity.

Definition 3.1. (1) The support of a bounded linear operator $T : H_M \to H_M$ is the complement of the set of points $(m, m') \in M \times M$ for which there exist g and g' in $C_0(M)$ such that

$$g'Tg = 0, \quad g(m) \neq 0 \quad \text{and} \quad g'(m') \neq 0;$$

(2) A bounded operator $T : H_M \to H_M$ has finite propagation if

$$\sup\{d(m, m') : (m, m') \in \text{supp}\,(T)\} < \infty;$$

(3) A bounded operator $T : H_M \to H_M$ is locally compact if the operators gT and Tg are compact for all $g \in C_0(M)$.

Definition 3.2. The Roe algebra $C^*(M)$ is the operator norm closure of the $*$-algebra of all locally compact, finite propagation operators acting on H_M.

$C^*(M)$ is independent of the choice of H_M (up to $*$-isomorphism). In particular, we can choose H_M to be $l^2(Z) \otimes H_0$, where Z is a countable dense subset of M, H_0 is a separable and infinite-dimensional Hilbert space, and

$C_0(M)$ acts on H_M by: $g(h_1 \otimes h_2) = gh_1 \otimes h_2$ for all $g \in C_0(M), h_1 \in l^2(Z), h_2 \in H_0$ (g acts on $l^2(Z)$ by pointwise multiplication). Such a choice of H_M will be useful later on in this paper.

Let Γ be a locally finite discrete metric space (a metric space is called locally finite if every ball contains finitely many elements).

Definition 3.3. For each $d \geq 0$, the Rips complex $P_d(\Gamma)$ is the simplicial polyhedron where the set of all vertices is Γ, and a finite subset $\{\gamma_0, \ldots, \gamma_n\} \subseteq \Gamma$ spans a simplex iff $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.

Endow $P_d(\Gamma)$ with the spherical metric. Recall that the spherical metric is the maximal metric whose restriction to each simplex $\{\sum_{i=0}^n t_i\gamma_i : t_i \geq 0, \sum_{i=0}^n t_i = 1\}$ is the metric obtained by identifying the simplex with S^n_+ through the map:

$$\sum_{i=0}^{n} t_i\gamma_i \to \left(\frac{t_0}{\sqrt{\sum_{i=0}^n t_i^2}}, \cdots, \frac{t_n}{\sqrt{\sum_{i=0}^n t_i^2}} \right),$$

where $S^n_+ = \{(x_0, \cdots, x_n) \in \mathbb{R}^{\ltimes+\nVdash} : x_i \geq 0, \sum_{i=0}^n x_i^2 = 1\}$ is endowed with the standard Riemannian metric. The distance of a pair of points in different connected components of $P_d(\Gamma)$ is defined to be infinity. Use of the spherical metric is necessary in Sect. 4 to avoid certain pathological phenomena when d goes to infinity (for the simplicial metric, the distance between the barycenters of two faces of an n-simplex goes to zero as n goes to infinity).

Conjecture 3.4 (The Coarse Baum–Connes Conjecture). *If Γ is a discrete metric space with bounded geometry, then the index map from $\lim_{d\to\infty} K_*(P_d(\Gamma))$ to $\lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$ is an isomorphism, where $K_*(P_d(\Gamma)) = KK_*(C_0(P_d(\Gamma)), \mathbb{C})$ is the locally finite K-homology group of $P_d(\Gamma)$.*

It should be pointed out that $\lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$ is isomorphic to $K_*(C^*(\Gamma))$.

We remark that Conjecture 3.4 is false if the bounded geometry condition is dropped [28]. In the case of a finitely generated group, the descent principle states that the coarse Baum–Connes conjecture for the group as a metric space with a word-length metric implies the strong Novikov conjecture if the group has a finite CW complex as its classifying space [23]. If Γ is a discrete subspace of a complete Riemannian manifold M such that there exists $c > 0$ satisfying $d(x, \Gamma) \leq c$ for all $x \in M$, then the coarse Baum–Connes conjecture for Γ provides a way of computing higher indices of elliptic operators on M. In particular, by the Lichnerowicz argument, the coarse Baum–Connes conjecture implies Gromov's conjecture stating that a uniformly contractible complete Riemannian manifold can not have uniformly positive scalar curvature [22]. It also implies the zero-in-the spectrum conjecture tating that the spectrum of the Laplacian acting on the space of all L^2-forms on a uniformly contractible complete Riemannian manifold contains zero [22].

4 The localization algebra

The localization algebra introduced in [27] plays an important role in the proof of our main result. For the convenience of the readers, we shall briefly recall its definition and its relation with K-homology. The concept of localization algebra is motivated by the heat kernel approach to index theory of elliptic operators.

Let M be a proper metric space.

Definition 4.1. The localization algebra $C_L^*(M)$ is the norm-closure of the algebra of all uniformly bounded and uniformly norm-continuous functions $f : [0, \infty) \to C^*(M)$ such that

$$\sup\{d(m, m') : (m, m') \in \operatorname{supp}(f(t))\} \to 0$$

as $t \to \infty$.

There exists a local index map (cf. [27]):

$$\operatorname{ind}_L : K_*(M) \to K_*(C_L^*(M)).$$

The evaluation homomorphism e from $C_L^*(M)$ to $C^*(M)$ is defined by:

$$e(f) = f(0).$$

If Γ is a locally finite discrete metric space, we have the following commuting diagram:

$$\begin{array}{ccc} & \lim\limits_{d\to\infty} K_*(P_d(\Gamma)) & \\ \operatorname{ind}_L \swarrow & & \searrow \operatorname{ind} \\ \lim\limits_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*(C^*(P_d(\Gamma))). \end{array}$$

Theorem 4.2 ([27]). *If Γ has bounded geometry, then the local index map*

$$ind_L : K_*(P_d(\Gamma)) \to K_*(C_L^*(P_d(\Gamma))),$$

is an isomorphism for every $d \geq 0$.

In general, the local index map $\operatorname{ind}_L : K_*(X) \to K_*(C_L^*(X))$, is an isomorphism for every finite-dimensional simplicial complex X endowed with the spherical metric. For the convenience of readers, we give an overview of the proof of this fact given in [27]. Let X_1 and X_2 be two simplicial subcomplexes of X such that X_1 and X_2 are endowed with metrics inherited from the metric of X. The local nature of the localization algebra can be used to prove a Mayer–Vietoris sequence for the K-groups of the localization algebras for $X_1 \cup X_2$, X_1, X_2 and $X_1 \cap X_2$ (cf. Proposition 3.11 of [27]), and certain (strong) Lipschitz homotopy invariance of the K-theory of the localization algebra (cf. Lemma 6.5 in this paper, Proposition 3.7 of [27] and also page 332 of [28] for a notational correction). Now the fact that the local index map $\mathrm{ind}_L : K_*(X) \to K_*(C_L^*(X))$, is an isomorphism follows from an induction argument on the dimension of skeletons of X using the Mayer–Vietoris sequence and the (strong) Lipschitz homotopy invariance for K-theory of localization algebras (cf. page 315–316 of [27]).

Theorem 4.2 implies that in order to prove the coarse Baum–Connes conjecture, it is enough to show that

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$$

is an isomorphism.

5 Twisted Roe algebras and twisted localization algebras

In this section, we shall introduce certain twisted Roe algebras and twisted localization algebras for bounded geometry spaces which admit a uniform embedding into Hilbert space. These algebras play important roles in the proof of our main result. I thank one of the referees and Vincent Lafforgue for suggesting the current more friendly definition of twisted Roe algebras.

As suggested by one of the referees, we shall first introduce the twisted Roe algebras and the twisted localization algebras in the finite-dimensional case to motivate the infinite-dimensional construction. The readers are encouraged to go through the paper in the finite-dimensional case first since the finite-dimensional case is considerably simpler.

Let H be a finite-dimensional Euclidean space. Denote by $\mathcal{C}(H)$ the $\mathbb{Z}/2$-graded C^*-algebra of continuous functions from H into the complexified Clifford algebra of H which vanish at infinity. Let $\mathcal{S} = C_0(\mathbb{R})$, graded according to even and odd functions. We define $\mathcal{A}(H)$ to be the graded tensor product of $\mathcal{S}$ with $\mathcal{C}(H)$.

Let Γ be a discrete metric space with bounded geometry. Let $f : \Gamma \to H$, be a uniform embedding. For each $d \geq 0$, we extend f to $P_d(\Gamma)$ by:

$$f(x) = \sum c_\gamma f(\gamma),$$

for all $x = \sum c_\gamma \gamma \in P_d(\Gamma)$, where $\gamma \in \Gamma$, $c_\gamma > 0$ and $\sum c_\gamma = 1$.

Choose a countable dense subset Γ_d of $P_d(\Gamma)$ for each $d > 0$ such that $\Gamma_{d_1} \subseteq \Gamma_{d_2}$ if $d_2 \geq d_1$.

Let $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ be the set of all functions T on $\Gamma_d \times \Gamma_d$ such that

(1) $T(x, y) \in \mathcal{A}(H)\widehat{\otimes}K$ for all $x, y \in \Gamma_d$, where K is the algebra of compact operators;

(2) $\exists M > 0$ and $L > 0$ such that $\|T(x, y)\| \leq M$ for all $x, y \in \Gamma_d$, and for each $y \in \Gamma_d$, $\#\{x : T(x, y) \neq 0\} \leq L$, $\#\{x : T(y, x) \neq 0\} \leq L$;
(3) $\exists r_1 > 0$ and $r_2 > 0$ such that
   (a) if $d(x, y) > r_1$, then $T(x, y) = 0$;
   (b) $\text{support}(T(x, y)) \subseteq B(f(x), r_2)$ for all $x, y \in \Gamma_d$, where $\text{support}(T(x, y))$ is defined to be the support $\{(s, h) \in \mathbb{R} \times H : (T(x, y))(s, h) \neq 0\}$ of the function $T(x, y) : \mathbb{R} \times H \to \text{Cliff}(H)\widehat{\otimes}K$, ($\text{Cliff}(H)$ is the complexified Clifford algebra of H), f is the uniform embedding, and $B(f(x), r_2) = \{(s, h) \in \mathbb{R} \times H : s^2 + \|h - f(x)\|^2 < r_2^2\}$;
(4) $\exists c > 0$ such that $D_Y(T(x, y))$ exists in $\mathcal{A}(H)\widehat{\otimes}K$, and $\|D_Y(T(x, y))\| \leq c$ for all $x, y \in \Gamma_d$ and $Y = (s, h) \in \mathbb{R} \times H$ satisfying $\|Y\| = \sqrt{s^2 + \|h\|^2} \leq 1$, where $D_Y(T(x, y))$ is the derivative of the function $T(x, y) : \mathbb{R} \times H \to \text{Cliff}(H)\widehat{\otimes}K$, in the direction of Y.

We define a product structure on $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ by:

$$(T_1 T_2)(x, y) = \sum_{z \in \Gamma_d} T_1(x, z) T_2(z, y).$$

Let

$$E = \left\{ \sum_{x \in \Gamma_d} a_x[x] : a_x \in \mathcal{A}(H)\widehat{\otimes}K, \sum_{x \in \Gamma_d} a_x^* a_x \text{ converges in norm} \right\}.$$

E is a Hilbert module over $\mathcal{A}(H)\widehat{\otimes}K$:

$$\left\langle \sum_{x \in \Gamma_d} a_x[x], \sum_{x \in \Gamma_d} b_x[x] \right\rangle = \sum_{x \in \Gamma_d} a_x^* b_x,$$

$$\left(\sum_{x \in \Gamma_d} a_x[x] \right) a = \sum_{x \in \Gamma_d} a_x a[x]$$

for all $a \in \mathcal{A}(H)\widehat{\otimes}K$.

$C^*_{alg}(P_d(\Gamma), \mathcal{A})$ acts on E by:

$$T\left(\sum_{x \in \Gamma_d} a_x[x] \right) = \sum_{y \in \Gamma_d} \left(\sum_{x \in \Gamma_d} T(y, x) a_x \right) [y],$$

where $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, $\sum a_x[x] \in E$. One can easily verify that T is a module homomorphism which has an adjoint module homomorphism.

Definition 5.1 (The finite-dimensional case). $C^*(P_d(\Gamma), \mathcal{A})$ is the operator norm closure of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ in $B(E)$, the C^*-algebra of all module homomorphisms from E to E for which there is an adjoint module homomorphism.

Let $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ be the set of all uniformly norm-continuous and uniformly bounded functions $g : \mathbb{R}_+ = [0, \infty) \to C^*_{alg}(P_d(\Gamma), \mathcal{A})$ such that

(1) $\exists$ a bounded function $r(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} r(t) = 0$ and if $d(x, y) > r(t)$, then $(g(t))(x, y) = 0$;

(2) $\exists R > 0$ such that support$((g(t))(x, y)) \subseteq B(f(x), R)$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$, where f is the uniform embedding, and $B(f(x), R) = \{(s, h) \in \mathbb{R} \times H : s^2 + \|h - f(x)\|^2 < R^2\}$;

(3) $\exists C > 0$ such that $\|D_Y((g(t))(x, y))\| \leq C$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$ and $Y \in \mathbb{R} \times H$ satisfying $\|Y\| \leq 1$.

Definition 5.2 (The finite-dimensional case). $C^*_L(P_d(\Gamma), \mathcal{A})$ is the norm closure of $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$, where $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ is endowed with the norm:

$$\|g\| = \sup_{t\in\mathbb{R}_+} \|g(t)\|_{C^*(P_d(\Gamma),\mathcal{A})}.$$

Next we shall introduce the twisted Roe algebras and twisted localization algebras in the infinite-dimensional case. We shall first recall an algebra associated to an infinite-dimensional Euclidean space introduced by Higson, Kasparov and Trout [15]. Let V be a countably infinite-dimensional Euclidean space. Denote by V_a, V_b, so on, the finite-dimensional, affine subspaces of V. Denote by V_a^0 the finite-dimensional linear subspace of V consisting of differences of elements in V_a. Let $\mathcal{C}(V_a)$ be the $\mathbb{Z}/2$-graded C^*-algebra of continuous functions from V_a into the complexified Clifford algebra of V_a^0 which vanish at infinity. Let $\mathcal{A}(V_a)$ be the graded tensor product of $\mathcal{S}$ with $\mathcal{C}(V_a)$.

If $V_a \subset V_b$, we have a decomposition:

$$V_b = V_{ba}^0 + V_a,$$

where V_{ba}^0 is the orthogonal complement of V_a^0 in V_b^0. For each $v_b \in V_b$, we have a corresponding decomposition: $v_b = v_{ba} + v_a$, where $v_{ba} \in V_{ba}^0$ and $v_a \in V_a$. Every function h on V_a can be extended to a function $\tilde{h}$ on V_b by the formula: $\tilde{h}(v_b) = h(v_a)$.

Definition 5.3. (1) If $V_a \subset V_b$, denote by C_{ba} the Clifford algebra-valued function on V_b which maps v_b to $v_{ba} \in V_{ba}^0 \subset \text{Cliff}\,(V_b^0)$. Define a homomorphism $\beta_{ba} : \mathcal{A}(V_a) \to \mathcal{A}(V_b)$, by

$$\beta_{ba}(g\widehat{\otimes}h) = g(X\widehat{\otimes}1 + 1\widehat{\otimes}C_{ba})(1\widehat{\otimes}\tilde{h})$$

for all $g \in \mathcal{S}$ and $h \in \mathcal{C}(V_a)$, where X is the function of multiplication by x on $\mathbb{R}$, considered as a degree one and unbounded multiplier of $\mathcal{S}$, and $g(X\widehat{\otimes}1 + 1\widehat{\otimes}C_{ba})$ is defined by functional calculus.

(2) We define a C^*-algebra $\mathcal{A}(V)$ by:

$$\mathcal{A}(V) = \varinjlim \mathcal{A}(V_a),$$

where the direct limit is over the directed set of all finite-dimensional affine subspaces $V_a \subset V$, using the homomorphism β_{ba} in (1).

Let H be a separable and infinite-dimensional Hilbert space. Let Γ be a discrete metric space with bounded geometry. Assume that Γ admits a uniform embedding $f : \Gamma \to H$. For each $n \in \mathbb{N}$, $x \in \Gamma$, define $W_n(x)$ to be the finite-dimensional Euclidean subspace of H spanned by $\{f(y) \in H : d(y, x) \leq n^2\}$. Let $W(x) = \bigcup_{n\in\mathbb{N}} W_n(x)$.

We have the following:

(1) $W_n(x) \subseteq W_{n+1}(x)$ for all $n \in \mathbb{N}$, $x \in \Gamma$;

(2) for each $r > 0$, there exists $N > 0$ such that $W_n(x) \subset W_{n+1}(y)$ for all $n \geq N$, x and y in Γ satisfying $d(x, y) \leq r$.

(3) $\forall\, n \in \mathbb{N}$, $\exists\, d_n > 0$ such that $\dim W_n(x) \leq d_n$ for all $x \in \Gamma$;

(4) there exists an Euclidean subspace V of H such that $W(x) = V$ for all $x \in \Gamma$.

Note that (3) follows from the bounded geometry property of Γ. We remark that (2) will be useful in the proof of Lemma 7.2.

For every $x \in P_d(\Gamma)$, write $x = \sum_{\substack{\gamma\in\Gamma \\ c_\gamma>0}} c_\gamma \gamma$. Define $W_n(x)$ to be the Euclidean subspace of H spanned by $W_n(\gamma)$ for all γ such that $c_\gamma > 0$. We extend f to $P_d(\Gamma)$ by:

$$f(x) = \sum c_\gamma f(\gamma).$$

Without loss of generality we assume that V is dense in H (otherwise we can take H to be the norm closure of V).

Throughout the rest of this paper, $\mathbb{R}_+ \times H$ is endowed with the weakest topology for which the projection to H is weakly continuous and the function $t^2 + \|h\|^2$ is continuous ($(t, h) \in \mathbb{R}_+ \times H$) (cf. the proof of Proposition 4.2 in [14]). Note that, for all $x \in H$ and $r > 0$, $B(x, r) = \{(t, h) \in \mathbb{R}_+ \times H : t^2 + ||h-x||^2 < r^2\}$ is an open subset of $\mathbb{R}_+ \times H$. For each finite-dimensional affine subspace V_a of V, the center of $\mathcal{A}(V_a)$ contains $C_0(\mathbb{R}_+ \times V_a)$. If $V_a \subset V_b$, then β_{ba} takes $C_0(\mathbb{R}_+ \times V_a)$ into $C_0(\mathbb{R}_+ \times V_b)$, where β_{ba} is as in Definition 5.3. The C^*-subalgebra $\varinjlim C_0(\mathbb{R}_+ \times V_a)$ is $*$-isomorphic to $C_0(\mathbb{R}_+ \times H)$, the algebra of all continuous functions on $\mathbb{R}_+ \times H$ which vanish at infinity, where the direct limit is over the directed set of all finite-dimensional affine subspaces $V_a \subset V$ using the homomorphism β_{ba}, and $\mathbb{R}_+ \times H$ is the locally compact space with the topology given as above. Hence the center of $\mathcal{A}(V)$ contains $C_0(\mathbb{R}_+ \times H)$. The support of an element $a \in \mathcal{A}(V)$ is defined to be the complement (in $\mathbb{R}_+ \times H$) of all (t, h) for which there exists $g \in C_0(\mathbb{R}_+ \times H)$ such that $ag = 0$ and $g(t, h) \neq 0$. $C_0(\mathbb{R}_+ \times H)$ acts on $\mathcal{A}(V)\widehat{\otimes}K$ by: $g(a\widehat{\otimes}k) = ga\widehat{\otimes}k$ for all $g \in C_0(\mathbb{R}_+ \times H)$, $a \in \mathcal{A}(V)$,

$k \in K$. Hence we can define the support of an element in $\mathcal{A}(V)\widehat{\otimes}K$ in a similar way.

Let $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ be the set of all functions T on $\Gamma_d \times \Gamma_d$ such that

(1) $\exists$ an integer N such that $T(x, y) \in (\beta_N(x))(\mathcal{A}(W_N(x))\widehat{\otimes}K) \subseteq \mathcal{A}(V)\widehat{\otimes}K$ for all $x, y \in \Gamma_d$, where $\beta_N(x) : \mathcal{A}(W_N(x))\widehat{\otimes}K \to \mathcal{A}(V)\widehat{\otimes}K$, is the $*$-homomorphism associated to the inclusion of $W_N(x)$ into V as in Definition 5.3, and K is the algebra of compact operators;
(2) $\exists M > 0$ and $L > 0$ such that $\|T(x, y)\| \le M$ for all $x, y \in \Gamma_d$, and for each $y \in \Gamma_d$, $\#\{x : T(x, y) \neq 0\} \le L$, $\#\{x : T(y, x) \neq 0\} \le L$;
(3) $\exists r_1 > 0$ and $r_2 > 0$ such that
  (a) if $d(x, y) > r_1$, then $T(x, y) = 0$;
  (b) $\text{support}(T(x, y)) \subseteq B(f(x), r_2)$ for all $x, y \in \Gamma_d$, where f is the uniform embedding, and $B(f(x), r_2) = \{(s, h) \in \mathbb{R}_+ \times H : s^2 + \|h - f(x)\|^2 < r_2^2\}$;
(4) $\exists c > 0$ such that $D_Y(T_1(x, y))$ exists in $\mathcal{A}(W_N(x))\widehat{\otimes}K$, and $\|D_Y(T_1(x, y))\| \le c$ for all $x, y \in \Gamma_d$ and $Y = (s, h) \in \mathbb{R} \times W_N(x)$ satisfying $\|Y\| = \sqrt{s^2 + \|h\|^2} \le 1$, where $(\beta_N(x))(T_1(x, y)) = T(x, y)$, and $D_Y(T_1(x, y))$ is the derivative of the function $T_1(x, y) : \mathbb{R} \times W_N(x) \to \text{Cliff}\,(W_N(x))\widehat{\otimes}K$, in the direction of Y.

It will become clear in the proof of Lemma 7.2 why we require condition (4) in the above definition. Note also that condition (4) is preserved under the $*$-homomorphism $\beta_{N'N}(x) : \mathcal{A}(W_N(x))\widehat{\otimes}K \to \mathcal{A}(W_{N'}(x))\widehat{\otimes}K$, associated to the inclusion of $W_N(x)$ into $W_{N'}(x)$ for any $N' > N$.

We define a product structure on $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ by:

$$(T_1T_2)(x, y) = \sum_{z \in \Gamma_d} T_1(x, z)T_2(z, y).$$

Let

$$E = \left\{ \sum_{x \in \Gamma_d} a_x[x] : a_x \in \mathcal{A}(V)\widehat{\otimes}K, \sum_{x \in \Gamma_d} a_x^* a_x \text{ converges in norm} \right\}.$$

E is a Hilbert module over $\mathcal{A}(V)\widehat{\otimes}K$:

$$\left\langle \sum_{x \in \Gamma_d} a_x[x], \sum_{x \in \Gamma_d} b_x[x] \right\rangle = \sum_{x \in \Gamma_d} a_x^* b_x,$$

$$\left(\sum_{x \in \Gamma_d} a_x[x] \right) a = \sum_{x \in \Gamma_d} a_x a[x]$$

for all $a \in \mathcal{A}(V)\widehat{\otimes}K$.

$C^*_{alg}(P_d(\Gamma), \mathcal{A})$ acts on E by:

$$T\left(\sum_{x\in\Gamma_d} a_x[x]\right) = \sum_{y\in\Gamma_d}\left(\sum_{x\in\Gamma_d} T(y,x)a_x\right)[y],$$

where $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, $\sum a_x[x] \in E$. One can easily verify that T is a module homomorphism which has an adjoint module homomorphism.

Definition 5.4. $C^*(P_d(\Gamma), \mathcal{A})$ is the operator norm closure of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ in $B(E)$, the C^*-algebra of all module homomorphisms from E to E for which there is an adjoint module homomorphism.

Let $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ be the set of all uniformly norm-continuous and uniformly bounded functions $g : \mathbb{R}_+ \to C^*_{alg}(P_d(\Gamma), \mathcal{A})$ such that

(1) $\exists N$ such that $(g(t))(x, y) \in (\beta_N(x))(\mathcal{A}(W_N(x))\widehat{\otimes}K) \subseteq \mathcal{A}(V)\widehat{\otimes}K$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$;

(2) $\exists$ a bounded function $r(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to\infty} r(t) = 0$ and if $d(x, y) > r(t)$, then $(g(t))(x, y) = 0$;

(3) $\exists R > 0$ such that $\mathrm{support}((g(t))(x, y)) \subseteq B(f(x), R)$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$, where f is the uniform embedding, and $B(f(x), R) = \{(s, h) \in \mathbb{R}_+ \times H : s^2 + \|h - f(x)\|^2 < R^2\}$;

(4) $\exists C > 0$ such that $\|D_Y((g_1(t))(x, y))\| \le C$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$ and $Y \in \mathbb{R} \times W_N(x)$ satisfying $\|Y\| \le 1$, where $(\beta_N(x))((g_1(t))(x, y)) = (g(t))(x, y)$.

Definition 5.5. $C^*_L(P_d(\Gamma), \mathcal{A})$ is the norm closure of $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$, where $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ is endowed with the norm:

$$\|g\| = \sup_{t\in\mathbb{R}_+} \|g(t)\|_{C^*(P_d(\Gamma),\mathcal{A})}.$$

6 K-Theory of twisted Roe algebras and twisted localization algebras

In this section, we shall study the K-theory of twisted Roe algebras and twisted localization algebras. In particular, we show that the evaluation map from the K-theory of twisted localization algebra is isomorphic to the K-theory of twisted Roe algebra for bounded geometry spaces which admit a uniform embedding into Hilbert space. This establishes "the twisted coarse Baum–Connes conjecture" for bounded geometry spaces which admit a uniform embedding into Hilbert space.

Definition 6.1. (1) The support of an element T in $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ is defined to be

$$\{(x, y, u) \in \Gamma_d \times \Gamma_d \times (\mathbb{R}_+ \times H) : u \in \mathrm{support}(T(x, y))\},$$

where $u = (t, h) \in \mathbb{R}_+ \times H$;

(2) The support of an element g in $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ is defined to be

$$\bigcup_{t \in \mathbb{R}_+} \mathrm{support}(g(t)).$$

Let O be an open subset of $\mathbb{R}_+ \times H$. Define $C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$ to be the subalgebra of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ consisting of all elements whose supports are contained in $\Gamma_d \times \Gamma_d \times O$. Define $C^*(P_d(\Gamma), \mathcal{A})_O$ to be the norm closure of $C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$. We can similarly define $C^*_L(P_d(\Gamma), \mathcal{A})_O$.

Lemma 6.2. *Let O and O' be open subsets of $\mathbb{R}_+ \times H$. If $O \subseteq O'$, then $C^*(P_d(\Gamma), \mathcal{A})_O$ and $C^*_L(P_d(\Gamma), \mathcal{A})_O$ are respectively closed, two sided ideals of $C^*(P_d(\Gamma), \mathcal{A})_{O'}$ and $C^*_L(P_d(\Gamma), \mathcal{A})_{O'}$.*

The proof of Lemma 6.2 is straightforward and is therefore omitted.

Lemma 6.3. *Let $B(x, r) = \{(t, h) \in \mathbb{R}_+ \times H : t^2 + \|h - x\|^2 < r^2\}$ for each $r > 0$ and $x \in H$; let $X_{i,j}$ and $X'_{i,j}$ be subsets of Γ for all $1 \leq i \leq i_0$ and $1 \leq j \leq j_0$. If $O_{r,j} = \cap_{i=1}^{i_0}(\cup_{\gamma \in X_{i,j}} B(f(\gamma), r))$ and $O'_{r,j} = \cap_{i=1}^{i_0}(\cup_{\gamma \in X'_{i,j}} B(f(\gamma), r))$ for each $1 \leq j \leq j_0$ (f is the uniform embedding), $O_r = \cup_{j=1}^{j_0} O_{r,j}$ and $O'_r = \cup_{j=1}^{j_0} O'_{r,j}$, then for each $r_0 > 0$ we have*

(1)
$$\lim_{r<r_0, r\to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O_r} + \lim_{r<r_0, r\to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O'_r} = \lim_{r<r_0, r\to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O_r \cup O'_r},$$
$$\lim_{r<r_0, r\to r_0} C^*_L(P_d(\Gamma), \mathcal{A})_{O_r} + \lim_{r<r_0, r\to r_0} C^*_L(P_d(\Gamma), \mathcal{A})_{O'_r} = \lim_{r<r_0, r\to r_0} C^*_L(P_d(\Gamma), \mathcal{A})_{O_r \cup O'_r};$$

(2)
$$\lim_{r<r_0, r\to r_0} (C^*(P_d(\Gamma), \mathcal{A})_{O_r} \cap C^*(P_d(\Gamma), \mathcal{A})_{O'_r}) = \lim_{r<r_0, r\to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O_r \cap O'_r},$$
$$\lim_{r<r_0, r\to r_0} (C^*_L(P_d(\Gamma), \mathcal{A})_{O_r} \cap C^*_L(P_d(\Gamma), \mathcal{A})_{O'_r}) = \lim_{r<r_0, r\to r_0} C^*_L(P_d(\Gamma), \mathcal{A})_{O_r \cap O'_r}.$$

Proof. We shall only prove the first identity. The rest of the identities can be proved in a similar way. It is enough to show that

$$\lim_{r<r_0, r\to r_0} C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_r \cup O'_r} \subseteq \lim_{r<r_0, r\to r_0} C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_r} + \lim_{r<r_0, r\to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O'_r}.$$

Given $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_r \cup O'_r}$ for some $r_0 > r > 0$, by the uniform embedding property and property (3) in the definition of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$, there exists $R > 0$ such that

$$\text{support}(T(x, y)) \subseteq \left(\cup_{j=1}^{j_0} \cap_{i=1}^{i_0} \left(\cup_{\gamma \in X_{i,j}, d(\gamma,x) \leq R} B(f(\gamma), r)\right)\right) \cup \left(\cup_{j=1}^{j_0} \cap_{i=1}^{i_0} \left(\cup_{\gamma \in X'_{i,j}, d(\gamma,x) \leq R} B(f(\gamma), r)\right)\right)$$

for all $x, y \in \Gamma_d$.

For each $k \in \mathbb{N}$, let $c_{r,k}$ be an even function in $\mathcal{S}$ such that (1) $0 \leq c_{r,k} \leq 1$, $\text{support}(c_{r,k}) \subseteq (-r - \frac{1}{k}, r + \frac{1}{k})$, $c_{r,k}|_{(-r-\frac{1}{2k}, r+\frac{1}{2k})} = 1$; (2) $c_{r,k}$ is differentiable and its derivative function is continuous. For each $\gamma \in \Gamma$, let $g_{r,k,\gamma} = (\beta(\gamma))(c_{r,k})$, where $\beta(\gamma)$ is the $*$ homomorphism: $\mathcal{S} = \mathcal{A}(0) \to \mathcal{A}(V)$, associated to the inclusion of the zero-dimensional affine space 0 into V by mapping 0 to $f(\gamma)$, where f is the uniform embedding. Note that $g_{r,k,\gamma} \in C_0(\mathbb{R}_+ \times H)$.

For each $x \in \Gamma_d$, $k \in \mathbb{N}$, let $h_{k,x}$ and $h'_{k,x}$ be functions in $C_0(\mathbb{R}_+ \times H)$ defined by:

$$h_{k,x} = \sum_{j=1}^{j_0} \Pi_{i=1}^{i_0} \left(\sum_{\gamma \in X_{i,j}, d(\gamma,x) \leq R} g_{r,k,\gamma} \right),$$

$$h'_{k,x} = \sum_{j=1}^{j_0} \Pi_{i=1}^{i_0} \left(\sum_{\gamma \in X'_{i,j}, d(\gamma,x) \leq R} g_{r,k,\gamma} \right).$$

Choose $k \in \mathbb{N}$ such that $r + \frac{1}{k} < r_0$. We define g_x and g'_x in $C_0(\mathbb{R}_+ \times H)$ by:

$$g_x = \frac{h_{2k,x}}{h_{k,x} + h'_{k,x}},$$

$$g'_x = \frac{h'_{2k,x}}{h_{k,x} + h'_{k,x}}.$$

Define T_1 and T_2 in $C^*(P_d(\Gamma), \mathcal{A})$ by:

$$T_1(x, y) = g_x T(x, y),$$
$$T_2(x, y) = g'_x T(x, y).$$

We have

$$T = T_1 + T_2.$$

By the properties of g_x and g'_x, and the bounded geometry property of Γ, it is not difficult to verify that

$$T_1 \in \lim_{r<r_0, r\to r_0} C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_r},$$
$$T_2 \in \lim_{r<r_0, r\to r_0} C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O'_r}.$$

□

Let e be the evaluation homomorphism from $C^*_L(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), \mathcal{A})$ defined by:

$$e(g) = g(0).$$

Lemma 6.4. *If O is the union of a family of open subsets $\{O_i\}_{i\in J}$ in $\mathbb{R}_+ \times H$ such that*

(1) $O_i \cap O_j = \emptyset$ if $i \neq j$;

(2) $\exists r > 0$, $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r)$ for all i, where $B(f(\gamma_i), r) = \{(t, h) \in \mathbb{R}_+ \times H : t^2 + \|h - f(\gamma_i)\|^2 < r^2\}$, and f is the uniform embedding,

then

$$e_* : \lim_{d\to\infty} K_*(C^*_L(P_d(\Gamma), \mathcal{A})_O) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma), \mathcal{A})_O),$$

is an isomorphism.

We need some preparations before we can prove Lemma 6.4. We shall first introduce two C^*-algebras which will be useful in the proof of Lemma 6.4.

Let $\{O_i\}_{i\in J}$ and $\{\gamma_i\}_{i\in J}$ be as in Lemma 6.4, let $\{X_i\}_{i\in J}$ be a family of closed subspaces of $P_d(\Gamma)$ for some d such that

(1) $\gamma_i \in X_i$ for every $i \in J$;

(2) $\{X_i\}_{i\in J}$ is uniformly bounded in the sense that there exists $r_0 > 0$ for which $diameter(X_i) \leq r_0$ for every $i \in J$.

Let $\mathcal{A}(V)_{O_i}$ be the C^*-subalgebra of $\mathcal{A}(V)$ generated by elements whose supports are contained in O_i. We define $A^*(X_i : i \in J)$ to be the C^*-subalgebra of

$$\{\oplus_{i\in J} b_i : b_i \in \mathcal{A}(V)_{O_i} \widehat{\otimes} C^*(X_i), sup_{i\in J}||b_i|| < \infty\}$$

generated by elements $\oplus_{i\in J} b_i$ for which there exist $N > 0$, $c_0 > 0$ such that, for each i, there is $b'_i \in \mathcal{A}(W_N(\gamma_i))\widehat{\otimes} C^*(X_i)$ for which $(\beta_N(\gamma_i))(b'_i) = b_i$, $D_Y(b'_i)$ exists in $\mathcal{A}(W_N(\gamma_i))\widehat{\otimes} C^*(X_i)$ and $\|D_Y(b'_i)\| \leq c_0$ for all $Y = (s, h) \in \mathbb{R} \times W_N(\gamma_i)$ satisfying $\|Y\| \leq 1$, where $\beta_N(\gamma_i)$ is the $*$-homomorphism:

$$\mathcal{A}(W_N(\gamma_i))\widehat{\otimes} C^*(X_i) \to \mathcal{A}(V)\widehat{\otimes} C^*(X_i),$$

induced by the inclusion of $W_N(\gamma_i)$ into V.

Similarly we define $A^*_L(X_i : i \in J)$ to be the C^*-subalgebra of

$$\{\oplus_{i\in J} b_i : b_i \in \mathcal{A}(V)_{O_i}\widehat{\otimes} C^*_L(X_i), sup_{i\in J}||b_i|| < \infty\}$$

generated by elements $\oplus_{i\in J} b_i$ such that

(1) the function

$$\oplus_{i\in J} b_i(t) : \mathbb{R}_+ \to \oplus_{i\in J}(\mathcal{A}(V)_{O_i}\widehat{\otimes} C^*(X_i)),$$

is uniformly norm-continuous in t;

(2) there exists a bounded function $c(t)$ on $\mathbb{R}_+$ for which

$$\lim_{t\to\infty} c(t) = 0 \ \text{ and } \ \sup\{d(x, y) : (x, y) \in \operatorname{supp}(b_i(t))\} \le c(t)$$

for all $i \in J$, where $\operatorname{supp}(b_i(t))$ is defined to be the complement (in $X_i \times X_i$) of all $(x, y) \in X_i \times X_i$ for which there exist functions h_1 and h_2 in $C(X_i)$, the algebra of all complex-valued continuous functions on X_i, such that $h_1(x) \neq 0$, $h_2(y) \neq 0$, and

$$(a_1\widehat{\otimes} h_1)(b_i(t))(a_2\widehat{\otimes} h_2) = 0$$

for all $a_1, a_2 \in \mathcal{A}(V)_{O_i}$;

(3) there exist $N > 0$, $c_0 > 0$ such that, for each i, there is $b'_i \in \mathcal{A}(W_N(\gamma_i))\widehat{\otimes} C^*_L(X_i)$ for which $(\beta_N(\gamma_i))(b'_i) = b_i$, $D_Y(b'_i)$ exists in $\mathcal{A}(W_N(\gamma_i))\widehat{\otimes} C^*_L(X_i)$ and $\|D_Y(b'_i)\| \le c_0$ for all $Y = (s, h) \in \mathbb{R} \times W_N(\gamma_i)$ satisfying $\|Y\| \le 1$.

Next we shall recall the concepts of Lipschitz maps and strong Lipschitz homotopy equivalence [27].

Let $\{Y_i\}_{i\in J}$ be another family of closed subspaces of $P_d(\Gamma)$ for some d such that

(1) $\gamma_i \in Y_i$ for every $i \in J$;

(2) $\{Y_i\}_{i\in J}$ is uniformly bounded.

A map

$$g : \bigsqcup_{i\in J} X_i \to \bigsqcup_{i\in J} Y_i$$

is said to be Lipschitz ($\bigsqcup_{i\in J} X_i$ and $\bigsqcup_{i\in J} Y_i$, are respectively the disjoint unions of $\{X_i\}_{i\in J}$ and $\{Y_i\}_{i\in J}$) if

(1) $g(x) \in Y_i$ for all $x \in X_i$;

(2) there exists a constant c such that

$$d(g(x), g(y)) \le cd(x, y)$$

for all x and y in X_i, where c is independent of i, and i is any element in J.

Let g_i $(i = 1, 2)$ be Lipschitz maps from $\bigsqcup_{i\in J} X_i$ to $\bigsqcup_{i\in J} Y_i$. g_1 is said to be strongly Lipschitz homotopy equivalent to g_2 if there exists a continuous map

$$F : [0, 1] \times \left(\bigsqcup_{i\in J} X_i\right) \to \bigsqcup_{i\in J} Y_i$$

such that

(1) $F(0,x)=g_1(x)$, $F(1,x)=g_2(x)$ for all $x\in\bigsqcup_{i\in J}X_i$;

(2) there exists a constant c for which

$$d(F(t,x),F(t,y))\le cd(x,y)$$

for all x and y in X_i, $t\in[0,1]$, where i is any element in J;

(3) F is equicontinuous in t, i.e. for any $\epsilon>0$, there exists $\delta>0$ such that

$$d(F(t_1,x),F(t_2,x))<\epsilon$$

for all $x\in\bigsqcup_{i\in J}X_i$ if $|t_1-t_2|<\delta$.

$\{X_i\}_{i\in J}$ is said to be strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i\in J}$ if there exist Lipschitz maps $g_1:\bigsqcup_{i\in J}X_i\to\bigsqcup_{i\in J}Y_i$, and $g_2:\bigsqcup_{i\in J}Y_i\to\bigsqcup_{i\in J}X_i$, such that g_1g_2 and g_2g_1 are respectively strongly Lipschitz homotopy equivalent to identity maps.

Define $A^*_{L,0}(X_i:i\in J)$ to be the C^*-subalgebra of $A^*_L(X_i:i\in J)$ consisting of elements $\oplus_{i\in J}b_i(t)$ satisfying $b_i(0)=0$ for all $i\in J$. When necessary (for example in the proof of the next lemma), we use $A^*_{L,0}(X_i,H_{X_i}:i\in J)$ to denote $A^*_{L,0}(X_i:i\in J)$, where H_{X_i} is the separable Hilbert space (equipped with a faithful and nondegenerate $*$-representation of $C(X_i)$ whose range contains no non-zero compact operator) in the definition of $C^*(X_i)$ (cf. Definition 3.1). Note that $A^*_{L,0}(X_i,H_{X_i}:i\in J)$ is independent of the choice of $\{H_{X_i}\}_{i\in J}$ (up to $*$-isomorphism).

The following result and its proof are essentially from [27].

Lemma 6.5. *If $\{X_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i\in J}$, then $K_*(A^*_{L,0}(X_i:i\in J))$ is isomorphic to $K_*(A^*_{L,0}(Y_i:i\in J))$.*

Proof. For any Lipschitz map g from $\bigsqcup_{i\in J}X_i$ to $\bigsqcup_{i\in J}Y_i$, we shall first recall the construction of a homomorphism from $K_*(A^*_{L,0}(X_i:i\in J))$ to $K_*(A^*_{L,0}(Y_i:i\in J))$ associated to g [27].

Let $\{\epsilon_k\}_{k=1}^\infty$ be a sequence of positive numbers such that $\lim_{k\to\infty}\epsilon_k=0$. For each k, there exists an isometry

$$U_{k,i}:H_{X_i}\to H_{Y_i}$$

such that

$$\operatorname{supp}(U_{k,i})\subseteq\{(x,y)\in X_i\times Y_i:d(g(x),y)<\epsilon_k\},$$

where H_{X_i} is the separable Hilbert space (equipped with a faithful and nondegenerate $*$-representation of $C(X_i)$ whose range contains no non-zero compact operator) in the definition of $C^*(X_i)$ (cf. Definition 3.1), and $\operatorname{supp}(U_{k,i})$ is the complement (in $X_i\times Y_i$) of the set of all points $(x,y)\in X_i\times Y_i$ for which there exist $g_1\in C(X_i)$ and $g_2\in C(Y_i)$ such that $g_2U_{k,i}g_1=0$, $g_1(x)\neq0$ and $g_2(y)\neq0$.

Let

$$U_k = \oplus_{i\in J} U_{k,i}.$$

Define a family of isometries

$$V_g(t) : (\oplus_{i\in J} H_{X_i}) \oplus (\oplus_{i\in J} H_{X_i}) \to (\oplus_{i\in J} H_{Y_i}) \oplus (\oplus_{i\in J} H_{Y_i})$$

($t \in [0, \infty)$) by:

$$V_g(t) = R(t-k+1)\begin{pmatrix} U_k & 0 \\ 0 & U_{k+1} \end{pmatrix} R^*(t-k+1)$$

for all natural numbers k and $k-1 \leq t < k$, where

$$R(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

We define a $*$-homomorphism

$$\mathrm{Ad}(V_g) : A^*_{L,0}(X_i : i \in J)^+ \to A^*_{L,0}(Y_i : i \in J)^+ \otimes M_2(\mathbb{C})$$

by:

$$\mathrm{Ad}(V_g)(b + cI) = V_g(t)\begin{pmatrix} b(t) & 0 \\ 0 & 0 \end{pmatrix} V_g^*(t) + cI,$$

where $b \in A^*_{L,0}(X_i : i \in J)$, $c \in \mathbb{C}$, $A^*_{L,0}(X_i : i \in J)^+$ and $A^*_{L,0}(Y_i : i \in J)^+$ are respectively obtained from $A^*_{L,0}(X_i : i \in J)$ and $A^*_{L,0}(Y_i : i \in J)$ by adjoining the identity.

Note that $\mathrm{Ad}(V_g)$ is well defined although $V_g(t)$ is not norm-continuous at $t = k$. $\mathrm{Ad}(V_g)$ induces a homomorphism

$$\mathrm{Ad}(V_g)_* : K_*(A^*_{L,0}(X_i : i \in J)) \to K_*(A^*_{L,0}(Y_i : i \in J)).$$

It is not difficult to verify that $\mathrm{Ad}(V_g)_*$ is independent of the choices of $\{\epsilon_k\}_{k=1}^{\infty}$ and V_g.

By assumption there exist Lipschitz maps

$$g_1 : \bigsqcup_{i\in J} X_i \to \bigsqcup_{i\in J} Y_i,$$

and

$$g_2 : \bigsqcup_{i\in J} Y_i \to \bigsqcup_{i\in J} X_i,$$

such that $g_1 g_2$ and $g_2 g_1$ are respectively strongly Lipschitz homotopy equivalent to identity maps. We have

$$\mathrm{Ad}(V_{g_1})_* \mathrm{Ad}(V_{g_2})_* = \mathrm{Ad}(V_{g_1 g_2})_*,$$

$$\mathrm{Ad}(V_{g_2})_*\mathrm{Ad}(V_{g_1})_* = \mathrm{Ad}(V_{g_2g_1})_*.$$

Hence it is enough to show that

$$\mathrm{Ad}(V_{g_1g_2})_* = Id_*, \quad \mathrm{Ad}(V_{g_2g_1})_* = Id_*.$$

Next we shall prove the second identity (the first identity can be proved in the same way).

Let $g = g_2g_1$. There is a strong Lipschitz homotopy F such that $F(0,x) = g(x)$ and $F(1,x) = x$ for all $x \in \bigsqcup_{i\in J} X_i$. Choose $\{t_{k,l}\}_{k\geq 0, l\geq 0} \subseteq [0,1]$ such that

(1) $t_{0,l} = 0$, $t_{k,l+1} \leq t_{k,l}$, $t_{k+1,l} \geq t_{k,l}$;

(2) there exists a sequence $N_l \to \infty$ such that $t_{k,l} = 1$ for all $k \geq N_l$, and $N_{l+1} \geq N_l$ for all l;

(3) there exists a sequence of positive numbers $\epsilon'_l \to 0$ such that

$$d(F(t_{k+1,l}, x), F(t_{k,l}, x)) < \epsilon'_l$$

and

$$d(F(t_{k,l+1}, x), F(t_{k,l}, x)) < \epsilon'_l$$

for all non-negative integer k and l, and all $x \in \bigsqcup_{i\in J} X_i$.

For example, we can take

$$t_{k,l} = \begin{cases} \frac{k}{l+1} & \text{if } k \leq l+1; \\ 1 & \text{otherwise}. \end{cases}$$

Let $U_{k,l,i}$ be an isometry from H_{X_i} to H_{X_i} such that

$$\mathrm{supp}(U_{k,l,i}) \subset \{(x,y) \in X_i \times X_i : d(F(t_{k,l}, x), y) < \epsilon_l\}$$

and

$$U_{k,l,i} = I$$

if $F(t_{k,l}, x) = x$ for all $x \in X_i$. Let

$$V_{k,l} = \oplus_{i\in J} U_{k,l,i}.$$

Define a family of isometries

$$V_k(t) : (\oplus_{i\in J} H_{X_i}) \oplus (\oplus_{i\in J} H_{X_i}) \to (\oplus_{i\in J} H_{X_i}) \oplus (\oplus_{i\in J} H_{X_i})$$

$(t \in [0,\infty))$ by:

$$V_k(t) = R(t-l+1) \begin{pmatrix} V_{k,l-1} & 0 \\ 0 & V_{k,l} \end{pmatrix} R^*(t-l+1)$$

for all natural numbers l and $l-1 \leq t < l$. V_k induces a homomorphism

$$\mathrm{Ad}(V_k) : A^*_{L,0}(X_i : i \in J)^+ \to A^*_{L,0}(X_i : i \in J)^+ \otimes M_2(\mathbb{C})$$

by

$$\mathrm{Ad}(V_k)(b+cI) = V_k(t)\begin{pmatrix} b(t) & 0 \\ 0 & 0 \end{pmatrix} V_k^*(t) + cI$$

for all $b \in A^*_{L,0}(X_i : i \in J)$ and $c \in \mathbb{C}$.

Note that $A^*_{L,0}(X_i : i \in J)$ is stable. Hence any element in $K_1(A^*_{L,0}(X_i : i \in J))$ can be represented by a unitary u in $A^*_{L,0}(X_i : i \in J)^+$. We define

$$a = \oplus_{k\geq 0}(\mathrm{Ad}(V_k)(u))\begin{pmatrix} u^{-1} & 0 \\ 0 & I \end{pmatrix},$$

$$b = \oplus_{k\geq 0}(\mathrm{Ad}(V_{k+1})(u))\begin{pmatrix} u^{-1} & 0 \\ 0 & I \end{pmatrix},$$

$$c = I \oplus_{k\geq 1} (\mathrm{Ad}(V_k)(u))\begin{pmatrix} u^{-1} & 0 \\ 0 & I \end{pmatrix}.$$

By the definition of V_k, it is not difficult to verify that a, b and c are invertible elements in $A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J)^+ \otimes M_2(\mathbb{C})$. Again by the definitions of V_k and V_{k+1}, a is equivalent to b in $K_1(A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J))$. Note that b is equivalent to c in $K_1(A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J))$. Hence $(\mathrm{Ad}(V_g)(u))(u^{-1} \oplus I) \oplus_{k\geq 1} I = ac^{-1}$ is equivalent to $\oplus_{k\geq 0} I$ in $K_1(A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J))$. This implies that $\mathrm{Ad}(V_g)(u) \oplus_{k\geq 1} I$ is equivalent to $u \oplus_{k\geq 1} I$ in $K_1(A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J))$. But

$$j_* : K_1(A^*_{L,0}(X_i : i \in J)) \to K_1(A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J))$$

is an isomorphism, where

$$j : A^*_{L,0}(X_i : i \in J) \to A^*_{L,0}(X_i, \oplus_{k\geq 0} H_{X_i} : i \in J)$$

is the homomorphism defined by:

$$\oplus_{i\in J} b_i \to \oplus_{i\in J}(b_i \oplus_{k\geq 1} 0).$$

Therefore $\mathrm{Ad}(V_g)(u)$ is equivalent to u in $K_1(A^*_{L,0}(X_i : i \in J))$. The case of K_0 can be dealt with in a similar way using a suspension argument. □

Let e be the evaluation homomorphism from $A^*_L(X_i : i \in J)$ to $A^*(X_i : i \in J)$ defined by:

$$\oplus_{i\in J} b_i(t) \to \oplus_{i\in J} b_i(0).$$

The following result is essentially from [27].

Lemma 6.6. *Let $\{\gamma_i\}_{i\in J}$ be as in Lemma 6.4. If $\{\triangle_i\}_{i\in J}$ is a family of simplices in $P_d(\Gamma)$ for some d such that $\gamma_i \in \triangle_i$ for all $i \in J$, then*

$$e_* : K_*(A^*_L(\triangle_i : i \in J)) \to K_*(A^*(\triangle_i : i \in J))$$

is an isomorphism.

Proof. We have a short exact sequence:

$$0 \to A^*_{L,0}(\triangle_i : i \in J) \to A^*_L(\triangle_i : i \in J) \to A^*(\triangle_i : i \in J) \to 0.$$

Hence it is enough to show

$$K_*(A^*_{L,0}(\triangle_i : i \in J)) = 0.$$

Note that $\{\triangle_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{\{\gamma_i\}\}_{i\in J}$. Hence by Lemma 6.5 it is enough to show

$$K_*(A^*_{L,0}(\{\gamma_i\} : i \in J)) = 0.$$

We shall prove

$$K_1(A^*_{L,0}(\{\gamma_i\} : i \in J)) = 0.$$

The K_0-case can be dealt with in a similar way using a suspension argument.

Note that $A^*_{L,0}(\{\gamma_i\} : i \in J)$ is stable. Hence any element in $K_1(A^*_{L,0}(\{\gamma_i\} : i \in J))$ can be represented by a unitary u in $A^*_{L,0}(\{\gamma_i\} : i \in J)^+$. For each $s \in [0, \infty)$, we define

$$u_s(t) = \begin{cases} I & \text{if} \quad 0 \le t \le s; \\ u(t-s) & \text{if} \quad s \le t < \infty. \end{cases}$$

Consider

$$w(s) = (\oplus_{k=0}^{\infty} u_k \oplus I)(I \oplus_{k=1}^{\infty} u_{k-s}^{-1} \oplus I),$$

where $s \in [0,1]$. Notice that $w(s)$ is an element in $A^*_{L,0}(\{\gamma_i\}, \oplus_{k=0}^{\infty} H_{\{\gamma_i\}} \oplus H_{\{\gamma_i\}} : i \in J)^+$ for each $s \in [0,1]$. We have

$$w(0) = u \oplus_{k=1}^{\infty} I \oplus I,$$

$$w(1) = (\oplus_{k=0}^{\infty} u_k \oplus I)(I \oplus_{k=1}^{\infty} u_{k-1}^{-1} \oplus I).$$

$w(1)$ is obviously equivalent to $\oplus_{k=0}^{\infty} I \oplus I$ in $K_1(A^*_{L,0}(\{\gamma_i\}, \oplus_{k=0}^{\infty} H_{\{\gamma_i\}} \oplus H_{\{\gamma_i\}} : i \in J))$. This, together with the above two identities, implies that $u \oplus_{k=1}^{\infty} I \oplus I$ is equivalent to $\oplus_{k=0}^{\infty} I \oplus I$ in $K_1(A^*_{L,0}(\{\gamma_i\}, \oplus_{k=0}^{\infty} H_{\{\gamma_i\}} \oplus H_{\{\gamma_i\}} : i \in J))$. But

$$j_* : K_1(A^*_{L,0}(\{\gamma_i\} : i \in J)) \to K_1(A^*_{L,0}(\{\gamma_i\}, \oplus_{k=0}^{\infty} H_{\{\gamma_i\}} \oplus H_{\{\gamma_i\}} : i \in J))$$

is an isomorphism, where

$$j : A^*_{L,0}(\{\gamma_i\} : i \in J) \to A^*_{L,0}(\{\gamma_i\}, \oplus_{k=0}^{\infty} H_{\{\gamma_i\}} \oplus H_{\{\gamma_i\}} : i \in J)$$

is the homomorphism defined by:

$$\oplus_{i \in J} b_i \to \oplus_{i \in J} (b_i \oplus_{k \geq 1} 0 \oplus 0).$$

Hence u is equivalent to the zero element in $K_1(A^*_{L,0}(\{\gamma_i\} : i \in J))$. $\square$

Proof of Lemma 6.4. Let $\mathcal{A}(V)_O$ be the C^*-subalgebra of $\mathcal{A}(V)$ generated by elements whose supports are contained in O. The support of an element $\sum a_x[x]$ in E is defined to be

$$\{(x, u) \in \Gamma_d \times (\mathbb{R}_+ \times H) : u \in \text{support}(a_x)\},$$

where E is as in Definition 5.4. Let E_O be the closure of the set of all elements in E whose supports are contained in $\Gamma_d \times O$. E_O is a Hilbert module over $\mathcal{A}(V)_O \widehat{\otimes} K$. $C^*(P_d(\Gamma), \mathcal{A})_O$ has a faithful representation on E_O. We have a decomposition:

$$E_O = \bigoplus_{i \in J} E_{O_i}.$$

By the uniform embedding property and property (3) in the definition of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$, each element $a \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$ has a corresponding decomposition:

$$a = \bigoplus_{i \in J} a_i$$

such that there exists $R > 0$ for which each a_i is supported on

$$\{(x, y, u) : u \in O_i, x \in \Gamma_d, y \in \Gamma_d, d(x, \gamma_i) \leq R, d(y, \gamma_i) \leq R\}.$$

Let

$$O_i(R) = \{(x, u) : u \in O_i, x \in \Gamma_d, d(x, \gamma_i) \leq R\}.$$

Define $E_{O_i(R)}$ to be the closure of the set of all elements in E whose supports are contained in $O_i(R)$. a_i lives in the image of the injective homomorphism from $B(E_{O_i(R)})$ to $B(E_{O_i})$:

$$\psi_i : b \to \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix},$$

with respect to the decomposition

$$E_{O_i} = E_{O_i(R)} \oplus E'_{O_i(R)}$$

for some Hilbert submodule $E'_{O_i(R)}$ of E_{O_i} (such Hilbert submodule exists in this case).

Note that

$$E_{O_i(R)} = (\mathcal{A}(V)_{O_i} \widehat{\otimes} K) \widehat{\otimes} \ell^2(\{x \in \Gamma_d : d(x, \gamma_i) \leq R\}).$$

Hence $B(E_{O_i(R)})$ can be identified with

$$\mathcal{A}(V)_{O_i} \widehat{\otimes} K \widehat{\otimes} B(l^2(\{x \in \Gamma_d : d(x, \gamma_i) \leq R\})).$$

Using this identification it is not difficult to verify that the map:

$$a \to \bigoplus_{i \in J} a_i,$$

gives a $*$-isomorphism from $C^*(P_d(\Gamma), \mathcal{A})_O$ to the C^*-algebra

$$\lim_{R \to \infty} A^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J).$$

Similarly $C_L^*(P_d(\Gamma), \mathcal{A})_O$ is $*$-isomorphic to the C^*-algebra

$$\lim_{R \to \infty} A_L^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J).$$

For each natural number k, let $\triangle_i^{(k)}$ be the simplex with vertices $\{\gamma \in \Gamma : d(\gamma, \gamma_i) \leq k\}$. By the above discussions, we know that $\lim_{d \to \infty} C^*(P_d(\Gamma), \mathcal{A})_O$ and $\lim_{d \to \infty} C_L^*(P_d(\Gamma), \mathcal{A})_O$ are respectively $*$-isomorphic to $\lim_{k \to \infty} A^*(\triangle_i^{(k)} : i \in J)$ and $\lim_{k \to \infty} A_L^*(\triangle_i^{(k)} : i \in J)$. This, together with Lemma 6.6, implies Lemma 6.4. □

Lemma 6.7. *Given $r > 0$, let $B(x, r) = \{(t, h) \in \mathbb{R}_+ \times H : t^2 + \|h - x\|^2 < r^2\}$ for each $x \in H$. If Γ has bounded geometry, then there exists an integer l_0 such that whenever $\bigcap_{k=1}^{\ell} B(f(\gamma_k), r) \neq \emptyset$ for distinct elements γ_k in Γ, we have $\ell \leq l_0$, where f is the uniform embedding.*

Proof. $\bigcap_{k=1}^{\ell} B(f(\gamma_k), r) \neq \emptyset$ implies that

$$|| f(\gamma_k) - f(\gamma_1)|| \leq 2r$$

for all $1 \leq k \leq l$. Hence, by the uniform embedding property, there exists $R > 0$ such that $d(\gamma_k, \gamma_1) \leq R$ for all $1 \leq k \leq l$, where R depends only on r. This, together with the bounded geometry property of Γ, implies Lemma 6.7. □

Now we are ready to prove "the twisted coarse Baum–Connes conjecture" for bounded geometry spaces which admit a uniform embedding into Hilbert space. This result plays a key role in the proof of the main theorem of this paper.

Theorem 6.8. *If Γ has bounded geometry, then*

$$e_* : \lim_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \to \lim_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})),$$

is an isomorphism.

Proof. Let $B(x,r) = \{(t,h) \in \mathbb{R}_+ \times H : t^2 + \|h-x\|^2 < r^2\}$ for all $r > 0, x \in H$. We define O_r by:

$$O_r = \bigcup_{\gamma \in \Gamma} B(f(\gamma), r),$$

where f is the uniform embedding.

We have

$$C_L^*(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} C_L^*(P_d(\Gamma), \mathcal{A})_{O_r},$$
$$C^*(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} C^*(P_d(\Gamma), \mathcal{A})_{O_r}.$$

Hence it is enough to show that

$$e_* : \lim_{d \to \infty} K_*(\lim_{r<r_0, r \to r_0} C_L^*(P_d(\Gamma), \mathcal{A})_{O_r})$$
$$\to \lim_{d \to \infty} K_*(\lim_{r<r_0, r \to r_0} C^*(P_d(\Gamma), \mathcal{A})_{O_r}),$$

is an isomorphism for every $r_0 > 0$. By Lemma 6.7, for each $r_0 > 0$, there exist finitely many subsets $\{J_k\}_{k=1}^{k_0}$ of Γ such that $O_r = \bigcup_{k=1}^{k_0} O_{r,k}$ for all $r < r_0$, where each $O_{r,k}$ is the disjoint union of $\{B(f(\gamma), r)\}_{\gamma \in J_k}$ for all $r < r_0$. Now Theorem 6.8 follows from Lemmas 6.4, 6.2, 6.3, and a Mayer–Vietoris sequence argument (cf. the K-theory Mayer–Vietoris sequence in Sect. 3 of [18]). □

7 Proof of the main theorem

In this section, we shall prove the main theorem of this paper. The proof of the main theorem is considerably simpler for the finite-dimensional case. For this reason, the readers are encouraged to go through the proof for the finite-dimensional case first. Note that the main theorem is not trivial in the finite-dimensional case.

We shall first prove a geometric analogue of the infinite-dimensional Bott periodicity introduced by Higson, Kasparov and Trout in [15]. The geometric analogue of infinite-dimensional Bott periodicity is then used to reduce the coarse Baum–Connes conjecture to the twisted coarse Baum–Connes conjecture for bounded geometry spaces which admit a uniform embedding into Hilbert space.

To prove the Bott periodicity in our context, we need to recall the definitions of certain elliptic operators and construct their associated asymptotic morphisms. For convenience of the readers, we shall introduce these constructions in the finite-dimensional case before we discuss their infinite-dimensional counterparts.

Let H be a finite-dimensional Euclidean space. Denote by $\mathcal{H}$ the Hilbert space of square integrable functions from H into Cliff (H), the complexified

Clifford algebra of H. Let s be the Schwartz subspace of $\mathcal{H}$. We define the Dirac operator D, an unbounded operator on $\mathcal{H}$ with domain s, to be:

$$D\xi = \sum_{i=1}^{n} (-1)^{\deg \xi} \frac{\partial \xi}{\partial x_i} v_i,$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis for H and $\{x_1, \ldots, x_n\}$ are the dual coordinates to $\{v_1, \ldots, v_n\}$.

Next we shall recall certain infinite-dimensional elliptic operators introduced by Higson, Kasparov and Trout [15], which play the role of the Dirac operator in the infinite-dimensional case.

Let H be a separable and infinite-dimensional Hilbert space. Let V be the countably infinite-dimensional Euclidean subspace of H defined in Sect. 5. If $V_a \subset V$ is a finite-dimensional affine subspace, denote by $\mathcal{H}_a$ the Hilbert space of square integrable functions from V_a into $\mathrm{Cliff}\,(V_a^0)$. If $V_a \subset V_b$, then there exists an isomorphism:

$$\mathcal{H}_b \cong \mathcal{H}_{ba} \widehat{\otimes} \mathcal{H}_a,$$

where $\mathcal{H}_{ba}$ is the Hilbert space associated to V_{ba}^0. Let $\xi_0 \in \mathcal{H}_{ba}$ be the unit vector defined by:

$$\xi_0(v_{ba}) = \pi^{-n_{ba}/4} \exp\left(-\frac{1}{2} \|v_{ba}\|^2 \right),$$

where $v_{ba} \in V_{ba}^0$, $n_{ba} = \dim\,(V_{ba}^0)$. We consider $\mathcal{H}_a$ as included in $\mathcal{H}_b$ via the isometry $\xi \to \xi_0 \widehat{\otimes} \xi$. We define

$$\mathcal{H} = \varinjlim \mathcal{H}_a,$$

where the Hilbert space direct limit is taken over the direct system of finite-dimensional affine subspaces of V.

Let $s = \varinjlim s_a$ be the algebraic direct limit of the Schwartz subspaces $s_a \subset \mathcal{H}_a$. If $V_a \subset V$ is a finite-dimensional affine subspace, we define the Dirac operator D_a, an unbounded operator on $\mathcal{H}$ with domain s, to be:

$$D_a \xi = \sum_{i=1}^{n} (-1)^{\deg \xi} \frac{\partial \xi}{\partial x_i} v_i,$$

where $\{v_1, \ldots, v_n\}$ is an orthonormal basis for V_a^0 and $\{x_1, \ldots, x_n\}$ are the dual coordinates to $\{v_1, \ldots, v_n\}$. If V_a is a linear subspace, then we define the Clifford operator C_a acting on s by:

$$(C_a \xi)(v_b) = v_a \xi(v_b)$$

for any $\xi \in s_b$ and $v_b \in V_b$ for some V_b satisfying $V_a \subseteq V_b$, where v_a is the vector projection of v_b onto V_a, and $v_a \xi(v_b)$ is the Clifford product of

v_a and $\xi(v_b)$. Note that C_a is a well-defined operator on s. Let $V_n(x) = W_{n+1}(x) \ominus W_n(x)$ if $n \geq 1$, $V_0(x) = W_1(x)$, where $x \in P_d(\Gamma)$, and $W_n(x)$ is as in Sect. 5. We have the algebraic decomposition:

$$V = V_0(x) \oplus V_1(x) \oplus \cdots \oplus V_n(x) \oplus \cdots .$$

For each n, define an unbounded operator $B_{n,t}(x)$ on $\mathcal{H}$ associated to the above decomposition by:

$$\begin{aligned} B_{n,t}(x) = t_0 D_0 + t_1 D_1 + \cdots + t_{n-1} D_{n-1} + t_n(D_n + C_n) \\ + t_{n+1}(D_{n+1} + C_{n+1}) + \cdots \end{aligned}$$

where $t_j = 1 + t^{-1} j$, D_n and C_n are respectively the Dirac operator and Clifford operator associated to $V_n(x)$.

We remark that $B_{n,t}(x)$ plays the role of the Dirac operator in the infinite-dimensional case. By Lemma 5.8 in [15], $B_{n,t}(x)$ is essentially selfadjoint.

We need to introduce another C^*-algebra before we can construct an asymptotic morphism associated to the Dirac operator D in the finite-dimensional case and the Higson–Kasparov–Trout operator $B_{n,t}$ in the infinite-dimensional case.

We define $C^*_{alg}(P_d(\Gamma), K)$ to be the algebra of all functions T on $\Gamma_d \times \Gamma_d$ such that

(1) $T(x, y) \in K(\mathcal{H})\widehat{\otimes}K$ for all $x, y \in \Gamma_d$, where $\mathcal{H}$ is respectively as in the definitions of the Dirac operator D in the finite-dimensional case and the Higson–Kasparov–Trout operator $B_{n,t}$ in the infinite-dimensional case, and $K(\mathcal{H})$ is the algebra of all compact operators acting on $\mathcal{H}$;

(2) $\exists M > 0$ and $L > 0$ such that $\|T(x, y)\| \leq M$ for all $x, y \in \Gamma_d$, and for each $y \in \Gamma_d$, $\#\{x \in \Gamma_d : T(x, y) \neq 0\} \leq L$, $\#\{x \in \Gamma_d : T(y, x) \neq 0\} \leq L$;

(3) $\exists r > 0$ such that if $d(x, y) > r$, then $T(x, y) = 0$.

The product structure on $C^*_{alg}(P_d(\Gamma), K)$ is defined by:

$$(T_1 \cdot T_2)(x, y) = \sum_{z \in \Gamma_d} T_1(x, z) T_2(z, y).$$

Let $E = \ell^2(\Gamma_d)\widehat{\otimes}\mathcal{H}\widehat{\otimes}H_0$, where H_0 is a separable and infinite-dimensional Hilbert space with a faithful $*$-representation of K.

$C^*_{alg}(P_d(\Gamma), K)$ acts on E by:

$$T(\delta_x \widehat{\otimes} h \widehat{\otimes} h_0) = \sum_{y \in \Gamma_d} \delta_y \widehat{\otimes} T(y, x)(h \widehat{\otimes} h_0)$$

for all $x \in \Gamma_d$, $h \in \mathcal{H}$, $h_0 \in H_0$, where δ_x and δ_y are respectively Dirac functions at x and y.

Definition 7.1. $C^*(P_d(\Gamma), K)$ is defined to be the operator norm closure of $C^*_{alg}(P_d(\Gamma), K)$ with respect to the above $*$-representation.

Now we are ready to construct an asymptotic morphism α from $C^*(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), K)$ associated to the Dirac operator in the finite-dimensional case and the Higson–Kasparov–Trout operator in the infinite-dimensional case. We remark that the asymptotic morphism α is adapted from [14].

If H is a finite-dimensional Euclidean space, we define

$$\theta_t : \mathcal{A}(H)\widehat{\otimes}K \to K(\mathcal{H})\widehat{\otimes}K,$$

by:

$$\theta_t((g\widehat{\otimes}h)\widehat{\otimes}k) = g_t(D)\pi(h_t)\widehat{\otimes}k$$

for every $g \in \mathcal{S}$, $h \in \mathcal{C}(H)$, $k \in K$, and $t \geq 1$, where $g_t(s) = g(t^{-1}s)$ for all $t \geq 1$ and $s \in \mathbb{R}$, $h_t(v) = h(t^{-1}v)$ for all $t \geq 1$ and $v \in H$, D is the Dirac operator as above, and $\pi(h_t)$ acts on $\mathcal{H}$ by pointwise multiplication.

We define

$$\alpha_t : C^*_{alg}(P_d(\Gamma), \mathcal{A}) \to C^*(P_d(\Gamma), K),$$

by:

$$(\alpha_t(T))(x, y) = \theta_t(T(x, y)),$$

for every $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, all $(x, y) \in \Gamma_d \times \Gamma_d$ and $t \geq 1$.

If H is a separable and infinite-dimensional Hilbert space, let V be the countably infinite-dimensional Euclidean subspace of H as in Sect. 5. For each $x \in P_d(\Gamma)$, let $B_{n,t}(x)$ be the infinite-dimensional elliptic operator of Higson–Kasparov–Trout defined in this section, let $V = V_0(x) \oplus V_1(x) \oplus \cdots \oplus V_n(x) \oplus \cdots$ and $W_n(x) = V_0(x) \oplus V_1(x) \oplus \cdots \oplus V_{n-1}(x)$ be as in the definition of $B_{n,t}(x)$.

For every non-negative integer n and $x \in \Gamma_d$, we define

$$\theta_t^n(x) : \mathcal{A}(W_n(x))\widehat{\otimes}K \to K(\mathcal{H})\widehat{\otimes}K$$

by:

$$(\theta_t^n(x))((g\widehat{\otimes}h)\widehat{\otimes}k) = g_t(B_{n,t}(x))\pi(h_t)\widehat{\otimes}k$$

for every $g \in \mathcal{S}$, $h \in \mathcal{C}(W_n(x))$, $k \in K$, and $t \geq 1$, where $g_t(s) = g(t^{-1}s)$ for all $t \geq 1$ and $s \in \mathbb{R}$, $h_t(v) = h(t^{-1}v)$ for all $t \geq 1$ and $v \in W_n(x)$, $\pi(h_t)$ acts on $\mathcal{H}$ by pointwise multiplication (if $V_a = W_n(x) \subseteq V_b$, we define $(\pi(h_t))\xi = \tilde{h}_t\xi$ for all $\xi \in \mathcal{H}_b$, where $\tilde{h}_t$ is as in Definition 5.3, $\mathcal{H}_a$ and $\mathcal{H}_b$ are as in the definition of $\mathcal{H}$). The fact that $(\theta_t^n(x))((g\widehat{\otimes}h)\widehat{\otimes}k)$ is in $K(\mathcal{H})\widehat{\otimes}K$ follows from Lemma 5.8 in [15]. Note that $\theta_t^n(x)$ is defined on a dense subalgebra of $\mathcal{A}(W_n(x))\widehat{\otimes}K$.

We define

$$\alpha_t : C^*_{alg}(P_d(\Gamma), \mathcal{A}) \to C^*(P_d(\Gamma), K),$$

by:

$$(\alpha_t(T))(x,y) = (\theta_t^{n_0}(x))(T_1(x,y))$$

for every $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$ and $t \geq 1$, where n_0 is a non-negative integer such that, for every pair x and y in Γ_d, there exists $T_1(x,y) \in \mathcal{A}(W_{n_0}(x))\widehat{\otimes}K$ satisfying $(\beta_{n_0}(x))(T_1(x,y)) = T(x,y)$. Note that, by the proof of Proposition 4.2 in [15], $\alpha_t(T)$ does not asymptotically depend on the choice of n_0, i.e. the difference of the values of $\alpha_t(T)$ for two different choices of n_0 goes to 0 in norm as t goes to ∞.

Lemma 7.2. *α extends to an asymptotic morphism from $C^*(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), K)$.*

We need some preparations before we can prove Lemma 7.2.

Given a non-negative integer m and an algebraic decopmosition:

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n \oplus \cdots,$$

define an unbounded operator B_t on $\mathcal{H}$ by

$$\begin{aligned} B_t = {} & (1+t^{-1}m)(D_0 + C_0) + (1+t^{-1}(m+1))(D_1+C_1) \\ & + \cdots (1+t^{-1}(m+n))(D_n+C_n) + \cdots, \end{aligned}$$

where each V_n is a finite-dimensional Euclidean subspace of V, D_n and C_n are respectively the Dirac operator and Clifford operator associated to V_n.

The following result is a modified version of Lemma 5.9 in [15].

Lemma 7.3. *Let B'_t be the unbounded operator on $\mathcal{H}$:*

$$\begin{aligned} B'_t = {} & (1+t^{-1}m)(D'_0 + C'_0) + (1+t^{-1}(m+1))(D'_1+C'_1) \\ & + \cdots (1+t^{-1}(m+n))(D'_n+C'_n) + \cdots, \end{aligned}$$

associated to a second decomposition

$$V = V'_0 \oplus V'_1 \oplus \cdots \oplus V'_n \oplus \cdots,$$

where D'_n and C'_n are respectively the Dirac operator and Clifford operator associated to the finite-dimensional Euclidean space V'_n. If

$$V_0 \oplus \cdots \oplus V_n \subseteq V'_0 \oplus \cdots \oplus V'_{n+1}$$

and

$$V'_0 \oplus \cdots \oplus V'_n \subseteq V_0 \oplus \cdots \oplus V_{n+1}$$

for all n, then for any $\epsilon > 0$, $R > 0$, $c > 0$, there exists $t_0 > 0$ such that

$$||g(sB'_t) - g(sB_t)|| < \epsilon$$

for all $t > t_0$, $s \in \mathbb{R}$, *and all* $g \in C_0(\mathbb{R})$ *for which (1) support*$(g) \subseteq [-R, R]$, $|g(t)| \leq c$ *for all* $t \in \mathbb{R}$*; (2) the derivative function* g' *of* g *is continuous, and* $|g'(t)| \leq c$ *for all* $t \in \mathbb{R}$.

Proof. Let

$$g_1(\theta) = g\left(\frac{2}{e^{i\theta} - i} - i\right).$$

By assumption, there exists $c' > 0$ such that

(1) c' depends only on R and c;

(2) $\frac{dg_1}{d\theta}$ is continuous, $|(\frac{dg_1}{d\theta})(\theta)| \leq c'$ for all $\theta \in [0, 2\pi]$.

Claim. There exists

$$p(\theta) = \sum_{k=-k_0}^{k_0} c_k e^{ik\theta}$$

such that

$$|g_1(\theta) - p(\theta)| < \epsilon/10$$

for all $\theta \in [0, 2\pi]$, where

(1) k_0 is a non-negative integer which depends only on c' and ϵ;

(2) $|c_k| \leq c$ for all $-k_0 \leq k \leq k_0$.

Proof of Claim. Let

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} g_1(\theta)e^{-ik\theta}\, d\theta,$$

$$a_k = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{dg_1}{d\theta}\right)(\theta)e^{-ik\theta}\, d\theta.$$

We have

$$a_k = ikc_k$$

for all k. Hence

$$\left|g_1(\theta) - p(\theta)\right| = \left|\sum_{|k|>|k_0|} \frac{a_k}{k}e^{ik\theta}\right|$$

$$\leq \left(\sum_{|k|>k_0} \frac{1}{k^2}\right)\left(\sum_{k=-\infty}^{\infty} |a_k|^2\right) \leq 2\pi(c')^2 \sum_{|k|>k_0} \frac{1}{k^2}.$$

The above inequality implies our Claim.

Substituting $e^{i\theta}$ by $\frac{2}{x+i}+i$ in the above Claim, we obtain

$$\left|g(x)-\left(c_0+\sum_{k=1}^{k_0}\left(c_k 2^k\left(\frac{1}{x+i}+\frac{i}{2}\right)^k+c_{-k}2^k\left(\frac{1}{x-i}-\frac{i}{2}\right)^k\right)\right)\right|<\epsilon/10$$

for all $x\in\mathbb{R}$, where

(1) k_0 depends only on R, c and ϵ;

(2) $|c_k|\leq c$ for all $-k_0\leq k\leq k_0$.

If

$$V_0\oplus\cdots\oplus V_n\subseteq V_0'\oplus\cdots\oplus V_n'$$

and

$$V_0'\oplus\cdots\oplus V_n'\subseteq V_0\oplus\cdots\oplus V_{n+1}$$

for all n, Lemma 7.3 follows from the above approximation result and the following inequalities

$$||(sB_t'+i)^{-1}-(sB_t+i)^{-1}||\leq t^{-1}$$

and

$$||(sB_t'-i)^{-1}-(sB_t-i)^{-1}||\leq t^{-1},$$

which follow from the estimations in the proof of Lemma 5.9 in [15].

The general case can be reduced to the above special case as follows. Let $\{\tilde{V}_n\}_{n=0}^{\infty}$ be the family of finite-dimensional Euclidean subspaces of V such that $\tilde{V}_0\oplus\cdots\oplus\tilde{V}_n$ is spanned by $V_0\oplus V_1\oplus\cdots\oplus V_n$ and $V_0'\oplus V_1'\oplus\cdots\oplus V_n'$ for all n. We have

$$V_0\oplus\cdots\oplus V_n\subseteq\tilde{V}_0\oplus\cdots\oplus\tilde{V}_n$$

and

$$\tilde{V}_0\oplus\cdots\oplus\tilde{V}_n\subseteq V_0\oplus\cdots\oplus V_{n+1}$$

for all n;

$$V_0'\oplus\cdots\oplus V_n'\subseteq\tilde{V}_0\oplus\cdots\oplus\tilde{V}_n$$

and

$$\tilde{V}_0\oplus\cdots\oplus\tilde{V}_n\subseteq V_0'\oplus\cdots\oplus V_{n+1}'$$

for all n. Now the general case follows by applying the special case twice. □

Let E be a finite-dimensional Euclidean space. Given a decomposition

$$E = E_0 \oplus \cdots \oplus E_{n-1},$$

define an unbounded operator $D(t)$ on the Hilbert space of all square integrable functions from E into $\mathrm{Cliff}(E)$ by

$$D(t) = D_0 + (1 + t^{-1})D_1 + \cdots + (1 + t^{-1}(n-1))D_{n-1},$$

where E_i is a Euclidean subspace of E and D_i is the Dirac operator associated to E_i for all $0 \leq i < n$.

Lemma 7.4. *Let*

$$E = E'_0 \oplus \cdots \oplus E'_{n-1},$$

be a second decomposition of E, let $D(t)'$ be the unbounded operator on the Hilbert space of all square integrable functions from E into Cliff(E) defined by

$$D(t)' = D'_0 + (1 + t^{-1})D'_1 + \cdots + (1 + t^{-1}(n-1))D'_{n-1},$$

where D'_i is the Dirac operator associated to the Euclidean space E'_i for all $0 \leq i < n$. For any $\epsilon > 0$, $R > 0$, $c > 0$, there exists $t_0 > 0$ such that

$$||g(sD(t)') - g(sD(t))|| < \epsilon$$

for all $t > t_0$ and $s \in \mathbb{R}$, and all $g \in C_0(\mathbb{R})$ for which (1) support$(g) \subseteq [-R, R]$, $|g(t)| \leq c$ for all $t \in \mathbb{R}$; (2) the derivative function g' is continuous, and $|g'(t)| \leq c$ for all $t \in \mathbb{R}$.

Proof. By the argument in the proof of Lemma 5.9 in [15], we have

$$||(sD(t)' + i)^{-1} - (sD(t) + i)^{-1}|| \leq nt^{-1},$$

$$||(sD(t)' - i)^{-1} - (sD(t) - i)^{-1}|| \leq nt^{-1}.$$

The above inequalities, together with the approximation argument in the proof of Lemma 7.3, imply Lemma 7.4. □

Lemma 7.5. *Let E be a finite-dimensional Euclidean space, let $D(t)$ be as in Lemma 7.4. For any $\epsilon > 0$, $R > 0$, $c > 0$, there exists $t_0 > 0$ such that*

$$||[g_t(D(t)), \pi(h_t)]|| < \epsilon$$

for all $t > t_0$, $g \in C_0(\mathbb{R})$ and $h \in \mathcal{C}(E)$ for which (1) support$(g) \subseteq [-R, R]$, $|g(t)| \leq c$ for all $t \in \mathbb{R}$; (2) the derivative function g' is continuous and $|g'(t)| \leq c$ for all $t \in \mathbb{R}$; (3) $D_Y h \in \mathcal{C}(E)$ for all $Y \in E$, and $||D_Y h|| \leq c$ for all $||Y|| \leq 1$, where [,] denotes the graded commutator and $\pi(h_t)$ denotes the multiplication operator by h_t, and $D_Y h$ is the derivative of the function: $h : E \to$ Cliff(E), in the direction of Y.

Proof. If $g(x) = (x+i)^{-1}$ or $(x-i)^{-1}$, by the computation in the proof of Lemma 2.9 in [15], we have

$$||[g_t(D(t)), \pi(h_t)]|| \leq t^{-2} dim^2(E)c,$$

where $dim(E)$ is the dimension of E. This, together with the approximation argument in the proof of Lemma 7.3, implies Lemma 7.5. □

Proof of Lemma 7.2. $\forall\ T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, define

$$\|T\|_{\max} = \sup_{\psi} \|\psi(T)\|,$$

where the sup is taken over all $*$-representation ψ of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$. Let $C^*_{\max}(P_d(\Gamma), \mathcal{A})$ be the completion of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ with respect to the norm $\|\ \ \|_{\max}$.

For any $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, there exists $r > 0$ such that $T(x, y) = 0$ if $d(x, y) > r$. By the definition of $W_n(x)$, there exists $N > 0$ such that $W_n(x) \subset W_{n+1}(y)$ for all $n \geq N$, x and y in Γ_d satisfying $d(x, y) \leq r$. Let $\beta_{W_{n+1}(y), W_n(x)}$ be the $*$-homomorphism from $\mathcal{A}(W_n(x))\widehat{\otimes}K$ to $\mathcal{A}(W_{n+1}(y))\widehat{\otimes}K$ induced by the inclusion of $W_n(x)$ into $W_{n+1}(y)$ if $n \geq N$ and $d(x, y) \leq r$. Let n_0 be a non-negative integer such that if $n \geq n_0$, then for every pair x and y in Γ_d there exists $T_1(x, y) \in \mathcal{A}(W_n(x))\widehat{\otimes}K$ satisfying $(\beta_n(x))(T_1(x, y)) = T(x, y)$. For each $n \geq max\{n_0, N\}$, x and y in Γ_d satisfying $d(x, y) \leq r$, let $B_{n+1,t}(x, y)$ be the Higson–Kasparov–Trout operator associated to the algebraic decomposition:

$$\begin{aligned} V = {} & V_0(x) \oplus \cdots \oplus V_{n-1}(x) \oplus (W_{n+1}(y) \ominus W_n(x)) \\ & \oplus V_{n+1}(y) \oplus V_{n+2}(y) \oplus \cdots . \end{aligned}$$

Define $\theta_t^{n+1}(x, y) : \mathcal{A}(W_{n+1}(y))\widehat{\otimes}K \to K(\mathcal{H})\widehat{\otimes}K$, by:

$$(\theta_t^{n+1}(x, y))((g\widehat{\otimes}h)\widehat{\otimes}k) = g_t(B_{n+1,t}(x, y))\pi(h_t)\widehat{\otimes}k$$

for every $(g\widehat{\otimes}h)\widehat{\otimes}k \in (\mathcal{S}\widehat{\otimes}\mathcal{C}(W_{n+1}(y)))\widehat{\otimes}K$, $n \geq max\{n_0, N\}$, x and y in Γ_d satisfying $d(x, y) \leq r$. By Lemma 7.3, the definition of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ and the proof of Proposition 4.2 in [15], for each $n \geq \{n_0, N\}$, we know that

$$(\theta_t^n(x))(T_1(x, y)) - (\theta_t^{n+1}(x, y))(\beta_{W_{n+1}(y), W_n(x)}(T_1(x, y)))$$

converges uniformly to 0 in norm on $\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, y) \leq r\}$ as t goes to ∞. But by Lemma 7.4 and the definition of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$, for each $n \geq \{n_0, N\}$,

$$(\theta_t^{n+1}(x, y))(\beta_{W_{n+1}(y), W_n(x)}(T_1(x, y))) - (\theta_t^{n+1}(y))(\beta_{W_{n+1}(y), W_n(x)}(T_1(x, y)))$$

converges uniformly to 0 in norm on $\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, y) \leq r\}$ as t goes to ∞. Hence for each $n \geq max\{n_0, N\}$, we know that

$$(\theta_t^n(x))(T_1(x, y)) - (\theta_t^{n+1}(y))(\beta_{W_{n+1}(y), W_n(x)}(T_1(x, y)))$$

converges uniformly to 0 in norm on $\{(x, y) \in \Gamma_d \times \Gamma_d : d(x, y) \leq r\}$ as t goes to ∞. This fact, together with Lemma 7.5 and the definition of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$, implies that α extends to an asymptotic morphism from $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), K)$. Hence α extends to an asymptotic morphism from $C^*_{max}(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), K)$. But by the uniform embedding property we have $C^*(P_d(\Gamma), \mathcal{A}) = C^*_{\max}(P_d(\Gamma), \mathcal{A})$. Therefore α extends to an asymptotic morphism from $C^*(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), K)$. $\square$

Let $C^*_{alg}(P_d(\Gamma))$ be the algebra of functions T on $\Gamma_d \times \Gamma_d$ such that

(1) $T(x, y) \in K$ for all x and $y \in \Gamma_d$;

(2) $\exists M > 0$ and $L > 0$ such that $\|T(x, y)\| \leq M$ for all $x, y \in \Gamma_d$, $\#\{x \in \Gamma_d : T(x, y) \neq 0\} \leq L$ and $\#\{x \in \Gamma_d : T(y, x) \neq 0\} \leq L$ for all $y \in \Gamma_d$;

(3) $\exists r > 0$ such that if $d(x, y) > r$, then $T(x, y) = 0$.

The product structure on $C^*_{alg}(P_d(\Gamma))$ is defined by:

$$(T_1 T_2)(x, y) = \sum_{z \in \Gamma_d} T_1(x, z) T_2(z, y).$$

$C^*_{alg}(P_d(\Gamma))$ has a $*$-representation on $\ell^2(\Gamma_d) \otimes H_0$, where H_0 is a separable and infinite-dimensional Hilbert space with a faithful $*$-representation of K. The operator norm completion of $C^*_{alg}(P_d(\Gamma))$ with respect to this $*$-representation is $*$-isomorphic to $C^*(P_d(\Gamma))$ when Γ has bounded geometry. Similarly we can define $C^*_{L,alg}(P_d(\Gamma))$, and the operator norm completion of $C^*_{L,alg}(P_d(\Gamma))$ is $*$-isomorphic to $C^*_L(P_d(\Gamma))$ when Γ has bounded geometry.

Next we shall construct an asymptotic morphism β from $\mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$, which plays the role of the Bott map in the classical Bott periodicity theorem.

If H is a finite-dimensional Euclidean space, for each $x \in \Gamma_d$, let

$$\beta(x) : \mathcal{S} \to \mathcal{A}(H),$$

be the $*$-homomorphism defined by:

$$(\beta(x))(g) = g(X\widehat{\otimes}1 + 1\widehat{\otimes}C_{H,x})$$

for all $g \in \mathcal{S}$, where X is the degree one and unbounded multiplier of $\mathcal{S}$ defined by: $(Xa)(s) = sa(s)$ for each $a \in \mathcal{S}$ with compact support and all $s \in \mathbb{R}$, $C_{H,x}$ is the Clifford algebra-valued function on H which maps v to $v - f(x) \in H \subset \mathrm{Cliff}(H)$ for all $v \in H$ (f is the uniform embedding), and $g(X\widehat{\otimes}1 + 1\widehat{\otimes}C_{H,x})$ is defined by functional calculus. We define

$$\beta_t : \mathcal{S}\widehat{\otimes}C^*_{alg}(P_d(\Gamma)) \to C^*(P_d(\Gamma), \mathcal{A})$$

by:

$$(\beta_t(g\widehat{\otimes}T))(x, y) = (\beta(x))(g_t)\widehat{\otimes}T(x, y)$$

for all $g \in \mathcal{S}$, $T \in C^*_{alg}(P_d(\Gamma))$ and $t \geq 1$, where $g_t(s) = g(t^{-1}s)$ for all $t \geq 1$ and $s \in \mathbb{R}$.

If H is a separable and infinite-dimensional Hilbert space, we define

$$\beta_t : \mathcal{S}\widehat{\otimes}C^*_{alg}(P_d(\Gamma)) \to C^*(P_d(\Gamma), \mathcal{A})$$

by:

$$(\beta_t(g\widehat{\otimes}T))(x, y) = (\beta(x))(g_t)\widehat{\otimes}T(x, y)$$

for all $g \in \mathcal{S}$, $T \in C^*_{alg}(P_d(\Gamma))$ and $t \geq 1$, where $g_t(s) = g(t^{-1}s)$ for all $t \geq 1$ and $s \in \mathbb{R}$, and $\beta(x) : \mathcal{S} = \mathcal{A}(0) \to \mathcal{A}(V)$, is the $*$-homomorphism associated to the inclusion of the zero-dimensional affine space 0 into V by mapping 0 to $f(x)$ (f is the uniform embedding).

Lemma 7.6. *β extends to an asymptotic morphism from $\mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$.*

Proof. Let E be as in Definition 5.4. For every $g \in \mathcal{S}$, we define a bounded module homomorphism $N_g : E \to E$, by:

$$N_g\left(\sum_{x\in\Gamma_d} a_x[x]\right) = \sum_{x\in\Gamma_d}((\beta(x))(g)\widehat{\otimes}1)a_x[x]$$

for all $\sum_{x\in\Gamma_d} a_x[x] \in E$. We have

$$\beta_t(g\widehat{\otimes}T) = N_{g_t} \cdot (1\widehat{\otimes}T)$$

for all $g \in \mathcal{S}$ and $T \in C^*_{alg}(P_d(\Gamma))$, where $1\widehat{\otimes}T$ is a bounded module homomorphism from E to E defined by:

$$(1\widehat{\otimes}T)\left(\sum_{x\in\Gamma_d} a_x[x]\right) = \sum_{y\in\Gamma_d}\left(\sum_{x\in\Gamma_d}(1\widehat{\otimes}T(y, x))a_x\right)[y]$$

for all $\sum_{x\in\Gamma_d} a_x[x] \in E$. It follows that β_t extends to a linear map from the algebraic tensor product $\mathcal{S}\widehat{\otimes}_{alg}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$ satisfying

$$||\beta_t(g\widehat{\otimes}T)|| \leq ||g||||T||$$

for all $g \in \mathcal{S}$ and $T \in C^*_{alg}(P_d(\Gamma))$.

Using the above fact, the definition of $C^*_{alg}(P_d(\Gamma))$ and the uniform embedding property, we can verify that β is an asymptotic morphism from the algebraic tensor product $\mathcal{S}\widehat{\otimes}_{alg}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$. Hence β extends to an asymptotic morphism from the maximal tensor product $\mathcal{S}\widehat{\otimes}_{max}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$. This fact, together with the nuclearity of $\mathcal{S}$, implies that β extends to an asymptotic morphism from $\mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma))$ to $C^*(P_d(\Gamma), \mathcal{A})$. ◻

We remark that the asymptotic morphism β is adapted from [14] and [15].

Note that α and β induce homomorphisms:

$$\alpha_* : K_*(C^*(P_d(\Gamma), \mathcal{A})) \to K_*(C^*(P_d(\Gamma))),$$
$$\beta_* : K_*(C^*(P_d(\Gamma))) \to K_*(C^*(P_d(\Gamma), \mathcal{A})).$$

The following result is a geometric analogue of the infinite-dimensional Bott periodicity introduced by Higson, Kasparov and Trout in [15].

Proposition 7.7. *$\alpha_* \circ \beta_*$ equals the identity homomorphism from $K_*(C^*(P_d(\Gamma)))$ to $K_*(C^*(P_d(\Gamma)))$.*

Proof. Let γ be the asymptotic morphism

$$\gamma : \mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma)) \to C^*(P_d(\Gamma), K)$$

defined by:

$$(\gamma_t(g\widehat{\otimes}T))(x, y) = g_{t^2}(B_{0,t}(x))\widehat{\otimes}T(x, y)$$

for every $g \in \mathcal{S}$, $T \in C^*_{alg}(P_d(\Gamma))$, $t \geq 1$, $(x, y) \in \Gamma_d \times \Gamma_d$.

For every $x \in \Gamma_d$, we define $\eta_t(x) : \mathcal{A}(W_1(x))\widehat{\otimes}K \to K(\mathcal{H})\widehat{\otimes}K$, by:

$$(\eta_t(x))((g\widehat{\otimes}h)\widehat{\otimes}k) = g_{t^2}(B_{1,t}(x))\pi(h_{t^2})\widehat{\otimes}k$$

for every $g \in \mathcal{S}$, $h \in \mathcal{C}(W_1(x))$, $k \in K$, and $t \geq 1$. Let γ' be the asymptotic morphism

$$\gamma' : \mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma)) \to C^*(P_d(\Gamma), K)$$

defined by:

$$(\gamma'_t(g\widehat{\otimes}T))(x, y) = (\eta_t(x))(\beta_{W_1(x)}(g)\widehat{\otimes}T(x, y))$$

for every $g \in \mathcal{S}$, $T \in C^*_{alg}(P_d(\Gamma))$, $t \geq 1$, $(x, y) \in \Gamma_d \times \Gamma_d$, where $\beta_{W_1(x)} : \mathcal{S} = \mathcal{A}(0) \to \mathcal{A}(W_1(x))$, is the $*$-homomorphism associated to the inclusion of the zero-dimensional linear space 0 into $W_1(x)$ by mapping 0 to 0. By the proof of Proposition 4.2 in [15], we know that γ is asymptotically equivalent to γ'. Hence we have $\gamma_* = \gamma'_*$.

For every $t \geq 1$ and $x \in \Gamma_d$, let $U_x(t)$ be the unitary operator acting on $\mathcal{H}$ induced by the translation operator $U_x(t)$ on V defined by:

$$(U_x(t))v = v + tf(x)$$

for all $v \in V$, where f is the uniform embedding. Recall that $C^*(P_d(\Gamma), K)$ has a faithful $*$-representation on $l^2(\Gamma_d)\widehat{\otimes}\mathcal{H}\widehat{\otimes}H_0$ (cf. Definition 7.1). For every $x \in \Gamma_d$, $s \in [0, 1]$ and $t \geq 1$, we define a unitary operator $U_{x,s}(t)$ acting on $(l^2(\Gamma_d)\widehat{\otimes}\mathcal{H}\widehat{\otimes}H_0) \oplus (l^2(\Gamma_d)\widehat{\otimes}\mathcal{H}\widehat{\otimes}H_0)$ by:

$$U_{x,s}(t) = R(s)\begin{pmatrix} 1\widehat{\otimes}U_x(t)\widehat{\otimes}1 & 0 \\ 0 & 1 \end{pmatrix} R^{-1}(s),$$

where

$$R(s) = \begin{pmatrix} \cos(\frac{\pi}{2}s) & \sin(\frac{\pi}{2}s) \\ -\sin(\frac{\pi}{2}s) & \cos(\frac{\pi}{2}s) \end{pmatrix}.$$

We define a homotopy of asymptotic morphisms

$$\gamma(s) : \mathcal{S}\widehat{\otimes}C^*(P_d(\Gamma)) \to C^*(P_d(\Gamma), K)\widehat{\otimes}M_2(\mathbb{C})$$

by:

$$((\gamma_t(s))(g\widehat{\otimes}T))(x, y) = U_{x,s}(t) \begin{pmatrix} (\gamma_t'(g\widehat{\otimes}T))(x, y) & 0 \\ 0 & 0 \end{pmatrix} U_{x,s}^{-1}(t)$$

for every $g \in \mathcal{S}$, $T \in C^*_{alg}(P_d(\Gamma))$, $t \geq 1$, $s \in [0, 1]$, and $(x, y) \in \Gamma_d \times \Gamma_d$, where $M_2(\mathbb{C})$ is the algebra of all 2×2 matrices over $\mathbb{C}$ endowed with the trivial grading. The fact that $\gamma(s)$ is an asymptotic morphism for each $s \in [0, 1]$ follows from straightforward computation using the definitions of $\gamma_t(s)$ and $C^*_{alg}(P_d(\Gamma))$, the uniform embedding property, and the fact that $f(x) \in W_1(x)$ and $U_x(t)$ commutes with $B_{1,t}(x)$ for all $x \in \Gamma_d$ and $t \geq 1$.

Again by the fact that $f(x) \in W_1(x)$ and $U_x(t)$ commutes with $B_{1,t}(x)$ for all $x \in \Gamma_d$ and $t \geq 1$, we can verify that

$$(\gamma_t(0))(g\widehat{\otimes}T) - \begin{pmatrix} \alpha_t(\beta_t(g\widehat{\otimes}T)) & 0 \\ 0 & 0 \end{pmatrix} \to 0$$

in norm as $t \to \infty$ for all $g \in \mathcal{S}$ and $T \in C^*_{alg}(P_d(\Gamma))$. It follows that

$$\gamma(0)_* = \alpha_* \circ \beta_*.$$

We also have

$$\gamma_t(1) = \begin{pmatrix} \gamma_t' & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\gamma(1)_* = \gamma_*'$. Therefore we have $\alpha_* \circ \beta_* = \gamma_*$. Replacing $B_{0,t}(x)$ with $s^{-1}B_{0,t}(x)$ in the definition of γ, for $0 < s \leq 1$, we obtain a homotopy between γ and the $*$ homomorphism: $g\widehat{\otimes}T \to g(0)P\widehat{\otimes}T$, where P is the projection onto the one-dimensional kernel of $B_{0,t}(x)$ and P does not depend on x (cf. Corollary 2.15 and the proof of Lemma 5.8 in [15]). It follows that γ_* equals the identity homomorphism. Therefore $\alpha_* \circ \beta_*$ equals the identity homomorphism. □

We can similarly construct asymptotic morphisms

$$\alpha_L : C^*_L(P_d(\Gamma), \mathcal{A}) \to C^*_L(P_d(\Gamma), K),$$
$$\beta_L : \mathcal{S}\widehat{\otimes}C^*_L(P_d(\Gamma)) \to C^*_L(P_d(\Gamma), \mathcal{A}),$$

where $C^*_L(P_d(\Gamma), K)$ is defined in a way similar to the definition of $C^*(P_d(\Gamma), K)$.

Proposition 7.8. *$(\alpha_L)_* \circ (\beta_L)_*$ equals the identity homomorphism from $K_*(C_L^*(P_d(\Gamma))$ to $K_*(C_L^*(P_d(\Gamma))$.*

The proof of Proposition 7.8 is similar to the proof of Proposition 7.7 and is therefore omitted.

Finally we are ready to prove the main result of this paper.

Proof of Theorem 1.1.

By Theorem 4.2, it is enough to show that

$$e_* : \lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))$$

is an isomorphism. Consider the following commuting diagram:

$$\begin{tikzcd}
\lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \arrow[r, "e_*"] \arrow[d, "(\beta_L)_*"'] & \lim_{d\to\infty} K_*(C^*(P_d(\Gamma))) \arrow[d, "\beta_*"'] \\
\lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \arrow[r, "e_*", "\cong"'] \arrow[d, "(\alpha_L)_*"'] & \lim_{d\to\infty} K_*(C^*(P_d(\Gamma), \mathcal{A})) \arrow[d, "\alpha_*"'] \\
\lim_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \arrow[r, "e_*"] & \lim_{d\to\infty} K_*(C^*(P_d(\Gamma)))
\end{tikzcd}$$

By Theorem 6.8 the middle horizontal homomorphism is an isomorphism (note that the uniform embedding condition is used here and the middle horizontal homomorphism is an isomorphism only after the inductive limits $\lim_{d\to\infty}$ are taken). By the geometric analogue of the infinite-dimensional Bott periodicity (Propositions 7.7 and 7.8), $(\beta_L)_* \circ (\alpha_L)_*$ and $\beta_* \circ \alpha_*$ are identity homomorphisms even before the inductive limits $\lim_{d\to\infty}$ are taken (note that the uniform embedding condition is also used here). The above facts, together with a diagram chasing argument, imply Theorem 1.1. □

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