arXiv:math/0208048v1 [math.RT] 6 Aug 2002

Dirac Cohomology for the Cubic Dirac Operator

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Abstract. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{r} \subset \mathfrak{g}$ be any reductive Lie subalgebra such that $B|\mathfrak{r}$ is nonsingular where B is the Killing form of $\mathfrak{g}$. Let $Z(r)$ and $Z(\mathfrak{g})$ be, respectively, the centers of the enveloping algebras of $\mathfrak{r}$ and $\mathfrak{g}$. Using a Harish-Chandra isomorphism one has a homomorphism $\eta : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ which, by a well-known result of H. Cartan, yields the the relative Lie algebra cohomology $H(\mathfrak{g}, \mathfrak{r})$.

Let V be any $\mathfrak{g}$- module. For the case where $\mathfrak{r}$ is a symmetric subalgebra, Vogan has defined the Dirac cohomology $Dir(V)$ of V. Using the cubic Dirac operator we extend his definition to the case where $\mathfrak{r}$ is arbitrary subject to the condition stated above. We then generalize results of Huang-Pandžić on a proof of a conjecture of Vogan. In particular $Dir(V)$ has a structure of a $Z(\mathfrak{r})$-module relative to a "diagonal" homomorphism $\gamma : Z(\mathfrak{r}) \to End\, Dir(V)$. In case V admits an infinitesimal character χ and I is the identity operator on Dir(V) we prove

$$\gamma \circ \eta = \chi\ I \qquad (A)$$

In addition we also prove that V always exists (in fact V can taken to be an object in Category O) such that $Dir(V) \neq 0$. If $\mathfrak{r}$ has the same rank as $\mathfrak{g}$ and V is irreducible and finite dimensional, then (A) generalizes a result of Gross-Kostant-Ramond-Sternberg.

0. Introduction

0.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let (x, y) be a nonsingular symmetric invariant bilinear form $B_{\mathfrak{g}}$ on $\mathfrak{g}$. Let $\mathfrak{r} \subset \mathfrak{g}$ be any reductive Lie subalgebra

* Research supported in part by NSF grant DMS-9625941 and in part by the KG&G Foundation

of $\mathfrak{g}$ such that $B_{\mathfrak{r}} = B_{\mathfrak{g}}|\mathfrak{r}$ is nonsingular. Let $\mathfrak{p}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$ so that $[\mathfrak{r}, \mathfrak{p}] \subset \mathfrak{p}$ and one has the direct sum $\mathfrak{g} = \mathfrak{r} + \mathfrak{p}$. Let $B_{\mathfrak{p}} = B_{\mathfrak{g}}|\mathfrak{p}$ so that $B_{\mathfrak{p}}$ is nonsingular and let $C(\mathfrak{p})$ be the Clifford algebra over $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. Then there exists a homomorphism $\nu_* : \mathfrak{r} \to C(\mathfrak{p})$ such that $[x, y] = [\nu_*(x), y]$ for $x \in \mathfrak{r}$ and $y \in \mathfrak{p}$ where the bracket on the right side is commutation in $C(\mathfrak{p})$. See §1.5 in [K1]. One then has a homomorphism

$$\zeta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}) \tag{0.1}$$

so that $\zeta(x) = x \otimes 1 + 1 \otimes \nu_*(x)$ for $x \in \mathfrak{r}$. This defines the structure of an $\mathfrak{r}$-module on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. We have defined an element $\square \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ in [K1] and have referred to $\square$ as the cubic Dirac operator. The definition of $\square$ is recalled in §2.2 of the present paper. In case $\mathfrak{r}$ is a symmetric subalgebra of $\mathfrak{g}$ the cubic term in $\square$ vanishes and $\square$ is the more familiar Dirac operator under consideration in [HP] and [P]. The result main result in [HP] works in the general case considered here and one has a unique homomorphism

$$\eta_{\mathfrak{r}} : Z(\mathfrak{g}) \to Z(\mathfrak{r})$$

so that for any $p \in Z(\mathfrak{g})$ one has

$$p \otimes 1 - \zeta(\eta_r(p)) = \square\,\omega + \omega\square \tag{0.2}$$

for some $\omega \in (U(\mathfrak{g}) \otimes C^{odd}(\mathfrak{p}))^{\mathfrak{r}}$. See Appendix. We will determine the homomorphism $\eta_{\mathfrak{r}}$ in the generality under consideration here.

A subspace $\mathfrak{s} \subset \mathfrak{g}$ will be said to be $(\mathfrak{r}, \mathfrak{p})$ split if $\mathfrak{s} = \mathfrak{s} \cap \mathfrak{r} + \mathfrak{s} \cap \mathfrak{p}$. In such a case we will write $\mathfrak{s}_{\mathfrak{r}} = \mathfrak{s} \cap \mathfrak{r}$ and $\mathfrak{s}_{\mathfrak{p}} = \mathfrak{s} \cap \mathfrak{p}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. Then $\mathfrak{b}$ and $\mathfrak{h}$ can (and will) be chosen so that they are both $(\mathfrak{r}, \mathfrak{p})$ split. Then $\mathfrak{b}_{\mathfrak{r}}$ is a Borel subalgebra of $\mathfrak{r}$ and $\mathfrak{h}_{\mathfrak{r}}$ is a Cartan subalgebra of $\mathfrak{r}$. It follows also that $\mathfrak{n}$ is $(\mathfrak{r}, \mathfrak{p})$-split. The subspace $\mathfrak{n}_{\mathfrak{p}} \subset \mathfrak{p}$ is isotropic with respect to $B_{\mathfrak{p}}$. Let $u \in C(\mathfrak{p})$ be the product, in any order, of a basis of $\mathfrak{n}_{\mathfrak{p}}$. Let $L = C(\mathfrak{p})\, u$ so that L is a left ideal in $C(\mathfrak{p})$ and is in particular a $C(\mathfrak{p})$-module with respect to left multiplication. Now assume that V is a $U(\mathfrak{g})$-module. The separate actions of $U(\mathfrak{g})$ and $C(\mathfrak{p})$ define a homomorphism

$$\xi_V : U(\mathfrak{g}) \otimes C(\mathfrak{p}) \to End\,(V \otimes L)$$

Extending Vogan's definition to the present case one defines the Dirac cohomology $H_D(V \otimes L)$ so that

$$H_D(V \otimes L) = Ker\,\Box_V/(Ker\,\Box_V \cap Im\,\Box_V) \tag{0.3}$$

where $\Box_V = \xi_V(\Box)$. The action of $\xi_V(Z(\mathfrak{g}) \otimes 1)$ on $V \otimes L$ defines a $Z(\mathfrak{g})$-module structure on $H_D(V \otimes L)$. Also if $\zeta_V = \xi_V \circ \zeta$ then the action of $\zeta_V(Z(\mathfrak{r}))$ on $V \otimes L$ defines a $Z(\mathfrak{r})$-module structure on $H_D(V \otimes L)$. As a consequence of (0.2) one has, for any $p \in Z(\mathfrak{g})$,

$$\xi_V(p \otimes 1) = \zeta_V(\eta_{\mathfrak{r}}(p)) \text{ on } H_D(V \otimes L) \tag{0.4}$$

This raises the question as to whether or not $H_D(V \otimes L)$ vanishes.

Let $\lambda \in \mathfrak{h}^*$ and let V_λ be the unique irreducible (Category O) $U(\mathfrak{g})$-module with highest weight λ (with respect to $\mathfrak{b}$). Let $0 \neq v_\lambda \in V_\lambda$ be a corresponding highest weight vector. If $dim\,\mathfrak{h}_{\mathfrak{p}} = k$ then $\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}_{\mathfrak{p}})u$ is a 2^k-dimensional subspace of $V_\lambda \otimes L$. Here $C(\mathfrak{h}_{\mathfrak{p}}) \subset C(\mathfrak{p})$ is the Clifford algebra over $\mathfrak{h}_{\mathfrak{p}}$. Let $\rho \in \mathfrak{h}^*$ have its usual meaning. In this paper we will prove

Theorem 0.1. *Choose λ so that $\lambda + \rho$ vanishes on $\mathfrak{h}_{\mathfrak{p}}$ (e.g., $\lambda = -\rho$ if $\mathfrak{r} = 0$). Then $\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}_{\mathfrak{p}})u \subset Ker\,\Box_{V_\lambda}$. Furthermore the map from cocycle to cohomology defines an injection*

$$\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}_{\mathfrak{p}})u \to H_D(V_\lambda \otimes L) \tag{0.5}$$

In particular $H_D(V_\lambda \otimes L) \neq 0$.

Since there is no restriction on $\lambda|\mathfrak{h}_{\mathfrak{r}}$ we can compute $\eta_{\mathfrak{r}}$.

Let $\phi_o : \mathfrak{h} \to \mathfrak{h}_{\mathfrak{r}}$ be the projection relative to the decomposition $\mathfrak{h} = \mathfrak{h}_{\mathfrak{r}} + \mathfrak{h}_{\mathfrak{p}}$. Then ϕ_o extends to a homomorphism $S(\mathfrak{h}) \to S(\mathfrak{h}_{\mathfrak{r}})$ and clearly induces a homomorphism

$$\phi : S(\mathfrak{h})^{W_{\mathfrak{g}}} \to S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}}$$

where $W_{\mathfrak{g}}$ and $W_{\mathfrak{r}}$ are the respective Weyl groups of $\mathfrak{h}$ relative to $\mathfrak{g}$ and $\mathfrak{h}_{\mathfrak{r}}$ relative to $\mathfrak{r}$. Given the fact that $\mathfrak{r}$ is essentially an arbitrary reductive Lie subalgebra of $\mathfrak{g}$ the following result established here is a strong generalization of Theorem 5.5 in [HP]. As will be noted in §5 in this paper it is also a generalization of Proposition 3.43, (5.18) and (5.19) in [K1].

Theorem 0.2. *The map $\eta_{\mathfrak{r}} : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ is uniquely determined so that the following diagram is commutative. In the diagram the vertical maps are the Harish-Chandra isomorphisms.*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\eta_{\mathfrak{r}}} & Z(\mathfrak{r}) \\ \big\downarrow A_{\mathfrak{g}} & & \big\downarrow A_{\mathfrak{r}} \\ S(\mathfrak{h})^{W_{\mathfrak{g}}} & \xrightarrow{\phi} & S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}} \end{array}$$

The map ϕ is well known in the theory of the cohomology of compact homogeneous spaces. Actually what is utilized in that theory is the map $S(\mathfrak{h}^*)^{W_{\mathfrak{g}}} \to S(\mathfrak{h}_{\mathfrak{r}}^*)^{W_{\mathfrak{r}}}$ induced by restriction of functions. However this is same as ϕ if $\mathfrak{h}$ and $\mathfrak{h}^*$ are identified and $\mathfrak{h}_{\mathfrak{r}}$ and $\mathfrak{h}_{\mathfrak{r}}^*$ are identified using $B_{\mathfrak{g}}$. Assume G is a compact connected semisimple Lie group and $\mathfrak{g}$ is the complexification of $Lie\, G$. Let $R \subset G$ be any connected compact subgroup and let $\mathfrak{r}$ be the complexification of $Lie\, R$. Obviously we can choose $B_{\mathfrak{g}}$ so that $B_{\mathfrak{g}}|\mathfrak{r}$ is nonsingular (e.g., let $B_{\mathfrak{g}}$ be the Killing form). The map $\eta_{\mathfrak{r}}$ induces the structure of a $Z(\mathfrak{g})$-module on $Z(\mathfrak{r})$. On the other hand the infinitesimal character for the module V_λ when $\lambda = -\rho$ defines the structure of a $Z(\mathfrak{g})$-module on $\mathbb{C}$. As a consequence of a well-known theorem of H. Cartan (see §9 in [C]) one has

Theorem 0.3. *There exists an isomorphism*

$$H^*(G/R,\mathbb{C}) \cong Tor_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})) \tag{0.6}$$

In §5 we reformulate certain results in [K1] using Dirac cohomology.

0.2. We wish to thank David Vogan for many profitable conversations and for introducing us to his Dirac cohomology concept in the case where $\mathfrak{r}$ is a symmetric subalgebra of $\mathfrak{g}$. We also wish to acknowledge the strong impact made upon us by the main result in [HP].

1. Preliminaries

1.1. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let $B_{\mathfrak{g}}$ be a nonsingular *ad*-invariant symmetric bilinear form (x,y) on $\mathfrak{g}$. Let $\mathfrak{r} \subset \mathfrak{g}$ be a reductive Lie subalgebra and assume that $B_{\mathfrak{r}} = B_{\mathfrak{g}}|\mathfrak{r}$ is nonsingular. Let $\mathfrak{p}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{r}$ in $\mathfrak{g}$ and let $B_{\mathfrak{p}} = B_{\mathfrak{g}}|\mathfrak{p}$. Then of course

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{p}$$

and $[\mathfrak{r},\mathfrak{p}] \subset \mathfrak{p}$. Let $\mathfrak{h}_{\mathfrak{r}}$ be a Cartan subalgebra of $\mathfrak{r}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}_{\mathfrak{r}}$. Of course $B_{\mathfrak{r}}|\mathfrak{h}_r$ and $B_{\mathfrak{g}}|\mathfrak{h}$ are nonsingular. Let $\mathfrak{h}_{\mathfrak{p}}$ be the orthocomplement of $\mathfrak{h}_{\mathfrak{r}}$ in $\mathfrak{h}$ so that

$$\mathfrak{h} = \mathfrak{h}_{\mathfrak{r}} + \mathfrak{h}_{\mathfrak{p}} \tag{1.1}$$

and

$$\mathfrak{h}_{\mathfrak{r}} = \mathfrak{h} \cap \mathfrak{r}$$

$$\mathfrak{h}_{\mathfrak{p}} = \mathfrak{h} \cap \mathfrak{p}$$

Let $\Delta \subset \mathfrak{h}^*$ be the set of roots for $(\mathfrak{h},\mathfrak{g})$ and for each $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. We normalize the choice so that $(e_\varphi, e_{-\varphi}) = 1$. Let $\mathfrak{g}^0$ be the centralizer of $\mathfrak{h}_\mathfrak{r}$ in $\mathfrak{g}$ and let $\Delta^0 = \{\varphi \in \Delta \mid \varphi(x) = 0, \forall x \in \mathfrak{h}_\mathfrak{r}\}$ so that

$$\mathfrak{g}^0 = \mathfrak{h} + \sum_{\varphi \in \Delta^0} \mathbb{C}\, e_\varphi \tag{1.2}$$

Let $\mathfrak{h}^\#$ be the real space of hyperbolic elements in $\mathfrak{h}$ and let $\kappa : \mathfrak{h} \to \mathfrak{h}^\#$ be the real projection which vanishes on $i\mathfrak{h}^\#$. Since $\mathfrak{h}_\mathfrak{r}$ is complex there clearly exists $f_\mathfrak{r} \in \kappa(\mathfrak{h}_\mathfrak{r})$ such that if $\mathfrak{g}^{f_\mathfrak{r}}$ is the centralizer of $f_\mathfrak{r}$ in $\mathfrak{g}$ then $\mathfrak{g}^{f_\mathfrak{r}} = \mathfrak{g}^0$. But $f_\mathfrak{r}$ defines a parabolic Lie subalgebra of $\mathfrak{q}$ of $\mathfrak{g}$ where $\mathfrak{g}^0$ is a Levi factor of $\mathfrak{q}$ and the *nilrad* $\mathfrak{q}$ is the span of all eigenvectors of $ad\, f_\mathfrak{r}$ with positive eigenalues. Clearly $ad\, f_\mathfrak{r}$ stabilizes both $\mathfrak{r}$ and $\mathfrak{p}$ and hence

$$nilrad\ \mathfrak{q} = \mathfrak{n}_\mathfrak{r} + \mathfrak{p}^+ \tag{1.3}$$

where $\mathfrak{n}_\mathfrak{r} = \mathfrak{r} \cap nilrad\ \mathfrak{q}$ and $\mathfrak{p}^+ = \mathfrak{p} \cap nilrad\ \mathfrak{q}$. Since clearly

$$\mathfrak{g}^0 \cap \mathfrak{r} = \mathfrak{h}_\mathfrak{r} \tag{1.4}$$

it follows that

$$\mathfrak{b}_\mathfrak{r} = \mathfrak{h}_\mathfrak{r} + \mathfrak{n}_\mathfrak{r} \tag{1.5}$$

is a Borel subalgebra of $\mathfrak{r}$ and $\mathfrak{n}_\mathfrak{r}$ is the nilradical of $\mathfrak{b}_\mathfrak{r}$. Furthermore (1.4) implies that

$$\mathfrak{g}^0 = \mathfrak{h}_\mathfrak{r} + \mathfrak{p}^0 \tag{1.6}$$

where $\mathfrak{p}^0 = \mathfrak{g}^0 \cap \mathfrak{p}$, is an orthogonal decomposition with respect to the (obviously) nonsingular bilinear form $B_\mathfrak{g}|\mathfrak{g}^0$. Let $\mathfrak{c} = Cent\,\mathfrak{g}^0$. Since of course $\mathfrak{h}_\mathfrak{r} \subset \mathfrak{c}$ one has

$$\mathfrak{c} = \mathfrak{h}_\mathfrak{r} + \mathfrak{c}_\mathfrak{p} \tag{1.7}$$

where $\mathfrak{c}_\mathfrak{p} = \mathfrak{c} \cap \mathfrak{p}$, is an orthogonal decomposition with respect to the (obviously) nonsingular bilinear form $B_\mathfrak{g}|\mathfrak{c}$. Of course $\mathfrak{c} \subset \mathfrak{h}$ so that $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{h}^\mathfrak{p}$. Let $\mathfrak{d}_\mathfrak{p}$ be the orthocomplement of $\mathfrak{c}_\mathfrak{p}$ in $\mathfrak{h}_\mathfrak{p}$ so that

$$\mathfrak{h}_\mathfrak{p} = \mathfrak{c}_\mathfrak{p} + \mathfrak{d}_\mathfrak{p} \tag{1.8}$$

is an orthogonal decomposition.

Remark 1.1 Note that (1.2) and (1.4) imply that

$$\mathfrak{p}^0 = \mathfrak{h}_{\mathfrak{p}} + \sum_{\varphi \in \Delta^0} \mathbb{C} e_\varphi \tag{1.9}$$

and that $\mathfrak{p}^0$ is a reductive Lie subalgebra of $\mathfrak{g}$ which happens to lie in $\mathfrak{p}$. Furthermore $\mathfrak{h}_{\mathfrak{p}}$ is a Cartan subalgebra of $\mathfrak{p}^0$ and $\mathfrak{c}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{p}}$ is the center of $\mathfrak{p}^0$. In particular

$$\mathfrak{p}^0 = \mathfrak{c}_{\mathfrak{p}} + [\mathfrak{p}^0, \mathfrak{p}^0] \tag{1.10}$$

is an orthogonal decomposition and (1.8) implies that $\mathfrak{d}_{\mathfrak{p}}$ is a Cartan subalgebra of the semisimple Lie algebra $[\mathfrak{p}^0, \mathfrak{p}^0]$. Obviously

$$[\mathfrak{p}^0, \mathfrak{p}^0] = \mathfrak{d}_{\mathfrak{p}} + \sum_{\varphi \in \Delta^0} \mathbb{C} e_\varphi \tag{1.11}$$

and (1.11) is the decomposition of $[\mathfrak{p}^0, \mathfrak{p}^0]$ as the sum of a Cartan subalgebra and corresponding root spaces. Let $\mathfrak{p}'$ be the orthocomplement of $\mathfrak{p}^0$ in $\mathfrak{p}$. Clearly $\mathfrak{p}'$ is stable under $ad\,\mathfrak{h}_{\mathfrak{r}}$ and hence $\mathfrak{p}'$ is stable under $ad\,f_{\mathfrak{r}}$. Let $\mathfrak{p}^-$ be the span of all eigenvectors of $ad\,f_{\mathfrak{r}}$ in $\mathfrak{p}'$ with negative eigenvalues. Obviously $\mathfrak{p}' = \mathfrak{p}^+ + \mathfrak{p}^-$ so that one has direct sums

$$\begin{aligned} \mathfrak{p} &= \mathfrak{p}^0 + \mathfrak{p}' \\ &= \mathfrak{p}^0 + \mathfrak{p}^+ + \mathfrak{p}^- \end{aligned} \tag{1.12}$$

Let $\Gamma \subset \mathfrak{h}_{\mathfrak{r}}^*$ be the set of all weights for the adjoint action of $\mathfrak{h}_{\mathfrak{r}}$ on $\mathfrak{p}'$ and for any $\mu \in \Gamma$ let $\mathfrak{p}^\mu \subset \mathfrak{p}'$ be the corresponding weight space so that one has the direct sum

$$\mathfrak{p}' = \sum_{\mu \in \Gamma} \mathfrak{p}^\mu \tag{1.13}$$

It is clear that any $\mu \in \Gamma$ extends uniquely to a linear functional (to be identified with μ) on the complex subspace of $\mathfrak{h}$ spanned by $\kappa(\mathfrak{h}_{\mathfrak{r}})$. One has a partition $\Gamma = \Gamma_+ \cup \Gamma_-$ where

$$\Gamma_+\,(\text{resp.}\,\Gamma_-) = \{\mu \in \Gamma \mid \mu(f_{\mathfrak{r}}) > 0\,(\text{resp.}\,\mu(f_{\mathfrak{r}}) < 0)\}$$

Remark 1.2. By considering the action of $ad\,\mathfrak{h}_r$ a standard argument implies that for $\mu,\nu\in\Gamma$ one has that $\mathfrak{p}^{\mu}$ is $B_{\mathfrak{p}}$-orthogonal to $\mathfrak{p}^{\nu}$ if $\nu\neq-\mu$. But since $B_{\mathfrak{p}}|\mathfrak{p}'$ is clearly nonsingular one has that $\Gamma=-\Gamma$ and $\mathfrak{p}^{\mu}$ is nonsingularly paired to $\mathfrak{p}^{-\mu}$ for any $\mu\in\Gamma$. It then follows that $\Gamma_-=-\Gamma_+$ and

$$\begin{aligned}\mathfrak{p}^+ &= \sum_{\mu\in\Gamma_+}\mathfrak{p}^{\mu}\\ \mathfrak{p}^- &= \sum_{\mu\in\Gamma_+}\mathfrak{p}^{-\mu}\end{aligned}\tag{1.14}$$

Obviously there exists a closed Weyl chamber $C\ \subset\ \mathfrak{h}^{\#}$ such that $f_{\mathfrak{r}}\in C$. Let $f\in\mathfrak{h}^{\#}$ be an element in the interior of C so that, in particular, $f\in\mathfrak{h}$ is regular and hyperbolic. One defines a choice of positive roots $\Delta_+\ \subset\ \Delta$ by putting $\Delta_+=\{\varphi\in\Delta\mid\varphi(f)>0\}$. Let $\Delta_-\ =-\Delta_-$. Let $\mathfrak{b}\ \subset\ \mathfrak{g}$ be the Borel subalgebra defined by putting

$$\mathfrak{b}=\mathfrak{h}+\sum_{\varphi\in\Delta_+}\mathbb{C}\,e_{\varphi}$$

Let $\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]$ be the nilradical of $\mathfrak{b}$. Since $f_{\mathfrak{r}}\in C$ one readily has

$$\begin{aligned}\mathfrak{b}\ &\subset\ \mathfrak{q}\\ nilrad\,\mathfrak{q}\ &\subset\ \mathfrak{n}\end{aligned}\tag{1.15}$$

Let $\mathfrak{n}^0=\mathfrak{n}\cap\mathfrak{p}^0$, $\Delta^0_+=\Delta_+\cap\Delta^0$ and $\Delta^0_-\ =-\Delta^0_+$ so that

$$\mathfrak{n}^0=\sum_{\varphi\in\Delta^0_+}\mathbb{C}\,e_{\varphi}\tag{1.16}$$

It then follows from (1.3),(1.5) and (1.15) that

$$\mathfrak{n}=\mathfrak{n}_{\mathfrak{r}}+\mathfrak{n}^0+\mathfrak{p}^+\tag{1.17}$$

noting now that $\mathfrak{n}_{\mathfrak{r}}$, the nilradical of the Borel subalgebra $\mathfrak{b}_{\mathfrak{r}}$ of $\mathfrak{r}$, is given by

$$\mathfrak{n}_r=\mathfrak{n}\cap\mathfrak{r}\tag{1.18}$$

and also

$$\mathfrak{n}^0+\mathfrak{p}^+=\mathfrak{n}\cap\mathfrak{p}\tag{1.19}$$

2. Dirac cocycles

2.1. Let $C(\mathfrak{p})$ be the Clifford algebra over $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$. As in §1.5 of [K1] we identify the underlying linear spaces of $C(\mathfrak{p})$ and the exterior algebra $\wedge\mathfrak{p}$ and understand that there are two multiplications in $C(\mathfrak{p})$. If $w, z \in C(\mathfrak{p})$ then $w\,z$ denotes the Clifford product and $w \wedge z$ the exterior product of w and z. If $w \in \wedge^k\mathfrak{p}$ and $z \in \wedge^{k'}\mathfrak{p}$ then one knows

$$w\,z - w \wedge z \in \sum_{j=0}^{k+k'-2} \wedge^j\mathfrak{p} \tag{2.1}$$

(for an argument see e.g., (1.6) in [K1]). The bilinear form $B_{\mathfrak{p}}$ on $\mathfrak{p}$ extends to an nonsingular bilinear form (w,z) on $C(\mathfrak{p})$, to be denoted by $B_{C(\mathfrak{p})}$, so that if $w \in \wedge^k\mathfrak{p}$ and $z \in \wedge^{k'}\mathfrak{p}$ then $(w,z) = 0$ if $k \neq k'$. If $k = k'$ then $(w,z) = det\,(w_i, z_j)$ where $w = w_1 \wedge \cdots \wedge w_k$ and $z = z_1 \wedge \cdots \wedge z_k$ for $z_i, w_j \in \mathfrak{p}$. It is immediate then that

$$\mathfrak{m}_{\mathfrak{p}} = \mathfrak{n}^0 + \mathfrak{p}^+ \tag{2.2}$$

is a $B_{\mathfrak{p}}$-isotropic subspace of $\mathfrak{p}$. However since $\mathfrak{m}_{\mathfrak{p}}$ is $B_{\mathfrak{p}}$-isotropic it follows that Clifford product and exterior product are the same for elements in $\mathfrak{m}_{\mathfrak{p}}$. Let u_0 be the product of all the root vectors e_φ for $\varphi \in \Delta_+^0$ in some order and let u_+ be the product of a basis of $\mathfrak{p}^+$ in some order. Put $u = u_0\,u_+$ so that, in $C(\mathfrak{p})$,

$$z\,u = 0 \quad \forall z \in \mathfrak{m}_p \tag{2.3}$$

Let $L \subset C(\mathfrak{p})$ be the left ideal

$$L = C(\mathfrak{p})\,u \tag{2.4}$$

In particular L is a $C(\mathfrak{p})$-module under left multiplication. Let $C(\mathfrak{h}_{\mathfrak{r}})$ be the Clifford algebra over $\mathfrak{h}_{\mathfrak{r}}$ so that $C(\mathfrak{h}_{\mathfrak{r}})$ is a subalgebra of $C(\mathfrak{p})$.

Proposition 2.1. *The map*

$$C(\mathfrak{h}_{\mathfrak{r}}) \to L, \qquad a \mapsto au$$

is injective. Furthermore for any $z \in \mathfrak{m}_{\mathfrak{p}}$ and $a \in C(\mathfrak{h}_{\mathfrak{r}})$ one has

$$z\,a\,u = 0 \tag{2.5}$$

Proof. The first statement is a consequence of (2.1) and the fact that $\mathfrak{h}_{\mathfrak{r}} \cap \mathfrak{m}_{\mathfrak{p}} = 0$. The equation (2.5) follows from (2.3) and the fact that $\mathfrak{h}_{\mathfrak{r}}$ is $B_{\mathfrak{p}}$-orthogonal to $\mathfrak{m}_{\mathfrak{p}}$. QED

2.2. Let $U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$ where $\mathfrak{a} \subset \mathfrak{g}$ is any Lie subalgebra. We are mainly concerned here with the algebra tensor product $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. If $x, y \in \mathfrak{p}$ then no confusion should arise from $x \otimes y$ as an element in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. The left factor x is taken to be in $U(\mathfrak{g})$ and the right factor y is taken to be in $C(\mathfrak{p})$. In §2.1 of [K1] we introduced an element $\square \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$ which we referred to as a cubic Dirac operator (see §0.23 in [K1]). We recall the definition of $\square$. Let I be an index set having cardinality equal to $dim\,\mathfrak{p}$. Then $\square = \square' + \square''$ where if $\{z_i\}$, $i \in I$, is an orthonormal basis of $\mathfrak{p}$ with respect to $B_{\mathfrak{p}}$ one has

$$\square' = \sum_{i \in I} z_i \otimes z_i \tag{2.6}$$

and

$$\square'' = 1 \otimes v \tag{2.7}$$

where $v \in \wedge^3 \mathfrak{p}$ is such that for any $x, x', x'' \in \mathfrak{p}$ one has

$$([x, x'], x'') = -2(v, x \wedge x' \wedge x'') \tag{2.8}$$

See (1.20) in [K1].

Now let $\lambda \in \mathfrak{h}^*$ be arbitrary and let V_λ be the unique irreducible highest module for $U(\mathfrak{g})$ with highest weight (relative to $\mathfrak{b}$) λ. We recall that λ extends uniquely to a character on $\mathfrak{b}$, $z \mapsto \lambda(z)$, which necessarily vanishes on $\mathfrak{n}$, and if $\mathbb{C}_\lambda$ is the corresponding 1-dimensional $U(\mathfrak{b})$-module then V_λ is the quotient of the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ by the unique maximal proper submodule. Let $0 \neq v_\lambda \in V_\lambda$ be a highest weight vector so that $z\,v_\lambda = \lambda(z)v_\lambda$ for any $z \in \mathfrak{b}$. Now let $V_{\lambda,L} = V_\lambda \otimes L$ so the action of $U(\mathfrak{g})$ on V_λ and $C(\mathfrak{p})$ on L defines an algebra homomorphism

$$\xi_\lambda : U(\mathfrak{g}) \otimes C(\mathfrak{p}) \to End\, V_{\lambda,L} \tag{2.9}$$

Let $a \in C(\mathfrak{h}_\mathfrak{r}$ and put

$$v_{\lambda,a} = v_\lambda \otimes a\,u \tag{2.10}$$

The element $v_{\lambda,a} \in V_{\lambda,L}$ is nonzero, by Proposition 2.1, if $a \neq 0$. Our principal goal now is to compute $\xi_\lambda(\Box)v_{\lambda,a}$.

For any $\nu \in \mathfrak{h}^*$ let $z_\nu \in \mathfrak{h}$ be the element corresponding to ν with respect to the isomorphism $\mathfrak{h}^* \to \mathfrak{h}$ defined by $B_\mathfrak{g}|\mathfrak{h}$. Thus $(z, z_\nu) = \nu(z)$ for any $z \in \mathfrak{h}$. But now, by (1.1), there uniquely exists $x_\nu \in \mathfrak{h}_\mathfrak{r}$ and $y_\nu \in \mathfrak{h}_\mathfrak{p}$ such that

$$z_\nu = x_\nu + y_\nu \tag{2.11}$$

We will first deal with $\xi_\lambda(\Box')v_{\lambda,a}$. Let $\{b_i\}$, $i \in I$, be any basis of $\mathfrak{p}$ and let $\{d_i\}$, $i \in I$, be the dual basis with respect to $B_\mathfrak{p}$. It is clear from (2.6) that $\Box'$ can be rewritten as the sum

$$\Box' = \sum_{i \in I} b_i \otimes d_i \tag{2.12}$$

We will now partition the index set I, first, as a union of three parts

$$I = I^\mathfrak{h} \cup I^0 \cup I' \tag{2.13}$$

where $\{b_j\}$, $j \in I^\mathfrak{h}$, is an orthonormal basis of $\mathfrak{h}_\mathfrak{p}$. Next $\{b_k\}$, $k \in I^0, = \{e_\varphi\}$, $\varphi \in \Delta^0$, and $\{b_m\}$, $m \in I'$, is a basis of $\mathfrak{p}'$. We further refine the choice of the basis by partitioning

$$I' = \bigcup_{\mu \in \Gamma} I^\mu \tag{2.14}$$

so that $\{b_m\}$, $m \in I^\mu$, is a basis of $\mathfrak{p}^\mu$. By Remark 1.2 we can make the choice so that

$$\{b_{m'}\},\ m' \in I^{-\mu}, = \{d_m\},\ m \in I^\mu, \tag{2.15}$$

Note also that if $j \in I^0$ and $b_j = e_\varphi$ for $\varphi \in \Delta^0$ then necessarily one has

$$d_j = e_{-\varphi} \tag{2.16}$$

In addition for any $j \in I^{\mathfrak{h}}$ one then clearly has

$$d_j = b_j \tag{2.17}$$

Remark 2.2. Note that, by (2.15), (2.16) and (2.17), setwise

$$\{b_i\},\ j \in I, = \{d_i\},\ i \in I,$$

Also for any $i \in I^0 \cup I'$ one has, by (2.2),

$$\text{either } b_i \in \mathfrak{m}_{\mathfrak{p}} \text{ or } d_i \in \mathfrak{m}_{\mathfrak{p}} \tag{2.18}$$

Lemma 2.3. *For any $\lambda \in \mathfrak{h}^*$ and $a \in C(\mathfrak{h}_{\mathfrak{p}})$ one has*

$$\xi_\lambda(\square')v_{\lambda,a} = v_\lambda \otimes y_\lambda\, a\, u \tag{2.19}$$

Proof. By (1.17) and (2.2) one has

$$\mathfrak{m}_{\mathfrak{p}} \ \subset \ \mathfrak{n} \tag{2.20}$$

It then follows from (2.5) and (2.18) that, for any $i \in I^0 \cup I'$,

$$\xi_\lambda(b_i \otimes d_i)v_{\lambda,a} = 0 \tag{2.21}$$

However for $j \in I^{\mathfrak{h}}$ one clearly has, by (2.5),

$$\xi_\lambda(b_j \otimes d_j)v_{\lambda,a} = \lambda(b_j)v_\lambda \otimes b_j\, a\, u \tag{2.22}$$

But clearly $\sum_{j\in I^{\mathfrak{h}}} \lambda(b_j)\, b_j = y_\lambda$. This proves (2.19). QED

2.3. We will now compute $\xi_\lambda(\Box'')v_{\lambda,a}$. To do so we first introduce a simple ordering in I. We will choose the ordering so that $i<j$ if $i\in I^{\mathfrak{h}}$ and $j\in I^0$, and also if $i\in I^0$ and $j\in I'$. Also we fix the order so that if $i,j\in I^0$ and $b_i=e_\varphi$, $b_j=e_{\varphi'}$ for some $\varphi,\varphi'\in\Delta^0$ then $i<j$ if $\varphi\in\Delta^0_+$ and $\varphi'\in\Delta^0_-$. In addition if $i,j\in I'$ then $i<j$ if $b_i\in\mathfrak{p}^+$ and $b_j\in\mathfrak{p}_-$. Let T' be the set of all ordered triples $\{i,j,k\}$ where $i,j,k\in I$ and $i<j<k$. The set $\{b_i\wedge b_j\wedge b_k\}$, $\{i,j,k\}\in T'$, is of course a basis of $\wedge^3\mathfrak{p}$. The dual basis with respect to $B_{C(\mathfrak{p})}|\wedge^3\mathfrak{p}$ is clearly $\{d_i\wedge d_j\wedge d_k\}$, $\{i,j,k\}\in T'$. But now we can write

$$v=\sum_{\{i,j,k\}\in T'} c_{ijk}\; b_i\wedge b_j\wedge b_k \tag{2.23}$$

for $c_{ijk}\in\mathbb{C}$. But then for any $\{i,j,k\}\in T'$ one has

$$([d_i,d_j],d_k)=-2\,c_{ijk} \tag{2.24}$$

by (2.8).

But it is clear from our choice of basis that, for any $i\in I$, b_i is a weight vector for some weight $\gamma_i\in\mathfrak{h}_r^*$ with respect to the action of $ad\,\mathfrak{h}_r$ on $\mathfrak{p}$. Also it is clear that $\gamma_i\in\Gamma\cup\{0\}$. Note then it follows from Remark 2.2 that d_i is a weight vector with weight $-\gamma_i$. But then it follows from (2.24) that $c_{ijk}\neq 0$ implies that $\gamma_i+\gamma_j+\gamma_k=0$. Thus if $T=\{\{i,j,k\}\in T'\mid \gamma_i+\gamma_j+\gamma_k=0\}$ then one has

$$v=\sum_{\{i,j,k\}\in T} c_{ijk}\; b_i\wedge b_j\wedge b_k \tag{2.25}$$

Let $i\in I$. Then obviously

$$\begin{aligned}\gamma_i=0 \quad&\Longleftrightarrow\quad b_i\in\mathfrak{p}^0\\ &\Longleftrightarrow\quad i\in I^{\mathfrak{h}}\cup I^0\\ \gamma_i\neq 0\quad&\Longleftrightarrow\quad b_i\in\mathfrak{p}'\\ &\Longleftrightarrow\quad i\in I'\end{aligned} \tag{2.26}$$

Now let $T_0 = \{\{i,j,k\} \in T \mid \gamma_i = \gamma_j = \gamma_k = 0\}$ so that if $\{i,j,k\} \in T$ then $\{i,j,k\} \in T_0$ if and only if $\{b_i, b_j, b_k\} \subset \mathfrak{p}^0$. Let T_1 be the complement of T_0 in T. For $\varepsilon = 0, 1$, let

$$v^{(\varepsilon)} = \sum_{\{i,j,k\} \in T_\varepsilon} c_{ijk}\; b_i \wedge b_j \wedge b_k$$

so that

$$v = v^{(0)} + v^{(1)} \tag{2.27}$$

and

$$\Box'' = \Box''_0 + \Box''_1 \tag{2.28}$$

where $\Box''_\varepsilon = 1 \otimes v^{(\varepsilon)}$. Let $\rho_0 = \frac{1}{2}\sum_{\varphi \in \Delta^0_+} \varphi$.

Lemma 2.4. *For any $\lambda \in \mathfrak{h}^*$ and $a \in C(\mathfrak{h}_\mathfrak{p})$ one has*

$$\xi_\lambda(\Box''_0) v_{\lambda,a} = v_\lambda \otimes y_{\rho_0}\, a\, u \tag{2.29}$$

Proof.. Let $\{i,j,k\} \in T_0$. Now if $i \in I^0$ then $b_i = e_{\varphi_1}$ for $\varphi_1 \in \Delta^0$. But then by the order relation on I one has $b_j = e_{\varphi_2}$ and $b_k = e_{\varphi_3}$ where also $\{\varphi_2, \varphi_3\} \subset \Delta_0$. But then by (2.16) and (2.24) one has $c_{ijk} = 0$ if $\varphi_1 + \varphi_2 + \varphi_3 \neq 0$. However if $\varphi_1 + \varphi_2 + \varphi_3 = 0$ then by the ordering one has $\varphi_1 \in \Delta^0_+$. But then $b_i \wedge b_j \wedge b_k = b_j\, b_k\, b_i$ since the elements b_i, b_j and b_k are mutually orthogonal so that exterior and Clifford multiplication are the same. But then $\xi_\lambda(b_i \wedge b_j \wedge b_k) v_{\lambda,a} = 0$ by (2.5). Thus in computing the left side of (2.29) we can ignore all the terms in the definition of $v^{(0)}$ for which $i \in I^0$. Now assume that $\{i,j,k\} \in T_0$ and $i \in I^\mathfrak{h}$. But then if also $j \in I^\mathfrak{h}$ one has $c_{ijk} = 0$ by (2.17) and (2.24). If $j \in I^0$ then also $k \in I^0$ so that for some $\varphi, \varphi' \in \Delta^0$ one has $b_j = e_\varphi$ and $b_k = e_{\varphi'}$. But then $c_{ijk} \neq 0$ implies $\varphi + \varphi' = 0$ by (2.16) and (2.24). But then $\varphi \in \Delta^0_+$ and $\varphi' = -\varphi$. Moreover (2.16) and (2.17) imply

$$c_{ijk} = \frac{1}{2}\,\varphi(b_i) \tag{2.30}$$

But since b_i is orthogonal to e_β for any $\beta \in \Delta$ one has

$$v^{(0)}\, a\, u = \sum_{i \in I^{\mathfrak{h}},\, \varphi \in \Delta_+^0} \frac{1}{2}\, \varphi(b_i)\; b_i(e_\varphi \wedge e_{-\varphi})\, a\, u \tag{2.31}$$

But now if $z, w \in \mathfrak{p}$ and $(z, w) = 1$ then

$$z \wedge w = -w\, z + 1 \tag{2.32}$$

by e.g., (1.6) in [K1]. Thus $e_\varphi \wedge e_{-\varphi} = e_\varphi\, e_\varphi + 1$ in (2.31). However $e_\varphi\, a\, u = 0$ by (2.5). Consequently

$$\begin{aligned} v^{(0)}\, a\, u &= (\sum_{i \in I^{\mathfrak{h}},\, \varphi \in \Delta_+^0} \frac{1}{2}\, \varphi(b_i)\; b_i)\, a\, u \\ &= (\sum_{i \in I^{\mathfrak{h}}} \rho_0(b_i)\; b_i)\, a\, u \\ &= y_{\rho_0}\; a\, u \end{aligned} \tag{2.33}$$

But of course (2.33) implies (2.29). QED

2.4. We will now determine $v^{(1)}\, a\, u$. We have assumed that $a \in C(\mathfrak{h}_{\mathfrak{p}})$. However for later purposes we want a to be more general. Let $C(\mathfrak{p}^0) \subset C(\mathfrak{p})$ be the Clifford algebra over $\mathfrak{p}^0$ with respect to $B_{\mathfrak{p}}|\mathfrak{p}^0$. The argument establishing (2.5) also establishes

$$z\, a\, u = 0 \tag{2.34}$$

for $z \in \mathfrak{p}^+$ and $a \in C(\mathfrak{p}^0)$.

Lemma 2.5. *Assume $a \in C(\mathfrak{p}^0)$. Let $\{i, j, k\} \in T_1$. Then if $(b_i \wedge b_j \wedge b_k)\, a\, u \neq 0$ one has $b_i \in \mathfrak{p}^0$ (i.e., $i \in I^{\mathfrak{h}} \cup I^0$), $j \in I^\mu$, for some $\mu \in \Gamma_+$ and $b_k = d_j$. Furthermore in such a case*

$$(b_i \wedge b_j \wedge b_k)\, a\, u = b_i\, a\, u$$

Moreover in this case

$$c_{ijk} = \frac{1}{2}([d_i, b_j], d_j) \tag{2.35}$$

Proof. One must have $j \in I'$, since otherwise $b_j \in \mathfrak{p}^0$ in which case $b_i \in \mathfrak{p}^0$ (by the ordering). However this implies $b_k \in \mathfrak{p}^0$ since

$$\gamma(i) + \gamma(j) + \gamma(k) = 0 \tag{2.36}$$

But then $\{i, j, k\} \in T_0$ which is a contradiction. Thus $j \in I'$ and hence $k \in I'$. But then if $i \in I'$ the relation (2.35) and the ordering implies that $b_i \in \mathfrak{p}^+$ and b_i, b_j and b_k are mutually orthognal so that $b_i \wedge b_j \wedge b_k = b_j\, b_k\, b_i$. But $b_i\, a\, u = 0$ by (2.34). Hence our nonvanishing assumption implies that $b_i \in \mathfrak{p}^0$. But now $j \in I^\mu$, by (2.14), for some $\mu \in \Gamma$. But then $k \in I^{-\mu}$ by (2.36). Hence $\mu \in \Gamma_+$ by (1.14) and the ordering in I. In particular $b_j \in \mathfrak{p}^+$. Thus the nonvanishing assumption in the lemma implies

$$b_i \wedge b_j \wedge b_k = b_i(b_j \wedge b_k)$$

But if b_j is orthogonal to b_k then $b_j \wedge b_k = -b_k\, b_j$. But $b_j\, a\, u = 0$ by (2.34). In the remaining case where b_k is not orthogonal to b_j one has $b_k = d_j$ by (2.15). However $b_j \wedge d_j = -d_j b_j + 1$ by (2.32). But $b_j\, a\, u = 0$ by (2.34). With the exception of (2.35) this proves the lemma. But the equality $b_k = d_j$ implies $b_j = d_k$ again by (2.15). But then (2.24) implies that $c_{ijk} = -\frac{1}{2}([d_i, d_j], b_j)$. But, by the invariance of $B_{\mathfrak{g}}$, $([d_i, d_j], b_j) = -([d_i, b_j], d_j)$. This proves (2.35). QED

Now let $\Delta_+^1 = \{\varphi \in \Delta_+ \mid \varphi|\mathfrak{h}_{\mathfrak{r}} \in \Gamma_+\}$ and let $\Delta_+^2 = \{\varphi \in \Delta_+ \mid \varphi|\mathfrak{h}_{\mathfrak{r}} \notin \Gamma_+ \cup 0\}$. Then one has a partition

$$\Delta_+ = \Delta_+^0 \cup \Delta_+^1 \cup \Delta_0^2 \tag{2.37}$$

We have already defined ρ_0. Let ρ, ρ_1 and ρ_2 be defined similarly where Δ_+, Δ_+^1 and Δ_+^2 respectively replace Δ_+^0. Thus

$$\rho = \rho_0 + \rho_1 + \rho_2 \tag{2.38}$$

Hence

$$y_\rho = y_{\rho_0} + y_{\rho_1} + y_{\rho_2} \tag{2.39}$$

Lemma 2.6. *One has*

$$y_{\rho_2} = 0 \tag{2.40}$$

Proof. Let $\varphi \in \Delta_+^2$ and let $e_\varphi^{\mathfrak{r}} \in \mathfrak{r}$ and $e_\varphi^{\mathfrak{p}} \in \mathfrak{p}$ be such that $e_\varphi = e_\varphi^{\mathfrak{r}} + e_\varphi^{\mathfrak{p}}$. But since $ad\,\mathfrak{h}_{\mathfrak{r}}$ stabilizes both $\mathfrak{r}$ and $\mathfrak{p}$ it follows that $e_\varphi^{\mathfrak{r}}$ and $e_\varphi^{\mathfrak{p}}$ are $ad\,\mathfrak{h}_{\mathfrak{r}}$-weight vectors for the weight $\varphi|\mathfrak{h}_{\mathfrak{r}}$. But, by definition, $\varphi|\mathfrak{h}_{\mathfrak{r}} \notin \Gamma_+ \cup \{0\}$. But since, clearly, $\varphi(h_{\mathfrak{r}}) \geq 0$ one has $e_\varphi^{\mathfrak{p}} = 0$. Hence $e_\varphi \in \mathfrak{r}$. But $[y, e_\varphi] = \varphi(y)\, e_\varphi \in \mathfrak{r}$ for any $y \in \mathfrak{h}_{\mathfrak{p}}$. But $[\mathfrak{r}, \mathfrak{p}] \subset \mathfrak{p}$. Thus

$$\varphi|\mathfrak{h}_{\mathfrak{p}} = 0, \qquad \forall \varphi \in \Delta_+^2 \tag{2.41}$$

This proves (2.40). QED

Now for any $\mu \in \Gamma$ let $\Delta_\mu^1 = \{\varphi \in \Delta \mid \varphi|\mathfrak{h}_{\mathfrak{r}} = \mu\}$. Also for any $\mu \in \Gamma$ let $\mathfrak{g}^\mu$ be the weight space in $\mathfrak{g}$ corresponding to the weight μ with respect to the action of $ad\,\mathfrak{h}_r$ on $\mathfrak{g}$. Clearly $\mathfrak{g}^\mu$ and $\mathfrak{g}^{-\mu}$ are nonsingularly paired by $B_{\mathfrak{g}}$. In fact $\mathfrak{g}^\mu$ is clearly stable under the adjoint action of $\mathfrak{g}^0$ and, since $\mathfrak{h} \subset \mathfrak{g}^0$, it follows immediately that $\{e_\varphi\}$, $\varphi \in \Delta_\mu^1$, is a basis of $\mathfrak{g}^\mu$ and $\{e_{-\varphi}\}$, $\varphi \in \Delta_\mu^1$, is the $B_{\mathfrak{g}}$-dual basis in $\mathfrak{g}^{-\mu}$. It follows therefore that, for any $z \in \mathfrak{g}^0$,

$$\frac{1}{2} tr\, ad\, z|\mathfrak{g}^\mu = \frac{1}{2} \sum_{\varphi \in \Delta_\mu^1} ([z, e_\varphi], e_{-\varphi}) \tag{2.42}$$

In particular if $\rho_1^\mu = \frac{1}{2} \sum_{\varphi \in \Delta_\mu^1} \varphi$ and if $z \in \mathfrak{h}_{\mathfrak{p}}$, then

$$\frac{1}{2}\, tr\, ad\, z|\mathfrak{g}^\mu = \rho_1^\mu(z) \tag{2.43}$$

But clearly $\sum_{\mu \in \Gamma_+} \rho_1^\mu = \rho_1$ since one readily has the partition

$$\Delta_+^1 = \bigcup_{\mu \in \Gamma_+} \Delta_\mu^1$$

Thus for $z \in \mathfrak{h}_{\mathfrak{p}}$,

$$\frac{1}{2}\sum_{\mu\in\Gamma_+} tr\, ad\, z|\mathfrak{g}^{\mu} = \rho_1(z) \tag{2.44}$$

For any $\mu \in \Gamma$ let $\mathfrak{r}^{\mu} = \mathfrak{r} \cap \mathfrak{g}^{\mu}$.

Lemma 2.7. *For any $\mu \in \Gamma$ one has*

$$\mathfrak{g}^{\mu} = \mathfrak{r}^{\mu} + \mathfrak{p}^{\mu} \tag{2.45}$$

Proof. Obviously the right side of (2.45) is contained in the left side. Conversely let $w \in \mathfrak{g}^{\mu}$ and let $w_{\mathfrak{r}} \in \mathfrak{r}$ and $w_{\mathfrak{p}}$ be such that $w = w_{\mathfrak{r}} + w_{\mathfrak{p}}$. But clearly both components $w_{\mathfrak{r}}$ and $w_{\mathfrak{p}}$ are also weight vectors of $ad\, \mathfrak{h}_r$ with weight μ. Thus $w_{\mathfrak{p}} \in \mathfrak{p}^{\mu}$ and obviously $w_{\mathfrak{r}} \in \mathfrak{r}^{\mu}$. QED

Let $\mu \in \Gamma_+$. Obviously $\mathfrak{r}^{\mu}$ is $B_{\mathfrak{g}}$-nonsingularly paired to $r^{-\mu}$. We already know the same is true if $\mathfrak{p}^{\mu}$ and $\mathfrak{p}^{-\mu}$ replace $\mathfrak{r}^{\mu}$ and $\mathfrak{r}^{-\mu}$. If $z \in \mathfrak{p}^0 \subset \mathfrak{g}^0$, to compute $tr\, ad\, z|\mathfrak{g}^{\mu}$ instead of using a basis of root vectors as we did in (2.42) we can use the basis $\{b_j\}$, $j \in I^{\mu}$, of $\mathfrak{p}^{\mu}$ together with some basis of $\mathfrak{r}^{\mu}$. But since $[\mathfrak{p}, \mathfrak{r}] \subset \mathfrak{p}$ it follows that $[z, \mathfrak{r}^{\mu}] \subset \mathfrak{p}^{\mu}$. Thus we need consider only the basis $\{b_j\}$, $j \in I^{\mu}$, of $\mathfrak{p}^{\mu}$ to compute the trace. Thus for any $i \in I^{\mathfrak{h}} \cup I^0$ (so that $b_i, d_i \in \mathfrak{p}^0$) and $\mu \in \Gamma_+$ one has

$$tr\, ad\ d_i|\mathfrak{g}^{\mu} = \sum_{j\in I^{\mu}} ([d_i, b_j], d_j) \tag{2.46}$$

But then, by Lemma 2.5, for $a \in C(\mathfrak{p}^0)$ one has

$$\frac{1}{2}\sum_{\mu\in\Gamma_+} (\sum_{i\in I^{\mathfrak{h}}\cup I^0} (tr\, ad\ d_i|\mathfrak{g}^{\mu}) b_i)\, a\, u = v^{(1)}\, a\, u \tag{2.47}$$

But $d_i \in [\mathfrak{g}^0, \mathfrak{g}^0]$ for $i \in I^0$ by (1.11) and (2.16). Hence $tr\, ad\ d_i|\mathfrak{g}^{\mu} = 0$ for $i \in I^0$ and any $\mu \in \Gamma_+$. Thus, by (2.15), (2.47) simplifies to

$$\frac{1}{2}\sum_{\mu\in\Gamma_+} (\sum_{i\in I^{\mathfrak{h}}} (tr\, ad\ b_i|\mathfrak{g}^{\mu}) b_i)\, a\, u = v^{(1)}\, a\, u \tag{2.48}$$

But then by (2.44) one has

$$\begin{aligned}\frac{1}{2}\sum_{\mu\in\Gamma_+}(\sum_{i\in I^{\mathfrak{h}}}(tr\,ad\;b_i|\mathfrak{g}^{\mu})b_i) &= \sum_{i\in I^{\mathfrak{h}}}\rho_1(b_i)b_i\\ &= y_{\rho_1}\end{aligned}$$

Recalling (2.47) we have proved

Lemma 2.8. *Let $a\in C(\mathfrak{p}^0)$. Then*

$$v^{(1)}\,a\,u = y_{\rho_1}\,a\,u$$

Recall that $\Box = \Box' + \Box''$ (see (2.6) and (2.7)). We now find a condition on λ to insure that $v_{\lambda,a}$ is a Dirac cocyle (assuming that $a\in C(\mathfrak{h}_{\mathfrak{p}})$).

Theorem 2.9. *Let $\lambda\in\mathfrak{h}^*$ and let $a\in C(\mathfrak{h}_{\mathfrak{p}})$. Recall $v_{\lambda,a} = v_\lambda\otimes a\,u\in V_\lambda\otimes L$. Then*

$$\xi_\lambda(\Box'')v_{\lambda,a} = v_\lambda\otimes y_\rho\,a\,u \tag{2.49}$$

Furthermore

$$\xi_\lambda(\Box)v_{\lambda,a} = v_\lambda\otimes y_{\lambda+\rho}\,a\,u \tag{2.50}$$

In particular

$$\xi_\lambda(\Box)v_{\lambda,a} = 0 \tag{2.51}$$

in case $(\lambda+\rho)|\mathfrak{h}_{\mathfrak{p}} = 0$.

Proof. Equation (2.49) follows from (2.27), (2.33), Lemma 2.8, (2.39) and (2.40). But then (2.50) follows from Lemma 2.3 and (2.48). The equation (2.51) is immediate from the definition of $y_{\lambda+\rho}$ (see (2.11)). QED

3. Non-vanishing Dirac cohomology

3.1. Henceforth we assume that $\lambda \in \mathfrak{h}^*$ is an arbitrary element satisfying $(\lambda+\rho)|\mathfrak{h}_{\mathfrak{p}} = 0$. We will establish that $v_{\lambda,a}$, for any $0 \neq a \in C(\mathfrak{h}_{\mathfrak{p}})$ defines a nonzero Dirac cohomology class.

Let $\mathfrak{n}_-^0$ be the span of the root vectors $e_{-\varphi}$ for $\varphi \in \Delta_+^0$ so that one has the triangular decomposition

$$\mathfrak{p}^0 = \mathfrak{h}_{\mathfrak{p}} + \mathfrak{n}^0 + \mathfrak{n}_-^0 \tag{3.1}$$

Put $\mathfrak{m}_{\mathfrak{p}}^- = \mathfrak{n}_-^0 + \mathfrak{p}_-$ so that (see (2.2)) one also has the direct sum

$$\mathfrak{p} = \mathfrak{h}_{\mathfrak{p}} + \mathfrak{m}_{\mathfrak{p}} + \mathfrak{m}_{\mathfrak{p}}^- \tag{3.2}$$

If $\mathfrak{n}_-$ is the span of all root vectors $e_{-\varphi}$ for $\varphi \in \Delta_+$ note that

$$\mathfrak{m}_{\mathfrak{p}}^- \ \subset \ \mathfrak{n}_- \tag{3.3}$$

since $\mathfrak{p}_-$ is spanned by eigenvectors of $ad\, f_{\mathfrak{r}}$ for negative eigenvalues. For any subspace $\mathfrak{a} \ \subset \ \mathfrak{p}$ let $C(\mathfrak{a}) \ \subset \ C(\mathfrak{p})$ be the Clifford algebra (with respect to $B_{\mathfrak{p}}$) generated by $\mathfrak{a}$ (and of course 1). Clearly $C(\mathfrak{a}) = \wedge \mathfrak{a}$. Since $\mathfrak{m}_{\mathfrak{p}}^-$ is obviously isotropic it follows that exterior and Clifford multiplication are the same in $C(\mathfrak{m}_{\mathfrak{p}}^-)$. Now by (2.3), (2.4) and (3.2) note that the map

$$C(\mathfrak{h}_{\mathfrak{r}} + \mathfrak{m}_{\mathfrak{p}}^-) \to L, \qquad w \mapsto w\,u \tag{3.4}$$

is a linear isomorphism. Let $C_*(\mathfrak{m}_{\mathfrak{p}}^-)$ be the ideal in $C(\mathfrak{m}_{\mathfrak{p}}^-)$ generated by $\mathfrak{m}_{\mathfrak{p}}^-$. It then follows from (3.4) that one has a direct sum

$$L = C(\mathfrak{h}_{\mathfrak{p}})u \oplus C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u \tag{3.5}$$

Now if $\lambda', \lambda'' \in \mathfrak{h}^*$ we will say that λ'' is less than λ' (or λ' is greater than λ'') and write $\lambda' > \lambda''$ in case $\lambda' - \lambda''$ is a nontrivial sum of positive roots. Let $V_{\lambda,*}$ be the span of all weight vectors, of some weight λ' in V_λ, where $\lambda > \lambda'$. Then clearly one has

$$V_\lambda = \mathbb{C}\, v_\lambda \oplus V_{\lambda,*} \tag{3.6}$$

For notational convenience let $M \;\subset\; V_\lambda \otimes L$ be defined by putting

$$M = (V_\lambda \otimes C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u) \;+\; V_{\lambda,*} \otimes L \tag{3.7}$$

It then follows from (3.5) and (3.6) that

$$\begin{aligned} V_{\lambda,L} &= V_\lambda \otimes L \\ &= (\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}^{\mathfrak{p}})u) \oplus M \end{aligned} \tag{3.8}$$

Let $H_D(V_{\lambda,L})$ be the Dirac cohomology defined by $\xi_\lambda(\Box)$. See (0.3). By Theorem (2.9) the map from a cocyle to the corresponding cohomology class defines a linear map

$$\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}^{\mathfrak{p}})u \to H_D(V_{\lambda,L}) \tag{3.9}$$

We will show that (3.9) is injective.

Proposition 3.1. *To show that (3.9) is injective it suffices to prove that M is stable under the action of $\xi_\lambda(\Box)$.*

Proof. This is immediate from (3.8) since $\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}^{\mathfrak{p}})u \;\subset\; Ker\, \xi_\lambda(\Box)$ by Theorem 2.9. QED

3.2. To show that M is stable under $\xi_\lambda(\Box)$ we first establish

Lemma 3.2. *The space M is stable under $\xi_\lambda(\Box')$.*

Proof. We use the notation of §2.2 where $\Box'$ is given by (2.12) and the basis b_i, $i \in I$, is defined as in §2.2. To prove the lemma it suffices to show that M is stable under $\xi_\lambda(b_i \otimes d_i)$ for any $i \in I$. Assume first that $i \in I^0 \cup I'$. It is obvious from (2.18) that

$$\text{either } b_i \in \mathfrak{m}_{\mathfrak{p}}^- \text{ or } d_i \in \mathfrak{m}_{\mathfrak{p}}^- \tag{3.10}$$

If $b_i \in \mathfrak{m}_{\mathfrak{p}}^-$ then $b_i \in \mathfrak{n}_-$ by (3.3) so that clearly

$$\begin{aligned} \xi_\lambda(b_i \otimes d_i)(M) &\subset \xi_\lambda(b_i \otimes d_i)(V_\lambda \otimes L) \\ &\subset V_{\lambda,*} \otimes L \\ &\subset M \end{aligned}$$

If $d_i \in \mathfrak{m}_{\mathfrak{p}}^-$ then by (3.4)

$$\begin{aligned} \xi_\lambda(b_i \otimes d_i)(M) &\subset \xi_\lambda(b_i \otimes d_i)(V_\lambda \otimes L) \\ &\subset V_\lambda \otimes C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u \\ &\subset M \end{aligned}$$

Now assume $i \in I^{\mathfrak{h}}$ so that $b_i \otimes d_i = b_i \otimes b_i$ where $b_i \in \mathfrak{h}_{\mathfrak{p}}$ by (2.17). But obviously

$$\begin{aligned} b_i\, V_{\lambda,*} &\subset V_{\lambda,*} \\ b_i\; C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u &\subset C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u \end{aligned}$$

so that M is stable under $\xi_\lambda(b_i \otimes d_i)$ in this case as well. QED

3.3. It remains only to show that M is stable under $\xi_\lambda(\Box'')$. But now for $z \in V_\lambda$ it is obvious that $z \otimes L$ is stable under $\xi_\lambda(\Box'')$. The question is then reduced to considering only L. In fact one immediately has

Lemma 3.3. *If*

$$v\, C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u \subset C_*(\mathfrak{m}_{\mathfrak{p}}^-)C(\mathfrak{h}_{\mathfrak{p}})u \tag{3.11}$$

then M is stable under $\xi_\lambda(\Box'')$.

We now proceed to establish the inclusion (3.11).

Let $SO(\mathfrak{p})$ be the special orthogonal group with respect to $B_{\mathfrak{p}}$. One has a homomorphism

$$\nu : \mathfrak{r} \to Lie\, SO(\mathfrak{p}) \tag{3.12}$$

where if $y \in \mathfrak{p}$ and $x \in \mathfrak{r}$ then $\nu(x)y = [x, y]$. Now $\wedge^2\mathfrak{p} \subset C(p)$ has the structure of a Lie algebra under Clifford commutation and one has a Lie algebra isomorphism $\wedge^2\mathfrak{p} \cong Lie\, SO(\mathfrak{p})$. See (1.7) in [K1]. Furthermore there exists a Lie algebra homomorphism $\nu_* : \mathfrak{r} \to \wedge^2\mathfrak{p}$ so that for $x \in \mathfrak{r}$ and $y \in \mathfrak{p}$ one has, using commutation in $C(\mathfrak{p})$,

$$\begin{aligned} \nu(x)y &= [\nu_*(x), y] \\ &= -2\,\iota(y)\nu_*(x) \end{aligned} \tag{3.13}$$

where $\iota(y)$ is the operator of interior product of $\wedge\,\mathfrak{p}$ by y. See (1.8) and (1.11) in [K1]. Let $\{b_i\}$ and $\{d_i\}$, for $i \in I$, be the basis and dual basis of $\mathfrak{p}$ defined as in §2.1. Let $I^+ = \{i \in I \mid b_i \in \mathfrak{p}^+\}$. See (1.14). Note $\{b_i\}$, $i \in I^+$, is a basis of $\mathfrak{p}^+$ and $\{d_i\}$, $i \in I^+$, is a basis of $\mathfrak{p}^-$ (see (2.15)) so that taken together $\{b_i\} \cup \{d_i\}$, $i \in I+$, is a basis of $\mathfrak{p}'$.

Proposition 3.4. *For any $x \in \mathfrak{h}_{\mathfrak{r}}$ one has, using the notation of (2.26),*

$$\nu_*(x) = \frac{1}{2} \sum_{i \in I^+} \gamma_i(x)\, b_i \wedge d_i \tag{3.14}$$

Proof. Let $w \in \wedge^2\mathfrak{p}$ be given by the right side of (3.8). Then clearly, for $i \in I^+$, $-2\iota(d_i)w = -\gamma_i(x)d_i$ and $-2\iota(b_i)w = \gamma_i(x)b_i$. On the other hand $-2\iota(y)w = 0$ if $y \in \mathfrak{p}^0$. But, by (3.13), these same equations are satisfied if $\nu_*(x)$ replaces w. This proves $w = \nu_*(x)$. QED

Let $\Lambda \subset \mathfrak{h}_{\mathfrak{r}}^*$ be the real space of all (complex) linear functionals β on $\mathfrak{h}^{\mathfrak{r}}$ such that there exists $\gamma \in \mathfrak{h}^*$ with the property that (1) $\gamma|\mathfrak{h}_{\mathfrak{r}} = \beta$ and (2) $\gamma(\mathfrak{h}^{\#}) \subset \mathbb{R}$. It is immediate that any $\beta \in \Lambda$ extends uniquely, as a linear functional, on the complex subspace of $\mathfrak{h}$ spanned by $\kappa(\mathfrak{h}_{\mathfrak{r}})$. (See §1.1). In particular $\beta(f_{\mathfrak{r}})$ is well-defined for

$\beta \in \Lambda$. Obviously (2.45) implies that $\Gamma \subset \Lambda$. It follows therefore that $\rho_{\mathfrak{p}} \in \Lambda$ where for any $x \in \mathfrak{h}_{\mathfrak{r}}$

$$\begin{aligned}\rho_{\mathfrak{p}}(x) &= \frac{1}{2}\, tr\ \nu(x)|\mathfrak{p}^+ \\ &= \frac{1}{2}\sum_{i\in I^+}\gamma_i(x)\end{aligned} \tag{3.15}$$

Proposition 3.5. *Let $x \in \mathfrak{h}_{\mathfrak{r}}$. Then*

$$\nu_*(x)\, u = \rho_{\mathfrak{p}}(x)\, u \tag{3.16}$$

Proof. Let $i \in I^+$. Then $b_i \wedge d_i = -d_i\, b_i + 1$ by (2.32). But $b_i\, u = 0$ by (2.3). But then (3.16) follows from (3.14) and (3.15). QED

Now consider the action of $\mathfrak{h}_{\mathfrak{r}}$ on L defined, for $x \in \mathfrak{h}_{\mathfrak{r}}$, by left multiplication on L by $\nu_*(x)$.

Lemma 3.6. *If $s \in L$ is an $\mathfrak{h}_{\mathfrak{r}}$-weight vector with weight $\beta \in \mathfrak{h}_{\mathfrak{r}}^*$ and $i \in I$, then $b_i\, s_i$ is an $\mathfrak{h}_{\mathfrak{r}}$-weight vector with weight $\gamma_i + \beta$.*

Proof. Let $x \in \mathfrak{h}_{\mathfrak{r}}$. Then $[\nu_*(x), b_i] = \gamma_i(x)\, b_i$ by (3.13). But

$$\begin{aligned}\nu_*(x)\, b_i\, s &= [\nu_*(x), b_i]\, s + b_i\, \nu_*(x) s \\ &= (\gamma_i(x) + \beta(x)) b_i\, s\end{aligned}$$

QED

It is immediate from (3.4), Proposition 3.5 and Lemma 3.6 that if β is an $\mathfrak{h}_{\mathfrak{r}}$-weight in L then $\beta \in \Lambda$. Let $\Lambda_L \subset \Lambda$ be the set of all $\mathfrak{h}_{\mathfrak{r}}$-weights in L and for any $\beta \in \Lambda_L$ let L^β be the corresponding weight space so that one has the direct sum

$$L = \sum_{\beta\in\Lambda_L} L^\beta \tag{3.17}$$

Proposition 3.7. *For any $\beta \in \Lambda_L$ the weight space L^β is stable under left multiplication by the element $v \in \wedge^3\mathfrak{p}$.*

Proof. It is immediate from (2.8) that v is invariant under the representation θ_ν of $\mathfrak{h}_\mathfrak{r}$ on $\wedge\mathfrak{p}$ using the notation of (1.12) in [K1]. But by (1.12) in [K1] this implies that, with respect to Clifford multiplication, v commutes with $\nu_*(x)$ for all $x \in \mathfrak{h}_\mathfrak{r}$. The proposition then follows immediately. QED

For notational convenience let $L_- = C_*(\mathfrak{m}_\mathfrak{p}^-)C(\mathfrak{h}_\mathfrak{p})u$ so that by (3.4) one has the direct sum

$$L = C(\mathfrak{h}_\mathfrak{p})\,u \oplus L_- \tag{3.18}$$

Our problem is to show that L_- is stable under left multiplication by v. Let $C_*(\mathfrak{n}_-^0)$ be the ideal in $C(\mathfrak{n}_-^0)$ generated by $\mathfrak{n}_-^0$ and let $C_*(\mathfrak{p}_-)$ be the ideal in $C(\mathfrak{p}_-)$ generated by p_-. Note that L_- can be written

$$L_- = (C_*(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u) \oplus (C_*(\mathfrak{p}_-)C(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u) \tag{3.19}$$

Now for $\beta, \beta' \in \Lambda$ we will write $\beta > \beta'$ (and say that β is higher than β') in case $\beta(f_\mathfrak{r}) > \beta'(f_\mathfrak{r})$. It is then immediate from Proposition 3.5, Lemma 3.6, (3.18) and (3.19) that $\rho_\mathfrak{p}$ is the highest $\mathfrak{h}_\mathfrak{r}$ weight in L and

$$\begin{aligned} C(\mathfrak{n}_0)C(\mathfrak{h}_\mathfrak{r})u &= L^{\rho_\mathfrak{p}} \\ C_*(\mathfrak{p}_-)C(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u &= \sum_{\beta\in\Lambda_L,\ \rho_\mathfrak{p}>\beta} L^\beta \end{aligned} \tag{3.20}$$

We can now simplify our problem.

Proposition 3.8. *To prove that $v\,L_- \subset L_-$ it suffices only to show that the subspace $C_*(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by v.*

Proof. By (3.20) and Proposition 3.7 it follows that $C_*(\mathfrak{p}_-)C(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by v. But then Proposition 3.8 follows from (3.19). QED

Now recall (see (2.27)) we have written $v = v^{(0)} + v^{(1)}$.

Proposition 3.9. *The space $C_*(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by $v^{(1)}$.*

Proof. Since $y_{\rho_1} \in \mathfrak{h}_\mathfrak{p}$ the proof follows immediately from Lemma 2.8. QED

We are reduced finally to showing that $C_*(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by $v^{(0)}$. Let $L_0 = C(\mathfrak{p}^0)u$ so that L_0 is a cyclic $C(\mathfrak{p}^0)$-module under left multiplication. On the other hand recalling the triangular decomposition (3.1) and recalling the definition of $u = u_0 u_+$ in §2.1 one has

$$L_0 = C(\mathfrak{h}_\mathfrak{p})u \oplus C_*(\mathfrak{n}_-^0)C(\mathfrak{h}_\mathfrak{p})u \tag{3.21}$$

Remark 3.10. Note that L_0 is stable under left multiplication by $v^{(0)}$ since, by definition, $v^{(0)} \in C(\mathfrak{p}^0)$.

3.4. Recalling that $\mathfrak{p}^0$ is a reductive Lie algebra and $B_\mathfrak{p}|\mathfrak{p}^0$ is nonsingular let $\sigma : \mathfrak{h}_\mathfrak{p} \to Lie\, SO(\mathfrak{p}^0)$ be defined so that for $y \in \mathfrak{h}_\mathfrak{p}$ and $z \in \mathfrak{p}^0$ one has $\sigma(y)z = [y, z]$. Going back again to §1.5 in [K1] one has a Lie algebra homomorphism $\sigma_* : \mathfrak{h}_\mathfrak{p} \to \wedge^2\mathfrak{p}^0$ so that for $y \in \mathfrak{h}_\mathfrak{p}$ and $z \in \mathfrak{p}^0$ one has $\sigma(y)z = [\sigma_*(y), z]$. Noting that $\sigma(y)z = 0$ for $z \in \mathfrak{h}_\mathfrak{p}$ the argument establishing (3.14) readily establishes

Proposition 3.11. *For any $y \in \mathfrak{h}_\mathfrak{p}$ one has*

$$\sigma_*(y) = \frac{1}{2} \sum_{\varphi \in \Delta_+^0} \varphi(y)\, e_\varphi \wedge e_{-\varphi} \tag{3.22}$$

Now, recalling the definition of ρ_0 in §2.2 (on the line following (2.28)), one has $\rho_0 = \frac{1}{2}\sum_{\varphi \in \Delta_+^0} \varphi$. The argument establishing Proposition 3.5 yields

Proposition 3.12. *Let $y \in \mathfrak{h}_\mathfrak{p}$. Then*

$$\sigma_*(y)\, u = \rho_0(y)\, u \tag{3.23}$$

The nonsingularity of $B_{\mathfrak{p}}|\mathfrak{p}^0$ implies the nonsingularity of $B_{\mathfrak{p}}|[\mathfrak{p}^0,\mathfrak{p}^0]$. But then, recalling (1.11), one has the nonsingularity of $B_{\mathfrak{p}}|\mathfrak{d}_{\mathfrak{p}}$ since $\mathfrak{d}_{\mathfrak{p}}$ is a Cartan subalgebra of the semisimple Lie algebra $[\mathfrak{p}^0,\mathfrak{p}^0]$. Recalling (1.8) one has $\mathfrak{d}_{\mathfrak{p}} \subset \mathfrak{h}_{\mathfrak{p}}$. Let $\mathfrak{e}_{\mathfrak{p}}$ be the $B_{\mathfrak{g}}$-orthocomplement of $\mathfrak{d}_{\mathfrak{p}}$ in $\mathfrak{h}$ so that

$$\mathfrak{h} = \mathfrak{d}_{\mathfrak{p}} + \mathfrak{e}_{\mathfrak{p}} \tag{3.24}$$

is a $B_{\mathfrak{g}}$-orthogonal direct sum. Let $\varphi \in \Delta^+$. Then one must have $[e_\varphi, e_{-\varphi}] \in \mathfrak{d}_{\mathfrak{p}}$ so that

$$\varphi|\mathfrak{e}_{\mathfrak{p}} = 0 \tag{3.25}$$

With respect to the decomposition (3.19) let $f_{\mathfrak{p}} \in \mathfrak{d}_{\mathfrak{p}}$ be the component in $\mathfrak{d}_{\mathfrak{p}}$ of the regular hyperbolic element $f \in \mathfrak{h}^{\#}$. But then for any $\varphi \in \Delta^0$ one has $\varphi(f_{\mathfrak{p}}) > 0$ or $\varphi(f_{\mathfrak{p}}) < 0$ according as $\varphi \in \Delta^0_+$ or $\varphi \in \Delta^0_-$. If $\delta, \delta' \in \mathfrak{h}_{\mathfrak{p}}^*$ we will say that δ is higher than δ' and write $\delta > \delta'$ if $(\delta - \delta'')(f_{\mathfrak{p}})$ is a positive real number. Now let $D \subset \mathfrak{h}_{\mathfrak{p}}^*$ be the set of weights for the action of $\mathfrak{h}_{\mathfrak{p}}$ on L_0 where $y \in \mathfrak{h}_{\mathfrak{p}}$ operates as left multiplication by $\sigma_*(y)$. For any $\delta \in D$ let L_0^δ be the weight space for the weight δ. If $\varphi \in \Delta^0$ and $t \in L_0$ is a weight vector with weight δ, then the argument establishing Lemma 3.6 also establishes that $e_\varphi\, t$ is a weight vector with weight $\tilde{\varphi} + \delta$ where $\tilde{\varphi} = \varphi|\mathfrak{h}_{\mathfrak{p}}$. But then Proposition 3.12 and (3.21) imply

Proposition 3.13. *Let $\tilde{\rho_0} = \rho_0|\mathfrak{h}_{\mathfrak{p}}$. Then $\tilde{\rho_0} \in D$ and $\tilde{\rho_0}$ is the highest weight. Moreover*

$$\begin{aligned} C(\mathfrak{h}_{\mathfrak{p}})u &= L_0^{\tilde{\rho_0}} \\ C_*(n_-^0)C(\mathfrak{h}_{\mathfrak{p}})u &= \sum_{\delta \in D,\, \tilde{\rho_0} > \delta} L_0^\delta \end{aligned} \tag{3.26}$$

3.5. We can establish the final step.

Proposition 3.14. *The space $C_*(n_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by $v^{(0)}$.*

Proof. It is clear from the definition of $v^{(1)}$ (see (2.27) that

$$(v^{(1)}, y \wedge y' \wedge y'') = 0$$

for any $y, y', y \in \mathfrak{p}^0$. But $v^{(0)} \in \wedge^3\mathfrak{p}^0$ and hence

$$([y, y'], y'') = -2(v^{(0)}, y \wedge y' \wedge y'') \tag{3.27}$$

for any $y, y', y \in \mathfrak{p}^0$ by (2.8). But then it follows immediately from (3.27) that $v^{(0)}$ is invariant under $\theta_\sigma(z)$ for any $z \in \mathfrak{h}_\mathfrak{p}$ using the notation of (1.12) in [K1]. But then $v^{(0)}$ commutes with $\sigma_*(z)$ in $C(\mathfrak{p}^0)$ for any $z \in \mathfrak{h}_\mathfrak{p}$ by (1.12) in [K1]. It follows therefore that any weight space L_0^δ is stable under left multiplication by $v^{(0)}$. But by Proposition 3.13 this implies that $C_*(n_-^0)C(\mathfrak{h}_\mathfrak{p})u$ is stable under left multiplication by $v^{(0)}$. QED

We have proved

Theorem 3.15. *If $\lambda \in \mathfrak{h}^*$ is such that $\lambda + \rho$ vanishes on $\mathfrak{h}_\mathfrak{p}$ then $V_{\lambda,L} = V_\lambda \otimes L$ has nonvanishing Dirac cohomology. In fact the map (3.9)*

$$\mathbb{C}v_\lambda \otimes C(\mathfrak{h}_\mathfrak{p})u \to H_D(V_{\lambda,L})$$

is injective. One notes that $dim\, C(\mathfrak{h}_\mathfrak{p})u = 2^k$ where $k = dim\, \mathfrak{h}_\mathfrak{p}$.

4. Consequences of Theorem 3.15

4.1. The homomorphism $\nu_* : \mathfrak{r} \to \wedge^2\mathfrak{p} \subset C(\mathfrak{p})$ (see (3.12) and (3.13)) defines a homomorphism

$$\zeta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes C(\mathfrak{p}) \tag{4.1}$$

where if $x \in \mathfrak{r}$ then $\zeta(x) = x\otimes 1 + 1\otimes \nu_*(x)$. See §2.15 in [K1]. This defines the structure of an $\mathfrak{r}$-module on $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Let $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^{\mathfrak{r}}$ denote the algebra of $\mathfrak{r}$-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$. Let $Z(\mathfrak{g})$ and $Z(\mathfrak{r})$, respectively, be the centers of $U(\mathfrak{g})$ and $U(\mathfrak{r})$. One notes that $\Box \in (U(\mathfrak{g}) \otimes C(\mathfrak{p}))^{\mathfrak{r}}$ and also $Z(\mathfrak{g}) \otimes 1$ and $\zeta(Z(\mathfrak{r}))$ are subalgebras of $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^{\mathfrak{r}}$. In case $\mathfrak{r}$ is symmetric the cubic term in $\Box$ vanishes. The main result in [HP] (Theorem 3.4) is a statement for the case where $\mathfrak{r}$ is symmetric. However, as noted in the Appendix, the proof in [HP] is valid in the general case considered here (i.e., the case where $\mathfrak{r}$ is arbitrary, subject only to the condition that $\mathfrak{r}$ is reductive and $B_{\mathfrak{g}}|\mathfrak{r}$ is nonsingular and the cubic Dirac operator $\Box$ replaces the more familiar Dirac operator in [HP] and [P]). In addition Corollary 3.5 in [HP] is also valid in the general case considered here. That is, one has a unique map

$$\eta_{\mathfrak{r}} : Z(\mathfrak{g}) \to Z(\mathfrak{r}) \tag{4.2}$$

with the property that, for $p \in Z(\mathfrak{g})$, there exists $\omega \in (U(\mathfrak{g}) \otimes C^{odd}(\mathfrak{p}))^{\mathfrak{r}}$ such that

$$p \otimes 1 - \zeta(\eta_{\mathfrak{r}}(p)) = \Box\, \omega + \omega\, \Box \tag{4.3}$$

Furthermore $\eta_{\mathfrak{r}}$ is an algebra homomorphism. But $H_D(V_{\lambda,L})$ is a module for both $Z(\mathfrak{g}) \otimes 1$ and $\zeta(Z(\mathfrak{r}))$. But (4.3) implies that for any $p \in Z(\mathfrak{g})$ one has

$$p \otimes 1 = \zeta(\eta_{\mathfrak{r}}(p)) \text{ on } H_D(V_{\lambda,L}) \tag{4.4}$$

Let $W_{\mathfrak{g}}$ be the Weyl group of the pair $(\mathfrak{h}, \mathfrak{g})$ operating on the symmetric algebra $S(\mathfrak{h})$ over $\mathfrak{h}$ and let $W_{\mathfrak{r}}$ be the Weyl group of the pair $(\mathfrak{h}_{\mathfrak{r}}, \mathfrak{r})$ operating on the symmetric algebra $S(\mathfrak{h}_{\mathfrak{r}})$ over $\mathfrak{r}$. One has the Harish-Chandra algebra isomorphisms

$$\begin{aligned} A_{\mathfrak{g}} &: Z(\mathfrak{g}) \to S(\mathfrak{h})^{W_{\mathfrak{g}}} \\ A_{\mathfrak{r}} &: Z(\mathfrak{r}) \to S(\mathfrak{h}_{\mathfrak{r}})^{W_{\mathfrak{r}}} \end{aligned} \tag{4.5}$$

Let $\rho_{\mathfrak{r}} \in \mathfrak{h}_{\mathfrak{r}}^*$ be defined so that for $x \in \mathfrak{h}_{\mathfrak{r}}$ one has $\rho_{\mathfrak{r}}(x) = \frac{1}{2}\, tr\ ad\, x|\mathfrak{n}_{\mathfrak{r}}$ (see 1.18)). It follows immediately from (1.17) that on $\mathfrak{h}_{\mathfrak{r}}^*$,

$$\rho_{\mathfrak{p}} + \rho_{\mathfrak{r}} = \rho|\mathfrak{h}_{\mathfrak{r}} \tag{4.6}$$

Let $E_{\lambda,L} \subset H_D(V_{\lambda,L})$ be the image of (3.9). On the other hand the elements of $\mathbb{C}\, v_\lambda \otimes C(\mathfrak{h}_\mathfrak{r})u$ are $\mathfrak{g}\otimes 1$ highest weight vectors with highest weight λ and these elements are highest weight vectors for $\zeta(\mathfrak{r})$ with highest weight $\lambda|\mathfrak{h}_\mathfrak{r} + \rho_\mathfrak{p}$. But this establishes the following generalization of Theorem 5.5 in [HP] (case where $\mathfrak{r}$ is symmetric).

Theorem 4.1. *Let $p \in Z(\mathfrak{g})$ and $q \in Z(\mathfrak{r})$. Then $p \otimes 1$ reduces to the scalar $A_\mathfrak{g}(p)(\lambda + \rho)$ on $E_{\lambda,L}$ and (by 4.6) $\zeta(q)$ reduces to the scalar $A_\mathfrak{r}(q)((\lambda + \rho)|\mathfrak{h}_\mathfrak{r})$ on $E_{\lambda,L}$.*

But now since $(\lambda + \rho)|\mathfrak{h}_\mathfrak{p} = 0$ and there is no restriction on $\lambda|\mathfrak{h}_\mathfrak{r}$, this completely determines the map $\eta_\mathfrak{r}$. In fact let $\phi_o : \mathfrak{h} \to \mathfrak{h}_\mathfrak{r}$ be the projection relative to the decomposition $\mathfrak{h} = \mathfrak{h}_\mathfrak{r} + \mathfrak{h}_\mathfrak{p}$. Then ϕ_o extends to a homomorphism $S(\mathfrak{h}) \to S(\mathfrak{h}_\mathfrak{r})$ and clearly induces a homomorphism

$$\phi : S(\mathfrak{h})^{W_\mathfrak{g}} \to S(\mathfrak{h}_\mathfrak{r})^{W_\mathfrak{r}} \tag{4.7}$$

Given the Harish-Chandra isomorphisms $A_\mathfrak{g}$ and $A_\mathfrak{r}$ the map $\eta_\mathfrak{r}$ is given by completing a commutative diagram. Since one must have $A_\mathfrak{g}(p)(\lambda+\rho) = A_\mathfrak{r}(\eta_\mathfrak{r}(p))((\lambda+\rho)|\mathfrak{h}_\mathfrak{r})$ for all $p \in Z(\mathfrak{g})$ and all $\lambda \in \mathfrak{h}^*$ such that $(\lambda + \rho)|\mathfrak{h}_\mathfrak{p} = 0$ we have established the following generalization of Theorem 5.5 in [HP]. It follows from the observations in §5 below that it is also a generalization of Proposition 3.43, (5.18) and (5.19) in [K1].

Theorem 4.2. *The map $\eta_\mathfrak{r} : Z(\mathfrak{g}) \to Z(\mathfrak{r})$ is uniquely determined so that the following diagram is commutative*

$$\begin{array}{ccc}
Z(\mathfrak{g}) & \xrightarrow{\eta_\mathfrak{r}} & Z(\mathfrak{r}) \\
\big\downarrow{\scriptstyle A_\mathfrak{g}} & & \big\downarrow{\scriptstyle A_\mathfrak{r}} \\
S(\mathfrak{h})^{W_\mathfrak{g}} & \xrightarrow{\phi} & S(\mathfrak{h}_\mathfrak{r})^{W_\mathfrak{r}}
\end{array}$$

The map ϕ is well known in the theory of the cohomology of compact homogeneous spaces. Actually what is utilized in that theory is the map $S(\mathfrak{h}^*)^{W_\mathfrak{g}} \to S(\mathfrak{h}_\mathfrak{r}^*)^{W_\mathfrak{r}}$

induced by restriction of functions. However this is same as ϕ if $\mathfrak{h}$ and $\mathfrak{h}^*$ are identified and $\mathfrak{h}_{\mathfrak{r}}$ and $\mathfrak{h}_{\mathfrak{r}}^*$ are identified using $B_{\mathfrak{g}}$. Assume G is a compact connected semisimple Lie group and $\mathfrak{g}$ is the complexification of $Lie\, G$. Let $R \subset G$ be any connected compact subgroup and let $\mathfrak{r}$ be the complexification of $Lie\, R$. Obviously we can choose $B_{\mathfrak{g}}$ so that $B_{\mathfrak{g}}|\mathfrak{r}$ is nonsingular (e.g., let $B_{\mathfrak{g}}$ be the Killing form). The map $\eta_{\mathfrak{r}}$ induces the structure of a $Z(\mathfrak{g})$-module on $Z(\mathfrak{r})$. On the other hand the infinitesimal character for the module V_λ when $\lambda = -\rho$ defines the structure of a $Z(\mathfrak{g})$-module on $\mathbb{C}$. As a consequence of a well-known theorem of H. Cartan (see §9 in [C]) one has

Theorem 4.3. *There exists an isomorphism*

$$H^*(G/R, \mathbb{C}) \cong Tor_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})) \tag{4.8}$$

5. The case where $rank\, \mathfrak{r} = rank\, \mathfrak{g}$ and $dim\, V_\lambda < \infty$

5.1. Let the notation be as §0.1 so that V is an arbitrary $\mathfrak{g}$-module. It is clear that $\zeta(\mathfrak{r})$ commutes with $\Box$ so that the Dirac cohomology, $H_D(V \otimes L)$ has the structure of an $\mathfrak{r}$-module. Of course $Ker\, \Box_V \subset Ker\, \Box_V^2$. Note that the special case

$$Ker\, \Box_V = Ker\, \Box_V^2 \tag{5.1}$$

occurs if and only if

$$Ker\, \Box_V \cap Im\, \Box_V = 0 \tag{5.2}$$

If (5.1), or equivalently (5.2), occurs then we may regard $H_D(V \otimes L) \subset V \otimes L$ where in fact one has

$$\begin{aligned} H_D(V \otimes L) &= Ker\, \Box_V \\ &= Ker\, \Box_V^2 \end{aligned} \tag{5.3}$$

In this section we would like to formulate results in [K1] and [K2], especially results beginning with §3 in [K1], in terms of Dirac cohomology. Assume, as in §3 of [K1], that $rank\, r = rank\, \mathfrak{g}$ so that $\mathfrak{h} = \mathfrak{h}_{\mathfrak{r}}$ and $\mathfrak{h}_{\mathfrak{p}} = 0$. Note that, in this case, the restriction on λ in Theorem 3.15 disappears. Also in this case

$$S = L \tag{5.4}$$

where S is the $C(\mathfrak{p})$-spin module of §3.1 in [K1]. See (3.11) in [K1]. Next assume that λ is dominant with respect to $\mathfrak{b}$ and integral with respect to $\mathfrak{g}$. But then V_λ is finite dimensional and $\mathfrak{g}$-irreducible. Consider $H_D(V_\lambda \otimes S)$. Using the notation of (4.8) let $R \subset G$ be any connected compact subgroup having the same rank as G. Up to conjugacy we can take R to be defined as in §5.21 in [K1] so that $\mathfrak{r}$ is the complexification of $Lie\, R$. In this section, as in [K1], let d be the Euler characteristic of G/R. We have written W for the Weyl group $W_{\mathfrak{g}}$ in [K1]. One has $W_{\mathfrak{r}} \subset W$ and one knows that the index of $W_{\mathfrak{r}}$ in W is d. See e.g., (5.32) in [K1]. Let $W^1 \subset W$ be the set of representatives of the right cosets of $W_{\mathfrak{r}}$ in W defined as in (3.24) in [K1] so that $d = card\, W^1$. For any $\tau \in W^1$ let $\tau \bullet \lambda = \tau(\lambda + \rho) - \rho_{\mathfrak{r}}$. Then $\tau \bullet \lambda$ is dominant with respect to $\mathfrak{b}_{\mathfrak{r}}$ and integral for the simply-connected covering group of R. In particular if $Z_{\tau\bullet\lambda}$ is an irreducible $\mathfrak{r}$-module with highest weight $\tau \bullet \lambda$ then $Z_{\tau\bullet\lambda}$ is finite dimensional. Also $Z_{\tau_i\bullet\lambda}$, $i = 1, 2$, are inequivalent for $\tau_i \in W^1$ where $\tau_1 \neq \tau_2$. See §3.22 in [K1]. But now Theorems 4.17 and 4.24 in [K1] imply

Theorem 5.1. *Assume $rank\, \mathfrak{r} = rank\, \mathfrak{g}$ and $\lambda \in \mathfrak{h}^*$ is dominant and integral with respect to G. Then $Z_{\tau\bullet\lambda}$ occurs with multiplicity one in $V_\lambda \otimes S$, for any $\tau \in W^1$, so that we can unambiguously regard $Z_{\tau\bullet\lambda} \subset V_\lambda \otimes S$. Furthermore the condition (5.1) is satisfied and (recalling (5.3))*

$$H_D(V_\lambda \otimes S) = \sum_{\tau \in W^1} Z_{\tau\bullet\lambda} \tag{5.5}$$

In particular $H_D(V_\lambda \otimes S)$, as an $\mathfrak{r}$-module, is multiplicity-free and decomposes into a sum of d irreducible components, where d is the Euler number of G/R.

Remark 5.2. In the case where $\mathfrak{r}$ is the Levi factor of a parabolic subalgebra of $\mathfrak{g}$ we have shown in [K2] that Theorems 4.17 and 4.24 in [K1] imply the Bott-Borel-Weil theorem (BBW). This may formulated in terms of Dirac cohomology. In case $\mathfrak{r}$ is the Levi factor of a parabolic subalgebra of $\mathfrak{g}$ the argument in [K2] shows that BBW is a consequence of Theorem 5.1 together with the construction of $Z_{\tau\bullet\lambda}$ given in Theorem 4.17 of [K1].

5.2. As mentioned above, Theorem 4.2, for the case where $rank\,\mathfrak{r} = rank\,\mathfrak{g}$ appears in [K1]. See §5, especially equations (5.18) and (5.19), in [K1]. In more detail, the map ϕ in the present case, is injection so that the map $\eta_{\mathfrak{r}}$ is injective. The image of $Z(\mathfrak{g})$ in $Z(\mathfrak{r})$ has been denoted by $Z_{\mathfrak{g}}(\mathfrak{r})$ in [K1]. Let the notation be as in Theorem 5.1. The set $\{Z_{\tau\bullet\lambda}\}$, $\tau \in W^1$, of representations of $\mathfrak{r}$ is referred to in [K1] as a multiplet. Recalling (5.5), a verification of equation (4.4) is the statement that the infinitesmal character of $Z(\mathfrak{r})$ for all the members of a multiplet remains the same when restricted to $Z_{\mathfrak{g}}(\mathfrak{r})$ and that, furthermore, the restriction is given by the infinitesmal character of $Z(\mathfrak{g})$ for the $\mathfrak{g}$-representation V_λ. But this and more is stated in Proposition 3.43 of [K1] together with (5.18) and (5.19).

Remark 5.3. In a certain sense matters have come full circle. Consider the case where $\mathfrak{g}$ is of type F_4 and $\mathfrak{r}$ is of type B_4 (i.e., $R \cong Spin\,9$). In that case $d = 3$ so the multiplets are triplets. That the members of each triplet had remarkable properties in common was the empirical discovery of the physicists Ramond and Pengpan. This discovery inspired the paper [GKRS] which in turn led to [K1]. One of the properties discovered by Ramond and Pengpan, in the terminology above, is, in retrospect, the statement that $Z_{\mathfrak{g}}(\mathfrak{r})$ operates the same way on each member of any triplet. (We use the term "in retrospect" since Ramond and Pengpan were dealing only with $\mathfrak{r}$ and were unaware of the role of $\mathfrak{g}$.) But, with the notion of Dirac cohomology, this behavior of $Z_{\mathfrak{g}}(\mathfrak{r})$ is necessarily the case since (see (5.5)) $H_D(V_\lambda \otimes S)$ is just the sum of the members of that triplet which corresponds to λ.

Appendix

A.1. One of the properties of $\square$ used in [HP] to establish the main theorem, Theorem 3.4 in [HP], was Lemma 3.1 in [HP]. Let $Cas_{\mathfrak{g}} \in Z(\mathfrak{g})$ be the $\mathfrak{g}$-Casimir element with respect to $B_{\mathfrak{g}}$ and let $Cas_{\mathfrak{r}} \in Z(\mathfrak{r})$ be the $\mathfrak{r}$-Casimir element with respect to $B_{\mathfrak{r}}$. Recall that $\mathfrak{r}$ is assumed to be symmetric in [HP]. Lemma 3.1 in [HP] asserts

$$\square^2 = Cas_{\mathfrak{g}} \otimes 1 - \zeta(Cas_{\mathfrak{r}}) + constant \tag{A.1}$$

It should be noted that the definition of Dirac operator in [HP] differs from its definition here and in [K1], in the symmetric case, by a factor of i. This factor clearly plays no significant role in our concerns here.

The equation (A.1) is used in [HP] to define a $\mathbb{Z}_2$-graded differential complex in $(U(\mathfrak{g}) \otimes C(\mathfrak{p}))^{\mathfrak{r}}$ with $ad\,\square$ as the coboundary operator. Here commutation with $\square$ is taken in the $\mathbb{Z}_2$-graded sense. However in the general case we are considering, where $\square$ is the cubic Dirac operator, we have established (A.1). See Theorem 2.16 in [K1]. The validity of (A.1) enables one to define this complex in the general case using the cubic Dirac operator. Also, as in [HP], $Z(\mathfrak{g}) \otimes 1$ and $\zeta(Z(\mathfrak{r}))$ are, in the general case, spaces of cocyles. Theorem 3.4 in [HP] asserts $\zeta(Z(\mathfrak{r}))$ is isomorphic to the cohomology of this complex. But again the same argument yields the same result in the general case. The idea in [HP] is to replace $U(\mathfrak{g})$ by the symmetric algebra $S(\mathfrak{g})$ and to replace $ad\,\square$ by its symbol. A computation of the symbol leads to the Koszul complex. The proof then follows from the acyclicity of the Koszul complex. The reason why this argument works in the general case is that one obtains the same symbol. This is because the cubic term has no affect on the symbol. It should be noted that our result, for the case where $\mathfrak{r} = 0$, appears in [AM].

The validity of Theorem 3.4 in [HP], for the general case, leads to the map (4.2), which one easily shows, is a homomorphism of algebras. Theorem 4.1 here determines the map (4.2) in the general case.

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