7

Non-galoisian Galois theory

This chapter presents a further generalization of the Galois theorem. Our new theorem holds for every effective descent morphism, not necessarily of Galois descent.

This generalized Galois theorem applies in particular to the case of an arbitrary field extension, not necessarily galoisian. But as the reader can guess, there is a price to pay for this generalization: the classical Galois group is now replaced by a weaker structure, namely, by what we call a "Galois pregroup".

The famous Joyal–Tierney theorem extending the Galois theory of Grothendieck to the context of toposes enters the scope of the present chapter.

7.1 Internal presheaves on an internal groupoid

In view of further generalizations, it is now useful to make more precise the notions of internal categories, groupoids and presheaves already introduced in section 4.6.

An internal category $\mathbb{C}$ in a category $\mathcal{C}$ with pullbacks consists in giving

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd}$$

where

(G1) the square

$$\begin{tikzcd} C_2 \arrow[r, "f_0"] \arrow[d, "f_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd}$$

is a pullback,

(G2) the triangles

$$\begin{tikzcd} C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd}$$

are commutative,

(G3) the squares

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "f_0"'] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "f_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

are commutative,

(G4) the triangles

$$\begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} n \circ d_0 \\ \mathsf{id}_{C_1} \end{pmatrix}"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} \mathsf{id}_{C_1} \\ n \circ d_1 \end{pmatrix}"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd}$$

are commutative,

(G5) considering further the pullback

$$\begin{tikzcd} C_3 \arrow[r, "g_0"] \arrow[d, "g_1"'] & C_2 \arrow[d, "f_1"] \\ C_2 \arrow[r, "f_0"'] & C_1 \end{tikzcd}$$

the diagram

$$\begin{tikzcd}[column sep=large, row sep=large] C_3 \arrow[r, "\begin{pmatrix} f_0 \circ g_0 \\ m \circ g_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} m \circ g_0 \\ f_1 \circ g_1 \end{pmatrix}"'] & C_2 \arrow[d, "m"] \\ C_2 \arrow[r, "m"'] & C_1 \end{tikzcd}$$

is commutative.

In the case $\mathcal{C} = \mathsf{Set}$, this reduces to the usual definition of a small category $\mathbb{C}$, with

- C_0 the set of objects,
- C_1 the set of arrows,
- C_2 the set of composable pairs

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C,$$

- C_3 the set of composable triples

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D,$$

Using the notation

- $d_0(\alpha) = A$, $d_1(\alpha) = B$,
- $n(A) = \mathsf{id}_A$,
- $f_0(\alpha, \beta) = \alpha$, $f_1(\alpha, \beta) = \beta$,
- $m(\alpha, \beta) = \beta \circ \alpha$.

the axioms reduce then to the following assertions:

(G1) is just the definition of C_2;
(G2) id_A is an arrow from A to A;
(G3) $\beta \circ \alpha$ is an arrow from A to C;

(G4) $\mathsf{id}_B \circ \alpha = \alpha$, $\alpha \circ \mathsf{id}_A = \alpha$;

(G5) $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$.

An internal groupoid $\mathbb{G}$ is an internal category $\mathbb{C}$ as above, together with an additional datum, namely, a morphism

$$\tau \colon C_1 \longrightarrow C_1$$

such that

(G6) the triangles

$$\begin{tikzcd} C_1 \arrow[r, "\tau"] \arrow[dr, "d_1"'] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\tau"] \arrow[dr, "d_0"'] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd}$$

are commutative,

(G7) the squares

$$\begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} \mathsf{id}_{C_1} \\ \tau \end{pmatrix}"] \arrow[d, "d_0"'] & C_2 \arrow[d, "m"] \\ C_0 \arrow[r, "n"'] & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} \tau \\ \mathsf{id}_{C_1} \end{pmatrix}"] \arrow[d, "d_1"'] & C_2 \arrow[d, "m"] \\ C_0 \arrow[r, "n"'] & C_1 \end{tikzcd}$$

are commutative.

These conditions imply at once $\tau \circ \tau = \mathsf{id}_{C_1}$.

In the case $\mathcal{C} = \mathsf{Set}$ and using again, without further notice, the previous notation, the additional requirements for a groupoid become

- $\tau(\alpha) = \alpha^{-1}$,

with the axioms

(G6) α^{-1} is an arrow from B to A,

(G7) $\alpha^{-1} \circ \alpha = \mathsf{id}_A$, $\alpha \circ \alpha^{-1} = \mathsf{id}_B$.

It is also well known, and left to the reader as an exercise, that an internal category is an internal groupoid precisely when the two commutative squares in (G3) turn out to be pullbacks. Thus for an internal category, being an internal groupoid is a property, not an additional structure.

A covariant internal presheaf on the internal category $\mathbb{C}$ consists in giving the data

$$P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{p_0} C_0$$

where

(P1) the object P_1 is defined by the pullback

$$\begin{tikzcd} P_1 \arrow[r, "\delta_0"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd}$$

(P2) the square

$$\begin{tikzcd} P_1 \arrow[r, "\delta_1"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

is commutative,

(P3) the triangle

$$\begin{tikzcd} P_0 \arrow[rr, "\eta = \begin{pmatrix} \mathrm{id}_{P_0} \\ n \circ p_0 \end{pmatrix}"] \arrow[drr, equal] & & P_1 \arrow[d, "\delta_1"] \\ & & P_0 \end{tikzcd}$$

is commutative,

$$\begin{tikzcd}
P_0 \arrow[d, "p_0"'] \arrow[dr, dotted, "\eta"] \arrow[drr, equal] & & \\
C_0 \arrow[dr, "n"'] & P_1 \arrow[r, "\delta_0"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\
& C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

Diagram 7.1

(P4) given the pullback

$$\begin{tikzcd}
P_2 \arrow[r, "\varphi_0"] \arrow[d, "p_2"'] & P_1 \arrow[d, "p_1"] \\
C_2 \arrow[r, "f_0"'] & C_1
\end{tikzcd}$$

the diagram

$$\begin{tikzcd}
P_2 \arrow[r, "\mu = \begin{pmatrix} \delta_0 \circ \varphi_0 \\ m \circ p_2 \end{pmatrix}"] \arrow[d, "\varphi_1 = \begin{pmatrix} \delta_1 \circ \varphi_0 \\ f_1 \circ p_2 \end{pmatrix}"'] & P_1 \arrow[d, "\delta_1"] \\
P_1 \arrow[r, "\delta_1"'] & P_0
\end{tikzcd}$$

is commutative.

Diagrams 7.1, 7.2 and 7.3 make explicit the way η, μ and φ_1 are defined, as unique factorizations making those diagrams commutative.

In the case $\mathcal{C} = \mathsf{Set}$, again using the previous notation, this reduces to the usual notion of a functor $P\colon \mathbb{C} \longrightarrow \mathsf{Set}$:

- $P_0 = \{(a, A) | a \in P(A)\}$;

$$
\begin{tikzcd}
P_2 \arrow[r, "\varphi_0"] \arrow[d, "p_2"'] \arrow[dr, dotted, "\mu"] & P_1 \arrow[dr, "\delta_0"] & \\
C_2 \arrow[dr, "m"'] & P_1 \arrow[r, "\delta_0"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\
 & C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
$$

Diagram 7.2

$$
\begin{tikzcd}
P_2 \arrow[r, "\varphi_0"] \arrow[d, "p_2"'] \arrow[dr, dotted, "\varphi_1"] & P_1 \arrow[dr, "\delta_1"'] & \\
C_2 \arrow[dr, "f_1"'] & P_1 \arrow[r, "\delta_0"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\
 & C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
$$

Diagram 7.3

- $p_0(a, A) = A$;
- $P_1 = \{(a, \alpha) \,|\, \alpha\colon A \longrightarrow B,\ \ a \in P(A)\}$;
- $p_1(a, \alpha) = \alpha$; $\delta_0(a, \alpha) = (a, A)$;
- $\delta_1(a, \alpha) = P(\alpha)(a)$;
- $P_2 = \{(a, \alpha, \beta) \,|\, A \xrightarrow{\alpha} B \xrightarrow{\beta} C,\ \ a \in P(A)\}$;
- $p_2(a, \alpha, \beta) = (\alpha, \beta)$, $\varphi_0(a, \alpha, \beta) = (a, \alpha)$;
- $\eta(a, A) = (a, \mathsf{id}_A)$, $\mu(a, \alpha, \beta) = (a, \beta \circ \alpha)$;
- $\varphi_1(a, \alpha, \beta) = \big(P(\alpha)(a), \beta\big)$.

The axioms reduce to the following assertions:

(P1) this is just the definition of P_1;
(P2) $P(\alpha)(a)$ is an element of $P(B)$;

$$\begin{tikzcd}
P_1 \arrow[r, "\delta_0"] \arrow[dr, dotted, "\theta_1"] \arrow[ddr, "p_1"'] & P_0 \arrow[dr, "\theta_0"] \arrow[ddr, "p_0"] & \\
& P_1' \arrow[r, "\delta_0'"] \arrow[d, "p_1'"] & P_0' \arrow[d, "p_0'"] \\
& C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

Diagram 7.4

(P3) $P(\mathsf{id}_A)(a) = a$;
(P4) $P(\beta \circ \alpha)(a) = P(\beta)P(\alpha)(a)$.

It remains to write down the definition of an internal natural transformation $\theta\colon P \Rightarrow P'$ between two internal presheaves on the internal category $\mathbb{C}$. This consists in giving a morphism $\theta_0\colon P_0 \longrightarrow P_0'$ such that the following conditions hold:

(N1) the triangle

$$\begin{tikzcd}
P_0 \arrow[rr, "\theta_0"] \arrow[dr, "p_0"'] & & P_0' \arrow[dl, "p_0'"] \\
& C_0 &
\end{tikzcd}$$

is commutative;

(N2) considering the factorization θ_1 in the commutative diagram 7.4, the following diagram is commutative:

$$\begin{tikzcd}
P_1 \arrow[r, "\theta_1"] \arrow[d, "\delta_1"'] & P_1' \arrow[d, "\delta_1'"] \\
P_0 \arrow[r, "\theta_0"'] & P_0'
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=large]
P_2 \arrow[r, shift left=2, "\varphi_0"] \arrow[r, "\mu"] \arrow[r, shift right=2, "\varphi_1"'] \arrow[d, "p_2"'] & P_1 \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, "\delta_1"'] \arrow[d, "p_1"'] & P_0 \arrow[l, "\eta"'] \arrow[d, "p_0"] \\
C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[loop below, "\tau"] & C_0 \arrow[l, "n"']
\end{tikzcd}
$$

Diagram 7.5

Again in the case $\mathcal{C} = \mathsf{Set}$ and two presheaves $P, P' \colon \mathbb{C} \rightrightarrows \mathsf{Set}$, this reduces to the definition of an ordinary natural transformation $\theta \colon P \Rightarrow P'$:

- $\theta_0(a, A) = \big(\theta_A(a), A\big)$,
- $\theta_1(a, \alpha) = \big(\theta_A(a), \alpha\big)$,

with axioms becoming

(N1) $\theta_A(a)$ is an element of $P'(A)$,
(N2) $P'(\alpha)\big(\theta_A(a)\big) = \theta_B\big(P(\alpha)(a)\big)$.

The notion of internal presheaf on an internal groupoid $\mathbb{G}$ is simply the notion of internal presheaf on the underlying internal category $\mathbb{C}$. But in the case of an internal groupoid, internal presheaves admit a much more elegant characterization, given by the following lemma.

Lemma 7.1.1 *With the previous notation, a covariant presheaf on the internal groupoid $\mathbb{G}$ given by the bottom line of diagram 7.5 consists in the other data constituting the diagram, where all the squares of arrows "corresponding to each other by the notation" are pullbacks. More precisely, the squares*

$$p_1 \circ \varphi_i = f_i \circ p_2, \quad p_0 \circ \delta_i = d_i \circ p_1, \quad p_1 \circ \mu = m \circ p_2, \quad p_1 \circ \eta = n \circ p_0$$

are pullbacks.

Proof Since the problem is entirely expressed in terms of commutative diagrams and pullbacks, it suffices to give a proof of the result in the category of sets. Indeed, write $\mathcal{D} \subseteq \mathcal{C}$ for the full subcategory of $\mathcal{C}$ obtained as closure under pullbacks of the finitely many objects which enter the problem. We can equivalently develop the proof in this category $\mathcal{D}$, which is now small and still has pullbacks. But the Yoneda embedding

$$\mathcal{D} \xrightarrow{Y} [\mathcal{D}^*, \mathsf{Set}], \quad D \mapsto \mathcal{D}(-, D)$$

is full and faithful and preserves and reflects pullbacks. It thus suffices to prove the result in the category $[\mathcal{D}^*, \mathsf{Set}]$. But since pullbacks and commutativities in $[\mathcal{D}^*, \mathsf{Set}]$ are pointwise, it suffices to prove the result pointwise in the category of sets.

Thus we use the previous notation, but assuming $\mathcal{C} = \mathsf{Set}$. For simplicity, we use the standard notation $m(\alpha, \beta) = \beta \circ \alpha$, $n(A) = \mathsf{id}_A$ and $\tau(\alpha) = \alpha^{-1}$.

Let us start with an internal presheaf on the groupoid $\mathbb{G}$ and let us prove that the squares of the statement are pullbacks. Two of them are pullbacks by definition of P_1 and P_2. We have already observed that when $\mathcal{C} = \mathsf{Set}$, the internal presheaf reduces to an ordinary covariant functor $P\colon \mathbb{G} \longrightarrow \mathsf{Set}$ and we use the corresponding set theoretical notation, made explicit above in this section.

Let us verify first that the square

$$\begin{tikzcd}
P_1 \arrow[r, "\delta_1"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\
C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}$$

is a pullback. If

$$(\alpha\colon A \longrightarrow B) \in C_1, \quad b \in P(B), \quad d_1(\alpha) = p_0(b),$$

then $P(\alpha^{-1})(b) \in A$ and thus

$$\Big(P(\alpha^{-1})(b), \alpha\Big) \in P_1,$$

with the expected properties

$$p_1\Big(P(\alpha^{-1})(b), \alpha\Big) = \alpha, \quad \delta_1\Big(P(\alpha^{-1})(b), \alpha\Big) = P(\alpha)P(\alpha^{-1})(b) = b.$$

The pair $\big(P(\alpha^{-1})(b), \alpha\big)$ is unique with these properties. Indeed given

$$(a, \beta) \in P_1, \quad p_1(a, \beta) = \alpha, \quad \delta_1(a, \beta) = b,$$

we get at once $\alpha = \beta$ and $b = P(\beta)(a)$, that is, $a = P(\beta^{-1})(a) = P(\alpha^{-1})(a)$.

To verify that the square

$$\begin{tikzcd}
P_1 \arrow[d, "p_1"'] & P_0 \arrow[l, "\eta"'] \arrow[d, "p_0"] \\
C_1 & C_0 \arrow[l, "n"]
\end{tikzcd}$$

is a pullback, choose

$$(a, \alpha) \in P_1, \quad A \in C_0, \quad p_1(a, \alpha) = n(A).$$

This implies $\alpha = \mathsf{id}_A$ and one has indeed

$$(a, A) \in P_0, \quad \eta(a, A) = (a, \mathsf{id}_A) = (a, \alpha), \quad p_0(a, A) = A.$$

Such a pair (a, A) is unique, since given

$$(b, B) \in P_0, \quad \eta(b, B) = (a, \alpha), \quad p_0(b, B) = A,$$

we get at once $A = B$ and $(b, \mathsf{id}_A) = \eta(b) = (a, \alpha)$ and thus $(b, B) = (a, A)$.

We prove now that the square

$$\begin{tikzcd}
P_2 \arrow[r, "\mu"] \arrow[d, "p_2"'] & P_1 \arrow[d, "p_1"] \\
C_2 \arrow[r, "m"'] & C_1
\end{tikzcd}$$

is a pullback. Given

$$(\alpha, \beta) \in C_2, \quad (a, \gamma) \in P_1, \quad m(\alpha, \beta) = p_1(a, \gamma),$$

one has $\gamma = \beta \circ \alpha$ and thus

$$(a, \alpha, \beta) \in P_2, \quad \mu(a, \alpha, \beta) = (a, \beta \circ \alpha) = (a, \gamma), \quad p_2(a, \alpha, \beta) = (\alpha, \beta).$$

The uniqueness of a triple with these properties holds, since given

$$(b, \delta, \varepsilon) \in P_2, \quad \mu(b, \delta, \varepsilon) = (a, \gamma), \quad p_2(b, \delta, \varepsilon) = (\alpha, \beta),$$

we deduce

$$(a, \gamma) = \mu(b, \delta, \varepsilon) = (b, \varepsilon \circ \delta), \quad (\alpha, \beta) = p_2(b, \delta, \varepsilon) = (\delta, \varepsilon),$$

from which $a = b$, $\alpha = \delta$ and $\beta = \epsilon$.

There remains the case of the square

$$\begin{tikzcd}
P_2 \arrow[r, "\varphi_1"] \arrow[d, "p_2"'] & P_1 \arrow[d, "p_1"] \\
C_2 \arrow[r, "f_1"'] & C_1
\end{tikzcd}$$

Consider

$$(\alpha, \beta) \in C_2, \quad (b, \gamma) \in P_1, \quad f_1(\alpha, \beta) = p_1(b, \gamma).$$

This forces $\beta = \gamma$ and thus

$$\Big(P(\alpha^{-1})(b), \alpha, \beta\Big) \in P_2$$

with the expected properties

$$\varphi_1\Big(P(\alpha^{-1})(b), \alpha, \beta\Big) = \Big(P(\alpha)P(\alpha^{-1})(b), \beta\Big) = (b, \beta) = (b, \gamma),$$
$$p_2\Big(P(\alpha^{-1})(b), \alpha, \beta\Big) = (\alpha, \beta).$$

To prove the uniqueness of such a triple, consider

$$(a, \delta, \varepsilon) \in P_2, \quad \varphi_1(a, \delta, \varepsilon) = (b, \gamma), \quad p_2(a, \delta, \varepsilon) = (\alpha, \beta).$$

This yields

$$(b, \gamma) = \varphi_1(a, \delta, \varepsilon) = \big(P(\delta)(a), \varepsilon\big), \quad (\alpha, \beta) = p_2(a, \delta, \varepsilon) = (\delta, \varepsilon)$$

and therefrom the equalities $\alpha = \delta$, $\beta = \varepsilon$, $b = P(\delta)(a)$. This also implies $a = P(\delta^{-1})(b) = P(\alpha^{-1})(b)$.

We must now prove the converse implication. Thus we start from the situation of the statement and prove that (P_0, p_0, δ_1) is an internal presheaf. By assumption, we have at once conditions (P1) and (P2). Considering the pullbacks

$$\begin{tikzcd}
P_0 \arrow[r, "\eta"] \arrow[d, "p_0"'] & P_1 \arrow[r, "\delta_1"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\
C_0 \arrow[r, "n"'] & C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "\mu"] \arrow[d, "p_2"'] & P_1 \arrow[r, "\delta_1"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\ C_2 \arrow[r, "m"'] & C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

$$\begin{tikzcd} P_2 \arrow[r, "\varphi_1"] \arrow[d, "p_2"'] & P_1 \arrow[r, "\delta_1"] \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"] \\ C_2 \arrow[r, "f_1"'] & C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

Diagram 7.6

with $d_1 \circ n = \mathsf{id}_{C_0}$, we get $\delta_1 \circ \eta = \mathsf{id}_{P_0}$, which is condition (P3). Finally consider the pullbacks of diagram 7.6. The equality $d_1 \circ m = d_1 \circ f_1$ indicates that those pullbacks have the same bottom composite, from which $\delta_1 \circ \mu = \delta_1 \circ \varphi_1$, which is condition (P4). $\square$

Lemma 7.1.2 *Let $\mathcal{C}$ be a category with pullbacks and $\mathbb{G}$ an internal groupoid in $\mathcal{C}$. Given two presheaves P, P' on $\mathbb{G}$, an internal natural transformation $\theta\colon P \Rightarrow P'$ is equivalently given by three arrows $\theta_i\colon P_i \longrightarrow P'_i$ such that the following triangles commute –*

$$\begin{tikzcd} P_i \arrow[rr, "\theta_i"] \arrow[dr, "p_i"'] & & P'_i \arrow[dl, "p'_i"] \\ & C_i & \end{tikzcd} \qquad i = 0, 1, 2$$

– and, in diagram 7.7 where the notation is borrowed from axioms (P3) *and* (P4)*, all the squares of "corresponding arrows" are commutative.*

Proof An internal natural transformation $\theta\colon P \Rightarrow P'$ yields the equality

$$p'_1 \circ (\theta_1 \circ \varphi_0) = p_1 \circ \varphi_0 = f_0 \circ p_2.$$

$$\begin{tikzcd}
P_2 \arrow[r, shift left=3, "\varphi_0"] \arrow[r, "\mu"] \arrow[r, shift right=3, "\varphi_1"'] \arrow[d, "\theta_2"'] & P_1 \arrow[r, shift left=3, "\delta_0"] \arrow[r, shift right=3, "\delta_1"'] \arrow[d, "\theta_1"'] & P_0 \arrow[l, "\eta"'] \arrow[d, "\theta_0"] \\
P'_2 \arrow[r, shift left=3, "\varphi'_0"] \arrow[r, "\mu'"] \arrow[r, shift right=3, "\varphi'_1"'] & P'_1 \arrow[r, shift left=3, "\delta'_0"] \arrow[r, shift right=3, "\delta'_1"'] & P'_0 \arrow[l, "\eta'"']
\end{tikzcd}$$

Diagram 7.7

$$\begin{tikzcd}
P_2 \arrow[r, "\varphi_0"] \arrow[dr, dotted, "\theta_2"] \arrow[ddr, "p_2"'] & P_1 \arrow[dr, "\theta_1"] \arrow[ddr, "p_1"] & \\
& P'_2 \arrow[r, "\varphi'_0"] \arrow[d, "p'_2"'] & P'_1 \arrow[d, "p'_1"] \\
& C_2 \arrow[r, "f_0"'] & C_1
\end{tikzcd}$$

Diagram 7.8

This implies the existence of a unique factorization θ_2 through the pullback defining P_2, as in diagram 7.8. This yields at once the commutativity of diagram 7.7.

Conversely given three arrows $\theta_i\colon P_i \longrightarrow P'_i$ which make commutative diagrams 7.7 and 7.8, we get at once an internal natural transformation $\theta\colon P \Rightarrow P'$ determined by the morphism θ_0. Since θ_1 and θ_2 are uniquely determined by θ_0, this concludes the proof. $\square$

Corollary 7.1.3 *Let $\mathcal{C}$ be a category with pullbacks and let*

$$\begin{tikzcd}
C_2 \arrow[r, shift left=3, "f_0"] \arrow[r, "m"] \arrow[r, shift right=3, "f_1"'] & C_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[loop below, "\tau"] & C_0 \arrow[l, "n"']
\end{tikzcd}$$

be an internal groupoid in $\mathcal{C}$. The category of internal presheaves on this internal groupoid is, in the category CAT *of all categories and functors, the two dimensional limit of the diagram*

$$\mathcal{C}/C_2 \overset{\xleftarrow{f_0^*}}{\underset{\xleftarrow{f_1^*}}{\xleftarrow{m^*}}} \mathcal{C}/C_1 \overset{\xleftarrow{d_0^*}}{\underset{\xleftarrow{d_1^*}}{\xrightarrow{n^*}}} \mathcal{C}/C_0$$

where, as usual, f^* indicates the functor "pullback along f".

Before proving this corollary, we comment on the notion of "two dimensional limit". Roughly speaking, it is the usual notion of limit "where equalities are replaced by isomorphisms". A formal theory is developed in section 7.9, but the following more intuitive approach will be sufficient to handle the generalized Galois theory of section 7.5.

Most constructions in category theory allow us just to define a category up to an equivalence and a functor up to an isomorphism, as in the statement of corollary 7.1.3, where the pullback functors are only defined up to isomorphism. It is sensible to expect that in a diagram of categories and functors of which one computes the limit, replacing the categories by equivalent ones and the functors by isomorphic ones will finally yield a limit category which is equivalent to the original one. Unfortunately, this sensible expectation is just wrong!

Write $\mathbf{1}$ for the unit category, with one object and the identity on it. The equalizer of twice the identity on $\mathbf{1}$ is again the category $\mathbf{1}$. Now replace $\mathbf{1}$ by the equivalent category $\mathbf{1}_2$, having two isomorphic objects. Write $F_1, F_2 \colon \mathbf{1}_2 \rightrightarrows \mathbf{1}_2$ for the two possible constant functors, both isomorphic to the identity on $\mathbf{1}_2$. This time, $\mathsf{Ker}\,(F_1, F_2)$ is the empty category, which is not at all equivalent to the original equalizer $\mathbf{1}$.

An object in a limit category $\mathsf{lim}_{i\in I}\,\mathcal{X}_i$ is a compatible family $(X_i)_{i\in I}$ of objects $X_i \in \mathcal{X}_i$, where the compatibility means that given a functor $F \colon \mathcal{X}_i \longrightarrow \mathcal{X}_j$ in the diagram, $F(X_i) = X_j$. Such an equality is to be considered somehow unnatural in category theory, since most often F has only been defined up to isomorphism, like the pullback functors f^* in the statement of 7.1.3. Thus requiring an isomorphism $F(X_i) \cong X_j$ in the category $\mathcal{X}_j$ would be more sensible. This is the spirit of what a two dimensional limit is.

Now observe that given a commutative diagram

$$\begin{tikzcd} \mathcal{X}_i \arrow[r, "F"] \arrow[dr, "H"'] & \mathcal{X}_j \arrow[d, "G"] \\ & \mathcal{X}_k \end{tikzcd}$$

in a diagram of categories and functors, we have of course

$$F(X_i) = X_j \text{ and } G(X_j) = X_k \Rightarrow H(X_i) = X_k.$$

But when F, G, H have been defined "up to an isomorphism", as in corollary 7.1.3, it no longer makes sense to expect an actual commutative diagram, but only a commutativity "up to an isomorphism", like the equality $d_0 \circ m = d_0 \circ f_0$ in 7.1.3 which yields the isomorphism (not the equality) $m^* \circ d_0^* \cong f_0^* \circ d_0^*$. So imagine now that an isomorphism

$$\theta\colon G \circ F \Rightarrow H$$

is given in the previous triangle, instead of plain commutativity. In our two dimensional limit we require isomorphisms

$$\alpha\colon F(X_i) \longrightarrow X_j,\quad \beta\colon G(X_j) \longrightarrow X_k,\quad \gamma\colon H(X_i) \longrightarrow X_k.$$

We must of course require a coherence condition between these various isomorphisms, namely, the commutativity of the diagram

$$\begin{tikzcd} GF(X_i) \arrow[r, "G(\alpha)"] \arrow[d, "\theta_{X_i}"'] & G(X_j) \arrow[d, "\beta"] \\ H(X_i) \arrow[r, "\gamma"'] & X_k \end{tikzcd}$$

Such a coherence condition is obviously satisfied in the special case where θ, α, β, γ are identities, that is, in the case of an ordinary limit.

In corollary 7.1.3, the diagram of pullback functors must thus be understood as a diagram of categories and functors, together with commutativities "up to isomorphisms" between some composite functors. That is, every equality like $d_0 \circ m = d_0 \circ f_0$ in the original diagram yields a corresponding "commutativity up to an isomorphism" in the diagram of pullback functors. The two dimensional limit of this diagram is then constituted of the families $(P_i, p_i) \in \mathcal{C}/C_i$, together with an isomorphism $\alpha\colon h^*(P_i, p_i) \longrightarrow (P_j, p_j)$ for every arrow $h\colon C_j \longrightarrow C_i$ in the diagram we consider; those isomorphisms are required to satisfy coherence conditions with respect to all "commutativities up to isomorphisms".

Such a two dimensional limit has the advantage of being independent (up to an equivalence) of isomorphic choices of the pullback functors h^*.

Of course there is a corresponding notion of two dimensional colimit of a diagram $(\mathcal{X}_i)_{i \in I}$ of categories and functors. One first considers the coproduct $\coprod_{i\in I} \mathcal{X}_i$ of the categories involved and, for every functor $F\colon \mathcal{X}_i \longrightarrow \mathcal{X}_j$ in the diagram, one forces the existence of coherent isomorphisms (instead of equalities) between each $X \in \mathcal{X}_i$ and the corresponding object $F(X) \in \mathcal{X}_j$. We refer again to 7.9 for a formal approach.

With these comments in mind, we switch now to the proof of corollary 7.1.3.

Proof of corollary An object of the two dimensional limit thus yields three arrows

$$\begin{tikzcd} P_2 \arrow[d, "p_2"'] & P_1 \arrow[d, "p_1"'] & P_0 \arrow[d, "p_0"'] \\ C_2 & C_1 & C_0 \end{tikzcd}$$

together with compatibility isomorphisms

$$p_2 \cong f_0^*(p_1) \cong m^*(p_1) \cong f_1^*(p_1),$$
$$p_1 \cong d_0^*(p_0) \cong d_1^*(p_0),$$
$$p_0 \cong n^*(p_1).$$

This means precisely giving arrows φ_0, μ, φ_1, δ_0, δ_1, η yielding pullbacks as in lemma 7.1.1. Thus the objects of the two dimensional limit category correspond precisely to the internal presheaves on the internal groupoid.

The case of morphisms follows at once from lemma 7.1.2. □

7.2 Internal precategories and their presheaves

Very roughly speaking, a precategory is what remains from the definition of an internal category when you cancel all references to pullbacks.

Definition 7.2.1 An internal precategory $\mathbb{C}$ in a category $\mathcal{C}$ consists in giving the data

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd}$$

where the following relations hold:

$$d_0 \circ f_1 = d_1 \circ f_0, \quad d_1 \circ m = d_1 \circ f_1, \quad d_0 \circ m = d_0 \circ f_0,$$
$$d_0 \circ n = \mathsf{id}_{C_0}, \quad d_1 \circ n = \mathsf{id}_{C_0}.$$

For simplicity of notation, let us write $\mathbb{P}$ for the (ordinary) category with three objects P_0, P_1, P_2 and generated by six arrows

$$\begin{tikzcd} P_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & P_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & P_0 \arrow[l, "n"'] \end{tikzcd}$$

on which the following conditions are imposed:

$$d_0 \circ f_1 = d_1 \circ f_0, \quad d_1 \circ m = d_1 \circ f_1, \quad d_0 \circ m = d_0 \circ f_0,$$
$$d_0 \circ n = \mathsf{id}_{C_0}, \quad d_1 \circ n = \mathsf{id}_{C_0}.$$

A precategory $\mathbb{C}$ in a category $\mathcal{C}$ is thus simply a functor $\mathbb{C}\colon \mathbb{P} \longrightarrow \mathcal{C}$. A morphism of precategories is then a natural transformation. We shall write $\mathsf{PreCat}(\mathcal{C})$ for the category of precategories in $\mathcal{C}$.

Definition 7.2.2 Let $\mathcal{C}$ be a category with pullbacks and $\mathbb{C}\colon \mathbb{P} \longrightarrow \mathcal{C}$ an internal precategory. The category $\mathcal{C}^{\mathbb{C}}$ of internal covariant presheaves on $\mathbb{C}$ is, by definition, the two dimensional limit of the diagram

$$\begin{tikzcd} \mathcal{C}/\mathbb{C}(P_2) & \mathcal{C}/\mathbb{C}(P_1) \arrow[l, shift right=2, "f_0^*"'] \arrow[l, "m^*"'] \arrow[l, shift left=2, "f_1^*"] \arrow[r, "n^*"] & \mathcal{C}/\mathbb{C}(P_0). \arrow[l, shift right=2, "d_0^*"'] \arrow[l, shift left=2, "d_1^*"] \end{tikzcd}$$

Let us make a strong point that definition 7.2.2 is not at all a generalization of the definition of internal presheaf on an internal category. But when the internal category turns out to be an internal groupoid, corollary 7.1.3 indicates that definition 7.2.2 is equivalent to the usual definition.

Now given a category $\mathcal{A}$ and a precategory $\mathbb{C}$ in $[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$, for every object $A \in \mathcal{A}$ we can consider the composite

$$\mathbb{P} \xrightarrow{\ \mathbb{C}\ } [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}] \xrightarrow{\ \mathsf{ev}_A\ } \mathsf{CAT}$$

where ev_A is the evaluation functor at the object A. This yields a diagram in CAT of which we can take the two dimensional limit, written $(\mathsf{Lim}\,\mathbb{C})(A)$, or the two dimensional colimit, written $(\mathsf{Colim}\,\mathbb{C})(A)$. This yields two functors

$$\mathsf{Colim}\,\mathbb{C}\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad A \mapsto (\mathsf{Colim}\,\mathbb{C})(A),$$
$$\mathsf{Lim}\,\mathbb{C}\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad A \mapsto (\mathsf{Lim}\,\mathbb{C})(A),$$

and thus finally two other functors

$$\mathsf{Colim}\colon \mathsf{PreCat}[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}] \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}],$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F(A) \arrow[r, shift left=2, "\alpha_A"] \arrow[r, shift right=2, "\beta_A"'] \arrow[r, phantom, "\Downarrow\ \theta_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, shift left=2, "\alpha_B"] \arrow[r, shift right=2, "\beta_B"'] \arrow[r, phantom, "\Downarrow\ \theta_B"] & G(B)
\end{tikzcd}$$

Diagram 7.9

$$\mathsf{Lim}\colon \mathsf{PreCat}[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}] \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$$

sending a precategory $\mathbb{C}\colon \mathbb{P} \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$ to its "pointwise 2-dimensional colimit" or its "pointwise two dimensional limit".

Let us now recall that given two functors $F, G\colon \mathcal{A} \rightrightarrows \mathsf{CAT}$ and two natural transformations $\alpha, \beta\colon F \Rightarrow G$, there is a notion of modification $\theta\colon \alpha \rightsquigarrow \beta$. For each object $A \in \mathcal{A}$, α and β induce two functors $\alpha_A, \beta_A\colon F(A) \rightrightarrows G(A)$. A modification θ consists in a natural transformation $\theta_A\colon \alpha_A \Rightarrow \beta_A$ for each $A \in \mathcal{A}$, with the naturality requirement $\theta_B \star 1_{F(f)} = 1_{G(f)} \star \theta_A$ for each arrow $f\colon A \longrightarrow B$ in $\mathcal{A}$, where $\star$ indicates the horizontal composition of natural transformations, also called "Godement product" (see diagram 7.9). The natural transformations between F and G and the modifications between them constitute a category which we still write $\mathsf{Nat}(F, G)$.

That situation can easily be generalized to the context of "pseudo-functors". The prototype of a contravariant pseudo-functor on a category $\mathcal{C}$ with pullbacks is given by

$$Z_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathsf{CAT}, \quad Z_{\mathcal{C}}(C) = \mathcal{C}/C, \quad Z_{\mathcal{C}}(f) = f^*.$$

Each $Z_{\mathcal{C}}(f)$ is only defined up to an isomorphism, thus the requirements $Z_{\mathcal{C}}(f \circ g) = Z_{\mathcal{C}}(g) \circ Z_{\mathcal{C}}(f)$ for a functor do not hold, but must be replaced by corresponding coherent isomorphisms $Z_{\mathcal{C}}(f \circ g) \cong Z_{\mathcal{C}}(g) \circ Z_{\mathcal{C}}(f)$. We have considered such a situation already many times. Observe that in this specific case, we can make the canonical choice that pulling back along an identity is the identity functor. A pseudo-functor $F\colon \mathcal{C} \longrightarrow \mathsf{CAT}$ on a category $\mathcal{C}$ generalizes the previous situation: in the definition of a functor, one replaces the preservation of the composition law by a preservation up to coherent isomorphisms. The notions of natural transformation and modification extend at once to this context. We write $\mathsf{Ps}(\mathcal{C}, \mathsf{CAT})$ for the 2-category (see definition 7.9.1) of pseudo-functors, pseudo-natural-transformations and modifications.

The reader who needs a more precise description should refer to section 7.9, where these notions are formally defined in a more general context.

The following Yoneda lemma for pseudo-functors is classical (see for example [8], volume 2, chapter 8).

Lemma 7.2.3 *Consider a pseudo-functor $F\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$, an object $A \in \mathcal{A}$ and the corresponding "representable" functor*

$$\mathcal{A}(-,A)\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad B \mapsto \mathcal{A}(B,A)$$

where $\mathcal{A}(B,A)$ is viewed as a discrete category. The following equivalence of categories holds –

$$F(A) \approx \mathsf{PsNat}\big(\mathcal{A}(-,A), F\big)$$

– between the category $F(A)$ and the category of pseudo-natural-transformations and modifications.

Proof In one direction, one considers the functor

$$M\colon F(A) \longrightarrow \mathsf{PsNat}\big(\mathcal{A}(-,A), F\big)$$

which sends an object $X \in F(A)$ to the pseudo-natural-transformation $M(X)$ determined by the functors

$$M(X)_B\colon \mathcal{A}(B,A) \longrightarrow F(B), \quad f \mapsto F(f)(X).$$

Conversely, one considers the functor

$$N\colon \mathsf{PsNat}\big(\mathcal{A}(-,A), F\big) \longrightarrow F(A)$$

which sends $\alpha\colon \mathcal{A}(-,A) \Rightarrow F$ to $\alpha_A(\mathsf{id}_A)$ and a modification $\theta\colon \alpha \rightsquigarrow \beta$ to $(\theta_A)_{\mathsf{id}_A}$. It is routine to check that this determines an equivalence of categories. □

Proposition 7.2.4 *Consider*

- *a category $\mathcal{A}$,*
- *a precategory $\mathbb{C}\colon \mathbb{P} \longrightarrow [\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$,*
- *a pseudo-functor $F\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$.*

There is an equivalence of categories

$$\mathsf{PsNat}(\mathsf{Colim}\,\mathbb{C}, F) \approx \operatorname*{Lim}_i \mathsf{PsNat}\big(\mathbb{C}(P_i), F\big).$$

Proof Since $\mathsf{Colim}\,\mathbb{C}$ is the two dimensional colimit in $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$ of the diagram

$$\begin{tikzcd} \mathbb{C}(P_2) \arrow[r, shift left=2, "\mathbb{C}(f_0)"] \arrow[r, "\mathbb{C}(m)" description] \arrow[r, shift right=2, "\mathbb{C}(f_1)"'] & \mathbb{C}(P_1) \arrow[r, shift left=2, "\mathbb{C}(d_0)"] \arrow[r, shift right=2, "\mathbb{C}(d_1)"'] & \mathbb{C}(P_0) \arrow[l, "\mathbb{C}(n)" description] \end{tikzcd}$$

the result reduces to saying that the "representable" functor

$$\mathsf{PsNat}(-, F) \colon \mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}) \longrightarrow \mathsf{CAT}$$

transforms this two dimensional colimit into a two dimensional limit, which is a classical result (see [8], volume 2). □

Corollary 7.2.5 *Let us consider*

- *a category $\mathcal{A}$,*
- *an internal precategory $\mathbb{C} \colon \mathbb{P} \longrightarrow \mathcal{A}$,*
- *the "Yoneda embedding" $Y_{\mathcal{A}} \colon \mathcal{A} \longrightarrow \mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$ mapping $A \in \mathcal{A}$ onto the discrete representable functor $\mathcal{A}(-, A)$ of lemma 7.2.3,*
- *a pseudo-functor $F \colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$.*

The following equivalence of categories holds:

$$\mathsf{PsNat}(\mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}, F) \approx \mathsf{Lim}(F \circ \mathbb{C}).$$

Proof In proposition 7.2.4, it suffices to observe that the equivalence

$$\mathsf{PsNat}\Big(\mathcal{A}(-, \mathbb{C}(P_i)), P\Big) \approx F\big(\mathbb{C}(P_i)\big)$$

holds by the Yoneda lemma 7.2.3. □

Definition 7.2.6 Let us consider

- a category $\mathcal{A}$,
- an internal precategory $\mathbb{C} \colon \mathbb{P} \longrightarrow \mathcal{A}$,
- a pseudo-functor $F \colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$.

The category $\mathsf{Lim}(F \circ \mathbb{C})$ will be written $F^{\mathbb{C}}$ and called "category of covariant internal F-presheaves on $\mathbb{C}$".

Lemma 7.2.7 *Let us consider*

- *a category $\mathcal{A}$ with pullbacks,*
- *an internal groupoid $\mathbb{C} \colon \mathbb{P} \longrightarrow \mathcal{A}$.*

The category $Z_{\mathcal{A}}^{\mathbb{C}}$ is equivalent to the usual category of covariant internal presheaves on the internal groupoid $\mathbb{C}$, where

$$Z_{\mathcal{A}} \colon \mathcal{A} \longrightarrow \mathsf{CAT}, \quad Z_{\mathcal{A}}(A) = \mathcal{A}/A, \quad Z_{\mathcal{A}}(f) = f^*$$

is the "pulling back" pseudo-functor.

Proof This is a reformulation of corollary 7.1.3, with the notation of definition 7.2.2. □

Definition 7.2.8 Let $\mathcal{A}$ be a category. The discrete internal (pre)category on an object $A \in \mathcal{A}$, written $\mathbb{C}_A$, is the precategory

$$\begin{tikzcd} A \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & A \arrow[r, shift left=3] \arrow[r, leftarrow] \arrow[r, shift right=3] & A \end{tikzcd}$$

where all six arrows are identity on A.

Lemma 7.2.9 *Let us consider*

- *a category $\mathcal{A}$,*
- *the discrete precategory $\mathbb{C}_A \colon \mathbb{P} \longrightarrow \mathcal{A}$ on an object $A \in \mathcal{A}$,*
- *a pseudo-functor $F \colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$.*

In these conditions, one has the equivalence

$$F^{\mathbb{C}_A} \approx F(A).$$

Proof The pseudo-functor $F \circ \mathbb{C}_A$ is the constant pseudo-functor on $F(A)$ and the diagram $\mathbb{P}$ is connected. This argument extends easily to the two dimensional aspects. □

7.3 A factorization system for functors

This section introduces a factorization system on CAT, the category of categories and functors, in the spirit of section 5.3.

Proposition 7.3.1 *Every functor $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ between categories factors, uniquely up to an equivalence, as $F = H \circ G$:*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{B} \\ & \mathcal{I} \arrow[ur, "H"'] & \end{tikzcd}$$

where

(i) *G is essentially surjective on objects, that is, every object of $\mathcal{I}$ is isomorphic to an object of the form $G(A)$,*
(ii) *H is full and faithful.*

Proof It suffices to take for $\mathcal{I}$ the full subcategory of $\mathcal{B}$ generated by the objects of the form $G(A)$, for $A \in \mathcal{A}$. Notice that if

$$\mathcal{A} \xrightarrow{G'} \mathcal{I}' \xrightarrow{H'} \mathcal{B}$$

is another such factorization of F, the category $H'(\mathcal{I}')$ is equivalent to $\mathcal{I}'$ because H' is full and faithful. But the objects of $H'(\mathcal{I}')$ are exactly all the objects of the form $F(A)$ and, by assumption on G', possibly some objects isomorphic to an object of the form $F(A)$. Thus $H'(\mathcal{I}')$ is equivalent to $\mathcal{I}$ and therefore $\mathcal{I}'$ is equivalent to $\mathcal{I}$. ☐

Proposition 7.3.1 yields in fact a factorization system "up to an equivalence" in the spirit of definition 5.3.1.

Proposition 7.3.2 *In the category* CAT *of all categories and functors, let us write*

- *$\mathcal{E}$ for the class of those functors which are essentially surjective on objects,*
- *$\mathcal{M}$ for the class of fully faithful functors.*

The following properties hold:

(i) *$\mathcal{E}$ and $\mathcal{M}$ contain all equivalences;*
(ii) *$\mathcal{E}$ and $\mathcal{M}$ are closed under composition;*
(iii) *every functor F factors as $F = H \circ G$, with $H \in \mathcal{M}$ and $G \in \mathcal{E}$;*
(iv) *consider a square*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G \in \mathcal{E}"] \arrow[d, "K"'] & \mathcal{B} \arrow[d, "L"] \arrow[dl, dotted, "M"'] \\
\mathcal{C} \arrow[r, "H \in \mathcal{M}"'] & \mathcal{D}
\end{tikzcd}$$

where $G \in \mathcal{E}$, $H \in \mathcal{M}$ and the square commutes up to an isomorphism; in these conditions, there exists a functor M, unique up to an isomorphism, which makes the whole diagram commutative up to isomorphisms.

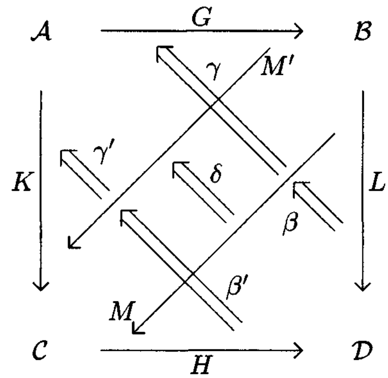

Diagram 7.10

Let us make more explicit condition (iv) of this statement; we refer to diagram 7.10. Since only isomorphisms are involved, we find it convenient to write them down in the direction which makes the formulæ as simple as possible. First, an isomorphic natural transformation $\alpha\colon L \circ G \xrightarrow{\cong} H \circ K$ is given. Together with M, we require the existence of two isomorphic natural transformations $\beta\colon L \xrightarrow{\cong} H \circ M$ and $\gamma\colon M \circ G \xrightarrow{\cong} K$ with the property $(H \star \gamma) \circ (\beta \star G) = \alpha$. If M', β', γ' have analogous properties, then there exists an isomorphic natural transformation $\delta\colon M \xrightarrow{\cong} M'$ such that $\beta' = (H \star \delta) \circ \beta$, $\gamma = \gamma' \circ (\delta \star G)$.

Proof of proposition Conditions (i) and (ii) are obviously satisfied, while condition (iii) follows from proposition 7.3.1. For condition (iv), observe that when $B \in \mathcal{B}$, then $B \cong G(A)$ for some $A \in \mathcal{A}$, because $G \in \mathcal{E}$. We put $M(B) = K(A)$, from which we get an isomorphism $K(A) = M(B) \cong MG(A)$. Also observe that

$$L(B) \cong LG(A) \cong HK(A) \cong HMG(A) \cong HM(B).$$

Defining M on the arrows reduces to giving a map

$$\begin{aligned}\mathcal{B}(B, B') \longrightarrow \mathcal{C}\big(M(B), M(B')\big) &\cong \mathcal{D}\big(HM(B), HM(B')\big)\\ &\cong \mathcal{D}\big(L(B), L(B')\big)\end{aligned}$$

since H is full and faithful. This composite is defined as being the action of the functor L. One concludes the proof with routine verifications. $\square$

Corollary 7.3.3 *With the notation of proposition 7.3.2, if*

$$(G\colon \mathcal{A} \longrightarrow \mathcal{B}) \in \mathcal{E}, \quad (H\colon \mathcal{C} \longrightarrow \mathcal{D}) \in \mathcal{M},$$

the following square is a two dimensional pullback, that is, commutes up to an isomorphism and is universal for that property.

$$\begin{tikzcd}
{[\mathcal{B},\mathcal{C}]} \arrow[r, "{[\mathsf{id}_{\mathcal{B}}, H]}"] \arrow[d, "{[G, \mathsf{id}_{\mathcal{C}}]}"'] & {[\mathcal{B},\mathcal{D}]} \arrow[d, "{[G, \mathsf{id}_{\mathcal{D}}]}"] \\
{[\mathcal{A},\mathcal{C}]} \arrow[r, "{[\mathsf{id}_{\mathcal{A}}, H]}"'] & {[\mathcal{A},\mathcal{D}]}.
\end{tikzcd}$$

Here, $[\mathcal{A},\mathcal{B}]$ denotes the category of functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them.

Proof This is just a reformulation of condition 7.3.2(iv). □

The previous results translate to the categories $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$, that is, the categories of pseudo-functors on a given category $\mathcal{A}$.

Proposition 7.3.4 *For a category $\mathcal{A}$, we consider the corresponding 2-category $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$ (see section 7.9) of pseudo-functors, pseudo-natural-transformations and modifications between them. We denote by*

- *$\mathcal{E}$ the class of pseudo-natural-transformations in $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$ all of whose components are functors essentially surjective on objects,*
- *$\mathcal{M}$ the class of pseudo-natural-transformations in $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$ all of whose components are full and faithful functors.*

The classes $\mathcal{E}$, $\mathcal{M}$ of $[\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}]$ satisfy all axioms (i), (ii), (iii), (iv) *of proposition 7.3.2.*

Proof The proof is just a routine pointwise application of proposition 7.3.2. □

Corollary 7.3.5 *With the notation of proposition 7.3.4, if*

$$(\varepsilon\colon F \Rightarrow G) \in \mathcal{E}, \quad (\mu\colon H \Rightarrow K) \in \mathcal{M}$$

in $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$, the following square is a two dimensional pullback:

$$\begin{tikzcd}
\mathsf{PsNat}(G,H) \arrow[r, "{(\mathsf{id}_G,\mu)}"] \arrow[d, "{(\varepsilon,\mathsf{id}_H)}"'] & \mathsf{PsNat}(G,K) \arrow[d, "{(\varepsilon,\mathsf{id}_K)}"] \\
\mathsf{PsNat}(F,H) \arrow[r, "{(\mathsf{id}_F,\mu)}"'] & \mathsf{PsNat}(F,K)
\end{tikzcd}$$

where PsNat *denotes the category of pseudo-natural-transformations and modifications between them.*

Proof By a pointwise application of corollary 7.3.3, since two dimensional limits in $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}},\mathsf{CAT})$ are computed pointwise. □

Proposition 7.3.6 *Let us consider*

- *a category* $\mathcal{A}$,
- *a precategory* $\mathbb{C}\colon \mathbb{P} \longrightarrow \mathsf{Ps}(\mathcal{A}^{\mathsf{op}},\mathsf{CAT})$.

The canonical morphism

$$s_0\colon \mathbb{C}(P_0) \Longrightarrow \mathsf{Colim}\,\mathbb{C}$$

is, in each component, essentially surjective on the objects.

Proof The composite

$$\mathbb{P} \xrightarrow{\mathbb{C}} \mathsf{Ps}(\mathcal{A}^{\mathsf{op}},\mathsf{CAT}) \xrightarrow{\mathsf{ev}_A} \mathsf{CAT}$$

yields a diagram in CAT which we write simply as

$$\begin{tikzcd}
C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0. \arrow[l, "n"']
\end{tikzcd}$$

We consider the two dimensional colimit of this diagram in CAT. For every object $A \in \mathcal{A}$ we must prove that the canonical functor

$$s_{0,A}\colon \mathbb{C}(P_0)(A) \longrightarrow (\mathsf{Colim}\,\mathbb{C})(A)$$

is essentially surjective on the objects.

This is an immediate consequence of the construction of a 2-dimensional colimit in CAT. Since there are morphisms

$$C_2 \longrightarrow C_1 \longrightarrow C_0,$$

every object of C_2 or C_1 is, up to isomorphism, isomorphic in the two dimensional colimit to an object of C_0. Thus every object of the two dimensional colimit is isomorphic to an object arising from C_0. □

7.4 Generalized descent theory

In the category of sets or, more generally, in a regular category (see [4]), the image factorization of a morphism $f\colon A \longrightarrow B$ is obtained by taking the quotient of A by the kernel pair of f, that is, in the case of sets, by the equivalence relation

$$R_f = \{(a, a') \mid a, a' \in A, \quad f(a) = f(a')\}.$$

Thus this yields the situation

$$\begin{tikzcd} R_f \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & A \arrow[rr, "f"] \arrow[dr, two heads, "q"'] & & B \\ & & A/R_f \arrow[ur, tail, "i"'] & \end{tikzcd}$$

Writing Δ_A, Δ_B for the diagonals of A and B – which are themselves the kernel pairs of id_A and id_B – the image factorization of f can be rewritten

$$A/\Delta_A \xrightarrow{\;q\;} \!\!\!\!\rightarrow A/R_f \rightarrowtail\!\xrightarrow{\;i\;} B/\Delta_B$$

and these morphisms are induced by the corresponding arrows between the kernel pairs

$$\Delta_A \rightarrowtail R_f \xrightarrow{f \times f} \Delta_B.$$

In example 4.6.2 we have seen how to view a kernel pair as an internal groupoid. Thus finally, the image factorization of f can be translated in terms of a factorization property involving the groupoids Δ_A, R_f and Δ_B.

Definition 7.4.1 generalizes this situation, allowing internal precategories instead of internal groupoids.

Definition 7.4.1 Let $\sigma\colon S \longrightarrow R$ be an arrow in a category $\mathcal{A}$. A precategorical decomposition of σ is a factorization in $\mathsf{PreCat}(\mathcal{A})$

$$\begin{tikzcd}
S \arrow[r, equal] \arrow[d, "i_2"'] & S \arrow[r, equal] \arrow[d, "i_1"'] & S \arrow[d, "i_0"] \\
C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] \arrow[d, "\pi_2"'] & C_1 \arrow[d, "\pi_1"'] & C_0 \arrow[l, "n"'] \arrow[d, "\pi_0"] \\
R \arrow[r, equal] & R \arrow[r, equal] & R
\end{tikzcd}$$

$$C_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} C_0$$

Diagram 7.11

$$\begin{tikzcd}
\mathbb{C}_S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}_R \\
& \mathbb{C} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

of the morphism σ, seen as an internal functor between the discrete internal categories $\mathbb{C}_S$ and $\mathbb{C}_R$ (see diagram 7.11), with, moreover, the two additional requirements

$$C_0 = S, \quad i_0 = \mathsf{id}_S.$$

The factorization of σ takes place in the category of internal precategories; thus in diagram 7.11, all vertical composites $\pi_k \circ i_k$ are equal to σ and all squares are commutative. The two additional requirements $C_0 = S$, $i_0 = \mathsf{id}_S$ extend the idea that, in the example of the relation R_f at the beginning of this section, R_f is a relation on A, the domain of f.

Lemma 7.4.2 *Let $\sigma\colon S \longrightarrow R$ be a morphism in a category $\mathcal{A}$ with pullbacks. Consider a precategorical decomposition of σ*

$$\begin{tikzcd}
\mathbb{C}_S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}_R \\
& \mathbb{C} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
S \arrow[dr, dotted, "i_2"] \arrow[drr, bend left, "n"] \arrow[ddr, bend right, "n"'] & & \\
& C_2 \arrow[r, "f_0"] \arrow[d, "f_1"] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "d_1"] \\
& C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

Diagram 7.12

in which $\mathbb{C}$ *turns out to be an actual internal category. In these conditions,* i *and* π *are uniquely determined by the knowledge of* σ *and* $\mathbb{C}$*.*

Proof With the notation of definition 7.4.1, we get at once, by commutativity of diagram 7.11,

$$\begin{aligned}
i_0 &= \mathrm{id}_S \quad \text{(by assumption)},\\
i_1 &= n \circ i_0 = n,\\
\pi_0 &= \pi_0 \circ \mathrm{id}_S = \pi_0 \circ i_0 = \sigma,\\
\pi_1 &= \pi_0 \circ d_0 = \sigma \circ d_0,\\
\pi_2 &= \pi_1 \circ f_0 = \sigma \circ d_0 \circ f_0.
\end{aligned}$$

There remains the case of i_2, which will use the fact that $\mathbb{C}$ is an internal category. We have at once

$$f_0 \circ i_2 = i_1 = n, \quad f_1 \circ i_2 = i_1 = n.$$

This proves that i_2 is the unique factorization making diagram 7.12 commutative, where the square is now a pullback by assumption. Thus i_2 is entirely determined by $\mathbb{C}$. □

Definition 7.4.3 Let us consider

- a category $\mathcal{A}$,
- a morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{A}$,
- a pseudo-functor $F \colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$,
- a precategorical decomposition of σ

$$
\begin{tikzcd}
\mathbb{C}_S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}_R \\
& \mathbb{C} \arrow[ur, "\pi"'] &
\end{tikzcd}
$$

The data $\big(\sigma, (i, \mathbb{C}, \pi)\big)$ constitute an effective descent structure with respect to F when the functor

$$F^{\pi} \colon F^{\mathbb{C}_R} \longrightarrow F^{\mathbb{C}}$$

is an equivalence of categories.

Let us recall that we have the situation

$$
\begin{tikzcd}
\mathbb{P}^{\mathsf{op}} \arrow[r, shift left=1.5, "\mathbb{C}_R^{\mathsf{op}}"] \arrow[r, shift right=1.5, "\mathbb{C}^{\mathsf{op}}"'] \arrow[r, phantom, "\Downarrow \pi^{\mathsf{op}}"] & \mathcal{A}^{\mathsf{op}} \arrow[r, "F"] & \mathsf{CAT}
\end{tikzcd}
$$

while, by definition 7.2.6,

$$F^{\mathbb{C}_R} = \mathsf{Lim}\, F \circ \mathbb{C}_R^{\mathsf{op}}, \quad F^{\mathbb{C}} = \mathsf{Lim}\, F \circ \mathbb{C}^{\mathsf{op}}.$$

The existence of the natural transformation

$$F \star \pi^{\mathsf{op}} \colon F \circ \mathbb{C}_R^{\mathsf{op}} \longrightarrow F \circ \mathbb{C}^{\mathsf{op}}$$

induces the existence of the factorization

$$F^{\pi} \colon \ \mathsf{Lim}\, F \circ \mathbb{C}_R^{\mathsf{op}} \longrightarrow \mathsf{Lim}\, F \circ \mathbb{C}^{\mathsf{op}}$$

involved in the statement. Let us recall further that, by lemma 7.2.9, one has an equivalence $F^{\mathbb{C}_R} \approx F(R)$.

Now we make more explicit, and develop further, the example sketched at the beginning of this section. At the same time, we justify our extension of the terminology "effective descent" in definition 7.4.3.

Proposition 7.4.4 *Consider an arrow $\sigma \colon S \longrightarrow R$ in a category $\mathcal{A}$ with pullbacks and the pseudo-functor*

$$Z_{\mathcal{A}} \colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad A \mapsto \mathcal{A}/A, \quad f \mapsto f^*$$

of lemma 7.2.7. The kernel pair of σ, seen as an internal groupoid $\mathbb{G}_\sigma$ as in example 4.6.2, determines a precategorical decomposition of σ

$$\begin{tikzcd}
S \arrow[r, equal] \arrow[d, "\nu"'] & S \arrow[r, equal] \arrow[d, "n"'] & S \arrow[d, equal] \\
(S \times_R S) \times_S (S \times_R S) \arrow[r, shift left=2, "f_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "f_1"'] \arrow[d, "\sigma \circ d_i \circ f_j"'] & S \times_R S \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "\sigma \circ d_i"'] & S \arrow[l, "n" description] \arrow[d, "\sigma"] \\
R \arrow[r, equal] & R \arrow[r, equal] & R
\end{tikzcd}$$

Diagram 7.13

$$\begin{tikzcd}
\mathbb{C}_S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}_R \\
& \mathbb{G}_\sigma \arrow[ur, "\pi"'] &
\end{tikzcd}$$

and the following conditions are equivalent:

(i) $\big(\sigma, (i, \mathbb{G}_\sigma, \pi)\big)$ *is an effective descent structure with respect to the pseudo-functor* $Z_{\mathcal{A}}$*;*

(ii) $\sigma \colon S \longrightarrow R$ *is an effective descent morphism in the sense of definition 4.4.1.*

Proof The precategorical decomposition of σ is simply given by diagram 7.13. Notice that this notation with some unspecified indices makes sense. Indeed $\sigma \circ d_0 = \sigma \circ d_1$, because (d_0, d_1) is by definition the kernel pair of σ. Moreover $d_0 \circ f_1 = d_1 \circ f_0$ since the corresponding square is a pullback by definition. So all $\sigma \circ d_i \circ f_j$, for all combinations of $i, j \in \{0, 1\}$, are the same. On the other hand n is the diagonal and ν is the unique factorization through the pullback, resulting from the relation $d_0 \circ n = \mathrm{id}_S = d_1 \circ n$. The commutativity of the diagram follows at once from the fact that m is given by the first and the fourth projections.

Lemma 7.2.7 implies that the category $Z_{\mathcal{A}}^{\mathbb{G}_\sigma}$ is equivalent to the category $\mathcal{A}^{\mathbb{G}_\sigma}$ of the usual internal presheaves on the internal groupoid $\mathbb{G}_\sigma$. On the other hand proposition 4.6.1 asserts that this category $\mathcal{A}^{\mathbb{G}_\sigma}$ is

$$\begin{tikzcd}
P \arrow[r, "k"] \arrow[d, "h'"'] & X \arrow[d, "h"] \\
S \times_R S \arrow[r, "d_0"] \arrow[d, "d_1"'] & S \arrow[d, "\sigma"] \\
S \arrow[r, "\sigma"'] & R
\end{tikzcd}$$

Diagram 7.14

monadic over $\mathcal{A}/S$, the functorial part of the corresponding monad $\mathbb{T}$ being the composite

$$T\colon \mathcal{A}/S \xrightarrow{d_0^*} \mathcal{A}/S \times_R S \xrightarrow{\Sigma_{d_1}} \mathcal{A}/S.$$

Thus this yields the following situation:

$$\begin{tikzcd}
Z_{\mathcal{A}}^{\mathbb{C}_R} \cong Z_{\mathcal{A}}(R) \cong \mathcal{A}/R \arrow[rr, "Z_{\mathcal{A}}^{\pi}"] \arrow[dr, shift right, "\sigma^*"'] & & (\mathcal{A}/S)^{\mathbb{T}} \cong \mathcal{A}^{\mathbb{G}_\sigma} \cong Z_{\mathcal{A}}^{\mathbb{G}_\sigma} \arrow[dl, shift right, "U"'] \\
& \mathcal{A}/S \arrow[ul, shift right, "\Sigma_\sigma"'] \arrow[ur, shift right, "F"'] &
\end{tikzcd}$$

where $F \dashv U$ is the Eilenberg–Moore adjunction of the monad $\mathbb{T}$.

Let us first observe that

$$\sigma^* \circ \Sigma_\sigma \cong T \cong \Sigma_{d_1} \circ d_0^*.$$

Indeed, consider $h\colon X \longrightarrow S$. In diagram 7.14, where the squares are pullbacks, we have

$$\begin{array}{rclcrcl}
(P, h') & \cong & d_0^*(X, h), & \qquad & (P, d_1 \circ h') & \cong & (\Sigma_{d_1} \circ d_0^*)(X, h), \\
(X, \sigma \circ h) & \cong & \Sigma_\sigma(X, h), & & (P, d_1 \circ h') & \cong & (\sigma^* \circ \Sigma_\sigma)(X, h).
\end{array}$$

This proves already that the functors T and $\sigma^* \circ \Sigma_\sigma$ are isomorphic on the objects, from which routine verifications show that the monad $\mathbb{T}$ is isomorphic to the monad induced by the adjunction $\Sigma_\sigma \dashv \sigma^*$.

Next we observe that $U \circ Z^{\pi}_{\mathcal{A}} \cong \sigma^*$. Writing for simplicity

$$G_2 = (S \times_R S) \times_S (S \times_R S), \quad G_1 = S \times_R S, \quad G_0 = S,$$

the category $Z^{\mathbb{G}_\sigma}_{\mathcal{A}}$ of lemma 7.2.7 is the two dimensional limit $\mathcal{L}$ of the bottom line in the following diagram:

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, "p_2"'] \arrow[d, "p_1"] \arrow[dr, "p_0"] & \\
\mathcal{A}/G_2 & \mathcal{A}/G_1 \arrow[l, shift right=3, "f_0^*"'] \arrow[l, "m^*"'] \arrow[l, shift left=3, "f_1^*"] \arrow[r, "n^*"] & \mathcal{A}/G_0 \arrow[l, shift right=3, "d_0^*"'] \arrow[l, shift left=3, "d_1^*"]
\end{tikzcd}$$

The forgetful functor U maps an object $L \in \mathcal{L}$ onto $p_0(L)$. On the other hand the category $Z^{\mathbb{C}_R}_{\mathcal{A}}$ yields the trivial two dimensional limit

$$\begin{tikzcd}
 & \mathcal{A}/R \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{A}/R & \mathcal{A}/R \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \arrow[r] & \mathcal{A}/R \arrow[l, shift right=3] \arrow[l, shift left=3]
\end{tikzcd}$$

where all arrows are identities. The factorization $Z^{\pi}_{\mathcal{A}} \colon \mathcal{A}/R \longrightarrow \mathcal{L}$ induced by this situation is such that

$$\begin{aligned}
p_0 \circ Z^{\pi}_{\mathcal{A}} &\cong \sigma^*, \\
p_1 \circ Z^{\pi}_{\mathcal{A}} &\cong (\sigma \circ d_i)^*, \\
p_2 \circ Z^{\pi}_{\mathcal{A}} &\cong (\sigma \circ d_i \circ f_j)^*.
\end{aligned}$$

The first of these relations is precisely the expected relation $U \circ Z^{\pi}_{\mathcal{A}} = \sigma^*$.

Since the adjunction $\Sigma_\sigma \dashv \sigma^*$ induces the monad $\mathbb{T}$ and $U \circ Z^{\pi}_{\mathcal{A}} = \sigma^*$, it follows at once that σ^* is monadic precisely when the comparison functor $Z^{\pi}_{\mathcal{A}}$ is an equivalence of categories. This yields the result, via definitions 4.4.1 and 7.4.3. □

7.5 Generalized Galois theory

This section establishes a Galois theorem for effective descent structures in the sense of definition 7.4.3, without any further assumption of Galois descent.

Definition 7.5.1 Let us consider

- a category $\mathcal{A}$,
- two pseudo-functors $F, G\colon \mathcal{A}^{\mathrm{op}} \rightrightarrows \mathsf{CAT}$,
- a pseudo-natural-transformation $\alpha\colon F \Rightarrow G$,
- a morphism $\sigma\colon S \longrightarrow R$ in $\mathcal{A}$.

An object $M \in G(R)$ is split by σ relatively to α, when there exist an object $X \in F(S)$ and an isomorphism $G(\sigma)(M) \cong \alpha_S(X)$ in $G(S)$.

Let us point out that since F and G are contravariant on $\mathcal{A}$, we have the following situation:

$$\begin{tikzcd}
F(R) \arrow[r, "F(\sigma)"] \arrow[d, "\alpha_R"'] & F(S) \ni X \arrow[d, "\alpha_S"] \\
M \in G(R) \arrow[r, "G(\sigma)"'] & G(S) \ni G(\sigma)(M) \cong \alpha_S(X)
\end{tikzcd}$$

We shall write $\mathsf{Split}_\alpha(\sigma)$ for the full subcategory of $G(R)$ whose objects are those split by σ relatively to α.

The corresponding Galois theorem is then

Theorem 7.5.2 (Galois theorem) *Let us consider*

- *a category $\mathcal{A}$,*
- *two pseudo-functors $F, G\colon \mathcal{A}^{\mathrm{op}} \rightrightarrows \mathsf{CAT}$,*
- *a pseudo-natural-transformation $\alpha\colon F \Rightarrow G$, all of whose components α_A are full and faithful,*
- *a morphism $\sigma\colon S \longrightarrow R$ in $\mathcal{A}$,*
- *a precategorical decomposition of σ*

$$\begin{tikzcd}
\mathbb{C}^S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}^R \\
& \mathbb{C} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

When $(\sigma, (i, \mathbb{C}, \pi))$ is an effective descent structure with respect to the pseudo-functor G, one has an equivalence of categories

$$\mathsf{Split}_\alpha(\sigma) \approx F^{\mathbb{C}}.$$

Proof First of all, by definition 7.5.1, an object M is in $\mathsf{Split}_\alpha(\sigma)$ when an isomorphism $G(\sigma)(M) \cong \alpha_S(X)$ exists in $G(S)$ for some object $X \in F(S)$. In other words, the left hand square below is a two dimensional pullback (a "bipullback").

$$\begin{tikzcd} \mathsf{Split}_\alpha(\sigma) \arrow[r, tail] \arrow[d] & G(R) \arrow[d, "G(\sigma)"] \\ F(S) \arrow[r, "\alpha_S"'] & G(S) \end{tikzcd} \qquad \begin{tikzcd} F^{\mathbb{C}} \arrow[r, tail] \arrow[d] & G(R) \arrow[d, "G(\sigma)"] \\ F(S) \arrow[r, "\alpha_S"'] & G(S) \end{tikzcd}$$

Notice that up to an equivalence, this is a special case of the construction given in the proof of proposition 7.9.6. It remains to prove that $F^{\mathbb{C}}$ can make the right hand square a two dimensional pullback as well.

By assumption, we have a structure of effective descent, thus in particular, by definition 7.4.3 and lemma 7.2.9, a commutative diagram, up to isomorphisms,

$$\begin{tikzcd} G(R) \arrow[r, "\approx"] \arrow[d, "G(\sigma)"'] & G^{\mathbb{C}_R} \arrow[r, "\approx", "G^\pi"'] \arrow[d, "G^\sigma"'] & G^{\mathbb{C}} \arrow[dl, "G^i"] \\ G(S) \arrow[r, "\approx"] & G^{\mathbb{C}_S} & \end{tikzcd}$$

where all horizontal arrows are equivalences of categories. The equivalence $\mathsf{Split}_\alpha(\sigma) \approx F^{\mathbb{C}}$ thus reduces to proving the existence of a two dimensional pullback

$$\begin{tikzcd} F^{\mathbb{C}} \arrow[r] \arrow[d] & G^{\mathbb{C}} \arrow[d, "G^i"] \\ F(S) \arrow[r, "\alpha_S"'] & G^{\mathbb{C}_S} \end{tikzcd}$$

For this we apply successively the Yoneda lemma of 7.2.3, definition 7.2.6 and corollary 7.2.5, writing

$$Y_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT}), \quad A \mapsto \mathcal{A}(-, A)$$

for the "discrete Yoneda embedding" of $\mathcal{A}$, as in 7.2.5. We get

$$F(S) \approx \mathsf{PsNat}\big(Y_{\mathcal{A}}(S), F\big),$$
$$F^{\mathbb{C}} \approx \mathsf{Lim}\, F \circ \mathbb{C} \approx \mathsf{PsNat}(\mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}, F),$$

and similarly for G. The desired equivalence can then reduce further to proving that the following commutative square is a two dimensional pullback.

$$\begin{tikzcd}
\mathsf{PsNat}(\mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}, F) \arrow[r, "{(\mathsf{id}, \alpha)}"] \arrow[d, "{(\varepsilon, \mathsf{id})}"'] & \mathsf{PsNat}(\mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}, G) \arrow[d, "{(\varepsilon, \mathsf{id})}"] \\
\mathsf{Nat}\big(Y_{\mathcal{A}}(S), F\big) \arrow[r, "{(\mathsf{id}, \alpha)}"'] & \mathsf{Nat}\big(Y_{\mathcal{A}}(S), G\big)
\end{tikzcd}$$

In this square, $\varepsilon : Y_{\mathcal{A}}(S) \longrightarrow \mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}$ is the canonical morphism

$$(Y_{\mathcal{A}} \circ \mathbb{C})(P_0) \longrightarrow \mathsf{Colim}\, Y_{\mathcal{A}} \circ \mathbb{C}$$

of proposition 7.3.6, in the case of the precategory $Y_{\mathcal{A}} \circ \mathbb{C}$ in the category $\mathsf{Ps}(\mathcal{A}^{\mathsf{op}}, \mathsf{CAT})$. By proposition 7.3.6, each component of this morphism is surjective on the objects. On the other hand, by assumption, each component of α is full and faithful. Our square is thus a two dimensional pullback, by corollary 7.3.5. □

Corollary 7.5.3 *Let us consider*

- *two categories* $\mathcal{A}$, $\mathcal{X}$,
- *a functor* $H\colon \mathcal{A} \longrightarrow \mathcal{X}$,
- *two pseudo-functors* $K\colon \mathcal{X}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$ *and* $G\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$,
- *a pseudo-natural-transformation* $\alpha\colon K \circ H \Rightarrow G$, *all of whose components* α_A *are full and faithful,*
- *a morphism* $\sigma\colon S \longrightarrow R$ *in* $\mathcal{A}$,
- *a precategorical decomposition of* σ

$$\begin{tikzcd} \mathbb{C}^S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}^R \\ & \mathbb{C} \arrow[ur, "\pi"'] & \end{tikzcd}$$

When $(\sigma, (i, \mathbb{C}, \pi))$ is an effective descent structure with respect to the pseudo-functor G, one has an equivalence of categories

$$\mathsf{Split}_\alpha(\sigma) \approx K^{H\circ\mathbb{C}}.$$

Proof By definition 7.2.6 one has

$$K^{H\circ\mathbb{C}} = \mathsf{Lim}\, K \circ H \circ \mathbb{C} = (K \circ H)^{\mathbb{C}}$$

and it remains to apply theorem 7.5.2 with $F = K \circ H$. □

7.6 Classical Galois theories

First we further particularize corollary 7.5.3 to get a "Galois theorem without Galois assumption", in the context of chapter 5. We keep writing

$$Z_{\mathcal{C}} \colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad C \mapsto \mathcal{C}/C, \ f \mapsto f^*$$

for the "pulling back" pseudo-functor on a category $\mathcal{C}$ with pullbacks.

Theorem 7.6.1 *Let us consider*

- *two categories $\mathcal{A}$, $\mathcal{P}$ with pullbacks,*
- *a functor $\mathcal{S} \colon \mathcal{A} \longrightarrow \mathcal{P}$,*
- *a pseudo-natural-transformation $\alpha \colon Z_{\mathcal{P}} \circ \mathcal{S} \Rightarrow Z_{\mathcal{A}}$ all of whose components*

  $$\alpha_A \colon \mathcal{P}/\mathcal{S}(A) \longrightarrow \mathcal{A}/A$$

  are full and faithful,
- *$\sigma \colon S \longrightarrow R$ a morphism of $\mathcal{A}$ which is of effective descent in the sense of definition 4.4.1,*
- *$\mathbb{G}_\sigma$ the kernel pair of σ, seen as an internal groupoid in $\mathcal{A}$ (see example 4.6.2).*

When $\mathcal{S}(\mathbb{G}_\sigma)$ remains an internal groupoid in $\mathcal{P}$, the following equivalence of categories holds:

$$\mathsf{Split}_\alpha(\sigma) \approx \mathcal{P}^{\mathcal{S}(\mathbb{G}_\sigma)}$$

where the right hand side denotes the category of internal presheaves on $\mathbb{G}$ in the usual sense of section 4.6.

Proof In corollary 7.5.3, we put $H = \mathcal{S}$, $K = Z_{\mathcal{P}}$ and $G = Z_{\mathcal{A}}$. Proposition 7.4.4 implies the existence of a precategorical decomposition

$$\begin{tikzcd} \mathbb{C}^S \arrow[rr, "\sigma"] \arrow[dr, "i"'] & & \mathbb{C}^R \\ & \mathbb{G}_\sigma \arrow[ur, "\pi"'] & \end{tikzcd}$$

where $\mathbb{G}_\sigma$ is the kernel pair of σ. Proposition 7.4.4 also asserts that $\big(\sigma, (i, \mathbb{G}_\sigma, \pi)\big)$ is an effective descent structure relatively to the pseudo-functor $Z_{\mathcal{A}}$. Corollary 7.5.3 then yields the equivalence

$$\mathsf{Split}_\alpha(\sigma) \approx Z_{\mathcal{P}}^{\mathcal{S}(\mathbb{G}_\sigma)}.$$

The desired equivalence follows at once from corollary 7.1.3 and definition 7.2.6. $\square$

Proposition 7.6.2 *Let us consider*

- *a category $\mathcal{A}$ with pullbacks,*
- *a semi-left-exact reflection $\mathcal{S} \dashv i \colon \mathcal{P} \rightleftarrows \mathcal{A}$,*
- *an effective descent morphism $\sigma \colon S \longrightarrow R$ in $\mathcal{A}$,*
- *the kernel pair $\mathbb{G}_\sigma$ of σ, seen as a groupoid in $\mathcal{A}$.*

In these conditions, there is an equivalence of categories

$$\mathsf{Split}_R(\sigma) \approx Z_{\mathcal{P}}^{\mathcal{S}(\mathbb{G}_\sigma)}$$

where $\mathsf{Split}_R(\sigma) \subseteq \mathcal{A}/R$ is the full subcategory of split objects in the sense of definition 5.1.7, with $\overline{\mathcal{A}}$ the class of all arrows in $\mathcal{A}$ and $\overline{\mathcal{P}}$ the class of all arrows in $\mathcal{P}$. If moreover $\mathcal{S}(\mathbb{G}_\sigma)$ is still a groupoid in $\mathcal{P}$, one obtains an equivalence of categories

$$\mathsf{Split}_R(\sigma) \approx \mathcal{P}^{\mathcal{S}(\mathbb{G}_\sigma)}.$$

Proof By proposition 5.5.2, for every object $A \in \mathcal{A}$ we have a full and faithful functor

$$\alpha_A \colon \mathcal{P}/\mathcal{S}(A) \longrightarrow \mathcal{A}/A$$

with left adjoint $\mathcal{S}_A$. This can be rewritten

$$\alpha_A = i_A \colon (Z_{\mathcal{P}} \circ \mathcal{S})(A) \longrightarrow Z_{\mathcal{A}}(A)$$

and the pseudo-naturality of $\alpha\colon Z_{\mathcal{P}} \circ \mathcal{S} \Rightarrow Z_{\mathcal{A}}$ is obvious.

To conclude the proof by theorem 7.6.1, it remains to observe that the two notions of split object

$$\mathsf{Split}_{\alpha}(\sigma), \quad \mathsf{Split}_{R}(\sigma)$$

coincide, a fact which is attested by lemma 5.1.12. □

At this stage it is certainly enlightening to compare more carefully the present proposition 7.6.2 and the Galois theorem of Janelidze (see 5.1.24).

Considering classes $\overline{\mathcal{A}}$ and $\overline{\mathcal{P}}$ in theorem 5.1.24 reduces, in theorem 7.5.2, to defining

$$F(A) = \overline{\mathcal{P}}/\mathcal{S}(A), \quad G(A) = \overline{\mathcal{A}}/A.$$

Lemma 5.1.4 still yields a pseudo-natural-transformation α with components

$$\alpha_A\colon \overline{\mathcal{P}}/\mathcal{S}(A) \longrightarrow \overline{\mathcal{A}}/A.$$

A first difference occurs in the fact that in theorem 5.1.24, one works under the minimal assumption

α_S is full and faithful

while in theorem 7.5.2, it is required that

each α_A is full and faithful.

This difference is rather unessential.

Indeed, in theorem 7.5.2, let us write

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd}$$

for the precategory $\mathbb{C}$ which is involved. Let us also write

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[dr, "\beta"'] & & G \\ & F' \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

for the factorization of $\alpha\colon F \Rightarrow G$ in α', full and faithful in each component, and β, essentially surjective on the objects in each component. By theorem 7.5.2, we obtain an equivalence

$$\mathsf{Split}_{\alpha'}(\sigma) \approx F'^{\mathbb{C}}.$$

If we assume simply that α_{C_0} is full and faithful, then β_{C_0} is an equivalence, from which we get at once the equivalence

$$\mathsf{Split}_{\alpha'}(\sigma) \approx \mathsf{Split}_{\alpha}(\sigma),$$

via definition 7.5.1. If moreover we assume that α_{C_1} and α_{C_2} are also full and faithful, then β_{C_1} and β_{C_2} become equivalences as well and by definition 7.2.6, we get another equivalence

$$F'^{\mathbb{C}} \approx F^{\mathbb{C}}.$$

So the conclusion of theorem 7.5.2 still holds, namely the existence of an equivalence

$$\mathsf{Split}_{\alpha}(\sigma) \approx F^{\mathbb{C}}.$$

Now when the corresponding precategory $\mathbb{C}$ has a rather particular form, as in theorem 5.1.24 where it is the kernel pair of the morphism σ,

$$\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[r, shift left=2, "f_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "f_1"'] & S \times_R S \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & S, \arrow[l, "n" description]
\end{tikzcd}$$

a further weakening of the assumptions can occur (in the presence of the assumption of Galois descent) to arrive finally at the single assumption of α_S being full and faithful. We shall not insist on these variations on the minimal hypothesis.

It is more important to comment on the presence in theorem 5.1.24, and the absence in proposition 7.6.2, of a *Galois* descent assumption. In fact,

- theorem 5.1.24 uses the assumption

  $\sigma\colon S \longrightarrow R$ is of Galois descent,

- proposition 7.6.2 uses the assumption

  $\mathcal{S}(\mathbb{G}_\sigma)$ is a groupoid.

We know by lemma 5.1.22 that the first of these assumptions implies the second one. But let us observe with more details the argument in both

$$
\begin{tikzcd}
\mathsf{Split}_R(\sigma) \arrow[r, tail] \arrow[d, "S \otimes -"'] & R\text{-}\mathsf{Alg} \arrow[d, "S \otimes -"] & \mathsf{Prof}^{\mathsf{Sp}(\mathbb{G}_\sigma)} \arrow[dd, "U"] \\
\mathsf{Split}_S(S) \arrow[r, tail] \arrow[d, "\cong"'] & S\text{-}\mathsf{Alg} & \\
\mathsf{Prof}/\mathsf{Sp}(S) & & \mathsf{Prof}/\mathsf{Sp}(S)
\end{tikzcd}
$$

Diagram 7.15

proofs. For simplicity, we write the argument in the context of rings and algebras, as in section 4.7.

In theorem 4.7.15, a special case of theorem 5.1.24, one in fact considers the situation of diagram 7.15. Of course, several arguments were formally developed in the opposite categories of algebras, but for simplicity we avoid writing the exponent $(-)^{\mathsf{op}}$ and thus allow some functors to be contravariant. The fact that σ is of Galois descent is used to prove that the functor $(S \otimes -)$ restricts to the categories of split algebras. Therefore $\mathsf{Split}_R(\sigma)$ becomes monadic over $\mathsf{Prof}/\mathsf{Sp}(S)$ because $(S \otimes -)$ is monadic by assumption, as a morphism of effective descent. The fact that σ is of Galois descent is used a second time to prove that $\mathsf{Sp}(\mathbb{G}_\sigma)$ is a groupoid, from which a classical result implies that $\mathsf{Prof}^{\mathsf{Sp}(\mathbb{G}_\sigma)}$ is monadic over $\mathsf{Prof}/\mathsf{Sp}(S)$. Since both monads involved are isomorphic, one gets the expected equivalence.

In proposition 7.6.2 and thus theorem 7.5.2, again expressed in the special context of theorem 4.7.15 one considers instead the squares of diagram 7.16, which become two dimensional pullbacks. Using the effective descent assumption on σ, one gets

$$R\text{-}\mathsf{Alg} \cong R\text{-}\mathsf{Alg}^{\mathbb{G}_\sigma} \quad \text{and thus} \quad \mathsf{Split}_R(\sigma) \cong \mathsf{Prof}^{\mathsf{Sp}(\mathbb{G}_\sigma)},$$

where the Galois groupoid $\mathbb{G}_\sigma$ is an internal groupoid in the opposite category of R-algebras (where it becomes discrete). The essential point in the approach of the present chapter has been to observe that the left hand square yields a Galois theorem as long as $R\text{-}\mathsf{Alg} \cong R\text{-}\mathsf{Alg}^{\mathbb{G}_\sigma}$ (condition of effective descent), just by describing $\mathsf{Prof}^{\mathsf{Sp}(\mathbb{G}_\sigma)}$ via the second square, which is again a two dimensional pullback.

$$
\begin{tikzcd}
\mathsf{Split}_R(\sigma) \arrow[r, tail] \arrow[d, "S \otimes -"'] & R\text{-}\mathsf{Alg} \arrow[d, "S \otimes -"] \\
\mathsf{Split}_S(\sigma) \arrow[r, tail] \arrow[d, "\cong"'] & S\text{-}\mathsf{Alg} \\
\mathsf{Prof}/\mathsf{Sp}(S) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{Prof}^{\mathsf{Sp}(\mathbb{G}_\sigma)} \arrow[r, tail] \arrow[d] & R\text{-}\mathsf{Alg}^{\mathbb{G}_\sigma} \arrow[d] \\
\mathsf{Prof}/\mathsf{Spec}(S) \arrow[r, tail, "\alpha_S"'] \arrow[d, "\cong"'] & S\text{-}\mathsf{Alg} \\
\mathsf{Split}_S(\sigma) &
\end{tikzcd}
$$

Diagram 7.16

Let us recall in particular (corollary 4.4.5) that every field extension is an effective descent morphism. Proposition 7.6.2 is thus a Galois theorem for every field extension. But this Galois theorem for a non-Galois extension of fields refers to a "Galois pregroup" and not a "Galois group". Indeed, if $K \subseteq L$ is the field extension, one has $\mathsf{Sp}(L) = \{*\}$ and the corresponding pregroup has the form

$$
\begin{tikzcd}
\mathsf{Sp}\Big((L \otimes_K L) \otimes_L (L \otimes_K L)\Big) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathsf{Sp}(L \otimes_K L) \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathsf{Sp}(L) \arrow[l] \cong \{*\}.
\end{tikzcd}
$$

Let us conclude this section with a historical remark. The present section, allowing a Galois theorem for non-Galois extensions, is inspired by [48], but such a theorem can already be found in [41], under a slightly different form. However, the case of commutative rings was partly understood already by Magid (see [67]) in 1974 and, in some sense, even by Grothendieck a long time ago. In particular Grothendieck knew that the fundamental group of $\mathbb{Q}$ can be described as $\pi_0(\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C})$, although $\mathbb{Q} \subset \mathbb{C}$ is not a Galois extension.

7.7 Grothendieck toposes

The rest of this chapter intends to show that the Galois theorem for toposes, due to Joyal and Tierney (see [57]), is a special instance of the theory developed in section 7.5. Doing this requires of course a deep knowledge of topos theory and we want to avoid it, since this book is not a priori intended for specialists in topos theory. In fact Joyal and

Tierney develop their theory over an arbitrary base topos: we shall only handle the case of Grothendieck toposes over the base topos of sets.

To achieve our goal, we devote this section to an elementary and rather unusual course on Grothendieck toposes. We give short but nevertheless explicit proofs, omitting only the routine calculations which can be found in every textbook on this topic, for example [55], [65] or [8], volume 3.

We recall that given a small category $\mathcal{C}$, a presheaf on $\mathcal{C}$ is a functor $P\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$. We write $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ for the category of presheaves on $\mathcal{C}$ and natural transformations between them.

Definition 7.7.1 A Grothendieck topos is a localization of a category of presheaves. More precisely, $\mathcal{E}$ is a Grothendieck topos when it can be presented as a full reflective subcategory

$$\mathcal{E} \underset{i}{\overset{a}{\rightleftarrows}} [\mathcal{C}^{\mathsf{op}}, \mathsf{Set}], \quad a \dashv i$$

where $\mathcal{C}$ is a small category and the reflection a preserves finite limits.

Among Grothendieck toposes, we find the categories of sheaves on a locale (see 4.2.9), which are called the localic toposes. We can view a locale L as a category $\mathcal{L}$ whose objects are the elements of L and where a unique morphism $u \longrightarrow v$ between two elements exists precisely when $u \le v$. The category of presheaves on the locale L is the usual category of contravariant presheaves on the corresponding category $\mathcal{L}$. When $u \le v$ in L, that is, when there is a morphism $u \longrightarrow v$ in $\mathcal{L}$, the image of this morphism by a presheaf P is simply written as

$$P(u \le v)\colon P(v) \longrightarrow P(u), \quad x \mapsto x|_u.$$

Definition 7.7.2 A sheaf on a locale L is a presheaf F on L which satisfies the following axiom:

$$\begin{aligned}
&\text{if} \quad u = \textstyle\bigvee_{i\in I} u_i \quad \text{in} \quad L, \\
&\qquad \forall i \in I \quad x_i \in F(u_i), \\
&\qquad \forall i,j \in I \quad x_i|_{u_i \wedge u_j} = x_j|_{u_i \wedge u_j} \\
&\text{then } \exists! x \in F(u) \quad \forall i \in I \quad x|_{u_i} = x_i.
\end{aligned}$$

A family $(x_i)_{i\in I}$ as above is called a compatible family in P along the covering $u = \bigvee_{i\in I} u_i$.

When in the axiom above, only the uniqueness of x holds, not necessarily the existence, F is called a separated presheaf.

Theorem 7.7.3 *The category* $\mathsf{Sh}(L)$ *of sheaves on a locale is a localization of the category of presheaves on* L*, thus is a Grothendieck topos.*

Definition 7.7.4 A localic topos is a category which is equivalent to the topos of sheaves on a locale.

Proof of theorem Consider a presheaf P on L. Given $u \in L$, define $P'(u)$ as the set of all compatible families $(x_i)_{i\in I}$ in P (see 7.7.2) for all possible coverings $u = \bigvee_{i\in I} u_i$. Two compatible families $\big(x_i \in P(u_i)\big)_{i\in I}$, $\big(y_j \in P(v_j)\big)_{j\in J}$ are considered equivalent when

$$\forall i \in I \;\; \forall j \in J \;\; x_i|_{u_i\wedge v_j} = y_j|_{u_i\wedge v_j}.$$

The set $\widetilde{P}(u)$ is then the quotient of $P'(u)$ by the equivalence relation generated by these pairs.

The presheaf structure on P induces at once a presheaf structure on $\widetilde{P}$. From its definition, it follows at once that $\widetilde{P}$ is separated. You are certainly convinced that, from its definition, $\widetilde{P}$ must in fact trivially be a sheaf; for your peace of mind, spend an hour in unsuccessful efforts to write down the details of this evidence, and then five minutes to produce an easy counterexample. What is straightforward is the fact that when P is already separated, then $\widetilde{P}$ is a sheaf. Putting together these two observations, we conclude that $\widetilde{\widetilde{P}}$ is a sheaf. It is again routine to verify that it is the sheaf reflection of the presheaf P.

It remains to prove the left exactness of the sheaf reflection. The terminal sheaf $\mathbf{1}$ is such that $\mathbf{1}(u)$ is a singleton for each $u \in L$. A subpresheaf $R \rightarrowtail \mathbf{1}$ is thus entirely determined by those $u \in L$ for which $R(u)$ is a singleton; let us write $\mathsf{Supp}(R)$ for the family of those elements. The fact that R is a presheaf reduces to

$$\big(u \in \mathsf{Supp}(R) \;\text{ and }\; v \leq u\big) \Rightarrow \big(v \in \mathsf{Supp}(R)\big).$$

Given a compatible family $(x_i)_{i\in I}$ in a presheaf P along a covering $u = \bigvee_{i\in I} u_i$, the compatibility of the family implies that it extends to a compatible family on all $u \in L$ which are smaller than some u_i. Thus in the construction of $\widetilde{P}$, we can equivalently restrict our attention to those families $(u_i)_{i\in I}$ which are downward directed, that is, which correspond to subpresheaves of $\mathbf{1}$. Now given a covering $\mathsf{Supp}(R) = (u_i)_{i\in I}$ of u corresponding to a subpresheaf $R \rightarrowtail \mathbf{1}$, a compatible family of elements in P along this covering is just a natural transformation $R \Rightarrow P$, since $R(u_i)$ is a singleton for each $i \in I$ and is empty otherwise. Therefore

$\widetilde{P}(u)$ can be defined as the colimit

$$\widetilde{P}(u) = \operatorname*{colim}_R \mathsf{Nat}(R, P)$$

where R runs through those subobjects of $\mathbf{1}$ which correspond to coverings of u. If R, S correspond respectively to the coverings $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ of u, then $R \cap S$ corresponds to the family $(u_i \wedge v_j)_{(i,j)\in I\times J}$. Since L is a locale,

$$\bigvee_{i\in I,\ j\in J} u_i \wedge v_j = \bigvee_{i\in i} \left(u_i \wedge \bigvee_{j\in J} v_j \right) = \bigvee_{i\in I} (u_i \wedge u) = \bigvee_{i\in I} u_i = u.$$

Thus $R\cap S$ corresponds again to a covering of u, proving that the colimit defining $\widetilde{P}$ is filtered. Since filtered colimits of sets commute with finite limits, the functor $\widetilde{(-)}$ preserves finite limits, and thus also the associated sheaf functor $a = \widetilde{\widetilde{(-)}}$ does. □

Proposition 7.7.5 *In the category of sheaves on a locale L, the representable presheaves are exactly all the subsheaves of the terminal sheaf $\mathbf{1}$. They constitute a locale isomorphic to L.*

Proof Consider a locale L viewed as a category $\mathcal{L}$. We have already considered the subpresheaves $R \rightarrowtail \mathbf{1}$ of the terminal sheaf in the proof of 7.7.3; each of them corresponds to a downward directed family $\mathsf{Supp}(R)$ of elements. Assume now that R is a sheaf. Since R takes only values "singleton" and "empty set", the family of all existing elements $* \in R(u)$ is trivially compatible, from which there exists a unique glueing of that family at the level $u = \bigvee \mathsf{Supp}(R)$, by the sheaf axiom. This means exactly that $R(u)$ is a singleton and u is the largest element of L with that property. But then R is just the representable functor $\mathcal{L}(-,u)$. And since all representable functors $\mathcal{L}(-,u)$ are obviously sheaves, we conclude that the subsheaves of $\mathbf{1}$ are exactly the representable functors, thus constitute a locale isomorphic to L. □

The next proposition lists various interesting properties of limits and colimits in a topos.

Proposition 7.7.6 *A Grothendieck topos $\mathcal{E}$ is complete and cocomplete. Moreover,*

(i) *colimits are universal,*
(ii) *finite limits commute with filtered colimits,*

(iii) *each functor* $A \times (-) \colon \mathcal{E} \longrightarrow \mathcal{E}$ *preserves colimits,*

(iv) *coproducts are disjoint,*

(v) *a union of a downward directed family of subobjects is their colimit,*

(vi) *the subobjects of every object constitute a locale.*

Proof With the notation of definition 7.7.1, the category of sets is complete and cocomplete, thus so is the category $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ of presheaves, where limits and colimits are computed pointwise. The Grothendieck topos $\mathcal{E}$ is thus complete as well, with limits computed as in $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$; it is also cocomplete, the colimit of a diagram being obtained by applying the reflection a to the colimit in $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$.

A colimit is universal when, pulling it back along any morphism, one gets again a colimit. A coproduct is disjoint when the canonical injections of the coproduct are monomorphisms and the intersection of two of them is the initial object.

We use again the notation of 7.7.1. Conditions (i) to (iv) express properties relating some colimits and some finite limits. These properties are valid in Set, thus they are valid in $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ where colimits and (finite) limits are computed pointwise. Since a preserves colimits, as a left adjoint, and finite limits, by assumption, all these properties transfer to $\mathcal{E}$.

Given $E \in \mathcal{E}$ and a monomorphism $f \colon P \rightarrowtail E$ in $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$, we get at once a commutative triangle

$$
\begin{tikzcd}
P \arrow[r, tail, "\eta_P"] \arrow[dr, tail, "f"'] & a(P) \arrow[d, tail, "a(f)"] \\
& E
\end{tikzcd}
$$

where $a(f)$ is a monomorphism by left exactness of a and i, and the unit η_P of the adjunction is a monomorphism since f is. The universal property of $a(P)$ indicates at once that it is the smallest subobject of E in $\mathcal{E}$ which contains P. But the union of a family of subobjects is the smallest subobject which contains them. Thus unions in $\mathcal{E}$ are obtained by first taking the set theoretical pointwise union in $[\mathcal{C}^{\mathsf{op}}, \mathsf{Set}]$ and next applying the associated sheaf functor.

The previous remark implies conditions (v) and (vi), since again these conditions are valid in Set, thus are valid pointwise in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, and finally are preserved by a. □

Next, we focus on properties of monomorphisms and epimorphisms. The reader familair with the notion of "regular category" will in particular recognize that proposition 7.7.7 below implies that a topos is a regular category. Let us mention that a topos is even an *exact* category in the sense of [4], but we shall not need this result.

Proposition 7.7.7 *In a Grothendieck topos $\mathcal{E}$:*

(i) *every monomorphism is regular,*

(ii) *a morphism which is both a monomorphism and an epimorphism is an isomorphism,*

(iii) *every epimorphism is regular,*

(iv) *epimorphisms are stable under change of base,*

(v) *the product of two epimorphisms is still an epimorphism,*

(vi) *every morphism factors as an epimorphism followed by a monomorphism.*

Proof We use the notation of definition 7.7.1. In the category of sets, every monomorphism is the equalizer of its cokernel pair, that is, its pushout with itself. The same conclusion thus applies pointwise in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. But a monomorphism $f\colon A \rightarrowtail B$ in $\mathcal{E}$ remains a monomorphism in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, thus f is an equalizer $f = \mathsf{Ker}\,(u,v)$ in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. Applying the functor a, we get $f = \mathsf{Ker}\,\big(a(u), a(v)\big)$.

In the same situation, if f is also an epimorphism in $\mathcal{E}$, then from $a(u) \circ f = a(v) \circ f$ we get $a(u) = a(v)$ and therefore $f = \mathsf{Ker}\,\big(a(u), a(v)\big)$ is an isomorphism.

Using again the notation of 7.7.1, we prove now the existence of images. Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{E}$, we compute its kernel pair (u,v) in $\mathcal{E}$, that is, the pullback of f with itself, which is thus also its kernel pair in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$:

$$\begin{tikzcd} P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[r, "f"] \arrow[d, two heads, "p"'] & B \\ & A/P \arrow[ur, hook, "s"'] & \end{tikzcd}$$

Next in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ we consider the coequalizer p of (u, v) and the factorization s from this coequalizer, resulting from the equality $f \circ u = f \circ v$. In the category of sets, s would be injective since this is the standard image factorization of f. Thus the same conclusion holds pointwise in the category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, and therefore p is an epimorphism and s a monomorphism in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. Applying the reflection a, which preserves colimits as a left adjoint and finite limits by assumption, we obtain the following diagram in $\mathcal{E}$:

$$\begin{tikzcd}
P \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[r, two heads, "f"] \arrow[d, two heads, "a(p)"'] & B \\
& a(A/P) \arrow[ur, tail, "a(s)"'] &
\end{tikzcd}$$

with $a(p)$ an epimorphism and $a(s)$ a monomorphism.

With the same notation, let now $f \colon A \twoheadrightarrow B$ be an epimorphism in $\mathcal{E}$. We must prove that f is a regular epimorphism, that is, a coequalizer. Going back to the last diagram, the monomorphism $a(s)$ is now an epimorphism as well, because so is f. By the first part of the proof, $a(s)$ is an isomorphism and f is isomorphic to $a(p)$, which is the coequalizer of (u, v).

Finally given two epimorphisms $f \colon A \twoheadrightarrow B$ and $g \colon C \twoheadrightarrow D$, the morphism $f \times g$ is the upper composite in the following diagram, where both parallelograms are pullbacks:

$$\begin{tikzcd}
& A \times C \arrow[r, two heads, "f \times \mathsf{id}_C"] \arrow[dl, "p_A"'] & B \times C \arrow[r, two heads, "\mathsf{id}_B \times g"] \arrow[dl, "p_B"'] \arrow[dr, "p_C"] & B \times D \arrow[dr, "p_D"] & \\
A \arrow[r, two heads, "f"'] & B & & C \arrow[r, two heads, "g"'] & D
\end{tikzcd}$$

Since f and g are epimorphisms, by pulling back, $f \times \mathsf{id}_C$ and $\mathsf{id}_B \times g$ are epimorphisms as well. $\square$

Lemma 7.7.8 *Let $f \colon A \longrightarrow B$ be a morphism in a Grothendieck topos. The map*

$$f^{-1} \colon \mathsf{SubObj}(B) \longrightarrow \mathsf{SubObj}(A), \quad S \mapsto f^{-1}(S)$$

$$\begin{tikzcd}
R \arrow[rr, tail, "r"] \arrow[dr, dotted, "\alpha"] \arrow[ddd, two heads, "p"'] & & A \arrow[ddd, "f"] \\
& f^{-1}(S) \arrow[ur, tail, "t"'] \arrow[d, "g"] & \\
& S \arrow[dr, tail, "s"] & \\
\exists_f(R) \arrow[ur, dotted, "\beta"] \arrow[rr, tail, "i"'] & & B
\end{tikzcd}$$

Diagram 7.17

between locales of subobjects has both a left adjoint written $\exists_f$ *and a right adjoint written* $\forall_f$.

Proof In the topos **Set** of sets, given a map $f\colon A \longrightarrow B$ and subobjects $R \rightarrowtail A$, $S \rightarrowtail B$, we define

$$\begin{aligned}
\exists_f(R) &= \big\{b \in B \big| \exists a \in A \;\; f(a) = b,\; a \in R\big\} = f(R), \\
\forall_f(R) &= \big\{b \in B \big| \forall a \in A \;\; f(a) = b \Rightarrow a \in R\big\}.
\end{aligned}$$

It is immediate that

$$\exists_f(R) = f(R) \subseteq S \Leftrightarrow R \subseteq f^{-1}(S)$$

proving $\exists_f \dashv f^{-1}$. On the other hand

$$S \subseteq \forall_f(R) \Leftrightarrow \big(f(a) \in S \Rightarrow a \in R\big) \Leftrightarrow f^{-1}(S) \subseteq R$$

proves the adjunction $f^{-1} \dashv \forall_f$. This example justifies the notation and fixes the intuition.

In the case of a Grothendieck topos, given a subobject $r\colon R \rightarrowtail A$ and a morphism $f\colon A \longrightarrow B$, let us define $\exists_f(R)$ as the image factorization of the composite $f \circ r$ (see 7.7.7(vi)). Consider the diagram 7.17 where the right hand quadrilateral is thus a pullback. If $R \subseteq f^{-1}(S)$, using the set theoretical notation, we get a morphism α making the diagram commutative. Thus $f \circ r$ factors through the subobject S; since the image of $f \circ r$ is the smallest subobject through which $f \circ r$ factorizes, $\exists_f(R) \subseteq S$. Conversely if $\exists_f(R) \subseteq S$, we get a morphism β making the diagram commutative. This yields a factorization α through the pullback defining $f^{-1}(S)$, proving $R \subseteq f^{-1}(S)$. This proves already the adjunction $\exists_f \dashv f^{-1}$.

Next we define

$$\forall_f(R) = \bigcup \{S | S \subseteq B, \ \ f^{-1}(S) \subseteq R\}.$$

The family of those S is obviously downward directed, thus the corresponding union is a colimit and therefore is universal (see 7.7.6). Thus

$$\begin{aligned} f^{-1}(\forall_f(R)) &= f^{-1}\Big(\bigcup \{S | S \subseteq B, \ \ f^{-1}(S) \subseteq R\}\Big) \\ &= \bigcup \{f^{-1}(S) | S \subseteq B, \ \ f^{-1}(S) \subseteq R\} \\ &\subseteq R. \end{aligned}$$

It follows at once that

$$\begin{aligned} f^{-1}(S) \subseteq R &\Rightarrow S \subseteq \forall_f(R), \\ S \subseteq \forall_f(R) &\Rightarrow f^{-1}(S) \subseteq f^{-1}(\forall_f(R)) \subseteq R, \end{aligned}$$

which proves the adjunction $f^{-1} \dashv \forall_f$. □

7.8 Geometric morphisms

We now turn our attention to the morphisms of toposes.

Definition 7.8.1 Given two categories $\mathcal{E}$ and $\mathcal{F}$ with finite limits, a geometric morphism $f\colon \mathcal{E} \longrightarrow \mathcal{F}$ between them is a pair of adjoint functors

$$f_*\colon \mathcal{E} \longrightarrow \mathcal{F}, \quad f^*\colon \mathcal{F} \longrightarrow \mathcal{E}, \quad f^* \dashv f_*$$

with the additional property that f^* preserves finite limits.

We shall use this definition of geometric morphism both when $\mathcal{E}$, $\mathcal{F}$ are toposes, and when they are locales, viewed as categories.

Given two categories $\mathcal{A}$, $\mathcal{B}$ with finite limits, we write $\mathsf{Geom}(\mathcal{A}, \mathcal{B})$ for the category whose objects are the geometric morphisms $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ and whose morphisms $\alpha\colon f \Rightarrow g$ are the natural transformations $\alpha\colon f^* \Rightarrow g^*$. It is routine to observe that one gets an equivalent category by choosing $\alpha\colon g^* \Rightarrow f^*$, but this is unnecessary for our purposes.

Lemma 7.8.2 *Given two locales L, M, the categories* $\mathsf{Geom}(\mathcal{L}, \mathcal{M})$ *and* $\mathsf{Geom}\big(\mathsf{Sh}(L), \mathsf{Sh}(M)\big)$ *of geometric morphisms are equivalent.*

Proof Consider first a geometric morphism $f\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$ between the toposes of sheaves. The functor f_* preserves the subobjects of $\mathbf{1}$ because it has a left adjoint f^*, while f^* has the same property because it preserves finite limits. Therefore, applying proposition 7.7.5, f induces a geometric morphism of locales

$$\sigma(f) = (f_*, f^*)\colon \mathcal{L} \longrightarrow \mathcal{M}.$$

And trivially, given another geometric morphism $g\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$, every natural transformation $\alpha\colon f^* \Rightarrow g^*$ restricts to a corresponding natural transformation $\sigma(\alpha)\colon \sigma(f)^* \Rightarrow \sigma(g)^*$.

Consider now a geometric morphism of locales $f\colon \mathcal{L} \longrightarrow \mathcal{M}$. Given a sheaf F on L, the composite

$$\mathcal{M}^{\mathsf{op}} \xrightarrow{f^*} \mathcal{L}^{\mathsf{op}} \xrightarrow{F} \mathsf{Set}$$

is a sheaf on M. Indeed, a covering in M is just a colimit in the corresponding category $\mathcal{M}$. Thus f^* preserves coverings, because it has a right adjoint f_*. But given a covering $v = \bigvee_{j \in J} v_j$ in M, a compatible family in $F \circ f^*$ along that covering is also a compatible family in F along the covering $f^*(v) = \bigvee_{j \in J} f^*(v_j)$, thus has a unique glueing in $F\big(f^*(v)\big)$. Composing with f^* thus yields a functor

$$\mathsf{Sh}(f)_*\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M), \quad F \mapsto F \circ f^*.$$

We must prove the existence of a left adjoint $\mathsf{Sh}(f)^*$ to this functor $\mathsf{Sh}(f)_*$. It is well known that a presheaf on every category is a colimit of representable presheaves (see [8], volume 1). Since on a locale L the representable functors are already sheaves (see 7.7.5), every sheaf G on the locale M is thus a colimit of representable sheaves. More precisely,

$$G = \operatorname*{colim}_{v \in M,\ y \in G(v)} \mathcal{M}(-, v).$$

We define

$$\mathsf{Sh}(f)^*(G) = \operatorname*{colim}_{v \in M,\ y \in G(v)} \mathcal{L}\big(-, f^*(v)\big).$$

The functoriality of $\mathsf{Sh}(f)^*(G)$ is obvious and the expected adjunction is proved as follows, using the previous notation:

$$\mathsf{Nat}\big(\mathsf{Sh}(f)^*(G), F\big)\big) \cong \mathsf{Nat}\Big(\mathsf{colim}_{v,y}\, \mathcal{L}\big(-, f^*(v)\big), F\Big)$$

$$\begin{aligned}
&\cong \mathsf{lim}_{v,y}\,\mathsf{Nat}\Big(\mathcal{L}\big(-,f^*(v)\big),F\Big)\\
&\cong \mathsf{lim}_{v,y}\,F\big(f^*(v)\big)\\
&\cong \mathsf{lim}_{v,y}\,\mathsf{Sh}(f)_*(F)(v)\\
&\cong \mathsf{lim}_{v,y}\,\mathsf{Nat}\big(\mathcal{M}(-,v),\mathsf{Sh}_*(f)(F)\big)\\
&\cong \mathsf{Nat}\big(\mathsf{colim}_{v,y}\,\mathcal{M}(-,v),\mathsf{Sh}_*(f)(F)\big)\\
&\cong \mathsf{Nat}\big(G,\mathsf{Sh}_*(f)(F)\big).
\end{aligned}$$

Finally, to have a geometric morphism $\mathsf{Sh}(f)$, it remains to prove that $\mathsf{Sh}(f)^*$ preserves finite limits. We have the following situation, given an element $u \in L$:

$$\begin{tikzcd}
\mathsf{Elts}(G) \arrow[r, "\phi_G"] & \mathcal{M} \arrow[r, "\mathcal{L}(u,f^*(-))"] \arrow[d, "G"'] & \mathsf{Set} \\
& \mathsf{Set} &
\end{tikzcd}$$

with $\mathsf{Elts}(G)$ the category of elements of G, that is, the category of those pairs (v,y), with $y \in G(v)$ indexing the previous colimit. The functor ϕ_G is the forgetful functor $\phi_G(v,y) = v$. Given an element $u \in L$, let us define

$$\mathsf{Sh}(f)'(G)(u) = \underset{v,y}{\mathsf{colim}}\,\mathcal{L}\big(u,f^*(v)\big) \in \mathsf{Set}.$$

This set is the so-called tensor product of the contravariant presheaf G with the covariant presheaf $\mathcal{L}\big(u,f^*(-)\big)$; it is a classical result (see for example [8], volume 1) that this tensor product is commutative. More explicitly, consider now the situation

$$\begin{tikzcd}
\mathsf{Elts}\Big(\mathcal{L}\big(u,f^*(-)\big)\Big) \arrow[r, "\phi"] & \mathcal{M} \arrow[r, "G"] \arrow[d, "\mathcal{L}(u,f^*(-))"'] & \mathsf{Set} \\
& \mathsf{Set} &
\end{tikzcd}$$

where $\mathsf{Elts}\Big(\mathcal{L}\big(u,f^*(-)\big)\Big)$ is now the category of elements of $\mathcal{L}\big(u,f^*(-)\big)$, that is the category of pairs (v,z) with $z \in \mathcal{L}\big(u,f^*(v)\big)$. But $\mathcal{L}\big(u,f^*(v)\big)$ is a singleton when $u \leq f^*(v)$ and the empty set otherwise. So our category of elements reduces to the category of those $v \in M$ such that $u \leq f^*(v)$. The commutativity of the tensor product of presheaves thus yields

$$\mathsf{Sh}(f)'(G)(u) = \operatorname*{colim}_{v,y} \mathcal{L}\big(u, f^*(v)\big) \in \mathsf{Set} = \operatorname*{colim}_{v,\ u \leq f^*(v)} G(v).$$

Since f^* preserves finite infima,

$$u \leq f^*(v) \text{ and } u \leq f^*(v') \Rightarrow u \leq f^*(v) \wedge f^*(v') = f^*(v \wedge v').$$

This proves that the second colimit is filtered, thus using it to define $\mathsf{Sh}(f)'$ implies at once that the functor

$$\mathsf{Sh}(f)' \colon \mathsf{Sh}(M) \longrightarrow [\mathcal{L}^{\mathsf{op}}, \mathsf{Set}]$$

preserves finite limits. Indeed, finite limits commute in $[\mathcal{L}^{\mathsf{op}}, \mathsf{Set}]$ with filtered colimits (see 7.7.6). Now the colimit defining $\mathsf{Sh}(f)^*(G)$ in $\mathsf{Sh}(L)$ is obtained by considering the first colimit defining $\mathsf{Sh}(f)'(G)$ and applying the associated sheaf functor to it. In other words, $\mathsf{Sh}(f)^*$ is the composite

$$\mathsf{Sh}(f)^* \colon \mathsf{Sh}(M) \xrightarrow{\mathsf{Sh}(f)'} [\mathcal{L}^{\mathsf{op}}, \mathsf{Set}] \xrightarrow{\quad a \quad} \mathsf{Sh}(L).$$

Since both $\mathsf{Sh}(f)'$ and a preserve finite limits, $\mathsf{Sh}(f)^*$ preserves finite limits as well and $\mathsf{Sh}(f)$ is a geometric morphism of locales.

It is obvious that given a second geometric morphism $g \colon \mathcal{L} \longrightarrow \mathcal{M}$, a natural transformation $\alpha \colon f^* \longrightarrow g^*$ extends uniquely, by unique factorization through the corresponding colimits, as a natural transformation $\mathsf{Sh}(\alpha) \colon \mathsf{Sh}(f)^* \Rightarrow \mathsf{Sh}(g)^*$.

It is also obvious that both constructions yield the expected equivalence. $\square$

Proposition 7.8.3 *The category* Loc *of locales and geometric morphisms has finite limits.*

Proof To prove the existence of finite limits, it suffices to prove the existence of a terminal object and pullbacks (see [8], volume 1).

The terminal topological space is the singleton, whose locale of open subsets is isomorphic to the lattice $\{0 < 1\}$. Let us prove that the locale $\{0 < 1\}$ is the terminal object of the category Loc of locales and geometric morphisms. Given a locale L, the following data obviously define a geometric morphism $h \colon L \longrightarrow \{0 < 1\}$:

$$h_*(u) = 1 \quad \Leftrightarrow \quad u = 1,$$

$$h^*(1) = 1, \qquad h^*(0) = 0.$$

Since h^* must preserve the initial and the terminal object, its definition is imposed, proving the uniqueness.

If X and Y are topological spaces, with locales $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ of open subsets, the topological product $X \times Y$ is not at all such that $\mathcal{O}(X \times Y) = \mathcal{O}(X) \times \mathcal{O}(Y)$. In fact, an open subset in $X \times Y$ has the form

$$\bigcup_{i \in I} U_i \times V_i, \quad U_i \in \mathcal{O}(X),\ V_i \in \mathcal{O}(Y).$$

That form of an open subset is clearly not unique; for example one has formulæ like

$$\left(\bigcup_{i \in I} U_i\right) \times V = \bigcup_{i \in I} (U_i \times V)$$

and analogously in the second variable. Now if we consider a pullback of topological spaces

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

the topology of P is that induced by the topology of $X \times Y$. Given open subsets $U \in \mathcal{O}(X)$, $V \in \mathcal{O}(Y)$ and $W \in \mathcal{O}(Z)$, one also has

$$P \cap \Big(\big(U \cap f^{-1}(W)\big) \times V \Big) = P \cap \Big(U \times \big(g^{-1}(W) \cap V\big) \Big).$$

Indeed, given $(x, y) \in P$, this means just

$$x \in U,\ f(x) \in W,\ y \in V \Leftrightarrow x \in U,\ g(y) \in W,\ y \in V$$

which is obvious since $f(x) = g(y)$. This suggests how to construct the corresponding pullback of the locales of open subsets and, more generally, a pullback of locales.

Let us thus consider three locales L, M, N and two geometric morphisms f, g as in the following diagram.

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "h"'] & L \arrow[d, "f"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

We want to construct the corresponding pullback P. With the previous intuition in mind and using a technique analogous to that for constructing the tensor product of modules, we consider first all the formal expressions

$$\bigvee_{i\in I} l_i \otimes m_i, \quad I \in \mathsf{Set},\ l_i \in L,\ m_i \in M.$$

The locale P is then the quotient of the set of all these formal expressions by the congruence generated by

$$\begin{aligned} \left(\bigvee_{i\in I} l_i\right) \otimes m &= \bigvee_{i\in I}(l_i \otimes m), \\ l \otimes \left(\bigvee_{i\in I} m_i\right) &= \bigvee_{i\in I}(l \otimes m_i), \\ \big(l \wedge f^*(n)\big) \otimes m &= l \otimes \big(g^*(n) \wedge m\big) \end{aligned}$$

with $l, l_i \in L$; $m, m_i \in M$; $n \in N$. The morphisms h and k are then simply determined by

$$k^*(l) = l \otimes 1, \quad h^*(m) = 1 \otimes m$$

with 1 denoting the top element. The rest is routine verifications which can be found in [8], volume 3. □

Let us now observe a peculiar property of Grothendieck toposes with respect to geometric morphisms.

Lemma 7.8.4 *For every Grothendieck topos $\mathcal{E}$, there exists a unique (up to isomorphism) geometric morphism $g\colon \mathcal{E} \longrightarrow \mathsf{Set}$.*

Proof The functor $g^*\colon \mathsf{Set} \longrightarrow \mathcal{E}$ must preserve finite limits, but also colimits since it is a left adjoint. Therefore

$$g^*(X) \cong g^*\left(\coprod_X \mathbf{1}\right) \cong \coprod_X g^*(\mathbf{1}) \cong \coprod_X \mathbf{1},$$

which proves the uniqueness.

For the existence, let us use the notation of 7.7.1. One has the following situation:

$$\begin{tikzcd} \mathcal{E} \arrow[r, shift right, "i"'] & {[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]} \arrow[l, shift right, "a"'] \arrow[r, shift left=2, "\mathsf{colim}"] \arrow[r, shift right=2, "\mathsf{lim}"'] & \mathsf{Set} \arrow[l, "\Delta"'] \end{tikzcd}$$

where

- $a \dashv i$ with a preserving finite limits,
- Δ maps the set X onto the constant functor on X,
- colim maps a presheaf onto its colimit,
- lim maps a presheaf onto its limit.

The adjunctions $\mathsf{colim} \dashv \Delta \dashv \mathsf{lim}$ hold just by definition of a colimit or a limit (see [8], volume 1); in particular Δ preserves limits. This shows that (i, a) and (lim, Δ) constitute geometric morphisms, from which we get a geometric morphism

$$(\mathsf{lim} \circ i, a \circ \Delta) : \mathcal{E} \longrightarrow \mathsf{Set}. \qquad \square$$

It can be easily shown that if the category $\mathcal{C}$ considered in the proof of 7.8.4 is connected, then $\mathsf{colim} \dashv \Delta \dashv \mathsf{lim}$ considered there is a special case of $I \dashv H \dashv \Gamma$ as in the comments following 6.2.1.

Theorem 7.8.5 *Consider a Grothendieck topos $\mathcal{E}$. There exist a locale L and a geometric morphism*

$$h : \mathcal{E} \longrightarrow \mathsf{Sh}(L)$$

with the properties:

(i) *h^* is full and faithful,*

(ii) *up to an equivalence, h^* identifies $\mathsf{Sh}(L)$ with the full subcategory of $\mathcal{E}$ of those objects which are the unions of their atomic subobjects (an object is atomic when it is a subobject of $\mathbf{1}$).*

The locale L and the geometric morphism h are unique, up to isomorphisms, for the previous properties. The locale L is isomorphic to the locale of subobjects of $\mathbf{1}$ in $\mathcal{E}$. Via condition (ii), *$\mathsf{Sh}(L)$ as a subcategory of $\mathcal{E}$ is saturated for subobjects.*

Definition 7.8.6 A geometric morphism which satisfies the conditions of theorem 7.8.5 is called hyperconnected.

Proof of theorem In every category, if $u\colon U \rightarrowtail \mathbf{1}$ is a subobject of the terminal object, two morphisms $\alpha, \beta\colon X \rightrightarrows U$ are necessarily equal since $u \circ \alpha = u \circ \beta$, with u a monomorphism. Therefore every morphism $f\colon U \longrightarrow Y$ is a monomorphism.

Let us also recall an argument already used in the proof of lemma 7.8.2: every sheaf F on the locale L can be written as a colimit

$$F = \operatorname*{colim}_{u \in L,\ x \in F(u)} \mathcal{L}(-, u)$$

of representable sheaves.

Let us now turn to the proof of our theorem. If two localic toposes $\mathsf{Sh}(L)$ and $\mathsf{Sh}(M)$ are equivalent, then by lemma 7.8.2, the locales L and M are equivalent. But in a locale, viewed as a category, only the identities are isomorphisms. Thus equivalent locales L and M are in fact isomorphic. The uniqueness requirement in the statement thus follows at once from conditions (i) and (ii).

To prove the existence, let us choose for L the locale of subobjects of $\mathbf{1}$ in $\mathcal{E}$; as usual, we write $\mathcal{L}$ when we view L as a category. We first define the functor

$$h^*\colon \mathsf{Sh}(L) \longrightarrow \mathcal{E}.$$

Given a sheaf $F \in \mathsf{Sh}(L)$, we write it as a colimit of representable sheaves

$$F = \operatorname*{colim}_{U \in L,\ x \in F(U)} \mathcal{L}(-, U) \in \mathsf{Sh}(L)$$

and we define in $\mathcal{E}$

$$h^*(F) = \operatorname*{colim}_{U \in L,\ x \in F(U)} U.$$

This definition extends immediately to the morphisms $f\colon F \longrightarrow G$ in $\mathsf{Sh}(L)$: for each term of the colimit presenting F one has the composite

$$\mathcal{L}(-, U) \xrightarrow{s_{U,x}} F \xrightarrow{f} G$$

which is a term of the colimit presenting G; here $s_{U,x}$ denotes the natural transformation corresponding to $x \in F(U)$ by the Yoneda lemma. This yields a natural transformation from the diagram defining $h^*(F)$ to the diagram defining $h^*(G)$ and thus a corresponding factorization $h^*(f)\colon h^*(F) \longrightarrow h^*(G)$ between the colimits. The observant reader will have recognized the classical formula defining the Kan extension of $\mathcal{L} \rightarrowtail \mathcal{E}$ along $\mathcal{L} \rightarrowtail \mathsf{Sh}(L)$, when the locale L is identified respectively with the categories of subobjects of $\mathbf{1}$ in $\mathcal{E}$ and $\mathsf{Sh}(L)$.

$$\begin{tikzcd} \mathcal{L} \arrow[r, tail] \arrow[d, tail] & \mathsf{Sh}(L) \arrow[dl, dotted, "h^*"] \\ \mathcal{E} & \end{tikzcd}$$

Observe at once that given $U \in L$, the colimit diagram expressing $\mathcal{L}(-,U)$ as colimit of representable sheaves admits precisely $\mathcal{L}(-,U)$ as the image of the terminal object. Thus the corresponding diagram defining $h^*\big(\mathcal{L}(-,U)\big)$ admits U as the image of the terminal object, proving that $h^*\big(\mathcal{L}(-,U)\big) = U$. In other words, h^* induces an isomorphism between the categories of subobjects of $\mathbf{1}$ in $\mathsf{Sh}(L)$ and $\mathcal{E}$.

It is rather easy to guess what the possible right adjoint h_* of h^* can be. Given $U \in L$ and $E \in \mathcal{E}$, one must have, applying the Yoneda lemma, the expected adjunction $h^* \dashv h_*$ and the definition of h^*,

$$h_*(E)(U) \cong \mathsf{Nat}\big(\mathcal{L}(-,U), h_*(E)\big) \cong \mathcal{E}\Big(h^*\big(\mathcal{L}(-,U)\big), E\Big) \cong \mathcal{E}(U,E).$$

We thus define $h_*\colon \mathcal{E} \longrightarrow \mathsf{Sh}(L)$ by the formula

$$h_*(E)\colon \mathcal{L}^{\mathsf{op}} \longrightarrow \mathsf{Set}, \quad U \mapsto \mathcal{E}(U,E)$$

with obvious action on the morphisms. Observe that $h_*(E)$ is a sheaf. Indeed, using the preservation of colimits by representable functors (see [8], volume 1), for a downward directed family $(U_i)_{i\in I}$ in L, one has

$$\begin{aligned} h_*(E)\left(\bigvee_{i\in I} U_i\right) &\cong \mathcal{E}\left(\bigvee_{i\in I} U_i, E\right) \\ &\cong \mathcal{E}\left(\operatorname*{colim}_{i\in I} U_i, E\right) \\ &\cong \lim_{i\in I} \mathcal{E}(U_i, E) \\ &\cong \lim_{i\in I} h_*(E)(U_i). \end{aligned}$$

So the elements of $h_*(E)(\bigvee_{i\in I} U_i)$ are in bijection with the elements of $\lim_{i\in I} h_*(E)(U_i)$, that is, with the compatible families in the sense of definition 7.7.2. This proves that $h^*(E)$ is a sheaf on L.

Proving the adjunction $h^* \dashv h_*$ is also easy. With previous notation and arguments, choosing $F \in \mathsf{Sh}(L)$ and $E \in \mathcal{E}$,

$$\mathcal{E}\big(h^*(F), E\big) \cong \mathcal{E}\left(\operatorname*{colim}_{U\in L,\ x\in F(U)} U, E\right)$$

$$\begin{aligned}
&\cong \lim_{U\in L,\ x\in F(U)} \mathcal{E}(U,E)\\
&\cong \lim_{U\in L,\ x\in F(U)} h_*(E)(U)\\
&\cong \lim_{U\in L,\ x\in F(U)} \mathsf{Nat}\big(\mathcal{L}(-,U),h_*(E)\big)\\
&\cong \mathsf{Nat}\Big(\operatorname*{colim}_{U\in L,\ x\in F(U)} \mathcal{L}(-,U),h_*(E)\Big)\\
&\cong \mathsf{Nat}\big(F,h_*(E)\big).
\end{aligned}$$

The unit of the adjunction $h^* \dashv h_*$ is thus given, for every sheaf $F \in \mathsf{Sh}(L)$ and every $V \in L$, by a map

$$\eta_{F,V}\colon F(V) \longrightarrow \mathcal{E}\big(V,h^*(F)\big).$$

Via the Yoneda lemma and the equality $V = h^*\big(\mathcal{L}(-,V)\big)$, this map is just

$$\mathsf{Nat}\big(\mathcal{L}(-,V),F\big) \longrightarrow \mathcal{E}\big(V,h^*(F)\big), \quad \alpha \mapsto h^*(\alpha).$$

We want now to prove that these maps are bijective. This will prove that the unit of the adjunction $h^* \dashv h_*$ is an isomorphism, and therefore that h^* is full and faithful (see [8], volume 1).

For this we construct an inverse

$$\mu_{F,V}\colon \mathcal{E}\big(V,h^*(F)\big) \longrightarrow \mathsf{Nat}\big(\mathcal{L}(-,V),F\big)$$

to $\eta_{F,V}$. We consider again the presentation of F as colimit of representable functors,

$$s_{U,x}\colon \mathcal{L}(-,U) \longrightarrow F, \quad U \in L,\ x \in F(U),$$

and the corresponding colimit defining $h^*(F)$,

$$\sigma_{U,x}\colon U \longrightarrow h^*(F), \quad U \in L,\ x \in F(U).$$

Given $\beta\colon V \longrightarrow h^*(F)$, we compute the pullbacks in $\mathcal{E}$

$$\begin{tikzcd}
W_{U,x} \arrow[r, tail] \arrow[d, tail] & U \arrow[d, tail, "\sigma_{U,x}"] \\
V \arrow[r, tail, "\beta"'] & h^*(F)
\end{tikzcd}$$

where $W_{U,x} \subseteq V$ is thus a subobject of $\mathbf{1}$. By universality of colimits (see 7.7.6),

$$V = \operatorname*{colim}_{U,x} W_{U,x} = \bigcup_{U,x} W_{U,x}$$

in $\mathcal{E}$, thus also in $\mathsf{Sh}(L)$, since h^* induces an isomorphism at the level of subobjects of $\mathbf{1}$. It is also clear that

$$\mathcal{L}(-, W_{U,x}) \rightarrowtail \mathcal{L}(-, U)$$

is a compatible family of morphisms in $\mathsf{Sh}(L)$ and thus induces a factorization between the corresponding colimits, which we choose as

$$\mu_{F,V}(\beta)\colon \mathcal{L}(-, V) \longrightarrow F.$$

Proving that $\eta_{F,V}$ and $\mu_{F,V}$ are mutually inverse is just an easy routine. An arrow $\alpha\colon \mathcal{L}(-, V) \longrightarrow F$ corresponds to an element $a \in F(V)$ by the Yoneda lemma and thus $\alpha = s_{V,a}$. By construction of h^*, $h^*(\alpha) = h^*(s_{V,a}) = \sigma_{V,a}$. By construction of $\mu_{F,V}$, the left hand pullback in

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, equal] & V \arrow[d, tail, "\sigma_{V,a}"] \\
V \arrow[r, tail, "\sigma_{V,a}"'] & h^*(F)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}(-,V) \arrow[r, equal] \arrow[d, equal] & \mathcal{L}(-,V) \arrow[d, tail, "s_{V,a}"] \\
\mathcal{L}(-,V) \arrow[r, tail, "\mu_{F,V}(\sigma_{V,a})"'] & F
\end{tikzcd}$$

yields the commutativity of the right hand square. This proves

$$(\mu_{F,V} \circ \eta_{F,V})(\alpha) = \mu_{F,V}(\sigma_{V,a}) = s_{V,a} = \alpha.$$

Conversely, let us start with $\beta\colon V \longrightarrow h^*(F)$ in $\mathcal{E}$. We use the notation above for defining $\mu_{F,V}(\beta)$. By definition of h^*, every time the left hand diagram below is commutative –

$$\begin{tikzcd}
\mathcal{L}(-,W) \arrow[r, tail] \arrow[d, tail] \arrow[dr, "s_{W,x|W}"] & \mathcal{L}(-,U) \arrow[d, tail, "s_{U,x}"] \\
\mathcal{L}(-,V) \arrow[r, tail, "\mu_{F,V}(\beta)"'] & F
\end{tikzcd}
\qquad
\begin{tikzcd}
W \arrow[r, tail] \arrow[d, tail] \arrow[dr, "\sigma_{W,x|W}"] & U \arrow[d, tail, "\sigma_{U,x}"] \\
V \arrow[r, tail, "(h^* \circ \mu_{F,V})(\beta)"'] & h^*(F)
\end{tikzcd}$$

– the right hand diagram is commutative as well. But among those W we have all the $W_{U,x}$, by definition of $\mu_{F,V}$. Thus β and $(h^* \circ \mu_{F,V})(\beta)$ coincide on all $W_{U,x}$ whose union is V. Therefore $\beta = (h^* \circ \mu_{F,V})(\beta)$, which concludes the proof that $\eta_{F,V}$ and $\mu_{F,V}$ are inverses to each other.

Let us consider now the counit of the adjunction $h^* \dashv h_*$. Given $E \in \mathcal{E}$, $(h^* \circ h_*)(E)$ is obtained by considering the diagram of all

$$\mathcal{L}(-,U) \rightarrowtail \mathcal{E}(-,E), \qquad \text{with } U \rightarrowtail \mathbf{1}.$$

By the Yoneda lemma, this reduces to considering all the elements of $\mathcal{E}(U,E)$ for all possible U, that is, all the atomic subobjects of E. The object $(h^* \circ h_*)(E)$ is then the colimit of all these atomic subobjects of $\mathbf{1}$. Since this family of atomic subobjects is trivially downward directed, the colimit is in fact a union. Thus $(h^* \circ h_*)(E)$ is the union of all atomic subobjects of E, which is of course a subobject of E. This proves that the counit of the adjunction is a monomorphism.

Identifying $\mathsf{Sh}(L)$ via h^* with a full subcategory of $\mathcal{E}$, an object $E \in \mathcal{E}$ is in $\mathsf{Sh}(L)$ precisely when at E, the counit of the adjunction is an isomorphism, that is, when E is the union of its atomic subobjects.

Now if $E = \bigcup_{i \in I} U_i \in \mathcal{E}$ expresses E as union of its atomic subobjects, for every subobject $S \rightarrowtail E$ in $\mathcal{E}$, one gets

$$S = S \cap E = S \cap \left(\bigcup_{i \in I} U_i \right) = \bigcup_{i \in I} S \cap U_i$$

in the locale of subobjects of E (see 7.7.6(vi)). But since each U_i is atomic, the same holds for $S \cap U_i$ and thus S is a union of atomic subobjects, thus a fortiori the union of all its atomic subobjects. This proves that $\mathsf{Sh}(L)$ is saturated in $\mathcal{E}$ on subobjects.

It remains to prove that h^* preserves finite limits, that is, that $\mathsf{Sh}(L)$, viewed as a full subcategory of $\mathcal{E}$, is closed under finite limits. We have already seen that h^* is an isomorphism at the level of subobjects of $\mathbf{1}$, thus the terminal object of $\mathcal{E}$ is also that of $\mathsf{Sh}(L)$. Next if $\alpha, \beta \colon F \rightrightarrows G$ are two morphisms in $\mathsf{Sh}(L)$, their equalizer $k \colon K \rightarrowtail A$ in $\mathcal{E}$ is in $\mathsf{Sh}(L)$, since this subcategory is saturated on subobjects.

Finally we must consider the case of binary products. First each functor $(- \times X)$ preserves colimits and therefore unions of downward directed families (see 7.7.6). Next, if U, V are subobjects of $\mathbf{1}$, then $U \times V$ is still a subobject of $\mathbf{1} \times \mathbf{1} = \mathbf{1}$. Now choose $E = \bigcup_{i \in I} U_i$ and $F = \bigcup_{j \in J} V_j$ two objects of $\mathsf{Sh}(L)$, expressed in $\mathcal{E}$ as unions of their atomic subobjects. One has at once

$$E \times F = \left(\bigcup_{i \in I} U_i\right) \times \left(\bigcup_{j \in J} V_j\right) = \bigcup_{i \in I,\ j \in J} (U_i \times V_j),$$

proving that $E \times F$ is a union of atomic subobjects, thus the union of all its atomic subobjects. □

Corollary 7.8.7 *Let $\mathcal{E}$ be a Grothendieck topos and $\sigma(\mathcal{E})$ its locale of subobjects of* $\mathbf{1}$. *For every locale M, there is an equivalence of categories*

$$\mathsf{Geom}\Big(\mathcal{E}, \mathsf{Sh}(M)\Big) \approx \mathsf{Geom}\Big(\mathsf{Sh}\big(\sigma(\mathcal{E})\big), \mathsf{Sh}(M)\Big).$$

Proof Given the Grothendieck topos $\mathcal{E}$, theorem 7.8.5 yields a geometric morphism $h\colon \mathcal{E} \longrightarrow \mathsf{Sh}\big(\sigma(\mathcal{E})\big)$. For simplicity of notation, let us keep writing L for the locale $\sigma(\mathcal{E})$.

If $f\colon \mathcal{E} \longrightarrow \mathsf{Sh}(M)$ is a geometric morphism, then both f^* and f_* map a subobject of $\mathbf{1}$ onto a subobject on $\mathbf{1}$: f^* by definition of a geometric morphism and f_* because it has a left adjoint f^*. Therefore f induces a geometric morphism of locales which for simplicity we write

$$\varphi = (\varphi_*, \varphi^*)\colon L = \sigma(\mathcal{E}) \longrightarrow M,$$

where φ_* and φ^* are thus the restrictions of f_* and f^*. By lemma 7.8.2, this defines a geometric morphism $\mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$, up to isomorphism.

Conversely, every geometric morphism $g\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$ yields, by composition with h, a geometric morphism $g \circ h\colon \mathcal{E} \longrightarrow \mathsf{Sh}(M)$.

Starting from $g\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$ above, restricting it to the subobjects of $\mathbf{1}$ and re-extending this restriction by colimits to the toposes of sheaves obviously yields a geometric morphism isomorphic to g.

Conversely, starting from $f\colon \mathcal{E} \longrightarrow \mathsf{Sh}(M)$ and the corresponding geometric morphism φ as above, we must prove that the triangle below is commutative up to an isomorphism:

$$\begin{tikzcd}
\sigma(\mathcal{E}) \arrow[d, "\varphi"'] & & \mathcal{E} \arrow[r, "h"] \arrow[dr, "f"'] & \mathsf{Sh}\big(\sigma(\mathcal{E})\big) \arrow[d, "\mathsf{Sh}(\varphi)"] \\
\sigma(\mathcal{E}) & & & \mathsf{Sh}(M)
\end{tikzcd}$$

By the definitions of $\mathsf{Sh}(\varphi)^*$, given $w \in M$,

$$\begin{aligned}\mathsf{Sh}(\varphi)^*\big(\mathcal{M}(-,w)\big) &= \operatorname*{colim}_{v\in M,\ y\in\mathcal{M}(v,w)} \mathcal{L}\big(-,f^*(v)\big)\\ &= \operatorname*{colim}_{v\in M,\ v\leq w} \mathcal{L}\big(-,f^*(v)\big)\\ &= \mathcal{L}\big(-,f^*(w)\big)\end{aligned}$$

since $\mathcal{L}\big(-,f^*(w)\big)$ is the terminal object of the diagram defining the last colimit. Moreover

$$h^*\Big(\mathcal{L}\big(-,f^*(w)\big)\Big) = f^*(w).$$

This proves that $h^* \circ \varphi^*$ and f^* coincide on the subobjects of $\mathbf{1}$. Since every sheaf on M is a colimit of subobjects of $\mathbf{1}$ and all functors h^*, φ^*, f^* preserve colimits (they have right adjoints), the isomorphism $h^* \circ \varphi^* \cong f^*$ holds. The isomorphism $\varphi_* \circ h_* \cong f_*$ follows at once by adjunction.

It is now routine to extend these arguments to the case of natural transformations to get the expected equivalence. □

7.9 Two dimensional category theory

This section formalizes in particular the notions of pseudo-functor and two dimensional limit used previously in this chapter. The two dimensional limits of the previous sections were always bilimits in the sense of the present section, and often even pseudo-limits.

In this section we allow a category to have a class of objects and, between two objects, a class of morphisms.

The theorem of Joyal and Tierney presented in the following section requires a slight generalization of our Galois theory of section 7.5, due to the fact that the category Groth of Grothendieck toposes does not admit kernel pairs, but "bikernel pairs". The slogan for defining a bilimit could be

Replace all equalitites by isomorphisms.

More precisely, one has to work with categories in which it makes sense to speak of isomorphic arrows: the prototype is CAT, the category of categories and functors, where natural transformations can be defined between functors, allowing us to speak of isomorphic functors.

Now the generalization is not just a triviality. Indeed, for example, a diagram entirely constituted of identities is of course commutative.

But a diagram entirely constituted of isomorphisms has no reason to be commutative. Thus replacing identities by isomorphisms requires at the same time introducing compatibility axioms between the various isomorphisms which enter the problem. This is the essence of two dimensional category theory, whose details can be found in [59] or [8], volume 1. We just sketch here its essential ingredients.

Bearing in mind the example of CAT, we state the following definition; in the example of CAT, the objects are the categories and given two categories $\mathcal{A}$, $\mathcal{B}$, the category $\mathsf{CAT}(\mathcal{A}, \mathcal{B})$ is that of functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them.

Definition 7.9.1 A 2-category $\mathcal{C}$ consists of

- a class of objects,
- for each pair (A, B) of objects, a category $\mathcal{C}(A, B)$,
- for each triple (A, B, C) of objects, a composition functor

$$\Gamma_{A,B,C} \colon \mathcal{C}(A, B) \times \mathcal{C}(B, C) \longrightarrow \mathcal{C}(A, C).$$

The following terminology and notation are classical:

- the objects of $\mathcal{A}(A, B)$ are called the arrows of $\mathcal{C}$ and are written as $f \colon A \longrightarrow B$;
- the arrows of $\mathcal{A}(A, B)$ are called the 2-cells of $\mathcal{C}$ and are written as $\alpha \colon f \Rightarrow g$
- the composition of 2-cells in the categories $\mathcal{C}(A, B)$ is called the vertical composition and written as $\alpha \circ \beta$ or just $\alpha\beta$;
- for the vertical composition, the identity 2-cell on an arrow f is written as 1_f;
- the composition of arrows given by the functors $\Gamma_{A,B,C}$ is written as $g \circ f$ or just gf –

$$\Gamma_{A,B,C}(f, g) = g \circ f;$$

- the composition of 2-cells via the functors $\Gamma_{A,B,C}$ is called the horizontal composition and as written $\gamma \star \delta$.

The axioms for a 2-category are just the usual associativity and identity axioms on the functors $\Gamma_{A,B,C}$, namely

(i) given four objects (A, B, C, D), the following square commutes –

$$\begin{tikzcd}
\mathcal{C}(A,B)\times\mathcal{C}(B,C)\times\mathcal{C}(C,D) \arrow[r, "\Gamma_{A,B,C}\times\mathsf{id}"] \arrow[d, "\mathsf{id}\times\Gamma_{B,C,D}"'] & \mathcal{C}(A,C)\times\mathcal{C}(C,D) \arrow[d, "\Gamma_{A,C,D}"] \\
\mathcal{C}(A,B)\times\mathcal{C}(B,D) \arrow[r, "\Gamma_{A,B,D}"'] & \mathcal{C}(A,D)
\end{tikzcd}$$

(ii) for each object $A\in\mathcal{C}$, there exists an arrow $\mathsf{id}_A\colon A\longrightarrow A$ which is an identity for the composition of arrows, while 1_{id_A} is an identity for the horizontal composition of 2-cells.

The examples of 2-categories we use in this book are just

(i) the 2-category CAT of categories, functors and natural transformations,
(ii) given a category $\mathcal{A}$, the 2-category $[\mathcal{A}^{\mathsf{op}},\mathsf{CAT}]$ of functors, natural transformations and modifications,
(iii) the 2-categories Groth of Grothendieck toposes, LOC of localic toposes and Loc of locales, with the geometric morphisms as arrows and the natural transformations $\alpha\colon f^*\Rightarrow g^*$ as 2-cells,
(iv) the ordinary categories $\mathcal{C}$, viewed as 2-categories in which each category $\mathcal{C}(A,B)$ is the discrete category on the set $\mathcal{C}(A,B)$ of arrows.

There are corresponding obvious notions of 2-functor, 2-natural-transformation and 2-modification, but we shall not need them explicitly. What we need in fact is the possibility, due to the presence of 2-cells, of weakening the notions of functor and limit by replacing equalities by isomorphic 2-cells. Again we do not need this in full generality, so that we introduce these notions here only in the special case where they will be useful in this book.

Definition 7.9.2 Let $\mathcal{P}$ be an ordinary category and $\mathcal{C}$ a 2-category. A pseudo-functor $F\colon\mathcal{P}\longrightarrow\mathcal{C}$ consists in giving

- for each object $P\in\mathcal{P}$, an object $F(P)\in\mathcal{C}$,
- for each arrow $f\colon A\longrightarrow B$ in $\mathcal{P}$, an arrow $F(f)\colon F(A)\longrightarrow F(B)$ in $\mathcal{P}$,
- for each pair $A\xrightarrow{f}B\xrightarrow{g}C$ of composable arrows in $\mathcal{P}$, an isomorphic 2-cell $\gamma_{g,f}\colon F(g)\circ F(f)\Rightarrow F(g\circ f)$;
- for each object $A\in\mathcal{P}$ an isomorphic 2-cell $\iota_A\colon F(\mathsf{id}_A)\Rightarrow\mathsf{id}_{F(A)}$.

These data must satisfy the expected coherence axioms respectively to composition and identities, namely

(i) given composable arrows $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in $\mathcal{P}$,

$$\gamma_{h,gf} \circ \left(1_{F(h)} \star \gamma_{g,f}\right) = \gamma_{hg,f} \circ \left(\gamma_{h,g} \star 1_{F(f)}\right),$$

(ii) given arrows $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{P}$,

$$\iota_B \star 1_{F(f)} = \gamma_{\mathsf{id}_B, f}, \quad 1_{F(g)} \star \iota_B = \gamma_{g, \mathsf{id}_B}.$$

When all the 2-cells $\gamma_{g,f}$ and ι_A are identities, we thus recapture exactly the ordinary notion of functor.

Of course, this notion of pseudo-functor could be stated in a more general context, namely, when $\mathcal{P}$ is itself a 2-category.

It is now a little bit fastidious, but in any case straightforward, to write down the coherence axioms for a pseudo-natural-transformation between pseudo-functors. Ignoring the technical axioms in the two following definitions will certainly not prevent the reader from understanding the use we make of these notions.

Definition 7.9.3 Consider two pseudo-functors $F, F' \colon \mathcal{P} \rightrightarrows \mathcal{C}$ from a category $\mathcal{P}$ to a 2-category $\mathcal{C}$. A pseudo-natural-transformation $\alpha \colon F \Rightarrow F'$ consists in giving

- for each object $A \in \mathcal{P}$, an arrow $\alpha_A \colon F(A) \longrightarrow F'(A)$ in $\mathcal{C}$,
- for each arrow $f \colon A \longrightarrow B$ in $\mathcal{P}$, an isomorphic 2-cell $\tilde{\alpha}_f \colon \alpha_B \circ F(f) \Rightarrow F'(f) \circ \alpha_A$.

These data must satisfy the obvious coherent axioms, namely, given arrows $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ in $\mathcal{P}$:

(i) $\left(\gamma'_{g,f} \star 1_{\alpha_A}\right) \circ \left(1_{F'(g)} \star \tilde{\alpha}_f\right) \circ \left(\tilde{\alpha}_g \star 1_{F(f)}\right) = \tilde{\alpha}_{g \circ f} \circ \left(1_{\alpha_C} \star \gamma_{g,f}\right)$;
(ii) $\left(\iota'_A \star 1_{\alpha_A}\right) \circ \tilde{\alpha}_{\mathsf{id}_A} = 1_{\alpha_A} \star \iota_A$.

Finally, to be complete, it remains to write down the axioms for a modification.

Definition 7.9.4 Consider two pseudo-functors $F, F' \colon \mathcal{P} \rightrightarrows \mathcal{C}$ from a category $\mathcal{P}$ to a 2-category $\mathcal{C}$ and two pseudo-natural-transformations $\alpha, \alpha' \colon F \Rightarrow F'$. A modification $\theta \colon \alpha \rightsquigarrow \alpha'$ consists in giving

- for each object $a \in \mathcal{P}$, a 2-cell $\theta_A \colon \alpha_A \Rightarrow \alpha'_A$.

These data must satisfy the obvious coherence axiom, namely, given an arrow $f \colon A \longrightarrow B$ in $\mathcal{P}$

(i) $\tilde{\alpha'}_f \circ \big(\theta_B \star 1_{F(f)}\big) = \big(1_{F'(f)} \star \theta_A\big) \circ \tilde{\alpha}_f$.

To extend the analogy with the case of ordinary categories, observe that every object $A \in \mathcal{C}$ of a 2-category determines a contravariant functor

$$\mathcal{C}(-,A)\colon \mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{CAT}, \quad B \mapsto \mathcal{C}(B,A)$$

yielding the corresponding Yoneda embedding

$$Y_{\mathcal{C}}\colon \mathcal{C} \longrightarrow [\mathcal{C}^{\mathsf{op}}, \mathsf{CAT}], \quad A \mapsto \mathcal{C}(-,A),$$

which is full and faithful in the sense that there exists an isomorphism of categories

$$\mathcal{C}(A,B) \xrightarrow{\cong} \text{2-}\mathsf{Nat}\Big(\mathcal{C}(-,A), \mathcal{C}(-,B)\Big)$$

between the category $\mathcal{C}(A,B)$ and the category of 2-natural-transformations and modifications between $\mathcal{C}(-,A)$ and $\mathcal{C}(-,B)$. This should be put in parallel with the considerations in corollary 7.2.5.

The notion of bilimit transposes in the same spirit the ordinary notion of limit. Given an ordinary functor $F\colon \mathcal{P} \longrightarrow \mathcal{C}$ between ordinary categories, one writes $\Delta_C = \Delta(C)\colon \mathcal{P} \longrightarrow \mathcal{C}$ for the constant functor corresponding to object C. A cone with vertex C over F is exactly a natural transformation $\sigma\colon \Delta_C \Rightarrow F$. A limit natural transformation is then a cone $\pi\colon \Delta_L \Rightarrow F$ such that, for every object $C \in \mathcal{C}$, the composition with π

$$\mathcal{C}(C,L) \longrightarrow \mathsf{Nat}(\Delta_C, F), \quad h \mapsto \pi \circ \Delta_h$$

is a bijection, with Δ_h the constant natural transformation corresponding to h.

Definition 7.9.5 Consider a pseudo-functor $F\colon \mathcal{P} \longrightarrow \mathcal{C}$ from a category $\mathcal{P}$ to a 2-category $\mathcal{C}$. The bilimit of F, when it exists, is a pair (L,π) where L is an object of $\mathcal{C}$ and $\pi\colon \Delta_L \Rightarrow F$ is a pseudo-natural-transformation such that composition with π

$$\mathcal{C}(C,L) \longrightarrow \mathsf{PsNat}(\Delta_C, F), \quad h \mapsto \pi \circ \Delta_h$$

induces an equivalence between the category $\mathcal{C}(C,L)$ and the category of pseudo-natural-transformations and modifications from Δ_C to F.

In particular, since everything is now "pseudo", all the triangles which normally commute in an ordinary limit are replaced by triangles which commute "up to an isomorphic 2-cell". Moreover, the definition requires an equivalence of categories, not an isomorphism: this implies again that the factorization through the bilimit is no longer unique, but unique up to an isomorphic 2-cell. Of course, this last fact implies that the bilimit, when it exists, is no longer defined up to an isomorphism, but only up to an equivalence. That is, two bilimits L and L' are connected by arrows $L \rightleftarrows L'$ such that both composites are isomorphic to the identity.

Just for information, let us mention that when one requires an isomorphism of categories in the previous definition, one recaptures the notion of pseudo-limit.

It is a classical result that CAT admits all small bilimits. But we shall just need the following result.

Proposition 7.9.6 *The 2-category* CAT *of categories, functors and natural transformations admits bipullbacks, thus in particular bikernel pairs.*

Proof The bipullback of two functors F, G is easily seen to be given by

$$\begin{tikzcd} \mathcal{P} \arrow[r, "F'"] \arrow[d, "G'"'] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{C} \end{tikzcd}$$

where the objects of $\mathcal{P}$ are the triples

$$(A, \alpha, C, \beta, B)) \text{ with } \alpha\colon F(A) \xrightarrow{\cong} C, \quad \beta\colon G(B) \xrightarrow{\cong} C$$

where thus $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$ and α, β isomorphisms in $\mathcal{C}$. □

The following proposition is a deep result in topos theory. We refer to [55] for a proof.

Proposition 7.9.7 *The 2-category* Groth *of Grothendieck toposes and geometric morphisms, with 2-cells as above, has bipullbacks, thus in particular bikernel pairs.* □

Corollary 7.9.8 *The 2-category* LOC *of localic toposes and geometric morphisms is stable for finite bilimits in the 2-category* Groth *of Grothendieck toposes and geometric morphisms.*

$$\begin{tikzcd}
\mathcal{P} \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{E} \arrow[dd, "f"] \\
& \mathcal{E}\times\mathcal{F} \arrow[ur, "p_{\mathcal{E}}"] \arrow[dl, "p_{\mathcal{F}}"'] & \\
\mathcal{F} \arrow[rr, "g"'] & & \mathcal{G}
\end{tikzcd}$$

Diagram 7.18

Proof Corollary 7.8.7 can be rephrased as the fact that the 2-category LOC of localic toposes is "bireflective" in the 2-category Groth of Grothendieck toposes. Extending a classical argument for ordinary limits, this at once implies the result. □

Applying lemma 7.8.2, it is easy to observe that a finite bilimit of localic toposes is obtained as the topos of sheaves on the limit of the corresponding diagram of locales. The form of limits in the category of locales (see proposition 7.8.3) indicates at once that a bilimit of localic toposes, and thus a bilimit of Grothendieck toposes, is not at all computed like the usual limit of underlying categories.

The next result extends a little bit the stability property of corollary 7.9.8.

Corollary 7.9.9 *In the category* Groth *of Grothendieck toposes and geometric morphisms, the bipullback of two localic toposes over a Grothendieck topos is localic as well.*

Proof We refer to diagram 7.18, where the toposes $\mathcal{E}$ and $\mathcal{F}$ are localic; we must prove that the topos $\mathcal{P}$ is localic as well. The bipullback can be reconstructed via the biproduct $\mathcal{E}\times\mathcal{F}$ and the biequalizer $\mathsf{Ker}\,(f\circ p_{\mathcal{E}}, g\circ p_{\mathcal{F}})$. By corollary 7.9.8, $\mathcal{E}\times\mathcal{F}$ is localic. Applying again theorem 7.8.5 and its corollary 7.8.7, we get the diagram

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "k"] \arrow[dr, "h"'] & \mathcal{E}\times\mathcal{F} \arrow[r, shift left, "f\circ p_{\mathcal{E}}"] \arrow[r, shift right, "g\circ p_{\mathcal{F}}"'] & \mathcal{G} \\
& \mathsf{Sh}(\sigma(\mathcal{P})) \arrow[u, "q"'] &
\end{tikzcd}$$

which commutes up to an isomorphism. So k is a "biequalizer" in Groth, which has bipullbacks. A classical argument for equalizers extends to prove that since k is a biequalizer, h is a biequalizer as well. But since the inverse image of h is full and faithful, it follows at once that h is a "biepimorphism" in Groth. Thus h is both an epimorphism and a biequalizer, and again a classical argument extends to prove that h is an equivalence. Thus $\mathcal{P}$ is localic. $\square$

It remains to apply and generalize all this to the case of precategories, introduced in definition 7.2.1.

For this, consider again the category $\mathbb{P}$ defined in section 7.2:

$$\begin{tikzcd} P_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & P_1 & P_0 \arrow[l, "n"'] \end{tikzcd} \quad P_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} P_0$$

with the conditions

$$d_0 \circ f_1 = d_1 \circ f_0, \qquad d_1 \circ m = d_1 \circ f_1, \qquad d_0 \circ m = d_0 \circ f_0,$$
$$d_0 \circ n = \mathsf{id}_{C_0}, \qquad d_1 \circ n = \mathsf{id}_{C_0}.$$

Definition 7.9.10 An internal pseudo-precategory in a 2-category $\mathcal{C}$ is a pseudo-functor $\mathbb{C}\colon \mathbb{P} \longrightarrow \mathcal{C}$.

Definition 7.9.11 Let us consider

- a 2-category $\mathcal{C}$,
- an internal pseudo-precategory $\mathbb{C}\colon \mathbb{P} \longrightarrow \mathcal{C}$,
- a pseudo-functor $F\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$.

The category $F^{\mathbb{C}}$ of covariant internal F-presheaves on $\mathcal{C}$ is the bilimit of the composite $F \circ \mathbb{C}$.

All the results of sections 7.1 to 7.5 admit expected straightforward translations to the present context of 2-categories. We shall freely use those results without further proofs.

7.10 The Joyal–Tierney theorem

Again we restrict our attention to the case of Grothendieck toposes and sketch the various proofs, referring to the works in the bibliography for a detailed account.

Let us first state the Galois theorem of Joyal and Tierney in the special case of Grothendieck toposes. This section is entirely devoted to "proving" this theorem, or at least to showing how it enters the context of our Galois theory of section 7.5.

Theorem 7.10.1 (Galois theorem) *For every Grothendieck topos $\mathcal{E}$, there exists an open localic groupoid $\mathbb{G}$ such that $\mathcal{E}$ is equivalent to the category of étale presheaves on $\mathbb{G}$.*

First of all, we have to explain the terminology used in the statement of this Galois theorem.

Definition 7.10.2 A geometric morphism $f\colon L \longrightarrow M$ of locales is open when

(i) f^* admits a left adjoint $f_!$,
(ii) $\forall u \in L\ \forall v \in M\quad f_!\big(u \wedge f^*(v)\big) = f_!(u) \wedge v.$

Condition (ii) is called the Frœbenius condition.

Example 7.10.3 Consider a continuous map $f\colon X \longrightarrow Y$ of topological spaces. Write $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ for the corresponding locales of open subsets. One gets at once a geometric morphism

$$(f^*, f_*)\colon \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$$

by defining, for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$,

$$\begin{aligned} f^*(V) &= f^{-1}(V), \\ f_*(U) &= \bigcup\big\{V \big| f^{-1}(V) \subseteq U\big\}. \end{aligned}$$

These definitions imply at once

$$f^*(V) \subseteq U \Leftrightarrow f^{-1}(V) \subseteq U \Leftrightarrow V \subseteq f_*(U).$$

Moreover the inverse image process preserves arbitrary unions and finite intersections of open subsets, since these are computed set theoretically. Now when f turns out to be an open map, that is, when f maps an open subset onto an open subset, we further define $f_!(U) = f(U)$ and obviously

$$f_!(U) \subseteq V \Leftrightarrow f(U) \subseteq V \Leftrightarrow U \subseteq f^{-1}(V).$$

This proves $f_! \dashv f^*$. Moreover

$$\begin{aligned} y \in f(U) \cap V &\Leftrightarrow \exists x \in U\ \ y = f(x) \in V \\ &\Leftrightarrow \exists x \in U\ \ x \in f^{-1}(V)\ \ y = f(x) \\ &\Leftrightarrow y \in f\big(U \cap f^{-1}(V)\big) \end{aligned}$$

which is the required Frœbenius condition. □

Definition 7.10.4 A geometric morphism $f\colon L \longrightarrow M$ of locales is étale when

- f is an open morphism,
- there exists a covering $1 = \bigvee_{i\in I} u_i$ of the top element $1 \in L$ such that for each index $i \in I$, the restriction

$$h_!\colon \{u \in L \mid u \le u_i\} \longrightarrow \{v \in M \mid v \le h_!(u_i)\}$$

is an isomorphism of locales.

Of course, in the context of example 7.10.3, a local homeomorphism between topological spaces, which is automatically open, yields a corresponding étale morphism of locales.

Definition 7.10.5

(i) A localic groupoid is a groupoid in the category of locales.
(ii) A localic groupoid is open when the "domain" and "codomain" operations $d_0, d_1\colon G_1 \rightrightarrows G_0$ are open.

It should be noticed that, in 7.10.5.(ii), the openness of d_0 is equivalent to the openness of d_1.

Definition 7.10.6 Let $\mathbb{G}$ be an open localic groupoid. With the notation of section 7.1, an internal presheaf (P_0, p_0, δ_1) on $\mathbb{G}$ is étale when the morphism $p_0\colon P_0 \longrightarrow G_0$ is étale.

This finishes the explanation of the terminology used in 7.10.1. Our intention is now to deduce this Galois theorem from the Galois theory of section 7.5, via the considerations of section 7.9. To achieve this, we shall freely use the following "localic covering" theorem, whose proof occupies a big part of the work of Joyal and Tierney (see [57]). Again we state the theorem only for Grothendieck toposes.

Theorem 7.10.7 (Localic covering) *For every Grothendieck topos $\mathcal{E}$, there exist a locale L and a geometric morphism $\lambda\colon \mathsf{Sh}(L) \longrightarrow \mathcal{E}$ which is open and of effective descent.* □

Of course we shall not prove this theorem here, but again we shall now explain the terminology used in this statement.

Definition 7.10.8 A geometric morphism $p\colon \mathcal{E} \longrightarrow \mathcal{F}$ between Grothendieck toposes is open when p^* commutes with the functors $\forall_f$ of lemma 7.7.8. More explicitly, given a morphism $f\colon A \longrightarrow B$ in $\mathcal{F}$ and a subobject $S \rightarrowtail A$, one has $p^*\big(\forall_f(S)\big) = \forall_{p^*(f)}\big(p^*(S)\big)$.

The relation with definition 7.10.2 is not clear a priori, but let us mention that a morphism of locales $f\colon L \longrightarrow M$ is open in the sense of definition 7.10.2 precisely when the corresponding geometric morphism $\mathsf{Sh}(f)\colon \mathsf{Sh}(L) \longrightarrow \mathsf{Sh}(M)$ is open in the sense of definition 7.10.8 (see [65]).

Definition 7.10.9 Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism between Grothendieck toposes. We consider the bikernel pair $\mathbb{G}_\varphi$ of φ in the 2-category Groth of Grothendieck toposes and view it as a pseudo-precategory, in the spirit of example 4.6.2,

$$\begin{tikzcd} (\mathcal{F} \times_{\mathcal{E}} \mathcal{F}) \times_{\mathcal{F}} (\mathcal{F} \times_{\mathcal{E}} \mathcal{F}) \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \mathcal{F} \arrow[l, "n"'] \end{tikzcd}$$

using the usual pullback notation to denote bipullbacks in Groth. A descent datum for φ is a pair (F, θ) where

(i) F is an object of $\mathcal{F}$,
(ii) $\theta\colon d_1^*(F) \longrightarrow d_0^*(F)$ is a morphism of $\mathcal{F} \times_{\mathcal{E}} \mathcal{F}$,
(iii) $n^*(\theta) \cong \mathsf{id}_F$,
(iv) $m^*(\theta) \cong f_0^*(\theta) \circ f_1^*(\theta)$,

where conditions (iii) and (iv) are convenient abbreviations for the commutativity of the diagrams 7.19 and 7.20, involving the isomorphisms appearing in the structure of pseudo-functors of n^*, m^*, f_0^* and f_1^*. A morphism of descent data $(F, \theta) \longrightarrow (F', \theta')$ is a morphism $f\colon F \longrightarrow F'$ in $\mathcal{F}$ which commutes with the descent data, that is, $\theta' \circ d_1^*(f) = d_0^*(f) \circ \theta$. We shall write $\mathsf{Desc}(\varphi)$ for this category of descent data.

Condition (iii) in 7.10.9 means the commutativity of diagram 7.19 while condition (iv) expresses the commutativity of the hexagon in diagram 7.2, which people familiar with classical descent theory will certainly recognize.

Definition 7.10.10 A geometric morphism $\varphi\colon \mathcal{F} \longrightarrow \mathcal{E}$ between Grothendieck toposes is of effective descent when the functor

$$\mathcal{E} \longrightarrow \mathsf{Desc}(\varphi), \quad E \mapsto \Big(\varphi^*(E), (d_1^* \circ \varphi^*)(E) \xrightarrow{\cong} (d_0^* \circ \varphi^*)(E)\Big)$$

$$\begin{tikzcd}
n^*d_1^*(F) \arrow[r, "n^*(\theta)"] \arrow[d, leftrightarrow, "\cong"'] & n^*d_0^*(F) \arrow[d, leftrightarrow, "\cong"] \\
F \arrow[r, equal] & F
\end{tikzcd}$$

Diagram 7.19

$$\begin{tikzcd}
& f_0^*d_1^*(F) \arrow[r, leftrightarrow, "\cong"] \arrow[dl, "f_0^*(\theta)"'] & f_1^*d_0^*(F) & \\
f_0^*d_0^*(F) \arrow[dr, leftrightarrow, "\cong"'] & & & f_1^*d_1^*(F) \arrow[ul, "f_1^*(\theta)"'] \arrow[dl, leftrightarrow, "\cong"] \\
& m^*d_0^*(F) & m^*d_1^*(F) \arrow[l, "m^*(\theta)"] &
\end{tikzcd}$$

Diagram 7.20

is an equivalence of categories.

Lemma 7.10.11 *Let $\varphi\colon \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes which is of effective descent. The category* $\mathsf{Desc}(\varphi)$ *of descent data is equivalent to the bilimit in* CAT *of the diagram*

$$\begin{tikzcd}
(\mathcal{F} \times_{\mathcal{E}} \mathcal{F}) \times_{\mathcal{F}} (\mathcal{F} \times_{\mathcal{E}} \mathcal{F}) & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \arrow[l, shift right=3, "f_0^*"'] \arrow[l, "m^*"] \arrow[l, shift left=3, "f_1^*"] \arrow[r, "n^*"] & \mathcal{F} \arrow[l, shift right=3, "d_0^*"'] \arrow[l, shift left=3, "d_1^*"]
\end{tikzcd}$$

where the pullback notation is used to denote bipullbacks in Groth.

Proof The equivalence in definition 7.10.10 shows that in the present case, for every descent datum (F, θ), the arrow θ is an isomorphism. The bilimit of the present statement has for objects those $F \in \mathcal{F}$ whose images along the various functors of the diagram constitute a compatible family up to compatible isomorphism. In particular we must have isomorphisms $\theta\colon d_1^*(F) \xrightarrow{\cong} d_0^*(F)$ and the various required compatibilities reduce precisely to the definition of a descent datum. □

We are now ready to exhibit the connection with the Galois theory of section 7.5. We shall apply corollary 7.5.3 – or more precisely, its straightforward generalization to the two dimensional context – to the following data, where we use freely the notation of 7.8.5 and 7.8.7:

- $\mathcal{A} = \mathsf{Groth}$ is the 2-category of Grothendieck toposes, geometric morphisms and natural transformations between them (see 7.8.1);
- $\mathcal{X} = \mathsf{Loc}$ is the 2-category of locales, geometric morphisms and natural transformations between them (see 7.8.1);
- $H\colon \mathcal{A} \longrightarrow \mathcal{X}$ is the functor $\sigma\colon \mathsf{Groth} \longrightarrow \mathsf{Loc}$, which maps a Grothendieck topos $\mathcal{F}$ onto its locale $\sigma(\mathcal{F})$ of subobjects of $\mathbf{1}$;
- $G\colon \mathcal{A}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$ is the forgetful functor

  $$\mathsf{Groth} \longrightarrow \mathsf{CAT}, \quad \mathcal{F} \mapsto \mathcal{F}, \quad f \mapsto f^*$$

  which is indeed contravariant;
- $K\colon \mathcal{X}^{\mathsf{op}} \longrightarrow \mathsf{CAT}$ is the functor

  $$\mathsf{Loc} \longrightarrow \mathsf{CAT}, \quad L \mapsto \mathsf{Sh}(L), \quad f \mapsto f^*$$

  which is also indeed contravariant;
- $\alpha\colon K \circ H \Rightarrow G$ is given, for each Grothendieck topos $\mathcal{E}$, by

  $$\alpha_{\mathcal{E}}\colon \mathsf{Sh}\big(\sigma(\mathcal{E})\big) \longrightarrow \mathcal{E}, \quad \alpha_{\mathcal{E}} = h^*$$

  where h is the hyperconnected geometric morphism of 7.8.5;
- fixing a Grothendieck topos $\mathcal{E}$, $\sigma\colon S \longrightarrow R$ is the localic covering $\lambda\colon \mathsf{Sh}(L) \longrightarrow \mathcal{E}$ of theorem 7.10.7;
- the required precategorical decomposition is replaced by a pseudo-precategorical decomposition; in the spirit of proposition 7.4.4 and via proposition 7.9.7, we use for this the bikernel pair $\mathbb{G}_\lambda$ of the localic covering $\lambda\colon \mathsf{Sh}(L) \longrightarrow \mathcal{E}$, viewed as a pseudo-precategory; thus this yields the situation of diagram 7.21;
- let us finally recall that, in view of theorem 7.10.7 and definition 7.10.10, we have an equivalence of categories $\mathcal{E} \approx \mathsf{Desc}(\lambda)$.

Lemma 7.10.12 *In the above situation, the two categories*

$$\mathsf{Split}_\alpha(\lambda) = \mathcal{E}$$

are equal, where $\mathsf{Split}_\alpha(\lambda)$ *is defined as after definition 7.5.1.*

$$
\begin{tikzcd}
& \mathsf{Sh}(L) \arrow[r, equal] \arrow[d, "i_2"'] & \mathsf{Sh}(L) \arrow[r, equal] \arrow[d, "i_1"'] & \mathsf{Sh}(L) \arrow[d, equal, "i_0"] \\
\big(\mathsf{Sh}(L)\times_{\mathcal{E}}\mathsf{Sh}(L)\big)\times_{\mathsf{Sh}(L)}\big(\mathsf{Sh}(L)\times_{\mathcal{E}}\mathsf{Sh}(L)\big) \arrow[rr, shift left=2, "f_0"] \arrow[rr, "m"] \arrow[rr, shift right=2, "f_1"'] \arrow[d, "\pi_2"'] & & \mathsf{Sh}(L)\times_{\mathcal{E}}\mathsf{Sh}(L) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[loop below, "\tau"] \arrow[d, "\pi_1"'] & \mathsf{Sh}(L) \arrow[l, "n"'] \arrow[d, "\pi_0"] \\
& \mathcal{E} \arrow[r, equal] & \mathcal{E} \arrow[r, equal] & \mathcal{E}
\end{tikzcd}
$$

Diagram 7.21

Proof $\mathsf{Split}_\alpha(\lambda)$ is the full subcategory of $G(\mathcal{E}) = \mathcal{E}$ generated by the objects $E \in \mathcal{E}$ such that there exists $F \in KH\big(\mathsf{Sh}(L)\big) \cong \mathsf{Sh}(L)$ with $\lambda^*(E) \cong G(\lambda)(E) \cong \alpha_{\mathsf{Sh}(L)}(F)$. But $\alpha_{\mathsf{Sh}(L)}$ is just the identity on $\mathsf{Sh}(L)$, thus every object of $F \in \mathsf{Sh}(L)$ has the form $\alpha_{\mathsf{Sh}(L)}(F)$. This holds in particular for the objects $\lambda^*(E)$, proving that all objects of $\mathcal{E}$ are in $\mathsf{Split}_\alpha(\lambda)$. □

Lemma 7.10.13 *In the above situation, the functor $\sigma\colon \mathsf{Groth} \longrightarrow \mathsf{Loc}$ transforms the pseudo-precategory $\mathbb{G}_\lambda$, bikernel pair of λ, into an open localic groupoid.*

Proof Applying corollary 7.9.9, we know that all toposes appearing in the definition of $\mathbb{G}_\lambda$ are localic. Lemma 7.8.2 then implies at once that $\sigma(\mathbb{G}_\lambda)$ is a "bigroupoid" in Loc. But since in a locale, only the identities are isomorphisms, two isomorphic geometric morphisms between locales must be equal. Therefore $\sigma(\mathbb{G}_\lambda)$ is an actual groupoid in Loc. □

Lemma 7.10.14 *Let $\mathbb{G}$ be an open localic groupoid. The category of étale internal presheaves on $\mathbb{G}$ is equivalent to the bilimit, in* CAT, *of the diagram*

$$
\begin{tikzcd}
\mathsf{Sh}(G_1 \times_{G_0} G_1) & \mathsf{Sh}(G_1) \arrow[l, shift right=2, "f_0^*"'] \arrow[l, "m^*"'] \arrow[l, shift left=2, "f_1^*"] \arrow[r, "n^*"] & \mathsf{Sh}(G_0), \arrow[l, shift right=2, "d_0^*"'] \arrow[l, shift left=2, "d_1^*"]
\end{tikzcd}
$$

$$\begin{tikzcd}
\mathsf{Loc}/G_1 \times_{G_0} G_1 & \mathsf{Loc}/G_1 \arrow[l, shift right=2, "f_0^*"'] \arrow[l, "m^*"] \arrow[l, shift left=2, "f_1^*"] \arrow[r, "n^*"] & \mathsf{Loc}/G_0 \arrow[l, shift right=2, "d_0^*"'] \arrow[l, shift left=2, "d_1^*"] \\
\mathsf{Et}/G_1 \times_{G_0} G_1 \arrow[u, tail] & \mathsf{Et}/G_1 \arrow[u, tail] \arrow[l, shift right=2, "f_0^*"'] \arrow[l, "m^*"] \arrow[l, shift left=2, "f_1^*"] \arrow[r, "n^*"] & \mathsf{Et}/G_0 \arrow[u, tail] \arrow[l, shift right=2, "d_0^*"'] \arrow[l, shift left=2, "d_1^*"]
\end{tikzcd}$$

Diagram 7.22

that is, with the previous notation and that of definition 7.2.6, to the category $K^{\mathbb{G}}$.

Proof Étale morphism are stable under pulling back, from which we get the commutative diagram 7.22, where Et denotes the category of locales and étale geometric morphisms between them; it is a classical result that for a given locale L, the subcategory $\mathsf{Et}/L \subseteq \mathsf{Loc}/L$ is full. The limit of the top line is the category of internal presheaves on $\mathbb{G}$, while the limit of the bottom line is the category of étale presheaves on $\mathbb{G}$. But this bottom line is, up to equivalences, precisely that of the statement: indeed, it is another classical result that the topos of sheaves on a locale L is equivalent to the category Et/L. This concludes the sketch of the proof. We refer to [8], volume 3, for the details. □

Lemma 7.10.15 *In the above situation,* $\big(\lambda, (i, \mathbb{G}_\lambda, \pi)\big)$ *is an effective descent structure with respect to the functor* G*, in the sense of definition 7.4.3.*

Proof By lemma 7.2.9, $G^{\mathbb{G}_\mathcal{E}} = G(\mathcal{E}) = \mathcal{E}$. On the other hand the category $G^{\mathbb{G}_\lambda}$ is the limit in CAT of the diagram (see 7.2.6)

$$\begin{tikzcd}
(\mathcal{F} \times_\mathcal{E} \mathcal{F}) \times_\mathcal{F} (\mathcal{F} \times_\mathcal{E} \mathcal{F}) & \mathcal{F} \times_\mathcal{E} \mathcal{F} \arrow[l, shift right=2, "f_0^*"'] \arrow[l, "m^*"] \arrow[l, shift left=2, "f_1^*"] \arrow[r, "n^*"] & \mathcal{F} \arrow[l, shift right=2, "d_0^*"'] \arrow[l, shift left=2, "d_1^*"]
\end{tikzcd}$$

and we have observed in lemma 7.10.11 that this limit is equivalent to the category $\mathsf{Desc}(\lambda)$ of descent data, which is itself equivalent to $\mathcal{E}$ (see definition 7.10.10). □

We are now able to conclude the proof of the expected Galois theorem 7.10.1.

$$\begin{tikzcd}
n^*d_1^*(P,p) \arrow[r, "n^*(\theta)"] \arrow[d, leftrightarrow, "\cong"'] & n^*d_0^*(P,p) \arrow[d, leftrightarrow, "\cong"] \\
(P,p) \arrow[r, equal] & (P,p)
\end{tikzcd}$$

Diagram 7.23

Proof of the Galois theorem 7.10.1 This is now an immediate consequence of corollary 7.5.3 and lemmas 7.10.12 to 7.10.15:

$$\mathcal{E} \cong \mathsf{Split}_\alpha(\lambda) \cong K^{H \circ \mathbb{G}_\lambda}$$

and this last category is equivalent to that of étale presheaves on $H(\mathbb{G}_\lambda)$, which is an open localic groupoid. □

In fact, to conclude that our theorem 7.10.1 yields exactly the theorem stated by Joyal and Tierney, one point remains to be checked. Indeed, Joyal and Tierney, instead of the usual definition of internal presheaf of section 7.1, use the following definition.

Definition 7.10.16 Consider a category $\mathcal{C}$ with pullbacks and an internal groupoid $\mathbb{G}$ in $\mathcal{C}$:

$$\begin{tikzcd}
G_2 \arrow[r, shift left=4, "f_0"] \arrow[r, "m"] \arrow[r, shift right=4, "f_1"'] & G_1 \arrow[loop below, "\tau"] \arrow[r, shift left=4, "d_0"] \arrow[r, shift right=4, "d_1"'] & G_0 \arrow[l, "n"']
\end{tikzcd}$$

A covariant internal presheaf on $\mathbb{G}$ in the sense of Joyal and Tierney is a triple (P,p,θ) where

(i) $(P,p) \in \mathcal{C}/G_0$,
(ii) $\theta\colon d_1^*(P,p) \longrightarrow d_0^*(P,p)$ in $\mathcal{C}/G_1$,
(iii) $n^*(\theta) \cong \mathsf{id}$,
(iv) $m^*(\theta) \cong f_0^*(\theta) \circ f_1^*(\theta)$,

where conditions (iii) and (iv) are convenient abbreviations for the commutativity of diagrams 7.23 and 7.24, involving the isomorphisms appearing in the structure of pseudo-functors of n^*, m^*, f_0^* and f_1^*.

One should compare this with definition 7.10.9. Axiom (iii) in 7.10.16 means the commutativity of diagram 7.23 while axiom (iv) again yields the classical hexagonal commutative diagram 7.24.

$$
\begin{tikzcd}
& f_0^* d_1^*(P,p) \arrow[r, leftrightarrow, "\cong"] \arrow[dl, "f_0^*(\theta)"'] & f_1^* d_0^*(P,p) & \\
f_0^* d_0^*(P,p) \arrow[dr, leftrightarrow, "\cong"'] & & & f_1^* d_1^*(P,p) \arrow[ul, "f_1^*(\theta)"'] \arrow[dl, leftrightarrow, "\cong"] \\
& m^* d_0^*(P,p) & m^* d_1^*(P,p) \arrow[l, "m^*(\theta)"] &
\end{tikzcd}
$$

Diagram 7.24

Lemma 7.10.17 *In the situation of the Galois theorem 7.10.1, consider the open localic groupoid $H(\mathbb{G}_\lambda)$ of lemma 7.10.13. On this groupoid, the category of étale internal presheaves in the sense of Joyal and Tierney is equivalent to the category of étale internal presheaves of definition 7.10.6.*

Proof The internal presheaf (P, p, θ) of Joyal and Tierney is étale when p is étale. In both cases, an étale presheaf yields an étale morphism of locales

$$p\colon P \longrightarrow \sigma\big(\mathsf{Sh}(L)\big) = L.$$

But the category of étale morphisms over L is equivalent to the category of sheaves on L. Thus giving the pair (P, p) with p étale reduces to giving a sheaf F on L. The structure of an internal presheaf on (P, p) in the sense of Joyal and Tierney reduces to that of a descent datum on the sheaf F, in the sense of definition 7.10.9. The category of étale presheaves in the sense of Joyal and Tierney is thus equivalent to the bilimit of lemma 7.10.11. Via lemma 7.10.13, this bilimit is also that of lemma 7.10.14, where the localic groupoid is chosen to be $H(\mathbb{G}_\lambda)$. This lemma 7.10.14 says precisely that this bilimit is a category equivalent to that of étale internal presheaves in the sense of definition 7.10.6. $\square$