The Bernstein-Gel′fand-Gel′fand complex and Kasparov theory for $\mathrm{SL}(3,\mathbb{C})$

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Abstract

In the case of the group $\mathrm{SL}(3,\mathbb{C})$, we describe how the Bernstein-Gel′fand-Gel′fand complex can be used to construct an element of Kasparov's equivariant K-homology. The resulting construction is a model for the γ-element. The key to the construction is the introduction of a lattice of operator ideals associated to the natural fibrations of the complete flag variety for $\mathrm{SL}(3,\mathbb{C})$.

1 Introduction

Kasparov's analytic K-homology is a blend of topology and analysis: a typical source for the construction of a K-homology cycle is an elliptic differential complex. If the elliptic complex is equivariant with respect to the action of a group G, and if moreover the group action satisfies some additional conformality property with respect to some Hermitian metric, then one can construct an element of equivariant K-homology (see [Kas84] for a classic example of this). Unfortunately, if G is a semisimple Lie group of rank greater than one, it seems probable that non-trivial examples of such complexes cannot exist. This paper describes a means of constructing an equivariant K-homology class from the Bernstein-Gel′fand-Gel′fand complex for $\mathrm{SL}(3,\mathbb{C})$, a differential complex which is neither elliptic nor conformal, but which satisfies some weaker ('directional') form of these conditions.

The motivation for this construction comes from the Baum-Connes conjecture. Although an understanding of the conjecture is not essential to this paper, it is useful for perspective. The Baum-Connes conjecture (with coefficients) states that the analytic assembly map of [BCH94],

$$\mu_{\mathsf{G},A} : K_j^{\mathsf{G}}(\underline{E}\mathsf{G}; A) \to K_j(A \rtimes_r \mathsf{G}).$$

is an isomorphism, for any coefficient G-C^*-algebra A. See [Hig98] for an overview of the conjecture and its many consequences. One of the major outstanding cases of the conjecture is the case of a simple Lie group of real-rank greater than one, such as the group $\mathsf{G} = \mathrm{SL}(3,\mathbb{C})$.

Let us recall the method of proof for the real-rank one simple groups. If G is a semisimple Lie group and K its maximal compact algebra, then the conjecture for G with coefficient algebra $A = C_0(\mathsf{G}/\mathsf{K})$ is straightforward to prove. On the other hand, the conjecture with trivial coefficients $A = \mathbb{C}$ implies the conjecture with any other coefficients. It follows that if the algebras $C_0(\mathsf{G}/\mathsf{K})$ and $\mathbb{C}$ are KK^{G}-equivalent then the Baum-Connes conjecture holds for all closed subgroups of G. Kasparov demonstrated the existence of elements $\alpha \in KK^{\mathsf{G}}(C_0(\mathsf{G}/\mathsf{K}), \mathbb{C})$ (the 'Dirac element') and $\beta \in KK^{\mathsf{G}}(\mathbb{C}, C_0(\mathsf{G}/\mathsf{K}))$ (the 'dual-Dirac element') for which $\alpha\beta = 1 \in KK^{\mathsf{G}}(C_0(\mathsf{G}/\mathsf{K}), C_0(\mathsf{G}/\mathsf{K}))$. The reverse composition, $\beta\alpha$ is a canonical idempotent in $KK^{\mathsf{G}}(\mathbb{C}, \mathbb{C})$, called γ_{G}, or just γ.

It has been proven that $\gamma_{\mathsf{G}} = 1$ for $\mathrm{SO}_0(n,1)$ in [Kas84], and for $\mathrm{SU}(n,1)$ in [JK95]. Importantly, though, the KK^{G}-cycles representing γ which enable these proofs are built using the compact homogeneous space G/B, where B is the maximal parabolic subgroup, rather than G/K. The following theorem summarizes Kasparov's method for identifying such a model of γ_{G}.

Theorem 1.1. *Let $\iota : \mathbb{C} \to C(\mathsf{G}/\mathsf{B})$ be the inclusion of the constant functions. Suppose $\theta \in KK^{\mathsf{G}}(C(\mathsf{G}/\mathsf{B}), \mathbb{C})$ is such that $\mathrm{Res}^{\mathsf{G}}_{\mathsf{K}}(\iota^*\theta) = 1 \in KK^{\mathsf{K}}(\mathbb{C}, \mathbb{C})$. Then $\iota^*\theta = \gamma_{\mathsf{G}}$.*

It was observed by N. Higson that, in each of the rank-one cases, the construction of such a θ was made by using some close variant of the Bernstein-Gel′fand-Gel′fand (BGG) complex for G. For complex semisimple G, the BGG complex is described as follows[1] .

Theorem 1.2. ([BGG75, Theorem 10.1])
There exists a complex consisting of (smooth section spaces of) direct sums of G-homogeneous line bundles over G/B, and G-equivariant differential operators between them, which resolves the trivial representation of G.

Although the action of G on this elliptic complex is not conformal, it is separately conformal on each line-bundle summand. In juxtaposition with Theorem 1.1, this suggests that the γ-element for a general complex semisimple group might be constructible from the BGG resolution. The purpose of this paper is to demonstrate that this is indeed possible for the group $\mathrm{SL}(3, \mathbb{C})$.

We note from the outset that it is known that $\gamma_{\mathsf{G}} \neq 1$ for any group G which has Kazhdan's property T. Therefore, a direct translation of Kasparov's method cannot prove the Baum-Connes conjecture for simple Lie groups of rank greater than one—some subtle variation of Kasparov's argument would be required. Nevertheless, it is hoped that a construction of this kind will be useful for further study of the Baum-Connes conjecture.

It is instructive to keep in mind the much easier case of the rank two semisimple group $\mathsf{G} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. Let us briefly sketch the construction of γ for this group. First, recall that the γ-element for $\mathrm{SL}(2, \mathbb{C})$ is obtained by taking the Dolbeault operator for $\mathbb{C}P^1$

$$L^2\Omega^{0,0}\mathbb{C}P^1 \xrightarrow{\overline{\partial}} L^2\Omega^{0,1}\mathbb{C}P^1,$$

which is precisely the BGG complex for $\mathrm{SL}(2, \mathbb{C})$, and replacing the operator $\overline{\partial}$ by its phase (in the sense of polar decomposition of unbounded operators). For

[1] Bernstein, Gel′fand and Gel′fand's formulation of this result is purely algebraic. See [ČSS01, Appendix A] for a discussion of the geometric interpretation.

$\mathsf{G} = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, the BGG resolution is again the Dolbeault complex for $\mathsf{G}/\mathsf{B} \cong \mathbb{C}P^1 \times \mathbb{C}P^1$, but decomposed into its 'directional' components as follows:

$$\begin{tikzcd}
 & L^2\Omega^{0,1}\mathbb{C}P^1 \otimes L^2\Omega^{0,0}\mathbb{C}P^1 \arrow[dr, "-1\otimes\overline{\partial}"] \arrow[d, dotted, no head] & \\
L^2\Omega^{0,0}\mathbb{C}P^1 \otimes L^2\Omega^{0,0}\mathbb{C}P^1 \arrow[ur, "\overline{\partial}\otimes 1"] \arrow[dr, "1\otimes\overline{\partial}"'] & \oplus \arrow[d, dotted, no head] & L^2\Omega^{0,1}\mathbb{C}P^1 \otimes L^2\Omega^{0,1}\mathbb{C}P^1 \\
 & L^2\Omega^{0,0}\mathbb{C}P^1 \otimes L^2\Omega^{0,1}\mathbb{C}P^1 \arrow[ur, "\overline{\partial}\otimes 1"'] &
\end{tikzcd}$$

If one replaces each of these four differential operators by its phase, one obtains the starting data for taking the Kasparov product of two copies of the gamma element of $\mathrm{SL}(2,\mathbb{C})$. One must now use an application of the Kasparov Technical Theorem to transform this into a genuine Fredholm module. Note that this final step involves in a crucial way the lattice of operator ideals

$$\begin{tikzcd}
 & \mathcal{B} \otimes \mathcal{B} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathcal{K} \otimes \mathcal{B} \arrow[dr, no head] & & \mathcal{B} \otimes \mathcal{K} \arrow[dl, no head] \\
 & \mathcal{K} \otimes \mathcal{K} &
\end{tikzcd}$$

where $\mathcal{B}$ and $\mathcal{K}$ are the algebras of bounded and compact operators, respectively, on $L^2\Omega^{0,\bullet}\mathbb{C}P^1$. We will not go into the details of this example further, but leave the reader to ponder the analogy with the structures introduced here for $\mathrm{SL}(3,\mathbb{C})$.

The structure of this paper is as follows. In Section 2 we set up notation and recall some basic facts concerning homogeneous line bundles over the flag manifold G/B for $\mathsf{G} = \mathrm{SL}(3,\mathbb{C})$. Section 3 contains the crucial definitions of a lattice of C^*-ideals associated to the fibrations of G/B. We describe some of the basic properties of these ideals. In Section 4 we study the bounded operators which will take the place of the differential operators of the BGG resolution[2]. In Section 5, we combine the results of the previous sections to give the construction of the γ-element for $\mathrm{SL}(3,\mathbb{C})$.

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2 Notation and Preliminaries

2.1 Lie groups

Throughout this paper G will denote the group $\mathrm{SL}(3,\mathbb{C})$, and we use the following notation for various subgroups: K is the maximal compact subgroup $\mathrm{SU}(3)$; B is the minimal parabolic subgroup of invertible upper triangular matrices; N and Z are the nilpotent subgroups of upper and lower triangular unipotent matrices, respectively; M is the group of diagonal matrices with entries of modulus

[2] In this paper, we do not use the BGG resolution itself for the construction, but build the corresponding KK^{G}-cycle directly using noncommutative harmonic analysis. The BGG resolution serves only as very strong guidance for the construction.

one; A is the group of diagonal matrices with positive real entries; $\mathsf{H} = \mathsf{MA}$ is the Cartan subgroup. The corresponding Lie algebras are $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$, $\mathfrak{n}$, $\mathfrak{z}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{h}$.

We set

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_\rho = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \in \mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g}.$$

We use X' to denote the conjugate transpose of an element $X \in \mathfrak{k}_{\mathbb{C}}$.

The dual of a complex vector space V will be denoted by $V^\dagger$. By extending the inclusions of $\mathfrak{m}$ and $\mathfrak{a}$ in $\mathfrak{h}$ to $\mathbb{C}$-linear identifications $\mathfrak{m}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}} \cong \mathfrak{h}$, we will identify infinitesimal characters of $\mathfrak{m}$ and $\mathfrak{a}$ with elements of $\mathfrak{h}^\dagger$. Characters of $\mathfrak{h}$ will be denoted by $\chi = \chi_{\mathsf{M}} \oplus \chi_{\mathsf{A}}$ where χ_{M} and χ_{A} are the restrictions of χ to M and A, respectively. Infinitesimal characters of $\mathfrak{b}$ will be identified with their restrictions to $\mathfrak{h}$.

The set of roots of K will be denoted by Δ. We let $\alpha_1, \alpha_2 \in \mathfrak{h}^\dagger$ be the weights of $X_1, X_2 \in \mathfrak{k}_{\mathbb{C}}$, respectively, and fix these as the simple roots. Also put $\rho = \alpha_1 + \alpha_2$. The weight lattice of K is denoted by Λ_W. We use δ_j $(j = 1, 2, 3)$ to denote the characters

$$\delta_j : \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \mapsto t_j$$

of M. Thus, δ_1 and $-\delta_3$ are the fundamental weights for K.

The Weyl group of G is denoted W.

2.2 Homogeneous bundles over the flag manifold

Throughout this paper, $\mathcal{X}$ will denote the homogeneous space $\mathcal{X} = \mathsf{G}/\mathsf{B} = \mathsf{K}/\mathsf{M}$.

For each weight μ of K, we let E_μ denote the G-homogeneous bundle over $\mathcal{X}$ induced from $\mu \oplus \rho$. Thus, sections of E_μ are identified with functions $s : \mathsf{G} \to \mathbb{C}$ which satisfy the B-equivariance property

$$s(xb) = e^{-(\mu \oplus \rho)}(b) s(x) \qquad \text{for all } x \in \mathsf{G},\ b \in \mathsf{B}. \tag{2.1}$$

Equivalently,

$$B_L s = -(\mu \oplus \rho)(B)\, s \qquad \text{for all } B \in \mathfrak{b}, \tag{2.2}$$

where B_L denotes the left-invariant differential operator on G determined by B. The space of continuous sections of E_μ will be denoted $C(\mathcal{X}; E_\mu)$. The group G acts on $C(\mathcal{X}; E_\mu)$ by pull-back:

$$(g \cdot s)(x) = s(g^{-1}x). \tag{2.3}$$

Thanks to the Iwasawa decomposition $\mathsf{G} = \mathsf{KAN}$,

$$\begin{aligned} C(\mathcal{X}; E_\mu) &\cong \{ s \in C(\mathsf{K}) \mid s(km) = e^{-\mu}(m) s(k) \quad \forall k \in \mathsf{K},\ m \in \mathsf{M} \} \\ &= \{ s \in C(\mathsf{K}) \mid M_L s = -\mu(M) s \quad \forall M \in \mathfrak{m} \}, \end{aligned}$$

The completion of $C(\mathcal{X}; E_\mu)$ with respect to the inner product

$$\langle s_1, s_2 \rangle = \int_{\mathsf{K}} \overline{s_1(k)} s_2(k) dk$$

will be denoted $L^2(\mathcal{X}; E_\mu)$. The pull-back action (2.3) of G on $C(\mathcal{X}; E_\mu)$ extends to a unitary representation of G on $L^2(\mathcal{X}; E_\mu)$, which we denote by U_μ.

The product of two sections $s \in C(\mathcal{X}; E_\mu)$ and $t \in C(\mathcal{X}; E_\nu)$ belongs to $C(\mathcal{X}; E_{\mu+\nu})$, and multiplication by s extends to a bounded linear map

$$s : L^2(\mathcal{X}; E_\nu) \to L^2(\mathcal{X}; E_{\mu+\nu}); \qquad t \mapsto st, \tag{2.4}$$

whose norm is the L^∞-norm of s.

We will also occasionally need to refer to the 'nilpotent' (or 'noncompact') picture of E_μ. Using the almost everywhere defined decomposition $\mathsf{G} = \mathsf{ZMAN}$, sections of E_μ are determined on an open dense subset of $\mathcal{X}$ by their restriction to Z. This restriction yields a trivializing chart for E_μ over $\mathsf{Z} \hookrightarrow \mathcal{X}$, and taking G-translates of this chart yields a trivializing atlas.

2.3 The Peter-Weyl transform

Let $\hat{\mathsf{K}}$ denote the set of (equivalence classes of) irreducible unitary representations of K. For each representation $\sigma \in \hat{\mathsf{K}}$, let V^σ denote its representation space, and $|\sigma|$ its dimension. Let $\sigma^\dagger$ be the contragredient representation, acting on $V^{\sigma\dagger}$. The pairing of $V^{\sigma\dagger}$ with V^σ will be denoted by $(\,\cdot\,,\,\cdot\,)$.

For any weight $\mu \in \Lambda_W$, the Peter-Weyl isomorphism

$$\begin{aligned} \bigoplus_{\sigma\in\hat{\mathsf{K}}} V^{\sigma\dagger} \otimes V^\sigma \quad &\to \quad L^2(\mathsf{K}) \\ \xi^\dagger \otimes \xi \quad &\mapsto \quad [\, k \mapsto |\sigma|^{\frac{1}{2}} (\xi^\dagger, \sigma(k)\xi) \,], \end{aligned} \tag{2.5}$$

restricts to an isomorphism

$$L^2(\mathcal{X}; E_{-\mu}) \cong \bigoplus_{\sigma\in\hat{\mathsf{K}}} V^{\sigma\dagger} \otimes (V^\sigma)_\mu,$$

where $(V^\sigma)_\mu$ is the μ-weight space of σ. We refer to this isomorphism as the Peter-Weyl transform. Under the Peter-Weyl transform, the multiplication operators (2.4) are described in terms of Clebsch-Gordan-type rules for tensor products of SU(3)-representations, while the group representation $U_{-\mu}$ of K becomes $\bigoplus_{\sigma\in\hat{\mathsf{K}}} (\sigma^\dagger \otimes 1)$. The full representation $U_{-\mu}(\mathsf{G})$ is harder to describe—see Section 3.3.

2.4 K-invariant differential operators

Any element T of the complexified universal enveloping algebra $\mathcal{U}(\mathfrak{k}_\mathbb{C})$ defines a left-invariant differential operator on $C^\infty(\mathsf{K})$. If T is a homogeneous element of weight β, then T maps smooth sections of $E_{-\mu}$ to sections of $E_{-(\mu+\beta)}$, for any $\mu \in \Lambda_W$. Thus T defines an unbounded operator between the corresponding L^2-section spaces. Here, let us use the space of K-finite vectors in $L^2(\mathcal{X}; E_{-\mu})$ as the initial domain of definition. Under the Peter-Weyl transform, T acts as $\bigoplus_{\sigma\in\hat{\mathsf{K}}} 1 \otimes \sigma(T)$.

Remark 2.1. It is important to note that while T maps M-equivariant functions of K to M-equivariant functions, it does not, in general, map B-equivariant functions of G to B-equivariant functions. Thus T is a well-defined operator on $L^2(\mathcal{X}; E_{mu})$ only if defined using the compact picture.

The conjugate transpose map $X \to X' \overset{\mathrm{def}}{=} \overline{X^t}$, for $(X \in \mathfrak{k}_{\mathbb{C}} \cong \mathfrak{g})$, extends to an algebra anti-automorphism of $\mathcal{U}(\mathfrak{k}_{\mathbb{C}})$. For any $T \in \mathcal{U}(\mathfrak{k}_{\mathbb{C}})$, the operators T and T' are formally adjoint, and the operators $T'T$ on $L^2(\mathcal{X}; E_{-\mu})$, and

$$\begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}$$

on $L^2(\mathcal{X}; E_{-\mu} \oplus E_{-(\mu+\beta)})$ are essentially self-adjoint. We use $|T|$ to denote the absolute value of T, and $\mathrm{Ph}(T)$ or T to denote the phase, as defined by the polar decomposition, $T = T|T|$.

3 C^*-algebras associated to the fibrations of $\mathcal{X}$

3.1 Spectral decompositions of $L^2(\mathcal{X}; E_\mu)$

Associated to the simple roots α_1, α_2 are the parabolic subgroups

$$\mathsf{P}_1 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}, \quad \mathsf{P}_2 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \subseteq \mathsf{G},$$

and the corresponding homogeneous spaces $\mathcal{Y}_i = \mathsf{G}/\mathsf{P}_i$ $(i = 1, 2)$. Let $\mathsf{M}_i = \mathsf{P}_i \cap \mathsf{K}$, so that also $\mathcal{Y}_i = \mathsf{K}/\mathsf{M}_i$. Note that $\mathsf{M}_i = \mathsf{K}_i \times \mathsf{K}'_i$, where

$$\mathsf{K}_1 = \begin{pmatrix} \mathrm{SU}(2) & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{K}'_1 = \left\{ \begin{pmatrix} \omega.I & & 0 \\ & & 0 \\ 0 & 0 & \omega^{-2} \end{pmatrix} \;\middle|\; |\omega| = 1 \right\},$$

and

$$\mathsf{K}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{SU}(2) & \\ 0 & & \end{pmatrix}, \quad \mathsf{K}'_2 = \left\{ \begin{pmatrix} \omega^{-2} & 0 & 0 \\ 0 & \omega.I & \\ 0 & & \end{pmatrix} \;\middle|\; |\omega| = 1 \right\}.$$

Suppose now that ϖ is a representation of K on a Hilbert space H. Fix $i = 1$ or 2. One may decompose H first into isotypical subspaces for the restriction of ϖ to M_i, and then further into isotypical components for the restriction to M. The isotypical subspaces of H for M are, of course, the weight spaces of H, so this double decomposition yields a decomposition of each weight space of H into M_i-types. In fact, for a fixed weight $\mu \in \Lambda_W$, the action of K'_i on the μ-weight space H_μ is fixed, so the decomposition of H_μ can be equivalently given in terms K_i-types. With $i = 1$ or 2, we use π_k throughout to denote the irreducible representation of $\mathsf{K}_i \cong \mathrm{SU}(2)$ with highest weight $k \in \mathbb{N}$.

Definition 3.1. We use $P_k^{(H_\mu, i)}$, or more concisely $P_k^{(i)}$, to denote the orthogonal projection of H_μ onto the component of K_i-type π_k.

For $A \subseteq \mathbb{N}$ we write $P_A^{(i)} = \sum_{k \in A} P_k^{(i)}$. In particular, if $k_1 \leq k_2$, we abbreviate $P^{(i)}_{\{k_1, \ldots, k_2\}}$ as $P^{(i)}_{[k_1, k_2]}$.

The domain H_μ will usually be implicitly assumed, and suppressed from the notation.

The most relevant example is $H = L^2(\mathsf{K})$ with ϖ being the right regular representation. Then the weight spaces are the spaces $L^2(\mathcal{X}; E_{-\mu})$, for which we get projections $P_k^{(i)} \in \mathcal{B}(L^2(\mathcal{X}; E_{-\mu}))$.

Remark 3.2. In this case, the projections $P_k^{(i)}$ are precisely the spectral projections of the tangential Hodge-Laplace operators $\Delta_i = X_i' X_i$ on $E_{-\mu}$, tangential along the fibration $\mathcal{X} \to \mathcal{Y}_i$. The analysis that follows can be understood as a study of the simultaneous spectral theory of these non-commuting differential operators.

A section $s \in L^2(\mathcal{X}; E_{-\mu})$ will be said to be of *right* K_i*-type* k if it is in the range of $P_k^{(i)}$. Equivalently, s has right K_i-type k if has a Peter-Weyl transform

$$\sum (\xi^* \otimes \xi) \in \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma \dagger} \otimes (V^\sigma)_\mu$$

where each ξ_m belongs to an irreducible K_i-subrepresentation of highest weight k.

Any K-homogeneous vector bundle over $\mathcal{X}$ admits a K-equivariant decomposition $E = \oplus_\mu E_\mu$ into line bundles. We thus define projections $P_k^{(i)}$ on $L^2(\mathcal{X}; E) = \bigoplus_\mu L^2(\mathcal{X}; E_\mu)$ to be the direct sum of the corresponding projections $P_k^{(i)}$ on each component.

A section of E_0 has K_i-type 0 if and only if, as a function on K, it is invariant under M_i. Thus, $P_0^{(i)} L^2(\mathcal{X}; E_0) = L^2(\mathcal{Y}_i)$. Since $\pi_k \otimes \pi_0 = \pi_k$, for any $k \in \mathbb{N}$, it follows that the spaces $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ are preserved by pointwise multiplication by continuous sections $f \in C(\mathcal{Y}_i)$.

Now let π be an irreducible representation of M_i on a vector space W^π, and let $F_{\pi^\dagger}$ be the K-homogenous bundle over $\mathcal{Y}_i$ induced from the contragredient representation $\pi^\dagger$. Thus

$$L^2(\mathcal{Y}_i; F_{\pi^\dagger}) = \{ f \in L^2(\mathsf{K}, W^{\pi\dagger}) \mid f(km_1) = \pi^\dagger(m_1)^{-1} f(k) \quad \forall k \in \mathsf{K}, m_1 \in \mathsf{M}_1 \}. \tag{3.1}$$

These spaces are, of course, also modules over $C(\mathcal{Y}_i)$. Their relationship with the modules $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ is as follows.

Lemma 3.3. *Let π be a representation of M_i whose restriction to K_i is π_k, and let $\mu \in \mathfrak{m}^\dagger$ be a weight of M_i appearing with nonzero multiplicity in π. Then $P_k^{(i)} L^2(\mathcal{X}; E_{-\mu})$ is $C(\mathcal{Y}_i)$-linearly isomorphic to $L^2(\mathcal{Y}_i; F_{\pi^\dagger})$.*

Proof. By the Peter-Weyl theorem,

$$L^2(\mathsf{K}, W^{\pi\dagger}) \cong \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma\dagger} \otimes V^\sigma \otimes W^{\pi\dagger} \cong \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma\dagger} \otimes \mathrm{End}(W^\pi, V^\sigma).$$

The M_1-equivariance condition in (3.1) implies that

$$L^2(\mathcal{Y}_i; F_\pi) \cong \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma\dagger} \otimes \mathrm{Hom}_{\mathsf{M}_i}(W^\pi, V^\sigma),$$

where $\mathrm{Hom}_{\mathsf{M}_i}$ denotes the space of intertwiners of M_i-representations.

Let $v \in W^\pi$ be of weight μ. The map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathsf{M}_i}(W^\pi, V^\sigma) & \to & (V^\sigma)_\mu \\ A & \mapsto & Av \end{array}$$

is an isomorphism, by the irreducibility of W^π. Therefore the map

$$\begin{array}{rcl} 1 \otimes 1 \otimes v : \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma\dagger} \otimes V^\sigma \otimes W^{\pi\dagger} & \to & \bigoplus_{\sigma \in \hat{\mathsf{K}}} V^{\sigma\dagger} \otimes V^\sigma \\ \xi^\dagger \otimes \eta \otimes w^\dagger & \mapsto & (w^\dagger, v)\xi^\dagger \otimes \eta, \end{array}$$

which is clearly $C(\mathcal{Y}_i)$-linear, restricts to an isomorphism

$$L^2(\mathcal{Y}_i; F_{\pi^\dagger}) \to P_k^{(i)} L^2(\mathcal{X}; E_{-\mu}).$$

□

This implies the following useful finite generation property.

Corollary 3.4. *Fix $\mu \in \Lambda_W$, $k \in \mathbb{N}$. There exists a finite collection of continuous sections $s_1, \ldots, s_n \in P_k^{(i)} C(\mathcal{X}; E_\mu)$ and bounded linear maps $\varphi_1, \ldots, \varphi_n : P_k^{(i)} L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{Y}_i)$ such that $\sum_{j=1}^n s_j \varphi_j(s) = s$ for all $s \in P_k^{(i)} L^2(\mathcal{X}; E_\mu)$.*

3.2 Spectrally proper and spectrally finite operators

Definition 3.5. Let $\alpha = \alpha_i$ be a simple root of K. Let $\mu, \nu \in \Lambda_W$, and let $T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ be a bounded operator. The *α-support* of T is defined as

$$\alpha\text{-}\operatorname{Supp} T = \{(k, k') \mid P_k^{(i)} T P_{k'}^{(i)} \neq 0\} \quad \subseteq \quad \mathbb{N} \times \mathbb{N}.$$

(In this, and other similar expressions, we use the domain and range of the operator T to specify the domains of the projections $P_k^{(i)}$ and $P_{k'}^{(i)}$.)

Recall that a subset S of $\mathbb{N} \times \mathbb{N}$ is *proper* if for every $k \in \mathbb{N}$, the sets

$$\{k' \in \mathbb{N} \mid (k, k') \in S\}$$

and

$$\{k' \in \mathbb{N} \mid (k', k) \in S\}$$

are finite.

Definition 3.6. A bounded operator $T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ is

- *spectrally proper for α* if $\alpha\text{-}\operatorname{Supp} T$ is a proper subset of $\mathbb{N} \times \mathbb{N}$.
- *spectrally finite for α* if $\alpha\text{-}\operatorname{Supp} T$ is finite.

Definition 3.7. With the above notation, set

$$\mathcal{A}_i(E_\mu, E_\nu) = \overline{\{T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu) \text{ spectrally proper for } \alpha_i\}}^{\|\cdot\|},$$

$$\mathcal{K}_i(E_\mu, E_\nu) = \overline{\{T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu) \text{ spectrally finite for } \alpha_i\}}^{\|\cdot\|}.$$

(Closures are taken in the operator norm.) Define also

$$\mathcal{A}(E_\mu, E_\nu) = \mathcal{A}_1(E_\mu, E_\nu) \cap \mathcal{A}_2(E_\mu, E_\nu),$$

and

$$\mathcal{K}(E_\mu, E_\nu) = \mathcal{K}_1(E_\mu, E_\nu) \cap \mathcal{K}_2(E_\mu, E_\nu).$$

In Proposition 3.17 we will see that $\mathcal{K}(E_\mu, E_\nu)$ is in fact the space of compact operators from $L^2(\mathcal{X}; E_\mu)$ to $L^2(\mathcal{X}; E_\nu)$, justifying the notation.

In the case $\mu = \nu$, the spectrally proper operators for α_i form an algebra, and the spectrally finite operators for α_i form an ideal in that algebra. Their norm-closures are therefore a C^*-algebra and a C^*-ideal, respectively. If μ and ν are allowed to vary, it is convenient to think of Definition 3.7 as defining C^*-categories $\mathcal{A}_i$, $\mathcal{K}_i$, $\mathcal{A}$ and $\mathcal{K}$. (For definitions and properties of C^*-categories, we refer the reader to [Mit02].) We similarly use the notation $\mathcal{B}(E_\mu, E_\nu)$ to denote the set of bounded linear operators between section spaces $L^2(\mathcal{X}; E_\mu)$ and $L^2(\mathcal{X}; E_\nu)$.

Remark 3.8. The reader familiar with Roe algebras will recognize spectral properness as a propagation condition—the same as that which would appear in the definition of the Roe algebra of the spectrum $\mathbb{N}$ if it were endowed with the indiscrete coarse structure (Example 2.8 of [Roe03]). From this point of view, $\mathcal{K}_i$ should be compared with the ideal of that Roe algebra which is associated to the subspace $\{0\}$ of $\mathbb{N}$, as in [HRY93, Section 5].

In the previous section it was noted that the projections $P_k^{(i)}$ can be defined on section spaces of K-homogeneous vector bundles of $\mathcal{X}$ of arbitrary dimension. Using this allows one to extend the definition of the above C^*-categories in a similar fashion. Equivalently, a bounded operator between L^2 sections of $E = \oplus_\mu E_\mu$ and $E' = \oplus_\nu E_\nu$ belongs to $\mathcal{A}_i(E, E')$ if and only if each of the entries in its matrix representation with respect to those direct sums belongs to $\mathcal{A}_i(E_\mu, E_\nu)$, for the appropriate μ and ν. The analogous statement also holds for the other C^*-categories defined above.

We now describe some alternative characterizations of these C^*-categories.

Lemma 3.9. *Let $i = 1$ or 2 and let $T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ be a bounded linear map. The following are equivalent:*

(i) $T \in \mathcal{K}_i$,

(ii) $(P_{[0,k]}^{(i)})^\perp T \to 0$ *and* $T(P_{[0,k]}^{(i)})^\perp \to 0$ *in norm as* $k \to \infty$,

(iii) $P_{[0,k]}^{(i)} T P_{[0,k]}^{(i)} \to T$ *in norm as* $k \to \infty$.

Proof. Property *(ii)* is immediate if T is spectrally finite for α_i, and hence holds for all $T \in \mathcal{K}_i$ by density. The implications *(ii)⇒(iii)* and *(iii)⇒(i)* are straightforward. □

Lemma 3.10. *Let $i = 1$ or 2 and let $T : L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ be a bounded linear map. The following are equivalent:*

(i) $T \in \mathcal{A}_i$,

(ii) *For any* $k \in \mathbb{N}$, $(P^{(i)}_{[0,l]})^{\perp} T P^{(i)}_{[0,k]} \to 0$ *and* $P^{(i)}_{[0,k]} T (P^{(i)}_{[0,l]})^{\perp} \to 0$ *in norm as* $l \to \infty$,

(iii) T *is a multiplier of* $\mathcal{K}_i$, *ie* TK *and* KT *are in* $\mathcal{K}_i$ *for all* $K \in \mathcal{K}_i$ *(when appropriately composable).*

Proof. If T is spectrally proper for α_i then *(ii)* is immediate, so by density, *(ii)* holds for all $T \in \mathcal{A}_i$. If T satisfies *(ii)*, then TK and KT satisfy *(ii)* of Lemma 3.9 for any K which is spectrally finite for α_i, from which *(iii)* follows by density again.

Finally, let T be a multiplier of $\mathcal{K}_i$. We will show that for any $\epsilon > 0$, we can approximate T within ϵ in norm by an operator S which is spectrally proper for α_i. We construct S by an inductive process: starting with $S_0 = T$, we will construct multipliers S_n of $\mathcal{K}_i$ such that

$$\|S_n - S_{n-1}\| < \epsilon.2^{-n}, \tag{3.2}$$

as well as a strictly increasing sequence $(a_n) \subseteq \mathbb{N}$ such that

$$(P^{(i)}_{[0,a_k]})^{\perp} S_n P^{(i)}_{[0,k]} = 0 \tag{3.3}$$

and

$$P^{(i)}_{[0,k]} S_n (P^{(i)}_{[0,a_k]})^{\perp} = 0, \tag{3.4}$$

for each $0 \leq k \leq n-1$. The norm-limit of these S_n will then be the desired approximating operator S.

Suppose we have defined S_{n-1}. Both $S_{n-1}P^{(i)}_{[0,n]}$ and $P^{(i)}_{[0,n]}S_{n-1}$ are in $\mathcal{K}_i$, so by Lemma 3.9 there is an integer a_n (larger than a_{n-1}) such that the operators

$$U_n = (P^{(i)}_{[0,a_n]})^{\perp} S_{n-1} P^{(i)}_{[0,n]}$$

and

$$V_n = P^{(i)}_{[0,n]} S_{n-1} (P^{(i)}_{[0,a_n]})^{\perp},$$

have norm less than $\epsilon.2^{-n-1}$. If we now put

$$S_n = S_{n-1} - U_n - V_n,$$

one can directly check the properties (3.2), (3.3) and (3.4).

□

3.3 Multiplication operators and group representations

Proposition 3.11. *Let* $s \in C(\mathcal{X}; E_{-\lambda})$, *for some weight* λ. *For any* $\mu \in \Lambda_W$, *the multiplication operator* $s : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\lambda)})$ *belongs to* $\mathcal{A}$.

Proof. Fix $i = 1$ or 2. Suppose first that $s(k) = (\xi^{\dagger}, \sigma(k)\xi)$, is a matrix unit, with $\sigma \in \hat{\mathsf{K}}$, $\xi^{\dagger} \in V_{\sigma}^{\dagger}$ and $\xi \in (V_{\sigma})_{\lambda}$. Suppose further that ξ is of K_i-type j. For each $k \in \mathbb{N}$, the K_i-types appearing in $\pi_j \otimes \pi_k$ lie between $|k-j|$ and $k+j$. Thus,

$$\alpha\text{-}\operatorname{Supp}(s) \subseteq \{(k,k') \mid |k-k'| \leq j\},$$

which is proper. Such s span a dense subspace of $C(\mathcal{X}; E_{\lambda})$, so we are done.

□

To prove the analogous result for the group representations we need a realization of the operators $U_{-\mu}(g)$ ($g \in \mathsf{G}$) under the Peter-Weyl transform. If $g = k \in \mathsf{K}$ this is immediate: $U_{-\mu}(k)$ is given by the left-regular representation of K on $L^2(\mathsf{K})$, so it preserves right K_i-types for both $i = 1$ and 2, and hence lies in $\mathcal{A}$. Thanks to the decomposition, $\mathsf{G} = \mathsf{KAK}$, it now suffices to understand the operators $U_{-\mu}(a)$, for $a \in \mathsf{A}$. It is easier to work with the infinitesimal operators $U_{-\mu}(A)$ with $A \in \mathfrak{a}$.

Let $s(k) = (\xi^\dagger, \sigma(k)\xi)$ in $L^2(\mathcal{X}; E_{-\mu})$ be a matrix unit. Using the Iwasawa decomposition, we define functions κ, a and n by

$$g = \kappa(g)\mathsf{a}(g)\mathsf{n}(g) \in \mathsf{KAN}.$$

For $a \in \mathsf{A}$, the B-equivariance property of (2.1) gives

$$\begin{aligned} U_{-\mu}(a)s(k) &= s(k\, k^{-1}a^{-1}k) \\ &= e^{-\rho}(\mathsf{a}(k^{-1}a^{-1}k))\; s(k\,\kappa(k^{-1}a^{-1}k)) \\ &= e^{-\rho}(\mathsf{a}(k^{-1}a^{-1}k))\; (\xi^\dagger, \sigma(k)\sigma(\kappa(k^{-1}a^{-1}k))\xi), \end{aligned}$$

for any $k \in \mathsf{K}$. Now put $a = \exp(tA)$, and take the derivative with respect to t at $t = 0$. We get

$$\begin{aligned} U_{-\mu}(A)s(k) &= \rho(d\mathsf{a}_e(\operatorname{Ad} k^{-1}(A)))\; (\xi^\dagger, \sigma(k)\xi) \\ &\qquad - (\xi^\dagger, \sigma(k)\sigma(d\kappa_e(\operatorname{Ad} k^{-1}(A)))\xi). \end{aligned} \tag{3.5}$$

The derivatives $d\kappa_e$ and $d\mathsf{a}_e$ at the identity $e \in \mathsf{G}$ are the projections of $\mathfrak{g}$ onto the $\mathfrak{k}$ and $\mathfrak{a}$ parts, respectively, of the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, respectively. However, the formula (3.5) uses $d\kappa_e$ and $d\mathsf{a}_e$ only on the real subspace $\mathfrak{p}$ of self-adjoint matrices in $\mathfrak{g}$, since $A \in \mathfrak{p}$ and the adjoint representation preserves $\mathfrak{p}$. This observation allows us to replace the maps $d\kappa_e$ and $d\mathsf{a}_e$, which are only $\mathbb{R}$-linear on $\mathfrak{g}$, by more convenient $\mathbb{C}$-linear maps, which we now describe.

For each root α, let us denote by X_α the elementary matrix in the α-root space of $\mathfrak{k}_{\mathbb{C}}$, *ie*, in the notation of Section 2.1, $X_{\alpha_1} = X_1$, $X_{-\alpha_1} = X_1'$, *etc.* The roots of K divide into positive and negative, and we define $\operatorname{sign}(\alpha)$ to be $+1$ or -1 accordingly. Let $H_1, H_2 \in \mathfrak{a}$ be a basis for the Cartan subalgebra $\mathfrak{h}$. Denote by $X_\alpha^\dagger$, $H_j^\dagger$ the elements of the basis of $\mathfrak{g}^\dagger$ dual to the above.

Lemma 3.12. *On the subspace $\mathfrak{p} \subseteq \mathfrak{g}$, $d\kappa_e$ and $d\mathsf{a}_e$ agree with the maps*

$$-\sum_{\alpha \in \Delta} \operatorname{sign}(\alpha) X_\alpha \otimes X_\alpha^\dagger \quad \in \quad \mathfrak{g} \otimes \mathfrak{g}^\dagger = \operatorname{End}(\mathfrak{g})$$

and

$$\sum_{i=1,2} H_i \otimes H_i^\dagger \quad \in \quad \mathfrak{g} \otimes \mathfrak{g}^\dagger = \operatorname{End}(\mathfrak{g}),$$

respectively.

Proof. One can this check directly on the basis

$$\begin{array}{ll} X_\alpha + X_{-\alpha}, & (\alpha \in \Delta^+) \\ iX_\alpha - iX_{-\alpha}, & (\alpha \in \Delta^+) \\ H_i & (i = 1, 2). \end{array}$$

for $\mathfrak{p}$. $\square$

We obtain the following formula for the group representation:

$$\begin{aligned}
U_{-\mu}(A)s(k) &= \sum_{i=1,2} \rho(H_i)\ (H_i^\dagger, \mathrm{Ad}\, k^{-1}(A))(\xi^\dagger, \sigma(k)\xi) \\
&\qquad + \sum_{\alpha\in\Delta} \mathrm{sign}(\alpha)\ (X_\alpha^\dagger, \mathrm{Ad}\, k^{-1}(A))\ (\xi^\dagger, \sigma(k)\sigma(X_\alpha)\xi). \\
&= (\xi^\dagger \otimes A, (\sigma\otimes \mathrm{Ad}^\dagger)(k)\ \Xi(\xi)), \qquad (3.6)
\end{aligned}$$

where

$$\Xi(\xi) = \sum_{i=1,2} \rho(H_i)\ \xi\otimes H_i^\dagger + \sum_{\alpha\in\Delta} \mathrm{sign}(\alpha)\ (X_\alpha\xi)\otimes X_\alpha^\dagger \quad \in \quad V^\sigma\otimes\mathfrak{g}^\dagger.$$

(We are now suppressing explicit mention of σ for notational convenience.) Thus, in the Peter-Weyl picture, the representation of $\mathfrak{a}$ is described in terms of tensor products with the co-adjoint representation $\mathrm{Ad}^\dagger$ of K.

In what follows, we will use the trace form

$$B_0(W_1, W_2) = \mathrm{Tr}(W_1W_2), \qquad (W_1, W_2\in\mathfrak{g})$$

to identify $\mathfrak{g}$ and $\mathfrak{g}^\dagger$. In particular, $X_\alpha^\dagger$ corresponds to $X_{-\alpha} = X'_\alpha$.

Proposition 3.13. *For any $g\in\mathsf{G}$, and any weight μ of $\mathfrak{k}$, $U_{-\mu}(g)\in\mathcal{A}$.*

Proof. We prove that $U_{-\mu}(g)\in\mathcal{A}_1$. The proof that $U_{-\mu}(g)\in\mathcal{A}_2$ is analogous.

Let $A\in\mathfrak{a}$. Given the formula (3.6), the effect of $U_\mu(A)$ on right K_1-types is governed by the maps

$$\Xi : V^\sigma \to V^\sigma\otimes\mathfrak{g}^\dagger$$

above. Suppose $\xi\in V^\sigma$ is a vector of K_1-type k. We will split Ξ into two pieces to be analyzed separately. Let us fix a choice of $H_1, H_2\in\mathfrak{a}$, namely,

$$H_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \qquad H_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix},$$

which are orthonormal for the trace form. Note also that H_1 is of K_1-type 2, and H_2 is of K_1-type 0.

We first claim that the vector

$$\Xi_1(\xi) = \rho(H_1)\xi\otimes H_1^\dagger + (X_1\xi)\otimes X_1^\dagger - (X'_1\xi)\otimes(X'_1)^\dagger$$

has a norm-bound depending only on the K_1-type of ξ. This is clear since H_1, X_1 and X'_1 all belong to $\mathfrak{k}_1$, and hence act with fixed norm on vectors of a given K_1-type. In fact, if we identify $\mu|_{\mathsf{K}_1} = m\in\mathbb{Z}$, then the well-known formulae for irreducible unitary representations of $\mathfrak{su}(2)$ give

$$\begin{aligned}
\|\rho(H_1)\xi\| &= \frac{1}{\sqrt{2}}|m|\|\xi\|, \\
\|\sigma(X_1)\xi\| &= \frac{1}{2}\sqrt{(k-m)(k+m+2)}\|\xi\|, \\
\|\sigma(X'_1)\xi\| &= \frac{1}{2}\sqrt{(k-m+2)(k+m)}\|\xi\|,
\end{aligned}$$

and hence

$$\|\Xi_1(\xi)\| \leq (\frac{1}{\sqrt{2}}k + \frac{1}{2}(k+1) + \frac{1}{2}(k+1))\|\xi\| \leq (2k+1)\|\xi\|.$$

So the norm-bound actually depends linearly on k. Note also that the vectors H_1, X_1, and X_1' all have K_1-type 2. By the fusion rules for SU(2)-representations, this implies that only possible K_1-types appearing in the vector $\Xi_1(\xi)$ are $k-2$, k and $k+2$.

We remark that this estimate on $\Xi_1(\xi) \in V^\sigma \otimes \mathfrak{g}^\dagger$ does not immediately carry over to a norm estimate on the corresponding part of $U_{-\mu}(A)s \in L^2(\mathcal{X}; E_{-\mu})$ because of the factor $|\sigma|^{\frac{1}{2}}$ which appears in the Peter-Weyl transform (2.5). However, the irreducible representations σ' appearing in $\sigma \otimes \mathrm{Ad}^\dagger$ have dimension $|\sigma'| \geq \frac{1}{18}|\sigma|$, as we will argue shortly. The section $k \mapsto (\xi^\dagger \otimes A, (\sigma \otimes \mathrm{Ad}^\dagger)(k)\Xi_1(\xi))$ therefore has L^2-norm bounded by $\sqrt{18}(2k+1)\|s\|$.

To obtain the stated dimension bound, suppose that σ has highest weight $\beta = b_1\delta_1 - b_2\delta_3$. Then $|\sigma| = \frac{1}{2}(b_1+1)(b_2+1)(b_1+b_2+2)$. The highest weight of σ' is $\beta + \alpha$, for $\alpha \in \Delta \cup \{0\}$. Note that these weights α have $\alpha = a_1\delta_1 - a_2\delta_3$, with $|a_1|, |a_2| \leq 2$, from which the estimate can be readily deduced.

Next, we claim that the remaining part of $\Xi(\xi)$,

$$\Xi_2(\xi) = \rho(H_2)\xi \otimes H_2 + X_2\xi \otimes X_2' - X_2'\xi \otimes X_2 + X_\rho\xi \otimes X_\rho' - X_\rho'\xi \otimes X_\rho$$

is of K_1-type k only. For the first term on the right this is immediate, since H_2 has K_1-type 0. For the latter four terms, a computation must be done. A vector has a unique K_1-type if and only if it is an eigenvector of the Casimir operator $\Omega_{\mathsf{K}_1} = 2X_1'X_1 + \frac{1}{2}H_1^2 + H_1$ for K_1, and the K_1-type is then uniquely determined by its eigenvalue. Since we are working in a fixed weight space for K, the element H_1 acts as a fixed scalar, so it suffices to consider the action of $X_1'X_1$.

One can compute

$$\begin{aligned}
&(\sigma \otimes \mathrm{ad})(X_1'X_1)(X_2\xi \otimes X_2' - X_2'\xi \otimes X_2 + X_\rho\xi \otimes X_\rho' - X_\rho'\xi \otimes X_\rho) \\
&\quad = \; X_1'X_1X_2\xi \otimes X_2' - X_1'X_1X_2'\xi \otimes X_2 \\
&\qquad + X_1'X_1X_\rho\xi \otimes X_\rho' - X_1'X_1X_\rho'\xi \otimes X_\rho \\
&\qquad - X_1'X_2'\xi \otimes X_\rho - X_1'X_\rho\xi \otimes X_2' - X_1X_2\xi \otimes X_\rho' - X_1X_\rho'\xi \otimes X_2 \\
&\qquad - X_2'\xi \otimes X_2 + X_\rho\xi \otimes X_\rho'.
\end{aligned} \tag{3.7}$$

The first four terms on the right hand side of (3.7) can be rewritten as

$$\begin{aligned}
&X_2X_1'X_1\xi \otimes X_2' - X_2'X_1'X_1\xi \otimes X_2 + X_\rho X_1'X_1\xi \otimes X_\rho' - X_\rho'X_1'X_1\xi \otimes X_\rho \\
&\qquad + X_1'X_\rho\xi \otimes X_2' + X_\rho'X_1\xi \otimes X_2 + X_2X_1\xi \otimes X_\rho' + X_1'X_2'\xi \otimes X_\rho.
\end{aligned}$$

Hence, (3.7) equals

$$X_2X_1'X_1\xi \otimes X_2' - X_2'X_1'X_1\xi \otimes X_2 + X_\rho X_1'X_1\xi \otimes X_\rho' - X_\rho'X_1'X_1\xi \otimes X_\rho.$$

We see that $\Xi_2(\xi)$ is an eigenvector of $X_1'X_1$ with exactly the same eigenvalue as ξ, as claimed.

Therefore, $U_{-\mu}(A)$ satisfies the hypotheses on Q in the following lemma, which will complete the proof.

□

Lemma 3.14. *Fix $i = 1$ or 2. Let Q be an unbounded skew-adjoint operator on $L^2(\mathcal{X}; E_\mu)$, such that*

(i) $P_j^{(i)} Q P_k^{(i)} = 0$ *whenever* $|j-k| > 2$, *and*

(ii) $P_k^{(i)} Q P_{k+2}^{(i)}$ *and* $P_{k+2}^{(i)} Q P_k^{(i)}$ *are bounded operators for each* $k \in \mathbb{N}$, *and their norms are bounded by* $C(k+1)$ *for some universal constant* C.

Then $e^Q \in \mathcal{A}_i$.

Remark 3.15. Such an operator Q has the flavour of a 'discrete wave operator' on the space $\mathbb{N}$, or more accurately, on the space $\log \mathbb{N}$. The following proof is an obvious generalization of the proof of finite propagation speed for ordinary wave operators.

Proof. We will prove that for any $\epsilon > 0$ and any $k \in \mathbb{N}$, there exists $k' > k$ such that

$$\|(P_{[0,k']}^{(i)})^\perp e^Q P_{[0,k]}^{(i)}\| < \epsilon.$$

We may also apply this with $-Q$ in place of Q, and thus we will obtain property *(ii)* of Lemma 3.10 for e^Q.

Fix $k \in \mathbb{N}$ and choose any unit vector $u \in P_{[0,k]}^{(i)} L^2(\mathcal{X}; E_\mu)$. Put

$$u_s = e^{sQ} u \qquad\qquad (0 \leq s \leq 1).$$

Let $h_n = \sum_{j=1}^n j^{-1}$ be the nth harmonic sum, and define $\phi : \mathbb{N} \to [0,1]$ by

$$\phi(n) = \begin{cases} 1, & n \leq k \\ \max\{0, 1 - \frac{\epsilon^2}{8C}(h_n - h_k)\}, & n > k \end{cases}.$$

Define an operator on $L^2(\mathcal{X}; E_\mu)$, diagonal with respect to the K_i-type decomposition, by

$$\Phi = \sum_{n \in \mathbb{N}} \phi(n) P_n^{(i)}.$$

Now we decompose Q into its diagonal and off-diagonal components. For brevity, let us put

$$Q_{m,n} = P_m^{(i)} Q P_n^{(i)}.$$

Then $Q = Q_- + Q_d + Q_+$, where

$$Q_- = \sum_{n \in \mathbb{N}} Q_{n,n+2}, \qquad Q_d = \sum_{n \in \mathbb{N}} Q_{n,n}, \qquad Q_+ = \sum_{n \in \mathbb{N}} Q_{n+2,n}.$$

(Note that all K_i-types appearing nontrivially in $L^2(\mathcal{X}; E_{-\mu})$ have the same parity as $\mu|_{\mathsf{K}_i}$.) The diagonal component Q_d commutes with Φ. Meanwhile,

$$\|[Q_{n,n+2}, \Phi]\| = \|(\phi(n+2) - \phi(n))\, Q_{n,n+2}\| \leq \frac{\epsilon^2}{4C} \frac{1}{(n+1)} \|Q_{n,n+2}\|,$$

so

$$\|[Q_-, \Phi]\| = \sup_{n \in \mathbb{N}} \|[Q_{n,n+2}, \Phi]\| \leq \frac{1}{4}\epsilon^2.$$

Similarly for Q_+. Hence, $\|[Q,\Phi]\| \le \frac{1}{2}\epsilon^2$.

We now have

$$|\frac{d}{ds}\langle \Phi u_s, u_s\rangle| = |\langle \Phi Q u_s, u_s\rangle + \langle \Phi u_s, Q u_s\rangle| = |\langle [\Phi, Q] u_s, u_s\rangle| \le \frac{1}{2}\epsilon^2,$$

for all $s \in [0,1]$. Therefore,

$$\langle \Phi u_1, u_1\rangle = \langle \Phi u_0, u_0\rangle + \int_0^1 \frac{d}{ds}\langle \Phi u_s, u_s\rangle \, ds \;\ge\; 1 - \frac{1}{2}\epsilon^2.$$

Let k' be the smallest integer for which $\phi(k') < \frac{1}{2}$. Put $v = P^{(i)}_{[0,k']} u_1$ and $w = (P^{(i)}_{[0,k']})^\perp u_1$. Then $\|v\|^2 + \|w\|^2 = 1$, but also,

$$\|v\|^2 + \frac{1}{2}\|w\|^2 > \langle \Phi v, v\rangle + \langle \Phi w, w\rangle = \langle \Phi u, u\rangle \ge 1 - \frac{1}{2}\epsilon^2.$$

Therefore, $\|w\| = \|(P^{(i)}_{[0,k']})^\perp e^Q u\| \le \epsilon$. Since u was arbitrary in $P^{(i)}_{[0,k]} L^2(\mathcal{X}; E_\mu)$, this proves the claim.

□

3.4 Relationship with compact operators

In this section we will need to make use of the Gel′fand-Tsetlin bases for irreducible representations of SU(3). We a brief overview here for the sake of fixing notation, and refer the reader to [Mol06] for a full introduction.

Integral weights for the Lie group U(3) are parameterized by triples of integers:

$$\mu = (\mu_1, \mu_2, \mu_3) = \sum_{j=1}^{3} \mu_j \delta_j.$$

Dominant weights are those for which $\mu_1 \ge \mu_2 \ge \mu_3$. We use the same notation for the weights of SU(3), acknowledging now that two triples represent the same weight if their difference is in $\mathbb{Z}(1,1,1)$.

The Gel′fand-Tsetlin basis vectors of an irreducible representation of U(3) with highest weight μ are indexed by patterns of integers

$$M = \begin{pmatrix} m_{31} & & m_{32} & & m_{33} \\ & m_{21} & & m_{22} & \\ & & m_{11} & & \end{pmatrix}$$

where $m_{3k} = \mu_k$, and the entries satisfy the 'betweenness conditions' $m_{j+1,k} \ge m_{jk} \ge m_{j+1,k+1}$. We denote the unit vector corresponding to a pattern M by (M). (If M is a pattern which does not satisfy the betweenness conditions, we take (M) to denote the zero vector.) Identifying U(1) and U(2) with the 'upper-left' subgroups

$$\begin{pmatrix} \mathrm{U}(1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} \multicolumn{2}{c}{\mathrm{U}(2)} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of U(3), the Gel′fand-Tsetlin vector (M) is defined, up to phase, by the property that for each $j = 1, 2, 3$, (M) belongs to an irreducible U(j)-subrepresentation with highest weight equal to the jth row of M. If S_j is the sum of the entries of the jth row of M (and $S_0 = 0$ by convention), then the weight of (M) is $(S_1 - S_0, S_2 - S_1, S_3 - S_2)$. When working instead with $\mathsf{SU}(3)$-representations, two Gel′fand-Tsetlin patterns describe the same vector if they differ in each entry by an overall constant.

One could also define a Gel′fand-Tsetlin-type basis using the 'lower-right' subgroups

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{U}(1) \end{pmatrix} \subseteq \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \multirow{2}{*}{$\mathrm{U}(2)$} \\ 0 & \end{pmatrix} \subseteq \mathrm{U}(3).$$

This is most easily achieved as follows. Let

$$\tilde{w} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathsf{G},$$

which is a representative of the Weyl group element

$$w \cdot \mu = (\mu_1, \mu_2, \mu_3) \mapsto (\mu_3, \mu_2, \mu_1)$$

Note that conjugation by $\tilde{w}$ interchanges the upper-left and lower-right subgroups. Given a representation σ of G, let σ' be the representation

$$\sigma'(g) = \sigma(\tilde{w} g \tilde{w}^{-1}),$$

which is isomorphic to σ. The ordinary Gel′fand-Tsetlin basis for σ' is an alternative basis for V^σ, whose basis vectors we denote by $(M)'$. The weight of $(M)'$ is $(S_3 - S_2, S_2 - S_1, S_1 - S_0)$.

Explicit formulae for the irreducible representations of $\mathfrak{g}$ in the Gel′fand-Tsetlin basis can be found, for instance, in [Mol06].

Lemma 3.16. *Fix $\mu \in \Lambda_W$. For any finite sets $A, B \subseteq \mathbb{N}$, $P^{(2)}_B P^{(1)}_A$ is a compact operator on $L^2(\mathcal{X}; E_\mu)$.*

Proof. It suffices to prove the result with $A = \{k\}$ and $B = \{l\}$ singleton sets. We begin with the case $\mu = 0$, $l = 0$, but k arbitrary.

Suppose $s \in P^{(1)}_k L^2(\mathcal{X}; E_0)$ and $t \in P^{(2)}_0 L^2(\mathcal{X}; E_0)$. Under the Peter-Weyl transform, $t \mapsto \sum_p \eta_p^\dagger \otimes \eta_p$, with each η_p of weight 0 and K_2-type 0. The only such vectors are, up to scalar multiple,

$$\eta_{n,0} \stackrel{\text{def}}{=} \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & 0 & & 0 & \\ & & 0 & & \end{matrix} \right)\!\!\right)' \in V^{(n,0,-n)},$$

with $n = 0, 1, \ldots$ Likewise, $s \mapsto \sum_p \xi_p^\dagger \otimes \xi_p$, with each ξ_p of weight 0 and K_1-type k. The Gel′fand-Tsetlin vectors of weight 0 in $V^{(n,0,-n)}$ are

$$\xi_{n,2j} \stackrel{\text{def}}{=} \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j & \\ & & 0 & & \end{matrix} \right)\!\!\right),$$

for $j = 0, \ldots, n$, and here $\xi_{n,2j}$ is of K_1-type $2j$. In particular, if k is odd then $P_0^{(2)} P_k^{(1)} = 0$ on $L^2(\mathcal{X}; E_0)$. We therefore restrict attention to the even case, $k = 2j$.

Let us write $\eta_{n,0} = \sum_{j=0}^{n} c_{n,j} \xi_{n,2j}$, for some coefficients $c_{n,j} \in \mathbb{C}$. Being of K_2-type 0, $\eta_{n,0}$ is annihilated by X_2. By the Gel′fand-Tsetlin formulae,

$$X_2 \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j & \\ & & 0 & & \end{matrix} \right)\!\!\right) = (j+1) \left(\frac{(n-j)(n+j+2)}{(2j+1)(2j+2)} \right)^{\frac{1}{2}} \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j{+}1 & & -j & \\ & & 0 & & \end{matrix} \right)\!\!\right)$$
$$+ j \left(\frac{(n-j+1)(n+j+1)}{2j(2j+1)} \right)^{\frac{1}{2}} \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j{+}1 & \\ & & 0 & & \end{matrix} \right)\!\!\right).$$

Solving for the coefficients $c_{n,j}$ we obtain, up to phase,

$$\eta_{n,0} = \sum_{j=0}^{n} \frac{\sqrt{2j+1}}{n+1} \xi_{n,2j}. \tag{3.8}$$

Therefore,

$$|\langle \xi_{n,2j}, \eta_{n,0} \rangle| = \frac{\sqrt{2j+1}}{n+1}.$$

Let $R_N \in \mathcal{B}(L^2(\mathcal{X}; E_0))$ be the projection onto the subspace spanned by sections of K-type $(n, 0, -n)$, for $n = 0, \ldots, N$. Note that R_N commutes with $P_k^{(1)}$ and $P_0^{(2)}$. If $s \in P_k^{(1)} L^2(\mathcal{X}; E_0)$, $t \in P_0^{(2)} L^2(\mathcal{X}; E_0)$, then by (3.4),

$$|\langle R_N^{\perp} t, s \rangle| \leq \frac{\sqrt{k+1}}{N+1} \|s\| \|t\|.$$

Therefore $\|R_N^{\perp} P_0^{(2)} P_k^{(1)}\| \leq \sqrt{k+1}/(N+1)$. Since N is arbitrary and R_N is finite rank, this proves that $P_0^{(2)} P_k^{(1)}$ is compact.

Now let μ, k and l be arbitrary. Use Lemma 3.4 to find a finite collection of continuous sections $s_j \in P_k^{(1)} C(\mathcal{X}; E_\mu)$, $t_{j'} \in P_l^{(2)} C(\mathcal{X}; E_\mu)$ and bounded linear maps

$$\varphi_j : P_k^{(1)} L^2(\mathcal{X}; E_\mu) \to P_0^{(1)} L^2(\mathcal{X}; E_0),$$
$$\psi_{j'} : P_l^{(2)} L^2(\mathcal{X}; E_\mu) \to P_0^{(2)} L^2(\mathcal{X}; E_0),$$

such that $\sum_j s_j \varphi_j = \mathrm{Id}$, $\sum_{j'} t_{j'} \varphi_{j'} = \mathrm{Id}$. Then

$$P_l^{(2)} P_k^{(1)} = \sum_{j,j'} P_l^{(2)} \psi_{j'}^* P_0^{(2)} \overline{t_{j'}} s_j P_0^{(1)} \varphi_j P_k^{(1)}.$$

For each j, j', we have $\overline{t_{j'}} s_j \in C(\mathcal{X}; E_0) \subseteq \mathcal{A}$. Thus, $\overline{t_{j'}} s_j P_0^{(1)}$ can be approximated arbitrarily well in norm by $P_{k'}^{(1)} \overline{t_{j'}} s_j P_0^{(1)}$, for some $k' \in \mathbb{N}$. Since $P_0^{(2)} P_{k'}^{(1)}$ is compact, the result follows.

□

Proposition 3.17. *The C^*-category $\mathcal{K} = \mathcal{K}_1 \cap \mathcal{K}_2$ is the category of compact operators between the section spaces $L^2(\mathcal{X}; E_\mu)$.*

Proof. For $i = 1$ or 2, the projections $(P^{(i)}_{[0,k]})^\perp$ converge strongly to 0 as $k \to \infty$. Thus $(P^{(i)}_{[0,k]})^\perp K$ and $K(P^{(i)}_{[0,k]})^\perp$ converge to zero in norm for any compact operator K. By Lemma 3.9, the compact operators belong to $\mathcal{K}_1 \cap \mathcal{K}_2$.

If $T_1 \in \mathcal{K}_1(E_\lambda, E_\mu)$ and $T_2 \in \mathcal{K}_2(E_\mu, E_\nu)$, are spectrally finite for α_1 and α_2, respectively, then for some $k_1, k_2 \in \mathbb{N}$, $T_2T_1 = T_2P^{(2)}_{[0,k_2]}P^{(1)}_{[0,k_1]}T_1$, which is compact by the previous lemma. A density argument completes the proof. □

A consequence of Proposition 3.17 is that elements of $\mathcal{K}_1$ are multipliers of $\mathcal{K}_2$. Therefore, by Lemma 3.10, $\mathcal{K}_1 \subseteq \mathcal{A}_2$ and hence $\mathcal{K}_1 \lhd \mathcal{A}$. Similarly, $\mathcal{K}_2 \lhd \mathcal{A}$. Summarizing, we have the following important lattice of ideals:

$$\begin{tikzcd}[arrows=dash]
& \mathcal{A} \arrow[dl] \arrow[dr] & \\
\mathcal{K}_1 \arrow[dr] & & \mathcal{K}_2 \arrow[dl] \\
& \mathcal{K} &
\end{tikzcd}$$

4 Normalized differential operators

4.1 Tangential pseudodifferential operators

In this section we will show how the ideals $\mathcal{K}_i$ relate to Connes' foliation C^*-algebra for the fibrations $\mathcal{X}_i \mapsto \mathcal{Y}_i = \mathsf{G}/\mathsf{P}_i$. For background to the material used here, we refer to [MS06].

For $i = 1$ or 2, let $\mathcal{F}_i$ denote the foliation of $\mathcal{X}$ associated to the fibration $\mathcal{X}_i \mapsto \mathcal{Y}_i$. Let E and E' be bundles over $\mathcal{X}$. We denote the algebra of order zero pseudodifferential operators from E to E', tangential along $\mathcal{F}_i$, by $\Psi^0_i(E, E')$. If $E = E'$ we abbreviate this to $\Psi^0_i(E)$. Let $S^*\mathcal{F}_i$ be the cosphere bundle of the foliation $\mathcal{F}_i$. Recall ([Con79]) that the tangential principal symbol map

$$\mathrm{Symb}_0 : \Psi^0_i(E) \to C(S^*\mathcal{F}_i, \mathrm{End}(E))$$

extends to the norm-closure $\overline{\Psi^0_i}(E)$ in $\mathcal{B}(L^2(\mathcal{X}; E))$, and yields a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*_r(\mathcal{G}_i) \longrightarrow \overline{\Psi^0_i}(E) \xrightarrow{\mathrm{Symb}_0} C(S^*\mathcal{F}_i, \mathrm{End}(E)) \longrightarrow 0.$$

Here $C^*_r(\mathcal{G}_i)$ is the C^*-algebra of the foliation groupoid $\mathcal{G}_i$ of $\mathcal{F}_i$, represented on sections of the bundle E.

Since $\mathcal{F}_i$ is in fact a fibration, $C^*_r(\mathcal{G}_i)$ has a realization in terms of Hilbert module operators. The section space $C(\mathcal{X}; E)$ becomes a pre-Hilbert module over $C(\mathcal{Y}_i)$ if we define a $C(\mathcal{Y}_i)$-valued inner product by L^2-integration along the fibres. Specifically, in the case $E = E_\mu$ ($\mu \in \Lambda_W$), the inner product is

$$\langle s_1, s_2\rangle_{C(\mathcal{Y}_i)}(k) = \int_{\mathsf{K}_i} \overline{s_1(kk_1)}s_2(kk_1)dk_1 \qquad (k \in \mathsf{K}),$$

for $s_1, s_2 \in C(\mathcal{X}; E_\mu)$. Denote the Hilbert module completion by $\mathcal{E}_i(\mathcal{X}; E_\mu)$. If $E = \oplus_\mu E_\mu$, then $\mathcal{E}_i(\mathcal{X}; E) = \oplus_\mu \mathcal{E}_i(\mathcal{X}; E_\mu)$. The algebra $C^*_r(\mathcal{G}_i)$ is precisely the algebra of compact Hilbert-module operators on $\mathcal{E}_i(\mathcal{X}; E)$.

The next proposition describes the relationship between $C^*_r(\mathcal{G}_i)$ and $\mathcal{K}_i(E, E)$.

Proposition 4.1. *Let $\mu, \nu \in \Lambda_W$. If $T : \mathcal{E}_i(\mathcal{X}; E_\mu) \to \mathcal{E}_i(\mathcal{X}; E_\nu)$ is a compact operator in the sense of Hilbert $C(\mathcal{Y}_i)$-modules, then its extension to an operator $L^2(\mathcal{X}; E_\mu) \to L^2(\mathcal{X}; E_\nu)$ belongs to $\mathcal{K}_i(E_\mu, E_\nu)$.*

Proof. If $t_1 \in C(\mathcal{X}; E_\mu)$, $t_2 \in C(\mathcal{X}; E_\nu)$ are each of a single $\mathcal{K}_i$-type, then the 'rank-one' operator

$$s \mapsto \langle t_1, s\rangle_{C(\mathcal{Y}_i)}\, t_2$$

clearly satisfies condition *(iii)* of Lemma 3.9. Such operators span a dense subspace of the $C(\mathcal{Y}_i)$-compact operators, and extension of compact Hilbert module operators to bounded L^2-operators is continuous with respect to the norm topologies.

□

4.2 Normalized BGG operators

In this section we prove various properties of the phases X_i^n of the K-invariant differential operators $X_i^n : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+n\alpha_i)})$. To begin with, we note that most of the analysis will reduce to the case $n = 1$. This is because $X_i^n = (X_i)^n$, where the right-hand side here denotes the composition of the operators $X_i : L^2(\mathcal{X}; E_{-(\mu+(j-1)\alpha_i)}) \to L^2(\mathcal{X}; E_{-(\mu+j\alpha_i)})$ for $j = 1, \ldots, n$.

Fix $i = 1$ or 2, and fix a weight $\mu \in \Lambda_W$. Let D_i be the essentially self-adjoint tangentially elliptic first-order differential operator

$$D_i = \begin{pmatrix} 0 & X_i' \\ X_i & 0 \end{pmatrix}$$

on sections of $E = E_{-\mu} \oplus E_{-(\mu+\alpha_i)}$. In what follows, we will make use of symbolic calculus, and since the operators $\mathrm{Ph}(D_i)$ are defined using functional calculus, some remarks are in order. Firstly, since the spectrum of D_i is discrete, there is some smooth function $\phi : \mathbb{R} \to [-1, 1]$ such that $\mathrm{Ph}(D_i) = \phi(D_i)$. Applying Theorem XII.1.3 of [Tay81] fibrewise, we see that $\mathrm{Ph}(D_i)$ is a tangential pseudodifferential operator of order zero, and moreover that its principal symbol is the order zero homogeneous part of $\phi(\mathrm{Symb}(D_i))$, *ie*, $\mathrm{Symb}(D_i)|\,\mathrm{Symb}(D_i)|^{-1}$.

Let N_i (respectively, Z_i) be the element of $\mathfrak{n}$ (respectively, $\mathfrak{z}$) which is represented by the same matrix as defines X_i (respectively, X_i') in $\mathfrak{k}_\mathbb{C}$. Let J denote the operation of multiplication by $\sqrt{-1}$ in $\mathfrak{g}$. For any $V \in \mathfrak{g}$, let

$$V^h = \frac{1}{2}(V - iJV), \qquad \overline{V^h} = \frac{1}{2}(V + iJV) \quad \in \quad \mathfrak{g}_\mathbb{C}.$$

Recall that in Section 3.3 we defined functions κ, a and n by the Iwasawa decomposition: $g = \kappa(g)\mathsf{a}(g)\mathsf{n}(g) \in \mathsf{KAN}$. We abbreviate $\mathsf{an}(g) = \mathsf{a}(g)\mathsf{n}(g)$.

Lemma 4.2. *For each $g \in \mathsf{G}$, the differential operator $X_i : C^\infty(\mathcal{X}; E_{-\mu}) \to C^\infty(\mathcal{X}; E_{-(\mu+\alpha_i)})$ satisfies*

$$U_{-(\mu+\alpha_i)}(g) X_i U_{-\mu}(g^{-1}) = c_g\, X_i + d_g,$$

where $c_g(k) = e^{\alpha_i}(\mathsf{a}(g^{-1}k))$ for $k \in \mathsf{K}$, and d_g is some smooth section of $E_{-\alpha_i}$.

Note that c_g is a smooth positive function on $\mathcal{X}$ and is independent of μ.

Remark 4.3. If one puts a holomorphic structure on the bundles E_μ by transferring the natural holomorphic structure from the complex Lie group Z, using the nilpotent picture, then both X_i and $U_{-(\mu+\alpha_i)}(g)X_iU_{-\mu}(g^{-1})$ are anti-holomorphic differential operators along the complex one-dimensional fibres of $\mathcal{F}_i$. Granted this, the lemma is then trivial, at least for some positive function c_g. However, the independence of c_g with respect to μ is important to us, so we include a proof which calculates it.

Proof. Let $s \in C^\infty(\mathcal{X}; E_{-\mu})$, considered as a B-equivariant function on G. Since $X_i = N_i^h - \overline{Z_i^h}$, and s is N-invariant, we have $X_i s = -\overline{Z_i^h} s$. Recall from Remark 2.1 that to realize the section $X_i s \in C^\infty(\mathcal{X}; E_{-(\mu+\alpha_i)})$ on G, one needs to extend $X_i s$ by B-equivariance from K:

$$X_i s(g) = e^{-\rho}(\mathsf{a}(g))\; X_i s(\kappa(g)).$$

For any $k \in \mathsf{K}$ and $g \in \mathsf{G}$,

$$\begin{aligned}(Z_iU_{-\mu}(g^{-1})s)(\kappa(g^{-1}k)) &= \frac{d}{dt} s(g\,\kappa(g^{-1}k)e^{tZ_i})|_{t=0} \\ &= \frac{d}{dt} s(k\,\mathsf{an}(g^{-1}k)^{-1}e^{tZ_i})|_{t=0} \\ &= e^{\rho}(\mathsf{a}(g^{-1}k))\;(\mathrm{Ad}(\mathsf{an}(g^{-1}k)^{-1})Z_i)s(k).\end{aligned}$$

For any $a \in \mathsf{A}$ and $n \in \mathsf{N}$, $\mathrm{Ad}(an)Z_i = e^{-\alpha_i}(a)Z_i + B$, for some $B \in \mathfrak{b}$. By the B-equivariance of s, we obtain

$$(Z_iU_{-\mu}(g^{-1})s)(\kappa(g^{-1}k)) \;=\; e^{\rho}(\mathsf{a}(g^{-1}k))\; e^{\alpha_i}(\mathsf{a}(g^{-1}k))\; Z_i s(k) \;+\; d'_g(k)s(k)$$

for some smooth function d'_g on K. Therefore,

$$(U_{-(\mu+\alpha_i)}(g)Z_iU_{-\mu}(g^{-1})s)(k) = e^{\alpha_i}(\mathsf{a}(g^{-1}k))\, Z_i s(k) \;+\; d''_g(k)s(k),$$

for some smooth d''_g on K. The same is true with JZ_i in place of Z_i (if we replace d''_g by id''_g), and the result follows. ∎

Let $\varphi_\mu : \mathsf{Z} \otimes \mathbb{C} \to E_\mu$ be the trivializing coordinate patch of E_μ given by the nilpotent picture, which is to say $\varphi_\mu^* s = s|_{\mathsf{Z}}$ for $s \in C(\mathcal{X}; E_\mu)$. We refer to φ as the 'standard chart' on E_μ. The next lemma describes X_i in these coordinates.

Lemma 4.4. *For $s \in C^\infty(\mathcal{X}; E_\mu)$,*

$$\varphi^*_{-(\mu+\alpha_i)} X_i s = (-f_0 \overline{Z_i^h} + f_1)\varphi^*_{-\mu} s,$$

for some smooth functions f_0, f_1 on Z. Moreover, f_0 is everywhere positive, and is independent of μ.

Proof. At the identity in G, $X_i s = -\overline{Z_i^h} s$, as observed in the proof of the previous lemma. In the standard chart, the representation $U_{-\mu}(z)$, with $z \in \mathsf{Z}$, is just left-translation by z. Now apply the preceding lemma to see that the left-invariant vector field $-\overline{Z_i^h}$ is everywhere linearly related to the differential operator X_i in these coordinates.

∎

Proposition 4.5. *Fix $\mu, \nu \in \Lambda_W$. Let $s \in C(\mathcal{X}; E_{-\nu})$ define a multiplication operator*

$$s : L^2(\mathcal{X}; E_{-\mu} \oplus E_{-(\mu+\alpha_i)}) \to L^2(\mathcal{X}; E_{-(\mu+\nu)} \oplus E_{-(\mu+\nu+\alpha_i)}).$$

Then $sD_i - D_i s \in \mathcal{K}_i$.

Proof. In the standard chart for these bundles, s is multiplication by $\varphi_{-\nu}^* s$ (independent of μ). By Lemma 4.4, the principal symbol of D_i, and hence of $\mathrm{Ph}(D_i)$, is the same on both the domain and range of s. Thus, $\mathrm{Symb}_0(s\,\mathrm{Ph}(D_i) - \mathrm{Ph}(D_i)s) = 0$ on the standard chart, which has dense image. $\square$

Proposition 4.6. *Let $\mu \in \Lambda_W$. For any $g \in \mathsf{G}$,*

$$[(U_{-\mu} \oplus U_{-(\mu+\alpha_i)})(g), D_i] \in \mathcal{K}_i.$$

Proof. For brevity, let us write $U = U_{-\mu} \oplus U_{-(\mu+\alpha_i)}$. From Lemma 4.2, $U(g)D_iU(g^{-1}) = c_g D_i + R_g$, for some order zero operator R_g on $E_{-\mu} \oplus E_{-(\mu+\alpha_i)}$. Therefore, the principal symbol of $U(g)\,\mathrm{Ph}(D_i)U(g^{-1}) = \mathrm{Ph}(U(g)D_iU(g^{-1}))$ is

$$c_g\,\mathrm{Symb}(D_i)\,|c_g\,\mathrm{Symb}(D_i)|^{-1} = \mathrm{Symb}(D_i)|\,\mathrm{Symb}(D_i)|^{-1} = \mathrm{Symb}(D_i).$$

Thus, $U(g)\,\mathrm{Ph}(D_i)U(g^{-1}) - \mathrm{Ph}(D_i) \in \mathcal{K}_i$, and since $U(g) \in \mathcal{A}_i$, we are done. $\square$

Looking at the matrix entries of D_i, the preceding two lemmas imply that the operators $sX_i - X_i s$ and $U_{-(\mu+\alpha_i)}(g)X_i - X_iU_{-\mu}(g)$ are also in $\mathcal{K}_i$.

Proposition 4.7. *The operator*

$$X_i : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\alpha_i)})$$

belongs to $\mathcal{A}$.

Proof. We prove the case $i = 1$, the other case being analogous. Since X_1 preserves K_1-types, it is clearly in $\mathcal{A}_i$, and we need only show that it belongs to $\mathcal{A}_2$. For this, we begin with a specific computation showing that, for the operator $X_1 : L^2(\mathcal{X}; E_0) \to L^2(\mathcal{X}; E_{\alpha_1})$, given any $\epsilon > 0$ there is $k' \in \mathbb{N}$ such that

$$\|(P^{(2)}_{[0,k']})^{\perp} X_1 P^{(2)}_0\| < \epsilon. \tag{4.1}$$

Let $V^{(\mu_1,\mu_2,\mu_3)}$ denote V^σ, where σ is the irreducible representation of K with highest weight (μ_1, μ_2, μ_3). Put

$$\xi_{n,j} = \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j & \\ & & 0 & & \end{matrix}\right)\!\!\right) \quad \text{and} \quad \eta_{n,j} = \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j & \\ & & 0 & & \end{matrix}\right)\!\!\right)',$$

which are the Gel′fand-Tsetlin vectors in $(V^{(n,0,-n)})_0$ of K_1-type $2j$ and K_2-type $2j$, respectively (see Section 3.4). Similarly, we have two Gel′fand-Tsetlin bases for the weight spaces $(V^{(n,0,-n)})_{\alpha_1}$, comprised respectively of the vectors

$$\xi'_{n,j} = \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j & & -j & \\ & & 1 & & \end{matrix}\right)\!\!\right) \qquad \text{and} \qquad \eta'_{n,j} = \left(\!\!\left(\begin{matrix} n & & 0 & & -n \\ & j-1 & & -j & \\ & & 0 & & \end{matrix}\right)\!\!\right)',$$

for $j = 1, \ldots, n$. We begin by estimating the quantity $\langle \eta'_{n,j}, X_1 \eta_{n,0} \rangle$.

From Equation (3.8),

$$\eta_{n,0} = \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{2a+1}\, \xi_{n,j}$$

(up to phase[3]). From the Gel′fand-Tsetlin formulae,

$$X_1 \xi_{n,a} = \sqrt{a(a+1)}\, \xi'_{n,a} = \sqrt{a(a+1)}\; X_1 \xi_{n,a}.$$

The automorphism $g \mapsto \tilde{w} g \tilde{w}^{-1}$, which was used to define the 'lower-right' Gel′fand-Tsetlin basis in Section 3.4, interchanges X'_1 and X_2. Therefore, using the Gel′fand-Tsetlin formula for the action of X_2,

$$\begin{aligned}
\langle \eta'_{n,b}, X_1 \eta_{n,0} \rangle &= \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{2a+1} \langle \eta'_{n,b}, X_1 \xi_{n,a} \rangle \\
&= \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{\frac{2a+1}{a(a+1)}} \langle X'_1 \eta'_{n,b}, \xi_{n,a} \rangle \\
&= \frac{1}{n+1} \sum_{a=0}^{n} \sqrt{\frac{2a+1}{a(a+1)}} \sqrt{\frac{b}{2}(n-b+1)(n+b+1)} \\
&\qquad \times \left(\frac{1}{\sqrt{2b+1}} \langle \eta_{n,b}, \xi_{n,a} \rangle + \frac{1}{\sqrt{2b-1}} \langle \eta_{n,b-1}, \xi_{n,a} \rangle \right) \\
&= \frac{1}{n+1} \sqrt{\frac{b}{2}(n-b+1)(n+b+1)} \sum_{a=0}^{n} \frac{(2a+1)}{\sqrt{a(a+1)}} (x_{n,a,b} - x_{n,a,b-1}), \quad (4.2)
\end{aligned}$$

where we have put $x_{n,a,b} = (2a+1)^{-\frac{1}{2}} (2b+1)^{-\frac{1}{2}} \langle \eta_{n,b}, \xi_{n,a} \rangle$. Let us now estimate $x_{n,a,b}$ for large n.

By the Gel′fand-Tsetlin formulae,

$$\begin{aligned}
&a(a+1) x_{n,a,b} \\
&\quad = (2a+1)^{-\frac{1}{2}} (2b+1)^{-\frac{1}{2}} \langle X'_1 X_1 \xi_{n,a}, \eta_{n,b} \rangle \\
&\quad = (2a+1)^{-\frac{1}{2}} (2b+1)^{-\frac{1}{2}} \langle \xi_{n,a}, X'_1 X_1 \eta_{n,b} \rangle \\
&\quad = \frac{b(n-b+1)(n+b+1)}{2(2b+1)} x_{n,a,b-1} + \frac{1}{2}(n(n+2) - b(b+1)) x_{n,a,b} \\
&\qquad + \frac{(b+1)(n-b)(n+b+2)}{2(2b+1)} x_{n,a,b+1}.
\end{aligned}$$

This yields the recurrence relation

$$\begin{aligned}
&b(n-b+1)(n+b+1)\, x_{n,a,b-1} \\
&\quad + (2b+1)(n(n+2) - b(b+1) - 2a(a+1))\, x_{n,a,b} \\
&\qquad + (b+1)(n-b)(n+b+2)\, x_{n,a,b+1} = 0, \quad (4.3)
\end{aligned}$$

which can solved, in principle, from the initial condition $x_{n,a,0} = (n+1)^{-1}$ of Equation (3.8). (One initial condition suffices since when $b = 0$, the first term in

[3] To avoid repeating this phrase throughout we may adjust the phase of the highest-weight vector in the 'lower right' Gel′fand-Tsetlin basis to correct the phase error.

(4.3) vanishes.) However, if n is large in comparison with b (and a is arbitrary), then after dividing by n^2, (4.3) is well approximated by

$$b\, x_{n,a,b-1} + (2b+1)(1-2n^{-2}a(a+1))\, x_{n,a,b} + (b+1)\, x_{n,a,b+1} = 0. \tag{4.4}$$

The solution to (4.4) is $x_{n,a,b} = (-1)^b(n+1)^{-1}P_b(1-2n^{-2}a(a+1))$, where P_b is the bth Legendre polynomial.

Therefore,

$$\lim_{n\to\infty}\sum_{a=0}^{n}\frac{(2a+1)}{\sqrt{a(a+1)}}x_{n,a,b} = (-1)^b\int_0^1 2P_b(1-2t^2)\,dt = (-1)^b\frac{2}{2b+1}.$$

By (4.2), then,

$$\lim_{n\to\infty}|\langle\eta'_{n,b}, X_1\eta_{n,0}\rangle| = \sqrt{\frac{b}{2}}\left(\frac{2}{2b-1}-\frac{2}{2b+1}\right) = \sqrt{\frac{1}{(2b-1)^2}-\frac{1}{(2b+1)^2}}.$$

So, for any $k'\in\mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty}\|(P^{(2)}_{[0,k']})^\perp X_1\eta_{n,0}\|^2 &= \lim_{n\to\infty}\left(\|X_1\eta_{n,0}\|^2 - \sum_{b=0}^{k'}|\langle\eta'_{n,b}, X_1\eta_{n,0}\rangle|^2\right)\\ &= \frac{1}{(2k'-1)^2}. \end{aligned}\tag{4.5}$$

Looking now to sections, we note that a section s in $P^{(2)}_0L^2(\mathcal{X};E_0)$ has Peter-Weyl transform $\sum_n \beta^\dagger_n\otimes\eta_{n,0}$, for some vectors $\beta^\dagger_n \in V^{(n,0,-n)\dagger}$. Therefore, $X_1s = \sum_n\beta^\dagger_n\otimes X_1\eta_{n,0}$. Let R_N denote the finite-rank projection onto the subspace of $L^2(\mathcal{X};E_{-\alpha_1})$ spanned by sections of K-type $(n,0,-n)$, for $n = 0,\dots,N$. By the above computation, for any $\epsilon>0$, one can choose N and k' large enough such that $\|(R_N)^\perp(P^{(2)}_{[0,k']})^\perp X_1P^{(2)}_0\| < \epsilon$ on $L^2(\mathcal{X};E_0)$. But the K-representation of type $(n,0,-n)$ contains K_2-types only up to $2n$, so if we enlarge k' to be greater than $2N$, we have $\|(P^{(2)}_{[0,k']})^\perp X_1P^{(2)}_0\| < \epsilon$ on $L^2(\mathcal{X};E_0)$, as claimed.

With this base case completed, we now let $\mu\in\Lambda_W$, and $k\in\mathbb{N}$ be arbitrary. Choose sections $s_j\in P^{(2)}_kC(\mathcal{X};E_{-\mu})$ and maps $\phi_j : P^{(2)}_kL^2(\mathcal{X};E_{-\mu})\to P^{(2)}_0L^2(\mathcal{X};E_0)$ as in Lemma 3.4. Then

$$\begin{aligned}X_1P^{(2)}_k &= \sum_j X_1s_jP^{(2)}_0\phi_jP^{(2)}_k\\ &= \sum_j(s_jX_1+[X_1,s_j])P^{(2)}_0\phi_jP^{(2)}_k\end{aligned}$$

Let $\epsilon>0$. Choose k' satisfying (4.1). By Propositions 3.11 and 4.5, we can also find k'' sufficiently large such that $\|(P^{(2)}_{k''})^\perp s_jP^{(2)}_{k'}\| < \epsilon$ and $\|(P^{(2)}_{k''})^\perp[X_1,s_j]\| < \epsilon$, for each j. Thus, as an operator from $L^2(\mathcal{X};E_0)$ to $L^2(\mathcal{X};E_{-(\mu+\alpha_1)})$,

$$\begin{aligned}&\|(P^{(2)}_{k''})^\perp(s_jX_1+[X_1,s_j])P^{(2)}_0\|\\ &\quad\leq \|(P^{(2)}_{k''})^\perp s_jP^{(2)}_{k'}X_1P^{(2)}_0\| + \|(P^{(2)}_{k''})^\perp s_j(P^{(2)}_{k'})^\perp X_1P^{(2)}_0\|\\ &\qquad +\|(P^{(2)}_{k''})^\perp[X_1,s_j]P^{(2)}_0\|\\ &\quad< (\|s_j\|+2)\epsilon,\end{aligned}$$

and hence, as an operator from $L^2(\mathcal{X}; E_{-\mu})$ to $L^2(\mathcal{X}; E_{-(\mu+\alpha_1)})$.

$$\|(P^{(2)}_{k''})^{\perp} X_1 P^{(2)}_{k}\| < C\epsilon,$$

for some constant C.

The same kind of estimates can be proven for $X_1' = {X_1}^*$ in place of X_1. (The analogue of (4.5) for X_1' can be most easily obtained by switching the representations $V^{(n,0,-n)}$ with their (unitarily equivalent) contragredient representations, which transforms X_1' to $-X_1$. The rest of the argument goes through by a mere substitution of X_1' for X_1.) Therefore, by Lemma 3.9, $X_1 \in \mathcal{A}(E_{-\mu}, E_{-(\mu+\alpha)})$ for each $\mu \in \Lambda_W$.

□

Proposition 4.8. *Let $i = 1$ or 2, $\mu, \nu \in \Lambda_W$, and $n \in \mathbb{N}$. For any $s \in L^2(\mathcal{X}; E_\nu)$, and any $g \in \mathfrak{g}$, the 'commutators'*

$$X_i^n s - s X_i^n : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\nu+n\alpha)}),$$

and

$${X_i}^n U_{-\mu} - U_{-(\mu+\alpha_i)} {X_i}^n : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-(\mu+\alpha)})$$

belong to $\mathcal{K}_i$.

Proof. Expand $X_i^n s - s X_i^n = \sum_{j=1}^n {X_i}^{j-1}(X_i s - s X_i){X_i}^{n-j}$, and apply Lemma 4.5. The group representation case follows in a similar fashion from Lemma 4.6. □

We conclude this section by remarking that Lemma 5.5 of [AS68] applies to tangential pseudodifferential operators:

Lemma 4.9. *Let $A \in \mathfrak{g}$. The one-parameter family of operators*

$$t \mapsto U_{-(\mu+n\alpha_i)}(\exp(tA)) X_i U_{-\mu}(\exp(-tA))$$

is continuous in the norm topology.

In the language of Kasparov, X_i is G-continuous.

4.3 Weyl commutation relations

The representations U_μ are unitary principal series representations of G. The representation U_μ is irreducible for any $\mu \in \Lambda_W$. Moreover, two such representations, U_μ and $U_{\mu'}$ are unitarily equivalent if and only if μ and μ' are in the same orbit of the Weyl group. If $\mu' = w_i \cdot \mu$, where $w_i \in \mathsf{W}$ is the reflection associated to the simple root α_i, then the intertwining operator between the two representations can be described explicitly.

Proposition 4.10. *Suppose $\mu' = w_i \cdot \mu$, and let n be the integer such that $\mu - \mu' = n\alpha_i$. The unitary intertwiner*

$$I = I_{\mu,\alpha_i} : L^2(\mathcal{X}; E_{-\mu}) \to L^2(\mathcal{X}; E_{-\mu'})$$

is $I = X_i^n$ if $n \geq 0$, and $I = (X_i')^n$ if $n \leq 0$.

These intertwining operators satisfy the 'commutation relation'

$$I_{w_2w_1\mu,\alpha_1}I_{w_1\mu,\alpha_2}I_{\mu,\alpha_1} = I_{w_1w_2\mu,\alpha_2}I_{w_2\mu,\alpha_1}I_{\mu,\alpha_2}, \tag{4.6}$$

for any $\mu \in \Lambda_W$ ([KS61]). In particular,

$$X_1X_2^2X_1 = X_2X_1^2X_2 : L^2(\mathcal{X};E_\rho) \to L^2(\mathcal{X};E_{-\rho}). \tag{4.7}$$

Note that the weights $\pm\rho$ defining the domain and range are crucial in this identity. Nevertheless, if the domain is changed, a weaker commutation property still holds.

Proposition 4.11. *As an operator from $L^2(\mathcal{X};E_0)$ to $L^2(\mathcal{X};E_{-2\rho})$,*

$$X_1X_2^2X_1 - X_2X_1^2X_2 \in \mathcal{K}_1 + \mathcal{K}_2.$$

Proof. Using a trivializing partition of unity for E_ρ, one can find a finite collection of sections $s_1,\ldots,s_2 \in C(\mathcal{X};E_\rho)$ such that $\sum_{j=1}^n \overline{s_j}s_j = 1 \in C(\mathcal{X})$. Then, as an operator from $L^2(\mathcal{X};E_0)$ to $L^2(\mathcal{X};E_{-2\rho})$,

$$\begin{aligned} X_1X_2^2X_1 &= \sum_{j=1}^n X_1X_2^2X_1\overline{s_j}s_j \\ &= \sum_{j=1}^n \overline{s_j}X_1X_2^2X_1s_j \qquad (\text{modulo } \mathcal{K}_1+\mathcal{K}_2), \end{aligned}$$

by repeatedly applying Proposition 3.11. In this last equation, $X_1X_2^2X_1$ is an operator from $L^2(\mathcal{X};E_\rho)$ to $L^2(\mathcal{X};E_{-\rho})$, so is equal to $X_2X_1^2X_2$. Reversing the same process, we see that

$$X_1X_2^2X_1 = X_2X_1^2X_2 : L^2(\mathcal{X};E_0) \to L^2(\mathcal{X};E_{-2\rho})$$

modulo $\mathcal{K}_1+\mathcal{K}_2$.

□

5 Construction of the gamma element

Let $n \in \mathbb{Z}^+$ and $\mu \in \Lambda_W$. Using the standard formulae for irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$, it is readily observed that the operators $X^n : L^2(\mathcal{X};E_{-\mu}) \to L^2(\mathcal{X};E_{-(\mu+n\alpha_i)})$ and $X'^n : L^2(\mathcal{X};E_{-(\mu+n\alpha_i)}) \to L^2(\mathcal{X};E_{-\mu})$ are inverses modulo $\mathcal{K}_i$, since their product in either order equals $P^{(i)}_{[0,k]}$ for some k. Let us define operators

$$\begin{aligned} T_1 &= -X_1^2X_2X_1' : L^2(\mathcal{X};E_{-\alpha_1}) \to L^2(\mathcal{X};E_{-(2\alpha_1+\alpha_2)}), \\ T_2 &= -X_2^2X_1X_2' : L^2(\mathcal{X};E_{-\alpha_2}) \to L^2(\mathcal{X};E_{-(\alpha_1+2\alpha_2)}). \end{aligned}$$

Thanks to Proposition 4.11, these operators are defined precisely so that each of the four diamonds

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

in the diagram

$$\begin{tikzcd}
 & L^2(\mathcal{X}; E_{-\alpha_1}) \arrow[r, "T_1"] \arrow[dr, "X_2^2" near end] & L^2(\mathcal{X}; E_{-(2\alpha_1+\alpha_2)}) \arrow[dr, "X_2"] & \\
L^2(\mathcal{X}; E_0) \arrow[ur, "X_1"] \arrow[dr, "X_2"'] & & & L^2(\mathcal{X}; E_{-(2\alpha_1+2\alpha_2)}) \\
 & L^2(\mathcal{X}; E_{-\alpha_2}) \arrow[r, "T_2"'] \arrow[ur, "X_1^2"' near end] & L^2(\mathcal{X}; E_{-(\alpha_1+2\alpha_2)}) \arrow[ur, "X_1"'] &
\end{tikzcd} \tag{5.1}$$

anticommutes modulo $\mathcal{K}_1 + \mathcal{K}_2$.

Let $\Upsilon = \{w\cdot(-\rho)+\rho \mid w \in \mathsf{W}\} \subseteq \Lambda_W$. These are the negatives of the weights appearing in the diagram above. Recall that there is a length function $l : \mathsf{W} \to \mathbb{N}$ on the Weyl group, defined by letting $l(w)$ be the length of the shortest word in simple reflections which represents w. We partition Υ according to the length of the Weyl group elements: $\Upsilon_p = \{w \cdot (-\rho) + \rho \mid w \in \mathsf{W},\ l(w) = p\}$. Let H_p denote the Hilbert space

$$H_p = \bigoplus_{\mu\in\Upsilon_p} L^2(\mathcal{X}; E_{-\mu}),$$

and let $H = \bigoplus_{p=0}^3 H_p$, endowed with the $\mathbb{Z}/2\mathbb{Z}$-grading coming from the parity of p.

If $\mu, \nu \in \Upsilon$, we identify an operator in $\mathcal{B}(E_{-\mu}, E_{-\nu})$ with the operator on H obtained by extending it trivially on each $L^2(\mathcal{X}; E_{-\mu'})$ for $\mu' \neq \mu$. Let $\mathcal{A}(H)$, $\mathcal{K}_1(H), \mathcal{K}_2(H), \mathcal{K}(H)$ denote the above defined C^*-algebras of operators on $H = \bigoplus_{\mu\in\Upsilon} L^2(\mathcal{X}; E_{-\mu})$.

Denote by Q_μ the projection onto a component $L^2(\mathcal{X}; E_{-\mu})$ of H. Let the continuous functions $f \in C(\mathcal{X})$ act 'diagonally' as multiplication operators on each component of H, and let group elements be represented on H by $U(g) = \bigoplus_{\mu\in\Upsilon} U_{-\mu}(g)$.

The following is a simple generalization of a key step in the construction of the Kasparov product (see [HR00, Proposition 9.2.5]).

Lemma 5.1. *There exist positive operators, $M_1, M_2 \in \mathcal{B}(H)$ with $M_1^2+M_2^2 = 1$, such that*

- *(i)* *$M_i\mathcal{K}_i(H) \subseteq \mathcal{K}(H)$ for $i = 1, 2$,*
- *(ii)* *M_1 and M_2 commute, modulo $\mathcal{K}(H)$, with all multiplication operators $f \in C(\mathcal{X})$, all representations of group elements $U(g)$ for $g \in \mathsf{G}$, and all operators appearing in the normalized BGG complex (5.1).*
- *(iii)* *M_1 and M_2 commute on the nose with the representation of the compact group $U(\mathsf{K})$, and with the projections Q_μ.*

Proof. Let $\mathcal{S} \subseteq \mathcal{B}(H)$ be the set consisting of all multiplication operators $f \in C(\mathcal{X})$, all $U(g)$ for $g \in \mathsf{G}$, all operators appearing in the normalized BGG complex, and all projections Q_μ for $\mu \in \Upsilon$. With $i = 1, 2$, let $\mathcal{K}_i^0(H)$ be the smallest C^*-subalgebra of $\mathcal{B}(H)$ which contains the projections $P_k^{(i)}$ and is derived by $\mathcal{S}$, *ie*, such that $[S, \mathcal{K}_i^0(H)] \subseteq \mathcal{K}_i^0$ for all $S \in \mathcal{S}$. Note that this is a separable C^*-algebra, and that $\mathcal{K}_i^0(H) \subseteq \mathcal{K}_i(H)$. Moreover, $\mathcal{K}_i^0(H)\mathcal{K}_i(H)$ contains all operators which are spectrally finite for α_i, and hence is dense in $\mathcal{K}_i(H)$. So it suffices to prove property *(i)* with the algebras $\mathcal{K}_i^0(H)$ in place of $\mathcal{K}_i(H)$.

By the Kasparov Technical Theorem ([HR00, Theorem 3.8.1]), there exists a self-adjoint operator $Z \in \mathcal{B}(H)$, with $0 \leq Z \leq 1$, such that $Z \cdot \mathcal{K}_1^0(H) \subseteq \mathcal{K}(H)$, $(1-Z)\cdot\mathcal{K}_2^0(H) \subseteq \mathcal{K}(H)$, and $[Z, \mathcal{S}] \subseteq \mathcal{K}(H)$. Standard averaging tricks, as in the proof of [HR00, Proposition 9.2.5] ensure that we can choose Z satisfying the on-the-nose commutation properties of *(iii)*. Now put $M_1 = Z^{\frac{1}{2}}$, $M_2 = (1-Z)^{\frac{1}{2}}$. $\square$

Amend the diagram (5.1) as follows:

$$\begin{tikzcd}
 & L^2(\mathcal{X}; E_{-\alpha_1}) \arrow[r, "M_1M_2T_1"] \arrow[drr, "M_2^2X_2^2"] & L^2(\mathcal{X}; E_{-(2\alpha_1+\alpha_2)}) \arrow[dr, "M_2X_2"] & \\
L^2(\mathcal{X}; E_0) \arrow[ur, "M_1X_1"] \arrow[dr, "M_2X_2"'] & & & L^2(\mathcal{X}; E_{-(2\alpha_1+2\alpha_2)}) \\
 & L^2(\mathcal{X}; E_{-\alpha_2}) \arrow[r, "M_1M_2T_2"'] \arrow[urr, "M_1^2X_1^2"] & L^2(\mathcal{X}; E_{-(\alpha_1+2\alpha_2)}) \arrow[ur, "M_1X_1"'] &
\end{tikzcd} \tag{5.2}$$

Let F denote the operator on H obtained by adding together all the operators of this diagram, plus their adjoints.

Theorem 5.2. *The operator F on the graded Hilbert space H, together with the multiplication representation of $C(\mathcal{X})$ and the unitary representation $U(\mathsf{G})$, defines a cycle θ of $KK^{\mathsf{G}}(C(\mathcal{X}), \mathbb{C})$.*

Proof. Since $[X_i, f]$, $[X_i, U(g)] \in \mathcal{K}_i(H)$ for all $f \in C(\mathcal{X})$, $g \in \mathsf{G}$, it is straightforward to see that each component of F commutes with each f and $U(g)$ modulo compact operators. The diamonds in (5.2) anti-commute modulo $\mathcal{K}(H)$, and hence the components of F^2 which alter the degree p are all compact. We consider now those components which preserve the degree. Let $\sim$ denote equality modulo compact operators.

- The component of F^2 preserving $H_0 = L^2(\mathcal{X}; E_0)$ is

$$X_1' M_1^2 X_1 + X_2' M_2^2 X_2 \;\sim\; M_1^2\,(X_1'X_1 - 1) + M_2^2\,(X_2'X_2 - 1) + 1 \;\sim\; 1.$$

- The component preserving $H_1 = L^2(\mathcal{X}; E_{\alpha_1}) \oplus L^2(\mathcal{X}; E_{\alpha_2})$ is given by a 2×2-matrix. Modulo compacts, the component mapping $L^2(\mathcal{X}; E_{\alpha_1})$ to itself is

$$M_1^2 X_1 X_1' + M_2^4 X_2^2 {X_2'}^2 + M_1^2 M_2^2 T_1' T_1 \;\sim\; M_1^2 + M_2^4 + M_1^2 M_2^2 = 1.$$

and the component mapping $L^2(\mathcal{X}; E_{\alpha_1})$ to $L^2(\mathcal{X}; E_{\alpha_2})$ is

$$\begin{aligned} &M_1 M_2 X_2 X_1' + M_1^3 M_2 {X_1'}^2 T_1 + M_1 M_2^3 T_2' X_2^2 \\ &\quad = M_1^2(M_1M_2(X_2X_1' + {X_1'}^2 T_1)) + M_2^2(M_1M_2(X_2X_1' + T_2'X_2^2)) \sim 0. \end{aligned}$$

The other two components can be computed similarly, with the result that the diagonal components equal 1 and off-diagonal components equal 0 modulo compacts.

- On H_2 and H_3, analogues of the above calculations similarly show that F^2 equals the identity modulo compact operators.

Finally, by Lemma 4.9, F is G-continuous.

$\square$

Proposition 5.3. *The* K*-equivariant index of* θ *is* 1*, and hence, by Theorem 1.1, it is a model for* $\gamma \in KK^{\mathsf{G}}(\mathbb{C}, \mathbb{C})$.

Proof. The operator F is K-equivariant on the nose, so its K-index is the sum of the K-indices of each K-isotypical component. Since these components are all finite-dimensional, this amounts to simply determining the graded dimension of each component of H. These graded dimensions can be deduced immediately by observing that, as $U(\mathsf{K})$-representations, the spaces $L^2(\mathcal{X}; E_{-\mu})$ appearing in (5.2) are exactly the same as (the L^2-completions of) those appearing in the BGG resolution of the trivial representation for G.

$\square$

Remark 5.4. If one does not wish to appeal to the BGG resolution, one can instead make a direct computation of the graded dimensions by using the Weyl character formula or the more elementary remarks of [FH91, p. 184].

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