6
Covering maps

Those Galois theories which involve adjunctions with the category of sets are especially simple and natural; in particular their Galois groups are really the automorphism groups of extensions. The main purpose of this chapter is to describe the classical theory of covering maps of locally connected topological spaces as such a Galois theory.

6.1 Categories of abstract families

For a family $(A_i)_{i\in I}$ of objects of a category $\mathcal{A}$ we will write

$$(A_i)_{i\in I} = A = (A_i)_{i\in I(A)}$$

considering I as a functor

$$I\colon \mathsf{Fam}(\mathcal{A}) \longrightarrow \mathsf{Set}$$

from the category of all families of objects in $\mathcal{A}$ to the category of sets.

According to that notation, a morphism $\alpha\colon A \longrightarrow B$ in $\mathsf{Fam}(\mathcal{A})$ consists of a map $I(\alpha)\colon I(A) \longrightarrow I(B)$ and an $I(A)$-indexed family of morphisms $\alpha_i\colon A_i \longrightarrow B_{I(\alpha)(i)}$. That is, in fact the category $\mathsf{Fam}(\mathcal{A})$ and the functor I are constructed simultaneously.

The categories of families often occur in geometry:

Proposition 6.1.1 *The category* $\mathsf{Fam}(\mathsf{CTop})$, *where* CTop *is the category of connected topological spaces, is equivalent to the category of topological spaces with open connected components.*

Proof Let $A = \coprod_{i\in I} A_i$ be a coproduct in the category Top of topological spaces, that is, A is a topological space which is the disjoint union of the open subsets A_i. Assume that each A_i is connected. To give a continuous map from A to a topological space is the same as to give a continuous map from each A_i to that space. On the other hand every continuous map from a connected space to A has its image contained in one of the A_i. Therefore the proof is straightforward: any topological space with open connected components corresponds to the family of its connected components. $\square$

This and other similar examples of $\mathsf{Fam}(\mathcal{A})$, some of which will be mentioned at the end of this section, suggest various categorical properties for the general $\mathsf{Fam}(\mathcal{A})$, related to the notion of connectedness. As we will see, they easily follow from the fact that each $A \in \mathsf{Fam}(\mathcal{A})$ can be considered as the coproduct

$$A = \coprod_{i \in I(A)} A_i$$

where A_i $\big(i \in I(A)\big)$ are considered as one member families. In particular this implies that $\mathsf{Fam}(\mathcal{A})$ admits all (small) coproducts, and gives the following.

Proposition 6.1.2 *For an object C in $\mathsf{Fam}(\mathcal{A})$, the following conditions are equivalent:*

(i) *the functor*

$$\mathsf{Hom}(C, -)\colon \mathsf{Fam}(\mathcal{A}) \longrightarrow \mathsf{Set}$$

preserves coproducts;

(ii) *$C \cong X \amalg Y$ implies either X or Y is the empty family (= the initial object 0 in $\mathsf{Fam}(\mathcal{A})$), and $C \neq 0$;*

(iii) *$C \cong X \amalg Y$ implies either X or Y is canonically isomorphic to C, and $C \neq 0$;*

(iv) *$I(C) = 1$ (i.e. C is a one member family).* $\square$

The objects C satisfying the equivalent conditions of proposition 6.1.2 are called *connected.* If we replace $\mathsf{Fam}(\mathcal{A})$ by an arbitrary category $\mathcal{C}$, then the connectedness is to be defined via a condition similar to 6.1.2(i), i.e. we introduce

Definition 6.1.3 An object C in a category $\mathcal{C}$ with coproducts is said to be connected if the functor $\mathsf{Hom}(C, -)\colon \mathcal{C} \longrightarrow \mathsf{Set}$ preserves coproducts.

And we obtain

Proposition 6.1.4 *A topological space is connected in the usual sense if and only if it is a connected object in* Top. □

Proposition 6.1.5 *The following conditions on a category $\mathcal{C}$ are equivalent:*

(i) *$\mathcal{C}$ admits coproducts and every object in $\mathcal{C}$ can be presented as a coproduct of connected objects;*
(ii) *the same condition, requiring in addition that the presentation is unique up to an isomorphism;*
(iii) *$\mathcal{C}$ is equivalent to a category of the form* $\mathsf{Fam}(\mathcal{A})$*;*
(iv) *$\mathcal{C}$ is canonically equivalent (up to choice of coproducts) to the category* $\mathsf{Fam}(\mathcal{A})$*, where $\mathcal{A}$ is the category of connected objects in $\mathcal{C}$.* □

Since not every topological space has open connected components, the category Top does not satisfy these equivalent conditions. For example the subspace $\{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\}$ of $\mathbb{R}$ is not the coproduct of its connected components (all of which are one element sets).

Readers are invited to deduce proposition 6.1.1 from proposition 6.1.5 and to check that the following categories also satisfy the equivalent conditions of 6.1.5:

- the category of small categories and various "similar" categories, such as the category of n-categories (strict or weak, whatever the term "weak" means), n-groupoids or multiple categories or groupoids, also with ω- instead of n-, etc.; indeed, coproducts in these cases are disjoint unions;
- categories of graphs, oriented or not (and again n-, ω-, multiple, etc.);
- various categories of relational systems like preorders, posets, sets provided with a reflexive or arbitrary relation, etc.;
- any category of the form $\mathsf{Set}^{\mathcal{K}}$, where $\mathcal{K}$ is a small category; in particular the category of sets itself, or of simplicial sets, or of presheaves over a topological space or a locale, or of sets provided with a monoid action (to investigate these examples, observe that a functor $F\colon \mathcal{K} \longrightarrow \mathsf{Set}$ is connected in $\mathsf{Set}^{\mathcal{K}}$ when its category of elements is connected in Cat);
- any category of the form $\mathcal{C}/B$ where $\mathcal{C}$ satisfies the equivalent conditions of 6.1.5, and B is an arbitrary ("base") object in $\mathcal{C}$.

6.2 Some limits in $\mathsf{Fam}(\mathcal{A})$

We begin with

Proposition 6.2.1 *For a category $\mathcal{A}$ the following conditions are equivalent:*

(i) *$\mathcal{A}$ has a terminal object;*
(ii) *$\mathsf{Fam}(\mathcal{A})$ has a connected terminal object;*
(iii) *$\mathsf{Fam}(\mathcal{A})$ has a terminal object which is a terminal object in $\mathcal{A}$ (regarding the objects of $\mathcal{A}$ as one member families).* □

Assuming from now on that these equivalent conditions hold, we obtain the functors

$$\begin{tikzcd} \mathsf{Fam}(\mathcal{A}) \arrow[r, shift left=3, "I"] \arrow[r, shift right=3, "\Gamma"] & \mathsf{Set}, \arrow[l, "H"'] \end{tikzcd}$$

where H is the right adjoint of I and Γ is the right adjoint of H. Explicitly

$$H(S) = S \cdot 1 = \coprod_S 1,$$

i.e. H sends a set S to the coproduct of "S copies" of the terminal object 1 (= to the S-indexed family of terminal objects) and

$$\Gamma(A) = \mathsf{Hom}(1, A)$$

for every $A \in \mathsf{Fam}(\mathcal{A})$.

Remark 6.2.2 The adjunctions $I \dashv H \dashv \Gamma$ can of course be established and all details checked by a straightforward calculation. However for readers familiar with 2-categories, it would also be nice to look at them from the 2-categorical viewpoint.

(i) $\mathsf{Fam}(\mathcal{A})$ can be described as a "free coproduct completion" of $\mathcal{A}$, and this makes $\mathsf{Fam}\colon \mathsf{Cat} \longrightarrow \mathsf{Cat}$ a 2-functor (it is strict due to the explicit construction of Fam we are using).
(ii) $\mathcal{A}$ has a terminal object if and only if there exists an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left] & \mathbf{1} \arrow[l, shift left] \end{tikzcd}$$

where $\mathbf{1}$ is the terminal object in CAT, that is, the category $\circlearrowright\!\bullet$.

(iii) Fam, being a 2-functor, preserves adjunctions and therefore gives the induced adjunction

$$\mathsf{Fam}(\mathcal{A}) \rightleftarrows \mathsf{Fam}(\mathbf{1})$$

which is just $I \dashv H$ (note that $\mathsf{Fam}(\mathbf{1})$ is nothing but Set).

(iv) For every category $\mathcal{C}$ with coproducts (in fact it is sufficient to have copowers) and every object $C \in \mathcal{C}$ there exists a unique (up to an isomorphism) adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{Set}, \quad F \dashv G$$

with $F(1) = C$; this can be described as $F(S) = S \cdot 1$ and $G = \mathsf{Hom}(C, -)$, and so $H \dashv \Gamma$ is a special case of it.

(v) Moreover identifying (up to an equivalence) $\mathsf{Fam}(\mathcal{A})$ with a full subcategory of $\mathsf{Set}^{\mathcal{A}^{\mathrm{op}}}$, one could present $I \dashv H$ and $H \dashv \Gamma$ as special cases of the same construction. □

Now we are going to describe certain pullbacks in $\mathsf{Fam}(\mathcal{A})$. The first lemma however looks better when it is formulated for general limits first:

Lemma 6.2.3 *Let $\mathcal{K}$ be a small category and $D\colon \mathcal{K} \longrightarrow \mathsf{Fam}(\mathcal{A})$ a functor. For every $x = (x_K)_{K \in \mathcal{K}} \in \lim ID$ let D_x be the functor $\mathcal{K} \longrightarrow \mathcal{A}$ defined by*

$$\begin{array}{ccc} K & & D(K)_{x_K} \\ {\scriptstyle f}\big\downarrow & \mapsto & \big\downarrow{\scriptstyle D(f)_{x_K}} \\ K' & & D(K')_{x_{K'}} \end{array}$$

If each D_x has a limit

$$\lim D_x = \Big(L_x, \big(\pi_{x,K}\colon L_x \longrightarrow D_x(K)\big)_{K \in \mathcal{K}}\Big),$$

then D also has a limit, which can be described as

$$\lim D = \Big(L, \big(\pi_K\colon L \longrightarrow D(K)\big)_{K \in \mathcal{K}}\Big)$$

where

- *$I(L) = \lim ID$,*
- *for $x \in I(L)$, L_x is as above,*

- $I(\pi_K)$ *is the projection* $\lim ID \longrightarrow ID(K)$, *i.e.* $I(\pi_K)(x_K)_{K\in\mathcal{K}} = x_K$,
- $(\pi_K)_x = \pi_{x,K} \colon L_x \longrightarrow D(K)_{x_K}$.

Proof is again straightforward, but since so many letters and indices are involved, it is better to sketch it.

The fact that the collection of π_K, $K \in \mathcal{K}$, forms a cone reduces to the commutativity of the diagrams

$$\begin{tikzcd} & L_x \arrow[dl, "(\pi_K)_x"'] \arrow[dr, "(\pi_{K'})_x"] & \\ D(K)_{x_K} \arrow[rr, "D(f)_{x_K}"'] & & D(K')_{x_{K'}} \end{tikzcd}$$

(for all f and x), which are the same as the diagrams

$$\begin{tikzcd} & L_x \arrow[dl, "\pi_{x,K}"'] \arrow[dr, "\pi_{x,K'}"] & \\ D_x(K) \arrow[rr, "D_x(f)"'] & & D_{x(K')} \end{tikzcd}$$

and these commute because each of the collections of $\pi_{x,K}$ forms a cone for the corresponding D_x.

Now, for proving the universal property of the limit, we take another cone

$$\Big(L', \big(\pi'_K \colon L' \longrightarrow D(K)\big)_{K\in\mathcal{K}}\Big)$$

and it is easy to check that the existence and uniqueness of a morphism from it to the cone described above reduce to the existence and uniqueness of morphisms from the cone

$$\Big(L'_i, \big((\pi'_K)_i \colon L'_i \longrightarrow D(K)_{I(\pi'_K)(i)}\big)_{K\in\mathcal{K}}\Big)$$

$\big($over D_x, where $x_K = I(\pi'_K)(i)\big)$ to the limiting cone $\lim D_x$. □

Corollary 6.2.4 *Let*

$$\begin{array}{ccc} & & V \\ & & \big\downarrow{\scriptstyle v} \\ U & \xrightarrow[u]{} & W \end{array}$$

be a diagram in Fam($\mathcal{A}$) *such that for each* $(s,t) \in I(U) \times_{I(W)} I(V)$ *there exists a pullback*

$$\begin{array}{ccc} P_{(s,t)} & \xrightarrow{q_{s,t}} & V_t \\ {\scriptstyle p_{s,t}}\big\downarrow & & \big\downarrow{\scriptstyle v_t} \\ U_s & \xrightarrow[u_s]{} & W_{I(u)(s)} \ \big(= W_{I(v)(t)}\big) \end{array}$$

in $\mathcal{A}$. *Then the family* P *of all* $P_{(s,t)}$ *together with the morphisms* $p_{s,t}$ *and* $q_{s,t}$ *forms a pullback of* u *and* v. □

In particular consider a pullback of the form

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\pi_2} & H(X) \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle H(\varphi)} \\ B & \xrightarrow[\eta_B]{} & HI(B), \end{array}$$

where

- $\eta\colon 1 \longrightarrow HI$ is the unit of the adjunction $I \dashv H$,
- B any object in Fam($\mathcal{A}$),
- X any set and φ a map from X to $I(B)$.

Identifying HI with the identity functor of Set, we can describe the object $B \times_{HI(B)} H(X)$ as follows.

Corollary 6.2.5 $B \times_{HI(B)} H(X)$ *is the family* $\big(B_{\varphi(x)}\big)_{x\in X}$ *where*

- *the projection* $\pi_1\colon \big(B_{\varphi(x)}\big)_{x\in X} \longrightarrow (B_i)_{i\in I(B)}$ *is determined by the map* $\varphi\colon X \longrightarrow I(B)$ *and the family* $B_{\varphi(x)} \longrightarrow B_{\varphi(x)}$, $x \in X$, *of identity morphisms,*

- *the projection* $\pi_2\colon \big(B_{\varphi(x)}\big)_{x\in X} \longrightarrow (1)_{x\in X}$ *is determined by the identity map* $X \longrightarrow X$ *and the family* $B_{\varphi(x)} \longrightarrow 1$, $x \in X$, *of morphisms.*

In particular that pullback always exists. $\square$

If $X = 1$, i.e. X is a one element set, then φ is determined by its image, which is a one element subset $\{i\}$ in $I(B)$; we will write $\varphi = \tilde{i}$. The inclusion $\{i\} \longrightarrow I(B)$ together with the identity morphism $B_i \longrightarrow B_i$ determines a morphism $[i]\colon B_i \longrightarrow B$, where B_i is considered as a one member family. Applying corollary 6.2.5 (up to an isomorphism) we obtain

Corollary 6.2.6 *The diagram*

$$\begin{tikzcd} B_i \arrow[r] \arrow[d, "{[i]}"'] & 1 = H(1) \arrow[d, "H(\tilde{i})"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

is a pullback for each $i \in I(B)$. *That is, the "connected components" of* B *can be described as the pullbacks of* η_B *along (*H*-images of) all possible maps from a (fixed) one element set* $I(B)$. $\square$

Finally, let us mention the "most straightforward" consequence of lemma 6.2.3:

Corollary 6.2.7 *The canonical embedding* $\mathcal{A} \longrightarrow \mathsf{Fam}(\mathcal{A})$ *preserves all limits which exist in* $\mathcal{A}$. $\square$

Readers familiar with fibrations of categories will of course recognize that all these observations on limits are special cases of simple and more elegant results (on general fibrations). Indeed, the functor $I\colon \mathsf{Fam}(\mathcal{A}) \longrightarrow \mathsf{Set}$ of 6.1 is the basic example of a fibration.

6.3 Involving extensivity

Having in mind the adjunction $I \dashv H$ between $\mathsf{Fam}(\mathcal{A})$ and Set, let us recall (with an adapted notation) from chapter 5:

For an abstract adjunction

$$\mathcal{C} \underset{H}{\overset{I}{\rightleftarrows}} \mathcal{X}, \quad \eta \colon \mathsf{id}_{\mathcal{C}} \Rightarrow HI, \quad \varepsilon \colon IH \Rightarrow \mathsf{id}_{\mathcal{X}}$$

between categories $\mathcal{C}$ and $\mathcal{X}$ with pullbacks, and an object B in $\mathcal{C}$, there is an induced adjunction

$$\mathcal{C}/B \underset{H^B}{\overset{I^B}{\rightleftarrows}} \mathcal{X}/I(B), \quad \eta^B \colon \mathsf{id}_{\mathcal{C}/\mathcal{B}} \Rightarrow H^B I^B, \quad \varepsilon^B \colon I^B H^B \Rightarrow \mathsf{id}_{\mathcal{X}/I(B)}$$

in which

- $I^B(A,\alpha) = \big(I(A), I(\alpha)\big)$, i.e. I^B sends a morphism $\alpha \colon A \longrightarrow B$, considered as an object (A,α) in $\mathcal{C}/B$, to $I(\alpha) \colon I(A) \longrightarrow I(B)$ considered as $\big(I(A), I(\alpha)\big) \in \mathcal{X}/I(B)$,
- $H^B(X,\varphi) = \big(B \times_{HI(B)} H(X), \pi_1\big)$ constructed as the pullback

$$\begin{tikzcd} B \times_{HI(B)} H(X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(X) \arrow[d, "H(\varphi)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

- $\eta^B_{(A,\alpha)} = \langle \alpha, \eta_A \rangle \colon A \longrightarrow B \times_{HI(B)} HI(A)$, i.e. $\eta^B_{(A,\alpha)}$ is the morphism making diagram 6.1 commute,
- $\varepsilon^B_{(X,\varphi)}$ is the composite

$$I\big(B \times_{HI(B)} H(X)\big) \xrightarrow{I(\pi_2)} IH(X) \xrightarrow{\varepsilon_X} X$$

in the notation above.

Corollary 6.2.5 indicates very clearly that in our special case $(I \dashv H) \colon \mathsf{Fam}(\mathcal{A}) \longrightarrow \mathsf{Set}$, this description is to be simplified. In order to do this nicely, we need to recall

Definition 6.3.1 A category $\mathcal{C}$ with coproducts is said to be (infinitary) extensive if for every family $(C_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathcal{C}$ the coproduct functor

$$\coprod \colon \prod_{\lambda\in\Lambda} \mathcal{C}/C_\lambda \longrightarrow \mathcal{C} \Big/ \coprod_{\lambda\in\Lambda} C_\lambda,$$

$$\begin{tikzcd}
A \arrow[dr, "\eta^B_{(A,\alpha)}"] \arrow[drr, "\eta_A"] \arrow[ddr, "\alpha"'] & & \\
& B \times_{HI(B)} HI(A) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & HI(A) \arrow[d, "HI(\alpha)"] \\
& B \arrow[r, "\eta_B"'] & HI(B)
\end{tikzcd}$$

Diagram 6.1

$$(f_\lambda\colon A_\lambda \to C_\lambda)_{\lambda\in\Lambda} \mapsto \left(\coprod_{\lambda\in\Lambda} f_\lambda\colon \coprod_{\lambda\in\Lambda} A_\lambda \to \coprod_{\lambda\in\Lambda} C_\lambda\right)$$

is a category equivalence.

Proposition 6.3.2 *Every category of the form* $\mathsf{Fam}(\mathcal{A})$ *is extensive.*

Proof Let us just recall the obvious construction of the inverse functor: Since $I\left(\coprod_{\lambda\in\Lambda} C_\lambda\right)$ is the disjoint union of the sets $I(C_\lambda)$ $(\lambda \in \Lambda)$, for a given object

$$f\colon A \longrightarrow \coprod_{\lambda\in\Lambda} C_\lambda$$

in $\mathcal{C}/\coprod_{\lambda\in\Lambda} C_\lambda$, we define A_λ as the "subfamily"

$$A_\lambda = (A_i)_{i\in I(f)^{-1}\left(I(C_\lambda)\right)}$$

of A, and define

$$f_\lambda\colon A_\lambda \longrightarrow C_\lambda$$

as the restriction of f to that subfamily. The correspondence

$$(A, f) \mapsto \big((A_\lambda, f_\lambda)\big)_{\lambda\in\Lambda}$$

determines the desired inverse functor from $\mathcal{C}/\coprod_{\lambda\in\Lambda} C_\lambda$ to $\prod_{\lambda\in\Lambda} \mathcal{C}/C_\lambda$. It is a routine calculation to check that both of the composites are identity functors (up to isomorphism of course). $\square$

In the next proposition we will write $I_{\mathcal{A}} \dashv H_{\mathcal{A}}$ instead of $I \dashv H$, which we must do because not just $\mathsf{Fam}(\mathcal{A})$ but also $\mathsf{Fam}(\mathcal{A}/B_i)$ (see below) is involved. However, we will keep using $I^B \dashv H^B$.

Proposition 6.3.3 *There is a canonical equivalence of adjunctions*

$$\begin{tikzcd}
\mathsf{Fam}(\mathcal{A})/B \arrow[r, shift left, "I^B"] \arrow[d, leftrightarrow, "\sim"'] & \mathsf{Set}/I(B) \arrow[l, shift left, "H^B"] \arrow[d, leftrightarrow, "\sim"] \\
\prod_{i\in I(B)} \mathsf{Fam}(\mathcal{A}/B_i) \arrow[r, shift left, "\prod_{i\in I(B)} I_{\mathcal{A}/B_i}"] & \prod_{i\in I(B)} \mathsf{Set}. \arrow[l, shift left, "\prod_{i\in I(B)} H_{\mathcal{A}/B_i}"]
\end{tikzcd}$$

Proof Since $\mathsf{Fam}(\mathcal{A})$ is extensive we have

$$\begin{aligned}
\mathsf{Fam}(\mathcal{A})/B &\sim \mathsf{Fam}(\mathcal{A}) \Big/ \coprod_{i\in I(B)} B_i \\
&\sim \prod_{i\in I(B)} \mathsf{Fam}(\mathcal{A})/B_i \sim \prod_{i\in I(B)} \mathsf{Fam}(\mathcal{A}/B_i)
\end{aligned}$$

and since $\mathcal{A}/B_i$ has a terminal object, $I_{\mathcal{A}/B_i}$ and $H_{\mathcal{A}/B_i}$ are well defined. The comutativity of the square involving I^B and $\prod_{i\in I(B)} I_{\mathcal{A}/B_i}$ is obvious. □

There are various conclusions and remarks to be made. Let us put them as

Conclusions 6.3.4

(i) The alternative description of the adjunction $I^B \dashv H^B$ given by proposition 6.3.3 should be considered as the "external version" of the description given above. This is a good example of a situation where the external description is simpler than the internal, although the internal one also works in more general situations (namely, for arbitrary adjunctions).

(ii) Proposition 6.3.3 appropriately reformulated tells us that $\mathsf{Fam}(\mathcal{A})$ always has those pullbacks which are needed to construct $I^B \dashv H^B$ for any $B \in \mathsf{Fam}(\mathcal{A})$, which also follows from corollary 6.2.5.

On the other hand since the pullback constructed in 6.2.5 is involved in the construction of H^B, one can deduce 6.2.5 from 6.3.3 without using lemma 6.2.3 (or corollary 6.2.4).

(iii) Proposition 6.3.3 and corollary 6.2.5 also independently tell us that the counit $\varepsilon^B\colon I^B H^B \Rightarrow \mathsf{id}$ is an isomorphism for any $\mathcal{A}$ and any $B \in \mathsf{Fam}(\mathcal{A})$.

(iv) The fact that "the extensivity helped us to remove pullbacks from the construction of $I^B \dashv H^B$" is not at all surprising: just note that the right adjoint of the coproduct functor used in definition 6.3.1 is given by the family of the pullback functors along all coproduct injections $C_\lambda \longrightarrow \coprod_{\lambda\in\Lambda} C_\lambda$ (and those pullbacks therefore exist in any extensive category).

(v) The whole story can of course be repeated in the more general context where the existence of a terminal object in $\mathcal{A}$ is not required. Although the functor H itself would not exist, we will still have the adjunctions $I^B \dashv H^B$, $B \in \mathcal{C}$, defined now via the equivalence of proposition 6.3.3, since each $\mathcal{A}/B_i$ has a terminal object and therefore we can use $I_{\mathcal{A}/B_i} \dashv H_{\mathcal{A}/B_i}$, $i \in I(B)$. $\square$

6.4 Local connectedness and étale maps

The category Top of topological spaces is extensive, but not of the form $\mathsf{Fam}(\mathcal{A})$. On the other hand the category of topological spaces with open connected components (see proposition 6.1.1), which is the "obvious best replacement" of Top by a category of the form $\mathsf{Fam}(\mathcal{A})$, does not have pullbacks. In this section we consider the category LoCo with objects all locally connected topological spaces (see 6.4.3 below) and morphisms all étale maps (= local homeomorphisms) between them. This category has "all nice properties" and is still large enough to contain all covering maps which categorical Galois theory is to be applied for.

The étale maps were already considered in the previous chapters (see definition 4.2.13); let us begin here with

Proposition 6.4.1 *A continuous map $\alpha\colon A \longrightarrow B$ is étale if and only if it is open, and locally injective, i.e. every $a \in A$ has an open neighbourhood U such that the restriction $\alpha|_U\colon U \longrightarrow B$ is injective.* $\square$

Also note that for any family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets of a topological space B, the canonical map

$$\coprod_{\lambda\in\Lambda} U_\lambda \longrightarrow B$$

is étale; we call this map the map associated with the family $(U_\lambda)_{\lambda\in\Lambda}$. In fact it is more convenient to make this construction "up to a homeomorphism", i.e. to replace a family of open subsets by a family of spaces equipped with injective homeomorphisms into the space B.

Proposition 6.4.2 *For every étale map $p\colon E\longrightarrow B$ there exist a family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets in B and a factorization*

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda} U_\lambda \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

in which f is the map associated with the family $(U_\lambda)_{\lambda\in\Lambda}$ and g is a surjective étale map. Moreover, g also is the map associated with $(U_\lambda)_{\lambda\in\Lambda}$ (up to a homeomorphism).

Proof Just take $(V_\lambda)_{\lambda\in\Lambda}$ to be the family of all open subsets in E for which the restriction p is injective, and take $U_\lambda = p(V_\lambda)$ for all $\lambda\in\Lambda$. □

The next proposition explains the reason why the category LoCo is better for our purposes than the category Top:

Proposition 6.4.3 *For a topological space B the following conditions are equivalent:*

(i) *B is locally connected, i.e. every open subset in B has open connected components;*
(ii) *for every étale map $\alpha\colon A\longrightarrow B$, the space A has open connected components;*
(iii) *for every étale map $\alpha\colon A\longrightarrow B$, the space A is locally connected.*

Proof (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is trivial; (i) $\Rightarrow$ (ii) follows from the obvious fact that a topological space which is a union of open subsets with open connected components itself has the same property; (ii) $\Rightarrow$ (iii) follows from (i) $\Leftrightarrow$ (ii) and the (easy) fact that the class of étale maps is closed under composition. □

We will also need

Lemma 6.4.4 *Let*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{h} Z$$

be a coequalizer diagram in Top*, in which f and g are open maps. Then the map h is also open.*

Proof Just note that since h is a quotient map, a subset Z' in Z is open if and only if its inverse image $h^{-1}(Z')$ is open. Whenever Y' is open in Y, so are also $fg^{-1}(Y')$ and $gf^{-1}(Y')$ and $hh^{-1}(Y')$ is obtained from Y' by taking the union of all possible iterations $fg^{-1}(-)$ and $gf^{-1}(-)$. □

Lemma 6.4.5 *Let*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

be a commutative diagram in Top *with f étale. Then*

(i) *if g is étale, so is h,*
(ii) *if h is open and surjective, then g and h are étale.*

Proof (i) For a point x in X choose open subsets $X' \subseteq X$ and $Y' \subseteq Y$ such that x is in X', $h(x)$ in Y', and the restrictions $f|_{X'}$ and $g|_{Y'}$ are injective. Then take $U = X' \cap h^{-1}(Y')$ and consider the commutative diagram

$$\begin{tikzcd} U \arrow[rr, "h'"] \arrow[dr, "f|_U"'] & & Y' \arrow[dl, "g|_{Y'}"] \\ & Z & \end{tikzcd}$$

in which h' is induced by h.

Since U is a subset of X', the map $f|_U$ is injective, and therefore so is h'. On the other hand since $g|_{Y'}$ is also injective, for every $V \subseteq U$ we have

$$h'(V) = (g|_{Y'})^{-1} f|_U(V),$$

and so since $f|_U$ is open, so is h'.

Since U is open (which uses the fact that h is continuous!), and this can be done for any x in X, we conclude that h is étale.

(ii) Of course we only need to prove that g is étale. For any $y \in Y$ take $x \in X$ with $h(x) = y$ and an open neighbourhood X' of x for which f induces a homeomorphism $X' \longrightarrow f(X')$. Then $h(X')$ is an open neighbourhood of y for which $g|_{h(X')}$ is injective. That is g is locally injective. On the other hand since f is open and h surjective, we know that g is open: indeed, just note that for every $Y' \subseteq Y$ we have $g(Y') = f\big(h^{-1}(Y')\big)$. $\square$

Lemma 6.4.6 *If*

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a pullback diagram in Top *with α étale, then π_1 also is étale.*

Proof For any $(e, a) \in E \times_B A$ choose an open neighbourhood U of a for which α induces a homemorphism $U \longrightarrow \alpha(U)$; then $\pi_2^{-1}(U) \approx E \times_B U$ is open in $E \times_B A$ and π_1 induces a homeomorphism

$$\pi_2^{-1}(U) \longrightarrow p^{-1}\big(\alpha(U)\big) \approx E \times_B \alpha(U). \qquad \square$$

Proposition 6.4.7 *Let* Étale *be the subcategory of* Top *with the same objects and morphisms all étale maps. Then for every topological space B we have*

(i) Étale$/B$ *is a full subcategory in* Top$/B$ *closed under colimits and finite limits,*

(ii) *the forgetful functor* Étale$/B \longrightarrow$ Set *reflects isomorphisms and preserves colimits and pullbacks.*

Proof (i) The fact that $\mathsf{\acute{E}tale}/B$ is a full subcategory of Top/B is just a reformulation of 6.4.5(i). It is obviously closed under coproducts, closed under coequalizers by 6.4.4 and 6.4.5(ii), and therefore closed under all colimits. It also contains the terminal object $(B, 1_B)$ and is closed under pullbacks by 6.4.6 – and so it is closed under all finite limits.

(ii) The reflection of isomorphisms is obvious, and the preservation of colimits and finite limits follows from (i). □

Finally, let us list "all the nice properties" of LoCo, which easily follow from the previous results:

Proposition 6.4.8

(i) LoCo *is canonically equivalent to* $\mathsf{Fam}(\mathsf{CLoCo})$, *where* CLoCo *is the full subcategory in* LoCo *with objects all connected locally connected spaces.*

(ii) *For every locally connected space* B, LoCo/B *is a full subcategory in* Top/B *closed under colimits and finite limits, and the forgetful functor* $\mathsf{LoCo}/B \longrightarrow \mathsf{Set}$ *reflects isomorphisms and preserves colimits and pullbacks.*

(iii) *All pullback functors in* LoCo *(and also in* $\mathsf{\acute{E}tale}$*) preserve colimits.* □

A fundamental property definitely missed in this list is the fact that each LoCo/B is a topos. However, we prefer not to use it in order to make clear that the theory of covering spaces can be developed categorically without any use of topos theory.

6.5 Localization and covering morphisms

Let $\mathcal{M}$ be a class of continuous maps between topological spaces. Classically one says that a map $\alpha\colon A \longrightarrow B$ is locally in $\mathcal{M}$ if every point b in B has an open neighbouhood U such that the map $\alpha^{-1}(U) \longrightarrow U$ induced by α belongs to $\mathcal{M}$. Equivalently, there exists a family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets in B such that $B = \bigcup_{\lambda\in\Lambda} U_\lambda$ and each $\alpha^{-1}(U_\lambda) \longrightarrow U_\lambda$ belongs to $\mathcal{M}$.

Let us make some simple remarks on that notion of "being locally in $\mathcal{M}$", always assuming

Convention *Throughout this section,* $\mathcal{M}$ *denotes a pullback stable class of morphisms in the category* Top *of topological spaces and continuous maps.*

Proposition 6.5.1 *Let*

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

be a pullback diagram in Top. *Then*

(i) *if α is locally in $\mathcal{M}$, then so is π_1,*

(ii) *if p is surjective and étale, and π_1 is locally in $\mathcal{M}$, then α also is locally in $\mathcal{M}$.*

Proof (i) Just take an appropriate family $(U_\lambda)_{\lambda\in\Lambda}$ of open subsets in B and note that the diagram

$$\begin{tikzcd} \pi_1^{-1}(p^{-1}(U_\lambda)) \arrow[r] \arrow[d] & \alpha^{-1}(U_\lambda) \arrow[d] \\ p^{-1}(U_\lambda) \arrow[r] & U_\lambda \end{tikzcd}$$

is a pullback for each $\lambda \in \Lambda$.

(ii) For each point b in B we choose

- e in E with $p(e) = b$,
- an open neighbourhood U of e for which p induces a homemorphism $U \longrightarrow p(U)$,
- an open neighbourhood V of e for which $\pi_1^{-1}(V) \longrightarrow V$ is in $\mathcal{M}$,
- $W = p(U \cap V)$, which is open and contains b.

Since the diagram

$$\begin{tikzcd} \pi_1^{-1}(U \cap V) \arrow[r, "\subseteq"] \arrow[d] & \pi_1^{-1}(V) \arrow[d] \\ U \cap V \arrow[r, "\subseteq"'] & V \end{tikzcd}$$

is a pullback, $\pi_1^{-1}(U \cap V) \longrightarrow U \cap V$ is in $\mathcal{M}$. The homeomorphism $U \longrightarrow p(U)$ induces a homeomorphism $U \cap V \longrightarrow p(U \cap V) = W$ and hence also a homeomorphism $\pi_1^{-1}(U \cap V) \longrightarrow \alpha^{-1}(W)$. The "trivial pullback"

$$\begin{tikzcd}
\alpha^{-1}(W) \arrow[r] \arrow[d] & \pi_1^{-1}(U \cap V) \arrow[d] \\
W \arrow[r] & U \cap V
\end{tikzcd}$$

then shows that $\alpha^{-1}(W) \longrightarrow W$ is in $\mathcal{M}$. □

Now let $\Sigma\mathcal{M}$ be the class of all continuous maps (isomorphic to the maps) of the form

$$\coprod_{\lambda \in \Lambda} f_\lambda \colon \coprod_{\lambda \in \Lambda} X_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Y_\lambda,$$

where each $f_\lambda \colon X_\lambda \longrightarrow Y_\lambda$ is in $\mathcal{M}$; this new class obviously contains $\mathcal{M}$ and is pullback stable.

Proposition 6.5.2 *For a continuous map $\alpha \colon A \longrightarrow B$ the following conditions are equivalent:*

(i) *α is locally in $\mathcal{M}$;*

(ii) *α is locally in $\Sigma\mathcal{M}$;*

(iii) *there exists a surjective étale map $p \colon E \longrightarrow B$ such that the pullback $\pi_1 \colon E \times_B A \longrightarrow E$ of α along p is in $\Sigma\mathcal{M}$.*

Proof Since (i) ⇒ (ii) is trivial and (iii) ⇒ (ii) follows from 6.5.1(ii), it suffices to prove (ii) ⇒ (i) and (i) ⇒ (iii).

(ii) ⇒ (i) The condition (ii) tells us that every $b \in B$ has an open neighbourhood U such that $\alpha^{-1}(U) \longrightarrow U$ can be identified with some

$$\coprod_{\lambda \in \Lambda} f_\lambda \colon \coprod_{\lambda \in \Lambda} X_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} Y_\lambda,$$

with each f_λ in $\mathcal{M}$. Then b belongs to one of the Y_λ, and since the diagram

$$\begin{tikzcd} X_\lambda \arrow[r] \arrow[d, "f_\lambda"'] & \coprod_{\lambda\in\Lambda} X_\lambda \arrow[d, "\coprod_{\lambda\in\Lambda} f_\lambda"] \\ Y_\lambda \arrow[r] & \coprod_{\lambda\in\Lambda} Y_\lambda \end{tikzcd}$$

is a pullback and therefore $\alpha^{-1}(Y_\lambda) \longrightarrow Y_\lambda$ can be identified with f_λ, we conclude that $\alpha^{-1}(Y_\lambda) \longrightarrow Y_\lambda$ is in $\mathcal{M}$.

(i) $\Rightarrow$ (iii) Let $(U_\lambda)_{\lambda\in\Lambda}$ be a family of open subsets in B such that each $\alpha^{-1}(U_\lambda) \longrightarrow U_\lambda$ is in $\mathcal{M}$. We take $p\colon E \longrightarrow B$ to be the map $\coprod_{\lambda\in\Lambda} U_\lambda \longrightarrow B$ considered in the previous section, and note that the diagram

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda} \alpha^{-1}(U_\lambda) \arrow[r] \arrow[d] & A \arrow[d, "\alpha"] \\ \coprod_{\lambda\in\Lambda} U_\lambda \arrow[r] & B \end{tikzcd}$$

is a pullback, i.e. $\pi_1\colon E \times_B A \longrightarrow E$ can be identified with

$$\coprod_{\lambda\in\Lambda} \alpha^{-1}(U_\lambda) \longrightarrow \coprod_{\lambda\in\Lambda} U_\lambda.$$

Therefore π_1 is in $\Sigma\mathcal{M}$. □

That is, the notion "is locally in", which is an instance of the general idea of "localization", can be described using the pullbacks along surjective étale maps instead of the inverse images of open neighbourhoods. In order to make this language of étale maps purely categorical (in LoCo) let us prove

Proposition 6.5.3 *An étale map $p\colon E \longrightarrow B$ between locally connected topological spaces is an effective descent morphism in* LoCo *if and only if it is surjective.*

Proof As follows from the "general theory" (see section 4.4) and 6.4.8(iii), p is an effective descent morphism if and only if the functor $p^*\colon \mathsf{LoCo}/B \longrightarrow \mathsf{LoCo}/E$ reflects isomorphisms.

The image under p^* of the morphism

$$\begin{tikzcd} p(E) \arrow[rr, "\subseteq"] \arrow[dr, "\subseteq"'] & & B \arrow[dl, equal] \\ & B & \end{tikzcd}$$

is an isomorphism in LoCo/E, and so if p^* reflects isomorphisms, then $p(E) = B$, i.e. p is surjective. The converse follows from 6.4.8(ii) and the fact that any pullback functor along a surjective map in Set reflects isomorphisms. □

The next step is to arrive at the categorical notion of a covering morphism. Let us begin by recalling the classical definition:

Definition 6.5.4 A continuous map $\alpha\colon A \longrightarrow B$ of topological spaces is said to be

(i) a trivial covering map if A is a disjoint union of open subsets each of which is mapped homeomorphically onto B by α,

(ii) a covering map if every point in B has an open neighbourhood whose inverse image is a disjoint union of open subsets each of which is mapped homeomorphically onto it by α.

One might also look again at the simplest standard example of a non trivial covering map, which is the canonical map $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$, the "covering of a circle by a line".

Lemma 6.5.5 *In* Top, *every covering map is étale.* □

Proposition 6.5.6 *We have the following:*

(i) *$\alpha\colon A \longrightarrow B$ is a trivial covering map if and only if it is (up to an isomorphism) the projection $B \times X \longrightarrow B$ for some discrete X;*

(ii) *α is a covering map if and only if it is locally a trivial covering map (i.e. it is locally in the class of trivial covering maps).*

Together with 6.5.2 and 6.5.3 this gives

Proposition 6.5.7 *Let $\mathcal{M}$ be the class of trivial covering maps between locally connected topological spaces, and $\alpha\colon A \longrightarrow B$ is a morphism in* LoCo. *The following hold:*

(i) *α is in the class $\Sigma\mathcal{M}$ if and only if for every connected component B_i (with $i \in I(B)$ in the notation of section 6.1) the induced map $\alpha^{-1}(B_i) \longrightarrow B_i$ is of the form $B_i \times X_i \longrightarrow B_i$ as in 6.5.6*(i)*;*

(ii) *α is a covering map if and only if there exists an effective descent morphism $p\colon E \longrightarrow B$ (in* LoCo*) such that the pullback of α along p is in $\Sigma\mathcal{M}$.* □

Here 6.5.7(i) tells us that for a given $B \in \mathsf{LoCo}$, in order to build up a morphism $\alpha\colon A \longrightarrow B$ that belongs to $\Sigma\mathcal{M}$, we should

- choose a family $(X_i)_{i\in I(B)}$ of sets indexed by the set $I(B)$ of connected components of B,
- define $\alpha_i\colon A_i \longrightarrow B_i$ for $i \in I(B)$ as the projection $B_i \times X_i \longrightarrow B_i$ (or, equivalently as the canonical map

$$\coprod_{x\in X_i} B_i \longrightarrow B_i$$

from the coproduct of "X_i copies" of B_i to B_i),
- and then define α as

$$\alpha = \coprod_{i\in I(B)} \alpha_i\colon \coprod_{i\in I(B)} A_i \longrightarrow \coprod_{i\in I(B)} B_i = B.$$

However, as we see from proposition 6.3.3 and conclusion 6.3.4(v), this is exactly how the functor H^B from $\mathsf{Set}/I(B)$ (which is equivalent to the category $\prod_{i\in I(B)} \mathsf{Set}$ of $I(B)$-indexed families of sets) to $\mathsf{Fam}(\mathcal{A})/B$ (which is LoCo/B here) is defined. That is, since H^B is full and faithful, we have

Proposition 6.5.8 *For a morphism $\alpha\colon A \longrightarrow B$ in* LoCo *the following conditions are equivalent:*

(i) *α is in $\Sigma\mathcal{M}$ (where $\mathcal{M}$ is as in 6.5.7);*

(ii) *(A, α) is (up to an isomorphism) in the image of*

$$H^B\colon \mathsf{Set}/I(B) \longrightarrow \mathsf{LoCo}/B;$$

(iii) *the canonical morphism $(A, \alpha) \longrightarrow H^B I^B(A, \alpha)$ is an isomorphism.*

If we replace LoCo *by* LoCo/B*, or more generally by* $\mathsf{Fam}(\mathcal{A})$ *with a terminal object in $\mathcal{A}$, then this condition* (iii) *becomes the assertion that the diagram*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\alpha"'] & HI(A) \arrow[d, "HI(\alpha)"] \\ B \arrow[r, "\eta_B"'] & HI(B) \end{tikzcd}$$

(where η is the unit of $I \dashv H$) is a pullback. □

And in this context we introduce

Definition 6.5.9 A morphism $\alpha\colon A \longrightarrow B$ in $\mathcal{C} = \mathsf{Fam}(\mathcal{A})$ is said to be a covering morphism if there exists an effective descent morphism $p\colon E \longrightarrow B$ such that (A, α) is split by p in the sense of definition 5.1.7, i.e. the diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\eta_{E \times_B A}"] \arrow[d, "\pi_1"'] & HI(E \times_B A) \arrow[d, "HI(\pi_1)"] \\ E \arrow[r, "\eta_E"'] & HI(E) \end{tikzcd}$$

is a pullback.

And from 6.5.7 and 6.5.8 we obtain

Theorem 6.5.10 *An étale map $\alpha\colon A \longrightarrow B$ of locally connected topological spaces is a covering map (in the classical sense) if and only if it is a covering morphism in* LoCo/B *in the sense of definition 6.5.9.* □

6.6 Classification of coverings

We are going to describe the category $\mathsf{Cov}(B)$ of pairs (A, α), where $\alpha\colon A \longrightarrow B$ is a covering morphism with fixed $B \in \mathcal{C}$. The context we have in mind is as in definition 6.5.9 although all arguments of this section can be repeated in the context of an abstract adjunction as at the beginning of section 6.3, provided we assume H^C to be full and faithful (or, equivalently, $\varepsilon^C\colon I^C H^C \longrightarrow 1$ to be an isomorphism for every $C \in \mathcal{C}$). For simplicity we assume that $\mathcal{C}$ has pullbacks.

The full subcategory in $\mathsf{Cov}(B)$ with objects all (A, α) split by a fixed effective descent morphism $p \colon E \longrightarrow B$ will be denoted by $\mathsf{Split}_B(p)$ in accordance with the notation of section 5.1. We can simply write

$$\mathsf{Cov}(B) = \bigcup_p \mathsf{Split}_B(p) \qquad (*)$$

and say that each $\mathsf{Split}_B(p)$ can be described via the categorical Galois theory; but it is important to know that $(*)$ is in fact a "good union", and that under the existence of what we call a universal covering it reduces to its largest member. Furthermore we will show that $\mathsf{Split}_B(p)$, which is thus the largest member in which p is actually a universal covering of B, also admits p as a morphism of Galois descent (the definition will be recalled below), and therefore can be described as in the Galois theorem 5.1.24 (with $\overline{\mathcal{P}} = \mathcal{P}$ in the notation of section 5.1). As we will show in the next sections, this extends the classical classification theorem of covering spaces over a "good space".

First we need

Lemma 6.6.1 *The class of effective descent morphisms in $\mathcal{C}$ is closed under composition, pullback stable, and has the right cancellation property, i.e. if it contains p_1p_2, then it contains p_1.*

Proof Let us restrict ourselves to the situation where the pullback functors in $\mathcal{C}$ preserve coequalizers (whose existence we also assume in this proof). In particular this condition holds in LoCo (see 6.4.8(iii)) and in many other important examples. As we already mentioned in the proof of proposition 6.5.3, under this condition, p is an effective descent morphism if and only if $p^* \colon \mathcal{C}/B \longrightarrow \mathcal{C}/E$ reflects isomorphisms.

The closedness under composition and the right cancellation property are now obvious since $(p_1p_2)^* = p_2^*p_1^*$, and the pullback stability easily follows from the so-called Beck–Chevalley property ("condition"): if the first square below is a pullback diagram, the second square commutes up to isomorphism.

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{C}/D \arrow[d, "\delta!"'] & \mathcal{C}/A \arrow[l, "q^*"'] \arrow[d, "\alpha!"] \\ \mathcal{C}/E & \mathcal{C}/B \arrow[l, "p^*"] \end{tikzcd}$$

$$\begin{tikzcd}
 & \mathcal{X}/I(D) \arrow[ld, "H^D"'] & & \mathcal{X}/I(A) \arrow[ll, "I(q)^*"'] \arrow[ld, "H^A"'] \\
\mathcal{C}/D & & \mathcal{C}/A \arrow[ll, "q^*"'] & \\
 & \mathcal{X}/I(E) \arrow[uu, "I(\delta)^*"] \arrow[ld, "H^E"'] & & \mathcal{X}/I(B) \arrow[ll, "I(p)^*"'] \arrow[uu, "I(\alpha)^*"'] \arrow[ld, "H^B"'] \\
\mathcal{C}/E \arrow[uu, "\delta^*"] & & \mathcal{C}/B \arrow[ll, "p^*"] \arrow[uu, "\alpha^*"'] &
\end{tikzcd}$$

Diagram 6.2

Here $\alpha!$ is the composition with α, which obviously reflects isomorphisms. $\square$

Corollary 6.6.2 *The class of covering morphisms of $\mathcal{C}$ is pullback stable.*

Proof Although lemma 6.6.1 makes this straightforward, let us explain the details.

For a given pullback diagram as in the proof of lemma 6.6.1, consider diagram 6.2. "Up to isomorphism" we can say

- the diagram commutes,
- the image of H^B coincides with the category of "trivial coverings" of B, i.e. the category of those objects in $\mathcal{C}/B$ which are split by the identity morphism of B,
- since the diagram commutes, p^*, q^*, α^* and δ^* send trivial coverings to trivial coverings,
- $\mathsf{Split}_B(p)$ is the category of all those objects in $\mathcal{C}/B$ which are sent to trivial coverings by p^*, and of course the same is true for $\mathsf{Split}_A(q)$ (with $q^*\colon \mathcal{C}/A \longrightarrow \mathcal{C}/D$ instead of $p^*\colon \mathcal{C}/B \longrightarrow \mathcal{C}/E$),
- therefore α^* restricts to a functor $\mathsf{Split}_B(p) \longrightarrow \mathsf{Split}_A(q)$.

Together with $(*)$ this tells us that the class of covering morphisms is pullback stable as desired. $\square$

Proposition 6.6.3 *The union* $(*)$ *has the following properties:*

(i) $p = p'p''$ *implies* $\mathsf{Split}_B(p') \subseteq \mathsf{Split}_B(p)$*;*

(ii) *the union is directed, i.e. for any* $p_i \colon E_i \longrightarrow B$*,* $i = 1, 2$*, there exists* $p \colon E \longrightarrow B$ *with* $\mathsf{Split}_B(p_i) \subseteq \mathsf{Split}_B(p)$*,* $i = 1, 2$*.*

Proof (i) Given a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p''"'] & & B \\ & E' \arrow[ur, "p'"'] & \end{tikzcd}$$

we have ("up to isomorphism" as in the proof of corollary 6.6.2) the following:

- if (A, α) is in $\mathsf{Split}_B(p')$, then $p'^*(A, \alpha)$ is in the image of $H^{E'}$ and therefore $p^*(A, \alpha) = p''^* p'^*(A, \alpha)$ is in the image of H^E;
- if $p^*(A, \alpha)$ is in the image of H^E, then (A, α) is in $\mathsf{Split}_B(p)$ (by definition).

Now (ii) follows from (i), since we can form the pullback

$$\begin{tikzcd} E_1 \times_B E_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "p_2"] \\ E_1 \arrow[r, "p_1"'] & B \end{tikzcd}$$

and use lemma 6.6.1 to show that $p_1\pi_1 = p_2\pi_2$ is an effective descent morphism (when so are p_1 and p_2). □

Remark 6.6.4 The morphism p'' in 6.6.3(i) may not be an effective descent morphism, which is important for various applications.

Now we introduce

Definition 6.6.5

(i) An object E in $\mathcal{C}$ is said to be Galois closed if it has no non trivial coverings, i.e. every covering morphism $E' \longrightarrow E$ is split by the identity morphism of E.

(ii) A covering morphism $p\colon E \longrightarrow B$ is said to be a universal covering (of B) if it is an effective descent morphism, and E is Galois closed.

Proposition 6.6.6 *Every universal covering morphism $p\colon E \longrightarrow B$ has the following properties:*

(i) *it is a morphism of Galois descent, i.e. $(E,p) \in \mathsf{Split}_B(p)$;*
(ii) $\mathsf{Cov}(B) = \mathsf{Split}_B(p)$.

Proof Consider the pullback

$$\begin{array}{ccc} E \times_B A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

where $\alpha\colon A \longrightarrow B$ is a covering morphism. Corollary 6.6.2 tells us that $\pi_1 : E \times_B A \longrightarrow E$ is a covering morphism, and since E is Galois closed we conclude that (A, α) is in $\mathsf{Split}_B(p)$. This proves (ii) and therefore also (i). $\square$

Now let us describe more precisely the relationship between the (terminology and) notation of this section and of section 5.1.

The data

$$\overline{\mathcal{A}} \subseteq \mathcal{A} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathcal{P} \supseteq \overline{\mathcal{P}}$$

of section 5.1 have now $\overline{\mathcal{A}} = \mathcal{A}$ and $\overline{\mathcal{P}} = \mathcal{P}$, i.e. $\overline{\mathcal{A}}$ and $\overline{\mathcal{P}}$ are the classes of all morphisms in $\mathcal{A}$ and $\mathcal{P}$ respectively ($\overline{\mathcal{P}} = \mathcal{P}$ was already mentioned before lemma 6.6.1) with the following correspondence for the notation:

Section 5.1	This section
$\mathcal{A}$	$\mathcal{C} = \mathsf{Fam}(\mathcal{A})$
$\mathcal{P}$	$\mathcal{X} = \mathsf{Set}$
$\mathcal{C}$	H
$\mathcal{S}$	I

Accordingly, definition 5.1.8 is to be translated as follows: a morphism $p\colon E \longrightarrow B$ is of Galois descent (we do not say "relative" since now $\overline{\mathcal{A}} = \mathcal{A}$ and $\overline{\mathcal{P}} = \mathcal{P}$, just as in chapter 5) if

- p is an effective descent morphism,
- $\varepsilon^E : I^E H^E \longrightarrow 1$ is an isomorphism – however, we can omit this condition since now ε^C is an isomorphism for every object $C \in \mathcal{C}$ –
- $\Sigma_p H^E(X, \varphi) \in \mathsf{Split}_B(p)$ for every object (X, φ) in $\mathcal{C}/I(E)$ – however, this is equivalent to $(E, p) \in \mathsf{Split}_B(p)$ required in 6.6.6(i) (the equivalence follows from proposition 5.5.6 and lemma 5.5.7).

For such a $p \colon E \longrightarrow B$ the Galois groupoid $\mathsf{Gal}\,[p]$ is displayed as

$$\begin{tikzcd} I(E \times_B E) \times_{I(E)} I(E \times_B E) \arrow[r] & I(E \times_B E) \arrow[loop below] \arrow[r, shift left=2] \arrow[r, shift right=2] & I(E) \arrow[l] \end{tikzcd},$$

or, equivalently, as

$$\begin{tikzcd} I(E \times_B E \times_B E) \arrow[r] & I(E \times_B E) \arrow[loop below] \arrow[r, shift left=2] \arrow[r, shift right=2] & I(E) \arrow[l] \end{tikzcd},$$

where

- the domain and codomain morphisms $I(E \times_B E) \longrightarrow I(E)$ are the I-images of the two projections $E \times_B E \longrightarrow E$,
- the composition morphism $I(E \times_B E \times_B E) \longrightarrow I(E \times_B E)$ is the I-image of the morphism $E \times_B E \times_B E \longrightarrow E \times_B E$ induced by the first and the third projection $E \times_B E \times_B E \longrightarrow E$,
- the identity morphism $I(E) \longrightarrow I(E \times_B E)$ is the I-image of the diagonal $E \longrightarrow E \times_B E$,
- the inverse $I(E \times_B E) \longrightarrow I(E \times_B E)$ is the I-image of the "interchange morphism" $E \times_B E \longrightarrow E \times_B E$.

After this, from the Galois theorem 5.1.24 and proposition 6.6.6, we obtain the following classification theorem for coverings.

Theorem 6.6.7 *If $p \colon E \longrightarrow B$ is a universal covering morphism, then there exists a category equivalence*

$$\mathsf{Cov}(B) \approx \mathsf{Set}^{\mathsf{Gal}\,[p]}. \qquad \square$$

6.7 The Chevalley fundamental group

There are two classical definitions of the fundamental group of a topological space which give isomorphic groups for certain "good" spaces.

- The Poincaré fundamental group $\pi_1(B, b)$ is the group of homotopy classes of paths in B from b to b. Recall that although this group depends on the choice of the base point b, it is determined uniquely up to isomorphism when B is path connected.
- The Chevalley fundamental group $\mathsf{Aut}(p) = \mathsf{Aut}_B(E, p)$ is defined only for connected spaces B which admit a universal covering map $p \colon E \longrightarrow B$ with connected E, and of course depends on it, but again, different p produce isomorphic groups. It is the group of all automorphisms u of E with $pu = p$.

The Chevalley definition is less useful for calculations, but it can be literally repeated in our general context, and in this section we are going to show that its generalized form "follows" from theorem 6.6.7, which means

- if B is connected and admits a universal covering, then there exists a universal covering $p \colon E \longrightarrow B$ with connected E,
- if p is as above and therefore $\mathsf{Gal}\,[p]$ is a group (since E is connected), then the groups $\mathsf{Gal}\,[p]$ and $\mathsf{Aut}(p)$ are isomorphic, and moreover, the isomorphism formally follows from theorem 6.6.7.

Accordingly:

Convention *In this section, the ground category $\mathcal{C}$ is supposed to be a category of the form $\mathsf{Fam}(\mathcal{A})$, and with pullbacks.*

We will need a number of simple observations listed in

Lemma 6.7.1 *Let $p \colon E \longrightarrow B$ be a universal covering morphism with connected B. Then the following hold:*

(i) $\mathsf{Gal}\,[p]$ *is a connected groupoid, i.e. for any two objects x and y in it, there exists a morphism $x \longrightarrow y$;*

(ii) *there exists a group G such that the category $\mathsf{Cov}(B)$ is equivalent to Set^G;*

(iii) *if*

$$\begin{tikzcd} A \arrow[r, "\alpha_2'"] \arrow[d, "\alpha_1'"'] & A_2 \arrow[d, "\alpha_2"] \\ A_1 \arrow[r, "\alpha_1"'] & B \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}/B$ in which α_1 and α_2 are covering morphisms and A_1 and A_2 are not initial objects (i.e. are non-empty families), then A also is not initial;

(iv) *for every covering morphism $\alpha\colon A \longrightarrow B$, in which A is not an initial object in $\mathcal{C}$, there exists a morphism $(E,p) \longrightarrow (A,\alpha)$ in $\mathcal{C}/B$, and therefore α is an effective descent morphism.*

Proof (i) Since B is connected, i.e. $I(B)$ is a one element set, the connectedness of $\mathsf{Gal}\,[p]$ is equivalent to the fact that

$$I(E \times_B E) \rightrightarrows I(E) \longrightarrow I(B)$$

is a coequalizer diagram. However, this follows from the fact that I is a left adjoint and

$$E \times_B E \rightrightarrows E \longrightarrow B \qquad (**)$$

is a coequalizer diagram. The last assertion here will be explained for readers not familiar with descent theory.

First of all it uses the existence of coequalizers in $\mathcal{C}$, otherwise we could only prove that $(**)$ is a coequalizer diagram in $\mathcal{C}/B$ (regarding B, E, $E \times_B E$ as objects in $\mathcal{C}/B$ in the obvious way) – but we could then use I^B instead of I. Now, as soon as we are in $\mathcal{C}/B$, we could either use the description of the left adjoint of the comparison functor, or directly check that the image of $(**)$ under $p^*\colon \mathcal{C}/B \longrightarrow \mathcal{C}/E$ is a split coequalizer and apply the Beck criterion as formulated in section 4.4.

Part (ii) follows from (i) and theorem 6.6.7 since every connected groupoid is equivalent to a one object groupoid, i.e. to a group.

(iii) It suffices to show that there exists a non-initial object (A',α') in $\mathcal{C}/B$ from which there are a morphism to (A_1,α_1) and a morphism to (A_2,α_2). However, such an object does exist already in $\mathsf{Cov}(B)$ – this follows from (ii) and the fact that $B = (B, 1_B)$ corresponds to a one element G-set under the equivalence $\mathsf{Cov}(B) \approx \mathsf{Set}^G$, since the product of two non-empty G-sets is always non-empty.

(iv) The existence of a morphism $(E,p) \longrightarrow (A,\alpha)$ is equivalent to the existence of a right inverse for the projection $\pi_1\colon E \times_B A \longrightarrow A$. On the other hand as we see from definition 6.5.9, π_1 has a right inverse if and only if the map

$$I(\pi_1)\colon I(E \times_B A) \longrightarrow I(E)$$

is surjective. Since

$$E \times_B A \approx \coprod_{i \in I(E)} E_i \times_B A$$

(where E_i, $i \in I(E)$, are the connected components of E) the surjectivity of $I(\pi_1)$ follows from the fact that each $E_i \times_B A$ is non-initial, which itself follows from (iii) since each $E_i \longrightarrow B$ obviously is a covering morphism.

The fact that α is an effective descent morphism follows now from lemma 6.6.1. $\square$

Corollary 6.7.2 *Under the assumptions of lemma 6.7.1, the following conditions on a covering morphism $\alpha\colon A \longrightarrow B$ are equivalent:*

(i) *α is a universal covering morphism;*

(ii) *A is non-initial and there exists a morphism $(A, \alpha) \longrightarrow (E, p)$ in $\mathcal{C}/B$;*

(iii) *A is non trivial and for every covering morphism $\alpha'\colon A' \longrightarrow B$ with non-initial A', there exists a morphism $(A, \alpha) \longrightarrow (A', \alpha')$;*

(iv) *(A, α) corresponds to a non-empty free G-set under the equivalence $\mathsf{Cov}(B) \approx \mathsf{Set}^G$ of 6.7.1*(ii)*;*

(v) *each connected component of (A, α) determines a universal covering morphism to B.*

Proof (i) $\Rightarrow$ (ii) follows from 6.7.1(iv): just replace (E, p) and (A, α) by each other.

(ii) $\Rightarrow$ (iii) also follows from 6.7.1(iv) and (iii) $\Rightarrow$ (ii) is trivial.

(iii) $\Rightarrow$ (iv) follows from the fact that only a free G-set can have a morphism to any other (non-empty) G-set.

(iv) $\Rightarrow$ (i) follows from the fact that (A, α) and (E, p) have isomorphic connected components (i.e. each connected component of (A, α) is isomorphic to each connected component of (E, p)), since (E, p) also corresponds to a free G-set by (i) $\Rightarrow$ (iv) applied to (E, p). However, we should also observe that α is an effective descent morphism by 6.7.1(iv).

Thus the conditions (i)–(iv) are equivalent to each other; this also implies that they are equivalent to (v). $\square$

Since there exists exactly one (up to an isomorphism) connected free G-set – namely G itself considered as a G-set – we also obtain

Corollary 6.7.3 *Every connected object B in $\mathcal{C}$ which admits a universal covering does admit a unique (up to isomorphism) connected universal covering, i.e. a universal covering $p\colon E \longrightarrow B$ with connected E.* $\square$

And finally we have

Theorem 6.7.4 *Let $p\colon E \longrightarrow B$ be a universal covering morphism with connected E (and therefore connected B). Then $\mathsf{Gal}\,[p]$ is a group isomorphic to $\mathsf{Aut}(p)$, the group of all automorphisms u of E with $pu = p$.*

Proof First we recall that $\mathsf{Gal}\,[p]$ is a group, i.e. one object groupoid, simply by the definition, since its set of objects is $I(E)$, which is a one element set when E is connected. We then take G as in 6.7.1(ii) and in 6.7.2(iv), and since

$$\mathsf{Set}^{\mathsf{Gal}\,[p]} \approx \mathsf{Cov}(B) \approx \mathsf{Set}^G,$$

we have $\mathsf{Gal}\,[p] \approx G$. On the other hand since (E, p) corresponds to the G-set G via the equivalence $\mathsf{Cov}(B) \approx \mathsf{Set}^G$, we obtain

$$\mathsf{Aut}(p) \approx \mathsf{Aut}(G) \approx G$$

– here $\mathsf{Aut}(G)$ is the automorphism group of G as a G-set, which is isomorphic to the group G of course.

That is, $\mathsf{Gal}\,[p] \approx \mathsf{Aut}(p)$ as desired. □

Remarks 6.7.5

(i) Since in the assumptions of theorem 6.7.4 there is a category equivalence

$$\mathsf{Cov}(B) \approx \mathsf{Set}^{\mathsf{Aut}(p)}$$

one easily establishes the standard Galois correspondence between the quotients of (E, p) which are coverings and the subgroups of $\mathsf{Aut}(p)$ – which we leave as a simple exercise for the reader.

(ii) According to the usual terminology in algebraic topology, only connected (universal) coverings should have been called universal coverings. However, this becomes less convenient in a general context, especially in the most general context of section 5.1.

6.8 Path and simply connected spaces

An obvious question which occurs when we apply the general results of previous sections to topological spaces (i.e. to $\mathcal{C} = \mathsf{LoCo}/B$ for some space B) is: given a covering map $p\colon E \longrightarrow B$, how can we find out if E is closed in the sense of definition 6.6.5? Since this is the case if and only if E has no non-trivial coverings in the classical sense, the classical theory provides a (well-known) partial answer, which is theorem 6.8.12 below. All the arguments we use in our detailed proof are very simple and can be found in any standard textbook which has a chapter on covering spaces, but they are generally mixed up in those books with a lot of other arguments needed for the "geometrical" proof of the appropriate special case of theorem 6.6.7. We will also recall here all the elementary notions of homotopy theory which we will need for that proof.

A path in a topological space B is a continuous map

$$f\colon [0,1] \longrightarrow B,$$

where

$$[0,1] = \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$$

is the (closed) interval of all real numbers between 0 and 1.

Two points b and b' in B are said to be connected by a path, if there exists a path f in B with $f(0) = b$ and $f(1) = b'$. This is clearly an equivalence relation, and the equivalence classes are called path connected components.

Two paths f and g are (homotopy) equivalent if there exists a continuous map

$$h\colon [0,1] \times [0,1] \longrightarrow B$$

with

$$\begin{gathered} h(t,0) = f(t), \quad h(t,1) = g(t), \\ f(0) = h(0,t) = g(0), \\ f(1) = h(1,t) = g(1), \end{gathered}$$

and we then write $f \sim g$ or $[f] = [g]$, denoting the equivalence class of f by $[f]$; the relation $\sim$ is clearly an equivalence relation.

Definition 6.8.1 A topological space B is said to be

(i) path connected, if it has exactly one path connected component,
(ii) simply connected, if it is path connected and every two paths f and g in B with $f(0) = g(0)$ and $f(1) = g(1)$ are equivalent,
(iii) locally path connected, if every open subset in B has open path connected components.

Consider the lifting problem

$$\begin{tikzcd} & (A,a) \arrow[d, "\alpha"] \\ (S,s) \arrow[ur, dotted, "\psi"] \arrow[r, "\varphi"'] & (B,b) \end{tikzcd}$$

which is to find, for given $\alpha\colon A \longrightarrow B$ with $\alpha(a) = b$ and $\varphi\colon S \longrightarrow B$ with $\varphi(s) = b$, a map $\psi\colon S \longrightarrow A$ with $\alpha\psi = \varphi$ and $\psi(s) = a$. Here α, φ, ψ are supposed to be continuous maps of topological spaces, and we call ψ a solution of the lifting problem. If for a given $\alpha\colon A \longrightarrow B$ and S such a lifting problem has a solution for all possible a, b, s, α, φ then we say that α has the lifting property with respect to S. In particular if this is the case for $S = [0,1]$, then we say that α has the path-lifting property. We will also speak of the unique (path-) lifting property if the ψ above is uniquely determined. Considering path-lifting we will always use the following.

Remark 6.8.2 For the path-lifting and the unique path-lifting properties we can assume that $s = 0$ in $S = [0,1]$.

Proof This follows from the fact that for every s with $0 < s < 1$ there are

(i) a canonical coequalizer diagram

$$\{s\} \rightrightarrows [0,s] \sqcup [s,1] \longrightarrow [0,1],$$

(ii) homeomorphisms $[0,1] \longrightarrow [0,s]$ and $[0,1] \longrightarrow [s,1]$, both with $0 \mapsto s$ (and there exists a homeomorphism $[0,1] \longrightarrow [0,1]$ with $0 \mapsto 1$). □

Investigating the lifting property, we begin with

Proposition 6.8.3 *Every trivial covering map (as defined in 6.5.4(i)) has the unique lifting property with respect to every connected space.*

Proof In the notation above, let

$$A = \coprod_{\lambda \in \Lambda} A_\lambda$$

be a disjoint union of open subsets, each of which is mapped homeomorphically onto B by α, and let λ_0 be a fixed element in Λ for which

$a \in A_{\lambda_0}$. If S is connected, then since A is a coproduct of A_λ, every continuous map $\psi\colon S \longrightarrow A$ with $\psi(s) = a \in A_{\lambda_0}$ must have $\psi(S) \subseteq A_{\lambda_0}$. Therefore the problem of the unique lifting reduces to the trivial case of a homeomorphism (namely of $A_{\lambda_0} \longrightarrow B$). □

The general covering maps do not have the same property; moreover, we have

Proposition 6.8.4 *Let $\alpha\colon A \longrightarrow B$ be a covering map of connected topological spaces. If B has a universal covering, then the following conditions are equivalent:*

(i) *α has the unique lifting property with respect to every connected space;*
(ii) *α has the unique lifting property with respect to B;*
(iii) *α has the lifting property with respect to B;*
(iv) *α is a split epimorphism, that is, there exists a continuous map $\beta\colon B \longrightarrow A$ with $\alpha \circ \beta = 1_B$;*
(v) *α is an isomorphism (i.e. a homeomorphism).*

Proof Since the implications (v) ⇒ (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are trivial (note that (iii) ⇒ (iv) is trivial just because A is non-empty), we only need to prove (iv) ⇒ (v). However, (iv) ⇒ (v) follows from 6.7.1(ii) since

- the connectedness in the topological sense coincides with the categorical connectedness not only in Top but also in $\mathsf{Cov}(B)$;
- under the equivalence of categories $\mathsf{Cov}(B) \approx \mathsf{Set}^G$, the morphism $\alpha\colon (A, \alpha) \longrightarrow (B, 1_B)$ in $\mathsf{Cov}(B)$ corresponds to a map from a connected G-set to a one element G-set. □

Yet, every covering map has the unique lifting property with respect to some special spaces S, in particular to $S = [0,1]^n$, which gives

- a trivial result for $n = 0$,
- the unique path-lifting property for $n = 1$,
- the unique path-equivalence-lifting property (described below) for $n = 2$.

We will prove this with all auxiliary results just for $n = 2$ since the path-equivalence-lifting is exactly what we need in order to show that certain spaces are Galois closed. However, the reader can easily reformulate all arguments for general n.

Lemma 6.8.5 *Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of open subsets in a topological space B with $B = \bigcup_{\lambda\in\Lambda} B_\lambda$ and $\varphi\colon [0,1]^2 \longrightarrow B$ is a continuous map. Then there exist finite sequences $x_1, \dots, x_k$ and $y_1, \dots, y_l$ of real numbers with*

(i) $0 = x_1 < \cdots < x_k = 1$ *and* $0 = y_1 < \cdots < y_l = 1$,
(ii) *for every* $(i,j) \in \{1,\dots,k-1\} \times \{1,\dots,l-1\}$ *there exists* $\lambda \in \Lambda$ *with*

$$f\big([x_i, x_{i+1}] \times [y_j, y_{j+1}]\big) \subseteq B_\lambda.$$

Proof We have to make a two step choice.

Step 1. For each $(x,y) \in [0,1]^2$ we choose four numbers x^-, x^+, y^-, y^+ in $[0,1]$ with the following properties:

- $x^- \leq x \leq x^+$ and $y^- \leq y \leq y^+$;
- $x^- = x$ only for $x = 0$, and $y^- = y$ only for $y = 0$;
- $x^+ = x$ only for $x = 1$, and $y^+ = y$ only for $y = 1$;
- there exists $\lambda \in \Lambda$ with

$$f\big([x^-, x^+] \times [y^-, y^+]\big) \subseteq B_\lambda.$$

This is possible because each (x,y) belongs to some $f^{-1}(B_\lambda)$, and each $f^{-1}(B_\lambda)$ is an open subset in $[0,1]^2$.

Step 2. Choose finite sequences $s_1, \dots, s_m$ and $t_1, \dots, t_n$ in $[0,1]$ with

$$[0,1]^2 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \mathsf{int}[s_i^-, s_i^+] \times [t_j^-, t_j^+],$$

using the following notation: for $u, v \in [0,1]$, $\mathsf{int}[u,v]$ is the interior of $[u,v]$ in $[0,1]$, i.e.

$$w \in \mathsf{int}[u,v] \Leftrightarrow \begin{cases} 0 \leq w < v & \text{when} \quad u = 0 \quad \text{and} \quad v \neq 1, \\ u < w < v & \text{when} \quad u \neq 0 \quad \text{and} \quad v \neq 1, \\ u < w \leq 1 & \text{when} \quad u \neq 0 \quad \text{and} \quad v = 1 \end{cases}$$

(note that the case $u = v$ never occurs above). This is possible since

$$[0,1]^2 = \bigcup_{(x,y)\in[0,1]^2} \mathsf{int}[x^-, x^+] \times \mathsf{int}[y^-, y^+]$$

and $[0,1]^2$ is a compact space.

After that we just take $x_1, \dots, x_k$ to be the set of all real numbers which occur either as s_i^- or as s_i^+, with the indices determined by the condition (i), and similarly define $y_1, \dots, y_l$ via t_j^- and t_j^+. Indeed, the condition (ii) holds since every rectangle of the form $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ is a subset of some rectangle of the form $[s_{i'}^-, s_{i'}^+] \times [t_{i'}^-, t_{i'}^+]$. $\square$

Corollary 6.8.6 *Let $\alpha\colon A \longrightarrow B$ be a covering map of topological spaces and $\varphi\colon [0,1]^2 \longrightarrow B$ a continuous map. Then there exist finite sequences $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ of real numbers with*

(a) $0 = x_1 < \cdots < x_k = 1$ *and* $0 = y_1 < \cdots < y_l = 1$,

(b) for every $(i,j) \in \{1, \ldots, k-1\} \times \{1, \ldots, l-1\}$ there exists an open subset B' in B such that

$$f\big([x_i, x_{i+1}] \times [y_j, y_{j+1}]\big) \subseteq B'$$

and the map $\alpha^{-1}(B') \longrightarrow B'$ induced by α is a trivial covering map. □

Lemma 6.8.7 *Let*

$$\begin{tikzcd}
(S_0, s_0) \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & (S_2, s_2) \arrow[d, "\tau_2"] & (A, \alpha) \arrow[d, "\alpha"] \\
(S_1, s_1) \arrow[r, "\tau_1"'] & (S, s) \arrow[r, "\varphi"'] \arrow[ur, dotted, "\psi"] & (B, b)
\end{tikzcd}$$

be a diagram in the category of pointed topological spaces in which the square part is a pushout and the triangle represents a lifting problem. Then this lifting problem has a unique solution provided the lifting problems

$$\begin{tikzcd}
 & (A, a) \arrow[d, "\alpha"] \\
(S_i, s_i) \arrow[r, "\varphi\tau_i"'] \arrow[ur, dotted, "\psi_i"] & (B, b)
\end{tikzcd}$$

($i = 0, 1, 2$, where τ_0 is defined as $\tau_1\sigma_1 = \tau_2\sigma_2$) have a unique solution.

Proof is obvious – just note ψ_1 and ψ_2 must agree on S_0 (i.e. $\psi_1\sigma_1 = \psi_2\sigma_2$) by the uniqueness of ψ_0. □

Now consider the lifting problem

$$\begin{tikzcd}
 & (A,a) \arrow[d, "\alpha"] \\
\big([0,1]^2,(0,0)\big) \arrow[r, "\varphi"'] \arrow[ur, dotted, "\psi"] & (B,b)
\end{tikzcd}$$

where $\alpha\colon A \longrightarrow B$ is a covering map.

We can choose $x_1, \ldots, x_k, y_1, \ldots, y_l$ as in corollary 6.8.6 and find a unique solution ψ using proposition 6.8.3 and lemma 6.8.7 as follows:

- we first restrict the original problem from $[0,1]^2$ to the "first small rectangle", i.e. to $[x_1, x_2] \times [y_1, y_2]$ (recall that $x_1 = 0 = y_1$), and then proposition 6.8.3 gives a unique solution ψ_{11};
- if $k \neq 2$, we obtain similarly a unique solution for

$$\begin{tikzcd}
 & \big(A, \psi_{11}(x_2, y_1)\big) \arrow[d, "\alpha"] \\
\big([x_2, x_3] \times [y_1, y_2], (x_2, y_1)\big) \arrow[r] \arrow[ur, dotted, "\psi_{21}"] & \big(B, \varphi(x_2, y_1)\big)
\end{tikzcd}$$

- then we apply lemma 6.8.7 to obtain a unique solution for the lifting problem above extended from $[x_2, x_3] \times [y_1, y_2]$ to $[x_1, x_3] \times [y_1, y_2]$ – and since that solution also extends ψ_{11}, it is also a unique solution for the original problem restricted to $[x_1, x_3] \times [y_1, y_2]$;
- then repeating similar arguments "horizontally", i.e. moving from $[x_1, x_i]$ to $[x_1, x_{i+1}]$, $i = 2, \ldots, k_1$, we extend the (unique) lifting to $[0,1] \times [y_1, y_2]$;
- finally we repeat the same "vertically", i.e. using the rectangles $[0,1] \times [y_i, y_{i+1}]$, $j = 1, \ldots, l-1$.

Moreover, using now a "two dimensional version" of remark 6.8.2 (or similar arguments directly), we can do the same with $\big([0,1]^2, (u,v)\big)$ instead of $\big([0,1]^2, (0,0)\big)$. That is, we obtain

Lemma 6.8.8 *Every covering map has the unique lifting property with respect to* $[0,1]^2$. □

Of course it is even simpler to carry out the same arguments for $[0,1]$ instead of $[0,1]^2$ and therefore obtain the unique path-lifting property for the covering maps. However, it also formally follows from lemma 6.8.8 (see the assertion (i) in corollary 6.8.10 below).

We need one more obvious lemma:

Lemma 6.8.9 *If a continuous map $\alpha\colon A \longrightarrow B$ has the unique lifting property with respect to a space S, then it has the same property with respect to any retract of S.* □

Corollary 6.8.10 *Let $\alpha\colon A \longrightarrow B$ be a continuous map which has the unique lifting property with respect to $[0,1]^2$. Then α also has*

(i) *the unique path-lifting property;*

(ii) *the path-equivalence-lifting property, i.e. if f and g are paths in A with $f(0) = g(0)$ and $\alpha f \sim \alpha g$, then $f \sim g$ (and in particular $f(1) = g(1)$).*

Proof Part (i) immediately follows from lemma 6.8.9 since $[0,1]$ is a retract of $[0,1]^2$.

(ii) Let S be the quotient space $[0,1]^2/R$, where $(x,y)R(x',y')$ if either $x = 0 = x'$, or $x = 1 = x'$ (or $(x,y) = (x',y')$), and let $\kappa_i\colon [0,1] \longrightarrow S$, $i = 0,1$, be the composites

$$[0,1] \xrightarrow{\;x \mapsto (x,i)\;} [0,1]^2 \xrightarrow{\;\text{canonical map}\;} S.$$

Then $\alpha f \sim \alpha g$ produces a continous map $\varphi\colon S \longrightarrow B$ with $\varphi\kappa_0 = \alpha f$ and $\varphi\kappa_1 = \alpha g$, and since S is obviously a retract of $[0,1]^2$ lemma 6.8.9 gives a (unique) solution of the lifting problem

$$\begin{tikzcd}
 & (A,a) \arrow[d, "\alpha"] \\
(S, \mathsf{cls}_R(0,0)) \arrow[ur, dotted, "\psi"] \arrow[r, "\varphi"'] & (B,b)
\end{tikzcd}$$

where $f(0) = a = g(0)$ and $b = \alpha(a)$; $\mathsf{cls}_R(0,0)$ denotes the equivalence class of $(0,0)$ for the relation R. We also have $\psi\kappa_0 = f$ by (a) (since both $\psi\kappa_0$ and f are liftings of $\varphi\kappa_0 = \alpha f$), and similarly $\varphi\kappa_1 = g$. Therefore ψ makes $f \sim g$. □

Corollary 6.8.11 *If* $\alpha\colon A \longrightarrow B$ *is as in corollary 6.8.10, A is path connected, and B is simply connected, then* α *is a bijection.*

Proof Injectivity: suppose $\alpha(a) = \alpha(a')$. Since A is path connected there is a path f in A with $f(0) = a$ and $f(1) = a'$. Compare f with the constant path $g\colon t \mapsto a$, $\big(t \in [0,1]\big)$: since their composites with α coincide on 0 and 1 and B is simply connected, we conclude that those composites are equivalent and so $a' = f(1) = g(1) = a$ by 6.8.10(ii).

Surjectivity is also obvious since $\alpha(A) \neq \emptyset$, B is path connected, and α has the path-lifting property. □

As we see from this result, in order to prove Galois closedness, we need a property of a space B which will make any connected A path connected when there is a covering map $A \longrightarrow B$. For, we observe,

- if $\alpha\colon A \longrightarrow B$ is a covering map (or just étale), and B is locally path connected, then so is A,
- if A is locally path connected, then it is a disjoint union of path connected open subsets,
- therefore if A is connected and locally path connected, then it is path connected.

Theorem 6.8.12 *Every simply connected locally path connected topological space is Galois closed.* □