Chapter Twelve

Index Theory for Foliations

In this final chapter we shall consider another example of higher index theory, in which the noncommutative 'parameter space' is the space of leaves of a *foliation*. Foliation index theory was the first context in which Connes developed his noncommutative geometry [15, 19, 16], and it provides a rich source of ideas and examples.

12.1 DEFINITION AND EXAMPLES OF FOLIATIONS

Let M be a smooth manifold of dimension $n = p + q$. A p-dimensional *foliation atlas* for M is an atlas of smooth charts $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ (where U_α is open in M) such that the transition functions

$$\phi_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1},$$

which are diffeomorphisms from one open subset of $\mathbb{R}^p \times \mathbb{R}^q$ to another, have the form

$$\phi_{\alpha\beta}(x, y) = (f(x, y), g(y))$$

where $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. In other words, the transition functions are required to respect the relation of "having the same $\mathbb{R}^q$ coordinate". We say that two foliation atlases are *equivalent* if their union is also a foliation atlas.

12.1 Definition. A p-dimensional *foliation* F on M is an equivalence class of p-dimensional foliation atlases. A manifold equipped with a foliation is called a *foliated manifold*. The charts appearing in a foliation atlas are called *flowboxes* for the foliation.

Let (M, F) be a foliated manifold. A *plaque* is a subset of M that is of the form $\mathbb{R}^p \times \{y\}$ in some foliation chart. We can define a new topology and manifold structure on M by taking the plaques as coordinate neighborhoods. In this topology M becomes a p-dimensional manifold with uncountably many connected components (except in trivial cases). These connected components are called the *leaves* of the foliation. Each leaf can be considered as a submanifold of M with its usual topology, and the partition of M into leaves looks locally like the partition of $\mathbb{R}^p \times \mathbb{R}^q$ into plaques. However, the global structure of the foliation may be extremely complicated: the leaves need not be compact, they may wind densely around in M (returning many times to the same foliation chart), and so on.

12.2 Example. A submersion $\pi \colon E \to B$ (see Definition 11.13)gives rise to a foliation, whose leaves are the fibers of π. By definition, every foliation has this structure locally.

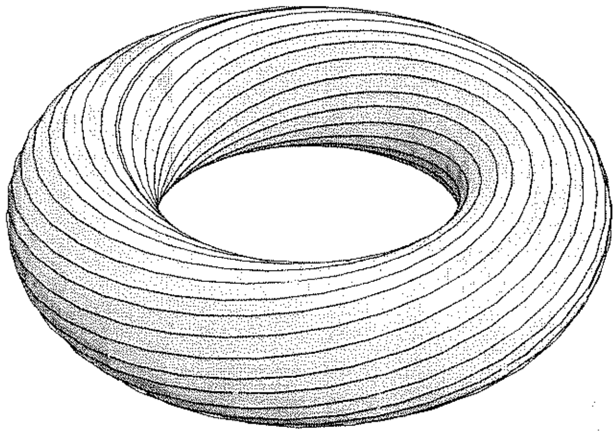

Figure 12.1 The irrational slope foliation.

12.3 Example. A very significant example in noncommutative geometry has been the *irrational slope foliation* of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, in which the leaves are the images of the lines $y = \alpha x + c$ for some fixed irrational α. Because α is irrational, each leaf is diffeomorphic to $\mathbb{R}$ and winds densely around the torus. See Figure 12.1, which shows a portion of one leaf of the irrational slope foliation. This foliation does not come from any submersion (because its leaves are not compact).

12.4 Example. The above construction can be generalized as follows: let Γ be a discrete group whose classifying space $B\Gamma$ is a compact manifold, and let $E\Gamma$ be the universal cover of $B\Gamma$. Suppose that Γ also acts on some other compact manifold N. Then the 'balanced product' $E\Gamma \times_\Gamma N$ — that is, the quotient of $E\Gamma \times N$ by the diagonal Γ-action — is a compact manifold and it is foliated by the images of $E\Gamma \times \{y\}$, $y \in N$. If the action of Γ on N is free, the leaves are all diffeomorphic to the universal cover $E\Gamma$.

F is not being used in the same way as in the original definition of 12.1

12.5 Example. If a smooth manifold M is equipped with a locally free, smooth action[1] of a Lie group H then a foliated manifold is obtained by defining F to be the bundle of tangent vectors on M which are tangent to the orbits of the action. Once again, we may obtain the irrational slope foliation on the torus as a particular case of this construction.

12.6 Example. A classical construction, due to Reeb, shows that there are 2-dimensional foliations of S^3. The construction depends on noting that S^3 can be

[1] An action is locally free if its isotropy subgroups are discrete.

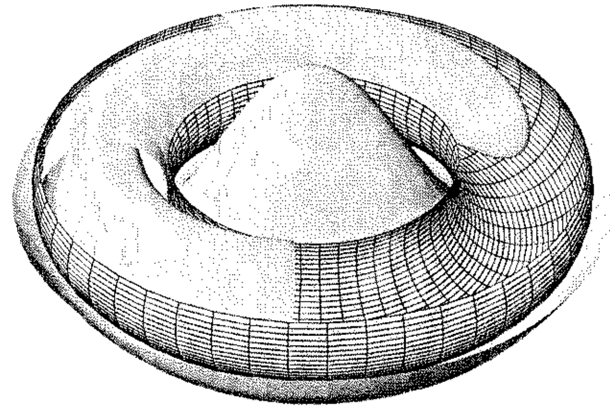

Figure 12.2 Parts of some leaves in the Reeb foliation.

obtained by identifying two solid tori along their boundaries. We regard each solid torus as the quotient of a solid cylinder by a discrete group of translations. Each such solid cylinder (such as $x^2 + y^2 \leq 1$, in Cartesian coordinates) can be foliated by translates of the surface obtained by rotating the curve $z = f(x)$ around the z-axis, where f is a smooth even function which increases to $+\infty$ as $x \to \pm 1$. The quotient of this foliation by the group of translations gives a foliation of the solid torus, having the torus itself as a boundary leaf; and by joining two copies of this foliation we get a foliation of S^3. Note that all the leaves except one are diffeomorphic to $\mathbb{R}^2$; the exceptional leaf is a 2-torus. See Figure 12.2.

12.7 Remark. If (M, F) is a foliated manifold, the tangent vectors to the leaves form a subbundle TF of TM. Clearly, this subbundle has the *integrability property*: if X, Y are sections of TF, then their Lie bracket $[X, Y]$ is a section of TF also. Conversely, a classical theorem of Frobenius states that any subbundle of TM that has the integrability property is tangent to a foliation.

i.e. [illegible] (Frobenius), and [illegible] condition

Let (M, F) be a foliated manifold.

12.8 Definition. A differential operator on M is said to be a *leafwise operator* if it restricts to a differential operator on each leaf. To put this another way, a differential operator is a leafwise operator if, when represented in a foliation chart $\mathbb{R}^p \times \mathbb{R}^q$, it only involves differentiation in the $\mathbb{R}^p$ directions.

Since TF is a subbundle of TM, T^*F is a quotient bundle of T^*M. The symbol of a leafwise operator vanishes on the kernel of $T^*M \to T^*F$, so it is a function on

[illegible] thing [illegible], of course. [illegible] globally elliptic operator on M [illegible] degenerate?

T^*F (with values in the endomorphisms of the bundle on which the operator acts). We'll call this function the *leafwise symbol*.

12.9 Definition. A leafwise operator is *leafwise elliptic* if its restriction to each leaf is elliptic. That is to say, a leafwise elliptic operator is one whose leafwise symbol is elliptic in the sense of Definition 5.14.

In Example 12.2, the leafwise elliptic operators are simply the elliptic families that we discussed in the previous chapter. In Example 12.4, leafwise elliptic operators are closely related to higher index theory. We are going to study the index theory of leafwise elliptic operators in a way that will include both of these examples. As usual, the first stage is to construct a suitable groupoid that will reflect the geometry of the situation.

12.2 FOLIATION GROUPOIDS

Let (M, F) be a foliated manifold. The partition of M into leaves defines an equivalence relation, which we call *leaf equivalence*. We wish to define a smooth groupoid which, in the quotient space picture of groupoids, represents the quotient of M by the leaf-equivalence relation, and which in the families picture represents the family which assigns to each point of M the leaf passing through that point. Actually, a small modification is necessary in order to define a manifold structure on this groupoid. We shall assemble the groupoid not just from pairs of leaf-equivalent points, but from *leafwise paths* connecting leaf-equivalent points.

12.10 Definition. A *leafwise path* in a foliated manifold (M, F) is a piecewise smooth path in M that lies in a single leaf (or, equivalently, whose tangent vector is everywhere tangent to the foliation).

Leafwise paths have a kind of 'foliated tubular neighborhood property' which is very important. To explain it, let us agree that if U is a flowbox (an open set in M foliated diffeomorphic to $\mathbb{R}^p \times \mathbb{R}^q$) then $\mathcal{P}(U)$ will denote the plaque set of U, that is the quotient of U by the relation of "having the same $\mathbb{R}^q$ coordinate". The plaque set $\mathcal{P}(U)$ is of course diffeomorphic to $\mathbb{R}^q$, and $\mathcal{P}$ is a functor on the category of flowboxes and inclusion maps: if $U \subseteq V$ are flowboxes, then there is a natural diffeomorphism of $\mathcal{P}(U)$ into $\mathcal{P}(V)$. In particular, suppose that V and V' are flowboxes that have nonempty intersection which is itself a flowbox. Then $\mathcal{P}(V \cap V')$ is diffeomorphic both to a subset of V and to a subset of V', and thus we obtain a diffeomorphism from an open subset of $\mathcal{P}(V)$ to an open subset of $\mathcal{P}(V')$.

Now suppose that γ is a leafwise path from m_0 to m_1. We can cover γ by flowboxes $U_0, U_1, \ldots, U_n$ such that $U_i \cap U_{i+1}$ is a flowbox, and by iterating the above construction we get a diffeomorphism from an open subset of $\mathcal{P}(U_0)$ to an open subset of $\mathcal{P}(U_n)$. That is, we obtain the *germ* of a diffeomorphism from the space of plaques near m_0 to the space of plaques near m_1.

Standard arguments prove

12.11 Lemma. *The germ of the diffeomorphism obtained above is well-defined (independent of the choices involved in the construction). Moreover, if γ and γ' are leafwise paths which are leafwise homotopic (keeping endpoints fixed) then the germ associated to γ is the same as the germ associated to γ'.* □

12.12 Definition. This germ is called the *holonomy* of the path γ. Two leafwise paths in a foliated manifold with the same beginning and end points are *holonomy equivalent* if their holonomies are equal.

Holonomy is an equivalence relation; according to Lemma 12.11, it is a coarsening of the leafwise homotopy relation. Quite frequently it is much coarser. For example, it may be shown that for a dense G_δ set of beginning and final points, all paths are holonomy-equivalent to one another.

Holonomy is compatible with concatenation of paths: if γ is a leafwise path from m_0 to m_1, and if η is a leafwise path from m_1 to m_2, then the holonomy class of the concatenated path $\eta \vee \gamma$ from m_0 to m_2 depends only on the holonomy classes of γ and η. It follows that the set of holonomy classes of leafwise paths from a point m to itself is a group under concatenation.

12.13 Exercise. Let M be a smooth manifold equipped with a smooth, locally free action of a connected Lie group H, and let (M, F) be the associated foliated manifold (see Example 12.5). Assume that on a dense subset of M the action of H is in fact free. Show that the holonomy group of $p \in M$ is isomorphic to the isotropy group for the action. ??

12.14 Definition. Let (M, F) be a foliated manifold. Its *holonomy groupoid* $G(M, F)$ is given as follows:

- The space $G(M, F)$ of morphisms is the set of all holonomy classes of leafwise paths γ in M.
- The space of objects is the manifold M.
- The source and range maps assign to a path γ its initial and final points.
- Composition is given by concatenation of paths.

The identity morphisms are the constant paths; the inverse of a path $\gamma\colon [0, 1] \to M$ is obtained by composing with an orientation-reversing diffeomorphism of $[0, 1]$.

The space $G(M, F)$ is made into a manifold in the following way. Let γ be a leafwise path from m to m', and let U and U' be flowboxes containing m and m' and small enough that the holonomy of γ provides a diffeomorphism $\phi\colon \mathcal{P}(U) \cong \mathcal{P}(U')$. Let $U \times_\phi U'$ denote the set of pairs $(u, u') \in U \times U'$ such that

$$\phi(\pi(u)) = \pi'(u'),$$

where $\pi\colon U \to \mathcal{P}(U)$ and $\pi'\colon U' \to \mathcal{P}(U')$ are the obvious quotient maps. Clearly, $U \times_\phi U'$ is diffeomorphic to $\mathbb{R}^{2p+q}$. Moreover, each point of $U \times_\phi U'$ corresponds to a nonempty holonomy class of leafwise paths: those which start at u, finish at u', and remain close to γ. Thus $U \times_\Phi U'$ is identified with a subset of G. We can define a topology and a manifold structure on G by using sets of the form $U \times_\Phi U'$ as local coordinate charts. (It is of course necessary to check that the transition functions between any two such sets are smooth; this is a routine matter.)

12.15 Example. If our foliation is given by a submersion $\pi\colon E \to B$ (Example 12.2), then the groupoid of the foliation is just the groupoid G_π associated to the submersion in Definition 11.16.

12.16 Example. In the case of the foliated manifold (M, F) obtained from an effective, locally free action of a connected Lie group on a manifold M, the foliation groupoid $G(M, F)$ may be identified with the transformation groupoid $H \ltimes M$.

What do [illegible] mean?

A Haar system for the groupoid $G(M, F)$ is given by a smoothly varying family of Lebesgue measures on the leaves; such Haar systems always exist. Using the techniques of Chapter 9, we can now define the *foliation C^*-algebra* $C^*(M, F)$ to be the C^*-algebra of the groupoid $G(M, F)$. We think of this as the "function algebra" of the noncommutative space M/F — the *space of leaves* of the foliation.

A leafwise elliptic operator on (M, F) is the same thing as an equivariant elliptic operator on the groupoid G. From Theorem 9.56 we therefore obtain

12.17 Proposition. *The functional calculus for a leafwise elliptic operator* D *gives a $*$-homomorphism*

$$f \mapsto f(D)\colon \mathcal{S} \to C^*(M, F).$$

Consequently, a leafwise elliptic operator has a well-defined longitudinal index *in* $K(C^*(M, F))$. $\square$

In the case of the foliation given by a submersion, this construction reduces to the families index of the previous chapter. Notice that we have again suppressed explicit mention of the bundle on which D acts; compare Remark 11.18.

12.18 Remark. Although the topology that we have defined always makes $G(M, F)$ into a smooth manifold, it is not necessarily Hausdorff[2]. When G is not Hausdorff, the foliation groupoid C^*-algebra $C^*_\lambda(M, F)$ is still defined as a completion of a dense subalgebra $C_c^\infty(G(M, F))$, but the latter is now defined to be the linear span of all smooth, compactly supported functions on coordinate charts as defined above (these functions are extended by zero so as to obtain functions defined on all of $G(M, F)$, but these extended functions need not be continuous on $G(M, F)$). The algebraic operations, and the regular representations are defined as before.

12.19 Exercise. Show that the holonomy group of the toral leaf of the Reeb foliation is equal to its homotopy group $\mathbb{Z}^2$.

Let G denote the holonomy groupoid of the Reeb foliation. Let p be a point of the toral leaf, let γ be a constant path at p, and let γ' be a meridian of the torus that begins and ends at p. Show that the two paths γ and γ' define different points of G, but any two neighborhoods of them in G intersect. Thus G is not Hausdorff.

[2] It *is* Hausdorff in many important cases, such as when all the leaves are simply connected, or when the foliation is not merely smooth but real analytic.

12.3 THE LONGITUDINAL INDEX THEOREM FOR FOLIATIONS

A leafwise elliptic operator on the foliated manifold (M, F) has a *leafwise symbol*, which gives rise to a K-theory class for the cotangent bundle T^*F to the foliation (see Definition 12.9). Using the same tangent groupoid techniques that have been employed in the previous chapter, we can construct an asymptotic morphism that gives rise to an *analytical (longitudinal) index map*

$$K(T^*F) \longrightarrow K(C^*(M, F))$$

which sends the symbol class of any leafwise elliptic operator to its longitudinal index. The *longitudinal index problem* for foliations is to find a means to compute this map.

We shall approach this problem by analogy with the other index theorems that were considered in the previous chapter. Thus, we plan to construct a topologically defined index group $\mathfrak{I}(M, F)$ and a topological index map $K(TF) \to \mathfrak{I}(M, F)$, in such a way that the analytical index factors through the topological index.

How should the index group be constructed? We want to think of the longitudinal index as a families index with base space equal to the "space of leaves M/F". In the case of the ordinary families index, with base B, our index group was defined by organizing all the possible families of operators over B — given by submersions $E \to B$ — into a group, by means of the colimit construction. Analogously, to define $\mathfrak{I}(M, F)$ we shall organize "families over the noncommutative space M/F", the space of leaves of the foliation, into a group. Our first task, then, is to describe what should be meant by such a "family" and, specifically, what should be the correct notion of a *submersion* of an ordinary manifold to M/F.

12.20 Definition. Let (M, F) be a foliated manifold and let W be another manifold. A *smooth map from W to M/F* is described by a smooth cocycle of W with values in $G(M, F)$. This means that there are given an open cover $\{W_i\}$ of W and smooth maps $\gamma_{ij} \colon W_i \cap W_j \to G$ such that

$$\gamma_{jk}\gamma_{ij} = \gamma_{ik}$$

where defined. These cocycles are subject to an equivalence relation (cohomology) which we shall not detail explicitly here.

Note in particular that from the cocycle identity, $\gamma_{ii}(x)$ is a *unit* of G — that is, a point of M. Thus the maps $\phi_i = \gamma_{ii}$ are smooth maps from W_i to M, with $\phi_i(x)$ and $\phi_j(x)$ being the source and target of $\gamma_{ij}(x)$.

12.21 Example. There is a canonical 'projection' map $M \to M/F$: in the above description, we can think of it as given by a cover of M by a single open set, with the map ϕ on this open set being the identity map. Of course, any map (in the ordinary sense) of a manifold W to M may be composed with this projection to yield a map of W to M/F. A particularly important case will arise when W is a *transversal* to the foliation — a q-dimensional submanifold whose tangent bundle is everywhere transverse to TF.

12.22 Example. Suppose that $\pi\colon M \to B$ is a submersion, with M and B compact. We regard it as a foliation F of M. Using the inverse function theorem, it is not hard to show that each point of B has a neighborhood U over which there exists a smooth section of the projection map $\pi^{-1}(U) \to U$. We can construct a map $B \to M/F$ by covering B by such open subsets, and then using a local section over each such subset to map B into M. The holonomy groups in this foliation are all trivial, so the 'transition functions' γ_{ij} exist and are uniquely determined.

Since $\pi\colon M \to B$ need not have any *global* sections, this example shows that maps to M/F need not factor through maps to M.

12.23 Definition. A map $f\colon W \to M/F$ (defined as above) is a *submersion* if the composite maps

$$TW_i \xrightarrow{d\phi_i} TM \longrightarrow TM/TF$$

are surjective on each fiber. When f is a submersion, there is a well-defined *pullback foliation* f^*F on W whose leaves map to the leaves of F.

12.24 Example. The inclusion of a transversal is a submersion to M/F, and the pullback foliation is the trivial one whose leaves are single points.

Now let $\mathcal{M}$ be the category whose objects are smooth manifolds W equipped with submersions to M/F, and whose morphisms are the smooth embeddings $W \to W'$ that make the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[dr, "f"'] & W' \arrow[d, "f'"] \\ & M/F \end{tikzcd} \tag{12.1}$$

commute[3]. We can define a functor on M by associating to each $f\colon W \to M/F$ the K-theory group $K(T^*(f^*F))$, and to each diagram as above the induced wrong way map $K(T^*(f^*F)) \to K(T^*((f')^*F))$ on K-theory.

12.25 Definition. The *index group* $\mathfrak{I}(M,F)$ is the colimit of this functor.

By construction there is a *topological index map* $K(T^*(f^*F)) \to \mathfrak{I}(M,F)$ associated to every submersion $f\colon W \to M/F$.

Now we are going to construct the analytical index map on $\mathfrak{I}(M,F)$. The first stage is the following.

12.26 Lemma. *Let* $f\colon W \to M/F$ *be a submersion. Then there is a natural homomorphism*

$$f_!\colon K(C^*(W, f^*F)) \to K(C^*(M,F))$$

from the K-theory of the C^**-algebra of the pullback foliation* f^*F *to the* K*-theory of the* C^**-algebra of* F *itself.* □

[3]The two composite 'maps' in this diagram are defined by certain cocyles, and 'commutativity' means that the cocycles are cohomologous.

We shall sketch the construction of this map. It is helpful to keep in mind the following trivial example: the foliation F consists of a single leaf, which is the whole manifold M, and W is a single point. In that case $C^*(W, f^*F) = \mathbb{C}$, $C^*(M, F) = \mathcal{K}$, and the map in question is the 'stabilization' isomorphism $K(\mathbb{C}) \to K(\mathcal{K})$. The construction will realize this map by considering $H = L^2(M)$ as a Hilbert $\mathcal{K}(H)$-module, whose algebra of 'compact' endomorphisms is $\mathbb{C}$. Compare Exercise 3.32.

In general, construct a manifold Z as follows. Choose an open cover $\mathcal{W} = \{W_i\}$ of W over which the cocycle $\{\gamma_{ij}\}$ representing f is defined. For each $W_i \in \mathcal{W}$, define $\widetilde{W_i}$ to be the set of pairs $\alpha = (x, \gamma)$, where $x \in W_i$, $\gamma \in G = G(M, F)$, and $s(\gamma) = \gamma_{ii}(x)$. Then Z is obtained from the disjoint union of the $\widetilde{W_i}$ by identifying $(x, \gamma) \in \widetilde{W_j}$ with $(x, \gamma_{ij}\gamma)$ in $\widetilde{W_i}$ whenever $x \in W_i \cap W_j$.

The groupoid $G = G(M, F)$ acts on Z on the right. The space of smooth, compactly supported functions on Z can therefore be equipped with a $C^*(M, F)$-valued inner product, defined by

$$\langle g_1, g_2 \rangle(\gamma) = \int g_1(\alpha) g_2(\alpha\gamma)\, d\alpha.$$

Completing $C_c^\infty(Z)$ we obtain a $C^*(M, F)$-Hilbert module, denoted E.

On the other hand, the groupoid $G(W, f^*F)$ acts on Z on the left, in a way that commutes with the action of $G(M, F)$ on the right. Thus $G(W, f^*F)$ acts on $C_c^\infty(Z)$ by $C^*(M, F)$-module endomorphisms. Simple estimates show that this action extends to an action of $C^*(W, f^*F)$ on E by bounded, adjointable $C^*(M, F)$-module operators, and in fact that these operators are *compact* in the Hilbert module sense. Thus we get a homomorphism of C^*-algebras from $C^*(W, f^*F)$ to $\mathcal{K}(E)$. Combining this with the stabilization map $K(\mathcal{K}(E)) \to K(C^*(M, F))$ gives the construction that we require.

12.27 Example. Suppose that W is a closed transversal to the foliation F. Then, near any point of W, one can find a flowbox $\mathbb{R}^p \times \mathbb{R}^q$ in which the leaves of the foliation are given by $\mathbb{R}^p \times \{y\}$ and the transversal W is given by $\{0\} \times \mathbb{R}^q$. Fix a smoothing kernel $k(y', y'')$ which defines a rank one projection compactly supported on $\mathbb{R}^q \times \mathbb{R}^q$. This projection can be thought of as an element of $C^*(G_{|U})$. The construction is easily globalized to yield a projection $p_W \in C_r^*(G)$ associated to the closed transversal W, and for all $g \in C(W)$ the element $gp \in C_r^*(G)$ is well-defined. Moreover, $g \mapsto gp$ gives a homomorphism

$$C(W) \to C_r^*(G).$$

This homomorphism yields a K-theory map $K(W) \to K(C^*(M, F))$ which is independent of the choices made and is in fact the map $f_!$ constructed above from the natural $f\colon W \to M/F$. (Notice that since the pull-back foliation on W is the foliation by points, $C^*(W, f^*F) = C(W)$.)

Recall now that for the foliation f^*F of W we have an *analytical index map* $K(T^*(f^*F)) \to K(C^*(W, f^*F))$ which sends the symbol of each leafwise elliptic operator to its longitudinal index. Combining with the map of Lemma 12.26 above we obtain a composite

$$K(T^*(f^*F)) \xrightarrow{\mathrm{Ind}_a} K(C^*(W, f^*F)) \longrightarrow K(C^*(M, F))$$

which we shall again denote by $f_!$. Now, once again, the key to the index theorem is that we have compatibility between these wrong way maps and those used in the definition of the index group:

12.28 Lemma. *Consider a commutative diagram of the form*

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[dr, "f"'] & W' \arrow[d, "f'"] \\ & M/F \end{tikzcd}$$

where i *is an embedding and* f *and* f' *are submersions. Then the diagram*

$$\begin{tikzcd} K(T^*(f^*F)) \arrow[r, "i_!"] \arrow[dr, "f_!"'] & K(T^*((f')^*F) \arrow[d, "f'_!"] \\ & K(C^*(M, F)) \end{tikzcd}$$

also commutes. □

Thus, the maps $f_!$ form a compatible system of homomorphisms and so pass to the colimit, where they define a homomorphism $\mathfrak{I}(M, F) \to K(C^*(M, F))$ that we shall call the (universal) analytical index. By construction we have

12.29 Corollary. *For any submersion* $f \colon W \to M/F$ *there is a commutative diagram*

$$\begin{tikzcd} K(T^*(f^*F)) \arrow[d, "\mathrm{Ind}_t"'] \arrow[dr, "\mathrm{Ind}_a"] & \\ \mathfrak{I}(M, F) \arrow[r] & K(C^*(M, F)) \end{tikzcd}$$

(Thus, the analytical index of a leafwise elliptic operator depends only on its topological index.)

We can think of this as an abstract index theorem for foliations. As in other examples, the usefulness of this result will depend on the extent to which we can compute the index group.

12.4 THE BAUM–CONNES CONJECTURE FOR FOLIATIONS

As a simple but helpful example, we consider the case of a foliation all of whose leaves are *contractible.* (In particular, such a foliation will not have any non-trivial holonomy.)

12.30 Lemma. *Let* (M, F) *be a compact foliated manifold with all leaves contractible. Then any map* $W \to M/F$ *lifts to a map* $W \to M$*, and any two such lifts of the same map are homotopic.*

Any Riemannian metric on TM restricts to a metric on TF, and thus all the leaves of F become Riemannian manifolds. It is not hard to check that the leaves are all *complete* in their inherited Riemannian metrics. In particular, if the leaves are simply connected and have nonpositive curvature in their induced metric, then they are contractible (by a well-known theorem of Hadamard). We are going to prove the lemma only for foliations of this sort, because for these one can give a simple and explicit geometric construction of the desired lift.

Proof. We need the following geometric fact about simply connected manifolds of nonpositive curvature: given such a manifold L and a $(q+1)$-tuple $x_0, \ldots, x_q)$ of points of L, there is a well-defined 'geodesic simplex' in L with the given vertices, that is an embedding

$$\Delta^q \to L,$$

where Δ^q is the standard q-simplex, whose vertices are the given points and whose edges are geodesics. (One way to construct such a simplex is to send a point $(\lambda_0, \ldots, \lambda_q) \in \Delta^q$ to the unique point x that minimizes the sum

$$\sum_{i=0}^{q} \frac{d(x, x_i)^2}{\lambda_i^2};$$

nonpositive curvature is used to show that this function has a unique minimum.) It is convenient to denote the point of the geodesic simplex that is the image of $(\lambda_0, \ldots, \lambda_q)$ by the symbol $\sum \lambda_i x_i$ Now let $f\colon W \to M/F$ be a map. By construction, W has a cover $\mathcal{W} = \{W_i\}$ by open sets on each of which f is represented by a map $f_i\colon W_i \to M$. If a point x belongs to several W_i, the corresponding $f_i(x)$ are all on the same leaf. Let $\{\chi_i\}$ be a locally finite partition of unity subordinated to $\mathcal{W}$. Using negative curvature to form 'convex combinations' of points in the same leaf, as above, define a map $F\colon W \to M$ by

$$F(x) = \sum \chi_i(x) f_i(x).$$

Then F is a smooth map and it lifts f. The homotopy uniqueness is proved by a similar construction. □

From the lemma we may deduce

12.31 Proposition. *Suppose that* (M, F) *is a compact foliated manifold with simply-connected leaves of nonpositive curvature. Then* $\mathfrak{I}(M, F) = K(T^*F)$.

Proof. The proof is similar to our previous computations of index groups. Let $f\colon W \to M/F$ be a submersion. Thanks to the lemma it can be lifted (uniquely up to homotopy) to a map $g\colon W \to M$, and $T^*(f^*F) \cong g^*(T^*F)$. The wrong way map associated to g gives a homomorphism

$$g_!\colon K(T^*(f^*F)) \to K(T^*F).$$

We want to show that these homomorphisms form a compatible system, and thus pass to a map on the colimit

$$\mathfrak{I}(M,F) \to K(T^*F)$$

which is inverse to the topological index map for (M,F). To check this compatibility notice that we can factor g as

$$W \xrightarrow{\iota} M \times \mathbb{R}^n \xrightarrow{\pi} M$$

where the first map is an embedding and the second is the canonical projection. Now $\iota_!$ is the wrong way map associated to an embedding of the sort appearing in diagram 12.1, and $\pi_!$ is the inverse of an (invertible) wrong way map of this sort. Thus, the desired compatibility follows from the functorial property of the wrong way map construction (Proposition ??). □

12.32 Remark. There is a construction of the *classifying space* BG for the holonomy groupoid of a foliation, generalizing the classifying space for a discrete group that we considered in the previous chapter. In terms of this construction, Lemma 12.30 may be interpreted as saying that the classifying space for a foliation with contractible leaves is just the ambient manifold. The proposition may be generalized to compute the index group in terms of the K-theory of the classifying space.

Baum and Connes' original formulation of their conjecture was that the analytical index map

$$\mathfrak{I}(M,F) \to K(C^*(M,F))$$

should be an isomorphism. In the case of foliations whose leaves have strictly negative curvature, this is now known to be true. Hence we obtain the following geometrical statement.

12.33 Proposition. *Let (M,F) be a spin foliation*[4] *of a compact manifold, having simply-connected leaves. If F admits a metric in which every leaf has strictly negative sectional curvature, it does not also admit a metric in which every leaf has strictly positive sectional curvature.*

Proof. The index group is just $K(T^*F)$, and the symbol of the Dirac operator is nonzero in this group (in fact it is the Thom element). But if there is a metric with positive (leafwise) scalar curvature, the index of the Dirac operator must vanish, by the usual Lichnerowicz argument. This contradicts the injectivity of the analytical index map $K(T^*F) = \mathfrak{I}(M,F) \to K(C^*(M,F))$. □

12.5 TRACES AND NUMERICAL INDICES

The foliation index, and the higher index that we discussed in the last chapter, take their values in certain C^*-algebra K-theory groups. If one can construct interesting

[4] This means that the tangent bundle TF admits a spin structure.

linear functionals on these K-theory groups, one can compose with them to obtain numerical indices. In this section and the next we shall look at the simplest of these constructions, which makes use of the notion of a *trace.*

12.34 Definition. A (finite) *trace* on a C^*-algebra A is a positive linear functional $\tau\colon A \to \mathbb{C}$ such that $\tau(aa') = \tau(a'a)$ for all $a, a' \in A$.

The usual trace on $M_n(\mathbb{C})$ is of course an example. For a commutative algebra $C(X)$, where X is a compact space, a trace is just a positive linear functional — that is, a measure. More relevant to higher index theory is the following:

12.35 Example. Let $A = C^*_r(\Gamma)$, where Γ is a discrete group. By construction, A is an algebra of operators on the Hilbert space $H = \ell^2(\Gamma)$, which has an orthonormal basis $\{e_\gamma : \gamma \in \Gamma\}$. Let

$$\tau(a) = \langle a \cdot e_1, e_1 \rangle$$

for $a \in A$, where e_1 is the basis element corresponding to the identity in Γ. It is easy to check that τ is a trace.

12.36 Proposition. *Let τ be a trace on the unital C^*-algebra A. Let $p = (p_{ij})$ be a projection in $M_n(A)$. Then the sum $\sum \tau(p_{ii})$ is real and nonnegative and depends only on the K-theory class represented by p. Moreover, this process extends to a homomorphism $\tau_*\colon K(A) \to \mathbb{R}$.* □

12.37 Exercise. Prove Proposition 12.36.

When we deal with non-unital algebras, the natural examples of traces are often *unbounded* (think about Lebesgue measure as a functional on $C_0(\mathbb{R})$). We can generalize our definition to cope with this as follows.

12.38 Definition. Let A be a C^*-algebra. An *(unbounded) trace* on A is a positive linear functional τ defined on a dense hereditary ideal $\mathrm{dom}(\tau)$ in A, which satisfies $\tau(xy) = \tau(yx)$ when $x \in \mathrm{dom}(\tau)$, $y \in A$. The statement that $\mathrm{dom}(\tau)$ is *hereditary* means that if x is a positive element of $\mathrm{dom}(\tau)$ and $0 \le y \le x$, then $y \in \mathrm{dom}(\tau)$ also.

The standard examples are Lebesgue measure on $\mathbb{R}$ (a positive linear functional on $C_0(\mathbb{R})$, finite on the dense ideal $C_c(\mathbb{R})$ of compactly supported functions) and the trace on $\mathcal{K}$ given by

$$\mathrm{Tr}(T) = \sum_i \langle Te_i, e_i \rangle$$

for an orthonormal basis e_i (a positive linear functional finite on the ideal of finite-rank operators). If A is any C^*-algebra with a trace τ, one can define a trace $\tau \otimes \mathrm{Tr}$ on $A \otimes \mathcal{K}$ by

$$(\tau \otimes \mathrm{Tr})(a \otimes T) = \tau(a)\,\mathrm{Tr}(T).$$

12.39 Exercise. Show that a unital C^*-algebra has no nontrivial dense ideals. Thus, 'unbounded trace' is a proper generalization of 'finite trace' only for non-unital C^*-algebras.

Elements close to 1 are invertible

12.40 Remark. At one point below we shall need to note that a trace τ on A extends to the unitalization $\widetilde{A}$ of A by defining

$$\tilde{\tau}(a + \lambda 1) = \tau(a).$$

The domain of definition of this extended trace $\tilde{\tau}$ is $\mathrm{dom}(\tilde{\tau}) = \mathrm{dom}(\tau) + \mathbb{C}$. This is of course not an ideal, but we still have the trace property in the form

$$x \in \mathrm{dom}(\tilde{\tau}), y \in A \Rightarrow xy - yx \in \mathrm{dom}(\tilde{\tau}) \text{ and } \tilde{\tau}(xy - yx) = 0.$$

12.41 Proposition. *Let A be a C^*-algebra equipped with an unbounded trace. For any $*$-homomorphism $\alpha\colon C_0(\mathbb{R}) \to A$, and any $f \in C_c(\mathbb{R})$, the image $\alpha(f)$ belongs to* $\mathrm{dom}(\tau)$.

Proof. We may assume $f \geq 0$. Let $a_0 = \alpha(f)$. There is a positive $a_1 \in A$ such that $a_0 a_1 = a_0$; just take $a_1 = \alpha(g)$ where $g \in C_0(\mathbb{R})$ is positive and equal to 1 on the support of f. By density, there is $b \in \mathrm{dom}(\tau)$ with $\|a_1 - b\| < \frac{1}{2}$. Then

$$a_0 = a_0^{1/2} a_1 a_0^{1/2} = a_0^{1/2}(a_1 - b)a_0^{1/2} + a_0^{1/2} b a_0^{1/2} \leq \tfrac{1}{2} a_0 + a_0^{1/2} b a_0^{1/2},$$

so $a_0 \leq 2a_0^{1/2} b a_0^{1/2} \in \mathrm{dom}(\tau)$. □

Now recall the 'difference construction' of K-theory elements from Chapter 5. We saw there that any graded $*$-homomorphism

$$\alpha\colon \mathcal{S} \to A \otimes M_2(\mathcal{K})$$

(where we consider $M_2(\mathcal{K})$ as a graded algebra, with grading operator $\varepsilon = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) gives rise to a class in $K(A)$. In fact, $K(A)$ can be identified with the set of homotopy classes of such $*$-homomorphisms (Proposition 5.20).

Suppose now that A is equipped with a trace τ, which we extend to $A \otimes \mathcal{K}$ in the way explained above. To simplify notation, we shall write the extended trace just as τ again, rather than the more correct $\tau \otimes \mathrm{Tr}$. Let $h \in C_c(\mathbb{R})$ be an even function with $h(0) = 0$. Then there is an odd function $g \in C_c(\mathbb{R})$ with $g^2 = f$. By the proposition above, $\alpha(h)$ and $\alpha(g)$ belong to $\mathrm{dom}(\tau)$. Now consider

$$\tau(\varepsilon\alpha(g)^2) = \tau(\alpha(g)\varepsilon\alpha(g)) = -\tau(\varepsilon\alpha(g)^2)$$

where we have used the fact that ε anticommutes with the odd element $\alpha(g)$ together with the law $\tau(xy) = \tau(yx)$. Thus $\tau(\varepsilon\alpha(h)) = 0$.

It now follows that, given α and τ as above, there is a (real) constant $\tau_*[\alpha]$ such that

$$\tau(\varepsilon\alpha(f)) = \tau_*[\alpha]\, f(0)$$

for every even $f \in C_c(\mathbb{R})$. Indeed, if $f_1(0) = f_2(0)$, put $h = f_1 - f_2$ and apply the result of the previous paragraph.

12.42 Proposition. *The constant $\tau_*(\alpha)$ defined above depends only on the homotopy class of the $*$-homomorphism $\alpha\colon \mathcal{S} \to A \otimes M_2(\mathcal{K})$. Moreover, the assignment $\alpha \mapsto \tau_*[\alpha]$ gives a homomorphism*

$$\tau_*\colon K(A) \to \mathbb{R}.$$

If A is unital, this homomorphism agrees with the one defined in Proposition 12.36 above.

Proof. Choose a compactly supported even f with $f(0) = 1$ and let g be an odd function such that $g^2 = 1 - f^2$. Let

$$F = \left(2\alpha(f^2) - 1\right)\varepsilon + 2\alpha(fg)$$

which is an element of $M_2(\widetilde{A \otimes \mathcal{K}})$. It is self-adjoint and

$$F^2 = (2\alpha(f^2) - 1)^2 + 4\alpha(f^2g^2) = 1 + 4\alpha(f^4 - f^2 + f^2g^2) = 1.$$

(A self-adjoint operator with square 1 is called a *symmetry*.) Moreover,

$$\tau_*[\alpha] = \tau(\varepsilon\alpha(f^2)) = \tfrac{1}{2}\tilde{\tau}(F)$$

where $\tilde{\tau}$ is the extension of the trace to the unitalization discussed in remark 12.40.

Let α_t be a homotopy of $*$-homomorphisms. Fixing f and g, we obtain a continuous path F_t of symmetries. Now it is easy to prove that if s and s' are two symmetries in a unital C^*-algebra, and $\|s - s'\| < 2$, then there is an invertible x such that $x^{-1}sx = s'$; just take $x = \frac{1}{2}(1 + s's)$. By subdividing the path F_t into steps of length < 2 in norm we therefore see that there is an invertible X such that $X^{-1}F_0X = F_1$. It follows that $\tilde{\tau}(F_0) = \tilde{\tau}(F_1)$, which is the desired homotopy invariance.

It is an easy exercise for the reader to show that τ_* is a homomorphism. To show that our definition matches up with the earlier one, remember (Example 5.9) that to a formal difference $[p] - [q]$ of projections we associate the $*$-homomorphism α that sends $f \in \mathcal{S}$ to

$$\begin{pmatrix} f(0)p & 0 \\ 0 & f(0)q \end{pmatrix}.$$

The trace of $\varepsilon\alpha(f)$ is thus clearly $f(0)(\tau(p) - \tau(q))$. □

In applications to index theory one is often interested in traces that arise from a certain extension principle. Let $\mathcal{A}$ be a $*$-algebra of operators on a Hilbert space H. We don't assume that A is closed: we are thinking of something like the algebra of smoothing operators on $L^2(M)$, for some compact manifold M. A positive linear functional $\phi\colon \mathcal{A} \to \mathbb{C}$ is *tracial* if $\phi(ab) = \phi(ba)$ for $a, b \in \mathcal{A}$. We shall say that a tracial functional ϕ is *normal* if, for fixed $a, a' \in \mathcal{A}$, the functional

$$b \mapsto \phi(aba')$$

is continuous relative to the strong topology on (norm) bounded subsets of $\mathcal{A}$. (Recall that the strong topology is defined by the seminorms $T \mapsto \|T\xi\|$, for $\xi \in H$.)

Let ϕ be a normal, tracial positive functional on $\mathcal{A}$, and let A denote the C^*-algebra closure of $\mathcal{A}$ in $\mathcal{B}(H)$. For each $a \in \mathcal{A}$ the functional $x \mapsto \phi(a^*xa)$ has a unique continuous extension to a positive linear functional on A. We shall continue to use the notation $\phi(a^*xa)$ for this extension.

Now let $x \in A$ be any positive element. Define

$$\tau(x) = \sup\{\phi(a^*u^*xua) : a \in \mathcal{A}, \|a\| < 1,\ u \in \widetilde{A}, uu^* = u^*u = 1\}.$$

Note that $\tau(x)$ may be a positive real number or $+\infty$.

12.43 Lemma. *If* $x = y^4$, *with* $y = y^* \in \mathcal{A}$, *then* $\tau(x) = \phi(x)$.

Thus τ extends ϕ from a dense subcone of the positive cone of $\mathcal{A}$. In particular, $\{x \in A^+ : \tau(x) < \infty\}$ is dense in A^+.

Proof. Suppose that $x = y^4$. First note the identity

$$\phi(a^*u^*y^4ua) = \phi(y^2uaa^*u^*y^2).$$

This holds for $u \in \mathcal{A}$ by the tracial property of ϕ, and both sides depend (norm) continuously on u because ϕ is normal.

Since $uaa^*u^* \leq 1$, positivity gives

$$\phi(a^*u^*y^4ua) \leq \phi(y^2.1.y^2) = \phi(x).$$

Thus $\tau(x) \leq \phi(x)$.

On the other hand, still taking $x = y^4$, let a_n be a sequence of elements in A, of norm less than 1, such that $a_n y \to y$ in norm as $n \to \infty$. Such a sequence can easily be constructed using the functional calculus for y and the density of $\mathcal{A}$. We have

$$\phi(a_n^* y^4 a_n) = \phi(y \cdot y a_n a_n^* y \cdot y) \to \phi(y \cdot y^2 \cdot y) = \phi(x)$$

as $n \to \infty$. This shows that $\tau(x) \geq \phi(x)$, completing the proof. □

12.44 Proposition. *Consider τ as an extended-real-valued functional defined on the positive cone of* A. *Then we have:*

(a) $\tau(\alpha x) = \alpha\tau(x)$ *for* $\alpha \in \mathbb{R}^+$,

(b) $\tau(x+y) = \tau(x) + \tau(y)$,

(c) $\tau(u^*xu) = \tau(x)$, *for* u *unitary in* $\widetilde{A}$.

Proof. Easy. □

Standard results in C^*-algebra theory now prove that the linear span of $\{x \in A^+ : \tau(x) < \infty\}$ is a dense hereditary ideal and that τ is a trace (with domain equal to this linear span) in the sense of our definition 12.38.

We have not yet made any use of the strong topology (we have only used the weaker assertion that $b \mapsto \phi(aba')$ is *norm* continuous). We shall do so now. Suppose that H is $\mathbb{Z}_2$-graded, and that D is an odd, unbounded, self-adjoint operator on H that is *affiliated* to A (this is to say that $f(D) \in A$ for all $f \in C_0(\mathbb{R})$). The functional calculus for D then defines a graded $*$-homomorphism

$$\mathcal{S} \to A, \qquad f \mapsto f(D),$$

and therefore an index D) $\in K(A)$. By construction (Proposition 12.42), we have

$$\tau_*(\text{Index}(D)) = \tau(\varepsilon f(D))$$

for any even $f \in C_c(\mathbb{R})$ with $f(0) = 1$.

12.45 Proposition. *Let $\mathcal{A}$ be an algebra of operators on a graded Hilbert space* H*, and let* A *be the* C^**-algebra generated by* $\mathcal{A}$*. Let* D *be an odd, self-adjoint operator affiliated to* A*. Let* ϕ *be a normal, tracial positive linear functional on* $\mathcal{A}$*, extended to a densely defined trace* τ *on* A *in the manner described above. Suppose that* $h(D) \in \mathcal{A}$ *for just one function* f *having* $h(0) = 1$.

If $\tau_*[\text{Index}(D)] \neq 0$*, then* $\ker(D)$ *is a non-trivial subspace of* H*. In more detail, if* $\tau_*[\text{Index}(D)] > 0$*, then* $\text{Kernel}(D) \cap H^+$ *is non-trivial; if* $\tau_*[\text{Index}(D)] < 0$*, then* $\text{Kernel}(D) \cap H^-$ *is non-trivial.*

Proof. Choose a sequence $\{f_n\}$ of even functions $f \in C_c(\mathbb{R})$ with $f(0) = 1$ and decreasing pointwise to the characteristic function of $\{0\}$. The operators $f_n(D)$ then tend strongly to the orthogonal projection P onto the kernel of D. Let h be as in the statement of the proposition. Then $\tau(\varepsilon h(D) f_n(D) h(D)) = \tau_*[\text{Index}(D)]$ for each n. If $P = 0$, then strong continuity gives

$$\tau(\varepsilon h(D) f_n(D) h(D)) \to \tau(\varepsilon h(D) P h(D)) = 0.$$

Thus the index vanishes. The last statement of the proposition is proved by a slight generalization of this argument (consider strong convergence on H^+ and H^- separately.) □

12.6 THE L^2 INDEX THEOREM

Now we shall apply this machinery to index theory. Our first application is to the higher index defined in Section 11.4. There, we considered *principal Γ-manifolds* X, which are manifolds equipped with a free and proper action of a discrete group Γ. A Γ-equivariant elliptic operator D on X (which is the same thing as an ordinary elliptic operator on $M = X/\Gamma$) has an index in $K(C_r^*(G))$ where G is the *fundamental groupoid* of Definition 11.36. This index can be defined by the $*$-homomorphism

$$\mathcal{S} \to C_r^*(G), \qquad f \mapsto f(D)$$

given by the functional calculus for D.

We are going to define an (unbounded) trace on $C_r^*(G)$. Recall that the space of objects of the groupoid G is the manifold M, which we assume is compact. Thus there is an inclusion $i\colon M \to G$ and every $k \in C_c^\infty(G)$ restricts to a function i^*k on M.

12.46 Definition. Define $\phi\colon C_c^\infty(G) \to \mathbb{C}$ by

$$\phi(k) = \int_M i^*k.$$

12.47 Remark. When Γ is the trivial group, G is the pair groupoid of M and we have

$$\phi(k) = \int_M k(x,x)\,dx$$

for a smoothing kernal k. The integral should be taken with respect to the same choice of smooth measure on M as is used to define a Haar system on G.

12.48 Lemma. *The functional ϕ is tracial.*

Proof. Let $k_1, k_2 \in C_c^\infty(G)$. We may regard them as functions on $X \times X$ which are Γ-periodic in the sense that $k_i(x\gamma, y\gamma) = k_i(x, y)$ for all $\gamma \in \Gamma$. In this interpretation, ϕ is given by

$$\phi(k) = \int_F k(x, x)\,dx,$$

where F is a fundamental domain for the action of Γ on X.

The commutator $[k_1, k_2]$ is given by the kernel

$$k(x, z) = \int_X (k_1(x, y)k_2(y, z) - k_2(x, y)k_1(y, z))\,dy.$$

Decomposing $X = \bigcup_{\gamma \in \Gamma} F\gamma$, we obtain

$$\phi(k) = \sum K(\gamma)$$

where

$$K(\gamma) = \iint_{F \times F} (k_1(x, y\gamma)k_2(y\gamma, z) - k_2(x, y\gamma)k_1(y\gamma, z))\,dy\,dx.$$

But, using the equivariance of k_1 and k_2, and reversing the order of integration, it is easy to see that $K(\gamma) = -K(\gamma^{-1})$. Thus $\phi(k) = -\phi(k)$, and so $\phi(k) = 0$ as required. $\square$

12.49 Lemma. *The functional ϕ is normal (with respect to the regular representation of the groupoid algebra $C_c^\infty(G)$).*

Proof. Fix $a, a' \in C_c^\infty(G)$ with kernels k, k'. For $x \in X$ let ξ_x denote the L^2 function $k(x, \cdot)$ and let η_x denote the L^2 function $k'(\cdot, x)$. Let $T \in C_c^\infty(G)$ be an operator with kernel t. Then by definition

$$\phi(aTa') = \int_F \iint_{X \times X} k(x, y)t(y, z)k'(z, x)\,dy\,dz\,dx = \int_F \langle \xi_x, T\eta_x \rangle\,dx.$$

If T_n is a norm bounded sequence converging strongly to T, then the functions $x \mapsto \langle \xi_x, T_n\eta_x \rangle$ converge pointwise to $\langle \xi_x, T\eta_x \rangle$ and are uniformly bounded. Therefore their integrals converge, by Lebesgue's dominated convergence theorem. $\square$

Let τ be the unbounded trace on $C_r^*(G)$ that is obtained be extending the tracial functional ϕ according to the procedure of the last section.

12.50 Definition. Let X be a principal Γ-manifold and let D be an elliptic operator on the compact manifold $M = \Gamma \backslash X$. The real number $\tau_*(\mathrm{Ind}(D))$, where τ is as above and $\mathrm{Ind}(D) \in K(C_r^*(G)$ is the higher index of D, is called the L^2-*index* of D.

Atiyah's L^2 index theorem is the following statement.

12.51 Theorem. *Let X be a principal Γ-manifold and let D be an elliptic operator on the compact manifold $M = \Gamma \backslash X$. Then the ordinary index of D is equal to its L^2 index.*

We shall sketch here a topological proof of the L^2-index theorem that has recently been presented by Chatterji and Mislin. First of all, notice that both the ordinary index and the L^2-index can be thought of as homomorphisms

$$K(T^*M) \to \mathbb{R}.$$

It is easy to check that these homomorphisms agree when $X = \Gamma$. The two indices are both natural and thus they pass to two linear maps

$$\mathfrak{I}(\Gamma) \to \mathbb{R}$$

on the index group. Our observation above that the L^2-index theorem is true for $X = \Gamma$ shows that the two homomorphisms agree on the one-dimensional subspace of $\mathfrak{I}(\Gamma)$ spanned by $X = \Gamma$.

Let us now consider what happens when we enlarge the group Γ.

12.52 Proposition. *Suppose that Γ is a subgroup of a group Γ'. There is a natural "induction" functor from principal Γ-manifolds to principal Γ'-manifolds, and this induction functor preserves both the index and the L^2-index.*

Proof. The induction functor is

$$X \mapsto \Gamma' \times_\Gamma X$$

where the balanced product is the quotient of $\Gamma' \times X$ by the equivalence relation $(\gamma'\gamma, x) \sim (\gamma', \gamma x)$. □

The proof of the L^2-index theorem is now completed by the following theorem, essentially due to Kan and Thurston:

12.53 Proposition. *Any (countable discrete) group Γ can be embedded as a subgroup of a group Γ' for which $\mathfrak{I}(\Gamma') \otimes \mathbb{R}$ is one-dimensional.* □

12.54 Remark. A different argument for the L^2-index theorem, much closer to the original one, uses the so-called 'heat equation' method. One considers the index formula

$$\tau_*(\text{Index } D) = \tau\left(\varepsilon e^{-tD^2}\right)$$

which follows from Proposition 12.42. As $t \searrow 0$ the operator e^{-tD^2} becomes localized — the value of its kernel at a point becomes determined by the geometry in a neighborhood of that point. Since M and X have exactly the same local geometry, the result is proved.

Here is a simple application.

12.55 Proposition. *Let M be a compact oriented surface of genus at least 2, equipped with some Riemannian metric. Then the universal cover of M has an infinite-dimensional space of L^2 harmonic 1-forms.*

According to the uniformization theorem, such an M has a metric of constant negative curvature, for which the universal cover is the Poincaré disk. In that case, an infinite-dimensional space of harmonic 1-forms can be described explicitly. The result, however, applies to *any* metric.

Proof. Apply the L^2-index theorem and Proposition 12.45 to the de Rham operator $D = d + d^*$, whose ordinary index is the Euler characteristic (see Proposition ??). This gives the nontriviality of the kernel of the de Rham operator on the universal cover, that is, the existence of *some* nonzero L^2 harmonic 1-forms. To get an infinite-dimensional space of such forms we need an additional argument, using equivariance and the fact that $\pi_1(M)$ is infinite. □

12.7 MEASURED FOLIATIONS AND THE INDEX THEOREM

Alain Connes' first version of the index theorem for foliated manifolds was analogous to the L^2-index theorem for coverings. The key concept here is that of an (invariant) *transverse measure* on a foliation.

12.56 Definition. A *transverse measure* on a foliated manifold (M, F) is a map which assigns to each flowbox U a Borel measure μ_U on the plaque set $\mathcal{P}(U)$, in such a way that if $U \subseteq V$ is an inclusion of flowboxes then μ_U is the restriction of μ_V to $\mathcal{P}(U)$.

Informally, a transverse measure is a measure on transversals that is invariant under holonomy.

12.57 Example. If F has a compact leaf L, then we can define a transverse measure on (M, F) by assigning a plaque with mass 1 if it belongs to L and with mass 0 otherwise. (This is called the *counting measure* associated to L.) In the case of the Reeb foliation, for example, it can be shown that the only transverse measures are scalar multiples of the counting measure associated to the unique compact leaf.

Let μ be a transverse measure on (M, F), and assume that M is compact and that F is oriented (that is, TF is oriented as a vector bundle over M). Let α be a p-form on M, where $p = \dim F$. Find a finite cover of M by flowboxes $\{U_i\}$, and choose a partition of unity $\{\phi_i\}$ subordinate to $\{U_i\}$. We can define a function on the plaque set $\mathcal{P}(U_i)$ by sending a plaque P to $\int_P \phi_i \alpha$ — notice that α is a p-form and P is an oriented p-manifold, so the integral is well-defined. Now define

$$C_\mu(\alpha) = \sum_i \int_{\mathcal{P}(U_i)} \left(\int_P \phi_i \alpha \right) d\mu_{U_i}(P).$$

12.58 Lemma. *The definition of $C(\alpha)$ is independent of the choices made in its construction. C_μ is a linear functional on the space of p-forms (a* p-current*). Moreover, C_μ is* closed*: that is, $C_\mu(d\beta) = 0$ for all $(p-1)$-forms β.* □

The proof of this lemma is elementary. The current C_μ is called the *Ruelle-Sullivan* current associated to the transverse measure μ. Because C_μ is closed it defines a linear functional on the de Rham cohomology group $H^p(M;\mathbb{R})$; that is, it defines a *homology class* $[C_\mu] \in H_p(M;\mathbb{R})$.

Using the Ruelle-Sullivan current we can define a trace on the groupoid algebra $C_c^\infty(G(M, F))$. Recall that the construction of the groupoid algebra makes implicit use of a *Haar system* on the groupoid. For the groupoid $G(M, F)$, a Haar system is simply a smooth family of Lebesgue measures on the leaves of F. Equivalently (since we are assuming that F is oriented) a Haar system is given by a p-form ω_F on M which restricts to a volume form on every leaf of F. Thus, from the transverse measure μ and the Haar system, we can construct an ordinary Borel measure ν on M by the formula

$$\int g\,d\nu = C_\mu(g\omega_F)$$

for a continuous function g on M.

Since M is the space of objects of the groupoid $G = G(M, F)$, there is a 'restriction to the diagonal' linear map

$$\delta\colon C_c^\infty(G) \to C^\infty(M).$$

Given a transverse measure μ, construct the linear functional ϕ_μ

$$k \mapsto \int \delta(k)\,d\nu = C_\mu(\delta(k)\omega_F)$$

on $C_c^\infty(G)$.

12.59 Lemma. *The functional ϕ_μ is tracial.*

Proof (sketch). Consider the case where the foliation is given by a product $M = F \times B$. In this case a transverse measure is simply a measure μ on B, and a Haar system is a family of Lebesgue measures λ_b on F, depending smoothly on $b \in B$. The groupoid algebra is the algebra of smooth functions on $F \times F \times B$, with composition law

$$k_1 \circ k_2(x, z, b) = \int k_1(x, y, b)k_2(y, z, b)\,d\lambda_b(y).$$

The functional ϕ_μ is given by the formula

$$\phi_\mu(k) = \iint k(x, x, b)\,d\lambda_b(x)\,d\mu(b).$$

Thus

$$\phi_\mu(k_1 \circ k_2) = \iiint k_1(x, y, b)k_2(y, x, b)\,d\lambda_b(y)\,d\lambda_b(x)\,d\mu(b),$$

and this is symmetrical in k_1 and k_2 by Fubini's theorem.

The general case is reduced to this one using foliation charts and partitions of unity. □

12.60 Lemma. *The functional ϕ_μ is normal.*

Proof. Exactly as in Lemma 12.49. □

Extend ϕ_μ to an unbounded trace τ_μ on $C^*(M, F)$ according to the procedure of Section 12.5. In turn, τ_μ defines a dimension function $\tau_{\mu*}\colon K(C^*(M, F)) \to \mathbb{R}$.

12.61 Exercise. Let (M, F) be a foliation equipped with a transverse measure μ. According to Example 12.27, a closed transversal W to F determines a homomorphism $K(W) \to K(C^*(M, F))$. Show that the diagram

$$\begin{tikzcd} K(W) \arrow[r] \arrow[dr] & K(C^*(M,F)) \arrow[d] \\ & \mathbb{R} \end{tikzcd}$$

commutes, where the right-hand vertical arrow is the dimension function $\tau_{\mu*}$ defined above and the left-hand vertical arrow is the dimension function defined by the restriction of the transverse measure to W (this induces a measure on W and therefore a trace on $C(W)$).

We can now state the *measured foliation index theorem*, which is the foliation analog of Atiyah's L^2 index theorem.

12.62 Theorem. *Let (M, F) be a compact foliated manifold equipped with a transverse measure μ, and let D be a leafwise elliptic operator on M. Denote by $\mathrm{Ind}(D)$ the longitudinal index of D in $K(C^*(M, F))$. Then*

$$\tau_{\mu*}(\mathrm{Ind}(D)) = (-1)^{\dim(F)}\langle [C_\mu], \mathrm{ch}(\sigma_D)\,\mathrm{Todd}(TF \otimes \mathbb{C})\rangle$$

where C_μ denotes the Ruelle-Sullivan current associated to μ. □

Connes' first proof of this result used the heat equation method that we have discussed in Remark 12.54. The virtue of the heat equation method (when it works) is that it produces a *local formula* for an index. This is particularly useful in a situation (such as the foliation case) where local structure is simple although global structure may be extremely complicated. A later proof of the measured foliation index theorem, by Connes and Skandalis, derives it from the foliation index theorem that we have discussed in this chapter. The key idea is to use an embedding into Euclidean space to reduce the theorem to the result proved in Exercise 12.61.

12.63 Example. Consider the Reeb foliation, with μ being the counting measure associated to the compact (toral) leaf. The holonomy cover of this compact leaf is simply the universal cover. Restricting to the compact leaf thus gives a $*$-homomorphism from the C^*-algebra of the foliation to the C^*-algebra of the homotopy groupoid of the compact leaf, and μ_* factors through this homomorphism. We see that in this case the measured foliation index theorem reduces to Atiyah's L^2-index theorem for the compact leaf.

Using Proposition 12.45 we obtain a geometric result on foliations by surfaces.

12.64 Proposition. *Let F be an oriented 2-dimensional foliation of a compact manifold M, equipped with an invariant transverse measure μ. Suppose that F is given a Riemannian metric, and let Ω be the Gauss curvature 2-form of the leaves. If $\langle [C_\mu], \Omega\rangle > 0$ then the foliation has some compact leaves (indeed, the set of compact leaves has positive μ measure).*

Proof. We apply the measured index theorem to the leafwise de Rham operator: that is, the operator $D = d + d^*$ acting on the space of leafwise differential forms, graded by the degree of forms. The measured index is just a constant times $\langle [C_\mu], \Omega \rangle$ in this case. Since this is greater than 0, the positively graded part of the kernel of D must be nonzero, by Proposition 12.45. Thus there must be some L^2 harmonic functions on the leaves (the spaces of harmonic functions and harmonic 2-forms are identified by Poincaré duality). But a complete Riemannian manifold (such as a leaf) admits L^2 harmonic functions if and only if it is compact. □

12.8 CODA

We have seen that one can interpret various kinds of generalized index theorems as a manifestation of a correspondence between the topology and the analysis of objects associated to smooth groupoids, expressed by the Baum-Connes conjecture. This method is extremely powerful but it has recently become clear that it also has certain limitations. In fact, there are examples of foliations for which the Baum-Connes conjecture fails. This is not the place to go into detail about the construction of these examples, but the basic idea is rather simple and involves a fundamental point which has already appeared both in our discussion of tensor products (Chapter 3) and in our discussion of group C^*-algebras (Chapter 11).

The point is that in general there may be *more than one way* to complete a given $*$-algebra to a C^*-algebra. The group (or groupoid) algebras appearing in Baum-Connes conjecture use completions relative to the regular representation of that group (or groupoid). The disadvantage of these completions is that they are not functorial: if $\phi\colon G \to H$ is a group homomorphism, the representation of G induced by ϕ from the regular representation of H may be entirely unrelated to the regular representation of H itself. Indeed, it is possible to give specific examples (related to Kazhdan's property T) where this lack of functoriality can be detected and analyzed.

On the other hand, the left hand side of the conjecture — the K-theory of BG — involves no analytic questions about representations and is completely functorial. Thus the conjecture itself predicts that $K(C^*_r(G))$ should depend functorially on G, but this would be a functoriality for which one could not give an analytic explanation. The counterexamples that are now known to various versions of the Baum-Connes conjecture use constructions based on property T to show that no such inscrutable functoriality can exist.

Such constructions deepen the mystery surrounding the Baum-Connes conjecture. Since we now know that it is not universally true, the enormously wide scope of its validity becomes still more intriguing.

12.9 NOTES AND REFERENCES

The basic reference for Connes' ideas about foliations and operator algebras is [15]. The general index theorem for foliations is proved in [19]. A further very deep development related to the *transverse* (rather than leafwise) structure of a foliation is in [16].

For the classical theory of foliations and many examples see the two-volume work by Candel and Conlon [11, 12].

Transverse measures were introduced in various contexts in the 1970s: see [41, 47]. They do not exist for all foliations. A proof of the index theorem for measured foliations which minimizes the use of C^*-algebraic ideas is in [44].

For counterexamples to various forms of the Baum-Connes conjecture see [26].

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