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K-Theory and Noncommutative Geometry

Guillermo Cortiñas
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Editors

European Mathematical Society

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Previously published:

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2000 Mathematics Subject Classification: 19-06, 58-06; 14A22, 14Fxx, 46Lxx, 53D55, 58Bxx

Key words: Isomorphism conjectures, Waldhausen K-theory, twisted K-theory, equivariant cyclic homology, index theory, Bost–Connes algebra, C*-manifolds, T-duality, deformation of gerbes, torsion theories, Parshin conjecture, Bloch–Kato conjecture

ISBN 978-3-03719-060-9

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

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CH-8092 Zürich
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Email: info@ems-ph.org
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Typeset using the authors' TEX files: I. Zimmermann, Freiburg
Printed in Germany

9 8 7 6 5 4 3 2 1

Preface

This volume contains the proceedings of VASBI, the ICM2006 satellite on K-theory and Noncommutative Geometry which took place in Valladolid, Spain, from August 31 to September 6, 2006. The conference's Scientific Committee was composed of Joachim Cuntz, Max Karoubi, Ryszard Nest and Charles A. Weibel. The conference was made possible through a grant by Caja Duero; additional funding was provided by the University of Valladolid. Funding to cover expenses of US based participants was provided by NSF, through a grant to C. A. Weibel, who was responsible, first of all, for preparing a succesful funding application, and then for managing the funds. The local committee, composed of N. Abad and E. Ellis, were in charge of conference logistics.

I am indebted to all these people and institutions for their support.

Guillermo Cortiñas, Organizer

Contents

Introduction

Since its inception 50 years ago, K-theory has been a tool for understanding a wide-ranging family of mathematical structures and their invariants: topological spaces, rings, algebraic varieties and operator algebras are the dominant examples. The invariants range from characteristic classes in cohomology, through determinants of matrices, to Chow groups of varieties, as well as to traces and indices of elliptic operators. Thus K-theory is notable for its connections with other branches of mathematics.

Noncommutative geometry, on the other hand, develops tools which allow one to think of noncommutative algebras in the same footing as commutative ones; as algebras of functions on (noncommutative) spaces. The algebras in question come from problems in various areas of mathematics and mathematical physics; typical examples include algebras of pseudodifferential operators, group algebras, and other algebras arising from quantum field theory.

To study noncommutative geometric problems one considers invariants of the relevant noncommutative algebras. These invariants include algebraic and topological K-theory, and also cyclic homology, discovered independently by Alain Connes and Boris Tsygan, which can be regarded both as a noncommutative version of de Rham cohomology, and as an additive version of K-theory. There are primary and secondary Chern characters which pass from K-theory to cyclic homology. These characters are relevant both to noncommutative and commutative problems, and have applications ranging from index theorems to the detection of singularities of commutative algebraic varieties.

The contributions to this volume represent this range of connections between K-theory, noncommmutative geometry, and other branches of mathematics.

An important connection between K-theory, topology, geometric group theory and noncommutative geometry is given by the isomorphism conjectures, such as those due to Baum–Connes, Bost–Connes and Farrell–Jones, which predict certain topological formulas for various kinds of K-theory of a crossed product, in both the C^*-algebraic and the purely algebraic contexts. These problems have received a lot of attention over the past twenty years. Not surprisingly, then, three of the articles in this volume discuss these problems. The first of these, by R. Meyer, is a survey on bivariant Kasparov theory and E-theory. Both bivariant K-theories are approached via their universal properties and equipped with extra structure such as a tensor product and a triangulated category structure. The construction of the Baum–Connes assembly map via localisation of categories is reviewed and the relation with the purely topological construction by Davis and Lück is explained. In the second article, A. Bartels, S. Echterhoff and W. Lück investigate when Isomorphism Conjectures, such as the ones due to Baum–Connes, Bost and Farrell–Jones, are stable under colimits of groups over directed sets (with not necessarily injective structure maps). They show in particular that both the K-theoretic Farrell–Jones Conjecture and the Bost Conjecture with coefficients hold for those groups for which Higson, Lafforgue and Skandalis have disproved the Baum–Connes Conjecture with coefficients. H. Emerson and R. Meyer study in their contribution an equivariant co-assembly map that is dual to the usual Baum–Connes assembly map and closely related to coarse geometry, equivariant Kasparov theory, and the existence of dual Dirac morphisms. As applications, they prove the existence of dual Dirac morphisms for groups with suitable compactifications, that is, satisfying the Carlsson–Pedersen condition, and study a K-theoretic counterpart to the proper Lipschitz cohomology of Connes, Gromov and Moscovici.

Many applications of K-theory to geometric topology come from the K-theory of Waldhausen categories. The article by F. Muro and A. Tonks is about the K-theory of a Waldhausen category. They give a simple representation of all elements in K_1 of such a category and prove relations between these representatives which hold in K_1.

Another connection between K-theory, topology and analysis comes from twisted K-theory, which has received renewed attention in the last decade, in the light of new developments inspired by Mathematical Physics. The next article is a survey on this subject, written by M. Karoubi, one of its founders. The author also proves some new results in the subject: a Thom isomorphism, explicit computations in the equivariant case and new cohomology operations.

As mentioned above, one of the most interesting points of contact between K-theory and noncommutative geometry comes through cyclic homology. C. Voigt's contribution is about this theory. He defines equivariant periodic cyclic homology for bornological quantum groups. Generalizing corresponding results from the group case, he shows that the theory is homotopy invariant, stable, and satisfies excision in both variables. Along the way he proves Radford's formula for the antipode of a bornological quantum group. Moreover he discusses anti-Yetter–Drinfeld modules and establishes an analogue of the Takesaki–Takai duality theorem in the setting of bornological quantum groups.

A central theme in noncommutative geometry is index theory. A basic construction in this area consists of associating a C^*-algebra to a geometric problem, and then compute an index using K-theory. P. Carrillo Rouse's article deals with the problem of index theory on singular spaces, and in particular with the construction of algebras and indices between the enveloping C^*-algebra and the convolution algebra of compactly supported functions.

The article by J. Cuntz is also about C^*-algebras. It presents a C^*-algebra which is naturally associated to the $ax + b$-semigroup over $\mathbb{N}$. It is simple and purely infinite and can be obtained from the algebra considered by Bost and Connes by adding one unitary generator which corresponds to addition. Its stabilization can be described as a crossed product of the algebra of continuous functions, vanishing at infinity, on the space of finite adeles for $\mathbb{Q}$ by the natural action of the $ax + b$-group over $\mathbb{Q}$.

W. Werner applies C^*-algebraic methods in infinite dimensional differential geometry in his contribution. It deals with a special class of 'non-compact' hermitian infinite dimensional symmetric spaces, generically denoted by U. The author calculates their invariant connection very explicitly and uses the concept of a Hilbert C*-manifold so that the Banach manifold in question is of the form $\operatorname{Aut} U/H$, where $\operatorname{Aut} U$ is the automorphism group of the Hilbert C*-manifold. Using results previously obtained with D. Blecher, he characterizes causal structure on U that comes from interpreting the elements of U as bounded Hilbert space operators.

The subject of the next article, by U. Bunke, T. Schick, M. Spitzweck and A. Thom, connected with group theory, topology, as well as to mathematical physics and noncommutative geometry, is Pontrjagin duality. The authors extend Pontrjagin duality from topological abelian groups to certain locally compact group stacks. To this end they develop a sheaf theory on the big site of topological spaces $\mathbf{S}$ in order to prove that the sheaves $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})$, $i = 1, 2$, vanish, where $\underline{G}$ is the sheaf represented by a locally compact abelian group and $\mathbb{T}$ is the circle. As an application of the theory they interpret topological T-duality of principal $\mathbb{T}^n$-bundles in terms of Pontrjagin duality of abelian group stacks.

An important source of examples and problems in noncommutative geometry comes from deformations of commutative structures. In this spirit, the article by P. Bressler, A. Gorokhovsky, R. Nest and B. Tsygan investigates the 2-groupoid of deformations of a gerbe on a C^∞ manifold. They identify the latter with the Deligne 2-groupoid of a corresponding twist of the differential graded Lie algebra of local Hochschild cochains on C^∞ functions.

The next article, by G. Garkusha and M. Prest, provides a bridge between the noncommutative and the commutative worlds. A common approach in noncommutative geometry is to study abelian or triangulated categories and to think of them as the replacement of an underlying scheme. This idea goes back to work of Grothendieck and Gabriel and continues to be of interest. The approach is justified by the fact that a (commutative noetherian) scheme can be reconstructed from the abelian category of quasi-coherent sheaves (Gabriel) or from the category of perfect complexes (Balmer). A natural question to ask, is whether the hypothesis on the scheme to be noetherian is really necessary. This is precisely the theme of recent work of Garkusha and Prest. They show how to deal with affine schemes, without imposing any finiteness assumptions. More specifically, the article in this volume discusses for module categories over commutative rings the classification of torsion classes of finite type. For instance, the prime ideal spectrum can be reconstructed from this classification.

The final two contributions are about K-theory and (commutative) algebraic geometry.

T. Geisser's article is about Parshin's conjecture. The latter states that $K_i(X)_{\mathbb{Q}} = 0$ for $i > 0$ and X smooth and projective over a finite field $\mathbb{F}_q$. The purpose of this article is to break up Parshin's conjecture into several independent statements, in the hope that each of them is easier to attack individually. If $CH_n(X, i)$ is Bloch's higher Chow group of cycles of relative dimension n, then in view of $K_i(X)_{\mathbb{Q}} \cong \bigoplus_n CH_n(X, i)_{\mathbb{Q}}$, Parshin's conjecture is equivalent to Conjecture $P(n)$ for all n, stating that $CH_n(X, i)_{\mathbb{Q}} = 0$ for $i > 0$, and all smooth and projective X. Assuming resolution of singularities, the author shows that Conjecture $P(n)$ is equivalent to the conjunction of three conjectures $A(n)$, $B(n)$ and $C(n)$, and gives several equivalent versions of these conjectures.

Finally, C. Weibel's article gives an axiomatic framework for proving that the norm residue map is an isomorphism (i.e., for settling the motivic Bloch–Kato conjecture).

This framework is a part of the Voevodsky–Rost program.

In conclusion, this volume presents a good sample of the wide range of aspects of current research in K-theory, noncommutative geometry and their applications.

Guillermo Cortiñas, Joachim Cuntz, Max Karoubi,
Ryszard Nest, and Charles A. Weibel

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Categorical aspects of bivariant K-theory

Ralf Meyer

1 Introduction

Non-commutative topology deals with topological properties of C^*-algebras. Already in the 1970s, the classification of AF-algebras by K-theoretic data [15] and the work of Brown–Douglas–Fillmore on essentially normal operators [6] showed clearly that topology provides useful tools to study C^*-algebras. A breakthrough was Kasparov's construction of a bivariant K-theory for separable C^*-algebras. Besides its applications within C^*-algebra theory, it also yields results in classical topology that are hard or even impossible to prove without it. A typical example is the Novikov conjecture, which deals with the homotopy invariance of certain invariants of smooth manifolds with a given fundamental group. This conjecture has been verified for many groups using Kasparov theory, starting with [31]. The C^*-algebraic formulation of the Novikov conjecture is closely related to the Baum–Connes conjecture, which deals with the computation of the K-theory $K_*(C^*_{\mathrm{red}}G)$ of reduced group C^*-algebras and has been one of the centres of attention in non-commutative topology in recent years.

The Baum–Connes conjecture in its original formulation [4] only deals with a single K-theory group; but a better understanding requires a different point of view. The approach by Davis and Lück in [14] views it as a natural transformation between two homology theories for G-CW-complexes. An analogous approach in the C^*-algebra framework appeared in [38]. These approaches to the Baum–Connes conjecture show the importance of studying not just single C^*-algebras, but categories of C^*-algebras and their properties. Older ideas like the universal property of Kasparov theory are of the same nature. Studying categories of objects instead of individual objects is becoming more and more important in algebraic topology and algebraic geometry as well.

Several mathematicians have suggested, therefore, to apply general constructions with categories (with additional structure) like generators, Witt groups, the centre, and support varieties to the C^*-algebra context. Despite the warning below, this seems a promising project, where little has been done so far. To prepare for this enquiry, we summarise some of the known properties of categories of C^*-algebras; we cover tensor products, some homotopy theory, universal properties, and triangulated structures. In addition, we examine the Universal Coefficient Theorem and the Baum–Connes assembly map.

Despite many formal similarities, the homotopy theory of spaces and non-commutative topology have a very different focus.

On the one hand, most of the complexities of the stable homotopy category of spaces vanish for C^*-algebras because only very few homology theories for spaces have a non-commutative counterpart: any functor on C^*-algebras satisfying some reasonable assumptions must be closely related to K-theory. Thus special features of topological K-theory become more transparent when we work with C^*-algebras.

On the other hand, analysis may create new difficulties, which appear to be very hard to study topologically. For instance, there exist C^*-algebras with vanishing K-theory which are nevertheless non-trivial in Kasparov theory; this means that the Universal Coefficient Theorem fails for them. I know no non-trivial topological statement about the subcategory of the Kasparov category consisting of C^*-algebras with vanishing K-theory; for instance, I know no compact objects.

It may be necessary, therefore, to restrict attention to suitable "bootstrap" categories in order to exclude pathologies that have nothing to do with classical topology. More or less by design, the resulting categories will be localisations of purely topological categories, which we can also construct without mentioning C^*-algebras. For instance, we know that the Rosenberg–Schochet bootstrap category is equivalent to a full subcategory of the category of BU-module spectra. But we can hope for more interesting categories when we work equivariantly with respect to, say, discrete groups.

2 Additional structure in C^*-algebra categories

We assume that the reader is familiar with some basic properties of C^*-algebras, including the definition (see for instance [2], [13]). As usual, we allow non-unital C^*-algebras. We define some categories of C^*-algebras in §2.1 and consider group C^*-algebras and crossed products in §2.2. Then we discuss C^*-tensor products and mention the notions of nuclearity and exactness in §2.3. The upshot is that $\mathfrak{C}^*\mathfrak{alg}$ and G-$\mathfrak{C}^*\mathfrak{alg}$ carry two structures of symmetric monoidal category, which coincide for nuclear C^*-algebras. We prove in §2.4 that $\mathfrak{C}^*\mathfrak{alg}$ and G-$\mathfrak{C}^*\mathfrak{alg}$ are bicomplete, that is, all diagrams in them have both a limit and a colimit. We equip morphism spaces between C^*-algebras with a canonical base point and topology in §2.5; thus the category of C^*-algebras is enriched over the category of pointed topological spaces. In §2.6, we define mapping cones and cylinders in categories of C^*-algebras; these rudimentary tools suffice to carry over some basic homotopy theory.

2.1 Categories of C^*-algebras

Definition 1. The category of C^*-algebras is the category $\mathfrak{C}^*\mathfrak{alg}$ whose objects are the C^*-algebras and whose morphisms $A \to B$ are the *-homomorphisms $A \to B$; we denote this set of morphisms by $\mathrm{Hom}(A, B)$.

A C^*-algebra is called *separable* if it has a countable dense subset. We often restrict attention to the full subcategory $\mathfrak{C}^*\mathfrak{sep} \subseteq \mathfrak{C}^*\mathfrak{alg}$ of separable C^*-algebras.

Examples of C^*-algebras are group C^*-algebras and C^*-crossed products. We briefly recall some relevant properties of these constructions. A more detailed discussion can be found in many textbooks such as [44].

Definition 2. We write $A \Subset \mathfrak{C}$ to denote that A is an object of the category $\mathfrak{C}$. The notation $f \in \mathfrak{C}$ means that f is a morphism in $\mathfrak{C}$; but to avoid confusion we always specify domain and target and write $f \in \mathfrak{C}(A, B)$ instead of $f \in \mathfrak{C}$.

2.2 Group actions, and crossed products. For any locally compact group G, we have a *reduced group* C^**-algebra* $\mathrm{C}^*_{\mathrm{red}}(G)$ and a *full group* C^**-algebra* $\mathrm{C}^*(G)$. Both are defined as completions of the group Banach algebra $(L^1(G), *)$ for suitable C^*-norms. They are related by a canonical surjective *-homomorphism $\mathrm{C}^*(G) \to \mathrm{C}^*_{\mathrm{red}}(G)$, which is an isomorphism if and only if G is *amenable*.

The norm on $\mathrm{C}^*(G)$ is the maximal C^*-norm, so that any strongly continuous unitary representation of G on a Hilbert space induces a *-representation of $\mathrm{C}^*(G)$. The norm on $\mathrm{C}^*_{\mathrm{red}}(G)$ is defined using the regular representation of G on $L^2(G)$; hence a representation of G only induces a *-representation of $\mathrm{C}^*_{\mathrm{red}}(G)$ if it is weakly contained in the regular representation. For reductive Lie groups and reductive p-adic groups, these representations are exactly the tempered representations, which are much easier to classify than all unitary representations.

Definition 3. A G-C^**-algebra* is a C^*-algebra A with a strongly continuous representation of G by C^*-algebra automorphisms. The category of G-C^*-algebras is the category $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ whose objects are the G-C^*-algebras and whose morphisms $A \to B$ are the G-equivariant *-homomorphisms $A \to B$; we denote this morphism set by $\mathrm{Hom}_G(A, B)$.

Example 4. If $G = \mathbb{Z}$, then a G-C^*-algebra is nothing but a pair (A, α) consisting of a C^*-algebra A and a *-automorphism $\alpha\colon A \to A$: let α be the action of the generator $1 \in \mathbb{Z}$.

Equipping C^*-algebras with a trivial action provides a functor

$$\tau\colon \mathfrak{C}^*\mathfrak{alg} \to G\text{-}\mathfrak{C}^*\mathfrak{alg}, \quad A \mapsto A_\tau. \tag{1}$$

Since $\mathbb{C}$ has only the identity automorphism, the trivial action is the only way to turn $\mathbb{C}$ into a G-C^*-algebra.

The *full and reduced* C^**-crossed products* are versions of the full and reduced group C^*-algebras with coefficients in G-C^*-algebras (see [44]). They define functors

$$G \ltimes \text{␣}, G \ltimes_{\mathrm{r}} \text{␣}\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}, \quad A \mapsto G \ltimes A, \quad G \ltimes_{\mathrm{r}} A,$$

such that $G \ltimes \mathbb{C} = \mathrm{C}^*(G)$ and $G \ltimes_{\mathrm{r}} \mathbb{C} = \mathrm{C}^*_{\mathrm{red}}(G)$.

Definition 5. A diagram $I \to E \to Q$ in $\mathfrak{C}^*\mathfrak{alg}$ is an *extension* if it is isomorphic to the canonical diagram $I \to A \to A/I$ for some ideal I in a C^*-algebra A; extensions in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ are defined similarly, using G-invariant ideals in G-C^*-algebras. We write $I \rightarrowtail E \twoheadrightarrow Q$ to denote extensions.

Although C^*-algebra extensions have some things in common with extensions of, say, modules, there are significant differences because $\mathfrak{C}^*\mathfrak{alg}$ is not Abelian, not even additive.

Proposition 6. *The full crossed product functor* $G \ltimes \llcorner\!\lrcorner \colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}$ *is* exact *in the sense that it maps extensions in* $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ *to extensions in* $\mathfrak{C}^*\mathfrak{alg}$.

Proof. This is Lemma 4.10 in [21]. □

Definition 7. A locally compact group G is called *exact* if the reduced crossed product functor $G \ltimes_{\mathrm{r}} \llcorner\!\lrcorner \colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}$ is exact.

Although this is not apparent from the above definition, *exactness* is a geometric property of a group: it is equivalent to Yu's property (A) or to the existence of an amenable action on a compact space [43].

Most groups you know are exact. The only source of non-exact groups known at the moment are Gromov's random groups. Although exactness might remind you of the notion of flatness in homological algebra, it has a very different flavour. The difference is that the functor $G \ltimes_{\mathrm{r}} \llcorner\!\lrcorner$ always preserves injections and surjections. What may go wrong for non-exact groups is exactness in the *middle* (compare the discussion before Proposition 18). Hence we cannot study the lack of exactness by derived functors.

Even for non-exact groups, there is a class of extensions for which reduced crossed products are always exact:

Definition 8. A *section* for an extension

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q \tag{2}$$

in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ is a map (of sets) $s\colon Q \to E$ with $p \circ s = \mathrm{id}_Q$. We call (2) *split* if there is a section that is a G-equivariant *-homomorphism. We call (2) *G-equivariantly cp-split* if there is a G-equivariant, completely positive, contractive, linear section.

Sections are also often called *lifts*, *liftings*, or *splittings*.

Proposition 9. *Both the reduced and the full crossed product functors map split extensions in* $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ *again to split extensions in* $\mathfrak{C}^*\mathfrak{alg}$ *and G-equivariantly cp-split extensions in* $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ *to cp-split extensions in* $\mathfrak{C}^*\mathfrak{alg}$.

Proof. Let $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be an extension in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$. Proposition 6 shows that $G \ltimes K \rightarrowtail G \ltimes E \twoheadrightarrow G \ltimes Q$ is again an extension. Since reduced and full crossed products are functorial for equivariant completely positive contractions, this extension is split or cp-split if the original extension is split or equivariantly cp-split, respectively. This yields the assertions for full crossed products.

Since a *-homomorphism with dense range is automatically surjective, the induced map $G \ltimes_{\mathrm{r}} p\colon G \ltimes_{\mathrm{r}} E \to G \ltimes_{\mathrm{r}} Q$ is surjective. It is evident from the definition of reduced crossed products that $G \ltimes_{\mathrm{r}} i$ is injective. What is unclear is whether the range of $G \ltimes_{\mathrm{r}} i$ and the kernel of $G \ltimes_{\mathrm{r}} p$ coincide. As for the full crossed product, a G-equivariant completely positive contractive section $s\colon Q \to E$ induces a completely positive contractive section $G \ltimes_{\mathrm{r}} s$ for $G \ltimes_{\mathrm{r}} p$. The linear map

$$\varphi := \mathrm{id}_{G \ltimes_{\mathrm{r}} E} - (G \ltimes_{\mathrm{r}} s) \circ (G \ltimes_{\mathrm{r}} p)\colon G \ltimes_{\mathrm{r}} E \to G \ltimes_{\mathrm{r}} E$$

is a retraction from $G \ltimes_{\mathrm{r}} E$ onto the kernel of $G \ltimes_{\mathrm{r}} p$ by construction. Furthermore, it maps the dense subspace $L^1(G, E)$ into $L^1(G, K)$. Hence it maps all of $G \ltimes_{\mathrm{r}} E$ into $G \ltimes_{\mathrm{r}} K$. This implies $G \ltimes_{\mathrm{r}} K = \ker(G \ltimes_{\mathrm{r}} p)$ as desired. □

2.3 Tensor products and nuclearity. Most results in this section are proved in detail in [42], [60]. Let A_1 and A_2 be two C*-algebras. Their (algebraic) tensor product $A_1 \otimes A_2$ is still a *-algebra. A *C*-tensor product* of A_1 and A_2 is a C*-completion of $A_1 \otimes A_2$, that is, a C*-algebra that contains $A_1 \otimes A_2$ as a dense *-subalgebra. A C*-tensor product is determined uniquely by the restriction of its norm to $A_1 \otimes A_2$. A norm on $A_1 \otimes A_2$ is allowed if it is a C*-*norm*, that is, multiplication and involution have norm 1 and $\|x^*x\| = \|x\|^2$ for all $x \in A_1 \otimes A_2$.

There is a maximal C*-norm on $A_1 \otimes A_2$. The resulting C*-tensor product is called *maximal* C*-*tensor product* and denoted $A_1 \otimes_{\max} A_2$. It is characterised by the following universal property:

Proposition 10. *There is a natural bijection between non-degenerate *-homomorphisms $A_1 \otimes_{\max} A_2 \to \mathbb{B}(\mathcal{H})$ and pairs of commuting non-degenerate *-homomorphisms $A_1 \to \mathbb{B}(\mathcal{H})$ and $A_2 \to \mathbb{B}(\mathcal{H})$; here we may replace $\mathbb{B}(\mathcal{H})$ by any multiplier algebra $\mathcal{M}(D)$ of a* C*-*algebra D.*

A *-representation $A \to \mathbb{B}(\mathcal{H})$ is *non-degenerate* if $A \cdot \mathcal{H}$ is dense in $\mathcal{H}$; we need this to get representations of A_1 and A_2 out of a representation of $A_1 \otimes_{\max} A_2$ because, for non-unital algebras, $A_1 \otimes_{\max} A_2$ need not contain copies of A_1 and A_2.

The maximal tensor product is *natural*, that is, it defines a bifunctor

$$\otimes_{\max} \colon \mathfrak{C}^*\mathfrak{alg} \times \mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}.$$

If A_1 and A_2 are G-C*-algebras, then $A_1 \otimes_{\max} A_2$ inherits two group actions of G by naturality; these are again strongly continuous, so that $A_1 \otimes_{\max} A_2$ becomes a $G \times G$-C*-algebra. Restricting the action to the diagonal in $G \times G$, we turn $A_1 \otimes_{\max} A_2$ into a G-C*-algebra. Thus we get a bifunctor

$$\otimes_{\max} \colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \times G\text{-}\mathfrak{C}^*\mathfrak{alg} \to G\text{-}\mathfrak{C}^*\mathfrak{alg}.$$

The following lemma asserts, roughly speaking, that this tensor product has the same formal properties as the usual tensor product for vector spaces:

Lemma 11. *There are canonical isomorphisms*

$$\begin{gathered}(A \otimes_{\max} B) \otimes_{\max} C \cong A \otimes_{\max} (B \otimes_{\max} C),\\ A \otimes_{\max} B \cong B \otimes_{\max} A,\\ \mathbb{C} \otimes_{\max} A \cong A \cong A \otimes_{\max} \mathbb{C}\end{gathered}$$

for all objects of G-$\mathfrak{C}^*\mathfrak{alg}$ (and, in particular, of $\mathfrak{C}^*\mathfrak{alg}$). These define a structure of symmetric monoidal category on G-$\mathfrak{C}^*\mathfrak{alg}$ (see [35], [52]).

A functor between symmetric monoidal categories is called *symmetric monoidal* if it is compatible with the tensor products in a suitable sense [52]. A trivial example is the functor $\tau\colon \mathfrak{C}^*\mathfrak{alg} \to G\text{-}\mathfrak{C}^*\mathfrak{alg}$ that equips a C*-algebra with the trivial G-action.

It follows from the universal property that $\otimes_{\max}$ is compatible with *full* crossed products: if $A \Subset G\text{-}\mathfrak{C}^*\mathfrak{alg}$, $B \Subset \mathfrak{C}^*\mathfrak{alg}$, then there is a natural isomorphism

$$G \ltimes \bigl(A \otimes_{\max} \tau(B)\bigr) \cong (G \ltimes A) \otimes_{\max} B. \tag{3}$$

Like full crossed products, the maximal tensor product may be hard to describe because it involves a maximum of all possible C*-tensor norms. There is another C*-tensor norm that is defined more concretely and that combines well with *reduced* crossed products.

Recall that any C*-algebra A can be represented faithfully on a Hilbert space. That is, there is an injective *-homomorphism $A \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$; here $\mathbb{B}(\mathcal{H})$ denotes the C*-algebra of bounded operators on $\mathcal{H}$. If A is separable, we can find such a representation on the separable Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$. The tensor product of two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ carries a canonical inner product and can be completed to a Hilbert space, which we denote by $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ support faithful representations of C*-algebras A_1 and A_2, then we get an induced *-representation of $A_1 \otimes A_2$ on $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$.

Definition 12. The *minimal tensor product* $A_1 \otimes_{\min} A_2$ is the completion of $A_1 \otimes A_2$ with respect to the operator norm from $\mathbb{B}(\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2)$.

It can be check that this is well-defined, that is, the C*-norm on $A_1 \otimes A_2$ does not depend on the chosen faithful representations of A_1 and A_2. The same argument also yields the naturality of $A_1 \otimes_{\min} A_2$. Hence we get a bifunctor

$$\otimes_{\min}\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \times G\text{-}\mathfrak{C}^*\mathfrak{alg} \to G\text{-}\mathfrak{C}^*\mathfrak{alg};$$

it defines another symmetric monoidal category structure on $G\text{-}\mathfrak{C}^*\mathfrak{alg}$.

We may also call $A_1 \otimes_{\min} A_2$ the *spatial* tensor product. It is *minimal* in the sense that it is dominated by any C*-tensor norm on $A_1 \otimes A_2$ that is compatible with the given norms on A_1 and A_2. In particular, we have a canonical surjective *-homomorphism

$$A_1 \otimes_{\max} A_2 \to A_1 \otimes_{\min} A_2. \tag{4}$$

Definition 13. A C*-algebra A_1 is *nuclear* if the map in (4) is an isomorphism for all C*-algebras A_2.

The name comes from an analogy between nuclear C*-algebras and nuclear locally convex topological vector spaces (see [20]). But this is merely an analogy: the only C*-algebras that are nuclear as locally convex topological vector spaces are the finite-dimensional ones.

Many important C*-algebras are nuclear. This includes the following examples:

- commutative C*-algebras;
- matrix algebras and algebras of compact operators on Hilbert spaces;

- group C*-algebras of *amenable* groups (or groupoids);
- C*-algebras of type I and, in particular, continuous trace C*-algebras.

If A is nuclear, then there is only one reasonable C*-algebra completion of $A \otimes B$. Therefore, if we can write down any, it must be equal to both $A \otimes_{\min} B$ and $A \otimes_{\max} B$.

Example 14. For a compact space X and a C*-algebra A, we let $\mathcal{C}(X, A)$ be the C*-algebra of all continuous functions $X \to A$. If X is a *pointed* compact space, we let $\mathcal{C}_0(X, A)$ be the C*-algebra of all continuous functions $X \to A$ that vanish at the base point of X; this contains $\mathcal{C}(X, A)$ as a special case because $\mathcal{C}(X, A) \cong \mathcal{C}_0(X_+, A)$, where $X_+ = X \sqcup \{\star\}$ with base point $\star$. We have

$$\mathcal{C}_0(X, A) \cong \mathcal{C}_0(X) \otimes_{\min} A \cong \mathcal{C}_0(X) \otimes_{\max} A.$$

Example 15. There is a unique C*-norm on $\mathbb{M}_n \otimes A = \mathbb{M}_n(A)$ for all $n \in \mathbb{N}$.

For a Hilbert space $\mathcal{H}$, let $\mathbb{K}(\mathcal{H})$ be the C*-algebra of compact operators on $\mathcal{H}$. Then $\mathbb{K}(\mathcal{H}) \otimes A$ contains copies of $\mathbb{M}_n(A)$, $n \in \mathbb{N}$, for all finite-dimensional subspaces of $\mathcal{H}$. These carry a unique C*-norm. The C*-norms on these subspaces are compatible and extend to the unique C*-norm on $\mathbb{K}(\mathcal{H}) \otimes A$.

The class of nuclear C*-algebras is closed under ideals, quotients (by ideals), extensions, inductive limits, and crossed products by actions of amenable locally compact groups. In particular, this covers crossed products by automorphisms (see Example 4).

C*-subalgebras of nuclear C*-algebras need not be nuclear any more, but they still enjoy a weaker property called *exactness*:

Definition 16. A C*-algebra A is called *exact* if the functor $A \otimes_{\min} \text{\textvisiblespace}$ preserves C*-algebra extensions.

It is known [33], [43] that a discrete group is exact (Definition 7) if and only if its group C*-algebra is exact (Definition 16), if and only if the group has an amenable action on some compact topological space.

Example 17. Let G be the non-Abelian free group on 2 generators. Let G act freely and properly on a tree as usual. Let X be the ends compactification of this tree, equipped with the induced action of G. This action is known to be amenable, so that G is an exact group. Since the action is amenable, the crossed product algebras $G \ltimes_{\mathrm{r}} \mathcal{C}(X)$ and $G \ltimes \mathcal{C}(X)$ coincide and are nuclear. The embedding $\mathbb{C} \to \mathcal{C}(X)$ induces an embedding $\mathrm{C}^*_{\mathrm{red}}(G) \to G \ltimes \mathcal{C}(X)$. But G is not amenable. Hence the C*-algebra $\mathrm{C}^*_{\mathrm{red}}(G)$ is exact but not nuclear.

As for crossed products, $\otimes_{\min}$ respects injections and surjections. The issue with exactness in the middle is the following. Elements of $A \otimes_{\min} B$ are limits of tensors of the form $\sum_{i=1}^n a_i \otimes b_i$ with $a_1, \dots, a_n \in A$, $b_1, \dots, b_n \in B$. If an element in $A \otimes_{\min} B$ is annihilated by the map to $(A/I) \otimes_{\min} B$, then we can approximate it by such finite sums for which $\sum_{i=1}^n (a_i \bmod I) \otimes b_i$ goes to 0. But this does not suffice to find approximations in $I \otimes B$. Thus the kernel of the projection map $A \otimes_{\min} B \twoheadrightarrow (A/I) \otimes_{\min} B$ may be strictly larger than $I \otimes_{\min} B$.

Proposition 18. *The functor* $\otimes_{\max}$ *is exact in each variable, that is,* $\text{␣} \otimes_{\max} D$ *maps extensions in* $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ *again to extensions for each* $D \Subset G\text{-}\mathfrak{C}^*\mathfrak{alg}$.

Both $\otimes_{\min}$ *and* $\otimes_{\max}$ *map split extensions to split extensions and (equivariantly) cp-split extensions again to (equivariantly) cp-split extensions.*

The proof is similar to the proofs of Propositions 6 and 9.

If A or B is nuclear, we simply write $A \otimes B$ for $A \otimes_{\max} B \cong A \otimes_{\min} B$.

2.4 Limits and colimits

Proposition 19. *The categories* $\mathfrak{C}^*\mathfrak{alg}$ *and* $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ *are bicomplete, that is, any (small) diagram in these categories has both a limit and a colimit.*

Proof. To get general limits and colimits, it suffices to construct equalisers and coequalisers for pairs of parallel morphisms $f_0, f_1 \colon A \rightrightarrows B$, direct products and coproducts $A_1 \times A_2$ and $A_1 \sqcup A_2$ for any pair of objects and, more generally, for arbitrary sets of objects.

The equaliser and coequaliser of $f_0, f_1 \colon A \rightrightarrows B$ are

$$\ker(f_0 - f_1) = \{a \in A \mid f_0(a) = f_1(a)\} \subseteq A$$

and the quotient of A_1 by the closed *-ideal generated by the range of $f_0 - f_1$, respectively. Here we use that quotients of C*-algebras by closed *-ideals are again C*-algebras. Notice that $\ker(f_0 - f_1)$ is indeed a C*-subalgebra of A.

The direct product $A_1 \times A_2$ is the usual direct product, equipped with the canonical C*-algebra structure. We can generalise the construction of the direct product to infinite direct products: let $\prod_{i \in I} A_i$ be the set of all *norm-bounded* sequences $(a_i)_{i \in I}$ with $a_i \in A_i$ for all $i \in I$; this is a C*-algebra with respect to the obvious *-algebra structure and the norm $\|(a_i)\| := \sup_{i \in I} \|a_i\|$. It has the right universal property because any *-homomorphism is norm-contracting. (A similar construction with Banach algebras would fail at this point.)

The coproduct $A_1 \sqcup A_2$ is also called *free product* and denoted $A_1 * A_2$; its construction is more involved. The free $\mathbb{C}$-algebra generated by A_1 and A_2 carries a canonical involution, so that it makes sense to study C*-norms on it. It turns out that there is a maximal such C*-norm. The resulting C*-completion is the free product C*-algebra. In the equivariant case, $A_1 \sqcup A_2$ inherits an action of G, which is strongly continuous. The resulting object of $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ has the correct universal property for a coproduct.

An *inductive system* of C*-algebras $(A_i, \alpha_i^j)_{i \in I}$ is called *reduced* if all the maps $\alpha_i^j \colon A_i \to A_j$ are injective; then they are automatically isometric embeddings. We may as well assume that these maps are identical inclusions of C*-subalgebras. Then we can form a *-algebra $\bigcup A_i$, and the given C*-norms piece together to a C*-norm on $\bigcup A_i$. The resulting completion is $\varinjlim(A_i, \alpha_i^j)$. In particular, we can construct an infinite coproduct as the inductive limit of its finite sub-coproducts. Thus we get infinite coproducts. $\square$

The category of commutative C*-algebras is equivalent to the opposite of the category of pointed compact spaces by the Gelfand–Naimark Theorem. It is frequently convenient to replace a *pointed compact* space X with base point $\star$ by the *locally compact* space $X \setminus \{\star\}$. A continuous map $X \to Y$ extends to a pointed continuous map $X_+ \to Y_+$ if and only if it is *proper*. But there are more pointed continuous maps $f\colon X_+ \to Y_+$ than proper continuous maps $X \to Y$ because points in X may be mapped to the point at infinity $\infty \in Y_+$. For instance, the zero homomorphism $\mathcal{C}_0(Y) \to \mathcal{C}_0(X)$ corresponds to the constant map $x \mapsto \infty$.

Example 20. If $U \subseteq X$ is an open subset of a locally compact space, then $\mathcal{C}_0(U)$ is an ideal in $\mathcal{C}_0(X)$. No map $X \to U$ corresponds to the embedding $\mathcal{C}_0(U) \to \mathcal{C}_0(X)$.

Example 21. Products of commutative C*-algebras are again commutative and correspond by the Gelfand–Naimark Theorem to coproducts in the category of pointed compact spaces. The coproduct of a set of pointed compact spaces is the *Stone–Čech compactification* of their wedge sum. Thus infinite products in $\mathfrak{C}^*\mathfrak{alg}$ and G-$\mathfrak{C}^*\mathfrak{alg}$ do not behave well for the purposes of homotopy theory.

The coproduct of two non-zero C*-algebras is never commutative and hence has no analogue for (pointed) compact spaces. The smash product for pointed compact spaces corresponds to the tensor product of C*-algebras because

$$\mathcal{C}_0(X \wedge Y) \cong \mathcal{C}_0(X) \otimes_{\min} \mathcal{C}_0(Y).$$

2.5 Enrichment over pointed topological spaces. Let A and B be C*-algebras. It is well-known that a *-homomorphism $f\colon A \to B$ is automatically norm-contracting and induces an isometric embedding $A/\ker f \to B$ with respect to the quotient norm on $A/\ker f$. The reason for this is that the norm for self-adjoint elements in a C*-algebra agrees with the spectral radius and hence is determined by the algebraic structure; by the C*-condition $\|a\|^2 = \|a^*a\|$, this extends to all elements of a C*-algebra.

It follows that $\mathrm{Hom}(A, B)$ is an *equicontinuous* set of linear maps $A \to B$. We always equip $\mathrm{Hom}(A, B)$ with the topology of pointwise norm-convergence. Its subbasic open subsets are of the form

$$\{f\colon A \to B \mid \|(f - f_0)(a)\| < 1 \text{ for all } a \in S\}$$

for $f_0 \in \mathrm{Hom}(A, B)$ and a finite subset $S \subseteq A$. Since $\mathrm{Hom}(A, B)$ is equicontinuous, this topology agrees with the topology of uniform convergence on compact subsets, which is generated by the corresponding subsets for compact S. This is nothing but the compact-open topology on mapping spaces. But it differs from the topology defined by the operator norm. We shall never use the latter.

Lemma 22. *If A is separable, then* $\mathrm{Hom}(A, B)$ *is metrisable for any* B.

Proof. There exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A with $\lim a_n = 0$ whose closed linear span is all of A. The metric

$$d(f_1, f_2) = \sup\{\|f_1(a_n) - f_2(a_n)\| \mid n \in \mathbb{N}\}$$

defines the topology of uniform convergence on compact subsets on $\mathrm{Hom}(A, B)$ because the latter is equicontinuous. $\square$

There is a distinguished element in $\mathrm{Hom}(A, B)$ as well, namely, the *zero homomorphism* $A \to 0 \to B$. Thus $\mathrm{Hom}(A, B)$ becomes a pointed topological space.

Proposition 23. *The above construction provides an enrichment of* $\mathfrak{C}^*\mathfrak{alg}$ *over the category of pointed Hausdorff topological spaces.*

Proof. It is clear that $0 \circ f = 0$ and $f \circ 0 = 0$ for all morphisms f. Furthermore, we must check that composition of morphisms is *jointly* continuous. This follows from the equicontinuity of $\mathrm{Hom}(A, B)$. $\square$

This enrichment allows us to carry over some important definitions from categories of spaces to $\mathfrak{C}^*\mathfrak{alg}$. For instance, a *homotopy* between two *-homomorphisms $f_0, f_1 : A \to B$ is a continuous path between f_0 and f_1 in the topological space $\mathrm{Hom}(A, B)$. In the following proposition, $\mathrm{Map}_+(X, Y)$ denotes the space of morphisms in the category of pointed topological spaces, equipped with the compact-open topology.

Proposition 24 (compare Proposition 3.4 in [28]). *Let A and B be* C**-algebras and let X be a pointed compact space. Then*

$$\mathrm{Map}_+\bigl(X, \mathrm{Hom}(A, B)\bigr) \cong \mathrm{Hom}\bigl(A, \mathcal{C}_0(X, B)\bigr)$$

as pointed topological spaces.

Proof. If we view A and B as pointed topological spaces with the norm topology and base point 0, then $\mathrm{Hom}(A, B) \subseteq \mathrm{Map}_+(A, B)$ is a topological subspace with the same base point because $\mathrm{Hom}(A, B)$ also carries the compact-open topology. Since X is compact, standard point set topology yields homeomorphisms

$$\begin{aligned}\mathrm{Map}_+\bigl(X, \mathrm{Map}_+(A, B)\bigr) &\cong \mathrm{Map}_+(X \wedge A, B)\\ &\cong \mathrm{Map}_+\bigl(A, \mathrm{Map}_+(X, B)\bigr) = \mathrm{Map}_+\bigl(A, \mathcal{C}_0(X, B)\bigr).\end{aligned}$$

These restrict to the desired homeomorphism. $\square$

In particular, a homotopy between two *-homomorphisms $f_0, f_1 : A \rightrightarrows B$ is equivalent to a *-homomorphism $f : A \to \mathcal{C}([0, 1], B)$ with $\mathrm{ev}_t \circ f = f_t$ for $t = 0, 1$, where ev_t denotes the *-homomorphism

$$\mathrm{ev}_t : \mathcal{C}([0, 1], B) \to B, \quad f \mapsto f(t).$$

We also have

$$\mathcal{C}_0\bigl(X, \mathcal{C}_0(Y, A)\bigr) \cong \mathcal{C}_0(X \wedge Y, A)$$

for all pointed compact spaces X, Y and all G-C*-algebras A. Thus a homotopy between two homotopies can be encoded by a *-homomorphism

$$A \to \mathcal{C}\bigl([0, 1], \mathcal{C}([0, 1], B)\bigr) \cong \mathcal{C}([0, 1]^2, B).$$

These constructions work only for pointed *compact* spaces. If we enlarge the category of C*-algebras to a suitable category of projective limits of C*-algebras as in [28], then we can define $\mathcal{C}_0(X, A)$ for any pointed *compactly generated* space X. But we lose some of the nice analytic properties of C*-algebras. Therefore, I prefer to stick to the category of C*-algebras itself.

2.6 Cylinders, cones, and suspensions. The following definitions go back to [54], where some more results can be found. The description of homotopies above leads us to define the *cylinder* over a C*-algebra A by

$$\mathrm{Cyl}(A) := \mathcal{C}([0,1], A).$$

This is compatible with the cylinder construction for spaces because

$$\mathrm{Cyl}\big(\mathcal{C}_0(X)\big) \cong \mathcal{C}\big([0,1], \mathcal{C}_0(X)\big) \cong \mathcal{C}_0([0,1]_+ \wedge X)$$

for any pointed compact space X; if we use locally compact spaces, we get $[0,1] \times X$ instead of $[0,1]_+ \wedge X$.

The universal property of $\mathrm{Cyl}(A)$ is dual to the usual one for spaces because the identification between pointed compact spaces and commutative C*-algebras is contravariant.

Similarly, we may define the *cone* $\mathrm{Cone}(A)$ and the *suspension* $\mathrm{Sus}(A)$ by

$$\mathrm{Cone}(A) := \mathcal{C}_0\big([0,1] \setminus \{0\}, A\big), \quad \mathrm{Sus}(A) := \mathcal{C}_0\big([0,1] \setminus \{0,1\}, A\big) \cong \mathcal{C}_0(\mathbb{S}^1, A),$$

where $\mathbb{S}^1$ denotes the pointed 1-sphere, that is, circle. These constructions are compatible with the corresponding ones for spaces as well, that is,

$$\mathrm{Cone}\big(\mathcal{C}_0(X)\big) \cong \mathcal{C}_0([0,1] \wedge X), \quad \mathrm{Sus}\big(\mathcal{C}_0(X)\big) \cong \mathcal{C}_0(\mathbb{S}^1 \wedge X).$$

Here $[0,1]$ has the base point 0.

Definition 25. Let $f : A \to B$ be a morphism in $\mathfrak{C}^*\mathfrak{alg}$ or G-$\mathfrak{C}^*\mathfrak{alg}$. The *mapping cylinder* $\mathrm{Cyl}(f)$ and the *mapping cone* $\mathrm{Cone}(f)$ of f are the limits of the diagrams

$$A \xrightarrow{f} B \xleftarrow{\mathrm{ev}_1} \mathrm{Cyl}(B), \quad A \xrightarrow{f} B \xleftarrow{\mathrm{ev}_1} \mathrm{Cone}(B).$$

More concretely,

$$\begin{aligned} \mathrm{Cone}(f) &= \big\{(a,b) \in A \times \mathcal{C}_0\big((0,1], B\big) \mid f(a) = b(1)\big\}, \\ \mathrm{Cyl}(f) &= \big\{(a,b) \in A \times \mathcal{C}_0([0,1], B\big) \mid f(a) = b(1)\big\}. \end{aligned}$$

If $f : X \to Y$ is a morphism of pointed compact spaces, then the mapping cone and mapping cylinder of the induced *-homomorphism $\mathcal{C}_0(f) : \mathcal{C}_0(Y) \to \mathcal{C}_0(X)$ agree with $\mathcal{C}_0\big(\mathrm{Cyl}(f)\big)$ and $\mathcal{C}_0\big(\mathrm{Cone}(f)\big)$, respectively.

The cylinder, cone, and suspension functors are exact for various kinds of extensions: they map extensions, split extensions, and cp-split extensions again to extensions, split extensions, and cp-split extensions, respectively. Similar remarks apply to mapping cylinders and mapping cones: for any morphism of extensions

$$\begin{tikzcd} I \arrow[r, tail] \arrow[d, "\alpha"] & E \arrow[r, two heads] \arrow[d, "\beta"] & Q \arrow[d, "\gamma"] \\ I' \arrow[r, tail] & E' \arrow[r, two heads] & Q', \end{tikzcd} \tag{5}$$

we get extensions

$$\mathrm{Cyl}(\alpha) \rightarrowtail \mathrm{Cyl}(\beta) \twoheadrightarrow \mathrm{Cyl}(\gamma), \quad \mathrm{Cone}(\alpha) \rightarrowtail \mathrm{Cone}(\beta) \twoheadrightarrow \mathrm{Cone}(\gamma); \tag{6}$$

if the extensions in (5) are split or cp-split, so are the resulting extensions in (6).

The familiar maps relating mapping cones and cylinders to cones and suspensions continue to exist in our case. For any morphism $f\colon A \to B$ in G-$\mathfrak{C}^*\mathfrak{alg}$, we get a morphism of extensions

$$\begin{tikzcd} \mathrm{Sus}(B) \arrow[r, tail] \arrow[d] & \mathrm{Cone}(f) \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\ \mathrm{Cone}(B) \arrow[r, tail] & \mathrm{Cyl}(f) \arrow[r, two heads] & A. \end{tikzcd}$$

The bottom extension splits and the maps $A \leftrightarrow \mathrm{Cyl}(f)$ are inverse to each other up to homotopy. By naturality, the composite map $\mathrm{Cone}(f) \to A \to B$ factors through $\mathrm{Cone}(\mathrm{id}_B) \cong \mathrm{Cone}(B)$ and hence is homotopic to the zero map.

3 Universal functors with certain properties

When we study topological invariants for C*-algebras, we usually require homotopy invariance and some exactness and stability conditions. Here we investigate these conditions and their interplay and describe some universal functors.

We discuss homotopy invariant functors on G-$\mathfrak{C}^*\mathfrak{alg}$ and the homotopy category $\mathrm{Ho}(G\text{-}\mathfrak{C}^*\mathfrak{alg})$ in §3.1. This is parallel to classical topology. We turn to Morita–Rieffel invariance and C*-stability in §3.2–3.3. We describe the resulting localisation using correspondences. By the way, in a C*-algebra context, *stability* usually refers to algebras of compact operators instead of suspensions. §3.4 deals with various exactness conditions: split-exactness, half-exactness, and additivity.

Whereas each of the above properties in itself seems rather weak, their combination may have striking consequences. For instance, a functor that is both C*-stable and split-exact is automatically homotopy invariant and satisfies Bott periodicity.

Throughout this section, we consider functors $\mathfrak{S} \to \mathfrak{C}$ where $\mathfrak{S}$ is a full subcategory of $\mathfrak{C}^*\mathfrak{alg}$ or G-$\mathfrak{C}^*\mathfrak{alg}$ for some locally compact group G. The target category $\mathfrak{C}$ may be arbitrary in §3.1–§3.3; to discuss exactness properties, we require $\mathfrak{C}$ to be an exact category or at least additive. Typical choices for $\mathfrak{S}$ are the categories of separable or separable nuclear C*-algebras, or the subcategory of all separable nuclear G-C*-algebras with an amenable (or a proper) action of G.

Definition 26. Let P be a property for functors defined on $\mathfrak{S}$. A *universal functor* with P is a functor $u\colon \mathfrak{S} \to \mathfrak{Univ}_P(\mathfrak{S})$ such that

- $\bar{F} \circ u$ has P for each functor $\bar{F}\colon \mathfrak{Univ}_P(\mathfrak{S}) \to \mathfrak{C}$;
- any functor $F\colon \mathfrak{S} \to \mathfrak{C}$ with P factors uniquely as $F = \bar{F} \circ u$ for some functor $\bar{F}\colon \mathfrak{Univ}_P(\mathfrak{S}) \to \mathfrak{C}$.

Of course, a universal functor with P need not exist. If it does, then it restricts to a bijection between objects of $\mathfrak{S}$ and $\mathfrak{Univ}_P(\mathfrak{S})$. Hence we can completely describe it by the sets of morphisms $\mathfrak{Univ}_P(A, B)$ from A to B in $\mathfrak{Univ}_P(\mathfrak{S})$ and the maps $\mathfrak{S}(A, B) \to \mathfrak{Univ}_P(A, B)$ for $A, B \Subset \mathfrak{S}$; the universal property means that for any functor $F\colon \mathfrak{S} \to \mathfrak{C}$ with P there is a unique functorial way to extend the maps $\mathrm{Hom}_G(A, B) \to \mathfrak{C}\big(F(A), F(B)\big)$ to $\mathfrak{Univ}_P(A, B)$. There is no *a priori* reason why the morphism spaces $\mathfrak{Univ}_P(A, B)$ for $A, B \Subset \mathfrak{S}$ should be independent of $\mathfrak{S}$; but this happens in the cases we consider, under some assumption on $\mathfrak{S}$.

3.1 Homotopy invariance. The following discussion applies to any full subcategory $\mathfrak{S} \subseteq G\text{-}\mathfrak{C}^*\mathfrak{alg}$ that is closed under the cylinder functor.

Definition 27. Let $f_0, f_1\colon A \rightrightarrows B$ be two parallel morphisms in $\mathfrak{S}$. We write $f_0 \sim f_1$ and call f_0 and f_1 *homotopic* if there is a *homotopy* between f_0 and f_1, that is, a morphism $f\colon A \to \mathrm{Cyl}(B) = \mathcal{C}([0,1], B)$ with $\mathrm{ev}_t \circ f = f_t$ for $t = 0, 1$.

It is easy to check that homotopy is an equivalence relation on $\mathrm{Hom}_G(A, B)$. We let $[A, B]$ be the set of equivalence classes. The composition of morphisms in $\mathfrak{S}$ descends to maps

$$[B, C] \times [A, B] \to [A, C], \quad \big([f], [g]\big) \mapsto [f \circ g],$$

that is, $f_1 \sim f_2$ and $g_1 \sim g_2$ implies $f_1 \circ f_2 \sim g_1 \circ g_2$. Thus the sets $[A, B]$ form the morphism sets of a category, called *homotopy category of* $\mathfrak{S}$ and denoted $\mathrm{Ho}(\mathfrak{S})$. The identity maps on objects and the canonical maps on morphisms define a *canonical functor* $\mathfrak{S} \to \mathrm{Ho}(\mathfrak{S})$. A morphism in $\mathfrak{S}$ is called a *homotopy equivalence* if it becomes invertible in $\mathrm{Ho}(\mathfrak{S})$.

Lemma 28. *The following are equivalent for a functor* $F\colon \mathfrak{S} \to \mathfrak{C}$*:*

(a) $F(\mathrm{ev}_0) = F(\mathrm{ev}_1)$ *as maps* $F\big(\mathcal{C}([0,1], A)\big) \to F(A)$ *for all* $A \Subset \mathfrak{S}$*;*

(b) $F(\mathrm{ev}_t)$ *induces isomorphisms* $F\big(\mathcal{C}([0,1], A)\big) \to F(A)$ *for all* $A \Subset \mathfrak{S}$, $t \in [0,1]$*;*

(c) *the embedding as constant functions* $\mathrm{const}\colon A \to \mathcal{C}([0,1], A)$ *induces an isomorphism* $F(A) \to F\big(\mathcal{C}([0,1], A)\big)$ *for all* $A \Subset \mathfrak{S}$*;*

(d) F *maps homotopy equivalences to isomorphisms;*

(e) *if two parallel morphisms* $f_0, f_1\colon A \rightrightarrows B$ *are homotopic, then* $F(f_0) = F(f_1)$*;*

(f) F *factors through the canonical functor* $\mathfrak{S} \to \mathrm{Ho}(\mathfrak{S})$.

Furthermore, the factorisation $\mathrm{Ho}(\mathfrak{S}) \to \mathfrak{C}$ *in* (f) *is necessarily unique.*

Proof. We only mention two facts that are needed for the proof. First, we have $\mathrm{ev}_t \circ \mathrm{const} = \mathrm{id}_A$ and $\mathrm{const} \circ \mathrm{ev}_t \sim \mathrm{id}_{\mathcal{C}([0,1],A)}$ for all $A \in\in \mathfrak{S}$ and all $t \in [0,1]$. Secondly, an isomorphism has a unique left and a unique right inverse, and these are again isomorphisms. □

The equivalence of (d) and (f) in Lemma 28 says that $\mathrm{Ho}(\mathfrak{S})$ is the localisation of $\mathfrak{S}$ at the family of homotopy equivalences.

Since $\mathcal{C}\big([0,1], \mathcal{C}_0(X)\big) \cong \mathcal{C}_0([0,1] \times X)$ for any locally compact space X, our notion of homotopy restricts to the usual one for pointed compact spaces. Hence the opposite of the homotopy category of pointed compact spaces is equivalent to a full subcategory of $\mathrm{Ho}(\mathfrak{C}^*\mathfrak{alg})$.

The sets $[A, B]$ inherit a base point $[0]$ and a quotient topology from $\mathrm{Hom}(A, B)$; thus $\mathrm{Ho}(\mathfrak{S})$ is enriched over pointed topological spaces as well. This topology on $[A, B]$ is not so useful, however, because it need not be Hausdorff.

A similar topology exists on Kasparov groups and can be defined in various ways, which turn out to be equivalent [12].

Let $F : G\text{-}\mathfrak{C}^*\mathfrak{alg} \to H\text{-}\mathfrak{C}^*\mathfrak{alg}$ be a functor with natural isomorphisms

$$F\big(\mathcal{C}([0,1], A)\big) \cong \mathcal{C}\big([0,1], F(A)\big)$$

that are compatible with evaluation maps for all A. The universal property implies that F descends to a functor $\mathrm{Ho}(G\text{-}\mathfrak{C}^*\mathfrak{alg}) \to \mathrm{Ho}(H\text{-}\mathfrak{C}^*\mathfrak{alg})$. In particular, this applies to the suspension, cone, and cylinder functors and, more generally, to the functors $A \otimes_{\max} {\llcorner\!\lrcorner}$ and $A \otimes_{\min} {\llcorner\!\lrcorner}$ on $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ for any $A \in\in G\text{-}\mathfrak{C}^*\mathfrak{alg}$ because both tensor product functors are associative and commutative and

$$\mathcal{C}([0,1], A) \cong \mathcal{C}([0,1]) \otimes_{\max} A \cong \mathcal{C}([0,1]) \otimes_{\min} A.$$

The same reasoning applies to the reduced and full crossed product functors $G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}$.

We may stabilise the homotopy category with respect to the *suspension* functor and consider a *suspension-stable homotopy category* with morphism spaces

$$\{A, B\} := \varinjlim_{k\to\infty} [\mathrm{Sus}^k A, \mathrm{Sus}^k B]$$

for all $A, B \in\in G\text{-}\mathfrak{C}^*\mathfrak{alg}$. We may also enlarge the set of objects by adding formal desuspensions and generalising the notion of *spectrum*. This is less interesting for C*-algebras than for spaces because most functors of interest satisfy Bott Periodicity, so that suspension and desuspension become equivalent.

3.2 Morita–Rieffel equivalence and stable isomorphism. One of the basic ideas of non-commutative geometry is that $G \ltimes \mathcal{C}_0(X)$ (or $G \ltimes_{\mathrm{r}} \mathcal{C}_0(X)$) should be a substitute for the quotient space $G\backslash X$, which may have bad singularities. In the special case of a free and proper G-space X, we expect that $G \ltimes \mathcal{C}_0(X)$ and $\mathcal{C}_0(G\backslash X)$ are "equivalent" in a suitable sense. Already the simplest possible case $X = G$ shows that we cannot expect an isomorphism here because

$$G \ltimes \mathcal{C}_0(G) \cong G \ltimes_{\mathrm{r}} \mathcal{C}_0(G) \cong \mathbb{K}(L^2 G).$$

The right notion of equivalence is a C*-version of Morita equivalence due to Marc A. Rieffel ([46], [47], [48]); therefore, we call it Morita–Rieffel equivalence.

The definition of Morita–Rieffel equivalence involves Hilbert modules over C*-algebras and the C*-algebras of compact operators on them; these notions are crucial for Kasparov theory as well. We refer to [34] for the definition and a discussion of their basic properties.

Definition 29. Two G-C*-algebras A and B are called *Morita–Rieffel equivalent* if there are a full G-equivariant Hilbert B-module $\mathcal{E}$ and a G-equivariant *-isomorphism $\mathbb{K}(\mathcal{E}) \cong A$.

It is possible (and desirable) to express this definition more symmetrically: $\mathcal{E}$ is an A, B-bimodule with two inner products taking values in A and B, satisfying various conditions [46]. Morita–Rieffel equivalent G-C*-algebras have equivalent categories of G-equivariant Hilbert modules via $\mathcal{E} \otimes_B \text{\textvisiblespace}$. The converse is unclear.

Example 30. The following is a more intricate example of a Morita–Rieffel equivalence. Let Γ and P be two subgroups of a locally compact group G. Then Γ acts on G/P by left translation and P acts on $\Gamma\backslash G$ by right translation. The corresponding orbit space is the double coset space $\Gamma\backslash G/P$. Both $\Gamma \ltimes \mathcal{C}_0(G/P)$ and $P \ltimes \mathcal{C}_0(\Gamma\backslash G)$ are non-commutative models for this double coset space. They are indeed Morita–Rieffel equivalent; the bimodule that implements the equivalence is a suitable completion of $\mathcal{C}_c(G)$, the space of continuous functions with compact support on G.

These examples suggest that Morita–Rieffel equivalent C*-algebras describe the *same* non-commutative space. Therefore, we expect that reasonable functors on $\mathfrak{C}^*\mathfrak{alg}$ should not distinguish between Morita–Rieffel equivalent C*-algebras. (We will slightly weaken this statement below.)

Definition 31. Two G-C*-algebras A and B are called *stably isomorphic* if there is a G-equivariant *-isomorphism $A \otimes \mathbb{K}(\mathcal{H}_G) \cong B \otimes \mathbb{K}(\mathcal{H}_G)$, where $\mathcal{H}_G := L^2(G \times \mathbb{N})$ is the direct sum of countably many copies of the regular representation of G; we let G act on $\mathbb{K}(\mathcal{H}_G)$ by conjugation, of course.

The following technical condition is often needed in connection with Morita–Rieffel equivalence.

Definition 32. A C*-algebra is called *σ-unital* if it has a countable approximate identity or, equivalently, contains a strictly positive element.

Example 33. All separable C*-algebras and all unital C*-algebras are σ-unital; the algebra $\mathbb{K}(\mathcal{H})$ is σ-unital if and only if $\mathcal{H}$ is separable.

Theorem 34 ([7]). *Two σ-unital G-C^*-algebras are G-equivariantly Morita–Rieffel equivalent if and only if they are stably isomorphic.*

In the non-equivariant case, this theorem is due to Brown–Green–Rieffel [7]. A simpler proof that carries over to the equivariant case appeared in [41].

3.3 C*-stable functors. The definition of C^*-stability is more intuitive in the non-equivariant case:

Definition 35. Fix a rank-one projection $p \in \mathbb{K}(\ell^2\mathbb{N})$. The resulting embedding $A \to A \otimes \mathbb{K}(\ell^2\mathbb{N})$, $a \mapsto a \otimes p$, is called a *corner embedding* of A.

A functor $F: \mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}$ is called C^*-*stable* if any corner embedding induces an isomorphism $F(A) \cong F\big(A \otimes \mathbb{K}(\ell^2\mathbb{N})\big)$.

The correct equivariant generalisation is the following:

Definition 36 ([36]). A functor $F: G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}$ is called C^*-*stable* if the canonical embeddings $\mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \leftarrow \mathcal{H}_2$ induce isomorphisms

$$F\big(A \otimes \mathbb{K}(\mathcal{H}_1)\big) \xrightarrow{\cong} F\big(A \otimes \mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2)\big) \xleftarrow{\cong} F\big(A \otimes \mathbb{K}(\mathcal{H}_2)\big)$$

for all non-zero G-Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

Of course, it suffices to require $F\big(A \otimes \mathbb{K}(\mathcal{H}_1)\big) \xrightarrow{\cong} F\big(A \otimes \mathbb{K}(\mathcal{H}_1 \oplus \mathcal{H}_2)\big)$. It is not hard to check that Definitions 35 and 36 are equivalent for trivial G.

Remark 37. We have argued in §3.2 why C^*-stability is an essential property for any decent homology theory for C^*-algebras. Nevertheless, it is tempting to assume less because C^*-stability together with split-exactness has very strong implications.

One reasonable way to weaken C^*-stability is to replace $\mathbb{K}(\ell^2\mathbb{N})$ by $\mathbb{M}_n$ for $n \in \mathbb{N}$ in Definition 35 (see [57]). If two *unital* C^*-algebras are Morita–Rieffel equivalent, then they are also Morita equivalent as rings, that is, the equivalence is implemented by a *finitely generated* projective module. This implies that a matrix-stable functor is invariant under Morita–Rieffel equivalence for *unital* C^*-algebras.

Matrix-stability also makes good sense in $G\text{-}\mathfrak{C}^*\mathfrak{alg}$ for a *compact* group G: simply require $\mathcal{H}_1$ and $\mathcal{H}_2$ in Definition 36 to be finite-dimensional. But we seem to run into problems for non-compact groups because they may have few finite-dimensional representations and we lack a finite-dimensional version of the equivariant stabilisation theorem.

Our next goal is to describe the *universal* C^*-stable functor. We abbreviate $A_{\mathbb{K}} := \mathbb{K}(L^2G) \otimes A$.

Definition 38. A *correspondence* from A to B (or $A \dashrightarrow B$) is a G-equivariant Hilbert $B_{\mathbb{K}}$-module $\mathcal{E}$ together with a G-equivariant essential (or non-degenerate) *-homomorphism $f: A_{\mathbb{K}} \to \mathbb{K}(\mathcal{E})$.

Given correspondences $\mathcal{E}$ from A to B and $\mathcal{F}$ from B to C, their *composition* is the correspondence from A to C with underlying Hilbert module $\mathcal{E} \mathbin{\bar{\otimes}_{B_{\mathbb{K}}}} \mathcal{F}$ and map $A_{\mathbb{K}} \to \mathbb{K}(\mathcal{E}) \to \mathbb{K}(\mathcal{E} \mathbin{\bar{\otimes}_{B_{\mathbb{K}}}} \mathcal{F})$, where the last map sends $T \mapsto T \otimes 1$; this yields compact operators because $B_{\mathbb{K}}$ maps to $\mathbb{K}(\mathcal{F})$. See [34] for the definition of the relevant completed tensor product of Hilbert modules.

Up to isomorphism, the composition of correspondences is associative and the identity maps $A \to A = \mathbb{K}(A)$ act as unit elements. Hence we get a category $\mathfrak{Corr}_G$ whose morphisms are the *isomorphism classes* of correspondences. It may have advantages to treat $\mathfrak{Corr}_G$ as a 2-category.

Any *-homomorphism $\varphi\colon A \to B$ yields a correspondence $f\colon A \to \mathbb{K}(\mathcal{E})$ from A to B, so that we get a canonical functor $\natural\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{Corr}_G$. We let $\mathcal{E}$ be the right ideal $\varphi(A_{\mathbb{K}}) \cdot B_{\mathbb{K}}$ in $B_{\mathbb{K}}$, viewed as a Hilbert B-module. Then $f(a) \cdot b := \varphi(a) \cdot b$ restricts to a compact operator $f(a)$ on $\mathcal{E}$ and $f\colon A \to \mathbb{K}(\mathcal{E})$ is essential. It can be checked that this construction is functorial.

In the following proposition, we require that the category of G-C*-algebras $\mathfrak{S}$ be closed under Morita–Rieffel equivalence and consist of σ-unital G-C*-algebras. We let $\mathfrak{Corr}_{\mathfrak{S}}$ be the full subcategory of $\mathfrak{Corr}_G$ with object class $\mathfrak{S}$.

Proposition 39. *The functor $\natural\colon \mathfrak{S} \to \mathfrak{Corr}_{\mathfrak{S}}$ is the universal* C**-stable functor on $\mathfrak{S}$; that is, it is* C**-stable, and any other such functor factors uniquely through $\natural$.*

Proof. First we sketch the proof in the non-equivariant case.

We verify that $\natural$ is C*-stable. The Morita–Rieffel equivalence between $\mathbb{K}(\ell^2\mathbb{N}) \otimes A \cong \mathbb{K}\bigl(\ell^2(\mathbb{N}, A)\bigr)$ and A is implemented by the Hilbert module $\ell^2(\mathbb{N}, A)$, which yields a correspondence $\bigl(\mathrm{id}, \ell^2(\mathbb{N}, A)\bigr)$ from $\mathbb{K}(\ell^2\mathbb{N}) \otimes A$ to A; this is inverse to the correspondence induced by a corner embedding $A \to \mathbb{K}(\ell^2\mathbb{N}) \otimes A$.

A Hilbert B-module $\mathcal{E}$ with an essential *-homomorphism $A \to \mathbb{K}(\mathcal{E})$ is countably generated because A is assumed σ-unital. Kasparov's Stabilisation Theorem yields an isometric embedding $\mathcal{E} \to \ell^2(\mathbb{N}, B)$. Hence we get *-homomorphisms

$$A \to \mathbb{K}(\ell^2\mathbb{N}) \otimes B \leftarrow B.$$

This diagram induces a map $F(A) \to F(\mathbb{K}(\ell^2\mathbb{N}) \otimes B) \cong F(B)$ for any C*-stable functor F. Now we should check that this well-defines a functor $\bar{F}\colon \mathfrak{Corr}_{\mathfrak{S}} \to \mathfrak{C}$ with $\bar{F} \circ \natural = F$, and that this yields the only such functor. We omit these computations.

The generalisation to the equivariant case uses the crucial property of the left regular representation that $L^2(G) \otimes \mathcal{H} \cong L^2(G \times \mathbb{N})$ for any countably infinite-dimensional G-Hilbert space $\mathcal{H}$. Since we replace A and B by $A_{\mathbb{K}}$ and $B_{\mathbb{K}}$ in the definition of correspondence right away, we can use this to repair a possible lack of G-equivariance; similar ideas appear in [36]. $\square$

Example 40. Let u be a G-invariant multiplier of B with $u^*u = 1$; such u are also called *isometries*. Then $b \mapsto ubu^*$ defines a *-homomorphism $B \to B$. The resulting correspondence $B \dashrightarrow B$ is isomorphic as a correspondence to the identity correspondence: the isomorphism is given by left multiplication with u, which defines a G-equivariant unitary operator from B to the closure of $uBu^* \cdot B = u \cdot B$.

Hence inner endomorphisms act trivially on C*-stable functors. Actually, this is one of the computations that we have omitted in the proof above; the argument can be found in [11].

Now we make the definition of a correspondence more concrete if A is unital. We have an essential *-homomorphism $\varphi\colon A \to \mathbb{K}(\mathcal{E})$ for some G-equivariant Hilbert B-module $\mathcal{E}$. Since A is unital, this means that $\mathbb{K}(\mathcal{E})$ is unital and φ is a unital *-homomorphism. Then $\mathcal{E}$ is finitely generated. Thus $\mathcal{E} = B^\infty \cdot p$ for some projection $p \in \mathbb{M}_\infty(B)$ and φ is a *-homomorphism $\varphi\colon A \to \mathbb{M}_\infty(B)$ with $\varphi(1) = p$. Two *-homomorphisms $\varphi_1, \varphi_2\colon A \rightrightarrows \mathbb{M}_\infty(B)$ yield isomorphic correspondences if and only if there is a partial isometry $v \in \mathbb{M}_\infty(B)$ with $v\varphi_1(x)v^* = \varphi_2(x)$ and $v^*\varphi_2(a)v = \varphi_1(a)$ for all $a \in A$.

Finally, we combine homotopy invariance and C*-stability and consider the universal C*-stable homotopy-invariant functor. This functor is much easier to characterise: the morphisms in the resulting universal category are simply the homotopy classes of G-equivariant *-homomorphisms $\mathbb{K}\big(L^2(G \times \mathbb{N})\big) \otimes A \to \mathbb{K}\big(L^2(G \times \mathbb{N})\big) \otimes B)$ (see [36], Proposition 6.1). Alternatively, we get the same category if we use homotopy classes of correspondences $A \dashrightarrow B$ instead.

3.4 Exactness properties. Throughout this subsection, we consider functors $F : \mathfrak{S} \to \mathfrak{C}$ with values in an exact category $\mathfrak{C}$. If $\mathfrak{C}$ is merely additive to begin with, we can equip it with the trivial exact category structure for which all extensions split. We also suppose that $\mathfrak{S}$ is closed under the kinds of C*-algebra extensions that we consider; depending on the notion of exactness, this means: direct product extensions, split extensions, cp-split extensions, or all extensions, respectively. Recall that split extensions in G-$\mathfrak{C}^*\mathfrak{alg}$ are required to split by a G-equivariant *-homomorphism.

3.4.1 Additive functors. The most trivial split extensions in G-$\mathfrak{C}^*\mathfrak{alg}$ are the *product extensions* $A \rightarrowtail A \times B \twoheadrightarrow B$ for two objects A, B. In this case, the coordinate embeddings and projections provide maps

$$A \leftrightarrows A \times B \leftrightarrows B. \tag{7}$$

Definition 41. We call F *additive* if it maps product diagrams (7) in $\mathfrak{S}$ to direct sum diagrams in $\mathfrak{C}$.

There is a partially defined addition on *-homomorphisms: call two parallel *-homomorphisms $\varphi, \psi : A \rightrightarrows B$ *orthogonal* if $\varphi(a_1) \cdot \psi(a_2) = 0$ for all $a_1, a_2 \in A$. Equivalently, $\varphi + \psi : a \mapsto \varphi(a) + \psi(a)$ is again a *-homomorphism.

Lemma 42. *The functor F is additive if and only if, for all $A, B \in\in \mathfrak{S}$, the maps* $\mathrm{Hom}(A, B) \to \mathfrak{C}\big(F(A), F(B)\big)$ *satisfy $F(\varphi + \psi) = F(\varphi) + F(\psi)$ for all pairs of orthogonal parallel *-homomorphisms φ, ψ.*

Alternatively, we may also require additivity for coproducts (that is, free products). Of course, this only makes sense if $\mathfrak{S}$ is closed under coproducts in G-$\mathfrak{C}^*\mathfrak{alg}$. The coproduct $A \sqcup B$ in G-$\mathfrak{C}^*\mathfrak{alg}$ comes with canonical maps $A \leftrightarrows A \sqcup B \leftrightarrows B$ as well; the maps $\iota_A \colon A \to A \sqcup B$ and $\iota_B \colon B \to A \sqcup B$ are the coordinate embeddings, the maps $\pi_A \colon A \sqcup B \to A$ and $\pi_B \colon A \sqcup B \to B$ restrict to $(\mathrm{id}_A, 0)$ and $(0, \mathrm{id}_B)$ on A and B, respectively.

Definition 43. We call F *additive on coproducts* if it maps coproduct diagrams $A \leftrightarrows A \sqcup B \leftrightarrows B$ to direct sum diagrams in $\mathfrak{C}$.

The coproduct and product are related by a canonical G-equivariant *-homomorphism $\varphi \colon A \sqcup B \twoheadrightarrow A \times B$ that is compatible with the maps to and from A and B, that is, $\varphi \circ \iota_A = \iota_A$, $\pi_A \circ \varphi = \pi_A$, and similarly for B. There is no map backwards, but there is a *correspondence* $\psi \colon A \times B \dashrightarrow A \sqcup B$, which is induced by the G-equivariant *-homomorphism

$$A \times B \to \mathbb{M}_2(A \sqcup B), \quad (a, b) \mapsto \begin{pmatrix} \iota_A(a) & 0 \\ 0 & \iota_B(b) \end{pmatrix}.$$

It is easy to see that the composite correspondence $\varphi \circ \psi$ is equal to the identity correspondence on $A \times B$. The other composite $\psi \circ \varphi$ is not the identity correspondence, but it is homotopic to it (see [9], [10]). This yields:

Proposition 44. *If F is* C**-stable and homotopy invariant, then the canonical map $F(\varphi) \colon F(A \sqcup B) \to F(A \times B)$ is invertible. Therefore, additivity and additivity for coproducts are equivalent for such functors.*

The correspondence ψ exists because the stabilisation creates enough room to replace ι_A and ι_B by homotopic homomorphisms with orthogonal ranges. We can achieve the same effect by a suspension (shift ι_A and ι_B to the open intervals $(0, 1/2)$ and $(1/2, 1)$, respectively). Therefore, any homotopy invariant functor satisfies $F\bigl(\mathrm{Sus}(A \sqcup B)\bigr) \cong F\bigl(\mathrm{Sus}(A \times B)\bigr)$.

3.4.2 Split-exact functors

Definition 45. We call F *split-exact* if, for any split extension $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ with section $s \colon Q \to E$, the map $\bigl(F(i), F(s)\bigr) \colon F(K) \oplus F(Q) \to F(E)$ is invertible.

It is clear that split-exact functors are additive.

Split-exactness is useful because of the following construction of Joachim Cuntz [9].

Let $B \triangleleft E$ be a G-invariant ideal and let $f_+, f_- \colon A \rightrightarrows E$ be G-equivariant *-homomorphisms with $f_+(a) - f_-(a) \in B$ for all $a \in A$. Equivalently, f_+ and f_- both lift the same morphism $\bar{f} \colon A \to E/B$. The data (A, f_+, f_-, E, B) is called a *quasi-homomorphism* from A to B.

Pulling back the extension $B \rightarrowtail E \twoheadrightarrow E/B$ along $\bar{f}$, we get an extension $B \rightarrowtail E' \twoheadrightarrow A$ with two sections $f'_+, f'_- \colon A \rightrightarrows E'$. The split-exactness of F shows that

$F(B) \rightarrowtail F(E') \twoheadrightarrow F(A)$ is a split extension in $\mathfrak{C}$. Since both $F(f'_-)$ and $F(f'_+)$ are sections for it, we get a map $F(f'_+) - F(f'_-)\colon F(A) \to F(B)$. *Thus a quasi-homomorphism induces a map $F(A) \to F(B)$ if F is split-exact.* The formal properties of this construction are summarised in [11].

Given a C*-algebra A, there is a *universal quasi-homomorphism* out of A. Let $Q(A) := A \sqcup A$ be the coproduct of two copies of A and let $\pi_A\colon Q(A) \to A$ be the *folding homomorphism* that restricts to id_A on both factors. Let $q(A)$ be its kernel. The two canonical embeddings $A \to A \sqcup A$ are sections for the folding homomorphism. Hence we get a quasi-homomorphism $A \rightrightarrows Q(A) \rhd q(A)$. The universal property of the free product shows that any quasi-homomorphism yields a G-equivariant *-homomorphism $q(A) \to B$.

Theorem 46. *Suppose $\mathfrak{S}$ is closed under split extensions and tensor products with $\mathcal{C}([0,1])$ and $\mathbb{K}(\ell^2\mathbb{N})$. If $F\colon \mathfrak{S} \to \mathfrak{C}$ is* C**-stable and split-exact, then F is homotopy invariant.*

This is a deep result of Nigel Higson [24]; a simple proof can be found in [11]. Besides basic properties of quasi-homomorphisms, it uses that inner endomorphisms act identically on C*-stable functors (Example 40).

Actually, the literature only contains Theorem 46 for functors on $\mathfrak{C}^*\mathfrak{alg}$. But the proof in [11] works for functors on categories $\mathfrak{S}$ as above.

3.4.3 Exact functors

Definition 47. We call F *exact* if $F(K) \to F(E) \to F(Q)$ is exact (at $F(E)$) for any extension $K \rightarrowtail E \twoheadrightarrow Q$ in $\mathfrak{S}$. More generally, given a class $\mathcal{E}$ of extensions in $\mathfrak{S}$ like, say, the class of equivariantly cp-split extensions, we define exactness for extensions in $\mathcal{E}$.

It is easy to see that exact functors are additive.

Most functors we are interested in satisfy homotopy invariance and Bott periodicity, and these two properties prevent a non-zero functor from being exact in the stronger sense of being *left* or *right* exact. This explains why our notion of exactness is much weaker than usual in homological algebra.

It is reasonable to require that a functor be part of a *homology* theory, that is, a sequence of functors $(F_n)_{n\in\mathbb{Z}}$ together with natural long exact sequences for all extensions [54]. We do not require this additional information because it tends to be hard to get *a priori* but often comes for free *a posteriori*:

Proposition 48. *Suppose that F is homotopy invariant and exact (or exact for equivariantly cp-split extensions). Then F has long exact sequences of the form*

$$\cdots \to F\bigl(\mathrm{Sus}(K)\bigr) \to F\bigl(\mathrm{Sus}(E)\bigr) \to F\bigl(\mathrm{Sus}(Q)\bigr) \to F(K) \to F(E) \to F(Q)$$

for any (equivariantly cp-split) extension $K \rightarrowtail E \twoheadrightarrow Q$. In particular, F is split-exact.

See §21.4 in [5] for the proof.

There probably exist exact functors that are not split-exact. It is likely that the algebraic K_1-functor provides a counterexample: it is exact but not split-exact on the category of rings [49]; but I do not know a counterexample to its split-exactness involving only C^*-algebras.

Proposition 48 and Bott periodicity yield long exact sequences that are infinite in *both* directions. Thus an exact homotopy invariant functor that satisfies Bott periodicity is part of a homology theory in a canonical way.

For universal constructions, we should replace a single functor by a *homology theory*, that is, a sequence of functors. The universal functors in this context are non-stable versions of E-theory and KK-theory. We refer to [27] for details.

A weaker property than exactness is the existence of *Puppe exact sequences* for mapping cones. The Puppe exact sequence is the special case of the long exact sequence of Proposition 48 for extensions of the form $\mathrm{Sus}(B) \rightarrowtail \mathrm{Cone}(f) \twoheadrightarrow A$ for a morphism $f\colon A \to B$. In practice, the exactness of a functor is often established by reducing it to the Puppe exact sequence. Let $K \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be an extension. The Puppe exact sequence yields the long exact sequence for the extension $\mathrm{Cone}(p) \rightarrowtail \mathrm{Cyl}(p) \twoheadrightarrow Q$. There is a canonical morphism of extensions

$$\begin{tikzcd} K \arrow[r, tail] \arrow[d] & E \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ \mathrm{Cone}(p) \arrow[r, tail] & \mathrm{Cyl}(p) \arrow[r, two heads] & Q, \end{tikzcd}$$

where the vertical map $E \to \mathrm{Cyl}(p)$ is a homotopy equivalence. Hence a functor with Puppe exact sequences is exact for $K \rightarrowtail E \twoheadrightarrow Q$ if and only if it maps the vertical map $K \to \mathrm{Cone}(p)$ to an isomorphism.

4 Kasparov theory

We define KK^G as the universal split-exact C^*-stable functor on G-$\mathfrak{C}^*\mathfrak{sep}$; since split-exact and C^*-stable functors are automatically homotopy invariant, KK^G is the universal split-exact C^*-stable homotopy functor as well. The universal property of Kasparov theory due to Cuntz and Higson asserts that this is equivalent to Kasparov's definition. We examine some basic properties of Kasparov theory and, in particular, show how to get functors between Kasparov categories.

We let E^G be the universal exact C^*-stable homotopy functor on G-$\mathfrak{C}^*\mathfrak{sep}$ or, equivalently, the universal exact, split-exact, and C^*-stable functor.

Kasparov's own definition of his theory is inspired by previous work of Atiyah [1] on K-homology; later, he also interacted with the work of Brown–Douglas–Fillmore [6] on extensions of C^*-algebras. A construction in abstract homotopy theory provides a homology theory for spaces that is dual to K-theory. Atiyah realized that certain abstract elliptic differential operators provide cycles for this dual theory; but he did not know the equivalence relation to put on these cycles. Brown–Douglas–Fillmore studied extensions of $\mathcal{C}_0(X)$ (and more general C*-algebras) by the compact operators and found that the resulting structure set is naturally isomorphic to a K-homology group.

Kasparov unified and vastly generalised these two results, defining a bivariant functor $\mathrm{KK}_*(A, B)$ that combines K-theory and K-homology and that is closely related to the classification of extensions $B \otimes \mathbb{K} \rightarrowtail E \twoheadrightarrow A$ (see [29], [30]). A deep theorem of Kasparov shows that two reasonable equivalence relations for these cycles coincide; this clarifies the homotopy invariance of the extension groups of Brown–Douglas–Fillmore. Furthermore, he constructed an equivariant version of his theory in [31] and applied it to prove the Novikov conjecture for discrete subgroups of Lie groups.

The most remarkable feature of Kasparov theory is an associative product on KK called *Kasparov product*. This generalises various known product constructions in K-theory and K-homology and allows to view KK as a category.

In applications, we usually need some non-obvious KK-element, and we must compute certain Kasparov products explicitly. This requires a concrete description of Kasparov cycles and their products. Since both are somewhat technical, we do not discuss them here and merely refer to [5] for a detailed treatment and to [56] for a very useful survey article. Instead, we use Higson's characterisation of KK by a universal property [23], which is based on ideas of Cuntz ([10], [9]). The extension to the equivariant case is due to Thomsen [58]. A simpler proof of Thomsen's theorem and various related results can be found in [36].

We do not discuss KK^G for $\mathbb{Z}/2$-graded G-C*-algebras here because it does not fit so well with the universal property approach, which would simply yield $\mathrm{KK}^{G\times\mathbb{Z}/2}$ because $\mathbb{Z}/2$-graded G-C*-algebras are the same as $G \times \mathbb{Z}/2$-C*-algebras. The relationship between the two theories is explained in [36], following Ulrich Haag [22]. The graded version of Kasparov theory is often useful because it allows us to treat even and odd KK-cycles simultaneously.

Fix a locally compact group G. The Kasparov groups $\mathrm{KK}_0^G(A, B)$ for $A, B \in G\text{-}\mathfrak{C}^*\mathfrak{sep}$ form the morphism sets $A \to B$ of a category, which we denote by $\mathfrak{KK}^G$; the composition in $\mathfrak{KK}^G$ is the Kasparov product. The categories $G\text{-}\mathfrak{C}^*\mathfrak{sep}$ and $\mathfrak{KK}^G$ have the same objects. We have a canonical functor

$$\mathrm{KK}^G\colon G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{KK}^G$$

that acts identically on objects. This functor contains all information about equivariant Kasparov theory for G.

Definition 49. A G-equivariant *-homomorphism $f\colon A \to B$ is called a KK^G-*equivalence* if $\mathrm{KK}^G(f)$ is invertible in $\mathfrak{KK}^G$.

Theorem 50. *Let $\mathfrak{S}$ be a full subcategory of $G\text{-}\mathfrak{C}^*\mathfrak{sep}$ that is closed under suspensions, G-equivariantly cp-split extensions, and Morita–Rieffel equivalence. Let $\mathfrak{KK}^G(\mathfrak{S})$ be the full subcategory of $\mathfrak{KK}^G$ with object class $\mathfrak{S}$ and let $\mathrm{KK}^G_{\mathfrak{S}}\colon \mathfrak{S} \to \mathfrak{KK}^G(\mathfrak{S})$ be the restriction of KK^G.*

The functor $\mathrm{KK}^G_{\mathfrak{S}}\colon \mathfrak{S} \to \mathfrak{KK}^G_{\mathfrak{S}}$ *is the universal split-exact* C^**-stable functor; in particular,* $\mathfrak{KK}^G(\mathfrak{S})$ *is an additive category. In addition, it has the following properties and is, therefore, universal among functors on* $\mathfrak{S}$ *with some of these extra properties:*

- *it is homotopy invariant;*
- *it is exact for* G*-equivariantly cp-split extensions;*
- *it satisfies Bott periodicity, that is, in* $\mathfrak{KK}^G$ *there are natural isomorphisms* $\mathrm{Sus}^2(A) \cong A$ *for all* $A \in\in \mathfrak{KK}^G$.

Corollary 51. *Let* $F\colon \mathfrak{S} \to \mathfrak{C}$ *be split-exact and* C^**-stable. Then* F *factors uniquely through* $\mathrm{KK}^G_{\mathfrak{S}}$*, is homotopy invariant, and satisfies Bott periodicity. A* KK^G*-equivalence* $A \to B$ *in* $\mathfrak{S}$ *induces an isomorphism* $F(A) \to F(B)$.

We will view the universal property of Theorem 50 as a definition of $\mathfrak{KK}^G$ and thus of the groups $\mathrm{KK}^G_0(A, B)$. We also let

$$\mathrm{KK}^G_n(A, B) := \mathrm{KK}^G\bigl(A, \mathrm{Sus}^n(B)\bigr);$$

since the Bott periodicity isomorphism identifies $\mathrm{KK}^G_2 \cong \mathrm{KK}^G_0$, this yields a $\mathbb{Z}/2$-graded theory.

Now we describe $\mathrm{KK}^G_0(A, B)$ more concretely. Recall $A_{\mathbb{K}} := A \otimes \mathbb{K}(L^2G)$.

Proposition 52. *Let* A *and* B *be two* G-C^**-algebras. There is a natural bijection between the morphism sets* $\mathrm{KK}^G_0(A, B)$ *in* $\mathfrak{KK}^G$ *and the set* $[q(A_{\mathbb{K}}), B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})]$ *of homotopy classes of* G*-equivariant* **-homomorphisms from* $q(A_{\mathbb{K}})$ *to* $B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})$.

Proof. The canonical functor $G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{KK}^G$ is C^*-stable and split-exact, and therefore homotopy invariant by Theorem 46 (this is already asserted in Theorem 50). Proposition 44 yields that it is additive for coproducts. Split-exactness for the split extension $q(A) \rightarrowtail Q(A) \twoheadrightarrow A$ shows that $\mathrm{id}_A * 0\colon Q(A) \to A$ restricts to a KK^G-equivalence $q(A) \sim A$. Similarly, C^*-stability yields KK^G-equivalences $A \sim A_{\mathbb{K}}$ and $B \sim B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})$. Hence homotopy classes of *-homomorphisms from $q(A_{\mathbb{K}})$ to $B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})$ yield classes in $\mathrm{KK}^G_0(A, B)$. Using the concrete description of Kasparov cycles, which we have not discussed, it is checked in [36] that this map yields a bijection as asserted. □

Another equivalent description is

$$\mathrm{KK}^G_0(A, B) \cong [q(A_{\mathbb{K}}) \otimes \mathbb{K}(\ell^2\mathbb{N}), q(B_{\mathbb{K}}) \otimes \mathbb{K}(\ell^2\mathbb{N})];$$

in this approach, the Kasparov product becomes simply the composition of morphisms. Proposition 52 suggests that $q(A_{\mathbb{K}})$ and $B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})$ may be the cofibrant and fibrant replacement of A and B in some model category related to KK^G. But it is not clear whether this is the case. The model category structure constructed in [28] is certainly quite different.

By the universal property, K-theory descends to a functor on $\mathfrak{KK}$, that is, we get canonical maps

$$\mathrm{KK}_0(A, B) \to \mathrm{Hom}\bigl(\mathrm{K}_*(A), \mathrm{K}_*(B)\bigr)$$

for all separable C^*-algebras A, B, where the right hand side denotes grading-preserving group homomorphisms. For $A = \mathbb{C}$, this yields a map $\mathrm{KK}_0(\mathbb{C}, B) \to \mathrm{Hom}\bigl(\mathbb{Z}, \mathrm{K}_0(B)\bigr) \cong \mathrm{K}_0(B)$. Using suspensions, we also get a corresponding map $\mathrm{KK}_1(\mathbb{C}, B) \to \mathrm{K}_1(B)$.

Theorem 53. *The maps* $\mathrm{KK}_*(\mathbb{C}, B) \to \mathrm{K}_*(B)$ *constructed above are isomorphisms for all* $B \Subset \mathfrak{C}^*\mathfrak{sep}$.

Thus Kasparov theory is a bivariant generalisation of K-theory. Roughly speaking, $\mathrm{KK}_*(A, B)$ is the place where maps between K-theory groups live. Most constructions of such maps, say, in index theory can, in fact, be improved to yield elements of $\mathrm{KK}_*(A, B)$. One reason why this has to be so is the *Universal Coefficient Theorem* (UCT), which computes $\mathrm{KK}_*(A, B)$ from $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ for many C^*-algebras A, B. If A satisfies the UCT, then any grading preserving group homomorphism $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$ lifts to an element of $\mathrm{KK}_0(A, B)$.

4.1 Extending functors and identities to $\mathfrak{KK}^G$. We can use the universal property to extend various functors G-$\mathfrak{C}^*\mathfrak{sep} \to H$-$\mathfrak{C}^*\mathfrak{sep}$ to functors $\mathfrak{KK}^G \to \mathfrak{KK}^H$. We explain this by an example:

Proposition 54. *The full and reduced crossed product functors*

$$G \ltimes_{\mathrm{r}} \text{␣}, G \ltimes \text{␣}\colon G\text{-}\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}$$

extend to functors $G \ltimes_{\mathrm{r}} \text{␣}, G \ltimes \text{␣}\colon \mathfrak{KK}^G \to \mathfrak{KK}$ *called* descent functors.

Gennadi Kasparov [31] constructs these functors directly using the concrete description of Kasparov cycles. This requires a certain amount of work; in particular, checking functoriality involves knowing how to compute Kasparov products. The construction via the universal property is formal:

Proof. We only write down the argument for *reduced* crossed products, the other case is similar. It is well-known that $G \ltimes_{\mathrm{r}} \bigl(A \otimes \mathbb{K}(\mathcal{H})\bigr) \cong (G \ltimes_{\mathrm{r}} A) \otimes \mathbb{K}(\mathcal{H})$ for any G-Hilbert space $\mathcal{H}$. Therefore, the composite functor

$$G\text{-}\mathfrak{C}^*\mathfrak{sep} \xrightarrow{G \ltimes_{\mathrm{r}} \text{␣}} \mathfrak{C}^*\mathfrak{sep} \xrightarrow{\mathrm{KK}} \mathfrak{KK}$$

is C^*-stable. Proposition 9 shows that this functor is split-exact as well (regardless of whether G is an exact group). Now the universal property provides an extension to a functor $\mathfrak{KK}^G \to \mathfrak{KK}$. □

Similarly, we get functors

$$A \otimes_{\min} \text{␣}, A \otimes_{\max} \text{␣}\colon \mathfrak{KK}^G \to \mathfrak{KK}^G$$

for any G-C*-algebra A. Since these extensions are natural, we even get bifunctors

$$\otimes_{\min}, \otimes_{\max} \colon \mathfrak{KK}^G \times \mathfrak{KK}^G \to \mathfrak{KK}^G.$$

The associativity, commutativity, and unit constraints in G-$\mathfrak{C}^*\mathfrak{alg}$ induce corresponding constraints in $\mathfrak{KK}^G$, so that both $\otimes_{\min}$ and $\otimes_{\max}$ turn $\mathfrak{KK}^G$ into a symmetric monoidal category.

Another example is the functor $\tau \colon \mathfrak{C}^*\mathfrak{alg} \to G$-$\mathfrak{C}^*\mathfrak{alg}$ that equips a C*-algebra with the trivial G-action; it extends to a functor $\tau \colon \mathfrak{KK} \to \mathfrak{KK}^G$.

The universal property also allows us to *prove identities* between functors. For instance, we have natural isomorphisms $G \ltimes_{\mathrm{r}} (\tau(A) \otimes_{\min} B) = A \otimes_{\min} (G \ltimes_{\mathrm{r}} B)$ for all G-C*-algebras B. Naturality means, to begin with, that the diagram

$$\begin{array}{ccc}
G \ltimes_{\mathrm{r}} (\tau(A_1) \otimes_{\min} B_1) & \xrightarrow{\cong} & A_1 \otimes_{\min} (G \ltimes_{\mathrm{r}} B_1) \\
{\scriptstyle G \ltimes_{\mathrm{r}} \tau(\alpha) \otimes_{\min} \beta}\downarrow & & \downarrow{\scriptstyle \alpha \otimes_{\min} (G \ltimes_{\mathrm{r}} \beta)} \\
G \ltimes_{\mathrm{r}} (\tau(A_2) \otimes_{\min} B_2) & \xrightarrow{\cong} & A_2 \otimes_{\min} (G \ltimes_{\mathrm{r}} B_2)
\end{array}$$

commutes if $\alpha \colon A_1 \to A_2$ and $\beta \colon B_1 \to B_2$ are a *-homomorphism and a G-equivariant *-homomorphism, respectively. Two applications of the uniqueness part of the universal property show that this diagram remains commutative in $\mathfrak{KK}$ if $\alpha \in \mathrm{KK}_0(A_1, A_2)$ and $\beta \in \mathrm{KK}_0^G(B_1, B_2)$. Similar remarks apply to the natural isomorphism $G \ltimes (\tau(A) \otimes_{\max} B) \cong A \otimes_{\max} (G \ltimes B)$ and hence to the isomorphisms $G \ltimes \tau(A) \cong \mathrm{C}^*(G) \otimes_{\max} A$ and $G \ltimes_{\mathrm{r}} \tau(A) \cong \mathrm{C}^*_{\mathrm{red}}(G) \otimes_{\min} A$.

Adjointness relations in Kasparov theory are usually proved most easily by constructing the unit and counit of the adjunction. For instance, if G is a compact group then the functor τ is left adjoint to $G \ltimes \text{␣} = G \ltimes_{\mathrm{r}} \text{␣}$, that is, for all $A \Subset \mathfrak{KK}$ and $B \Subset \mathfrak{KK}^G$, we have natural isomorphisms

$$\mathrm{KK}^G_*(\tau(A), B) \cong \mathrm{KK}_*(A, G \ltimes B). \tag{8}$$

This is also known as the *Green–Julg Theorem.* For $A = \mathbb{C}$, it specialises to a natural isomorphism $\mathrm{K}^G_*(B) \cong \mathrm{K}_*(G \ltimes B)$; this was one of the first appearances of noncommutative algebras in topological K-theory.

Proof of (8). We already know that τ and $G \ltimes \text{␣}$ are functors between $\mathfrak{KK}$ and $\mathfrak{KK}^G$. It remains to construct natural elements

$$\alpha_A \in \mathrm{KK}_0\big(A, G \ltimes \tau(A)\big), \quad \beta_B \in \mathrm{KK}^G_0(\tau(G \ltimes B), B)$$

for all $A \Subset \mathfrak{KK}$, $B \Subset \mathfrak{KK}^G$ that satisfy the conditions for unit and counit of adjunction [35].

The main point is that $\tau(G \ltimes B)$ is the G-fixed point subalgebra of $B_{\mathbb{K}} = B \otimes \mathbb{K}(L^2G)$. The embedding $\tau(G \ltimes B) \to B_{\mathbb{K}}$ provides a G-equivariant correspondence β_B from $\tau(G \ltimes B)$ to B and thus an element of $\mathrm{KK}^G_0(\tau(G \ltimes B), B)$. This construction is certainly natural for G-equivariant *-homomorphisms and hence for KK^G-morphisms by the uniqueness part of the universal property of KK^G.

Let $e_\tau\colon \mathbb{C} \to \mathrm{C}^*(G)$ be the embedding that corresponds to the trivial representation of G. Recall that $G \ltimes \tau(A) \cong \mathrm{C}^*(G) \otimes A$. Hence the exterior product of the identity map on A and $\mathrm{KK}(e_\tau)$ provides $\alpha_A \in \mathrm{KK}_0\bigl(A, G \ltimes \tau(A)\bigr)$. Again, naturality for *-homomorphisms is clear and implies naturality for morphisms in KK.

Finally, it remains to check that

$$\tau(A) \xrightarrow{\tau(\alpha_A)} \tau\bigl(G \ltimes \tau(A)\bigr) \xrightarrow{\beta_{\tau(A)}} \tau(A),$$
$$G \ltimes B \xrightarrow{\alpha_{G\ltimes B}} G \ltimes \tau(G \ltimes B) \xrightarrow{G\ltimes\beta_B} G \ltimes B$$

are the identity morphisms in KK^G. Then we get the desired adjointness using a general construction from category theory (see [35]). In fact, both composites are equal to the identity already as *correspondences*, so that we do not have to know anything about Kasparov theory except its C^*-stability to check this. □

A similar argument yields an adjointness relation

$$\mathrm{KK}_0^G\bigl(A, \tau(B)\bigr) \cong \mathrm{KK}_0(G \ltimes A, B) \tag{9}$$

for a *discrete* group G. More conceptually, (9) corresponds via Baaj–Skandalis duality [3] to the Green–Julg Theorem for the dual quantum group of G, which is *compact* because G is discrete. But we can also write down unit and counit of adjunction directly.

The trivial representation $\mathrm{C}^*(G) \to \mathbb{C}$ yields natural *-homomorphisms

$$G \ltimes \tau(B) \cong \mathrm{C}^*(G) \otimes_{\max} B \to B$$

and hence $\beta_B \in \mathrm{KK}_0(G \ltimes \tau(B), B)$. The canonical embedding $A \to G \ltimes A$ is G-equivariant if we let G act on $G \ltimes A$ by conjugation; but this action is inner, so that $G \ltimes A$ and $\tau(G \ltimes A)$ are G-equivariantly Morita–Rieffel equivalent. Thus the canonical embedding $A \to G \ltimes A$ yields a correspondence $A \dashrightarrow \tau(G \ltimes A)$ and $\alpha_A \in \mathrm{KK}_0^G\bigl(A, \tau(G \ltimes A)\bigr)$. We must check that the composites

$$G \ltimes A \xrightarrow{G\ltimes\alpha_A} G \ltimes \tau(G \ltimes A) \xrightarrow{\beta_{G\ltimes A}} G \ltimes A,$$
$$\tau(B) \xrightarrow{\alpha_{\tau(B)}} \tau\bigl(G \ltimes \tau(B)\bigr) \xrightarrow{\tau(\beta_B)} \tau(B)$$

are identity morphisms in KK and KK^G, respectively. Once again, this holds already on the level of correspondences.

4.2 Triangulated category structure. We turn $\mathfrak{KK}^G$ into a triangulated category by extending standard constructions for topological spaces [38]. Some arrows change direction because the functor $\mathcal{C}_0$ from spaces to C^*-algebras is contravariant. We have already observed that $\mathfrak{KK}^G$ is additive. The suspension is given by $\Sigma^{-1}(A) := \mathrm{Sus}(A)$. Since $\mathrm{Sus}^2(A) \cong A$ in $\mathfrak{KK}^G$ by Bott periodicity, we have $\Sigma = \Sigma^{-1}$. Thus we do not need formal desuspensions as for the stable homotopy category.

Definition 55. A triangle $A \to B \to C \to \Sigma A$ in $\mathfrak{KK}^G$ is called *exact* if it is isomorphic as a triangle to the *mapping cone triangle*

$$\mathrm{Sus}(B) \to \mathrm{Cone}(f) \to A \xrightarrow{f} B$$

for some G-equivariant *-homomorphism f.

Alternatively, we can use G-equivariantly cp-split extensions in G-$\mathfrak{C}^*\mathfrak{sep}$. Any such extension $I \rightarrowtail E \twoheadrightarrow Q$ determines a class in $\mathrm{KK}_1^G(Q, I) \cong \mathrm{KK}_0^G(\mathrm{Sus}(Q), I)$, so that we get a triangle $\mathrm{Sus}(Q) \to I \to E \to Q$ in $\mathfrak{KK}^G$. Such triangles are called *extension triangles*. A triangle in $\mathfrak{KK}^G$ is exact if and only if it is isomorphic to the extension triangle of a G-equivariantly cp-split extension [38].

Theorem 56. *With the suspension automorphism and exact triangles defined above, $\mathfrak{KK}^G$ is a triangulated category. So is $\mathfrak{KK}^G(\mathfrak{S})$ if $\mathfrak{S} \subseteq G$-$\mathfrak{C}^*\mathfrak{sep}$ is closed under suspensions, G-equivariantly cp-split extensions, and Morita–Rieffel equivalence as in Theorem* 50*.*

Proof. That $\mathfrak{KK}^G$ is triangulated is proved in detail in [38]. We do not discuss the triangulated category axioms here. Most of them amount to properties of mapping cone triangles that can be checked by copying the corresponding arguments for the stable homotopy category (and reverting arrows). These axioms hold for $\mathfrak{KK}^G(\mathfrak{S})$ because they hold for $\mathfrak{KK}^G$. The only axiom that requires more care is the existence axiom for exact triangles; it requires any morphism to be part of an exact triangle. We can prove this as in [38] using the concrete description of $\mathrm{KK}_0^G(A, B)$ in Proposition 52. For some applications like the generalisation to $\mathfrak{KK}^G(\mathfrak{S})$, it is better to use extension triangles instead. Any $f \in \mathrm{KK}_0^G(A, B) \cong \mathrm{KK}_1^G(\mathrm{Sus}(A), B)$ can be represented by a G-equivariantly cp-split extension $\mathbb{K}(\mathcal{H}_B) \rightarrowtail E \twoheadrightarrow \mathrm{Sus}(A)$, where $\mathcal{H}_B$ is a full G-equivariant Hilbert B-module, so that $\mathbb{K}(\mathcal{H}_B)$ is G-equivariantly Morita–Rieffel equivalent to B. The extension triangle of this extension contains f and belongs to $\mathfrak{KK}^G(\mathfrak{S})$ by our assumptions on $\mathfrak{S}$. □

Since model category structures related to C*-algebras are rather hard to get (compare [28]), triangulated categories seem to provide the most promising formal setup for extending results from classical spaces to C*-algebras. An earlier attempt can be found in [54]. Triangulated categories clarify the basic bookkeeping with long exact sequences. Mayer–Vietoris exact sequences and inductive limits are discussed from this point of view in [38]. More importantly, this framework sheds light on more advanced constructions like the Baum–Connes assembly map. We will briefly discuss this below.

4.3 The Universal Coefficient Theorem. There is a very close relationship between K-theory and Kasparov theory. We have already seen that $\mathrm{K}_*(A) \cong \mathrm{KK}_*(\mathbb{C}, A)$ is a special case of KK. Furthermore, KK inherits deep properties of K-theory such as Bott periodicity. Thus we may hope to express $\mathrm{KK}_*(A, B)$ using only the K-theory of

A and B – at least for many A and B. This is the point of the *Universal Coefficient Theorem*.

The Kasparov product provides a canonical homomorphism of graded groups

$$\gamma\colon \mathrm{KK}_*(A,B) \to \mathrm{Hom}_*\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr),$$

where Hom_* denotes the $\mathbb{Z}/2$-graded Abelian group of all group homomorphisms $\mathrm{K}_*(A) \to \mathrm{K}_*(B)$. There are topological reasons why γ cannot always be invertible: since Hom_* is not exact, the bifunctor $\mathrm{Hom}_*\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr)$ would not be exact on cp-split extensions. A construction of Lawrence Brown provides another natural map

$$\kappa\colon\ \ker\gamma \to \mathrm{Ext}_*\bigl(\mathrm{K}_{*+1}(A),\mathrm{K}_*(B)\bigr).$$

The following theorem is due to Jonathan Rosenberg and Claude Schochet [53], [51]; see also [5].

Theorem 57. *The following are equivalent for a separable* C^**-algebra* A*:*

(a) $\mathrm{KK}_*(A,B) = 0$ *for all* $B \Subset \mathfrak{KK}$ *with* $\mathrm{K}_*(B) = 0$*;*

(b) *the map* γ *is surjective and* κ *is bijective for all* $B \Subset \mathfrak{KK}$*;*

(c) *for all* $B \Subset \mathfrak{KK}$*, there is a short exact sequence of* $\mathbb{Z}/2$*-graded Abelian groups*

$$\mathrm{Ext}_*\bigl(\mathrm{K}_{*+1}(A),\mathrm{K}_*(B)\bigr) \rightarrowtail \mathrm{KK}_*(A,B) \twoheadrightarrow \mathrm{Hom}_*\bigl(\mathrm{K}_*(A),\mathrm{K}_*(B)\bigr).$$

(d) *A belongs to the smallest class of* C^**-algebras that contains* $\mathbb{C}$ *and is closed under* KK*-equivalence, suspensions, countable direct sums, and cp-split extensions;*

(e) *A is* KK*-equivalent to* $\mathcal{C}_0(X)$ *for some pointed compact metrisable space* X*.*

If these conditions are satisfied, then the extension in (c) *is natural and splits, but the section is not natural.*

The class of C^*-algebras with these properties is also called the *bootstrap* class because of description (d). Alternatively, we may say that they satisfy the *Universal Coefficient Theorem* because of (c). Since commutative C^*-algebras are nuclear, (e) implies that the natural map $A \otimes_{\max} B \to A \otimes_{\min} B$ is a KK-equivalence if A or B belongs to the bootstrap class [55]. This fails for some A, so that the Universal Coefficient Theorem does not hold for all A. Remarkably, this is the only obstruction to the Universal Coefficient Theorem known at the moment: we know no nuclear C^*-algebra that does not satisfy the Universal Coefficient Theorem. As a result, we can express $\mathrm{KK}_*(A,B)$ using only $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ for many A and B.

When we restrict attention to nuclear C^*-algebras, then the bootstrap class is closed under various operations like tensor products, arbitrary extensions and inductive limits (without requiring any cp-sections), and under crossed products by torsion-free amenable groups. Remarkably, there are no general results about crossed products by finite groups.

The Universal Coefficient Theorem and the universal property of KK imply that very few homology theories for (pointed compact metrisable) spaces can extend to the non-commutative setting. More precisely, if we require the extension to be split-exact, C^*-stable, and additive for countable direct sums, then only K-theory with coefficients is possible. Thus we rule out most of the difficult (and interesting) problems in stable homotopy theory. But if we only want to study K-theory, anyway, then the operator algebraic framework usually provides very good analytical tools. This is most valuable for equivariant generalisations of K-theory.

Jonathan Rosenberg and Claude Schochet [50] have also constructed a spectral sequence that, in favourable cases, computes $\mathrm{KK}^G(A, B)$ from $\mathrm{K}_*^G(A)$ and $\mathrm{K}_*^G(B)$; they require G to be a compact Lie group with torsion-free fundamental group and A and B to belong to a suitable bootstrap class. This equivariant UCT is clarified in [39], [40].

4.4 E-theory and asymptotic morphisms. Recall that Kasparov theory is only exact for (equivariantly) cp-split extensions. E-theory is a similar theory that is exact for all extensions.

Definition 58. We let $\mathrm{E}^G : G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathfrak{E}^G$ be the universal C^*-stable, exact homotopy functor.

Lemma 59. *The functor* E^G *is split-exact and factors through* $\mathrm{KK}^G : G\text{-}\mathfrak{C}^*\mathfrak{sep} \to \mathrm{KK}^G$. *Hence it satisfies Bott periodicity.*

Proof. Proposition 48 shows that any exact homotopy functor is split-exact. The remaining assertions now follow from Corollary 51. □

The functor E (for trivial G) was first defined as above by Nigel Higson [25]. Then Alain Connes and Nigel Higson [8] found a more concrete description using *asymptotic morphisms*. This is what made the theory usable. The equivariant generalisation of the theory is due to Erik Guentner, Nigel Higson, and Jody Trout [21].

We write $\mathrm{E}_n^G(A, B)$ for the space of morphisms $A \to \mathrm{Sus}^n(B)$ in $\mathfrak{E}^G$. Bott periodicity shows that there are only two different groups to consider.

Definition 60. The *asymptotic algebra* of a C^*-algebra B is the C^*-algebra

$$\mathrm{Asymp}(B) := \mathcal{C}_b(\mathbb{R}_+, B)/\mathcal{C}_0(\mathbb{R}_+, B).$$

An *asymptotic morphism* $A \to B$ is a *-homomorphism $f : A \to \mathrm{Asymp}(B)$.

Representing elements of $\mathrm{Asymp}(B)$ by bounded functions $[0, \infty) \to B$, we can represent f by a family of maps $f_t : A \to B$ such that $f_t(a) \in \mathcal{C}_b(\mathbb{R}_+, B)$ for each $a \in A$ and the map $a \mapsto f_t(a)$ satisfies the conditions for a *-homomorphism asymptotically for $t \to \infty$. This provides a concrete description of asymptotic morphisms and explains the name.

If a locally compact group G acts on B, then $\mathrm{Asymp}(B)$ inherits an action of G by naturality (which need not be strongly continuous).

Definition 61. Let A and B be two G-C*-algebras for a locally compact group G. A *G-equivariant asymptotic morphism* from A to B is a G-equivariant *-homomorphism $f: A \to \mathrm{Asymp}(B)$. We write $[\![A, B]\!]$ for the set of homotopy classes of G-equivariant asymptotic morphisms from A to B. Here a homotopy is a G-equivariant *-homomorphism $A \to \mathrm{Asymp}\bigl(\mathcal{C}([0,1], B)\bigr)$.

The asymptotic algebra fits, by definition, into an extension

$$\mathcal{C}_0(\mathbb{R}_+, B) \rightarrowtail \mathcal{C}_b(\mathbb{R}_+, B) \twoheadrightarrow \mathrm{Asymp}(B).$$

Notice that $\mathcal{C}_0(\mathbb{R}_+, B) \cong \mathrm{Cone}(B)$ is contractible. If $f: A \to \mathrm{Asymp}(B)$ is a G-equivariant asymptotic morphism, then we can use it to pull back this extension to an extension $\mathrm{Cone}(B) \rightarrowtail E \twoheadrightarrow A$ in G-$\mathfrak{C}^*\mathfrak{alg}$; the G-action on E is automatically strongly continuous. If F is exact and homotopy invariant, then $F\bigl(\mathrm{Sus}^n(E)\bigr) \to F\bigl(\mathrm{Sus}^n(A)\bigr)$ is an isomorphism for all $n \geq 1$ by Proposition 48. The evaluation map $\mathcal{C}_b(\mathbb{R}_+, B) \to B$ at some $t \in \mathbb{R}_+$ pulls back to a morphism $E \to B$, and these morphisms for different t are all homotopic. Hence we get a well-defined map $F\bigl(\mathrm{Sus}^n(A)\bigr) \cong F\bigl(\mathrm{Sus}^n(E)\bigr) \to F\bigl(\mathrm{Sus}^n(B)\bigr)$ for each asymptotic morphism $A \to B$. This explains how asymptotic morphisms are related to exact homotopy functors. This observation leads to the following theorem:

Theorem 62. *There are natural bijections*

$$\mathrm{E}_0^G(A, B) \cong [\![\mathrm{Sus}\bigl(A_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})\bigr), \mathrm{Sus}\bigl(B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})\bigr)]\!]$$

for all separable G-C-algebras A, B.*

An important step in the proof of Theorem 62 is the *Connes–Higson construction*, which to an extension $I \rightarrowtail E \twoheadrightarrow Q$ in $\mathfrak{C}^*\mathfrak{sep}$ associates an asymptotic morphism $\mathrm{Sus}(Q) \to I$. A G-equivariant generalisation of this construction is discussed in [59]. Thus any extension in G-$\mathfrak{C}^*\mathfrak{sep}$ gives rise to an exact triangle $\mathrm{Sus}(Q) \to I \to E \to Q$ in $\mathfrak{E}^G$.

This also leads to the triangulated category structure of $\mathfrak{E}^G$. As for $\mathfrak{KK}^G$, we can define it using mapping cone triangles or extension triangles – both approaches yield the same class of exact triangles. The canonical functor $\mathfrak{KK}^G \to \mathfrak{E}^G$ is exact because it evidently preserves mapping cone triangles.

Now that we have two bivariant homology theories with apparently very similar formal properties, we must ask which one we should use. It may seem that the better exactness properties of E-theory raise it above KK-theory. But actually, these strong exactness properties have a drawback: for a general group G, the reduced crossed product functor need not be exact, so that there is no guarantee that it extends to a functor $\mathfrak{E}^G \to \mathfrak{E}$. Only full crossed products exist for all groups by Proposition 6; the construction of $G \ltimes \llcorner: \mathfrak{E}^G \to \mathfrak{E}$ is the same as in KK-theory. Similar problems occur with $\otimes_{\min}$ but not with $\otimes_{\max}$.

Furthermore, since $\mathfrak{KK}^G$ has a weaker universal property, it acts on more functors, so that results about $\mathfrak{KK}^G$ have stronger consequences. A good example of a functor that is split-exact but probably not exact is *local cyclic cohomology* (see [37], [45]). Therefore, the best practice seems to prove results in $\mathfrak{KK}^G$ if possible.

Many applications can be done with either E or KK, we hardly notice any difference. An explanation for this is the work of Houghton-Larsen and Thomsen ([27], [59]), which describes $\mathrm{KK}_0^G(A,B)$ in the framework of asymptotic morphisms. Recall that asymptotic morphisms $A \to B$ generate extensions $\mathrm{Cone}(B) \rightarrowtail E \twoheadrightarrow A$. If this extension is G-equivariantly cp-split, then the projection map $E \twoheadrightarrow A$ is a KK^G-equivalence. A G-equivariant completely positive contractive section for the extension exists if and only if we can represent our asymptotic morphism by a continuous family of G-equivariant, completely positive contractions $f_t \colon A \to B$, $t \in [0,\infty)$.

Definition 63. Let $[\![A,B]\!]_{\mathrm{cp}}$ be the set of homotopy classes of asymptotic morphisms from A to B that can be lifted to a G-equivariant, completely positive, contractive map $A \to \mathcal{C}_b(\mathbb{R}_+, B)$; of course, we only use homotopies with the same kind of lifting.

Theorem 64 ([59]). *There are natural bijections*

$$\mathrm{KK}_0^G(A,B) \cong [\![\mathrm{Sus}\big(A_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})\big), \mathrm{Sus}\big(B_{\mathbb{K}} \otimes \mathbb{K}(\ell^2\mathbb{N})\big)]\!]_{\mathrm{cp}}$$

for all separable G-C^*-algebras A, B; the canonical functor $\mathfrak{KK}^G \to \mathfrak{E}^G$ corresponds to the obvious map that forgets the additional constraints.

Corollary 65. *If A is a nuclear C^*-algebra, then* $\mathrm{KK}_*(A,B) \cong \mathrm{E}_*(A,B)$.

Proof. The Choi–Effros Lifting Theorem asserts that any extension of A has a completely positive contractive section. □

In the equivariant case, the same argument yields $\mathrm{KK}_*^G(A,B) \cong \mathrm{E}_*^G(A,B)$ if A is nuclear and G acts properly on A (see also [55]). It should be possible to weaken properness to amenability here, but I am not aware of a reference for this.

5 The Baum–Connes assembly map for spaces and operator algebras

The Baum–Connes conjecture is a guess for the K-theory $\mathrm{K}_*(\mathrm{C}^*_{\mathrm{red}}G)$ of reduced group C^*-algebras [4]. We shall compare the approach of Davis and Lück [14] using homotopy theory for G-spaces and its counterpart in bivariant K-theory formulated in [38]. To avoid technical difficulties, we *assume that the group G is discrete*.

The first step in the Davis–Lück approach is to embed the groups of interest such as $\mathrm{K}_*(\mathrm{C}^*_{\mathrm{red}}G)$ in a *G-homology theory*, that is, a homology theory on the category of (spectra of) G-CW-complexes. For the Baum–Connes assembly map we need a homology theory for G-CW-complexes with $F_*(G/H) \cong \mathrm{K}_*(\mathrm{C}^*_{\mathrm{red}}H)$. This amounts to finding a G-equivariant spectrum with appropriate homotopy groups [14] and is the most difficult part of the construction. Other interesting invariants like the algebraic K- and L-theory of group rings can be treated using other spectra instead.

In the world of C^*-algebras, we cannot treat algebraic K- and L-theory; but we have much better tools to study $K_*(C^*_{red} G)$. We do not need a G-homology theory but a homological functor on the triangulated category $\mathfrak{KK}^G$. More precisely, we need a homological functor that takes the value $K_*(C^*_{red} H)$ on $\mathcal{C}_0(G/H)$ for all subgroups H. The functor $A \mapsto K_*(G \ltimes_r A)$ works fine here because $G \ltimes_r \mathcal{C}_0(G/H, A)$ is Morita–Rieffel equivalent to $H \ltimes_r A$ for any H-C^*-algebra A. The corresponding assertion for full crossed products is known as *Green's Imprimitivity Theorem*; reduced crossed products can be handled similarly. Thus a topological approach to the Baum–Connes conjecture forces us to consider $K_*(G \ltimes_r A)$ for all G-C^*-algebras A, which leads to the *Baum–Connes conjecture with coefficients*.

5.1 Assembly maps via homotopy theory. Recall that a homology theory on pointed CW-complexes is determined by its value on $\mathbb{S}^0$. Similarly, a G-homology theory F is determined by its values $F_*(G/H)$ on homogeneous spaces for all subgroups $H \subseteq G$. This does not help much because these groups – which are $K_*(C^*_{red} H)$ in the case of interest – are very hard to compute.

The idea behind assembly maps is to approximate a given homology theory by a simpler one that only depends on $F_*(G/H)$ for $H \in \mathcal{F}$ for some family of subgroups $\mathcal{F}$. The Baum–Connes assembly map uses the family of finite subgroups here; other families like virtually cyclic subgroups appear in isomorphism conjectures for other homology theories. We now fix a family of subgroups $\mathcal{F}$, which we assume to be closed under conjugation and subgroups.

A *$G, \mathcal{F}$-CW-complex* is a G-CW-complex in which the stabilisers of cells belong to $\mathcal{F}$. The *universal $G, \mathcal{F}$-CW-complex* is a $G, \mathcal{F}$-CW-complex $\mathcal{E}(G, \mathcal{F})$ with the property that, for any $G, \mathcal{F}$-CW-complex X there is a G-map $X \to \mathcal{E}(G, \mathcal{F})$, which is unique up to G-homotopy. This universal property determines $\mathcal{E}(G, \mathcal{F})$ uniquely up to G-homotopy. It is easy to see that $\mathcal{E}(G, \mathcal{F})$ is H-equivariantly contractible for any $H \in \mathcal{F}$. Conversely, a $G, \mathcal{F}$-CW-complex with this property is universal.

Example 66. Let $G = \mathbb{Z}$ and let $\mathcal{F}$ be the family consisting only of the trivial subgroup; this agrees with the family of finite subgroups because G is torsion-free. A $G, \mathcal{F}$-CW-complex is essentially the same as a CW-complex with a free cellular action of $\mathbb{Z}$. It is easy to check that $\mathbb{R}$ with the action of $\mathbb{Z}$ by translation and the usual cell decomposition is a universal $G, \mathcal{F}$-CW-complex.

Given any G-CW-complex X, the canonical map $\mathcal{E}(G, \mathcal{F}) \times X \to X$ has the following properties:

- $\mathcal{E}(G, \mathcal{F}) \times X$ is a $G, \mathcal{F}$-CW-complex;
- if Y is a $G, \mathcal{F}$-CW-complex, then any G-map $Y \to X$ lifts uniquely up to G-homotopy to a map $Y \to \mathcal{E}(G, \mathcal{F}) \times X$;
- for any $H \in \mathcal{F}$, the map $\mathcal{E}(G, \mathcal{F}) \times X \to X$ becomes a homotopy equivalence in the category of H-spaces.

The first two properties make precise in what sense $\mathcal{E}(G,\mathcal{F}) \times X$ is the best approximation to X among $G,\mathcal{F}$-CW-complexes.

Definition 67. The *assembly map* with respect to $\mathcal{F}$ is the map $F_*\big(\mathcal{E}(G,\mathcal{F})\big) \to F_*(\star)$ induced by the constant map $\mathcal{E}(G,\mathcal{F}) = \mathcal{E}(G,\mathcal{F}) \times \star \to \star$.

More generally, the assembly map with coefficients in a pointed G-CW-complex (or spectrum) X is the map $F_*(\mathcal{E}(G,\mathcal{F})_+ \wedge X) \to F_*(\mathbb{S}^0 \wedge X) = F_*(X)$ induced by the map $\mathcal{E}(G,\mathcal{F})_+ \to \star_+ = \mathbb{S}^0$.

In the stable homotopy category of pointed G-CW-complexes (or spectra), we get an exact triangle $\mathcal{E}(G,\mathcal{F})_+ \wedge X \to X \to N \to \mathbb{S}^1 \wedge \mathcal{E}(G,\mathcal{F})_+ \wedge X$, where N is H-equivariantly contractible for each $H \in \mathcal{F}$. This means that the domain of the assembly map $F_*(\mathcal{E}(G,\mathcal{F})_+ \wedge X)$ is the *localisation* of F_* at the class of all objects that are H-equivariantly contractible for each $H \in \mathcal{F}$.

Thus the assembly map is an isomorphism for all X if and only if $F_*(N) = 0$ whenever N is H-equivariantly contractible for each $H \in \mathcal{F}$. Thus an isomorphism conjecture can be interpreted in two equivalent ways. First, it says that we can reconstruct the homology theory from its restriction to $G,\mathcal{F}$-CW-complexes. Secondly, it says that the homology theory vanishes for spaces that are H-equivariantly contractible for $H \in \mathcal{F}$.

5.2 From spaces to operator algebras. We can carry over the construction of assembly maps above to bivariant Kasparov theory; we continue to assume G discrete to simplify some statements. From now on, we let $\mathcal{F}$ be the family of finite subgroups. This is the family that appears in the Baum–Connes assembly map. Other families of subgroups can also be treated, but some proofs have to be modified and are not yet written down.

First we need an analogue of $G,\mathcal{F}$-CW-complexes. These are constructible out of simpler "cells" which we describe first, using the *induction functors*

$$\mathrm{Ind}_H^G : \mathfrak{KK}^H \to \mathfrak{KK}^G$$

for subgroups $H \subseteq G$. For a finite group H, $\mathrm{Ind}_H^G(A)$ is the H-fixed point algebra of $\mathcal{C}_0(G, A)$, where H acts by $h \cdot f(g) = \alpha_h\big(f(gh)\big)$. For infinite H, we have

$$\begin{aligned}\mathrm{Ind}_H^G(A) = \{f \in \mathcal{C}_b(G,A) \mid \alpha_h f(gh) = f(g) \text{ for all } g \in G, h \in H,\\ \text{and } gH \mapsto \|f(g)\| \text{ is in } \mathcal{C}_0(G/H)\};\end{aligned}$$

the group G acts by translations on the left.

This construction is functorial for equivariant *-homomorphisms. Since it commutes with C^*-stabilisations and maps split extensions again to split extensions, it descends to a functor $\mathfrak{KK}^H \to \mathfrak{KK}^G$ by the universal property (compare §4.1).

We also have the more trivial *restriction functors* $\mathrm{Res}_G^H : \mathfrak{KK}^G \to \mathfrak{KK}^H$ for subgroups $H \subseteq G$. The induction and restriction functors are adjoint:

$$\mathrm{KK}^G(\mathrm{Ind}_H^G A, B) \cong \mathrm{KK}^H(A, \mathrm{Res}_G^H B)$$

for all $A \in\in \mathfrak{KK}^G$; this can be proved like the similar adjointness statements in §4.1, using the embedding $A \to \operatorname{Res}_G^H \operatorname{Ind}_H^G(A)$ as functions supported on $H \subseteq G$ and the correspondence $\operatorname{Ind}_H^G \operatorname{Res}_G^H(A) \cong \mathcal{C}_0(G/H, A) \to \mathbb{K}(\ell^2 G/H) \otimes A \sim A$. It is important here that $H \subseteq G$ is an *open* subgroup. By the way, if $H \subseteq G$ is a *cocompact* subgroup (which means finite index in the discrete case), then Res_G^H is the left-adjoint of Ind_H^G instead.

Definition 68. We let $\mathcal{CI}$ be the subcategory of all objects of $\mathfrak{KK}^G$ of the form $\operatorname{Ind}_H^G(A)$ for $A \in\in \mathfrak{KK}^H$ and $H \in \mathcal{F}$. Let $\langle \mathcal{CI} \rangle$ be the smallest class in $\mathfrak{KK}^G$ that contains $\mathcal{CI}$ and is closed under KK^G-equivalence, countable direct sums, suspensions, and exact triangles.

Equivalently, $\langle \mathcal{CI} \rangle$ is the *localising subcategory* generated by $\mathcal{CI}$. This is our substitute for the category of $(G, \mathcal{F})$-CW-complexes.

Definition 69. Let $\mathcal{CC}$ be the class of all objects of $\mathfrak{KK}^G$ with $\operatorname{Res}_G^H(A) \cong 0$ for all $H \in \mathcal{F}$.

Theorem 70. *If $P \in \langle \mathcal{CI} \rangle$, $N \in \mathcal{CC}$, then $\mathrm{KK}^G(P, N) = 0$. Furthermore, for any $A \in\in \mathfrak{KK}^G$ there is an exact triangle $P \to A \to N \to \Sigma P$ with $P \in \langle \mathcal{CI} \rangle$, $N \in \mathcal{CC}$.*

Definitions 68–69 and Theorem 70 are taken from [38]. The map $F_*(P) \to F_*(A)$ for a functor $F\colon \mathfrak{KK}^G \to \mathfrak{C}$ is analogous to the assembly map in Definition 67 and deserves to be called the *Baum–Connes assembly map for F*.

We can use the tensor product in $\mathfrak{KK}^G$ to simplify the proof of Theorem 70: once we have a triangle $P_\mathbb{C} \to \mathbb{C} \to N_\mathbb{C} \to \Sigma P_\mathbb{C}$ with $P_\mathbb{C} \in \langle \mathcal{CI} \rangle$, $N_\mathbb{C} \in \mathcal{CC}$, then

$$A \otimes P_\mathbb{C} \to A \otimes \mathbb{C} \to A \otimes N_\mathbb{C} \to \Sigma A \otimes P_\mathbb{C}$$

is an exact triangle with similar properties for A. It makes no difference whether we use $\otimes_{\min}$ or $\otimes_{\max}$ here. The map $P_\mathbb{C} \to \mathbb{C}$ in $\mathrm{KK}^G(P_\mathbb{C}, \mathbb{C})$ is analogous to the map $\mathcal{E}(G, \mathcal{F}) \to \star$. It is also called a *Dirac morphism* for G because the K-homology classes of Dirac operators on smooth spin manifolds provided the first important examples [31].

The two assembly map constructions with spaces and C*-algebras are not just analogous but provide the same Baum–Connes assembly map. To see this, we must understand the passage from the homotopy category of spaces to $\mathfrak{KK}$. Usually, we map spaces to operator algebras using the commutative C*-algebra $\mathcal{C}_0(X)$. But this construction is only functorial for *proper* continuous maps, and the functoriality is contravariant. The assembly map for, say, $G = \mathbb{Z}$ is related to the non-proper map $p\colon \mathbb{R} \to \star$, which does not induce a map $\mathbb{C} \to \mathcal{C}_0(\mathbb{R})$; even if it did, this map would still go in the wrong direction. The *wrong-way functoriality* in KK provides an element $p_! \in \mathrm{KK}_1(\mathcal{C}_0(\mathbb{R}), \mathbb{C})$ instead, which is the desired Dirac morphism up to a shift in the grading. This construction only applies to manifolds with a Spin^c-structure, but it can be generalised as follows.

On the level of Kasparov theory, we can define another functor from suitable spaces to $\mathfrak{KK}$ that is a *covariant* functor for *all* continuous maps. The definition uses a notion of duality due to Kasparov [31] that is studied more systematically in [17]. It requires yet another version $\mathrm{RKK}^G_*(X; A, B)$ of Kasparov theory that is defined for a locally compact space X and two G-C*-algebras A and B. Roughly speaking, the cycles for this theory are G-equivariant families of cycles for $\mathrm{KK}_*(A, B)$ parametrised by X. The groups $\mathrm{RKK}^G_*(X; A, B)$ are contravariantly functorial and homotopy invariant in X (for G-equivariant continuous maps).

We have $\mathrm{RKK}^G_*(\star; A, B) = \mathrm{KK}^G_*(A, B)$ and, more generally, $\mathrm{RKK}^G_*(X; A, B) \cong \mathrm{KK}^G_*\bigl(A, \mathcal{C}(X, B)\bigr)$ if X is *compact*. The same statement holds for non-compact X, but the algebra $\mathcal{C}(X, B)$ is not a C*-algebra any more: it is an inverse system of C*-algebras.

Definition 71 ([17]). A G-C*-algebra P_X is called an *abstract dual* for X if, for all second countable locally compact G-spaces Y and all separable G-C*-algebras A and B, there are natural isomorphisms

$$\mathrm{RKK}^G(X \times Y; A, B) \cong \mathrm{RKK}^G(Y; P_X \otimes A, B)$$

that are compatible with tensor products.

Abstract duals exist for many spaces. For trivial reasons, $\mathbb{C}$ is an abstract dual for the one-point space. For a smooth manifold X with an isometric action of G, both $\mathcal{C}_0(T^*X)$ and the algebra of $\mathcal{C}_0$-sections of the Clifford algebra bundle on X are abstract duals for X; if X has a G-equivariant Spin^c-structure – as in the example of $\mathbb{Z}$ acting on $\mathbb{R}$ – we may also use a suspension of $\mathcal{C}_0(X)$. For a finite-dimensional simplicial complex with a simplicial action of G, an abstract dual is constructed by Gennadi Kasparov and Georges Skandalis in [32] and in more detail in [17]. It seems likely that the construction can be carried over to infinite-dimensional simplicial complexes as well, but this has not yet been written down.

There are also spaces with no abstract dual. A prominent example is the Cantor set: it has no abstract dual, even for trivial G (see [17]).

Let $\mathcal{D}$ be the class of all G-spaces that admit a dual. Recall that $X \mapsto \mathrm{RKK}^G(X \times Y; A, B)$ is a contravariant homotopy functor for continuous G-maps. Passing to corepresenting objects, we get a *covariant* homotopy functor

$$\mathcal{D} \to \mathfrak{KK}^G, \quad X \mapsto P_X.$$

This functor is very useful to translate constructions from homotopy theory to bivariant K-theory. An instance of this is the comparison of the Baum–Connes assembly maps in both setups:

Theorem 72. *Let $\mathcal{F}$ be the family of finite subgroups of a discrete group G, and let $\mathcal{E}(G, \mathcal{F})$ be the universal $(G, \mathcal{F})$-CW-complex. Then $\mathcal{E}(G, \mathcal{F})$ has an abstract dual P, and the map $\mathcal{E}(G, \mathcal{F}) \to \star$ induces a Dirac morphism in* $\mathrm{KK}^G_0(P, \mathbb{C})$.

Theorem 72 should hold for all families of subgroups $\mathcal{F}$, but only the above special case is treated in [17], [38].

5.3 The Dirac-dual-Dirac method and geometry. Let us compare the approaches in §5.1 and §5.2! The bad thing about the C*-algebraic approach is that it applies to fewer theories. The good thing about it is that Kasparov theory is so flexible that any canonical map between K-theory groups has a fair chance to come from a morphism in $\mathfrak{KK}^G$ which we can construct explicitly.

For some groups, the Dirac morphism in $\mathrm{KK}^G(P, \mathbb{C})$ is a KK-equivalence:

Theorem 73 (Higson–Kasparov [26]). *Let the group G be amenable or, more generally, a-T-menable. Then the Dirac morphism for G is a KK^G-equivalence, so that G satisfies the Baum–Connes conjecture with coefficients.*

The class of groups for which the Dirac morphism has a *one-sided* inverse is even larger. This is the point of the *Dirac-dual-Dirac method*. The following definition in [38] is based on a simplification of this method:

Definition 74. A *dual Dirac morphism* for G is an element $\eta \in \mathrm{KK}^G(\mathbb{C}, P)$ with $\eta \circ D = \mathrm{id}_P$.

If such a dual Dirac morphism exists, then it provides a section for the assembly map $F_*(P \otimes A) \to F_*(A)$ for any functor $F : \mathfrak{KK}^G \to \mathfrak{C}$ and any $A \in\in \mathfrak{KK}^G$, so that the assembly map is a split monomorphism. Currently, we know no group without a dual Dirac morphism. It is shown in [16], [18], [19] that the existence of a dual Dirac morphism is a *geometric property* of G because it is related to the invertibility of another assembly map that only depends on the *coarse* geometry of G (in the torsion-free case).

Instead of going into this construction, we briefly indicate another point of view that also shows that the existence of a dual Dirac morphism is a geometric issue. Let P be an abstract dual for some space X (like $\mathcal{E}(G, \mathcal{F})$). The duality isomorphisms in Definition 71 are determined by two pieces of data: a *Dirac morphism* $D \in \mathrm{KK}^G(P, \mathbb{C})$ and a *local dual Dirac morphism* $\Theta \in \mathrm{RKK}^G(X; \mathbb{C}, P)$. The notation is motivated by the special case of a Spin^c-manifold X with $P = \mathcal{C}_0(X)$, where D is the K-homology class defined by the Dirac operator and η is defined by a local construction involving pointwise Clifford multiplications. If $X = \mathcal{E}(G, \mathcal{F})$, then it turns out that $\eta \in \mathrm{KK}^G(\mathbb{C}, P)$ is a dual Dirac morphism if and only if the canonical map $\mathrm{KK}^G(\mathbb{C}, P) \to \mathrm{RKK}^G(X; \mathbb{C}, P)$ maps $\eta \mapsto \Theta$. Thus the issue is to *globalise* the local construction of Θ. This is possible if we know, say, that X has non-positive curvature. This is essentially how Kasparov proves the Novikov conjecture for fundamental groups of non-positively curved smooth manifolds in [31].

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Inheritance of isomorphism conjectures under colimits

Arthur Bartels, Siegfried Echterhoff, and Wolfgang Lück

1 Introduction

1.1 Assembly maps. We want to study the following *assembly maps*:

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{K}_R) \to H_n^G(\{\bullet\};\mathbf{K}_R) = K_n(R \rtimes G); \tag{1.1}$$

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{KH}_R) \to H_n^G(\{\bullet\};\mathbf{KH}_R) = KH_n(R \rtimes G); \tag{1.2}$$

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(\{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(R \rtimes G); \tag{1.3}$$

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}_{A,l^1}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}_{A,l^1}^{\mathrm{top}}) = K_n(A \rtimes_{l^1} G); \tag{1.4}$$

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}_{A,r}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}_{A,r}^{\mathrm{top}}) = K_n(A \rtimes_r G); \tag{1.5}$$

$$\mathrm{asmb}_n^G\colon H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}_{A,m}^{\mathrm{top}}) \to H_n^G(\{\bullet\};\mathbf{K}_{A,m}^{\mathrm{top}}) = K_n(A \rtimes_m G). \tag{1.6}$$

Some explanations are in order. A *family of subgroups of G* is a collection of subgroups of G which is closed under conjugation and taking subgroups. Examples are the family $\mathcal{F}\mathrm{in}$ of finite subgroups and the family $\mathcal{VC}\mathrm{yc}$ of virtually cyclic subgroups.

Let $E_{\mathcal{F}}(G)$ be the *classifying space associated to $\mathcal{F}$*. It is uniquely characterized up to G-homotopy by the properties that it is a G-CW-complex and that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$. For more information about these spaces $E_{\mathcal{F}}(G)$ we refer for instance to the survey article [29].

Given a group G acting on a ring (with involution) by structure preserving maps, let $R \rtimes G$ be the twisted group ring (with involution) and denote by $K_n(R \rtimes G)$, $KH_n(R \rtimes G)$ and $L_n^{\langle -\infty\rangle}(R \rtimes G)$ its *algebraic K-theory* in the non-connective sense (see Gersten [17] or Pedersen–Weibel [34]), *its homotopy K-theory* in the sense of Weibel [41], and its *L-theory with decoration $-\infty$* in the sense of Ranicki [37, Chapter 17]. Given a group G acting on a C^*-algebra A by automorphisms of C^*-algebras, let $A \rtimes_{l^1} G$ be the Banach algebra obtained from $A \rtimes G$ by completion with respect to the l^1-norm, let $A \rtimes_r G$ be the reduced crossed product C^*-algebra, and let $A \rtimes_m G$ be the maximal crossed product C^*-algebra and denote by $K_n(A \rtimes_{l^1} G)$, $K_n(A \rtimes_r G)$ and $K_n(A \rtimes_m G)$ their *topological K-theory*.

The source and target of the assembly maps are given by G-homology theories (see Definition 2.1 and Theorem 6.1) with the property that for every subgroup $H \subseteq G$

and $n \in \mathbb{Z}$

$$\begin{aligned}
H_n^G(G/H; \mathbf{K}_R) &\cong K_n(R \rtimes H);\\
H_n^G(G/H; \mathbf{KH}_R) &\cong KH_n(R \rtimes H);\\
H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) &\cong L_n^{\langle -\infty \rangle}(R \rtimes H);\\
H_n^G(G/H; \mathbf{K}_{A,l^1}^{\mathrm{top}}) &\cong K_n(A \rtimes_{l^1} H);\\
H_n^G(G/H; \mathbf{K}_{A,r}^{\mathrm{top}}) &\cong K_n(A \rtimes_r H);\\
H_n^G(G/H; \mathbf{K}_{A,r}^{\mathrm{top}}) &\cong K_n(A \rtimes_m H).
\end{aligned}$$

All the assembly maps are induced by the projection from $E_{\mathcal{F}\mathrm{in}}(G)$ or $E_{\mathcal{VC}\mathrm{yc}}(G)$ respectively to the one-point-space $\{\bullet\}$.

Remark 1.7. It might be surprising to the reader that we restrict to C*-algebra coefficients A in the assembly map (1.4). Indeed, our main results rely heavily on the validity of the conjecture for hyperbolic groups, which, so far, is only known for C*-algebra coefficients. Moreover we also want to study the passage from the l^1-setting to the C^*-setting. Hence we decided to restrict ourselves to the case of C^*-coefficients throughout. We mention that on the other hand the assembly map (1.4) can also be defined for Banach algebra coefficients [33].

1.2 Conventions. Before we go on, let us fix some conventions. A group G is always discrete. Hyperbolic group is to be understood in the sense of Gromov (see for instance [11], [12], [18], [19]). Ring means associative ring with unit and ring homomorphisms preserve units. Homomorphisms of Banach algebras are assumed to be norm decreasing.

1.3 Isomorphism conjectures. The *Farrell–Jones conjecture* for algebraic K-theory for a group G and a ring R with G-action by ring automorphisms says that the assembly map (1.1) is bijective for all $n \in \mathbb{Z}$. Its version for homotopy K-theory says that the assembly map (1.2) is bijective for all $n \in \mathbb{Z}$. If R is a ring with involution and G acts on R by automorphism of rings with involutions, the L-theoretic version of the Farrell–Jones conjecture predicts that the assembly map (1.3) is bijective for all $n \in \mathbb{Z}$. The Farrell–Jones conjecture for algebraic K- and L-theory was originally formulated in the paper by Farrell–Jones [15, 1.6 on page 257] for the trivial G-action on R. Its homotopy K-theoretic version can be found in [4, Conjecture 7.3], again for trivial G-action on R.

Let G be a group acting on the C^*-algebra A by automorphisms of C^*-algebras. The *Bost conjecture with coefficients* and the *Baum–Connes conjecture with coefficients* respectively predict that the assembly map (1.4) and (1.5) respectively are bijective for all $n \in \mathbb{Z}$. The original statement of the Baum–Connes conjecture with trivial coefficients can be found in [9, Conjecture 3.15 on page 254].

Our formulation of these conjectures follows the homotopy theoretic approach in [13]. The original assembly maps are defined differently. We do not give the proof that our maps agree with the original ones but at least refer to [13, page 239], where the Farrell–Jones conjecture is treated and to Hambleton–Pedersen [21], where such identification is given for the Baum–Connes conjecture with trivial coefficients.

1.4 Inheritance under colimits. The main purpose of this paper is to prove that these conjectures are inherited under colimits over directed systems of groups (with not necessarily injective structure maps). We want to show:

Theorem 1.8 (Inheritance under colimits). *Let $\{G_i \mid i \in I\}$ be a directed system of groups with (not necessarily injective) structure maps $\phi_{i,j}\colon G_i \to G_j$. Let $G = \operatorname{colim}_{i\in I} G_i$ be its colimit with structure maps $\psi_i\colon G_i \to G$. Let R be a ring (with involution) and let A be a C^*-algebra with structure preserving G-action. Given $i \in I$ and a subgroup $H \subseteq G_i$, we let H act on R and A by restriction with the group homomorphism $(\psi_i)|_H\colon H \to G$. Fix $n \in \mathbb{Z}$. Then:*

(i) *If the assembly map*

$$\operatorname{asmb}_n^H\colon H_n^H(E_{\mathcal{VC}yc}(H);\mathbf{K}_R) \to H_n^H(\{\bullet\};\mathbf{K}_R) = K_n(R \rtimes H)$$

of (1.1) *is bijective for all $n \in \mathbb{Z}$, all $i \in I$ and all subgroups $H \subseteq G_i$, then for every subgroup $K \subseteq G$ of G the assembly map*

$$\operatorname{asmb}_n^K\colon H_n^K(E_{\mathcal{VC}yc}(K);\mathbf{K}_R) \to H_n^K(\{\bullet\};\mathbf{K}_R) = K_n(R \rtimes K)$$

of (1.1) *is bijective for all $n \in \mathbb{Z}$.*

The corresponding version is true for the assembly maps given in (1.2)*,* (1.3)*,* (1.4)*, and* (1.6)*.*

(ii) *Suppose that all structure maps $\phi_{i,j}$ are injective and that the assembly map*

$$\operatorname{asmb}_n^{G_i}\colon H_n^{G_i}(E_{\mathcal{VC}yc}(G_i);\mathbf{K}_R) \to H_n^{G_i}(\{\bullet\};\mathbf{K}_R) = K_n(R \rtimes G_i)$$

of (1.1) *is bijective for all $n \in \mathbb{Z}$ and $i \in I$. Then the assembly map*

$$\operatorname{asmb}_n^G\colon H_n^G(E_{\mathcal{VC}yc}(G);\mathbf{K}_R) \to H_n^G(\{\bullet\};\mathbf{K}_R) = K_n(R \rtimes G)$$

of (1.1) *is bijective for all $n \in \mathbb{Z}$.*

The corresponding statement is true for the assembly maps given in (1.2)*,* (1.3)*,* (1.4)*,* (1.5)*, and* (1.6)*.*

Theorem 1.8 will follow from Theorem 4.5 and Lemma 6.2 as soon as we have proved Theorem 6.1. Notice that the version (1.5) does not appear in assertion (i). A counterexample will be discussed below. The (fibered) version of Theorem 1.8 (i) in the case of algebraic K-theory and L-theory with coefficients in $\mathbb{Z}$ with trivial G-action has been proved by Farrell–Linnell [16, Theorem 7.1].

1.5 Colimits of hyperbolic groups. In [23, Section 7] Higson, Lafforgue and Skandalis construct counterexamples to the *Baum–Connes conjecture with coefficients*, actually with a commutative C^*-algebra as coefficients. They formulate precise properties for a group G which imply that it does *not* satisfy the Baum–Connes conjecture with coefficients. Gromov [20] describes the construction of such a group G as a colimit over a directed system of groups $\{G_i \mid i \in I\}$, where each G_i is hyperbolic.

This construction did raise the hope that these groups G may also be counterexamples to the Baum–Connes conjecture with trivial coefficients. But – to the authors' knowledge – this has not been proved and no counterexample to the Baum–Connes conjecture with trivial coefficients is known.

Of course one may wonder whether such counterexamples to the Baum–Connes conjecture with coefficients or with trivial coefficients respectively may also be counterexamples to the Farrell–Jones conjecture or the Bost conjecture with coefficients or with trivial coefficients respectively. However, this can be excluded by the following result.

Theorem 1.9. *Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of groups* (*with not necessarily injective structure maps*)*. Suppose that each G_i is hyperbolic. Let $K \subseteq G$ be a subgroup. Then:*

(i) *The group K satisfies for every ring R on which K acts by ring automorphisms the Farrell–Jones conjecture for algebraic K-theory with coefficients in R, i.e., the assembly map* (1.1) *is bijective for all $n \in \mathbb{Z}$.*

(ii) *The group K satisfies for every ring R on which K acts by ring automorphisms the Farrell–Jones conjecture for homotopy K-theory with coefficients in R, i.e., the assembly map* (1.2) *is bijective for all $n \in \mathbb{Z}$.*

(iii) *The group K satisfies for every C^*-algebra A on which K acts by C^*-automorphisms the Bost conjecture with coefficients in A, i.e., the assembly map* (1.4) *is bijective for all $n \in \mathbb{Z}$.*

Proof. If G is the colimit of the directed system $\{G_i \mid i \in I\}$, then the subgroup $K \subseteq G$ is the colimit of the directed system $\{\psi_i^{-1}(K) \mid i \in I\}$, where $\psi_i : G_i \to G$ is the structure map. Hence it suffices to prove Theorem 1.9 in the case $G = K$. This case follows from Theorem 1.8 (i) as soon as one can show that the Farrell–Jones conjecture for algebraic K-theory, the Farrell–Jones conjecture for homotopy K-theory, or the Bost conjecture respectively holds for every subgroup H of a hyperbolic group G with arbitrary coefficients R and A respectively.

Firstly we prove this for the Bost conjecture. Mineyev and Yu [30, Theorem 17] show that every hyperbolic group G admits a G-invariant metric $\hat{d}$ which is weakly geodesic and strongly bolic. Since every subgroup H of G clearly acts properly on G with respect to any discrete metric, it follows that H belongs to the class $\mathcal{C}'$ as described by Lafforgue in [27, page 5] (see also the remarks at the top of page 6 in [27]). Now the claim is a direct consequence of [27, Theorem 0.0.2].

The claim for the Farrell–Jones conjecture is proved for algebraic K-theory and homotopy K-theory in Bartels–Lück–Reich [6] which is based on the results of [5]. $\square$

There are further groups with unusual properties that can be obtained as colimits of hyperbolic groups. This class contains for instance a torsion-free non-cyclic group all whose proper subgroups are cyclic constructed by Ol'shanskii [32]. Further examples are mentioned in [31, p.5] and [38, Section 4].

We mention that if one can prove the L-theoretic version of the Farrell–Jones conjecture for subgroups of hyperbolic groups with arbitrary coefficients, then it is also true for subgroups of colimits of hyperbolic groups by the argument above.

1.6 Discussion of (potential) counterexamples. If G is an infinite group which satisfies Kazhdan's property (T), then the assembly map (1.6) for the maximal group C^*-algebra fails to be an isomorphism if the assembly map (1.5) for the reduced group C^*-algebra is injective (which is true for a very large class of groups and in particular for all hyperbolic groups by [25]). The reason is that a group has property (T) if and only if the trivial representation 1_G is isolated in the dual $\widehat{G}$ of G. This implies that $C_m^*(G)$ has a splitting $\mathbb{C} \oplus \ker(1_G)$, where we regard 1_G as a representation of $C_m^*(G)$. If G is infinite, then the first summand is in the kernel of the regular representation $\lambda\colon C_m^*(G) \to C_r^*(G)$ (see for instance [14]), hence the K-theory map $\lambda\colon K_0(C_m^*(G)) \to K_0(C_r^*(G))$ is not injective. For a finite group H we have $A \rtimes_r H = A \rtimes_m H$ and hence we can apply [13, Lemma 4.6] to identify the domains of (1.5) and (1.6). Under this identification the composition of the assembly map (1.6) with λ is the assembly map (1.5) and the claim follows.

Hence the Baum–Connes conjecture for the maximal group C^*-algebras is not true in general since the Baum–Connes conjecture for the reduced group C^*-algebras has been proved for some groups with property (T) by Lafforgue [26] (see also [39]). So in the sequel our discussion refers always to the Baum–Connes conjecture for the reduced group C^*-algebra.

One may speculate that the Baum–Connes conjecture with trivial coefficients is less likely to be true for a given group G than the Farrell–Jones conjecture or the Bost conjecture. Some evidence for this speculation comes from lack of functoriality of the reduced group C^*-algebra. A group homomorphism $\alpha\colon H \to G$ induces in general not a C^*-homomorphism $C_r^*(H) \to C_r^*(G)$, one has to require that its kernel is amenable. Here is a counterexample, namely, if F is a non-abelian free group, then $C_r^*(F)$ is a simple algebra [35] and hence there is no unital algebra homomorphism $C_r^*(F) \to C_r^*(\{1\}) = \mathbb{C}$. On the other hand, any group homomorphism $\alpha\colon H \to G$ induces a homomorphism

$$H_n^H(E_{\mathcal{F}\mathrm{in}}(H); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}}) \xrightarrow{\mathrm{ind}_\alpha} H_n^G(\alpha_* E_{\mathcal{F}\mathrm{in}}(H); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}}) \xrightarrow{H_n^G(f)} H_n^G(E_{\mathcal{F}\mathrm{in}}(G); \mathbf{K}_{\mathbb{C},r}^{\mathrm{top}})$$

where G acts trivially on $\mathbb{C}$ and $f : \alpha_* E_{\mathcal{F}\mathrm{in}}(H) \to E_{\mathcal{F}\mathrm{in}}(G)$ is the up to G-homotopy unique G-map. Notice that the induction map ind_α exists since the isotropy groups of $E_{\mathcal{F}\mathrm{in}}(H)$ are finite. Moreover, this map is compatible under the assembly maps for H and G with the map $K_n(C_r^*(\alpha))\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$ provided that α has amenable kernel and hence $C_r^*(\alpha)$ is defined. So the Baum–Connes conjecture implies that every group homomorphism $\alpha\colon H \to G$ induces a group homomorphism $\alpha_*\colon K_n(C_r^*(H)) \to K_n(C_r^*(G))$, although there may be no C^*-homomorphism $C_r^*(H) \to C_r^*(G)$ induced by α. No such direct construction of α_* is known in general.

Here is another failure of the reduced group C^*-algebra. Let G be the colimit of the directed system $\{G_i \mid i \in I\}$ of groups (with not necessarily injective structure maps). Suppose that for every $i \in I$ and preimage H of a finite group under the canonical map $\psi_i\colon G_i \to G$ the Baum–Connes conjecture for the maximal group C^*-algebra holds (This is for instance true by [22] if $\ker(\psi_i)$ has the Haagerup property). Then

$$\begin{aligned}\operatorname{colim}_{i\in I} H_n^{G_i}(E_{\mathcal{F}\mathrm{in}}(G_i);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},m}) &\xrightarrow{\cong} \operatorname{colim}_{i\in I} H_n^{G_i}(E_{\psi_i^*\mathcal{F}\mathrm{in}}(G_i);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},m})\\ &\xrightarrow{\cong} H_n^{G}(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},m})\end{aligned}$$

is a composition of two isomorphisms. The first map is bijective by the Transitivity Principle 4.3, the second by Lemma 3.4 and Lemma 6.2. This implies that the following composition is an isomorphism

$$\begin{aligned}\operatorname{colim}_{i\in I} H_n^{G_i}(E_{\mathcal{F}\mathrm{in}}(G_i);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},r}) &\to \operatorname{colim}_{i\in I} H_n^{G_i}(E_{\psi_i^*\mathcal{F}\mathrm{in}}(G_i);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},r})\\ &\to H_n^{G}(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{\mathbb{C},r})\end{aligned}$$

Namely, these two compositions are compatible with the passage from the maximal to the reduced setting. This passage induces on the source and on the target isomorphisms since $E_{\mathcal{F}\mathrm{in}}(G_i)$ and $E_{\mathcal{F}\mathrm{in}}(G)$ have finite isotropy groups, for a finite group H we have $C_r^*(H) = C_m^*(H)$ and hence we can apply [13, Lemma 4.6]. Now assume furthermore that the Baum–Connes conjecture for the reduced group C^*-algebra holds for G_i for each $i \in I$ and for G. Then we obtain an isomorphism

$$\operatorname{colim}_{i\in I} K_n(C_r^*(G_i)) \xrightarrow{\cong} K_n(C_r^*(G)).$$

Again it is in general not at all clear whether there exists such a map in the case, where the structure maps $\psi_i\colon G_i \to G$ do not have amenable kernels and hence do not induce maps $C_r^*(G_i) \to C_r^*(G)$.

These arguments do not apply to the Farrell–Jones conjecture or the Bost conjecture. Namely any group homomorphism $\alpha\colon H \to G$ induces maps $R \rtimes H \to R \rtimes G$, $A \rtimes_{l^1} H \to A \rtimes_{l^1} G$, and $A \rtimes_m H \to A \rtimes_m G$ for a ring R or a C^*-algebra A with structure preserving G-action, where we equip R and A with the H-action coming from α. Moreover we will show for a directed system $\{G_i \mid i \in I\}$ of groups (with not necessarily injective structure maps) and $G = \operatorname{colim}_{i\in I} G_i$ that there are canonical isomorphisms (see Lemma 6.2)

$$\begin{aligned}
\operatorname{colim}_{i\in I} K_n(R\rtimes G_i) &\xrightarrow{\cong} K_n(R\rtimes G);\\
\operatorname{colim}_{i\in I} KH_n(R\rtimes G_i) &\xrightarrow{\cong} KH_n(R\rtimes G);\\
\operatorname{colim}_{i\in I} L_n^{\langle -\infty\rangle}(R\rtimes G_i) &\xrightarrow{\cong} L_n^{\langle -\infty\rangle}(R\rtimes G);\\
\operatorname{colim}_{i\in I} K_n(A\rtimes_{l^1} G_i) &\xrightarrow{\cong} K_n(A\rtimes_{l^1} G);\\
\operatorname{colim}_{i\in I} K_n(A\rtimes_{m} G_i) &\xrightarrow{\cong} K_n(A\rtimes_{m} G).
\end{aligned}$$

Let A be a C^*-algebra with G-action by C^*-automorphisms. We can consider A as a ring only. Notice that we get a commutative diagram

$$\begin{tikzcd}
H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{K}_A) \arrow[r] \arrow[d] & KH_n(A\rtimes G) \arrow[d] \\
H_n^G(E_{\mathcal{VC}\mathrm{yc}}(G);\mathbf{KH}_A) \arrow[r] & KH_n(A\rtimes G) \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{KH}_A) \arrow[r] \arrow[u, "\cong"'] \arrow[d] & KH_n(A\rtimes G) \arrow[u, "\mathrm{id}"'] \arrow[d] \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{A,l^1}) \arrow[r] \arrow[d, "\cong"'] & K_n(A\rtimes_{l^1} G) \arrow[d] \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{A,m}) \arrow[r] \arrow[d, "\cong"'] & K_n(A\rtimes_{m} G) \arrow[d] \\
H_n^G(E_{\mathcal{F}\mathrm{in}}(G);\mathbf{K}^{\mathrm{top}}_{A,r}) \arrow[r] & K_n(A\rtimes_{r} G),
\end{tikzcd}$$

where the horizontal maps are assembly maps and the vertical maps are change of theory and rings maps or induced by the up to G-homotopy unique G-map $E_{\mathcal{F}\mathrm{in}}(G)\to E_{\mathcal{VC}\mathrm{yc}}(G)$. The second left vertical map, which is marked with $\cong$, is bijective. This is shown in [4, Remark 7.4] in the case, where G acts trivially on R, the proof carries directly over to the general case. The fourth and fifth vertical left arrow, which are marked with $\cong$, are bijective, since for a finite group H we have $A\rtimes H = A\rtimes_{l^1} H = A\rtimes_r H = A\rtimes_m H$ and hence we can apply [13, Lemma 4.6]. In particular the Bost conjecture and the Baum–Connes conjecture together imply that the map $K_n(A\rtimes_{l^1} G)\to K_n(A\rtimes_r G)$ is bijective, the map $K_n(A\rtimes_{l^1} G)\to K_n(A\rtimes_m G)$ is split injective and the map $K_n(A\rtimes_m G)\to K_n(A\rtimes_r G)$ is split surjective.

The upshot of this discussions is:

- The counterexamples of Higson, Lafforgue and Skandalis [23, Section 7] to the Baum–Connes conjecture with coefficients are not counterexamples to the Farrell–Jones conjecture or the Bost conjecture.

- The counterexamples of Higson, Lafforgue and Skandalis [23, Section 7] show that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is in general not bijective.

- The passage from the topological K-theory of the Banach algebra $l^1(G)$ to the reduced group C^*-algebra is problematic and may cause failures of the Baum–Connes conjecture.

- The Bost conjecture and the Farrell–Jones conjecture are more likely to be true than the Baum–Connes conjecture.

- There is – to the authors' knowledge – no promising candidate of a group G for which a strategy is in sight to show that the Farrell–Jones conjecture or the Bost conjecture are false. (Whether it is reasonable to believe that these conjectures are true for all groups is a different question.)

1.7 Homology theories and spectra. The general strategy of this paper is to present most of the arguments in terms of equivariant homology theories. Many of the arguments for the Farrell–Jones conjecture, the Bost conjecture or the Baum–Connes conjecture become the same, the only difference lies in the homology theory we apply them to. This is convenient for a reader who is not so familiar with spectra and prefers to think of K-groups and not of K-spectra.

The construction of these equivariant homology theories is a second step and done in terms of spectra. Spectra cannot be avoided in algebraic K-theory by definition and since we want to compare also algebraic and topological K-theory, we need spectra descriptions here as well. Another nice feature of the approach to equivariant topological K-theory via spectra is that it yields a theory which can be applied to all G-CW-complexes. This will allow us to consider in the case $G = \operatorname{colim}_{i \in I} G_i$ the equivariant K-homology of the G_i-CW-complex $\psi_i^* E_{\mathcal{F}\mathrm{in}}(G) = E_{\psi^*\mathcal{F}\mathrm{in}}(G_i)$ although $\psi_i^* E_{\mathcal{F}\mathrm{in}}(G)$ has infinite isotropy groups if the structure map $\psi_i : G_i \to G$ has infinite kernel.

Details of the constructions of the relevant spectra, namely, the proof of Theorem 8.1, will be deferred to [2]. We will use the existence of these spectra as a black box. These constructions require some work and technical skills, but their details are not at all relevant for the results and ideas of this paper and their existence is not at all surprising.

1.8 Twisting by cocycles. In the L-theory case one encounters also non-orientable manifolds. In this case twisting with the first Stiefel–Whitney class is required. In a more general setup one is given a group G, a ring R with involution and a group homomorphism $w \colon G \to \operatorname{cent}(R^\times)$ to the center of the multiplicative group of units in R. So far we have used the standard involution on the group ring RG, which is given by $\overline{r \cdot g} = \overline{r} \cdot g^{-1}$. One may also consider the w-twisted involution given by

$\overline{r \cdot g} = \overline{r} w(g) \cdot g^{-1}$. All the results in this paper generalize directly to this case since one can construct a modified L-theory spectrum functor (over G) using the w-twisted involution and then the homology arguments are just applied to the equivariant homology theory associated to this w-twisted L-theory spectrum.

Acknowledgements. The work was financially supported by the Sonderforschungsbereich 478 – Geometrische Strukturen in der Mathematik – and the Max-Planck-Forschungspreis of the third author.

2 Equivariant homology theories

In this section we briefly explain basic axioms, notions and facts about equivariant homology theories as needed for the purposes of this article. The main examples which will play a role in connection with the Bost, the Baum–Connes and the Farrell–Jones conjecture will be presented later in Theorem 6.1.

Fix a group G and a ring Λ. In most cases Λ will be $\mathbb{Z}$. The following definition is taken from [28, Section 1].

Definition 2.1 (G-homology theory). A *G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules* is a collection of covariant functors $\mathcal{H}_n^G$ from the category of G-CW-pairs to the category of Λ-modules indexed by $n \in \mathbb{Z}$ together with natural transformations $\partial_n^G(X, A)\colon \mathcal{H}_n^G(X, A) \to \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$ for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- *G-homotopy invariance.* If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$.

- *Long exact sequence of a pair.* Given a pair (X, A) of G-CW-complexes, there is a long exact sequence

$$\cdots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \\ \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \cdots,$$

  where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions.

- *Excision.* Let (X, A) be a G-CW-pair and let $f\colon A \to B$ be a cellular G-map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f)\colon (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_n^G(F, f)\colon \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B).$$

- *Disjoint union axiom.* Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i : X_i \to \coprod_{i \in I} X_i$ the canonical inclusion. Then the map
$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i) \colon \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\Big(\coprod_{i \in I} X_i\Big)$$
is bijective.

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be G-homology theories. A *natural transformation* $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ *of G-homology theories* is a sequence of natural transformations $T_n \colon \mathcal{H}_n^G \to \mathcal{K}_n^G$ of functors from the category of G-CW-pairs to the category of Λ-modules which are compatible with the boundary homomorphisms.

Lemma 2.2. *Let $T_* \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a natural transformation of G-homology theories. Suppose that $T_n(G/H)$ is bijective for every homogeneous space G/H and $n \in \mathbb{Z}$.*
Then $T_n(X, A)$ is bijective for every G-CW-pair (X, A) and $n \in \mathbb{Z}$.

Proof. The disjoint union axiom implies that both G-homology theories are compatible with colimits over directed systems indexed by the natural numbers (such as the system given by the skeletal filtration $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \cup_{n \geq 0} X_n = X$). The argument for this claim is analogous to the one in [40, 7.53]. Hence it suffices to prove the bijectivity for finite-dimensional pairs. Using the axioms of a G-homology theory, the five lemma and induction over the dimension one reduces the proof to the special case $(X, A) = (G/H, \emptyset)$. □

Next we present a slight variation of the notion of an equivariant homology theory introduced in [28, Section 1]. We have to treat this variation since we later want to study coefficients over a fixed group Γ which we will then pullback via group homomorphisms with Γ as target. Namely, fix a group Γ. A group (G, ξ) over Γ is a group G together with a group homomorphism $\xi \colon G \to \Gamma$. A map $\alpha \colon (G_1, \xi_1) \to (G_2, \xi_2)$ of groups over Γ is a group homomorphisms $\alpha \colon G_1 \to G_2$ satisfying $\xi_2 \circ \alpha = \xi_1$.

Let $\alpha \colon H \to G$ be a group homomorphism. Given an H-space X, define the *induction of X with α* to be the G-space denoted by $\alpha_* X$ which is the quotient of $G \times X$ by the right H-action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha \colon H \to G$ is an inclusion, we also write ind_H^G instead of α_*. If (X, A) is an H-CW-pair, then $\alpha_*(X, A)$ is a G-CW-pair.

Definition 2.3 (Equivariant homology theory over a group Γ). An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over a group* Γ assigns to every group (G, ξ) over Γ a G-homology theory $\mathcal{H}_*^G$ with values in Λ-modules and comes with the following so called *induction structure*: given a homomorphism $\alpha \colon (H, \xi) \to (G, \mu)$ of groups over Γ and an H-CW-pair (X, A), there are for each $n \in \mathbb{Z}$ natural homomorphisms
$$\operatorname{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^G(\alpha_*(X, A)) \tag{2.4}$$
satisfying

- *Compatibility with the boundary homomorphisms.* $\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H$.
- *Functoriality.* Let $\beta\colon (G,\mu) \to (K,\nu)$ be another morphism of groups over Γ. Then we have for $n \in \mathbb{Z}$

$$\mathrm{ind}_{\beta\circ\alpha} = \mathcal{H}_n^K(f_1) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K((\beta\circ\alpha)_*(X,A)),$$

  where $f_1\colon \beta_*\alpha_*(X,A) \xrightarrow{\cong} (\beta\circ\alpha)_*(X,A)$, $(k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism.
- *Compatibility with conjugation.* Let (G,ξ) be a group over Γ and let $g \in G$ be an element with $\xi(g) = 1$. Then the conjugation homomorphisms $c(g)\colon G \to G$ defines a morphism $c(g)\colon (G,\xi) \to (G,\xi)$ of groups over Γ. Let $f_2\colon (X,A) \to c(g)_*(X,A)$ be the G-homeomorphism which sends x to $(1,g^{-1}x)$ in $G \times_{c(g)} (X,A)$.

  Then for every $n \in \mathbb{Z}$ and every G-CW-pair (X,A) the homomorphism

$$\mathrm{ind}_{c(g)}\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_n^G(c(g)_*(X,A))$$

  agrees with $\mathcal{H}_n^G(f_2)$.
- *Bijectivity.* If $\alpha\colon (H,\xi) \to (G,\mu)$ is a morphism of groups over Γ such that the underlying group homomorphism $\alpha\colon H \to G$ is an inclusion of groups, then $\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(\{\bullet\}) \to \mathcal{H}_n^G(\alpha_*\{\bullet\}) = \mathcal{H}_n^G(G/H)$ is bijective for all $n \in \mathbb{Z}$.

Definition 2.3 reduces to the one of an equivariant homology in [28, Section 1] if one puts $\Gamma = \{1\}$.

Lemma 2.5. *Let $\alpha\colon (H,\xi) \to (G,\mu)$ be a morphism of groups over Γ. Let (X,A) be an H-CW-pair such that* $\ker(\alpha)$ *acts freely on $X - A$. Then*

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^G(\alpha_*(X,A))$$

is bijective for all $n \in \mathbb{Z}$.

Proof. Let $\mathcal{F}$ be the set of all subgroups of H whose intersection with $\ker(\alpha)$ is trivial. Obviously, this is a family, i.e., closed under conjugation and taking subgroups. A H-CW-pair (X,A) is called a $\mathcal{F}$-H-CW-pair if the isotropy group of any point in $X - A$ belongs to $\mathcal{F}$. A H-CW-pair (X,A) is a $\mathcal{F}$-H-CW-pair if and only if $\ker(\alpha)$ acts freely on $X - A$.

The n-skeleton of $\alpha_*(X,A)$ is α_* applied to the n-skeleton of (X,A). Let (X,A) be an H-CW-pair and let $f\colon A \to B$ be a cellular H-map of H-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a H-CW-pair. Then there is an obvious natural isomorphism of G-CW-pairs

$$\alpha_*(X \cup_f B, B) \xrightarrow{\cong} (\alpha_* X \cup_{\alpha_* f} \alpha_* B, \alpha_* B).$$

Now we proceed as in the proof of Lemma 2.2 but now considering the transformations

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^G(\alpha_*(X,A))$$

only for $\mathcal{F}$-H-CW-pairs (X,A). Thus we can reduce the claim to the special case $(X,A) = H/L$ for some subgroup $L \subseteq H$ with $L \cap \ker(\alpha) = \{1\}$. This special case follows from the following commutative diagram whose vertical arrows are bijective by the axioms and whose upper horizontal arrow is bijective since α induces an isomorphism $\alpha|_L\colon L \to \alpha(L)$:

$$\begin{tikzcd}
\mathcal{H}_n^L(\{\bullet\}) \arrow[r, "\mathrm{ind}_{\alpha|_L\colon L\to\alpha(L)}"] \arrow[d, "\mathrm{ind}_L^H"] & \mathcal{H}_n^{\alpha(L)}(\{\bullet\}) \arrow[d, "\mathrm{ind}_{\alpha(L)}^G"] \\
\mathcal{H}_n^H(H/L) \arrow[r, "\mathrm{ind}_\alpha"] & \mathcal{H}_n^G(\alpha_* H/L) = \mathcal{H}_n^G(G/\alpha(L)).
\end{tikzcd}$$ □

3 Equivariant homology theories and colimits

Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ.

Let X be a G-CW-complex. Let $\alpha\colon H \to G$ be a group homomorphism. Denote by α^*X the H-CW-complex obtained from X by *restriction with* α. We have already introduced the induction α_*Y of an H-CW-complex Y. The functors α_* and α^* are adjoint to one another. In particular the adjoint of the identity on α^*X is a natural G-map

$$f(X,\alpha)\colon \alpha_*\alpha^*X \to X. \tag{3.1}$$

It sends an element in $G \times_\alpha \alpha^*X$ given by (g,x) to gx.

Consider a map $\alpha\colon (H,\xi) \to (G,\mu)$ of groups over Γ. Define the Λ-map

$$a_n = a_n(X,\alpha)\colon \mathcal{H}_n^H(\alpha^*X) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}_n^G(\alpha_*\alpha^*X) \xrightarrow{\mathcal{H}_n^G(f(X,\alpha))} \mathcal{H}_n^G(X).$$

If $\beta\colon (G,\mu) \to (K,\nu)$ is another morphism of groups over Γ, then by the axioms of an induction structure the composite $\mathcal{H}_n^H(\alpha^*\beta^*X) \xrightarrow{a_n(\beta^*X,\alpha)} \mathcal{H}_n^G(\beta^*X) \xrightarrow{a_n(X,\beta)} \mathcal{H}_n^K(X)$ agrees with $a_n(X,\beta\circ\alpha)\colon \mathcal{H}_n^H(\alpha^*\beta^*X) = \mathcal{H}_n^H((\beta\circ\alpha)^*X) \to \mathcal{H}_n^G(X)$ for a K-CW-complex X.

Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i\colon G_i \to G$ for $i \in I$ and $\phi_{i,j}\colon G_i \to G_j$ for $i,j \in I, i \le j$. We obtain for every G-CW-complex X a system of Λ-modules $\{\mathcal{H}^{G_i}(\psi_i^*X) \mid i \in I\}$ with structure maps $a_n(\psi_j^*X,\phi_{i,j})\colon \mathcal{H}^{G_i}(\psi_i^*X) \to \mathcal{H}^{G_j}(\psi_j^*X)$. We get a map of Λ-modules

$$t_n^G(X,A) \ := \ \operatorname{colim}_{i\in I} a_n(X,\psi_i)\colon \operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(\psi_i^*(X,A)) \ \to \ \mathcal{H}_n^G(X,A). \tag{3.2}$$

The next definition is an extension of [4, Definition 3.1].

Definition 3.3 ((Strongly) continuous equivariant homology theory). An equivariant homology theory $\mathcal{H}^?_*$ over Γ is called *continuous* if for every group (G,ξ) over Γ and every directed system of subgroups $\{G_i \mid i \in I\}$ of G with $G = \bigcup_{i\in I} G_i$ the Λ-map (see (3.2))

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

An equivariant homology theory $\mathcal{H}^?_*$ over Γ is called *strongly continuous* if for every group (G,ξ) over Γ and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i : G_i \to G$ for $i \in I$ the Λ-map

$$t_n^G(\{\bullet\})\colon \operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(\{\bullet\}) \to \mathcal{H}_n^G(\{\bullet\})$$

is an isomorphism for every $n \in \mathbb{Z}$.

Here and in the sequel we view G_i as a group over Γ by $\xi \circ \psi_i : G_i \to \Gamma$ and $\psi_i : G_i \to G$ as a morphism of groups over Γ.

Lemma 3.4. *Let (G,ξ) be a group over Γ. Consider a directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i : G_i \to G$ for $i \in I$. Let (X, A) be a G-CW-pair. Suppose that $\mathcal{H}^?_*$ is strongly continuous.*

Then the Λ-homomorphism (see (3.2)*)*

$$t_n^G(X,A)\colon \operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(\psi_i^*(X,A)) \xrightarrow{\cong} \mathcal{H}_n^G(X,A)$$

is bijective for every $n \in \mathbb{Z}$.

Proof. The functor sending a directed systems of Λ-modules to its colimit is an exact functor and compatible with direct sums over arbitrary index maps. If (X, A) is a pair of G-CW-complexes, then $(\psi_i^* X, \psi_i^* A)$ is a pair of G_i-CW-complexes. Hence the collection of maps $\{t_n^G(X,A) \mid n \in \mathbb{Z}\}$ is a natural transformation of G-homology theories of pairs of G-CW-complexes which satisfy the disjoint union axiom. Hence in order to show that $t_n^G(X,A)$ is bijective for all $n \in \mathbb{Z}$ and all pairs of G-CW-complexes (X, A), it suffices by Lemma 2.2 to prove this in the special case $(X,A) = (G/H, \emptyset)$.

For $i \in I$ let $k_i : G_i/\psi_i^{-1}(H) \to \psi_i^*(G/H)$ be the G_i-map sending $g_i\psi_i^{-1}(H)$ to $\psi_i(g_i)H$. Consider a directed system of Λ-modules $\{\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \mid i \in I\}$ whose structure maps for $i, j \in I, i \leq j$ are given by the composite

$$\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \xrightarrow{\operatorname{ind}_{\phi_{i,j}}} \mathcal{H}_n^{G_j}(G_j \times_{\phi_{i,j}} G_i/\psi_i^{-1}(H)) \xrightarrow{\mathcal{H}_n^{G_j}(f_{i,j})} \mathcal{H}_n^{G_j}(G_j/\psi_j^{-1}(H))$$

for the G_j-map $f_{i,j} : G_j \times_{\phi_{i,j}} G_i/\psi_i^{-1}(H) \to G_j/\psi_j^{-1}(H)$ sending $(g_j, g_i\psi_i^{-1}(H))$

to $g_j\phi_{i,j}(g_i)\psi_j^{-1}(H)$. Then the following diagram commutes:

$$\begin{tikzcd}
\operatorname{colim}_{i\in I}\mathcal{H}_n^{\psi_i^{-1}(H)}(\{\bullet\}) \arrow[r, "\operatorname{colim}_{i\in I}\operatorname{ind}_{\psi_i^{-1}(H)}^{G_i}", "\cong"'] \arrow[dd, "t_n^H(\{\bullet\})"', "\cong"] & \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \arrow[d, "\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(k_i)"] \\
& \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*(G/H)) \arrow[d, "t_n^G(G/H)"] \\
\mathcal{H}_n^H(\{\bullet\}) \arrow[r, "\operatorname{ind}_H^G", "\cong"'] & \mathcal{H}_n^G(G/H),
\end{tikzcd}$$

where the horizontal maps are the isomorphism given by induction. For the directed system $\{\psi_i^{-1}(H) \mid i \in I\}$ with structure maps $\phi_{i,j}|_{\psi_i^{-1}(H)}\colon \psi_i^{-1}(H) \to \psi_j^{-1}(H)$, the group homomorphism $\operatorname{colim}_{i\in I}\psi_i|_{\psi_i^{-1}(H)}\colon \operatorname{colim}_{i\in I}\psi_i^{-1}(H) \to H$ is an isomorphism. This follows by inspecting the standard model for the colimit over a directed system of groups. Hence the left vertical arrow is bijective since $\mathcal{H}_*^?$ is strongly continuous by assumption. Therefore it remains to show that the map

$$\operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(k_i)\colon \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(G_i/\psi_i^{-1}(H)) \to \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H) \tag{3.5}$$

is surjective.

Notice that the map given by the direct sum of the structure maps

$$\bigoplus_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H) \to \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is surjective. Hence it remains to show for a fixed $i \in I$ that the image of the structure map

$$\mathcal{H}_n^{G_i}(\psi_i^*G/H) \to \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is contained in the image of the map (3.5).

We have the decomposition of the G_i-set ψ_i^*G/H into its G_i-orbits

$$\coprod_{G_i(gH)\in G_i\backslash(\psi_i^*G/H)} G_i/\psi_i^{-1}(gHg^{-1}) \xrightarrow{\cong} \psi_i^*G/H, \quad g_i\psi_i^{-1}(gHg^{-1}) \mapsto \psi_i(g_i)gH.$$

It induces an identification of Λ-modules

$$\bigoplus_{G_i(gH)\in G_i\backslash(\psi_i^*G/H)} \mathcal{H}_n^{G_i}\big(G_i/\psi_i^{-1}(gHg^{-1})\big) = \mathcal{H}_n^{G_i}(\psi_i^*G/H).$$

Hence it remains to show for fixed elements $i \in I$ and $G_i(gH) \in G_i\backslash(\psi_i^*G/H)$ that the obvious composition

$$\mathcal{H}_n^{G_i}\big(G_i/\psi_i^{-1}(gHg^{-1})\big) \subseteq \mathcal{H}_n^{G_i}(\psi_i^*G/H) \to \operatorname{colim}_{i\in I}\mathcal{H}_n^{G_i}(\psi_i^*G/H)$$

is contained in the image of the map (3.5).

Choose an index j with $j \geq i$ and $g \in \operatorname{im}(\psi_j)$. Then the structure map for $i \leq j$ is a map $\mathcal{H}_n^{G_i}(\psi_i^* G/H) \to \mathcal{H}_n^{G_j}(\psi_j^* G/H)$ which sends the summand corresponding to $G_i(gH) \in G_i \backslash (\psi_i^* G/H)$ to the summand corresponding to $G_j(1H) \in G_j \backslash (\psi_j^* G/H)$ which is by definition the image of

$$\mathcal{H}_n^{G_j}(k_j)\colon \mathcal{H}_n^{G_j}(G_j/\psi_j^{-1}(H)) \to \mathcal{H}_n^{G_j}(\psi_j^* G/H).$$

Obviously the image of composite of the last map with the structure map

$$\mathcal{H}_n^{G_j}(\psi_j^* G/H) \to \operatorname{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* G/H)$$

is contained in the image of the map (3.5). Hence the map (3.5) is surjective. This finishes the proof of Lemma 3.4. □

4 Isomorphism conjectures and colimits

A *family $\mathcal{F}$ of subgroups of* G is a collection of subgroups of G which is closed under conjugation and taking subgroups. Let $E_{\mathcal{F}}(G)$ be the *classifying space associated to* $\mathcal{F}$. It is uniquely characterized up to G-homotopy by the properties that it is a G-CW-complex and that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$. For more information about these spaces $E_{\mathcal{F}}(G)$ we refer for instance to the survey article [29]. Given a group homomorphism $\phi\colon K \to G$ and a family $\mathcal{F}$ of subgroups of G, define the family $\phi^*\mathcal{F}$ of subgroups of K by

$$\phi^*\mathcal{F} \quad = \quad \{H \subseteq K \mid \phi(H) \in \mathcal{F}\}. \tag{4.1}$$

If ϕ is an inclusion of subgroups, we also write $\mathcal{F}|_K$ instead of $\phi^*\mathcal{F}$.

Definition 4.2 (Isomorphism conjecture for $\mathcal{H}_*^?$). Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ.

A group (G, ξ) over Γ together with a family of subgroups $\mathcal{F}$ of G satisfies the *isomorphism conjecture* (*for* $\mathcal{H}_*^?$) if the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \to \{\bullet\}$ to the one-point-space $\{\bullet\}$ induces an isomorphism

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \xrightarrow{\cong} \mathcal{H}_n^G(\{\bullet\})$$

for all $n \in \mathbb{Z}$.

From now on fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ over Γ.

Theorem 4.3 (Transitivity principle). *Let (G, ξ) be a group over Γ. Let $\mathcal{F} \subseteq \mathcal{G}$ be families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group $(H, \xi|_H)$ over Γ satisfies the isomorphism conjecture for $\mathcal{F}|_H$.*

Then the up to G-homotopy unique map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$ induces an isomorphism $\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$ for all $n \in \mathbb{Z}$. In particular, (G, ξ) satisfies the isomorphism conjecture for $\mathcal{G}$ if and only if (G, ξ) satisfies the isomorphism conjecture for $\mathcal{F}$.

Proof. The proof is completely analogous to the one in [4, Theorem 2.4, Lemma 2.2], where only the case $\Gamma = \{1\}$ is treated. $\square$

Theorem 4.4. *Let (G, ξ) be a group over Γ. Let $\mathcal{F}$ be a family of subgroups of G.*

(i) *Let G be the directed union of subgroups $\{G_i \mid i \in I\}$. Suppose that $\mathcal{H}^?_*$ is continuous and for every $i \in I$ the isomorphism conjecture holds for $(G_i, \xi|_{G_i})$ and $\mathcal{F}|_{G_i}$.*

Then the isomorphism conjecture holds for (G, ξ) and $\mathcal{F}$.

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i : G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and for every $i \in I$ the isomorphism conjecture holds for $(G_i, \xi \circ \psi_i)$ and $\psi_i^*\mathcal{F}$.*

Then the isomorphism conjecture holds for (G, ξ) and $\mathcal{F}$.

Proof. (i) The proof is analogous to the one in [4, Proposition 3.4].

(ii) This follows from the following commutative square whose horizontal arrows are bijective because of Lemma 3.4 and the identification $\psi_i^* E_{\mathcal{F}}(G) = E_{\psi_i^*\mathcal{F}}(G_i)$:

$$\begin{tikzcd}
\operatorname{colim}_{i\in I} \mathcal{H}^{G_i}_n(E_{\psi_i^*\mathcal{F}}(G_i)) \arrow[r, "t^G_n(E_{\mathcal{F}}(G))", "\cong"'] \arrow[d] & \mathcal{H}^G_n(E_{\mathcal{F}}(G)) \arrow[d] \\
\operatorname{colim}_{i\in I} \mathcal{H}^{G_i}_n(\{\bullet\}) \arrow[r, "t^G_n(\{\bullet\})", "\cong"'] & \mathcal{H}^G_n(\{\bullet\}).
\end{tikzcd}$$
$\square$

Fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the class of finite groups or the class of virtually cyclic groups. For a group G let $\mathcal{C}(G)$ be the family of subgroups of G which belong to $\mathcal{C}$.

Theorem 4.5. *Let (G, ξ) be a group over Γ.*

(i) *Let G be the directed union $G = \bigcup_{i\in I} G_i$ of subgroups G_i Suppose that $\mathcal{H}^?_*$ is continuous and that the isomorphism conjecture is true for $(G_i, \xi|_{G_i})$ and $\mathcal{C}(G_i)$ for all $i \in I$.*

Then the isomorphism conjecture is true for (G, ξ) and $\mathcal{C}(G)$.

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i : G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and that the isomorphism conjecture is true for $(H, \mathcal{C}(H))$ for every $i \in I$ and every subgroup $H \subseteq G_i$.*

Then for every subgroup $K \subseteq G$ the isomorphism conjecture is true for $(K, \xi|_K)$ and $\mathcal{C}(K)$.

Proof. (i) This follows from Theorem 4.4 (i) since $\mathcal{C}(G_i) = \mathcal{C}(G)|_{G_i}$ holds for $i \in I$.

(ii) If G is the colimit of the directed system $\{G_i \mid i \in I\}$, then the subgroup $K \subseteq G$ is the colimit of the directed system $\{\psi_i^{-1}(K) \mid i \in I\}$. Hence we can assume $G = K$ without loss of generality.

Since $\mathcal{C}$ is closed under quotients by assumption, we have $\mathcal{C}(G_i) \subseteq \psi_i^*\mathcal{C}(G)$ for every $i \in I$. Hence we can consider for any $i \in I$ the composition

$$H_n^{G_i}(E_{\mathcal{C}(G_i)}(G_i)) \to H_n^{G_i}(E_{\psi_i^*\mathcal{C}(G)}(G_i)) \to H_n^{G_i}(\{\bullet\}).$$

Because of Theorem 4.4 (ii) it suffices to show that the second map is bijective. By assumption the composition of the two maps is bijective. Hence it remains to show that the first map is bijective. By Theorem 4.3 this follows from the assumption that the isomorphism conjecture holds for every subgroup $H \subseteq G_i$ and in particular for any $H \in \psi_i^*\mathcal{C}(G)$ for $\mathcal{C}(G_i)|_H = \mathcal{C}(H)$. □

5 Fibered isomorphism conjectures and colimits

In this section we also deal with the fibered version of the isomorphism conjectures. (This is not directly needed for the purpose of this paper and the reader may skip this section.) This is a stronger version of the Farrell–Jones conjecture. The Fibered Farrell–Jones conjecture does imply the Farrell–Jones conjecture and has better inheritance properties than the Farrell–Jones conjecture.

We generalize (and shorten the proof of) the result of Farrell–Linnell [16, Theorem 7.1] to a more general setting about equivariant homology theories as developed in Bartels–Lück [3].

Definition 5.1 (Fibered isomorphism conjecture for $\mathcal{H}_*^?$). Fix a group Γ and an equivariant homology theory $\mathcal{H}_*^?$ with values in Λ-modules over Γ. A group (G, ξ) over Γ together with a family of subgroups $\mathcal{F}$ of G satisfies the *fibered isomorphism conjecture* (*for* $\mathcal{H}_*^?$) if for each group homomorphism $\phi\colon K \to G$ the group $(K, \xi \circ \phi)$ over Γ satisfies the isomorphism conjecture with respect to the family $\phi^*\mathcal{F}$.

Theorem 5.2. *Let (G, ξ) be a group over Γ. Let $\mathcal{F}$ be a family of subgroups of G. Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i\in I} G_i$ and structure maps $\psi_i : G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and for every $i \in I$ the fibered isomorphism conjecture holds for $(G_i, \xi \circ \psi_i)$ and $\psi_i^*\mathcal{F}$.*

Then the fibered isomorphism conjecture holds for (G, ξ) and $\mathcal{F}$.

Proof. Let $\mu\colon K \to G$ be a group homomorphism. Consider the pullback of groups

$$\begin{tikzcd} K_i \arrow[r, "\mu_i"] \arrow[d, "\overline{\psi}_i"'] & G_i \arrow[d, "\psi_i"] \\ K \arrow[r, "\mu"] & G. \end{tikzcd}$$

Explicitly $K_i = \{(k, g_i) \in K \times G_i \mid \mu(k) = \psi_i(g_i)\}$. Let $\overline{\phi}_{i,j} \colon K_i \to K_j$ be the map induced by $\phi_{i,j} \colon G_i \to G_j$, id_K and id_G and the pullback property. One easily checks by inspecting the standard model for the colimit over a directed set that we obtain a directed system $\overline{\phi}_{i,j} \colon K_i \to K_j$ of groups indexed by the directed set I and the system of maps $\overline{\psi_i} \colon K_i \to K$ yields an isomorphism $\operatorname{colim}_{i \in I} K_i \xrightarrow{\cong} K$. The following diagram commutes:

$$\begin{tikzcd}
\operatorname{colim}_{i\in I} \mathcal{H}_n^{K_i}\left(\overline{\psi_i}^{\,*}\mu^* E_{\mathcal{F}}(G)\right) \arrow[r, "t_n^K(\mu^* E_{\mathcal{F}}(G))", "\cong"'] \arrow[d] & \mathcal{H}_n^K(\mu^* E_{\mathcal{F}}(G)) \arrow[d] \\
\operatorname{colim}_{i\in I} \mathcal{H}_n^{K_i}(\{\bullet\}) \arrow[r, "t_n^K(\{\bullet\})", "\cong"'] & \mathcal{H}_n^K(\{\bullet\}),
\end{tikzcd}$$

where the vertical arrows are induced by the obvious projections onto $\{\bullet\}$ and the horizontal maps are the isomorphisms from Lemma 3.4. Notice that $\overline{\psi_i}^{\,*}\mu^* E_{\mathcal{F}}(G)$ is a model for $E_{\overline{\psi_i}^{\,*}\mu^*\mathcal{F}}(K_i) = E_{\mu_i^*\psi_i^*\mathcal{F}}(K_i)$. Hence each map $\mathcal{H}_n^{K_i}\left(\overline{\psi_i}^{\,*}\mu^* E_{\mathcal{F}}(G)\right) \to \mathcal{H}_n^{K_i}(\{\bullet\})$ is bijective since $(G_i, \xi \circ \psi_i)$ satisfies the fibered isomorphism conjecture for $\psi_i^* \mathcal{F}$ and hence $(K_i, \xi \circ \psi_i \circ \mu_i)$ satisfies the isomorphism conjecture for $\mu_i^* \psi_i^* \mathcal{F}$. This implies that the left vertical arrow is bijective. Hence the right vertical arrow is an isomorphism. Since $\mu^* E_{\mathcal{F}}(G)$ is a model for $E_{\mu^*\mathcal{F}}(K)$, this means that $(K, \xi \circ \mu)$ satisfies the isomorphism conjecture for $\mu^*\mathcal{F}$. Since $\mu \colon K \to G$ is any group homomorphism, (G, ξ) satisfies the fibered isomorphism conjecture for $\mathcal{F}$. □

The proof of the following results are analogous to the one in [3, Lemma 1.6] and [4, Lemma 1.2], where only the case $\Gamma = \{1\}$ is treated.

Lemma 5.3. *Let (G, ξ) be a group over Γ and let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that (G, ξ) satisfies the fibered isomorphism conjecture for the family $\mathcal{F}$.*

Then (G, ξ) satisfies the fibered isomorphism conjecture for the family $\mathcal{G}$.

Lemma 5.4. *Let (G, ξ) be a group over Γ. Let $\phi \colon K \to G$ be a group homomorphism and let $\mathcal{F}$ be a family of subgroups of G. If (G, ξ) satisfies the fibered isomorphism conjecture for the family $\mathcal{F}$, then $(K, \xi \circ \phi)$ satisfies the fibered isomorphism conjecture for the family $\phi^* \mathcal{F}$.*

For the remainder of this section fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the families $\mathcal{F}$in or $\mathcal{VC}$yc.

Lemma 5.5. *Let (G, ξ) be a group over Γ. Suppose that the fibered isomorphism conjecture holds for (G, ξ) and $\mathcal{C}(G)$. Let $H \subseteq G$ be a subgroup.*

Then the fibered isomorphism conjecture holds for $(H, \xi|_H)$ and $\mathcal{C}(H)$.

Proof. This follows from Lemma 5.4 applied to the inclusion $H \to G$ since $\mathcal{C}(H) = \mathcal{C}(G)|_H$. □

Theorem 5.6. *Let (G, ξ) be a group over Γ.*

(i) *Let G be the directed union $G = \bigcup_{i \in I} G_i$ of subgroups G_i. Suppose that $\mathcal{H}_*^?$ is continuous and that the fibered isomorphism conjecture is true for $(G_i, \xi|_{G_i})$ and $\mathcal{C}(G_i)$ for all $i \in I$.*

Then the fibered isomorphism conjecture is true for (G, ξ) and $\mathcal{C}(G)$.

(ii) *Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i : G_i \to G$. Suppose that $\mathcal{H}_*^?$ is strongly continuous and that the fibered isomorphism conjecture is true for $(G_i, \xi \circ \psi_i)$ and $\mathcal{C}(G_i))$ for all $i \in I$.*

Then the fibered isomorphism conjecture is true for (G, ξ) and $\mathcal{C}(G)$.

Proof. (i) The proof is analogous to the one in [4, Proposition 3.4], where the case $\Gamma = \{1\}$ is considered.

(ii) Because $\mathcal{C}$ is closed under taking quotients we conclude $\mathcal{C}(G_i) \subseteq \psi_i^* \mathcal{C}(G)$. Now the claim follows from Theorem 5.2 and Lemma 5.3. □

Corollary 5.7. (i) *Suppose that $\mathcal{H}_*^?$ is continuous. Then the (fibered) isomorphism conjecture for (G, ξ) and $\mathcal{C}(G)$ is true for all groups (G, ξ) over Γ if and only if it is true for all such groups where G is a finitely generated group.*

(ii) *Suppose that $\mathcal{H}_*^?$ is strongly continuous. Then the fibered isomorphism conjecture for (G, ξ) and $\mathcal{C}(G)$ is true for all groups (G, ξ) over Γ if and only if it is true for all such groups where G is finitely presented.*

Proof. Let (G, ξ) be a group over Γ where G is finitely generated. Choose a finitely generated free group F together with an epimorphism $\psi : F \to G$. Let K be the kernel of ψ. Consider the directed system of finitely generated subgroups $\{K_i \mid i \in I\}$ of K. Let $\overline{K_i}$ be the smallest normal subgroup of K containing K_i. Explicitly $\overline{K_i}$ is given by elements which can be written as finite products of elements of the shape $f k_i f^{-1}$ for $f \in F$ and $k \in K_i$. We obtain a directed system of groups $\{F/\overline{K_i} \mid i \in I\}$, where for $i \leq j$ the structure map $\phi_{i,j} : F/\overline{K_i} \to F/\overline{K_j}$ is the canonical projection. If $\psi_i : F/\overline{K_i} \to F/K = G$ is the canonical projection, then the collection of maps $\{\psi_i \mid i \in I\}$ induces an isomorphism $\operatorname{colim}_{i \in I} F/\overline{K_i} \xrightarrow{\cong} G$. By construction for each $i \in I$ the group $F/\overline{K_i}$ is finitely presented and the fibered isomorphism conjecture holds for $(F/\overline{K_i}, \xi \circ \psi_i)$ and $\mathcal{C}(F/\overline{K_i})$ by assumption. Theorem 5.6 (ii) implies that the fibered Farrell–Jones conjecture for (G, ξ) and $\mathcal{C}(G)$ is true. □

6 Some equivariant homology theories

In this section we will describe the relevant homology theories over a group Γ and show that they are (strongly) continuous. (We have defined the notion of an equivariant homology theory over a group in Definition 2.3.)

6.1 Desired equivariant homology theories. We will need the following

Theorem 6.1 (Construction of equivariant homology theories). *Suppose that we are given a group Γ and a ring R (with involution) or a C^*-algebra A respectively on which Γ acts by structure preserving automorphisms. Then:*

(i) *Associated to these data there are equivariant homology theories with values in $\mathbb{Z}$-modules over the group Γ*

$$
\begin{aligned}
&H^?_*(-;\mathbf{K}_R)\\
&H^?_*(-;\mathbf{KH}_R)\\
&H^?_*(-;\mathbf{L}_R^{\langle -\infty\rangle}),\\
&H^?_*(-;\mathbf{K}^{\mathrm{top}}_{A,l^1}),\\
&H^?_*(-;\mathbf{K}^{\mathrm{top}}_{A,r}),\\
&H^?_*(-;\mathbf{K}^{\mathrm{top}}_{A,m}),
\end{aligned}
$$

where in the case $H^?_*(-;\mathbf{K}^{\mathrm{top}}_{A,r})$ we will have to impose the restriction to the induction structure that a homomorphisms $\alpha\colon (H,\xi)\to(G,\mu)$ over Γ induces a transformation $\mathrm{ind}_\alpha\colon \mathcal{H}^H_n(X,X_0)\to\mathcal{H}^G_n(\alpha_*(X,X_0))$ *only if the kernel of the underlying group homomorphism $\alpha\colon H\to G$ acts with amenable isotropy on $X\setminus X_0$.*

(ii) *If (G,μ) is a group over Γ and $H\subseteq G$ is a subgroup, then there are for every $n\in\mathbb{Z}$ identifications*

$$
\begin{aligned}
H^H_n(\{\bullet\};\mathbf{K}_R)&\cong H^G_n(G/H;\mathbf{K}_R)\cong K_n(R\rtimes H);\\
H^H_n(\{\bullet\};\mathbf{KH}_R)&\cong H^G_n(G/H;\mathbf{KH}_R)\cong KH_n(R\rtimes H);\\
H^H_n(\{\bullet\};\mathbf{L}_R^{\langle -\infty\rangle})&\cong H^G_n(G/H;\mathbf{L}_R^{\langle -\infty\rangle})\cong L_n^{\langle -\infty\rangle}(R\rtimes H);\\
H^H_n(\{\bullet\};\mathbf{K}^{\mathrm{top}}_{A,l^1})&\cong H^G_n(G/H;\mathbf{K}^{\mathrm{top}}_{A,l^1})\cong K_n(A\rtimes_{l^1} H);\\
H^H_n(\{\bullet\};\mathbf{K}^{\mathrm{top}}_{A,r})&\cong H^G_n(G/H;\mathbf{K}^{\mathrm{top}}_{A,r})\cong K_n(A\rtimes_r H);\\
H^H_n(\{\bullet\};\mathbf{K}^{\mathrm{top}}_{A,m})&\cong H^G_n(G/H;\mathbf{K}^{\mathrm{top}}_{A,r})\cong K_n(A\rtimes_m H).
\end{aligned}
$$

Here H and G act on R and A respectively via the given Γ-action, $\mu\colon G\to\Gamma$ and the inclusion $H\subseteq G$, $K_n(R\rtimes H)$ is the algebraic K-theory of the twisted group ring $R\rtimes H$, $KH_n(R\rtimes H)$ is the homotopy K-theory of the twisted group ring $R\rtimes H$, $L_n^{\langle -\infty\rangle}(R\rtimes H)$ is the algebraic L-theory with decoration $\langle -\infty\rangle$ of the twisted group ring with involution $R\rtimes H$, $K_n(A\rtimes_{l^1} H)$ is the topological K-theory of the crossed product Banach algebra $A\rtimes_{l^1} H$, $K_n(A\rtimes_r H)$ is the topological K-theory of the reduced crossed product C^*-algebra $A\rtimes_r H$, and $K_n(A\rtimes_m H)$ is the topological K-theory of the maximal crossed product C^*-algebra $A\rtimes_m H$;

(iii) *Let* $\zeta\colon \Gamma_0 \to \Gamma_1$ *be a group homomorphism. Let* R *be a ring* (*with involution*) *and* A *be a* C^**-algebra on which* Γ_1 *acts by structure preserving automorphisms. Let* (G,μ) *be a group over* Γ_0. *Then in all cases the evaluation at* (G,μ) *of the equivariant homology theory over* Γ_0 *associated to* ζ^*R *or* ζ^*A *respectively agrees with the evaluation at* $(G,\zeta\circ\mu)$ *of the equivariant homology theory over* Γ_1 *associated to* R *or* A *respectively.*

(iv) *Suppose the group* Γ *acts on the rings* (*with involution*) R *and* S *or on the* C^**-algebras* A *and* B *respectively by structure preserving automorphisms. Let* $\xi\colon R \to S$ *or* $\xi\colon A \to B$ *be a* Γ*-equivariant homomorphism of rings* (*with involution*) *or* C^**-algebras respectively. Then* ξ *induces natural transformations of homology theories over* Γ

$$\xi_*^?\colon H_*^?(-;\mathbf{K}_R) \to H_*^?(-;\mathbf{K}_S);$$

$$\xi_*^?\colon H_*^?(-;\mathbf{KH}_R) \to H_*^?(-;\mathbf{KH}_S);$$

$$\xi_*^?\colon H_*^?(-;\mathbf{L}_R^{\langle-\infty\rangle}) \to H_*^?(-;\mathbf{L}_S^{\langle-\infty\rangle});$$

$$\xi_*^?\colon H_*^?(-;\mathbf{K}_{A,l^1}^{\mathrm{top}}) \to H_*^?(-;\mathbf{K}_{B,l^1}^{\mathrm{top}});$$

$$\xi_*^?\colon H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}}) \to H_*^?(-;\mathbf{K}_{B,r}^{\mathrm{top}});$$

$$\xi_*^?\colon H_*^?(-;\mathbf{K}_{A,m}^{\mathrm{top}}) \to H_*^?(-;\mathbf{K}_{B,m}^{\mathrm{top}}).$$

They are compatible with the identifications appearing in assertion (ii).

(v) *Let* Γ *act on the* C^**-algebra* A *by structure preserving automorphisms. We can consider* A *also as a ring with structure preserving* G*-action. Then there are natural transformations of equivariant homology theories with values in* $\mathbb{Z}$*-modules over* Γ

$$\begin{aligned} H_*^?(-;\mathbf{K}_A) \to H_*^?(-;\mathbf{KH}_A) \to H_*^?(-;\mathbf{K}_{A,l^1}^{\mathrm{top}}) \\ \to H_*^?(-;\mathbf{K}_{A,m}^{\mathrm{top}}) \to H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}}). \end{aligned}$$

They are compatible with the identifications appearing in assertion (ii).

6.2 (Strong) continuity. Next we want to show

Lemma 6.2. *Suppose that we are given a group* Γ *and a ring* R (*with involution*) *or a* C^**-algebra* A *respectively on which* G *acts by structure preserving automorphisms.*

Then the homology theories with values in $\mathbb{Z}$*-modules over* Γ

$$H_*^?(-;\mathbf{K}_R),\ H_*^?(-;\mathbf{KH}_R),\quad H_*^?(-;\mathbf{L}_R^{\langle-\infty\rangle}),\quad H_*^?(-;\mathbf{K}_{A,l^1}^{\mathrm{top}}),\quad \textit{and}\quad H_*^?(-;\mathbf{K}_{A,m}^{\mathrm{top}})$$

(*see Theorem* 6.1) *are strongly continuous in the sense of Definition* 3.3, *whereas*

$$H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}})$$

is only continuous.

Proof. We begin with $H^?_*(-;\mathbf{K}_R)$ and $H^?_*(-;\mathbf{KH}_R)$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ together with a map $\mu\colon G \to \Gamma$ that the canonical maps

$$\operatorname{colim}_{i\in I} K_n(R \rtimes G_i) \to K_n(R \rtimes G);$$
$$\operatorname{colim}_{i\in I} KH_n(R \rtimes G_i) \to KH_n(R \rtimes G),$$

are bijective for all $n \in \mathbb{Z}$. Obviously $R \rtimes G$ is the colimit of rings $\operatorname{colim}_{i\in I} R \rtimes G_i$. Now the claim follows for $K_n(R \rtimes G)$ for $n \geq 0$ from [36, (12) on page 20].

Using the Bass–Heller–Swan decomposition one gets the results for $K_n(R \rtimes G)$ for all $n \in \mathbb{Z}$ and that the map

$$\operatorname{colim}_{i\in I} N^pK_n(R \rtimes G_i) \to N^pK_n(R \rtimes G)$$

is bijective for all $n \in \mathbb{Z}$ and all $p \in \mathbb{Z}$, $p \geq 1$ for the Nil-groups $N^pK_n(RG)$ defined by Bass [8, XII]. Now the claim for homotopy K-theory follows from the spectral sequence due to Weibel [41, Theorem 1.3].

Next we treat $H^?_*(-;\mathbf{L}_R^{\langle -\infty\rangle})$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i\in I} G_i$ together with a map $\mu\colon G \to \Gamma$ that the canonical map

$$\operatorname{colim}_{i\in I} L_n^{\langle -\infty\rangle}(R \rtimes G_i) \to L_n^{\langle -\infty\rangle}(R \times G)$$

is bijective for all $n \in \mathbb{Z}$. Recall from [37, Definition 17.1 and Definition 17.7] that

$$\begin{aligned} L_n^{\langle -\infty\rangle}(R \rtimes G) &= \operatorname{colim}_{m\to\infty} L_n^{\langle -m\rangle}(R \rtimes G);\\ L_n^{\langle -m\rangle}(R \rtimes G) &= \operatorname{coker}\left(L_{n+1}^{\langle -m+1\rangle}(R \rtimes G) \to L_{n+1}^{\langle -m+1\rangle}(R \rtimes G[\mathbb{Z}])\right) \quad \text{for } m \geq 0. \end{aligned}$$

Since $L_n^{\langle 1\rangle}(R \rtimes G)$ is $L_n^h(R \rtimes G)$, it suffices to show that

$$\omega_n\colon \operatorname{colim}_{i\in I} L_n^h(R \rtimes G_i) \to L_n^h(R \rtimes G)$$

is bijective for all $n \in \mathbb{Z}$. We give the proof of surjectivity for $n = 0$ only, the proofs of injectivity for $n = 0$ and of bijectivity for the other values of n are similar.

The ring $R \rtimes G$ is the colimit of rings $\operatorname{colim}_{i\in I} R \rtimes G_i$. Let $\psi_i\colon R \rtimes G_i \to R \rtimes G$ and $\phi_{i,j}\colon R \rtimes G_i \to R \rtimes G_j$ for $i, j \in I, i \leq j$ be the structure maps. One can define $R \rtimes G$ as the quotient of $\coprod_{i\in I} R \rtimes G_i/\sim$, where $x \in R \rtimes G_i$ and $y \in R \rtimes G_j$ satisfy $x \sim y$ if and only if $\phi_{i,k}(x) = \phi_{j,k}(y)$ holds for some $k \in I$ with $i, j \leq k$. The addition and multiplication is given by adding and multiplying representatives belonging to the same $R \rtimes G_i$. Let $M(m,n; R \rtimes G)$ be the set of (m,n)-matrices with entries in $R \rtimes G$. Given $A_i \in M(m,n; R \rtimes G_i)$, define $\phi_{i,j}(A_i) \in M(m,n; R \rtimes G_j)$ and $\psi_i(A_i) \in M(m,n; R \rtimes G)$ by applying $\phi_{i,j}$ and ψ_i to each entry of the matrix A_i. We need the following key properties which follow directly from inspecting the model for the colimit above:

(i) Given $A \in M(m,n; R \rtimes G)$, there exists $i \in I$ and $A_i \in M(m,n; R \rtimes G_i)$ with $\psi_i(A_i) = A$.

(ii) Given $A_i \in M(m,n; R \rtimes G_i)$ and $A_j \in M(m,n; R \rtimes G_j)$ with $\psi_i(A_i) = \psi_j(A_j)$, there exists $k \in I$ with $i, j \leq k$ and $\phi_{i,k}(A_i) = \phi_{j,k}(A_j)$.

An element $[A]$ in $L_0^h(R \rtimes G)$ is represented by a quadratic form on a finitely generated free $R \rtimes G$-module, i.e., a matrix $A \in GL_n(R \rtimes G)$ for which there exists a matrix $B \in M(n,n; R \rtimes G)$ with $A = B + B^*$, where B^* is given by transposing the matrix B and applying the involution of R elementwise. Fix such a choice of a matrix B. Choose $i \in I$ and $B_i \in M(n,n; R \rtimes G_i)$ with $\psi_i(B_i) = B$. Then $\psi_i(B_i + B_i^*) = A$ is invertible. Hence we can find $j \in I$ with $i \leq j$ such that $A_j := \phi_{i,j}(B_i + B_j^*)$ is invertible. Put $B_j = \phi_{i,j}(B_i)$. Then $A_j = B_j + B_j^*$ and $\psi_j(A_j) = A$. Hence A_j defines an element $[A_j] \in L_n^h(R \rtimes G_j)$ which is mapped to $[A]$ under the homomorphism $L_n^h(R \rtimes G_j) \to L_n(R \rtimes G)$ induced by ψ_j. Hence the map ω_0 of (6.2) is surjective.

Next we deal with $H_*^?(-; \mathbf{K}_{A,l^1}^{\mathrm{top}})$. We have to show for every directed systems of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ together with a map $\mu\colon G \to \Gamma$ that the canonical map

$$\operatorname{colim}_{i \in I} K_n(A \rtimes_{l^1} G_i) \to K_n(A \rtimes_{l^1} G)$$

is bijective for all $n \in \mathbb{Z}$. Since topological K-theory is a continuous functor, it suffices to show that the colimit (or sometimes also called inductive limit) of the system of Banach algebras $\{A \rtimes_{l^1} G_i \mid i \in I\}$ in the category of Banach algebras with norm decreasing homomorphisms is $A \rtimes_{l^1} G$. So we have to show that for any Banach algebra B and any system of (norm deceasing) homomorphisms of Banach algebras $\alpha_i\colon A \rtimes_{l^1} G_i \to B$ compatible with the structure maps $A \rtimes_{l^1} \phi_{i,j}\colon A \rtimes_{l^1} G_i \to A \rtimes_{l^1} G_j$ there exists precisely one homomorphism of Banach algebras $\alpha\colon A \rtimes_{l^1} G \to B$ with the property that its composition with the structure map $A \rtimes_{l^1} \psi_i\colon A \rtimes_{l^1} G_i \to A \rtimes_{l^1} G$ is α_i for $i \in I$.

It is easy to see that in the category of $\mathbb{C}$-algebras the colimit of the system $\{A \rtimes G_i \mid i \in I\}$ is $A \rtimes G$ with structure maps $A \rtimes \psi_i\colon A \rtimes G_i \to A \rtimes G$. Hence the restrictions of the homomorphisms α_i to the subalgebras $A \rtimes G_i$ yields a homomorphism of central $\mathbb{C}$-algebras $\alpha'\colon A \rtimes G \to B$ uniquely determined by the property that the composition of α' with the structure map $A \rtimes \psi_i\colon A \rtimes G_i \to A \rtimes G$ is $\alpha_i|_{A \rtimes G_i}$ for $i \in I$. If α exists, its restriction to the dense subalgebra $A \rtimes G$ has to be α'. Hence α is unique if it exists. Of course we want to define α to be the extension of α' to the completion $A \rtimes_{l^1} G$ of $A \rtimes G$ with respect to the l^1-norm. So it remains to show that $\alpha'\colon A \rtimes G \to B$ is norm decreasing. Consider an element $u \in A \rtimes G$ which is given by a finite formal sum $u = \sum_{g \in F} a_g \cdot g$, where $F \subset G$ is some finite subset of G and $a_g \in A$ for $g \in F$. We can choose an index $j \in I$ and a finite set $F' \subset G_j$ such that $\psi_j|_{F'}\colon F' \to F$ is one-to-one. For $g \in F$ let $g' \in F'$ denote the inverse image of g under this map. Consider the element $v = \sum_{g' \in F'} a_g \cdot g'$ in $A \rtimes G_j$. By construction we have $A \rtimes \psi_j(v) = u$ and $\|v\| = \|u\| = \sum_{i=1}^n \|a_i\|$. We conclude

$$\|\alpha'(u)\| = \|\alpha' \circ (A \rtimes \psi_j)(v)\| = \|\alpha_j(v)\| \leq \|v\| = \|u\|.$$

The proof for $H_*^?(-; \mathbf{K}_{A,m}^{\mathrm{top}})$ follows similarly, using the fact that by definition of the norm on $A \rtimes_m G$ every $*$-homomorphism of $A \rtimes G$ into a C^*-algebra B extends uniquely to $A \rtimes_m G$. The proof for the continuity of $H_*^?(-;\mathbf{K}_{A,r}^{\mathrm{top}})$ follows from Theorem 4.1 in [10]. □

Notice that we have proved all promised results of the introduction as soon as we have completed the proof of Theorem 6.1 which we have used as a black box so far.

7 From spectra over groupoids to equivariant homology theories

In this section we explain how one can construct equivariant homology theories from spectra over groupoids.

A *spectrum* $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps* $\sigma(n)\colon E(n) \wedge S^1 \to E(n+1)$. A (*strong*) *map* of spectra (sometimes also called function in the literature) $\mathbf{f}\colon \mathbf{E} \to \mathbf{E}'$ is a sequence of maps $f(n)\colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ for all $n \in \mathbb{Z}$. This should not be confused with the notion of a map of spectra in the stable category (see [1, III.2.]). Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) := \operatorname{colim}_{k\to\infty} \pi_{i+k}(E(k)),$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)_*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism and the homomorphism induced by the structure map. We denote by $\mathsf{Spectra}$ the category of spectra.

A *weak equivalence* of spectra is a map $\mathbf{f}\colon \mathbf{E} \to \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a small groupoid $\mathcal{G}$, denote by $\mathsf{Groupoids} \downarrow \mathcal{G}$ the category of small groupoids over $\mathcal{G}$, i.e., an object is a functor $F_0\colon \mathcal{G}_0 \to \mathcal{G}$ with a small groupoid as source and a morphism from $F_0\colon \mathcal{G}_0 \to \mathcal{G}$ to $F_1\colon \mathcal{G}_1 \to \mathcal{G}$ is a functor $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ satisfying $F_1 \circ F = F_0$. We will consider a group Γ as a groupoid with one object and Γ as set of morphisms. An *equivalence* $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ of groupoids is a functor of groupoids F for which there exists a functor of groupoids $F'\colon \mathcal{G}_1 \to \mathcal{G}_0$ such that $F' \circ F$ and $F \circ F'$ are naturally equivalent to the identity functor. A functor $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ of small groupoids is an equivalence of groupoids if and only if it induces a bijection between the isomorphism classes of objects and for any object $x \in \mathcal{G}_0$ the map $\mathrm{aut}_{\mathcal{G}_0}(x) \to \mathrm{aut}_{\mathcal{G}_1}(F(x))$ induced by F is an isomorphism of groups.

Lemma 7.1. *Let Γ be a group. Consider a covariant functor*

$$\mathbf{E}\colon \mathsf{Groupoids} \downarrow \Gamma \to \mathsf{Spectra}$$

which sends equivalences of groupoids to weak equivalences of spectra.

Then we can associate to it an equivariant homology theory $\mathcal{H}_*^?(,-;\mathbf{E})$ (*with values in* $\mathbb{Z}$*-modules*) *over* Γ *such that for every group* (G,μ) *over* Γ *and subgroup* $H \subseteq G$ *we have a natural identification*

$$\mathcal{H}_n^H(\{\bullet\};\mathbf{E}) = \mathcal{H}_n^G(G/H,\mathbf{E}) = \pi_n(\mathbf{E}(H)).$$

If $\mathbf{T}\colon \mathbf{E} \to \mathbf{F}$ *is a natural transformation of such functors* $\mathsf{Groupoids} \downarrow \Gamma \to \mathsf{Spectra}$*, then it induces a transformation of equivariant homology theories over* Γ

$$\mathcal{H}_*^?(-;\mathbf{T})\colon \mathcal{H}_*^?(,-;\mathbf{E}) \to \mathcal{H}_*^?(,-;\mathbf{F})$$

such that for every group (G,μ) *over* Γ *and subgroup* $H \subseteq G$ *the homomorphism* $\mathcal{H}_n^H(\{\bullet\};\mathbf{T})\colon \mathcal{H}_n^H(\{\bullet\};\mathbf{E}) \to \mathcal{H}_n^H(\{\bullet\};\mathbf{F})$ *agrees under the identification above with* $\pi_n(\mathbf{T}(H))\colon \pi_n(\mathbf{E}(H)) \to \pi_n(\mathbf{F}(H))$.

Proof. We begin with explaining how we can associate to a group (G,μ) over Γ a G-homology theory $\mathcal{H}_*^G(-;\mathbf{E})$ with the property that for every subgroup $H \subseteq G$ we have an identification

$$\mathcal{H}_n^G(G/H,\mathbf{E}) = \pi_n(\mathbf{E}(H)).$$

We just follow the construction in [13, Section 4]. Let $\mathsf{Or}(G)$ be the *orbit category* of G, i.e., objects are homogenous spaces G/H and morphisms are G-maps. Given a G-set S, the associated *transport groupoid* $t^G(S)$ has S as set of objects and the set of morphisms from $s_0 \in S$ to $s_1 \in S$ consists of the subset $\{g \in G \mid gs_1 = s_2\}$ of G. Composition is given by the group multiplication. A G-map of sets induces a functor between the associated transport groupoids in the obvious way. In particular the projection $G/H \to G/G$ induces a functor of groupoids $\mathrm{pr}_S\colon t^G(S) \to t^G(G/G) = G$. Thus $t^G(S)$ becomes an object in $\mathsf{Groupoids} \downarrow \Gamma$ by the composite $\mu \circ \mathrm{pr}_S$. We obtain a covariant functor $t^G\colon \mathsf{Or}(G) \to \mathsf{Groupoids} \downarrow \Gamma$. Its composition with the given functor $\mathbf{E}$ yields a covariant functor

$$\mathbf{E}^G := \mathbf{E} \circ t^G\colon \mathsf{Or}(G) \to \mathsf{Spectra}.$$

Now define

$$\mathcal{H}_*^G(X,A;\mathbf{E}) := \mathcal{H}_*^G(-;\mathbf{E}^G),$$

where $\mathcal{H}_*^G(-;\mathbf{E}^G)$ is the G-homology theory which is associated to $\mathbf{E}^G\colon \mathsf{Or}(G) \to \mathsf{Spectra}$ and defined in [13, Section 4 and 7]. Namely, if X is a G-CW-complex, we can assign to it a contravariant functor $\mathrm{map}_G(G/?,X)\colon \mathsf{Or}(G) \to \mathsf{Spaces}$ sending G/H to $\mathrm{map}_G(G/H,X) = X^H$ and put $\mathcal{H}_n^G(X;\mathbf{E}^G) := \pi_n(\mathrm{map}_G(G/?,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G)$ for the spectrum $\mathrm{map}_G(G/?,X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G$ (which is denoted in [13] by $\mathrm{map}_G(G/?,X)_+ \otimes_{\mathsf{Or}(G)} \mathbf{E}^G$).

Next we have to explain the induction structure. Consider a group homomorphism $\alpha\colon (H,\xi) \to (G,\mu)$ of groups over Γ and an H-CW-complex X. We have to construct a homomorphism

$$\mathcal{H}_n^H(X;\mathbf{E}) \to \mathcal{H}_n^G(\alpha_* X;\mathbf{E}).$$

This will be done by constructing a map of spectra

$$\operatorname{map}_H(H/?, X)_+ \wedge_{\mathsf{Or}(H)} \mathbf{E}^H \to \operatorname{map}_G(G/?, \alpha_* X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G.$$

We follow the constructions in [13, Section 1]. The homomorphism α induces a covariant functor $\mathsf{Or}(\alpha)\colon \mathsf{Or}(H) \to \mathsf{Or}(G)$ by sending H/L to $\alpha_*(H/L) = G/\alpha(L)$. Given a contravariant functor $Y\colon \mathsf{Or}(H) \to \mathsf{Spaces}$, we can assign to it its induction with $\mathsf{Or}(\alpha)$ which is a contravariant functor $\alpha_* Y\colon \mathsf{Or}(G) \to \mathsf{Spaces}$. Given a contravariant functor $Z\colon \mathsf{Or}(G) \to \mathsf{Spaces}$, we can assign to it its restriction which is the contravariant functor $\alpha^* Z := Z \circ \mathsf{Or}(\alpha)\colon \mathsf{Or}(G) \to \mathsf{Spaces}$. Induction α_* and α^* form an adjoint pair. Given an H-CW-complex X, there is a natural identification $\alpha_*(\operatorname{map}_H(H/?, X)) = \operatorname{map}_G(G/?, \alpha_* X)$. Using [13, Lemma 1.9] we get for an H-CW-complex X a natural map of spectra

$$\operatorname{map}_H(H/?, X)_+ \wedge_{\mathsf{Or}(H)} \alpha^*\mathbf{E}^G \to \operatorname{map}_G(G/?, \alpha_* X)_+ \wedge_{\mathsf{Or}(G)} \mathbf{E}^G.$$

Given an H-set S, we obtain a functor of groupoids $t^H(S) \to t^G(\alpha_* S)$ sending $s \in S$ to $(1, s) \in G \times_\alpha S$ and a morphism in $t^H(S)$ given by a group element h to the one in $t^G(\alpha_* S)$ given by $\alpha(h)$. This yields a natural transformation of covariant functors $\mathsf{Or}(H) \to \mathsf{Groupoids} \downarrow \Gamma$ from $t^H \to t^G \circ \mathsf{Or}(\alpha)$. Composing with the functor $\mathbf{E}$ gives a natural transformation of covariant functors $\mathsf{Or}(H) \to \mathsf{Spectra}$ from $\mathbf{E}^H$ to $\alpha^*\mathbf{E}^G$. It induces a map of spectra

$$\operatorname{map}_H(H/?, X)_+ \wedge_{\mathsf{Or}(H)} \mathbf{E}^H \to \operatorname{map}_H(H/?, X)_+ \wedge_{\mathsf{Or}(H)} \alpha^*\mathbf{E}^G.$$

Its composition with the maps of spectra constructed beforehand yields the desired map of spectra $\operatorname{map}_H(H/?, X)_+ \otimes_{\mathsf{Or}(H)} \mathbf{E}^H \to \operatorname{map}_G(G/?, \alpha_* X)_+ \otimes_{\mathsf{Or}(G)} \mathbf{E}^G$.

We omit the straightforward proof that the axioms of an induction structure are satisfied. This finishes the proof of Theorem 6.1.

The statement about the natural transformation $\mathbf{T}\colon \mathbf{E} \to \mathbf{F}$ is obvious. □

8 Some K-theory spectra associated to groupoids

The last step in completing the proof of Theorem 6.1 is to prove the following Theorem 8.1 because then we can apply it in combination with Lemma 7.1. (Actually we only need the version of Theorem 8.1, where $\mathcal{G}$ is given by a group Γ.) Let $\mathsf{Groupoids}^{\mathsf{finker}} \downarrow \mathcal{G}$ be the subcategory of $\mathsf{Groupoids} \downarrow \mathcal{G}$ which has the same objects and for which a morphism from $F_0\colon \mathcal{G}_0 \to \mathcal{G}$ to $F_1\colon \mathcal{G}_1 \to \mathcal{G}$ given by a functor $F\colon \mathcal{G}_0 \to \mathcal{G}_1$ satisfying $F_1 \circ F = F_0$ has the property that for every object $x \in \mathcal{G}_0$ the group homomorphism $\operatorname{aut}_{\mathcal{G}_0}(x) \to \operatorname{aut}_{\mathcal{G}_1}(F(x))$ induced by F has a finite kernel. Denote by Rings, $*$-Rings, and C^*-$C^*\mathsf{Algebras}$ the categories of rings, rings with involution and C^*-algebras.

Theorem 8.1. *Let $\mathcal{G}$ be a fixed groupoid. Let $R\colon \mathcal{G} \to \mathsf{Rings}$, $R\colon \mathcal{G} \to *\text{-}\mathsf{Rings}$, or $A\colon \mathcal{G} \to C^*\mathsf{Algebras}$ respectively be a covariant functor. Then there exist covariant* *functors*

$$\begin{aligned}
\mathbf{K}_R &\colon \mathsf{Groupoids} \downarrow \mathcal{G} \to \mathsf{Spectra};\\
\mathbf{KH}_R &\colon \mathsf{Groupoids} \downarrow \mathcal{G} \to \mathsf{Spectra};\\
\mathbf{L}_R^{\langle -\infty\rangle} &\colon \mathsf{Groupoids} \downarrow \mathcal{G} \to \mathsf{Spectra};\\
\mathbf{K}^{\mathrm{top}}_{A,l^1} &\colon \mathsf{Groupoids} \downarrow \mathcal{G} \to \mathsf{Spectra};\\
\mathbf{K}^{\mathrm{top}}_{A,r} &\colon \mathsf{Groupoids}^{\mathsf{finker}} \downarrow \mathcal{G} \to \mathsf{Spectra};\\
\mathbf{K}^{\mathrm{top}}_{A,m} &\colon \mathsf{Groupoids} \downarrow \mathcal{G} \to \mathsf{Spectra},
\end{aligned}$$

together with natural transformations

$$\begin{aligned}
\mathbf{I}_1 &\colon \mathbf{K} \to \mathbf{KH};\\
\mathbf{I}_2 &\colon \mathbf{KH} \to \mathbf{K}^{\mathrm{top}}_{A,l^1};\\
\mathbf{I}_3 &\colon \mathbf{K}^{\mathrm{top}}_{A,l^1} \to \mathbf{K}^{\mathrm{top}}_{A,m};\\
\mathbf{I}_4 &\colon \mathbf{K}^{\mathrm{top}}_{A,m} \to \mathbf{K}^{\mathrm{top}}_{A,r},
\end{aligned}$$

of functors from Groupoids $\downarrow \mathcal{G}$ *or* Groupoids$^{\mathsf{finker}}$ $\downarrow \mathcal{G}$ *respectively to* Spectra *such that the following holds:*

(i) *Let $F_i \colon \mathcal{G}_i \to \mathcal{G}$ be objects for $i = 0,1$ and $F \colon F_0 \to F_1$ be a morphism between them in* Groupoids $\downarrow \mathcal{G}$ *or* Groupoids$^{\mathsf{finker}}$ $\downarrow \mathcal{G}$ *respectively such that the underlying functor of groupoids $F \colon \mathcal{G}_0 \to \mathcal{G}_1$ is an equivalence of groupoids. Then the functors send F to a weak equivalences of spectra.*

(ii) *Let $F_0 \colon \mathcal{G}_0 \to \mathcal{G}$ be an object in* Groupoids $\downarrow \mathcal{G}$ *or* Groupoids$^{\mathsf{finker}}$ $\downarrow \mathcal{G}$ *respectively such that the underlying groupoid $\mathcal{G}_0$ has only one object x. Let $G = \mathrm{mor}_{\mathcal{G}_0}(x,x)$ be its automorphisms group. We obtain a ring $R(y)$, a ring $R(y)$ with involution, or a C^*-algebra $B(y)$ with G-operation by structure preserving maps from the evaluation of the functor R or A respectively at $y = F(x)$. Then:*

$$\begin{aligned}
\pi_n(\mathbf{K}_R(F)) &= K_n(R(y) \rtimes G);\\
\pi_n(\mathbf{KH}_R(F)) &= KH_n(R(y) \rtimes G);\\
\pi_n(\mathbf{L}_R^{\langle -\infty\rangle}(F)) &= L_n^{\langle -\infty\rangle}(R(y) \rtimes G);\\
\pi_n(\mathbf{K}^{\mathrm{top}}_{A(y),l^1}(F)) &= K_n(A(y) \rtimes_{l^1} G);\\
\pi_n(\mathbf{K}^{\mathrm{top}}_{A(y),r}(F)) &= K_n(A(y) \rtimes_r G);\\
\pi_n(\mathbf{K}^{\mathrm{top}}_{A(y),m}(F)) &= K_n(A(y) \rtimes_m G),
\end{aligned}$$

where $K_n(R(y) \rtimes G)$ is the algebraic K-theory of the twisted group ring $R(y) \rtimes G$, $KH_n(R(y) \rtimes G)$ is the homotopy K-theory of the twisted group ring $R(y) \rtimes G$,

$L_n^{\langle -\infty \rangle}(R(y) \rtimes G)$ is the algebraic L-theory with decoration $\langle -\infty \rangle$ of the twisted group ring with involution $R(y) \rtimes G$, $K_n(A(y) \rtimes_{l^1} G)$ is the topological K-theory of the crossed product Banach algebra $A(y) \rtimes_{l^1} G$, $K_n(A(y) \rtimes_r G)$ is the topological K-theory of the reduced crossed product C^*-algebra $A(y) \rtimes_r G$, and $K_n(A(y) \rtimes_m G)$ is the topological K-theory of the maximal crossed product C^*-algebra $A(y) \rtimes_m G$.

The natural transformations $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_4$ become under this identifications the obvious change of rings and theory homomorphisms.

(iii) *These constructions are in the obvious sense natural in R and A respectively and in $\mathcal{G}$.*

We defer the details of the proof of Theorem 8.1 in [2]. Its proof requires some work but there are many special cases which have already been taken care of. If we would not insist on groupoids but only on groups as input, these are the standard algebraic K- and L-theory spectra or topological K-theory spectra associated to group rings, group Banach algebras and group C^*-algebras. The construction for the algebraic K- and L-theory and the topological K-theory in the case, where G acts trivially on a ring R or a C^*-algebra are already carried out or can easily be derived from [4], [13], and [24] except for the case of a Banach algebra. The case of the K-theory spectrum associated to an additive category with G-action has already been carried out in [7]. The main work which remains to do is to treat the Banach case and to construct the relevant natural transformation from $\mathbf{KH}$ to $\mathbf{K}^{\mathrm{top}}_{A,l^1}$

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Coarse and equivariant co-assembly maps

Heath Emerson and Ralf Meyer*

1 Introduction

This is a sequel to the articles [5], [7], which deal with a coarse co-assembly map that is dual to the usual coarse assembly map. Here we study an equivariant co-assembly map that is dual to the Baum–Connes assembly map for a group G.

A rather obvious choice for such a dual map is the map

$$p_{\mathcal{E}G}^*\colon \mathrm{KK}_*^G(\mathbb{C},\mathbb{C}) \to \mathrm{RKK}_*^G(\mathcal{E}G;\mathbb{C},\mathbb{C}) \tag{1}$$

induced by the projection $p_{\mathcal{E}G}\colon \mathcal{E}G \to \text{point}$. This map and its application to the Novikov conjecture go back to Kasparov ([10]). Nevertheless, (1) is not quite the map that we consider here. Our map is closely related to the coarse co-assembly map of [5]. It is an isomorphism if and only if the Dirac-dual-Dirac method applies to G. Hence there are many cases – groups with $\gamma \neq 1$ – where our co-assembly map is an isomorphism and (1) is not.

Most of our results only work if the group G is (almost) totally disconnected and has a G-compact universal proper G-space $\mathcal{E}G$. We impose this assumption throughout the introduction.

First we briefly recall some of the main ideas of [5], [7]. The new ingredient in the coarse co-assembly map is the *reduced stable Higson corona* $\mathfrak{c}^{\mathfrak{red}}(X)$ of a coarse space X. Its definition resembles that of the usual Higson corona, but its K-theory behaves much better. The coarse co-assembly map is a map

$$\mu\colon \mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(X)\big) \to \mathrm{KX}^*(X), \tag{2}$$

where $\mathrm{KX}^*(X)$ is a coarse invariant of X that agrees with $\mathrm{K}^*(X)$ if X is uniformly contractible.

If $|G|$ is the coarse space underlying a group G, then there is a commuting diagram

$$\begin{tikzcd}
\mathrm{KK}_*^G\big(\mathbb{C}, C_0(G)\big) \arrow[r, "p_{\mathcal{E}G}^*"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{RKK}_*^G\big(\mathcal{E}G;\mathbb{C}, C_0(G)\big) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(|G|)\big) \arrow[r, "\mu"] & \mathrm{KX}^*(|G|).
\end{tikzcd} \tag{3}$$

*This research was partially carried out at the Westfälische Wilhelms-Universität Münster and supported by the EU-Network *Quantum Spaces and Noncommutative Geometry* (Contract HPRN-CT-2002-00280) and the *Deutsche Forschungsgemeinschaft* (SFB 478).

In this situation, $\mathrm{KX}^*(|G|) \cong \mathrm{K}^*(\mathcal{E}G)$ because $|G|$ is coarsely equivalent to $\mathcal{E}G$, which is uniformly contractible. The commuting diagram (3), coupled with the reformulation of the Baum–Connes assembly map in [11], is the source of the relationship between the coarse co-assembly map and the Dirac-dual-Dirac method mentioned above.

If G is a torsion-free discrete group with finite classifying space BG, the coarse co-assembly map is an isomorphism if and only if the Dirac-dual-Dirac method applies to G. A similar result for groups with torsion is available, but this requires working equivariantly with respect to compact subgroups of G.

In this article, we work equivariantly with respect to the whole group G. The action of G on its underlying coarse space $|G|$ by isometries induces an action on $\mathfrak{c}^{\mathfrak{red}}(G)$. We consider a G-equivariant analogue

$$\mu\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) \to \mathrm{K}_*(C_0(\mathcal{E}G) \rtimes G) \tag{4}$$

of the coarse co-assembly map (2); here $\mathrm{K}^{\mathrm{top}}_*(G, A)$ denotes the domain of the Baum–Connes assembly map for G with coefficients A. We avoid $\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(X) \rtimes G)$ and $\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}(X) \rtimes_{\mathrm{r}} G)$ because we can say nothing about these two groups. In contrast, the group $\mathrm{K}^{\mathrm{top}}_*\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr)$ is much more manageable. The only *analytical* difficulties in this group come from coarse geometry.

There is a commuting diagram similar to (3) that relates (4) to equivariant Kasparov theory. To formulate this, we need some results of [11]. There is a certain G-C*-algebra P and a class $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ called *Dirac morphism* such that the Baum–Connes assembly map for G is equivalent to the map

$$\mathrm{K}_*\bigl((A \otimes \mathsf{P}) \rtimes_{\mathrm{r}} G\bigr) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G)$$

induced by Kasparov product with D. The Baum–Connes conjecture holds for G with coefficients in $\mathsf{P} \otimes A$ for any A. The Dirac morphism is a *weak equivalence*, that is, its image in $\mathrm{KK}^H(\mathsf{P}, \mathbb{C})$ is invertible for each compact subgroup H of G.

The existence of the Dirac morphism allows us to localise the (triangulated) category KK^G at the multiplicative system of weak equivalences. The functor from KK^G to its localisation turns out to be equivalent to the map

$$p^*_{\mathcal{E}G}\colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B).$$

One of the main results of this paper is a commuting diagram

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\bigr) \arrow[r] \arrow[d, leftrightarrow, "\cong"] & \mathrm{K}^*(\mathcal{E}G) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \arrow[r, "p^*_{\mathcal{E}G}"] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}).
\end{tikzcd} \tag{5}$$

In other words, the equivariant coarse co-assembly (4) is equivalent to the map

$$p^*_{\mathcal{E}G}\colon \mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}).$$

This map is our proposal for a dual to the Baum–Connes assembly map.

We should justify why we prefer the map (4) over (1). Both maps have isomorphic targets:

$$\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathsf{P}) \cong \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C}) \cong \mathrm{K}_*(C_0(\mathcal{E}G) \rtimes G).$$

Even in the usual Baum–Connes assembly map, the analytical side involves a choice between full and reduced group C*-algebras and crossed products. Even though the full group C*-algebra has better functoriality properties and is sometimes preferred because it gives potentially finer invariants, the reduced one is used because its K-theory is closer to $\mathrm{K}_*^{\mathrm{top}}(G)$. In formulating a dual version of the assembly map, we are faced with a similar situation. Namely, the topological object that is dual to $\mathrm{K}_*^{\mathrm{top}}(G)$ is $\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. For the analytical side, we have some choices; we prefer $\mathrm{KK}_*^G(\mathbb{C}, \mathsf{P})$ over $\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C})$ because the resulting co-assembly map is an isomorphism in more cases.

Of course, we must check that this choice is analytical enough to be useful for applications. The most important of these is the Novikov conjecture. Elements of $\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ yield maps $\mathrm{K}_*^{\mathrm{top}}(G) \to \mathbb{Z}$, which are analogous to higher signatures. In particular, (4) gives rise to such objects. The maps $\mathrm{K}_*^{\mathrm{top}}(G) \to \mathbb{Z}$ that come from a class in the range of (1) are known to yield homotopy invariants for manifolds because there is a pairing between $\mathrm{KK}_*^G(\mathbb{C}, \mathbb{C})$ and $\mathrm{K}_*(C^*_{\max} G)$ (see [8]). But since (4) factors through (1), the former also produces homotopy invariants.

In particular, surjectivity of (4) implies the Novikov conjecture for G. More is true: since $\mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ is the home of a *dual-Dirac morphism*, (5) yields that G has a dual-Dirac morphism and hence a *γ-element* if and only if (4) is an isomorphism. This observation can be used to give an alternative proof of the main result of [7].

We call elements in the range of (4) *boundary classes*. These automatically form a graded ideal in the $\mathbb{Z}/2$-graded unital ring $\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. In contrast, the range of the unital ring homomorphism (1) need not be an ideal because it always contains the unit element of $\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. We describe two important constructions of boundary classes, which are related to compactifications and to the proper Lipschitz cohomology of G studied in [3], [4].

Let $\mathcal{E}G \subseteq Z$ be a G-equivariant compactification of $\mathcal{E}G$ that is compatible with the coarse structure in a suitable sense. Since there is a map

$$\mathrm{K}_*^{\mathrm{top}}\big(G, C(Z \setminus \mathcal{E}G)\big) \to \mathrm{K}_*^{\mathrm{top}}\big(G, \mathfrak{c}^{\mathfrak{red}}(\mathcal{E}G)\big),$$

we get boundary classes from the boundary $Z \setminus \mathcal{E}G$. This construction also shows that G has a dual-Dirac morphism if it satisfies the Carlsson–Pedersen condition. This improves upon a result of Nigel Higson ([9]), which shows split injectivity of the Baum–Connes assembly map with coefficients under the same assumptions.

Although we have discussed only $\mathrm{KK}_*^G(\mathbb{C}, \mathsf{P})$ so far, our main technical result is more general and can also be used to construct elements in Kasparov groups of the form $\mathrm{KK}_*^G\big(\mathbb{C}, C_0(X)\big)$ for suitable G-spaces X. If X is a proper G-space, then we can use such classes to construct boundary classes in $\mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. This provides a

K-theoretic counterpart of the proper Lipschitz cohomology of G defined by Connes, Gromov, and Moscovici in [3]. Our approach clarifies the geometric parts of several constructions in [3]; thus we substantially simplify the proof of the homotopy invariance of Gelfand–Fuchs cohomology classes in [3].

2 Preliminaries

2.1 Dirac-dual-Dirac method and Baum–Connes conjecture. First, we recall the *Dirac-dual-Dirac method* of Kasparov and its reformulation in [11]. This is a technique for proving injectivity of the *Baum–Connes assembly map*

$$\mu\colon \mathrm{K}_*^{\mathrm{top}}(G, B) \to \mathrm{K}_*\big(C_r^*(G, B)\big), \tag{6}$$

where G is a locally compact group and B is a C^*-algebra with a strongly continuous action of G or, briefly, a G-C^**-algebra*.

This method requires a proper G-C^*-algebra A and classes

$$d \in \mathrm{KK}^G(A, \mathbb{C}), \quad \eta \in \mathrm{KK}^G(\mathbb{C}, A), \quad \gamma := \eta \otimes_A d \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C}),$$

such that $p_{\mathcal{E}G}^*(\gamma) = 1_{\mathbb{C}}$ in $\mathrm{RKK}^G(\mathcal{E}G; \mathbb{C}, \mathbb{C})$. If these data exist, then the Baum–Connes assembly map (6) is injective for all B. If, in addition, $\gamma = 1_{\mathbb{C}}$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$, then the Baum–Connes assembly map is invertible for all B, so that G satisfies the Baum–Connes conjecture with arbitrary coefficients.

Let A and B be G-C^*-algebras. An element $f \in \mathrm{KK}^G(A, B)$ is called a *weak equivalence* in [11] if its image in $\mathrm{KK}^H(A, B)$ is invertible for each compact subgroup H of G.

The following theorem contains some of the main results of [11].

Theorem 1. *Let G be a locally compact group. Then there is a G-C^*-algebra P and a class $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ called* Dirac morphism *such that*

(a) D *is a weak equivalence;*

(b) *the Baum–Connes conjecture holds with coefficients in $A \otimes \mathsf{P}$ for any A;*

(c) *the assembly map* (6) *is equivalent to the map*

$$\mathsf{D}_*\colon \mathrm{K}_*(A \otimes \mathsf{P} \rtimes_{\mathrm{r}} G) \to \mathrm{K}_*(A \rtimes_{\mathrm{r}} G);$$

(d) *the Dirac-dual-Dirac method applies to G if and only if there is a class $\eta \in \mathrm{KK}^G(\mathbb{C}, \mathsf{P})$ with $\eta \otimes_{\mathbb{C}} D = 1_A$, if and only if the map*

$$\mathsf{D}^*\colon \mathrm{KK}_*^G(\mathbb{C}, \mathsf{P}) \to \mathrm{KK}_*^G(\mathsf{P}, \mathsf{P}), \quad x \mapsto D \otimes x, \tag{7}$$

is an isomorphism.

Whereas [7] studies the invertibility of (7) by relating it to (2), here we are going to study the map (7) itself.

It is shown in [11] that the localisation of the category KK^G at the weak equivalences is isomorphic to the category $\mathrm{RKK}^G(\mathcal{E}G)$ whose morphism spaces are the groups $\mathrm{RKK}^G(\mathcal{E}G; A, B)$ as defined by Kasparov in [10]. This statement is equivalent to the existence of a Poincaré duality isomorphism

$$\mathrm{KK}^G_*(A \otimes \mathsf{P}, B) \cong \mathrm{RKK}^G_*(\mathcal{E}G; A, B) \tag{8}$$

for all G-C*-algebras A and B (this notion of duality is analysed in [6]). The canonical functor from KK^G to the localisation becomes the obvious functor

$$p^*_{\mathcal{E}G} \colon \mathrm{KK}^G(A, B) \to \mathrm{RKK}^G(\mathcal{E}G; A, B).$$

Since D is a weak equivalence, $p^*_{\mathcal{E}G}(\mathsf{D})$ is invertible. Hence the maps in the following commuting square are isomorphisms for all G-C*-algebras A and B:

$$\begin{tikzcd}
\mathrm{RKK}^G_*(\mathcal{E}G; A, B \otimes \mathsf{P}) \arrow[r, "\mathsf{D}_*", "\cong"'] \arrow[d, "\mathsf{D}^*", "\cong"'] & \mathrm{RKK}^G_*(\mathcal{E}G; A, B) \arrow[d, "\mathsf{D}^*", "\cong"'] \\
\mathrm{RKK}^G_*(\mathcal{E}G; A \otimes \mathsf{P}, B \otimes \mathsf{P}) \arrow[r, "\mathsf{D}_*", "\cong"'] & \mathrm{RKK}^G_*(\mathcal{E}G; A \otimes \mathsf{P}, B).
\end{tikzcd}$$

Together with (8) this implies

$$\mathrm{KK}^G_*(A \otimes \mathsf{P}, B) \cong \mathrm{KK}^G_*(A \otimes \mathsf{P}, B \otimes \mathsf{P}).$$

In the following, it will be useful to turn the isomorphism

$$\mathrm{K}^{\mathrm{top}}_*(G, A) \cong \mathrm{K}_*\big((A \otimes \mathsf{P}) \rtimes_{\mathrm{r}} G\big)$$

in Theorem 1.(c) into a definition.

2.2 Group actions on coarse spaces. Let G be a locally compact group and let X be a right G-space and a coarse space. We always assume that G acts continuously and coarsely on X, that is, the set $\{(xg, yg) \mid g \in K, (x, y) \in E\}$ is an entourage for any compact subset K of G and any entourage E of X.

Definition 2. We say that G *acts by translations* on X if $\{(x, gx) \mid x \in X,\ g \in K\}$ is an entourage for all compact subsets $K \subseteq G$. We say that G acts by *isometries* if every entourage of X is contained in a G-invariant entourage.

Example 3. Let G be a locally compact group. Then G has a unique coarse structure for which the right translation action is isometric; the corresponding coarse space is denoted $|G|$. The generating entourages are of the form

$$\bigcup_{g \in G} Kg \times Kg = \{(xg, yg) \mid g \in G, x, y \in K\}$$

for compact subsets K of G. The *left* translation action is an action by translations for this coarse structure.

Example 4. More generally, any proper, G-compact G-space X carries a unique coarse structure for which G acts isometrically; its entourages are defined as in Example 3. With this coarse structure, the orbit map $G \to X$, $g \mapsto g \cdot x$, is a coarse equivalence for any choice of $x \in X$. If the G-compactness assumption is omitted, the result is a σ-coarse space. We always equip a proper G-space with this additional structure.

2.3 The stable Higson corona. We next recall the definition of the *stable Higson corona* of a coarse space X from [5], [7]. Let D be a C*-algebra.

Let $\mathcal{M}(D \otimes \mathbb{K})$ be the multiplier algebra of $D \otimes \mathbb{K}$, and let $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D)$ be the C*-algebra of norm-continuous, bounded functions $f : X \to \mathcal{M}(D \otimes \mathbb{K})$ for which $f(x) - f(y) \in D \otimes \mathbb{K}$ for all $x, y \in X$. We also let

$$\mathfrak{B}^{\mathfrak{red}}(X, D) := \bar{\mathfrak{B}}^{\mathfrak{red}}(X, D)/C_0(X, D \otimes \mathbb{K}).$$

Definition 5. A function $f \in \bar{\mathfrak{B}}^{\mathfrak{red}}(X, D)$ has *vanishing variation* if the function $E \ni (x, y) \mapsto \|f(x) - f(y)\|$ vanishes at ∞ for any closed entourage $E \subseteq X \times X$.

The *reduced stable Higson compactification* of X with coefficients D is the subalgebra $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D) \subseteq \bar{\mathfrak{B}}^{\mathfrak{red}}(X, D)$ of vanishing variation functions. The quotient

$$\mathfrak{c}^{\mathfrak{red}}(X, D) := \bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)/C_0(X, D \otimes \mathbb{K}) \subseteq \mathfrak{B}^{\mathfrak{red}}(X, D)$$

is called *reduced stable Higson corona* of X. This defines a functor on the coarse category of coarse spaces: a coarse map $f : X \to X'$ induces a map $\mathfrak{c}^{\mathfrak{red}}(X', D) \to \mathfrak{c}^{\mathfrak{red}}(X, D)$, and two maps $X \to X'$ induce the same map $\mathfrak{c}^{\mathfrak{red}}(X', D) \to \mathfrak{c}^{\mathfrak{red}}(X, D)$ if they are close. Hence a coarse equivalence $X \to X'$ induces an isomorphism $\mathfrak{c}^{\mathfrak{red}}(X', D) \cong \mathfrak{c}^{\mathfrak{red}}(X, D)$.

For some technical purposes, we must allow unions $\mathcal{X} = \bigcup X_n$ of coarse spaces such that the embeddings $X_n \to X_{n+1}$ are coarse equivalences; such spaces are called *σ-coarse spaces*. The main example is the *Rips complex* $\mathcal{P}(X)$ of a coarse space X, which is used to define its coarse K-theory. More generally, if X is a proper but not G-compact G-space, then X may be endowed with the structure of a σ-coarse space. For coarse spaces of the form $|G|$ for a locally compact group G with a G-compact universal proper G-space $\mathcal{E}G$, we may use $\mathcal{E}G$ instead of $\mathcal{P}(X)$ because $\mathcal{E}G$ is coarsely equivalent to G and uniformly contractible. Therefore, we do not need σ-coarse spaces much; they only occur in Lemma 7.

It is straightforward to extend the definitions of $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)$ and $\mathfrak{c}^{\mathfrak{red}}(X, D)$ to σ-coarse spaces (see [5], [7]). Since we do not use this generalisation much, we omit details on this.

Let H be a locally compact group that acts coarsely and properly on X. It is crucial for us to allow non-compact groups here, whereas [7] mainly needs equivariance for compact groups. Let D be an H-C*-algebra, and let $\mathbb{K}_H := \mathbb{K}(\ell^2\mathbb{N} \otimes L^2 H)$. Then H acts on $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D \otimes \mathbb{K}_H)$ by

$$(h \cdot f)(x) := h \cdot \big(f(xh)\big),$$

where we use the obvious action of H on $D \otimes \mathbb{K}_H$ and its multiplier algebra. The action of H on $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D \otimes \mathbb{K}_H)$ need not be continuous; we let $\bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D)$ be the subalgebra of H-continuous elements in $\bar{\mathfrak{B}}^{\mathfrak{red}}(X, D \otimes \mathbb{K}_H)$. We let $\bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D)$ be the subalgebra of vanishing variation functions in $\bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D)$. Both algebras contain $C_0(X, D \otimes \mathbb{K}_H)$ as an ideal. The corresponding quotients are denoted by $\mathfrak{B}^{\mathfrak{red}}_H(X, D)$ and $\mathfrak{c}^{\mathfrak{red}}_H(X, D)$. By construction, we have a natural morphism of extensions of H-C*-algebras

$$\begin{tikzcd}
C_0(X, D \otimes \mathbb{K}_H) \arrow[r, tail] \arrow[d, equal] & \bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D) \arrow[r, two heads] \arrow[d, "\subseteq"] & \mathfrak{c}^{\mathfrak{red}}_H(X, D) \arrow[d, "\subseteq"] \\
C_0(X, D \otimes \mathbb{K}_H) \arrow[r, tail] & \bar{\mathfrak{B}}^{\mathfrak{red}}_H(X, D) \arrow[r, two heads] & \mathfrak{B}^{\mathfrak{red}}_H(X, D).
\end{tikzcd} \tag{9}$$

Concerning the extension of this construction to σ-coarse spaces, we only mention one technical subtlety. We must extend the functor $\mathrm{K}^{\mathrm{top}}_*(H, \llcorner\!\lrcorner)$ from C*-algebras to σ-H-C*-algebras. Here we use the definition

$$\mathrm{K}^{\mathrm{top}}_*(H, A) \cong \mathrm{K}_*\big((A \otimes \mathsf{P}) \rtimes_{\mathrm{r}} H\big), \tag{10}$$

where $\mathsf{D} \in \mathrm{KK}^H(\mathsf{P}, \mathbb{C})$ is a Dirac morphism for H. The more traditional definition as a colimit of $\mathrm{KK}^G_*(C_0(X), A)$, where $X \subseteq \mathcal{E}G$ is G-compact, yields a wrong result if A is a σ-H-C*-algebra because colimits and limits do not commute.

Let H be a locally compact group, let X be a coarse space with an isometric, continuous, proper action of H, and let D be an H-C*-algebra. The H-equivariant coarse K-theory $\mathrm{KX}^*_H(X, D)$ of X with coefficients in D is defined in [7] by

$$\mathrm{KX}^*_H(X, D) := \mathrm{K}^{\mathrm{top}}_*\big(H, C_0(\mathcal{P}(X), D)\big). \tag{11}$$

As observed in [7], we have $\mathrm{K}^{\mathrm{top}}_*\big(H, C_0(\mathcal{P}(X), D)\big) \cong \mathrm{K}_*(C_0(\mathcal{P}(X), D) \rtimes H)$ because H acts properly on $\mathcal{P}(X)$.

For most of our applications, X will be equivariantly uniformly contractible for all compact subgroups $K \subseteq H$, that is, the natural embedding $X \to \mathcal{P}(X)$ is a K-equivariant coarse homotopy equivalence. In such cases, we simply have

$$\mathrm{KX}^*_H(X, D) \cong \mathrm{K}^{\mathrm{top}}_*\big(H, C_0(X, D)\big). \tag{12}$$

In particular, this applies if X is an H-compact universal proper H-space (again, recall that the coarse structure is determined by requiring H to act isometrically).

The *H-equivariant coarse co-assembly map for X with coefficients in D* is a certain map

$$\mu^* \colon \mathrm{K}^{\mathrm{top}}_{*+1}\big(H, \mathfrak{c}^{\mathfrak{red}}_H(X, D)\big) \to \mathrm{KX}^*_H(X, D)$$

defined in [7]. In the special case where we have (12), this is simply the boundary map for the extension $C_0(X, D \otimes \mathbb{K}_H) \rightarrowtail \bar{\mathfrak{c}}^{\mathfrak{red}}_H(X, D) \twoheadrightarrow \mathfrak{c}^{\mathfrak{red}}_H(X, D)$. We are implicitly using the fact that the functor $\mathrm{K}^{\mathrm{top}}_*(H, \llcorner\!\lrcorner)$ has long exact sequences for arbitrary extensions of H-C*-algebras, which is proved in [7] using the isomorphism

$$\mathrm{K}^{\mathrm{top}}_*(H, B) := \mathrm{K}_*\big((B \otimes \mathsf{P}) \rtimes_{\mathrm{r}} H\big) \cong \mathrm{K}_*\big((B \otimes_{\max} \mathsf{P}) \rtimes H\big)$$

and exactness properties of maximal C*-tensor products and full crossed products.

There is also an alternative picture of the co-assembly map as a forget-control map, provided X is uniformly contractible (see [7, §2.8]). We have the following equivariant version of this result:

Proposition 6. *Let G be a totally disconnected group with a G-compact universal proper G-space $\mathcal{E}G$. Then the G-equivariant coarse co-assembly map for G is equivalent to the map*

$$j_*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathrm{c}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) \to \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr)$$

induced by the inclusion $j\colon \mathrm{c}^{\mathfrak{red}}_G(\mathcal{E}G, D) \to \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)$.

The equivalence of the two maps means that there is a natural commuting diagram

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathrm{c}^{\mathfrak{red}}_G(|G|, D)\bigr) \arrow[r, "\mu^*"] \arrow[d, leftrightarrow, "\cong"] & \mathrm{KX}^G_*(|G|, D) \arrow[d, leftrightarrow, "\cong"] \\
\mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathrm{c}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) \arrow[r, "j^*"] & \mathrm{K}^{\mathrm{top}}_{*+1}\bigl(G, \mathfrak{B}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr).
\end{tikzcd}$$

Recall that j is induced by the inclusion $\bar{\mathrm{c}}^{\mathfrak{red}}_G(\mathcal{E}G, D) \to \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)$, which exactly forgets the vanishing variation condition. Hence j_* is a forget-control map.

Proof. We may replace $|G|$ by $\mathcal{E}G$ because $\mathcal{E}G$ is coarsely equivalent to $|G|$. The coarse K-theory of $\mathcal{E}G$ agrees with the usual K-theory of $\mathcal{E}G$ (see [7]). A slight elaboration of the proof of [7, Lemma 15] shows that

$$\mathrm{K}^H_*\bigl(\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) \cong \mathrm{KK}^H_*\bigl(\mathbb{C}, \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr)$$

vanishes for all compact subgroups H of G. This yields $\mathrm{K}^{\mathrm{top}}_*\bigl(G, \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\mathcal{E}G, D)\bigr) = 0$ by a result of [2]. Now the assertion follows from the Five Lemma and the naturality of the K-theory long exact sequence for (9) as in [7]. □

3 Classes in Kasparov theory from the stable Higson corona

In this section, we show how to construct classes in equivariant KK-theory from the K-theory of the stable Higson corona. The following lemma is our main technical device:

Lemma 7. *Let G and H be locally compact groups and let X be a coarse space equipped with commuting actions of G and H. Suppose that G acts by translations and that H acts properly and by isometries. Let A and D be H-C*-algebras, equipped with the trivial G-action. We abbreviate*

$$B_X := C_0(X, D \otimes \mathbb{K}_H \otimes_{\max} A) \rtimes H, \quad E_X := (\bar{\mathrm{c}}^{\mathfrak{red}}_H(X, D) \otimes_{\max} A) \rtimes H$$

and similarly for $\mathcal{P}(X)$ instead of X. There are extensions $B_X \rightarrowtail E_X \twoheadrightarrow E_X/B_X$ and $B_{\mathcal{P}(X)} \rightarrowtail E_{\mathcal{P}(X)} \twoheadrightarrow E_{\mathcal{P}(X)}/B_{\mathcal{P}(X)}$ with

$$E_{\mathcal{P}(X)}/B_{\mathcal{P}(X)} \cong E_X/B_X \cong (\mathfrak{c}_H^{\mathfrak{red}}(X, D) \otimes_{\max} A) \rtimes H,$$

and a natural commuting diagram

$$\begin{array}{ccc} \mathrm{K}_{*+1}(E_X/B_X) & \xrightarrow{\partial} & \mathrm{K}_*(B_{\mathcal{P}(X)}) \\ \downarrow \psi & & \downarrow \phi \\ \mathrm{KK}_*^G(\mathbb{C}, B_X) & \xrightarrow{p_{\mathcal{E}G}^*} & \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, B_X). \end{array} \tag{13}$$

Proof. The quotients E_X/B_X and $E_{\mathcal{P}(X)}/B_{\mathcal{P}(X)}$ are as asserted and agree because $X \to \mathcal{P}(X)$ is a coarse equivalence and because maximal tensor products and full crossed products are exact functors in complete generality, unlike spatial tensor products and reduced crossed products. We let ∂ be the K-theory boundary map for the extension $B_{\mathcal{P}(X)} \rightarrowtail E_{\mathcal{P}(X)} \twoheadrightarrow E_X/B_X$.

Since we have a natural map $\mathfrak{c}_H^{\mathfrak{red}}(X, D) \otimes_{\max} A \to \mathfrak{c}_H^{\mathfrak{red}}(X, D \otimes_{\max} A)$, we may replace the pair (D, A) by $(D \otimes_{\max} A, \mathbb{C})$ and omit A if convenient. Stabilising D by $\mathbb{K}_H$, we can further eliminate the stabilisations.

First we lift the K-theory boundary map for the extension $B_X \rightarrowtail E_X \twoheadrightarrow E_X/B_X$ to a map $\psi\colon \mathrm{K}_{*+1}(E_X/B_X) \to \mathrm{KK}_*^G(\mathbb{C}, B_X)$. The G-equivariance of the resulting Kasparov cycles follows from the assumption that G acts on X by translations.

We have to distinguish between the cases $* = 0$ and $* = 1$. We only write down the construction for $* = 0$. Since the algebra E_X/B_X is matrix-stable, $\mathrm{K}_1(E_X/B_X)$ is the homotopy group of unitaries in E_X/B_X without further stabilisation. A cycle for $\mathrm{KK}_0^G(\mathbb{C}, B_X)$ is given by two G-equivariant Hilbert modules $\mathcal{E}_\pm$ over B_X and a G-continuous adjointable operator $F\colon \mathcal{E}_+ \to \mathcal{E}_-$ for which $1 - FF^*$, $1 - F^*F$ and $gF - F$ for $g \in G$ are compact; we take $\mathcal{E}_\pm = B_X$ and let $F \in E_X \subseteq \mathcal{M}(B_X)$ be a lifting for a unitary $u \in E_X/B_X$. Since G acts on X by translations, the induced action on $\bar{\mathfrak{c}}^{\mathfrak{red}}(X, D)$ and hence on E_X/B_X is trivial. Hence u is a G-invariant unitary in E_X/B_X. For the lifting F, this means that

$$1 - FF^*, \quad 1 - F^*F, \quad gF - F \in B_X.$$

Hence F defines a cycle for $\mathrm{KK}_0^G(\mathbb{C}, B_X)$. We get a well-defined map $[u] \mapsto [F]$ from $\mathrm{K}_1(E_X/B_X)$ to $\mathrm{KK}_0^G(\mathbb{C}, B_X)$ because homotopic unitaries yield operator homotopic Kasparov cycles.

Next we have to factor the map $p_{\mathcal{E}G}^* \circ \psi$ in (13) through $\mathrm{K}_0(B_{\mathcal{P}(X)})$. The main ingredient is a certain continuous map $\bar{c}\colon \mathcal{E}G \times X \to \mathcal{P}(X)$. We use the same description of $\mathcal{P}(X)$ as in [7] as the space of positive measures on X with $1/2 < \mu(X) \leq 1$; this is a σ-coarse space in a natural way, we write it as $\mathcal{P}(X) = \bigcup \mathrm{P}_d(X)$.

There is a function $c\colon \mathcal{E}G \to \mathbb{R}_+$ for which $\int_{\mathcal{E}G} c(\mu g)\,\mathrm{d}g = 1$ for all $\mu \in \mathcal{E}G$ and $\operatorname{supp} c \cap Y$ is compact for G-compact $Y \subseteq \mathcal{E}G$. If $\mu \in \mathcal{E}G$, $x \in X$, then the condition

$$\langle \bar{c}(\mu, x), \alpha \rangle := \int_G c(\mu g)\alpha(g^{-1}x)\,\mathrm{d}g$$

for $\alpha \in C_0(X)$ defines a probability measure on X. Since such measures are contained in $\mathcal{P}(X)$, $\bar{c}$ defines a map $\bar{c}\colon \mathcal{E}G \times X \to \mathcal{P}(X)$. This map is continuous and satisfies $\bar{c}(\mu g, g^{-1}xh) = \bar{c}(\mu, x)h$ for all $g \in G$, $\mu \in \mathcal{E}G$, $x \in X$, $h \in H$.

For a C*-algebra Z, let $C(\mathcal{E}G, Z)$ be the σ-C*-algebra of all continuous functions $f\colon \mathcal{E}G \to Z$ without any growth restriction. Thus $C(\mathcal{E}G, Z) = \varprojlim C(K, Z)$, where K runs through the directed set of compact subsets of $\mathcal{E}G$.

We claim that $(\bar{c}^* f)(\mu)(x) := f\big(\bar{c}(\mu, x)\big)$ for $f \in C_0(\mathcal{P}(X), D)$ defines a continuous $*$-homomorphism

$$\bar{c}^*\colon C_0(\mathcal{P}(X), D) \to C\big(\mathcal{E}G, C_0(X, D)\big).$$

If $K \subseteq \mathcal{E}G$ is compact, then there is a compact subset $L \subseteq G$ such that $c(\mu \cdot g) = 0$ for $\mu \in K$ and $g \notin L$. Hence $\bar{c}(\mu, x)$ is supported in $L^{-1}x$ for $\mu \in K$. Since G acts on X by translations, such measures are contained in a filtration level $\mathrm{P}_d(X)$. Hence $\bar{c}^*(f)$ restricts to a C_0-function $K \times X \to D$ for all $f \in C_0(\mathcal{P}(X), D)$. This proves the claim. Since $\bar{c}$ is H-equivariant and G-invariant, we get an induced map

$$B_{\mathcal{P}(X)} = C_0(\mathcal{P}(X), D) \rtimes H \to (C\big(\mathcal{E}G, C_0(X, D)\big) \rtimes H)^G = C(\mathcal{E}G, B_X)^G,$$

where $Z^G \subseteq Z$ denotes the subalgebra of G-invariant elements. We obtain an induced $*$-homomorphism between the stable multiplier algebras as well.

An element of $\mathrm{K}_0\big(B_{\mathcal{P}(X)}\big)$ is represented by a self-adjoint bounded multiplier $F \in \mathcal{M}(B_{\mathcal{P}(X)} \otimes \mathbb{K})$ such that $1 - FF^*$ and $1 - F^*F$ belong to $B_{\mathcal{P}(X)} \otimes \mathbb{K}$. Now $\tilde{F} := \bar{c}^*(F)$ is a G-invariant bounded multiplier of $C(\mathcal{E}G, B_X \otimes \mathbb{K})$ and hence a G-invariant multiplier of $C_0(\mathcal{E}G, B_X \otimes \mathbb{K})$, such that $\alpha \cdot (1 - \tilde{F}\tilde{F}^*)$ and $\alpha \cdot (1 - \tilde{F}^*\tilde{F})$ belong to $C_0(\mathcal{E}G, B_X \otimes \mathbb{K})$ for all $\alpha \in C_0(\mathcal{E}G)$. This says exactly that $\tilde{F}$ is a cycle for $\mathrm{RKK}_0^G(\mathcal{E}G; \mathbb{C}, B_X)$. This construction provides the natural map

$$\phi\colon \mathrm{K}_0(B_{\mathcal{P}(X)}) \to \mathrm{RKK}_0^G(\mathcal{E}G; \mathbb{C}, B_X).$$

Finally, a routine computation, which we omit, shows that the two images of a unitary $u \in E_X/B_X$ differ by a compact perturbation. Hence the diagram (13) commutes. □

We are mainly interested in the case where A is the source P of a Dirac morphism for H. Then $\mathrm{K}_{*+1}(E_X/B_X) = \mathrm{K}_*^{\mathrm{top}}\big(H, \mathfrak{c}_H^{\mathfrak{red}}(X, D)\big)$, and the top row in (13) is the H-equivariant coarse co-assembly map for X with coefficients D. Since we assume H to act properly on X, we have a KK^G-equivalence $B_X \sim C_0(X, D) \rtimes H$, and similarly for $\mathcal{P}(X)$. Hence we now get a commuting square

$$\begin{array}{ccc}
\mathrm{K}_{*+1}^{\mathrm{top}}\big(H, \mathfrak{c}_H^{\mathfrak{red}}(X, D)\big) & \xrightarrow{\quad\partial\quad} & \mathrm{KX}_H^*(X, D) \\
\Big\downarrow \psi & & \Big\downarrow \phi \\
\mathrm{KK}_*^G(\mathbb{C}, C_0(X, D) \rtimes H) & \xrightarrow{\quad p_{\mathcal{E}G}^*\quad} & \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, C_0(X, D) \rtimes H).
\end{array} \tag{14}$$

If, in addition, $D = \mathbb{C}$ and the action of H on X is free, then we can further simplify this to

$$\begin{tikzcd} \mathrm{K}^{\mathrm{top}}_{*+1}\big(H, \mathfrak{c}^{\mathfrak{red}}_H(X)\big) \arrow[r, "\partial"] \arrow[d, "\psi"] & \mathrm{KX}^*_H(X) \arrow[d, "\phi"] \\ \mathrm{KK}^G_*\big(\mathbb{C}, C_0(X/H)\big) \arrow[r, "p^*_{\mathcal{E}G}"] & \mathrm{RKK}^G_*\big(\mathcal{E}G; \mathbb{C}, C_0(X/H)\big). \end{tikzcd} \tag{15}$$

We may also specialise the space X to $|G|$, with G acting by multiplication on the left, and with $H \subseteq G$ a compact subgroup acting on $|G|$ by right multiplication. This is the special case of (13) that is used in [7]. The following applications will require other choices of X.

3.1 Applications to Lipschitz classes. Now we use Lemma 7 to construct interesting elements in $\mathrm{KK}^G_*\big(\mathbb{C}, C_0(X)\big)$ for a G-space X. This is related to the method of Lipschitz maps developed by Connes, Gromov and Moscovici in [3].

3.1.1 Pulled-back coarse structures. Let X be a G-space, let Y be a coarse space and let $\alpha\colon X \to Y$ be a proper continuous map. We pull back the coarse structure on Y to a coarse structure on X, letting $E \subseteq X \times X$ be an entourage if and only if $\alpha_*(E) \subseteq Y \times Y$ is one. Since α is proper and continuous, this coarse structure is compatible with the topology on X. For this coarse structure, G *acts by translations* if and only if α satisfies the following *displacement condition* used in [3]: for any compact subset $K \subseteq G$, the set

$$\big\{\big(\alpha(gx), \alpha(x)\big) \in Y \times Y \mid x \in X,\ g \in K\big\}$$

is an entourage of Y. The map α becomes a coarse map. Hence we obtain a commuting diagram

$$\begin{tikzcd} \mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(Y)\big) \arrow[r, "\alpha^*"] \arrow[d, "\partial^Y"] & \mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(X)\big) \arrow[r, "\psi"] \arrow[d, "\partial^X"] & \mathrm{KK}^G_*\big(\mathbb{C}, C_0(X)\big) \arrow[d, "p^*_{\mathcal{E}G}"] \\ \mathrm{KX}^*(Y) \arrow[r, "\alpha^*"] & \mathrm{KX}^*(X) \arrow[r, "\phi"] & \mathrm{RKK}^G_*\big(\mathcal{E}G; \mathbb{C}, C_0(X)\big). \end{tikzcd}$$

with ψ and ϕ as in Lemma 7.

The constructions of [3, §I.10] only use $Y = \mathbb{R}^N$ with the Euclidean coarse structure. The coarse co-assembly map is an isomorphism for $\mathbb{R}^N$ because $\mathbb{R}^N$ is scalable. Moreover, $\mathbb{R}^N$ is uniformly contractible and has bounded geometry. Hence we obtain canonical isomorphisms

$$\mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\big) \cong \mathrm{KX}^*(\mathbb{R}^N) \cong \mathrm{K}^*(\mathbb{R}^N).$$

In particular, $\mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\big) \cong \mathbb{Z}$ with generator $[\partial\mathbb{R}^N]$ in $\mathrm{K}_{1-N}\big(\mathfrak{c}^{\mathfrak{red}}(\mathbb{R}^N)\big)$. This class is nothing but the usual dual-Dirac morphism for the locally compact group $\mathbb{R}^N$.

As a result, any map $\alpha\colon X \to \mathbb{R}^N$ that satisfies the displacement condition above induces
$$[\alpha] := \psi\bigl(\alpha^*[\partial\mathbb{R}^N]\bigr) \in \mathrm{KK}^G_{-N}\bigl(\mathbb{C}, C_0(X)\bigr).$$
The commutative diagram (13) computes $p^*_{\mathcal{E}G}[\alpha] \in \mathrm{RKK}^G_{-N}\bigl(\mathcal{E}G; \mathbb{C}, C_0(X)\bigr)$ in purely topological terms.

3.1.2 Principal bundles over coarse spaces. As in [3], we may replace a fixed map $X \to \mathbb{R}^N$ by a section of a vector bundle over X. But we need this bundle to have a G-equivariant spin structure. To encode this, we consider a G-equivariant $\mathrm{Spin}(N)$-principal bundle $\pi\colon E \to B$ together with actions of G on E and B such that π is G-equivariant and the action on E commutes with the action of $H := \mathrm{Spin}(N)$. Let $T := E \times_{\mathrm{Spin}(N)} \mathbb{R}^N$ be the associated vector bundle over B. It carries a G-invariant Euclidean metric and spin structure. As is well-known, sections $\alpha\colon B \to T$ correspond bijectively to $\mathrm{Spin}(N)$-equivariant maps $\alpha'\colon E \to \mathbb{R}^N$; here a section α corresponds to the map $\alpha'\colon E \to \mathbb{R}^N$ that sends $y \in E$ to the coordinates of $\alpha\pi(y)$ in the orthogonal frame described by y. Since the group $\mathrm{Spin}(N)$ is compact, the map α' is proper if and only if $b \mapsto \|\alpha(b)\|$ is a proper function on B.

As in §3.1.1, a $\mathrm{Spin}(N)$-equivariant proper continuous map $\alpha'\colon E \to Y$ for a coarse space Y allows us to pull back the coarse structure of Y to E; then $\mathrm{Spin}(N)$ acts by isometries. The group G acts by translations if and only if α' satisfies the displacement condition from §3.1.1. If $Y = \mathbb{R}^N$, we can rewrite this in terms of $\alpha\colon B \to T$: we need
$$\sup\{\|g\alpha(g^{-1}b) - \alpha(b)\| \mid b \in B,\ g \in K\}$$
to be bounded for all compact subsets $K \subseteq G$.

If the displacement condition holds, then we are in the situation of Lemma 7 with $H = \mathrm{Spin}(N)$ and $X = E$. Since H acts freely on E, $C_0(E) \rtimes H$ is G-equivariantly Morita–Rieffel equivalent to $C_0(B)$. We obtain canonical maps

$$\begin{aligned}\mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr) &\xrightarrow{(\alpha')^*} \mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(E)\bigr)\\ &\xrightarrow{\psi} \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(E) \rtimes \mathrm{Spin}(N)\bigr) \cong \mathrm{KK}^G_*\bigl(\mathbb{C}, C_0(B)\bigr).\end{aligned}$$

The $\mathrm{Spin}(N)$-equivariant coarse co-assembly map for $\mathbb{R}^N$ is an isomorphism by [7] because the group $\mathbb{R}^N \rtimes \mathrm{Spin}(N)$ has a dual-Dirac morphism. Using also the uniform contractibility of $\mathbb{R}^N$ and $\mathrm{Spin}(N)$-equivariant Bott periodicity, we get

$$\mathrm{K}^{\mathrm{Spin}(N)}_{*+1}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr) \cong \mathrm{KX}^*_{\mathrm{Spin}(N)}(\mathbb{R}^N) \cong \mathrm{K}^*_{\mathrm{Spin}(N)}(\mathbb{R}^N) \cong \mathrm{K}^{*+N}_{\mathrm{Spin}(N)}(\text{point}).$$

The class of the trivial representation in $\mathrm{Rep}(\mathrm{Spin} N) \cong \mathrm{K}^{\mathrm{Spin}(N)}_0(\mathbb{C})$ is mapped to the usual dual-Dirac morphism $[\partial\mathbb{R}^N] \in \mathrm{K}^{\mathrm{Spin}(N)}_{1-N}\bigl(\mathfrak{c}^{\mathfrak{red}}_{\mathrm{Spin}(N)}(\mathbb{R}^N)\bigr)$ for $\mathbb{R}^N$. As a result, any proper section $\alpha\colon B \to T$ satisfying the displacement condition induces
$$[\alpha] := \psi\bigl(\alpha^*[\partial\mathbb{R}^N]\bigr) \in \mathrm{KK}^G_{-N}\bigl(\mathbb{C}, C_0(B)\bigr).$$

Again, the commutative diagram (13) computes $p_{\mathcal{E}G}^*[\alpha] \in \mathrm{RKK}^G_{-N}\big(\mathcal{E}G; \mathbb{C}, C_0(X)\big)$ in purely topological terms.

3.1.3 Coarse structures on jet bundles. Let M be an oriented compact manifold and let $\mathrm{Diff}^+(M)$ be the infinite-dimensional Lie group of orientation-preserving diffeomorphisms of M. Let G be a locally compact group that acts on M by a continuous group homomorphism $G \to \mathrm{Diff}^+(M)$. The *Gelfand–Fuchs cohomology* of M is part of the group cohomology of $\mathrm{Diff}^+(M)$ and by functoriality maps to the group cohomology of G. It is shown in [3] that the range of Gelfand–Fuchs cohomology in the cohomology of G yields homotopy-invariant higher signatures.

This argument has two parts; one is geometric and concerns the construction of a class in $\mathrm{KK}^G_*\big(\mathbb{C}, C_0(X)\big)$ for a suitable space X; the other uses cyclic homology to construct linear functionals on $\mathrm{K}_*(C_0(X) \rtimes_{\mathrm{r}} G)$ associated to Gelfand–Fuchs cohomology classes. We can simplify the first step; the second has nothing to do with coarse geometry.

Let $\pi^k : \mathrm{J}^k_+(M) \to M$ be the *oriented k-jet bundle* over M. That is, a point in $\mathrm{J}^k_+(M)$ is the kth order Taylor series at 0 of an orientation-preserving diffeomorphism from a neighbourhood of $0 \in \mathbb{R}^n$ into M. This is a principal H-bundle over M, where H is a connected Lie group whose Lie algebra $\mathfrak{h}$ is the space of polynomial maps $p : \mathbb{R}^n \to \mathbb{R}^n$ of order k with $p(0) = 0$, with an appropriate Lie algebra structure. The maximal compact subgroup $K \subseteq H$ is isomorphic to $\mathrm{SO}(n)$, acting by isometries on $\mathbb{R}^n$. It acts on $\mathfrak{h}$ by conjugation.

Since our construction is natural, the action of G on M lifts to an action on $\mathrm{J}^k_+(M)$ that commutes with the H-action. We let H act on the right and G on the left. Define $X_k := \mathrm{J}^k_+(M)/K$. This is the bundle space of a fibration over M with fibres H/K. Gelfand–Fuchs cohomology can be computed using a chain complex of $\mathrm{Diff}^+(M)$-invariant differential forms on X_k for $k \to \infty$. Using this description, Connes, Gromov, and Moscovici associate to a Gelfand–Fuchs cohomology class a functional $\mathrm{K}_*(C_0(X_k) \rtimes_{\mathrm{r}} G) \to \mathbb{C}$ for sufficiently high k in [3].

Since $\mathrm{J}^k_+(M)/H \cong M$ is compact, there is a unique coarse structure on $\mathrm{J}^k_+(M)$ for which H acts isometrically (see §2.2). With this coarse structure, $\mathrm{J}^k_+(M)$ is coarsely equivalent to H. The compactness of $\mathrm{J}^k_+(M)/H \cong M$ also implies easily that G acts by translations. We have a Morita–Rieffel equivalence $C_0(X_k) \sim C_0(\mathrm{J}^k_+ M) \rtimes K$ because K acts freely on $\mathrm{J}^k_+(M)$. We want to study the map

$$\mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}_K(\mathrm{J}^k_+ M) \rtimes K) \xrightarrow{\psi} \mathrm{KK}^G_*(\mathbb{C}, C_0(\mathrm{J}^k_+ M) \rtimes K) \cong \mathrm{KK}^G_*\big(\mathbb{C}, C_0(X_k)\big)$$

produced by Lemma 7.

Since H is almost connected, it has a dual-Dirac morphism by [10]; hence the K-equivariant coarse co-assembly map for H is an isomorphism by the main result of [7]. Moreover, H/K is a model for $\mathcal{E}G$ by [1]. We get

$$\mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}_K(\mathrm{J}^k_+ M) \rtimes K) \cong \mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}_K(|H|) \rtimes K) \cong \mathrm{KX}^*_K(|H|) \cong \mathrm{K}^*_K(H/K).$$

Let $\mathfrak{h}$ and $\mathfrak{k}$ be the Lie algebras of H and K. There is a K-equivariant homeomorphism $\mathfrak{h}/\mathfrak{k} \cong H/K$, where K acts on $\mathfrak{h}/\mathfrak{k}$ by conjugation. Now we need to know whether there is a K-equivariant spin structure on $\mathfrak{h}/\mathfrak{k}$. One can check that this is the case if $k \equiv 0, 1 \bmod 4$. Since we can choose k as large as we like, we can always assume that this is the case. The spin structure allows us to use Bott periodicity to identify $\mathrm{K}_K^*(H/K) \cong \mathrm{K}^H_{*-N}(\mathbb{C})$, which is the representation ring of K in degree $-N$, where $N = \dim \mathfrak{h}/\mathfrak{k}$. Using our construction, the trivial representation of K yields a canonical element in $\mathrm{KK}^G_{-N}\big(\mathbb{C}, C_0(X_k)\big)$.

This construction is much shorter than the corresponding one in [3] because we use Kasparov's result about dual-Dirac morphisms for almost connected groups. Much of the corresponding argument in [3] is concerned with proving a variation on this result of Kasparov.

4 Computation of $\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P})$

So far we have merely used the diagram (13) to construct certain elements in $\mathrm{KK}^G_*(\mathbb{C}, B)$. Now we show that this construction yields an isomorphic description of $\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P})$. This assertion requires G to be a totally disconnected group with a G-compact universal proper G-space. We assume this throughout this section.

Lemma 8. *In the situation of Lemma 7, suppose that $X = |G|$ with G acting by left translations and that $H \subseteq G$ is a compact subgroup acting on X by right translations; here $|G|$ carries the coarse structure of Example 3. Then the maps ψ and ϕ are isomorphisms.*

Proof. We reduce this assertion to results of [7]. The C*-algebras $\mathfrak{c}^{\mathfrak{red}}_H(|G|, D) \rtimes H$ and $\mathfrak{c}^{\mathfrak{red}}_H(|G|, D)^H$ are strongly Morita equivalent, whence have isomorphic K-theory. It is shown in [7] that

$$\mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}_H(|G|, D)^H\big) \cong \mathrm{KK}^G_*(\mathbb{C}, \operatorname{Ind}_H^G D). \tag{16}$$

Finally, $\operatorname{Ind}_H^G(D) = C_0(G, D)^H$ is G-equivariantly Morita–Rieffel equivalent to $C_0(G, D) \rtimes H$. Hence we get

$$\begin{aligned} \mathrm{K}_{*+1}(\mathfrak{c}^{\mathfrak{red}}_H(|G|, D) \rtimes H) \cong \mathrm{K}_{*+1}\big(\mathfrak{c}^{\mathfrak{red}}_H(|G|, D)^H\big) &\cong \mathrm{KK}^G_*\big(\mathbb{C}, \operatorname{Ind}_H^G(D)\big) \\ &\cong \mathrm{KK}^G_*\big(\mathbb{C}, C_0(G, D) \rtimes H\big). \end{aligned} \tag{17}$$

It is a routine exercise to verify that this composition agrees with the map ψ in (13). Similar considerations apply to the map ϕ. $\square$

We now set $X = |G|$, and let $G = H$. The actions of G on $|G|$ on the left and right are by translations and isometries, respectively. Lemma 7 yields a map

$$\Psi_*\colon \mathrm{K}_{*+1}\big((\mathfrak{c}^{\mathfrak{red}}(|G|, D) \otimes_{\max} A) \rtimes G\big) \to \mathrm{KK}^G_*\big(\mathbb{C}, C_0(|G|, D \otimes_{\max} A) \rtimes G\big). \tag{18}$$

for all A, D, where we use the G-equivariant Morita-Rieffel equivalence between $C_0(|G|, D) \rtimes G$ and D. It fits into a commuting diagram

$$\begin{array}{ccc} \mathrm{K}_{*+1}\big((\mathrm{c}_G^{\mathfrak{red}}(|G|, D) \otimes_{\max} A) \rtimes G\big) & \xrightarrow{\partial} & \mathrm{K}_*(C_0(\mathcal{E}G, D \otimes_{\max} A) \rtimes G) \\ \downarrow \Psi_*^{D,A} & & \cong \downarrow \\ \mathrm{KK}_*^G(\mathbb{C}, D \otimes_{\max} A) & \xrightarrow{p_{\mathcal{E}G}^*} & \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, D \otimes_{\max} A). \end{array}$$

Lemma 9. *The class of G-C^*-algebras A for which $\Psi_*^{D,A}$ is an isomorphism for all D is triangulated and thick and contains all G-C^*-algebras of the form $C_0(G/H)$ for compact open subgroups H of G.*

Proof. The fact that this category of algebras is triangulated and thick means that it is closed under suspensions, extensions, and direct summands. These formal properties are easy to check.

Since $H \subseteq G$ is open, there is no difference between H-continuity and G-continuity. Hence

$$\begin{aligned} \big(\mathrm{c}_G^{\mathfrak{red}}(|G|, D) \otimes_{\max} C_0(G/H)\big) \rtimes G &\cong \big(\mathrm{c}_H^{\mathfrak{red}}(|G|, D) \otimes_{\max} C_0(G/H)\big) \rtimes G \\ &\sim \big(\mathrm{c}_H^{\mathfrak{red}}(|G|, D) \rtimes H, \end{aligned}$$

where $\sim$ means Morita-Rieffel equivalence. Similar simplifications can be made in other corners of the square. Hence the diagram for $A = C_0(G/H)$ and G acting on the right is equivalent to a corresponding diagram for trivial A and H acting on the right. The latter case is contained in Lemma 8. □

Theorem 10. *Let G be an almost totally disconnected group with G-compact $\mathcal{E}G$. Then for every $B \in \mathrm{KK}^G$, the map*

$$\Psi_* \colon \mathrm{K}_{*+1}^{\mathrm{top}}\big(G, \mathrm{c}^{\mathfrak{red}}(|G|, B)\big) \to \mathrm{KK}_*^G(\mathbb{C}, B \otimes \mathsf{P})$$

is an isomorphism and the diagram

$$\begin{array}{ccc} \mathrm{K}_{*+1}^{\mathrm{top}}\big(G, \mathrm{c}^{\mathfrak{red}}(|G|, B)\big) & \xrightarrow{\mu^*} & \mathrm{KX}_G^*(|G|, B) \\ \cong \downarrow \Psi_* & & \cong \downarrow \\ \mathrm{KK}_*^G(\mathbb{C}, B \otimes \mathsf{P}) & \xrightarrow{p_{\mathcal{E}G}^*} & \mathrm{RKK}_*^G(\mathcal{E}G; \mathbb{C}, B \otimes \mathsf{P}) \end{array}$$

commutes. In particular, $\mathrm{K}_{*+1}^{\mathrm{top}}\big(G, \mathrm{c}^{\mathfrak{red}}(|G|)\big)$ *is naturally isomorphic to* $\mathrm{KK}_*^G(\mathbb{C}, \mathsf{P})$.

Proof. It is shown in [7] that for such groups G, the algebra P belongs to the thick triangulated subcategory of KK^G that is generated by $C_0(G/H)$ for compact subgroups H of G. Hence the assertion follows from Lemma 9 and our definition of $\mathrm{K}^{\mathrm{top}}$. □

Corollary 11. *Let* $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$ *be a Dirac morphism for* G. *Then the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{K}^{\mathrm{top}}_{*+1}\big(G, \mathfrak{c}^{\mathfrak{red}}_G(|G|)\big) \arrow[r, "\partial"] \arrow[d, "\Psi_*", "\cong"'] \arrow[rr, bend left, "\mu^*"] & \mathrm{K}_*(C_0(\mathcal{E}G, \mathsf{P}) \rtimes G) \arrow[r, "\mathsf{D}_*", "\cong"'] \arrow[d, "\cong"'] & \mathrm{K}_*(C_0(\mathcal{E}G) \rtimes G) \arrow[d, "\cong"'] \\
\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \arrow[r, "p^*_{\mathcal{E}G}"] \arrow[dr, "\mathsf{D}_*"'] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}) \arrow[r, "\mathsf{D}_*", "\cong"'] & \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}) \\
& \mathrm{KK}^G_*(\mathbb{C}, \mathbb{C}), \arrow[ur, "p^*_{\mathcal{E}G}"'] &
\end{tikzcd}$$

where Ψ_* *is as in Theorem* 10 *and* μ^* *is the* G*-equivariant coarse co-assembly map for* $|G|$, *and the indicated maps are isomorphisms.*

Proof. This follows from Theorem 10 and the general properties of the Dirac morphism discussed in §2.1. □

Definition 12. We call $a \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$

(a) a *boundary class* if it lies in the range of

$$\mu^* : \mathrm{K}^{\mathrm{top}}_{*+1}\big(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\big) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C});$$

(b) *properly factorisable* if $a = p^*_{\mathcal{E}G}(b \otimes_A c)$ for some proper G-C^*-algebra A and some $b \in \mathrm{KK}^G_*(\mathbb{C}, A)$, $c \in \mathrm{KK}^G_*(A, \mathbb{C})$;

(c) *proper Lipschitz* if $a = p^*_{\mathcal{E}G}(b \otimes_{C_0(X)} c)$, where $b \in \mathrm{KK}^G_*\big(\mathbb{C}, C_0(X)\big)$ is constructed as in §3.1.1 and §3.1.2 and $c \in \mathrm{KK}^G_*(C_0(X), \mathbb{C})$ is arbitrary.

Proposition 13. *Let* G *be a totally disconnected group with* G*-compact* $\mathcal{E}G$.

(a) *A class* $a \in \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ *is properly factorisable if and only if it is a boundary class.*

(b) *Proper Lipschitz classes are boundary classes.*

(c) *The boundary classes form an ideal in the ring* $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$.

(d) *The class* $1_{\mathcal{E}G}$ *is a boundary class if and only if* G *has a dual-Dirac morphism; in this case, the* G*-equivariant coarse co-assembly map* μ^* *is an isomorphism.*

(e) *Any boundary class lies in the range of*

$$p^*_{\mathcal{E}G} : \mathrm{KK}^G_*(\mathbb{C}, \mathbb{C}) \to \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$$

and hence yields homotopy invariants for manifolds.

Proof. By Corollary 11, the equivariant coarse co-assembly map

$$\mu^*\colon \mathrm{K}^{\mathrm{top}}_{*+1}\big(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\big) \to \mathrm{K}_*\big(C_0(\mathcal{E}G \rtimes G)\big) \cong \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$$

is equivalent to the map

$$\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \xrightarrow{p^*_{\mathcal{E}G}} \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathsf{P}) \cong \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}).$$

If we combine this with the isomorphism $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}) \cong \mathrm{KK}^G_*(\mathsf{P}, \mathsf{P})$, the resulting map

$$\mathrm{KK}^G_*(\mathbb{C}, \mathsf{P}) \to \mathrm{KK}^G_*(\mathsf{P}, \mathsf{P})$$

is simply the product (on the left) with $\mathsf{D} \in \mathrm{KK}^G(\mathsf{P}, \mathbb{C})$; this map is known to be an isomorphism if and only if it is surjective, if and only if 1_{P} is in its range, if and only if the H-equivariant coarse co-assembly map

$$\mathrm{K}_*(\mathfrak{c}^{\mathfrak{red}}_H(|G|) \rtimes H) \to \mathrm{KX}^*_H(|G|)$$

is an isomorphism for all compact subgroups H of G by [7].

For any G-C*-algebra B, the $\mathbb{Z}/2$-graded group $\mathrm{K}^{\mathrm{top}}_*(G, B) \cong \mathrm{K}_*\big((B \otimes \mathsf{P}) \rtimes_{\mathrm{r}} G\big)$ is a graded module over the $\mathbb{Z}/2$-graded ring $\mathrm{KK}^G_*(\mathsf{P}, \mathsf{P}) \cong \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ in a canonical way; the isomorphism between these two groups is a ring isomorphism because it is the composite of the two ring isomorphisms

$$\mathrm{KK}^G_*(\mathsf{P}, \mathsf{P}) \underset{\cong}{\xrightarrow{p^*_{\mathcal{E}G}}} \mathrm{RKK}^G_*(\mathcal{E}G; \mathsf{P}, \mathsf{P}) \underset{\cong}{\xleftarrow{\_\otimes \mathsf{P}}} \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}).$$

Hence we get module structures on $\mathrm{K}^{\mathrm{top}}_*\big(G, \mathfrak{c}^{\mathfrak{red}}(|G|)\big)$ and

$$\mathrm{K}^{\mathrm{top}}_*\big(G, C_0(\mathcal{E}G, \mathbb{K})\big) \cong \mathrm{K}_*(C_0(\mathcal{E}G) \rtimes G) \cong \mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C}).$$

The latter isomorphism is a module isomorphism; thus $\mathrm{K}^{\mathrm{top}}_*\big(G, C_0(\mathcal{E}G, \mathbb{K})\big)$ is a free module of rank 1. The equivariant co-assembly map is natural in the formal sense, so that it is an $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$-module homomorphism. Hence its range is a submodule, that is, an ideal in $\mathrm{RKK}^G_*(\mathcal{E}G; \mathbb{C}, \mathbb{C})$ (since this ring is graded commutative, there is no difference between one- and two-sided graded ideals). This also yields that μ^* is surjective if and only if it is bijective, if and only if the unit class $1_{\mathcal{E}G}$ belongs to its range; we already know this from [7].

If A is a proper G-C*-algebra, then $\mathrm{id}_A \otimes \mathsf{D} \in \mathrm{KK}^G(A \otimes \mathsf{P}, A)$ is invertible ([11]). If $b \in \mathrm{KK}^G_*(\mathbb{C}, A)$ and $c \in \mathrm{KK}^G_*(A, \mathbb{C})$, then we can write the Kasparov product $b \otimes_A c$ as

$$\mathbb{C} \xrightarrow{b} A \underset{\cong}{\xleftarrow{\mathrm{id}_A \otimes \mathsf{D}}} A \otimes \mathsf{P} \xrightarrow{c \otimes \mathrm{id}_{\mathsf{P}}} \mathsf{P} \xrightarrow{\mathsf{D}} \mathbb{C},$$

where the arrows are morphisms in the category KK^G. Therefore, $b \otimes_A c$ factors through D and hence is a boundary class by Theorem 10. □

5 Dual-Dirac morphisms and the Carlsson–Pedersen condition

Now we construct boundary classes in $\mathrm{RKK}^G_*(\mathcal{E}G;\mathbb{C},\mathbb{C})$ from more classical boundaries. We suppose again that $\mathcal{E}G$ is G-compact, so that $\mathcal{E}G$ is a coarse space.

A *metrisable compactification* of $\mathcal{E}G$ is a metrisable compact space Z with a homeomorphism between $\mathcal{E}G$ and a dense open subset of Z. It is called *coarse* if all scalar-valued functions on Z have vanishing variation; this implies the corresponding assertion for operator-valued functions because $C(Z,D) \cong C(Z)\otimes D$. Equivalently, the embedding $\mathcal{E}G \to Z$ factors through the Higson compactification of $\mathcal{E}G$. A compactification is called *G-equivariant* if Z is a G-space and the embedding $\mathcal{E}G \to Z$ is G-equivariant. An equivariant compactification is called *strongly contractible* if it is H-equivariantly contractible for all compact subgroups H of G. The *Carlsson–Pedersen condition* requires that there should be a G-compact model for $\mathcal{E}G$ that has a coarse, strongly contractible, and G-equivariant compactification.

Typical examples of such compactifications are the Gromov boundary for a hyperbolic group (viewed as a compactification of the Rips complex), or the visibility boundary of a CAT(0) space on which G acts properly, isometrically, and cocompactly.

Theorem 14. *Let G be a locally compact group with a G-compact model for $\mathcal{E}G$ and let $\mathcal{E}G \subseteq Z$ be a coarse, strongly contractible, G-equivariant compactification. Then G has a dual-Dirac morphism.*

Proof. We use the C*-algebra $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)$ as defined in §2.3. Since Z is coarse, there is an embedding $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z) \subseteq \bar{\mathfrak{c}}^{\mathfrak{red}}_G(\mathcal{E}G)$. Let $\partial Z := Z \setminus \mathcal{E}G$ be the boundary of the compactification. Identifying

$$\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z) \cong \bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)/C_0(\mathcal{E}G,\mathbb{K}_G),$$

we obtain a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & C_0(\mathcal{E}G,\mathbb{K}_G) \arrow[r] & \bar{\mathfrak{c}}^{\mathfrak{red}}_G(\mathcal{E}G) \arrow[r] & \mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G) \arrow[r] & 0 \\
0 \arrow[r] & C_0(\mathcal{E}G,\mathbb{K}_G) \arrow[r] \arrow[u, equal] & \bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z) \arrow[r] \arrow[u, "\subseteq"] & \bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z) \arrow[r] \arrow[u, "\subseteq"] & 0.
\end{tikzcd}$$

Let H be a compact subgroup. Since Z is compact, we have $\bar{\mathfrak{B}}_G(Z) = C(Z,\mathbb{K})$. Since Z is H-equivariantly contractible by hypothesis, $\bar{\mathfrak{B}}_G(Z)$ is H-equivariantly homotopy equivalent to $\mathbb{C}$. Hence $\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)$ has vanishing H-equivariant K-theory. This implies $\mathrm{K}^{\mathrm{top}}_*\big(G,\bar{\mathfrak{B}}^{\mathfrak{red}}_G(Z)\big) = 0$ by [2], so that the connecting map

$$\mathrm{K}^{\mathrm{top}}_{*+1}\big(G,\bar{\mathfrak{B}}^{\mathfrak{red}}_G(\partial Z)\big) \to \mathrm{K}^{\mathrm{top}}_*\big(G,C_0(\mathcal{E}G)\big) \cong \mathrm{K}_*(C_0(\mathcal{E}G)\rtimes G) \tag{19}$$

is an isomorphism. This in turn implies that the connecting map

$$\mathrm{K}^{\mathrm{top}}_{*+1}\big(G,\mathfrak{c}^{\mathfrak{red}}_G(\mathcal{E}G)\big) \to \mathrm{K}^{\mathrm{top}}_*\big(G,C_0(\mathcal{E}G)\big)$$

is surjective. Thus we can lift $1 \in \mathrm{RKK}_0^G(\mathcal{E}G; \mathbb{C}, \mathbb{C}) \cong \mathrm{K}_0(C_0(\mathcal{E}G) \rtimes G)$ to

$$\alpha \in \mathrm{K}_1^{\mathrm{top}}\big(G, \mathfrak{c}_G^{\mathfrak{red}}(\mathcal{E}G)\big) \cong \mathrm{K}_1^{\mathrm{top}}\big(G, \mathfrak{c}_G^{\mathfrak{red}}(|G|)\big).$$

Then $\Psi_*(\alpha) \in \mathrm{KK}_0^G(\mathbb{C}, \mathsf{P})$ is the desired dual-Dirac morphism. □

The group $\mathrm{K}_*^{\mathrm{top}}\big(G, \overline{\mathfrak{B}}_G^{\mathfrak{red}}(\partial Z)\big)$ that appears in the above argument is a *reduced* topological G-equivariant K-theory for ∂Z and hence differs from $\mathrm{K}_*^{\mathrm{top}}\big(G, C(\partial Z)\big)$. The relationship between these two groups is analysed in [6].

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On K_1 of a Waldhausen category

Fernando Muro and Andrew Tonks*

Introduction

A general notion of K-theory, for a category $\mathbf{W}$ with cofibrations and weak equivalences, was defined by Waldhausen [10] as the homotopy groups of the loop space of a certain simplicial category $wS_\bullet\mathbf{W}$,

$$K_n\mathbf{W} = \pi_n\Omega\,|wS_\bullet\mathbf{W}| \cong \pi_{n+1}\,|wS_\bullet\mathbf{W}|\,, \quad n \geq 0.$$

Waldhausen K-theory generalizes the K-theory of an exact category $\mathbf{E}$, defined as the homotopy groups of $\Omega\,|Q\mathbf{E}|$, where $Q\mathbf{E}$ is a category defined by Quillen.

Gillet and Grayson defined in [2] a simplicial set $G_\bullet\mathbf{E}$ which is a model for $\Omega\,|Q\mathbf{E}|$. This allows one to compute $K_1\mathbf{E}$ as a fundamental group, $K_1\mathbf{E} = \pi_1\,|G_\bullet\mathbf{E}|$. Using the standard techniques for computing π_1 Gillet and Grayson produced algebraic representatives for arbitrary elements in $K_1\mathbf{E}$. These representatives were simplified by Sherman [8], [9] and further simplified by Nenashev [5]. Nenashev's representatives are pairs of short exact sequences on the same objects,

$$A \underset{\rightarrowtail}{\rightarrowtail} B \underset{\twoheadrightarrow}{\twoheadrightarrow} C.$$

Such a pair gives a loop in $|G_\bullet\mathbf{E}|$ which corresponds to a 2-sphere in $|wS_\bullet\mathbf{E}|$, obtained by pasting the 2-simplices associated to each short exact sequence along their common boundary.

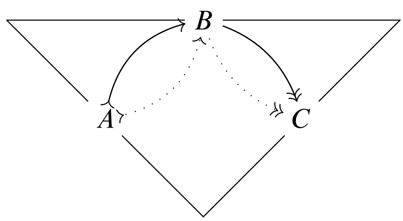

Figure 1. Nenashev's representative of an element in $K_1\mathbf{E}$.

Nenashev showed in [6] certain relations which are satisfied by these pairs of short exact sequences, and he later proved in [7] that pairs of short exact sequences together with these relations yield a presentation of $K_1\mathbf{E}$.

*The first author was partially supported by the MEC-FEDER grants MTM2004-01865 and MTM2004-03629, the MEC postdoctoral fellowship EX2004-0616, and a Juan de la Cierva research cantract; and the second author by the MEC-FEDER grant MTM2004-03629.

In this paper we produce representatives for all elements in K_1 of an arbitrary Waldhausen category $\mathbf{W}$ which are as close as possible to Nenashev's representatives for exact categories. They are given by pairs of cofiber sequences where the cofibrations have the same source and target. The cofibers, however, may be non-isomorphic, but they are weakly equivalent via a length 2 zig-zag.

$$\begin{tikzcd}[column sep=small, row sep=small] & & & C_1 & \\ A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[urr, two heads] \arrow[drr, two heads] & & & C. \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & & C_2 & \end{tikzcd}$$

This diagram is called a *pair of weak cofiber sequences*. It corresponds to a 2-sphere in $|wS.\mathbf{W}|$ obtained by pasting the 2-simplices associated to the cofiber sequences along the common part of the boundary. The two edges which remain free after this operation are filled with a disk made of two pieces corresponding to the weak equivalences.

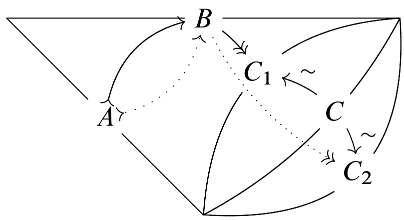

Figure 2. Our representative of an element in $K_1\mathbf{W}$.

In addition we show in Section 3 that pairs of weak cofiber sequences satisfy a certain generalization of Nenashev's relations.

Let $K_1^{\mathrm{wcs}}\,\mathbf{W}$ be the group generated by the pairs of weak cofiber sequences in $\mathbf{W}$ modulo the relations in Section 3. The results of this paper show in particular that there is a natural surjection

$$K_1^{\mathrm{wcs}}\,\mathbf{W} \twoheadrightarrow K_1\mathbf{W}$$

which, by [7], is an isomorphism in case $\mathbf{W} = \mathbf{E}$ is an exact category. See Section 6 for a comparison of $K_1^{\mathrm{wcs}}\mathbf{E}$ with Nenashev's presentation. We do not know whether this natural surjection is an isomorphism for an arbitrary Waldhausen category $\mathbf{W}$, but we prove here further evidence which supports the conjecture: the homomorphism

$$\cdot\,\eta\colon K_0\mathbf{W} \otimes \mathbb{Z}/2 \to K_1\mathbf{W}$$

induced by the action of the Hopf map $\eta \in \pi_*^s$ in the stable homotopy groups of spheres factors as

$$K_0\mathbf{W} \otimes \mathbb{Z}/2 \longrightarrow K_1^{\mathrm{wcs}}\,\mathbf{W} \twoheadrightarrow K_1\mathbf{W}.$$

We finish this introduction with some comments about the method of proof. We do not rely on any previous similar result because we do not know of any generalization of the Gillet–Grayson construction for an arbitrary Waldhausen category $\mathbf{W}$. The largest known class over which the Gillet–Grayson construction works is the class of pseudo-additive categories, which extends the class of exact categories but still does not cover all Waldhausen categories, see [3]. We use instead the algebraic model $\mathcal{D}_*\mathbf{W}$ defined in [4] for the 1-type of the Waldhausen K-theory spectrum $K\mathbf{W}$. The algebraic object $\mathcal{D}_*\mathbf{W}$ is a chain complex of non-abelian groups concentrated in dimensions $i = 0, 1$ whose homology is $K_i\mathbf{W}$,

$$\begin{tikzcd} & (\mathcal{D}_0\mathbf{W})^{\mathrm{ab}} \otimes (\mathcal{D}_0\mathbf{W})^{\mathrm{ab}} \arrow[d, "\langle\cdot,\cdot\rangle"'] & & \\ K_1\mathbf{W} \arrow[r, hook] & \mathcal{D}_1\mathbf{W} \arrow[r, "\partial"] & \mathcal{D}_0\mathbf{W} \arrow[r, two heads] & K_0\mathbf{W}, \end{tikzcd} \tag{A}$$

i.e. the lower row is exact. This non-abelian chain complex is equipped with a bilinear map $\langle\cdot,\cdot\rangle$ which determines the commutators. This makes $\mathcal{D}_*\mathbf{W}$ a stable quadratic module in the sense of [1]. This stable quadratic module is defined in [4] in terms of generators and relations. Generators correspond simply to objects, weak equivalences, and cofiber sequences in $\mathbf{W}$.

Acknowledgement. The authors are very grateful to Grigory Garkusha for asking about the relation between the algebraic model for Waldhausen K-theory defined in [4] and Nenashev's presentation of K_1 of an exact category in [7].

1 The algebraic model for K_0 and K_1

Definition 1.1. A *stable quadratic module* C_* is a diagram of group homomorphisms

$$C_0^{\mathrm{ab}} \otimes C_0^{\mathrm{ab}} \xrightarrow{\langle\cdot,\cdot\rangle} C_1 \xrightarrow{\partial} C_0$$

such that given $c_i, d_i \in C_i$, $i = 0, 1$,

1. $\partial\langle c_0, d_0\rangle = [d_0, c_0]$,
2. $\langle\partial(c_1), \partial(d_1)\rangle = [d_1, c_1]$,
3. $\langle c_0, d_0\rangle + \langle d_0, c_0\rangle = 0$.

Here $[x, y] = -x - y + x + y$ is the commutator of two elements $x, y \in K$ in any group K, and K^{ab} is the abelianization of K. It follows from the axioms that the image of $\langle\cdot,\cdot\rangle$ and $\operatorname{Ker}\partial$ are central in C_1, the groups C_0 and C_1 have nilpotency class 2, and $\partial(C_1)$ is a normal subgroup of C_0.

A *morphism* $f: C_* \to D_*$ of stable quadratic modules is given by group homomorphisms $f_i: C_i \to D_i$, $i = 0, 1$, compatible with the structure homomorphisms of C_* and D_*, i.e. $f_0\partial = \partial f_1$ and $f_1\langle\cdot,\cdot\rangle = \langle f_0, f_0\rangle$.

Stable quadratic modules were introduced in [1, Definition IV.C.1]. Notice, however, that we adopt the opposite convention for the homomorphism $\langle\cdot,\cdot\rangle$, i.e. we compose with the symmetry isomorphism on the tensor square.

Remark 1.2. There is a natural right action of C_0 on C_1 defined by

$$c_1^{c_0} = c_1 + \langle c_0, \partial(c_1) \rangle.$$

The axioms of a stable quadratic module imply that commutators in C_0 act trivially on C_1, and that C_0 acts trivially on the image of $\langle \cdot, \cdot \rangle$ and on Ker ∂.

The action gives $\partial \colon C_1 \to C_0$ the structure of a crossed module. Indeed a stable quadratic module is the same as a commutative monoid in the category of crossed modules such that the monoid product of two elements in C_0 vanishes when one of them is a commutator, see [4, Lemma 4.18].

A stable quadratic module can be defined by generators and relations in degree 0 and 1. In the appendix we give details about the construction of a stable quadratic module defined by generators and relations. This is useful to understand Definition 1.3 below from a purely group-theoretic perspective.

We assume the reader has certain familiarity with Waldhausen categories and related concepts. We refer to [12] for the basics, see also [11]. The following definition was introduced in [4].

Definition 1.3. Let $\mathbf{C}$ be a Waldhausen category with distinguished zero object $*$, and cofibrations and weak equivalences denoted by $\rightarrowtail$ and $\xrightarrow{\sim}$, respectively. A generic cofiber sequence is denoted by

$$A \rightarrowtail B \twoheadrightarrow B/A.$$

We define $\mathcal{D}_*\mathbf{C}$ as the stable quadratic module generated in dimension zero by the symbols

- $[A]$ for any object in $\mathbf{C}$,

and in dimension one by

- $[A \xrightarrow{\sim} A']$ for any weak equivalence,
- $[A \rightarrowtail B \twoheadrightarrow B/A]$ for any cofiber sequence,

such that the following relations hold.

(R1) $\partial[A \xrightarrow{\sim} A'] = -[A'] + [A]$.

(R2) $\partial[A \rightarrowtail B \twoheadrightarrow B/A] = -[B] + [B/A] + [A]$.

(R3) $[*] = 0$.

(R4) $[A \xrightarrow{1_A} A] = 0$.

(R5) $[A \xrightarrow{1_A} A \twoheadrightarrow *] = 0$, $[* \rightarrowtail A \xrightarrow{1_A} A] = 0$.

(R6) For any pair of composable weak equivalences $A \xrightarrow{\sim} B \xrightarrow{\sim} C$,

$$[A \xrightarrow{\sim} C] = [B \xrightarrow{\sim} C] + [A \xrightarrow{\sim} B].$$

(R7) For any commutative diagram in $\mathbf{C}$ as follows

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\sim"] & B \arrow[r, two heads] \arrow[d, "\sim"] & B/A \arrow[d, "\sim"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & B'/A' \end{tikzcd}$$

we have

$$\begin{aligned}[A \xrightarrow{\sim} A'] + [B/A \xrightarrow{\sim} B'/A']^{[A]} = &- [A' \rightarrowtail B' \twoheadrightarrow B'/A'] \\ &+ [B \xrightarrow{\sim} B'] + [A \rightarrowtail B \twoheadrightarrow B/A].\end{aligned}$$

(R8) For any commutative diagram consisting of four cofiber sequences in $\mathbf{C}$ as follows

$$\begin{tikzcd} & & C/B \\ & B/A \arrow[r, tail] & C/A \arrow[u, two heads] \\ A \arrow[r, tail] & B \arrow[r, tail] \arrow[u, two heads] & C \arrow[u, two heads] \end{tikzcd}$$

we have

$$\begin{aligned}&[B \rightarrowtail C \twoheadrightarrow C/B] + [A \rightarrowtail B \twoheadrightarrow B/A] \\ &\quad = [A \rightarrowtail C \twoheadrightarrow C/A] + [B/A \rightarrowtail C/A \twoheadrightarrow C/B]^{[A]}.\end{aligned}$$

(R9) For any pair of objects A, B in $\mathbf{C}$

$$\langle [A], [B] \rangle = -[B \overset{i_2}{\rightarrowtail} A \vee B \overset{p_1}{\twoheadrightarrow} A] + [A \overset{i_1}{\rightarrowtail} A \vee B \overset{p_2}{\twoheadrightarrow} B].$$

Here

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \vee B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

are the inclusions and projections of a coproduct in $\mathbf{C}$.

Remark 1.4. These relations are quite natural, and some illustration of their meaning is given in [4, Figure 2]. They are not however minimal: relation (R3) follows from (R2) and (R5), and (R4) follows from (R6). Also (R5) is equivalent to

(R5′) $[* \rightarrowtail * \twoheadrightarrow *] = 0.$

This equivalence follows from (R8) on considering the diagrams

$$
\begin{tikzcd}
 & & * \\
 & * \arrow[r, tail] & * \arrow[u, two heads] \\
A \arrow[r, tail, "1_A"] & A \arrow[u, two heads] \arrow[r, tail, "1_A"] & A \arrow[u, two heads],
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A \\
 & * \arrow[r, tail] & A \arrow[u, two heads, "1_A"'] \\
* \arrow[r, tail] & * \arrow[u, two heads] \arrow[r, tail] & A. \arrow[u, two heads, "1_A"']
\end{tikzcd}
$$

We now introduce the basic object of study of this paper.

Definition 1.5. A *weak cofiber sequence* in a Waldhausen category **C** is a diagram

$$A \rightarrowtail B \twoheadrightarrow C_1 \overset{\sim}{\leftarrow} C$$

given by a cofiber sequence followed by a weak equivalence in the opposite direction.

We associate the element

$$[A \rightarrowtail B \twoheadrightarrow C_1 \overset{\sim}{\leftarrow} C] = [A \rightarrowtail B \twoheadrightarrow C_1] + [C \overset{\sim}{\rightarrow} C_1]^{[A]} \in \mathcal{D}_*\mathbf{C} \tag{1.1}$$

to any weak cofiber sequence. By (R1) and (R2) we have

$$\partial[A \rightarrowtail B \twoheadrightarrow C_1 \overset{\sim}{\leftarrow} C] = -[B] + [C] + [A].$$

The following fundamental identity for weak cofiber sequences is a slightly less trivial consequence of the relations (R1)–(R9).

Proposition 1.6. *Consider a commutative diagram*

$$
\begin{tikzcd}
A' \arrow[r, tail, "j^A"] \arrow[d, tail, "j'"'] & A \arrow[r, two heads, "r^A"] \arrow[d, tail, "j"'] & A''' \arrow[d, tail, "j'''"'] & A'' \arrow[l, "\sim", "w^A"'] \arrow[d, tail, "j''"] \\
B' \arrow[r, tail, "j^B"] \arrow[d, two heads, "r'"'] & B \arrow[r, two heads, "r^B"] \arrow[d, two heads, "r"'] & B''' \arrow[d, two heads, "r'''"'] & B'' \arrow[l, "\sim", "w^B"'] \arrow[d, two heads, "r''"] \\
D' \arrow[r, tail, "j^D"] & D \arrow[r, two heads, "r^D"] & D''' & D'' \arrow[l, "\sim", "w^D"'] \\
C' \arrow[r, tail, "j^C"] \arrow[u, "w'", "\sim"'] & C \arrow[r, two heads, "r^C"] \arrow[u, "w", "\sim"'] & C''' \arrow[u, "w'''", "\sim"'] & C'' \arrow[l, "\sim", "w^C"'] \arrow[u, "w''", "\sim"']
\end{tikzcd}
$$

such that the induced map $j \cup_{A'} j^B \colon A \cup_{A'} B' \rightarrowtail B$ is a cofibration. Denote the horizontal weak cofiber sequences by l^A, l^B, l^D and l^C, and the vertical ones by l', l, l''' and l''. Then the following equation holds in $\mathcal{D}_*\mathbf{C}$.

$$-[l^A] - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} + [l'] = \langle [C'], [A''] \rangle.$$

Proof. Applying (R7) to the two obvious equivalences of cofiber sequences gives

$$[D' \rightarrowtail D \twoheadrightarrow D'''] = [w] + [l^C] - [w^C]^{[C']} - [w''']^{[C']} - [w'], \tag{a}$$

$$[A''' \rightarrowtail B''' \twoheadrightarrow D'''] = [w^B] + [l''] - [w'']^{[A'']} - [w^D]^{[A'']} - [w^A], \tag{b}$$

using the notation of (1.1).

Now consider the following commutative diagrams:

$$\begin{tikzcd}
A' \arrow[r, tail, "j^A"] \arrow[d, tail, "j'"'] \arrow[dr, phantom, "\text{push}"] & A \arrow[d] \arrow[ddr, tail, bend left, "j"] & \\
B' \arrow[r, tail] \arrow[drr, tail, bend right, "j^B"'] & X \arrow[dr, tail] & \\
& & B,
\end{tikzcd}
\qquad
\begin{tikzcd}
A' \arrow[r, tail, "j^A"] \arrow[d, tail, "j'"'] \arrow[dr, phantom, "\text{push}"] & A \arrow[d] \arrow[ddr, bend left, "i_1 r^A"] & \\
B' \arrow[r, tail] \arrow[drr, bend right, "i_2 r'"'] & X \arrow[dr, two heads] & \\
& & A''' \vee D'.
\end{tikzcd}$$

We now have four commutative diagrams of cofiber sequences as in (R8).

c)

$$\begin{tikzcd}
& & D' \\
& A''' \arrow[r, tail] & A''' \vee D' \arrow[u, two heads] \\
A' \arrow[r, tail] & A \arrow[r, tail] \arrow[u, two heads] & X \arrow[u, two heads]
\end{tikzcd}$$

d)

$$\begin{tikzcd}
& & A''' \\
& D' \arrow[r, tail] & A''' \vee D' \arrow[u, two heads] \\
A' \arrow[r, tail] & B' \arrow[r, tail] \arrow[u, two heads] & X \arrow[u, two heads]
\end{tikzcd}$$

e)

$$\begin{tikzcd}
& & D''' \\
& D' \arrow[r, tail] & D \arrow[u, two heads] \\
A \arrow[r, tail] & X \arrow[r, tail] \arrow[u, two heads] & B \arrow[u, two heads]
\end{tikzcd}$$

f)

$$\begin{tikzcd}
& & D''' \\
& A''' \arrow[r, tail] & B''' \arrow[u, two heads] \\
B' \arrow[r, tail] & X \arrow[r, tail] \arrow[u, two heads] & B \arrow[u, two heads]
\end{tikzcd}$$

Therefore we obtain the corresponding relations

$$
\begin{aligned}
&[A \rightarrowtail X \twoheadrightarrow D'] + [l^A] - [w^A]^{[A']} &&\text{(c)}\\
&\qquad = [A' \rightarrowtail X \twoheadrightarrow A''' \vee D'] + [A''' \rightarrowtail A''' \vee D' \twoheadrightarrow D']^{[A']},\\
&[B' \rightarrowtail X \twoheadrightarrow A'''] + [l'] - [w']^{[A']} &&\text{(d)}\\
&\qquad = [A' \rightarrowtail X \twoheadrightarrow A''' \vee D'] + [D' \rightarrowtail A''' \vee D' \twoheadrightarrow A''']^{[A']},\\
&[X \rightarrowtail B \twoheadrightarrow D'''] + [A \rightarrowtail X \twoheadrightarrow D'] &&\text{(e)}\\
&\qquad = [l] - [w]^{[A]} + [D' \rightarrowtail D \twoheadrightarrow D''']^{[A]}\\
&\qquad = [l] + ([l^C] - [w^C]^{[C']} - [w''']^{[C']} - [w'])^{[A]},\\
&[X \rightarrowtail B \twoheadrightarrow D'''] + [B' \rightarrowtail X \twoheadrightarrow A'''] &&\text{(f)}\\
&\qquad = [l^B] - [w^B]^{[B']} + [A''' \rightarrowtail B''' \twoheadrightarrow D''']^{[B']}\\
&\qquad = [l^B] + ([l''] - [w'']^{[A'']} - [w^D]^{[A'']} - [w^A])^{[B']}.
\end{aligned}
$$

Here we have used equations (a) and (b). Now $-$(c) + (d) and $-$(e) + (f) yield

$$
\begin{aligned}
&[w^A]^{[A']} - [l^A] - [A \rightarrowtail X \twoheadrightarrow D'] + [B' \rightarrowtail X \twoheadrightarrow A'''] + [l'] - [w']^{[A']} &&\text{(g)}\\
&\qquad = -[A''' \rightarrowtail A''' \vee D' \twoheadrightarrow D']^{[A']} + [D' \rightarrowtail A''' \vee D' \twoheadrightarrow A''']^{[A']}\\
&\qquad = \langle [D'], [A'''] \rangle,
\end{aligned}
$$

$$
\begin{aligned}
&-[A \rightarrowtail X \twoheadrightarrow D'] + [B' \rightarrowtail X \twoheadrightarrow A'''] &&\text{(h)}\\
&\qquad = [w']^{[A]} + a^{[C']+[A]} - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} - a^{[A'']+[B']} - [w^A]^{[B']},
\end{aligned}
$$

where we use (R9) and Remark 1.2, and we write $a = [w'''] + [w^C] = [w^D] + [w'']$, by (R6). In fact it is easy to check by using Remark 1.2 and the laws of stable quadratic modules that the terms $a^{[C']+[A]}$ and $a^{[A'']+[B']}$ cancel in this expression, since

$$\partial\big(- [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']}\big) = -([C'] + [A]) + ([A''] + [B']).$$

Substituting (h) into the left hand side of (g) we obtain

$$
\begin{aligned}
&[w^A]^{[A']} - [l^A] + [w']^{[A]} - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} - [w^A]^{[B']} + [l'] - [w']^{[A']}\\
&\qquad = \langle [D'], [A'''] \rangle,
\end{aligned}
$$

which, using again Definition 1.1 and Remark 1.2, may be rewritten as

$$
\begin{aligned}
&- [l^A] - [l^C]^{[A]} - [l] + [l^B] + [l'']^{[B']} + [l']\\
&\qquad = \; - [w']^{[A'']+[A']} - [w^A]^{[A']} - \langle [A'''], [D'] \rangle + [w']^{[A']} + [w^A]^{[C']+[A']}.
\end{aligned}
$$

The result then follows from the identity

$$\langle \partial[w^A], \partial[w'] \rangle - \langle [A''], \partial[w'] \rangle + \langle [C'], \partial[w^A] \rangle = \langle [C'], [A''] \rangle + \langle [A'''], [D'] \rangle. \quad \square$$

2 Pairs of weak cofiber sequences and K_1

Let **C** be a Waldhausen category. A *pair of weak cofiber sequences* is a diagram given by two weak cofiber sequences with the first, second, and fourth objects in common:

$$\begin{tikzcd}[row sep=small] & & C_1 & \\ A \arrow[r, shift left, tail] \arrow[r, shift right, tail] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 . & \end{tikzcd}$$

We associate to any such pair the following element

$$\left\{ \begin{tikzcd}[row sep=small] & & C_1 & \\ A \arrow[r, shift left, tail, "j_1"] \arrow[r, shift right, tail, "j_2"'] & B \arrow[ur, two heads, "r_1"] \arrow[dr, two heads, "r_2"'] & & C \arrow[ul, "\sim"', "w_1"] \arrow[dl, "\sim", "w_2"'] \\ & & C_2 & \end{tikzcd} \right\} = -[A \overset{j_1}{\rightarrowtail} B \overset{r_1}{\twoheadrightarrow} C_1 \underset{w_1}{\overset{\sim}{\leftarrow}} C] + [A \overset{j_2}{\rightarrowtail} B \overset{r_2}{\twoheadrightarrow} C_2 \underset{w_2}{\overset{\sim}{\leftarrow}} C] \tag{2.1}$$

$$= [A \overset{j_2}{\rightarrowtail} B \overset{r_2}{\twoheadrightarrow} C_2 \underset{w_2}{\overset{\sim}{\leftarrow}} C] - [A \overset{j_1}{\rightarrowtail} B \overset{r_1}{\twoheadrightarrow} C_1 \underset{w_1}{\overset{\sim}{\leftarrow}} C],$$

which lies in $K_1\mathbf{C} = \operatorname{Ker}\partial$, see diagram (A) in the introduction. For the second equality in (2.1) we use that $\operatorname{Ker}\partial$ is central, see Remark 1.2, so we can permute the terms cyclically.

The following theorem is one of the main results of this paper.

Theorem 2.1. *Any element in $K_1\mathbf{C}$ is represented by a pair of weak cofiber sequences.*

This theorem will be proved later in this paper. We first give a set of useful relations between pairs of weak cofiber sequences and develop a sum-normalized version of the model $\mathcal{D}_*\mathbf{C}$.

3 Relations between pairs of weak cofiber sequences

Suppose we have six pairs of weak cofiber sequences in a Waldhausen category **C**

$$\begin{tikzcd}[row sep=small] & & A''_1 & \\ A' \arrow[r, shift left, tail, "j_1^A"] \arrow[r, shift right, tail, "j_2^A"'] & A \arrow[ur, two heads, "r_1^A"] \arrow[dr, two heads, "r_2^A"'] & & A'', \arrow[ul, "\sim", "w_1^A"'] \arrow[dl, "\sim"', "w_2^A"] \\ & & A''_2 & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] & & B''_1 & \\ B' \arrow[r, shift left, tail, "j_1^B"] \arrow[r, shift right, tail, "j_2^B"'] & B \arrow[ur, two heads, "r_1^B"] \arrow[dr, two heads, "r_2^B"'] & & B'', \arrow[ul, "\sim", "w_1^B"'] \arrow[dl, "\sim"', "w_2^B"] \\ & & B''_2 & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] & & \check{C}''_1 & \\ C' \arrow[r, shift left, tail, "j_1^C"] \arrow[r, shift right, tail, "j_2^C"'] & C \arrow[ur, two heads, "r_1^C"] \arrow[dr, two heads, "r_2^C"'] & & C'', \arrow[ul, "\sim", "w_1^C"'] \arrow[dl, "\sim"', "w_2^C"] \\ & & \check{C}''_2 & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & & C'_1 & \\ A' \arrow[r, shift left, tail, "j'_1"] \arrow[r, shift right, tail, "j'_2"'] & B' \arrow[ur, two heads, "r'_1"] \arrow[dr, two heads, "r'_2"'] & & C', \arrow[ul, "\sim", "w'_1"'] \arrow[dl, "\sim"', "w'_2"] \\ & & C'_2 & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] & & C_1 & \\ A \arrow[r, shift left, tail, "j_1"] \arrow[r, shift right, tail, "j_2"'] & B \arrow[ur, two heads, "r_1"] \arrow[dr, two heads, "r_2"'] & & C, \arrow[ul, "\sim", "w_1"'] \arrow[dl, "\sim"', "w_2"] \\ & & C_2 & \end{tikzcd} \quad \begin{tikzcd}[row sep=small] & & \hat{C}''_1 & \\ A'' \arrow[r, shift left, tail, "j''_1"] \arrow[r, shift right, tail, "j''_2"'] & B'' \arrow[ur, two heads, "r''_1"] \arrow[dr, two heads, "r''_2"'] & & C'', \arrow[ul, "\sim", "w''_1"'] \arrow[dl, "\sim"', "w''_2"] \\ & & \hat{C}''_2 & \end{tikzcd}$$

that we denote for the sake of simplicity as λ^A, λ^B, λ^C, λ', λ, and λ'', respectively. Moreover, assume that for $i = 1, 2$ there exists a commutative diagram

$$\begin{tikzcd}
A' \arrow[r, tail, "j_i^A"] \arrow[d, tail, "j_i'"'] & A \arrow[r, two heads, "r_i^A"] \arrow[d, tail, "j_i"'] & A_i'' \arrow[d, tail, "\check{j}_i"'] & A'' \arrow[l, "\sim", "w_i^A"'] \arrow[d, tail, "j_i''"] \\
B' \arrow[r, tail, "j_i^B"] \arrow[d, two heads, "r_i'"'] & B \arrow[r, two heads, "r_i^B"] \arrow[d, two heads, "r_i"'] & B_i'' \arrow[d, two heads, "\check{r}_i"'] & B'' \arrow[l, "\sim", "w_i^B"'] \arrow[d, two heads, "r_i''"] \\
C_i' \arrow[r, tail, "\hat{j}_i"] & C_i \arrow[r, two heads, "\hat{r}_i"] & C_i'' & \hat{C}_i'' \arrow[l, "\sim", "\hat{w}_i"'] \\
C' \arrow[r, tail, "j_i^C"'] \arrow[u, "w_i'", "\sim"'] & C \arrow[r, two heads, "r_i^C"'] \arrow[u, "w_i", "\sim"'] & \check{C}_i'' \arrow[u, "\check{w}_i", "\sim"'] & C'' \arrow[l, "\sim", "w_i^C"'] \arrow[u, "w_i''", "\sim"']
\end{tikzcd}$$

such that the induced map $j_i \cup_{A'} j_i^B \colon A \cup_{A'} B' \rightarrowtail B$ is a cofibration. Notice that six of the eight weak cofiber sequences in this diagram are already in the six diagrams above. The other two weak cofiber sequences are just assumed to exist with the property of making the diagram commutative.

Theorem 3.1. *In the situation above the following equation holds*

$$\{\lambda^A\} - \{\lambda^B\} + \{\lambda^C\} = \{\lambda'\} - \{\lambda\} + \{\lambda''\}. \tag{S1}$$

Proof. For $i = 1, 2$ let λ_i^A, λ_i^B, λ_i^C, λ_i', λ_i, and λ_i'' be the upper and the lower weak cofiber sequences of the pairs above, respectively. By Proposition 1.6

$$\begin{aligned}
-[\lambda_1^A] - [\lambda_1^C]^{[A]} - [\lambda_1] + [\lambda_1^B] + [\lambda_1'']^{[B']} + [\lambda_1'] &= \langle [C'], [A''] \rangle \\
&= -[\lambda_2^A] - [\lambda_2^C]^{[A]} - [\lambda_2] + [\lambda_2^B] + [\lambda_2'']^{[B']} + [\lambda_2'].
\end{aligned}$$

Since $\partial(\lambda_1^C) = \partial(\lambda_2^C)$ and $\partial(\lambda_1'') = \partial(\lambda_2'')$ the actions cancel,

$$-[\lambda_1^A] - [\lambda_1^C] - [\lambda_1] + [\lambda_1^B] + [\lambda_1''] + [\lambda_1'] = -[\lambda_2^A] - [\lambda_2^C] - [\lambda_2] + [\lambda_2^B] + [\lambda_2''] + [\lambda_2'].$$

Now one can rearrange the terms in this equation, by using that pairs of weak cofiber sequences are central in $\mathcal{D}_1\mathbf{C}$, obtaining the equation in the statement. □

Theorem 3.1 encodes the most relevant relation satisfied by pairs of weak cofiber sequences in $K_1\mathbf{C}$. They satisfy a further relation which follows from the very definition in (2.1), but which we would like to record as a proposition by analogy with [6].

Proposition 3.2. *A pair of weak cofiber sequences given by two times the same weak cofiber sequence is trivial.*

$$\left\{\begin{tikzcd} & & C_1 & \\ A \arrow[r, tail, shift left, "j"] \arrow[r, tail, shift right, "j"'] & B \arrow[ur, two heads, "r"] \arrow[dr, two heads, "r"'] & & C \arrow[ul, "\sim"', "w"] \arrow[dl, "\sim", "w"'] \\ & & C_1 & \end{tikzcd}\right\} = 0. \tag{S2}$$

We establish a further useful relation in the following proposition.

Proposition 3.3. *Relations* (S1) *and* (S2) *imply that the sum of two pairs of weak cofiber sequences coincides with their coproduct.*

$$\left\{\begin{tikzcd} & & C_1 \vee \bar{C}_1 & \\ A \vee \bar{A} \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \vee \bar{B} \arrow[ur, two heads] \arrow[dr, two heads] & & C \vee \bar{C} \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 \vee \bar{C}_2 & \end{tikzcd}\right\} = \left\{\begin{tikzcd} & & C_1 & \\ A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd}\right\}$$

$$+ \left\{\begin{tikzcd} & & \bar{C}_1 & \\ \bar{A} \arrow[r, tail, shift left] \arrow[r, tail, shift right] & \bar{B} \arrow[ur, two heads] \arrow[dr, two heads] & & \bar{C} \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & \bar{C}_2 & \end{tikzcd}\right\}.$$

Proof. Apply relations (S1) and (S2) to the following pairs of weak cofiber sequences

$$\begin{tikzcd} & & C_1 & \\ A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C, \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd} \qquad \begin{tikzcd} & & C_1 \vee \bar{C}_1 & \\ A \vee \bar{A} \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \vee \bar{B} \arrow[ur, two heads] \arrow[dr, two heads] & & C \vee \bar{C}, \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 \vee \bar{C}_2 & \end{tikzcd}$$

$$\begin{tikzcd} & & \bar{C}_1 & \\ \bar{A} \arrow[r, tail, shift left] \arrow[r, tail, shift right] & \bar{B} \arrow[ur, two heads] \arrow[dr, two heads] & & \bar{C}, \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & \bar{C}_2 & \end{tikzcd} \qquad \begin{tikzcd} & & \bar{A} & \\ A \arrow[r, tail, shift left, "i_1"] \arrow[r, tail, shift right, "i_1"'] & A \vee \bar{A} \arrow[ur, two heads, "p_2"] \arrow[dr, two heads, "p_2"'] & & \bar{A}, \arrow[ul, "\sim", "1"'] \arrow[dl, "\sim"', "1"] \\ & & \bar{A} & \end{tikzcd}$$

$$\begin{tikzcd} & & \bar{B} & \\ B \arrow[r, tail, shift left, "i_1"] \arrow[r, tail, shift right, "i_1"'] & B \vee \bar{B} \arrow[ur, two heads, "p_2"] \arrow[dr, two heads, "p_2"'] & & \bar{B}, \arrow[ul, "\sim", "1"'] \arrow[dl, "\sim"', "1"] \\ & & \bar{B} & \end{tikzcd} \qquad \begin{tikzcd} & & \bar{C} & \\ C \arrow[r, tail, shift left, "i_1"] \arrow[r, tail, shift right, "i_1"'] & C \vee \bar{C} \arrow[ur, two heads, "p_2"] \arrow[dr, two heads, "p_2"'] & & \bar{C}. \arrow[ul, "\sim", "1"'] \arrow[dl, "\sim"', "1"] \\ & & \bar{C} & \end{tikzcd}$$

□

4 Waldhausen categories with functorial coproducts

Waldhausen categories admit finite coproducts $A \vee B$. The operation $\vee$ need not be strictly associative and unital. However, as we check below, any Waldhausen category can be replaced by an equivalent one with strict coproducts.

We now give an explicit definition of what we understand by a strict coproduct functor. Let $\mathbf{C}$ be a Waldhausen category endowed with a symmetric monoidal structure $+$ which is strictly associative

$$(A + B) + C = A + (B + C),$$

strictly unital with unit object $*$, the distinguished zero object,

$$* + A = A = A + *,$$

but not necessarily strictly commutative, such that

$$A = A + * \longrightarrow A + B \longleftarrow * + B = B$$

is always a coproduct diagram. Such a category will be called a *Waldhausen category with a functorial coproduct*. Then we define the sum-normalized stable quadratic module $\mathcal{D}_*^+\mathbf{C}$ as the quotient of $\mathcal{D}_*\mathbf{C}$ by the extra relation

(R10) $[B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] = 0.$

Remark 4.1. In $\mathcal{D}_*^+\mathbf{C}$, the relations (R2) and (R10) imply that

$$[A + B] = [A] + [B].$$

Furthermore, relation (R9) becomes equivalent to

(R9′) $\langle [A], [B] \rangle = [\tau_{B,A} \colon B + A \overset{\cong}{\longrightarrow} A + B].$

Here $\tau_{B,A}$ is the symmetry isomorphism of the symmetric monoidal structure. This equivalence follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r, tail, "i_2"] \arrow[d, equal] & B + A \arrow[r, two heads, "p_1"] \arrow[d, "\tau_{B,A}"', "\cong"] & B \arrow[d, equal] \\ A \arrow[r, tail, "i_1"'] & A + B \arrow[r, two heads, "p_2"'] & B \end{tikzcd}$$

together with (R4), (R7) and (R10).

Theorem 4.2. *Let $\mathbf{C}$ be a Waldhausen category with a functorial coproduct such that there exists a set $\mathbb{S}$ which freely generates the monoid of objects of $\mathbf{C}$ under the operation $+$. Then the projection*

$$p \colon \mathcal{D}_*\mathbf{C} \twoheadrightarrow \mathcal{D}_*^+\mathbf{C}$$

is a weak equivalence. Indeed, it is part of a strong deformation retraction.

We prove this theorem later in this section.

Waldhausen categories satisfying the hypothesis of Theorem 4.2 are general enough as the following proposition shows.

Proposition 4.3. *For any Waldhausen category* $\mathbf{C}$ *there is another Waldhausen category* $\mathrm{Sum}(\mathbf{C})$ *with a functorial coproduct whose monoid of objects is freely generated by the objects of* $\mathbf{C}$ *except from* $*$. *Moreover, there are natural mutually inverse exact equivalences of categories*

$$\mathrm{Sum}(\mathbf{C}) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathbf{C}.$$

Proof. The statement already says which are the objects of $\mathrm{Sum}(\mathbf{C})$. The functor φ sends an object Y in $\mathrm{Sum}(\mathbf{C})$, which can be uniquely written as a formal sum of non-zero objects in $\mathbf{C}$, $Y = X_1 + \cdots + X_n$, to an arbitrarily chosen coproduct of these objects in $\mathbf{C}$

$$\varphi(Y) = X_1 \vee \cdots \vee X_n.$$

For $n = 0, 1$ we make special choices, namely for $n = 0$, $\varphi(Y) = *$, and for $n = 1$ we set $\varphi(Y) = X_1$. Morphisms in $\mathrm{Sum}(\mathbf{C})$ are defined in the unique possible way so that φ is fully faithful. Then the formal sum defines a functorial coproduct on $\mathrm{Sum}(\mathbf{C})$. The functor ψ sends $*$ to the zero object of the free monoid and any other object in $\mathbf{C}$ to the corresponding object with a single summand in $\mathrm{Sum}(\mathbf{C})$, so that $\varphi\psi = 1$. The inverse natural isomorphism $1 \cong \psi\varphi$,

$$X_1 + \cdots + X_n \cong (X_1 \vee \cdots \vee X_n),$$

is the unique isomorphism in $\mathrm{Sum}(\mathbf{C})$ which φ maps to the identity. Cofibrations and weak equivalences in $\mathrm{Sum}(\mathbf{C})$ are the morphisms which φ maps to cofibrations and weak equivalences, respectively. This Waldhausen category structure in $\mathrm{Sum}(\mathbf{C})$ makes the functors φ and ψ exact. □

Let us recall the notion of homotopy in the category of stable quadratic modules.

Definition 4.4. Given morphisms $f, g\colon C_* \to C'_*$ of stable quadratic modules, a *homotopy* $f \overset{\alpha}{\simeq} g$ from f to g is a function $\alpha\colon C_0 \to C'_1$ satisfying

1. $\alpha(c_0 + d_0) = \alpha(c_0)^{f_0(d_0)} + \alpha(d_0)$,
2. $g_0(c_0) = f_0(c_0) + \partial\alpha(c_0)$,
3. $g_1(c_1) = f_1(c_1) + \alpha\partial(c_1)$.

The following lemma then follows from the laws of stable quadratic modules.

Lemma 4.5. *A homotopy* α *as in Definition* 4.4 *satisfies*

$$\alpha([c_0, d_0]) = -\langle f_0(d_0), f_0(c_0)\rangle + \langle g_0(d_0), g_0(c_0)\rangle, \tag{a}$$
$$\alpha(c_0) + g_1(c_1) = f_1(c_1) + \alpha(c_0 + \partial(c_1)). \tag{b}$$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. In order to define a strong deformation retraction,

$$\begin{tikzcd} \mathcal{D}_*\mathbf{C} \arrow[loop left, "\alpha"] \arrow[r, shift left, "p"] & \mathcal{D}_*^+\mathbf{C} \arrow[l, shift left, "j"] \end{tikzcd}, \quad 1 \overset{\alpha}{\simeq} jp, \ pj = 1,$$

the crucial step will be to define the homotopy $\alpha\colon \mathcal{D}_0\mathbf{C} \to \mathcal{D}_1\mathbf{C}$ on the generators $[A]$ of $\mathcal{D}_0\mathbf{C}$. Then one can use the equations

1. $\alpha(c_0 + d_0) = \alpha(c_0)^{d_0} + \alpha(d_0)$,
2. $jp(c_0) = c_0 + \partial\alpha(c_0)$,
3. $jp(c_1) = c_1 + \alpha\partial(c_1)$,

for $c_i, d_i \in \mathcal{D}_i\mathbf{C}$ to define the composite jp and the homotopy α on all of $\mathcal{D}_*\mathbf{C}$. It is a straightforward calculation to check that the map jp so defined is a homomorphism and that α is well defined. In particular, Lemma 4.5 (a) implies that α vanishes on commutators of length 3. In order to define j we must show that jp factors through the projection p, that is,

$$jp([B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A]) = 0.$$

If we also show that

$$p\alpha = 0$$

then equations (2) and (3) above say $pjp(c_i) = p(c_i) + 0$, and since p is surjective it follows that the composite pj is the identity.

The set of objects of $\mathbf{C}$ is the free monoid generated by objects $S \in \mathbb{S}$. Therefore we can define inductively the homotopy α by

$$\alpha([S + B]) = [B \overset{i_2}{\rightarrowtail} S + B \overset{p_1}{\twoheadrightarrow} S] + \alpha([B])$$

for $B \in \mathbf{C}$ and $S \in \mathbb{S}$, with $\alpha([S]) = 0$. We claim that a similar relation then holds for all objects A, B of $\mathbf{C}$,

$$\begin{aligned} \alpha([A + B]) &= [B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] + \alpha([A] + [B]) \\ &= [B \overset{i_2}{\rightarrowtail} A + B \overset{p_1}{\twoheadrightarrow} A] + \alpha([A])^{[B]} + \alpha([B]). \end{aligned} \tag{4.1}$$

If $A = *$ or $A \in \mathbb{S}$ then (4.1) holds by definition, so we assume inductively it holds for given A, B and show it holds for $S + A$, B also:

$$\begin{aligned} \alpha([S + A + B]) &= [A + B \rightarrowtail S + A + B \twoheadrightarrow S] + \alpha([A + B]) \\ &= [A + B \rightarrowtail S + A + B \twoheadrightarrow S] + [B \rightarrowtail A + B \twoheadrightarrow A] + \alpha([A])^{[B]} + \alpha([B]) \\ &= [B \rightarrowtail S + A + B \twoheadrightarrow S + A] + [A \rightarrowtail S + A \twoheadrightarrow S]^{[B]} + \alpha([A])^{[B]} + \alpha([B]) \\ &= [B \rightarrowtail S + A + B \twoheadrightarrow S + A] + \alpha([S + A])^{[B]} + \alpha([B]). \end{aligned}$$

Here we have used (R8) for the composable cofibrations

$$B \rightarrowtail A + B \rightarrowtail S + A + B.$$

It is clear from the definition of α that the relation $p\alpha = 0$ holds. It remains to see that jp factors through $\mathcal{D}_*^+\mathbf{C}$, that is,

$$jp(c_1) = 0 \quad \text{if } c_1 = [B \rightarrowtail A + B \twoheadrightarrow A].$$

Let $c_0 = [A + B]$ so that $c_0 + \partial(c_1) = [A] + [B]$. Then by Lemma 4.5 (b) we have

$$\begin{aligned} jp(c_1) &= -\alpha(c_0) + c_1 + \alpha(c_0 + \partial(c_1)) \\ &= -\alpha([A + B]) + [B \rightarrowtail A + B \twoheadrightarrow A] + \alpha([A] + [B]). \end{aligned}$$

This is zero by (4.1). $\square$

We finish this section with four lemmas which show useful relations in $\mathcal{D}_*^+\mathbf{C}$.

Lemma 4.6. *The following equality holds in* $\mathcal{D}_*^+\mathbf{C}$.

$$\begin{aligned} &[A + A' \rightarrowtail B + B' \twoheadrightarrow B/A + B'/A' \xleftarrow{\sim} C + C'] \\ &\quad = [A \rightarrowtail B \twoheadrightarrow B/A \xleftarrow{\sim} C]^{[B']} + [A' \rightarrowtail B' \twoheadrightarrow B'/A' \xleftarrow{\sim} C'] + \langle [A], [C'] \rangle. \end{aligned}$$

Proof. Use (R4), (R10) and Proposition 1.6 applied to the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & B'/A' \arrow[d, tail] & C' \arrow[l, "\sim"'] \arrow[d, tail] \\
A + A' \arrow[r, tail] \arrow[d, two heads] & B + B' \arrow[r, two heads] \arrow[d, two heads] & B/A + B'/A' \arrow[d, two heads] & C + C' \arrow[l, "\sim"'] \arrow[d, two heads] \\
A \arrow[r, tail] \arrow[d, equal] & B \arrow[r, two heads] \arrow[d, equal] & B/A \arrow[d, equal] & C \arrow[l, "\sim"'] \arrow[d, equal] \\
A \arrow[r, tail] & B \arrow[r, two heads] & B/A & C. \arrow[l, "\sim"']
\end{tikzcd}$$

$\square$

As special cases we have

Lemma 4.7. *The following equality holds in* $\mathcal{D}_*^+\mathbf{C}$.

$$\begin{aligned} &[A + A' \rightarrowtail B + B' \twoheadrightarrow B/A + B'/A'] \\ &\quad = [A \rightarrowtail B \twoheadrightarrow B/A]^{[B']} + [A' \rightarrowtail B' \twoheadrightarrow B'/A'] + \langle [A], [B'/A'] \rangle. \end{aligned}$$

Lemma 4.8. *The following equality holds in* $\mathcal{D}_*^+\mathbf{C}$.

$$[A + B \xrightarrow{\sim} A' + B'] = [A \xrightarrow{\sim} A']^{[B']} + [B \xrightarrow{\sim} B'].$$

We generalize (R9′) in the following lemma.

Lemma 4.9. *Let* $\mathbf{C}$ *be a Waldhausen category with a functorial coproduct and let* $A_1, \dots, A_n$ *be objects in* $\mathbf{C}$*. Given a permutation* $\sigma \in \mathrm{Sym}(n)$ *of* n *elements we denote by*

$$\sigma_{A_1,\dots,A_n} \colon A_{\sigma_1} + \cdots + A_{\sigma_n} \xrightarrow{\cong} A_1 + \cdots + A_n$$

the isomorphism permuting the factors of the coproduct. Then the following formula holds in $\mathcal{D}_*^+\mathbf{C}$*:*

$$[\sigma_{A_1,\dots,A_n}] = \sum_{\substack{i>j \\ \sigma_i<\sigma_j}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle.$$

Proof. The result holds for $n = 1$ by (R4). Suppose $\sigma \in \mathrm{Sym}(n)$ for $n \geq 2$, and note that the isomorphism $\sigma_{A_1,\dots,A_n}$ factors naturally as

$$\begin{aligned}(\sigma'_{A_1,\dots,A_{n-1}} + 1)(1 + \tau_{A_n,B}) \colon A_{\sigma_1} + \cdots + A_{\sigma_n} &\longrightarrow A_{\sigma'_1} + \cdots + A_{\sigma'_{n-1}} + A_n \\ &\longrightarrow A_1 + \cdots + A_{n-1} \; + \; A_n\end{aligned}$$

where $(\sigma'_1, \dots, \sigma'_{n-1}, n) = (\sigma_1, \dots, \widehat{\sigma_k}, \dots, \sigma_n, \sigma_k)$ and $B = A_{\sigma_{k+1}} + \cdots + A_{\sigma_n}$. Therefore by (R4), (R6) and Lemma 4.8,

$$\begin{aligned}[\sigma_{A_1,\dots,A_n}] &= [\sigma'_{A_1,\dots,A_{n-1}} + 1] + [1 + \tau_{A_n,B}] \\ &= ([\sigma'_{A_1,\dots,A_{n-1}}]^{[A_n]} + 0) + (0 + [\tau_{A_n,B}]).\end{aligned}$$

By induction, (R9′) and Remark 4.1 this is equal to

$$\sum_{\substack{p>q \\ \sigma'_p<\sigma'_q}} \langle [A_{\sigma'_p}], [A_{\sigma'_q}] \rangle + \langle [B], [A_n] \rangle = \sum_{\substack{i>j \\ \sigma_i<\sigma_j<n}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle + \sum_{\substack{i>j \\ \sigma_i<\sigma_j=n}} \langle [A_{\sigma_i}], [A_{\sigma_j}] \rangle$$

as required. □

5 Proof of Theorem 2.1

The morphism of stable quadratic modules $\mathcal{D}_*F \colon \mathcal{D}_*\mathbf{C} \to \mathcal{D}_*\mathbf{D}$ induced by an exact functor $F \colon \mathbf{C} \to \mathbf{D}$, see [4], takes pairs of weak cofiber sequences to pairs of weak cofiber sequences,

$$(\mathcal{D}_*F)\left\{\begin{tikzcd}[column sep=small] & & C_1 & \\ A \arrow[r, tail, shift left] \arrow[r, tail, shift right] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd}\right\} = \left\{\begin{tikzcd}[column sep=small] & & F(C_1) & \\ F(A) \arrow[r, tail, shift left] \arrow[r, tail, shift right] & F(B) \arrow[ur, two heads] \arrow[dr, two heads] & & F(C) \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & F(C_2) & \end{tikzcd}\right\}.$$

By [4, Theorem 3.2] exact equivalences of Waldhausen categories induce homotopy equivalences of stable quadratic modules, and hence isomorphisms in K_1. Therefore if the theorem holds for a certain Waldhausen category then it holds for all equivalent ones. In particular by Proposition 4.3 it is enough to prove the theorem for any Waldhausen category $\mathbf{C}$ with a functorial coproduct such that the monoid of objects is freely generated by a set $\mathbb{S}$. By Theorem 4.2 we can work in the sum-normalized construction $\mathcal{D}_*^+\mathbf{C}$ in this case.

Any element $x \in \mathcal{D}_1^+\mathbf{C}$ is a sum of weak equivalences and cofiber sequences with coefficients $+1$ or -1. By Lemma 4.8 modulo the image of $\langle\cdot,\cdot\rangle$ we can collect on the one hand all weak equivalences with coefficient $+1$ and on the other all weak equivalences with coefficient -1. Moreover, by Lemma 4.7 we can do the same for cofiber sequences. Therefore the following equation holds modulo the image of $\langle\cdot,\cdot\rangle$.

$$\begin{aligned}
x &= -[V_1 \xrightarrow{\sim} V_2] - [X_1 \rightarrowtail X_2 \twoheadrightarrow X_3]\\
&\quad + [Y_1 \rightarrowtail Y_2 \twoheadrightarrow Y_3] + [W_1 \xrightarrow{\sim} W_2]\\
\text{(R4), (R10), 4.8, 4.7} \quad &= -[V_1 + W_1 \xrightarrow{\sim} V_2 + W_1]\\
&\quad - [X_1 + Y_1 \rightarrowtail X_2 + Y_3 + Y_1 \twoheadrightarrow X_3 + Y_3]\\
&\quad + [X_1 + Y_1 \rightarrowtail X_3 + X_1 + Y_2 \twoheadrightarrow X_3 + Y_3]\\
&\quad + [V_1 + W_1 \xrightarrow{\sim} V_1 + W_2]\\
\text{renaming} \quad &= -[L \xrightarrow{\sim} L_1] - [A \rightarrowtail E_1 \twoheadrightarrow D]\\
&\quad + [A \rightarrowtail E_2 \twoheadrightarrow D] + [L \xrightarrow{\sim} L_2] \mod \langle\cdot,\cdot\rangle.
\end{aligned}$$

Suppose that $\partial(x) = 0$ modulo commutators. Then modulo $[\cdot,\cdot]$

$$\begin{aligned}
0 &= -[L] + [L_1] - [A] - [D] + [E_1]\\
&\quad - [E_2] + [D] + [A] - [L_2] + [L]\\
&= [L_1] + [E_1] - [E_2] - [L_2] \mod [\cdot,\cdot],\\
\text{i.e. } [L_1 + E_1] &= [L_2 + E_2] \mod [\cdot,\cdot].
\end{aligned} \tag{5.1}$$

The quotient of $\mathcal{D}_0^+\mathbf{C}$ by the commutator subgroup is the free abelian group on $\mathbb{S}$, hence by (5.1) there are objects $S_1, \ldots, S_n \in \mathbb{S}$ and a permutation $\sigma \in \mathrm{Sym}(n)$ such that

$$\begin{aligned}
L_1 + E_1 &= S_1 + \cdots + S_n,\\
L_2 + E_2 &= S_{\sigma_1} + \cdots + S_{\sigma_n},
\end{aligned}$$

so there is an isomorphism

$$\sigma_{S_1,\ldots,S_n} \colon L_2 + E_2 \xrightarrow{\sim} L_1 + E_1.$$

Again by Lemmas 4.8 and 4.7 modulo the image of $\langle\cdot,\cdot\rangle$

$$\begin{aligned}
x &= -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D]\\
&\quad + [A \rightarrowtail L_2 + E_2 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D]\\
\text{4.9} \quad &= -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D] +
\end{aligned}$$

$$
\begin{aligned}
& + [\sigma_{S_1,\dots,S_n} \colon L_2 + E_2 \xrightarrow{\sim} L_1 + E_1] \\
& + [A \rightarrowtail L_2 + E_2 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D] \\
\text{(R7)} \quad & = -[L + D \xrightarrow{\sim} L_1 + D] - [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_1 + D] \\
& + [A \rightarrowtail L_1 + E_1 \twoheadrightarrow L_2 + D] + [L + D \xrightarrow{\sim} L_2 + D] \\
\text{renaming} \quad & = -[C \xrightarrow{\sim} C_1] - [A \rightarrowtail B \twoheadrightarrow C_1] \\
& + [A \rightarrowtail B \twoheadrightarrow C_2] + [C \xrightarrow{\sim} C_2] \\
& = -[C \xrightarrow{\sim} C_1]^{[A]} - [A \rightarrowtail B \twoheadrightarrow C_1] \\
& + [A \rightarrowtail B \twoheadrightarrow C_2] + [C \xrightarrow{\sim} C_2]^{[A]} \\
& = -[A \rightarrowtail B \twoheadrightarrow C_1 \xleftarrow{\sim} C] + [A \rightarrowtail B \twoheadrightarrow C_2 \xleftarrow{\sim} C] \mod \langle \cdot, \cdot \rangle,
\end{aligned}
$$

i.e. there is $y \in \mathcal{D}_1^+ \mathbf{C}$ in the image of $\langle \cdot, \cdot \rangle$ such that

$$
x = \left\{ \begin{tikzcd}[row sep=small] & & C_1 & \\ A \arrow[r, shift left, tail] \arrow[r, shift right, tail] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd} \right\} + y.
$$

Now assume that $\partial(x) = 0$. Since the pair of weak cofiber sequences is also in the kernel of ∂ we have $\partial(y) = 0$. In order to give the next step we need a technical lemma.

Lemma 5.1. *Let C_* be a stable quadratic module such that C_0 is a free group of nilpotency class 2. Then any element $y \in \operatorname{Ker} \partial \cap \operatorname{Image}\langle \cdot, \cdot \rangle$ is of the form $y = \langle a, a \rangle$ for some $a \in C_0$.*

Proof. For any abelian group A let $\hat{\otimes}^2 A$ be the quotient of the tensor square $A \otimes A$ by the relations $a \otimes b + b \otimes a = 0$, $a, b \in A$, and let $\wedge^2 A$ be the quotient of $A \otimes A$ by the relations $a \otimes a = 0$, $a \in A$, which is also a quotient of $\hat{\otimes}^2 A$. The projection of $a \otimes b \in A \otimes A$ to $\hat{\otimes}^2 A$ and $\wedge^2 A$ is denoted by $a \hat{\otimes} b$ and $a \wedge b$, respectively.

There is a commutative diagram of group homomorphisms

$$
\begin{tikzcd}
 & C_0^{\mathrm{ab}} \otimes C_0^{\mathrm{ab}} \arrow[d, two heads] \arrow[dr, bend left, two heads] \arrow[dl, dashed, no head] & \\
C_0^{\mathrm{ab}} \otimes \mathbb{Z}/2 \arrow[r, hook, "\bar{\tau}"] & \hat{\otimes}^2 C_0^{\mathrm{ab}} \arrow[r, two heads, "q"] \arrow[d, "c_1"] & \wedge^2 C_0^{\mathrm{ab}} \arrow[d, "c_0"] \\
 & C_1 \arrow[r, "\partial"] & C_0
\end{tikzcd}
$$

where $\bar{\tau}(a \otimes 1) = a \hat{\otimes} a$, the factorization c_1 of $\langle \cdot, \cdot \rangle$ is given by $c_1(a \hat{\otimes} b) = \langle a, b \rangle$, which is well defined by Definition 1.1 (3), and $c_0(a \wedge b) = [b, a]$ is well known to be injective in the case C_0 is free of nilpotency class 2.

Moreover, C_0^{ab} is a free abelian group, hence the middle row is a short exact sequence, see [1, I.4]. Therefore any element $y \in C_1$ which is both in the image of c_1 and the kernel of ∂ is in the image of $c_1 \bar{\tau}$ as required. $\square$

The group $\mathcal{D}_0^+\mathbf{C}$ is free of nilpotency class 2, therefore by the previous lemma $y = \langle a, a\rangle$ for some $a \in \mathcal{D}_0^+\mathbf{C}$. By the laws of a stable quadratic module the element $\langle a, a\rangle$ only depends on a mod 2, compare [4, Definition 1.8], and so we can suppose that a is a sum of basis elements with coefficient $+1$. Hence by Remark 4.1 we can take $a = [M]$ for some object M in $\mathbf{C}$,

$$y = \langle [M], [M]\rangle.$$

The element $\langle [M], [M]\rangle$ is itself a pair of weak cofiber sequences

$$\langle [M], [M]\rangle = \left\{ \begin{tikzcd} & & M & \\ M \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & M+M \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & M \arrow[ul, "\sim"', "1"] \arrow[dl, "\sim", "1"'] \\ & & M & \end{tikzcd} \right\},$$

therefore x is also a pair of weak cofiber sequences, by Proposition 3.3, and the proof of Theorem 2.1 is complete.

6 Comparison with Nenashev's approach for exact categories

For any Waldhausen category $\mathbf{C}$ we denote by $K_1^{\mathrm{wcs}}\mathbf{C}$ the abelian group generated by pairs of weak cofiber sequences

$$\left\{ \begin{tikzcd} & & C_1 & \\ A \arrow[r, shift left, tail] \arrow[r, shift right, tail] & B \arrow[ur, two heads] \arrow[dr, two heads] & & C \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & C_2 & \end{tikzcd} \right\}$$

modulo the relations (S1) and (S2) in Section 3. Theorem 3.1 and Proposition 3.2 show the existence of a natural homomorphism

$$K_1^{\mathrm{wcs}}\mathbf{C} \twoheadrightarrow K_1\mathbf{C} \tag{6.1}$$

which is surjective by Theorem 2.1.

Given an exact category $\mathbf{E}$ Nenashev defines in [6] an abelian group $D(\mathbf{E})$ by generators and relations which surjects naturally to $K_1\mathbf{E}$. Moreover, he shows in [7] that this natural surjection is indeed a natural isomorphism

$$D(\mathbf{E}) \cong K_1\mathbf{E}. \tag{6.2}$$

Generators of $D(\mathbf{E})$ are pairs of short exact sequences

$$\left\{ \begin{tikzcd} A \arrow[r, shift left, tail, "j_1"] \arrow[r, shift right, tail, "j_2"'] & B \arrow[r, shift left, two heads, "r_1"] \arrow[r, shift right, two heads, "r_2"'] & C \end{tikzcd} \right\}.$$

They satisfy two kind of relations. The first, analogous to (S2), says that a generator vanishes provided $j_1 = j_2$ and $r_1 = r_2$. The second is a simplification of (S1): given six pairs of short exact sequences

$$A' \underset{j_2^A}{\overset{j_1^A}{\rightrightarrows}} A \underset{r_2^A}{\overset{r_1^A}{\rightrightarrows}} A'', \qquad B' \underset{j_2^B}{\overset{j_1^B}{\rightrightarrows}} B \underset{r_2^B}{\overset{r_1^B}{\rightrightarrows}} B'', \qquad C' \underset{j_2^C}{\overset{j_1^C}{\rightrightarrows}} C \underset{r_2^C}{\overset{r_1^C}{\rightrightarrows}} C'',$$

$$A' \underset{j_2'}{\overset{j_1'}{\rightrightarrows}} B' \underset{r_2'}{\overset{r_1'}{\rightrightarrows}} C', \qquad A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{r_2}{\overset{r_1}{\rightrightarrows}} C, \qquad A'' \underset{j_2''}{\overset{j_1''}{\rightrightarrows}} B'' \underset{r_2''}{\overset{r_1''}{\rightrightarrows}} C'',$$

denoted for simplicity as λ^A, λ^B, λ^C, λ', λ, and λ'', such that the diagram

$$\begin{tikzcd}
A' \arrow[r, tail, "j_i^A"] \arrow[d, tail, "j_i'"'] & A \arrow[r, two heads, "r_i^A"] \arrow[d, tail, "j_i"'] & A'' \arrow[d, tail, "j_i''"] \\
B' \arrow[r, tail, "j_i^B"] \arrow[d, two heads, "r_i'"'] & B \arrow[r, two heads, "r_i^B"] \arrow[d, two heads, "r_i"'] & B'i \arrow[d, two heads, "r_i''"] \\
C' \arrow[r, tail, "j_i^C"'] & C \arrow[r, two heads, "r_i^C"'] & C''
\end{tikzcd}$$

commutes for $i = 1, 2$ then

$$\{\lambda^A\} - \{\lambda^B\} + \{\lambda^C\} = \{\lambda'\} - \{\lambda\} + \{\lambda''\}.$$

Proposition 6.1. *For any exact category* $\mathbf{E}$ *there is a natural isomorphism*

$$D(\mathbf{E}) \cong K_1^{\mathrm{wcs}}\,\mathbf{E}.$$

Proof. There is a clear homomorphism $D(\mathbf{E}) \to K_1^{\mathrm{wcs}}\,\mathbf{E}$ defined by

$$\left\{ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{r_2}{\overset{r_1}{\rightrightarrows}} C \right\} \mapsto \left\{ \begin{tikzcd}[row sep=small] & & C & \\ A \arrow[r, shift left, tail, "j_1"] \arrow[r, shift right, tail, "j_2"'] & B \arrow[ur, two heads, "r_1"] \arrow[dr, two heads, "r_2"'] & & C \arrow[ul, "\sim"', "1"] \arrow[dl, "\sim", "1"'] \\ & & C & \end{tikzcd} \right\}.$$

Exact categories regarded as Waldhausen categories have the particular feature that all weak equivalences are isomorphisms, hence one can also define a homomorphism $K_1^{\mathrm{wcs}}\,\mathbf{E} \to D(\mathbf{E})$ as

$$\left\{ \begin{tikzcd}[row sep=small] & & C_1 & \\ A \arrow[r, shift left, tail, "j_1"] \arrow[r, shift right, tail, "j_2"'] & B \arrow[ur, two heads, "r_1"] \arrow[dr, two heads, "r_2"'] & & C \arrow[ul, "\cong"', "w_1"] \arrow[dl, "\cong", "w_2"'] \\ & & C_2 & \end{tikzcd} \right\} \mapsto \left\{ A \underset{j_2}{\overset{j_1}{\rightrightarrows}} B \underset{w_2^{-1} r_2}{\overset{w_1^{-1} r_1}{\rightrightarrows}} C \right\}.$$

We leave the reader to check the compatibility with the relations and the fact that these two homomorphisms are inverse of each other. □

Remark 6.2. The composite of the isomorphism in Proposition 6.1 with the natural epimorphism (6.1) for $\mathbf{C} = \mathbf{E}$ coincides with Nenashev's isomorphism (6.2), therefore the natural epimorphism (6.1) is an isomorphism when $\mathbf{C} = \mathbf{E}$ is an exact category.

7 Weak cofiber sequences and the stable Hopf map

After the previous section one could conjecture that the natural epimorphism

$$K_1^{\mathrm{wcs}}\,\mathbf{C} \twoheadrightarrow K_1\mathbf{C}$$

in (6.1) is an isomorphism not only for exact categories but for any Waldhausen category $\mathbf{C}$. In order to support this conjecture we show the following result.

Theorem 7.1. *For any Waldhausen category* $\mathbf{C}$ *there is a natural homomorphism*

$$\phi\colon K_0\mathbf{C} \otimes \mathbb{Z}/2 \longrightarrow K_1^{\mathrm{wcs}}\,\mathbf{C}$$

which composed with (6.1) *yields the homomorphism* $\cdot\,\eta\colon K_0\mathbf{C} \otimes \mathbb{Z}/2 \to K_1\mathbf{C}$ *determined by the action of the stable Hopf map* $\eta \in \pi_*^s$ *in the stable homotopy groups of spheres.*

For the proof we use the following lemma.

Lemma 7.2. *The following relation holds in* $K_1^{\mathrm{wcs}}\,\mathbf{C}$

$$\left\{\begin{tikzcd}[row sep=small] & & A'' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A'' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A \arrow[ul, "\sim"', "w_1'w_1"] \arrow[dl, "\sim", "w_2'w_2"'] \\ & & A'' & \end{tikzcd}\right\} = \left\{\begin{tikzcd}[row sep=small] & & A' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A \arrow[ul, "\sim"', "w_1"] \arrow[dl, "\sim", "w_2"'] \\ & & A' & \end{tikzcd}\right\}$$

$$+ \left\{\begin{tikzcd}[row sep=small] & & A'' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A'' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A' \arrow[ul, "\sim"', "w_1'"] \arrow[dl, "\sim", "w_2'"'] \\ & & A'' & \end{tikzcd}\right\}.$$

Proof. Use relations (S1) and (S2) applied to the following pairs of weak cofiber sequences

$$\begin{tikzcd}[row sep=small] & & * & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & * \arrow[ur, tail, two heads] \arrow[dr, tail, two heads] & & * \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & * & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & & A'' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A'' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A \arrow[ul, "\sim"', "w_1'w_1"] \arrow[dl, "\sim", "w_2'w_2"'] \\ & & A'' & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & & A' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A \arrow[ul, "\sim"', "w_1"] \arrow[dl, "\sim", "w_2"'] \\ & & A' & \end{tikzcd},$$

$$\begin{tikzcd}[row sep=small] & & * & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & * \arrow[ur, tail, two heads] \arrow[dr, tail, two heads] & & * \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\ & & * & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & & A'' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A'' \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A' \arrow[ul, "\sim"', "w_1'"] \arrow[dl, "\sim", "w_2'"'] \\ & & A'' & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & & A & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A \arrow[ur, tail, two heads, "1"] \arrow[dr, tail, two heads, "1"'] & & A \arrow[ul, "\sim"', "1"] \arrow[dl, "\sim", "1"'] \\ & & A & \end{tikzcd}.$$

□

Now we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. We define the homomorphism by

$$\phi[A]=\left\{\begin{tikzcd}[row sep=small] & & A & \\ A \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & A\vee A \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & A \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}\right\}=\left\{\begin{tikzcd}[row sep=small] & & A\vee A & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A\vee A \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A\vee A \arrow[ul, "1"', "\sim"] \arrow[dl, "\tau_{A,A}", "\sim"'] \\ & & A\vee A & \end{tikzcd}\right\}.$$

Here the second equality follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences

$$\begin{tikzcd}[row sep=small] & & A & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A\vee A & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A\vee A \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A\vee A, \arrow[ul, "1"', "\sim"] \arrow[dl, "\tau_{A,A}", "\sim"'] \\ & & A\vee A & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & & * & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & * \arrow[ur, two heads] \arrow[dr, two heads] & & *, \arrow[ul, "\sim"] \arrow[dl, "\sim"'] \\ & & * & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A & \\ A \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & A\vee A \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & A, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A & \\ A \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_2"'] & A\vee A \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_1"'] & & A. \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}$$

Given a cofiber sequence $A \overset{j}{\rightarrowtail} B \overset{r}{\twoheadrightarrow} C$ the equation $\phi[C]+\phi[A]=\phi[B]$ follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences

$$\begin{tikzcd}[row sep=small] & & A & \\ A \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & A\vee A \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & A, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & B & \\ B \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & B\vee B \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & B, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & B & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & C & \\ C \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & C\vee C \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & C, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & C & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & & C & \\ A \arrow[r, shift left, tail, "j"] \arrow[r, shift right, tail, "j"'] & B \arrow[ur, two heads, "r"] \arrow[dr, two heads, "r"'] & & C, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & C & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & C\vee C & \\ A\vee A \arrow[r, shift left, tail, "j\vee j"] \arrow[r, shift right, tail, "j\vee j"'] & B\vee B \arrow[ur, two heads, "r\vee r"] \arrow[dr, two heads, "r\vee r"'] & & C\vee C, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & C\vee C & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & C & \\ A \arrow[r, shift left, tail, "j"] \arrow[r, shift right, tail, "j"'] & B \arrow[ur, two heads, "r"] \arrow[dr, two heads, "r"'] & & C. \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & C & \end{tikzcd}$$

Given a weak equivalence $w\colon A \to A'$ the equation $\phi[A]=\phi[A']$ follows from (S1) and (S2) applied to the following pairs of weak cofiber sequences

$$\begin{tikzcd}[row sep=small] & & * & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & * \arrow[ur, two heads] \arrow[dr, two heads] & & *, \arrow[ul, "\sim"] \arrow[dl, "\sim"'] \\ & & * & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A' & \\ A' \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & A'\vee A' \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & A', \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A' & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A & \\ A \arrow[r, shift left, tail, "i_2"] \arrow[r, shift right, tail, "i_1"'] & A\vee A \arrow[ur, two heads, "p_1"] \arrow[dr, two heads, "p_2"'] & & A, \arrow[ul, "1"', "\sim"] \arrow[dl, "1", "\sim"'] \\ & & A & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & & A' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A' \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A, \arrow[ul, "w"', "\sim"] \arrow[dl, "w", "\sim"'] \\ & & A' & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A'\vee A' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A'\vee A' \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A\vee A, \arrow[ul, "w\vee w"', "\sim"] \arrow[dl, "w\vee w", "\sim"'] \\ & & A'\vee A' & \end{tikzcd}\quad \begin{tikzcd}[row sep=small] & & A' & \\ * \arrow[r, shift left, tail] \arrow[r, shift right, tail] & A' \arrow[ur, two heads, "1"] \arrow[dr, two heads, "1"'] & & A. \arrow[ul, "w"', "\sim"] \arrow[dl, "w", "\sim"'] \\ & & A' & \end{tikzcd}$$

The equation $2\phi[A] = 0$ follows from (S2) and Lemma 7.2 by using the second formula for $\phi[A]$ above and the fact that $\tau_{A,A}^2 = 1$.

We have already shown that ϕ is a well-defined homomorphism. The composite of ϕ and (6.1) coincides with the action of the stable Hopf map by the main result of [4].

□

Appendix. Free stable quadratic modules and presentations

Free stable quadratic modules, and also stable quadratic modules defined by generators and relations, can be characterized up to isomorphism by obvious universal properties. In this appendix we give more explicit constructions of these notions.

Let **squad** be the category of stable quadratic modules and let

$$U\colon \mathbf{squad} \longrightarrow \mathbf{Set} \times \mathbf{Set}$$

be the forgetful functor, $U(C_*) = (C_0, C_1)$. The functor U has a left adjoint F, and a stable quadratic module $F(E_0, E_1)$ is called a *free stable quadratic module* on the sets E_0 and E_1. In order to give an explicit description of $F(E_0, E_1)$ we fix some notation. Given a set E we denote by $\langle E\rangle$ the *free group* with basis E, and by $\langle E\rangle^{\mathrm{ab}}$ the *free abelian group* with basis E. The *free group of nilpotency class* 2 with basis E, denoted by $\langle E\rangle^{\mathrm{nil}}$, is the quotient of $\langle E\rangle$ by triple commutators. Given a pair of sets E_0 and E_1, we write $E_0 \cup \partial E_1$ for the set whose elements are the symbols e_0 and ∂e_1 for each $e_0 \in E_0, e_1 \in E_1$.

To define $F(E_0, E_1)$, consider the groups

$$\begin{aligned} F(E_0, E_1)_0 &= \langle E_0 \cup \partial E_1\rangle^{\mathrm{nil}},\\ F(E_0, E_1)_1 &= (\langle E_0\rangle^{\mathrm{ab}} \otimes \mathbb{Z}/2) \times \operatorname{Ker}\delta. \end{aligned}$$

Here $\delta\colon F(E_0, E_1)_0 \to \langle E_0\rangle^{\mathrm{ab}}$ is the homomorphism given by $\delta e_0 = e_0$ and $\delta\partial e_1 = 0$. In the notation of the proof of lemma 5.1 there are isomorphisms

$$\begin{aligned} \operatorname{Ker}\delta &\cong \wedge^2\langle E_0\rangle^{\mathrm{ab}} \times \langle E_0 \times E_1\rangle^{\mathrm{ab}} \times \langle E_1\rangle^{\mathrm{nil}},\\ (\langle E_0\rangle^{\mathrm{ab}} \otimes \mathbb{Z}/2) \times \operatorname{Ker}\delta &\cong \hat{\otimes}^2\langle E_0\rangle^{\mathrm{ab}} \times \langle E_0 \times E_1\rangle^{\mathrm{ab}} \times \langle E_1\rangle^{\mathrm{nil}}, \end{aligned}$$

and intuitively we think of $F(E_0, E_1)_1$ as a group generated by symbols $\langle e_0, e_0'\rangle$, $\langle e_0, \partial e_1\rangle$ and e_1. The symbol $\langle \partial e_1, \partial e_1'\rangle$ is unnecessary since it will be given by the commutator $[e_1', e_1]$.

We define structure homomorphisms on $F(E_0, E_1)$ as follows. The boundary

$$\partial\colon F(E_0, E_1)_1 \longrightarrow F(E_0, E_1)_0$$

is the projection onto $\operatorname{Ker}\delta$ followed by the inclusion of the kernel. The bracket

$$\langle\cdot,\cdot\rangle\colon F(E_0, E_1)_0^{\mathrm{ab}} \otimes F(E_0, E_1)_0^{\mathrm{ab}} \longrightarrow F(E_0, E_1)_1$$

is given by the product of the following two homomorphisms,

$$c' \colon F(E_0, E_1)_0^{\mathrm{ab}} \otimes F(E_0, E_1)_0^{\mathrm{ab}} \longrightarrow \langle E_0 \rangle^{\mathrm{ab}} \otimes \mathbb{Z}/2,$$

defined on the generators $x, y \in E_0 \cup \partial E_1$ by $c'(x, y) = x \otimes 1$ if $x = y \in E_0$ and $c'(x, y) = 0$ otherwise, and

$$c \colon F(E_0, E_1)_0^{\mathrm{ab}} \otimes F(E_0, E_1)_0^{\mathrm{ab}} \longrightarrow \operatorname{Ker} \delta$$

induced by the commutator bracket, $c(a, b) = [b, a]$.

It is now straightforward to define explicitly the stable quadratic module C_* presented by *generators* E_i and *relations* $R_i \subset F(E_0, E_1)_i$ in degrees $i = 0, 1$, by

$$\begin{aligned} C_0 &= F(E_0, E_1)_0 / N_0, \\ C_1 &= F(E_0, E_1)_1 / N_1. \end{aligned}$$

Here $N_0 \subset F(E_0, E_1)_0$ is the normal subgroup generated by the elements of R_0 and ∂R_1, and $N_1 \subset F(E_0, E_1)_1$ is the normal subgroup generated by the elements of R_1 and $\langle F(E_0, E_1)_0, N_0 \rangle$. The boundary and bracket on $F(E_0, E_1)$ induce a stable quadratic module structure on C_* which satisfies the following universal property: given a stable quadratic module C'_*, any pair of functions $E_i \to C'_i$ $(i = 0, 1)$ such that the induced morphism $F(E_0, E_1) \to C'_*$ annihilates R_0 and R_1 induces a unique morphism $C_* \to C'_*$.

In [1] Baues considers the *totally free* stable quadratic module C_* with basis given by a function $g \colon E_1 \to \langle E_0 \rangle^{\mathrm{nil}}$. In the language of this paper C_* is the stable quadratic module with generators E_i in degree $i = 0, 1$ and degree 0 relations $\partial(e_1) = g(e_1)$ for all $e_1 \in E_1$.

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Twisted K-theory – old and new

by Max Karoubi

Some history and motivation about this paper

The subject "K-theory with local coefficients", now called "twisted K-theory", was introduced by P. Donovan and the author in [19] almost forty years ago.[1] It associates to a compact space X and a "local coefficient system"

$$\alpha \in \mathrm{GBr}(X) = \mathbb{Z}/2 \times H^1(X;\mathbb{Z}/2) \times \mathrm{Tors}(H^3(X;\mathbb{Z}))$$

an abelian group $K^\alpha(X)$ which generalizes the usual Grothendieck–Atiyah–Hirzebruch K-theory of X when we restrict α being in $\mathbb{Z}/2$ (cf. [5]). This "graded Brauer group" $\mathrm{GBr}(X)$ has the following group structure: if $\alpha = (\varepsilon, w_1, W_3)$ and $\alpha' = (\varepsilon', w_1', W_3')$ are two elements, one defines the sum $\alpha+\alpha'$ as $(\varepsilon+\varepsilon', w_1+w_1', W_3+W_3'+\beta(w_1\cdot w_1'))$, where $\beta\colon H^2(X;\mathbb{Z}/2) \to H^3(X;\mathbb{Z})$ is the Bockstein homomorphism. With this definition, one has a generalized cup-product[2]:

$$K^\alpha(X) \times K^{\alpha'}(X) \to K^{\alpha+\alpha'}(X).$$

The motivation for this definition is to give in K-theory a satisfactory Thom isomorphism and Poincaré duality pairing which are analogous to the usual ones in cohomology with local coefficients. More precisely, as proved in [28], if V is a real vector bundle on a compact space X with a positive metric, the K-theory of the Thom space of V is isomorphic to a certain "algebraic" group $K^{C(V)}(X)$ associated to the Clifford bundle $C(V)$, viewed as a bundle of $\mathbb{Z}/2-$graded algebras. A careful analysis of this group shows that it depends only on the class of $C(V)$ in $\mathrm{GBr}(X)$, the three invariants being respectively the rank of V mod. 2, $w_1(V)$ and $\beta(w_2(V)) = W_3(V)$, where $w_1(V)$, $w_2(V)$ are the first two Stiefel–Whitney classes of V. In particular, if V is a ^{c}spinorial bundle of even rank, one recovers a well-known theorem of Atiyah and Hirzebruch. On the other hand, if X is a compact manifold, it is well known that such a Thom isomorphism theorem induces a pairing between K-groups

$$K^\alpha(X) \times K^{\alpha'}(X) \to \mathbb{Z}$$

if $\alpha + \alpha'$ is the class of $-C(V)$ in $\mathrm{GBr}(X)$, where V is the tangent bundle of X.

The necessity to revisit these ideas comes from a new interest in the subject because of its relations with Physics [43], as shown by the number of recent publications. However, for these applications, the first definition recalled above is not complete since the coefficient system is restricted to the torsion elements of $H^3(X;\mathbb{Z})$. As it was pointed out by J. Rosenberg [38] and later on by C. Laurent, J.-L. Tu, P. Xu

[1] See the appendix for a short history of the subject.

[2] Strictly speaking, this product is defined up to non canonical isomorphism ; see 2.1 for more details.

[35], M. F. Atiyah and G. Segal [8], this restriction is in fact not necessary. In order to avoid it, one may use for instance the Atiyah–Jänich theorem [2], [27] about the representability of K-theory by the space of Fredholm operators (already quoted in [19] for the cup-product which cannot be defined otherwise).

In the present paper, we would like to make a synthesis between different viewpoints on the subject: [19], [38], [35] and [8] (partially of course)[3]. We hope to have been "pedagogical" in some sense to the non experts.

However, this paper is not just historical. It presents the theory with another point of view and contains some new results. We extend the Thom isomorphism to this more general setting (see also [16]), which is important in order to relate the "ungraded" and "graded" twisted K-theories. We compute many interesting equivariant twisted K-groups, complementing the basic papers [35], [8] and [9]. For this purpose, we use the "Chern character" for finite group actions, as defined by Baum, Connes, Kuhn and Słomińska [11], [33], [41], together with our generalized Thom isomorphism. These last computations are related to some previous ones [31] and to the work of many authors. Finally, we introduce new cohomology operations which are complementary to those defined in [19] and [9].

We don't pretend to be exhaustive in a subject which has already many ramifications. In an appendix to this paper we try to give a short historical survey and a list of interesting contributions of many authors related to the results quoted here.

General plan of the paper

Let us first recall the point of view developed in [19], in order to describe the background material. We consider a locally trivial bundle of $\mathbb{Z}/2$-graded central simple complex algebras A, i.e. modelled on $M_{2n}(\mathbb{C})$ or $M_n(\mathbb{C}) \times M_n(\mathbb{C})$, with the obvious gradings[4]. Then $\mathcal{A}$ has a well-defined class α in the group $\mathrm{GBr}(X)$ (as introduced above). On the other hand, one may consider the category of "$\mathcal{A}$-bundles" , whose objects are vector bundles provided with an $\mathcal{A}$-module structure (fibrewise). We call this category $E^{\mathcal{A}}(X)$; the graded objects of this category are vector bundles which are modules over $\mathcal{A} \mathbin{\widehat{\otimes}} C^{0,1}$, where $C^{0,1}$ is the Clifford algebra $\mathbb{C} \times \mathbb{C} = \mathbb{C}[x]/(x^2 - 1)$. The group $K^{\alpha}(X)$ is now defined as the "Grothendieck group" of the forgetful functor

$$E^{\mathcal{A} \widehat{\otimes} C^{0,1}}(X) \to E^{\mathcal{A}}(X).$$

We refer the reader to [28], p. 191, for this definition which generalizes the usual Grothendieck group of a category. For our purpose, we make it quite explicit at the end of §1, using the concept of "grading".

Despite its algebraic simplicity, this definition of $K^{\alpha}(X)$ is not quite satisfactory for various reasons. For instance, it is not clear how to define in a simple way a

[3] As it was pointed out to me by J. Rosenberg, one should also add the following reference, in the spirit of [18]: Ellen Maycock Parker, The Brauer group of graded continuous trace C*-algebras, Trans. Amer. Math. Soc. 308 (1988), Nr. 1, 115–132.

[4] up to graded Morita equivalence.

"cup-product"

$$K^{\alpha}(X) \times K^{\alpha'}(X) \to K^{\alpha+\alpha'}(X)$$

as mentioned earlier (even if α and α' are in the much smaller group $\mathbb{Z}/2$).

To correct this defect, a second definition may be given in terms of Fredholm operators in a Hilbert space. More precisely, we consider graded Hilbert bundles E which are also graded $\mathcal{A}$-modules in an obvious sense, together with a continuous family of Fredholm operators

$$D \colon E \to E$$

with the following properties:

1. D is self-adjoint of degree 1,

2. D commutes with the action of $\mathcal{A}$ (in the graded sense).

One gets an abelian semi-group from the homotopy classes of such pairs (E, D), with the addition rule

$$(E, D) + (E', D') = (E \oplus E', D \oplus D').$$

The associated group gives the second definition of $K^{\alpha}(X)$ which is equivalent to the first one (see [19], p. 18, and [29], p. 88). We use also the notation $K^{\mathcal{A}}(X)$ instead of $K^{\alpha}(X)$ where α is the class of $\mathcal{A}$ in $\mathrm{GBr}(X)$ when we want to be more explicit.

With this new viewpoint, the cup-product alluded to above becomes obvious. It is defined by the following formula[5]:

$$(E, D) \cup (E', D') = (E \mathbin{\hat{\otimes}} E', D \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} D')$$

where the symbol $\hat{\otimes}$ denotes the graded tensor product of bundles or morphisms. It is a map from $K^{\mathcal{A}}(X) \times K^{\mathcal{A}'}(X)$ to $K^{\mathcal{A} \hat{\otimes} \mathcal{A}'}(X)$ and therefore it induces a (non canonical) map from $K^{\alpha}(X) \times K^{\alpha'}(X)$ to $K^{\alpha+\alpha'}(X)$.

For simplicity's sake, we have only considered *complex* K-theory. We could as well study the real case: one has to replace $\mathrm{GBr}(X)$ by

$$\mathrm{GBr}O(X) = \mathbb{Z}/8 \times H^1(X; \mathbb{Z}/2) \times H^2(X; \mathbb{Z}/2).$$

If we take for the coefficient system $\alpha = n$ to be in $\mathbb{Z}/8$, we get the usual groups $KO^n(X)$ as defined using Clifford algebras in [28] and [29], p. 88 (these groups being written $\overline{K}^n$ in the later reference).

In this paper, we essentially follow the same pattern, but with bundles of infinite dimension in the spirit of [38]. As a matter of fact, all the technical tools are already present in [19], [29] and [38], for instance the Fredholm operator machinery which is necessary to define the cup product. However, this paper is not a rewriting of these papers, since we take a more synthetic view point and have other applications in mind. For instance, the K-theory of Banach algebras $K_n(A)$ and its graded version, denoted

[5] See the appendix about the origin of such a formula.

here by $\mathrm{Gr}K_n(A)$, are more systematically used. On the other hand, since the equivariant K-theory and its relation with cohomology have been studied carefully in [17], [35] and [9], we limit ourselves to the applications in this case. One of them is the definition of operations in twisted K-theory in the graded and ungraded situations.

Here is the contents of the paper:

1. K-theory of $\mathbb{Z}/2$-graded Banach algebras
2. Ungraded twisted K-theory in the finite and infinite-dimensional cases
3. Graded twisted K-theory in the finite and infinite-dimensional cases
4. The Thom isomorphism
5. General equivariant K-theory
6. Some computations in the equivariant case
7. Operations in twisted K-theory

Appendix. A short historical survey of twisted K-theory

Acknowledgments. I would like to thank A. Adem, A. Carey, P. Hu, I. Kriz, C. Leruste, V. Mathai, J. Rosenberg, J.-L. Tu, P. Xu and the referee for their remarks and suggestions after various drafts of this paper.

1 K-theory of $\mathbb{Z}/2$-graded Banach algebras

Higher K-theory of real or complex Banach algebras A is well known (cf. [28] or [12] for instance). Starting from the usual Grothendieck group $K(A) = K_0(A)$, there are many equivalent ways to define "derived functors" $K_n(A)$, for $n \in \mathbb{Z}$, such that any exact sequence of Banach algebras

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

induces an exact sequence of abelian groups

$$\cdots \longrightarrow K_{n+1}(A) \longrightarrow K_{n+1}(A") \longrightarrow K_n(A') \longrightarrow K_n(A) \longrightarrow K_n(A") \longrightarrow \cdots .$$

Moreover, by Bott periodicity, these groups are periodic of period 2 in the complex case and 8 in the real case.

The K-theory of $\mathbb{Z}/2$-graded Banach algebras A (in the real or complex case) is less well known[6] and for the purpose of this paper we shall recall its definition which is already present but not systematically used in [28] and [19]. We first introduce $C^{p,q}$ as the Clifford algebra of $\mathbb{R}^{p+q}$ with the quadratic form

$$-(x_1)^2 - \cdots - (x_p)^2 + (x_{p+1})^2 + \cdots + (x_{p+q})^2.$$

[6]This is of course included in the general KK-theory of Kasparov which was introduced later than our basic references, at least for C*-algebras.

It is naturally $\mathbb{Z}/2$-graded. If A is an arbitrary $\mathbb{Z}/2$-graded Banach algebra, $A^{p,q}$ is the graded tensor product $A \hat{\otimes} C^{p,q}$. For A unital, we now define the graded K-theory of A (denoted by $\mathrm{Gr}K(A)$) as the K-theory of the forgetful functor:

$$\phi\colon \mathcal{P}(A^{0,1}) \to \mathcal{P}(A)$$

(see [28] or 1.4 for a concrete definition). Here $\mathcal{P}(A)$ denotes in general the category of finitely generated projective left A-modules. One may remark that the objects of $\mathcal{P}(A^{0,1})$ are graded objects of the category $\mathcal{P}(A)$. The functor ϕ simply "forgets" the grading. One should also notice that $\mathrm{Gr}K(A^{p,q})$ is naturally isomorphic to $\mathrm{Gr}K(A^{p+1,q+1})$, since $A^{p,q}$ is Morita equivalent to $A^{p+1,q+1}$ (in the graded sense).

If A is not unital, we define $\mathrm{Gr}K(A)$ by the usual method, as the kernel of the augmentation map

$$\mathrm{Gr}K(A^+) \to \mathrm{Gr}K(k)$$

where A^+ is the k-algebra A with a unit added ($k = \mathbb{R}$ or $\mathbb{C}$, according to the theory, with the trivial grading).

There is a "suspension functor" on the category of graded algebras, associating to A the graded tensor product $A^{0,1} = A \hat{\otimes} C^{0,1}$. One of the fundamental results[7] in [28], p. 210, is the fact that this suspension functor is compatible with the usual one: in other words, we have a well-defined isomorphism

$$\mathrm{Gr}K(A^{0,1}) \equiv \mathrm{Gr}K(A(\mathbb{R}))$$

where $A(T)$ denotes in general the algebra of continuous maps $f(t)$ on the locally compact space T with values in A, such that $f(t) \to 0$ when t goes to infinity. As a consequence, we have an isomorphism between the following groups (for $n \geq 0$):

$$\mathrm{Gr}K(A^{0,n}) \equiv \mathrm{Gr}K(A(\mathbb{R}^n))$$

which we call $\mathrm{Gr}K_n(A)$. More generally, we put $\mathrm{Gr}K_n(A) = \mathrm{Gr}K(A^{p,q})$ for $q - p = n \in \mathbb{Z}$. These groups $\mathrm{Gr}K_n(A)$ satisfy the same exactness property as the groups $K_n(A)$ above, from which they are naturally derived. They are of course linked with them by the following exact sequence (for all $n \in \mathbb{Z}$):

$$K_{n+1}(A \hat{\otimes} C^{0,1}) \longrightarrow K_{n+1}(A) \longrightarrow \mathrm{Gr}K_n(A) \longrightarrow K_n(A \hat{\otimes} C^{0,1}) \longrightarrow K_n(A).$$

In particular, if we start with an ungraded Banach algebra A, we see that Bott periodicity follows from these previous considerations, thanks to the periodicity of Clifford algebras up to graded Morita equivalence: this was the main theme developed in [6], [44] and [28], in order to give a more conceptual proof of the periodicity theorems.

When A is unital, it is technically important to describe the group $\mathrm{Gr}K(A)$ in a more concrete way. If E is an object of $\mathcal{P}(A)$, a *grading* of E is given by an involution ε which commutes (resp. anticommutes) with the action of the elements of degree 0

[7] Strictly speaking, one has to replace the category $\mathcal{C}^{p,q}$ with an arbitrary graded category. However, the proof of Theorem 2.2.2 in [28] easily extends to this case.

(resp. 1) in A. In this way, E with a grading ε may be viewed as a module over the algebra $A\hat{\otimes}C^{0,1}$. We now consider triples $(E, \varepsilon_1, \varepsilon_2)$, where ε_1 and ε_2 are two gradings of E. The homotopy classes of such triples obviously form a semi-group. Its quotient by the semi-group of "elementary" triples (i.e. such that $\varepsilon_1 = \varepsilon_2$) is isomorphic to $\mathrm{Gr}K(A)$.

2 Ungraded twisted K-theory in the finite and infinite-dimensional cases

Let X be a compact space and let us consider bundles of algebras $\mathcal{A}$ with fiber $M_n(\mathbb{C})$. As it was shown by Serre [25], such bundles are classified by Čech cocycles (up to Čech coboundaries):

$$g_{ji} : U_i \cap U_j \to PU(n)$$

where $PU(n)$ is $U(n)/S^1$, the projective unitary group. In other words, the bundle of algebras $\mathcal{A}$ may be obtained by gluing together the bundles of C*-algebras $(U_i \times M_n(\mathbb{C}))$, using the transition functions g_{ji} [Note that $U(n)$ acts on $M_n(\mathbb{C})$ by inner automorphisms and therefore induces an action of $PU(n)$ on the algebra $M_n(\mathbb{C})$]. The Brauer group of X denoted by $\mathrm{Br}(X)$ is the quotient of this semi-group of bundles (via the tensor product) by the following equivalence relation: $\mathcal{A}$ is equivalent to $\mathcal{A}'$ iff there exist vector bundles V and V' such that the bundles of algebras $\mathcal{A} \hat{\otimes} \mathrm{End}(V)$ and $A' \hat{\otimes} \mathrm{End}(V')$ are isomorphic. It was proved by Serre [25] that $\mathrm{Br}(X)$ is naturally isomorphic to the torsion subgroup of $H^3(X; \mathbb{Z})$.

The Serre–Swan theorem (cf. [28] for instance) may be easily translated in this situation to show that the category of finitely generated projective $\mathcal{A}$-module bundles E (as in [19]) is equivalent to the category $\mathcal{P}(A)$ of finitely generated projective modules over $A = \Gamma(X, \mathcal{A})$, the algebra of continuous sections of the bundle $\mathcal{A}$. The key observation for the proof is that E is a direct factor of a "trivial" $\mathcal{A}$-bundle; this is easily seen with finite partitions of unity, since X is compact. One should notice that if $\mathcal{A}$ is equivalent to $\mathcal{A}'$ the associated categories $\mathcal{P}(A)$ and $\mathcal{P}(A')$ are equivalent. Note however that this equivalence is non canonical since $\mathcal{A} \otimes \mathrm{End}(V)$ and $\mathcal{A}' \otimes \mathrm{End}(V')$ are not canonically isomorphic.

Definition 2.1. The ungraded[8] twisted K-theory $K^{(\mathcal{A})}(X)$ is by definition the K-theory of the ring A (which is the same as the K-theory of the category $E^{\mathcal{A}}(X)$ mentioned in the introduction). By abuse of notation, we shall simply call it $K(\mathcal{A})$. We also define $K_n^{(\mathcal{A})}(X)$ as the K_n-group of the Banach algebra $\Gamma(X, \mathcal{A})$. It only depends on the class of $\mathcal{A}$ in $\mathrm{Br}(X) = \mathrm{Tors}(H^3(X; \mathbb{Z}))$.

The key observation made by J. Rosenberg [38] is the following: we can "stabilize" the situation (in the C*-algebra sense), i.e. embed $M_n(\mathbb{C})$ into the algebra of compact operators $\mathcal{K}$ in a separable Hilbert space H, thanks to the split inclusion of $\mathbb{C}^n$ in $l^2(\mathbb{N})$.

[8] We use the notation $K^{(\mathcal{A})}(X)$, not to be confused with the graded twisted K-group $K^{\mathcal{A}}(X)$ which will be defined in the next section.

Now, a bigger group $PU(H) = U(H)/S^1$ is acting on $\mathcal{K}$ by inner automorphisms. If we take a Čech cocycle

$$g_{ji} : U_i \cap U_j \to PU(H)$$

we may use it to construct a bundle $\mathcal{A}$ of (non unital) C*-algebras with fiber $\mathcal{K}$.

Let us now consider the commutative diagram

$$\begin{array}{ccc} S^1 & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ U(n) & \longrightarrow & U(H) \\ \downarrow & & \downarrow \\ PU(n) & \longrightarrow & PU(H). \end{array}$$

Thanks to Kuiper's theorem [34], we remark that the classifying space of $U(H)$ is contractible. Therefore, the classifying space $BPU(H)$ of the topological group $PU(H)$, is a nice model of the Eilenberg–Mac Lane space $K(\mathbb{Z}, 3)$ (compare with the well-known paper of Dixmier and Douady [18]). Moreover, if we start with a finite-dimensional algebra bundle $\mathcal{A}$ over X with fiber $M_n(\mathbb{C})$, the diagram above shows how to associate to $\mathcal{A}$ another bundle of algebras $\mathcal{A}'$ with fiber $\mathcal{K}$, together with a C*-inclusion from $\mathcal{A}$ to $\mathcal{A}'$. We note that the invariant $W_3(A)$ in $\mathrm{Br}(X) = \mathrm{Tors}(H^3(X;\mathbb{Z}))$ defined in [25] is simply induced by the classifying map from X to $BPU(H)$ (which factors through $BPU(n)$). In this finite example, it is an n-torsion class since one has another commutative diagram

$$\begin{array}{ccc} \mu_n & \longrightarrow & S^1 \\ \downarrow & & \downarrow \\ SU(n) & \longrightarrow & U(n) \\ \downarrow & & \downarrow \\ PU(n) & = & PU(n). \end{array}$$

Theorem 2.2. *The inclusion from $\mathcal{A}$ to $\mathcal{A}'$ induces an isomorphism*

$$K_r(\mathcal{A}) = K_r(\Gamma(X, \mathcal{A})) \to K_r(\Gamma(X, \mathcal{A}')) = K_r(\mathcal{A}')$$

where the K_r define the classical topological K-theory of C-algebras.*

Proof. The proof is classical for a trivial algebra bundle, since $\mathbb{C}$ is Morita equivalent to $\mathcal{K}$ (in the C*-algebra sense). It extends to the general case by a no less classical Mayer–Vietoris argument. □

Definition 2.3. Let now $\mathcal{A}$ be an algebra bundle with fiber $\mathcal{K}$ on a compact space X *with structural group $PU(H)$*. We define $K^{(\mathcal{A})}(X)$ (also denoted by $K(\mathcal{A})$) as the K-theory of the (non unital) Banach algebra $\Gamma(X, \mathcal{A})$. This K-theory only depends of the class α of $\mathcal{A}$ in $H^3(X;\mathbb{Z})$ and we shall also call it $K^{(\alpha)}(X)$ [Due to 2.4, this is a generalization of definition 2.2].

Before treating the graded case in the next section, we would like to give an equivalent definition of $K(\mathcal{A})$ in terms of Fredholm operators, as was done in [19] for the torsion elements in $H^3(X;\mathbb{Z})$ and in [8] for the general case. The basic idea is to remark that $PU(H)$ acts not only on the C*-algebra of compact operators in H, but also on the ring of bounded operators $\mathrm{End}(H)$ and on the Calkin algebra $\mathrm{End}(H)/\mathcal{K}$ (with the norm topology[9])). Let us call $\mathcal{B}$ the algebra bundle with fiber $\mathrm{End}(H)$ associated to the cocycle defined in 2.1 and $\mathcal{B}/\mathcal{A}$ the quotient algebra bundle. Therefore, we have an exact sequence of C*-algebras bundles

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A} \longrightarrow 0.$$

which induces an exact sequence for the associated rings of sections (thanks to a partition of unity again)

$$0 \longrightarrow \Gamma(X,\mathcal{A}) \longrightarrow \Gamma(X,\mathcal{B}) \longrightarrow \Gamma(X,\mathcal{B}/\mathcal{A})) \longrightarrow 0.$$

If $\mathcal{B}$ is trivial, it is well known that the algebra of continuous maps from X to $\mathrm{End}(H)$ has trivial K_n-groups because this algebra is flabby[10]. By a Mayer–Vietoris argument, it follows that $K_n(\Gamma(X,\mathcal{B}))$ is also trivial. Therefore the connecting homomorphism

$$K_1(\mathcal{B}/\mathcal{A}) = K_1(\Gamma(X,\mathcal{B}/\mathcal{A})) \to K_0(\Gamma(X,\mathcal{A})) \equiv K^{(\mathcal{A})}(X) = K(\mathcal{A})$$

is an isomorphism, a well-known observation in index theory.

Let us now consider the elements of $\mathcal{B}$ which map onto $(\mathcal{B}/\mathcal{A})^*$ via the map π. These elements form a bundle of Fredholm operators on H (the twist comes from the action of $PU(H)$). This subbundle of $\mathcal{B}$ will be denoted by $\mathrm{Fredh}(H)$. Therefore, we have a principal fibration

$$\Gamma(X,\mathcal{A}) \longrightarrow \Gamma(X,\mathrm{Fredh}(H)) \xrightarrow{\pi} \Gamma(X,(\mathcal{B}/\mathcal{A})^*)$$

with contractible fiber the Banach space $\Gamma(X,\mathcal{A})$ (this fibration admits a local section thanks to Michael's theorem [37]). Therefore, the space of sections of $\mathrm{Fredh}(H)$ has the homotopy type of $\Gamma(X,(\mathcal{B}/\mathcal{A})^*)$. In particular, the path components are in bijective correspondence via the map π. The following theorem is a generalization of a well-known theorem of Atiyah and Jänich [2], [27]:

Theorem 2.4 ([8]). *The set of homotopy classes of continuous sections of the fibration*

$$\mathrm{Fredh}(H) \longrightarrow X$$

is naturally isomorphic to $K^{(\mathcal{A})}(X)$.

[9] other topologies may be also considered, see [8].

[10] A unital Banach algebra Λ is called flabby if there exists a continuous functor τ from $\mathcal{P}(\Lambda)$ to itself such that $\tau + \mathrm{Id}$ is isomorphic to τ. For instance, $\Lambda = \mathrm{End}(H)$ is flabby since $\mathcal{P}(A)$ is equivalent to the category of Hilbert spaces which are isomorphic to direct factors in H; τ is then defined by the infinite Hilbert sum $\tau(E) = E \oplus \cdots \oplus E \oplus \cdots$.

Proof. As we have seen above, the two spaces $\Gamma(X, \mathrm{Fredh}(H))$ and $\Gamma(X, (\mathcal{B}/\mathcal{A})^*)$ have the same homotopy type. On the other hand, it is a well known consequence of Kuiper's theorem [34] than the (non unital) ring map $\Gamma(X, \mathcal{B}/\mathcal{A}) \to \Gamma(X, M_r(\mathcal{B}/\mathcal{A}))$ induces a bijection between the path components of the associated groups of invertible elements (see for instance [32], p. 93). Therefore, $\pi_0(\Gamma(X, \mathrm{Fredh}(H)))$ may be identified with

$$\varinjlim_{r} \pi_0(\Gamma(X, GL_r(\mathcal{B}/\mathcal{A}))) = K_1(\mathcal{B}/\mathcal{A})$$

and therefore with $K(\mathcal{A})$, as we already mentioned in 2.6. □

Remark 2.5. We may also consider the following "stabilized" bundle

$$\mathrm{Fredh}_s(H) = \varinjlim_{n} \mathrm{Fredh}(H^n)$$

and, without Kuiper's theorem, prove in the same way that the set of connected components of the space of sections of this bundle is isomorphic to $K^{(\mathcal{A})}(X)$.

There is an obvious ring homomorphism (since the Hilbert tensor product $H \otimes H$ is isomorphic to H)

$$\mathcal{K} \otimes \mathcal{K} \longrightarrow \mathcal{K}$$

If $\mathcal{A}$ and $\mathcal{A}'$ are bundles of algebras on X modelled on $\mathcal{K}$, we may use this homomorphism to get a new algebra bundle on X, which we denote by $\mathcal{A} \otimes \mathcal{A}'$. From the cocycle point of view, we have a commutative diagram, where the top arrow is induced by complex multiplication

$$\begin{array}{ccc}
S^1 \times S^1 & \longrightarrow & S^1 \\
\downarrow & & \downarrow \\
U(H) \times U(H) & \longrightarrow & U(H \otimes H) \\
\downarrow & & \downarrow \\
PU(H) \times PU(H) & \longrightarrow & PU(H \otimes H).
\end{array}$$

It follows that $W_3(\mathcal{A} \otimes \mathcal{A}') = W_3(\mathcal{A}) + W_3(\mathcal{A}')$ in $H^3(X; \mathbb{Z})$ and one gets a "cup-product"

$$K^{(\alpha)}(X) \times K^{(\alpha')}(X) \longrightarrow K^{(\alpha+\alpha')}(X).$$

(well defined up to non canonical isomorphism: see 2.1). This is a particular case of a "graded cup-product" which will be introduced in the next section.

3 Graded twisted K-theory in the finite and infinite-dimensional cases

We are going to change our point of view and now consider $\mathbb{Z}/2$-graded finite-dimensional algebras which are central and simple (in the graded sense). We are only interested in the complex case. The real case is treated with great details in [19] and does not seem to generalize in the infinite-dimensional framework [11].

In the complex case, there are just two "types" of such graded algebras (up to Morita equivalence[12]) which are $\mathbb{R} = \mathbb{C}$ and $\mathbb{C} \times \mathbb{C} = \mathbb{C}[x]/(x^2 - 1)$. For a type R of algebra, the graded inner automorphisms of $A = R \widehat{\otimes} \mathrm{End}(V_0 \oplus V_1)$ may be given by either an element of degree 0 or an element of degree 1 in A^*. This gives us an augmentation (whose kernel is denoted by $\mathrm{Aut}^0(A)$):

$$\mathrm{Aut}(A) \longrightarrow \mathbb{Z}/2.$$

Therefore, for bundles of $\mathbb{Z}/2$-graded algebras modelled on A, we already get an invariant in $H^1(X; \mathbb{Z}/2)$, called the "orientation" of A and which may be represented by a real line bundle. A typical example is the (complexified) Clifford bundle $C(V)$, associated to a real vector bundle V of rank n. Its orientation invariant is the first Stiefel–Whitney class associated to V (cf. [19]). Note that the type R of $C(V)$ is $\mathbb{C}$ if n is even and $\mathbb{C}[x]/(x^2 - 1) \cong \mathbb{C} \times \mathbb{C}$ if n is odd[13].

For the second invariant, let us start with $M_{2n}(\mathbb{C})$ as a basic graded algebra to fix the ideas and let us put a C*-algebra metric on A. We have an exact sequence of groups as in the ungraded case (where $PU^0(2n) = PU(2n) \cap \mathrm{Aut}^0(A)$)

$$1 \longrightarrow S^1 \longrightarrow U(n) \times U(n) \longrightarrow PU^0(2n) \longrightarrow 1.$$

Therefore, a bundle with structural group $PU^0(2n)$ also has a class in $H^2(X; S^1) = H^3(X; \mathbb{Z})$ which is easily seen to have order n as in the ungraded case. A similar argument holds if the graded algebra is $M_n(\mathbb{C}) \times M_n(\mathbb{C})$. It follows that the "graded Brauer group" $\mathrm{GBr}(X)$ is

$$\mathrm{GBr}(X) = \mathbb{Z}/2 \times H^1(X; \mathbb{Z}/2) \times \mathrm{Tors}(H^3(X; \mathbb{Z}))$$

as already quoted in the introduction (see [19] for the explicit group law on $\mathrm{GBr}(X)$). If $\mathcal{A}$ is a bundle of $\mathbb{Z}/2$-graded finite-dimensional algebras, we define the graded twisted K-theory $K^{\mathcal{A}}(X)$ as the *graded* K-theory of the graded algebra $\Gamma(X, \mathcal{A})$ as recalled in Section 1. This definition only depends on the class of $\mathcal{A}$ in $\mathrm{GBr}(X)$. We recover the definition in [19] by using again the Serre–Swan theorem as in 2.1. For instance, if we consider the bundle of (complex) Clifford algebras $C(V)$, associated to a real vector bundle V, the invariants we get are $w_1(V)$ and $W_3(V)$, as quoted in the introduction. If these invariants are trivial, the bundle V is ^{c}spinorial of even rank and $C(V)$ may be

[11] This is not quite true if we work in the context of "Real" K-theory in the sense of Atiyah [2]. We shall not consider this generalization here, although it looks interesting in the light of "equivariant twisted K-theory" as we shall show in §6.

[12] This means that we are allowed to take the graded tensor product with $\mathrm{End}(V_0 \oplus V_1)$ with the obvious grading.

[13] If V is oriented and even dimensional for instance, this does not imply that $C(V)$ is a bundle of graded algebras of type $M_2(\Lambda)$ for a certain bundle of ungraded algebras Λ. However, for suitable vector bundles V_0 and V_1, this is the case for the graded tensor product $C(V) \widehat{\otimes} \mathrm{End}(V_0 \oplus V_1)$.

identified with the bundle of endomorphisms $\mathrm{End}(S^+ \oplus S^-)$, where S^+ and S^- are the even and odd "spinors" associated to the Spin^c-structure.

In order to define graded twisted K-theory in the infinite-dimensional case, we follow the same pattern as in §2. For instance, let us take a graded bundle of algebras $\mathcal{A}$ modelled on $M_2(\mathcal{K})$: it has 2 invariants, one in $H^1(X;\mathbb{Z}/2)$, the other in $H^3(X;\mathbb{Z})$ (and not just in the torsion part of this group). We then define $K^{\mathcal{A}}(X)$ as the *graded* K-theory of the graded algebra $\Gamma(X, \mathcal{A})$, according to §1. The same definition holds for bundles of graded algebras modelled on $\mathcal{K} \times \mathcal{K} = \mathcal{K}[x]/(x^2 - 1) = \mathcal{K} \mathbin{\hat{\otimes}} \mathcal{K}x$.

If $C^{0,1}$ is the Clifford algebra $\mathbb{C} \times \mathbb{C}$ with its usual graded structure, the general results of §1 show that the group $K^{\mathcal{A}}(X)$ fits into an exact sequence:

$$K_1^{(\mathcal{A}\hat{\otimes}C^{0,1})}(X) \longrightarrow K_1^{(\mathcal{A})}(X) \longrightarrow K^{\mathcal{A}}(X) \longrightarrow K^{(\mathcal{A}\hat{\otimes}C^{0,1})}(X) \longrightarrow K^{(\mathcal{A})}(X)$$

where $K_i^{(\mathcal{A})}(X)$ denotes in general the K_i-group of the Banach algebra $\Gamma(X, \mathcal{A})$.

Let us now assume that $\mathcal{A}$ is oriented modelled on $M_2(\mathcal{K})$ (which means that the structural group of $\mathcal{A}$ may be reduced to $PU^0(H \oplus H)$; see below or 3.2 in the finite-dimensional situation). We are going to show that $\mathcal{A}$ may be written as $M_2(\mathcal{A}')$, with the obvious grading, $\mathcal{A}'$ being an ungraded bundle of algebras modelled on $\mathcal{K}$. For this purpose, we write the commutative diagram

$$\begin{array}{ccc}
S^1 & \longrightarrow & S^1 \\
\downarrow & & \downarrow \\
U(H) & \longrightarrow & U(H) \times U(H) \\
\downarrow & & \downarrow \\
PU(H) & \longrightarrow & PU^0(H \oplus H)
\end{array}$$

where the first horizontal map is the identity and the others are induced by the diagonal. This shows that $H^1(X;PU(H)) \equiv H^1(X;PU^0(H \oplus H))$, which is equivalent to saying that $\mathcal{A}$ may be written as $M_2(\mathcal{A}')$ for a certain bundle of algebras $\mathcal{A}'$.

Therefore, $\mathcal{A} \mathbin{\hat{\otimes}} C^{0,1}$ is Morita equivalent to $\mathcal{A}' \times \mathcal{A}'$ and $K^{\mathcal{A}}(X)$ is the K-theory of the ring homomorphism (more precisely the functor defined by the associated extension of the scalars, as we shall consider in other situations)

$$\mathcal{A}' \times \mathcal{A}' \longrightarrow M_2(\mathcal{A}')$$

defined by $(a,b) \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ (no grading). We should also note that $\mathcal{A}'$ is Morita equivalent to $\mathcal{A}$ as an ungraded bundle of algebras. Since the usual K-theory (resp. graded K-theory) is invariant under Morita equivalence (resp. graded Morita equivalence), the previous considerations lead to the following theorem:

Theorem 3.1. *Let $\mathcal{A}$ be an oriented bundle of graded algebras modelled on $M_2(\mathcal{K})$. Then $K^{\mathcal{A}}(X)$ is isomorphic to $K^{(\mathcal{A})}(X)$ via the identification above.*

The same method may be applied when $\mathcal{A}$ is an oriented bundle of graded algebras modelled on $\mathcal{K}\times\mathcal{K} = \mathcal{K}[x]/(x^2-1)$. The *graded oriented* automorphisms of $\mathcal{K}\times\mathcal{K}$ induced by $PU(H \oplus H)$ are diagonal matrices of the type

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This shows that $\mathcal{A}$ is isomorphic to $\mathcal{A}' \times \mathcal{A}'$ and $\mathcal{A} \hat{\otimes} C^{0,1}$ is isomorphic to $M_2(\mathcal{A}')$. Therefore, $K^{\mathcal{A}}(X)$ is the Grothendieck group of the ring homomorphism $\mathcal{A}' \to M_2(\mathcal{A}')$, defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

Hence we have the following theorem, analogous to 3.5:

Theorem 3.2. *Let $\mathcal{A}$ be an oriented bundle of graded algebras modelled on $\mathcal{K} \times \mathcal{K}$. Then $\mathcal{A}$ is isomorphic to $\mathcal{A}' \times \mathcal{A}'$ and the group $K^{\mathcal{A}}(X)$ is isomorphic to $K_1(\mathcal{A}')$ via the identification above.*

Remark 3.3. One may notice that if $\mathcal{A}$ is a bundle of oriented graded algebras modelled on $M_2(\mathcal{K})$, the associated bundle with fiber $M_2(\mathcal{K}^+)$[14] has a section ε of degree 0 and of square 1, which commutes (resp. anticommutes) with the elements of degree 0 (resp. 1); it is simply defined by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, we would like to give an equivalent definition of $K^{\mathcal{A}}(X)$ in terms of Fredholm operators as in §2. This was done in [19] if the class of $\mathcal{A}$ belongs to the torsion group of $H^3(X;\mathbb{Z})$ and in [8] for the general case. We shall give another treatment here, using again the K-theory of graded Banach algebras.

Following the general notations of §2, we have an exact sequence of bundles of graded Banach algebras

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{A} \longrightarrow 0.$$

Since the graded K-groups of $\mathcal{B}$ are trivial, we see as in 2.6 that $\mathrm{Gr}K(\Gamma(X,\mathcal{A}))$ is isomorphic to $\mathrm{Gr}K_1(\Gamma(X, B/\mathcal{A}))$.

In order to shorten the notations, we denote by B the graded Banach algebra $\Gamma(X,\mathcal{B})$, by Λ the graded Banach algebra $\Gamma(X, B/\mathcal{A})$ and by f the surjective map $B \to \Lambda$. The following lemma[15] may be proved in the same way as in [29], p. 78:

[14] $\mathcal{K}^+$ is the ring $\mathcal{K}$ which a unit added

[15] It might be helpful for a better understanding to notice that the category of finitely generated free $\mathrm{End}(H)$-modules is equivalent to the category of Hilbert spaces H^n for $n \in \mathbb{N}$. This "local" situation is twisted by the group $PU(H)$.

Lemma 3.4. *Any element of* $\mathrm{Gr}K_1(\Lambda)$ *may be written as the homotopy class of a pair* (E, ε) *where* E *is a free graded* B*-module and* ε *is a grading of degree* 1 *of the associated* Λ*-module* (*see* 1.4 *for the definition of a grading*).

By the well-known dictionary between modules and bundle theory, we may view ε as a grading of a suitable bundle of free $\mathrm{End}(H)/\mathcal{K}$-modules. By spectral theory, we may also assume that ε is self-adjoint. Finally, following [29], we define a *quasi-grading*[16] of E as a family of Fredholm endomorphisms D such that

1. $D^* = D$,
2. D is of degree 1.

The following theorem is the analogue in the graded case of Theorem 2.8 (cf [29], p. 78/79).

Theorem 3.5. *The* (*graded*) *twisted* K*-group* $K^{\mathcal{A}}(X)$ *is the Grothendieck group associated to the semi-group of homotopy classes of pairs* (E, D) *where* E *is a free* $\mathbb{Z}/2$*-graded* $\mathcal{B}$*-module and* D *is a family of Fredholm endomorphisms of* E *which are self-adjoint and of degree* 1.

Remark 3.6. Let us assume that $\mathcal{A}$ is oriented modelled on $M_2(\mathcal{K})$. The description above gives a Fredholm description of $\mathrm{Gr}K_1(\mathcal{A}) = K_1(\mathcal{A})$: we just take the homotopy classes of sections of the associated bundle of self-adjoint Fredholm operators $\mathrm{Fredh}^*(\mathcal{B})$ whose essential spectrum is divided into two non empty parts[17] in $\mathbb{R}^{+*}$ and $\mathbb{R}^{-*}$.

This Fredholm description of $K^{\mathcal{A}}(X)$ enables us to define a cup-product

$$K^{\mathcal{A}}(X) \times K^{\mathcal{A}'}(X) \longrightarrow K^{\mathcal{A}\hat{\otimes}\mathcal{A}'}(X)$$

where $\mathcal{A} \mathbin{\hat{\otimes}} \mathcal{A}'$ denotes the graded tensor product of $\mathcal{A}$ and $\mathcal{A}'$. This cup-product is given by the same formula as in [19], p. 19, and generalizes it:

$$(E, D) \cup (E', D') = (E \mathbin{\hat{\otimes}} E', D \mathbin{\hat{\otimes}} 1 + 1 \mathbin{\hat{\otimes}} D')$$

(see the appendix about the origin of this formula in usual topological K-theory).

To conclude this section, let us consider a locally compact space X and a bundle of graded algebras $\mathcal{A}$ on X. For technical reasons, we assume the existence of a compact space Z containing X as an open subset, such that $\mathcal{A}$ extends to a bundle (also called $\mathcal{A}$) on Z. There is an obvious definition of $K^{\mathcal{A}}(X)$ as a relative term in the following exact sequence (where $T = Z - X$ and $\mathcal{A}' = \mathcal{A} \mathbin{\hat{\otimes}} C^{0,1}$):

$$K^{\mathcal{A}'}(Z) \longrightarrow K^{\mathcal{A}'}(T) \longrightarrow K^{\mathcal{A}}(X) \longrightarrow K^{\mathcal{A}}(Z) \longrightarrow K^{\mathcal{A}}(T).$$

[16] "quasi-graduation" in French.

[17] See the appendix about the role of self-adjoint Fredholm operators in K-theory.

By the usual excision theorem in topological K-theory, one may prove that this definition of $K^{\mathcal{A}}(X)$ is independent from the choice of Z.

The method described before and also in [29], §3, shows how to generalize the definition of $K^{\mathcal{A}}(X)$ in this case: one takes homotopy classes of pairs (E, D) as in 3.12, with the added assumption that the family D is acyclic at infinity. In other words, there is a compact set $S \subset X$, such that D_x is an isomorphism when $x \notin S$ (see [29], p. 89–97 for the technical details of this approach). This Fredholm description of $K^{\mathcal{A}}(X)$ will be important in the next section for the description of the Thom isomorphism.

4 The Thom isomorphism in twisted *K*-theory[18]

Let V be a finite-dimensional real vector bundle on a locally compact space X which extends over a compactification of X as was assumed in 3.15. Then the complexified Clifford bundle $C(V)$ has a well-defined class in the graded Brauer group of X. If $\mathcal{A}$ is another twist on X, we can consider the graded tensor product $\mathcal{A} \mathbin{\widehat{\otimes}} C(V)$ and the associated group $K^{\mathcal{A}\widehat{\otimes}C(V)}(X)$. As was described in 3.15, it is the group[19] associated to pairs (E, D) where E is a graded bundle of free $\mathcal{B}$-modules and D is a family of Fredholm endomorphisms which are of degree 1, self-adjoint and acyclic at ∞. Let us now consider the projection $\pi : V \to X$. For simplicity's sake, we shall often call $E, \mathcal{A}, \mathcal{B}, \ldots$ the respective pull-backs of $E, \mathcal{A}, \mathcal{B}, \ldots$ via this projection. Since $\mathcal{B}$ and $C(V)$ are subbundles of $\mathcal{B} \mathbin{\widehat{\otimes}} C(V)$, E may be provided with the induced $\mathcal{B}$ and $C(V)$-modules structures.

Theorem 4.1. *Let $d(E, D)$be an element of $K^{\mathcal{A}\widehat{\otimes}C(V)}(X)$ with the notations above. We define an element $t(d(E, D))$ in the group $K^{\mathcal{A}}(V)$ as $d(\pi^*(E), D')$ where D' is the family of Fredholm operators on $\pi^*(E)$, defined over the point v in V (with projection x on X) as*

$$D'_{(x,v)} = D_x + \rho(v)$$

where $\rho(v)$ denotes the action of the element v of the vector bundle V considered as a subbundle of $C(V)$. The homomorphism

$$t\colon K^{\mathcal{A}\widehat{\otimes}C(V)}(X) \to K^{\mathcal{A}}(V)$$

(t for "Thom") defined above is then an isomorphism.

Proof.[20] We should first notice that V may be identified with the open unit ball bundle of the vector bundle V and is therefore an open subset of the closed unit ball bundle of V. Moreover, since V and $\mathcal{A}$ extends to a compactification of X, the required conditions in 3.15 for the definition of the twisted K-theory of X and V are fulfilled.

[18] See also [16].

[19] We should note that E is a $\mathcal{B}$-module, not an $\mathcal{A}$-module. Nevertheless, we shall keep the notation $K^{\mathcal{A}}$.

[20] According to a suggestion of J. Rosenberg, it should be possible to give a proof with the KK-theory of Kasparov by describing an explicit inverse to the homomorphism t. However, KK-theory is out of the scope of this paper.

We shall now provide two different proofs of the theorem.

The first one, more elementary in spirit, consists in using a Mayer–Vietoris argument which we can apply here since the two sides of the formula above behave as cohomology theories[21] with respect to the base X. Therefore, we may assume that $\mathcal{A}$ and V are trivial: this is a special case of the theorem stated in [30], pp. 211/212.

The second one is more subtle and may be generalized to the equivariant case. Let us first describe the Thom isomorphism for complex V. This is a slight modification of Atiyah's argument using the elliptic Dolbeault complex [3]. More precisely, we consider the composite map

$$\phi\colon K^{\mathcal{A}}(X) \to K^{\mathcal{A}\widehat{\otimes}C(V)}(X) \xrightarrow{t} K^{\mathcal{A}}(V).$$

The first map θ is the cup-product with the algebraic "Thom class" which is ΛV provided with the classical Clifford graded module structure. This map is an isomorphism from well-known algebraic considerations (Morita equivalence). Therefore t is an isomorphism if and only if ϕ is an isomorphism. On the other hand, ϕ is just the cup-product with the topological Thom class T_V which belongs to the usual topological K-theory $K(V)$ of Atiyah and Hirzebruch [5], [6]. In order to prove that ϕ is an isomorphism, we may now use the exact sequence in 3.3 to reduce ourselves to the ungraded twisted case. In other words, it is enough to show that the cup-product with T_V induces an isomorphism

$$\Psi\colon K^{(\mathcal{A})}(X) \to K^{(\mathcal{A})}(V).$$

In order to prove this last point, Atiyah defines a reverse map[22]

$$\Psi'\colon K^{(\mathcal{A})}(V) \to K^{(\mathcal{A})}(X).$$

He shows that $\Psi'\Psi = \mathrm{Id}$ and, by an ingenious argument, deduces that $\Psi\Psi' = \mathrm{Id}$ as well.

Now, as soon as Theorem 4.2 is proved for complex V, the general case follows from a trick already used in [30], p. 241: we consider the following three Thom homomorphisms which behave "transitively":

$$\begin{aligned}K^{\mathcal{A}\widehat{\otimes}C(V)\widehat{\otimes}C(V)\widehat{\otimes}C(V)}(X) &\longrightarrow K^{\mathcal{A}\widehat{\otimes}C(V)\widehat{\otimes}C(V)}(V)\\ &\longrightarrow K^{\mathcal{A}\widehat{\otimes}C(V)}(V\oplus V) \longrightarrow K^{\mathcal{A}}(V\oplus V\oplus V).\end{aligned}$$

We know that the composites of two consecutive arrows are isomorphisms since $V\oplus V$ carries a complex structure. It follows that the first arrow is an isomorphism, which is essentially the theorem stated (using Morita equivalence again). $\square$

[21] Strictly speaking, one has to "derive" the two members of the formula, which can be done easily since they are Grothendieck groups of graded Banach categories.

[22] More precisely, one has to define an index map parametrized by a Banach bundle, which is also classical [20].

Let $\mathcal{A}$ be a *graded* twist. As we have seen before, it has two invariants in $H^1(X;\mathbb{Z}/2)$ and in $H^3(X;\mathbb{Z})$. The first one provides a line bundle L in such a way that the graded tensor product $\mathcal{A}_1 = \mathcal{A} \mathbin{\widehat{\otimes}} C(L)$ is oriented. From Thom isomorphism and the considerations in 3.5/7, we deduce the following theorem which gives the relation between the ungraded and graded twisted K-groups.

Theorem 4.2. *Let $\mathcal{A}$ be a graded twist of type $M_2(\mathcal{K})$ and $\mathcal{A}_1 = \mathcal{A} \mathbin{\widehat{\otimes}} C(L)$, where L is the orientation bundle of $\mathcal{A}$. Then $\mathcal{A}_1$ may be written as $\mathcal{A}_2 \times \mathcal{A}_2$ where $\mathcal{A}_2$ is ungraded of type $\mathcal{K}$. Therefore, we have the following isomorphisms*

$$K^{\mathcal{A}}(X) \cong K^{\mathcal{A}\widehat{\otimes}C(L)\widehat{\otimes}C(L)}(X) \cong K^{\mathcal{A}_1}(L) \cong K_1^{(\mathcal{A}_2)}(L).$$

Let $\mathcal{A}$ be a graded twist of type $\mathcal{K} \times \mathcal{K}$. With the same notations, we have the following isomorphisms

$$K^{\mathcal{A}}(X) \cong K^{\mathcal{A}\widehat{\otimes}C(L)\widehat{\otimes}C(L)}(X) \cong K^{\mathcal{A}_1}(L) \cong K^{(\mathcal{A}_1)}(L).$$

5 General equivariant K-theory

Note: this section is inserted here for the convenience of the reader as an introduction to §6. It is mainly a summary of results found in [39], [12], [35] and [8].

Let A be a Banach algebra and G a compact Lie group acting on A via a continuous group homomorphism

$$G \longrightarrow \mathrm{Aut}(A)$$

where $\mathrm{Aut}(A)$ is provided with the norm topology. We are interested in the category whose objects are finitely generated projective A-modules E together with a continuous left action of G on E such that we have the following identity holds ($g \in G$, $a \in A$, $e \in E$)

$$g.(a.e) = (g.a).(g.e)$$

with an obvious definition of the dots. We define $K_G(A)$ as the Grothendieck group of this category $\mathcal{C} = \mathcal{P}_G(A)$. By the well-known dictionary between bundles and modules, we recover the usual equivariant K-theory defined by Atiyah and Segal [39] if A is the ring of continuous maps on a compact space X (thanks to the lemma below). It may also be defined as a suitable semi-direct product of G by A (cf. [12]).

More generally, if A is a $\mathbb{Z}/2$-graded algebra (where G acts by degree 0 automorphisms), we define the graded equivariant K-theory of A, denoted by $\mathrm{Gr}K_G(A)$, as the Grothendieck group of the forgetful functor

$$\mathcal{C}^{0,1} \longrightarrow \mathcal{C}$$

where $\mathcal{C}^{0,1}$ is the category of "graded objects" in $\mathcal{P}_G(A)$ which is defined as $\mathcal{P}_G(A \mathbin{\widehat{\otimes}} C^{0,1})$.

In the same spirit as in §1, we define "derived" groups $K_G^{p,q}(A)$ as $\mathrm{Gr}K_G(A \mathbin{\widehat{\otimes}} C^{p,q})$. They satisfy the same formal properties as the usual groups $K_n(A)$ (also denoted by

$K^{-n}(A)$ with $n = q - p$), for instance Bott periodicity. The following key lemma enables us to translate many general theorems of K-theory into the equivariant framework:

Lemma 5.1. *Let E be an object of the category $\mathcal{P}_G(A)$. Then E is a direct summand of an object of the type $A \otimes_{\mathbb{C}} M$ where M is a finite-dimensional G-module.*

Proof. ([39], p. 134). Let us consider the union Γ of all finite-dimensional invariant subspaces of the G-Banach space E. According to a version of the Peter-Weyl theorem quoted in [39], this union is dense in A. We now consider a set $e_1, \dots, e_n$ of generators of E as an A-module. Since Γ is dense in A and E is projective, one may choose these generators in the subspace Γ. Let $M_1, \dots, M_n$ be finite-dimensional invariant subspaces of E containing $e_1, \dots, e_n$ respectively and let M be the following direct sum:

$$M = M_1 \oplus \cdots \oplus M_n$$

We define an equivariant surjection

$$\phi\colon A \otimes_{\mathbb{C}} M \cong (A \otimes_{\mathbb{C}} M_1) \oplus \cdots \oplus (A \otimes_{\mathbb{C}} M_n) \to E.$$

between projective left A-modules by the formula

$$\phi(\lambda_1 \otimes m1, \dots, \lambda_n \otimes m_n) = \lambda_1 m_1 + \cdots + \lambda_n m_n.$$

This surjection admits a section, which we can average out thanks to a Haar measure in order to make it equivariant. Therefore, E is a direct summand in $A \otimes_{\mathbb{C}} M$ as stated in the lemma. $\square$

As for usual K-theory, one may also define equivariant K-theory for non unital rings and, using Lemma 5.2 above, prove that any equivariant exact sequence of rings on which G acts

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

induces an exact sequence of equivariant K-groups

$$K_G^{n-1}(A) \longrightarrow K_G^{n-1}(A'') \longrightarrow K_G^n(A') \longrightarrow K_G^n(A) \longrightarrow K_G^n(A'')$$

and an analogous exact sequence in the graded equivariant framework.

After these generalities, let us assume that G acts on a compact space X and let $\mathcal{A}$ be a bundle of algebras modelled on $\mathcal{K}$. We define the (ungraded) equivariant twisted K-group $K_G^{(\mathcal{A})}(X)$ as $K_G(A)$, where A is the Banach algebra of sections of the bundle $\mathcal{A}$. Similarly, if $\mathcal{A}$ is a bundle of graded algebras modelled on $\mathcal{K} \times \mathcal{K}$ or $M_2(\mathcal{K})$, we define the (graded) equivariant twisted K-group $K_G^{\mathcal{A}}(X)$ as $\mathrm{Gr}K_G(A)$, where A is the graded Banach algebra of sections of $\mathcal{A}$.

As seen in §3, it is also natural to consider free graded $\mathcal{B}$-modules together with a continuous action of G (compatible with the action on X) and a family of Fredholm operators D which are self-adjoint of degree 1, commuting with the action of G. With the same ideas as in §3, we can show that the Grothendieck group of this category is isomorphic to $K_G^{\mathcal{A}}(X)$. This is essentially the definition proposed in [8].

An example was in fact given at the beginning of the history of twisted K-theory. In [19], §8, we defined a "power operation"

$$P\colon K^{\mathcal{A}}(X) \to K_{S_n}^{\mathcal{A}^{\widehat{\otimes} n}}(X)$$

where S_n denotes the symmetric group on n letters acting on $\mathcal{A}^{\widehat{\otimes} n}$ (in the spirit of Atiyah's paper on power operations [2]). According to the general philosophy of Adams and Atiyah, one can deduce from this n^{th} power operation new "Adams operations" (n being odd and the group law in $\mathrm{GBr}(X)$ being written multiplicatively):

$$\Psi^n\colon K^{\alpha}(X) \to K^{\alpha^n}(X) \otimes_{\mathbb{Z}} \Omega_n$$

where α belongs to $\mathbb{Z}/2 \times H^1(X;\mathbb{Z}/2) \times H^3(X;\mathbb{Z})$ and Ω_n denotes the free $\mathbb{Z}$-module generated by the n^{th} roots of unity in $\mathbb{C}$ (the ring of cyclotomic integers[23]). One can prove, following [19], that these additive maps Ψ^n satisfy all the required properties proved by Adams. We shall come back to them in §7, making the link with operations recently defined by Atiyah and Segal [9].

In order to fix ideas, let us consider the graded version of equivariant twisted K-theory $K_G^{\mathcal{A}}(X)$, where $\mathcal{A}$ is modelled on $\mathbb{C} \times \mathbb{C}$ with the obvious grading. We have again a "Thom isomorphism":

$$t\colon K_G^{\mathcal{A}\widehat{\otimes} C(V)}(X) \to K_G^{\mathcal{A}}(V)$$

whose proof is the same as in the non equivariant case (using elliptic operators). On the other hand, if L is the orientation line bundle of $\mathcal{A}$, the group G acts on L in a way compatible with the action on X. Therefore, $\mathcal{A}_1 = \mathcal{A} \widehat{\otimes} C(L)$ is an algebra bundle with trivial orientation. and $\mathcal{A}_1 \widehat{\otimes} C^{0,1}$ is naturally isomorphic to $\mathcal{A}_1 \times \mathcal{A}_1 = \mathcal{A}_1[x]/(x^2-1)$. As in 4.4, this shows that the graded twisted group $K_G^{\mathcal{A}}(X)$ is naturally isomorphic to the ungraded twisted K-group $K_G^{\mathcal{A}_1}(L)$. The same method shows that we can reduce graded twisted K-groups to ungraded ones if $\mathcal{A}$ is a bundle of graded algebras modelled on $M_2(\mathcal{K})$.

Let us mention finally one of the main contributions of Atiyah and Segal to the subject ([8], §6, see also [17]), which we interpret in our language. One is interested in algebra bundles $\mathcal{A}$ on X modelled on $\mathcal{K}$, provided with a left G-action. The isomorphism classes of such bundles are in bijective correspondence with principal bundles P over $PU(H)$ (acting on the right) together with a left action of G.

Such an algebra bundle $\mathcal{T}$ is called "trivial" if $\mathcal{T}$ may be written as the bundle of algebras of compact operators in $\mathrm{End}(V)$, where V is a G-Hilbert bundle. Equivalently, this means that the structure group of $\mathcal{T}$ can be lifted equivariantly to $U(H)$ (in a way compatible with the G-action).

[23] The introduction of this ring is absolutely necessary in the graded case.

We now say that two such algebra bundles $\mathcal{A}$ and $\mathcal{A}'$ are equivalent if there exist two trivial algebra bundles $\mathcal{T}$ and $\mathcal{T}''$ such that $\mathcal{A} \otimes \mathcal{T}$ and $A' \otimes \mathcal{T}''$ are isomorphic as G-bundles of algebras. The quotient is a group since the dual of $\mathcal{A}$ is its inverse via the tensor product of principal $PU(H)$-bundles. This is the "equivariant Brauer group" $\mathrm{Br}_G(X)$.

A closely related definition (in the framework of C*-algebras and for a locally compact group G) is given in [17]. It is very likely that it coincides with this one for compact Lie groups, in the light of an interesting filtration described in this paper, probably associated to a spectral sequence.

Theorem 5.2 (cf. [8], Proposition 6.3). *Let $X_G = EG \times_G X$ be the Borel space associated to X. Then the natural map*

$$\mathrm{Br}_G(X) \to \mathrm{Br}(X_G)$$

is an isomorphism.

The interest of this theorem lies in the fact that the equivariant K-theories $K_G(\Gamma(X, \mathcal{A}))$ and $K_G(\Gamma(X, \mathcal{A}'))$ are isomorphic if $\mathcal{A}$ and $\mathcal{A}'$ are equivalent. This follows from the well-known Morita invariance in operator K-theory. We shall study concrete applications of this principle in the next section.

6 Some computations of twisted equivariant K-groups

Let us look at the particular case of the ungraded twisted K-groups $K_G^{(\mathcal{A})}(X)$ where G is a finite group acting on the trivial bundle of algebras $\mathcal{A} = X \times M_n(\mathbb{C})$ via a group homomorphism $G \to PU(n)$.[24] We define $\widetilde{G}$ as the pull-back diagram

$$\begin{tikzcd} \widetilde{G} \arrow[r] \arrow[d] & SU(n) \arrow[d] \\ G \arrow[r] & PU(n) \end{tikzcd}$$

Therefore, $\widetilde{G}$ is a central covering of G with fiber μ_n (whose elements are denoted by Greek letters such as λ). The following definition is already present in [31], §2.5 (for $n = 2$):

Definition 6.1. A finite-dimensional representation ρ of $\widetilde{G}$ is of "linear type" if $\rho(\lambda u) = \lambda\rho(u)$ for any $\lambda \in \mu_n$. We now consider the category $E_{\widetilde{G}}^{\mathcal{A}}(X)_l$ whose objects are $\widetilde{G}$-$\mathcal{A}$-modules as before, except that we request that the $\widetilde{G}$-action be of linear type and *commutes with the action of* $\mathcal{A}$. By Morita invariance, $E_{\widetilde{G}}^{\mathcal{A}}(X)_l$ is equivalent to the category $E_{\widetilde{G}}(X)_l$ of finite-dimensional $\widetilde{G}$-bundles on X, the action of $\widetilde{G}$ on the fibers being of linear type.

[24] However, we don't assume that G acts trivially on X in general.

Theorem 6.2.[25] *The (ungraded) twisted K-theory $K_G^{(\mathcal{A})}(X)$ is canonically isomorphic to the Grothendieck group of the category $E_{\widetilde{G}}(X)_l$.*

Proof. One just repeats the argument in the proof of Theorem 2.6 in [31], where A is a Clifford algebra $C(V)$ and $\mathbb{Z}/2$ plays the role of μ_n. We simply "untwist" the action of $\widetilde{G}$ thanks to the formula (F) written explicitly in the proof of 2.6 (loc. cit.). $\square$

For $\mathcal{A} = X \times A$ with $A = M_n(\mathbb{C})$, the previous argument shows that $K_G^{(\mathcal{A})}(X)$ is a subgroup of the usual equivariant K-theory $K_{\widetilde{G}}(X)$. From now on, we shall write $K_G^A(X)$ instead of $K_G^{(\mathcal{A})}(X)$. Similarly, in the graded case ($A = M_n(\mathbb{C}) \times M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$), we shall write $K_G^A(X)$ instead of $K_G^{\mathcal{A}}(X)$. If X is a point and G is finite, $K_G^{(A)}(X)$ is just the K-theory of the semi-direct product $G \ltimes A$.

Theorem 6.3. *Let G be a finite group acting on the algebra of matrices $A = M_n(\mathbb{C})$ and let $\widetilde{G}$ be the central extension G by μ_n described in* 6.1. *Then, for X reduced to a point, the group $K_G^{(\mathcal{A})}(X) = K(G \ltimes A)$ is a free abelian group of rank the number of conjugacy classes in G which split into n conjugacy classes in $\widetilde{G}$.*

Proof. We can apply the same techniques as the ones detailed in [31], §2.6/12 (for $n = 2$). By the theory of characters on $\widetilde{G}$, one is looking for functions f on $\widetilde{G}$ (which we call of "linear type") such that

1. $f(hgh^{-1}) = f(g)$,
2. $f(\lambda x) = \lambda f(x)$ if λ is an n^{th} root of the unity.

The $\mathbb{C}$-vector space of such functions is in bijective correspondence with the space of functions on the set of conjugacy classes of G which split into n conjugacy classes of $\widetilde{G}$. $\square$

Like the Brauer group of a space X, one may define in a similar way the Brauer group $\mathrm{Br}(G)$ of a finite group G by considering algebras $A = M_n(\mathbb{C})$ as above with a G-action (see [24] for a broader perspective ; this is also a special case of the general theory of Atiyah and Segal mentioned at the end of §5). From the diagram written in 6.1, one deduces a cohomology invariant

$$w_2(A) \in H^2(G; \mu_n)$$

and (via the Bockstein homomorphism) a second invariant $W_3(A) \in H^2(G; S^1) = H^2(G; \mathbb{Q}/\mathbb{Z}) = H^3(G; \mathbb{Z})$. It is easy to show that this correspondence induces a well-defined map

$$W_3 \colon \mathrm{Br}(G) \to H^3(G; \mathbb{Z}).$$

The following theorem is a special case of 5.9 in a more algebraic situation.

[25] There is an obvious generalization when $\mathcal{A}$ is infinite-dimensional. However, for our computations, we restrict ourselves to the finite-dimensional case.

Theorem 6.4. *Let G be a finite group. Then the previous homomorphism*

$$W_3 \colon \mathrm{Br}(G) \to H^3(G;\mathbb{Z})$$

is bijective.

Proof. First of all, we remark that $H^3(G;\mathbb{Z}) \cong H^2(G;\mathbb{Q}/\mathbb{Z})$ is the direct limit of the groups $H^2(G;\mu_m)$ through the maps $H^2(G;\mu_m) \to H^2(G;\mu_p)$ when m divides p. This stabilization process corresponds on the level of algebras to the tensor product $A \mapsto A \otimes \mathrm{End}(V)$, where V is a G-vector space of dimension p/m. Therefore, the map W_3 is injective.

The proof of the surjectivity is a little bit more delicate (see also 5.9). We can say first that it is a particular case of a much more general result proved by A. Fröhlich and C. T. C. Wall [24] about the equivariant Brauer group of an arbitrary field k: there is a split exact sequence (with their notations)

$$0 \longrightarrow \mathrm{Br}(k) \longrightarrow BM(k,G) \longrightarrow H^2(G;U(k)) \longrightarrow 0$$

where $\mathrm{Br}(k)$ is the usual Brauer group of k, $U(k)$ is the group of invertible elements in k and $BM(k,G)$ is a group built out of central simple algebras over k with a G-action. Since $\mathrm{Br}(\mathbb{C}) = 0$ and $H^2(G;U(k)) = H^2(G;\mathbb{Q}/\mathbb{Z})$, the theorem is an immediate consequence.

Here is an elementary proof suggested by the referee (for $k = \mathbb{C}$). If we start with a central extension

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

we consider the finite dimensional vector space

$$H = \{f \in L^2(\widetilde{G}) \text{ such that } f(\lambda\gamma) = \lambda^{-1} f(\gamma) \text{ with } (\lambda,\gamma) \in \mu_n \times \widetilde{G}\}.$$

Then $\widetilde{G}$ acts by left translation on H in such a way that this action is of linear type as described in 6.2. Therefore, we have a projective representation of G on H and $\mathrm{End}(H)$ is a matrix algebra where G acts. □

Remark 6.5. Let us consider an arbitrary central extension $\widetilde{G}$ of G by $\mu_n \cong \mathbb{Z}/n$ associated to a cohomology class $c \in H^2(G;\mathbb{Z}/n)$. We are interested in the set of elements g of G such that the conjugacy class $\langle g \rangle$ splits into n conjugacy classes in $\widetilde{G}_1$. This set only depends on the image of c in $H^3(G;\mathbb{Z})$ via the Bockstein homomorphism $H^2(G;\mathbb{Z}/n) \to H^3(G;\mathbb{Z})$ (cf. 6.6). In other words, two central extensions of G by $\mathbb{Z}/n$ with the same associated image by the Bockstein homomorphism have the same set of n-split conjugacy classes.

In order to show this fact, let us consider the following diagram:

$$\begin{tikzcd}
\mu_n \arrow[r] \arrow[d] & \mu_{nm} \arrow[d] \\
\widetilde{G}_1 \arrow[r] \arrow[d, "\pi"'] & \widetilde{G} \arrow[d] \\
G \arrow[r, equal] & G,
\end{tikzcd}$$

and an element g_1 of $\widetilde{G}_1$. The conjugacy class of $g = \pi(g_1)$ splits into n conjugacy classes in $\widetilde{G}_1$ if and only if there is a trace function f on $\widetilde{G}_1$ with values in $\mathbb{C}$ such that $f(g_1 c) = f(g_1)c$ when $c \in \mu_n$. Such a trace function extends obviously to $\widetilde{G}$, which yields to the result, since the direct limit of the groups $H^2(G;\mathbb{Z}/nm)$ is precisely $H^2(G;\mathbb{Q}/\mathbb{Z}) \cong H^3(G;\mathbb{Z})$.

This remark may be generalized as follows according to a suggestion of J.-P. Serre: let $\widetilde{G}_1$ and $\widetilde{G}$ be two group extensions (not necessary central) of G by abelian groups C_1 and C of orders m_1 and m respectively, such that the following diagram commutes (with α injective):

$$\begin{tikzcd}
C_1 \arrow[r, "\alpha"] \arrow[d] & C \arrow[d] \\
\widetilde{G}_1 \arrow[r] \arrow[d, "\pi"'] & \widetilde{G} \arrow[d] \\
G \arrow[r, equal] & G.
\end{tikzcd}$$

The previous argument shows that if an element g of G splits into m conjugacy classes in $\widetilde{G}$, it splits into m_1 conjugacy classes in $\widetilde{G}_1$: take trace functions f on $\widetilde{G}$ with values in C such that $f(gc) = f(g)c$ (we write multiplicatively the abelian group C). The converse is true if the extension of G by C is central.

Let us now assume that X is not reduced to a point. We can use the Baum–Connes–Kuhn–Słomińska Chern character [11], [33], [41] which is defined on $K_\Gamma(X)$ (for any finite group Γ), with values in the direct sum $\bigoplus_{\langle\gamma\rangle} H^{\text{even}}(X^g)^{C(\gamma)}$. In this formula, $\langle\gamma\rangle$ runs through all the conjugacy classes of G, $C(\gamma)$ being the centralizer of γ (the cohomology is taken with complex coefficients). One of the main features of this "Chern character"

$$K_\Gamma(X) \longrightarrow \bigoplus_{\langle\gamma\rangle} H^{\text{even}}(X^\gamma)^{C(\gamma)}$$

is the isomorphism it induces between $K_\Gamma(X) \otimes_{\mathbb{Z}} \mathbb{C}$ and the cohomology with complex coefficients on the right-hand side. If E is a Γ-vector bundle, the map is defined explicitly by Formula 1.13, p. 170 in [11].

Let us now take for Γ the group $\widetilde{G}$ previously considered and let us analyze the formula in this case. We shall view the right-hand side not just as a function on the set of conjugacy classes $\langle\gamma\rangle$, but as a function f on the full group Γ with certain extra properties which we shall explain now.

If we replace γ by γ' such that $\gamma' = \sigma\gamma\sigma^{-1}$, we have a canonical isomorphism $\sigma^* \colon H^*(X^{\gamma'})^{C(\gamma')} \to H^*(X^{\gamma})^{C(\gamma)}$ induced by $x \mapsto \sigma x$. This map exchanges $f(\gamma)$ and $f(\gamma')$ and we have the relation $f(\gamma) = \sigma^*(f(\gamma'))$, which says that f is essentially a function on the set of conjugacy classes.

On the other hand, if the action of $\widetilde{G}$ is of linear type, we have an extra relation, an easy consequence of the formula in [11], which is $f(\mu\gamma) = \mu f(\gamma)$ when μ is an n^{th} root of unity. To summarize, we get the following theorem.

Theorem 6.6. *Let G be a finite group and $A = M_n(\mathbb{C})$ with a G-action. Then the ungraded twisted equivariant K-theory $K_G^{(A)}(X)$ is a subgroup of the equivariant K-theory $K_{\widetilde{G}}(X)$, where $\widetilde{G}$ is the pull-back diagram*

$$\begin{array}{ccc}
\widetilde{G} & \longrightarrow & SU(n) \\
{\scriptstyle\pi}\big\downarrow & & \big\downarrow \\
G & \longrightarrow & PU(n).
\end{array}$$

More precisely, $K_G^{(A)}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ may be identified with the $\mathbb{C}$-vector space of functions f on $\Gamma = \widetilde{G}$ with $f(\gamma)$ in $H^{even}(X^g)^{C(\gamma)}$, $\pi(\gamma) = g$, such that the following two identities hold:

1. *If $\gamma' = \sigma\gamma\sigma^{-1}$, one has $f(\gamma) = \sigma^*(f(\gamma'))$, according to the formula above.*
2. *$f(\mu\gamma) = \mu f(\gamma)$ if μ is an n^{th} root of unity.*

In particular, if X is reduced to a point, we have $\sigma^* = \mathrm{Id}$ and $K_G^{(A)}(X)$ is free with rank the number of conjugacy classes of G which split into n conjugacy classes in $\widetilde{G}$ (as seen in 6.5.)

Remark 6.7. This theorem is not really new. In a closely related context, one finds similar results in [1] and [42]. We should also notice that the same ideas have been used in [31] for representations of "linear type". Finally, the theorem easily extends to locally compact spaces if we consider cohomology with compact supports on the right-hand side.

Theorem 6.8. *Let A be any finite-dimensional graded semi-simple complex algebra with a graded action of a finite group G. Then the graded K-theory $\mathrm{Gr}K_0(A') \oplus \mathrm{Gr}K_1(A')$ of the semi-direct product $A' = G \ltimes A$ is a non trivial free $\mathbb{Z}$-module. In particular, if V is a real finite-dimensional vector space with a G action, the group $K_G^A(V) \oplus K_G^A(V \oplus 1)$ is free non trivial thanks to the Thom isomorphism.*

Proof. The algebra $G \ltimes A$ is graded semi-simple over the complex numbers. Therefore, it is a direct sum of graded algebras Morita equivalent to $M_n(\mathbb{C}) \times M_n(\mathbb{C})$ or $M_{2n}(\mathbb{C})$. In both cases, the graded K-theory is non trivial. The last part of the theorem follows from 4.2. $\square$

The following theorem is a direct consequence of the previous considerations:

Theorem 6.9. *Let us now assume that $A = M_{2n}(\mathbb{C})$ is G-oriented as a graded algebra: in other words, there is an involutive element ε of A of degree 0 which commutes with the action of G and commutes (resp. anticommutes) with the elements of A of degree 0 (resp. 1). Then, the graded K-theory $\mathrm{Gr}K_*(A')$, with $A' = G \ltimes A$, is a finitely generated free module concentrated in degree 0. More precisely, one has $\mathrm{Gr}K_0(A') = K(A')$ and $\mathrm{Gr}K_1(A') = 0$. In particular, if V is an even-dimensional real vector space and if $A \mathbin{\hat{\otimes}} C(V)$ is G-oriented, we have (via the Thom isomorphism)*

$$K_G^A(V) = K_G^{A\hat{\otimes}C(V)}(P) = K(G \ltimes A \mathbin{\hat{\otimes}} C(V)) \text{ and } K_G^A(V \oplus 1) = 0$$

where P is a point. If we write $A \mathbin{\hat{\otimes}} C(V)$ as an algebra of matrices $M_r(\mathbb{C})$ with a representation ρ of G and call $\widetilde{G}$ the associated central extension by μ_r, the rank of $K_G^A(V)$ is the number of conjugacy classes of G which split into r conjugacy classes in $\widetilde{G}$.

In the abelian case, the following two theorems are related to results obtained by P. Hu and I. Kriz [26], using different methods.

Theorem 6.10. *Let us consider the algebra $A = M_n(\mathbb{C})$ provided with an action of an abelian group G and P a point. Then the ungraded twisted K-theory $K_G^{(A)}(P)_* = K_*(A')$, with $A' = G \ltimes A$, is concentrated in degree 0 and is a free $\mathbb{Z}$-module. If we tensor this group with the rationals and if we look at it as an $R(G) \otimes \mathbb{Q} = \mathbb{Q}[G]$-module, it may be identified with $R(G') \otimes \mathbb{Q}$ for a suitable subgroup G' of G. In particular, the rank of $K_0(A')$ divides the order of G.*

Proof. The first part of the theorem is a consequence of the previous more general considerations. As we have shown before, the algebra A' gives rise to the following commutative diagram:

$$\begin{tikzcd} \widetilde{G}_n \arrow[r] \arrow[d, "\pi"'] & SU(n) \arrow[d] \\ G \arrow[r] & PU(n), \end{tikzcd}$$

the fibers of the vertical maps being μ_n. The subset of elements $\tilde{g}$ in $\widetilde{G}_n$ such that $\pi(\tilde{g})$ splits into n conjugacy classes is just the center $Z(\widetilde{G}_n)$ of $\widetilde{G}_n$ (since G is abelian). Let us put $\Gamma_n = \pi(Z(\widetilde{G}_n))$. Then $K(A')$ may be written as $K_G^A(P)$ where P is a point. According to Theorem 6.10, this is the subgroup of the representation ring of $\widetilde{G}_n$ generated by representations of linear type. At this stage, it is convenient to make $n = \infty$ by extension of the roots of unity, so that we have an extension of G by $\mathbb{Q}/\mathbb{Z}$

$$\mathbb{Q}/\mathbb{Z} \longrightarrow \widetilde{G} \longrightarrow G$$

(the "linear type" finite-dimensional representations of $\widetilde{G}$ are the same as the original linear type finite-dimensional representations of $\widetilde{G}_n$). We call $R(\widetilde{G})_l)$ the associated

Grothendieck group. By the theory of characters, we see that $R(\widetilde{G})_l \otimes \mathbb{Q}$ is isomorphic to $R(Z(\widetilde{G}))_l \otimes \mathbb{Q}$, since the characters of such linear type representations of $\widetilde{G}$ vanish outside $Z(\widetilde{G})$. On the other hand, if we denote by G' the image of $Z(\widetilde{G})$ in G, the extension of abelian groups

$$\mathbb{Q}/\mathbb{Z} \longrightarrow Z(\widetilde{G}) \longrightarrow G'$$

splits (non canonically). This means that we can identify $R(\widetilde{G})_l \otimes \mathbb{Q}$ with the representation ring $R(G') \otimes \mathbb{Q}$ as an $R(G) \otimes \mathbb{Q}$-module. This proves the last part of the theorem. □

Theorem 6.11. *Let us consider a graded algebra $A = M_{2n}(\mathbb{C})$ provided with a non oriented action of an abelian group G (with respect to the grading). Then the graded twisted K-theory $K_G^A(P)_* = \mathrm{Gr}K_*(A')$, with $A' = G \ltimes A$, is concentrated in a single degree* (0 *or* 1) *and is a free $\mathbb{Z}$-module. If we tensor this group with the complex numbers and if we look at it as an $R(G) \otimes \mathbb{C} = \mathbb{C}[G]$-module, it may be identified with $R(G') \otimes \mathbb{Q}$ for a suitable subgroup G' of G. In particular, the rank of $K_G^A(P)_*$ divides the order of G.*

Proof. Let L be the orientation bundle of A (with respect to the action of G). If we change A into $A \mathbin{\hat{\otimes}} C(L)$ and if we apply the Thom isomorphism theorem, we have to compute $K_G^A(L)_*$ where G acts on A (resp. L) in an oriented way (resp. non oriented way).

Let us apply Theorem 6.10 in this situation: since G is abelian, the function f of the theorem must be equal to 0 on the elements of $\widetilde{G}$ which are not in $Z(\widetilde{G})$. Therefore, the relevant group $K_G^A(L)_*$ is reduced to $K_{G'}(L)$ (after tensoring with $\mathbb{C}$ and where G' is the image of $Z(G)$ in G). We now consider two cases:

1. the action ρ of G' on L is oriented, in which case we only find $R(G')$ (with a shift of dimension). Therefore, the dimension of $K_G^A(P) \otimes \mathbb{C}$ is the order of G' which divides the order of G.

2. the action ρ of G' on L is not oriented. We then find a direct sum of copies of $\mathbb{C}$, each one corresponding to an element of G' such that $\rho(g') = -1$. The dimension of $K_G^A(L) \otimes \mathbb{C}$ is therefore half the order of G', hence divides $|G|/2$. □

Remark 6.12. For an oriented action of G on $M_{2n}(\mathbb{C})$, Theorem 3.5 enables us to solve the analogous problem in ungraded twisted K-theory, which is done in 6.14.

On the other hand, as we already mentioned, the two last theorems are related to results of P. Hu and I. Kriz [26].

In [31] we also perform analogous types of computations when A is a Clifford algebra and G is any finite group, again using the Thom isomorphism. For instance, if G is the symmetric group S_n acting on the Clifford algebra of $\mathbb{R}^n$ via the canonical representation of S_n on $\mathbb{R}^n$, there is a nice relation between K-theory and the pentagonal identity of Euler (cf. [31], p. 532).

We would like to point out also that the theory $K_{\pm}(X)$ introduced recently by Atiyah and Hopkins [7] is a particular case of twisted equivariant K-theory. As a matter of fact, it was explicitly present in [28], §3, 40 years ago, before the formal introduction of twisted K-theory. Here is a detailed explanation of this identification.

According to [7], the definition of $K_{\pm}(X)$ (in the complex or real case) is the group $K_{\mathbb{Z}/2}(X \times \mathbb{R}^8)$, where $\mathbb{Z}/2$ acts on X and also on $\mathbb{R}^8 = \mathbb{R} \times \mathbb{R}^7$ by $(\lambda, \mu) \mapsto (-\lambda, \mu)$. According to the Thom isomorphism in equivariant K-theory (proved in the non spinorial case in [30]), it coincides with an explicit graded twisted K-group $K^{A}_{\mathbb{Z}/2}(X)$, as defined in [28]. Here A is the Clifford algebra $C(\mathbb{R}^2) = C^{1,1}$ of $\mathbb{R}^2$ provided with the quadratic form $x^2 - y^2$ and where $\mathbb{Z}/2$ acts via the involution $(\lambda, \mu) \to (-\lambda, \mu)$ on $\mathbb{R} \times \mathbb{R}$ (this is also mentioned briefly in [7], p. 2, footnote 1). This identification is valid as well in the real framework, where we have 8-periodicity.

These groups $K^{A}_{\mathbb{Z}/2}(X)$ were considered in [28], §3.3, in a broader context: A may be any Clifford algebra bundle $C(V)$ (where V is a *real* vector bundle provided with a non degenerate quadratic form) and $\mathbb{Z}/2$ may be replaced by any compact Lie group acting in a coherent way on X and V. The paper [31] gives a method to compute these equivariant twisted K-groups. As quoted in the appendix, the real and complex self-adjoint Fredholm descriptions (for the non twisted case) which play an important role in [7] were considered independently in [10] and [32].

7 Operations on twisted K-groups

This section is a partial synthesis of [19] and [9].

Let us start with the simple case of bundles of (ungraded) infinite C*-algebras modelled on $\mathcal{K}$, like in [9]. As it was shown in [2] and [19], we have a n^{th} power map

$$P\colon K^{\mathcal{A}}(X) \to K^{(\mathcal{A}^{\otimes n})}_{S_n}(X)$$

where the symmetric group S_n acts on $\mathcal{A}^{\otimes n}$ by permutation of the factors.

Lemma 7.1.[26] *The group $K^{(\mathcal{A}^{\otimes n})}_{S_n}(X)$ is isomorphic to the group $K^{(\mathcal{A}^{\otimes n})_0}_{S_n}(X)$ where the symbol* 0 *means that S_n is acting trivially on $\mathcal{A}^{\otimes n}$*

Proof. As we have shown many times in §6, this "untwisting" of the action of the symmetric group on $\mathcal{A}^{\otimes n}$ is due to the following fact: the standard representation

$$S_n \to PU(H^{\otimes n})$$

can be lifted into a representation $\rho\colon S_n \to U(H^{\otimes n})$ in a way compatible with the diagonal action of elements of $PU(H)$, a fact which is obvious to check. □

[26] We should note that this lemma is not true for $K_{S_n}(A^{\otimes n})$ for a general noncommutative ring A. Therefore, it is not possible to define λ-operations in this case. Twisted K-theory is somehow intermediary between the commutative case and the noncommutative one.

Remark 7.2. If we take a bundle of finite dimensional algebras modelled on $A = \mathrm{End}(E)$ where $E = \mathbb{C}^r$, there is another way to check (functorially) this untwisting of the action of S_n on $A^{\otimes n}$: we identify $A^{\otimes n}$ with $\mathrm{End}(E^{\otimes n})$ and $(A^{\otimes n})^*$ with $\mathrm{Aut}(E^{\otimes n})$. We have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{End}(E^{\otimes n})^* = \mathrm{Aut}(E^{\otimes n}) \arrow[d, "\pi"] \\ S_n \arrow[ur, "\theta"] \arrow[r] & \mathrm{End}(E^{\otimes n}) = \mathrm{End}(E)^{\otimes n}. \end{tikzcd}$$

The vertical map sends the invertible element α to the automorphism $(u \mapsto \alpha u \alpha^{-1})$. If σ is a permutation, the horizontal map sends σ to the automorphism

$$u_1 \otimes \cdots \otimes u_n \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

while the map θ sends σ to the automorphism of $E^{\otimes n}$ defined by

$$x_1 \otimes \cdots \otimes x_n \mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}.$$

Finally, the composition $\pi\theta$, computed on a decomposable tensor of $E^{\otimes n}$, gives the required result:

$$x_1 \otimes \cdots \otimes x_n \xrightarrow{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \xrightarrow{u} u_1(x_{\sigma(1)}) \otimes \cdots \otimes u_n(x_{\sigma(n)})$$

$$\xrightarrow{\sigma^{-1}} u_{\sigma(1)}(x_1) \otimes \cdots \otimes u_{\sigma(n)}(x_n).$$

As was shown in [2], a $\mathbb{Z}$-module map

$$R(S_n) \longrightarrow \mathbb{Z}$$

defines an operation in twisted K-theory by taking the composite of the following maps:

$$K^{(\mathcal{A})}(X) \longrightarrow K_{S_n}^{(\mathcal{A}^{\otimes n})}(X) \cong K_{S_n}^{(\mathcal{A}^{\otimes n})_0}(X) \cong K^{(\mathcal{A})^{\otimes n}}(X) \otimes R(S_n) \longrightarrow K^{(\mathcal{A})^{\otimes n}}(X).$$

This is essentially[27] what was done in [9], §10, in order to define the λ^n operation of Grothendieck in this context for instance.

Let us call (F, ∇) or even F (for short) a representative of the image of (E, D) by the composite of the maps (we now use the Fredholm description of twisted K-theory)

$$K^{(\mathcal{A})}(X) \longrightarrow K_{S_n}^{(\mathcal{A}^{\otimes n})}(X) \longrightarrow K_{A_n}^{(\mathcal{A}^{\otimes n})_0}(X).$$

We can define the Adams operations Ψ^n with the method described in [19] (which we intend to generalize later on). For this, we restrict the action of S_n to the cyclic group

[27] The second homomorphism was not explicitly given however.

$\mathbb{Z}/n$ identified with the group of n^{th} roots of the unity. Let us call F_r the subbundle of F where the action of $\mathbb{Z}/n$ is given by ω^r, ω being a fixed primitive root of the unity. Then $\Psi^n(E, D)$ is defined by the following sum:

$$\Psi^n(E, D) = \sum_0^{n-1} F_r \omega^r.$$

It belongs formally to $K^{(\mathcal{A})^{\otimes n}}(X) \otimes_{\mathbb{Z}} \Omega_n$ where Ω_n is the ring of n-cyclotomic integers. However, if n is prime, using the action of the symmetric group S_n, it is easy to check that F_r is isomorphic to F_1 if $r \neq 1$. Therefore we end up in $K^{(\mathcal{A})^{\otimes n}}(X)$, considered as a subgroup of $K^{(\mathcal{A})^{\otimes n}}(X) \underset{\mathbb{Z}}{\otimes} \Omega_n$, as it was expected. It is essentially proved in [2] that this definition of Ψ^n agrees with the classical one.

There is another operation in twisted K-theory which is "complex conjugation", classically denoted by Ψ^{-1}, which maps $K^{(\mathcal{A})}(X)$ to $K^{(\mathcal{A})^{-1}}(X)$ (if we write multiplicatively the group law in $\mathrm{Br}(X)$). It is shown in [9], §10, how we can combine this operation with the previous ones in order to get "internal" operations, i.e. mapping $K^{(\mathcal{A})}(X)$ to itself.

It is more tricky to define operations in *graded* twisted K-theory. If Λ is a $\mathbb{Z}/2$-graded algebra, it is no longer true in general that a graded involution of Λ is induced by an inner automorphism with an element of degree 0 and of order 2. A typical example is the Clifford algebra

$$(\mathbb{C} \oplus \mathbb{C})^{\hat{\otimes} 2} = (C^{0,1})^{\hat{\otimes} 2}$$

which may be identified with the graded algebra $M_2(\mathbb{C})$. If we put

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we see that there is no inner automorphism by an element of order 2 and degree 0 permuting e_1 and e_2.

In order to define such operations in the graded case, we may proceed in at least two ways. We first remark that if $\mathcal{A}'$ is an oriented bundle modelled on $M_2(\mathcal{K})$, the groups $K^{(\mathcal{A}')}(X)$ and $K^{\mathcal{A}'}(X)$ are isomorphic (see 3.5). Moreover, the ungraded tensor product $\mathcal{A}'^{\otimes n}$ is isomorphic to the graded one $\mathcal{A}'^{\hat{\otimes} n}$ (since $M_2(\mathbb{C}) \hat{\otimes} M_2(\mathbb{C})$ is isomorphic to $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$). Let us now assume that the bundle of algebras $\mathcal{A}$ is modelled on $\mathcal{K} \times \mathcal{K}$ but not necessarily oriented and let L be the orientation real line bundle of $\mathcal{A}$. Then $\mathcal{A}' = \mathcal{A} \hat{\otimes} C(L)$ is oriented modelled on $M_2(\mathcal{K})$ ($C(V)$ denotes in general the Clifford bundle associated to V). We may apply the previous method to define operations from $K^{(\mathcal{A}')}(Y) = K^{\mathcal{A}'}(Y)$ to $K^{\mathcal{A}'^{\hat{\otimes} n}}(Y)$. If we apply the Thom isomorphism to the vector bundle $Y = L$ with basis X (see §4), we deduce (*for n odd*) operations from $K^{\mathcal{A}}(X)$ to $K^{\mathcal{A}^{\hat{\otimes} n}}(X)$. However, if $\mathcal{A}$ is oriented, we loose some information since what we get are essentially the previous operations applied to the suspension of X. Note also that the same method may be applied to $K^{\mathcal{A}}(X)$, when $\mathcal{A}$ is modelled on $M_2(\mathcal{K})$.

Another way to proceed is to use a variation of the method developed in [19], p. 21, for any $\mathcal{A}$. We consider the following diagram (with X connected):

$$\begin{tikzcd}
 & F(X,S^1) \arrow[r] \arrow[d] & F(X,S^1) \arrow[d] \\
\Gamma_n \arrow[r, "u"] \arrow[d, "\pi"'] & U^0(\mathcal{A}^{\widehat{\otimes} n}) \arrow[r, "v"] \arrow[d] & U^0(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^{\widehat{\otimes} n} \arrow[d] \\
A_n \arrow[r] & \mathrm{Aut}^0(\mathcal{A}^{\widehat{\otimes} n}) \arrow[r] & \mathrm{Aut}^0(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^{\widehat{\otimes} n}.
\end{tikzcd}$$

In this diagram A_n is the alternating group (for $n \geq 3$), Γ_n is the cartesian product of A_n and $U^0(\mathcal{A}^{\widehat{\otimes} n})$ over $\mathrm{Aut}^0(\mathcal{A}^{\widehat{\otimes} n})$ and the horizontal maps between the U's and the Aut's are essentially given by $g \mapsto g \otimes g$. Note that these maps induce a map from $F(X, S^1)$ to itself which is $f(z) \mapsto f(z^2)$.

By the general theory developed in 7.2, we know that there exists a canonical map ϕ from A_n to $U^0(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^{\widehat{\otimes} n}$ such that its composite with the projection from $U^0(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^{\widehat{\otimes} n}$ to $\mathrm{Aut}^0(\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A})^{\widehat{\otimes} n}$ is the composite ζ of the last horizontal maps. Let us now define the subgroup C_n of Γ_n which elements x are defined by $\varphi(\pi(x)) = v(u(x))$. It is clear that C_n is a double cover of A_n which is either the product of A_n by $\mathbb{Z}/2$ or the Schur group C_n [40] (since $H^2(A_n; \mathbb{Z}/2)$ is isomorphic to $\mathbb{Z}/2$ when $n > 3$).

Using the methods of §6, we have therefore the following composite maps:

$$K^{\mathcal{A}}(X) \longrightarrow K_{A_n}^{\mathcal{A}^{\widehat{\otimes} n}}(X) \longrightarrow K_{(C_n)}^{\mathcal{A}^{\widehat{\otimes} n}}(X) \longrightarrow K^{\mathcal{A}^{\widehat{\otimes} n}}(X) \otimes R(C_n),$$

where the notation $K_{(C_n)}^{\mathcal{A}^{\widehat{\otimes} n}}(X)$ means that the group C_n acts trivially on $\mathcal{A}^{\widehat{\otimes} n}$. Following again Atiyah [2], we see then that any group homomorphism $R(C_n) \to \mathbb{Z}$ gives rise to an operation $K^{\mathcal{A}}(X) \to K^{\mathcal{A}^{\widehat{\otimes} n}}(X)$ Therefore, in the graded case, the Schur group C_n replaces the symmetric group S_n, which we have used in the ungraded case.

In order to define more computable operations, we may replace the alternating group A_n by the cyclic group $\mathbb{Z}/n$ (when n is odd) as it was already done in [19]. The reduction of the central extension C_n becomes trivial and we have a commutative diagram

$$\begin{tikzcd}
 & U^0(\mathcal{A}^{\widehat{\otimes} n}) \arrow[d] \\
\mathbb{Z}/n \arrow[ur] \arrow[r] & \mathrm{Aut}(\mathcal{A}^{\widehat{\otimes} n}).
\end{tikzcd}$$

The Adams operation Ψ^n is then given by the same formula as in 7.4 and [19]

$$\Psi^n \colon K^{\mathcal{A}}(X) \to K^{\mathcal{A}^{\otimes n}}(X) \otimes_{\mathbb{Z}} \Omega_n$$

where Ω_n is the ring of n-cyclotomic integers. We can show that this operation is additive and multiplicative up to canonical isomorphisms (see Theorem 30, p. 23 in [19]).

We should also notice, following [9], that we can define the Adams operation Ψ^{-1} in graded twisted K-theory as well and combine it with the Ψ^n's in order to define "internal" operations from $K^{\mathcal{A}}(X)$ to $K^{\mathcal{A}^{\widehat{\otimes}n}}(X) \underset{\mathbb{Z}}{\otimes} \Omega_n$.

The simplest non-trivial example is

$$\Psi^n \colon \mathbb{Z} \cong K^1(S^1) \to K^n(S^1) \underset{\mathbb{Z}}{\otimes} \Omega_n \cong K^1(S^1) \underset{\mathbb{Z}}{\otimes} \Omega_n$$

where n is a product of different odd primes. Since the operation Ψ^n on $K^2(S^2)$ is the multiplication by n, we deduce that $\theta = \sqrt{(-1)^{(n-1)/2}n}$ belongs to Ω_n (a well-known result due to Gauss) and that Ψ^n on $K^1(S^1)$ is essentially the inclusion of $\mathbb{Z}$ in Ω_n defined by $1 \mapsto \theta$.

As a concluding remark, we should notice that the image of Ψ^n as defined in 7.8 is not arbitrary. If k and n are coprimes, the multiplication by k on the group $\mathbb{Z}/n$ defines an element of the symmetric group S_n. The signature of this permutation is called the Legendre symbol $(\frac{k}{n})$. Moreover, this permutation conjugates the elements r and rk in the group $\mathbb{Z}/n$. If the Legendre symbol is 1, this permutation can be lifted to the Schur group C_n. Let us denote now by F_r (as in 7.4) the element of the twisted K-group associated to the eigenvalue $e^{2i\pi r}$. Then we see that F_r and F_{rk} are isomorphic if the Legendre symbol $(\frac{k}{n})$ is equal to 1 since r and rk are conjugate by an element of the Schur group. If n is prime for instance, $\Psi^n(E)$ may therefore be written in the following way:

$$\Psi^n(E) = F_0 + \sum_{(\frac{k}{n})=1} U\omega^k + \sum_{(\frac{k}{n})=-1} V\omega^k,$$

where U (resp. V) is any F_k with Legendre's symbol equal to 1 (resp. -1). This shows in particular that the element $\theta = \sqrt{(-1)^{(n-1)/2}n}$ in 7.9 is a "Gauss sum", a well-known result.

8 Appendix: A short historical survey of twisted K-theory

Topological K-theory was of course invented by Atiyah and Hirzebruch in 1961 [4] after the fundamental work of Grothendieck [14] and Bott [15]. However, twisted K-theory which is an elaboration of it took some time to emerge. One should quote first the work of Atiyah, Bott and Shapiro [6] where Clifford modules and Clifford bundles (in relation with the Dirac operator) where used to reinterpret Bott periodicity and Thom isomorphism in the presence of Spin or Spin^c structures. We then started to understand in [28], as quoted in the introduction, that K-theory of the Thom space may be defined almost algebraically as K-theory of a functor associated to Clifford bundles. Finally, it was realized in [19] that we can go a step further and consider general graded algebra bundles instead of Clifford bundles associated to vector bundles (with a suitable metric).

On the other hand, the theorem of Atiyah and Jänich [2], [27] showed that Fredholm operators play a crucial role in K-theory. It was discovered independently in [10] and [29], [32] that Clifford modules may also be used as a variation for the spectrum of (real and complex) K-theory. In particular, the cup-product from $K^n(X) \times K^p(Y)$ to $K^{n+p}(X \times Y)$ may be reinterpreted in a much simpler way with this variation. What appeared as a convenience for usual K-theory in these previous references became a necessity for the definition of the product in the twisted case, as it was shown in [19].

The next step was taken by J. Rosenberg [38], some twenty years later, in redefining $\mathrm{GBr}(X)$ for any class in $H^3(X;\mathbb{Z})$ (not just torsion classes), thanks to the new role played by C*-algebras in K-theory. The need for such a generalization was not clear in the 60's (although all the tools were there) since we were just interested in usual Poincaré pairing. In passing, we should mention that one can define more general twistings in any cohomology theory and therefore in K-theory (see for instance [8], p. 18). However, the geometric ones are the most interesting for the purpose of our paper.

We already mentioned the paper of E. Witten [43] in the introduction which made use of this more general twisting of J. Rosenberg, together with [35], [8], [9]. But we should also quote many other papers: [13] for the relations with the theory of gerbes and D-branes in Physics; [1] for the relation with orbifolds; [36] who used the Chern character defined by A. Connes and M. Karoubi in order to prove an isomorphism between twisted K-theory and a computable "twisted cohomology" (at least rationally); [22], [21] about the relation between loop groups and twisted K-theory; [23] (again) for the relation with equivariant cohomology and many other works which are mentioned inside the paper and in the references. In particular, M. F. Atiyah and M. Hopkins [7] introduced another type of K-theory, denoted by them $K_{\pm}X)$, linked to deep physical problems, which is a particular case of twisted equivariant K-theory. This definition of $K_{\pm}(X)$ was already given in [28], §3.3, in terms of Clifford bundles with a group action (see 6.16 and also [31] for the details).

We invite the interested reader to use the classical research tools on the Web for many other references in mathematics and physics.

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Equivariant cyclic homology for quantum groups

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1 Introduction

Equivariant cyclic homology can be viewed as a noncommutative generalization of equivariant de Rham cohomology. For actions of finite groups or compact Lie groups, different aspects of the theory have been studied by various authors [4], [5], [6], [17], [18]. In order to treat noncompact groups as well, a general framework for equivariant cyclic homology following the Cuntz–Quillen formalism [8], [9], [10] has been introduced in [25]. For instance, in the setting of discrete groups or totally disconnected groups this yields a new approach to classical constructions in algebraic topology [26]. However, in contrast to the previous work mentioned above, a crucial feature of the construction in [25] is the fact that the basic ingredient in the theory is not a complex in the usual sense of homological algebra. In particular, the theory does not fit into the traditional scheme of defining cyclic homology using cyclic modules or mixed complexes.

In this note we define equivariant periodic cyclic homology for quantum groups. This generalizes the constructions in the group case developed in [25]. Again we work in the setting of bornological vector spaces. Correspondingly, the appropriate notion of a quantum group in this context is the concept of a bornological quantum group introduced in [27]. This class of quantum groups includes all locally compact groups and their duals as well as all algebraic quantum groups in the sense of van Daele [24]. As in the theory of van Daele, an important ingredient in the definition of a bornological quantum group is the Haar measure. It is crucial for the duality theory and also explicitly used at several points in the construction of the homology theory presented in this paper. However, with some modifications our definition of equivariant cyclic homology could also be adapted to a completely algebraic setting using Hopf algebras with invertible antipodes instead.

From a conceptual point of view, equivariant cyclic homology should be viewed as a homological analogon to equivariant KK-theory [15], [16]. The latter has been extended by Baaj and Skandalis to coactions of Hopf-C^*-algebras [3]. However, in our situation it is more convenient to work with actions instead of coactions.

An important ingredient in equivariant cyclic homology is the concept of a covariant module [25]. In the present paper we will follow the terminology introduced in [12] and call these objects anti-Yetter–Drinfeld modules instead. In order to construct the natural symmetry operator on these modules in the general quantum group setting we prove a formula relating the fourth power of the antipode with the modular functions of a bornological quantum group and its dual. In the context of finite dimensional Hopf algebras this formula is a classical result due to Radford [23].

Although anti-Yetter–Drinfeld modules occur naturally in the constructions one should point out that our theory does not fit into the framework of Hopf-cyclic cohomology [13]. Still, there are relations to previous constructions for Hopf algebras by Akbarpour and Khalkhali [1], [2] as well as Neshveyev and Tuset [22]. Remark in particular that cosemisimple Hopf algebras or finite dimensional Hopf algebras can be viewed as bornological quantum groups. However, basic examples show that the homology groups defined in [1], [2], [22] only reflect a small part of the information contained in the theory described below.

Let us now describe how the paper is organized. In Section 2 we recall the definition of a bornological quantum group. We explain some basic features of the theory including the definition of the dual quantum group and the Pontrjagin duality theorem. This is continued in Section 3 where we discuss essential modules and comodules over bornological quantum groups as well as actions on algebras and their associated crossed products. We prove an analogue of the Takesaki–Takai duality theorem in this setting. Section 4 contains the discussion of Radford's formula relating the antipode with the modular functions of a quantum group and its dual. In Section 5 we study anti-Yetter–Drinfeld modules over bornological quantum groups and introduce the notion of a paracomplex. Section 6 contains a discussion of equivariant differential forms in the quantum group setting. After these preparations we define equivariant periodic cyclic homology in Section 7. In Section 8 we show that our theory is homotopy invariant, stable and satisfies excision in both variables. Finally, Section 9 contains a brief comparison of our theory with the previous approaches mentioned above.

Throughout the paper we work over the complex numbers. For simplicity we have avoided the use of pro-categories in connection with the Cuntz–Quillen formalism to a large extent.

2 Bornological quantum groups

The notion of a bornological quantum group was introduced in [27]. We will work with this concept in our approach to equivariant cyclic homology. For information on bornological vector spaces and more details we refer to [14], [20], [27]. All bornological vector spaces are assumed to be convex and complete.

A bornological algebra H is called essential if the multiplication map induces an isomorphism $H \hat{\otimes}_H H \cong H$. The multiplier algebra $M(H)$ of a bornological algebra H consists of all two-sided multipliers of H, the latter being defined by the usual algebraic conditions. There exists a canonical bounded homomorphism $\iota\colon H \to M(H)$. A bounded linear functional $\phi\colon H \to \mathbb{C}$ on a bornological algebra is called faithful if $\phi(xy) = 0$ for all $y \in H$ implies $x = 0$ and $\phi(xy) = 0$ for all x implies $y = 0$. If there exists such a functional the map $\iota\colon H \to M(H)$ is injective. In this case one may view H as a subset of the multiplier algebra $M(H)$.

In the sequel H will be an essential bornological algebra with a faithful bounded linear functional. For technical reasons we assume moreover that the underlying bornological vector space of H satisfies the approximation property.

A module M over H is called essential if the module action induces an isomorphism $H \hat{\otimes}_H M \cong M$. Moreover an algebra homomorphism $f: H \to M(K)$ is essential if f turns K into an essential left and right module over H. Assume that $\Delta: H \to M(H \hat{\otimes} H)$ is an essential homomorphism. The map Δ is called a comultiplication if it is coassociative, that is, if $(\Delta \hat{\otimes} \mathrm{id})\Delta = (\mathrm{id} \hat{\otimes} \Delta)\Delta$ holds. Moreover the Galois maps $\gamma_l, \gamma_r, \rho_l, \rho_r: H \hat{\otimes} H \to M(H \hat{\otimes} H)$ for Δ are defined by

$$\gamma_l(x \otimes y) = \Delta(x)(y \otimes 1), \quad \gamma_r(x \otimes y) = \Delta(x)(1 \otimes y),$$
$$\rho_l(x \otimes y) = (x \otimes 1)\Delta(y), \quad \rho_r(x \otimes y) = (1 \otimes x)\Delta(y).$$

Let $\Delta: H \to M(H \hat{\otimes} H)$ be a comultiplication such that all Galois maps associated to Δ define bounded linear maps from $H \hat{\otimes} H$ into itself. If ω is a bounded linear functional on H we define for every $x \in H$ a multiplier $(\mathrm{id} \hat{\otimes} \omega)\Delta(x) \in M(H)$ by

$$(\mathrm{id} \hat{\otimes} \omega)\Delta(x) \cdot y = (\mathrm{id} \hat{\otimes} \omega)\gamma_l(x \otimes y),$$
$$y \cdot (\mathrm{id} \hat{\otimes} \omega)\Delta(x) = (\mathrm{id} \hat{\otimes} \omega)\rho_l(y \otimes x).$$

In a similar way one defines $(\omega \hat{\otimes} \mathrm{id})\Delta(x) \in M(H)$. A bounded linear functional $\phi: H \to \mathbb{C}$ is called left invariant if

$$(\mathrm{id} \hat{\otimes} \phi)\Delta(x) = \phi(x)1$$

for all $x \in H$. Analogously one defines right invariant functionals.

Let us now recall the definition of a bornological quantum group.

Definition 2.1. A bornological quantum group is an essential bornological algebra H satisfying the approximation property with a comultiplication $\Delta: H \to M(H \hat{\otimes} H)$ such that all Galois maps associated to Δ are isomorphisms together with a faithful left invariant functional $\phi: H \to \mathbb{C}$.

The definition of a bornological quantum group is equivalent to the definition of an algebraic quantum group in the sense of van Daele [24] in the case that the underlying bornological vector space carries the fine bornology. The functional ϕ is unique up to a scalar and referred to as the left Haar functional of H.

Theorem 2.2. *Let H be a bornological quantum group. Then there exists an essential algebra homomorphism $\epsilon: H \to \mathbb{C}$ and a linear isomorphism $S: H \to H$ which is both an algebra antihomomorphism and a coalgebra antihomomorphism such that*

$$(\epsilon \hat{\otimes} \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \hat{\otimes} \epsilon)\Delta$$

and

$$\mu(S \hat{\otimes} \mathrm{id})\gamma_r = \epsilon \hat{\otimes} \mathrm{id}, \quad \mu(\mathrm{id} \hat{\otimes} S)\rho_l = \mathrm{id} \hat{\otimes} \epsilon.$$

Moreover the maps ϵ and S are uniquely determined.

Using the antipode S one obtains that every bornological quantum group is equipped with a faithful right invariant functional ψ as well. Again, such a functional is uniquely determined up to a scalar. There are injective bounded linear maps $\mathcal{F}_l, \mathcal{F}_r, \mathcal{G}_l, \mathcal{G}_r : H \to H' = \mathrm{Hom}(H, \mathbb{C})$ defined by the formulas

$$\mathcal{F}_l(x)(h) = \phi(hx), \quad \mathcal{F}_r(x)(h) = \phi(xh),$$
$$\mathcal{G}_l(x)(h) = \psi(hx), \quad \mathcal{G}_r(x)(h) = \psi(xh).$$

The images of these maps coincide and determine a vector space $\hat{H}$. Moreover, there exists a unique bornology on $\hat{H}$ such that these maps are bornological isomorphisms. The bornological vector space $\hat{H}$ is equipped with a multiplication which is induced from the comultiplication of H. In this way $\hat{H}$ becomes an essential bornological algebra and the multiplication of H determines a comultiplication on $\hat{H}$.

Theorem 2.3. *Let H be a bornological quantum group. Then $\hat{H}$ with the structure maps described above is again a bornological quantum group. The dual quantum group of $\hat{H}$ is canonically isomorphic to H.*

Explicitly, the duality isomorphism $P : H \to \hat{\hat{H}}$ is given by $P = \hat{\mathcal{G}}_l \mathcal{F}_l S$ or equivalently $P = \hat{\mathcal{F}}_r \mathcal{G}_r S$. Here we write $\hat{\mathcal{G}}_l$ and $\hat{\mathcal{F}}_r$ for the maps defined above associated to the dual Haar functionals on $\hat{H}$. The second statement of the previous theorem should be viewed as an analogue of the Pontrjagin duality theorem.

In [27] all calculations were written down explicitly in terms of the Galois maps and their inverses. However, in this way many arguments tend to become lengthy and not particularly transparent. To avoid this we shall use the Sweedler notation in the sequel. That is, we write

$$\Delta(x) = x_{(1)} \otimes x_{(2)}$$

for the coproduct of an element x, and accordingly for higher coproducts. Of course this has to be handled with care since expressions like the previous one only have a formal meaning. Firstly, the element $\Delta(x)$ is a multiplier and not contained in an actual tensor product. Secondly, we work with completed tensor products which means that even a generic element in $H \hat{\otimes} H$ cannot be written as a finite sum of elementary tensors as in the algebraic case.

3 Actions, coactions and crossed products

In this section we review the definition of essential comodules over a bornological quantum group and their relation to essential modules over the dual. Moreover we consider actions on algebras and their associated crossed products and prove an analogue of the Takesaki–Takai duality theorem.

Let H be a bornological quantum group. Recall from Section 2 that a module V over H is called essential if the module action induces an isomorphism $H \hat{\otimes}_H V \cong V$. A bounded linear map $f : V \to W$ between essential H-modules is called H-linear or

H-equivariant if it commutes with the module actions. We denote the category of essential H-modules and equivariant linear maps by H -Mod. Using the comultiplication of H one obtains a natural H-module structure on the tensor product of two H-modules and H -Mod becomes a monoidal category in this way.

We will frequently use the regular actions associated to a bornological quantum group H. For $t \in H$ and $f \in \widehat{H}$ one defines

$$t \rightharpoonup f = f_{(1)} f_{(2)}(t), \quad f \leftharpoonup t = f_{(1)}(t) f_{(2)}$$

and this yields essential left and right H-module structures on $\widehat{H}$, respectively.

Dually to the concept of an essential module one has the notion of an essential comodule. Let H be a bornological quantum group and let V be a bornological vector space. A coaction of H on V is a right H-linear bornological isomorphism $\eta\colon V \hat{\otimes} H \to V \hat{\otimes} H$ such that the relation

$$(\mathrm{id} \otimes \gamma_r)\eta_{12}(\mathrm{id} \otimes \gamma_r^{-1}) = \eta_{12}\eta_{13}$$

holds.

Definition 3.1. Let H be a bornological quantum group. An essential H-comodule is a bornological vector space V together with a coaction $\eta\colon V \hat{\otimes} H \to V \hat{\otimes} H$.

A bounded linear map $f\colon V \to W$ between essential comodules is called H-colinear if it is compatible with the coactions in the obvious sense. We write Comod- H for the category of essential comodules over H with H-colinear maps as morphisms. The category Comod- H is a monoidal category as well.

If the quantum group H is unital, a coaction is the same thing as a bounded linear map $\eta\colon V \to V \hat{\otimes} H$ such that $(\eta \hat{\otimes} \mathrm{id})\eta = (\mathrm{id} \hat{\otimes} \Delta)\eta$ and $(\mathrm{id} \hat{\otimes} \epsilon)\eta = \mathrm{id}$.

Modules and comodules over bornological quantum groups are related in the same way as modules and comodules over finite dimensional Hopf algebras.

Theorem 3.2. *Let H be a bornological quantum group. Every essential left H-module is an essential right $\widehat{H}$-comodule in a natural way and vice versa. This yields inverse isomorphisms between the category of essential H-modules and the category of essential $\widehat{H}$-comodules. These isomorphisms are compatible with tensor products.*

Since it is more convenient to work with essential modules instead of comodules we will usually prefer to consider modules in the sequel.

An essential H-module is called projective if it has the lifting property with respect to surjections of essential H-modules with bounded linear splitting. It is shown in [27] that a bornological quantum group H is projective as a left module over itself. This can be generalized as follows.

Lemma 3.3. *Let H be a bornological quantum group and let V be any essential H-module. Then the essential H-modules $H \hat{\otimes} V$ and $V \hat{\otimes} H$ are projective.*

Proof. Let V_τ be the space V equipped with the trivial H-action induced by the counit. We have a natural H-linear isomorphism $\alpha_l\colon H \hat{\otimes} V \to H \hat{\otimes} V_\tau$ given by $\alpha_l(x \otimes v) = x_{(1)} \otimes S(x_{(2)}) \cdot v$. Similarly, the map $\alpha_r\colon V \hat{\otimes} H \to V_\tau \hat{\otimes} H$ given by $\alpha_r(v \otimes x) = S^{-1}(x_{(1)}) \cdot v \otimes x_{(2)}$ is an H-linear isomorphism. Since H is projective this yields the claim. □

Using category language an H-algebra is by definition an algebra in the category H-Mod. We formulate this more explicitly in the following definition.

Definition 3.4. Let H be a bornological quantum group. An H-algebra is a bornological algebra A which is at the same time an essential H-module such that the multiplication map $A \hat{\otimes} A \to A$ is H-linear.

If A is an H-algebra we will also speak of an action of H on A. Remark that we do not assume that an algebra has an identity element. The unitarization A^+ of an H-algebra A becomes an H-algebra by considering the trivial action on the extra copy $\mathbb{C}$.

According to Theorem 3.2 we can equivalently describe an H-algebra as a bornological algebra A which is at the same time an essential $\hat{H}$-comodule such that the multiplication is $\hat{H}$-colinear.

Under additional assumptions there is another possibility to describe this structure which resembles the definition of a coaction in the setting of C^*-algebras. Let us call an essential bornological algebra A regular if it is equipped with a faithful bounded linear functional and satisfies the approximation property. If A is regular it follows from [27] that the natural bounded linear map $A \hat{\otimes} H \to M(A \hat{\otimes} H)$ is injective.

Definition 3.5. Let H be a bornological quantum group. An algebra coaction of H on a regular bornological algebra A is an essential algebra homomorphism $\alpha\colon A \to M(A \hat{\otimes} H)$ such that the coassociativity condition

$$(\alpha \hat{\otimes} \mathrm{id})\alpha = (\mathrm{id} \hat{\otimes} \Delta)\alpha$$

holds and the maps α_l and α_r from $A \hat{\otimes} H$ to $M(A \hat{\otimes} H)$ given by

$$\alpha_l(a \otimes x) = (1 \otimes x)\alpha(a), \quad \alpha_r(a \otimes x) = \alpha(a)(1 \otimes x)$$

induce bornological automorphisms of $A \hat{\otimes} H$.

It can be shown that an algebra coaction $\alpha\colon A \to M(A \hat{\otimes} H)$ on a regular bornological algebra A satisfies $(\mathrm{id} \hat{\otimes} \epsilon)\alpha = \mathrm{id}$. In particular, the map α is always injective.

Proposition 3.6. *Let H be a bornological quantum group and let A be a regular bornological algebra. Then every algebra coaction of $\hat{H}$ on A corresponds to a unique H-algebra structure on A and vice versa.*

Proof. Assume that α is an algebra coaction of $\hat{H}$ on A and define $\eta = \alpha_r$. By definition η is a right $\hat{H}$-linear automorphism of $A \hat{\otimes} \hat{H}$. Moreover we have for $a \in A$

and $f, g \in \hat{H}$ the relation

$$\begin{aligned}(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1})(a \otimes f \otimes g) &= (\mathrm{id} \hat{\otimes} \gamma_r)((\alpha(a) \otimes 1)(1 \otimes \gamma_r^{-1}(f \otimes g))) \\ &= (\mathrm{id} \hat{\otimes} \Delta)(\alpha(a))(1 \otimes \gamma_r \gamma_r^{-1}(f \otimes g)) \\ &= (\alpha \hat{\otimes} \mathrm{id})(\alpha(a))(1 \otimes f \otimes g) \\ &= \eta^{12}\eta^{13}(a \otimes f \otimes g)\end{aligned}$$

in $M(A \hat{\otimes} \hat{H} \hat{\otimes} \hat{H})$. Using that A is regular we deduce that

$$(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1})(a \otimes f \otimes g) = \eta^{12}\eta^{13}(a \otimes f \otimes g)$$

in $A \hat{\otimes} \hat{H} \hat{\otimes} \hat{H}$ and hence η defines a right $\hat{H}$-comodule structure on A. In addition we have

$$\begin{aligned}(\mu \hat{\otimes} \mathrm{id})\eta^{13}\eta^{23}(a \otimes b \otimes f) &= (\mu \hat{\otimes} \mathrm{id})\eta^{13}(a \otimes \alpha(b)(1 \otimes f)) \\ &= \alpha(a)\alpha(b)(1 \otimes f) = \alpha(ab)(1 \otimes f) = \eta(\mu \hat{\otimes} \mathrm{id})(a \otimes b \otimes f)\end{aligned}$$

and it follows that A becomes an H-algebra using this coaction.

Conversely, assume that A is an H-algebra implemented by the coaction $\eta\colon A \hat{\otimes} \hat{H} \to A \hat{\otimes} \hat{H}$. Define bornological automorphisms η_l and η_r of $A \hat{\otimes} \hat{H}$ by

$$\eta_l = (\mathrm{id} \hat{\otimes} S^{-1})\eta^{-1}(\mathrm{id} \hat{\otimes} S), \quad \eta_r = \eta.$$

The map η_l is left $\hat{H}$-linear for the action of $\hat{H}$ on the second tensor factor and η_r is right $\hat{H}$-linear. Since η is a compatible with the multiplication we have

$$\eta_r(\mu \hat{\otimes} \mathrm{id}) = (\mu \hat{\otimes} \mathrm{id})\eta_r^{13}\eta_r^{23}$$

and

$$\eta_l(\mu \hat{\otimes} \mathrm{id}) = (\mu \hat{\otimes} \mathrm{id})\eta_l^{23}\eta_l^{13}.$$

In addition one has the equation

$$(\mathrm{id} \hat{\otimes} \mu)\eta_l^{12} = (\mathrm{id} \hat{\otimes} \mu)\eta_r^{13}$$

relating η_l and η_r. These properties of the maps η_l and η_r imply that

$$\alpha(a)(b \otimes f) = \eta_r(a \otimes f)(b \otimes 1), \quad (b \otimes f)\alpha(a) = (b \otimes 1)\eta_l(a \otimes f)$$

defines an algebra homomorphism α from A to $M(A \hat{\otimes} \hat{H})$. As in the proof of Proposition 7.3 in [27] one shows that α is essential. Observe that we may identify the natural map $A \hat{\otimes}_A (A \hat{\otimes} \hat{H}) \to A \hat{\otimes} \hat{H}$ induced by α with $\eta_r^{13}\colon A \hat{\otimes}_A (A \hat{\otimes} \hat{H} \hat{\otimes}_{\hat{H}} \hat{H}) \to (A \hat{\otimes}_A A) \hat{\otimes} (\hat{H} \hat{\otimes}_{\hat{H}} \hat{H})$ since A is essential.

The maps α_l and α_r associated to the homomorphism α can be identified with η_l and η_r, respectively. Finally, the coaction identity $(\mathrm{id} \hat{\otimes} \gamma_r)\eta^{12}(\mathrm{id} \hat{\otimes} \gamma_r^{-1}) = \eta^{12}\eta^{13}$ implies $(\alpha \hat{\otimes} \mathrm{id})\alpha = (\mathrm{id} \hat{\otimes} \Delta)\alpha$. Hence α defines an algebra coaction of $\hat{H}$ on A.

It follows immediately from the constructions that the two procedures described above are inverse to each other. □

To every H-algebra A one may form the associated crossed product $A \rtimes H$. The underlying bornological vector space of $A \rtimes H$ is $A \hat{\otimes} H$ and the multiplication is defined by the chain of maps

$$A \hat{\otimes} H \hat{\otimes} A \hat{\otimes} H \xrightarrow{\gamma_r^{24}} A \hat{\otimes} H \hat{\otimes} A \hat{\otimes} H \xrightarrow{\mathrm{id} \hat{\otimes} \lambda \hat{\otimes} \mathrm{id}} A \hat{\otimes} A \hat{\otimes} H \xrightarrow{\mu \hat{\otimes} \mathrm{id}} A \hat{\otimes} H$$

where λ denotes the action of H on A. Explicitly, the multiplication in $A \rtimes H$ is given by the formula

$$(a \rtimes x)(b \rtimes y) = a x_{(1)} \cdot b \otimes x_{(2)} y$$

for $a, b \in A$ and $x, y \in H$. On the crossed product $A \rtimes H$ one has the dual action of $\hat{H}$ defined by

$$f \cdot (a \rtimes x) = a \rtimes (f \rightharpoonup x)$$

for all $f \in \hat{H}$. In this way $A \rtimes H$ becomes an $\hat{H}$-algebra. Consequently one may form the double crossed product $A \rtimes H \rtimes \hat{H}$. In the remaining part of this section we discuss the Takesaki–Takai duality isomorphism which clarifies the structure of this algebra.

First we describe a general construction which will also be needed later in connection with stability of equivariant cyclic homology. Assume that V is an essential H-module and that A is an H-algebra. Moreover let $b\colon V \times V \to \mathbb{C}$ be an equivariant bounded linear map. We define an H-algebra $l(b; A)$ by equipping the space $V \hat{\otimes} A \hat{\otimes} V$ with the multiplication

$$(v_1 \otimes a_1 \otimes w_1)(v_2 \otimes a_2 \otimes w_2) = b(w_1, v_2)\, v_1 \otimes a_1 a_2 \otimes w_2$$

and the diagonal H-action.

As a particular case of this construction consider the space $V = \hat{H}$ with the regular action of H given by $(t \rightharpoonup f)(x) = f(xt)$ and the pairing

$$\beta(f, g) = \hat{\psi}(fg).$$

We write $\mathcal{K}_H$ for the algebra $l(\beta; \mathbb{C})$ and $A \hat{\otimes} \mathcal{K}_H$ for $l(\beta; A)$. Remark that the action on $A \hat{\otimes} \mathcal{K}_H$ is not the diagonal action in general. We denote an element $f \otimes a \otimes g$ in this algebra by $|f\rangle \otimes a \otimes \langle g|$ in the sequel. Using the isomorphism $\hat{\mathcal{F}}_r S^{-1}\colon \hat{H} \to H$ we identify the above pairing with a pairing $H \times H \to \mathbb{C}$. The corresponding action of H on itself is given by left multiplication and using the normalization $\phi = S(\psi)$ we obtain the formula

$$\beta(x, y) = \beta(S\mathcal{G}_r S(x), S\mathcal{G}_r S(y)) = \beta(\mathcal{F}_l(x), \mathcal{F}_l(y)) = \phi(S^{-1}(y)x) = \psi(S(x)y)$$

for the above pairing expressed in terms of H.

Let H be a bornological quantum group and let A be an H-algebra. We define a bounded linear map $\gamma_A\colon A \rtimes H \rtimes \hat{H} \to A \hat{\otimes} \mathcal{K}_H$ by

$$\gamma_A(a \rtimes x \rtimes \mathcal{F}_l(y)) = |y_{(1)} S(x_{(2)})\rangle \otimes y_{(2)} S(x_{(1)}) \cdot a \otimes \langle y_{(3)}|$$

and it is easily verified that γ_A is an equivariant bornological isomorphism. In addition, a straightforward computation shows that γ_A is an algebra homomorphism. Consequently we obtain the following analogue of the Takesaki–Takai duality theorem.

Proposition 3.7. *Let H be a bornological quantum group and let A be an H-algebra. Then the map $\gamma_A\colon A \rtimes H \rtimes \widehat{H} \to A \hat{\otimes} \mathcal{K}_H$ is an equivariant algebra isomorphism.*

For algebraic quantum groups a discussion of Takesaki–Takai duality is contained in [11]. More information on similar duality results in the context of Hopf algebras can be found in [21].

If $H = \mathcal{D}(G)$ is the smooth convolution algebra of a locally compact group G then an H-algebra is the same thing as a G-algebra. As a special case of Proposition 3.7 one obtains that for every G-algebra A the double crossed product $A \rtimes H \rtimes \widehat{H}$ is isomorphic to the G-algebra $A \hat{\otimes} \mathcal{K}_G$ used in [25].

4 Radford's formula

In this section we prove a formula for the fourth power of the antipode in terms of the modular elements of a bornological quantum group and its dual. This formula was obtained by Radford in the setting of finite dimensional Hopf algebras [23].

Let H be a bornological quantum group. If ϕ is a left Haar functional on H there exists a unique multiplier $\delta \in M(H)$ such that

$$(\phi \hat{\otimes} \mathrm{id})\Delta(x) = \phi(x)\delta$$

for all $x \in H$. The multiplier δ is called the modular element of H and measures the failure of ϕ from being right invariant. It is shown in [27] that δ is invertible with inverse $S(\delta) = S^{-1}(\delta) = \delta^{-1}$ and that one has $\Delta(\delta) = \delta \otimes \delta$ as well as $\epsilon(\delta) = 1$. In terms of the dual quantum group the modular element δ defines a character, that is, an essential homomorphism from $\widehat{H}$ to $\mathbb{C}$. Similarly, there exists a unique modular element $\hat{\delta} \in M(\widehat{H})$ for the dual quantum group which satisfies

$$(\hat{\phi} \hat{\otimes} \mathrm{id})\hat{\Delta}(f) = \hat{\phi}(f)\hat{\delta}$$

for all $f \in \widehat{H}$.

The Haar functionals of a bornological quantum group are uniquely determined up to a scalar multiple. In many situations it is convenient to fix a normalization at some point. However, in the discussion below it is not necessary to keep track of the scaling of the Haar functionals. If ω and η are linear functionals we shall write $\omega \equiv \eta$ if there exists a nonzero scalar λ such that $\omega = \lambda\eta$. We use the same notation for elements in a bornological quantum group or linear maps that differ by some nonzero scalar multiple. Moreover we shall identify H with its double dual using Pontrjagin duality.

To begin with observe that the bounded linear functional $\delta \rightharpoonup \phi$ on H defined by

$$(\delta \rightharpoonup \phi)(x) = \phi(x\delta)$$

is faithful and satisfies

$$((\delta \rightharpoonup \phi) \hat{\otimes} \mathrm{id})\Delta(x) = (\phi \hat{\otimes} \mathrm{id})(\Delta(x\delta)(1 \otimes \delta^{-1})) = \phi(x\delta)\delta\delta^{-1} = (\delta \rightharpoonup \phi)(x).$$

It follows that $\delta \rightharpoonup \phi$ is a right Haar functional on H. In a similar way we obtain a right Haar functional $\phi \leftharpoonup \delta$ on H. Hence

$$\delta \rightharpoonup \phi \equiv \psi \equiv \phi \leftharpoonup \delta$$

by uniqueness of the right Haar functional which yields in particular the relations

$$\mathcal{F}_l(x\delta) \equiv \mathcal{G}_l(x), \quad \mathcal{F}_r(\delta x) \equiv \mathcal{G}_r(x)$$

for the Fourier transform.

According to Pontrjagin duality we have $x = \hat{\mathcal{G}}_l \mathcal{F}_l S(x)$ for all $x \in H$. Since $\hat{\mathcal{G}}_l(f) \equiv \hat{\mathcal{F}}_l(f\hat{\delta})$ for every $f \in \hat{H}$ as well as

$$\begin{aligned}(\mathcal{F}_l(S(x))\hat{\delta})(h) &= \phi(h_{(1)}S(x))\hat{\delta}(h_{(2)}) = \phi(hS(x_{(2)}))\hat{\delta}(\delta S^2(x_{(1)})) \\ &\equiv \mathcal{F}_l(S(x_{(2)}))(h)\hat{\delta}(x_{(1)}) = \mathcal{F}_l(S(x \leftharpoonup \hat{\delta}))(h)\end{aligned}$$

we obtain $x \equiv \hat{\mathcal{F}}_l \mathcal{F}_l S(x \leftharpoonup \hat{\delta})$ or equivalently

$$S^{-1}(\hat{\delta} \rightharpoonup x) \equiv \hat{\mathcal{F}}_l \mathcal{F}_l(x). \tag{4.1}$$

Using equation (4.1) and the formula $x \equiv S^{-1}\hat{\mathcal{F}}_l \mathcal{F}_r(x)$ obtained from Pontrjagin duality we compute

$$\hat{\mathcal{F}}_l \mathcal{F}_l(\hat{\delta}^{-1} \rightharpoonup S^2(x)) \equiv S^{-1}S^2(x) = S(x) \equiv \hat{\mathcal{F}}_l \mathcal{F}_r(x)$$

which implies

$$\mathcal{F}_l(S^2(x)) \equiv \mathcal{F}_r(\hat{\delta} \rightharpoonup x) \tag{4.2}$$

since $\hat{\mathcal{F}}_l$ is an isomorphism. Similarly, we have $\hat{\mathcal{F}}_l S \mathcal{G}_l \equiv \mathrm{id}$ and using $\hat{\mathcal{F}}_l(f) \equiv \hat{\mathcal{G}}_l(f\hat{\delta}^{-1})$ together with

$$\begin{aligned}(S\mathcal{G}_l(x)\hat{\delta}^{-1})(h) &= \psi(S(h_{(1)})x)\hat{\delta}^{-1}(h_{(2)}) \\ &\equiv \psi(S(h)x_{(2)})\hat{\delta}^{-1}(x_{(1)}) = S\mathcal{G}_l(x \leftharpoonup \hat{\delta}^{-1})(h)\end{aligned}$$

we obtain $\hat{\mathcal{G}}_l S \mathcal{G}_l(x) \equiv x \leftharpoonup \hat{\delta}$. According to the relation $\mathcal{F}_l(x\delta) \equiv \mathcal{G}_l(x)$ this may be rewritten as $S^{-1}\hat{\mathcal{F}}_r \mathcal{F}_l(x\delta) \equiv x \leftharpoonup \hat{\delta}$ which in turn yields

$$\hat{\mathcal{F}}_r \mathcal{F}_l(x) \equiv S((x\delta^{-1}) \leftharpoonup \hat{\delta}). \tag{4.3}$$

Due to Pontrjagin duality we have $x = \hat{\mathcal{F}}_r \mathcal{G}_r S(x)$ for all $x \in H$ and using $\mathcal{G}_r(x) \equiv \mathcal{F}_r(\delta x)$ we obtain

$$\hat{\mathcal{F}}_r \mathcal{F}_r(S(x)) \equiv \hat{\mathcal{F}}_r \mathcal{G}_r(\delta^{-1}S(x)) = x\delta. \tag{4.4}$$

According to equation (4.3) and equation (4.4) we have

$$\begin{aligned}\hat{\mathcal{F}}_r \mathcal{F}_l(S^{-2}(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta)) &\equiv S((S^{-2}(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta)\delta^{-1}) \leftharpoonup \hat{\delta}) \\ &\equiv S(S^{-2}(\delta^{-1}x)) = S^{-1}(\delta^{-1}x) \equiv \hat{\mathcal{F}}_r \mathcal{F}_r(x)\end{aligned}$$

and since $\widehat{\mathcal{F}}_r$ is an isomorphism this implies

$$\mathcal{F}_r(S^2(x)) \equiv \mathcal{F}_l(\delta^{-1}(x \leftharpoonup \hat{\delta}^{-1})\delta). \tag{4.5}$$

Assembling these relations we obtain the following result.

Proposition 4.1. *Let H be a bornological quantum group and let δ and $\hat{\delta}$ be the modular elements of H and $\widehat{H}$, respectively. Then*

$$S^4(x) = \delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta$$

for all $x \in H$.

Proof. Using equation (4.2) and equation (4.5) we compute

$$\mathcal{F}_l(S^4(x)) \equiv \mathcal{F}_r(\hat{\delta} \rightharpoonup S^2(x)) = \mathcal{F}_r(S^2(\hat{\delta} \rightharpoonup x)) \equiv \mathcal{F}_l(\delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta)$$

which implies

$$S^4(x) \equiv \delta^{-1}(\hat{\delta} \rightharpoonup x \leftharpoonup \hat{\delta}^{-1})\delta$$

for all $x \in H$ since $\mathcal{F}_l$ is an isomorphism. The claim follows from the observation that both sides of the previous equation define algebra automorphisms of H. □

5 Anti-Yetter–Drinfeld modules

In this section we introduce the notion of an anti-Yetter–Drinfeld module over a bornological quantum group. Moreover we discuss the concept of a paracomplex.

We begin with the definition of an anti-Yetter–Drinfeld module. In the context of Hopf algebras this notion was introduced in [12].

Definition 5.1. Let H be a bornological quantum group. An H-anti-Yetter–Drinfeld module is an essential left H-module M which is also an essential left $\widehat{H}$-module such that

$$t \cdot (f \cdot m) = (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot m).$$

for all $t \in H, f \in \widehat{H}$ and $m \in M$. A homomorphism $\xi\colon M \to N$ between anti-Yetter–Drinfeld modules is a bounded linear map which is both H-linear and $\widehat{H}$-linear.

We will not always mention explicitly the underlying bornological quantum group when dealing with anti-Yetter–Drinfeld modules. Moreover we shall use the abbreviations AYD-module and AYD-map for anti-Yetter–Drinfeld modules and their homomorphisms.

According to Theorem 3.2 a left $\widehat{H}$-module structure corresponds to a right H-comodule structure. Hence an AYD-module can be described equivalently as a bornological vector space M equipped with an essential H-module structure and an H-comodule structure satisfying a certain compatibility condition. Formally, this compatibility condition can be written down as

$$(t \cdot m)_{(0)} \otimes (t \cdot m)_{(1)} = t_{(2)} \cdot m_{(0)} \otimes t_{(3)} m_{(1)} S(t_{(1)})$$

for all $t \in H$ and $m \in M$.

We want to show that AYD-modules can be interpreted as essential modules over a certain bornological algebra. Following the notation in [12] this algebra will be denoted by $\mathsf{A}(H)$. As a bornological vector space we have $\mathsf{A}(H) = \widehat{H} \hat{\otimes} H$ and the multiplication is defined by the formula

$$(f \otimes x) \cdot (g \otimes y) = f(S^2(x_{(1)}) \rightharpoonup g \leftharpoonup S^{-1}(x_{(3)})) \otimes x_{(2)} y.$$

There exists an algebra homomorphism $\iota_H \colon H \to M(\mathsf{A}(H))$ given by

$$\iota_H(x) \cdot (g \otimes y) = S^2(x_{(1)}) \rightharpoonup g \leftharpoonup S^{-1}(x_{(3)}) \otimes x_{(2)} y$$

and

$$(g \otimes y) \cdot \iota_H(x) = g \otimes yx.$$

It is easily seen that ι_H is injective. Similarly, there is an injective algebra homomorphism $\iota_{\widehat{H}} \colon \widehat{H} \to M(\mathsf{A}(H))$ given by

$$\iota_{\widehat{H}}(f) \cdot (g \otimes y) = fg \otimes y$$

as well as

$$(g \otimes y) \cdot \iota_{\widehat{H}}(f) = g(S^2(y_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(y_{(3)})) \otimes y_{(2)}$$

and we have the following result.

Proposition 5.2. *For every bornological quantum group H the bornological algebra $\mathsf{A}(H)$ is essential.*

Proof. The homomorphism $\iota_{\widehat{H}}$ induces on $\mathsf{A}(H)$ the structure of an essential left $\widehat{H}$-module. Similarly, the space $\mathsf{A}(H)$ becomes an essential left H-module using the homomorphism ι_H. In fact, if we write $\widehat{H}_\tau$ for the space $\widehat{H}$ equipped with the trivial H-action the map $c \colon \mathsf{A}(H) \to \widehat{H}_\tau \hat{\otimes} H$ given by

$$c(f \otimes x) = S(x_{(1)}) \rightharpoonup f \leftharpoonup x_{(3)} \otimes x_{(2)}$$

is an H-linear isomorphism. Actually, the actions of $\widehat{H}$ and H on $\mathsf{A}(H)$ are defined in such a way that $\mathsf{A}(H)$ becomes an AYD-module. Using the canonical isomorphism $H \hat{\otimes}_H \mathsf{A}(H) \cong \mathsf{A}(H)$ one obtains an essential $\widehat{H}$-module structure on $H \hat{\otimes}_H \mathsf{A}(H)$ given explicitly by the formula

$$f \cdot (x \otimes g \otimes y) = x_{(2)} \otimes (S(x_{(1)}) \rightharpoonup f \leftharpoonup x_{(3)}) g \otimes y.$$

Correspondingly, we obtain a natural isomorphism $\widehat{H} \hat{\otimes}_{\widehat{H}} (H \hat{\otimes}_H \mathsf{A}(H)) \cong \mathsf{A}(H)$. It is straightforward to verify that the identity map $\widehat{H} \hat{\otimes} (H \hat{\otimes} \mathsf{A}(H)) \cong \widehat{H} \hat{\otimes} H \hat{\otimes} \widehat{H} \hat{\otimes} H \to \mathsf{A}(H) \hat{\otimes} \mathsf{A}(H)$ induces an isomorphism $\widehat{H} \hat{\otimes}_{\widehat{H}} (H \hat{\otimes}_H \mathsf{A}(H)) \to \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} \mathsf{A}(H)$. This yields the claim. □

We shall now characterize AYD-modules as essential modules over $\mathsf{A}(H)$.

Proposition 5.3. *Let H be a bornological quantum group. Then the category of* AYD*-modules over H is isomorphic to the category of essential left* $\mathsf{A}(H)$*-modules.*

Proof. Let $M \cong \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M$ be an essential $\mathsf{A}(H)$-module. Then we obtain a left H-module structure and a left $\hat{H}$-module structure on M using the canonical homomorphisms $\iota_H \colon H \to M(\mathsf{A}(H))$ and $\iota_{\hat{H}} \colon \hat{H} \to M(\mathsf{A}(H))$. Since the action of H on $\mathsf{A}(H)$ is essential we have natural isomorphisms

$$H \hat{\otimes}_H M \cong H \hat{\otimes}_H \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M \cong \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M \cong M$$

and hence M is an essential H-module. Similarly we have

$$\hat{H} \hat{\otimes}_{\hat{H}} M \cong \hat{H} \hat{\otimes}_{\hat{H}} \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} \hat{\otimes} M \cong \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M \cong M$$

since $\mathsf{A}(H)$ is an essential $\hat{H}$-module. These module actions yield the structure of an AYD-module on M.

Conversely, assume that M is an H-AYD-module. Then we obtain an $\mathsf{A}(H)$-module structure on M by setting

$$(f \otimes t) \cdot m = f \cdot (t \cdot m)$$

for $f \in \hat{H}$ and $t \in H$. Since M is an essential H-module we have a natural isomorphism $H \hat{\otimes}_H M \cong M$. As in the proof of Proposition 5.2 we obtain an induced essential $\hat{H}$-module structure on $H \hat{\otimes}_H M$ and canonical isomorphisms $\mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M \cong \hat{H} \hat{\otimes}_{\hat{H}} (H \hat{\otimes}_H M) \cong M$. It follows that M is an essential $\mathsf{A}(H)$-module.

The previous constructions are compatible with morphisms and it is easy to check that they are inverse to each other. This yields the assertion. □

There is a canonical operator T on every AYD-module which plays a crucial role in equivariant cyclic homology. In order to define this operator it is convenient to pass from $\hat{H}$ to H in the first tensor factor of $\mathsf{A}(H)$. More precisely, consider the bornological isomorphism $\lambda \colon \mathsf{A}(H) \to H \hat{\otimes} H$ given by

$$\lambda(f \otimes y) = \hat{\mathcal{F}}_l(f) \otimes y \leftharpoonup \hat{\delta}^{-1}$$

where $\hat{\delta} \in M(\hat{H})$ is the modular function of $\hat{H}$. The inverse map is given by

$$\lambda^{-1}(x \otimes y) = S\mathcal{G}_l(x) \otimes y \leftharpoonup \hat{\delta}.$$

It is straightforward to check that the left H-action on $\mathsf{A}(H)$ corresponds to

$$t \cdot (x \otimes y) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} y$$

and the left $\hat{H}$-action becomes

$$f \cdot (x \otimes y) = (f \rightharpoonup x) \otimes y$$

under this isomorphism. The right H-action is identified with

$$(x \otimes y) \cdot t = x \otimes y(t \leftharpoonup \hat{\delta}^{-1})$$

and the right $\widehat{H}$-action corresponds to

$$(x \otimes y) \cdot g = x_{(2)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})y_{(3)} \leftharpoonup \hat{\delta}^{-1}$$

where δ is the modular function of H. Using this description of $\mathsf{A}(H)$ we obtain the following result.

Proposition 5.4. *The bounded linear map* $T \colon \mathsf{A}(H) \to \mathsf{A}(H)$ *defined by*

$$T(x \otimes y) = x_{(2)} \otimes S^{-1}(x_{(1)})y$$

is an isomorphism of $\mathsf{A}(H)$*-bimodules.*

Proof. It is evident that T is a bornological isomorphism with inverse given by $T^{-1}(x \otimes y) = x_{(2)} \otimes x_{(1)}y$. We compute

$$\begin{aligned} T(t \cdot (x \otimes y)) &= T(t_{(3)}xS(t_{(1)}) \otimes t_{(2)}y) \\ &= t_{(5)}x_{(2)}S(t_{(1)}) \otimes S^{-1}(t_{(4)}x_{(1)}S(t_{(2)}))t_{(3)}y \\ &= t_{(3)}x_{(2)}S(t_{(1)}) \otimes t_{(2)}S^{-1}(x_{(1)})y \\ &= t \cdot T(x \otimes y) \end{aligned}$$

and it is clear that T is left $\widehat{H}$-linear. Consequently T is a left $\mathsf{A}(H)$-linear map. Similarly, we have

$$T((x \otimes y) \cdot t) = T(x \otimes y(t \leftharpoonup \hat{\delta}^{-1})) = x_{(2)} \otimes S^{-1}(x_{(1)})y(t \leftharpoonup \hat{\delta}^{-1}) = T(x \otimes y) \cdot t$$

and hence T is right H-linear. In order to prove that T is right $\widehat{H}$-linear we compute

$$\begin{aligned} &T^{-1}((x \otimes y) \cdot g) \\ &\quad = T^{-1}(x_{(2)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})y_{(3)} \leftharpoonup \hat{\delta}^{-1}) \\ &\quad = x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})x_{(2)}(y_{(3)} \leftharpoonup \hat{\delta}^{-1}) \\ &\quad = x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(S^{-2}(\hat{\delta} \rightharpoonup x_{(1)})\delta) \otimes \hat{\delta}(y_{(1)})(x_{(2)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \end{aligned}$$

which according to Proposition 4.1 is equal to

$$\begin{aligned} &\quad = x_{(3)}(S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)}))(\delta S^2(x_{(1)} \leftharpoonup \hat{\delta})) \otimes \hat{\delta}(y_{(1)})(x_{(2)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \\ &\quad = x_{(6)}(S^2(x_{(2)})S^2(y_{(2)}) \rightharpoonup g \leftharpoonup S^{-1}(y_{(4)})S^{-1}(x_{(4)}))(S^{-2}(x_{(5)})\delta) \\ &\qquad\quad \otimes \hat{\delta}(x_{(1)}y_{(1)})(x_{(3)}y_{(3)}) \leftharpoonup \hat{\delta}^{-1} \\ &\quad = (x_{(2)} \otimes x_{(1)}y) \cdot g \\ &\quad = T^{-1}(x \otimes y) \cdot g. \end{aligned}$$

We conclude that T is a right $\mathsf{A}(H)$-linear map as well. □

If M is an arbitrary AYD-module we define $T: M \to M$ by

$$T(F \otimes m) = T(F) \otimes m$$

for $F \otimes m \in \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M$. Due to Proposition 5.3 and Lemma 5.4 this definition makes sense.

Proposition 5.5. *The operator T defines a natural isomorphism $T: \mathrm{id} \to \mathrm{id}$ of the identity functor on the category of* AYD*-modules.*

Proof. It is clear from the construction that $T: M \to M$ is an isomorphism for all M. If $\xi: M \to N$ is an AYD-map the equation $T\xi = \xi T$ follows easily after identifying ξ with the map $\mathrm{id} \hat{\otimes} \xi: \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} M \to \mathsf{A}(H) \hat{\otimes}_{\mathsf{A}(H)} N$. This yields the assertion. □

If the bornological quantum group H is unital one may construct the operator T on an AYD-module M directly from the coaction $M \to M \hat{\otimes} H$ corresponding to the action of $\hat{H}$. More precisely, one has the formula

$$T(m) = S^{-1}(m_{(1)}) \cdot m_{(0)}$$

for every $m \in M$.

Using the terminology of [25] it follows from Proposition 5.5 that the category of AYD-modules is a para-additive category in a natural way. This leads in particular to the concept of a paracomplex of AYD-modules.

Definition 5.6. A paracomplex $C = C_0 \oplus C_1$ consists of AYD-modules C_0 and C_1 and AYD-maps $\partial_0: C_0 \to C_1$ and $\partial_1: C_1 \to C_0$ such that

$$\partial^2 = \mathrm{id} - T$$

where the differential $\partial: C \to C_1 \oplus C_0 \cong C$ is the composition of $\partial_0 \oplus \partial_1$ with the canonical flip map.

The morphism ∂ in a paracomplex is called a differential although it usually does not satisfy the relation $\partial^2 = 0$. As for ordinary complexes one defines chain maps between paracomplexes and homotopy equivalences. We always assume that such maps are compatible with the AYD-module structures. Let us point out that it does not make sense to speak about the homology of a paracomplex in general.

The paracomplexes we will work with arise from paramixed complexes in the following sense.

Definition 5.7. A paramixed complex M is a sequence of AYD-modules M_n together with AYD-maps b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and

$$[b, B] = bB + Bb = \mathrm{id} - T.$$

If T is equal to the identity operator on M this reduces of course to the definition of a mixed complex.

6 Equivariant differential forms

In this section we define equivariant differential forms and the equivariant X-complex. Moreover we discuss the properties of the periodic tensor algebra of an H-algebra. These are the main ingredients in the construction of equivariant cyclic homology.

Let H be a bornological quantum group. If A is an H-algebra we obtain a left action of H on the space $H \hat{\otimes} \Omega^n(A)$ by

$$t \cdot (x \otimes \omega) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega$$

for $t, x \in H$ and $\omega \in \Omega^n(A)$. Here $\Omega^n(A) = A^+ \hat{\otimes} A^{\hat{\otimes} n}$ for $n > 0$ is the space of noncommutative n-forms over A with the diagonal H-action. For $n = 0$ one defines $\Omega^0(A) = A$. There is a left action of the dual quantum group $\hat{H}$ on $H \hat{\otimes} \Omega^n(A)$ given by

$$f \cdot (x \otimes \omega) = (f \rightharpoonup x) \otimes \omega = f(x_{(2)}) x_{(1)} \otimes \omega.$$

By definition, the equivariant n-forms $\Omega^n_H(A)$ are the space $H \hat{\otimes} \Omega^n(A)$ together with the H-action and the $\hat{H}$-action described above. We compute

$$\begin{aligned} t \cdot (f \cdot (x \otimes \omega)) &= t \cdot (f(x_{(2)}) x_{(1)} \otimes \omega) \\ &= f(x_{(2)}) t_{(3)} x_{(1)} S(t_{(1)}) \otimes t_{(2)} \cdot \omega \\ &= (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(5)})) \cdot (t_{(4)} x S(t_{(2)}) \otimes t_{(3)} \cdot \omega) \\ &= (S^2(t_{(1)}) \rightharpoonup f \leftharpoonup S^{-1}(t_{(3)})) \cdot (t_{(2)} \cdot (x \otimes \omega)) \end{aligned}$$

and deduce that $\Omega^n_H(A)$ is an H-AYD-module. We let $\Omega_H(A)$ be the direct sum of the spaces $\Omega^n_H(A)$.

Now we define operators d and b_H on $\Omega_H(A)$ by

$$d(x \otimes \omega) = x \otimes d\omega$$

and

$$b_H(x \otimes \omega da) = (-1)^{|\omega|}(x \otimes \omega a - x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot a)\omega).$$

The map b_H should be thought of as a twisted version of the usual Hochschild operator. We compute

$$\begin{aligned} & b_H^2(x \otimes \omega da db) = (-1)^{|\omega|+1} b_H(x \otimes \omega dab - x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot b)\omega da) \\ & = (-1)^{|\omega|+1} b_H(x \otimes \omega d(ab) - x \otimes \omega a db - x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot b)\omega da) \\ & = -(x \otimes \omega ab - x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (ab)\omega - x \otimes \omega ab + x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot b)\omega a \\ & \quad - x_{(2)} \otimes (S^{-1}(x_{(1)}) \cdot b)\omega a + x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (ab)\omega) = 0 \end{aligned}$$

which shows that b_H^2 is a differential as in the nonequivariant situation. Let us discuss the compatibility of d and b_H with the AYD-module structure. It is easy to check that

d is an AYD-map and that the operator b_H is $\hat{H}$-linear. Moreover we compute

$$\begin{aligned} b_H(t\cdot(x\otimes\omega da)) &= (-1)^{|\omega|}(t_{(4)}xS(t_{(1)})\otimes(t_{(2)}\cdot\omega)(t_{(3)}\cdot a) \\ &\quad - t_{(6)}x_{(2)}S(t_{(1)})\otimes(S^{-1}(t_{(5)}x_{(1)}S(t_{(2)}))t_{(4)}\cdot a)(t_{(3)}\cdot\omega)) \\ &= (-1)^{|\omega|}(t_{(3)}xS(t_{(1)})\otimes t_{(2)}\cdot(\omega a) - t_{(4)}x_{(2)}S(t_{(1)})\otimes(t_{(2)}S^{-1}(x_{(1)})\cdot a)(t_{(3)}\cdot\omega)) \\ &= t\cdot b_H(x\otimes\omega da) \end{aligned}$$

and deduce that b_H is an AYD-map as well.

Similar to the non-equivariant case we use d and b_H to define an equivariant Karoubi operator κ_H and an equivariant Connes operator B_H by

$$\kappa_H = 1-(b_H d + d b_H)$$

and

$$B_H = \sum_{j=0}^{n}\kappa_H^j d$$

on $\Omega_H^n(A)$. Let us record the following explicit formulas. For $n>0$ we have

$$\kappa_H(x\otimes\omega da) = (-1)^{n-1}x_{(2)}\otimes(S^{-1}(x_{(1)})\cdot da)\omega$$

on $\Omega_H^n(A)$ and in addition $\kappa_H(x\otimes a) = x_{(2)}\otimes S^{-1}(x_{(1)})\cdot a$ on $\Omega_H^0(A)$. For the Connes operator we compute

$$B_H(x\otimes a_0da_1\dots da_n) = \sum_{i=0}^{n}(-1)^{ni}x_{(2)}\otimes S^{-1}(x_{(1)})\cdot(da_{n+1-i}\dots da_n).da_0\dots da_{n-i}$$

Furthermore, the operator T is given by

$$T(x\otimes\omega) = x_{(2)}\otimes S^{-1}(x_{(1)})\cdot\omega$$

on equivariant differential forms. Observe that all operators constructed so far are AYD-maps and thus commute with T according to Proposition 5.5.

Lemma 6.1. *On $\Omega_H^n(A)$ the following relations hold:*

a) $\kappa_H^{n+1}d = Td$,

b) $\kappa_H^n = T + b_H\kappa_H^n d$,

c) $\kappa_H^n b_H = b_H T$,

d) $\kappa_H^{n+1} = (\mathrm{id} - db_H)T$,

e) $(\kappa_H^{n+1}-T)(\kappa_H^n - T) = 0$,

f) $B_H b_H + b_H B_H = \mathrm{id} - T$.

Proof. a) follows from the explicit formula for κ_H. b) We compute

$$\begin{aligned}\kappa_H^n(x \otimes a_0 da_1 \dots da_n) &= x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (da_1 \dots da_n) a_0 \\ &= x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (a_0 da_1 \dots da_n) \\ &\quad + (-1)^n b_H(x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (da_1 \dots da_n) da_0) \\ &= x_{(2)} \otimes S^{-1}(x_{(1)}) \cdot (a_0 da_1 \dots da_n) + b_H \kappa_H^n d(x \otimes a_0 da_1 \dots da_n)\end{aligned}$$

which yields the claim. c) follows by applying b_H to both sides of b). d) Apply κ_H to b) and use a). e) is a consequence of b) and d). f) We compute

$$\begin{aligned}B_H b_H + b_H B_H &= \sum_{j=0}^{n-1} \kappa_H^j d b_H + \sum_{j=0}^{n} b_H \kappa_H^j d = \sum_{j=0}^{n-1} \kappa_H^j (d b_H + b_H d) + \kappa_H^n b_H d \\ &= \mathrm{id} - \kappa_H^n (1 - b_H d) = \mathrm{id} - \kappa_H^n (\kappa_H + d b_H) \\ &= \mathrm{id} - T + d b_H T - T d b_H = \mathrm{id} - T\end{aligned}$$

where we use d) and b). □

From the definition of B_H and the fact that $d^2 = 0$ we obtain $B_H^2 = 0$. Let us summarize this discussion as follows.

Proposition 6.2. *Let A be an H-algebra. The space $\Omega_H(A)$ of equivariant differential forms is a paramixed complex in the category of* AYD*-modules.*

We remark that the definition of $\Omega_H(A)$ for $H = \mathcal{D}(G)$ differs slightly from the definition of $\Omega_G(A)$ in [25] if the locally compact group G is not unimodular. However, this does not affect the definition of the equivariant homology groups.

In the sequel we will drop the subscripts when working with the operators on $\Omega_H(A)$ introduced above. For instance, we shall simply write b instead of b_H and B instead of B_H.

The n-th level of the Hodge tower associated to $\Omega_H(A)$ is defined by

$$\theta^n \Omega_H(A) = \bigoplus_{j=0}^{n-1} \Omega_H^j(A) \oplus \Omega_H^n(A)/b(\Omega_H^{n+1}(A)).$$

Using the grading into even and odd forms we see that $\theta^n \Omega_H(A)$ together with the boundary operator $B + b$ becomes a paracomplex. By definition, the Hodge tower $\theta \Omega_H(A)$ of A is the projective system $(\theta^n \Omega_H(A))_{n \in \mathbb{N}}$ obtained in this way.

From a conceptual point of view it is convenient to work with pro-categories in the sequel. The pro-category $\mathrm{pro}(\mathcal{C})$ over a category $\mathcal{C}$ consists of projective systems in $\mathcal{C}$. A pro-H-algebra is simply an algebra in the category $\mathrm{pro}(H\text{-}\mathsf{Mod})$. For instance, every H-algebra becomes a pro-H-algebra by viewing it as a constant projective system. More information on the use of pro-categories in connection with cyclic homology can be found in [10], [25].

Definition 6.3. Let A be a pro-H-algebra. The equivariant X-complex $X_H(A)$ of A is the paracomplex $\theta^1\Omega_H(A)$. Explicitly, we have

$$X_H(A)\colon\ \Omega^0_H(A) \underset{b}{\overset{d}{\rightleftarrows}} \Omega^1_H(A)/b(\Omega^2_H(A)).$$

We are interested in particular in the equivariant X-complex of the periodic tensor algebra $\mathcal{T}A$ of an H-algebra A. The periodic tensor algebra $\mathcal{T}A$ is the even part of $\theta\Omega(A)$ equipped with the Fedosov product given by

$$\omega \circ \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

for homogenous forms ω and η. The natural projection $\theta\Omega(A) \to A$ restricts to an equivariant homomorphism $\tau_A\colon \mathcal{T}A \to A$ and we obtain an extension

$$\mathcal{J}A \rightarrowtail \mathcal{T}A \xrightarrow{\tau_A} A$$

of pro-H-algebras.

The main properties of the pro-algebras $\mathcal{T}A$ and $\mathcal{J}A$ are explained in [20], [25]. Let us recall some terminology. We write $\mu^n\colon N^{\hat{\otimes} n} \to N$ for the iterated multiplication in a pro-H-algebra N. Then N is called locally nilpotent if for every equivariant pro-linear map $f\colon N \to C$ with constant range C there exists $n \in \mathbb{N}$ such that $f\mu^n = 0$. It is straightforward to check that the pro-H-algebra $\mathcal{J}A$ is locally nilpotent.

An equivariant pro-linear map $l\colon A \to B$ between pro-H-algebras is called a lonilcur if its curvature $\omega_l\colon A \hat{\otimes} A \to B$ defined by $\omega_l(a,b) = l(ab) - l(a)l(b)$ is locally nilpotent, that is, if for every equivariant pro-linear map $f\colon B \to C$ with constant range C there exists $n \in \mathbb{N}$ such that $f\mu^n_B\omega_l^{\hat{\otimes} n} = 0$. The term lonilcur is an abbreviation for "equivariant pro-linear map with *lo*cally *nil*potent *cur*vature". Since $\mathcal{J}A$ is locally nilpotent the natural map $\sigma_A\colon A \to \mathcal{T}A$ is a lonilcur.

Proposition 6.4. *Let A be an H-algebra. The pro-H-algebra $\mathcal{T}A$ and the lonilcur $\sigma_A\colon A \to \mathcal{T}A$ satisfy the following universal property. If $l\colon A \to B$ is a lonilcur into a pro-H-algebra B there exists a unique equivariant homomorphism $[\![l]\!]\colon \mathcal{T}A \to B$ such that $[\![l]\!]\sigma_A = l$.*

An important ingredient in the Cuntz–Quillen approach to cyclic homology [8], [9], [10] is the concept of a quasifree pro-algebra. The same is true in the equivariant theory.

Definition 6.5. A pro-H-algebra R is called H-equivariantly quasifree if there exists an equivariant splitting homomorphism $R \to \mathcal{T}R$ for the projection τ_R.

We state some equivalent descriptions of equivariantly quasifree pro-H-algebras.

Theorem 6.6. *Let H be a bornological quantum group and let R be a pro-H-algebra. The following conditions are equivalent:*

a) *R is H-equivariantly quasifree.*

b) *There exists an equivariant pro-linear map* $\nabla\colon \Omega^1(R) \to \Omega^2(R)$ *satisfying*

$$\nabla(a\omega) = a\nabla(\omega), \quad \nabla(\omega a) = \nabla(\omega)a - \omega da$$

for all $a \in R$ *and* $\omega \in \Omega^1(R)$.

c) *There exists a projective resolution* $0 \to P_1 \to P_0 \to R^+$ *of the* R*-bimodule* R^+ *of length* 1 *in* $\mathrm{pro}(H\text{-}\mathsf{Mod})$.

A map $\nabla\colon \Omega^1(R) \to \Omega^2(R)$ satisfying condition b) in Theorem 6.6 is also called an equivariant graded connection on $\Omega^1(R)$.

We have the following basic examples of quasifree pro-H-algebras.

Proposition 6.7. *Let* A *be any* H*-algebra. The periodic tensor algebra* $\mathcal{T}A$ *is* H*-equivariantly quasifree.*

An important result in theory of Cuntz and Quillen relates the X-complex of the periodic tensor algebra $\mathcal{T}A$ to the standard complex of A constructed using noncommutative differential forms. The comparison between the equivariant X-complex and equivariant differential forms is carried out in the same way as in the group case [25].

Proposition 6.8. *There is a natural isomorphism* $X_H(\mathcal{T}A) \cong \theta\Omega_H(A)$ *such that the differentials of the equivariant* X*-complex correspond to*

$$\begin{aligned} \partial_1 &= b - (\mathrm{id} + \kappa)d && \text{on } \theta\Omega_H^{\mathrm{odd}}(A) \\ \partial_0 &= -\sum_{j=0}^{n-1} \kappa^{2j} b + B && \text{on } \Omega_H^{2n}(A). \end{aligned}$$

Theorem 6.9. *Let* H *be a bornological quantum group and let* A *be an* H*-algebra. Then the paracomplexes* $\theta\Omega_H(A)$ *and* $X_H(\mathcal{T}A)$ *are homotopy equivalent.*

For the proof of Theorem 6.9 it suffices to observe that the corresponding arguments in [25] are based on the relations obtained in Proposition 6.1.

7 Equivariant periodic cyclic homology

In this section we define equivariant periodic cyclic homology for bornological quantum groups.

Definition 7.1. Let H be a bornological quantum group and let A and B be H-algebras. The equivariant periodic cyclic homology of A and B is

$$HP^H_*(A, B) = H_*(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}(A \rtimes H \rtimes \hat{H})), X_H(\mathcal{T}(B \rtimes H \rtimes \hat{H})))).$$

We write $\mathrm{Hom}_{\mathsf{A}(H)}$ for the space of AYD-maps and consider the usual differential for a Hom-complex in this definition. Using Proposition 5.5 it is straightforward to check that this yields indeed a complex. Remark that both entries in the above Hom-complex are only paracomplexes.

Let us consider the special case that $H = \mathcal{D}(G)$ is the smooth group algebra of a locally compact group G. In this situation the definition of HP^H_* reduces to the definition of HP^G_* given in [25]. This is easily seen using the Takesaki–Takai isomorphism obtained in Proposition 3.7 and the results from [27].

As in the group case HP^H_* is a bifunctor, contravariant in the first variable and covariant in the second variable. We define $HP^H_*(A) = HP^H_*(\mathbb{C}, A)$ to be the equivariant periodic cyclic homology of A and $HP^*_H(A) = HP^H_*(A, \mathbb{C})$ to be equivariant periodic cyclic cohomology. There is a natural associative product

$$HP^H_i(A, B) \times HP^H_j(B, C) \to HP^H_{i+j}(A, C), \quad (x, y) \mapsto x \cdot y$$

induced by the composition of maps. Every equivariant homomorphism $f\colon A \to B$ defines an element in $HP^H_0(A, B)$ denoted by $[f]$. The element $[\mathrm{id}] \in HP^H_0(A, A)$ is denoted 1 or 1_A. An element $x \in HP^H_*(A, B)$ is called invertible if there exists an element $y \in HP^H_*(B, A)$ such that $x \cdot y = 1_A$ and $y \cdot x = 1_B$. An invertible element of degree zero is called an HP^H-equivalence. Such an element induces isomorphisms $HP^H_*(A, D) \cong HP^H_*(B, D)$ and $HP^H_*(D, A) \cong HP^H_*(D, B)$ for all H-algebras D.

8 Homotopy invariance, stability and excision

In this section we show that equivariant periodic cyclic homology is homotopy invariant, stable and satisfies excision in both variables. Since the arguments carry over from the group case with minor modifications most of the proofs will only be sketched. More details can be found in [25].

We begin with homotopy invariance. Let B be a pro-H-algebra and consider the Fréchet algebra $C^\infty[0, 1]$ of smooth functions on the interval $[0, 1]$. We denote by $B[0, 1]$ the pro-H-algebra $B \hat{\otimes} C^\infty[0, 1]$ where the action on $C^\infty[0, 1]$ is trivial. A (smooth) equivariant homotopy is an equivariant homomorphism $\Phi\colon A \to B[0, 1]$ of H-algebras. Evaluation at the point $t \in [0, 1]$ yields an equivariant homomorphism $\Phi_t\colon A \to B$. Two equivariant homomorphisms from A to B are called equivariantly homotopic if they can be connected by an equivariant homotopy.

Theorem 8.1 (Homotopy invariance). *Let A and B be H-algebras and let $\Phi\colon A \to B[0, 1]$ be a smooth equivariant homotopy. Then the elements $[\Phi_0]$ and $[\Phi_1]$ in $HP^H_0(A, B)$ are equal. Hence the functor HP^H_* is homotopy invariant in both variables with respect to smooth equivariant homotopies.*

Recall that $\theta^2\Omega_H(A)$ is the paracomplex $\Omega^0_H(A) \oplus \Omega^1_H(A) \oplus \Omega^2_H(A)/b(\Omega^3_H(A))$ with the usual differential $B + b$ and the grading into even and odd forms for any pro-H-algebra A. There is a natural chain map $\xi^2\colon \theta^2\Omega_H(A) \to X_H(A)$.

Proposition 8.2. *Let A be an equivariantly quasifree pro-H-algebra. Then the map $\xi^2\colon \theta^2\Omega_H(A) \to X_H(A)$ is a homotopy equivalence.*

A homotopy inverse is constructed using an equivariant connection for $\Omega^1(A)$.

Now let $\Phi\colon A \to B[0,1]$ be an equivariant homotopy. The derivative of Φ is an equivariant linear map $\Phi'\colon A \to B[0,1]$. If we view $B[0,1]$ as a bimodule over itself the map Φ' is a derivation with respect to Φ in the sense that $\Phi'(ab) = \Phi'(a)\Phi(b) + \Phi(a)\Phi'(b)$ for $a, b \in A$. We define an AYD-map $\eta\colon \Omega^n_H(A) \to \Omega^{n-1}_H(B)$ for $n > 0$ by

$$\eta(x \otimes a_0 da_1 \dots da_n) = \int_0^1 x \otimes \Phi_t(a_0)\Phi'_t(a_1) d\Phi_t(a_2) \dots d\Phi_t(a_n) dt$$

and an explicit calculation yields the following result.

Lemma 8.3. *We have $X_H(\Phi_1)\xi^2 - X_H(\Phi_0)\xi^2 = \partial\eta + \eta\partial$. Hence the chain maps $X_H(\Phi_t)\xi^2\colon \theta^2\Omega_H(A) \to X_H(B)$ for $t = 0, 1$ are homotopic.*

Using the map Φ we obtain an equivariant homotopy $A \hat{\otimes} \mathcal{K}_H \to (B \hat{\otimes} \mathcal{K}_H)[0,1]$ which induces an equivariant homomorphism $\mathcal{T}(A \hat{\otimes} \mathcal{K}_H) \to \mathcal{T}((B \hat{\otimes} \mathcal{K}_H)[0,1])$. Together with the obvious homomorphism $\mathcal{T}((B \hat{\otimes} \mathcal{K}_H)[0,1]) \to \mathcal{T}(B \hat{\otimes} \mathcal{K}_H)[0,1]$ this yields an equivariant homotopy $\Psi\colon \mathcal{T}(A \hat{\otimes} \mathcal{K}_H) \to \mathcal{T}(B \hat{\otimes} \mathcal{K}_H)[0,1]$. Since $\mathcal{T}(A \hat{\otimes} \mathcal{K}_H)$ is equivariantly quasifree we can apply Proposition 8.2 and Lemma 8.3 to obtain $[\Phi_0] = [\Phi_1] \in HP^H_0(A, B)$. This finishes the proof of Theorem 8.1.

Homotopy invariance has several important consequences. Let us call an extension $0 \to J \to R \to A \to 0$ of pro-H-algebras with equivariant pro-linear splitting a universal locally nilpotent extension of A if J is locally nilpotent and R is equivariantly quasifree. In particular, $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$ is a universal locally nilpotent extension of A. Using homotopy invariance one shows that HP^H_* can be computed using arbitrary universal locally nilpotent extensions.

Let us next study stability. One has to be slightly careful to formulate correctly the statement of the stability theorem since the tensor product of two H-algebras is no longer an H-algebra in general.

Let H be a bornological quantum group and assume that we are given an essential H-module V together with an equivariant bilinear pairing $b\colon V \times V \to \mathbb{C}$. Moreover let A be an H-algebra. Recall from Section 3 that $l(b; A) = V \hat{\otimes} A \hat{\otimes} V$ is the H-algebra with multiplication

$$(v_1 \otimes a_1 \otimes w_1)(v_2 \otimes a_2 \otimes w_2) = b(w_1, v_2)\, v_1 \otimes a_1 a_2 \otimes w_2$$

and the diagonal H-action. We call the pairing b admissible if there exists an H-invariant vector $u \in V$ such that $b(u, u) = 1$. In this case the map $\iota_A\colon A \to l(b; A)$ given by $\iota(a) = u \otimes a \otimes u$ is an equivariant homomorphism.

Theorem 8.4. *Let H be a bornological quantum group and let A be an H-algebra. For every admissible equivariant bilinear pairing $b\colon V \times V \to \mathbb{C}$ the map $\iota\colon A \to l(b; A)$ induces an invertible element $[\iota_A] \in H_0(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}A), X_H(\mathcal{T}(l(b; A)))))$.*

Proof. The canonical map $l(b; A) \to l(b; \mathcal{T}A)$ induces an equivariant homomorphism $\lambda_A\colon \mathcal{T}(l(b; A)) \to l(b; \mathcal{T}A)$. Define the map $\mathrm{tr}_A\colon X_H(l(b; \mathcal{T}A)) \to X_H(\mathcal{T}A)$ by

$$\mathrm{tr}_A(x \otimes (v_0 \otimes a_0 \otimes w_0)) = b(S^{-1}(x_{(1)}) \cdot w_0, v_0)\, x_{(2)} \otimes a_0$$

and

$$\begin{aligned}\mathrm{tr}_A(x \otimes (v_0 \otimes a_0 \otimes w_0)\, d(v_1 \otimes a_1 \otimes w_1)) \\ = b((S^{-1}(x_{(1)}) \cdot w_1, v_0) b(w_0, v_1)\, x_{(2)} \otimes a_0 da_1.\end{aligned}$$

In these formulas we implicitly use the twisted trace $\mathrm{tr}_x \colon l(b) \to \mathbb{C}$ for $x \in H$ defined by $\mathrm{tr}_x(v \otimes w) = b((S^{-1}(x) \cdot w, v)$. The twisted trace satisfies the relation

$$\mathrm{tr}_x(T_0 T_1) = \mathrm{tr}_{x_{(2)}}((S^{-1}(x_{(1)}) \cdot T_1) T_0)$$

for all $T_0, T_1 \in l(b)$. Using this relation one verifies that tr_A defines a chain map. It is clear that tr_A is $\hat{H}$-linear and it is straightforward to check that tr_A is H-linear. Let us define $t_A = \mathrm{tr}_A \circ X_H(\lambda_A)$ and show that $[t_A]$ is an inverse for $[\iota_A]$. Using the fact that u is H-invariant one computes $[\iota_A] \cdot [t_A] = 1$. We have to prove $[t_A] \cdot [\iota_A] = 1$. Consider the following equivariant homomorphisms $l(b; A) \to l(b; l(b; A))$ given by

$$\begin{aligned}i_1(v \otimes a \otimes w) &= u \otimes v \otimes a \otimes w \otimes u, \\ i_2(v \otimes a \otimes w) &= v \otimes u \otimes a \otimes u \otimes w.\end{aligned}$$

As above we see $[i_1] \cdot [t_{l(b;A)}] = 1$ and we determine $[i_2] \cdot [t_{l(b;A)}] = [t_A] \cdot [\iota_A]$. Let h_t be the linear map from $l(b; A)$ into $l(b; l(b; A))$ given by

$$\begin{aligned}h_t(v \otimes a \otimes w) = \cos(\pi t/2)^2 u \otimes v \otimes a \otimes w \otimes u + \sin(\pi t/2)^2 v \otimes u \otimes a \otimes u \otimes w \\ - i \cos(\pi t/2) \sin(\pi t/2) u \otimes v \otimes a \otimes u \otimes w \\ + i \sin(\pi t/2) \cos(\pi t/2) v \otimes u \otimes a \otimes w \otimes u.\end{aligned}$$

The family h_t depends smoothly on t and we have $h_0 = i_1$ and $h_1 = i_2$. Since u is invariant the map h_t is in fact equivariant and one checks that h_t is a homomorphism. Hence we have indeed defined a smooth homotopy between i_1 and i_2. This yields $[i_1] = [i_2]$ and hence $[t_A] \cdot [\iota_A] = 1$. □

We derive the following general stability theorem.

Proposition 8.5 (Stability). *Let H be a bornological quantum group and let A be an H-algebra. Moreover let V be an essential H-module and let $b \colon V \times V \to \mathbb{C}$ be a nonzero equivariant bilinear pairing. Then there exists an invertible element in $HP_0^G(A, l(b; A))$. Hence there are natural isomorphisms*

$$HP_*^H(l(b; A), B) \cong HP_*^H(A, B) \quad HP_*^H(A, B) \cong HP_*^H(A, l(b; B))$$

for all H-algebras A and B.

Proof. Let us write $\beta \colon H \times H \to \mathbb{C}$ for the canonical equivariant bilinear pairing introduced in Section 3. Moreover we denote by b_τ the pairing $b \colon V_\tau \times V_\tau \to \mathbb{C}$ where V_τ is the space V equipped with the trivial H-action. We have an equivariant isomorphism $\gamma \colon l(b_\tau; l(\beta; A)) \cong l(\beta; l(b; A))$ given by

$$\gamma(v \otimes (x \otimes a \otimes y) \otimes w) = x_{(1)} \otimes x_{(2)} \cdot v \otimes a \otimes y_{(1)} \cdot w \otimes y_{(2)}$$

and using $\beta(x, y) = \psi(S(y)x)$ as well as the fact that ψ is right invariant one checks that γ is an algebra homomorphism. Now we can apply Theorem 8.4 to obtain the assertion. $\square$

We deduce a simpler description of equivariant periodic cyclic homology in certain cases. A bornological quantum group H is said to be of compact type if the dual algebra $\hat{H}$ is unital. Moreover let us call H of semisimple type if it is of compact type and the value of the integral for $\hat{H}$ on $1 \in \hat{H}$ is nonzero. For instance, the dual of a cosemisimple Hopf algebra $\hat{H}$ is of semisimple type.

Proposition 8.6. *Let H be a bornological quantum group of semisimple type. Then we have*

$$HP^H_*(A, B) \cong H_*(\mathrm{Hom}_{\mathsf{A}(H)}(X_H(\mathcal{T}A), X_H(\mathcal{T}B)))$$

for all H-algebras A and B.

Proof. Under the above assumptions the canonical bilinear pairing $\beta\colon \hat{H} \times \hat{H} \to \mathbb{C}$ is admissible since the element $1 \in \hat{H}$ is invariant. $\square$

Finally we discuss excision in equivariant periodic cyclic homology. Consider an extension

$$K \xrightarrow{\iota} E \xrightarrow{\pi} Q$$

of H-algebras equipped with an equivariant linear splitting $\sigma\colon Q \to E$ for the quotient map $\pi\colon E \to Q$.

Let $X_H(\mathcal{T}E : \mathcal{T}Q)$ be the kernel of the map $X_H(\mathcal{T}\pi) : X_H(\mathcal{T}E) \to X_G(\mathcal{T}Q)$ induced by π. The splitting σ yields a direct sum decomposition $X_H(\mathcal{T}E) = X_H(\mathcal{T}E : \mathcal{T}Q) \oplus X_H(\mathcal{T}Q)$ of AYD-modules. The resulting extension

$$X_H(\mathcal{T}E : \mathcal{T}Q) \rightarrowtail X_H(\mathcal{T}E) \twoheadrightarrow X_H(\mathcal{T}Q)$$

of paracomplexes induces long exact sequences in homology in both variables. There is a natural covariant map $\rho\colon X_H(\mathcal{T}K) \to X_H(\mathcal{T}E : \mathcal{T}Q)$ of paracomplexes and we have the following generalized excision theorem.

Theorem 8.7. *The map $\rho\colon X_H(\mathcal{T}K) \to X_H(\mathcal{T}E : \mathcal{T}Q)$ is a homotopy equivalence.*

This result implies excision in equivariant periodic cyclic homology.

Theorem 8.8 (Excision). *Let A be an H-algebra and let $(\iota, \pi)\colon 0 \to K \to E \to Q \to 0$ be an extension of H-algebras with a bounded linear splitting. Then there are two natural exact sequences*

$$\begin{tikzcd}
HP^H_0(A,K) \arrow[r] & HP^H_0(A,E) \arrow[r] & HP^H_0(A,Q) \arrow[d] \\
HP^H_1(A,Q) \arrow[u] & HP^H_1(A,E) \arrow[l] & HP^H_1(A,K) \arrow[l]
\end{tikzcd}$$

and

$$\begin{array}{ccccc} HP_0^H(Q,A) & \longrightarrow & HP_0^H(E,A) & \longrightarrow & HP_0^H(K,A) \\ \uparrow & & & & \downarrow \\ HP_1^H(K,A) & \longleftarrow & HP_1^H(E,A) & \longleftarrow & HP_1^H(Q,A). \end{array}$$

The horizontal maps in these diagrams are induced by the maps in the extension.

In Theorem 8.8 we only require a linear splitting for the quotient homomorphism $\pi\colon E \to Q$. Taking double crossed products of the extension given in Theorem 8.8 yields an extension

$$K \mathbin{\hat{\otimes}} \mathcal{K}_H \rightarrowtail E \mathbin{\hat{\otimes}} \mathcal{K}_H \twoheadrightarrow Q \mathbin{\hat{\otimes}} \mathcal{K}_H$$

of H-algebras with a linear splitting. Using Lemma 3.3 one obtains an equivariant linear splitting for this extension. Now we can apply Theorem 8.7 to this extension and obtain the claim by considering long exact sequences in homology.

For the proof of Theorem 8.7 one considers the left ideal $\mathfrak{L} \subset \mathcal{T}E$ generated by $K \subset \mathcal{T}E$. The natural projection $\tau_E\colon \mathcal{T}E \to E$ induces an equivariant homomorphism $\tau\colon \mathfrak{L} \to K$ and one obtains an extension

$$N \rightarrowtail \mathfrak{L} \xrightarrow{\tau} K$$

of pro-H-algebras. The pro-H-algebra N is locally nilpotent and Theorem 8.7 follows from the following assertion.

Theorem 8.9. *With the notations as above we have*

a) *The pro-H-algebra $\mathfrak{L}$ is equivariantly quasifree.*

b) *The inclusion map $\mathfrak{L} \to \mathcal{T}E$ induces a homotopy equivalence $\psi\colon X_H(\mathfrak{L}) \to X_H(\mathcal{T}E : \mathcal{T}Q)$.*

In order to prove the first part of Theorem 8.9 one constructs explicitly a projective resolution of $\mathfrak{L}$ of length one.

9 Comparison to previous approaches

In this section we discuss the relation of the theory developed above to the previous approaches due to Akbarpour and Khalkhali [1], [2] as well as Neshveyev and Tuset [22]. As a natural domain in which all these theories are defined we choose to work with actions of cosemisimple Hopf algebras over the complex numbers. Note that a cosemisimple Hopf algebra H is a bornological quantum group with the fine bornology. In the sequel all vector spaces are equipped with the fine bornology when viewed as bornological vector spaces.

In their study of cyclic homology of crossed products [1] Akbarpour and Khalkhali define a cyclic module $C_*^H(A)$ of equivariant chains associated to a unital H-module algebra A. The space $C_n^H(A)$ in degree n of this cyclic module is the coinvariant space of $H \otimes A^{\otimes n+1}$ with respect to a certain action of H. Recall that the coinvariant space V_H associated to an H-module V is the quotient of V by the linear subspace generated by all elements of the form $t \cdot v - \epsilon(t)v$. Using the natural identification

$$\Omega_H^n(A) = H \otimes A^{\otimes n+1} \oplus H \otimes A^{\otimes n}$$

the action considered by Akbarpour and Khalkhali corresponds precisely to the action of H on the first summand in this decomposition. Hence $C_n^H(A)$ can be identified as a direct summand in the coinvariant space $\Omega_H^n(A)_H$. Moreover, the cyclic module structure of $C_*^H(A)$ reproduces the boundary operators b and B of $\Omega_H(A)_H$. We point out that the relation $T = \mathrm{id}$ holds on the coinvariant space $\Omega_H(A)_H$ which means that the latter is always a mixed complex.

It follows that there is a natural isomorphism of the cyclic type homologies associated to the cyclic module $C_*^H(A)$ and the mixed complex $\Omega_H(A)_H$, respectively. Note also that the complementary summand of $C_*^H(A)$ in $\Omega_H(A)_H$ is obtained from the bar complex of A tensored with H. Since A is assumed to be a unital H-algebra, this complementary summand is contractible with respect to the differential induced from the Hochschild boundary of $\Omega_H(A)_H$.

In the cohomological setting Akbarpour and Khalkhali introduce a cocyclic module $C_H^*(A)$ for every unital H-module algebra. The definition of this cocyclic module given in [2] is not literally dual to the one of the cyclic module $C_*^H(A)$. In order to establish the connection to our constructions let first A be an arbitrary H-algebra. We define a modified action of H on $\Omega_H(A)$ by the formula

$$t \circ (x \otimes \omega) = t_{(2)} x S^{-1}(t_{(3)}) \otimes t_{(1)} \cdot \omega$$

and write $\Omega_H(A)^\mu$ for the space $\Omega_H(A)$ equipped with this action. Let us compare the modified action with the original action

$$t \cdot (x \otimes \omega) = t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega$$

introduced in Section 6. In the space $\Omega_H(A)_H$ of coinvariants with respect to the original action we have

$$\begin{aligned} t \circ (x \otimes \omega) &= t_{(2)} x S^{-1}(t_{(3)}) \otimes t_{(1)} \cdot \omega = t_{(4)} x S^{-1}(t_{(5)}) t_{(1)} S(t_{(2)}) \otimes t_{(3)} \cdot \omega \\ &= t_{(2)} \cdot (x S^{-1}(t_{(3)}) t_{(1)} \otimes \omega) = x S^{-1}(t_{(2)}) t_{(1)} \otimes \omega = \epsilon(t)\, x \otimes \omega \end{aligned}$$

which implies that the canonical projection $\Omega_H(A) \to \Omega_H(A)_H$ factorizes over the coinvariant space $\Omega_H(A)_H^\mu$ with respect to the modified action. Similarly, in the coinvariant space $\Omega_H(A)_H^\mu$ we have

$$\begin{aligned} t \cdot (x \otimes \omega) &= t_{(3)} x S(t_{(1)}) \otimes t_{(2)} \cdot \omega = t_{(3)} x S(t_{(1)}) t_{(5)} S^{-1}(t_{(4)}) \otimes t_{(2)} \cdot \omega \\ &= t_{(2)} \circ (x S(t_{(1)}) t_{(3)} \otimes \omega) = x S(t_{(1)}) t_{(2)} \otimes \omega = \epsilon(t)\, x \otimes \omega. \end{aligned}$$

As a consequence we see that the identity map on $\Omega_H(A)$ induces an isomorphism

$$\Omega_H(A)_H \cong \Omega_H(A)_H^\mu$$

between the coinvariant spaces.

We may view a linear map $\Omega_H(A) \to \mathbb{C}$ as a linear map $\Omega(A) \to F(H)$ where $F(H)$ denotes the linear dual space of H. Under this identification an element in $\mathrm{Hom}_H(\Omega_H(A)^\mu, \mathbb{C})$ corresponds to a linear map $f : \Omega(A) \to F(H)$ satisfying the equivariance condition

$$f(t \cdot \omega)(x) = f(\omega)(S(t_{(2)})xt_{(1)})$$

for all $t \in H$ and $\omega \in \Omega(A)$. In a completely analogous fashion to the case of homology discussed above, a direct inspection using the canonical isomorphisms

$$\mathrm{Hom}_H(\Omega_H(A)^\mu, \mathbb{C}) \cong \mathrm{Hom}(\Omega_H(A)^\mu_H, \mathbb{C}) \cong \mathrm{Hom}(\Omega_H(A)_H, \mathbb{C})$$

shows that the cyclic type cohomologies of the cocyclic module $C^*_H(A)$ agree for every unital H-algebra A with the ones associated to the mixed complex $\Omega_H(A)_H$. In particular, the definition in [2] is indeed obtained by dualizing the construction given in [1].

The main difference between the cocyclic module used by Akbarbour and Khalkhali and the definition in [22] is that Neshveyev and Tuset work with right actions instead of left actions. It is explained in [22] that the two approaches lead to isomorphic cocyclic modules and hence to isomorphic cyclic type cohomologies.

Now assume that H is a semisimple Hopf algebra. Then the coinvariant space $\Omega_H(A)_H$ is naturally isomorphic to the space $\Omega_H(A)^H$ of invariants. If A is a unital H-algebra then Theorem 6.9 and Proposition 8.6, together with the above considerations, yield a natural isomorphism

$$HP_*(C^H_*(A)) \cong HP^H_*(\mathbb{C}, A)$$

which identifies the periodic cyclic homology of the cyclic module $C^H_*(A)$ with the equivariant cyclic homology of A in the sense of Definition 7.1. Similarly, for a semisimple Hopf algebra H one obtains a natural isomorphism

$$HP^*(C^*_H(A)) \cong HP^H_*(A, \mathbb{C})$$

for every unital H-algebra A.

Both of these isomorphisms fail to hold more generally, even in the classical setting of group actions. Let Γ be a discrete group and consider the group ring $H = \mathbb{C}\Gamma$. For the H-algebra $\mathbb{C}$ with the trivial action one easily obtains

$$HP^*(C^*_H(\mathbb{C})) \cong \mathrm{Hom}_H(H_{\mathsf{ad}}, \mathbb{C})$$

located in degree zero where H_{ad} is the space $H = \mathbb{C}\Gamma$ viewed as an H-module with the adjoint action. On the other hand, a result in [25] shows that there is a natural isomorphism

$$HP^H_*(\mathbb{C}, \mathbb{C}) \cong HP^*(H)$$

which identifies the H-equivariant theory of the complex numbers with the periodic cyclic cohomology of $H = \mathbb{C}\Gamma$. This is the result one should expect from equivariant

KK-theory [16]. Hence, roughly speaking, the theory in [2], [22] only recovers the degree zero part of the group cohomology. A similar remark applies to the homology groups defined in [1].

We mention that Akbarbour and Khalkhali have obtained an analogue of the Green–Julg isomorphism

$$HP^H_*(\mathbb{C}, A) \cong HP_*(A \rtimes H)$$

if H is semisimple and A is a unital H-algebra [1]. This result holds in fact more generally in the case that H is the convolution algebra of a compact quantum group and A is an arbitrary H-algebra. Similarly, there is a dual version

$$HP^H_*(A, \mathbb{C}) \cong HP^*(A \rtimes H)$$

of the Green–Julg theorem for the convolution algebras of discrete quantum groups. The latter generalizes the identification of equivariant periodic cyclic cohomology for discrete groups mentioned above. These results will be discussed elsewhere.

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A Schwartz type algebra for the tangent groupoid

Paulo Carrillo Rouse

1 Introduction

The concept of groupoid is central in non commutative geometry. Groupoids generalize the concepts of spaces, groups and equivalence relations. It is clear nowadays that groupoids are natural substitutes of singular spaces. Many people have contributed to realizing this idea. We can find for instance a groupoid-like treatment in Dixmier's work on transformation group, [11], or in Brown–Green–Rieffel's work on orbit classification of relations, [3]. In foliation theory, several models for the leaf space of a foliation were realized using groupoids by several authors, for example by Haefliger ([12]) and Wilkelnkemper ([24]), to mention some of them. There is also the case of orbifolds, these can be seen indeed as étale groupoids (see for example Moerdijk's paper [17]). There are also some particular groupoid models for manifolds with corners and conic manifolds, worked on by e.g. Monthubert [18], Debord–Lescure–Nistor ([10]) and Aastrup–Melo–Monthubert–Schrohe ([1]).

The way we treat "singular spaces" in non commutative geometry is by associating to them algebras. In the case when the "singular space" is represented by a Lie groupoid, we can, for instance, consider the convolution algebra of differentiable functions with compact support over the groupoid (see Connes' or Paterson's books [8] and [22]). This last algebra plays the role of the algebra of smooth functions over the "singular space" represented by the groupoid. From the convolution algebra it is also possible to construct a C^*-algebra, $C^*(\mathscr{G})$, that plays, in some sense, the role of the algebra of continuous functions over the "singular space". The idea of associating algebras in this sense can be traced back in works of Dixmier ([11]) for transformation groups, Connes ([6]) for foliations and Renault ([23]) for locally compact groupoids, to mention some of them.

Using methods of non commutative geometry, we would like to get invariants of these algebras, and hence, of the spaces they represent. For that, Connes showed that many groupoids and algebras associated to them appeared as 'non commutative analogues' of spaces to which many tools of geometry (and topology) such as K-theory and characteristic classes could be applied ([7], [8]). One classical way to obtain invariants in classical geometry (topology) is through the index theory in the sense of Atiyah–Singer. In the Lie groupoid case there is a pseudodifferential calculus, developed by Connes ([6]), Monthubert–Pierrot ([19]) and Nistor–Weinstein–Xu ([21]) in general. Some interesting particular cases were treated in the groupoid-spirit by Melrose ([16]), Moroianu ([20]) and others (see [1]). Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ be a Lie groupoid. Then there is an analytic index morphism (see [19]),

$$\mathrm{ind}_a \colon K^0(A^*\mathscr{G}) \to K_0(C^*(\mathscr{G})),$$

where $A\mathscr{G}$ is the Lie algebroid of $\mathscr{G}$. The "C^*-index" ind_a is a homotopy invariant of the $\mathscr{G}$-pseudodifferential elliptic operators ($\mathscr{G}$-PDO) and has proved to be very useful in many different situations (see for example [2], [9], [10]). One way to define the above index map is using Connes' tangent groupoid associated to $\mathscr{G}$ as explained by Hilsum and Skandalis in [13] or by Monthubert and Pierrot in [19]. The tangent groupoid is a Lie groupoid

$$\mathscr{G}^T \rightrightarrows \mathscr{G}^{(0)} \times [0,1]$$

with $\mathscr{G}^T := A\mathscr{G} \times \{0\} \sqcup \mathscr{G} \times (0,1]$, and the groupoid structure is given by the groupoid structure of $A\mathscr{G}$ at $t=0$ and by the groupoid structure of $\mathscr{G}$ for $t \neq 0$. One of the main features of the tangent groupoid is that its C^*-algebra $C^*(\mathscr{G}^T)$ is a continuous field of C^*-algebras over the closed interval $[0,1]$, with associated fiber algebras

$$C_0(A^*\mathscr{G}) \text{ at } t=0, \quad \text{and} \quad C^*(\mathscr{G}) \text{ for } t \neq 0.$$

In fact, it gives a C^*-algebraic quantization of the Poisson manifold $A^*\mathscr{G}$ (in the sense of [14]), and this is the main point why it allows to define the index morphism as a sort of "deformation". Thus, the tangent groupoid construction has been very useful in index theory ([2], [10], [13]), but also for other purposes ([14], [21]).

Now, to understand the purpose of the present work, let us first say that the indices (in the sense of Atiyah–Singer–Connes) have not necessarily to be considered as elements in $K_0(C^*(\mathscr{G}))$. Indeed, it is possible to consider indices in $K_0(C_c^\infty(\mathscr{G}))$. The C_c^∞-indices are more refined, but they have several inconveniences (see Alain Connes' book Section 9.β for a discussion on this matter). Nevertheless this kind of indices have the great advantage that one can apply to them the existent tools (such as pairings with cyclic cocycles or Chern–Connes character) in order to obtain numerical invariants.

In this work we begin a study of more refined indices. In particular we are looking for indices between the C_c^∞ and the C^*-levels, trying to keep the advantages of both approaches (see [5] for a more complete discussion). In the case of Lie groupoids, this refinement could mean to forget for a moment the powerful tools of the theory of C^*-algebras, and instead working in a purely algebraic and geometric level. In the present article, we construct an algebra of C^∞ functions over $\mathscr{G}^T$, denoted by $\mathscr{S}_c(\mathscr{G}^T)$. This algebra is also a field of algebras over the closed interval $[0,1]$, with associated fiber algebras

$$\mathscr{S}(A\mathscr{G}) \text{ at } t=0, \quad \text{and} \quad C_c^\infty(\mathscr{G}) \text{ for } t \neq 0,$$

where $\mathscr{S}(A\mathscr{G})$ is the Schwartz algebra of the Lie algebroid. Furthermore, we will have

$$C_c^\infty(\mathscr{G}^T) \subset \mathscr{S}_c(\mathscr{G}^T) \subset C^*(\mathscr{G}^T), \tag{1}$$

as inclusions of algebras. Let us explain in some words why we define an algebra over the tangent groupoid such that in zero it is Schwartz: The Schwartz algebras have in general the good K-theory groups. For example we are interested in the symbols of $\mathscr{G}$-PDO, and, more precisely, in their homotopy classes in K-theory, that is, we are interested in the group $K^0(A^*\mathscr{G}) = K_0(C_0(A^*\mathscr{G}))$. Here it would not be enough to take the K-theory of $C_c^\infty(A\mathscr{G})$ (see the example [8], p. 142), however it is enough to consider the Schwartz algebra $\mathscr{S}(A^*\mathscr{G})$. Indeed, the Fourier transform shows that this last algebra is stable under holomorphic calculus on $C_0(A^*\mathscr{G})$ and so it has the "good" K-theory, meaning that $K^0(A^*\mathscr{G}) = K_0(\mathscr{S}(A^*\mathscr{G}))$. None of the inclusions in (1) is stable under holomorphic calculus, but that is precisely what we wanted because our algebra $\mathscr{S}_c(\mathscr{G}^T)$ has the remarkable property that its evaluation at zero is stable under holomorphic calculus, while its evaluation at one (for example) is not.

The algebra $\mathscr{S}_c(\mathscr{G}^T)$ is, as a vector space, a particular case of a more general construction that we do for "deformation to the normal cone manifolds" from which the tangent groupoid is a special case (see [4] and [13]). A deformation to the normal cone manifold (DNC for short) is a manifold associated to an injective immersion $X \hookrightarrow M$ that is considered as a sort of blow up in differential geometry. The construction of a DNC manifold has very nice functorial properties (Section 3) which we exploit to achieve our construction. We think that our construction could be used also for other purposes, for example, it seems that it could help to give more understanding in quantization theory (see again [4]).

The article is organized as follows. In the second section we recall the basic facts about Lie groupoids. We explain very briefly how to define the convolution algebra $C_c^\infty(\mathscr{G})$. In the third section we explain the "deformation to the normal cone" construction associated to an injective immersion. Even if this could be considered as classical material, we do it in some detail since we will use in the sequel very explicit descriptions that we could not find elsewhere. We also review some functorial properties associated to these deformations. A particular case of this construction is the tangent groupoid associated to a Lie groupoid. In the fourth section we start by constructing a vector space $\mathscr{S}_c(\mathscr{D}_X^M)$ for any deformation to the normal cone manifold $\mathscr{D}_X^M$; this space already exhibits the characteristic of being a field of vector spaces over the closed interval [0, 1] whose fiber at zero is a Schwartz space while for values different from zero it is equal to $C_c^\infty(M)$. We then define the algebra $\mathscr{S}_c(\mathscr{G}^T)$; the main result is that its product is well-defined and associative. The last section is devoted to motivate the construction of our algebra by explaining shortly some further developments that will immediately follow from this work. All the results of the present work are part of the author's PhD thesis.

I want to thank my PhD advisor, Georges Skandalis, for all the ideas that he shared with me, and for all the comments and remarks he made on the present work. I would also like to thank the referee for the useful comments he made for improving this paper.

2 Lie groupoids

Let us recall what a groupoid is.

Definition 2.1. A *groupoid* consists of the following data: two sets $\mathscr{G}$ and $\mathscr{G}^{(0)}$, and maps

- $s, r\colon \mathscr{G} \to \mathscr{G}^{(0)}$, called the source and target map respectively,

- $m\colon \mathscr{G}^{(2)} \to \mathscr{G}$, called the product map (where $\mathscr{G}^{(2)} = \{(\gamma,\eta) \in \mathscr{G} \times \mathscr{G} \colon s(\gamma) = r(\eta)\}$),
- $u\colon \mathscr{G}^{(0)} \to \mathscr{G}$ the unit map, and
- $i : \mathscr{G} \to \mathscr{G}$ the inverse map

such that, if we write $m(\gamma,\eta) = \gamma \cdot \eta$, $u(x) = x$ and $i(\gamma) = \gamma^{-1}$, we have

1. $\gamma \cdot (\eta \cdot \delta) = (\gamma \cdot \eta) \cdot \delta$ for all $\gamma, \eta, \delta \in \mathscr{G}$ when this is possible,
2. $\gamma \cdot x = \gamma$ and $x \cdot \eta = \eta$ for all $\gamma, \eta \in \mathscr{G}$ with $s(\gamma) = x$ and $r(\eta) = x$,
3. $\gamma \cdot \gamma^{-1} = u(r(\gamma))$ and $\gamma^{-1} \cdot \gamma = u(s(\gamma))$ for all $\gamma \in \mathscr{G}$,
4. $r(\gamma \cdot \eta) = r(\gamma)$ and $s(\gamma \cdot \eta) = s(\eta)$.

Generally, we denote a groupoid by $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ where the parallel arrows are the source and target maps and the other maps are given.

Now, a Lie groupoid is a groupoid in which every set and map appearing in the last definition is C^∞ (possibly with borders), and the source and target maps are submersions. For A, B subsets of $\mathscr{G}^{(0)}$ we use the notation $\mathscr{G}_A^B$ for the subset $\{\gamma \in \mathscr{G} : s(\gamma) \in A,\ r(\gamma) \in B\}$.

Throughout this paper, $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ is a Lie groupoid. We recall how to define an algebra structure in $C_c^\infty(\mathscr{G})$ using smooth Haar systems.

Definition 2.2. A *smooth Haar system* over a Lie groupoid consists of a family of measures μ_x in $\mathscr{G}_x$ for each $x \in \mathscr{G}^{(0)}$ such that

- for $\eta \in \mathscr{G}_x^y$ we have the following compatibility condition:

$$\int_{\mathscr{G}_x} f(\gamma)\, d\mu_x(\gamma) = \int_{\mathscr{G}_y} f(\gamma \circ \eta)\, d\mu_y(\gamma),$$

- for each $f \in C_c^\infty(\mathscr{G})$ the map

$$x \mapsto \int_{\mathscr{G}_x} f(\gamma)\, d\mu_x(\gamma)$$

belongs to $C_c^\infty(\mathscr{G}^{(0)})$.

A Lie groupoid always possesses a smooth Haar system. In fact, if we fix a smooth (positive) section of the 1-density bundle associated to the Lie algebroid we obtain a smooth Haar system in a canonical way. The advantage of using 1-densities is that the measures are locally equivalent to the Lebesgue measure. We suppose for the rest of the paper an underlying smooth Haar system given by 1-densities (for complete details see [22]). We can now define a convolution product on $C_c^\infty(\mathscr{G})$. For $f, g \in C_c^\infty(\mathscr{G})$ we set

$$(f * g)(\gamma) = \int_{\mathscr{G}_{s(\gamma)}} f(\gamma \cdot \eta^{-1}) g(\eta)\, d\mu_{s(\gamma)}(\eta).$$

This gives a well-defined associative product.

Remark 2.3. There is a way to avoid the Haar system when one works with Lie groupoids, using half densities (see Connes' book [8]).

3 Deformation to the normal cone

Let M be a C^∞ manifold and $X \subset M$ be a C^∞ submanifold. We denote by $\mathcal{N}_X^M$ the normal bundle to X in M, *i.e.*, $\mathcal{N}_X^M := T_X M/TX$. We define the following set

$$\mathcal{D}_X^M := \mathcal{N}_X^M \times 0 \sqcup M \times (0,1]. \tag{2}$$

The purpose of this section is to recall how to define a C^∞-structure with boundary in $\mathcal{D}_X^M$. This is more or less classical, for example it was extensively used in [13]. Here we are only going to do a sketch.

Let us first consider the case where $M = \mathbb{R}^n$ and $X = \mathbb{R}^p \times \{0\}$ (where we identify canonically $X = \mathbb{R}^p$). We denote by $q = n - p$ and by $\mathcal{D}_p^n$ for $\mathcal{D}_{\mathbb{R}^p}^{\mathbb{R}^n}$ as above. In this case we clearly have that $\mathcal{D}_p^n = \mathbb{R}^p \times \mathbb{R}^q \times [0,1]$ (as a set). Consider the bijection

$$\Psi \colon \mathbb{R}^p \times \mathbb{R}^q \times [0,1] \to \mathcal{D}_p^n \tag{3}$$

given by

$$\Psi(x,\xi,t) = \begin{cases} (x,\xi,0) & \text{if } t = 0, \\ (x,t\xi,t) & \text{if } t > 0, \end{cases}$$

whose the inverse is given explicitly by

$$\Psi^{-1}(x,\xi,t) = \begin{cases} (x,\xi,0) & \text{if } t = 0, \\ (x,\frac{1}{t}\xi,t) & \text{if } t > 0. \end{cases}$$

We can consider the C^∞-structure with border on $\mathcal{D}_p^n$ induced by this bijection.

In the general case, let $(\mathcal{U}, \phi)$ be a local chart in M and suppose it is an X-slice, so that it satisfies

1) $\phi \colon \mathcal{U} \xrightarrow{\cong} U \subset \mathbb{R}^p \times \mathbb{R}^q$;

2) if $\mathcal{V} = \mathcal{U} \cap X$, $V = U \cap (\mathbb{R}^p \times 0)$, then we have $\mathcal{V} = \phi^{-1}(V)$.

With this notation we have that $\mathcal{D}_V^U \subset \mathcal{D}_p^n$ is an open subset. We may define a function

$$\tilde{\phi} \colon \mathcal{D}_{\mathcal{V}}^{\mathcal{U}} \to \mathcal{D}_V^U$$

in the following way. For $x \in \mathcal{V}$ we have $\phi(x) \in \mathbb{R}^p \times \{0\}$. If we write $\phi(x) = (\phi_1(x), 0)$, then

$$\phi_1 \colon \mathcal{V} \to V \subset \mathbb{R}^p$$

is a diffeomorphism, where $V = U \cap (\mathbb{R}^p \times \{0\})$. Set $\tilde{\phi}(v, \xi, 0) = (\phi_1(v), d_N\phi_v(\xi), 0)$ and $\tilde{\phi}(u, t) = (\phi(u), t)$ for $t \neq 0$. Here $d_N\phi_v \colon \mathcal{N}_v \to \mathbb{R}^q$ is the normal component of the derivative $d\phi_v$ for $v \in \mathcal{V}$. It is clear that $\tilde{\phi}$ is also a bijection (in particular it induces a C^∞ structure with border over $\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}$).

Let us define, with the same notations as above, the following set:

$$\Omega_V^U = \{(x, \xi, t) \in \mathbb{R}^p \times \mathbb{R}^q \times [0,1] : (x, t \cdot \xi) \in U\}.$$

This is an open subset of $\mathbb{R}^p \times \mathbb{R}^q \times [0,1]$ and thus a C^∞ manifold (with border). It is immediate that $\mathcal{D}_V^U$ is diffeomorphic to Ω_V^U through the restriction of Ψ given in (3). Now we consider an atlas $\{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha \in \Delta}$ of M consisting of X-slices. It is clear that

$$\mathcal{D}_X^M = \bigcup_{\alpha \in \Delta} \mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \tag{4}$$

and if we take $\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \overset{\varphi_\alpha}{\to} \Omega_{V_\alpha}^{U_\alpha}$ defined as the composition

$$\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha} \overset{\phi_\alpha}{\to} \mathcal{D}_{V_\alpha}^{U_\alpha} \overset{\Psi_\alpha^{-1}}{\to} \Omega_{V_\alpha}^{U_\alpha}$$

then we obtain the following result.

Proposition 3.1. *$\{(\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha}, \varphi_\alpha)\}_{\alpha \in \Delta}$ is a C^∞ atlas with border over $\mathcal{D}_X^M$.*

In fact the proposition can be proved directly from the following elementary lemma.

Lemma 3.2. *Let $F \colon U \to U'$ be a C^∞ diffeomorphism where $U \subset \mathbb{R}^p \times \mathbb{R}^q$ and $U' \subset \mathbb{R}^p \times \mathbb{R}^q$ are open subsets. We write $F = (F_1, F_2)$ and we suppose that $F_2(x, 0) = 0$. Then the function $\tilde{F} \colon \Omega_V^U \to \Omega_{V'}^{U'}$ defined by*

$$\tilde{F}(x, \xi, t) = \begin{cases} (F_1(x,0), \frac{\partial F_2}{\partial \xi}(x,0) \cdot \xi, 0) & \text{if } t = 0, \\ (F_1(x, t\xi), \frac{1}{t} F_2(x, t\xi), t) & \text{if } t > 0. \end{cases}$$

is a C^∞ map.

Proof. Since the result will hold if and only if it is true in each coordinate, it is enough to prove that if we have a C^∞ map $F \colon U \to \mathbb{R}$ with $F(x, 0) = 0$, then the map $\tilde{F} \colon \Omega_V^U \to \mathbb{R}$ given by

$$\tilde{F}(x, \xi, t) = \begin{cases} \frac{\partial F}{\partial \xi}(x, 0) \cdot \xi & \text{if } t = 0, \\ \frac{1}{t} F(x, t\xi) & \text{if } t > 0 \end{cases}$$

is a C^∞ map. For that we write

$$F(x, \xi) = \frac{\partial F}{\partial \xi}(x, 0) \cdot \xi + h(x, \xi) \cdot \xi$$

with $h\colon U \to \mathbb{R}^q$ a C^∞ map such that $h(x, 0) = 0$. Then

$$\frac{1}{t}F(x, t\xi) = \frac{\partial F}{\partial \xi}(x, 0) \cdot \xi + h(x, t\xi) \cdot \xi$$

from which we immediately get the result. □

Definition 3.3 (DNC). Let $X \subset M$ be as above. The set $\mathcal{D}_X^M$ provided with the C^∞ structure with border induced by the atlas described in the last proposition is called the *deformation to normal cone associated to* $X \subset M$. We will often write DNC instead of deformation to the normal cone.

Remark 3.4. Following the same steps, it is possible to define a deformation to the normal cone associated to an injective immersion $X \hookrightarrow M$.

Let us mention some basic examples of DCN manifolds $\mathcal{D}_X^M$.

Examples 3.5. 1. Consider the case when $X = \emptyset$. We have that $\mathcal{D}_\emptyset^M = M \times (0, 1]$ with the usual C^∞ structure on $M \times (0, 1]$. We used this fact implicitly to cover $\mathcal{D}_X^M$ as in (4).

2. Consider the case when $X \subset M$ is an open subset. Then we do not have any deformation at zero and we immediately see from the definition that $\mathcal{D}_X^M$ is just the open subset of $M \times [0, 1]$ consisting of the union of $X \times [0, 1]$ and $M \times (0, 1]$.

The most important feature of the DNC construction is that it is in some sense functorial. More explicitly, let (M, X) and (M', X') be C^∞-couples as above and let $F\colon (M, X) \to (M', X')$ be a couple morphism, i.e., a C^∞ map $F\colon M \to M'$ with $F(X) \subset X'$. We define $\mathcal{D}(F)\colon \mathcal{D}_X^M \to \mathcal{D}_{X'}^{M'}$ by the following formulas:

$$\mathcal{D}(F)(x, \xi, 0) = (F(x), d_N F_x(\xi), 0) \quad \text{and} \quad \mathcal{D}(F)(m, t) = (F(m), t) \text{ for } t \neq 0,$$

where $d_N F_x$ is by definition the map

$$(\mathcal{N}_X^M)_x \xrightarrow{d_N F_x} (\mathcal{N}_{X'}^{M'})_{F(x)}$$

induced by $T_x M \xrightarrow{dF_x} T_{F(x)} M'$.

We have the following proposition, which is also an immediate consequence of the lemma above.

Proposition 3.6. *The map* $\mathcal{D}(F)\colon \mathcal{D}_X^M \to \mathcal{D}_{X'}^{M'}$ *is* C^∞.

Remark 3.7. If we consider the category $\mathcal{C}_2^\infty$ of C^∞ pairs given by a C^∞ manifold and a C^∞ submanifold and pair morphisms as above, we can reformulate the proposition and say that we have a functor

$$\mathcal{D}\colon \mathcal{C}_2^\infty \to \mathcal{C}^\infty$$

where $\mathcal{C}^\infty$ denotes the category of C^∞ manifolds with border.

3.1 The tangent groupoid

Definition 3.8 (Tangent groupoid). Let $\mathscr{G} \rightrightarrows \mathscr{G}^{(0)}$ be a Lie groupoid. The *tangent groupoid* associated to $\mathscr{G}$ is the groupoid that has $\mathscr{D}_{\mathscr{G}^{(0)}}^{\mathscr{G}}$ as the set of arrows and $\mathscr{G}^{(0)} \times [0,1]$ as the units with

- $s^T(x,\eta,0) = (x,0)$ and $r^T(x,\eta,0) = (x,0)$ at $t = 0$,
- $s^T(\gamma,t) = (s(\gamma),t)$ and $r^T(\gamma,t) = (r(\gamma),t)$ at $t \neq 0$,
- the product is given by $m^T((x,\eta,0),(x,\xi,0)) = (x,\eta+\xi,0)$ and $m^T((\gamma,t),(\beta,t)) = (m(\gamma,\beta),t)$ if $t \neq 0$ and if $r(\beta) = s(\gamma)$,
- the unit map $u^T : \mathscr{G}^{(0)} \to \mathscr{G}^T$ is given by $u^T(x,0) = (x,0)$ and $u^T(x,t) = (u(x),t)$ for $t \neq 0$.

We denote $\mathscr{G}^T := \mathscr{D}_{\mathscr{G}^{(0)}}^{\mathscr{G}}$.

As we have seen above $\mathscr{G}^T$ can be considered as a C^∞ manifold with border. As a consequence of the functoriality of the DNC construction we can show that the tangent groupoid is in fact a Lie groupoid. Indeed, it is easy to check that if we identify in a canonical way $\mathscr{D}_{\mathscr{G}^{(0)}}^{\mathscr{G}^{(2)}}$ with $(\mathscr{G}^T)^{(2)}$, then

$$m^T = \mathscr{D}(m), \quad s^T = \mathscr{D}(s), \quad r^T = \mathscr{D}(r), \quad u^T = \mathscr{D}(u)$$

where we are considering the following pair morphisms:

$$\begin{aligned} m &: ((\mathscr{G})^{(2)}, \mathscr{G}^{(0)}) \to (\mathscr{G}, \mathscr{G}^{(0)}), \\ s, r &: (\mathscr{G}, \mathscr{G}^{(0)}) \to (\mathscr{G}^{(0)}, \mathscr{G}^{(0)}), \\ u &: (\mathscr{G}^{(0)}, \mathscr{G}^{(0)}) \to (\mathscr{G}, \mathscr{G}^{(0)}). \end{aligned}$$

Finally, if $\{\mu_x\}$ is a smooth Haar system on $\mathscr{G}$, then, setting

- $\mu_{(x,0)} := \mu_x$ at $(\mathscr{G}^T)_{(x,0)} = T_x\mathscr{G}_x$, and
- $\mu_{(x,t)} := t^{-q} \cdot \mu_x$ at $(\mathscr{G}^T)_{(x,t)} = \mathscr{G}_x$ for $t \neq 0$, where $q = \dim \mathscr{G}_x$,

one obtains a smooth Haar system for the tangent groupoid (details may be found in [22]).

We finish this section with some interesting examples of groupoids and their tangent groupoids.

Examples 3.9. 1. *The tangent groupoid of a group.* Let G be a Lie group considered as a Lie groupoid, $\mathscr{G} := G \rightrightarrows \{e\}$. In this case the normal bundle to the inclusion $\{e\} \hookrightarrow G$ is of course identified with the Lie algebra of the group. Hence, the tangent groupoid is a deformation of the group in its Lie algebra:

$$\mathscr{G}^T = \mathfrak{g} \times \{0\} \sqcup G \times (0,1].$$

2. *The tangent groupoid of a smooth vector bundle.* Let $E \overset{p}{\to} X$ be a smooth vector bundle over a (connected) C^∞ manifold X. We can consider the Lie groupoid $E \rightrightarrows X$ induced by the vector structure of the fibers, *i.e.*, $s(\xi) = p(\xi) = r(\xi)$ and the composition is given by the vector sum $\xi \circ \eta = \xi + \eta$. In this case the normal vector bundle associated to the zero section can be identified to E itself. Hence, as a set the tangent groupoid is $E \times [0,1]$, but the C^∞-structure at zero is given locally as in (3).

3. *The tangent groupoid of a C^∞-manifold.* Let M a C^∞-manifold. We can consider the product groupoid $\mathscr{G}_M := M \times M \rightrightarrows M$. The tangent groupoid in this case takes the following form:

$$\mathscr{G}_M^T = TM \times \{0\} \sqcup M \times M \times (0,1].$$

This is called the tangent groupoid to M and it was introduced by Connes for giving a very conceptual proof of the Atiyah–Singer index theorem (see [8] and [10]).

4 An algebra for the tangent groupoid

In this section we will show how to construct an algebra for the tangent groupoid which consists of C^∞ functions that satisfy a rapid decay condition at zero while out of zero they satisfy a compact support condition. This algebra is the main construction in this work.

4.1 Schwartz type spaces for deformation to the normal cone manifolds. Our algebra for the tangent groupoid will be a particular case of a construction associated to any deformation to the normal cone. We start by defining a space for DNCs associated to open subsets of $\mathbb{R}^p \times \mathbb{R}^q$.

Definition 4.1. Let $p, q \in \mathbb{N}$ and $U \subset \mathbb{R}^p \times \mathbb{R}^q$ an open subset, and let $V = U \cap (\mathbb{R}^p \times \{0\})$.

(1) Let $K \subset U \times [0,1]$ be a compact subset. We say that K is a *conic compact* subset of $U \times [0,1]$ relative to V if

$$K_0 = K \cap (U \times \{0\}) \subset V.$$

(2) Let $g \in C^\infty(\Omega_V^U)$. We say that f has *compact conic support* K, if there exists a conic compact K of $U \times [0,1]$ relative to V such that if $t \neq 0$ and $(x, t\xi, t) \notin K$ then $g(x,\xi,t) = 0$.

(3) We denote by $\mathscr{S}_c(\Omega_V^U)$ the set of functions $g \in C^\infty(\Omega_V^U)$ that have compact conic support and that satisfy the following condition:

(s_1) for all $k, m \in \mathbb{N}$, $l \in \mathbb{N}^p$ and $\alpha \in \mathbb{N}^q$ there exists $C_{(k,m,l,\alpha)} > 0$ such that

$$(1 + \|\xi\|^2)^k \|\partial_x^l \partial_\xi^\alpha \partial_t^m g(x,\xi,t)\| \leq C_{(k,m,l,\alpha)}.$$

Now, the spaces $\mathcal{S}_c(\Omega_V^U)$ are invariant under diffeomorphisms. More precisely if $F\colon U \to U'$ is a C^∞ diffeomorphism as in Lemma 3.2 then we can prove the next result.

Proposition 4.2. *Let $g \in \mathcal{S}_c(\Omega_{V'}^{U'})$, then $\tilde{g} := g \circ \tilde{F} \in \mathcal{S}_c(\Omega_V^U)$.*

Proof. The first observation is that $\tilde{g} \in C^\infty(\Omega_V^U)$, thanks to Lemma 3.2. Let us check that it has compact conic support. For that, let $K' \subset U' \times [0,1]$ be the conic compact support of g. We let

$$K = (F^{-1} \times \mathrm{Id}_{[0,1]}) \subset U \times [0,1],$$

which is a conic compact subset of $U \times [0,1]$ relative to V, and it is immediate by definition that $\tilde{g}(x,\xi,t) = 0$ if $t \neq 0$ and $(x, t\cdot\xi, t) \notin K$, that is, $\tilde{g}$ has compact conic support K.

We now check the rapid decay property (s_1): To simplify the proof we first introduce some useful notation. Writing $F = (F_1, F_2)$ as in Lemma 3.2, we denote $F_1(x,\xi) = (A_1(x,\xi),\dots,A_p(x,\xi))$ and $F_2(x,\xi) = (B_1(x,\xi),\dots,B_q(x,\xi))$. We also write $w = w(x,\xi,t) = (A_1(x,t\xi),\dots,A_p(x,t\xi))$ and $\eta = \eta(x,\xi,t) = (\tilde{B_1}(x,\xi,t),\dots,\tilde{B_q}(x,\xi,t))$ where $\tilde{B_j}$ is as above, *i.e.*,

$$\tilde{B_j}(x,\xi,t) = \begin{cases} \frac{\partial B_j}{\partial \xi}(x,0)\cdot\xi & \text{if } t = 0, \\ \\ \frac{1}{t}B_j(x,t\xi) & \text{if } t \neq 0. \end{cases}$$

In particular by definition we have $\tilde{F}(x,\xi,t) = (w,\eta,t)$. We also write $z = (x,\xi,t)$ and $u = (\omega,\eta,t)$. Hence, what we would like is to find bounds for expressions of the following type:

$$\|\xi\|^k \|\partial_z^\alpha \tilde{g}(z)\|$$

for arbitrary $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^p \times \mathbb{N}^q \times \mathbb{N}$. A simple calculation shows that the derivatives $\partial_z^\alpha \tilde{g}(z)$ are of the following form:

$$\partial_z^\alpha \tilde{g}(z) = \sum_{|\beta| \leq |\alpha|} P_\beta(z)\partial_u^\beta g(u)$$

where $P_\beta(z)$ is a finite sum of products of the form

$$\partial_z^\gamma \omega_i(z) \cdot \partial_z^\delta \eta_j(z).$$

We are only interested to see what happens in the set $K_\Omega := \{z = (x,\xi,t) \in \Omega : (x, t\cdot\xi, t) \in K\}$ since outside of this set we have that g and all its derivatives vanish ($(x,t\xi,t) \in K$ iff $(w,t\eta,t) \in K'$). For a point $z = (x,\xi,t) \in K_\Omega$ we have that $(x, t\cdot\xi)$ is contained in a compact set, and then it follows that the expressions

$$\|\partial_z^\gamma \omega_i(z)\|$$

are bounded in K_Ω. For the expressions $\|\partial_z^\delta \eta_j(z)\|$, we proceed first by developing as in Lemma 3.2, that is,

$$\eta_j(x,\xi,t) = \left(\frac{\partial B_j}{\partial \xi}(x,0)\cdot\xi + h^j(x,t\xi)\right)\cdot\xi.$$

Now, since we are only considering points in K_Ω, it is immediate that we can find constants $C_j > 0$ such that

$$\|\partial_z^\delta \eta_j(z)\| \le C_j \cdot \|\xi\|^{m_\delta}.$$

In the same way (remember that F is a diffeomorphism) we can have constants $C_i > 0$ such that

$$\|\xi_i(\omega,\eta,t)\| \le C_i \cdot \|\eta\|.$$

Putting all together, and using the property (s_1) for g, we get bounds $C > 0$ such that

$$\|\xi\|^k \|\partial_z^\alpha \tilde{g}(z)\| \le C,$$

and this concludes the proof. □

Remark 4.3. We can resume the last invariance result as follows: If $(\mathcal{U},\mathcal{V})$ is a C^∞ pair diffeomorphic to (U,V) with $U \subset E$, an open subset of a vector space E, and $V = U \cap E$, then $\mathcal{S}_c(\mathcal{D}_\mathcal{V}^\mathcal{U})$ is well defined and does not depend on the pair diffeomorphism.

With the last compatibility result in hand we are ready to give the main definition in this work.

Definition 4.4. Let $g \in C^\infty(\mathcal{D}_X^M)$.

(a) We say that g has *compact conic support* K, if there exists a compact subset $K \subset M \times [0,1]$ with $K_0 := K \cap (M \times \{0\}) \subset X$ (conic compact relative to X) such that if $t \neq 0$ and $(m,t) \notin K$ then $g(m,t) = 0$.

(b) We say that g is *rapidly decaying* at zero if for every X-slice chart $(\mathcal{U},\phi)$ and for every $\chi \in C_c^\infty(\mathcal{U}\times[0,1])$, the map $g_\chi \in C^\infty(\Omega_V^U)$ given by

$$g_\chi(x,\xi,t) = (g\circ\varphi^{-1})(x,\xi,t)\cdot(\chi\circ p\circ\varphi^{-1})(x,\xi,t)$$

is in $\mathcal{S}_c(\Omega_V^U)$, where p is the projection $p\colon \mathcal{D}_X^M \to M\times[0,1]$ given by $(x,\xi,0)\mapsto(x,0)$, and $(m,t)\mapsto(m,t)$ for $t\neq 0$.

Finally, we denote by $\mathcal{S}_c(\mathcal{D}_X^M)$ the set of functions $g \in C^\infty(\mathcal{D}_X^M)$ that are rapidly decaying at zero with compact conic support.

Remark 4.5. (a) It follows from the definition of $\mathcal{S}_c(\mathcal{D}_X^M)$ that the latter contains $C_c^\infty(\mathcal{D}_X^M)$ as a vector subspace.

(b) It is clear that $C_c^\infty(M\times(0,1])$ can be considered as a subspace of $\mathcal{S}_c(\mathcal{D}_X^M)$ by extending by zero the functions at $\mathcal{N}_X^M$.

Following the lines of the last remark we are going to precise a possible decomposition of our space $\mathcal{S}_c(\mathcal{D}_X^M)$ that will be very useful in the sequel. Let $\{(\mathcal{U}_\alpha,\phi_\alpha)\}_{\alpha\in\Delta}$ a family of X-slices covering X. Consider the open cover of $M\times[0,1]$ consisting of $\{(\mathcal{U}_\alpha\times[0,1],\phi_\alpha)\}_{\alpha\in\Delta}$ together with $M\times(0,1]$. We can take a partition of the unity subordinated to the last cover,

$$\{\chi_\alpha,\lambda\}_{\alpha\in\Delta}.$$

That is, we have the following properties:

- $0\leq\chi_\alpha,\lambda\leq 1$,
- $\operatorname{supp}\chi_\alpha\subset\mathcal{U}_\alpha\times[0,1]$ and $\operatorname{supp}\lambda\subset M\times(0,1]$,
- $\sum_\alpha\chi_\alpha+\sum\lambda=1$.

Let $f\in\mathcal{S}_c(\mathcal{D}_X^M)$. If we denote

$$f_\alpha := f|_{\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha}}\cdot(\chi_\alpha\circ p)\in C^\infty(\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha})$$

and

$$f_\lambda := f|_{M\times(0,1]}\cdot(\lambda\circ p)\in C^\infty(M\times(0,1]),$$

then we obtain the following decomposition:

$$f=\sum_\alpha f_\alpha+f_\lambda.$$

Now, since f is conic compactly supported we can suppose, without lost of generality, that

- $f_\lambda\in C_c^\infty(M\times(0,1])$, and
- that χ_α is compactly supported in $\mathcal{U}_\alpha\times[0,1]$.

What we conclude of all this, is that we can decompose our space $\mathcal{S}_c(\mathcal{D}_X^M)$ as follows:

$$\mathcal{S}_c(\mathcal{D}_X^M)=\sum_{\alpha\in\Lambda}\mathcal{S}_c(\mathcal{D}_{\mathcal{V}_\alpha}^{\mathcal{U}_\alpha})+C_c^\infty(M\times(0,1]). \tag{5}$$

As we mentioned in the introduction, we want to see the space $\mathcal{S}_c(\mathcal{D}_X^M)$ as a field of vector spaces over the interval $[0,1]$, where at zero we talk about Schwartz spaces. In our case we are interested in Schwartz functions on the vector bundle $\mathcal{N}_X^M$. Let us first recall the notion of the Schwartz space associated to a vector bundle.

Definition 4.6. Let (E,p,X) be a smooth vector bundle over a C^∞ manifold X. We define the Schwartz space $\mathcal{S}(E)$ as the set of C^∞ functions $g\in C^\infty(E)$ such that g is a Schwartz function at each fiber (uniformly) and g has compact support in the direction of X, *i.e.*, if there exists a compact subset $K\subset X$ such that $g(E_x)=0$ for $x\notin K$.

The vector space $\mathcal{S}(E)$ is an associative algebra with the product given as follows: for $f, g \in \mathcal{S}(E)$, we put

$$(f * g)(\xi) = \int_{E_{p(\xi)}} f(\xi - \eta) g(\eta) \, d\mu_{p(\xi)}(\eta), \tag{6}$$

where μ_ξ is a smooth Haar system of the Lie groupoid $E \rightrightarrows X$. A classical Fourier argument can be applied to show that the algebra $\mathcal{S}(E)$ is isomorphic to $(\mathcal{S}(E^*), \cdot)$ (punctual product). In particular this implies that $K_0(\mathcal{S}(E)) \cong K^0(E^*)$.

In the case we are interested in, we have a couple (M, X) and a vector bundle associated to it, that is, the normal bundle over X, $\mathcal{N}_X^M$. The reason why we gave the last definition is because we get evaluation linear maps

$$e_0 \colon \mathcal{S}_c(\mathcal{D}_X^M) \to \mathcal{S}(\mathcal{N}_X^M), \tag{7}$$

and

$$e_t \colon \mathcal{S}_c(\mathcal{D}_X^M) \to C_c^\infty(M) \tag{8}$$

for $t \neq 0$. Consequently, we have that the vector space $\mathcal{S}_c(\mathcal{D}_X^M)$ is a field of vector spaces over the closed interval $[0, 1]$ whose fiber spaces are $\mathcal{S}(\mathcal{N}_X^M)$ at $t = 0$ and $C_c^\infty(M)$ for $t \neq 0$.

Let us finish this subsection by giving the examples of spaces $\mathcal{S}_c(\mathcal{D}_X^M)$ corresponding to the DCN manifolds in Examples 3.5.

Examples 4.7. 1. For $X = \emptyset$, we have that $\mathcal{S}_c(\mathcal{D}_\emptyset^M) \cong C_c^\infty(M \times (0, 1])$.

2. For $X \subset M$ an open subset we have that $\mathcal{S}_c(\mathcal{D}_X^M) \cong C_c^\infty(W)$ where $W \subset M \times [0, 1]$ is the open subset consisting of the union of $X \times [0, 1]$ and $M \times (0, 1]$.

4.2 Schwartz type algebra for the tangent groupoid. In this section we define an algebra structure on $\mathcal{S}_c(\mathcal{G}^T)$. We start by defining a function $m_c \colon \mathcal{S}_c(\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}^{(2)}}) \to \mathcal{S}_c(\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}})$ by the following formulas:

For $F \in \mathcal{S}_c(\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}^{(2)}})$, we let

$$m_c(F)(x, \xi, 0) = \int_{T_x \mathcal{G}_x} F(x, \xi - \eta, \eta, 0) \, d\mu_x(\eta)$$

and

$$m_c(F)(\gamma, t) = \int_{\mathcal{G}_{s(\gamma)}} F(\gamma \circ \delta^{-1}, \delta, t) t^{-q} \, d\mu_{s(\gamma)}(\delta).$$

If we canonically identify $\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}^{(2)}}$ with $(\mathcal{G}^T)^{(2)}$, the map above is nothing else than the integration along the fibers of $m^T \colon (\mathcal{G}^T)^{(2)} \to \mathcal{G}^T$. We have the following proposition:

Proposition 4.8. *$m_c \colon \mathcal{S}_c((\mathcal{G}^T)^{(2)}) \to \mathcal{S}_c(\mathcal{G}^T)$ is a well-defined linear map.*

The interesting part of the proposition is that the map is well defined since it is evidently be linear. Let us suppose for the moment that the proposition is true. Under this assumption, we will define the product in $\mathscr{S}_c(\mathscr{G}^T)$.

Definition 4.9. Let $f, g \in \mathscr{S}_c(\mathscr{G}^T)$, we define a function $f * g$ in $\mathscr{G}^T$ by

$$(f * g)(x, \xi, 0) = \int_{T_x \mathscr{G}_x} f(x, \xi - \eta, 0) g(x, \eta, 0)\, d\mu_x(\eta)$$

and

$$(f * g)(\gamma, t) = \int_{\mathscr{G}_{s(\gamma)}} f(\gamma \circ \delta^{-1}, t) g(\delta, t) t^{-q}\, d\mu_{s(\gamma)}(\delta)$$

for $t \neq 0$.

We can enounce our main result.

Theorem 4.10. $*$ *defines an associative product on* $\mathscr{S}_c(\mathscr{G}^T)$.

Proof. Remember that we are assuming for the moment the validity of Proposition 4.8. Let $f, g \in \mathscr{S}_c(\mathscr{G}^T)$. We let $F := (f, g)$ be the function in $(\mathscr{G}^T)^{(2)}$ defined by

$$(f, g)(x, \xi, \eta, 0) = f(x, \xi, 0) \cdot g(x, \eta, 0)$$

and

$$(f, g)((\gamma, t), (\delta, t)) = f(\gamma, t) \cdot g(\delta, t)$$

for $t \neq 0$. Now, from the Leibnitz formula for the derivative of a product it is immediate that $(f, g) \in \mathscr{S}_c((\mathscr{G}^T)^{(2)})$. Finally, by definition we have that

$$m_c((f, g)) = f * g,$$

hence, by Proposition 4.8, $f * g$ is a well-defined element in $\mathscr{S}_c(\mathscr{G}^T)$.

For the associativity of the product, let us remark that when one restricts the product to $C_c^\infty(\mathscr{G}^T)$, this coincides with the product classically considered on $C_c^\infty(\mathscr{G}^T)$ (which is associative, see for example [22]). The associativity for $\mathscr{S}_c(\mathscr{G}^T)$ is proved exactly in the same way that for $C_c^\infty(\mathscr{G}^T)$. $\square$

It remains to prove Proposition 4.8. We are going to start locally. Let $U \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q$ be an open set and $V = U \cap \mathbb{R}^p \times \{0\} \times \{0\}$. Let $P : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$ the canonical projection $(x, \eta, \xi) \mapsto (x, \eta)$. We set $U' = P(U) \in \mathbb{R}^p \times \mathbb{R}^q$, then U' is also an open subset, $V \cong U' \cap \mathbb{R}^p \times \{0\}$ and $P|_V = \mathrm{Id}_V$. We denote also by P the restriction $P : U \to U'$. As in Lemma 3.2 we have a C^∞ map $\tilde{P} : \Omega_V^U \to \Omega_V^{U'}$ which in this case is explicitly written as

$$\tilde{P}(x, \eta, \xi, t) = (x, \eta, t).$$

We define $\tilde{P}_c : \mathscr{S}_c(\Omega_V^U) \to \mathscr{S}_c(\Omega_V^{U'})$ as follows:

$$\tilde{P}_c(F)(x, \eta, t) = \int_{\{\xi \in \mathbb{R}^q : (x, \eta, \xi, t) \in \Omega_V^U\}} F(x, \eta, \xi, t)\, d\xi.$$

Let us prove a lemma.

Lemma 4.11. $\tilde{P}_c : \mathcal{S}_c(\Omega_V^U) \to \mathcal{S}_c(\Omega_V^{U'})$ *is well defined.*

Proof. The first observation is that the integral in the definition of $\tilde{P}_c$ is always well defined. Indeed, this is a consequence of the following two facts:

- for $t = 0$, $\xi \mapsto F(x, \eta, \xi, 0) \in \mathcal{S}(\mathbb{R}^q)$,
- for $t \neq 0$, $\xi \mapsto F(x, \eta, \xi, t) \in C_c^\infty(\mathbb{R}^q)$.

Taking derivatives under the integral sign, we obtain that $\tilde{P}_c(F) \in C^\infty(\Omega_V^{U'})$. Then, we just have to show that $\tilde{P}_c(F)$ verifies the two conditions of Definition 4.1. For the first, if $K \subset U \times [0,1]$ is the compact conic support of F, then it is enough to put

$$K' = (P \times \mathrm{Id}_{[0,1]})(K)$$

in order to obtain a conic compact subset of $U' \times [0,1]$ relative to V and to check that K' is the compact conic support of $\tilde{P}_c(F)$. Let us now verify the condition (s_1). Let $k, m \in \mathbb{N}$, $l \in \mathbb{N}^p$ and $\beta \in \mathbb{N}^q$. We want to find $C_{(k,m,l,\beta)} > 0$ such that

$$(1 + \|\eta\|^2)^k \|\partial_x^l \partial_\eta^\beta \partial_t^m \tilde{P}_c(F)(x, \eta, t)\| \le C_{(k,m,l,\alpha)}.$$

For $k' \ge k + \frac{q}{2}$ and $\alpha = (0, \beta) \in \mathbb{R}^q \times \mathbb{R}^q$ we have by hypothesis that there exists $C'_{(k',m,l,\alpha)} > 0$ such that

$$\|\partial_x^l \partial_\eta^\beta \partial_t^m F(x, \eta, \xi, t)\| \le C' \frac{1}{(1 + \|(\eta, \xi)\|^2)^{k'}}.$$

Then we also have that

$$\begin{aligned} \|\partial_x^l \partial_\eta^\beta \partial_t^m \tilde{P}_c(F)(x, \eta, t)\| &\le C' \int_{\{\xi \in \mathbb{R}^q : (x,\eta,\xi,t) \in \Omega_V^U\}} \frac{1}{(1 + \|(\eta, \xi)\|^2)^{k'}} \, d\xi \\ &\le C' \frac{1}{(1 + \|\eta\|^2)^{\frac{q}{2} - k'}} \int_{\{\xi \in \mathbb{R}^q\}} \frac{1}{(1 + \|\xi\|^2)^{k'}} \, d\xi \le C \frac{1}{(1 + \|\eta\|^2)^k} \end{aligned}$$

with

$$C = C' \cdot \int_{\{\xi \in \mathbb{R}^q\}} \frac{1}{(1 + \|\xi\|^2)^{k'}} \, d\xi.$$

□

We can now give the proof of Proposition 4.8.

Proof of Proposition 4.8. Let us first fix some notation. We suppose $\dim \mathcal{G} = p + q$ and $\dim \mathcal{G}^{(0)} = p$, in particular this implies that $\dim \mathcal{G}^{(2)} = p + q + q$. Let $(\mathcal{U}, \phi)$ and $(\mathcal{U}', \phi')$ be $\mathcal{G}^{(0)}$-slices in $\mathcal{G}^{(2)}$ and $\mathcal{G}$, respectively, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{m} & \mathcal{U}' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ U & \xrightarrow[P]{} & U', \end{array}$$

where $P : \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^p \times \mathbb{R}^q$ is the canonical projection (as above) and $P(U) = U'$. This is possible since m is a surjective submersion. Now, we apply the DNC construction to the diagram above to obtain, thanks to the functoriality of the construction, the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}_{\mathcal{V}}^{\mathcal{U}} \arrow[r, "\mathcal{D}(m)"] \arrow[d, "\tilde{\phi}"'] & \mathcal{D}_{\mathcal{V}'}^{\mathcal{U}'} \arrow[d, "\tilde{\phi}'"] \\
\mathcal{D}_{V}^{U} \arrow[r, "\mathcal{D}(P)"'] & \mathcal{D}_{V'}^{U'} \\
\Omega_{\phi} \arrow[r, "\tilde{P}"'] \arrow[u, "\Psi_{\phi}"] & \Omega_{\phi'}, \arrow[u, "\Psi_{\phi'}"']
\end{tikzcd}$$

where Ψ_ϕ and $\tilde{\phi}$ are as in Section 3. Let $g \in \mathcal{S}_c(\mathcal{D}_V^U)$ and define

$$P_c(g)(x,\eta,t) = \begin{cases} \int_{\mathbb{R}^q} g(x,\eta,\xi,0)\, d\xi & \text{if } t = 0, \\ \\ \int_{\{\xi \in \mathbb{R}^q : (x,\eta,\xi,t) \in \Omega_V^U\}} g(x,\eta,\xi,t) t^{-q}\, d\xi & \text{if } t \neq 0. \end{cases}$$

Then, from the last commutative diagram, we get that

$$P_c(g) = \tilde{P}_c(g \circ \Psi_\phi) \circ (\Psi_{\phi'})^{-1},$$

hence, thanks to Lemma 4.11, we can conclude that we have a well-defined linear map

$$P_c : \mathcal{S}_c(\mathcal{D}_V^U) \to \mathcal{S}_c(\mathcal{D}_{V'}^{U'}).$$

We now use the Proposition 4.2 to write

$$\mathcal{S}_c(\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}) = \{h \in C^\infty(\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}) : h \circ \tilde{\phi}^{-1} \in \mathcal{S}_c(\mathcal{D}_V^U)\},$$

and so for $h \in \mathcal{S}_c(\mathcal{D}_{\mathcal{V}}^{\mathcal{U}})$ we see that

$$P_c(h \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}' \in \mathcal{S}_c(\mathcal{D}_{\mathcal{V}'}^{\mathcal{U}'}).$$

We use again the second commutative diagram to see that

$$m_c(h) = P_c(h \circ \tilde{\phi}^{-1}) \circ \tilde{\phi}'.$$

We then have a well-defined linear map

$$m_c : \mathcal{S}_c(\mathcal{D}_{\mathcal{V}}^{\mathcal{U}}) \to \mathcal{S}_c(\mathcal{D}_{\mathcal{V}'}^{\mathcal{U}'}).$$

To pass to the global case we only have to use the decomposition of $\mathcal{S}_c(\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}^{(2)}})$ and of $\mathcal{S}_c(\mathcal{D}_{\mathcal{G}^{(0)}}^{\mathcal{G}})$ as in (5), and of course the invariance under diffeomorphisms (Proposition 4.2). $\square$

Recall that we have well-defined evaluation morphisms as in (7) and (8). In the case of the tangent groupoid they are by definition morphisms of algebras. Hence, the algebra $\mathcal{S}_c(\mathcal{G}^T)$ is a field of algebras over the closed interval [0, 1], with associated fiber algebras,

$$\mathcal{S}(A\mathcal{G}) \text{ at } t = 0, \quad \text{and} \quad C_c^\infty(\mathcal{G}) \text{ for } t \neq 0.$$

It is very interesting to see what this means in the examples given in 3.9.

5 Further developments

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ be a Lie groupoid. In index theory for Lie groupoids the tangent groupoid has been used to define the analytic index associated to the group, as a morphism $K^0(A^*\mathcal{G}) \to K_0(C_r^*(\mathcal{G}))$ (see [19]) or as a KK-element in $KK(C_0(A^*\mathcal{G}), C_r^*(\mathcal{G}))$ (see [13]), this can be done because one has the following short exact sequence of C^*-algebras

$$0 \to C_r^*(\mathcal{G} \times (0,1]) \longrightarrow C_r^*(\mathcal{G}^T) \xrightarrow{e_0} C_0(A^*\mathcal{G}) \longrightarrow 0, \tag{9}$$

and because of the fact that the K-groups of the algebra $C_r^*(\mathcal{G} \times (0,1])$ vanish (homotopy invariance). The index defined at the C^*-level has proven to be very useful (see for example [9]) but extracting numerical invariants from it, with the existent tools, is very difficult. In non commutative geometry, and also in classical geometry, the tools for obtaining more explicit invariants are more developed for the 'smooth objects'; in our case this means the convolution algebra $C_c^\infty(\mathcal{G})$, where we can for example apply Chern–Weil–Connes theory. Hence, in some way, the indices defined in $K_0(C_c^\infty(\mathcal{G}))$ are more refined objects and for some cases it would be preferable to work with them. Unfortunately these indices are not good enough, since for example they are not homotopy invariant; in [8] Alain Connes discusses this and also other reasons why it is not enough to keep with the C_c^∞-indices. The main reason to construct the algebra $\mathcal{S}_c(\mathcal{G}^T)$ is that it gives an intermediate way between the C_c^∞-level and the C^*-level and will allow us in [5] to define another analytic index morphism associated to the groupoid, with the advantage that this index will take values in a group that allows to do pairings with cyclic cocycles and in general to apply Chern–Connes theory to it. The way we are going to define our index is by obtaining first a short exact sequence analogous to (9), that is, a sequence of the following kind

$$0 \to J \longrightarrow \mathcal{S}_c(\mathcal{G}^T) \xrightarrow{e_0} \mathcal{S}(A^*\mathcal{G})) \longrightarrow 0. \tag{10}$$

The problem here will be that we do not dispose of the advantages of the K-theory for C^*-algebras, since the algebras we are considering are not of this type (we do not have for example homotopy invariance).

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C^*-algebras associated with the $ax + b$-semigroup over $\mathbb{N}$

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1 Introduction

In this note we present a C^*-algebra (denoted by $\mathcal{Q}_{\mathbb{N}}$) which is associated to the $ax+b$-semigroup over $\mathbb{N}$. It is in fact a natural quotient of the C^*-algebra associated with this semigroup by some additional relations (which make it simple and purely infinite). These relations are satisfied in representations related to number theory. The study of this algebra is motivated by the construction of Bost–Connes in [1]. Our C^*-algebra contains the algebra considered by Bost–Connes, but in addition a generator corresponding to translation by the additive group $\mathbb{Z}$.

As a C^*-algebra, $\mathcal{Q}_{\mathbb{N}}$ has an interesting structure. It is a crossed product of the Bunce–Deddens algebra associated to $\mathbb{Q}$ by the action of the multiplicative semigroup $\mathbb{N}^\times$. It has a unique canonical KMS-state. We also determine its K-theory, whose generators turn out to be determined by prime numbers.

On the other hand, $\mathcal{Q}_{\mathbb{N}}$ can also be obtained as a crossed product of the commutative algebra of continuous functions on the completion $\hat{\mathbb{Z}}$ by the natural action of the $ax+b$-semigroup over $\mathbb{N}$. More interestingly, its stabilization is isomorphic to the crossed product of the algebra $\mathcal{C}_0(\mathbb{A}_f)$ of continuous functions on the space of finite adeles by the natural action of the $ax+b$-group $P_{\mathbb{Q}}^+$ over $\mathbb{Q}$. Somewhat surprisingly one obtains exactly the same C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ (up to stabilization) working with the completion $\mathbb{R}$ of $\mathbb{Q}$ at the infinite place and taking the crossed product by the natural action of the $ax+b$-group $P_{\mathbb{Q}}^+$ on $\mathcal{C}_0(\mathbb{R})$.

In the last section we consider the analogous construction of a C^*-algebra replacing the multiplicative semigroup $\mathbb{N}^\times$ by $\mathbb{Z}^\times$, i.e., omitting the condition of positivity on the multiplicative part of the $ax+b$-(semi)group. We obtain a purely infinite C^*-algebra $\mathcal{Q}_{\mathbb{Z}}$ which can be written as a crossed product of $\mathcal{Q}_{\mathbb{N}}$ by $\mathbb{Z}/2$. The fixed point algebra for its canonical one-parameter group (λ_t) is a dihedral group analogue of the Bunce–Deddens algebra. Its K-theory involves a shift of parity from K_0 to K_1 and vice versa. Its stabilization is isomorphic to the natural crossed product $\mathcal{C}_0(\mathbb{A}_f) \rtimes P_{\mathbb{Q}}$.

2 A canonical representation of the $ax + b$-semigroup

We denote by $\mathbb{N}$ the set of natural numbers including 0. $\mathbb{N}$ will normally be regarded as a semigroup with addition.

*Research supported by the Deutsche Forschungsgemeinschaft.

We denote by $\mathbb{N}^\times$ the set of natural numbers excluding 0. $\mathbb{N}^\times$ will normally be regarded as a semigroup with multiplication.

The natural analogue, for a semigroup S, of an unitary representation of a group is a representation of S by isometries, i.e. by operators s_g, $g \in S$ on a Hilbert space that satisfy $s_g^* s_g = 1$.

On the Hilbert space $\ell^2(\mathbb{N})$ consider the isometries s_n, $n \in \mathbb{N}^\times$ and v^k, $k \in \mathbb{N}$ defined by

$$v^k(\xi_m) = \xi_{m+k}, \quad s_n(\xi_m) = \xi_{mn}$$

where $\xi_m, m \in \mathbb{N}$ denotes the standard orthonormal basis.

We have $s_n s_m = s_{nm}$ and $v^n v^m = v^{n+m}$, i.e. s and v define representations of $\mathbb{N}^\times$ and $\mathbb{N}$ respectively, by isometries. Moreover we have the following relation

$$s_n v^k = v^{nk} s_n$$

which expresses the compatibility between multiplication and addition.

In other words, the s_n and v^k define a representation of the $ax + b$-semigroup

$$P_{\mathbb{N}} = \left\{ \begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \mid n \in \mathbb{N}^\times,\ k \in \mathbb{N} \right\}$$

over $\mathbb{N}$, where the matrix $\left(\begin{smallmatrix} 1 & k \\ 0 & n \end{smallmatrix}\right)$ is represented by $(v^k s_n)^*$.

Note that, for each n, the operators $s_n, vs_n, \dots, v^{n-1}s_n$ generate a C^*-algebra isomorphic to $\mathcal{O}_n$, [4].

3 A purely infinite simple C^*-algebra associated with the $ax + b$-semigroup

The C^*-algebra A generated by the elements s_n and v considered in Section 2 contains the algebra $\mathcal{K}$ of compact operators on $\ell^2(\mathbb{N})$. Denote by u, resp. u^k the image of v, resp. v^k in the quotient $A/\mathcal{K}$. We also still denote by s_n the image of s_n in the quotient. Then the u^k are unitary and are defined also for $k \in \mathbb{Z}$ and they furthermore satisfy the characteristic relation $\sum_{k=0}^{n-1} u^k e_n u^{-k} = 1$ where $e_n = s_n s_n^*$ denotes the range projection of s_n. This relation expresses the fact that $\mathbb{N}$ is the union of the sets of numbers which are congruent to k mod n for $k = 0, \dots, n-1$.

We now consider the universal C^*-algebra generated by elements satisfying these relations.

Definition 3.1. We define the C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ as the universal C^*-algebra generated by isometries s_n, $n \in \mathbb{N}^\times$ with range projections $e_n = s_n s_n^*$, and by a unitary u satisfying the relations

$$s_n s_m = s_{nm}, \quad s_n u = u^n s_n, \quad \sum_{k=0}^{n-1} u^k e_n u^{-k} = 1$$

for $n, m \in \mathbb{N}^\times$.

Lemma 3.2. *In $\mathcal{Q}_\mathbb{N}$ we have*

(a) $e_n = \sum_{i=0}^{m-1} u^{in} e_{nm} u^{-in}$ *for all* $n, m \in \mathbb{N}^\times$*;*

(b) $e_p s_q = e_{pq} s_p = s_q e_p$ *and* $e_p e_q = e_{pq} = e_q e_p$ *when p and q are relatively prime;*

(c) $s_n^* s_m = s_m s_n^*$ *for n, m relatively prime.*

Proof. (a) This follows by conjugating the identity $1 = \sum u^i e_m u^{-i}$ by $s_n \sqcup s_n^*$ and using the fact that $s_n e_m s_n^* = e_{nm}$.

(b) Since the $u^i e_{pq} u^{-i}$ are pairwise orthogonal for $0 \le i < pq$, we see that $u^{lp} e_{pq} u^{-lp} \perp u^{kq} e_{pq} u^{-kq}$ if $0 < l < q, 0 < k < p$. Thus, using (a),

$$e_p s_q = \sum_{l=0}^{q-1} u^{lp} e_{pq} u^{-lp} \sum_{k=0}^{p-1} u^{kq} e_{pq} u^{-kq} s_q = e_{pq} s_q.$$

This obviously implies $e_p e_q = e_{pq}$ and, by symmetry $e_q e_p = e_{pq}$. In particular, e_p and e_q commute.

(c) Using (b) we get

$$s_p (s_p^* s_q) s_q^* = e_p e_q = e_{pq} = s_{pq} s_{pq}^* = s_p (s_q s_p^*) s_q^*.$$

Since s_p, s_q are isometries this implies that the expressions in parentheses on the left and right hand side are equal. Here we have, in a first step, assumed that p, q are prime. However any s_n is a product of s_p's with p prime. □

From Lemma 3.2 (a) it follows that any two of the projections $u^i e_n u^{-i}$ commute. We denote by D the commutative subalgebra of $\mathcal{Q}_\mathbb{N}$ generated by all these projections.

We also denote by $\mathcal{F}$ the subalgebra of $\mathcal{Q}_\mathbb{N}$ generated by u and the projections e_n, $n \in \mathbb{N}^\times$.

To analyze the structure of $\mathcal{Q}_\mathbb{N}$ further we write it as an inductive limit of the subalgebras B_n generated by $s_{p_1}, s_{p_2}, \dots, s_{p_n}$ and u, where $p_1, p_2, \dots, p_n$ denote the n first prime numbers. Each B_n contains a natural (maximal) commutative subalgebra D_n generated by all projections of the form $u^k e_m u^{*k}$ where m is a product of powers of the $p_1, \dots, p_n$ (i.e. a natural number that contains only the p_i as prime factors).

Lemma 3.3. *The spectrum* Spec D_n *of D_n can be identified canonically with the compact space*

$$\{0, \dots, p_1 - 1\}^\mathbb{N} \times \dots \times \{0, \dots, p_n - 1\}^\mathbb{N} \cong \hat{\mathbb{Z}}_{p_1} \times \dots \times \hat{\mathbb{Z}}_{p_n}$$

Proof. D_n is the inductive limit of the subalgebras $D_n^{(k)} \cong \mathbb{C}^{l_k}$ with $l_k = p_1^k p_2^k \dots p_n^k$. The algebra $D_n^{(k)}$ is generated by the pairwise orthogonal projections $u^i e_{l_k} u^{-i}, 0 \le i < l_k$ and in fact, by Lemma 3.2 (a), $D_n^{(k)}$ is the k-fold tensor product of $D_n^{(1)}$ by itself. □

Consider the action of $\mathbb{T}^n$ on B_n given by

$$\alpha_{(t_1,\dots,t_n)}(s_{p_i}) = t_i s_{p_i}$$

and denote by $\mathcal{F}_n$ the fixed-point algebra for α (i.e. $\mathcal{F}_n = \mathcal{F} \cap B_n$). There is a natural faithful conditional expectation $E\colon B_n \to \mathcal{F}_n$ defined by $E(x) = \int_{\mathbb{T}^n} \alpha_t(x)dt$.

Now D_n is the fixed point algebra, in $\mathcal{F}_n$, for the action β of $\mathbb{T}$ on $\mathcal{F}_n$ given by

$$\beta_t(e_k) = e_k, \quad \beta_t(u) = e^{it}u$$

and there is an associated expectation $F\colon \mathcal{F}_n \to D_n$ defined by $F(x) = \int_{\mathbb{T}} \beta_t(x)dt$. The composition $G = F \circ E$ gives a faithful conditional expectation $A_n \to D_n$. These conditional expectations extend to the inductive limit and thus give conditional expectations $E\colon \mathcal{Q}_{\mathbb{N}} \to \mathcal{F}$, $F\colon \mathcal{F} \to \mathcal{D}$ and $G\colon \mathcal{Q}_{\mathbb{N}} \to \mathcal{D}$.

Theorem 3.4. *The C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite.*

Proof. Since inductive limits of purely infinite simple C^*-algebras are purely infinite simple [8], 4.1.8 (ii), it suffices to show that each B_n is purely infinite simple.

For each N, denote, as above, by $D_n^{(N)}$ the subalgebra of D_n generated by $\{u^k e_l u^{-k},\ k \in \mathbb{Z}\}$ where $l = p_1^N p_2^N \dots p_n^N$. The natural map Spec $D_n \to$ Spec $D_n^{(N)}$ is surjective and, by the proof of Lemma 3.3, $D_n^{(N)} \cong \mathbb{C}^l$.

Choose $\xi_1, \xi_2, \dots, \xi_l$ in Spec D_n such that $\{\xi_1, \xi_2, \dots, \xi_l\} \to$ Spec $D_n^{(N)}$ is bijective and such that

$$\xi_i(s_k \sqcup s_k^*) \neq \xi_i(\sqcup)$$

for all k of the form $k = p_1^{m_1} \dots p_n^{m_n}$ with $m_i \leq N$. We can choose pairwise orthogonal projections $f_1, f_2, \dots, f_l$ in D_n with sufficiently small support around the ξ_i such that $f_i s_k f_i = 0$ for all $1 \leq i \leq l$ and k as above and such that $f_i x f_i = \xi_i(x) f_i$ for all $x \in D_n^N$. Then the map $\varphi\colon D_n^{(N)} \to C^*(f_1, \dots, f_l) \cong \mathbb{C}^l$ defined by $x \mapsto \sum f_i x f_i$ is an isomorphism.

We denote by $\mathcal{P}^{(N)}$ the set of linear combinations of all products of the form $u^k s_{p_1}^{m_1} \dots s_{p_n}^{m_n}$ with $m_i \leq N$. For $x \in \mathcal{P}^{(N)}$ we have that

$$\varphi(G(x)) = \sum f_i x f_i.$$

Let now $0 \leq x \in B_n$ be different from 0. Since G is faithful, $G(x) \neq 0$ and we normalize x such that $\|G(x)\| = 1$. Let $y \in \mathcal{P}^{(N)}$, for sufficiently large N, be such that $\|x - y\| < \varepsilon \leq 1$. We may also assume that $\|G(y)\| = 1$. Then there exists i_0, $1 \leq i_0 \leq l$ such that $f_{i_0} y f_{i_0} = f_{i_0}$. Moreover, there exists an isometry $s \in B_n$ (of the form $s = u^k s_m$) such that $s^* f_{i_0} s = 1$.

In conclusion, we have $s^* y s = 1$ and

$$\|s^*xs - 1\| = \|s^*xs - s^*ys\| = \|s^*(x - y)s\| < \varepsilon.$$

This shows that s^*xs is invertible. $\square$

As a consequence of the simplicity of $\mathcal{Q}_{\mathbb{N}}$ we see that the canonical representation on $\ell^2(\mathbb{Z})$ by

$$s_n(\xi_k) = \xi_{nk}, \quad u^n(\xi_k) = \xi_{k+n}$$

is faithful. Similarly, if we divide the C^*-algebra generated by the analogous isometries on $\ell^2(\mathbb{N})$, discussed in Section 2, by the canonical ideal $\mathcal{K}$, we get an algebra isomorphic to $\mathcal{Q}_{\mathbb{N}}$.

The subalgebra $\mathcal{F}$ is generated by u and the projections e_n, thus by a weighted shift. Thus we recognize $\mathcal{F}$ as the Bunce–Deddens algebra of the type where every prime appears with infinite multiplicity. This well known algebra has been introduced in [3] in exactly that form. It is simple and has a unique tracial state. It can also be represented as inductive limit of the inductive system $(M_n\mathcal{C}(S^1))$ with maps

$$M_n(\mathcal{C}(S^1)) \longrightarrow M_{nk}(\mathcal{C}(S^1))$$

mapping the unitary u generating $\mathcal{C}(S^1)$ to the $k \times k$-matrix

$$\begin{pmatrix} 0 & 0 & \dots & & u \\ 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ & & \dots & & \\ 0 & \dots & & 1 & 0 \end{pmatrix}.$$

From this description it immediately follows that $K_0(\mathcal{F}) = \mathbb{Q}$ and $K_1(\mathcal{F}) = \mathbb{Z}$.

Remark 3.5. (a) Just as $\mathcal{O}_n$ is a crossed product of a UHF-algebra by $\mathbb{N}$, we see that $\mathcal{Q}_{\mathbb{N}}$ is a crossed product $\mathcal{F} \rtimes \mathbb{N}^\times$ by the multiplicative semigroup $\mathbb{N}^\times$. The algebra $\mathcal{Q}_{\mathbb{N}}$ also contains the commutative subalgebra D. Since D is the inductive limit of the D_n, we see from Lemma 3.3 that Spec $D = \hat{\mathbb{Z}} := \prod_p \hat{\mathbb{Z}}_p$. The Bost–Connes algebra $C_{\mathbb{Q}}$ [1] can be described as a crossed product $\mathcal{C}(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^\times$ and is a natural subalgebra of $\mathcal{Q}_{\mathbb{N}}$ (using the natural inclusion $\mathcal{C}(\hat{\mathbb{Z}}) \to \mathcal{F}$).

We can also obtain $\mathcal{Q}_{\mathbb{N}}$ by adding to the generators $\mu_n, n \in \mathbb{N}$ (our s_n) and e_γ, $\gamma \in \mathbb{Q}/\mathbb{Z}$ for $C_{\mathbb{Q}}$, described in [1], Proposition 18, one additional unitary generator u satisfying

$$ue_\gamma = \gamma e_\gamma u \quad \text{and} \quad u^n \mu_n = \mu_n u$$

(here we identify an element γ of $\mathbb{Q}/\mathbb{Z}$ with the corresponding complex number of modulus 1).

(b) The algebra generated by s_n (for a single n) and u has also been considered in [5]. It has been shown there that this algebra is simple and contains a Bunce–Deddens algebra.

4 The canonical action of $\mathbb{R}$ on $\mathcal{Q}_{\mathbb{N}}$

So far, in our discussion, the isometries associated with each prime number appear as generators on the same footing and it is a priori not clear how to determine, for two prime numbers p and q, the size of p and q (or even the bigger number among p and q) from the corresponding generators s_p and s_q. In fact, the C^*-algebra generated by the s_n, $n \in \mathbb{N}$ is the infinite tensor product of one Toeplitz algebra for each prime number and in this C^*-algebra there is no way to distinguish the s_p for different p.

However, the fact that we have added u to the generators allows to retrieve the n from s_n using the KMS-condition.

Definition 4.1. Let (A, λ_t) be a C^*-algebra equipped with a one-parameter automorphism group (λ_t), τ a state on A and $\beta \in (0, \infty]$. We say that τ satisfies the β-KMS-condition with respect to (λ_t), if for each pair x, y of elements in A, there is a holomorphic function $F_{x,y}$, continuous on the boundary, on the strip $\{z \in \mathbb{C} \mid \operatorname{Im} z \in [0, \beta]\}$ such that

$$F_{x,y}(t) = \tau(x\lambda_t(y)), \quad F_{x,y}(t + i\beta) = \tau(\lambda_t(y)x)$$

for $t \in \mathbb{R}$.

Proposition 4.2. *Let τ_0 be the unique tracial state on $\mathcal{F}$ and define τ on $\mathcal{Q}_\mathbb{N}$ by $\tau = \tau_0 \circ E$. Let (λ_t) denote the one-parameter automorphism group on $\mathcal{Q}_\mathbb{N}$ defined by $\lambda_t(s_n) = n^{it}s_n$ and $\lambda_t(u) = u$. Then τ is a 1-KMS-state for (λ_t).*

Proof. According to [2] 5.3.1, it suffices to check that

$$\tau(x\lambda_i(y)) = \tau(yx)$$

for a dense *-subalgebra of analytic vectors for (λ_t). Here (λ_z) denotes the extension to complex variables $z \in \mathbb{C}$ of (λ_t) on the set of analytic vectors.

However it is immediately clear that this identity holds for x, y linear combinations of elements of the form as_n or s_m^*b, $a, b \in \mathcal{F}$. Such linear combinations are analytic and dense. □

Theorem 4.3. *There is a unique state τ on $\mathcal{Q}_\mathbb{N}$ with the following property.*

There exists a one-parameter automorphism group $(\lambda_t)_{t\in\mathbb{R}}$ for which τ is a 1-KMS-state and such that $\lambda_t(u) = u$, $\lambda_t(e_n) = e_n$ for all n and t. Moreover we have

- *τ is given by $\tau = \tau_0 \circ E$ where τ_0 is the canonical trace on $\mathcal{F}$,*
- *the one-parameter group for which τ is 1-KMS, is unique and is the standard automorphism group considered above, determined by*

$$\lambda_t(s_n) = n^{it}s_n \quad \text{and} \quad \lambda_t(u) = u.$$

Proof. Since λ_t acts as the identity on e_n and on u, it is the identity on $\mathcal{F} = C^*(u, \{e_n\})$. If τ is a KMS-state for (λ_t), it therefore has to be a trace on $\mathcal{F}$. It is well-known (and clear) that there is a unique trace τ_0 on $\mathcal{F}$. For instance, the relation

$$\sum_{0\le k<n} u^k e_n u^{-k} = 1$$

shows that $\tau_0(e_n) = 1/n$ and $\tau_0(u) = 0$.

The relation

$$\tau(s_n^* s_n) = n\tau(s_n s_n^*)$$

and the 1-KMS-condition show that $\lambda_t(s_n) = n^{it} s_n$. $\square$

5 The K-groups of $\mathcal{Q}_{\mathbb{N}}$

The K-groups of $\mathcal{Q}_{\mathbb{N}}$ can be computed using the fact that $\mathcal{Q}_{\mathbb{N}}$ is a crossed product of the Bunce–Deddens algebra $\mathcal{F}$ by the semigroup $\mathbb{N}^\times$. The K-groups of a Bunce–Deddens algebra are well known and easy to determine using its representation as an inductive limit of algebras of type $M_n(\mathcal{C}(S^1))$. Specifically, for $\mathcal{F}$, we have

$$K_0(\mathcal{F}) = \mathbb{Q}, \quad K_1(\mathcal{F}) = \mathbb{Z}.$$

We consider $\mathcal{Q}_{\mathbb{N}}$ as an inductive limit of the subalgebras $B_n = C^*(\mathcal{F}, s_{p_1}, \ldots, s_{p_n})$ where $2 = p_1 < p_2 < \cdots$ is the sequence of prime numbers in natural order. Now B_{n+1} can be considered as a crossed product of B_n by the action of the semigroup $\mathbb{N}$ given by conjugation by $s_{p_{n+1}}$ (in fact B_{n+1} is Morita equivalent to a crossed product by $\mathbb{Z}$ of an algebra Morita equivalent to B_n), just as in [4]. Thus the K-groups of B_n can be determined inductively using the Pimsner–Voiculescu sequence.

Theorem 5.1. *The K-groups of B_n are given by*

$$K_0(B_n) \cong \mathbb{Z}^{2^{n-1}}, \quad K_1(B_n) \cong \mathbb{Z}^{2^{n-1}}.$$

Proof. The first application of the Pimsner–Voiculescu sequence gives an exact sequence

$$\cdots \longrightarrow K_1(B_1) \longrightarrow \mathbb{Q} \xrightarrow{\mathrm{id}-\alpha_*} \mathbb{Q} \longrightarrow K_0(B_1) \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{id}-\alpha_*} \mathbb{Z} \longrightarrow K_1(B_1) \longrightarrow \cdots$$

where α_* is the map induced by $\mathrm{Ad}\, s_1$ on K_0, K_1.

Since $\alpha_* = 2$ on $K_0(\mathcal{F})$ and $\alpha_* = 1$ on $K_1(\mathcal{F})$ this gives $K_0(B_1) = \mathbb{Z}$, $K_1(B_1) = \mathbb{Z}$. In the following steps $\alpha_i = \mathrm{Ad}\, s_{p_i}$ induces 1 on K_0 and on K_1. Thus each consecutive prime p_i doubles the number of generators of K_0 and K_1. $\square$

Remark 5.2. It follows that the groups $K_0(\mathcal{Q}_{\mathbb{N}})$ and $K_1(\mathcal{Q}_{\mathbb{N}})$ are free abelian with infinitely many generators. By construction, the elements of $K_0(\mathcal{Q}_{\mathbb{N}})$ are obtained as Bott projections from the "commuting" elements u and $s_{p_1}, \ldots, s_{p_{2n-1}}$ where $p_1, \ldots p_{2n-1}$ is any choice of an odd number of different prime numbers. Similarly, the elements of $K_1(\mathcal{Q}_{\mathbb{N}})$ are obtained as a K-theory product of such elements in K_0 with one additional $s_{p_{2n}}$. Therefore the generators of $K_*(\mathcal{Q}_{\mathbb{N}})$ are labeled by the square free numbers of the form $p_1 p_2 \ldots p_k$ with $p_1, \ldots, p_k$ distinct primes.

6 Representations as crossed products

For each n, define the endomorphism φ_n of $\mathcal{Q}_{\mathbb{N}}$ by $\varphi_n(x) = s_n x s_n^*$. Since $\varphi_n\varphi_m = \varphi_{nm}$ this defines an inductive system.

Definition 6.1. We define $\overline{\mathcal{Q}}_{\mathbb{N}}$ as the inductive limit of the inductive system $(\mathcal{Q}_{\mathbb{N}}, \varphi_n)$.

By construction we have a family ι_n of natural inclusions of $\mathcal{Q}_{\mathbb{N}}$ into the inductive limit $\overline{\mathcal{Q}}_{\mathbb{N}}$ satisfying the relations $\iota_{nm}\varphi_n = \iota_m$. We denote by 1_k the element $\iota_k(1)$ of $\overline{\mathcal{Q}}_{\mathbb{N}}$. We have that $1_k = \iota_{kl}(e_l) \le 1_{kl}$. The union of the subalgebras $1_k\overline{\mathcal{Q}}_{\mathbb{N}}1_k$ is dense in $\overline{\mathcal{Q}}_{\mathbb{N}}$ and $1_k \le 1_{kl}$ for all k, l. In order to define a multiplier a of $\overline{\mathcal{Q}}_{\mathbb{N}}$ it therefore suffices to define $a1_k$ and $1_k a$ for all k.

We can extend the isometries s_n naturally to unitaries $\bar{s}_n$ in the multiplier algebra $\mathcal{M}(\overline{\mathcal{Q}}_{\mathbb{N}})$ of $\overline{\mathcal{Q}}_{\mathbb{N}}$ by requiring

$$\bar{s}_n 1_k = \iota_k(s_n), \quad 1_k \bar{s}_n = \iota_{kn}(s_n).$$

Note that this is well defined because, using the identities $s_n e_l = e_{nl} s_n$ and $\varphi_l(s_n) = s_n e_l$, we have

$$(1_k\bar{s}_n)1_l = \iota_{kn}(s_n)\iota_l(1) = \iota_{knl}(s_n e_l e_{kn}) = \iota_{knl}(e_{nl} s_n e_{kn}) = \iota_k(1)\iota_l(s_n) = 1_k(\bar{s}_n 1_l).$$

Proposition 6.2. *The elements $\bar{s}_n$, $n \in \mathbb{N}$, define unitaries in $\mathcal{M}(\overline{\mathcal{Q}}_{\mathbb{N}})$ such that $\bar{s}_n\bar{s}_m = \bar{s}_{nm}$ and such that $\bar{s}_n\bar{s}_m^* = \bar{s}_m^*\bar{s}_n$.*

Defining $\bar{s}_a = \bar{s}_n\bar{s}_m^*$ for $a = n/m \in \mathbb{Q}_+^\times$ we define unitaries in $\mathcal{M}(\overline{\mathcal{Q}}_{\mathbb{N}})$ such that $\bar{s}_a\bar{s}_b = \bar{s}_{ab}$ for $a, b \in \mathbb{Q}_+^\times$.

Proof. In order to show that $\bar{s}_n$ is unitary it suffices to show that $1_k\bar{s}_n\bar{s}_n^* = 1_k$ and $1_k\bar{s}_n^*\bar{s}_n = 1_k$ for all k, n. □

We can also extend the generating unitary u in $\mathcal{Q}_{\mathbb{N}}$ to a unitary in the multiplier algebra $\mathcal{M}(\overline{\mathcal{Q}}_{\mathbb{N}})$. We define the unitary $\bar{u}$ in $\mathcal{M}(\overline{\mathcal{Q}}_{\mathbb{N}})$ by the identity

$$\bar{u}\ 1_k = \iota_k(u^k).$$

We can also define fractional powers of $\bar{u}$ by setting

$$\bar{u}^{1/n}\ 1_{kn} = \iota_{kn}(u^k).$$

Proposition 6.3. *For all $a \in \mathbb{Q}_+^\times$ and $b \in \mathbb{Q}$ we have the identity*

$$\bar{s}_a\bar{u}^b = \bar{u}^{ab}\bar{s}_a.$$

Proof. Check that $\bar{s}_{1/n}\bar{u} = \bar{u}^{1/n}\bar{s}_{1/n}$ □

Following Bost–Connes we denote by $P_{\mathbb{Q}}^+$ the $ax+b$-group

$$P_{\mathbb{Q}}^+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Q}_+^\times,\ b \in \mathbb{Q} \right\}.$$

It follows from the previous proposition that we have a representation of $P_{\mathbb{Q}}^+$ in the unitary group of $\mathcal{M}(\bar{\mathcal{Q}}_{\mathbb{N}})$. Denote by $\mathbb{A}_f$ the locally compact space of finite adeles over $\mathbb{Q}$, i.e.,

$$\mathbb{A}_f = \{(x_p)_{p\in\mathcal{P}} \mid x_p \in \hat{\mathbb{Q}}_p \text{ and } x_p \in \hat{\mathbb{Z}}_p \text{ for almost all } p\}$$

where $\mathcal{P}$ is the set of primes in $\mathbb{N}$.

The canonical commutative subalgebra D of $\mathcal{Q}_{\mathbb{N}}$ is by the Gelfand transform isomorphic to $\mathcal{C}(X)$ where X is the compact space $\hat{\mathbb{Z}} = \prod_{p\in\mathcal{P}} \hat{\mathbb{Z}}_p$. It is invariant under the endomorphisms φ_n and we obtain an inductive system of commutative algebras (D, φ_n). The inductive limit $\bar{D}$ of this system is a canonical commutative subalgebra of $\bar{\mathcal{Q}}_{\mathbb{N}}$. It is isomorphic to $\mathcal{C}_0(\mathbb{A}_f)$. In fact the spectrum of $\bar{D}$ is the projective limit of the system (Spec $D, \hat{\varphi}_n$) and φ_n corresponds to multiplication by n on $\hat{\mathbb{Z}}$.

Theorem 6.4. *The algebra $\bar{\mathcal{Q}}_{\mathbb{N}}$ is isomorphic to the crossed product of $\mathcal{C}_0(\mathbb{A}_f)$ by the natural action of the $ax+b$-group $P_{\mathbb{Q}}^+$.*

Proof. Denote by B the crossed product and consider the projection $e \in \mathcal{C}_0(\mathbb{A}_f) \subset B$ defined by the characteristic function of the maximal compact subgroup $\prod_{p\in\mathcal{P}} \hat{\mathbb{Z}}_p \subset \mathbb{A}_f$. Consider also the multipliers $\bar{s}_n^*$ and $\bar{u}^*$ of B defined as the images of the elements $\left(\begin{smallmatrix}1 & 0\\ 0 & n\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ of $P_{\mathbb{Q}}^+$, respectively. Then the elements $s_n = \bar{s}_n e$ and $u = \bar{u}e$ satisfy the relations defining $\mathcal{Q}_{\mathbb{N}}$. Moreover, the C^*-algebra generated by the s_n and u contains the subalgebra $e\mathcal{C}_0(\mathbb{A}_f)$ of B. This shows that this subalgebra equals eBe and, by simplicity of $\mathcal{Q}_{\mathbb{N}}$, it follows that $eBe \cong \mathcal{Q}_{\mathbb{N}}$. □

It is a somewhat surprising fact that we get exactly the same C^*-algebra, even together with its canonical action of $\mathbb{R}$ if we replace the completion $\mathbb{A}_f$ of $\mathbb{Q}$ at the finite places by the completion $\mathbb{R}$ at the infinite place.

Theorem 6.5. *The algebra $\bar{\mathcal{Q}}_{\mathbb{N}}$ is isomorphic to the crossed product of $\mathcal{C}_0(\mathbb{R})$ by the natural action of the ax+b-group $P_{\mathbb{Q}}^+$. There is an isomorphism $\bar{\mathcal{Q}}_{\mathbb{N}} \to \mathcal{C}_0(\mathbb{R}) \rtimes P_{\mathbb{Q}}^+$ which carries the natural one parameter group λ_t to a one parameter group λ'_t such that $\lambda'_t(fu_g) = a^{-it} fu_g$ where u_g denotes the unitary multiplier of $\mathcal{C}_0(\mathbb{R}) \rtimes P_{\mathbb{Q}}^+$ associated with an element $g \in P_{\mathbb{Q}}^+$ of the form*

$$g = \begin{pmatrix}1 & b\\ 0 & a\end{pmatrix}$$

(*the KMS-state τ is carried to the state which extends Lebesgue measure on $\mathcal{C}_0(\mathbb{R})$*).

In order to prove this we need a little bit of preparation concerning the representation of the Bunce–Deddens algebra $\mathcal{F}$ as an inductive limit. As noted above it is an inductive limit of the inductive system $(M_n(\mathcal{C}(S^1)))$ with maps sending the unitary generator z of $\mathcal{C}(S^1)$ to a unitary v in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ satisfying $v^k = 1 \otimes z$. We observe now that such a v is unique up to unitary equivalence.

Lemma 6.6. *Let v_1, v_2 be two unitaries in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ such that $v_1^k = v_2^k = 1 \otimes z$. Then there is a unitary w in $M_k(\mathbb{C}) \otimes \mathcal{C}(S^1)$ such that $v_2 = wv_1w^*$.*

Proof. The spectral projections $p_i(t)$ for the different $k-th$ roots of $e^{2\pi it}1$, given by $v_1(t)$ and $v_2(t)$, have to be continuous functions of t. This implies that all the $p_i(t)$ are one-dimensional and that, after possibly relabelling, we must have the situation where

$$p_i(t+1) = p_{i+1}(t).$$

This means that the p_i combine to define a line bundle on the k-fold covering of S^1 by S^1. However any two such line bundles are unitarily equivalent. □

We now determine the crossed product $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}$ where $\mathbb{Q}$ acts by translation.

Lemma 6.7. *The crossed product $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}$ is isomorphic to the stabilized Bunce–Deddens algebra $\mathcal{K} \otimes \mathcal{F}$.*

Proof. The algebra $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}$ is an inductive limit of algebras of the form $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}$ with respect to the maps

$$\beta_k \colon \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z} \longrightarrow \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}$$

obtained from the embeddings $\mathbb{Z} \cong \mathbb{Z}\frac{1}{k} \hookrightarrow \mathbb{Q}$. It is well known that $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}$ is isomorphic to $\mathcal{K} \otimes \mathcal{C}(S^1)$. An explicit isomorphism is obtained from the map

$$\sum_{n\in\mathbb{Z}} f_n u^n \longmapsto \sum_{n\in\mathbb{Z},k\in\mathbb{Z}} \tau_k(f_n) e_{k,k+n}$$

where τ_k denotes translation by k, u^k denotes the unitary in the crossed product implementing this automorphism and e_{ij} denote the matrix units in $\mathcal{K} \cong \mathcal{K}(\ell^2\mathbb{Z})$.

This map sends $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}$ to the mapping torus algebra

$$\{f \in \mathcal{C}(\mathbb{R}, \mathcal{K}) \mid f(t+1) = Uf(t)U^*\} \cong \mathcal{K} \otimes \mathcal{C}(S^1)$$

where U is the multiplier of $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{Z}))$ given by the bilateral shift on $\ell^2(\mathbb{Z})$.

A projection p corresponding, under this isomorphism, to $e \otimes 1$ in $\mathcal{K} \otimes \mathcal{C}(S^1)$ with e a projection of rank 1 can be represented in the form $p = ug + f + gu^*$ with appropriate positive functions f and g with compact support on $\mathbb{R}$. Under the map $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z} \cong \mathcal{C}_0(\mathbb{R}) \rtimes k\mathbb{Z} \to \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}$, the projection p is mapped to $p' = u^k g_k + f_k + g_k u^{*k}$ where $g_k(t) := g(t/k)$, $f_k(t) := f(t/k)$. Now, p' corresponds to a projection of rank k in $\mathcal{K} \otimes \mathcal{C}(S^1)$.

Let z be the unitary generator of $\mathcal{C}(S^1)$. Then the element $e \otimes z$ corresponds the function $e^{2\pi it}(ug + f + gu^*)$ which is mapped to $v = e^{2\pi it/k}(u^k g_k + f_k + g_k u^{*k})$. Thus $v^k = p'$ and p' corresponds to a projection of rank k in $\mathcal{K} \otimes \mathcal{C}(S^1)$.

On the other hand $\mathcal{F} = \varinjlim A_n$ where $A_n = M_n(\mathcal{C}(S^1))$ and the inductive limit is taken relative to the maps $\alpha_k \colon M_n(\mathcal{C}(S^1)) \to M_{kn}(\mathcal{C}(S^1))$ which map the unitary generator z of $\mathcal{C}(S^1)$ to an element v such that $v^k = 1 \otimes z$. Compare this now to the inductive system $A'_n = \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Z}\frac{1}{n}$ with respect to the maps β_k considered above.

From our analysis of β_k and from Lemma 6.6 we conclude that there are unitaries W_k in $\mathcal{M}(A'_{kn})$ such that the following diagram commutes

$$\begin{tikzcd} A_n \arrow[r, "\alpha_k"] \arrow[d] & A_{kn} \arrow[d] \\ A'_n \arrow[r, "\mathrm{Ad}\, W_k \beta_k"] & A'_{kn} \end{tikzcd}$$

where the vertical arrows denote the natural inclusions $M_n(\mathcal{C}(S^1)) \to \mathcal{K} \otimes \mathcal{C}(S^1)$.

We conclude that these natural inclusions induce an isomorphism from $\mathcal{K} \otimes \mathcal{F} = \varinjlim \mathcal{K} \otimes A_n$ to $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q} = \varinjlim A'_n$. $\square$

Proof of Theorem 6.5. Consider the commutative diagram and the injection $\mathcal{F} \hookrightarrow \mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}$ constructed at the end of the proof of Lemma 6.7. Note also that the β_k are σ-unital and therefore extend to the multiplier algebra. This shows that the injection transforms α_p into β_p times an approximately inner automorphism, for each prime p. Therefore the injection transforms λ_t, which is the dual action on the crossed product for the character $n \mapsto n^{it}$ of $\mathbb{N}^\times$, to the restriction of the corresponding dual action on the crossed product $(\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes \mathbb{N}^\times$.

Finally, note that the endomorphisms β_n of $\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}$ are in fact automorphisms and that therefore β extends from a semigroup action to an action of the group $\mathbb{Q}_+^\times$. This shows that $(\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes \mathbb{N}^\times = (\mathcal{C}_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes \mathbb{Q}_+^\times = \mathcal{C}_0(\mathbb{R}) \rtimes P_{\mathbb{Q}}^+$. $\square$

Remark 6.8. If we consider the full space of adeles $\mathbb{Q}_{\mathbb{A}} = \mathbb{A}_f \times \mathbb{R}$ rather than the finite adeles $\mathbb{A}_f$ or the completion at the infinite place $\mathbb{R}$, we obtain the following situation. By [9], IV, §2, Lemma 2, $\mathbb{Q}$ is discrete in $\mathbb{Q}_{\mathbb{A}}$ and the quotient is the compact space $(\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z}$. Thus, the crossed product $\mathcal{C}_0(\mathbb{Q}_{\mathbb{A}}) \rtimes \mathbb{Q}$ which would be analogous to the Bunce–Deddens algebra is Morita equivalent to $\mathcal{C}((\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z})$. Now, $(\hat{\mathbb{Z}} \times \mathbb{R})/\mathbb{Z}$ has a measure which is invariant under the action of the multiplicative semigroup $\mathbb{N}^\times$. Therefore the crossed product $\mathcal{C}_0(\mathbb{Q}_{\mathbb{A}}) \rtimes P_{\mathbb{Q}}^+$ has a trace and is not isomorphic to $\mathcal{Q}_{\mathbb{N}}$.

7 The case of the multiplicative semigroup $\mathbb{Z}^\times$

We consider now the analogous C^*-algebras where we replace the multiplicative semigroup $\mathbb{N}^\times$ by the semigroup $\mathbb{Z}^\times$. This case is also important in view of generalizations from $\mathbb{Q}$ to more general number fields (in fact the construction of $\mathcal{Q}_{\mathbb{Z}}$ and much of the analysis of its structure carries over to the ring of integers in an arbitrary algebraic number field, at least if the class number is 1). Thus, on $\ell^2(\mathbb{Z})$ we consider the isometries $s_n, n \in \mathbb{Z}^\times$ and the unitaries $u^m, m \in \mathbb{Z}$ defined by $s_n(\xi_k) = \xi_{nk}$ and $u^m(\xi_k) = \xi_{k+m}$. As above, these operators satisfy the relations

$$s_n s_m = s_{nm}, \quad s_n u^m = u^{nm} s_n, \quad \sum_{i=0}^{n-1} u^i e_n u^{-i} = 1. \tag{1}$$

The operator s_{-1} plays a somewhat special role and we therefore denote it by f. Then the $s_n, n \in \mathbb{Z}$ and u generate the same C^*-algebra as the $s_n, n \in \mathbb{N}$ together with u and f. The element f is a selfadjoint unitary so that $f^2 = 1$ and we have the relations

$$fs_n = s_n f, \quad fuf = u^{-1}.$$

We consider now again the universal C^*-algebra generated by isometries $s_n, n \in \mathbb{Z}$ and a unitary u subject to the relations (1). We denote this C^*-algebra by $\mathcal{Q}_\mathbb{Z}$. We see from the discussion above that we get a crossed product $\mathcal{Q}_\mathbb{Z} \cong \mathcal{Q}_\mathbb{N} \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by the automorphism α of $\mathcal{Q}_\mathbb{N}$ that fixes the $s_n, n \in \mathbb{N}$ and $\alpha(u) = u^{-1}$.

Theorem 7.1. *The algebra $\mathcal{Q}_\mathbb{Z}$ is simple and purely infinite.*

Proof. Composing the conditional expectation $G\colon \mathcal{Q}_\mathbb{N} \to \mathcal{D}$ used in the proof of Theorem 3.4 with the natural expectation $\mathcal{Q}_\mathbb{Z} = \mathcal{Q}_\mathbb{N} \rtimes \mathbb{Z}/2 \to \mathcal{Q}_\mathbb{N}$ we obtain again a faithful expectation $G'\colon \mathcal{Q}_\mathbb{Z} \to \mathcal{D}$. The rest of the proof follows exactly the proof of Theorem 3.4, using in addition the fact that the f_i in that proof can be chosen such that $f_i f f_i = 0$. □

Denote by $P_\mathbb{Q}$ the full $ax+b$-group over $\mathbb{Q}$, i.e.

$$P_\mathbb{Q} = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Q}^\times,\ b \in \mathbb{Q} \right\}.$$

Theorem 7.2. *$\mathcal{Q}_\mathbb{Z}$ is isomorphic to the crossed product of $\mathcal{C}_0(\mathbb{A}_f)$ or of $\mathcal{C}_0(\mathbb{R})$ by the natural action of $P_\mathbb{Q}$.*

Proof. This follows from Theorems 6.4 and 6.5 since $P_\mathbb{Q} = P_\mathbb{Q}^+ \rtimes \mathbb{Z}/2$. □

On $\mathcal{Q}_\mathbb{Z}$ we can define the one-parameter group (λ_t) by $\lambda_t(s_n) = n^{it} s_n, n \in \mathbb{N}^\times$. The fixed point algebra is the crossed product $\mathcal{F} \rtimes \mathbb{Z}/2$ of the Bunce–Deddens algebra by $\mathbb{Z}/2$.

In order to compute the K-groups of $\mathcal{Q}_\mathbb{Z}$ we first determine the K-theory for $\mathcal{F}' = \mathcal{F} \rtimes \mathbb{Z}/2$. This algebra is the inductive limit of the subalgebras $A'_n = C^*(u, f, e_n)$.

Lemma 7.3. (a) *The C^*-algebra $C^*(u, f)$ is isomorphic to $C^*(D)$, where D is the dihedral group $D = \mathbb{Z} \rtimes \mathbb{Z}/2$ ($\mathbb{Z}/2$ acts on $\mathbb{Z}$ by $a \mapsto -a$). For each $n = 1, 2, \ldots,$ the algebra A'_n is isomorphic to $M_n(C^*(D))$.*

(b) *We have $K_0(A'_n) = K_0(C^*(D)) = \mathbb{Z}^3$ and $K_1(A'_n) = K_1(C^*(D)) = 0$ for all n. The generators of $K_0(C^*(D))$ are given by* [1] *and by the classes of the spectral projections $(uf)^+$ and f^+ of uf and f, for the eigenvalue 1.*

(c) *Let p be prime. If $p = 2$, then the map $K_0(A'_n) \to K_0(A'_{pn})$ is described, with respect to the basis* [1], $[(uf)^+]$ *and* $[f^+]$ *of $K_0(C^*(D)) \cong \mathbb{Z}^3$, by the matrix*

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

If p is odd, then the map $K_0(A'_n) \to K_0(A'_{pn})$ is described by the matrix

$$\begin{pmatrix} p & \frac{p-1}{2} & \frac{p-1}{2} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. (a) It is clear that the universal algebra generated by two unitaries u, f satisfying $f^2 = 1$ and $fuf = u^*$ is isomorphic to $C^*(D)$.

In the decomposition of $C^*(u, f, e_2)$ with respect to the orthogonal projections e_2 and ue_2u^{-1}, the elements u and f correspond to matrices

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_0 & 0 \\ 0 & f_1 \end{pmatrix}$$

where w is unitary and f_0, f_1 are symmetries (selfadjoint unitaries).

The relations between u and f imply that $wf_1 = f_0$ and that wf_1 is a selfadjoint unitary, whence $f_1wf_1 = w^*$. Thus A'_2 is isomorphic to $M_2(C^*(w, f_1))$ and w, f_1 satisfy the same relations as u, f.

If p is an odd prime, then in the decomposition of $C^*(u, f, e_p)$ with respect to the pairwise orthogonal projections $e_p, ue_pu^{-1}, \dots, u^{p-1}e_pu^{-(p-1)}$, u and f correspond to the following $p \times p$-matrices

$$\begin{pmatrix} 0 & 0 & \dots & & w \\ 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ & & \dots & & \\ 0 & \dots & & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_0 & 0 & \dots & & 0 \\ 0 & 0 & \dots & & f_1^* \\ 0 & 0 & \dots & f_2^* & 0 \\ & & \dots & & \\ 0 & 0 & f_2 & \dots & \\ 0 & f_1 & \dots & 0 & 0 \end{pmatrix}$$

where w and the $f_1, \dots, f_{\frac{p-1}{2}}$ are unitary and f_0 is a symmetry.

The relation $fuf = u^*$ implies that $f_0 = wf_1$, $f_1 = f_2 = \dots = f_{\frac{p-1}{2}}$ and $f_i^2 = 1$ for all i. From $(wf_1)^2 = 1$ we derive that $f_1wf_1 = w^*$. Thus $A'_p \cong M_p(C^*(w, f_1))$ and w, f_1 satisfy the same relations as u, f.

The case of general n is obtained by iteration using the fact that $C^*(u, f, e_{nm}) \cong M_m(C^*(u, f, e_n))$.

(b) This follows for instance from the well known fact that $C^*(D) \cong (\mathbb{C} \star \mathbb{C})^\sim$.

(c) Let $p = 2$. Using the description of $K_0(C^*(u, f))$ under (b) and the description of the map $C^*(u, f) \to C^*(u, f, e_2)$ from the proof of (a), we see that the generators $[1]$, $[(uf)^+]$ and $[f^+]$ are respectively mapped to $2[1]$, $[1]$ and $[f^+] + [(uf)^+]$.

Let p be an odd prime. The matrix corresponding to f in the proof of (a) is conjugate to the matrix where all the f_i, $i = 1, \dots, (p-1)/2$ are replaced by 1. Thus the class of f^+ is mapped to $[(wf_1)^+] + \frac{p-1}{2}[1]$.

The matrix corresponding to uf is conjugate to a selfadjoint matrix of the same form as the matrix representing the image of f, but with the upper left entry f_0 replaced by f_1. Thus the class of $(uf)^+$ is mapped to $[f_1^+] + \frac{p-1}{2}[1]$.

$\square$

Proposition 7.4. *The K-groups of the algebra $\mathcal{F}' = \mathcal{F} \rtimes \mathbb{Z}/2$ are given by $K_0(\mathcal{F}') = \mathbb{Q} \oplus \mathbb{Z}$ and $K_1(\mathcal{F}') = 0$.*

Proof. This follows from the fact that $\mathcal{F}'$ is the inductive limit of the algebras A'_n and the description of the maps $K_0(A'_n) \to K_0(A'_{pn})$ given in Lemma 7.3 (c). The element $[f_+] + [uf_+] - [1]$ is invariant under the map $K_0(A'_n) \to K_0(A'_{pn})$ and maps to the generator of $\mathbb{Z}$ in the inductive limit. □

Remark 7.5. Since $\mathcal{F}'$ is a simple C^*-algebra with unique trace in a classifiable class (in the sense of the classification program), this computation of its K-theory in combination with the classification result in [6] shows that $\mathcal{F}'$ is an AF-algebra. Thus we obtain an example of a simple AF-algebra $\mathcal{F}'$ admitting an action by $\mathbb{Z}/2$ such that the fixed point algebra is the Bunce–Deddens algebra $\mathcal{F}$ and thus not AF. This example had been considered before in [7].

In analogy to the case over $\mathbb{N}$ we denote by B'_n the C^*-subalgebra of $\mathcal{Q}_\mathbb{Z}$ generated by $u, f, s_{p_1}, \dots, s_{p_n}$. We now immediately deduce

Theorem 7.6. *We have*

$$K_0(B'_n) = \mathbb{Z}^{2^{n-1}}, \quad K_1(B'_n) = \mathbb{Z}^{2^{n-1}}$$

and

$$K_0(\mathcal{Q}_\mathbb{Z}) = \mathbb{Z}^\infty, \qquad K_1(\mathcal{Q}_\mathbb{Z}) = \mathbb{Z}^\infty.$$

Note added in proof. The arguments in the computation of the K-theory are not complete. This problem will be addressed in a subsequent paper.

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On a class of Hilbert C*-manifolds

Wend Werner

1 Introduction

This note had initially two objectives, an explicit calculation, for all vector fields, of the invariant connection on a certain type of infinite dimensional symmetric space and, using results from [1], to characterize those invariant cone fields on a similar kind of spaces that can be thought of as the result of some kind of 'quantization'. Both questions are related since the invariance of the cone fields is intimately connected to the behavior of parallel transport along geodesics.

Invariant connections for finite dimensional symmetric spaces have long been known to exist and to be unique. One has to be a little bit more careful in the infinite dimensional, Banach manifold setting since there the existence of a sufficient amount of smooth functions no longer can be proven. The type of calculations we are interested in here have been carried out under similar circumstances in [2].

Important for our approach is to use an invariant Hilbert C*-structure on the fibers of the tangent bundle. We show that the symmetric space we are dealing with can be defined in terms of the automorphism group of this structure. For the underlying invariant (operator) Finsler structure, the analogous result holds.

The theory of invariant cone fields on finite dimensional spaces is very well developed. A comprehensive account is [5]. We will see in the last section that the infinite dimensional theory might behave slightly differently.

2 Hilbert C*-manifolds

2.1. Recall that a (left) Hilbert C*-module over a C*-algebra $\mathfrak{A}$ is a complex vector space E which is a left $\mathfrak{A}$-module with a sesquilinear pairing $E \times E \to \mathfrak{A}$ satisfying, for $r, s \in E$ and $a \in \mathfrak{A}$, the following requirements:

(i) $\langle ar, s\rangle = a\langle r, s\rangle$;

(ii) $\langle r, s\rangle = \langle s, r\rangle^*$;

(iii) $\langle s, s\rangle > 0$ for $s \neq 0$;

(iv) equipped with the norm
$$\|s\| = \sqrt{\|\langle s, s\rangle\|},$$
E is a Banach space.

Right Hilbert C*-modules are defined similarly. Whenever we want to refer to the algebra $\mathfrak{A}$ explicitly, we speak of a Hilbert $\mathfrak{A}$-module.

2.2. The objects defined above coincide with the so called ternary rings of operators (TRO), which are intrinsically characterized in [9], [11]. On such a space E, a triple product $\{\cdot,\cdot,\cdot\}$ is given in such a way that E, up to (the obvious definition of) TRO-isomorphisms, is a subspace of a space of bounded Hilbert space operators $L(H)$, invariant under the triple product

$$\{x, y, z\} = xy^*z.$$

The relation to (left) Hilbert C*-modules is based on the equation

$$\{x, y, z\} = \langle x, y\rangle z,$$

connecting triple product to module action as well as scalar product of a Hilbert $\mathfrak{A}$-module. Here, $\mathfrak{A}$ is the algebra of linear mappings

$$\mathfrak{A} = EE^* = \overline{\operatorname{lin}}\{x \mapsto \{e_1, e_2, x\} \mid e_{1,2} \in E\},$$

which is independent of the chosen embedding. Note that the norm of an element $e \in E$ must coincide with $\|e\| = \|\{e, e, e\}\|^{1/3}$.

2.3. TRO-morphisms will be those mappings that preserve the product $\{\cdot,\cdot,\cdot\}$. These mappings differ in general from what is considered to be the natural choice for Hilbert $\mathfrak{A}$-morphisms, the so called adjointable maps. The latter are in particular $\mathfrak{A}$-module morphisms, a property that would be too restrictive for our purposes. We will hence stick in the following to TRO-morphisms.

Definition 2.4. Let M be a Banach manifold and $\mathfrak{A}$ a C*-algebra. M is said to be a (*right-, left-*) *Hilbert* $\mathfrak{A}$*-manifold* if on each tangent space $T_p(M)$ there is given the structure of a Hilbert $\mathfrak{A}$-module depending smoothly on base points.

Definition 2.5. Let M be a Hilbert C*-manifold. The group of automorphisms, Aut M consists of all diffeomorphisms $\Phi\colon M \to M$ so that $d\Phi$ is (pointwise) a TRO-morphism.

2.6. It is not clear under which circumstances a Banach manifold can be given the structure of a Hilbert C*-manifold. This is so because, first, no characterization seems to be known of Banach spaces that are (topologically linear) isomorphic to Hilbert C*-modules, and, second, because in general, there is no smooth partition of the unit that would permit the step from local to global.

We will use group actions instead. This will bring up homogeneous Hilbert C*-manifolds in the following sense.

Definition. A Banach manifold M is called *homogeneous* Hilbert C*-manifold iff it is a Hilbert C*-manifold for which Aut M acts transitively.

2.7. Suppose M is a homogeneous space with respect to a smooth Banach Lie group action. Fix a base point $o \in M$, and denote the isotropy subgroup at o by H. Suppose that $T_o(M)$ carries the structure of a Hilbert $\mathfrak{A}$-module with form $\langle\cdot,\cdot\rangle_o$, module map $x \mapsto a \cdot_o x$, and TRO-structure $\{x, y, z\}_o = \langle x, y\rangle_o \cdot_o z$. If for all $h \in H$, $x, y, z \in T_o(M)$ we have

$$d_o h\{x, y, z\}_o = \{d_o h(x), d_o h(y), d_o h(z)\}_o$$

then a Hilbert $\mathfrak{A}$-module on $T_p(M)$ for any $p = g(o) \in M$, is well-defined through

$$\{x, y, z\}_p = d_o g\{d_p g^{-1}(x), d_p g^{-1}(y), d_p g^{-1}(z)\}_o.$$

With this structure, M becomes a homogeneous Hilbert C*-manifold

2.8. Our example here is the following. Fix a TRO E and denote by U its open unit ball. If we follow the path laid out above, we find the following invariant Hilbert C*-structure on U. Define a triple product for $T_a M$ at $a \in U$ by

$$\{x, y, z\}_a = x(1 - a^*a)^{-1} y^*(1 - aa^*)^{-1} z,$$

so that

$$\langle x, y\rangle_a = (1 - aa^*)^{-1/2} x(1 - a^*a)^{-1} y^*(1 - aa^*)^{-1/2}$$

as well as

$$\gamma \cdot_a z = (1 - aa^*)^{1/2}\gamma(1 - aa^*)^{-1/2}, \quad \gamma \in EE^*.$$

We will refer to this structure as the *canonical Hilbert C*-structure* on U.

2.9. This definition is motivated in the following way. As shown in [4], $\operatorname{Hol} U$, the group of all biholomorphic automorphisms U, consists of mappings of the form $T \circ M_a$, where for any $a \in U$,

$$M_a(x) := (1 - aa^*)^{-1/2}(x + a)(1 + a^*x)^{-1}(1 - a^*a)^{1/2},$$

and T is a (linear) isometry of E, restricted to U. Then $\operatorname{Hol} U$ acts transitively on U, and the group of (linear) isometries is the isotropy subgroup at the point 0. For later use, we include here the fact that

$$d_x M_a(h) = (1 - aa^*)^{1/2}(1 + xa^*)^{-1} h(1 + a^*x)^{-1}(1 - a^*a)^{1/2}$$

as well as

$$\begin{aligned} &d_x^2 M_a(h_1, h_2) \\ &\quad = -(1 - aa^*)^{1/2}(1 + xa^*)^{-1} h_2 a^*(1 + xa^*) h_1 (1 + a^*x)^{-1}(1 - a^*a)^{1/2} \\ &\qquad - (1 - aa^*)^{1/2}(1 + xa^*)^{-1} h_1 (1 + a^*x)^{-1} a^* h_2 (1 + a^*x)^{-1}(1 - a^*a)^{1/2}. \end{aligned}$$

The Hilbert C*-structure from 2.8 is in fact constructed according to the construction in 2.6 for a group G, somewhat smaller than $\operatorname{Hol} U$.

Definition 2.10. Suppose X is a (closed) subspace of $L(H)$. Equip

$$M_n(X) = \{(x_{ij}) \mid x_{ij} \in X \text{ for } i, j = 1, \dots, n\}$$

with the norm it carries as a subspace of $M_n(L(H)) = L(H \oplus \cdots \oplus H)$, and, for a bounded operator $T\colon X \to X$, denote by $T^{(n)} = \mathrm{id}_{M_n(\mathbb{C})} \otimes X\colon M_n(X) \to M_n(X)$ the operator $(x_{ij}) \longmapsto (Tx_{ij})$. Then T is said to be completely bounded, iff

$$\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|T^{(n)}\| < \infty.$$

Similarly, T is called a complete isometry iff each of the maps $T^{(n)}$ is an isometry.

We have the following

Theorem 2.11 ([3], [9]). *Let E be a TRO. Then $M_n(E)$ carries a distinguished TRO-structure, and the group of TRO-automorphisms of E coincides with the group of complete isometries.*

Theorem 2.12. *Let U be the open unit ball of a TRO E, equipped with the canonical Hilbert C*-structure, and suppose $M_n(E)$ carries the standard TRO-structure for each $n \in \mathbb{N}$. Then a diffeomorphism $\Phi\colon U \to U$ is a Hilbert C*-automorphism iff $\Phi = T \circ M_a$, where $a \in U$, and T is the restriction of a linear and completely isometric mapping of E to U.*

Proof. That each map of the form $T \circ M_a$ is in $\operatorname{Aut} U$ follows from the construction of the Hilbert C*-structure on U as well as from Theorem 2.11. Since the derivative of any element $\Phi \in \operatorname{Aut} U$ is complex linear by definition, $\operatorname{Aut} U \subseteq \operatorname{Hol} U$, and hence $\Phi = T \circ M_a$ for some $a \in U$ and an isometry T. Because the linear map T fixes the origin, dT must be a TRO-automorphism, and the result follows, by another application of Theorem 2.11. $\square$

2.13. Let M again be a Hilbert C*-manifold. Then the tangent space at each point m of M carries an essentially unique norm, naturally connected to the TRO-structure by $\|x\|_m = \|\{x, x, x\}_m\|^{1/3}$. Since TROs embed completely isometrically into some space of bounded Hilbert space operators, this norm naturally extends to the spaces $M_n(T_m M)$. We will call a Banach manifold M an *operator Finsler manifold* in case each tangent space carries the structure of an operator space depending continuously on base points. In this sense, any Hilbert C*-manifold carries an operator Finsler structure in a natural way. If, as before, U is the open unit ball of a fixed TRO $(E, \{\cdot,\cdot,\cdot\}, \|\cdot\|)$, furnished with its invariant Hilbert C*-structure then, using that complete isometries and automorphisms of a TRO are the same mappings, we have

Theorem. *If the open unit ball of a TRO is equipped with the natural invariant operator Finsler structure its automorphism groups coincides with the automorphism group of the underlying homogeneous Hilbert C*-manifold.*

3 The invariant connection

3.1. The definition of a connection for Banach manifolds cannot be, due to the scarcity of smooth functions, the usual one. We follow [6], 1.5.1 Definition.

Definition 3.2. Let M be a manifold, modeled over the Banach space E, and denote the space of bounded bilinear mappings $E \times E \to E$ by $L^2(E, E)$. Then M is said to possess a connection iff there is an atlas $\mathcal{U}$ for M so that for each $U \in \mathcal{U}$ there is a smooth mapping $\Gamma\colon U \to L^2(E, E)$, called the *Christoffel symbol* of the connection on U, which under a change of coordinates Φ transforms according to

$$\Gamma(\Phi' X, \Phi' Y) = \Phi''(X, Y) + \Phi' \Gamma(X, Y).$$

The covariant derivative of a vector field Y in the direction of the vector field X is, locally, defined to be the principal part of

$$\nabla_X Y = dX(Y) - \Gamma(X, Y),$$

where, in a chart, the principal part of $(u, X) \in U \times T(U)$ is X.

The reader should note that this definition is equivalent to specifying a smooth vector subbundle H of TTM with the property that H_p is for each $p \in TM$ closed and complementary to the tangent space $\ker(d_p\pi)$ of the fiber $E_{\pi(p)}$ through p. It is not, however, equivalent to the requirement that ∇ be $C^\infty(M)$-linear in its first variable and a derivation w.r.t. the action of $C^\infty(M)$ on vector fields, although connections as defined above do have this property.

3.3. We can keep the notion of invariance of a connection under the smooth action of a (Banach) Lie group, however. In fact, if such a group G acts on M then, for each $g \in G$ a connection $g^*\nabla$ is defined by letting

$$g^*\nabla_X Y = \nabla_{g^*X} g^* Y, \quad g^*X(gm) = d_m g X(m).$$

Christoffel symbols then transform as in the definition above,

$$\Gamma_{g(m)}(g'(m)X(m), g'(m)Y(m)) = g''(m)(X(m), Y(m)) + g'(m)\Gamma_m(X(m), Y(m)),$$

and we call ∇ invariant under the action of G whenever $g^*\nabla = \nabla$ for all $g \in G$.

3.4. If M is the Hilbert C*-manifold U that was defined in the last section, there can be at most one connection which is invariant under the action of the group of biholomorphic self-maps of U. This is because the difference of two of them is the difference of their Christoffel symbols which have to vanish at each point according to the way they transform under the reflections $\sigma_a = M_a \sigma_o M_{-a}$, $\sigma_o(x) = -x$. To find one, we let

$$\Gamma_o(x, y) = 0.$$

Then, for any g leaving the origin fixed, $g'' = 0$ and so Γ_0 remains zero when transformed under g. Using the transformation rule for Christoffel symbols we find

Theorem 3.5. *On the Hilbert C*-manifold U there exists exactly one invariant connection whose Christoffel symbol at a is given by*

$$\begin{aligned}\Gamma_a(x,y) &= y(1-a^*a)^{-1/2}a^*(1-aa^*)^{-1/2}x \\ &\quad + x(1-a^*a)^{-1/2}a^*(1-aa^*)^{-1/2}y = 2\{x, M_a'(0)a, y\}_a^s,\end{aligned}$$

where $\{x,y,z\}^s = \frac{1}{2}(\{z,y,x\} + \{x,y,z\})$ *denotes the* (*symmetric*) *Jordan triple product on* E.

3.6. The reader should observe that the form $\{\cdot,\cdot,\cdot\}$ defined above is parallel for the invariant connection, i.e. for all vector fields X, Y, Z and W we have

$$\nabla_W\{X,Y,Z\} = \{\nabla_W XYZ\} + \{X\nabla_W YZ\} + \{XY\nabla_W Z\}.$$

This follows either from direct calculation (using the Jacobi identity for $\{\cdot,\cdot,\cdot\}$), or from the fact that the covariant derivative of $\{\cdot,\cdot,\cdot\}$ is invariant under reflections. This condition shows that ∇ behaves like the Levi-Civitá connection with respect to the Hilbert C*-structure on M. Under what conditions such a connection exists under more general circumstances, is investigated in [10].

4 Causality

4.1. It is customary in a number of physical theories to pass from a Lorentzian to a Riemannian manifold (‘Wick Rotation’). This is due to the difficulties one is faced with in a truly Lorentzian situation and which disappear in the Riemannian set-up. One peculiarity of Lorentzian manifolds can still be modeled in the Riemannian situation: Light cones in the fibers of the tangent bundle which specify those pairs of points on M that may interact with each other and are thus intimately connected to the notion of causality.

Definition 4.2. A cone field on a manifold M is, for each $m \in M$, an assignment of a cone $C_m \subseteq T_m(M)$.

Here a cone is always supposed to be pointed but not necessarily to be generating.

4.3. Our interest here is with cone fields that are invariant under the action of a group G, i.e. that $C_{gm} = d_m g(C_m)$ for all $m \in M$ and $g \in G$. In our case, G will be a certain subgroup of $\mathrm{Aut}\, U$. Though it might be tempting to see invariance under $\mathrm{Hol}\, U$ (or a large subgroup) as a substitute for diffeomorphism invariance in the complex case, the main motivation here comes from the property that the connection defined in the previous section has the property that for an invariant cone field (C_m) parallel transport along a curve γ from $\gamma(a)$ to $\gamma(b)$ ends in $C_{\gamma(b)}$ if it began in $C_{\gamma(a)}$. This is well known in the finite dimensional situation (see [7], [8]) and, in this regard, not much is changing in passing to infinite dimensions.

4.4. In the following we will suppose that 'space' is represented by a certain set D of self-adjoint operators, and that on this set there is defined an invariant cone field. The question we would like to answer is this: How can such fields be characterized, which come from interpreting the points of D as bounded Hilbert space operators?

In the following, denote by E a fixed (abstract) ternary ring of operators, and by U its unit ball.

4.5. On top of the causal structure of U we need 'selfadjointness', which for us will be the existence of a '*real form*' for U, compatible with the (almost) complex structure. We do this by requiring that E carries an involutory real automorphism '*' so that $(ix)^* = -ix^*$ for all $x \in E$. Since we will be studying TRO-embeddings into $L(H)$ that preserve the real form, it will be necessary to impose the additional condition that, for all $x, y, z \in E$, $\{x, y, z\}^* = \{z^*, y^*, x^*\}$. (Note that then each $x \in E$ has a unique decomposition into real and imaginary parts, with some norm estimates.) A ternary ring of operators E that meets all these conditions, will be called a *-ternary ring of operators. We will suppose in the following that E is a space of this kind.

4.6. The 'space manifold' here will be the open unit ball U_{sa} of the selfadjoint part of E. U_{sa} is itself a symmetric space. If G_{sa} consists of all elements in $\operatorname{Aut} U$ which leave U_{sa} invariant, then $U_{\text{sa}} = G_{\text{sa}}/H_{\text{sa}}$, where H_{sa} comprises the TRO-automorphisms that are *-selfadjoint. In fact, it follows from $\{x, y, z\}^* = \{z^*, y^*, x^*\}$ for all $x, y, z \in E$ (and an expansion into power series) that $M_a(x)^* = M_{a^*}(x^*)$ for all $x \in U$ and so $M_a \in G_{\text{sa}}$ iff $a^* = a$. A TRO-automorphism T is in H_{sa} iff $T(x^*)^* = T(x)$ for all $x \in U$, and so

$$G_{\text{sa}} = \{T \circ M_a \mid T \in H_{\text{sa}}, a \in U_{\text{sa}}\},$$

as well as $U_{\text{sa}} = G_{\text{sa}}/H_{\text{sa}}$.

4.7. In order to comply with the requirement that causality be invariant under parallel transport we have to impose the condition that the field of cones we fix in TU_{sa} must be invariant under the action of G_{sa}. We consider smooth embeddings $\Phi\colon U \to L(H)$ which preserve

- the Hilbert C*-structure,
- the complex structure as well as the (canonical) real forms,
- the action of the automorphism groups.

And we want to know: What characterizes the G_{sa} invariant cone fields that are pulled back to U via Φ? Whenever a cone field meets these properties we will call it *natural*.

4.8. Since a natural cone field is supposed to be invariant under the action of G_{sa}, we may restrict our attention to cones in $T_oU = E$. Furthermore, any cone in E that gives rise to a G_{sa} invariant field of cones has to be invariant under the action of H_{sa}. It can also be shown that under the assumptions made above, $d\Phi$ has to preserve the ternary structure of each tangent space T_pU. The question we were asking thus becomes: What properties must an H_{sa}-invariant cone in E_{sa} possess so that it is of the form $\Psi^{-1}(L(H)_+)$ for a *-ternary monomorphism $\Psi\colon E \to L(H)$? For the sake of simplicity, we restrict our attention to the case where E is a dual Banach space. Then

Theorem 4.9 ([1]). *Let E be a *-ternary ring of operators, which is a dual Banach space. Define the center of E by*

$$Z(E) = \{e \in E \mid exy = xye \text{ for all } x, y \in E\}$$

and call $u \in E$ tripotent whenever $\{u, u, u\} = u$. Then a cone $C \subseteq E_{\mathrm{sa}}$ is natural iff there is a central, selfadjoint tripotent element $u \in E$ so that

$$C = C_u := \{eue^* \mid e \in E\}.$$

Definition 4.10. If E is a TRO with real form *, and $u \in E_{\mathrm{sa}}$, then u is called *rigid* iff $\Phi(u) = u$ for all *-selfadjoint TRO-automorphisms Φ of E.

The following result is now easy to prove.

Theorem 4.11. *The only cones C_u in E that give rise to a G_{sa}-invariant causal structure on U_{sa}, coming from an embedding of E into some space $L(H)$, are those for which the central, selfadjoint element u is rigid.*

Note that the existence of a rigid selfadjoint central tripotent u is impossible in finite dimensions. In fact, $Z(E)$ is a weak*-closed commutative von Neumann algebra (see Lemma 4.10 in [1]) in which the tripotents are projections, which is invariant under any TRO-automorphism Φ of E, and for which $\Phi|_{Z(E)}$ is a C*-automorphism whose properties essential here are best understood by means of the underlying homeomorphism of the spectrum of $Z(E)$.

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Duality for topological abelian group stacks and T-duality

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Contents

1 Introduction

1.1 A sheaf theoretic version of Pontrjagin duality

1.1.1 A character of an abelian topological group G is a continuous homomorphism

$$\chi\colon G \to \mathbb{T}$$

from G to the circle group $\mathbb{T}$. The set of all characters of G will be denoted by $\hat{G}$. It is again a group under point-wise multiplication.

Assume that G is locally compact. The compact-open topology on the space of continuous maps $\mathsf{Map}(G, \mathbb{T})$ induces a compactly generated topology on this space of maps, and hence a topology on its subset $\hat{G} \subseteq \mathsf{Map}(G, \mathbb{T})$. The group $\hat{G}$ equipped with this topology is called the dual group of G.

1.1.2 An element $g \in G$ gives rise to a character $\mathsf{ev}(g) \in \hat{\hat{G}}$ defined by $\mathsf{ev}(g)(\chi) := \chi(g)$ for $\chi \in \hat{G}$. In this way we get a continuous homomorphism

$$\mathsf{ev}\colon G \to \hat{\hat{G}}.$$

The main assertion of Pontrjagin duality is

Theorem 1.1 (Pontrjagin duality). *If G is a locally compact abelian group, then $\mathsf{ev}\colon G \to \hat{\hat{G}}$ is an isomorphism of topological groups.*

Proofs of this theorem can be found e.g. in [Fol95] or [HM98].

1.1.3 In the present paper we use the language of sheaves in order to encode the topology of spaces and groups. Let X, B be topological spaces. The space X gives rise to a sheaf of sets $\underline{X}$ on B which associates to an open subset $U \subseteq B$ the set $\underline{X}(U) = C(U, X)$ of continuous maps from U to X. If G is a topological abelian group, then $\underline{G}$ is a sheaf of abelian groups on B.

The space X or the group G is not completely determined by the sheaf it generates over B. As an extreme example take $B = \{*\}$. Then X and the underlying discrete space X^δ induce isomorphic sheaves on B.

For another example, assume that B is totally disconnected. Then the sheaves generated by the spaces $[0, 1]$ and $\{*\}$ are isomorphic.

1.1.4 But one can do better and consider sheaves which are defined on all topological spaces or at least on a sufficiently big subcategory $\mathbf{S} \subset \mathsf{TOP}$. We turn $\mathbf{S}$ into a Grothendieck site determined by the pre-topology of open coverings (for details about the choice of $\mathbf{S}$ see Section 3). We will see that the topology in $\mathbf{S}$ is sub-canonical so that every object $X \in \mathbf{S}$ represents a sheaf $\underline{X} \in \mathsf{Sh}\mathbf{S}$. By the Yoneda Lemma the space X can be recovered from the sheaf $\underline{X} \in \mathsf{Sh}\mathbf{S}$ represented by X. The evaluation of the sheaf $\underline{X}$ on $A \in \mathbf{S}$ is defined by $\underline{X}(A) := \mathsf{Hom}_{\mathbf{S}}(A, X)$. Our site $\mathbf{S}$ will in particular contain all locally compact spaces.

1.1.5 We can reformulate Pontrjagin duality in sheaf theoretic terms. The circle group $\mathbb{T}$ belongs to $\mathbf{S}$ and gives rise to a sheaf $\underline{\mathbb{T}}$. Given a sheaf of abelian groups $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we define its dual by

$$D(F) := \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}})$$

and observe that

$$D(\underline{G}) \cong \underline{\hat{G}}$$

for an abelian group $G \in \mathbf{S}$.

The image of the natural pairing

$$F \otimes_{\mathbb{Z}} D(F) \to \underline{\mathbb{T}}$$

under the isomorphism

$$\begin{aligned}\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F \otimes_{\mathbb{Z}} D(F), \underline{\mathbb{T}}) &\cong \mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(F), \underline{\mathbb{T}})) \\ &= \mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, D(D(F)))\end{aligned}$$

gives the evaluation map

$$\underline{\mathsf{ev}_F} : F \to D(D(F)).$$

Definition 1.2. We call a sheaf of abelian groups $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ dualizable, if the evaluation map $\underline{\mathsf{ev}_F} : F \to D(D(F))$ is an isomorphism of sheaves.

The sheaf theoretic reformulation of Pontrjagin duality is now:

Theorem 1.3 (Sheaf theoretic version of Pontrjagin duality). *If G is a locally compact abelian group, then $\underline{G}$ is dualizable.*

1.2 Picard stacks

1.2.1 A group gives rise to a category BG with one object so that the group appears as the group of automorphisms of this object. A sheaf-theoretic analog is the notion of a gerbe.

1.2.2 A set can be identified with a small category which has only identity morphisms. In a similar way a sheaf of sets can be considered as a strict presheaf of categories. In the present paper a presheaf of categories on $\mathbf{S}$ is a lax contravariant functor $\mathbf{S} \to \mathsf{Cat}$. Thus a presheaf F of categories associates to each object $A \in \mathbf{S}$ a category $F(A)$, and to each morphism $f : A \to B$ a functor $f^* : F(B) \to F(A)$. The adjective lax means, that in addition for each pair of composable morphisms $f, g \in \mathbf{S}$ we have specified an isomorphism of functors $g^* \circ f^* \overset{\phi_{f,g}}{\cong} (f \circ g)^*$ which satisfies higher associativity relations. The sheaf of categories is called strict if these isomorphisms are identities.

1.2.3 A category is called a groupoid if all its morphisms are isomorphisms. A prestack on $\mathbf{S}$ is a presheaf of categories on $\mathbf{S}$ which takes values in groupoids. A prestack is a stack if it satisfies in addition descent conditions on the level of objects and morphisms. For details about stacks we refer to [Vis05].

1.2.4 A sheaf of groups $F \in \mathsf{Sh_{Ab}}\mathbf{S}$ gives rise to a prestack ${}^p\mathcal{B}F$ which associates to $U \in \mathbf{S}$ the groupoid ${}^p\mathcal{B}F(U) := BF(U)$. This prestack is not a stack in general. But it can be stackified (this is similar to sheafification) to the stack $\mathcal{B}F$.

1.2.5 For an abelian group G the category BG is actually a group object in Cat. The group operation is implemented by the functor $BG \times BG \to BG$ which is obvious on the level of objects and given by the group structure of G on the level of morphisms. In this case associativity and commutativity is strictly satisfied. In a similar manner, for $F \in \mathsf{Sh_{Ab}}\mathbf{S}$ the prestack ${}^p\mathcal{B}F$ on $\mathbf{S}$ becomes a group object in prestacks on $\mathbf{S}$.

1.2.6 A group G can of course also be viewed as a category G with only identity morphisms. This category is again a strict group object in Cat. The functor $G \times G \to G$ is given by the group operation on the level of objects, and in the obvious way on the level of morphisms.

1.2.7 In general, in order to define group objects in two-categories like Cat or the category of stacks on $\mathbf{S}$, one would relax the strictness of associativity and commutativity. For our purpose the appropriate relaxed notion is that of a Picard category which we will explain in detail in 2. The corresponding sheafified notion is that of a Picard stack. We let $\mathsf{PIC}\,\mathbf{S}$ denote the two-category of Picard stacks on $\mathbf{S}$.

1.2.8 A sheaf of groups $F \in \mathbf{S}_{\mathsf{Ab}}\mathbf{S}$ gives rise to a Picard stack in two ways.

First of all it determines the Picard stack F which associates to $U \in \mathbf{S}$ the Picard category $F(U)$ in the sense of 1.2.6. The other possibility is the Picard stack $\mathcal{B}F$ obtained as the stackification of the Picard prestack ${}^p\mathcal{B}F$.

1.2.9 Let $A \in \mathsf{PIC}\,\mathbf{S}$ be a Picard stack. The presheaf of its isomorphism classes of objects generates a sheaf of abelian groups which will be denoted by $H^0(A) \in \mathsf{Sh_{Ab}}\mathbf{S}$. The Picard stack A gives furthermore rise to the sheaf of abelian groups $H^{-1}(A) \in \mathsf{Sh_{Ab}}\mathbf{S}$ of automorphisms of the unit object.

A Picard stack $A \in \mathsf{PIC}\,\mathbf{S}$ fits into an extension (see 2.5.12)

$$\mathcal{B}H^{-1}(A) \to A \to H^0(A). \tag{1}$$

1.3 Duality of Picard stacks

1.3.1 It is an essential observation by Deligne that the two-category of Picard stacks $\mathsf{PIC}\,\mathbf{S}$ admits an interior $\underline{\mathsf{HOM}}_{\mathsf{PIC}\,\mathbf{S}}$. Thus for Picard stacks $A, B \in \mathsf{PIC}\,\mathbf{S}$ we have a Picard stack

$$\underline{\mathsf{HOM}}_{\mathsf{PIC}\,\mathbf{S}}(A, B) \in \mathsf{PIC}\mathbf{S}$$

of additive morphisms from A to B (see 2.4).

1.3.2 Since we can consider sheaves of groups as Picard stacks in two different ways one can now ask how Pontrjagin duality is properly reflected in the language of Picard stacks. It turns out that the correct dualizing object is the stack $\mathcal{B}\underline{\mathbb{T}} \in \mathsf{PIC}\,\mathbf{S}$, and not $\underline{\mathbb{T}} \in \mathsf{PIC}\,\mathbf{S}$ as one might guess first. For a Picard stack A we define its dual by

$$D(A) := \underline{\mathsf{HOM}}_{\mathsf{PIC}\,\mathbf{S}}(A, \mathcal{B}\underline{\mathbb{T}}).$$

We hope that using the same symbol for the dual sheaf and dual Picard stack does not introduce to much confusion (see the two footnotes attached to the formulas (2) and (3) below). One can ask what the duals of the Picard stacks F and $\mathcal{B}F$ look like. In general we have (see 5.5)

$$D(\mathcal{B}F) \cong D(F).^{1} \tag{2}$$

One could expect that

$$D(F) \cong \mathcal{B}D(F),^{2} \tag{3}$$

but this only holds under the condition that $\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(F, \underline{\mathbb{T}}) = 0$ (see 5.6). This condition is not always satisfied, e.g.

$$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{\oplus_{n\in\mathbb{N}}\mathbb{Z}/2\mathbb{Z}}, \underline{\mathbb{T}}) \neq 0$$

(see 4.29). But the main effort of the present paper is made to show that

$$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) = 0$$

for a large class of locally compact abelian groups (this condition is part of admissibility, see Definition 1.5).

1.3.3 Let $A \in \mathsf{PICS}$ be a Picard stack. There is a natural evaluation

$$\underline{\mathsf{ev}_A} \colon A \to D(D(A)).$$

In analogy to 1.2 we make the following definition.

Definition 1.4. We call a Picard stack dualizable if $\underline{\mathsf{ev}_A} \colon A \to D(D(A))$ is an equivalence of Picard stacks.

1.3.4 If G is a locally compact abelian group and

$$\mathsf{Ext}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong \mathsf{Ext}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{\hat{G}}, \underline{\mathbb{T}}) \cong 0, \tag{4}$$

then the evaluation maps

$$\underline{\mathsf{ev}_{\underline{G}}} \colon \underline{G} \to D(D(\underline{G})), \quad \underline{\mathsf{ev}_{\mathcal{B}\underline{G}}} \colon \mathcal{B}\underline{G} \to D(D(\mathcal{B}\underline{G}))$$

are isomorphisms by applying (2) and (3) twice, and using the sheaf-theoretic version of Pontrjagin duality 1.3. In other words, under the condition (4) above $\underline{G}$ and $\mathcal{B}\underline{G}$ are dualizable Picard stacks.

[1]On the right-hand side $D(F)$ is the dual sheaf as in 1.1.5, considered as a Picard stack as in 1.2.8.

[2]The symbol $D(F)$ on the left-hand side of this equation denotes the dual of the Picard stack given by the sheaf F, while $D(F)$ on the right-hand side is the dual sheaf.

1.3.5 Given the structure

$$\mathcal{B}H^{-1}(A) \to A \to H^0(A)$$

of $A \in \mathsf{PIC}\,\mathbf{S}$, we ask for a similar description of the dual $D(A) \in \mathsf{PIC}\,\mathbf{S}$.

We will see under the crucial condition

$$\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(H^0(A), \underline{\mathbb{T}}) \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(H^{-1}(A), \underline{\mathbb{T}}) \cong 0,$$

that $D(A)$ fits into

$$\mathcal{B}D(H^0(A)) \to D(A) \to D(H^{-1}(A)).$$

Without the condition the description of $H^{-1}(D(A))$ is more complicated, and we refer to (45) for more details.

1.3.6 This discussion now leads to one of the main results of the present paper.

Definition 1.5. We call the sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ admissible, iff

$$\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong 0.$$

Theorem 1.6 (Pontrjagin duality for Picard stacks). *If $A \in \mathsf{PIC}\,\mathbf{S}$ is a Picard stack such that $H^i(A)$ and $D(H^i(A))$ are dualizable and admissible for $i = -1, 0$, then A is dualizable.*

1.4 Admissible groups

1.4.1 Pontrjagin duality for locally compact abelian groups implies that the sheaf $\underline{G}$ associated to a locally compact abelian group G is dualizable. In order to apply Theorem 1.6 to Picard stacks $A \in \mathsf{PIC}\,\mathbf{S}$ whose sheaves $H^i(A), i = -1, 0$ are represented by locally compact abelian groups we must know for a locally compact group G whether the sheaf $\underline{G}$ is admissible.

Definition 1.7. We call a locally compact group G admissible, if the sheaf $\underline{G}$ is admissible.

1.4.2 Admissibility of a locally compact group is a complicated property. Not every locally compact group is admissible, e.g. a discrete group of the form $\oplus_{n\in\mathbb{N}}\mathbb{Z}/p\mathbb{Z}$ for some integer p is not admissible (see 4.29).

1.4.3 Admissibility is a vanishing condition

$$\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong 0.$$

This condition depends on the site $\mathbf{S}$. In the present paper we shall also consider the subsites $\mathbf{S}_{\text{lc-acyc}} \subset \mathbf{S}_{\text{lc}} \subset \mathbf{S}$ of locally compact locally acyclic spaces and locally compact spaces.

For these sites the $\underline{\mathsf{Ext}}$-functor commutes with restriction (we verify this property in 3.4). Admissibility thus becomes a weaker condition on a smaller site. We will refine our notion of admissibility by saying that a group G is admissible on the site $\mathbf{S}_{\mathrm{lc}}$ (or similarly for $\mathbf{S}_{\text{lc-acyc}}$), if the corresponding restrictions of the extension sheaves vanish, e.g.

$$\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})_{|\mathbf{S}_{\mathrm{lc}}} \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})_{|\mathbf{S}_{\mathrm{lc}}} \cong 0$$

in the case $\mathbf{S}_{\mathrm{lc}}$.

1.4.4 Some locally compact abelian groups are admissible on the site $\mathbf{S}$. This applies e.g. to finitely generated groups like $\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$, but also to $\mathbb{T}^n$ and $\mathbb{R}^n$.

In the case of profinite groups G we need the technical assumption that it does not have too much two-torsion and three-torsion.

Definition 1.8 (4.6). We say that the topological abelian group G satisfies the two-three condition, if

1. it does not admit $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ as a sub-quotient,
2. the multiplication by 3 on the component G_0 of the identity has finite cokernel.

We can show that a profinite abelian group which satisfies the two-three condition is also admissible. We conjecture that it is possible to remove the condition using other techniques.

A compact connected abelian group is divisible. Hence, if $p \in \mathbb{N}$ is a prime, then the multiplication $p\colon G \to G$ is set-theoretically surjective.

Definition 1.9. We say that G is locally topologically p-divisible, if the map p has a continuous local section. The group G is locally topologically divisible, if it is locally topologically p-divisable for all primes p.

For a general connected compact group which satisfies the two-three condition we can only show that it is admissible on $\mathbf{S}_{\text{lc-acyc}}$. If it is locally topologically divisible, then it is admissible on the larger site S_{lc}.

We think that the two-tree condition and the restriction to a locally compact site is of technical nature. The condition of local compactness enters the proof since at one place we want to calculate the cohomology of the sheaf $\underline{\mathbb{Z}}$ on the space $A \times G$ using a Künneth formula. For this reason we want that A is locally compact.

As our counterexample above shows, a general (infinitely generated) discrete group is not admissible unless we restrict to the site $\mathbf{S}_{\text{lc-acyc}}$. We do not know if the restriction to the site $\mathbf{S}_{\text{lc-acyc}}$ is really necessary for a general compact connected groups.

Using that the class of admissible groups is closed under finite products and extensions we get the following general theorem.

Theorem 1.10 (4.8). 1. *If G is a locally compact abelian group which satisfies the two-three condition, then it is admissible over* $\mathbf{S}_{\text{lc-acyc}}$.

2. *Assume that*

(a) *G satisfies the two-three condition,*

(b) *G admits an open subgroup of the form $C \times \mathbb{R}^n$ with C compact such that $G/C \times \mathbb{R}^n$ is finitely generated*

(c) *the connected component of the identity of G is locally topologically divisible.*

Then G is admissible over $\mathbf{S}_{\mathrm{lc}}$.

The whole Section 4 is devoted to the proof of this theorem. This section is very long and technical. A reader who is interested in the extension of Pontrjagin duality to topological abelian group stacks and applications to T-duality is advised to skip this section in a first reading and to take Theorem 4.8 for granted.

1.5 T-duality

1.5.1 The aim of topological T-duality is to model the underlying topology of mirror symmetry in algebraic geometry and T-duality in string theory. For a detailed motivation we refer to [BRS]. The objects of topological T-duality over a space B are pairs (E, G) of a $\mathbb{T}^n$-principal bundle $E \to B$ and a gerbe $G \to E$ with band $\mathbb{T}$. A gerbe with band $\mathbb{T}$ over a space E is a map of stacks $G \to E$ which is locally isomorphic to $\mathcal{B}\mathbb{T}_{|E} \to \underline{E}$. Topological T-duality associates to (E, G) dual pairs $(\widehat{E}, \widehat{G})$. For precise definitions we refer to [BRS] and 6.1.

1.5.2 The case $n = 1$ (n is the dimension of the fibre of $E \to B$) is quite easy to understand (see [BS05]). In this case every pair admits a unique T-dual (up to isomorphism, of course). The higher dimensional case is more complicated since on the one hand not every pair admits a T-dual, and on the other hand, in general a T-dual is not unique, compare [BS05], [BRS].

1.5.3 The general idea is that the construction of a T-dual pair of (E, G) needs the choice of an additional structure. This structure might not exist, and in this case there is no T-dual. On the other hand, if the additional structure exists, then it might not be unique, so that the T-dual is not unique, too.

1.5.4 The additional structure in [BRS] was the extension of the pair to a T-duality triple. We review this notion in 6.1.

One can also interpret the approach to T-duality via non-commutative topology [MR05], [MR06] in this way. The gerbe $G \to E$ determines an isomorphism class of a bundle of compact operators (by equality of the Dixmier–Douady classes). The additional structure which determines a T-dual is an $\mathbb{R}^n$-action on this bundle of compact operators which lifts the $\mathbb{T}^n$-action on E. Let us call a bundle of algebras of compact operators on E together with such a $\mathbb{R}^n$-action a dynamical triple.

The precise relationship between T-duality triples and dynamical triples is studied in the thesis of Ansgar Schneider [Sch07].

1.5.5 The initial motivation of the present paper was a third choice for the additional structure of the pair which came to live in a discussion with T. Pantev in spring 2006. It was motivated by the analogy with some constructions in algebraic geometry, see e.g. [DP].

The starting point is the observation that a T-dual exists if and only if the restriction of the gerbe $G \to E$ to $E|_{B^{(1)}}$, the restriction of E to a one-skeleton of B, is trivial. In particular, the restriction of G to the fibres of E has to be trivial. Of course, the T^n-bundle $E \to B$ is locally trivial on B, too. Therefore, locally on the base B, the sequence of maps $G \to E \to B$ is equivalent to

$$(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n)_{|B} \to \underline{\mathbb{T}}^n_{|B} \to \underline{B}_{|B},$$

where we identify spaces over B with the corresponding sheaves. The stack $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n)_{|B}$ is a Picard stack.

Our proposal for the additional structure on $G \to E \to B$ is that of a torsor over the Picard stack $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n)_{|B}$.

1.5.6 In order to avoid the definition of a torsor over a group object in the two-categorial world of stacks we use the following trick (see 6.2 for details). Note that a torsor X over an abelian group G can equivalently be described as an extension of abelian groups

$$0 \to G \to U \xrightarrow{p} \mathbb{Z} \to 0$$

so that $X \cong p^{-1}(1)$. We use the same trick to describe a torsor over the Picard stack $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n)_{|B}$ as an extension

$$(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n)_{|B} \to U \to \underline{\mathbb{Z}}_{|B}$$

of Picard stacks.

1.5.7 The sheaf of sections of $E \to B$ is a torsor over $\underline{\mathbb{T}}^n$. Let

$$0 \to \underline{\mathbb{T}}^n_{|B} \to \mathcal{E} \to \underline{\mathbb{Z}}_{|B} \to 0$$

be the corresponding extension of sheaves of abelian groups. The filtration (1) of the Picard stack U has the form

$$\mathcal{B}\underline{\mathbb{T}}_{|B} \to U \to \mathcal{E},$$

and locally on B

$$U \cong (\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n \times \underline{\mathbb{Z}})_{|B}. \tag{5}$$

1.5.8 The proposal 1.5.5 was motivated by the hope that the Pontrjagin dual $D(U)$ of U determines the T-dual pair. In fact, this can not be true directly since the structure of $D(U)$ (this uses admissibility of $\mathbb{T}^n$ and $\mathbb{Z}^n$) is given locally on B by

$$D(U) \cong (\mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{Z}}^n \times \underline{\mathbb{Z}})_{|B}.$$

Here the factor $\underline{\mathbb{Z}}$ in (5) gives rise to $\mathcal{B}\underline{\mathbb{T}}$, the factor $\mathcal{B}\underline{\mathbb{T}}$ yields $\underline{\mathbb{Z}}$, and $\underline{\mathbb{T}}^n$ yields $\mathcal{B}\underline{\mathbb{Z}}^n$ according to the rules (2) and (3). The problematic factor is $\mathcal{B}\mathbb{Z}^n$ in a place where we expect a factor $\underline{\mathbb{T}}^n$. The way out is to interpret the gerbe $\mathcal{B}\underline{\mathbb{Z}}^n$ as the gerbe of $\mathbb{R}^n$-reductions of the trivial principal bundle $\mathbb{T}^n \times B \to B$.

1.5.9 We have canonical isomorphisms $H^0(D(U)) \cong D(H^{-1}(U)) \cong D(\underline{\mathbb{T}}_{|B}) \cong \underline{\mathbb{Z}}_{|B}$ and let $D(U)_1 \subset D(U)$ be the pre-image of $\{1\}_{|B} \subset \underline{\mathbb{Z}}_{|B}$ under the natural map $D(U) \to H^0(D(U))$. The quotient $D(U)_1/(\mathcal{B}\underline{\mathbb{T}})_{|B}$ is a $\mathbb{Z}^n$-gerbe over B which we interpret as the gerbe of $\mathbb{R}^n$-reductions of a $\mathbb{T}^n$-principal bundle $\hat{E} \to B$ which is well-defined up to unique isomorphism. The full structure of $D(U)$ supplies the dual gerbe $\hat{G} \to \hat{E}$. The details of this construction are explained in 6.4.

1.5.10 In this way we use Pontrjagin duality of Picard stacks in order to construct a T-dual pair to (E, G). Schematically the picture is

$$(E, G) \overset{1}{\rightsquigarrow} U \overset{2}{\rightsquigarrow} D(U) \overset{3}{\rightsquigarrow} (\hat{E}, \hat{G}),$$

where the steps are as follows:

1. choice of the structure on G of a torsor over $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}^n})_{|B}$
2. Pontrjagin duality $D(U) := \underline{\mathsf{Hom}}_{\mathsf{PICS}}(U, \mathcal{B}\underline{\mathbb{T}})$
3. Extraction of the dual pair from $D(U)$ as explained in 1.5.9.

1.5.11 Consider a pair (E, G). In Subsection 6.4 we provide two constructions:

1. The construction Φ starts with the choice of a T-duality triple extending (E, G) and constructs the structure of a torsor over $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}^n})_{|B}$ on G.
2. The construction Ψ starts with the structure on G of a torsor over $(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}^n})_{|B}$ and constructs an extension of (E, G) to a T-duality triple.

Our main result Theorem 6.23 asserts that the constructions Ψ and Φ are inverses to each other. In other words, the theories of topological T-duality via Pontrjagin duality of Picard stacks and via T-duality triples are equivalent.

2 Sheaves of Picard categories

2.1 Picard categories

2.1.1 In a cartesian closed category one can define the notion of a group object in the standard way. It is given by an object, a multiplication, an identity, and an inversion morphism. The group axioms can be written as a collection of commutative diagrams involving these morphisms.

Stacks on a site **S** form a two-cartesian closed two-category. A group object in a two-cartesian closed two-category is again given by an object and the multiplication, identity and inversion morphisms. In addition each of the commutative diagrams from the category case is now filled by a two-morphisms. These two-morphisms must satisfy higher associativity relations.

Instead of writing out all these relations we will follow the exposition of Deligne SGA4, exposé XVIII [Del], which gives a rather effective way of working with group objects in two-categories. We will not be interested in the most general case. For our purpose it suffices to consider a notion which includes sheaves of abelian groups and gerbes with band given by a sheaf of abelian groups. We choose to work with sheaves of strictly commutative Picard categories.

2.1.2 Let C be a category,

$$F : C \times C \to C$$

be a bi-functor, and

$$\sigma\colon F(F(X,Y),Z) \stackrel{\sim}{\to} F(X,F(Y,Z))$$

be a natural isomorphism of tri-functors.

Definition 2.1. The pair (F,σ) is called an associative functor if the following holds. For every family $(X_i)_{i\in I}$ of objects of C let $e\colon I \to M(I)$ denote the canonical map into the free monoid (without identity) on I. We require the existence of a map $\underline{F}\colon M(I) \to \mathsf{Ob}(C)$ and isomorphisms $a_i : \underline{F}(e(i)) \stackrel{\sim}{\to} X_i$, and isomorphisms $a_{g,h}\colon \underline{F}(gh) \stackrel{\sim}{\to} F(\underline{F}(g),\underline{F}(h))$ such that the following diagram commutes:

$$\begin{tikzcd}
\underline{F}(f(gh)) \arrow[r, "a_{f,gh}"] \arrow[d, equal] & F(\underline{F}(f),\underline{F}(gh)) \arrow[r, "a_{g,h}"] & F(\underline{F}(f),F(\underline{F}(g),\underline{F}(h)) \\
\underline{F}((fg)h) \arrow[r, "a_{fg,h}"'] & F(\underline{F}(fg),\underline{F}(h)) \arrow[r, "a_{f,g}"'] & F(F(\underline{F}(f),\underline{F}(g)),\underline{F}(h)). \arrow[u, "\sigma"]
\end{tikzcd} \tag{6}$$

2.1.3 Let (F,σ) be as above. Let in addition be given a natural transformation of bi-functors

$$\tau\colon F(X,Y) \to F(Y,X).$$

Definition 2.2. (F,σ,τ) is called a commutative and associative functor if the following holds. For every family $(X_i)_{i\in I}$ of objects in C let $e\colon I \to N(I)$ denote the canonical map into the free abelian monoid (without identity) on I. We require that there exist $\underline{F}\colon N(I) \to C$, isomorphisms $a_i : \underline{F}(e(i)) \to X_i$ and isomorphisms $a_{g,h}\colon \underline{F}(gh) \stackrel{\sim}{\to} F(\underline{F}(g),\underline{F}(h))$ such that (6) and

$$\begin{tikzcd}
\underline{F}(gh) \arrow[r, "a_{g,h}"] \arrow[d, equal] & F(\underline{F}(g),\underline{F}(h)) \arrow[d, "\tau"] \\
\underline{F}(hg) \arrow[r, "a_{h,g}"'] & F(\underline{F}(h),\underline{F}(g))
\end{tikzcd} \tag{7}$$

commute.

2.1.4 We can now define the notion of a strict Picard category. Instead of F will use the symbol "$+$".

Definition 2.3. A Picard category is a groupoid P together with a bi-functor $+\colon P \times P \to P$ and natural isomorphisms σ and τ as above such that $(+, \sigma, \tau)$ is a commutative and associative functor, and such that for all $X \in P$ the functor $Y \mapsto X + Y$ is an equivalence of categories.

2.2 Examples of Picard categories

2.2.1 Let G be an abelian group. We consider G as a category with only identity morphisms. The addition $+\colon G \times G \to G$ is a bi-functor. For σ and τ we choose the identity transformations. Then $(G, +, \sigma, \tau)$ is a strict Picard category which we will denote again by G.

2.2.2 Let us now consider the abelian group G as a category BG with one object $*$ and $\mathsf{Mor}_{BG}(*, *) := G$. We let $+\colon BG \times BG \to BG$ be the bi-functor which acts as addition $\mathsf{Mor}_{BG}(*) \times \mathsf{Mor}_{BG}(*) \to \mathsf{Mor}_{BG}(*)$. For σ and τ we again choose the identity transformations. We will denote this strict Picard category $(BG, +, \sigma, \tau)$ shortly by BG.

2.2.3 Let G, H be abelian groups. We define the category $\mathsf{EXT}(G, H)$ as the category of short exact sequences

$$\mathcal{E}\colon 0 \to H \to E \to G \to 0.$$

Morphisms in $\mathsf{EXT}(G, H)$ are isomorphisms of complexes which reduce to the identity on G and H. We define the bi-functor $+\colon \mathsf{EXT}(G, H) \times \mathsf{EXT}(G, H) \to \mathsf{EXT}(G, H)$ as the Baer addition

$$\mathcal{E}_1 + \mathcal{E}_2\colon 0 \to H \to F \to G \to 0,$$

where $F := \tilde{F}/H$, $\tilde{F} \subset E_1 \times E_2$ is the pre-image of the diagonal in $G \times G$, and the action of H on $\tilde{F}$ is induced by the anti-diagonal action on $E_1 \times E_2$. The associativity morphism $\sigma\colon (\mathcal{E}_1 + \mathcal{E}_2) + \mathcal{E}_3 \to \mathcal{E}_1 + (\mathcal{E}_2 + \mathcal{E}_3)$ is induced by the canonical isomorphism $(E_1 \times E_2) \times E_3 \to E_1 \times (E_2 \times E_3)$. Finally, the transformation $\tau\colon \mathcal{E}_1 + \mathcal{E}_2 \to \mathcal{E}_2 + \mathcal{E}_1$ is induced by the flip $E_1 \times E_2 \to E_2 \times E_1$. The triple $(+, \sigma, \tau)$ defines on $\mathsf{EXT}(G, H)$ the structure of a Picard category.

2.2.4 Let G be a topological abelian group, e.g. $G := \mathbb{T}$. On a space B we consider the category of G-principal bundles $\mathcal{B}G(B)$. Given two G-principal bundles $Q \to B$, $P \to B$ we can define a new G-principal bundle $Q \otimes_G P := Q \times_B P/G$. The quotient is taken by the anti-diagonal action, i.e. we identify $(q, p) \sim (qg, pg^{-1})$. The G-principal structure on $Q \otimes_G P$ is induced by the action $[q, p]g := [q, pg]$.

We define the associativity morphism

$$\sigma\colon (Q \otimes_G P) \otimes_G R \to Q \otimes_G (P \otimes_G R)$$

as the map $[[q,p],r] \to [q,[p,r]]$. Finally we let $\tau\colon Q \otimes_G P \to P \otimes_G Q$ be given by $[q,p] \to [p,q]$.

We claim that $(\otimes_G, \sigma, \tau)$ is a strict Picard category. Let I be a collection of objects of $\mathcal{B}G(B)$. We choose an ordering of I. Then we can write each element of $j \in N(I)$ in a unique way as ordered product $f = P_1 P_2 \dots P_r$ with $P_1 \le P_2 \dots \le P_r$. We define $\underline{F}(f) := P_1 \otimes_G (P_2 \otimes_G \cdots (\cdots \otimes_G P_r) \cdots)$. We let $a_P\colon \underline{F}(P) \xrightarrow{\sim} P$, $P \in I$, be the identity. Furthermore, for $f, g \in N(I)$ we define $a_{f,g}\colon \underline{F}(fg) \xrightarrow{\sim} \underline{F}(f) \otimes_B \underline{F}(g)$ as the map induced the permutation which puts the factors of f and g in order (and so that the order to repeated factors is not changed). Commutativity of (6) is now clear. In order to check the commutativity of (7) observe that $\tau\colon Q \otimes_G Q \to Q \otimes_G Q$ is the identity, as $[qg,q] \mapsto [q,qg] = [qg,q]$.

2.3 Picard stacks and examples

2.3.1 We consider a site $\mathbf{S}$. A prestack on $\mathbf{S}$ is a lax associative functor $P\colon \mathbf{S}^{\mathrm{op}} \to$ groupoids, where from now on by a groupoid we understand an essentially small category[3] in which all morphisms are isomorphisms. Lax associative means that as a part of the data for each composable pair of maps f, g in $\mathbf{S}$ there is an isomorphism of functors $l_{f,g}\colon P(f) \circ P(g) \xrightarrow{\sim} P(g \circ f)$, and these isomorphisms satisfy higher associativity relation. A prestack is a stack if it satisfies the usual descent conditions one the level of objects and morphisms. For a reference of the language of stacks see e.g. [Vis05].

Definition 2.4. A Picard stack P on $\mathbf{S}$ is a stack P together with an operation $+\colon P \times P \to P$ and transformations σ, τ which induce for each $U \in \mathbf{S}$ the structure of a Picard category on $P(U)$.

2.3.2 Let $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ denote the category of sheaves of abelian groups on $\mathbf{S}$. Extending the example 2.2.1 to sheaves we can view each object $\mathcal{F} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ as a Picard stack, which we will again denote by $\mathcal{F}$.

2.3.3 We can also extend Example 2.2.2 to sheaves. In this way every sheaf $\mathcal{F} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ gives rise to a Picard stack $\mathcal{B}\mathcal{F}$.

2.3.4 We now extend the example 2.2.3 to sheaves. First of all note that one can define a Picard category $\mathsf{EXT}(\mathcal{G}, \mathcal{H})$ of extensions of sheaves $0 \to \mathcal{H} \to \mathcal{E} \to \mathcal{G} \to 0$ as a direct generalization of 2.2.3. Then we define a prestack $\mathcal{E}\mathsf{XT}(\mathcal{G}, \mathcal{H})$ which associates to $U \in \mathbf{S}$ the Picard category $\mathcal{E}\mathsf{XT}(\mathcal{G}, \mathcal{H})(U) := \mathsf{EXT}(\mathcal{G}_{|U}, \mathcal{H}_{|U})$. One checks that $\mathcal{E}\mathsf{XT}(\mathcal{G}, \mathcal{H})$ is a stack.

2.3.5 Let $\mathcal{G} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ be a sheaf of abelian groups. We consider $\mathcal{G}$ as a group object in the category $\mathsf{Sh}\,\mathbf{S}$ of sheaves of sets on $\mathbf{S}$. A $\mathcal{G}$-torsor is an object $\mathcal{T} \in \mathsf{Sh}\,\mathbf{S}$ with an action of $\mathcal{G}$ such that $\mathcal{T}$ is locally isomorphic to $\mathcal{G}$. We consider the category of $\mathcal{G}$-torsors $\mathsf{Tors}(\mathcal{G})$ and their isomorphisms. We define the functor $+\colon \mathsf{Tors}(\mathcal{G}) \times \mathsf{Tors}(\mathcal{G}) \to$

[3]Later, e.g. in 2.5.3, we need that the isomorphism classes in a groupoid form a set.

$\mathsf{Tors}(\mathcal{G})$ by $\mathcal{T}_1 + \mathcal{T}_2 := T_1 \times \mathcal{T}_2/\mathcal{G}$, where we take the quotient by the anti-diagonal action. The structure of a $\mathcal{G}$-torsor is induced by the action of $\mathcal{G}$ on the second factor $\mathcal{T}_2$. We let σ and τ be induced by the associativity transformation of the cartesian product and the flip, respectively. With these structures $\mathsf{Tors}(\mathcal{G})$ becomes a Picard category.

We define a Picard stack $\mathcal{T}\mathsf{ors}(\mathcal{G})$ by localization, i.e. we set $\mathcal{T}\mathsf{ors}(\mathcal{G})(U) := \mathsf{Tors}(\mathcal{G}_{|U})$.

We have a canonical map of Picard stacks

$$U : \mathcal{E}\mathsf{XT}(\underline{\mathbb{Z}}, \mathcal{G}) \to \mathcal{T}\mathsf{ors}(\mathcal{G})$$

which maps the extension $\mathcal{E} : 0 \to \mathcal{G} \to \mathcal{E} \stackrel{\pi}{\to} \underline{\mathbb{Z}} \to 0$ to the G-torsor $U(\mathcal{E}) = \pi^{-1}(1)$. Here $\underline{\mathbb{Z}}$ denotes the constant sheaf on $\mathbf{S}$ with value $\mathbb{Z}$. One can check that U is an equivalence of Picard stacks (see 2.4.3 for precise definitions).

2.3.6 The example 2.2.4 has a sheaf theoretic interpretation. We assume that $\mathbf{S}$ is some small subcategory of the category of topological spaces which is closed under taking open subspaces, and such that the topology is induced by the coverings of spaces by families of open subsets.

Let G be a topological abelian group. We have a Picard prestack ${}^{p}\mathcal{B}G$ on $\mathbf{S}$ which associates to each space $B \in \mathbf{S}$ the Picard category ${}^{p}\mathcal{B}G(B)$ of G-principal bundles on B. As in 2.2.4 the monoidal structure on ${}^{p}\mathcal{B}G(B)$ is given by the tensor product of G-principal bundles (this uses that G is abelian). We now define $\mathcal{B}G$ as the stackification of ${}^{p}\mathcal{B}G$.

The topological group G gives rise to a sheaf $\underline{G} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ which associates to $U \in \mathbf{S}$ the abelian group $\underline{G}(U) := C(U, G)$. We have a canonical transformation

$$\Gamma : \mathcal{B}G \to \mathcal{T}\mathsf{ors}(\underline{G}).$$

It is induced by a transformation ${}^{p}\Gamma : {}^{p}\mathcal{B}G \to \mathcal{T}\mathsf{ors}(\underline{G})$. We describe the functor ${}^{p}\Gamma_B : {}^{p}\mathcal{B}G(B) \to \mathcal{T}\mathsf{ors}(\underline{G})(B)$ for all $B \in \mathbf{S}$. Let $E \to B$ be a G-principal bundle. Then we define ${}^{p}\Gamma_B(E) \in \mathsf{Tors}(\underline{G}_{|B})$ to be the sheaf which associates to each $(\phi : U \to B) \in \mathbf{S}/B$ the set of sections of $\phi^* E \to U$.

One can check that Γ is an equivalence of Picard stacks (see 2.4.3 for precise definitions).

2.4 Additive functors

2.4.1 We shall now discuss the notion of an additive functor between Picard stacks $F : P_1 \to P_2$.

Definition 2.5. An additive functor between Picard categories is a functor $F : P_1 \to P_2$ and a natural transformation $F(x+y) \stackrel{\sim}{\to} F(x)+F(y)$ such that the following diagrams commute:

1.

$$\begin{array}{ccc} F(x+y) & \longrightarrow & F(x)+F(y) \\ \Big\downarrow{\scriptstyle F(\tau)} & & \Big\downarrow{\scriptstyle \tau} \\ F(y+x) & \longrightarrow & F(y)+F(x); \end{array}$$

2.

$$\begin{array}{ccccc} F((x+y)+z) & \longrightarrow & F(x+y)+F(z) & \longrightarrow & (F(x)+F(y))+F(z) \\ \Big\downarrow{\scriptstyle F(\sigma)} & & & & \Big\downarrow{\scriptstyle \sigma} \\ F(x+(y+z)) & \longrightarrow & F(x)+F(y+z)) & \longrightarrow & F(x)+(F(y)+F(z)). \end{array}$$

Definition 2.6. An isomorphism between additive functors $u\colon F \to G$ is an isomorphism of functors such that

$$\begin{array}{ccc} F(x+y) & \xrightarrow{u_{x+y}} & G(x+y) \\ \Big\downarrow & & \Big\downarrow \\ F(x)+F(y) & \xrightarrow{u_x+u_y} & G(x)+G(y) \end{array}$$

commutes.

Definition 2.7. We let $\mathsf{Hom}(P_1, P_2)$ denote the groupoid of additive functors from P_1 to P_2. By PIC we denote the two-category of Picard categories.

2.4.2 The groupoid $\mathsf{Hom}(P_1, P_2)$ has a natural structure of a Picard category. We set

$$(F_1+F_2)(x) := F_1(x)+F_2(x)$$

and define the transformation

$$(F_1+F_2)(x+y) \to (F_1+F_2)(x)+(F_1+F_2)(y)$$

such that

$$\begin{array}{ccccc} (F_1+F_2)(x+y) & & \longrightarrow & & (F_1+F_2)(x)+(F_1+F_2)(y) \\ \Big\| & & & & \Big\| \\ F_1(x+y)+F_2(x+y) & \longrightarrow & (F_1(x)+F_1(y))+(F_2(x)+F_2(y)) & \xrightarrow{\alpha\circ\tau\circ\alpha} & F_1(x)+F_2(x)+F_1(y)+F_2(y) \end{array}$$

commutes. The associativity and commutativity constraints are induced by those of P_2.

2.4.3 Let now P_1, P_2 be two Picard stacks on $\mathbf{S}$.

Definition 2.8. An additive functor $F: P_1 \to P_2$ is a morphism of stacks together with a two-morphism

$$\begin{tikzcd} P_1 \times P_1 \arrow[r, "+"] \arrow[d, "F\times F"'] \arrow[dr, Rightarrow] & P_1 \arrow[d, "F"] \\ P_2 \times P_2 \arrow[r, "+"] & P_2 \end{tikzcd}$$

which induces for each $U \in \mathbf{S}$ an additive functor $F_U: P_1(U) \to P_2(U)$ of Picard categories. An isomorphism between additive functors $u: F_1 \to F_2$ is a two-isomorphism of morphisms of stacks which induces for each $U \in \mathbf{S}$ an isomorphism of additive functors $u_U: F_{1,U} \to F_{2,U}$.

As in the case of Picard categories, the additive functors between Picard stacks form again a Picard category.

Definition 2.9. We let $\mathsf{Hom}(P_1, P_2)$ denote the groupoid of additive functors from P_1, P_2.We get a two-category $\mathsf{PIC}(\mathbf{S})$ of Picard stacks on $\mathbf{S}$.

Definition 2.10. We let $\mathsf{HOM}(P_1, P_2)$ denote the Picard category of additive functors between Picard stacks P_1 and P_2.

2.4.4 Let P, Q be Picard stacks on the site $\mathbf{S}$. By localization we define a Picard stack $\underline{\mathsf{HOM}}(P, Q)$.

Definition 2.11. The Picard stack $\underline{\mathsf{HOM}}(P, Q)$ is the sheaf of Picard categories given by

$$\underline{\mathsf{HOM}}_{\mathsf{PIC}(S)}(P, Q)(U) := \mathsf{HOM}(P_{|U}, Q_{|U}).$$

A priori this describes a prestack, but one easily checks the stack conditions.

2.5 Representation of Picard stacks by complexes of sheaves of groups

2.5.1 There is an obvious notion of a Picard prestack. Furthermore, there is an associated Picard stack construction a such that for a Picard prestack P and a Picard stack Q we have a natural equivalence

$$\mathsf{Hom}(aP, Q) \cong \mathsf{Hom}(P, Q) \tag{8}$$

2.5.2 Let $\mathbf{S}$ be a site. We follow Deligne, SGA 4.3, Expose XVIII, [Del]. Let $C(\mathbf{S})$ be the two-category of complexes of sheaves of abelian groups

$$K: 0 \to K^{-1} \stackrel{d}{\to} K^0 \to 0$$

which live in degrees $-1, 0$. Morphisms in $C(\mathbf{S})$ are morphisms of complexes, and two-isomorphisms are homotopies between morphisms.

To such a complex we associate a Picard stack $\mathsf{ch}(K)$ on $\mathbf{S}$ as follows. We first define a Picard prestack ${}^{\mathsf{p}}\mathsf{ch}(K)$.

1. For $U \in \mathbf{S}$ we define the set of objects of ${}^{\mathsf{p}}\mathsf{ch}(K)(U)$ as $K^0(U)$.
2. For $x, y \in K^0(U)$ we define the set of morphisms by
$$\mathsf{Hom}_{{}^{\mathsf{p}}\mathsf{ch}(K)(U)}(x, y) := \{f \in K^{-1}(U) \mid df = x - y\}.$$
3. The composition of morphisms is addition in $K^{-1}(U)$.
4. The functor $+\colon {}^{\mathsf{p}}\mathsf{ch}(K)(U) \times {}^{\mathsf{p}}\mathsf{ch}(K)(U) \to {}^{\mathsf{p}}\mathsf{ch}(K)(U)$ is given by addition of objects and morphisms. The associativity and the commutativity constraints are the identities.

Definition 2.12. We define $\mathsf{ch}(K)$ as the Picard stack associated to the prestack ${}^{\mathsf{p}}\mathsf{ch}(K)$, i.e. $\mathsf{ch}(K) := a^{\mathsf{p}}\mathsf{ch}(K)$.

2.5.3 If P is a Picard stack, then we can define a presheaf ${}^{p}H^0(P)$ which associates to $U \in \mathbf{S}$ the group of isomorphism classes of $P(U)$. We let $H^0(P)$ be the associated sheaf. Furthermore we have a sheaf $H^{-1}(P)$ which associates to $U \in \mathbf{S}$ the group of automorphisms $\mathsf{Aut}(e_U)$, where $e_U \in P(U)$ is a choice of a neutral object with respect to $+$. Since the neutral object e_U is determined up to unique isomorphism, the group $\mathsf{Aut}(e_U)$ is also determined up to unique isomorphism. Note that $\mathsf{Aut}(e_U)$ is abelian, as shows the following diagram

$$\begin{tikzcd} e+e \arrow[r, "\mathsf{id}+f"] \arrow[d, "\cong"] & e+e \arrow[r, "g+\mathsf{id}"] \arrow[d, "\cong"] & e+e \arrow[d, "\cong"] \\ e \arrow[r, "f"] & e \arrow[r, "g"] & e \end{tikzcd}$$

and the fact that $(\mathsf{id} + f) \circ (g + \mathsf{id}) = g + f = (g + \mathsf{id}) \circ (\mathsf{id} + f)$.

If $V \to U$ is a morphism in $\mathbf{S}$, then we have a unique isomorphism $f\colon (e_U)_{|V} \xrightarrow{\sim} e_V$ which induces a group homomorphism $f \circ \cdots \circ f^{-1}\colon \mathsf{Aut}(e_U)_{|V} \to \mathsf{Aut}(e_V)$. This homomorphism is the structure map of the sheaf $H^{-1}(P)$ which is determined up to unique isomorphism in this way.

We now observe that
$$H^{-1}(\mathsf{ch}(K)) \cong H^{-1}(K), \quad H^0(\mathsf{ch}(K)) \cong H^0(K). \tag{9}$$

2.5.4 A morphism of complexes $f\colon K \to L$ in $C(\mathbf{S})$, i.e. a diagram

$$\begin{tikzcd} 0 \arrow[r] & K^{-1} \arrow[r, "d_K"] \arrow[d, "f^{-1}"] & K^0 \arrow[r] \arrow[d, "f^0"] & 0 \\ 0 \arrow[r] & L^{-1} \arrow[r, "d_L"] & L^0 \arrow[r] & 0, \end{tikzcd}$$

induces an additive functor of Picard prestacks ${}^{\mathsf{p}}\mathsf{ch}(f)\colon {}^{\mathsf{p}}\mathsf{ch}(K) \to {}^{\mathsf{p}}\mathsf{ch}(L)$ and, by the functoriality of the associated stack construction, an additive functor
$$\mathsf{ch}(f)\colon \mathsf{ch}(K) \to \mathsf{ch}(L).$$
In view of (9), it is an equivalence if and only if f is a quasi isomorphism.

2.5.5 We consider two morphisms $f, g\colon K \to L$ of complexes in $C(\mathbf{S})$. A homotopy $H\colon f \to g$ is a morphism of sheaves $H\colon K^0 \to L^{-1}$ such that $f^0 - g^0 = d_L \circ H$ and $f^{-1} - g^{-1} = H \circ d_K$. It is easy to see that H induces an isomorphism of additive functors ${}^{\mathsf{p}}\mathsf{ch}(H)\colon {}^{\mathsf{p}}\mathsf{ch}(f) \to {}^{\mathsf{p}}\mathsf{ch}(g)$ and therefore

$$\mathsf{ch}(H)\colon \mathsf{ch}(f) \to \mathsf{ch}(g)$$

by the associated stack construction and (8).

One can show that the isomorphisms ${}^{\mathsf{p}}\mathsf{ch}(f) \to {}^{\mathsf{p}}\mathsf{ch}(g)$ correspond precisely to homotopies $H\colon f \to g$. This implies that the morphisms $\mathsf{ch}(f) \to \mathsf{ch}(g)$ also correspond bijectively to these homotopies.

2.5.6 We have the following result.

Lemma 2.13 ([Del, 1.4.13]). 1. *For every Picard stack P there exists a complex $K \in C(\mathbf{S})$ such that $P \cong \mathsf{ch}(K)$.*

2. *For every additive functor $F\colon \mathsf{ch}(K) \to \mathsf{ch}(L)$ there exists a quasi isomorphism $k\colon K' \to K$ and a morphism $l\colon K' \to L$ in $C(\mathbf{S})$ such that $F \cong \mathsf{ch}(l) \circ \mathsf{ch}(k)^{-1}$.*

2.5.7 Let $\mathsf{PIC}^\flat(\mathbf{S})$ be the category of Picard stacks obtained from the two-category $\mathsf{PIC}\,\mathbf{S}$ by identifying isomorphic additive functors.

Proposition 2.14 ([Del, 1.4.15]). *The construction* ch *gives an equivalence of categories*

$$D^{[-1,0]}(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}) \to \mathsf{PIC}^\flat(\mathbf{S}),$$

where $D^{[-1,0]}(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is the full subcategory of $D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ of objects whose cohomology is trivial in degrees $\notin \{-1, 0\}$.

Lemma 2.15 ([Del, 1.4.16]). *Let $K, L \in C(\mathbf{S})$ and assume that L^{-1} is injective.*

1. ${}^{\mathsf{p}}\mathsf{ch}(L)$ *is already a stack.*
2. *For every morphism $F \in \mathsf{Hom}_{\mathsf{PIC}(\mathbf{S})}(\mathsf{ch}(K), \mathsf{ch}(L))$ there exists a morphism $f \in \mathsf{Hom}_{C(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(K, L)$ such that $\mathsf{ch}(f) \cong F$.*

Let $C'(\mathbf{S}) \subseteq C(\mathbf{S})$ denote the full sub-two-category of complexes $0 \to K^{-1} \to K^0 \to 0$ with K^{-1} injective.

Lemma 2.16 ([Del, 1.4.17]). *The construction* ch *induces an equivalence of the two-categories $\mathsf{PIC}(\mathbf{S})$ and $C'(\mathbf{S})$.*

2.5.8 Finally we give a characterization of the Picard stack $\underline{\mathsf{Hom}}(\mathsf{ch}(K), \mathsf{ch}(L))$.

Lemma 2.17 ([Del, 1.4.18.1]). *Assume that L^{-1} is injective. Then we have an equivalence*

$$\mathsf{ch}(\tau_{\le 0}\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(K, L)) \xrightarrow{\sim} \underline{\mathsf{HOM}}_{\mathsf{PIC}(\mathbf{S})}(\mathsf{ch}(K), \mathsf{ch}(L)).$$

Lemma 2.18. *Let $K, L \in C(\mathbf{S})$. Then we have an isomorphism*

$$H^i(\underline{\mathsf{HOM}}_{\mathsf{PIC}(\mathbf{S})}(\mathsf{ch}(K), \mathsf{ch}(L))) \cong R^i \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}} \mathbf{S}}(K, L)$$

for $i = -1, 0$.

Proof. First observe that by the discussion above the left hand side, and by the definition of $R\,\underline{\mathsf{Hom}}$ the right side both only depend on the quasi-isomorphism type of the complex L of length 2. Without loss of generality we can therefore assume that L^{-1} is injective.

We now choose an injective resolution $I: 0 \to L^{-1} \to I^0 \to I^1 \dots$ of L starting with the choice of an embedding $L^0 \to I^0$. Then we have $\underline{\mathsf{Hom}}(K, I) \cong R\,\underline{\mathsf{Hom}}(K, L)$. We now observe that $H^i\underline{\mathsf{Hom}}(K, I) \cong H^i\underline{\mathsf{Hom}}(K, L)$ for $i = 0, -1$. While the case $i = -1$ is obvious, for $i = 0$ observe that a 0-cycle in $\underline{\mathsf{Hom}}(K, I)$ is a morphism of complexes and necessarily factors over $L \to I$. We thus have for $i \in \{-1, 0\}$

$$R^i\,\underline{\mathsf{Hom}}(K, L) \cong H^i\underline{\mathsf{Hom}}(K, L) \cong H^i(\underline{\mathsf{Hom}}(\mathsf{ch}(K), \mathsf{ch}(L))). \qquad \square$$

2.5.9 For $A, B \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we have canonical isomorphisms

$$\mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A) \cong R^0\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A[2]) \cong \mathsf{Hom}_{D(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2]).$$

In the following we recall two eventually equivalent ways how an exact complex

$$\mathcal{K}: 0 \to A \to X \to Y \to B \to 0$$

represents an element

$$Y(\mathcal{K}) \in \mathsf{Hom}_{D(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2]) \cong \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A)$$

(the letter Y stands for Yoneda who investigated this construction first). Let $\mathcal{K}_A$ be the complex

$$\mathcal{K}_A: 0 \to X \to Y \to B \to 0,$$

where B sits in degree 0. The obvious inclusion $\alpha: A[2] \to \mathcal{K}_A$ induced by $A \to X$ is a quasi-isomorphism. Furthermore, we have a canonical map $\beta: B \to \mathcal{K}_A$. The element $Y(\mathcal{K}) \in \mathsf{Hom}_{D(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2])$ is by definition the composition

$$Y(\mathcal{K}): B \xrightarrow{\beta} \mathcal{K}_A \xrightarrow{\alpha^{-1}} A[2]. \tag{10}$$

We can also consider the complex $\mathcal{K}_B$ given by

$$0 \to A \to X \to Y \to 0$$

where A is in degree -2. The projection $Y \to B$ induces a quasi-isomorphism $\gamma: \mathcal{K}_B \to B$. We furthermore have a canonical map $\delta: \mathcal{K}_B \to A[2]$. We consider the composition $Y'(\mathcal{K}) \in \mathsf{Hom}_{D(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2])$

$$Y'(\mathcal{K}): B \xrightarrow{\gamma^{-1}} \mathcal{K}_B \xrightarrow{\delta} A[2]. \tag{11}$$

Lemma 2.19. *In* $\mathsf{Hom}_{D(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2])$ *we have the equality*

$$Y(\mathcal{K}) = Y'(\mathcal{K}).$$

Proof. We consider the morphism of complexes $\phi\colon \mathcal{K}_B \to \mathcal{K}_A$ given by obvious maps in the diagram

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & B \\
A \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & Y\,. \arrow[u]
\end{tikzcd}$$

It fits into

$$\begin{tikzcd}
B \arrow[r, "\beta"] \arrow[d, equal] & \mathcal{K}_A & A[2] \arrow[l, "\alpha"'] \arrow[d, equal] \\
B & \mathcal{K}_B \arrow[l, "\gamma"] \arrow[r, "\delta"] \arrow[u, "\phi"] & A[2].
\end{tikzcd}$$

It suffices to show that the two squares commute. In fact we will show that ϕ is homotopic to $\beta \circ \gamma$ and $\alpha \circ \delta$. Note that $\phi - \beta \circ \gamma$ is given by

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & B \\
A \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] \arrow[ul, dotted, "\mathrm{id}"] & Y\,. \arrow[u, "0"'] \arrow[ul, dotted, "0"]
\end{tikzcd}$$

The dotted arrows in this diagram gives the zero homotopy of this difference.

Similarly $\phi - \alpha \circ \delta$ is given by

$$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & B \\
A \arrow[r] \arrow[u, "0"] & X \arrow[r] \arrow[u] \arrow[ul, dotted, "0"] & Y\,, \arrow[u] \arrow[ul, dotted, "\mathrm{id}"]
\end{tikzcd}$$

and we have again indicated the required zero homotopy. □

2.5.10 The equivalence between the two-categories $\mathsf{PIC}(\mathbf{S})$ and $C'(\mathbf{S})$ (see 2.16) allows us to classify equivalence classes of Picard stacks with fixed $H^i(P) \cong A_i$, $i = -1, 0$, $A_i \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. An equivalence of such Picard stacks is an equivalence which induces the identity on the cohomology.

Lemma 2.20. *The set* $\mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(A_0, A_{-1})$ *of equivalence classes of Picard stacks* P *with given isomorphisms* $H^i(P) \cong A_i$, $i = 0, -1$, $A_i \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ *is in bijection with* $\mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(A_0, A_{-1})$.

Proof. We define a map

$$\phi\colon \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(A_0, A_{-1}) \to \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(A_0, A_{-1}) \tag{12}$$

as follows.

Consider an exact complex

$$\mathcal{K}\colon 0 \to A_{-1} \to K^{-1} \to K^0 \to A_0 \to 0$$

such that K^{-1} is injective, and let $K \in C(\mathbf{S})$ be the complex

$$0 \to K^{-1} \to K^0 \to 0$$

Then we define $\phi(\mathsf{ch}(K)) := Y(\mathcal{K})$.

We must show that ϕ is well-defined. Indeed, if $\mathsf{ch}(K) \cong \mathsf{ch}(L)$, then (see 2.16) $K \cong L$ by an equivalence which induces the identity on the level of cohomology. It follows that $Y(\mathcal{K}) = Y(\mathcal{L})$.

Since $\mathsf{ch}\colon C'(\mathbf{S}) \to \mathsf{PIC}^\flat(\mathbf{S})$ is an equivalence, hence in particular surjective on the level of equivalence classes, the map ϕ is well-defined. Since every element of $\mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(A_{-1}, A_0)$ can be written as $Y(\mathcal{K})$ for a suitable complex $\mathcal{K}$ as above we conclude that ϕ is surjective. If $\phi(\mathsf{ch}(K)) \cong \phi(\mathsf{ch}(L))$, then there exists $M \in C(\mathbf{S})$ with given isomorphisms $H^i(M) \cong A_i$ together with quasi-isomorphisms $K \leftarrow M \to L$ inducing the identity on cohomology. But this diagram induces an equivalence $\mathsf{ch}(K) \cong \mathsf{ch}(M) \cong \mathsf{ch}(L)$. □

2.5.11 We continue with a further description of $\mathcal{B}F$ for $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, compare 1.2.8. Note that the sheaf of groups of automorphisms of $\mathcal{B}F$ is F, whereas the group of objects is trivial. Therefore we can represent $\mathcal{B}F$ as $\mathsf{ch}(F[1])$, where $F[1]$ is the complex $0 \to F \to 0 \to 0$ concentrated in degree -1.

2.5.12 Let $P \in \mathsf{PIC}(\mathbf{S})$ be a Picard stack and $G \subseteq H^{-1}(P)$. Let us assume that $P = \mathsf{ch}(K)$ for a suitable complex $K\colon 0 \to K^{-1} \to K^0 \to 0$. We have a natural injection $G \hookrightarrow \ker(K^{-1} \to K^0)$. We consider the quotient $\bar{K}^{-1}$ defined by the exact sequence $0 \to G \to K^{-1} \to \bar{K}^{-1} \to 0$ and obtain a new Picard stack $\bar{P} \cong \mathsf{ch}(\bar{K})$, where $\bar{K}\colon 0 \to \bar{K}^{-1} \to K^0 \to 0$. The following diagram represents a sequence of morphisms of Picard stacks

$$\mathcal{B}G \to P \to \bar{P},$$

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K^{-1} \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{K}^{-1} \arrow[r] & K^0 \arrow[r] & 0.
\end{tikzcd}$$

The Picard stack $\bar{P}$ can be considered as a quotient of P by $\mathcal{B}G$. We will employ this construction in 6.4.5.

3 Sheaf theory on big sites of topological spaces

3.1 Topological spaces and sites

3.1.1 In this paper a topological space will always be compactly generated and Hausdorff. We will define categorical limits and colimits in the category of compactly generated Hausdorff spaces. Furthermore, we will equip mapping spaces $\mathsf{Map}(X,Y)$ with the compactly generated topology obtained from the compact-open topology. In this category we have the exponential law

$$\mathsf{Map}(X \times Y, Z) \cong \mathsf{Map}(X, \mathsf{Map}(Y, Z)).$$

By $\mathsf{Map}(X,Y)^\delta$ we will denote the underlying set. For details on this *convenient category of topological spaces* we refer to [Ste67].

3.1.2 The sheaf theory of the present paper refers to the Grothendieck site $\mathbf{S}$. The underlying category of $\mathbf{S}$ is the category of compactly generated topological Hausdorff spaces. The covering families of a space $X \in \mathbf{S}$ are coverings by families of open subsets.

We will also need the sites $\mathbf{S}_{\mathrm{lc}}$ and $\mathbf{S}_{\text{lc-acyc}}$ given by the full subcategories of locally compact and locally compact locally acyclic spaces (see 3.4.6).

3.1.3 We let $\Pr \mathbf{S}$ and $\mathsf{Sh}\mathbf{S}$ denote the category of presheaves and sheaves of sets on $\mathbf{S}$. Then we have an adjoint pair of functors

$$i^\sharp \colon \Pr \mathbf{S} \Leftrightarrow \mathsf{Sh}\mathbf{S} : i$$

where i is the inclusion of sheaves into presheaves, and $i^\sharp$ is the sheafification functor. If $F \in \Pr \mathbf{S}$, then sometimes we will write $F^\sharp := i^\sharp F$.

As before, by $\Pr_{\mathsf{Ab}} \mathbf{S}$ and $\mathsf{Sh}_{\mathsf{Ab}} \mathbf{S}$ we denote the categories of presheaves and sheaves of abelian groups.

3.2 Sheaves of topological groups

3.2.1 In this subsection we collect some general facts about sheaves generated by spaces and topological groups. We formulate the results for the site $\mathbf{S}$. But they remain true if one replaces $\mathbf{S}$ by $\mathbf{S}_{\mathrm{lc}}$ or $\mathbf{S}_{\text{lc-acyc}}$.

3.2.2 Every object $X \in \mathbf{S}$ represents a presheaf $\underline{X} \in \Pr \mathbf{S}$ of sets defined by

$$\mathbf{S} \ni U \mapsto \underline{X}(U) := \mathsf{Hom}_{\mathbf{S}}(U, X) \in \mathsf{Sets}$$

on objects, and by

$$\mathsf{Hom}_{\mathbf{S}}(U, V) \ni f \mapsto f^* \colon \underline{X}(V) \to \underline{X}(U)$$

on morphisms.

Lemma 3.1. *$\underline{X}$ is a sheaf.*

Proof. Straightforward. □

Note that a topology is called sub-canonical if all representable presheaves are sheaves. Hence $\mathbf{S}$ carries a sub-canonical topology.

3.2.3 We will also need the relative version. For $Y \in \mathbf{S}$ we consider the relative site $\mathbf{S}/Y$ of spaces over Y. Its objects are morphisms $A \to Y$, and its morphisms are commutative diagrams

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & Y. & \end{tikzcd}$$

The covering families of $A \to Y$ are induced from the coverings of A by open subsets.

An object $X \to Y \in \mathbf{S}/Y$ represents the presheaf $\underline{X \to Y} \in \Pr \mathbf{S}/Y$ (by a similar definition as in the absolute case 3.2.2). The induced topology on $\mathbf{S}/Y$ is again sub-canonical. In fact, we have the following lemma.

Lemma 3.2. *For all $(X \to Y) \in \mathbf{S}$ the presheaf $\underline{X \to Y}$ is a sheaf.*

Proof. Straightforward. □

3.2.4 Let I be a small category and $X \in \mathbf{S}^I$. Then we have

$$\lim_{i \in I} \underline{X(i)} \cong \underline{\lim_{i \in I} X(i)}.$$

One can not expect a similar property for arbitrary colimits. But we have the following result.

Lemma 3.3. *Let I be a directed partially ordered set and $X \in \mathbf{S}^I$ be a direct system of discrete sets such that $X(i) \to X(j)$ is injective for all $i \leq j$. Then the canonical map*

$$\operatorname{colim}_{i \in I} \underline{X(i)} \to \underline{\operatorname{colim}_{i \in I} X(i)}$$

is an isomorphism.

Proof. Let ${}^p\operatorname{colim}_{i \in I} \underline{X(i)}$ denote the colimit in the sense of presheaves. Then we have an inclusion

$${}^p\operatorname{colim}_{i \in I} \underline{X(i)} \hookrightarrow \underline{\operatorname{colim}_{i \in I} X(i)}$$

(this uses injectivity of the structure maps of the system X). Since the target is a sheaf and sheafification preserves injections it induces an inclusion

$$\operatorname{colim}_{i \in I} \underline{X(i)} \hookrightarrow \underline{\operatorname{colim}_{i \in I} X(i)}. \tag{13}$$

Let now $A \in \mathbf{S}$ and $f \in \underline{\mathsf{colim}_{i\in I} X(i)}(A) = \mathsf{Hom}_{\mathbf{S}}(A, \mathsf{colim}_{i\in I} X(i))$. As $\mathsf{colim}_{i\in I} X(i)$ is discrete and f is continuous the family of subsets $\{f^{-1}(x) \subseteq A \mid x \in \mathsf{colim}_{i\in I} X(i)\}$ is an open covering of A by disjoint open subsets. The family

$$\prod_x f_{|f^{-1}(x)} \in \prod_x {}^p\mathsf{colim}_{i\in I} \underline{X(i)}(f^{-1}(x))$$

represents a section over A of the sheafification $\mathsf{colim}_{i\in I} \underline{X(i)}$ of ${}^p\mathsf{colim}_{i\in I} \underline{X(i)}$ which of course maps to f under the inclusion (13). Therefore (13) is also surjective. $\square$

3.2.5 If $G \in \mathbf{S}$ is a topological abelian group, then $\mathsf{Hom}_{\mathbf{S}}(U, G) := \underline{G}(U)$ has the structure of a group by point-wise multiplications. In this case $\underline{G}$ is a sheaf of abelian groups $\underline{G} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$.

3.2.6 We can pass back and forth between (pre)sheaves of abelian groups and (pre)sheaves of sets using the following adjoint pairs of functors

$${}^p\mathbb{Z}(\ldots) : \mathrm{Pr}\,\mathbf{S} \Leftrightarrow \mathrm{Pr}_{\mathsf{Ab}}\mathbf{S} : \mathcal{F}, \quad \mathbb{Z}(\ldots) : \mathsf{Sh}\mathbf{S} \Leftrightarrow \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} : \mathcal{F}.$$

The functor $\mathcal{F}$ forgets the abelian group structure. The functors ${}^p\mathbb{Z}(\ldots)$ and $\mathbb{Z}(\ldots)$ are called the linearization functors. The presheaf linearization associates to a presheaf of sets $H \in \mathrm{Pr}\,\mathbf{S}$ the presheaf ${}^p\mathbb{Z}(H) \in \mathrm{Pr}_{\mathsf{Ab}}\,\mathbf{S}$ which sends $U \in \mathbf{S}$ to the free abelian group $\mathbb{Z}(H)(U)$ generated by the set $H(U)$. The linearization functor for sheaves is the composition $\mathbb{Z}(\ldots) := i^\sharp \circ {}^p\mathbb{Z}(\ldots)$ of the presheaf linearization and the sheafification.

Lemma 3.4. *Consider an exact sequence of topological groups in* $\mathbf{S}$

$$1 \to G \to H \to L \to 1.$$

Then $1 \to \underline{G} \to \underline{H} \to \underline{L}$ *is an exact sequence of sheaves of groups.*

The map of topological spaces $H \to L$ *has local sections if and only if*

$$1 \to \underline{G} \to \underline{H} \to \underline{L} \to 1$$

is an exact sequence of sheaves of groups.

Proof. Exactness at $\underline{G}$ and $\underline{H}$ is clear.

Assume the existence of local sections of $H \to L$. This implies exactness at $\underline{L}$.

On the other hand, evaluating the sequence at the object $L \in \mathbf{S}$ we see that the exactness of the sequence of sheaves implies the existence of local sections to $H \to L$. $\square$

3.2.7 For sheaves $G, H \in \mathsf{Sh}\mathbf{S}$ we define the sheaf

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(G, H) \in \mathsf{Sh}\mathbf{S}, \quad \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(G, H)(U) := \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(G_{|U}, H_{|U}).$$

Again, this a priori defines a presheaf, but one checks the sheaf conditions in a straightforward manner.

3.2.8 If $X, H \in \mathbf{S}$ and H is in addition a topological abelian group, then $\mathsf{Map}(X, H)$ is again a topological abelian group. For a topological abelian group $G \in \mathbf{S}$ we let $\mathsf{Hom}_{\text{top-Ab}}(G, H) \subseteq \mathsf{Map}(G, H)$ be the closed subgroup of homomorphisms. Recall the construction of the internal $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\dots,\dots)$, compare 3.2.7.

Lemma 3.5. *Assume that $G, H \in \mathbf{S}$ are topological abelian groups. Then we have a canonical isomorphism of sheaves of abelian groups*

$$e\colon \underline{\mathsf{Hom}_{\text{top-Ab}}(G, H)} \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{H}).$$

Proof. We first describe the morphism e. Let $\phi \in \underline{\mathsf{Hom}(G, H)}(U)$. We define the element $e(\phi) \in \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{H})(U)$, i.e. a morphism of sheaves $e(\phi)\colon \underline{G}_{|U} \to \underline{H}_{|U}$, such that it sends $f \in \underline{G}(\sigma\colon V \to U)$ to $\{V \ni v \mapsto \phi(\sigma(v))f(v)\} \in \underline{H}(\sigma\colon V \to U)$.

Let us now describe the inverse e^{-1}. Let $\psi \in \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{H})(U)$ be given. Since $(\mathsf{pr}_U : U \times G \to U) \in \mathbf{S}/U$ it gives rise to a map

$$\psi_G\colon \underline{G}(\mathsf{pr}_U\colon U \times G \to U) \to \underline{H}(\mathsf{pr}_U\colon U \times G \to U).$$

We now consider $(\mathsf{pr}_G\colon U \times G \to G) \in \underline{G}(\mathsf{pr}_U\colon U \times G \to U)$ and set $\phi_G := \psi_G(\mathsf{pr}_G) \in \underline{H}(\mathsf{pr}_U\colon U \times G \to U) = \mathsf{Hom}_{\mathbf{S}}(U \times G, H)$. We now invoke the exponential law isomorphism $\exp\colon \mathsf{Hom}_{\mathbf{S}}(U \times G, H) \xrightarrow{\sim} \mathsf{Hom}_{\mathbf{S}}(U, \mathsf{Map}(G, H))$. Since ψ was a homomorphism and using the sheaf property, we actually get an element $e^{-1}(\psi) := \exp(\phi_G) \in \mathsf{Hom}_{\mathbf{S}}(U, \mathsf{Hom}_{\text{top-Ab}}(G, H)) = \underline{\mathsf{Hom}_{\text{top-Ab}}(G, H)}(U)$. (Details of the argument for this fact are left to the reader.) $\square$

If $\mathbf{S}$ is replaced by one of the sub-sites $\mathbf{S}_{\text{lc}}$ or $\mathbf{S}_{\text{lc-acyc}}$, then it may happen that $\mathsf{Hom}_{\text{top-Ab}}(G, H)$ does not belong to the sub-site. In this case Lemma 3.5 remains true if one interprets $\underline{\mathsf{Hom}_{\text{top-Ab}}(G, H)}$ as the restriction of the sheaf on $\mathbf{S}$ represented by $\mathsf{Hom}_{\text{top-Ab}}(G, H)$ to the corresponding sub-site.

3.3 Restriction

3.3.1 In this subsection we prove a general result in sheaf theory (Proposition 3.12) which is probably well-known, but which we could not locate in the literature. Let $\mathbf{S}$ be a site. For $Y \in \mathbf{S}$ we can consider the relative site $\mathbf{S}/Y$. Its objects are maps $(U \to Y)$ in $\mathbf{S}$, and its morphisms are diagrams

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & Y. & \end{tikzcd}$$

The covering families for $\mathbf{S}/Y$ are induced by the covering families of $\mathbf{S}$, i.e. for $(U \to Y) \in \mathbf{S}/Y$ a covering family $\tau := (U_i \to U)_{i \in I}$ induces in $\mathbf{S}/Y$ a covering family

$$\tau_{|Y} := \left(\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & U \arrow[dl] \\ & Y & \end{tikzcd} \right)_{i\in I} .$$

3.3.2 There is a canonical functor $f : \mathbf{S}/Y \to \mathbf{S}$ given on objects by $f(U \to Y) := U$. It induces adjoint pairs of functors

$$f_* : \mathsf{Sh}\mathbf{S}/Y \Leftrightarrow \mathsf{Sh}\mathbf{S} : f^*, \quad {}^pf_* : \mathrm{Pr}\,\mathbf{S}/Y \Leftrightarrow \mathrm{Pr}\,\mathbf{S} : {}^pf^*.$$

We will often write $f^*(F) =: F_{|Y}$ for the restriction functor. For $F \in \mathrm{Pr}\,\mathbf{S}$ it is given in explicit terms by

$$F_{|Y}(U \to Y) := F(U).$$

The functor f^* is in fact the restriction of ${}^pf^*$ to the subcategory of sheaves $\mathsf{Sh}\mathbf{S} \subseteq \mathrm{Pr}\,\mathbf{S}$. If $F \in \mathsf{Sh}\mathbf{S}$, then one must check that ${}^pf^*F$ is a sheaf. To this end note that the descent conditions for ${}^pf^*F$ with respect to the induced coverings described in 3.3.1 immediately follow from the descent conditions for F.

3.3.3 Let us now study how the restriction acts on representable sheaves. We assume that $\mathbf{S}$ has finite products. Let $X, Y \in \mathbf{S}$.

Lemma 3.6. *If* $\mathbf{S}$ *has finite products, then we have a natural isomorphism*

$$\underline{X}_{|Y} \cong \underline{X \times Y \to Y}.$$

Proof. By the universal property of the product for $\sigma : U \to Y$ we have

$$\begin{aligned} \underline{X \times Y \to Y}(U \to Y) &= \{h \in \mathsf{Hom}_{\mathbf{S}}(U, X \times Y) \mid \mathsf{pr}_Y \circ h = \sigma\} \\ &\cong \mathsf{Hom}_{\mathbf{S}}(U, X) \\ &= \underline{X}(U) \\ &= \underline{X}_{|Y}(U \to Y). \end{aligned}$$

□

3.3.4 Let $i^\sharp\colon \mathrm{Pr}\,\mathbf{S} \to \mathsf{Sh}\mathbf{S}$ and $i^\sharp_Y\colon \mathrm{Pr}\,\mathbf{S}/Y \to \mathsf{Sh}\mathbf{S}/Y$ be the sheafification functors.

Lemma 3.7. *We have a canonical isomorphism* $f^* \circ i^\sharp \cong i^\sharp_Y \circ {}^pf^*$.

Proof. The argument is similar to that in the proof of [BSS, Lemma 2.31]. The main point is that the sheafifications $i^\sharp F(U)$ and $i^\sharp_Y F_{|Y}(U \to Y)$ can be expressed in terms of the category of covering families $\mathsf{cov}_{\mathbf{S}}(U)$ and $\mathsf{cov}_{\mathbf{S}/Y}(U \to Y)$. Compared with [BSS, Lemma 2.31] the argument is simplified by the fact that the induction of covering families described in 3.3.1 induces an isomorphism of categories $\mathsf{cov}_{\mathbf{S}}(U) \to \mathsf{cov}_{\mathbf{S}/Y}(U \to Y)$. □

3.3.5 Recall that a functor is called exact if it commutes with limits and colimits.

Lemma 3.8. *The restriction functor* $f^*\colon \mathsf{Sh}\mathbf{S} \to \mathsf{Sh}\mathbf{S}/Y$ *is exact.*

Proof. The functor f^* is a right-adjoint and therefore commutes with limits. Since colimits of presheaves are defined object-wise, i.e. for a diagram

$$\mathcal{C} \to \mathrm{Pr}\,\mathbf{S}, \quad c \mapsto F_c$$

of presheaves we have

$$({}^p\mathsf{colim}_{c\in\mathcal{C}} F_c)(U) = \mathsf{colim}_{c\in\mathcal{C}} F_c(U),$$

it follows from the explicit description of ${}^pf^*$, that it commutes with colimits of presheaves. Indeed, for a diagram of sheaves $F\colon \mathcal{C} \to \mathsf{Sh}\mathbf{S}$ we have

$$\mathsf{colim}_{c\in\mathcal{C}} F_c = i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c.$$

By Lemma 3.7 we get $f^*\mathsf{colim}_{c\in\mathcal{C}} F_c \cong f^* i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c \cong i_Y^\sharp\, {}^pf^*\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c \cong i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}}\, {}^pf^* F_c \cong \mathsf{colim}_{c\in\mathcal{C}} f^* F_c$. □

3.3.6 Let $H \in \mathsf{Sh}\mathbf{S}$ and $X \in \mathbf{S}$.

Lemma 3.9. *If* $\mathbf{S}$ *has finite products, then for* $U \in \mathbf{S}$ *we have a natural bijection*

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{X}, H)(U) \cong H(U \times X).$$

Proof. We have the following chain of isomorphisms

$$\begin{aligned}\underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{X}, H)(U) &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{X}_{|U}, H_{|U})\\ &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{U \times X \to U}, H_{|U}) \quad \text{(by Lemma 3.6)}\\ &\cong H(U \times X).\end{aligned}$$ □

We will need the explicit description of this bijection. We define a map

$$\Psi\colon \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{X}, H)(U) \to H(U \times X)$$

as follows. An element $h \in \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}/U}(\underline{X}, H)(U) = \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(\underline{X}_{|U}, H_{|U})$ induces a map $\tilde{h}\colon \underline{X}_{|U}(U \times X \to U) \to H_{|U}(U \times X \to U)$. Now we have $\underline{X}_{|U}(U \times X \to U) = \underline{X}(U \times X) = \mathsf{Map}(U \times X, X)$ and $H_{|U}(U \times X \to U) = H(U \times X)$. Let $\mathsf{pr}_X\colon U \times X \to X$ be the projection. We define $\Psi(h) := \tilde{h}(\mathsf{pr}_X)$.

We now define $\Phi\colon H(U\times X) \to \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{X}, H)(U)$. Let $g \in H(U\times X)$. For each $(e\colon V \to U) \in \mathbf{S}/U$ we must define a map $\hat{g}_{(V\to U)}\colon \underline{X}_{|U}(V \to U) \to H_{|U}(V \to U)$. Note that $\underline{X}_{|U}(V \to U) = \underline{X}(V) = \mathsf{Map}(V, X)^\delta$. Let $x \in \mathsf{Map}(V, X)$. Then we have an induced map $(e, x)\colon V \to U \times X$. We define $\Psi(g) := H(e, x)(g)$. One checks that this construction is functorial in e and therefore defines a morphism of sheaves.

A straight forward calculation shows that Ψ and Φ are inverses to each other and induce the bijection above.

3.3.7 For a map $U \to V$ we have an isomorphism of sites $\mathbf{S}/U \cong (\mathbf{S}/V)/(U \to V)$. Several formulas below implicitly contain this identification. For example, we have a canonical isomorphism

$$F_{|U} \cong (F_{|V})_{|U\to V}$$

for $F \in \mathsf{Sh}\mathbf{S}$. For $G, H \in \mathsf{Sh}\mathbf{S}$ this induces isomorphisms

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(G,H)_{|V} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}/V}(G_{|V},H_{|V}). \tag{14}$$

3.3.8 We now consider a sheaf of abelian groups $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. If $\tau \in \mathsf{cov}_{\mathbf{S}}(U)$ is a covering family of U and $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, then we can define the Čech complex $\check{C}^*(\tau; H)$ (see [BSS, 2.3.5]).

Definition 3.10. The sheaf H is called flabby if $H^i(\check{C}^*(\tau; H)) \cong 0$ for all $i \geq 1$, $U \in \mathbf{S}$ and $\tau \in \mathsf{cov}_{\mathbf{S}}(U)$.

For $A \in \mathbf{S}$ we have the section functor

$$\Gamma(A;\dots)\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathsf{Ab}, \quad \Gamma(A;H) := H(A).$$

This functor is left exact and admits a right-derived functor $R\Gamma(A;\dots)$ which can be calculated using injective resolutions. Note that $\Gamma(A;\dots)$ is acyclic on flabby sheaves, i.e. $R^i\Gamma(A;H) \cong 0$ for $i \geq 1$ if H is flabby. Hence, the derived functor $R\Gamma(A;\dots)$ can also be calculated using flabby resolutions. Note that an injective sheaf is flabby.

Lemma 3.11. *The restriction functor* $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y$ *preserves flabby sheaves.*

Proof. Let $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ be flabby. For $(U \to Y) \in \mathbf{S}/Y$ and $\tau \in \mathsf{cov}_{\mathbf{S}}(U)$ we let $\tau_{|Y} \in \mathsf{cov}_{\mathbf{S}/Y}(U \to Y)$ be the induced covering family as in 3.3.1. Note that $\check{C}^*(\tau_{|Y}; H_{|Y}) \cong \check{C}^*(\tau; H)$ is acyclic. Since $\mathsf{cov}_{\mathbf{S}/Y}(U \to Y)$ is exhausted by families of the form $\tau_{|Y}$, $\tau \in \mathsf{cov}_{\mathbf{S}}(U)$ it follows that $H_{|Y}$ is flabby. □

3.3.9 We now consider sheaves of abelian groups $F, G \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$.

Proposition 3.12. *Assume that the site* $\mathbf{S}$ *has the property that for all* $U \in \mathbf{S}$ *the restriction* $\mathsf{Sh}\mathbf{S} \to \mathsf{Sh}\mathbf{S}/U$ *preserves representable sheaves. If* $Y \in \mathbf{S}$*, then in* $D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y)$ *there is a natural isomorphism*

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G)_{|Y} \cong R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y},G_{|Y}).$$

Proof. We choose an injective resolution $G \to I$ and furthermore an injective resolution $I_{|Y} \to J$. Note that by Lemma 3.11 restriction is exact and therefore $G_{|Y} \to J$ is a resolution of $G_{|Y}$. Then we have

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G)_{|Y} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,I)_{|Y} \overset{(14)}{\cong} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y},I_{|Y}).$$

Furthermore,

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y},G_{|Y}) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y},J)$$

and the map $I_{|Y} \to J$ induces

$$
\begin{aligned}
R\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(F,G)_{|Y} &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}/Y}(F_{|Y}, I_{|Y}) \\
&\stackrel{*}{\to} \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}/Y}(F_{|Y}, J) \qquad (15)\\
&\cong R\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}/Y}(F_{|Y}, G_{|Y}).
\end{aligned}
$$

If the restriction $f^*\colon \mathsf{Sh}\mathbf{S} \to \mathsf{Sh}\mathbf{S}/Y$ would preserve injectives, then $I_{|Y} \to J$ would be a homotopy equivalence and the marked map would be a quasi-isomorphism.

In the generality of the present paper we do not know whether f^* preserves injectives. Nevertheless we show that our assumption on $\mathbf{S}$ implies that the marked map in (15) is a quasi isomorphism for all sheaves $F \in \mathsf{Sh_{Ab}}\mathbf{S}$.

3.3.10 We first show a special case.

Lemma 3.13. *If F is representable, then the marked map in* (15) *is a quasi isomorphism.*

Proof. Let $(U \to Y) \in \mathbf{S}/Y$ and $(A \to Y) \in \mathbf{S}$. Then on the one hand we have

$$
\begin{aligned}
\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}/Y}(\mathbb{Z}(\underline{A})_{|Y}, I_{|Y})(U \to Y) &= \mathsf{Hom}_{(\mathsf{Sh}\mathbf{S}/Y)/(U\to Y)}((\underline{A}_{|Y})_{|(U\to Y)}, (I_{|Y})_{|U\to Y}) \\
&\quad \text{(by 3.2.7)} \\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{A}_{|U}, I_{|U}) \\
&\quad \text{(by 3.3.7)}.
\end{aligned}
$$

Since $\underline{A}_{|U}$ is representable by our assumption on $\mathbf{S}$ there exists $(B \to U) \in \mathbf{S}/U$ such that $\underline{A}_{|U} \cong \underline{B \to U}$.

Since the restriction functor is exact (Lemma 3.8) and the restriction of an injective sheaf from $\mathbf{S}$ to $\mathbf{S}/Y$ is still flabby (Lemma 3.11) we see that $G_{|U} \to I_{|U}$ is a flabby resolution. It follows that

$$
\begin{aligned}
\mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{A}_{|U}, I_{|U}) &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{B \to U}, I_{|U}) \\
&\cong I_{|U}(B \to U) \cong R\Gamma(B \to U, G_{|U}).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}/Y}(\mathbb{Z}(\underline{A})_{|Y}, J)(U \to Y) &\cong \mathsf{Hom}_{(\mathsf{Sh}\mathbf{S}/Y)/(U\to Y)}((\underline{A}_{|Y})_{|U\to Y}, J_{|U\to Y}) \\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{A}_{|U}, J_{|U}) \\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{B \to U}, J_{|U}) \\
&\cong J(B \to U) \\
&\cong R\Gamma(B \to U, G_{|U}).
\end{aligned}
$$

$\square$

3.3.11 We consider the functor which associates to $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ the cone

$$C(F) := \mathsf{Cone}\left(\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, I_{|Y}) \to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, J)\right).$$

In order to show that the marked map in (15) is a quasi isomorphism we must show that $C(F)$ is acyclic for all $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$.

The restriction functor $(\dots)_{|Y}$ is exact by Lemma 3.8. For $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y$ the functor $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(\dots, H)$ is a right-adjoint. As a contravariant functor it transforms colimits into limits and is left exact. In particular, the functors

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}((\dots)_{|Y}, I_{|Y}), \quad \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}((\dots)_{|Y}, J)$$

transform coproducts of sheaves into products of complexes. It follows that $F \to C(F)$ also transforms coproducts of sheaves into products of complexes. If $P \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is a coproduct of representable sheaves, then by Lemma 3.13 $C(A)$ is a product of acyclic complexes and hence acyclic.

We claim that $F \to C(F)$ transforms short exact sequences of sheaves to short exact sequences of complexes. Since J is injective and $(\dots)_{|Y}$ is exact, the functor $F \mapsto \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, J)$ is a composition of exact functors and thus has this property. Furthermore we have $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, I_{|Y}) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, I)_{|Y}$. Since I is injective, the functor $F \to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, I_{|Y})$ has this property, too. This implies the claim.

We now argue by induction. Let $n \in \mathbb{N}$ and assume that we have already shown that $H^i(C(F)) \cong 0$ for all $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and $i < n$.

Consider $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. Then there exists a coproduct of representables $P \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and an exact sequence

$$0 \to K \to P \to F \to 0.$$

In order to construct P observe that in general one has a canonical isomorphism

$$F \cong \mathsf{colim}_{\underline{A} \to F}\,\underline{A}.$$

We let $P := \bigoplus_{\underline{A} \to F} \underline{A}$. The collection of maps $\underline{A} \to F$ in the index category induces a canonical surjection

$$P = \bigoplus_{\underline{A} \to F} \underline{A} \to \mathsf{colim}_{\underline{A} \to F}\,\underline{A} \cong F.$$

The short exact sequence of complexes

$$0 \to C(F) \to C(P) \to C(K) \to 0$$

induces a long exact sequence in cohomology. Since $H^i(P) \cong 0$ for all $i \in \mathbb{Z}$ we conclude that $H^i(C(F)) \cong H^{i-1}(C(K))$ for all $i \in \mathbb{N}$. By our induction hypothesis we get $H^n(C(F)) \cong 0$.

By induction on n we show that $C(F)$ is acyclic for all $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, and this implies Proposition 3.12. □

3.3.12 Note that a site $\mathbf{S}$ which has finite products satisfies the assumption of Proposition 3.12. In fact, for $A \in \mathbf{S}$ we have by Lemma 3.6 that

$$\underline{A}_{|Y} \cong \underline{A \times Y \to Y}.$$

Therefore the restriction functor $(\dots)_{|Y}$ preserves representables.

Corollary 3.14. *Let* $\mathbf{S}$ *be as in Proposition* 3.12. *For sheaves* $F, G \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ *and* $Y \in \mathbf{S}$ *we have*

$$\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G)_{|Y} \cong \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\mathbf{S}/Y}(F_{|Y}, G_{|Y}).$$

Proof. Since $(\dots)_{|Y}$ is exact (Lemma 3.8) we have

$$\begin{aligned}
\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G)_{|Y} &\cong (H^i R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G))_{|Y}\\
&\cong H^i(R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,G)_{|Y})\\
&\cong H^i(R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/Y}(F_{|Y}, G_{|Y})) \quad \text{(by Proposition 3.12)}\\
&\cong \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\mathbf{S}/Y}(F_{|Y}, G_{|Y}).
\end{aligned}$$

□

3.3.13 Let us assume that $\mathbf{S}$ has finite fibre products. Fix $U \in \mathbf{S}$. Then we can define a morphism of sites ${}^U\times : \mathbf{S} \to \mathbf{S}/U$ which maps $A \in \mathbf{S}$ to $(U \times A \to U) \in \mathbf{S}/U$. One easily checks the conditions given in [Tam94, 1.2.2]. This morphism of sites induces an adjoint pair of functors

$${}^U\times_* : \mathsf{Sh}\mathbf{S} \Leftrightarrow \mathsf{Sh}\mathbf{S}/U : {}^U\times^* .$$

Let $f : \mathbf{S}/U \to \mathbf{S}$ be the restriction defined in 3.3.2 (with Y replaced by U) and let

$$f_* : \mathsf{Sh}\mathbf{S}/U \Leftrightarrow \mathsf{Sh}\mathbf{S} : f^*$$

be the corresponding pair of adjoint functors.

Lemma 3.15. *We assume that* $\mathbf{S}$ *has finite products. Then we have a canonical isomorphism* $f^* \cong {}^U\times_*$.

Proof. We first define this isomorphism on representable sheaves. Since every sheaf can be written as a colimit of representable sheaves, ${}^U\times_*$ commutes with colimits as a left-adjoint, and f^* commutes with colimits by Lemma 3.8, the isomorphism then extends to all sheaves. Let $W \in \mathbf{S}$ and $F \in \mathsf{Sh}\mathbf{S}/U$). Then we have

$$\begin{aligned}
\mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}({}^U\times_* \underline{W}, F) &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(\underline{W}, {}^U\times^*(F))\\
&\cong {}^U\times^*(F)(W)\\
&\cong F(W \times U \to U)\\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(\underline{(W \times U \to U)}, F) \quad \text{(by Lemma 3.6)}\\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/U}(f^*\underline{W}, F)
\end{aligned}$$

This isomorphism is represented by a canonical isomorphism

$${}^U\times_*(\underline{W}) \cong f^*(\underline{W})$$

in $\mathsf{Sh}\mathbf{S}/U$. □

3.3.14 Let $f : \mathbf{S} \to \mathbf{S}'$ be a morphism of sites. It induces an adjoint pair

$$f_* : \mathsf{Sh}\mathbf{S} \Leftrightarrow \mathsf{Sh}\mathbf{S}' : f^*.$$

Lemma 3.16. *For $F \in \mathsf{Sh}\mathbf{S}$ the functor f_* has the explicit description*

$$f_*(F) := \mathsf{colim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S}/F)} \underline{f(U)}.$$

Proof. In fact, for $G \in \mathsf{Sh}\mathbf{S}'$ we have

$$\begin{aligned}
\mathsf{Hom}_{\mathsf{Sh}\mathbf{S}'}(\mathsf{colim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{f(U)}, G) &\cong \mathsf{lim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}'}(\underline{f(U)}, G) \\
&\cong \mathsf{lim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} G(f(U)) \\
&\cong \mathsf{lim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} f^*G(U) \\
&\cong \mathsf{lim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}'}(\underline{U}, f^*G) \\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(\mathsf{colim}_{(\underline{U} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{U}, f^*G) \\
&\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(F, f^*G).
\end{aligned}$$

□

Lemma 3.17. *For $A \in \mathbf{S}$ we have*

$$f_*(\underline{A}) = \underline{f(A)}.$$

Proof. Since $(\mathsf{id}_A : A \to A) \in \mathbf{S}/A$ is final we have

$$\begin{aligned}
f_*(\underline{A}) &\cong \mathsf{colim}_{(\underline{U} \to \underline{A}) \in \mathsf{Sh}(\mathbf{S})/\underline{A}} \underline{f(U)} \\
&\cong \mathsf{colim}_{(U \to A) \in \mathbf{S}/A} \underline{f(U)} \\
&\cong \underline{f(A)}.
\end{aligned}$$

□

Lemma 3.18. *If $f : \mathbf{S} \to \mathbf{S}'$ is fully faithful, then we have for $A \in \mathbf{S}$ that $f^* f_* \underline{A} \cong \underline{A}$.*

Proof. We calculate for $U \in \mathbf{S}$ that

$$\begin{aligned}
f^* f_* \underline{A}(U) &\cong f^* \underline{f(A)}(U) \quad \text{(by Lemma 3.17)} \\
&\cong \underline{f(A)}(f(U)) \\
&\cong \mathsf{Hom}_{\mathbf{S}'}(f(U), f(A)) \\
&\cong \mathsf{Hom}_{\mathbf{S}}(U, A) \\
&\cong \underline{A}(U)
\end{aligned}$$

□

3.3.15 For $U \in \mathbf{S}$ we have an induced morphism of sites ${}_U f : \mathbf{S}/U \to \mathbf{S}'/f(U)$ which maps $(A \to U)$ to $(f(A) \to f(U))$.

If f is fully faithful, then ${}_U f$ is fully faithful for all $U \in \mathbf{S}$.

Lemma 3.19. *For $G \in \mathsf{Sh}\mathbf{S}'$ we have in $\mathbf{S}/U$ the identity*

$${}_U f^*(G_{|f(U)}) \cong (f^*G)_{|U}.$$

If we know in addition that $\mathbf{S}$, $\mathbf{S}'$ have finite products which are preserved by $f : \mathbf{S} \to \mathbf{S}'$, then for $F \in \mathbf{S}$ we have in $\mathbf{S}'/f(U)$ the identity

$${}_U f_* F_{|U} \cong (f_* F)_{|f(U)}.$$

Proof. Indeed, for $(V \to U) \in \mathbf{S}/U$ we have

$$\begin{aligned}(f^*G)_{|U}(V \to U) &= G(f(V)) \\ &= G_{|f(U)}(f(V) \to f(U)) = ({}_U f^* G_{|f(U)})(V \to U).\end{aligned}$$

In order to see the second identity note that we can write

$$F_{|U} \cong \mathrm{colim}_{(\underline{A} \to F_{|U}) \in \mathsf{Sh}(\mathbf{S}/U)/F_{|U}} \underline{A}.$$

Since ${}_U f_*$ is a left-adjoint it commutes with colimits. Furthermore the restriction functors $(\dots)_{|U}$ and $(\dots)_{f(U)}$ are exact by Lemma 3.8 and therefore also commute with colimits. Writing

$$F \cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{A}$$

we get

$$\begin{aligned}F_{|U} &\cong (\mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{A})_{|U} \cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{A}_{|U} \\ &\cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{A \times U \to A} \quad \text{(by Lemma 3.6).}\end{aligned}$$

Using that f preserves products in the isomorphism marked by (!) we calculate

$$\begin{aligned}{}_U f_* F_{|U} &\cong {}_U f_* \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{A \times U \to U} \\ &\cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \; {}_U f_* \underline{A \times U \to U} \\ &\cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{f(A \times U) \to f(U)} \quad \text{(by Lemma 3.17)} \\ &\overset{(!)}{\cong} \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{f(A) \times f(U) \to f(U)} \\ &\cong \mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{f(A)}_{|f(U)} \\ &\cong (\mathrm{colim}_{(\underline{A} \to F) \in \mathsf{Sh}(\mathbf{S})/F} \underline{f(A)})_{|f(U)} \\ &\cong (f_* F)_{|f(U)}. \quad \text{(by Lemma 3.16)}\end{aligned}$$

□

Lemma 3.20. *Assume that $\mathbf{S}, \mathbf{S}'$ have finite products which are preserved by $f : \mathbf{S} \to \mathbf{S}'$. For $F \in \mathsf{Sh}\mathbf{S}$ and $G \in \mathsf{Sh}\mathbf{S}'$ we have a natural isomorphism*

$$\underline{\mathrm{Hom}}_{\mathsf{Sh}\mathbf{S}}(F, f^*G) \cong f^* \underline{\mathrm{Hom}}_{\mathsf{Sh}\mathbf{S}'}(f_*F, G).$$

Proof. For $U \in \mathbf{S}$ we calculate

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathsf{Sh}\mathbf{S}}(F, f^*G)(U) &= \mathrm{Hom}_{\mathsf{Sh}\mathbf{S}/U}(F_{|U}, (f^*G)_{|U}) \\ &\cong \mathrm{Hom}_{\mathsf{Sh}\mathbf{S}/U}(F_{|U}, {}^U f^*(G_{|f(U)})) \quad \text{(by Lemma 3.19)} \\ &\cong \mathrm{Hom}_{\mathsf{Sh}\mathbf{S}'/f(U)}({}^U f_*(F_{|U}), G_{|f(U)}) \\ &\cong \mathrm{Hom}_{\mathsf{Sh}\mathbf{S}'/f(U)}((f_*F)_{|f(U)}, G_{|f(U)}) \quad \text{(by Lemma 3.19)} \\ &= \underline{\mathrm{Hom}}_{\mathsf{Sh}\mathbf{S}'}(f_*F, G)(f(U)) \\ &= f^* \underline{\mathrm{Hom}}_{\mathsf{Sh}\mathbf{S}'}(f_*F, G)(U)\end{aligned}$$

□

3.3.16 We now consider the derived version of Lemma 3.20. Let $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and $G \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}'$ be sheaves of abelian groups.

Proposition 3.21. *We make the following assumptions:*

1. *The sites* $\mathbf{S}, \mathbf{S}'$ *have finite products.*
2. *The morphism of sites* $f : \mathbf{S} \to \mathbf{S}'$ *preserves finite products.*
3. *We assume that* $f^* : \mathsf{Sh}\mathbf{S}' \to \mathsf{Sh}\mathbf{S}$ *is exact.*

Then in $D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ *there is a natural isomorphism*

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, f^*G) \cong f^*R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}'}(f_*F, G).$$

Proof. By [Tam94, Proposition 3.6.7] the conditions on the sites and f imply that the functor f_* is exact. It follows that f^* preserves injectives (see e.g. [Tam94, Proposition 3.6.2]). We choose an injective resolution $G \to I^\bullet$. Then $f^*G \to f^*I^\bullet$ is an injective resolution of f^*G. It follows that

$$\begin{aligned} f^*R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}'}(f_*F, G) &\cong f^*\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}'}(f_*F, I^\bullet) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, f^*I^\bullet) \cong R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, f^*G). \end{aligned}$$ □

Proposition 3.21 is very similar in spirit to Proposition 3.12. On the other hand, we can not deduce 3.12 from 3.21 since the functor $f : \mathsf{Sh}\mathbf{S}/U \to \mathbf{S}$ in general does not preserve products. In fact, if $\mathbf{S}$ has fibre products, then $(A \to U) \times (B \to U) = (A \times_U B \to U)$. Then $f((A \to U) \times (B \to U)) \cong A \times_U B$ while $f(A \to U) \times f(B \to U) \cong A \times B$.

3.3.17 Recall the notations ${}^W\!\times, {}_W\times$ from 3.3.13.

Lemma 3.22. *Let* $W \in \mathbf{S}$. *For every* $F \in \mathsf{Sh}\mathbf{S}$ *and* $A \in \mathbf{S}$ *we have*

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{W}, F)(A) \cong {}^W\times^*\,{}^W\times_*\,F(A).$$

Proof. For $A \in \mathbf{S}$ we consider $f : \mathbf{S}/A \to \mathbf{S}$ and get

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{W}, F)(A) &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/A}(\underline{W}_{|A}, F_{|A}) \\ &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/A}(f^*\underline{W}, f^*F) \\ &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}/A}({}^A\times_*(\underline{W}), f^*F) \quad \text{(by Lemma 3.15)} \\ &\cong \mathsf{Hom}_{\mathsf{Sh}\mathbf{S}}(\underline{W}, {}^A\times^* f^*(F)) \\ &\cong {}^A\times^*(f^*F)(W) \qquad (16) \\ &\cong f^*F(A \times W \to A) \\ &\cong F(A \times W) \\ &\cong {}^W\times^*(f^*F)(A) \\ &\cong {}^W\times^*({}^W\times_*(F))(A) \quad \text{(by Lemma 3.15).} \end{aligned}$$ □

3.3.18 To abbreviate, let us introduce the following notation.

Definition 3.23. If $\mathbf{S}$ has finite products, then for $W \in \mathbf{S}$ we introduce the functor

$$\mathcal{R}_W := {}^W\times^* \circ {}^W\times_* \colon \mathsf{Sh}\mathbf{S} \to \mathsf{Sh}\mathbf{S}.$$

We denote its restriction to the category of sheaves of abelian groups by the same symbol. Since ${}^W\times_*$ is exact and ${}^W\times^*$ is left-exact, it admits a right-derived functor $R\mathcal{R}_W \colon D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}) \to D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$.

Lemma 3.24. *Let $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and $W \in \mathbf{S}$. Then we have a canonical isomorphism* $R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}), F) \cong R\mathcal{R}_W(F)$.

Proof. This follows from the isomorphism of functors $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}), \dots) \cong \mathcal{R}_W(\dots)$ from $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ to $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. In fact, we have for $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ that

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}), F) &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}\mathbf{S}}(\underline{W}, \mathcal{F}(F)) \\ &\cong {}^W\times^* ({}^W\times_*(F)) \quad \text{(Equation 16)} \qquad (17) \\ &\cong \mathcal{R}_W(F) \quad \text{(Definition 3.23).} \end{aligned}$$

□

3.4 Application to sites of topological spaces

3.4.1 In this subsection we consider the site $\mathbf{S}$ of compactly generated topological spaces as in 3.1.2 and some of its sub-sites. We are interested in proving that restriction to sub-sites preserve $\underline{\mathsf{Ext}}^i$-sheaves.

We will further study properties of the functor $\mathcal{R}_W$. In particular, we are interested in results asserting that the higher derived functors $R^i\mathcal{R}_W(F)$, $i \geq 1$ vanish under certain conditions on F and W.

Lemma 3.25. *If $C \in \mathbf{S}$ is compact and H is a discrete space, then* $\mathsf{Map}(C, H)$ *is discrete, and*

$$\mathcal{R}_C(\underline{H}) = \underline{\mathsf{Map}(C, H)}.$$

Proof. We first show that $\mathsf{Map}(C, H)$ is a discrete space in the compact-open topology. Let $f \in \mathsf{Map}(C, H)$. Since C is compact, the image $f(C)$ is compact, hence finite. We must show that $\{f\} \subseteq \mathsf{Map}(C, H)$ is open. Let $h_1, \dots, h_r$ be the finite set of values of f. The sets $f^{-1}(h_i) \subseteq C$ are closed and therefore compact and their union is C. The sets $\{h_i\} \subseteq H$ are open. Therefore $U_i := \{g \in \mathsf{Map}(C, H) \mid g(f^{-1}(h_i)) \subseteq \{h_i\}\}$ are open subsets of $\mathsf{Map}(C, H)$. We now see that $\{f\} = \bigcap_{i=1}^r U_i$ is open.

We have by the exponential law

$$\begin{aligned} \mathcal{R}_C(\underline{H})(A) &\cong \underline{H}(A \times C) \cong \mathsf{Hom}_{\mathbf{S}}(A \times C, H) \\ &\cong \mathsf{Hom}_{\mathbf{S}}(A, \mathsf{Map}(C, H)) \cong \underline{\mathsf{Map}(C, H)}(A). \end{aligned}$$

□

3.4.2 A sheaf $G \in \mathsf{Sh_{Ab}}\mathbf{S}$ is called $\mathcal{R}_W$-acyclic if $R^i\mathcal{R}_W(G) \cong 0$ for $i \geq 1$.

Lemma 3.26. *If G is a discrete group, then $\underline{G}$ is $\mathcal{R}_{\mathbb{R}^n}$-acyclic.*

Proof. Let $\underline{G} \to I^\bullet$ be an injective resolution. Then we have for $A \in \mathbf{S}$ that

$$\mathcal{R}_{\mathbb{R}^n}(I^\bullet)(A) \overset{(16)}{\cong} I^\bullet(A \times \mathbb{R}^n).$$

Therefore $R^i\mathcal{R}_{\mathbb{R}^n}(\underline{G})$ is the sheafification of the presheaf

$$A \mapsto H^i(I^\bullet(A \times \mathbb{R}^n)).$$

Since G is discrete, the sheaf cohomology of $\underline{G}$ is homotopy invariant, and therefore

$$H^i(I^\bullet(A \times \mathbb{R}^n)) \cong H^i(A \times \mathbb{R}^n; \underline{G}) \overset{!}{\cong} H^i(A; \underline{G}).$$

To be precise this can be seen as follows. Let (A) denote the site of open subsets of A. It comes with a natural map $\nu_A \colon (A) \to \mathbf{S}$. The sheaf cohomology functor is the derived functor of the evaluation functor. In order to indicate on which category this evaluation functor is defined we temporarily use subscripts. If $I \in \mathsf{Sh_{Ab}}\mathbf{S}$ is injective, then $\nu_A^* I \in \mathsf{Sh_{Ab}}(A)$ is still flabby (see [BSS, Lemma 2.32]). This implies that $H^*_{\mathsf{Sh_{Ab}}\mathbf{S}}(A; \underline{G}) \cong H^*_{(A)}(A, \nu_A^*(\underline{G}))$.

The diagram

$$\begin{tikzcd} A \arrow[rr, "\mathsf{id}_A \times 0"] \arrow[dr, "\mathsf{id}_A"'] & & A \times \mathbb{R}^n \arrow[dl, "\mathsf{pr}_A"] \\ & A & \end{tikzcd}$$

is a homotopical isomorphism (in the sense of [KS94, 2.7.4]) $A \to A \times \mathbb{R}^n$ over A. We now apply [KS94, 2.7.7] which says that the natural map

$$R(\mathsf{pr}_A)_* \mathsf{pr}_A^*(\nu_A^*(\underline{G})) \to R(\mathsf{id}_A)_* \mathsf{id}_A^*(\nu_A^*(\underline{G}))$$

is an isomorphism in $D^+(\mathsf{Sh_{Ab}}(A))$. But since G is discrete we get $\mathsf{pr}_A^*(\nu_A^* \underline{G}) \cong \nu^*_{A \times \mathbb{R}^n}(\underline{G})$. If we apply the functor $R\Gamma_{(A)}(A, \dots)$ to this isomorphism and take cohomology we get the desired isomorphism marked by !.[4]

Now, the sheafification of the presheaf $\mathbf{S} \ni A \to H^i(A; \underline{G}) \in \mathsf{Ab}$ is exactly the ith cohomology sheaf of $I^\bullet$ which vanishes for $i \geq 1$. $\square$

Lemma 3.27. *The sheaf $\underline{\mathbb{R}}^n$ is $\mathcal{R}_W$-acyclic for every compact $W \in \mathbf{S}$.*

Proof. Let $\underline{\mathbb{R}}^n \to I^\bullet$ be an injective resolution. Then $R^i\mathcal{R}_W(\underline{\mathbb{R}}^n)$ is the sheafification of the presheaf

$$\mathbf{S} \ni A \mapsto H^i(\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}_{|A}), I^\bullet_{|A})) \cong H^i(I^\bullet(A \times W)).$$

[4] It is tempting to apply a Künneth formula to calculate $H^*(A \times \mathbb{R}^n, \underline{G})$. But since A is not necessarily compact it is not clear that the Künneth formula holds.

Let $[s] \in H^i(I^\bullet(A \times W))$ be represented by $s \in I^i(A \times W)$, and $a \in A$. Then we must find a neighbourhood $U \subseteq A$ of a such that $[s]_{|U \times W} = 0$, i.e. $s_{|U \times W} = dt$ for some $t \in I^{i-1}(U \times W)$, where $d\colon I^{i-1} \to I^i$ is the boundary operator of the resolution.

The ring structure of $\mathbb{R}$ induces on $\underline{\mathbb{R}}^n$ the structure of a sheaf of rings. In order to distinguish this sheaf of rings from the sheaf of groups $\underline{\mathbb{R}}^n$ we will use the notation $\mathcal{C}$. Note that $\underline{\mathbb{R}}^n$ is in fact a sheaf of $\mathcal{C}$-modules.

The forgetful functor $\mathsf{res}\colon \mathsf{Sh}_{\mathcal{C}\text{-mod}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ fits into an adjoint pair

$$\mathsf{ind} : \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \Leftrightarrow \mathsf{Sh}_{\mathcal{C}\text{-mod}}\mathbf{S} : \mathsf{res},$$

where ind is given by $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \ni V \to V \otimes_{\mathbb{Z}} \mathcal{C} \in \mathsf{Sh}_{\mathcal{C}\text{-mod}}$. Since $\mathcal{C}$ is a torsion-free sheaf it is flat. It follows that ind is exact and res preserves injectives.

We can now choose an injective resolution $\underline{\mathbb{R}}^n \to J^\bullet$ in $\mathsf{Sh}_{\mathcal{C}\text{-mod}}\mathbf{S}$ and assume that $I^\bullet = \mathsf{res}(J^\bullet)$.

Since the complex of sheaves $I^\bullet$ is exact we can find an open covering $(V_r)_{r \in R}$ of $A \times W$ such that $s_{|V_r} = dt_r$ for some $t_r \in I^{i-1}(V_r)$. Since W is compact (locally compact suffices), by [Ste67, Theorem 4.3]) the product topology on $A \times W$ is the compactly generated topology used for the product in $\mathbf{S}$. Hence after refining the covering (V_r) we can assume that $V_r = A_r \times W_r$ for open subsets $A_r \subseteq A$ and $W_r \subseteq W$ for all $r \in R$.

We define $R_a := \{r \in R \mid a \in A_r\}$. The family $(W_r)_{r \in R_a}$ is an open covering of W. Since W is compact we can choose a finite set $r_1, \dots, r_k \in R_a$ such that $\mathcal{W} := (W_{r_1}, \dots, W_{r_k})$ is still an open covering of W. The subset $U := \bigcap_{j=1}^k A_{r_j}$ is an open neighbourhood of $a \in A$.

Since I^{i-1} is injective we can choose[5] extensions $\tilde{t}_r \in I^{i-1}(A \times W)$ such that $(\tilde{t}_r)_{|V_r} = t_r$.

We choose a partition of unity $(\chi_1, \dots, \chi_v)$ subordinate to the finite covering $\mathcal{W}$. We take advantage of the fact that $I^\bullet = \mathsf{res}(J^\bullet)$ which implies that we can multiply sections by continuous functions, and that d commutes with this multiplication. We define

$$t := \sum_{k=1}^{v} \chi_k (\tilde{t}_{r_k})_{|U \times W} \in I^{i-1}(U \times W).$$

Note that $\chi_k(s - d\tilde{t}_{r_k})_{|U \times W} = 0$. In fact we have $\chi_k(s - d\tilde{t}_{r_k})_{|(U \times W) \cap V_{r_k}} = \chi_k(s - dt_{r_k})_{|(U \times W) \cap V_{r_k}} = 0$. Furthermore, there is a neighbourhood Z of the complement of $(U \times W) \cap V_{r_k}$ in $U \times W$ where χ_k vanishes. Therefore the restrictions $\chi_k(s - d\tilde{t}_{r_k})$ vanish on the open covering $\{Z, (U \times W) \cap V_{r_k}\}$ of $U \times W$, and this implies the

[5] Let $U \subseteq X$ be an open subset. Then we have an injection $\underline{U} \to \underline{X}$ and hence an injection $\mathbb{Z}(\underline{U}) \to \mathbb{Z}(\underline{X})$. For an injective sheaf I we get a surjection $\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{X}), I) \to \mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{U}), I)$. In other symbols, $I(X) \to I(U)$ is surjective.

assertion. We get

$$\begin{aligned} dt &= \sum_{j=1}^{v} d(\chi_k(\tilde{t}_{r_k})_{|U\times W}) = \sum_{j=1}^{v} \chi_k d(\tilde{t}_{r_k})_{|U\times W} \\ &= \sum_{j=1}^{v} \chi_k (d\tilde{t}_{r_k})_{|U\times W} = \sum_{j=1}^{v} \chi_k s_{|U\times W} \\ &= s_{|U\times W}. \end{aligned}$$

□

Corollary 3.28. 1. *If G is discrete, then we have $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{\mathbb{R}^n}), \underline{G}) \cong 0$ for all $i \geq 1$.*

2. *For every compact $W \in \mathbf{S}$ and $n \geq 1$ we have $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}), \underline{\mathbb{R}^n}) \cong 0$ for all $i \geq 1$.*

Lemma 3.29. *If W is a profinite space, then every sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is $\mathcal{R}_W$-acyclic. Consequently, $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{W}), F) \cong 0$ for $i \geq 1$.*

Proof. We first show the following intermediate result which is used in 3.4.3 in order to finish the proof of Lemma 3.29.

Lemma 3.30. *If W is a profinite space, then $\Gamma(W;\dots)$ is exact.*

Proof. A profinite topological space can be written as limit, $W \cong \lim_I W_n$, for an inverse system of finite spaces $(W_n)_{n\in I}$. Let $p_n : W \to W_n$ denote the projections. First of all, W is compact. Every covering of W admits a finite subcovering. Furthermore, a finite covering admits a refinement to a covering by pairwise disjoint open subsets of the form $\{p_n^{-1}(x)\}_{x\in W_n}$ for an appropriate $n \in I$. This implies the vanishing $\check{H}^p(W, F) \cong 0$ of the Čech cohomology groups for $p \geq 1$ and every presheaf $F \in \mathrm{Pr}_{\mathsf{Ab}}\mathbf{S}$.

Let $\mathcal{H}^q = R^q i$ be the derived functor of the embedding $i : \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathrm{Pr}_{\mathsf{Ab}}\mathbf{S}$ of sheaves into presheaves. We now consider the Čech cohomology spectral sequence [Tam94, 3.4.4] $(E_r, d_r) \Rightarrow R^*\Gamma(W, F)$ with $E_2^{p,q} \cong \check{H}^p(W; \mathcal{H}^q(F))$ and use [Tam94, 3.4.3] to the effect that $\check{H}^0(W; \mathcal{H}^q(F)) \cong 0$ for all $q \geq 1$. Combining these two vanishing results we see that the only non-trivial term of the second page of the spectral sequence is $E_2^{0,0} \cong \check{H}^0(W; \mathcal{H}^0(F)) \cong F(W)$. Vanishing of $R^i\Gamma(W;\dots)$ for $i \geq 1$ is equivalent to the exactness of $\Gamma(W;\dots)$. □

3.4.3 We now prove Lemma 3.29. Let $i \geq 1$. We use that $R^i\mathcal{R}_W(F)$ is the sheafification of the presheaf $\mathbf{S} \ni A \mapsto R^i\Gamma(A \times W; F) \in \mathsf{Ab}$. For every sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we have by some intermediate steps in (16)

$$\Gamma(A \times W; F) \cong \Gamma(W; \mathcal{R}_A(F)).$$

Let us choose an injective resolution $F \to I^\bullet$. Using Lemma 3.30 for the second isomorphism we get

$$H^i\Gamma(A; \mathcal{R}_W(I^\bullet)) \overset{s\mapsto\tilde{s}}{\cong} H^i\Gamma(A \times W; I^\bullet) \overset{\tilde{s}\mapsto\tilde{s}}{\cong} \Gamma(W; H^i\mathcal{R}_A(I^\bullet)). \qquad (18)$$

Consider a point $a \in A$ and $s \in H^i\Gamma(A; \mathcal{R}_W(I^\bullet))$. We must show that there exists a neighbourhood $a \in U \subseteq A$ of a such that $s_{|U} = 0$. Since $I^\bullet$ is an exact sequence of sheaves in degree ≥ 1 there exists an open covering $\{Y_r\}_{r\in R}$ of $A \times W$ such that $\tilde{s}_{|Y_r} = 0$. After refining this covering we can assume that $Y_r \cong U_r \times V_r$ for suitable open subsets $U_r \subseteq A$ and $V_r \subseteq W$. Consider the set $R_a := \{r \in R \mid a \in U_r\}$. Since W is compact the covering $\{V_r\}_{r\in R_a}$ of W admits a finite subcovering indexed by $Z \subseteq R_a$. The set $U := \bigcap_{r\in Z} U_r \subseteq A$ is an open neighbourhood of a. By further restriction we get $\tilde{s}_{|U\times V_r} = 0$ for all $r \in Z$. By (18) this means that $0 = \overline{s_{|U}}_{|V_r} \in \Gamma(W; H^i\mathcal{R}_U(I^\bullet))$. Therefore $\overline{s_{|U}}$ vanishes locally on W and therefore globally. This implies $s_{|U} = 0$. □

3.4.4 Let $f : \mathbf{S}_{\mathrm{lc}} \to \mathbf{S}$ be the inclusion of the full subcategory of locally compact topological spaces. Since an open subset of a locally compact space is again locally compact we can define the topology on $\mathbf{S}_{\mathrm{lc}}$ by

$$\mathsf{cov}_{\mathbf{S}_{\mathrm{lc}}}(A) := \mathsf{cov}_{\mathbf{S}}(A), \quad A \in \mathbf{S}_{\mathrm{lc}}.$$

The compactly generated topology and the product topology on products of locally compact spaces coincides. The same applies to fibre products. Furthermore, a fibre product of locally compact spaces is locally compact. The functor f preserves fibre products. In view of the definition of the topology $\mathbf{S}_{\mathrm{lc}}$ the inclusion functor $f : \mathbf{S}_{\mathrm{lc}} \to \mathbf{S}$ is a morphism of sites.

Lemma 3.31. *Restriction to the site* $\mathbf{S}_{\mathrm{lc}}$ *commutes with sheafification, i.e.*

$$i^\sharp \circ {}^p f^* \cong f^* \circ i^\sharp.$$

Proof. (Compare with the proof of Lemma 3.7.) This follows from $\mathsf{cov}_{\mathbf{S}_{\mathrm{lc}}}(A) \cong \mathsf{cov}_{\mathbf{S}}(A)$ for all $A \in \mathbf{S}$ and the explicit construction of $i^\sharp$ in terms of the set $\mathsf{cov}_{\mathbf{S}_{\mathrm{lc}}}$ (see [Tam94, Section 3.1]). □

Lemma 3.32. *The restriction* $f^* : \mathsf{Sh}\mathbf{S} \to \mathsf{Sh}\mathbf{S}_{\mathrm{lc}}$ *is exact.*

Proof. (Compare with the proof of Lemma 3.8.) The functor f^* is a right-adjoint and thus commutes with limits. Colimits of presheaves are defined object-wise, i.e. for a diagram

$$\mathcal{C} \to \mathrm{Pr}\,\mathbf{S}, \quad c \mapsto F_c$$

of presheaves we have

$$({}^p\mathsf{colim}_{c\in\mathcal{C}} F_c)(U) = \mathsf{colim}_{c\in\mathcal{C}} F_c(U).$$

It follows from the explicit description of ${}^p f^*$ that this functor commutes with colimits of presheaves. For a diagram of sheaves $F : \mathcal{C} \to \mathsf{Sh}\mathbf{S}$ we have

$$\mathsf{colim}_{c\in\mathcal{C}} F_c = i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c.$$

By Lemma 3.31 we get $f^*\mathsf{colim}_{c\in\mathcal{C}} F_c \cong f^* i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c \cong i^\sharp\, {}^p f^*\, {}^p\mathsf{colim}_{c\in\mathcal{C}} F_c \cong i^\sharp\, {}^p\mathsf{colim}_{c\in\mathcal{C}}\, {}^p f^* F_c \cong \mathsf{colim}_{c\in\mathcal{C}} f^* F_c$. □

3.4.5 We have now verified that $f : \mathbf{S}_{\mathrm{lc}} \to \mathbf{S}$ satisfies the assumptions of Proposition 3.21.

Corollary 3.33. *Let $f : \mathbf{S}_{\mathrm{lc}} \to \mathbf{S}$ be the inclusion of the site of locally compact spaces. For $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}$ and $G \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we have*

$$f^* R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(f_* F, G) \cong R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}}(F, f^* G).$$

In particular we have

$$f^* \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(f_* F, G) \cong \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}}(F, f^* G)$$

for all $k \geq 0$.

In fact, the first assertion implies the second since f^* is exact.

We need this result in the following special case. If $G \in \mathbf{S}$ is a locally compact group, then by abuse of notation we write $\underline{G}$ for the sheaves of abelian groups represented by G in both categories $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}$.

We have $f^*\underline{G} = \underline{G}$. By Lemma 3.17 we also have $f_*\underline{G} \cong \underline{G}$.

Corollary 3.34. *Let $G, H \in \mathbf{S}_{lc}$ be locally compact abelian groups. We have*

$$f^* \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{H}) \cong \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{H})$$

for all $k \geq 0$.

3.4.6 In some places we will need a second sub-site of $\mathbf{S}$, the site $\mathbf{S}_{\text{loc-acyc}}$ of locally acyclic spaces.

Definition 3.35. A space $U \in \mathbf{S}$ is called acyclic, if $H^i(U; \underline{H}) \cong 0$ for all discrete abelian groups H and $i \geq 1$.

By Lemma 3.30 all profinite spaces are acyclic. The space $\mathbb{R}^n$ is another example of an acyclic space. In fact, the homotopy invariance used in the proof of 3.26 shows that the inclusion $0 \to \mathbb{R}^n$ induces an isomorphism $H^i(\mathbb{R}^n; \underline{H}) \cong H^i(\{0\}; \underline{H})$, and a one-point space is clearly acyclic.

Definition 3.36. A space $A \in \mathbf{S}$ is called locally acyclic if it admits an open covering by acyclic spaces.

In general we do not know if the product of two locally acyclic spaces is again locally acyclic (the Künneth formula needs a compactness assumption). In order to ensure the existence of finite products we consider the combination of the conditions locally acyclic and locally compact.

Note that all finite-dimensional manifolds are locally acyclic and locally compact. An open subset of a locally acyclic locally compact space is again locally acyclic. We let $\mathbf{S}_{\text{lc-acyc}} \subset \mathbf{S}$ be the full subcategory of locally acyclic locally compact spaces. The topology on $\mathbf{S}_{\text{lc-acyc}}$ is given by

$$\mathsf{cov}_{\mathbf{S}_{\text{lc-acyc}}}(A) := \mathsf{cov}_{\mathbf{S}}(A).$$

Let $g : \mathbf{S}_{\text{lc-acyc}} \to \mathbf{S}$ be the inclusion. The proofs of Lemma 3.31, Lemma 3.32 and Corollary 3.33 apply verbatim.

Corollary 3.37. 1. *The restriction* $g^*\colon \mathsf{Sh}\,\mathbf{S} \to \mathsf{Sh}\,\mathbf{S}_{\text{lc-acyc}}$ *is exact.*

2. *For* $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}$ *and* $G \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ *we have*

$$g^* R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(g_*F, G) \cong R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(F, g^*G)$$

and

$$g^*\underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(g_*F, G) \cong \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(F, g^*G)$$

for all $k \geq 0$.

Corollary 3.38. *Let* $G, H \in \mathbf{S}_{\text{lc-acyc}}$ *be locally acyclic abelian groups. We have*

$$g^*\underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{H}) \cong \underline{\mathsf{Ext}}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{G}, \underline{H})$$

for all $k \geq 0$.

Note that the product $\prod_{\mathbb{N}} \mathbb{T}$ is compact but not locally acyclic.

3.5 $\mathbb{Z}_{\text{mult}}$-modules

3.5.1 We consider the multiplicative semigroup $\mathbb{Z}_{\text{mult}}$ of non-zero integers. Every abelian group G (written multiplicatively) has a tautological action of $\mathbb{Z}_{\text{mult}}$ by homomorphisms given by

$$\mathbb{Z}_{\text{mult}} \times G \to G, \quad (n, g) \mapsto g^n.$$

Definition 3.39. When we consider G with this action we write $G(1)$.

The semigroup $\mathbb{Z}_{\text{mult}}$ will therefore act on all objects naturally constructed from an abelian group G. We will in particular use the action of $\mathbb{Z}_{\text{mult}}$ on the group-homology and group-cohomology of G.

If G is a topological group, then $\mathbb{Z}_{\text{mult}}$ acts by continuous maps. It therefore also acts on the cohomology of G as a topological space. Also this action will play a role later.

The remainder of the present subsection sets up some language related to $\mathbb{Z}_{\text{mult}}$-actions.

Definition 3.40. A $\mathbb{Z}_{\text{mult}}$-module is an abelian group with an action of $\mathbb{Z}_{\text{mult}}$ by homomorphisms.

We will write this action as $\mathbb{Z}_{\text{mult}} \times G \ni (m, g) \mapsto \Psi^m(g) \in G$. Thus in the case of $G(1)$ we have $\Psi^m(g) = g^m$.

3.5.2 We let $\mathbb{Z}_{\text{mult}}$-mod denote the category of $\mathbb{Z}_{\text{mult}}$-modules. An equivalent description of this category is as the category of modules under the commutative semigroup ring $\mathbb{Z}[\mathbb{Z}_{\text{mult}}]$. The category $\mathbb{Z}_{\text{mult}}$-mod is an abelian tensor category.

We have an exact inclusion of categories

$$\mathsf{Ab} \hookrightarrow (\mathbb{Z}_{\text{mult}}\text{-}\mathsf{mod}), \quad G \mapsto G(1)$$

By

$$\mathcal{F} : (\mathbb{Z}_{\text{mult}}\text{-mod}) \to \mathsf{Ab}$$

we denote the forgetful functor.

Definition 3.41. Let G be an abelian group. For $k \in \mathbb{Z}$ we let $G(k) \in \mathbb{Z}_{\text{mult}}\text{-mod}$ denote the $\mathbb{Z}_{\text{mult}}$-module given by the action $\mathbb{Z}_{\text{mult}} \times G \ni (p, g) \mapsto \Psi^p(g) := g^{p^k} \in G$.

Observe that for abelian groups V, W we have a natural isomorphism

$$V(k) \otimes_{\mathbb{Z}} W(l) \cong (V \otimes_{\mathbb{Z}} W)(k + l). \tag{19}$$

3.5.3 Let V be a $\mathbb{Z}_{\text{mult}}$-module.

Definition 3.42. We say that V has weight k if there exists an isomorphism $V \cong \mathcal{F}(V)(k)$ of $\mathbb{Z}_{\text{mult}}$-modules.

If V has weight k, then every sub-quotient of V has weight k. Note that a $\mathbb{Z}_{\text{mult}}$-module can have many weights. We have e.g. isomorphisms of $\mathbb{Z}_{\text{mult}}$-modules $(\mathbb{Z}/2\mathbb{Z})(1) \cong (\mathbb{Z}/2\mathbb{Z})(k)$ for all $k \neq 0$.

3.5.4 Let $V \in \mathbb{Z}_{\text{mult}}\text{-mod}$. We say that $v \in V$ has weight k if it generates a submodule $\mathbb{Z} < v > \subset V$ of weight k. For $k \in \mathbb{N}$ we let $W_k : (\mathbb{Z}_{\text{mult}}\text{-mod}) \to \mathsf{Ab}$ be the functor which associates to $V \in \mathbb{Z}_{\text{mult}}\text{-mod}$ its subgroup of vectors of weight k. Then we have an adjoint pair of functors

$$(k) : \mathsf{Ab} \Leftrightarrow (\mathbb{Z}_{\text{mult}}\text{-mod}) : W_k.$$

Observe that the functor W_k is not exact. Consider for example a prime number $p \in \mathbb{N}$ and the sequence

$$0 \to \mathbb{Z}(1) \xrightarrow{p} \mathbb{Z}(1) \to (\mathbb{Z}/p\mathbb{Z})(p) \to 0.$$

The projection map is indeed $\mathbb{Z}_{\text{mult}}$-equivariant since $m^p \equiv m \bmod p$ for all $m \in \mathbb{Z}$. Then

$$0 \cong W_p(\mathbb{Z}(1)) \to W_p((\mathbb{Z}/p\mathbb{Z})(p)) \cong \mathbb{Z}/p\mathbb{Z}$$

is not surjective.

3.5.5 Let $V \in \mathsf{Ab}$ and $V(1) \in \mathbb{Z}_{\text{mult}}\text{-mod}$. Then we can form the graded tensor algebra

$$T^*_{\mathbb{Z}}(V(1)) = \mathbb{Z} \oplus V(1) \oplus V(1) \otimes_{\mathbb{Z}} V(1) \oplus \cdots .$$

We see that $T^k(V(1))$ has weight k. The elements $x \otimes x \in V(1) \otimes V(1)$ generate a homogeneous ideal I. Hence the graded algebra $\Lambda^*_{\mathbb{Z}}(V(1)) := T^*_{\mathbb{Z}}(V(1))/I$ has the property that $\Lambda^k_{\mathbb{Z}}(V(1))$ has weight k.

3.5.6 It makes sense to speak of a sheaf or presheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules on the site $\mathbf{S}$. We let $\mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S}$ and $\mathrm{Pr}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\,\mathbf{S}$ denote the corresponding abelian categories of sheaves and presheaves.

Definition 3.43. Let $V \in \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S}$ (or $V \in \mathrm{Pr}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\,\mathbf{S}$) and $k \in \mathbb{Z}$. We say that V is of weight k if the map $V \xrightarrow{m^k - m} V$ vanishes for all $m \in \mathbb{Z}_{\mathrm{mult}}$.

We define the functors $(k)\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S}$, $\mathcal{F}\colon \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, and $W_k\colon \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ (and their presheaf versions) object-wise. We also have a pair of adjoint functors

$$(k) : \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \Leftrightarrow \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S} : W_k$$

(and the corresponding presheaf version). A sheaf $V \in \mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathsf{mod}}\mathbf{S}$ of $\mathbb{Z}_{\mathrm{mult}}$-modules has weight $k \in \mathbb{Z}$ if $V \cong \mathcal{F}(V)(k) \cong W_k(V)(k)$.

4 Admissibility of sheaves represented by topological abelian groups

4.1 Admissible sheaves and groups

4.1.1 The main topic of the present paper is a duality theory for abelian group stacks (Picard stacks, see 2.4) on the site $\mathbf{S}$. A Picard stack $P \in \mathsf{PIC}(\mathbf{S})$ gives rise to the sheaf of objects $H^0(P)$ and the sheaf of automorphisms of the neutral object $H^{-1}(P)$. These are sheaves of abelian groups on $\mathbf{S}$.

We will define the notion of a dual Picard stack $D(P)$ (see 5.4). With the intention to generalize the Pontrjagin duality for locally compact abelian groups to group stacks we study the question under which conditions the natural map $P \to D(D(P))$ is an isomorphism. In Theorem 5.11 we see that this is the case if the sheaves are dualizable (see 5.2) and admissible (see 4.1). Dualizability is a sheaf-theoretic generalization of the classical Pontrjagin duality and is satisfied e.g. for the sheaves $\underline{G}$ for locally compact groups $G \in \mathbf{S}$ (see 5.3). Admissibility is more exotic and will be defined below (4.1). One of the main results of the present paper asserts that the sheaves $\underline{G}$ are admissible for a large (but not exhaustive) class of locally compact groups $G \in \mathbf{S}$, and for an even larger class if $\mathbf{S}$ is replaced by $\mathbf{S}_{\text{lc-acyc}}$.

The main tool in our computations is a certain double complex, which will be introduced in Section 4.2. Similar techniques were used by Lawrence Breen in [Bre69], [Bre76], [Bre78].

Definition 4.1. We call a sheaf of groups F admissible if $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong 0$ for $i = 1, 2$. A topological abelian group $G \in \mathbf{S}$ is called admissible if $\underline{G}$ is an admissible sheaf.

We will also consider the sub-sites $\mathbf{S}_{\text{lc-acyc}} \subset \mathbf{S}_{\text{lc}} \subset \mathbf{S}$ of locally compact spaces.

Definition 4.2. A locally compact abelian topological group $G \in \mathbf{S}_{\mathrm{lc}}$ is called admissible on $\mathbf{S}_{\mathrm{lc}}$ if $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{T}}) \cong 0$ for $i = 1, 2$ (and correspondingly for $\mathbf{S}_{\text{lc-acyc}}$).

Lemma 4.3. *Let $f : \mathbf{S}_{lc} \to \mathbf{S}$ be the inclusion. If $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\mathrm{lc}}$ and f_*F is admissible, then F is admissible on $\mathbf{S}_{lc}$.*

Proof. This is an application of Corollary 3.33. □

Corollary 4.4. *If $G \in \mathbf{S}_{lc}$ is admissible, then it is admissible on $\mathbf{S}_{lc}$.*

This is an application of Corollary 3.34.

Lemma 4.5. *The class of admissible sheaves is closed under finite products and extensions.*

Proof. For finite products the assertions follows from the fact that $\underline{\mathsf{Ext}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots, \underline{\mathbb{T}})$ commutes with finite products. Given an extension of sheaves

$$0 \to F \to G \to H \to 0$$

such that F and H are admissible, then also G is admissible. This follows immediately from the long exact sequence obtained by applying $\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots, \underline{\mathbb{T}})$. □

4.1.2 In this paragraph we formulate one of the main theorems of the present paper. Let G be a locally compact topological abelian group. We first recall some technical conditions.

Definition 4.6. We say that G satisfies the two-three condition, if

1. it does not admit $\prod_{n\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ as a subquotient,
2. the multiplication by 3 on the component G_0 of the identity has finite cokernel.

Definition 4.7. We say that G is locally topologically divisible if for all primes $p \in \mathbb{N}$ the multiplication map $p\colon G \to G$ has a continuous local section.

Using that the class of admissible groups is closed under finite products and extensions we get the following general theorem.

Theorem 4.8. 1. *If G is a locally compact abelian group which satisfies the two-three condition, then it is admissible over $\mathbf{S}_{\text{lc-acyc}}$.*

2. *Assume that*

(a) *G satisfies the two-three condition,*

(b) *G has an open subgroup of the form $C \times \mathbb{R}^n$ with C compact such that $G/C \times \mathbb{R}^n$ is finitely generated*

(c) *the connected component of the identity of G is locally topologically divisible.*

Then G is admissible over $\mathbf{S}_{\mathrm{lc}}$.

Proof. By [HM98, Theorem 7.57(i)] the group G has a splitting $G \cong H \times \mathbb{R}^n$ for some $n \in \mathbb{N}_0$, where H has a compact open subgroup U. The quotient $G/(U \times \mathbb{R}^n) \cong H/U$ is therefore discrete. Using Lemma 4.5 conclude that G is admissible over $\mathbf{S}_{\mathrm{lc}}$ if $\mathbb{R}^n$ and H are so.

Admissibility of $\mathbb{R}^n$ follows from Theorem 4.33 (which we prove later) in conjunction with 4.5 and 4.4.

Since $D := H/U$ is discrete, the exact sequence

$$0 \to U \to H \to H/U \to 0$$

has local sections and therefore induces an exact sequence of associated sheaves

$$0 \to \underline{U} \to \underline{H} \to \underline{D} \to 0$$

by Lemma 3.4. If D is finitely generated, then it is admissible by Theorem 4.18, and hence admissible on $\mathbf{S}_{\mathrm{lc}}$ by Lemma 4.4. Otherwise it is admissible on $\mathbf{S}_{\mathrm{lc\text{-}acyc}}$ by Theorem 4.28.

Therefore H is admissible on $\mathbf{S}_{\mathrm{lc}}$ (or $\mathbf{S}_{\mathrm{lc\text{-}acyc}}$, respectively) by Lemma 4.5 if U is admissible over $\mathbf{S}_{\mathrm{lc}}$ (or $\mathbf{S}_{\mathrm{lc\text{-}acyc}}$, respectively). The compact group U fits into an exact sequence

$$0 \to U_0 \to U \to P \to 0$$

where P is profinite and U_0 is closed and connected. By assumption U, U_0 and P satisfy the two-three condition. The connected compact group U_0 is admissible on $\mathbf{S}_{\mathrm{lc}}$ (or on $\mathbf{S}_{\mathrm{lc}}$ without the assumption that it is locally topologically divisible), by Theorem 4.79, and the profinite P is admissible by Theorem 4.66, and hence admissible on $\mathbf{S}_{\mathrm{lc}}$ by Lemma 4.4.

Note that

$$0 \to \underline{U_0} \to \underline{U} \to \underline{P} \to 0$$

is exact by Lemma 4.39. Now it follows from Lemma 4.5 that U is admissible on $\mathbf{S}_{\mathrm{lc}}$ (or $\mathbf{S}_{\mathrm{lc\text{-}acyc}}$, respectively). □

4.1.3 We conjecture that the assumption that G satisfies the two-three condition is only technical and forced by our technique to prove admissibility of compact groups.

In the remainder of this section, we will mainly be concerned with the proof of the statements used in the proof of Theorem 4.8 above.

4.1.4 Our proofs of admissibility for a sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ will usually be based on the following argument.

Lemma 4.9. *Assume that $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ satisfies*

1. $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{Z}}) \cong 0$ *for* $i = 2, 3$*;*
2. $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{R}}) \cong 0$ *for* $i = 1, 2$.

Then F is admissible.

Proof. We apply the functor $\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\dots)$ to the sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0$$

and get the following segments of the long exact sequence

$$\dots \to \underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\underline{\mathbb{R}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\underline{\mathbb{T}}) \to \underline{\mathsf{Ext}}^{i+1}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\underline{\mathbb{Z}}) \to \cdots .$$

We see that the assumptions on F imply that $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\underline{\mathbb{T}}) \cong 0$ for $i = 1, 2$. □

4.1.5 A space $W \in \mathbf{S}$ is called profinite if it can be written as the limit of an inverse system of finite spaces. Lemma 3.29 has as a special case the following theorem.

Theorem 4.10. *If $W \in \mathbf{S}$ is profinite, then $\mathbb{Z}(\underline{W})$ is admissible.*

4.2 A double complex

4.2.1 Let $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ be a sheaf of groups with underlying sheaf of sets $\mathcal{F}(H) \in \mathsf{Sh}\mathbf{S}$. Applying the linearization functor $\mathbb{Z}$ (see 3.2.6) we get again a sheaf of groups $\mathbb{Z}(\mathcal{F}(H)) \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. The group structure of H induces on $\mathbb{Z}(\mathcal{F}(H))$ a ring structure. We denote this sheaf of rings by $\mathbb{Z}[H]$. It is the sheafification of the presheaf ${}^p\mathbb{Z}[H]$ which associates to $A \in \mathbf{S}$ the integral group ring ${}^p\mathbb{Z}[H](A)$ of the group $H(A)$.

We consider the category $\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$ of sheaves of $\mathbb{Z}[H]$-modules. The trivial action of the sheaf of groups H on the sheaf $\underline{\mathbb{Z}}$ induces the structure of a sheaf of $\mathbb{Z}[H]$-modules on $\underline{\mathbb{Z}}$. With the notation introduced below we could (but refrain from doing this) write this sheaf of $\mathbb{Z}[H]$-modules as $\mathsf{coind}(\underline{\mathbb{Z}})$.

4.2.2 The forgetful functor $\mathsf{res}\colon \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ fits into an adjoint pair of functors

$$\mathsf{ind} : \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \Leftrightarrow \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S} : \mathsf{res}.$$

Explicitly, the functor ind is given by

$$\mathsf{ind}(V) := \mathbb{Z}[H] \otimes_{\mathbb{Z}} V.$$

Since $\mathbb{Z}[H]$ is a torsion-free sheaf and therefore a sheaf of flat $\underline{\mathbb{Z}}$-modules the functor ind is exact. Consequently the functor res preserves injectives.

4.2.3 We let $\mathsf{coinv}\colon \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S} \to \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ denote the coinvariants functor given by

$$\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S} \ni V \mapsto \mathsf{coinv}(V) := V \otimes_{\mathbb{Z}[H]} \underline{\mathbb{Z}} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}.$$

This functor fits into the adjoint pair

$$\mathsf{coinv} : \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S} \Leftrightarrow \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} : \mathsf{coind}$$

with the coinduction functor which maps $W \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ to the sheaf of $\mathbb{Z}[H]$-modules induced by the trivial action of H in W, formally this can be written as

$$\mathsf{coind}(W) := \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, W).$$

Lemma 4.11. *Every sheaf $F \in \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$ is a quotient of a flat sheaf.*

Proof. Indeed, the counits of the adjoint pairs $(\mathbb{Z}(\dots), \mathcal{F})$ and $(\mathsf{ind}, \mathsf{res})$ induce a surjection $\mathsf{ind}(\mathbb{Z}(\mathcal{F}(\mathsf{res}(F)))) \to F$. Explicitly it is given by the composition of the sum and action map (omitting to write some forgetful functors)

$$\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[F] \to \mathbb{Z}[H] \otimes F \to F.$$

Since $\mathbb{Z}[H]$ is a sheaf of unital rings this action is surjective. Moreover, for $A \in \mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$ we have

$$A \otimes_{\mathbb{Z}[H]} (\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[F]) \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[F].$$

Since $\mathbb{Z}[F]$ is a torsion-free sheaf of abelian groups the operation $A \to A \otimes_{\mathbb{Z}[H]} (\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[F])$ preserves exact sequences in $\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$. Therefore $\mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}[F]$ is a flat sheaf of $\mathbb{Z}[H]$-modules. □

Lemma 4.12. *The class of flat $\mathbb{Z}[H]$-modules is* coinv*-acyclic.*

Proof. Let $F^\bullet$ be an exact lower bounded homological complex of flat $\mathbb{Z}[H]$-modules. We choose a flat resolution $P^\bullet \to \underline{\mathbb{Z}}$ in $\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$ which exists by Lemma 4.11[6]. Since $F^\bullet$ consists of flat modules the induced map

$$F^\bullet \otimes_{\mathbb{Z}[H]} P^\bullet \to F^\bullet \otimes_{\mathbb{Z}[H]} \underline{\mathbb{Z}} = \mathsf{coinv}(F^\bullet)$$

is a quasi-isomorphism. Since $P^\bullet$ consists of flat modules, tensoring by $P^\bullet$ commutes with taking cohomology, so that we have

$$H^*(F^\bullet \otimes_{\mathbb{Z}[H]} P^\bullet) \cong H^*(H^*(F^\bullet) \otimes_{\mathbb{Z}[H]} P^\bullet) \cong 0.$$

Therefore $\mathsf{coinv}(\dots)$ maps acyclic complexes of flat $\mathbb{Z}[H]$-modules to acyclic complexes of $\underline{\mathbb{Z}}$-modules. □

Corollary 4.13. *We can calculate $L^*\mathsf{coinv}(\underline{\mathbb{Z}})$ using a flat resolution.*

4.2.4 We will actually work with a very special flat resolution of $\underline{\mathbb{Z}}$. The bar construction on the sheaf of groups H gives a sheaf $H^\bullet$ of simplicial sets with an action of H. We let $C^\bullet(H) := C(\mathbb{Z}(H^\bullet))$ be the sheaf of homological chain complexes associated to the sheaf of simplicial groups $\mathbb{Z}(H^\bullet)$. The H-action on $H^\bullet$ induces an H-action on $C^\bullet(H)$ and therefore the structure of a sheaf of $\mathbb{Z}[H]$-modules. In order to understand the structure of $C^\bullet(H)$ we first consider the presheaf version ${}^pC^\bullet(H) := C({}^p\mathbb{Z}(H^\bullet))$. In fact, we can write

$$H^i \cong H \times \underbrace{H \times \dots \times H}_{i \text{ factors}},$$

as a sheaf of H-sets, and therefore

$$ {}^pC^i(H) \cong {}^p\mathbb{Z}[H] \otimes^p_{\mathbb{Z}} {}^p\mathbb{Z}(\underbrace{H \times \dots \times H}_{i \text{ factors}}) \tag{20}$$

[6] Note that $P^\bullet$ is a homological complex, i.e. the differentials have degree -1.

The cohomology of the complex ${}^pC^\bullet(H)$ is given by

$$H^i({}^pC^\bullet(H)) \cong \begin{cases} {}^p\underline{\mathbb{Z}}, & i = 0, \\ 0, & i \geq 1, \end{cases}$$

where ${}^p\underline{\mathbb{Z}} \in \mathrm{Pr}_{\mathsf{Ab}}\mathbf{S}$ denotes the constant presheaf with value $\mathbb{Z}$. Since sheafification $i^\sharp$ is an exact functor and by definition $C^\bullet(H) = i^\sharp\, {}^pC^\bullet(H)$ we get

$$H^i(C^\bullet(H)) \cong \begin{cases} \underline{\mathbb{Z}}, & i = 0, \\ 0, & i \geq 1. \end{cases}$$

Furthermore,

$$C^i(H) \cong \mathbb{Z}[H] \otimes_{\mathbb{Z}} \mathbb{Z}(\underbrace{H \times \cdots \times H}_{i \text{ factors}}) = \mathsf{ind}(\mathbb{Z}(\underbrace{H \times \cdots \times H}_{i \text{ factors}})) \tag{21}$$

shows that $C^i(H)$ is flat. Let us write $C^\bullet := C^\bullet(H) = \mathsf{ind}(D^\bullet)$ with

$$D^i := \mathbb{Z}(\underbrace{H \times \cdots \times H}_{i \text{ factors}}).$$

Definition 4.14. For a sheaf $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we define the complex $U^\bullet := U^\bullet(H) := \mathsf{coinv}(C^\bullet)$.

It follows from the construction that $U^\bullet$ depends functorially on H. In particular, for $\alpha\colon H \to H'$ we have a map of complexes $U^\bullet(\alpha)\colon U^\bullet(H) \to U^\bullet(H')$.

4.2.5 The main tool in our proofs of admissibility of a sheaf F is the study of the sheaves

$$R^*\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \mathsf{coind}(W))$$

for $W = \underline{\mathbb{Z}}, \underline{\mathbb{R}}$. Let us write this in a more complicated way using the special flat resolution $C^\bullet(H) \to \underline{\mathbb{Z}}$ constructed in 4.2.4. We choose an injective resolution $\mathsf{coind}(W) \to I^\bullet$ in $\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}$. Using that $\mathsf{res} \circ \mathsf{coind} = \mathsf{id}$, and that $\mathsf{res}(I^\bullet)$ is injective in $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, we get[7]

$$\begin{aligned} R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \mathsf{coind}(W)) &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, I^\bullet) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(C^\bullet(H), I^\bullet) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\mathsf{ind}(D^\bullet), I^\bullet) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D^\bullet, \mathsf{res}(I^\bullet)) \\ &= \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D^\bullet, \mathsf{res}(\mathsf{coind}(\mathsf{res}(I^\bullet)))) \\ &= \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\mathsf{ind}(D^\bullet), \mathsf{coind}(\mathsf{res}(I^\bullet))) \end{aligned}$$

[7] Note that $D^\bullet$ is considered in this calculation as a sequence of sheaves, not as a complex. The differentials in the intermediate steps involving $D^\bullet$ are still induced via the isomorphisms from the differentials of $C^\bullet$ and $I^\bullet$.

$$\begin{aligned} &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(C^\bullet, \mathsf{coind}(\mathsf{res}(I^\bullet))) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{coinv}(C^\bullet), \mathsf{res}(I^\bullet)) \\ &\cong R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(L\mathsf{coinv}(\underline{\mathbb{Z}}), W). \end{aligned}$$

4.2.6 In general, the coinvariants functor $\mathsf{coinv}(\dots) = \dots \otimes_{\mathbb{Z}[H]} \underline{\mathbb{Z}}$ can be written in terms of the tensor product in the sense of presheaves composed with a sheafification. Furthermore, $C^\bullet(H)$ is the sheafification of ${}^pC^\bullet(H)$. Using the fact that the tensor product of presheaves commutes with sheafification

$$\begin{aligned} U^i &= \mathsf{coinv}(C^i) \\ &\cong (C^i(H) \otimes^p_{\mathbb{Z}[H]} \underline{\mathbb{Z}})^\sharp \quad \text{(Definition 4.14)} \\ &= (({}^pC^i(H))^\sharp \otimes^p_{({}^p\mathbb{Z}[H])^\sharp} ({}^p\underline{\mathbb{Z}})^\sharp)^\sharp \\ &\cong ({}^pC^i(H) \otimes^p_{{}^p\mathbb{Z}[H]} {}^p\underline{\mathbb{Z}})^\sharp \qquad (22) \\ &\cong ({}^p\mathbb{Z}[H] \otimes^p_{\mathbb{Z}} {}^p\mathbb{Z}(\underbrace{H \times \dots \times H}_{i \text{ factors}}) \otimes^p_{{}^p\mathbb{Z}[H]} {}^p\underline{\mathbb{Z}})^\sharp \quad \text{(Equation (20))} \\ &\cong ({}^p\mathbb{Z}(\underbrace{H \times \dots \times H}_{i \text{ factors}}))^\sharp \\ &= ({}^pD^i)^\sharp \end{aligned}$$

with

$${}^pD^i := {}^p\mathbb{Z}(\underbrace{H \times \dots \times H}_{i \text{ factors}}). \qquad (23)$$

In particular, we have

$$U^i = \mathbb{Z}(H^i). \qquad (24)$$

4.2.7 Let G be a group. Then we can form the standard reduced bar complex for the group homology with integer coefficients

$$G^\bullet \dots \to \mathbb{Z}(G^n) \to \mathbb{Z}(G^{n-1}) \to \dots \to \mathbb{Z} \to 0.$$

The abelian group $\mathbb{Z}(G^n)$ (sitting in degree n) is freely generated by the underlying set of G^n, and we write the generators in the form $[g_1|\dots|g_n]$. The differential is given by

$$d = \sum_{i=0}^{n} (-1)^i d_i : \mathbb{Z}(G^n) \to \mathbb{Z}(G^{n-1}),$$

where

$$d_i[g_1|\dots|g_n] := \begin{cases} [g_2|\dots|g_n], & i = 0, \\ [g_1|\dots|g_i g_{i+1}|\dots|g_n], & 1 \le i \le n-1, \\ [g_1|\dots|g_{n-1}], & i = n. \end{cases}$$

The cohomology of this complex is the group homology $H_*(G; \mathbb{Z})$.

4.2.8 For $A \in \mathbf{S}$ the complex ${}^pD^\bullet(A)$ (see (23)) is exactly the standard complex (see 4.2.7) for the group homology $H_*(H(A), \underline{\mathbb{Z}})$ of the group $H(A)$. The cohomology sheaves $H^*(U^\bullet)$ are thus the sheafifications of the cohomology presheaves

$$\mathbf{S} \ni A \mapsto H_*(H(A), \underline{\mathbb{Z}}) \in \mathsf{Ab}.$$

4.2.9 In this paragraph we collect some facts about the homology of abelian groups. An abelian group V is the same thing as a $\mathbb{Z}$-module. We define the graded $\mathbb{Z}$-algebra $\Lambda^*_{\mathbb{Z}} V$ as the quotient of the tensor algebra

$$T_{\mathbb{Z}} V := \bigoplus_{n \geq 0} \underbrace{V \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} V}_{n \text{ factors}}$$

by the graded ideal $I \subseteq T_{\mathbb{Z}} V$ generated by the elements $x \otimes x$, $x \in V$.

4.2.10 Let G be an abelian group. We refer to [Bro82, V.6.4] for the following fact.

Fact 4.15. There exists a canonical map

$$m \colon \Lambda^i_{\mathbb{Z}} G \to H_i(G; \mathbb{Z}).$$

It is an isomorphism for $i = 0, 1, 2$, and it becomes an isomorphism after tensoring with $\mathbb{Q}$ for all $i \geq 0$. If G is torsion-free, then it is an isomorphism $m \colon \Lambda^i_{\mathbb{Z}} G \xrightarrow{\sim} H_i(G; \mathbb{Z})$ for all $i \geq 0$.

4.2.11 Let $U^\bullet = U^\bullet(H)$ (see Definition 4.14) for $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. The cohomology sheaves $L^*\mathsf{coinv}(\underline{\mathbb{Z}})) \cong H^*(U^\bullet)$ are the sheafifications of the presheaves $H^*({}^pD^\bullet)$. By the fact 4.15 we have $H^i({}^pD^\bullet) \cong \Lambda^i_{\mathbb{Z}} H$ for $i = 0, 1, 2$ for the presheaves $\mathbf{S} \ni U \mapsto \Lambda^i_{\mathbb{Z}} H(U)$. In particular, we have $H^0(U^\bullet) \cong \underline{\mathbb{Z}}$ and $H^1(U^\bullet) \cong H$. If H is a torsion-free sheaf, then $H^i(U^\bullet) \cong (\Lambda^i_{\mathbb{Z}} H)^\sharp$ for all $i \geq 0$. Finally, for an arbitrary sheaf $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we have

$$(\Lambda^*_{\mathbb{Z}} H)^\sharp \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^*(U^\bullet) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

4.2.12 Our application of this relies on the study of the two spectral sequences converging to

$$\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(L\mathsf{coinv}(\underline{\mathbb{Z}}), \underline{\mathbb{Z}}) \cong \underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}).$$

We choose an injective resolution $\underline{\mathbb{Z}} \to I^\bullet$ in $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. Then

$$\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(L\mathsf{coinv}(\underline{\mathbb{Z}}), \underline{\mathbb{Z}}) \cong H^*(\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)).$$

The first spectral sequence denoted by (F_r, d_r) is obtained by taking the cohomology in the $U^\bullet$-direction first. Its second page is given by

$$F_2^{p,q} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(L^q\mathsf{coinv}(\underline{\mathbb{Z}}), \underline{\mathbb{Z}}).$$

This page contains the object of our interest, namely by 4.2.11 the sheaves of groups

$$F_2^{p,1} \cong \underline{\mathsf{Ext}}^p_{\mathsf{ShAb}\,\mathbf{S}}(H, \underline{\mathbb{Z}}).$$

The other spectral sequence (E_r, d_r) is obtained by taking the cohomology in the $I^\bullet$-direction first. In view of (22) its first page is given by

$$E_1^{p,q} \cong \underline{\mathsf{Ext}}^q_{\mathsf{ShAb}\,\mathbf{S}}(\mathbb{Z}(H^p), \underline{\mathbb{Z}}),$$

which can be evaluated easily in many cases.

Let us note that

$$H^* R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathbb{Z}[H]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$$

has the structure of a graded ring with multiplication given by the Yoneda product.

4.2.13 We now verify Assumption 2 of Lemma 4.9 for all compact groups.

Proposition 4.16. *Let $H \in \mathbf{S}$ be a compact group. Then we have $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{H}, \underline{\mathbb{R}}) \cong 0$ for $i = 1, 2$.*

As in Definition 4.14 let $U^\bullet := U^\bullet(\underline{H})$. Let $\underline{\mathbb{R}} \to I^\bullet$ be an injective resolution. Then we get a double complex $\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet)$.

We first take the cohomology in the $I^\bullet$-, and then in the $U^\bullet$-direction. We get a spectral sequence with first term

$$E_1^{p,q} \cong \underline{\mathsf{Ext}}^q_{\mathsf{ShAb}\,\mathbf{S}}(\mathbb{Z}(\mathcal{F}\,\underline{H}^p), \underline{\mathbb{R}}).$$

It follows from Corollary 3.28, 2., that $E_1^{p,q} \cong 0$ for $q \geq 1$.

We consider the complex

$$C^\bullet(H, \mathbb{R})\colon 0 \to \mathsf{Map}(H, \mathbb{R}) \to \cdots \to \mathsf{Map}(H^{p-1}, \mathbb{R}) \to \mathsf{Map}(H^p, \mathbb{R}) \to \cdots$$

of topological groups which calculates the continuous group cohomology $H^*_{\mathrm{cont}}(H; \mathbb{R})$ of H with coefficients in $\mathbb{R}$. Now observe that by the exponential law for $A \in \mathbf{S}$

$$\underline{\mathsf{Ext}}^0_{\mathsf{ShAb}\,\mathbf{S}}(\mathbb{Z}(\mathcal{F}\,\underline{H}^p), \underline{\mathbb{R}})(A) \overset{\text{Lemma 3.9}}{\cong} \mathsf{Hom}_{\mathbf{S}}(H^p \times A, \mathbb{R}) \cong \mathsf{Hom}_{\mathbf{S}}(A, \mathsf{Map}(H^p, \mathbb{R})).$$

Hence the complex $(E_1^{*,0}, d_1)(A)$ is isomorphic to the complex

$$\mathsf{Hom}_{\mathbf{S}}(A, C^\bullet(H, \mathbb{R})) = \underline{C^\bullet(H, \mathbb{R})}(A).$$

Since H is a compact group we have $H^i_{\mathrm{cont}}(H; \mathbb{R}) \cong 0$ for $i \geq 1$. Of importance for us is a particular continuous chain contraction $h^p\colon \mathsf{Map}(H^p, \mathbb{R}) \to \mathsf{Map}(H^{p-1}, \mathbb{R})$, $p \geq 1$, which is given by the following explicit formula. If $c \in \mathsf{Map}(H^p, \mathbb{R})$ is a cocycle, then we can define $h^p(c) := b \in \mathsf{Map}(H^{p-1}, \mathbb{R})$ by the formula

$$b(t_1, \ldots, t_{p-1}) := (-1)^p \int_H c(t_1, \ldots, t_{p-1}, t)\,dt,$$

where dt is the normalized Haar measure. Then we have $db = c$. The maps $(h^p)_{p>0}$ induce a chain contraction $(h^p_*)_{p>0}$ of the complex $\underline{C^\bullet(H,\mathbb{R})}$. Therefore $H^i \mathsf{Map}(A, \mathcal{C}^\bullet(H,\mathbb{R})) \cong H^i \underline{C^\bullet(H,\mathbb{R})}(A)$ for $i \geq 1$, too.

The spectral sequence thus degenerates from the second page on. We conclude that

$$H^i \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet) \cong 0, \quad i \geq 1.$$

We now take the cohomology of the double complex $\underline{\mathsf{Hom}}^i_{\mathsf{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$ in the other order, first in the $U^\bullet$-direction and then in the $I^\bullet$-direction. In order to calculate the cohomology of $U^\bullet$ in degree ≤ 2 we use the fact 4.15. We get again a spectral sequence with second term (for $q \leq 2$, using 4.2.11)

$$F_2^{p,q} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^q_{\mathbb{Z}} \underline{H})^\sharp, \underline{\mathbb{R}}).$$

We know by Corollary 3.28, 2., that

$$F_2^{p,0} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{R}}) \cong 0, \quad p \geq 1$$

(note that we can write $\underline{\mathbb{Z}} = \mathbb{Z}(\underline{\{*\}})$ for a one-point space). Furthermore note that

$$F_2^{0,1} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{R}}) \cong \underline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(H, \mathbb{R})} \cong 0$$

since there are no continuous homomorphisms $H \to \mathbb{R}$. The second page of the spectral sequence thus has the following structure.

2	$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}} \underline{H})^\sharp, \underline{\mathbb{R}})$				
1	0	$\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{R}})$	$\underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{R}})$		
0	$\underline{\mathbb{R}}$	0	0	0	0
	0	1	2	3	4

Since the spectral sequence must converge to zero in positive degrees we see that

$$\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{R}}) \cong 0.$$

We claim that $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}} \underline{H})^\sharp, \underline{\mathbb{R}}) \cong 0$. Note that for $A \in \mathbf{S}$ we have

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}} \underline{H})^\sharp, \underline{\mathbb{R}})(A) \cong \underline{\mathsf{Hom}}_{\mathsf{Pr_{Ab}}\mathbf{S}}(\Lambda^2_{\mathbb{Z}} \underline{H}, \underline{\mathbb{R}})(A) \cong \mathsf{Hom}_{\mathsf{Pr_{Ab}}\mathbf{S}/A}(\Lambda^2_{\mathbb{Z}} \underline{H}_{|A}, \underline{\mathbb{R}}_{|A}).$$

An element $\lambda \in \mathsf{Hom}_{\mathsf{Pr_{Ab}}\mathbf{S}/A}(\Lambda^2_{\mathbb{Z}} \underline{H}_{|A}, \underline{\mathbb{R}}_{|A})$ induces a family of a biadditive (antisymmetric) maps

$$\lambda^W : \underline{H}(W) \times \underline{H}(W) \to \underline{\mathbb{R}}(W)$$

for $(W \to A) \in \mathbf{S}/A$ which is compatible with restriction. Restriction to points gives continuous biadditive maps $H \times H \to \mathbb{R}$. Since H is compact the only such map is the constant map to zero. Therefore λ^W vanishes for all $(W \to A)$. This proves the claim.

Again, since the spectral sequence (F_r, d_r) must converge to zero in higher degrees we see that

$$F_2^{2,1} \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{R}}) \cong 0.$$

This finishes the proof of the lemma. $\square$

4.3 Discrete groups

4.3.1 In this subsection we study admissibility of discrete abelian groups. First we show the easy fact that a finitely generated discrete abelian group is admissible. In the second step we try to generalize this result using the representation of an arbitrary discrete abelian group as a colimit of its finitely generated subgroups. The functor $\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots,\underline{\mathbb{T}})$ does not commute with colimits because of the presence of higher $R\,\mathsf{lim}$-terms in the spectral sequence (26), below.

And in fact, not every discrete abelian group is admissible.

Lemma 4.17. *For $n \in \mathbb{N}$ we have $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}/n\underline{\mathbb{Z}},\underline{\mathbb{Z}}) \cong 0$ for $i \geq 2$.*

Proof. We apply the functor $\underline{\mathsf{Ext}}^*_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots,\underline{\mathbb{Z}})$ to the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}/n\underline{\mathbb{Z}} \to 0$$

and get the long exact sequence

$$\underline{\mathsf{Ext}}^{i-1}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}},\underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{i}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}/n\underline{\mathbb{Z}},\underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{i}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}},\underline{\mathbb{Z}}) \to \cdots .$$

Since by Theorem 4.10 $\underline{\mathsf{Ext}}^{i}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{Z}},\underline{\mathbb{Z}}) \cong 0$ for $i \geq 1$, the assertion follows. □

Theorem 4.18. *A finitely generated abelian group is admissible.*

Proof. The group $\mathbb{Z}/n\mathbb{Z}$ is admissible, since Assumption 2 of Lemma 4.9 follows from Proposition 4.16, while Assumption 1 follows from Lemma 4.17. The group $\mathbb{Z}$ is admissible by Theorem 4.10 since we can write $\underline{\mathbb{Z}} \cong \mathbb{Z}(\underline{\{*\}})$ for a point $\{*\} \in \mathbf{S}$. A finitely generated abelian group is a finite product of groups of the form $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ for various $n \in \mathbb{N}$. A finite product of admissible groups is admissible. □

4.3.2 We now try to extend this result to general discrete abelian groups using colimits. Let I be a filtered category (see [Tam94, 0.3.2] for definitions). The category $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is an abelian Grothendieck category (see [Tam94, Theorem I.3.2.1]). The category $\mathsf{Hom}_{\mathsf{Cat}}(I,\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is again abelian and a Grothendieck category (see [Tam94, Proposition 0.1.4.3]). We have the adjoint pair

$$\mathsf{colim} : \mathsf{Hom}_{\mathsf{Cat}}(I,\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}) \Leftrightarrow \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} : C_?,$$

where to $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ there is associated the constant functor $C_F \in \mathsf{Hom}_{\mathsf{Cat}}(I,\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ with value F by the functor $C_?$. The functor $C_?$ also has a right-adjoint lim:

$$C_? : \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \Leftrightarrow \mathsf{Hom}_{\mathsf{Cat}}(I,\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}) : \mathsf{lim}.$$

This functor is left-exact and admits a right-derived functor $R\,\mathsf{lim}$. Finally, for $F \in \mathsf{Hom}_{\mathsf{Cat}}(I,\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ we have the functor

$${}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F,\dots)\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}},\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$$

which fits into the adjoint pair

$$\int^I \cdots \otimes F : \mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}) \Leftrightarrow: \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} : {}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \dots), \tag{25}$$

where for $G \in \mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ the symbol $\int^I G \otimes F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ denotes the coend of the functor $I^{\mathrm{op}} \times I \to \mathsf{Sh}_{\mathsf{Ab}}$, $(i, j) \mapsto G(i) \otimes F(j)$; correspondingly $\int_I$ denotes the end of the appropriate functor. Indeed we have for all $A \in \mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ and $B \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ a natural isomorphism

$$\begin{aligned}
\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}\Big(\int^I A \otimes F, B\Big) &\cong \int_I^{I^{\mathrm{op}} \times I} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(A \otimes F, B)\\
&\cong \int_I^{I^{\mathrm{op}}} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(A, {}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, B))\\
&\cong \mathsf{Hom}_{\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(A, {}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, B))
\end{aligned}$$

The functor ${}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \dots)$ is therefore also left-exact and admits a right-derived version.

4.3.3 We say that $P \in \mathsf{Hom}_{\mathsf{Cat}}(I, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is I-free if there exists a collection of flat sheaves $(U(l))_{l \in I}$, $U(l) \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, such that $P(j) = \bigoplus_{l \to j} U(l)$, and $P(i \to j)\colon \bigoplus_{l \to i} U(l) \to \bigoplus_{l \to j} U(l)$ maps the summand $U(l)$ at $l \to i$ identically to the summand $U(l)$ at $l \to i \to j$.

Lemma 4.19. *If $P \in \mathsf{Hom}_{\mathsf{Cat}}(I, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is I-free, and if $J \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is injective, then ${}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(P, J) \in \mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is injective.*

Proof. We consider an exact sequence

$$(A^\bullet\colon 0 \to A^0 \to A^1 \to A^2 \to 0)$$

in $\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$. Then by Equation (25) we have

$$\mathsf{Hom}_{\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(A^\bullet, {}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(P, J)) \cong \mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}\left(\int^I A^\bullet \otimes P, J\right).$$

Exactness in $\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ is defined object-wise [Tam94, Theorem 0.1.3.1]. Therefore $0 \to A^0(i) \to A^1(i) \to A^2(i) \to 0$ is an exact complex of sheaves for all $i \in I$. Since $P(j)$ is flat for all $j \in I$ the complex $A^\bullet(i) \otimes P(j)$ is exact for all pairs $(i, j) \in I \times I$. The complex of sheaves $\int^I A^\bullet \otimes P$ is the complex of push-outs along the exact sequence of diagrams

$$\begin{tikzcd}
\bigsqcup_{(i \to j) \in \mathsf{Mor}(I)} A^\bullet(j) \otimes P(i) \arrow[r, "\mathsf{id} \otimes P(i \to j)"] \arrow[d, "A^\bullet(i \to j) \otimes \mathsf{id}"'] & \bigsqcup_{j \in I} A^\bullet(j) \otimes P(j) \\
\bigsqcup_{i \in I} A^\bullet(i) \otimes P(i). &
\end{tikzcd}$$

We claim that this $\int^I A^\bullet \otimes P \cong \bigoplus_{i\in I} A^\bullet(i) \otimes U(i)$. Since $U(i)$ is flat for all $i \in I$ this is a sum of exact sequences and hence exact. Since J is injective we conclude that $\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\int^I A^\bullet \otimes P, J)$ is exact. So $\mathsf{Hom}_{\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}},\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(\dots, {}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(P, J))$ preserves exactness, hence ${}^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(P, J)$ is an injective object in $\mathsf{Hom}_{\mathsf{Cat}}(I^{\mathrm{op}}, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$.

In order to finish the proof it remains to show the claim. We expand the push-out diagram by inserting the structure of P.

$$\begin{tikzcd}
\bigoplus_{i\to j} A^\bullet(j) \otimes \bigoplus_{l\to i} U(l) \arrow[r, "a:=\mathsf{id}\otimes(i\to j)"] \arrow[d, "b:=A^\bullet(i\to j)\otimes\mathsf{id}"'] & \bigoplus_{l\to j} A^\bullet(j) \otimes U(l) \arrow[d] \\
\bigoplus_{l\to i} A^\bullet(i) \otimes U(l) \arrow[r, "d:=A^\bullet(l\to i)\otimes\mathsf{id}"] & \bigoplus_{l\in I} A^\bullet(l) \otimes U(l).
\end{tikzcd}$$

It suffices to show that the lower right corner has the universal property. The lower horizontal map has a split $s\colon \bigoplus_{l\in I} A^\bullet(l) \otimes U(l) \to \bigoplus_{l\to i} A^\bullet(i) \otimes U(l)$ given by the inclusion of summands induced by $l \mapsto l \xrightarrow{\mathsf{id}} l$. Two maps ψ_l, ψ_r from the lower left and upper right corner to some sheaf V which satisfy $\psi_l \circ b = \psi_r \circ a$ must induce a unique map $\psi\colon \bigoplus_{l\in I} A^\bullet(l) \otimes U(l) \to V$ such that $\psi \circ d = \psi_l$ and $\psi \circ c = \psi_r$. We have now other choice than to define $\psi := \psi_l \circ s$, and this map has the required properties as can be checked by an easy diagram chaise. $\square$

Lemma 4.20. *Every sheaf $F \in \mathsf{Hom}_{\mathsf{Cat}}(I, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ has an I-free resolution.*

Proof. It suffices to show that there exists a surjection $P \to F$ from an I-free sheaf. We start with the surjection $\mathbb{Z}(F) \to F$ and note that $\mathbb{Z}(F)(i)$ is flat for all $i \in I$. Then we define the I-free sheaf P by $P(i) := \oplus_{l\to i}\mathbb{Z}(F)(l)$, and the surjection $P \to Z(F)$ by $(l \to i)_*\colon Z(F)(l) \to \mathbb{Z}(F)(i)$ on the summand of $P(i)$ with index $(l \to i)$. $\square$

Lemma 4.21. *We have a natural isomorphism in $D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$*

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}(F), H) \cong R\,\mathsf{lim} R^I\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, H).$$

Proof. Let $F \in \mathsf{Hom}_{\mathsf{Cat}}(I, \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$. By Lemma 4.20 we can choose an I-free resolution $P^\bullet$ of F. Let $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ and $H \to I^\bullet$ be an injective resolution. Then we have

$$R\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}\ F, H) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}\ F, I^\bullet).$$

Since the category $\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is a Grothendieck abelian category [Tam94, Theorem I.3.2.1] the functor colim is exact [Tam94, Theorem 0.3.2.1]. Therefore $\mathsf{colim}\, P^\bullet(F) \to \mathsf{colim}\, F$ is a quasi-isomorphism. It follows that

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}\ F, I^\bullet) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}\ P^\bullet(F), I^\bullet).$$

By Lemma 3.8 the restriction functor is exact and therefore commutes with colimits. We get

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathrm{colim}\ P^\bullet(F), I^\bullet)(A) &\cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{S}/A}((\mathrm{colim}\ P^\bullet(F))_{|A}, I^\bullet_{|A}) \\
&\cong \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{S}/A}(\mathrm{colim}(P^\bullet(F)_{|A}), I^\bullet_{|A}) \\
&\cong \lim{}^I \mathrm{Hom}_{\mathrm{Sh_{Ab}}\mathbf{S}/A}(P^\bullet(F)_{|A}, I^\bullet_{|A}) \\
&\cong \lim\,({}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet)(A)) \\
&\cong (\lim{}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet))(A),
\end{aligned}$$

where in the last step we use that a limit of sheaves is defined object-wise. In other words

$$\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathrm{colim}\ F, I^\bullet) \cong \underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathrm{colim}\ P^\bullet(F), I^\bullet) \cong \lim{}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet)$$

Applying Lemma 4.19 we see that ${}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet) \in \mathrm{Hom}_{\mathrm{Cat}}(I, \mathrm{Sh_{Ab}}\mathbf{S})$ is injective, and

$$\begin{aligned}
R\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathrm{colim}\ F, H) &\cong \lim{}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet) \\
&\cong R\lim{}^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(P^\bullet(F), I^\bullet) \\
&\cong R\lim R^I\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(F, H). \qquad \square
\end{aligned}$$

4.3.4 Lemma 4.21 implies the existence of s spectral sequence with second page

$$E_2^{p,q} \cong R^p\lim R^{q\,I}\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(F, H) \tag{26}$$

which converges to $\mathrm{Gr}R^*\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathrm{colim}\ F, H)$.

Lemma 4.22. *Let I be a category and $U \in \mathbf{S}$. Then we have a commutative diagram*

$$\begin{array}{ccc}
D^+((\mathrm{Sh_{Ab}}\mathbf{S})^{I^{\mathrm{op}}}) & \xrightarrow{R\lim} & D^+(\mathrm{Sh_{Ab}}\mathbf{S}) \\
\big\downarrow{\scriptstyle R\Gamma(U,\dots)} & & \big\downarrow{\scriptstyle R\Gamma(U,\dots)} \\
D^+(\mathrm{Ab}^{I^{\mathrm{op}}}) & \xrightarrow{R\lim} & D^+(\mathrm{Ab}).
\end{array}$$

Proof. Since the limit of a diagram of sheaves is defined object-wise we have $\Gamma(U,\dots)\circ\lim = \lim\circ\Gamma(U,\dots)$. We show that the two compositions $R\Gamma(U,\dots)\circ R\lim$, $R\lim\circ R\Gamma(U,\dots)$ are both isomorphic to $R(\Gamma(U,\dots)\circ\lim)$ by showing that $\lim$ and $\Gamma(U,\dots)$ preserve injectives.

The left-adjoint of the functor $\lim\colon (\mathrm{Sh_{Ab}}\mathbf{S})^{I^{\mathrm{op}}} \to \mathrm{Sh_{Ab}}\mathbf{S}$ is the constant diagram functor $C_{\dots}\colon \mathrm{Sh_{Ab}}\mathbf{S} \to (\mathrm{Sh_{Ab}}\mathbf{S})^{I^{\mathrm{op}}}$ which is exact. Therefore $\lim$ preserves injectives.

The left-adjoint of the functor $\Gamma(U,\dots)$ is the functor $\dots\otimes_{\mathbb{Z}}\mathbb{Z}(U)$. Since $\mathbb{Z}(U)$ is a sheaf of flat $\mathbb{Z}$-modules (it is torsion-free) this functor is exact (object-wise and therefore on diagrams). It follows that $\Gamma(U,\dots)$ preserves injectives. $\square$

4.3.5 The general remarks on colimits above are true for all sites with finite products (because of the use of Lemma 3.8). In particular we can replace $\mathbf{S}$ by the site $\mathbf{S}_{\text{lc-acyc}}$ of locally acyclic locally compact spaces.

Lemma 4.23. *Let $F \in (\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}})^{I^{\mathrm{op}}}$. If for all acyclic $U \in \mathbf{S}_{\text{lc-acyc}}$ the canonical map $\lim F(U) \to R\lim(F(U))$ is a quasi-isomorphism of complexes of abelian groups, then $\lim F \to R\lim F$ is a quasi-isomorphism.*

Proof. We choose an injective resolution $F \to I^\bullet$ in $(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}})^{I^{\mathrm{op}}}$. As in the proof of Lemma 4.22 for each $U \in \mathbf{S}_{\text{lc-acyc}}$ the complex $I^\bullet(U)$ is injective. If we assume that U is acyclic, the map $F(U) \to I^\bullet(U)$ is a quasi-isomorphism in $(\mathsf{Ab})^{I^{\mathrm{op}}}$, and $R\lim\,(F(U)) \cong \lim(I^\bullet(U))$. By assumption

$$(\lim F)(U) \cong \lim(F(U)) \to \lim(I^\bullet(U)) \cong (\lim I^\bullet)(U) \cong (R\lim F)(U)$$

is a quasi-isomorphism for all acyclic $U \in \mathbf{S}_{\text{lc-acyc}}$. An arbitrary object $A \in \mathbf{S}_{\text{lc-acyc}}$ can be covered by acyclic open subsets. Therefore $\lim F \to \lim I^\bullet$ is an quasi-isomorphism locally on A for each $A \in \mathbf{S}_{\text{lc-acyc}}$. Hence $\lim F \to R\lim F$ is an isomorphism in $D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}})$. □

4.3.6 Let D be a discrete group. Then we let I be the category of all finitely generated subgroups of D. This category is filtered. Let $F\colon I \to \mathsf{Ab}$ be the "identity" functor. Then we have a natural isomorphism

$$D \cong \operatorname{colim} F.$$

By Theorem 4.18 we know that a finitely generated group G is admissible, i.e.

$$R^q \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong 0, \quad q = 1, 2.$$

Note that by Lemma 3.3 we have $\operatorname{colim}\underline{F} = \underline{D}$. Using the spectral sequence (26) we get for $p = 1, 2$ that

$$R^p \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{D}, \underline{\mathbb{T}}) \cong R^p \lim \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{F}, \underline{\mathbb{T}}). \tag{27}$$

Let $g\colon \mathbf{S}_{\text{lc-acyc}} \to \mathbf{S}$ be the inclusion. Since $\mathbb{T}$ and D belong to $\mathbf{S}_{\text{lc-acyc}}$ using Lemma 3.38 we also have

$$\begin{aligned} g^* R^p \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{D}, \underline{\mathbb{T}}) &\cong R^p \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{D}, \underline{\mathbb{T}}) \\ &\cong R^p \lim \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{F}, \underline{\mathbb{T}}). \end{aligned} \tag{28}$$

4.3.7 We now must study the $R^p\lim$-term. First we make the index category I slightly smaller. Let $\bar{D} \subseteq D \otimes_{\mathbb{Z}} \mathbb{Q} =: D_{\mathbb{Q}}$ be the image of the natural map $i\colon D \to D_{\mathbb{Q}}$, $d \mapsto d \otimes 1$. We observe that $\bar{D}$ generates $D_{\mathbb{Q}}$. We choose a basis $B \subseteq \bar{D}$ of the $\mathbb{Q}$-vector space $D_{\mathbb{Q}}$ and let $\mathbb{Z}B \subseteq \bar{D}$ be the $\mathbb{Z}$-lattice generated by B. For a subgroup $A \subseteq D$ let $\bar{A} := i(A) \subset D_{\mathbb{Q}}$. We consider the partially ordered (by inclusion) set

$$J := \{A \subseteq D \mid A \text{ finitely generated and } \mathbb{Q}\bar{A} \cap \mathbb{Z}B \subset \bar{A} \text{ and } [\bar{A} : \bar{A} \cap \mathbb{Z}B] < \infty\}.$$

Here $[G : H]$ denotes the index of a subgroup H in a group G. We still let A denote the "identity" functor $A\colon J \to \mathsf{Ab}$.

Lemma 4.24. *We have $D \cong \mathsf{colim}\, A$.*

Proof. It suffices to show that $J \subseteq I$ is cofinal. Let $A \subseteq D$ be a finitely generated subgroup. Choose a finite subset $\bar{b} \subset B$ such that the $\mathbb{Q}$-vectorspace $\mathbb{Q}\bar{b} \subset D_{\mathbb{Q}}$ generated by $\bar{b}$ contains $\bar{A}$ and choose representatives b in D of the elements of $\bar{b}$. They generate a finitely generated group $U \subset D$ such that $\bar{U} = i(U) = \mathbb{Z}\bar{b}$. We now consider the group $G := \langle U, A\rangle$. This group is still finitely generated. Similarly, since $\mathbb{Q}\bar{G} = \mathbb{Q}\bar{U} + \mathbb{Q}\bar{A} = \mathbb{Q}\bar{b}$ we have

$$\mathbb{Q}\bar{G} \cap \mathbb{Z}B = \mathbb{Q}\bar{b} \cap \mathbb{Z}B = \bar{b} \subseteq \bar{U} \subseteq \bar{G}.$$

Moreover, since $\mathbb{Q}\bar{G} = \mathbb{Q}\bar{b} = \mathbb{Q}(\bar{G} \cap \mathbb{Z}B)$, $[\bar{G} : \bar{G} \cap \mathbb{Z}B] < \infty$, i.e. $G \subset J$. On the other hand, by construction $A \subseteq G$. □

On J we define the grading $w\colon J \to \mathbb{N}_0$ by

$$w(A) := |A_{\mathrm{tors}}| + \mathsf{rk}\,\bar{A} + [\bar{A} : \bar{A} \cap \mathbb{Z}B] \quad \text{for } A \in J.$$

Lemma 4.25. *The category J together with the grading $w\colon J \to \mathbb{N}_0$ is a direct category in the sense of Definition* 5.1.1 *in* [Hov99].

Proof. We must show that $A \subset G$ implies $w(A) \leq w(G)$, and that $A \subsetneq G$ implies $w(A) < w(G)$. First of all we have $A_{\mathrm{tors}} \subset G_{\mathrm{tors}}$ and therefore $|A_{\mathrm{tors}}| \leq |G_{\mathrm{tors}}|$. Moreover we have $\bar{A} \subseteq \bar{G}$, hence $\mathsf{rk}\,\bar{A} \leq \mathsf{rk}\,\bar{G}$. Finally, we claim that the canonical map

$$\bar{A}/(\bar{a} \cap \mathbb{Z}B) \to \bar{G}/(\bar{G} \cap \mathbb{Z}B)$$

is injective. In fact, we have $\bar{A} \cap (\bar{G} \cap \mathbb{Z}B) = (\bar{A} \cap \bar{G}) \cap \mathbb{Z}B = \bar{A} \cap \mathbb{Z}B$. It follows that

$$|\bar{A} : \bar{A} \cap \mathbb{Z}B| \leq |\bar{G} : \bar{G} \cap \mathbb{Z}B|.$$

Let now $A \subseteq G$ and $w(A) = w(G)$. We want to see that this implies $A = G$. First note that the inclusion of finite groups $A_{\mathrm{tors}} \to G_{\mathrm{tors}}$ is an isomorphism since both groups have the same number of elements. It remains to see that $\bar{A} \to \bar{G}$ is an isomorphism. We have $\mathsf{rk}\,\bar{A} = \mathsf{rk}\,\bar{G}$. Therefore $\bar{A} \cap \mathbb{Z}B = \mathbb{Q}\bar{A} \cap \mathbb{Z}B = \mathbb{Q}\bar{G} \cap B\mathbb{Z} = \bar{G} \cap B\mathbb{Z}$. Now the equality $[\bar{A} : \bar{A} \cap \mathbb{Z}B] = [\bar{G} : \bar{G} \cap \mathbb{Z}B]$ implies that $\bar{A} = \bar{G}$. □

4.3.8 The category $C(\mathsf{Ab})$ has the projective model structure whose weak equivalences are the quasi-isomorphisms, and whose fibrations are level-wise surjections. By Lemma 4.25 the category J^{op} with the grading w is an inverse category. On $C(\mathsf{Ab})^{J^{\mathrm{op}}}$ we consider the inverse model structure whose weak equivalences are the quasi-isomorphisms, and whose cofibrations are the object-wise ones. The fibrations are characterized by a matching space condition which we will explain in the following.

For $j \in J$ let $J_j \subseteq J$ be the category with non-identity maps all non-identity maps in J with codomain j. Furthermore consider the functor

$$M_j\colon C(\mathsf{Ab})^{J^{\mathrm{op}}} \xrightarrow{\text{restriction}} C(\mathsf{Ab})^{(J_j)^{\mathrm{op}}} \xrightarrow{\mathsf{lim}} C(\mathsf{Ab})$$

There is a natural morphism $F(j) \to M_j F$ for all $F \in C(\mathsf{Ab})^{J^{\mathrm{op}}}$. The matching space condition asserts that a map $F \to G$ in $C(\mathsf{Ab})^{J^{\mathrm{op}}}$ is a fibration if and only if

$$F(j) \to G(j) \times_{M_j G} M_j F$$

is a fibration in $C(\mathsf{Ab})$, i.e. a level-wise surjection, for all $j \in J$. In particular, F is fibrant if $F(j) \to M_j F$ is a level-wise surjection for all $j \in J$.

For $X \in C(\mathsf{Ab})^{J^{\mathrm{op}}}$ the fibrant replacement $X \to RX$ induces the morphism

$$\mathsf{lim}\, X \to \mathsf{lim}\, RX \cong R\mathsf{lim}\, X.$$

Let now again A denote the identity functor $A\colon J \to \mathsf{Ab}$ for the discrete abelian group D.

Lemma 4.26. 1. *For all $U \in \mathbf{S}$ which are acyclic the diagram of abelian groups $\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{A}, \underline{\mathbb{T}})(U) \in C(\mathsf{Ab})^{J^{\mathrm{op}}}$ is fibrant.*

2.

$$\mathsf{lim}\,\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{A}, \underline{\mathbb{T}}) \to R\mathsf{lim}\,\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{A}, \underline{\mathbb{T}}) \in C(\mathsf{Ab})^{J^{\mathrm{op}}}$$

is a quasi-isomorphism.

Proof. The assertion 1. for locally compact acyclic U verifies the assumption of Lemma 4.23. Hence 2. follows from 1. We now concentrate on 1. We must show that

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{A}(j), \underline{\mathbb{T}})(U) \to M_j \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{A}, \underline{\mathbb{T}})(U) \tag{29}$$

is surjective. We have

$$M_j \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{A}, \underline{\mathbb{T}}) \cong \mathsf{lim}\, \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{A}_{|J_j}, \underline{\mathbb{T}}) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}\, \underline{A}_{|J_j}, \underline{\mathbb{T}}).$$

The map (29) is induced by the map

$$\mathsf{colim} \underline{A}_{|J_j} \to \underline{F}(j). \tag{30}$$

By Lemma 3.3 we have $\mathsf{colim}\, \underline{A}_{|J_j} \cong \underline{\mathsf{colim}\, A_{|J_j}}$. The map $\mathsf{colim}\, A_{|J_j} \to A(j)$ is of course an injection.

We finish the argument by the following observation. Let $H \to G$ be an injective map of finitely generated groups (we apply this with $H := \mathsf{colim} A_{|J_j}$ and $G := F(j)$). Then for $U \in \mathbf{S}$

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})(U) \to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{H}, \underline{\mathbb{T}})(U)$$

is a surjection. In fact we have

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})(U) &\cong \underline{\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T})}(U) \quad \text{(by Lemma 3.5)} \\ &\cong \underline{\hat{G}}(U) \end{aligned}$$

(and a similar equation for H). Since $H \to G$ is injective, its Pontrjagin dual $\widehat{G} \to \widehat{H}$ is surjective. Because of the classification of discrete finitely generated abelian groups, $\widehat{G}$ and $\widehat{H}$ both are homeomorphic to finite unions of finite dimensional tori. Because U is acyclic, every map $U \to \widehat{H}$ lifts to $\widehat{G}$. Therefore $\underline{\widehat{G}}(U) \to \underline{\widehat{H}}(U)$ is surjective. □

Corollary 4.27. *We have* $R^p \lim \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}_{\text{lc-acyc}}}(\underline{A}, \underline{\mathbb{T}}) \cong 0$ *for all* $p \geq 1$.

Theorem 4.28. *Every discrete abelian group is admissible on* $\mathbf{S}_{\text{lc-acyc}}$.

Proof. This follows from (28) and Corollary 4.27. □

4.3.9 We now present an example which shows that not every discrete group D is admissible on $\mathbf{S}$ or $\mathbf{S}_{\text{lc}}$, using Corollary 4.41 to be established later.

Lemma 4.29. *Let I be an infinite set, $1 \neq n \in \mathbb{N}$ and $D := \oplus_I \mathbb{Z}/n\mathbb{Z}$. Then D is not admissible on* $\mathbf{S}$ *or* $\mathbf{S}_{\text{lc}}$.

Proof. We consider the sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{T} \xrightarrow{n} \mathbb{T} \to 0 \tag{31}$$

which has no global section, and the product of I copies of it

$$0 \to \prod_I \mathbb{Z}/n\mathbb{Z} \to \prod_I \mathbb{T} \to \prod_I \mathbb{T} \to 0. \tag{32}$$

This sequence of compact abelian groups does not have local sections. In fact, an open subset of $\prod_I \mathbb{T}$ always contains a subset of the form $U = \prod_{I'} \mathbb{T} \times V$, where $I' \subset I$ is cofinite and $V \subset \prod_{I \setminus I'} \mathbb{T}$ is open. A section $s \colon U \to \prod_I \mathbb{T}$ would consist of sections of the sequence (31) at the entries labeled with I'.

By Lemma 3.4 the sequence of sheaves associated to (32) is not exact. In view of Corollary 4.41 below, the group $\widehat{\prod_I \mathbb{Z}/n\mathbb{Z}} \cong \oplus_I \mathbb{Z}/n\mathbb{Z}$ is not admissible on $\mathbf{S}$ or $\mathbf{S}_{\text{lc}}$. □

4.4 Admissibility of the groups $\mathbb{R}^n$ and $\mathbb{T}^n$

Theorem 4.30. *The group $\mathbb{T}$ is admissible.*

Proof. Since $\mathbb{T}$ is compact, Assumption 2. of Lemma 4.9 follows from Proposition 4.16. It remains to show the first assumption of Lemma 4.9. As we will see this follows from the following result.

Lemma 4.31. *We have* $\underline{\mathsf{Ext}}^i_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) = 0$ *for* $i = 1, 2, 3$.

Let us assume this lemma for the moment. We apply $\underline{\mathsf{Ext}}^*_{\mathsf{Sh_{Ab}}\mathbf{S}}(\ldots, \underline{\mathbb{Z}})$ to the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0$$

and consider the following part of the resulting long exact sequence for $i = 2, 3$:

$$\underline{\mathrm{Ext}}^{i-1}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^{i}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{T}}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^{i}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) \to \underline{\mathrm{Ext}}^{i}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}).$$

The outer terms vanish since $R\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, F) \cong F$ for all $F \in \mathrm{Sh_{Ab}}\mathbf{S}$. Therefore by Lemma 4.31

$$\underline{\mathrm{Ext}}^{i}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{T}}, \underline{\mathbb{Z}}) \cong \underline{\mathrm{Ext}}^{i}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) \cong 0,$$

and this is Assumption 1. of 4.9.

It remains to prove Lemma 4.31.

4.4.1 Proof of Lemma 4.31. We choose an injective resolution $\underline{\mathbb{Z}} \to I^\bullet$. The sheaf $\underline{\mathbb{R}}$ gives rise to the homological complex $U^\bullet$ introduced in 4.14. We get the double complex $\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$ as in 4.2.12. As before we discuss the associated two spectral sequences which compute $\underline{\mathrm{Ext}}^{*}_{\mathrm{Sh}_{\mathbb{Z}[\underline{\mathbb{R}}]\text{-mod}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$.

4.4.2 We first take the cohomology in the $I^\bullet$-, and then in the $U^\bullet$-direction. In view of (24), the first page of the resulting spectral sequence is given by

$$E_1^{p,q} \cong \underline{\mathrm{Ext}}^{q}_{\mathrm{Sh_{Ab}}\mathbf{S}}(\mathbb{Z}(\mathcal{F}\underline{\mathbb{R}}^p), \underline{\mathbb{Z}}),$$

where $\mathcal{F}\underline{\mathbb{R}}$ denotes the underlying sheaf of sets of $\underline{\mathbb{R}}$. By Corollary 3.28, 1. we have $E_1^{p,q} \cong 0$ for $q \geq 1$.

We now consider the case $q = 0$. For $A \in \mathbf{S}$ we have $E_1^{p,0}(A) \cong \Gamma(A \times \mathbb{R}^p; \underline{\mathbb{Z}}) \cong \Gamma(A; \underline{\mathbb{Z}})$ for all p, i.e. $E_1^{p,0} \cong \underline{\mathbb{Z}}$. We can easily calculate the cohomology of the complex $(E_1^{*,0}, d_1)$ which is isomorphic to

$$0 \to \underline{\mathbb{Z}} \xrightarrow{0} \underline{\mathbb{Z}} \xrightarrow{\mathrm{id}} \underline{\mathbb{Z}} \xrightarrow{0} \underline{\mathbb{Z}} \xrightarrow{\mathrm{id}} \underline{\mathbb{Z}} \to \cdots .$$

We get

$$H^i(E_1^{*,0}, d_1) \cong \begin{cases} \underline{\mathbb{Z}}, & i = 0, \\ 0, & i \geq 1. \end{cases}$$

The spectral sequence (E_r, d_r) thus degenerates at the second term, and

$$H^i \underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet) \cong \begin{cases} \underline{\mathbb{Z}}, & i = 0, \\ 0, & i \geq 1. \end{cases} \tag{33}$$

4.4.3 We now consider the second spectral sequence $(F_r^{p,q}, d_r)$ associated to the double complex $\underline{\mathrm{Hom}}_{\mathrm{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$ which takes cohomology first in the $U^\bullet$ and then in the $I^\bullet$-direction. Its second page is given by

$$F_2^{p,q} \cong \underline{\mathrm{Ext}}^{p}_{\mathrm{Sh_{Ab}}\mathbf{S}}(H^q U^\bullet, \underline{\mathbb{Z}}).$$

Since $\underline{\mathbb{R}}$ is torsion-free we can apply 4.2.11 and get

$$F_2^{p,q} \cong \underline{\mathrm{Ext}}^{p}_{\mathrm{Sh_{Ab}}\mathbf{S}}((\Lambda^q_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}}).$$

In particular we have $\Lambda^0_{\mathbb{Z}}\underline{\mathbb{R}} \cong \underline{\mathbb{Z}}$ and thus

$$F_2^{p,0} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong 0$$

for $p \geq 1$ (recall that $\underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, H) \cong 0$ for every $H \in \mathsf{Sh_{Ab}}\mathbf{S}$ and $p \geq 1$). Furthermore, since $\Lambda^1_{\mathbb{Z}}\underline{\mathbb{R}} \cong \underline{\mathbb{R}}$, by Lemma 3.5 we have

$$F_2^{0,1} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(\mathbb{R}, \mathbb{Z})} \cong 0$$

since $\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(\mathbb{R}, \mathbb{Z}) \cong 0$.

Here is a picture of the relevant part of the second page.

3	$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^3_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}})$					
2	$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}})$	$\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}})$				
1	0	$\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$	$\underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$	$\underline{\mathsf{Ext}}^3_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$		
0	$\underline{\mathbb{Z}}$	0	0	0	0	0
	0	1	2	3	4	5

4.4.4 Let V be an abelian group. Recall the definition of $\Lambda^* V$ from 4.2.9. If V has the structure of a $\mathbb{Q}$-vector space, then $T^{\geq 1}_{\mathbb{Z}} V$ has the structure of a graded $\mathbb{Q}$-vector space, and $I \subseteq T^{\geq 1}_{\mathbb{Z}} V$ is a graded $\mathbb{Q}$-vector subspace. Therefore $\Lambda^{\geq 1}_{\mathbb{Z}} V$ has the structure of a graded $\mathbb{Q}$-vector space, too.

4.4.5 We claim that

$$F_2^{0,i} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}}) \cong 0$$

for $i \geq 1$. Note that

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Hom}}_{\mathsf{Pr_{Ab}}\mathbf{S}}(\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}}, \underline{\mathbb{Z}}).$$

Let $A \in \mathbf{S}$. An element

$$\lambda \in \underline{\mathsf{Hom}}_{\mathsf{Pr_{Ab}}\mathbf{S}}(\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}}, \underline{\mathbb{Z}})(A) \cong \mathsf{Hom}_{\mathsf{Pr_{Ab}}\mathbf{S}/A}(\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}}_{|A}, \underline{\mathbb{Z}}_{|A})$$

induces a homomorphism of groups $\lambda^W \colon \Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}}(W) \to \underline{\mathbb{Z}}(W)$ for every $(W \to A) \in \mathbf{S}/A$. Since $\Lambda^i_{\mathbb{Z}}\underline{\mathbb{R}}(W)$ is a $\mathbb{Q}$-vector space and $\underline{\mathbb{Z}}(W)$ does not contain divisible elements we see that $\lambda^W = 0$. This proves the claim.

The claim implies that $F_2^{2,1} \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$ survives to the limit of the spectral sequence. Because of (33) it must vanish. This proves Lemma 4.31 in the case $i = 2$.

The term $F_2^{1,1} \cong \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$ also survives to the limit and therefore also vanishes because of (33). This proves Lemma 4.31 in the case $i = 1$.

4.4.6 Finally, since $F_3^{0,3} \cong 0$, we see that $d_2\colon F_2^{1,2} \to F_2^{3,1}$ must be an isomorphism, i.e.

$$d_2\colon \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}}) \xrightarrow{\sim} \underline{\mathsf{Ext}}^3_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}}).$$

We will finish the proof of Lemma 4.31 for $i = 3$ by showing that $d_2 = 0$.

4.4.7 We consider the natural action of $\mathbb{Z}_{\mathrm{mult}}$ on $\mathbb{R}$ and hence on $\underline{\mathbb{R}}$. This turns $\underline{\mathbb{R}}$ into a sheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight 1 (see 3.5.1). It follows that $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, I^\bullet)$ is a complex of sheaves of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight 1. Finally we see that $\underline{\mathsf{Ext}}^i_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{R}}, \underline{\mathbb{Z}})$ are sheaves of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight 1 for $i \geq 0$.

Now observe (see 3.5.5) that $\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}}$ is a presheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight 2. Hence $(\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp$ and thus $\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{\mathbb{R}})^\sharp, \underline{\mathbb{Z}})$ are sheaves of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight 2.

Since $\underline{\mathbb{R}} \to U^\bullet(\underline{\mathbb{R}}) =: U^\bullet$ is a functor we get an action of $\mathbb{Z}_{\mathrm{mult}}$ on $U^\bullet$ and hence on the double complex $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$. This implies that the differentials of the associated spectral sequences commute with the $\mathbb{Z}_{\mathrm{mult}}$-actions (see 4.6.11 for more details). This in particular applies to d_2. The equality $d_2 = 0$ now directly follows from the following lemma.

Lemma 4.32. *Let $V, W \in \mathsf{Sh_{Ab}}\mathbf{S}$ be sheaves of $\mathbb{Z}_{\mathrm{mult}}$-modules of weights $k \neq l$. Assume that W has the structure of a sheaf of $\mathbb{Q}$-vector spaces. If $d \in \mathsf{Hom}_{\mathsf{Sh_{Ab}}\mathbf{S}}(V, W)$ is $\mathbb{Z}_{\mathrm{mult}}$-equivariant, then $d = 0$.*

Proof. Let $\alpha\colon \mathbb{Z}_{\mathrm{mult}} \to \mathsf{End}_{\mathsf{Sh_{Ab}}\mathbf{S}}(V)$ and $\beta\colon \mathbb{Z}_{\mathrm{mult}} \to \mathsf{End}_{\mathsf{Sh_{Ab}}\mathbf{S}}(W)$ denote the actions. Then we have $d \circ \alpha(q) - \beta(q) \circ d = 0$ for all $q \in \mathbb{Z}_{\mathrm{mult}}$. We consider $q := 2$. Then $(2^k - 2^l) \circ d = 0$. Since W is a sheaf of $\mathbb{Q}$-vector spaces and $(2^k - 2^l) \neq 0$ this implies that $d = 0$. □

Theorem 4.33. *The group $\mathbb{R}$ is admissible.*

Proof. The outer terms of the exact sequence

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0$$

are admissible by Theorem 4.30 and Theorem 4.18. We now apply Lemma 4.5, stability of admissibility under extensions. □

4.5 Profinite groups

Definition 4.34. A topological group G is called profinite if there exists a small left-filtered[8] poset[9] I and a system $F \in \mathsf{Ab}^I$ such that

1. for all $i \in I$ the group $F(i)$ is finite,
2. for all $i \leq j$ the morphism $F(i) \to F(j)$ is a surjection,
3. there exists an isomorphism $G \cong \lim_{i \in I} F(i)$ as topological groups.

[8] i.e. for every pair $i, k \in I$ there exists $j \in I$ with $j \leq i$ and $j \leq k$

[9] A poset is considered here as a category.

For the last statement we consider finite abelian groups as topological groups with the discrete topology. Note that the homomorphisms $G \to F(i)$ are surjective for all $i \in I$. We call the system $F \in \mathsf{Ab}^I$ an inverse system.

Lemma 4.35 ([HR63]). *The following assertions on a topological abelian group G are equivalent:*

1. *G is compact and totally disconnected.*
2. *Every neighbourhood $U \subseteq G$ of the identity contains a compact subgroup K such that G/K is a finite abelian group.*
3. *G is profinite.*

Lemma 4.36. *Let G be a profinite abelian group and $n \in \mathbb{Z}$. We define the groups K, Q as the kernel and cokernel of the multiplication map by n, i.e. by the exact sequence*

$$0 \to K \to G \stackrel{n}{\to} G \to Q \to 0. \tag{34}$$

Then K and Q are again profinite.

Proof. We write $G := \lim_{j \in J} G_j$ for an inverse system $(G_j)_{j \in J}$ of finite abelian groups. We define the system of finite subgroups $(K_j)_{j \in J}$ by the sequences

$$0 \to K_j \to G_j \stackrel{n}{\to} G_j .$$

Since taking kernels commutes with limits the natural projections $K \to K_j$, $j \in J$ represent K as the limit $K \cong \lim_{j \in J} K_j$.

Since cokernels do not commute with limits we will use a different argument for Q. Since G is compact and the multiplication by n is continuous, $nG \subseteq G$ is a closed subgroup. Therefore the group theoretic quotient Q is a topological group in the quotient topology.

A quotient of a profinite group is again profinite [HM98], Exercise E.1.13. Here is a solution of this exercise, using the following general structural result about compact abelian groups.

Lemma 4.37 ([HR63]). *If H is a compact abelian group, then for every open neighbourhood $1 \in U \subseteq H$ there exists a compact subgroup $C \subset U$ such that $H/C \cong \mathbb{T}^a \times F$ for some $a \in \mathbb{N}_0$ and a finite group F.*

Since G is compact, its quotient Q is compact, too. This lemma in particular implies that Q is the limit of the system of these quotients Q/C. In our case, since G is profinite, it can not have T^a as a quotient, i.e given a surjection

$$G \to Q \to \mathbb{T}^a \times F$$

we conclude that $a = 0$. Hence we can write Q as a limit of an inverse system of finite quotients. This implies that Q is profinite. $\square$

4.5.1 Let
$$0 \to K \to G \to H \to 0$$
be an exact sequence of profinite groups, where $K \to G$ is the inclusion of a closed subgroup.

Lemma 4.38. *The sequence of sheaves*
$$0 \to \underline{K} \to \underline{G} \to \underline{H} \to 0$$
is exact.

Proof. By [Ser02, Proposition 1] every surjection between profinite groups has a section. Hence we can apply Lemma 3.4. □

In this result one can in fact drop the assumptions that K and G are profinite. In our basic example the group K is the connected component $G_0 \subseteq G$ of the identity of G.

Lemma 4.39. *Let*
$$0 \to K \to G \to H \to 0$$
be an exact sequence of compact abelian groups such that H is profinite. Then the sequence of sheaves
$$0 \to \underline{K} \to \underline{G} \to \underline{H} \to 0$$
is exact.

Proof. We can apply [HM98, Theorem 10.35] which says that the projection $G \to H$ has a global section. Thus the sequence of sheaves is exact by 3.4 (even as sequence of presheaves). □

4.5.2 In order to show that certain discrete groups are not admissible we use the following lemma.

Lemma 4.40. *Let*
$$0 \to K \to G \to H \to 0$$
be an exact sequence of compact abelian groups. If the discrete abelian group $\hat{K}$ is admissible, then sequence of sheaves
$$0 \to \underline{K} \to \underline{G} \to \underline{H} \to 0$$
is exact.

Proof. If U is a compact group, then its Pontrjagin dual $\hat{U} := \mathsf{Hom}_{\text{top-Ab}}(U, \mathbb{T})$ is a discrete group. Pontrjagin duality preserves exact sequences. Therefore we have the exact sequence of discrete groups
$$0 \to \hat{H} \to \hat{G} \to \hat{K} \to 0.$$

The surjective map of discrete sets $\hat{G} \to \hat{K}$ has of course a section. Therefore by Lemma 3.4 the sequence of sheaves of abelian groups

$$0 \to \underline{\hat{H}} \to \underline{\hat{G}} \to \underline{\hat{K}} \to 0$$

is exact. We apply $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots, \underline{\mathbb{T}})$ and get the exact sequence

$$\begin{aligned} 0 \to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{K}}, \underline{\mathbb{T}}) &\to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{G}}, \underline{\mathbb{T}}) \\ &\to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{H}}, \underline{\mathbb{T}}) \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{K}}, \underline{\mathbb{T}}) \to \cdots . \end{aligned}$$

By Lemma 3.5 we have $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{G}}, \underline{\mathbb{T}}) \cong \underline{G}$ etc. Therefore this sequence translates to

$$0 \to \underline{K} \to \underline{G} \to \underline{H} \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{K}}, \underline{\mathbb{T}}) \to \cdots . \tag{35}$$

By our assumption $\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\hat{K}}, \underline{\mathbb{T}}) = 0$ so that $\underline{G} \to \underline{H}$ is surjective. □

4.5.3 The same argument does apply for the site $\mathbf{S}_{\mathrm{lc}}$. Evaluating the surjection $\underline{G} \to \underline{H}$ on H we conclude the following fact.

Corollary 4.41. *If $K \subset G$ is a closed subgroup of a compact abelian group such that $\hat{K}$ is admissible or admissible over $\mathbf{S}_{\mathrm{lc}}$, then the projection $G \to G/K$ has local sections.*

4.5.4 If G is an abelian group, then let $G_{\mathrm{tors}} \subseteq G$ denote the subgroup of torsion elements. We call G a torsion group, if $G_{\mathrm{tors}} = G$. If $G_{\mathrm{tors}} \cong 0$, then we say that G is torsion-free. If G is a torsion group and H is torsion-free, then $\mathsf{Hom}_{\mathsf{Ab}}(H, G) \cong 0$.

A presheaf $F \in \mathrm{Pr}_{\mathbf{S}}\,\mathbf{S}$ is called a presheaf of torsion groups if $F(A)$ is a torsion group for every $A \in \mathbf{S}$. The notion of a torsion sheaf is more complicated.

Definition 4.42. A sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is called a torsion sheaf if for each $A \in \mathbf{S}$ and $f \in F(A)$ there exists an open covering $(U_i)_{i \in I}$ of A such that $f_{|U_i} \in F(U_i)_{\mathrm{tors}}$.

The following lemma provides equivalent characterizations of torsion sheaves.

Lemma 4.43 ([Tam94], (9.1)). *Consider a sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. The following assertions are equivalent.*

1. *F is a torsion sheaf.*
2. *F is the sheafification of a presheaf of torsion groups.*
3. *The canonical morphism $\mathsf{colim}_{n\in\mathbb{N}}\,({}_nF) \to F$ is an isomorphism, where $({}_nF)_{n\in\mathbb{N}}$ is the direct system ${}_nF := \ker(F \stackrel{n!}{\to} F)$.*

Note that a subsheaf or a quotient of a torsion sheaf is again a torsion sheaf.

Lemma 4.44. *If H is a discrete torsion group, then $\underline{H}$ is a torsion sheaf.*

Proof. We write $H = \mathsf{colim}_{n\in\mathbb{N}}({}_nH)$, where ${}_nH := \ker(H \xrightarrow{n!} H)$. By Lemma 3.3 we have $\underline{H} = \mathsf{colim}_{n\in\mathbb{N}}\underline{{}_nH}$. Now $\mathsf{colim}_{n\in\mathbb{N}}\underline{{}_nH}$ is the sheafification of the presheaf ${}^p\mathsf{colim}_{n\in\mathbb{N}}\underline{{}_nH}$ of torsion groups and therefore a torsion sheaf. □

In general $\underline{H}$ is not a presheaf of torsion groups. Consider e.g. $H := \oplus_{n\in\mathbb{N}}\mathbb{Z}/n\mathbb{Z}$. Then the element $\mathsf{id} \in \underline{H}(H)$ is not torsion.

Definition 4.45. A sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is called torsion-free if the group $F(A)$ is torsion-free for all $A \in \mathbf{S}$.

A sheaf F is torsion-free if and only if for all $n \in \mathbb{N}$ the map $F \xrightarrow{n} F$ is injective. It suffices to test this for all primes.

Lemma 4.46. *If $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is a torsion sheaf and $E \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is torsion-free, then* $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, E) \cong 0$.

Proof. We have

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, E) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathsf{colim}_{n\in\mathbb{N}}({}_nF), E) \cong \lim_{n\in\mathbb{N}}\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}({}_nF, E).$$

On the one hand, via the first entry multiplication by $n!$ induces on $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}({}_nF, E)$ the trivial map. On the other hand it induces an injection since E is torsion-free. Therefore $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}({}_nF, E) \cong 0$ for all $n \in \mathbb{N}$. This implies the assertion. □

4.5.5 If $F \in \mathsf{Sh}\mathbf{S}$ and $C \in \mathbf{S}$, then we can form the sheaf $\mathcal{R}_C(F) \in \mathsf{Sh}\mathbf{S}$ by the prescription $\mathcal{R}_C(F)(A) := F(A \times C)$ for all $A \in \mathbf{S}$ (see 3.3.18). We will show that $\mathcal{R}_C$ preserves torsion sheaves provided C is compact.

Lemma 4.47. *If $C \in \mathbf{S}$ is compact and $H \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is a torsion sheaf, then $\mathcal{R}_C(H)$ is a torsion sheaf.*

Proof. Given $s \in \mathcal{R}_C(H)(A) = H(A \times C)$ and $a \in A$ we must find $n \in \mathbb{N}$ and a neighbourhood U of a such that $(ns)_{|U\times C} = 0$. Since C is compact and $A \in \mathbf{S}$ is compactly generated by assumption on $\mathbf{S}$, the compactly generated topology (this is the topology we use here) on the product $A\times C$ coincides with the product topology ([Ste67, Theorem 4.3]). Since H is torsion there exists an open covering $(W_i = A_i \times C_i)_{i\in I}$ of $A \times C$ and a family of non-zero integers $(n_i)_{i\in I}$ such that $(n_is)_{|W_i} = 0$.

The set of subsets of C

$$\{C_i \mid i \in I,\ a \in A_i\}$$

forms an open covering of C. Using the compactness of C we choose a finite set $i_1, \dots, i_r \in I$ with $a \in A_{i_k}$ such that $\{C_{i_k} \mid k = 1, \dots r\}$ is still an open covering of C. Then we define the open neighbourhood U of $a \in A$ by $U := \bigcap_{k=1}^r A_{i_k}$. Set $n := \prod_{k=1}^r n_{i_k}$. Then we have $(ns)_{U\times C} = 0$. □

Lemma 4.48. *Let H be a discrete torsion group and $G \in \mathbf{S}$ be a compactly generated*[10] *group. Then the sheaf $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{H})$ is a torsion sheaf.*

[10] i.e. generated by a compact subset

Proof. Let $C \subseteq G$ be a compact generating set of G. Precomposition with the inclusion $C \to G$ gives an inclusion $\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, H) \to \mathsf{Map}(C, H)$. We have for $A \in \mathbf{S}$

$$\begin{aligned}\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{H})(A) &\cong \underline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, H)}(A) \quad \text{(by Lemma 3.5)}\\ &\cong \mathsf{Hom}_{\mathbf{S}}(A, \mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, H))\\ &\subseteq \mathsf{Hom}_{\mathbf{S}}(A, \mathsf{Map}(C, H))\\ &= \mathsf{Hom}_{\mathbf{S}}(A \times C, H)\\ &\cong \underline{H}(A \times C)\\ &\cong \mathcal{R}_C(\underline{H})(A).\end{aligned}$$

By Lemma 4.44 the sheaf $\underline{H}$ is a torsion sheaf. It follows from Lemma 4.47 that $\mathcal{R}_C(\underline{H})$ is a torsion sheaf. By the calculation above $\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{H})$ is a sub-sheaf of the torsion sheaf $\mathcal{R}_C(\underline{H})$ and therefore itself a torsion sheaf. □

Lemma 4.49. *If G is a compact abelian group, then*

$$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong \underline{\hat{G}}.$$

Moreover, if G is profinite, then $\underline{\hat{G}}$ is a torsion sheaf.

Proof. We apply the functor $\underline{\mathsf{Ext}}^*_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \dots)$ to

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0$$

and get the following segment of a long exact sequence

$$\begin{aligned}\cdots \to \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{R}}) &\to \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{T}})\\ &\to \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{R}}) \to \cdots .\end{aligned}$$

Since G is compact we have $0 = \underline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, \mathbb{R})} \cong \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{R}})$ by Lemma 3.5. Furthermore, by Proposition 4.16 we have $\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{R}}) = 0$. Therefore we get

$$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong \underline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, \mathbb{T})} \cong \underline{\hat{G}},$$

again by Lemma 3.5. If G is profinite, then (and only then) its dual $\hat{G}$ is a discrete torsion group by [HM98, Corollary 8.5]. In this case, by Lemma 4.44 the sheaf $\underline{\hat{G}}$ is a torsion sheaf. □

4.5.6 Let H be a discrete group. For $A \in \mathbf{S}$ we consider the continuous group cohomology $H^i_{cont}(G; \mathsf{Map}(A, H))$, which is defined as the cohomology of the group cohomology complex

$$0 \to \mathsf{Map}(A, H) \to \mathsf{Hom}_{\mathbf{S}}(G, \mathsf{Map}(A, H)) \to \cdots \to \mathsf{Hom}_{\mathbf{S}}(G^i, \mathsf{Map}(A, H)) \to \cdots$$

with the differentials dual to the ones given by 4.2.7. The map

$$\mathbf{S} \ni A \mapsto H^i_{cont}(G; \mathsf{Map}(A, H))$$

defines a presheaf whose sheafification we will denote by $\mathcal{H}^i(G, H)$.

Lemma 4.50. *If G is profinite and H is a discrete group, then $\mathcal{H}^i(G,H)$ is a torsion sheaf for $i \geq 1$.*

Proof. We must show that for each section $s \in H^i_{\mathrm{cont}}(G; \mathsf{Map}(A,H))$ and $a \in A$ there exists a neighbourhood U of a and a number $l \in \mathbb{Z}$ such that $(ls)_{|U} = 0$. This additional locality is important. Note that by the exponential law we have

$$C^i_{\mathrm{cont}}(G, \mathsf{Map}(A,H)) := \mathsf{Hom}_{\mathbf{S}}(G^i, \mathsf{Map}(A,H)) \cong \mathsf{Hom}_{\mathbf{S}}(G^i \times A, H).$$

Let $s \in H^i_{\mathrm{cont}}(G; \mathsf{Map}(A,H))$ and $a \in A$. Let s be represented by a cycle $\hat{s} \in C^i_{\mathrm{cont}}(G, \mathsf{Map}(A,H))$. Note that $\hat{s}\colon G^i \times A \to H$ is locally constant. The sets $\{\hat{s}^{-1}(h) \mid h \in H\}$ form an open covering of $G^i \times A$.

Since G and therefore G^i are compact and $A \in \mathbf{S}$ is compactly generated by assumption on $\mathbf{S}$, the compactly generated topology (this is the topology we use here) on the product $G^i \times A$ coincides with the product topology ([Ste67, Theorem 4.3]).

Since $G^i \times \{a\} \subseteq G^i \times A$ is compact we can choose a finite set $h_1, \ldots, h_r \in H$ such that $\{\hat{s}^{-1}(h_i) \mid i = 1, \ldots, r\}$ covers $G^i \times \{a\}$. Now there exists an open neighbourhood $U \subseteq A$ of a such that $G^i \times U \subseteq \bigcup_{s=1}^r \hat{s}^{-1}(h_i)$. On $G^i \times U$ the function $\hat{s}$ has at most a finite number of values belonging to the set $\{h_1, \ldots, h_r\}$.

Since G is profinite there exists a finite quotient group $G \to \bar{G}$ such that $\hat{s}_{|G^i \times U}$ has a factorization $\bar{s}\colon \bar{G}^i \times U \to H$. Note that $\bar{s}$ is a cycle in $C^i_{\mathrm{cont}}(\bar{G}, \mathsf{Map}(U,H))$.

Now we use the fact that the higher (i.e. in degree ≥ 1) cohomology of a finite group with arbitrary coefficients is annihilated by the order of the group. Hence $|\bar{G}|\bar{s}$ is the boundary of some $\bar{t} \in C^{i-1}_{\mathrm{cont}}(\bar{G}, \mathsf{Map}(U,H))$. Pre-composing $\bar{t}$ with the projection $G^i \times U \to \bar{G}^i \times U$ we get $\hat{t} \in C^i_{\mathrm{cont}}(G, \mathsf{Map}(U,H))$ whose boundary is $\hat{s}$. This shows that $(|\bar{G}|s)_{|U} = 0$. □

4.5.7 Let G be a profinite group. We consider the complex $U^\bullet := U^\bullet(\underline{G})$ as defined in 4.14. Let $\underline{\mathbb{Z}} \to I^\bullet$ be an injective resolution. Then we get a double complex $\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet)$ as in 4.2.12.

Lemma 4.51. *For $i \geq 1$*

$$H^i\,\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet) \text{ is a torsion sheaf.} \tag{36}$$

Proof. We first take the cohomology in the $I^\bullet$-, and then in the $U^\bullet$-direction. The first page of the resulting spectral sequence is given by

$$E_1^{r,q} \cong \underline{\mathsf{Ext}}^q_{\mathsf{ShAb}\,\mathbf{S}}(\mathbb{Z}(\underline{G}^r), \underline{\mathbb{Z}}).$$

Since the G^r are profinite topological spaces, by Lemma 3.29 we get $E_1^{r,q} \cong 0$ for $q \geq 1$. We now consider the case $q = 0$. For $A \in \mathbf{S}$ we have

$$E_1^{r,0}(A) \cong \underline{\mathbb{Z}}(A \times G^r) \cong \mathsf{Hom}_{\mathbf{S}}(G^r \times A, \mathbb{Z}) \cong \mathsf{Hom}_{\mathbf{S}}(G^r, \mathsf{Map}(A,\mathbb{Z}))$$

for all $r \geq 0$. The differential of the complex $(E_1^{*,0}(A), d_1)$ is exactly the differential of the complex $C^i_{\mathrm{cont}}(G, \mathsf{Map}(A,\mathbb{Z}))$ considered in 4.5.6. By Lemma 4.50 we conclude that the cohomology sheaves $E_2^{i,0}$ are torsion sheaves. The spectral sequence degenerates at the second term. We thus have shown that $H^i\,\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet)$ is a torsion sheaf. □

4.5.8 For abelian groups V, W let $V *_{\mathbb{Z}} W := \mathsf{Tor}_1^{\mathbb{Z}}(V, W)$ denote the Tor-product. If V and W in addition are $\mathbb{Z}_{\mathrm{mult}}$-modules then $V * W$ is a $\mathbb{Z}_{\mathrm{mult}}$-module by the functoriality of the Tor-product.

Lemma 4.52. *If V, W are $\mathbb{Z}_{\mathrm{mult}}$-modules of weight k, l, then $V \otimes_{\mathbb{Z}} W$ and $V *_{\mathbb{Z}} W$ are of weight $k + l$.*

Proof. The assertion for the tensor product follows from (19). Let $P^\bullet \to V'$ and $Q^\bullet \to W'$ be projective resolutions of the underlying $\mathbb{Z}$-modules V', W' of V, W. Then $P^\bullet(k) \to V$ and $Q^\bullet(l) \to W$ are $\mathbb{Z}_{\mathrm{mult}}$-equivariant resolutions of V and W. We have by (19)

$$V *_{\mathbb{Z}} W \cong H^1(P^\bullet(k) \otimes_{\mathbb{Z}} Q^\bullet(l)) \cong H^1(P^\bullet \otimes_{\mathbb{Z}} Q^\bullet)(k + l). \qquad \square$$

4.5.9 The tautological action of $\mathbb{Z}_{\mathrm{mult}}$ on an abelian group G (we write $G(1)$ for this $\mathbb{Z}_{\mathrm{mult}}$-module) (see 3.5.1) induces an action of $\mathbb{Z}_{\mathrm{mult}}$ on the bar complex $\mathbb{Z}(G^\bullet)$ by diagonal action on the generators, and therefore on the group cohomology $H^*(G(1); \mathbb{Z})$ of G.

In the following lemma we calculate the cohomology of the group $\mathbb{Z}/p\mathbb{Z}(1)$ as a $\mathbb{Z}_{\mathrm{mult}}$-module.

Lemma 4.53. *Let $p \in \mathbb{N}$ be a prime. Then we have*

$$H^i(\mathbb{Z}/p\mathbb{Z}(1); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}(0), & i = 0, \\ 0, & i \text{ odd}, \\ \mathbb{Z}/p\mathbb{Z}(k), & i = 2k \geq 2. \end{cases}$$

Proof. The group cohomology of $\mathbb{Z}/p\mathbb{Z}$ can be identified as a ring with the ring $\mathbb{Z} \oplus c\mathbb{Z}/p\mathbb{Z}[c]$, where c has degree 2. Furthermore, for every group there is a canonical map $\widehat{G} \to H^2(G; \mathbb{Z})$ which in the case $G \cong \mathbb{Z}/p\mathbb{Z}$ happens to be an isomorphism. This implies that $H^2(\mathbb{Z}/p\mathbb{Z}(1); \mathbb{Z})$ has weight 1 as a $\mathbb{Z}_{\mathrm{mult}}$-module. Since the cup product in group cohomology is natural it is $\mathbb{Z}_{\mathrm{mult}}$-equivariant. Therefore, the power c^k generates a module of weight k. Hence $H^{2k}(\mathbb{Z}/p\mathbb{Z}(1); \mathbb{Z})$ has weight k. $\square$

4.5.10 For a $\mathbb{Z}_{\mathrm{mult}}$-module V (with $\mathbb{Z}_{\mathrm{mult}}$-action Ψ) let P_k^v be the operator $x \mapsto \Psi^v x - v^k x$. Note that this is a commuting family of operators. We let M_{23}^v be the monoid generated by P_2^v, P_3^v.

Definition 4.54. A $\mathbb{Z}_{\mathrm{mult}}$-module V is called a weight 2-3-extension if for all $x \in V$ and all $v \in \mathbb{Z}_{\mathrm{mult}}$ there is $P^v \in M_{23}^v$ such that

$$P^v x = 0. \tag{37}$$

A sheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules is called a weight 2-3-extension, if every section locally satisfies Equation (37) (with P^v depending on the section and the neighborhood).

In the tables below we will mark weight 2-3-extensions with attribute weight 2-3.

Lemma 4.55. *Every $\mathbb{Z}_{\mathrm{mult}}$-module of weight 2 or of weight 3 is a weight 2-3-extension. The class of weight 2-3-extensions is closed under extensions.*

Proof. This is a simple diagram chase, using the fact that for a $\mathbb{Z}_{\mathrm{mult}}$-module W of weight k, $\Psi^v x = v^k x$ for all $k \in \mathbb{Z}_{\mathrm{mult}}$ and $x \in W$. □

Lemma 4.56. *Let $n \in \mathbb{N}$ and V be a torsion-free $\mathbb{Z}$-module. For $i \in \{0, 2, 3\}$ the cohomology $H^i((\mathbb{Z}/p\mathbb{Z})^n(1); V)$ is a $\mathbb{Z}/p\mathbb{Z}$-module whose weight is given by the following table.*

i	0	2	3	4
weight	0	1	2	2-3

Moreover, $H^1((\mathbb{Z}/p\mathbb{Z})^n; V) \cong 0$.

Proof. We first calculate $H^i((\mathbb{Z}/p\mathbb{Z})^n; \mathbb{Z})$ using the Künneth formula and induction by $n \geq 1$. The start is Lemma 4.53. Let us assume the assertion for products with less than n factors. The cases $i = 0, 1, 2$ are straightforward. We further get

$$H^3((\mathbb{Z}/p\mathbb{Z})^n(1); \mathbb{Z}) \cong H^2(\mathbb{Z}/p\mathbb{Z}(1); \mathbb{Z}) *_{\mathbb{Z}} H^2((\mathbb{Z}/p\mathbb{Z})^{n-1}(1); \mathbb{Z}).$$

By Lemma 4.52 the $*_{\mathbb{Z}}$-product of two modules of weight 1 is of weight 2. Similarly, we have an exact sequence

$$\begin{aligned} 0 \to \bigoplus_{j=0}^{4} H^j(\mathbb{Z}/p\mathbb{Z}^{n-1}(1); \mathbb{Z}) \otimes H^{4-j}(\mathbb{Z}/p\mathbb{Z}; \mathbb{Z}) \to H^4((\mathbb{Z}/p\mathbb{Z})^n(1); \mathbb{Z}) \\ \to \bigoplus_{j=2}^{3} H^j(\mathbb{Z}/p\mathbb{Z}(1); \mathbb{Z}) *_{\mathbb{Z}} H^{5-j}((\mathbb{Z}/p\mathbb{Z})^{n-1}(1); \mathbb{Z}) \to 0. \end{aligned}$$

By Lemma 4.55 and induction we conclude that $H^4((\mathbb{Z}/p\mathbb{Z})^n(1); \mathbb{Z})$ is a weight 2-3-extension.

We can calculate the cohomology $H^i(G; V)$ of a group G in a trivial G-module by the standard complex $C^\bullet(G; V) := C(\mathsf{Map}(G^\bullet, V))$. If G is finite, then we have $C^\bullet(G; V) \cong C^\bullet(G; \mathbb{Z}) \otimes_{\mathbb{Z}} V$. If V is torsion-free, it is a flat $\mathbb{Z}$-module and therefore $H^i(C^\bullet(G; \mathbb{Z}) \otimes_{\mathbb{Z}} V) \cong H^i(C^\bullet(G; \mathbb{Z})) \otimes_{\mathbb{Z}} V$. Applying this to $G = (\mathbb{Z}/p\mathbb{Z})^n$ we get the assertion from the special case $\mathbb{Z} = V$. □

4.5.11 An abelian group G is a $\mathbb{Z}$-module. Let $p \in \mathbb{Z}$ be a prime. If $pg = 0$ for every $g \in G$, then we say that G is a $\mathbb{Z}/p\mathbb{Z}$-module.

Definition 4.57. A sheaf $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules if $F(A)$ is a $\mathbb{Z}/p\mathbb{Z}$-module for all $A \in \mathbf{S}$.

4.5.12 Below we consider profinite abelian groups which are also $\mathbb{Z}/p\mathbb{Z}$ modules. The following lemma describes their structure.

Lemma 4.58. *Let p be a prime number. If G is compact and a $\mathbb{Z}/p\mathbb{Z}$-module, then there exists a set S such that $G \cong \prod_S \mathbb{Z}/p\mathbb{Z}$. In particular, G is profinite.*

Proof. The dual group $\widehat{G}$ of the compact group G is discrete and also a $\mathbb{Z}/p\mathbb{Z}$-module, hence an $\mathbb{F}_p$-vector space. Let $S \subseteq \widehat{G}$ be an $\mathbb{F}_p$-basis. Then we can write $\widehat{G} \cong \oplus_S \mathbb{Z}/p\mathbb{Z}$. Pontrjagin duality interchanges sums and products. We get

$$G \cong \hat{\widehat{G}} \cong \widehat{\oplus_S \mathbb{Z}/p\mathbb{Z}} \cong \prod_S \widehat{\mathbb{Z}/p\mathbb{Z}} \cong \prod_S \mathbb{Z}/p\mathbb{Z}. \qquad \square$$

4.5.13 Assume that G is a compact group and a $\mathbb{Z}/p\mathbb{Z}$-module. Note that the construction $G \to U^\bullet := U^\bullet(G)$ is functorial in G (see 4.6.11 for more details). The tautological action of $\mathbb{Z}_{\mathrm{mult}}$ on G induces an action of $\mathbb{Z}_{\mathrm{mult}}$ on $U^\bullet$. We can improve Lemma 4.51 as follows.

Lemma 4.59. *Let $\underline{\mathbb{Z}} \to I^\bullet$ be an injective resolution. For $i \in \{2, 3, 4\}$ the sheaves*

$$H^i \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$$

are sheaves of $\mathbb{Z}/p\mathbb{Z}$-modules and $\mathbb{Z}_{\mathrm{mult}}$-modules whose weights are given by the following table.

i	0	2	3	4
weight	0	1	2	2-3

Moreover, $H^1 \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet) \cong 0$.

Proof. We argue with the spectral sequence as in the proof of 4.51. Since $E_1^{p,q} = 0$ for $q \geq 1$, it suffices to show that the sheaves $E_2^{i,0}$ are sheaves of $\mathbb{Z}/p\mathbb{Z}$-modules and $\mathbb{Z}_{\mathrm{mult}}$-modules of weight k, where k corresponds to i as in the table (and with the appropriate modification for $i = 4$).

We have for $A \in \mathbf{S}$

$$E_1^{i,0}(A) \cong \underline{\mathbb{Z}}(A \times G^i) \cong \mathsf{Hom}_{\mathbf{S}}(G^i, \mathsf{Map}(A, \mathbb{Z})) = C^i_{\mathrm{cont}}(G; \mathsf{Map}(A, \mathbb{Z})),$$

where for a topological G-module V the complex $C^\bullet_{\mathrm{cont}}(G; V)$ denotes the continuous group cohomology complex. We now consider the presheaf $\widetilde{X}^i$ defined by

$$A \mapsto \widetilde{X}^i(A) := H^i_{\mathrm{cont}}(G, \mathsf{Map}(A, \mathbb{Z})).$$

Then by definition $E_2^{i,0} := X^i := (\widetilde{X}^i)^\sharp$ is the sheafification of $\widetilde{X}^i$.

The action of $\mathbb{Z}_{\mathrm{mult}}$ on G induces an action $q \mapsto [q]$ on the presheaf $\widetilde{X}^i$, which descends to an action of $\mathbb{Z}_{\mathrm{mult}}$ on the associated sheaf X^i.

We fix $i \in \{0, 1, 2, 3\}$ and let k be the associated weight as in the table. For $i \geq 2$ we must show that for each section $s \in H^i_{\mathrm{cont}}(G; \mathsf{Map}(A, \mathbb{Z}))$ and $a \in A$ there exists a neighbourhood U of a such that $((q^k - [q])s)_{|U} = 0$ and $(ps)_{|U} = 0$ for all $q \in \mathbb{Z}_{\mathrm{mult}}$. For $i = 4$, instead of $((q^k - [q])s)_{|U} = 0$ we must find $P^v \in M^v_{23}$ (depending on U and s) (see 4.5.10 for notation) such that $P^v s_{|U} = 0$. For $i = 1$ we must show that $s_{|U} = 0$.

We perform the argument in detail for $i = 2, 3$. It is a refinement of the proof of Lemma 4.50. The cases $i = 1$ and $i = 4$ are very similar.

Let s be represented by a cycle $\hat{s} \in C^i_{\mathrm{cont}}(G, \mathsf{Map}(A, \mathbb{Z}))$. Note that $\hat{s} \colon G^i \times A \to \mathbb{Z}$ is locally constant. The sets $\{\hat{s}^{-1}(h) \mid h \in \mathbb{Z}\}$ form an open covering of $G^i \times A$. Since G and therefore G^i is compact and $A \in \mathbf{S}$ is compactly generated by assumption on $\mathbf{S}$, the compactly generated topology (this is the topology we use here) on the product $G^i \times A$ coincides with the product topology ([Ste67, Theorem 4.3]). This allows the following construction.

The set of subsets

$$\{\hat{s}^{-1}(h) \mid h \in \mathbb{Z}\}$$

forms an open covering of $G^i \times A$. Using the compactness of the subset $G^i \times \{a\} \subseteq G^i \times A$ we choose a finite set $h_1, \dots, h_r \in \mathbb{Z}$ such that $G^i \times \{a\} \subseteq \bigcup_{i=1}^r \hat{s}^{-1}(h_i)$. There exists a neighbourhood $U \subseteq A$ of a such that $G^i \times U \subseteq \bigcup_{i=1}^r \hat{s}^{-1}(h_i)$.

We now use that G is profinite by Lemma 4.58. Since $\hat{s}(G^i \times U)$ is a finite set (a subset of $\{h_1, \dots, h_r\}$) there exists a finite quotient group $G \to \bar{G}$ such that $\hat{s}_{|G^i \times U}$ has a factorization $\bar{s} \colon \bar{G}^i \times U \to \mathbb{Z}$. In our case $\bar{G} \cong (\mathbb{Z}/p\mathbb{Z})^r$ for a suitable $r \in \mathbb{N}$. Note that $\bar{s}$ is a cycle in $C^i_{\mathrm{cont}}(\bar{G}, \mathsf{Map}(U, \mathbb{Z}))$. Now by Lemma 4.56 we know that $(q^k - [q])\bar{s}$ and $p\bar{s}$ are the boundaries of some $\bar{t}, \bar{t}_1 \in C^{i-1}_{\mathrm{cont}}(\bar{G}, \mathsf{Map}(U, \mathbb{Z}))$. Pre-composing $\bar{t}, \bar{t}_1$ with the projection $G^i \times U \to \bar{G}^i \times U$ we get $\hat{t}, \hat{t}_1 \in C^i_{\mathrm{cont}}(G, \mathsf{Map}(U, \mathbb{Z}))$ whose boundaries are $(q^k - [q])\hat{s}$ and $p\hat{s}$, respectively. This shows that $(q^k - [q]s)_{|U} = 0$ and $(ps)_{|U} = 0$. $\square$

4.5.14 We consider a compact group G and form the double complex $\mathsf{Hom}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet)$ defined in 4.5.7. Taking the cohomology first in the $U^\bullet$ and then in the $I^\bullet$-direction we get a second spectral sequence $(F_r^{p,q}, d_r)$ with second page

$$F_2^{p,q} \cong \underline{\mathsf{Ext}}^p_{\mathsf{ShAb}\,\mathbf{S}}(H^q U^\bullet, \underline{\mathbb{Z}}).$$

In the following we calculate the term $F_2^{1,2}$ and show the vanishing of several other entries.

Lemma 4.60. *The left lower corner of $F_2^{p,q}$ has the following form.*

3	0					
2	0	$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})$				
1	0	$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$	$\underline{\mathsf{Ext}}^2_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$	$\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$		
0	$\underline{\mathbb{Z}}$	0	0	0	0	0
	0	1	2	3	4	5

Proof. The proof is similar to the corresponding argument in the proof of 4.16. Using 4.2.11 we get

$$F_2^{p,q} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^q_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})$$

for $q = 0, 1, 2$. In particular we have $\Lambda^0_{\mathbb{Z}}\underline{G} \cong \underline{\mathbb{Z}}$ and thus

$$F_2^{p,0} \cong \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \cong 0.$$

Furthermore, since $\Lambda^1_{\mathbb{Z}}\underline{G} \cong \underline{G}$ we have

$$F_2^{0,1} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \overset{\text{Lemma 3.5}}{\cong} \underline{\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{Z})} \cong 0$$

since $\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{Z}) \cong 0$ by compactness of G.

We claim that

$$F_2^{0,2} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}}) \cong 0.$$

Note that for $A \in \mathbf{S}$ we have

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})(A) \cong \underline{\mathsf{Hom}}_{\mathsf{Pr_{Ab}}\mathbf{S}}(\Lambda^2_{\mathbb{Z}}\underline{G}, \underline{\mathbb{Z}})(A) \cong \mathsf{Hom}_{\mathsf{Pr_{Ab}}\mathbf{S}/A}(\Lambda^2_{\mathbb{Z}}\underline{G}_{|A}, \underline{\mathbb{Z}}_{|A}).$$

An element $\lambda \in \mathsf{Hom}_{\mathsf{Pr_{Ab}}\mathbf{S}/A}(\Lambda^2_{\mathbb{Z}}\underline{G}_{|A}, \underline{\mathbb{Z}}_{|A})$ induces a family of biadditive (antisymmetric) maps

$$\lambda^W : \underline{G}(W) \times \underline{G}(W) \to \underline{\mathbb{Z}}(W)$$

for $(W \to A) \in \mathbf{S}/A$ which is compatible with restriction. Restriction to points gives biadditive maps $G \times G \to \mathbb{Z}$. Since G is compact the only such map is the constant map to zero. Therefore λ^W vanishes for all $(W \to A)$. This proves the claim. The same kind of argument shows that $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^q_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})(A) = 0$ for $q \geq 2$.

Let us finally show that $F_2^{0,3} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((H^3U^\bullet)^\sharp, \underline{\mathbb{Z}}) \cong 0$. By 4.15 we have an exact sequence (see (23) for the notation ${}^pD^\bullet$)

$$0 \to K \to \Lambda^3_{\mathbb{Z}}\underline{G} \to H^3({}^pD^\bullet) \to C \to 0,$$

where K and C are defined as the kernel and cokernel presheaf, and the middle map becomes an isomorphism after tensoring with $\mathbb{Q}$. This means that $0 \cong K \otimes \mathbb{Q}$ and $0 \cong C \otimes \mathbb{Q}$. Hence, these K and C are presheaves of torsion groups. Let $A \in \mathbf{S}$. Then an element $s \in \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((H^3U^\bullet)^\sharp, \underline{\mathbb{Z}})(A)$ induces by precomposition an element $\tilde{s} \in \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\Lambda^3_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})(A) \cong 0$. Therefore s factors over $\bar{s} \in \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(C^\sharp, \underline{\mathbb{Z}})(A)$. Since C is torsion we conclude that $\bar{s} = 0$ by Lemma 4.46. The same argument shows that $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((H^qU^\bullet)^\sharp, \underline{\mathbb{Z}}) \cong 0$ for $q \geq 3$. □

Lemma 4.61. *Assume that p is an odd prime number, $p \neq 3$. If G is a compact group and a $\mathbb{Z}/p\mathbb{Z}$-module, then $\underline{\mathsf{Ext}}^i_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}; \underline{\mathbb{Z}}) \cong 0$ for $i = 2, 3$.*

If $p = 3$, then at least $\underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$.

Proof. We consider the spectral sequence (F_r, d_r) introduced in 4.5.14. It converges to the graded sheaves associated to a certain filtrations of the cohomology sheaves of the total complex of the double complex $\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(U^\bullet, I^\bullet)$ defined in 4.5.7. In degree $2,3,4$ these cohomology sheaves are sheaves of $\mathbb{Z}/p\mathbb{Z}$-modules carrying actions of $\mathbb{Z}_{\mathrm{mult}}$ with weights determined in Lemma 4.59.

The left lower corner of the second page of the spectral sequence was evaluated in Lemma 4.60. Note that $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ is a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules with an action of $\mathbb{Z}_{\mathrm{mult}}$ of weight 1 for all $i \geq 0$. The term $F_2^{2,1} \cong \underline{\mathsf{Ext}}^2_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ survives to the limit of the spectral sequence and is a submodule of a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules of weight 2. On the other hand it has weight 1.

A $\mathbb{Z}/p\mathbb{Z}$-module V with an action of $\mathbb{Z}_{\mathrm{mult}}$ which has weights 1 and 2 at the same time must be trivial[11]. In fact, for every $q \in \mathbb{Z}_{\mathrm{mult}}$ we get the identity $q^2 - q = (q-1)q = 0$ in $\mathsf{End}_{\mathbb{Z}}(V)$, and this implies that $q \equiv 1 \pmod p$ for all $q \notin p\mathbb{Z}$. From this follows $p = 2$, and this case was excluded.

Similarly, the sheaf $\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})$ is a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules of weight 2 (see 3.5.5), and $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ is a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules of weight 1. Since p is odd, this implies that the differential $d_2^{1,3} \colon \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ is trivial. Hence, $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ survives to the limit. It is a subsheaf of a sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules which is a weight 2-3-extension. On the other hand it has weight 1. Substituting the weight 1-condition into equation (37), because then $P_2^v = v - v^2 = (1-v)v$ and $P_3^v = v - v^3 = (1-v)(1+v)v$ this implies that locally every section satisfies $(1-v)^n(1+v)^j v^n s = 0$ for suitable $n, j \in \mathbb{N}$ (depending on s and the neighborhood) and for all $v \in \mathbb{Z}_{\mathrm{mult}}$. If $p > 3$ we can choose v such that $(1-v)$, $(1+v)$, v are simultaneously units, and in this case the equation implies that locally every section is zero.

We conclude that $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$. □

Lemma 4.62. *Let G be a compact group which satisfies the two-three condition* (*see Definition* 4.6)*. Assume further that*

1. *G is profinite, or*

2. *G is connected and locally topologically divisible* (*Definition* 4.7)*.*

Then the sheaves $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ are torsion-free for $i = 2, 3$.

Proof. We must show that for all primes p and $i = 2, 3$ the maps of sheaves

$$\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \xrightarrow{p} \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$$

are injective. The multiplication by p can be induced by the multiplication

$$p \colon \underline{G} \to \underline{G}.$$

[11] Note that in general a $\mathbb{Z}/p\mathbb{Z}$-module can very well have different weights. E.g a module of weight k has also weight pk.

We consider the exact sequence

$$0 \to K \to G \xrightarrow{p} G \to C \to 0.$$

The groups K and C are groups of $\mathbb{Z}/p\mathbb{Z}$-modules and therefore profinite by 4.58.

We claim that the sequence of sheaves

$$0 \to \underline{K} \to \underline{G} \xrightarrow{p} \underline{G} \to \underline{C} \to 0$$

is exact. We first show the claim in the case that G is profinite. Since C is profinite, by Lemma 4.39 and Lemma 3.4 we know that

$$0 \to \underline{G/K} \to \underline{G} \to C \to 0$$

is exact. Furthermore G/K is profinite so that the projection $G \to G/K$ has local sections. This implies that $\underline{G/K} \cong \underline{G}/\underline{K}$, and hence the claim.

We now discuss the case of a connected locally topologically divisible G. In this case $C = \{1\}$. Since by assumption $p\colon G \to G$ has local sections we can again use Lemma 3.4 in order to conclude.

We decompose this sequence into two short exact sequences

$$0 \to \underline{K} \to \underline{G} \to Q \to 0, \quad 0 \to Q \to \underline{G} \to \underline{C} \to 0.$$

Now we apply the functor $\underline{\mathsf{Ext}}^*_{\mathsf{ShAb}\,\mathbf{S}}(\dots, \underline{\mathbb{Z}})$ to the first sequence and study the associated long exact sequence

$$\underline{\mathsf{Ext}}^{i-1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \mathsf{Ext}^{i-1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{K}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{i}_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{i}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \cdots.$$

Note that $\mathsf{Ext}^2_{\mathsf{ShAb}\,\mathbf{S}}(\underline{K}, \underline{\mathbb{Z}}) \cong 0$ by Lemma 4.61 if p is odd, and by Lemma 4.17 if $p = 2$ (the two-three condition ensures that K in this case is an at most finite product of copies of $\mathbb{Z}/2\mathbb{Z}$.). This implies that

$$\underline{\mathsf{Ext}}^{3}_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{3}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$$

is injective.

Next we show that

$$\underline{\mathsf{Ext}}^{2}_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{2}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$$

is injective. For this it suffices to see that

$$\underline{\mathsf{Ext}}^{1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^{1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{K}, \underline{\mathbb{Z}})$$

is surjective. But this follows from the diagram

$$\begin{tikzcd}
\underline{\mathsf{Ext}}^{1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \arrow[r] \arrow[d, "\cong"] & \underline{\mathsf{Ext}}^{1}_{\mathsf{ShAb}\,\mathbf{S}}(\underline{K}, \underline{\mathbb{Z}}) \arrow[d, "\cong"] \\
\underline{\widehat{G}} \arrow[r, "\text{surjective}"] & \underline{\widehat{K}}
\end{tikzcd},$$

where the vertical maps are the isomorphisms given by Lemma 4.49, and surjectivity of $\hat{G} \to \hat{K}$ follows from the fact that Pontrjagin duality preserves exact sequences.

Next we apply $\underline{\mathsf{Ext}}^*_{\mathsf{ShAb}\,\mathbf{S}}(\dots, \underline{\mathbb{Z}})$ to the sequence

$$0 \to Q \to \underline{G} \to \underline{C} \to 0$$

and get the long exact sequence

$$\mathsf{Ext}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{C}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}}) \to \cdots .$$

Again we use that $\mathsf{Ext}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{C}, \underline{\mathbb{Z}}) \cong 0$ for $i = 2, 3$ by Lemma 4.61 for $p > 3$, and by Lemma 4.17 if $p \in \{2, 3\}$ (in this case C is an at most finite product of copies of $\mathbb{Z}/p\mathbb{Z}$ by the two-three condition) in order to conclude that

$$\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}})$$

is injective for $i = 2, 3$. Therefore the composition

$$\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(Q, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$$

is injective for $i = 2, 3$. This is what we wanted to show. □

Lemma 4.63. *Let G be a compact connected group which satisfies the two-three condition. Then the sheaves $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{\text{lc-acyc}}}(\underline{G}, \underline{\mathbb{Z}})$ are torsion-free for $i = 2, 3$.*

Proof. The only point in the proof of Lemma 4.62 where we have used the condition of local topological divisibility was the exactness of the sequence

$$0 \to \underline{K} \to \underline{G} \xrightarrow{p} \underline{G} \to 0$$

of sheaves on $\mathbf{S}$. We show that this sequence is exact with this condition if we consider the sheaves on $\mathbf{S}_{\text{lc-acyc}}$ (by restriction). Let us start with the dual sequence

$$0 \to \hat{G} \xrightarrow{p} \hat{G} \to \hat{K} \to 0$$

of discrete groups. It gives an exact sequence of sheaves

$$0 \to \underline{\hat{G}} \xrightarrow{p} \underline{\hat{G}} \to \underline{\hat{K}} \to 0.$$

We apply $\mathsf{Hom}_{\mathsf{ShAb}\,\mathbf{S}_{\text{lc-acyc}}}(\dots, \underline{\mathbb{T}})$ and get, using Lemma 3.5, the long exact sequence

$$0 \to \underline{K} \to \underline{G} \xrightarrow{p} \underline{G} \to \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}_{\text{lc-acyc}}}(\underline{\hat{K}}, \underline{\mathbb{T}}) \to \cdots .$$

By Theorem 4.28 we have $\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}_{\text{lc-acyc}}}(\underline{\hat{K}}, \underline{\mathbb{T}}) = 0$. Now we can argue as in the proof of 4.62. □

Lemma 4.64. *Let G be a profinite abelian group. Then the following assertions are equivalent.*

1. $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ *is torsion-free for* $i = 2, 3$

2. $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$ *for* $i = 2, 3$.

Proof. It is clear that 2. implies 1. Therefore let us show that $\underline{\mathsf{Ext}}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$ for $i = 2, 3$ under the assumption that we already know that it is torsion-free.

We consider the double complex $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet)$ introduced in 4.5.7. By Lemma 4.51 we know that the cohomology sheaves $H^i(\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet))$ of the associated total complex are torsion sheaves.

The spectral sequence (F_r, d_r) considered in 4.5.14 calculates the associated graded sheaves of a certain filtration of $H^i(\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(U^\bullet, I^\bullet))$. The left lower corner of its second page was already evaluated in Lemma 4.60.

The term $F_2^{2,1} \cong \underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ survives to the limit of the spectral sequence. On the one hand by our assumption it is torsion-free. On the other hand by Lemma 4.51, case $i = 2$, it is a subsheaf of a torsion sheaf. It follows that

$$\underline{\mathsf{Ext}}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0.$$

This settles the implication 1. $\Rightarrow$ 2. in case $i = 2$ of Lemma 4.64.

We now claim that $\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})$ is a torsion sheaf. Let us assume the claim and finish the proof of Lemma 4.64. Since $F_2^{3,1} \cong \underline{\mathsf{Ext}}^3_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ is torsion-free by assumption the differential d_2 must be trivial by Lemma 4.46. Since also $F_3^{0,3} \cong 0$ the sheaf $\underline{\mathsf{Ext}}^3_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ survives to the limit of the spectral sequence. By Lemma 4.51, case $i = 3$, it is a subsheaf of a torsion sheaf and therefore itself a torsion sheaf. It follows that $\underline{\mathsf{Ext}}^3_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ is a torsion sheaf and a torsion-free sheaf at the same time, hence trivial. This is the assertion 1. $\Rightarrow$ 2. of Lemma 4.64 for $i = 3$.

We now show the claim. We start with some general preparations. If F, H are two sheaves on some site, then one forms the presheaf $\mathbf{S} \ni A \mapsto (F \otimes^p_{\mathbb{Z}} H)(A) := F(A) \otimes_{\mathbb{Z}} H(A) \in \mathsf{Ab}$. The sheaf $F \otimes_{\mathbb{Z}} H$ is by definition the sheafification of $F \otimes^p_{\mathbb{Z}} H$. We can write the definition of the presheaf $\Lambda^2_{\mathbb{Z}}\underline{G}$ in terms of the following exact sequence of presheaves

$$0 \to K \to {}^p\mathbb{Z}(\mathcal{F}(\underline{G})) \xrightarrow{\alpha} \underline{G} \otimes^p_{\mathbb{Z}} \underline{G} \to \Lambda^2_{\mathbb{Z}}\underline{G} \to 0,$$

where K is by definition the kernel. For $W \in \mathbf{S}$ the map $\alpha_W : {}^p\mathbb{Z}(\underline{G}(W)) \to \underline{G}(W) \otimes_{\mathbb{Z}} \underline{G}(W)$ is defined on generators by $\alpha_W(x) = x \otimes x$, $x \in \underline{G}(W)$. Sheafification is an exact functor and thus gives

$$0 \to ({}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K)^\sharp \to \underline{G} \otimes_{\mathbb{Z}} \underline{G} \to (\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp \to 0.$$

We now apply the functor $\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\dots, \underline{\mathbb{Z}})$ and consider the following segment of the associated long exact sequence

$$\begin{aligned} &\to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, \underline{\mathbb{Z}}) \to \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(({}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K)^\sharp, \underline{\mathbb{Z}}) \\ &\quad \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, \underline{\mathbb{Z}}) \to \end{aligned} \tag{38}$$

The following two facts imply the claim:

1. $\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, \underline{\mathbb{Z}})$ is torsion.

2. $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\,{}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K)^\sharp, \underline{\mathbb{Z}})$ vanishes.

Let us start with 1. Let $\underline{\mathbb{Z}} \to I^\bullet$ be an injective resolution. We study

$$H^1\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, I^\bullet).$$

We have

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, I^\bullet) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, I^\bullet)).$$

Let $K^\bullet = \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, I^\bullet)$. Since $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong \overline{\mathsf{Hom}_{\mathsf{top\text{-}Ab}}(G, \mathbb{Z})} \cong 0$ by the compactness of G the map $d^0 \colon K^0 \to K^1$ is injective. Let $d^1 \colon K^1 \to K^2$ be the second differential. Now

$$H^1\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, I^\bullet) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \ker(d^1))/\mathsf{im}(d^0_*),$$

where $d^0_* \colon \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, K^0) \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \ker(d^1))$ is induced by d^0. Since d^0 is injective we have

$$\mathsf{im}(d^0_*) = \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \mathsf{im}(d^0)).$$

Applying $\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \dots)$ to the exact sequence

$$0 \to K^0 \to \ker(d^1) \to \ker(d^1)/\mathsf{im}(d^0) \to 0$$

and using $\ker(d^1)/\mathsf{im}(d^0) \cong \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})$ we get

$$\begin{aligned} 0 \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, K^0) &\xrightarrow{d^0_*} \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \ker(d^1)) \to \\ &\to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})) \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, K^0) \to \cdots . \end{aligned}$$

In particular,

$$\begin{aligned} \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G} \otimes_{\mathbb{Z}} \underline{G}, \underline{\mathbb{Z}}) &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \ker(d^1))/\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, K^0) \\ &\subseteq \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}})). \end{aligned}$$

By Lemma 4.49 we know that $\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong \underline{\widehat{G}}$. Since G is compact and profinite, the group $\widehat{G}$ is discrete and torsion. It follows by Lemma 4.48 that

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}))$$

is a torsion sheaf. A subsheaf of a torsion sheaf is again a torsion sheaf. This finishes the argument for the first fact.

We now show the fact 2.

Note that

$$\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}((\,{}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K)^\sharp, \underline{\mathbb{Z}}) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(\mathbb{Z}(\mathcal{F}(\underline{G}))/K^\sharp, \underline{\mathbb{Z}}). \tag{39}$$

Consider $A \in \mathbf{S}$ and $\phi \in \underline{\mathrm{Hom}}_{\mathrm{ShAb}\,\mathbf{S}}(\mathbb{Z}(\mathcal{F}(\underline{G}))/K^\sharp, \underline{\mathbb{Z}})(A)$. We must show that each $a \in A$ has a neighbourhood $U \subseteq A$ such that $\phi_{|U} = 0$. Pre-composing ϕ with the projection $\mathbb{Z}(\mathcal{F}(\underline{G})) \to \mathbb{Z}(\mathcal{F}(\underline{G}))/K^\sharp$ we get an element

$$\bar{\phi} \in \underline{\mathrm{Hom}}_{\mathrm{ShAb}\,\mathbf{S}}(\mathbb{Z}(\mathcal{F}(\underline{G})), \underline{\mathbb{Z}})(A) \cong \underline{\mathrm{Hom}}_{\mathrm{Sh}\,\mathbf{S}}(\mathcal{F}(\underline{G}), \underline{\mathbb{Z}})(A) \overset{\text{Lemma 3.9}}{\cong} \underline{\mathbb{Z}}(A \times G).$$

We are going to show that $\bar{\phi} = 0$ after restriction to a suitable neighbourhood of $a \in A$ using the fact that it annihilates $K^\sharp$. Let us start again on the left-hand side of (39). We have

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathrm{ShAb}\,\mathbf{S}}(({}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K)^\sharp, \underline{\mathbb{Z}})(A) &\cong \underline{\mathrm{Hom}}_{\mathrm{PrAb}\,\mathbf{S}}(({}^p\mathbb{Z}(\mathcal{F}(\underline{G}))/K), \underline{\mathbb{Z}})(A) \\ &\cong \mathrm{Hom}_{\mathrm{PrAb}\,\mathbf{S}/A}({}^p\mathbb{Z}(\mathcal{F}(\underline{G}_{|A}))/K_{|A}, \underline{\mathbb{Z}}_{|A}). \end{aligned}$$

For $(A \times G \to A) \in \mathbf{S}/A$ the morphism ϕ gives rise to a group homomorphism

$$\hat{\phi}\colon \mathbb{Z}(\mathcal{F}(\underline{G}(A \times G)))/K(A \times G) \to \underline{\mathbb{Z}}(A \times G).$$

The symbol $\mathcal{F}(\underline{G}(A\times G))$ denotes the underlying set of the group $\underline{G}(A\times G)$. We have $(\mathrm{pr}_G\colon A\times G \to G) \in \underline{G}(A\times G)$ and by the explicit description of the element $\bar{\phi}$ given after the proof of Lemma 3.9 we see $\bar{\phi} = \hat{\phi}(\{[\mathrm{pr}_G]\})$, where $[\mathrm{pr}_G] \in \mathbb{Z}(\mathcal{F}(\underline{G}(A\times G)))$ denotes the generator corresponding to $\mathrm{pr}_G \in \underline{G}(A \times G)$, and $\{\dots\}$ indicates that we take the class modulo $K(A \times G)$.

The homomorphism $\hat{\phi} \in \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}\mathcal{F}(\underline{G}(A \times G))/K(A \times G), \underline{\mathbb{Z}}(A \times G))$ is represented by a homomorphism

$$\tilde{\phi}\colon \mathbb{Z}\mathcal{F}(\underline{G}(A \times G)) \to \underline{\mathbb{Z}}(A \times G),$$

and we have $\bar{\phi} = \tilde{\phi}([\mathrm{pr}_G])$. By definition of K we have the exact sequence

$$0 \to K(A\times G) \to \mathbb{Z}(\mathcal{F}(\underline{G}(A\times G))) \to \underline{G}(A\times G)\otimes_{\mathbb{Z}}\underline{G}(A\times G) \to \Lambda^2_{\mathbb{Z}}\underline{G}(A\times G) \to 0.$$

For $x \in \underline{G}(A \times G)$ let $[x] \in \mathbb{Z}(\mathcal{F}(\underline{G}(A \times G)))$ denote the corresponding generator. Since $(nx) \otimes (nx) = n^2(x \otimes x)$ in $\underline{G}(A \times G) \otimes_{\mathbb{Z}} \underline{G}(A \times G)$ we have $n^2[x] - [nx] \in K(A \times G)$ for $n \in \mathbb{Z}$. It follows that $\tilde{\phi}$ must satisfy the relation $\tilde{\phi}(n^2[x]) = \tilde{\phi}([nx])$ for all $n \in \mathbb{Z}$ and $x \in \underline{G}(A \times G)$. Let us now apply this reasoning to $x := [\mathrm{pr}_G]$. For every $n \in \mathbb{Z}$ we have a map $A \times G \xrightarrow{\mathrm{id}_A \times n =: \psi_n} A \times G$, and by naturality the map $\tilde{\phi}$ respects this action. Moreover, the element $[nx] \in \mathbb{Z}(\mathcal{F}(\underline{G}(A \times G)))$ is obtained from $[x]$ via this action, since

$$\begin{tikzcd} A \times G \arrow[r, "\psi_n"] \arrow[d, "\mathrm{pr}_G"] & A \times G \arrow[d, "\mathrm{pr}_G"] \\ G \arrow[r, "n"] & G \end{tikzcd}$$

commutes. It follows that

$$n^2\bar{\phi} = \tilde{\phi}(n^2[\mathrm{pr}_G]) = \tilde{\phi}([n\mathrm{pr}_G]) = \tilde{\phi}(\mathbb{Z}\mathcal{F}(\overline{G}(\psi_n))([\mathrm{pr}_G])) = \underline{\mathbb{Z}}(\psi_n)(\bar{\phi}) = n^*\bar{\phi},$$

where we write $n^* := \underline{\mathbb{Z}}(\psi_n)$. For $k \in \mathbb{Z}$ we define the open subset $V_k := \bar{\phi}^{-1}(k) \subseteq A \times G$. The family $(V_k)_{k\in K}$ forms an open pairwise disjunct covering of $A \times G$. Since G is compact the compactly generated topology of $A \times G$ is the product topology ([Ste67, Theorem 4.3]). Since G is compact, we can choose a finite sequence $k_1, \dots, k_r \in \mathbb{Z}$ such that $G \times \{a\} \subseteq \bigcup_{i=1}^r V_k$. Furthermore, we can find a neighbourhood $U \subseteq A$ of a such that $G \times U \subseteq \bigcup_{i=1}^r V_k$.

Note that $\bar{\phi}_{|G\times U}$ has at most finitely many values. Therefore there exists a finite quotient $G \to F$ such that $\bar{\phi}_{|U\times G}$ factors through $f : U \times F \to \mathbb{Z}$.

The action of $\mathbb{Z}$ on G is compatible with the corresponding action of $\mathbb{Z}$ on F, and we have an action $n^* \colon \mathsf{Hom}_{\mathbf{S}}(U \times F, \mathbb{Z}) \to \mathsf{Hom}_{\mathbf{S}}(U \times F, \mathbb{Z})$. The element $f \in \mathsf{Hom}_{\mathbf{S}}(U \times F, \mathbb{Z})$ still satisfies $n^2 f = n^* f$. We now take $n := |F| + 1$. Then $n^* = \mathsf{id}$ so that $(n^2 - 1) f = 0$, hence $f = 0$. This implies $\bar{\phi}_{|U\times G} = 0$.

This finishes the proof of the second fact. $\square$

4.5.15 The Lemma 4.62 verifies Assumption 1. in 4.64 for a large class of profinite groups.

Corollary 4.65. *If G is a profinite group which satisfies the two-three condition* 4.6*, then we have* $\mathsf{Ext}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$ *for* $i = 2, 3$.

Theorem 4.66. *A profinite abelian group which satisfies the two-three condition is admissible.*

4.6 Compact connected abelian groups

4.6.1 Let G be a compact abelian group. We shall use the following fact shown in [HM98, Corollary 8.5].

Fact 4.67. G is connected if and only if $\widehat{G}$ is torsion-free.

4.6.2 For a space $X \in \mathbf{S}$ and $F \in \mathrm{Pr}_{\mathsf{Ab}}\mathbf{S}$ let $\check{H}^*(X; F)$ denote the Čech cohomology of X with coefficients in F. It is defined as follows. To each open covering $\mathcal{U}$ one associates the Čech complex $\check{C}^\bullet(\mathcal{U}; F)$. The open coverings form a left-filtered category whose morphisms are refinements. The Čech complex depends functorially on the covering, i.e. if $\mathcal{V} \to \mathcal{U}$ is a refinement, then we have a functorial chain map $\check{C}^\bullet(\mathcal{U}; F) \to \check{C}^\bullet(\mathcal{V}; F)$. We define

$$\check{C}^\bullet(X; F) := \mathsf{colim}_{\mathcal{U}} \check{C}^\bullet(\mathcal{U}; F)$$

and

$$\check{H}^*(X; F) := \mathsf{colim}_{\mathcal{U}} H^*(\check{C}^\bullet(\mathcal{U}; F)) \cong H^*(\mathsf{colim}_{\mathcal{U}} \check{C}^\bullet(\mathcal{U}; F)) \cong H^*(\check{C}^\bullet(X; F)).$$

4.6.3 Fix a discrete group H. Let ${}^p\underline{H}$ be the constant presheaf with values H. Note that then the sheafification ${}^p\underline{H}^\sharp$ is isomorphic to $\underline{H}$ as defined in 3.2.5. Moreover

$$\check{C}^\bullet(X;\, {}^p\underline{H}) \cong \check{C}^\bullet(X, \underline{H}). \tag{40}$$

In general Čech cohomology differs from sheaf cohomology. The relation between these two is given by the Čech cohomology spectral sequence (E_r, d_r) (see [Tam94, 3.4.4]) converging to $H^*(X; F)$. Let $\mathcal{H}^*(F) := R^*i(F)$ denote the right derived functors of the inclusion $i\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S} \to \mathsf{Pr}_{\mathsf{Ab}}\mathbf{S}$. Then the second page of the spectral sequence is given by

$$E_2^{p,q} \cong \check{H}^p(X; \mathcal{H}^q(F)).$$

By [Tam94, 3.4.7] the edge homomorphism

$$\check{H}^p(X; F) \to H^p(X; F)$$

is an isomorphism for $p = 0, 1$ and injective for $p = 2$.

4.6.4 We now observe that Čech cohomology transforms strict inverse limits of compact spaces into colimits of cohomology groups.

Lemma 4.68. *Let H be a discrete abelian group. If $(X_i)_{i\in I}$ is an inverse system of compact spaces in $\mathbf{S}$ such that $X = \mathsf{lim}_{i\in I} X_i$, then*

$$\check{H}^p(X; \underline{H}) \cong \mathsf{colim}_{i\in I}\, \check{H}^p(X_i; \underline{H}).$$

Proof. We first show that the system of open coverings of X contains a cofinal system of coverings which are pulled back from the quotients $p_i\colon X \to X_i$. Let $\mathcal{U} = (U_r)_{r\in R}$ be a covering by open subsets. For each r there exists a family $I_r \subset I$ and subsets $U_{r,i} \subseteq X_i, i \in I_r$ such that $U_r = \cup_{i\in I_r} p_i^{-1}(U_{r,i})$. The set of open subsets $\{p_i^{-1}(U_{r,i}) \mid r \in R,\ i \in I_r\}$ is an open covering of X. Since X is compact we can choose a finite subcovering $\mathcal{V} := \{U_{r_1,i_1}, \ldots U_{r_k,i_k}\}$ which can naturally be viewed as a refinement of $\mathcal{U}$. Since I is left filtered we can choose $j \in I$ such that $j < i_d$ for $d = 1, \ldots, k$. For $j \le i$ let $p_{ji}\colon X_j \to X_i$ be the structure map of the system which are all surjective by the strictness assumption on the inverse system. Therefore $\mathcal{V}' := \{p_{j,i_d}^{-1}(U_{r_d,i_d}) \mid d = 1, \ldots, k\}$ is an open covering of X_j, and $\mathcal{V} = p_j^{-1}(\mathcal{V}')$.

Now we observe that $\check{C}^\bullet(p^*\mathcal{V}';\, {}^p\underline{H}) \cong \check{C}^\bullet(\mathcal{V}';\, {}^p\underline{H})$. Therefore

$$\begin{aligned}
\check{C}^\bullet(X; \underline{H}) &\cong \check{C}^\bullet(X;\, {}^p\underline{H}) \quad \text{(by Equation (40))}\\
&\cong \mathsf{colim}_{\mathcal{U}} \check{C}^\bullet(\mathcal{U};\, {}^p\underline{H})\\
&= \mathsf{colim}_{i\in I}\mathsf{colim}_{\text{coverings } \mathcal{U} \text{ of } X_i} \check{C}^\bullet(\mathcal{U};\, {}^p\underline{H})\\
&\cong \mathsf{colim}_{i\in I} \check{C}^\bullet(X_i;\, {}^p\underline{H})\\
&\cong \mathsf{colim}_{i\in I} \check{C}^\bullet(X_i; \underline{H}) \quad \text{(by Equation (40))}
\end{aligned}$$

since one can interchange colimits. Since filtered colimits are exact and therefore commute with taking cohomology this implies the lemma. □

4.6.5 Next we show that Čech cohomology has a Künneth formula.

Lemma 4.69. *Let H be a discrete ring of finite cohomological dimension. Assume that X, Y are compact. Then there exists a Künneth spectral sequence with second term*

$$E^2_{p,q} := \bigoplus_{i+j=q} \mathsf{Tor}^H_p(\check{H}^i(X;\underline{H}), \check{H}^j(Y;\underline{H}))$$

which converges to $\check{H}^{p+q}(X \times Y;\underline{H})$.

Proof. Since X is compact the topology on $X \times Y$ is the product topology [Ste67, Theorem 4.3]. Using again the compactness of X and Y we can find a cofinal system of coverings of $X \times Y$ of the form $p^*\mathcal{U} \cap q^*\mathcal{V}$ for coverings $\mathcal{U}$ of X and $\mathcal{V}$ of Y, where $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ denote the projections, and the intersection of coverings is the covering by the collection of all cross intersections. We have

$$\check{C}^*(p^*\mathcal{U} \cap q^*\mathcal{V};\ {}^p\underline{H}) \cong \check{C}^*(p^*\mathcal{U};\ {}^p\underline{H}) \otimes_H \check{C}^*(q^*\mathcal{V};\ {}^p\underline{H}).$$

Since the tensor product over H commutes with colimits we get

$$\check{C}^*(X \times Y;\ {}^p\underline{H}) \cong \check{C}^*(X;\ {}^p\underline{H}) \otimes_H \check{C}^*(Y;\ {}^p\underline{H}),$$

and hence by (40)

$$\check{C}^*(X \times Y;\underline{H}) \cong \check{C}^*(X;\underline{H}) \otimes_H \check{C}^*(Y;\underline{H}).$$

The Künneth spectral sequence is the spectral sequence associated to this double complex of flat (as colimits of free) H-modules. □

4.6.6 We now recall that sheaf cohomology also transforms strict inverse limits of spaces into colimits, and that it has a Künneth spectral sequence. The following is a specialization of [Brd97, II.14.6] to compact spaces.

Lemma 4.70. *Let $(X_i)_{i\in I}$ be an inverse system of compact spaces and $X = \mathsf{lim}_{i\in I} X_i$, and H let be a discrete abelian group. Then*

$$H^*(X;\underline{H}) \cong \mathsf{colim}_{i\in I} H^*(X_i;\underline{H}).$$

For simplicity we formulate the Künneth formula for the sheaf $\underline{\mathbb{Z}}$ only. The following is a specialization of [Brd97, II.18.2] to compact spaces and the sheaf $\underline{\mathbb{Z}}$.

Lemma 4.71. *For compact spaces X, Y we have a Künneth spectral sequence with second term*

$$E^2_{p,q} = \bigoplus_{i+j=q} \mathsf{Tor}^{\mathbb{Z}}_p(H^i(X;\underline{\mathbb{Z}}), H^j(Y;\underline{\mathbb{Z}}))$$

which converges to $H^{p+q}(X \times Y;\underline{\mathbb{Z}})$.

Of course, this formulation is much too complicated since $\mathbb{Z}$ has cohomological dimension one. In fact, the spectral sequence decomposes into a collection of short exact sequences.

Lemma 4.72. *If G is a compact connected abelian group, then $H^*(G;\underline{\mathbb{Z}}) \cong \check{H}^*(G;\underline{\mathbb{Z}})$.*

Proof. The Čech cohomology spectral sequence provides the map

$$\check{H}^*(G;\underline{\mathbb{Z}}) \to H^*(G;\underline{\mathbb{Z}}).$$

We now use that fact that G is the projective limit of groups isomorphic to T^a, $a \in \mathbb{N}$. Since the compact abelian group G is connected, by Fact 4.67 we know that $\widehat{G}$ is torsion-free. Since $\widehat{G}$ is torsion-free it is the filtered colimit of its finitely generated subgroups $\widehat{F}$. If $\widehat{F} \subset \widehat{G}$ is finitely generated, then $\widehat{F} \cong \mathbb{Z}^a$ for some $a \in \mathbb{N}$, therefore $F := \widehat{\widehat{F}} \cong T^a$. Pontrjagin duality transforms the filtered colimit $\widehat{G} \cong \mathrm{colim}_{\widehat{F}} \widehat{F}$ into a strict limit $G \cong \lim_{\widehat{F}} F$.

Since T^a is a manifold it admits a cofinal system of good open coverings where all multiple intersections are contractible. Therefore we get the isomorphism

$$\check{H}^*(T^a;\underline{\mathbb{Z}}) \xrightarrow{\sim} H^*(T^a;\underline{\mathbb{Z}}).$$

Since both cohomology theories commute with limits of compact spaces we get

$$\check{H}^*(G;\underline{\mathbb{Z}}) \xrightarrow{\sim} H^*(G;\underline{\mathbb{Z}}).$$ □

4.6.7 We now use the calculation of cohomology of the underlying space of a connected compact abelian group G. By [HM98, Theorem 8.83] we have

$$\check{H}^*(G,\underline{\mathbb{Z}}) \cong \Lambda^*_{\mathbb{Z}}\widehat{G} \tag{41}$$

as a graded Hopf algebra. This result uses the two properties 4.68 and 4.69 of Čech cohomology in an essential way.

By Lemma 4.72 we also have

$$H^*(G;\underline{\mathbb{Z}}) \cong \Lambda^*_{\mathbb{Z}}\widehat{G}.$$

Note that $\Lambda^*_{\mathbb{Z}}\widehat{G}$ is torsion free. In fact, $\Lambda^*_{\mathbb{Z}}\widehat{G} \cong \mathrm{colim}_{\widehat{F}} \Lambda^*_{\mathbb{Z}}\widehat{F}$, where the colimit is taken over all finitely generated subgroups $\widehat{F}$ of $\widehat{G}$. We therefore get a colimit of injections of torsion-free abelian groups, which is itself torsion-free.

4.6.8 Recall the notation related to $\mathbb{Z}_{\mathrm{mult}}$-actions introduced in 3.5.

Lemma 4.73. *If $V, W \in \mathsf{Ab}$ with W torsion-free and $k, l \in \mathbb{Z}$, $k \neq l$, then*

$$\mathrm{Hom}_{\mathbb{Z}_{\mathrm{mult}}\text{-}\mathrm{mod}}(V(k), W(l)) = 0.$$

Proof. Let $\phi \in \mathsf{Hom}_{\mathsf{Ab}}(V, W)$ and $v \in V$. Then for all $m \in \mathbb{Z}_{\mathrm{mult}}$ we have

$$m^k \phi(v) = \Psi^m(\phi(v)) = \phi(\Psi^m(v)) = \phi(m^l v) = m^l \phi(v),$$

i.e. each $w \in \mathsf{im}(\phi) \subseteq W(l)$ satisfies $(m^k - m^l)w = 0$ for all $m \in \mathbb{Z}_{\mathrm{mult}}$. This set of equations implies that $w = 0$, since W is torsion-free. □

Lemma 4.74. *If $V, W \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ with W torsion-free and $k, l \in \mathbb{Z}$, $k \neq l$, then*

$$\mathsf{Hom}_{\mathsf{Sh}_{\mathbb{Z}_{\mathrm{mult}}\text{-mod}}\mathbf{S}}(V(k), W(l)) = 0.$$

We leave the easy proof of this sheaf version of Lemma 4.73 to the interested reader. Note that a subquotient of a sheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules of weight k also has weight k.

4.6.9 By $\mathbf{S}_{\mathrm{lc}} \subset \mathbf{S}$ we denote the sub-site of locally compact objects. The restriction to locally compact spaces becomes necessary because of the use of the Künneth (or base change) formula below.

The first page of the spectral sequence (E_r, d_r) introduced in 4.2.12 is given by

$$E_1^{q,p} = \underline{\mathsf{Ext}}^p_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\mathbb{Z}(\underline{G}^q), \mathbb{Z}).$$

This sheaf is the sheafification of the presheaf

$$\mathbf{S} \ni A \mapsto H^p(A \times G^q, \underline{\mathbb{Z}}) \in \mathsf{Ab}.$$

In order to calculate this cohomology we use the Künneth formula [Brd97, II.18.2] and that $H^*(G, \underline{\mathbb{Z}})$ is torsion-free (4.67). We get for $A \in \mathbf{S}_{\mathrm{lc}}$ that

$$H^*(A \times G^q, \underline{\mathbb{Z}}) \cong H^*(A, \underline{\mathbb{Z}}) \otimes_{\mathbb{Z}} (\Lambda^* \hat{G})^{\otimes_{\mathbb{Z}} q}$$

for locally compact A. The sheafification of $\mathbf{S}_{\mathrm{lc}} \ni A \to H^i(A, \underline{\mathbb{Z}})$ vanishes for $i \geq 1$, and gives $\underline{\mathbb{Z}}$ for $i = 0$. Since sheafification commutes with the tensor product with a fixed group we get

$$(E_1^{q,*})_{|\mathbf{S}_{\mathrm{lc}}} \cong \underline{(\Lambda^* \hat{G})^{\otimes_{\mathbb{Z}} q}}. \tag{42}$$

4.6.10 We now consider the tautological action of the multiplicative semigroup $\mathbb{Z}_{\mathrm{mult}}$ on G of weight 1. As before we write $G(1)$ for the group G with this action. Observe that this action is continuous. Applying the duality functor we get an action of $\mathbb{Z}_{\mathrm{mult}}$ on the dual group $\hat{G}$ which is also of weight 1. Therefore

$$\widehat{G(1)} = \hat{G}(1).$$

The calculation of the cohomology (41) of the topological space G with $\mathbb{Z}$-coefficients is functorial in G. We conclude that $H^*(G(1), \mathbb{Z}) \cong \Lambda^*_{\mathbb{Z}}(\hat{G}(1))$ is a decomposable $\mathbb{Z}_{\mathrm{mult}}$-module. By 3.5.5 the group

$$H^p(G^q(1), \mathbb{Z}) \cong [(\Lambda^* \hat{G}(1))^{\otimes_{\mathbb{Z}} q}]^p$$

is of weight p.

4.6.11 We now observe that the spectral sequence (E_r, d_r) is functorial in G. To this end we use the fact that $G \mapsto U^\bullet(\underline{G})$ is a covariant functor from groups $G \in \mathbf{S}$ to complexes of sheaves on $\mathbf{S}$. If $G_0 \to G_1$ is a homomorphism of topological groups in $\mathbf{S}$, then we get an induced map $U^q(G_0) \to U^q(G_1)$, namely the map $\mathbb{Z}(\underline{G_0^q}) \to \mathbb{Z}(\underline{G_1^q})$. Under the identification made in 4.6.9 the induced map

$$\underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\,\mathbf{S}_{lc}}(\mathbb{Z}(\underline{G_1^q}), \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^p_{\mathsf{Sh_{Ab}}\,\mathbf{S}_{lc}}(\mathbb{Z}(\underline{G_0^q}), \underline{\mathbb{Z}})$$

goes to the map

$$\underline{H^p(G_1^q; \underline{\mathbb{Z}})} \to \underline{H^p(G_0^q; \underline{\mathbb{Z}})}$$

induced by the pull-back

$$H^p(G_1^q; \underline{\mathbb{Z}}) \to H^p(G_0^q; \underline{\mathbb{Z}})$$

associated to the map of spaces $G_0^q \to G_1^q$.

4.6.12 The discussion in 4.6.11 and 4.6.10 shows that the $\mathbb{Z}_{\mathrm{mult}}$-module structure $G(1)$ induces one on the spectral sequence (E_r, d_r), and we see that $E_1^{q,p}$ has weight p (which is the number of factors $\widehat{G}(1)$ contributing to this term).

We introduce the notation $H^k := H^k(\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(U^\bullet, I^\bullet))_{|\mathbf{S}_{lc}}$. Note that H^k has a filtration

$$0 = F^{-1}H^k \subseteq F^0 H^k \subseteq \cdots \subseteq F^k H^k = H^k$$

which is preserved by the action of $\mathbb{Z}_{\mathrm{mult}}$. The spectral sequence (E_r, d_r) converges to the associated graded sheaf $\mathsf{Gr}(H^k)$.

Note that $\mathsf{Gr}^p(H^k)$ is a subquotient of $E_2^{k-p,p}$. Since a subquotient of a sheaf of weight p also has weight p (see the our remark after Lemma 4.74) we get the following conclusion.

Corollary 4.75. $\mathsf{Gr}^p(H^k)$ *has weight* p.

4.6.13 We now study some aspects of the second page $E_2^{q,p}$ of the spectral sequence in order to show the following lemma.

Lemma 4.76. 1. *For* $k \geq 1$ *we have* $F^0 H^k \cong 0$.
2. *For* $k \geq 2$ *we have* $F^1 H^k \cong 0$.

Proof. We start with 1. We see that $E_2^{*,0}$ is the cohomology of the complex $E_1^{*,0} \cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(U^\bullet, \underline{\mathbb{Z}})$. Explicitly, using in the last step the fact that G and therefore G^q are connected,

$$\begin{aligned} E_1^{q,0} &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(U^q, \underline{\mathbb{Z}}) \overset{(24)}{\cong} \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(\mathbb{Z}(\underline{G^q}), \underline{\mathbb{Z}}) \overset{(17)}{\cong} \mathcal{R}_{G^q}(\underline{\mathbb{Z}}) \\ &\cong \underline{\mathsf{Map}(G^q, \mathbb{Z})} \cong \underline{\mathbb{Z}} \qquad \text{(by Lemma 3.25).} \end{aligned}$$

An inspection of the formulas 4.2.7 for the differential of the complex $U^\bullet$ shows that $d_1\colon E_1^{q,0} \to E_1^{q+1,0}$ vanishes for even q, and is the identity for odd q. In other words this complex is isomorphic to

$$0 \to \underline{\mathbb{Z}} \xrightarrow{0} \underline{\mathbb{Z}} \xrightarrow{\mathrm{id}} \underline{\mathbb{Z}} \xrightarrow{0} \underline{\mathbb{Z}} \xrightarrow{\mathrm{id}} \underline{\mathbb{Z}} \xrightarrow{0} \cdots .$$

This implies that $E_2^{q,0} = 0$ for $q \geq 1$ and therefore the assertion of the lemma.

We now turn to 2. We calculate

$$H^1(G^q;\underline{\mathbb{Z}}) \cong [(\Lambda^*\hat{G})^{\otimes_{\mathbb{Z}} q}]^1 = \underbrace{\hat{G} \oplus \cdots \oplus \hat{G}}_{q \text{ summands}}.$$

By (42) we get

$$E_1^{q,1} \cong \underline{\hat{G}^q}.$$

We see that $(E_1^{*,1}, d_1)$ is the sheafification of the complex of discrete groups

$$\hat{G}^\bullet\colon 0 \to \hat{G} \to \hat{G}^2 \to \hat{G}^3 \to \cdots , \tag{43}$$

where the differential is the dual of the differential of the complex

$$0 \leftarrow G \leftarrow G^2 \leftarrow G^3 \leftarrow \cdots$$

induced by the maps given in 4.2.7. Let us describe the differential more explicitly. If $\chi \in \hat{G}$, and $\mu\colon G \times G \to G$ is the multiplication map, then we have $\mu^*\chi = (\chi,\chi) \in \widehat{G \times G} \cong \hat{G} \times \hat{G}$. We can write $\partial\colon \hat{G}^q \to \hat{G}^{q+1}$ as

$$\partial = \sum_{i=0}^{q+1} (-1)^i \partial_i .$$

Using the formulas of 4.2.7 we get

$$\partial_i(\chi_1,\ldots,\chi_q) = (\chi_1,\ldots,\chi_i,\chi_i,\ldots,\chi_q)$$

for $i = 1,\ldots,q$. Furthermore

$$\partial_0(\chi_1,\ldots,\chi_q) = (0,\chi_1,\ldots,\chi_q)$$

and

$$\partial_{q+1}(\chi_1,\ldots,\chi_q) = (\chi_1,\ldots,\chi_q,0).$$

Note that $0 = \partial\colon \hat{G}^0 \to \hat{G}^1$. We see that we can write the higher (≥ 1) degree part of the complex (43) in the form

$$K^\bullet \otimes_{\mathbb{Z}} \hat{G},$$

where $K^q = \mathbb{Z}^q$ for $q \geq 1$ and $\partial\colon \mathbb{Z}^q \to \mathbb{Z}^{q+1}$ is given by the same formulas as above. One now shows[12] that

$$H^q(K^\bullet) = \begin{cases} \mathbb{Z}, & q = 1, \\ 0, & q \geq 2. \end{cases}$$

[12] We leave this as an exercise in combinatorics to the interested reader.

Since $\widehat{G}$ is torsion-free and hence a flat $\mathbb{Z}$-module we have

$$H^q(\widehat{G}^\bullet) \cong H^q(K^\bullet \otimes_{\mathbb{Z}} \widehat{G}) \cong H^q(K^\bullet) \otimes_{\mathbb{Z}} \widehat{G}.$$

Therefore $H^q(\widehat{G}^\bullet) = 0$ for $q \geq 2$. This implies the assertion of the lemma in the case 2. □

Lemma 4.77. *Let $F^0 \subseteq F^1 \subseteq \cdots \subseteq F^{k-1} \subseteq F^k = F$ be a filtered sheaf of $\mathbb{Z}_{\mathrm{mult}}$-modules such that $\mathsf{Gr}^l(F)$ has weight l, and such that $F^0 = F^1 = 0$. If $V \subseteq F$ is a torsion-free sheaf of weight 1, then $V = 0$.*

Proof. Assume that $V \neq 0$. We show by induction (downwards) that $F^l \cap V \neq 0$ for all $l \geq 1$. The case $l = 1$ gives the contradiction. Assume that $l > 1$. We consider the exact sequence

$$0 \to F^{l-1} \cap V \to F^l \cap V \to \mathsf{Gr}_l(F).$$

First of all, by induction assumption, the sheaf $V \cap F^l$ is non-trivial, and a as a subsheaf of V it is torsion-free of weight 1. Since $\mathsf{Gr}_l(F)$ has weight $l \neq 1$, the map $V \cap F_l \to \mathsf{Gr}_l(F)$ can not be injective. Otherwise its image would be a torsion-free sheaf of two different weights l and 1, and this is impossible by Lemma 4.74. Hence $F^{l-1} \cap V \neq 0$. □

4.6.14 We now show that Lemma 4.64 extends to connected compact groups.

Lemma 4.78. *Let G be a compact connected abelian group. Then the following assertions are equivalent.*

1. $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{lc}}(\underline{G}, \underline{\mathbb{Z}})$ *is torsion-free for* $i = 2, 3$.
2. $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{lc}}(\underline{G}, \underline{\mathbb{Z}}) \cong 0$ *for* $i = 2, 3$.

Proof. The non-trivial direction is that 1. implies 2. Therefore let us assume that $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ is torsion-free for $i = 2, 3$. We now look at the spectral sequence (F_r, d_r). The left lower corner of its second page was calculated in 4.60. We see that $F_2^{1,2} = \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}})$ has weight 2, while $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ has weight 1. Since $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ is torsion-free by assumption, it follows from Lemma 4.74 that the differential $d_2^{1,2} \colon \underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}((\Lambda^2_{\mathbb{Z}}\underline{G})^\sharp, \underline{\mathbb{Z}}) \to \underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ is trivial.

We conclude that $\underline{\mathsf{Ext}}^2_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ and $\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ survive to the limit of the spectral sequence. We see that $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}})$ are torsion-free sub-sheaves of weight 1 of H^{i+1} for $i = 2, 3$. Using the structure of H^{i+1} given in 4.75 in conjunction with Lemmas 4.76 and 4.77 we get $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}_{\mathrm{lc}}}(\underline{G}, \underline{\mathbb{Z}}) = 0$ for $i = 2, 3$. □

4.6.15 The combination of Lemma 4.62 (resp. Lemma 4.62) and Lemma 4.78 gives the following result.

Theorem 4.79. 1. *A compact connected abelian group G which satisfies the two-three condition is admissible on the site $\mathbf{S}_{\text{lc-acyc}}$.*

2. *If G is in addition locally topologically divisible, then it is admissible on $\mathbf{S}_{\mathrm{lc}}$.*

5 Duality of locally compact group stacks

5.1 Pontrjagin Duality

5.1.1 In this subsection we extend Pontrjagin duality for locally compact groups to abelian group stacks whose sheaves of objects and automorphisms are represented by locally compact groups. In algebraic geometry a parallel theory has been considered in [DP].

The site $\mathbf{S}$ denotes the site of compactly generated spaces as in 3.1.2 or one of its sub-sites $\mathbf{S}_{\text{lc}}$, $\mathbf{S}_{\text{lc-acyc}}$. The reason for considering these sub-sites lies in the fact that certain topological groups are only admissible on these sub-sites (see the Definitions 4.1, 4.2 and Theorem 4.8).

5.1.2 Let $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$.

Definition 5.1. We define the dual sheaf of F by

$$D(F) := \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}).$$

Definition 5.2. We call F dualizable, if the canonical evaluation morphism

$$c\colon F \to D(D(F)) \tag{44}$$

is an isomorphism of sheaves.

Lemma 5.3. *If G is a locally compact group which together with its Pontrjagin dual $\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T})$ is contained in $\mathbf{S}$, then $\underline{G} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ is dualizable.*

Proof. By Lemma 3.5 we have isomorphisms

$$\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}) \cong \underline{\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T})}$$

and

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{G}, \underline{\mathbb{T}}), \underline{\mathbb{T}}) &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T})}, \underline{\mathbb{T}}) \\ &\cong \underline{\mathsf{Hom}_{\text{top-Ab}}(\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T}), \mathbb{T})}. \end{aligned}$$

The morphism c in (44) is induced by the evaluation map

$$G \to \mathsf{Hom}_{\text{top-Ab}}(\mathsf{Hom}_{\text{top-Ab}}(G, \mathbb{T}), \mathbb{T})$$

which is an isomorphism by the classical Pontrjagin duality of locally compact abelian groups [Fol95], [HM98]. $\square$

5.1.3 The sheaf of abelian groups $\underline{\mathbb{T}} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ gives rise to a Picard stack $\mathcal{B}\underline{\mathbb{T}} \in \mathsf{PIC}(\mathbf{S})$ as explained in 2.3.3. Recall from 2.5.11 the following alternative description of $\mathcal{B}\underline{\mathbb{T}}$. Let $\underline{\mathbb{T}}[1]$ be the complex with the only non-trivial entry $\underline{\mathbb{T}}[1]^{-1} := \underline{\mathbb{T}}$. Then we have $\mathcal{B}\underline{\mathbb{T}} \cong \mathsf{ch}(\underline{\mathbb{T}}[1])$ in the notation of 2.12.

5.1.4 Let now $P \in \mathsf{PIC}(\mathbf{S})$ be a Picard stack. Recall Definition 2.11, where we define the internal $\underline{\mathsf{HOM}}$ between two Picard stacks.

Definition 5.4. We define the dual stack by

$$D(P) := \underline{\mathsf{HOM}}_{\mathsf{PIC}(\mathbf{S})}(P, \mathcal{B}\underline{\mathbb{T}}).$$

We hope that using the same symbol D for the dual in the case of Picard stacks and the case of a sheaf of abelian groups will not cause confusion.

5.1.5 This definition is compatible with Definition 5.1 of the dual of a sheaf of abelian groups in the following sense.

Lemma 5.5. *If $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$, then we have a natural isomorphism*

$$\mathsf{ch}(D(F)) \cong D(\mathcal{B}F).$$

Proof. First observe that by definition $D(\mathcal{B}F) = \underline{\mathsf{HOM}}_{\mathsf{PIC}(\mathbf{S})}(\mathsf{ch}(F[1]), \mathsf{ch}(\underline{\mathbb{T}}[1]))$. We use Lemma 2.18 in order to calculate $H^i(D(\mathcal{B}F))$. We have

$$H^{-1}(D(\mathcal{B}F)) \cong R^{-1}\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F[1], \underline{\mathbb{T}}[1]) \cong 0$$

and

$$H^0(D(\mathcal{B}F)) \cong R^0\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F[1], \underline{\mathbb{T}}[1]) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong D(F).$$

The composition of this isomorphism with the projection $D(\mathcal{B}F) \to \mathsf{ch}(H^0(D(\mathcal{B}F)))$ from the stack $D(\mathcal{B}F)$ onto its sheaf of isomorphism classes (considered as Picard stack) provides the asserted natural isomorphism. $\square$

5.1.6 A sheaf of groups $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ can also be considered as a complex $F \in C(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$ with non-trivial entry $F^0 := F$. It thus gives rise to a Picard stack $\mathsf{ch}(F)$. Recall the definition of an admissible sheaf 4.1.

Lemma 5.6. *We have natural isomorphisms*

$$H^{-1}(D(\mathsf{ch}(F))) \cong D(F), \quad H^0(D(\mathsf{ch}(F))) \cong \underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}).$$

In particular, if F is admissible, then $D(\mathsf{ch}(F)) \cong \mathcal{B}(D(F))$.

Proof. We use again Lemma 2.18. We have

$$H^{-1}(D(\mathsf{ch}(F))) \cong R^{-1}\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}[1]) \cong \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong D(F).$$

Furthermore,

$$H^0(D(\mathsf{ch}(F))) \cong R^0\underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}[1]) \cong \mathsf{Ext}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}). \qquad \square$$

5.1.7 Assume now that F is dualizable and admissible (at this point we only need $\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(F, \underline{\mathbb{T}}) \cong 0$). Then we have

$$\begin{aligned} D(D(\mathsf{ch}(F))) &\cong D(\mathcal{B}(D(F))) \quad \text{(by Lemma 5.6)} \\ &\cong \mathsf{ch}(D(D(F)) \quad \text{(by Lemma 5.5)} \\ &\cong \mathsf{ch}(F). \end{aligned}$$

Similarly, if F is dualizable and $D(F)$ admissible (again we only need the weaker condition that $\underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(D(F), \underline{\mathbb{T}}) \cong 0$), then we have, again by Lemmas 5.5 and 5.6,

$$D(D(\mathcal{B}F)) \cong D(\mathsf{ch}(D(F))) \cong \mathcal{B}D(D(F)) \cong \mathcal{B}F.$$

5.1.8 Let us now formalize this observation. Let $P \in \mathsf{PIC}(\mathbf{S})$ be a Picard stack.

Definition 5.7. We call P dualizable if the natural evaluation morphism $P \to D(D(P))$ is an isomorphism.

The discussion of 5.1.7 can now be formulated as follows.

Corollary 5.8. 1. *If F is dualizable and admissible, then* $\mathsf{ch}(F)$ *is dualizable.*
2. *If F is dualizable and $D(F)$ is admissible, then $\mathcal{B}F$ is dualizable.*

The goal of the present subsection is to extend this kind of result to more general Picard stacks.

5.1.9 Let $P \in \mathsf{PIC}(\mathbf{S})$.

Lemma 5.9. *We have*

$$H^{-1}(D(P)) \cong D(H^0(P))$$

and an exact sequence

$$0 \to \underline{\mathsf{Ext}}^1_{\mathsf{Sh_{Ab}}\mathbf{S}}(H^0(P), \underline{\mathbb{T}}) \to H^0(D(P)) \to D(H^{-1}(P)) \to \underline{\mathsf{Ext}}^2_{\mathsf{Sh_{Ab}}\mathbf{S}}(H^0(P), \underline{\mathbb{T}}). \tag{45}$$

Proof. By Lemma 2.13 we can choose $K \in C(\mathsf{Sh_{Ab}}\mathbf{S})$ such that $P \cong \mathsf{ch}(K)$. We now get

$$\begin{aligned} H^{-1}(D(P)) &\cong R^{-1}\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(K, \underline{\mathbb{T}}[1]) \quad \text{(by Lemma 2.18)} \\ &\cong R^0\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(K, \underline{\mathbb{T}}) \\ &\cong \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(H^0(K), \underline{\mathbb{T}}) \\ &\cong D(H^0(P)). \end{aligned}$$

Again by Lemma 2.18 we have

$$H^0(D(P)) \cong R^0\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(K, \underline{\mathbb{T}}[1]) \cong R^1\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\mathbf{S}}(K, \underline{\mathbb{T}}).$$

In order to calculate this sheaf in terms of the cohomology sheaves $H^i(K)$ of K we choose an injective resolution $\underline{\mathbb{T}} \to I$. Then we have

$$R\underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(K, \underline{\mathbb{T}}) \cong \underline{\mathsf{Hom}}_{\mathsf{ShAb}\,\mathbf{S}}(K, I).$$

We must calculate the first total cohomology of this double complex. We first take the cohomology in the K-direction, and then in the I-direction. The second page of the associated spectral sequence has the following form.

1	$\underline{\mathsf{Ext}}^0_{\mathsf{ShAb}\,\mathbf{S}}(H^{-1}(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(H^{-1}(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^2_{\mathsf{ShAb}\,\mathbf{S}}(H^{-1}(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(H^{-1}(P), \underline{\mathbb{T}})$
0	$\underline{\mathsf{Ext}}^0_{\mathsf{ShAb}\,\mathbf{S}}(H^0(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^1_{\mathsf{ShAb}\,\mathbf{S}}(H^0(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^2_{\mathsf{ShAb}\,\mathbf{S}}(H^0(P), \underline{\mathbb{T}})$	$\underline{\mathsf{Ext}}^3_{\mathsf{ShAb}\,\mathbf{S}}(H^0(P), \underline{\mathbb{T}})$
	0	1	2	3

The sequence (45) is exactly the edge sequence for the total degree-1 term. □

5.1.10 The appearance in (45) of the groups $\underline{\mathsf{Ext}}^i_{\mathsf{ShAb}\,\mathbf{S}}(H^0(P), \underline{\mathbb{T}})$ for $i = 1, 2$ was the motivation for the introduction of the notion of an admissible sheaf in 4.1.

Corollary 5.10. *Let $P \in \mathsf{PIC}(\mathbf{S})$ be such that $H^0(P)$ is admissible. Then we have*

$$H^0(D(P)) \cong D(H^{-1}(P)), \quad H^{-1}(D(P)) \cong D(H^0(P)).$$

Theorem 5.11. *Let $P \in \mathsf{PIC}(\mathbf{S})$ and assume that*

1. *$H^0(P)$ and $H^{-1}(P)$ are dualizable;*
2. *$H^0(P)$ and $D(H^{-1}(P))$ are admissible.*

Then P is dualizable.

Proof. In view of 2.14 it suffices to show that the evaluation map $c\colon P \to D(D(P))$ induces isomorphisms

$$H^i(c)\colon H^i(P) \to H^i(D(D(P)))$$

for $i = -1, 0$. Consider first the case $i = 0$. Then by Corollary 5.10 we have an isomorphism

$$h^0\colon H^0(D(D(P))) \xrightarrow{\sim} D(H^{-1}(D(F)) \xrightarrow{\sim} D(D(H^0(P))).$$

One now checks that the map

$$h^0 \circ H^0(c)\colon H^0(P) \to D(D(H^0(P)))$$

is the evaluation map (44). Since we assume that $H^0(P)$ is dualizable this map is an isomorphism. Hence $H^0(c)$ is an isomorphism, too.

For $i = -1$ we use the isomorphism

$$h^{-1}\colon H^{-1}(D(D(P))) \xrightarrow{\sim} D(H^0(D(P))) \xrightarrow{\sim} D(D(H^{-1}(P)))$$

and the fact that $h^{-1} \circ H^{-1}(c)\colon H^{-1}(P) \to D(D(H^{-1}(P)))$ is an isomorphism. □

5.1.11 If G is a locally compact group which together with its Pontrjagin dual belongs to $\mathbf{S}$, then by 5.3 the sheaf $\underline{G}$ is dualizable. By Theorem 4.8 we know a large class of locally compact groups which are admissible on $\mathbf{S}$ or at least after restriction to $\mathbf{S}_{\mathrm{lc}}$ or $\mathbf{S}_{\text{lc-acyc}}$.

We get the following result.

Theorem 5.12. *Let $G_0, G_{-1} \in \mathbf{S}$ be two locally compact abelian groups. We assume that their Pontrjagin duals belong to $\mathbf{S}$, and that G_0 and DG_{-1} are admissible on $\mathbf{S}$. If $P \in \mathsf{PIC}(\mathbf{S})$ has $H^i(P) \cong \underline{G}_i$ for $i = -1, 0$, then P is dualizable.*

Let us specialize to the case which is important for the application to T-duality. Note that a group of the form $\mathbb{T}^n \times \mathbb{R}^m \times F$ for a finitely generated abelian group F is admissible by Theorem 4.8. This class of groups is closed under forming Pontrjagin duals.

Corollary 5.13. *If $P \in \mathsf{PIC}(\mathbf{S})$ has $H^i(P) \cong \underline{\mathbb{T}^{n_i}} \times \underline{\mathbb{R}^{m_i}} \times \underline{F_i}$ for some finitely generated abelian groups F_i for $i = -1, 0$, then P is dualizable.*

For more general groups one may have to restrict to the sub-site $\mathbf{S}_{\mathrm{lc}}$ or even to $\mathbf{S}_{\text{lc-acyc}}$.

5.2 Duality and classification

5.2.1 Let $A, B \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$. By Lemma 2.20 we know that the isomorphism classes $\mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(A, B)$ of Picard stacks with $H^0(P) \cong B$ and $H^{-1}(P) \cong A$ are classified by a characteristic class

$$\phi\colon \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(B, A) \xrightarrow{\sim} \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A). \tag{46}$$

Lemma 5.14. *If A is dualizable and $D(A)$, B are admissible, then there is a natural isomorphism*

$$\mathcal{D}\colon \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A) \xrightarrow{\sim} \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A), D(B))$$

such that

$$\phi(D(P)) = \mathcal{D}(\phi(P)) \quad \text{for all } P \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(A, B). \tag{47}$$

Proof. In order to define $\mathcal{D}$ we use the identifications

$$\mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A) \cong \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B, A[2]),$$
$$\mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A), D(B)) \cong \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(D(A), D(B)[2]).$$

We choose an injective resolution $\underline{\mathbb{T}} \to \mathcal{I}$. For a complex of sheaves F we define $RD(F) := \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(F, \mathcal{I})$. The map $\underline{\mathbb{T}} \to \mathcal{I}$ induces a map $D(F) \to RD(F)$.

Note that RD descends to a functor between derived categories $RD\colon D^b(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})^{\mathrm{op}} \to D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})$. We now consider the following web of maps:

$$\begin{tikzcd}
\mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B,A[2]) \arrow[r, "RD"] \arrow[dr, dotted, "\mathcal{D}'"] \arrow[ddr, dotted, "\mathcal{D}"'] & \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(RD(A[-2]),RD(B)) \arrow[d, "D(A)\to RD(A)"] \\
& \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(D(A),RD(B)[2]) \\
& \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(D(A),D(B)[2]). \arrow[u, "u"', "D(B)\to RD(B)"]
\end{tikzcd}$$

Since B is admissible the map $D(B) \to RD(B)$ is an isomorphism in cohomology in degree $0,1,2$ (because $D(B)$ is concentrated in degree 0 and $H^k(RD(B)) = R^k\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B,\underline{\mathbb{T}}) = \mathsf{Ext}^k_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B,\underline{\mathbb{T}})$). Since $D(A)$ is acyclic in non-zero degree, the map u is an isomorphism. For this, observe that $D(B) \to RD(B)$ can be replaced up to quasi-isomorphism by an embedding $0 \to \widetilde{RD(B)} \to RD(B) \to Q \to 0$ such that the quotient is zero in degrees $0,1,2$. The statement then follows from the long exact Ext-sequence for $\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A),-)$, because $\mathsf{Ext}^i_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A),B) = 0$ for $i = 0,1,2$. Therefore the diagram defines the map

$$\mathcal{D} := u^{-1} \circ \mathcal{D}'\colon \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B,A[2]) \to \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(D(A),D(B)[2]).$$

5.2.2 We now show the relation (47). By Lemma 2.13 it suffices to show (47) for $P \in \mathsf{PIC}(\mathbf{S})$ of the form $P = \mathsf{ch}(K)$ for complexes

$$\mathcal{K}\colon 0 \to A \to X \to Y \to B \to 0, \quad K\colon 0 \to X \to Y \to 0.$$

As in 2.5.9 we consider

$$\mathcal{K}_A\colon 0 \to X \to Y \to B \to 0$$

with B in degree 0. Then by Definition (12) of the map ϕ and the Yoneda map Y, the element $\phi(\mathsf{ch}(K)) \in \mathsf{Hom}_{D^+(\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S})}(B,A[2])$ is represented by the composition (see (10))

$$Y(\mathcal{K})\colon B \stackrel{\beta}{\to} \mathcal{K}_A \stackrel{\alpha}{\leftarrow} A[2].$$

Since RD preserves quasi-isomorphisms we get $RD(\alpha^{-1}) = RD(\alpha)^{-1}$. It follows that

$$RD(Y(\mathcal{K})) : RD(A[2]) \xrightarrow{RD(\alpha)^{-1}} RD(\mathcal{K}_A) \xrightarrow{RD(\beta)} RD(B).$$

We read off that

$$\mathcal{D}'(Y(\mathcal{K}))\colon D(A) \to RD(A) \xrightarrow{RD(\alpha)^{-1}} RD(\mathcal{K}_A)[2] \xrightarrow{RD(\beta)} RD(B)[2].$$

Let $\underline{\mathbb{T}} \to I^0 \to I^1 \to \dots$ be the injective resolution $\mathcal{I}$ of $\underline{\mathbb{T}}$. We define $J := \ker(I^1 \to I^2)$ and $I := I^0$. Then we have $\mathcal{B}\underline{\mathbb{T}} = \mathsf{ch}(L)$ with $L\colon 0 \to I \to J \to 0$ with J in degree zero. Note that I is injective. Then by Lemma 2.17 we have

$$D(P) \cong \mathsf{ch}(H), \tag{48}$$

where $H := \tau_{\le 0}\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(K, L)$. Let

$$Q := \ker(\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(X, I) \oplus \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, J) \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(X, J)).$$

Then H is the complex

$$H\colon 0 \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I) \to Q \to 0.$$

There is a natural map $D(B) \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I)$ (induced by $\underline{\mathbb{T}} \to I$ and $Y \to B$), and a projection $Q \to D(A)$ induced by $A \to X$ and passage to cohomology. Since B is admissible the complex

$$\mathcal{H}\colon 0 \to D(B) \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I) \xrightarrow{d} Q \to D(A) \to 0$$

is exact. Note that $\ker(d) = H^{-1}\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(K, I)$ and $\mathsf{coker}(d) = H^0\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(K, I)$. We get

$$\phi(D(P)) \overset{(48)}{=} \phi(\mathsf{ch}(H)) \overset{(12)}{=} Y(\mathcal{H}) \overset{\text{Lemma 2.19}}{=} Y'(\mathcal{H}).$$

Explicitly, in view of (11) the map $Y'(\mathcal{H}) \in \mathsf{Hom}_{D^+(\mathsf{Sh_{Ab}}\,\mathbf{S})}(D(A), D(B)[2])$ is given by the composition

$$Y'(\mathcal{H})\colon D(A) \xrightarrow{\gamma^{-1}} \mathcal{H}_{D(A)} \xrightarrow{\delta} D(B)[2],$$

where

$$\mathcal{H}_{D(A)}\colon 0 \to D(B) \to \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I) \to Q \to 0$$

with Q in degree 0, the map $\gamma\colon \mathcal{H}_{D(A)} \to D(A)$ is the quasi-isomorphism induced by the projection $Q \to D(A)$, and $\delta\colon \mathcal{H}_{D(A)} \to D(B)[2]$ is the canonical projection. Since B is admissible we have $R^1\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(B, \underline{\mathbb{T}}) \cong 0$ and hence a quasi-isomorphism $\mathcal{H}_{D(A)} \to \mathcal{H}'_{D(A)}$ fitting into the following larger diagram.

$$\begin{tikzcd}
\mathcal{H}_{D(A)}\colon 0 \arrow[r] \arrow[d, "\sim"] & D(B) \arrow[r] \arrow[d] & \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\
\mathcal{H}'_{D(A)}\colon 0 \arrow[r] \arrow[d] & \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(B, I) \arrow[r] \arrow[d] & \begin{array}{c}\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(B, J)\\ \oplus \\ \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I)\end{array} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
RD(\mathcal{K}_A)\colon 0 \arrow[r] & \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(B, I) \arrow[r] & \begin{array}{c}\underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(B, I^1)\\ \oplus \\ \underline{\mathsf{Hom}}_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(Y, I)\end{array} \arrow[r] & \underline{\mathsf{Hom}}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}}(\mathcal{K}_A, \mathcal{I}) \arrow[r] & \cdots
\end{tikzcd}$$

5.2.3 The final step in the verification of (47) follows from a consideration of the diagram

$$\begin{tikzcd}
D(B)[2] \arrow[rr] & & RD(B)[2] \\
\mathcal{H}_{D(A)} \arrow[u, "\delta"] \arrow[r, "\sim"] \arrow[d, "\gamma", "\sim"'] & \mathcal{H}'_{D(A)} \arrow[r] \arrow[d, "\sim"] & RD(\mathcal{K}_A) \arrow[u, "RD(\beta)"'] \arrow[d, "RD(\alpha)"', "\sim"] \\
D(A) \arrow[r, equal] \arrow[uu, bend left, "Y'(\mathcal{H})"] \arrow[uurr, bend left, "u\circ Y'(\mathcal{H})"] & D(A) \arrow[r] \arrow[uur, bend right, "\mathcal{D}'(Y(\mathcal{K}))"'] & RD(A) \arrow[uu, bend right, "RD(Y(\mathcal{K}))"']
\end{tikzcd}$$

showing the marked equality in

$$\phi(D(P)) = Y'(\mathcal{H}) \stackrel{!}{=} \mathcal{D}(Y(\mathcal{K})) = \mathcal{D}(\phi(P)).$$

It remains to show that

$$\mathcal{D}\colon \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A) \to \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A), D(B))$$

is an isomorphism.

To this end we look at the following commutative web of maps

$$\begin{tikzcd}
R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A), D(B)) \arrow[d, "\cong"', "u"] & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, A) \arrow[l, dotted, "\mathcal{D}"'] \arrow[r, "\cong"] \arrow[dl, dotted, "\mathcal{D}'"] \arrow[ddl, dotted] \arrow[ddr, dotted] & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, D(D(A))) \arrow[d, "\cong"', "v"] \\
R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A), RD(B)) \arrow[r, "\cong", "RD"'] & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(D(A) \otimes^L B, \mathbb{T}) & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, RD(D(A))) \arrow[l, "z"', "\cong"] \\
R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(RD(A), RD(B)) \arrow[u] \arrow[r, "\cong"] & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(RD(A) \otimes^L B, \mathbb{T}) \arrow[u] & R^2\mathsf{Hom}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(B, RD(RD(A))). \arrow[u] \arrow[l, "\cong"']
\end{tikzcd}$$

The horizontal isomorphisms in the two lower rows are given by the derived adjointness of the tensor product and the internal homomorphisms. The horizontal isomorphism in the first row is induced by the isomorphism $A \to D(D(A))$. The maps u and v are isomorphisms since we assume that $D(A)$ is admissible, compare the corresponding argument in the proof of Lemma 5.14. The map z is induced by the canonical map $A \to RD(RD(A))$. $\square$

6 T-duality of twisted torus bundles

6.1 Pairs and topological T-duality

6.1.1 The goal of this subsection is to introduce the main objects of topological T-duality and review the structure of the theory. Let B be a topological space.

Definition 6.1. A pair (E, H) over B consists of a locally trivial principal $\mathbb{T}^n$-bundle $E \to B$ and a gerbe $H \to E$ with band $\mathbb{T}_{|E}$. An isomorphism of pairs is a diagram

$$\begin{tikzcd} H \arrow[r, "\psi"] \arrow[d] & H' \arrow[d] \\ E \arrow[r, "\phi"] \arrow[d] & E' \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

consisting of an isomorphism of $\mathbb{T}^n$-principal bundles ϕ and an isomorphism of $\mathbb{T}$-banded gerbes ψ. By $P(B)$ we denote the set of isomorphism classes of pairs over B. If $f : B \to B'$ is a continuous map, then pull-back induces a functorial map $P(f)\colon P(B') \to P(B)$.

6.1.2 The functor
$$P : \mathsf{TOP}^{\mathrm{op}} \to \mathsf{Sets}$$
has been introduced in [BS05] for $n = 1$ and in [BRS] for $n \geq 1$ in connection with the study of topological T-duality.

The main result of [BS05] in the case $n = 1$ is the construction of an involutive natural isomorphism $T\colon P \to P$, the T-duality isomorphism, which associates to each isomorphism class of pairs $t \in P(B)$ the class of the T-dual pair $\hat{t} := T(t) \in P(B)$.

In the higher-dimensional case $n \geq 2$ the situation is more complicated. First of all, not every pair $t \in P(B)$ admits a T-dual. Furthermore, a T-dual, if it exists, may not be unique. The results of [BRS] are based on the notion of a T-duality triple. In the following we recall the definition of a T-duality triple.

6.1.3 As a preparation we recall the following two facts. Let $\pi\colon E \to B$ be a $\mathbb{T}^n$-principal bundle. Associated to the decomposition of the global section functor $\Gamma(E, \dots)$ as composition $\Gamma(E, \dots) = \Gamma(B, \dots) \circ \pi_*\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/E \to \mathsf{Ab}$ there is a decreasing filtration

$$\cdots \subseteq F^n H^*(E; \underline{\mathbb{Z}}) \subseteq F^{n-1} H^*(E; \underline{\mathbb{Z}}) \subseteq \cdots \subseteq F^0 H^*(E; \underline{\mathbb{Z}})$$

of the cohomology groups $H^*(E; \underline{\mathbb{Z}}) \cong R^*\Gamma(E; \underline{\mathbb{Z}})$ and a Serre spectral sequence with second term $E_2^{i,j} \cong H^i(B; R^j\pi_*\underline{\mathbb{Z}})$ which converges to $\mathsf{Gr} H^*(E; \underline{\mathbb{Z}})$.

6.1.4 The isomorphism classes of gerbes over X with band $\underline{\mathbb{T}}_{|X}$ are classified by $H^2(X;\underline{\mathbb{T}}) \cong H^3(X;\underline{\mathbb{Z}})$. The class associated to such a gerbe $H \to X$ is called Dixmier–Douady class $d(X) \in H^3(X;\underline{\mathbb{Z}})$.

If $H \to X$ is a gerbe with band $\underline{\mathbb{T}}_{|X}$ over some space X, then the automorphisms of H (as a gerbe with band $\underline{\mathbb{T}}_{|X}$) are classified by $H^1(X;\underline{\mathbb{T}}) \cong H^2(X;\underline{\mathbb{Z}})$.

6.1.5 In the definition of a T-duality triple we furthermore need the following notation. We let $y \in H^1(\mathbb{T};\underline{\mathbb{Z}})$ be the canonical generator. If $\mathsf{pr}_i \colon \mathbb{T}^n \to \mathbb{T}$ is the projection onto the ith factor, then we set $y_i := \mathsf{pr}_i^* y \in H^1(\mathbb{T}^n;\underline{\mathbb{Z}})$. Let $E \to B$ be a $\mathbb{T}^n$-principal bundle and $b \in B$. We consider its fibre E_b over b. Choosing a base point $e \in E_b$ we use the $\mathbb{T}^n$-action in order to fix a homeomorphism $a_e \colon E \xrightarrow{\sim} \mathbb{T}^n$ such that $a_e(et) = t$ for all $t \in \mathbb{T}^n$. The classes $x_i := a_e^*(y_i) \in H^1(E_b;\underline{\mathbb{Z}})$ are independent of the choice of the base point. Applying this definition to the bundle $\hat{E} \to B$ below gives the classes $\hat{x}_i \in H^1(\hat{E}_b;\underline{\mathbb{Z}})$ used in Definition 6.2.

Definition 6.2. A T-duality triple $t := ((E,H),(\hat{E},\hat{H}),u)$ over B consists of two pairs $(E,H),(\hat{E},\hat{H})$ over B and an isomorphism $u \colon \hat{p}^*\hat{H} \to p^*H$ of gerbes with band $\underline{\mathbb{T}}_{|E\times_B\hat{E}}$ defined by the diagram

$$\begin{tikzcd}
 & p^*H \arrow[dl] \arrow[dr] & & \hat{p}^*\hat{H} \arrow[ll, "u"'] \arrow[dl] \arrow[dr] & \\
H \arrow[dr] & & E\times_B\hat{E} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & & \hat{H} \arrow[dl] \\
 & E \arrow[dr, "\pi"'] & & \hat{E} \arrow[dl, "\hat{\pi}"] & \\
 & & B, & &
\end{tikzcd} \tag{49}$$

where all squares are two-cartesian. The following conditions are required:

1. The Dixmier–Douady classes of the gerbes satisfy $d(H) \in F^2H^3(E;\underline{\mathbb{Z}})$ and $d(\hat{H}) \in F^2H^3(\hat{E};\underline{\mathbb{Z}})$.
2. The isomorphism of gerbes u satisfies the condition [BRS, (2.7)] which says the following. If we restrict the diagram (49) to a point $b \in B$, then we can trivialize the restrictions of gerbes $H_{|E_b}$, $\hat{H}_{|\hat{E}_b}$ to the fibres $E_b, \hat{E}_b$ of the $\mathbb{T}^n$-bundles over b such that the induced isomorphism of trivial gerbes u_b over $E_b \times \hat{E}_b$ is classified by $\sum_{i=1}^n \mathsf{pr}_{E_b}^* x_i \cup \mathsf{pr}_{\hat{E}}^* \hat{x}_i \in H^2(E_b \times \hat{E}_b;\underline{\mathbb{Z}})$.

6.1.6 There is a natural notion of an isomorphism of T-duality triples. For a map $f \colon B' \to B$ and a T-duality triple over B there is a natural construction of a pull-back triple over B'.

Definition 6.3. We let $\mathsf{Triple}(B)$ denote the set of isomorphism classes of T-duality triples over B. For $f: B' \to B$ we let $\mathsf{Triple}(f): \mathsf{Triple}(B) \to \mathsf{Triple}(B')$ be the map induced by the pull-back of T-duality triples.

6.1.7 In this way we define a functor

$$\mathsf{Triple}: \mathsf{TOP}^{\mathrm{op}} \to \mathsf{Sets}.$$

This functor comes with specializations

$$s, \hat{s}: \mathsf{Triple} \to P$$

given by $s((E,H),(\widehat{E},\widehat{H}),u) := (E,H)$ and $\hat{s}((E,H),(\widehat{E},\widehat{H}),u) = (\widehat{E},\widehat{H})$.

Definition 6.4. A pair (E,H) is called dualizable if there exists a triple $t \in \mathsf{Triple}(B)$ such that $s(t) \cong (E,H)$. The pair $\hat{s}(t) = (\widehat{E},\widehat{H})$ is called a T-dual of (E,H).

Thus the choice of a triple $t \in s^{-1}(E,H)$ encodes the necessary choices in order to fix a T-dual. One of the main results of [BRS] is the following characterization of dualizable pairs.

Theorem 6.5. *A pair (E,H) is dualizable in the sense of Definition* 6.4 *if and only if* $d(H) \in F^2H^3(E;\underline{\mathbb{Z}})$.

Further results of [BRS, Theorem 2.24] concern the classification of the set of duals of a given pair (E,H).

6.1.8 For the purpose of the present paper it is more natural to interpret the isomorphism of gerbes $u: \hat{p}^*\widehat{H} \to p^*H$ in a T-duality triple $((E,H),(\widehat{E},\widehat{H}),u)$ in a different, but equivalent manner. To this end we introduce the notion of a dual gerbe.

6.1.9 First we recall the definition of the tensor product $w: H \otimes_X H' \to X$ of gerbes $u: H \to X$ and $u': H' \to X$ with band $\underline{\mathbb{T}}_{|X}$ over $X \in \mathbf{S}$. Consider first the fibre product of stacks $(u,u'): H \times_X H' \to X$. Let $T \in \mathbf{S}/X$. An object $s \in H \times_X H'(T)$ is given by a triple (t,t',ϕ) of objects $t \in H(T)$, $t' \in H'(T)$ and an isomorphism $\phi: u(t) \to u'(t')$. By the definition of a $\mathbb{T}_{|X}$-banded gerbe the group of automorphisms of t relative to u is the group $\mathsf{Aut}_{H(T)/\mathsf{rel}(u)}(t) \cong \underline{\mathbb{T}}(T)$. We thus have an isomorphism $\mathsf{Aut}_{H\times_X H'/\mathsf{rel}((u,u'))}(s) \cong \underline{\mathbb{T}}(T) \times \underline{\mathbb{T}}(T)$. Similarly, for $s_0, s_1 \in H \times_X H'(T)$ the set $\mathsf{Hom}_{H\times_X H'/\mathsf{rel}((u,u'))}(s_0,s_1)$ is a torsor over $\underline{\mathbb{T}}(T) \times \underline{\mathbb{T}}(T)$.

We now define a prestack $H \otimes_X^p H'$. By definition the groupoid $H \otimes_X^p H'(T)$ has the same objects as $H \times_X H'(T)$, but the morphism sets are factored by the anti-diagonal $\underline{\mathbb{T}}(T) \overset{\mathsf{antidiag}}{\subseteq} \underline{\mathbb{T}}(T) \times \underline{\mathbb{T}}(T)$, i.e.

$$\mathsf{Hom}_{H\otimes_X^p H'/\mathsf{rel}(w)}(s_0,s_1) = \mathsf{Hom}_{H\times_X H'/\mathsf{rel}((u,u'))}(s_0,s_1)/\mathsf{antidiag}(\underline{\mathbb{T}}(T)).$$

The stack $H \otimes_X H'$ is defined as the stackification of the prestack $H \otimes_X^p H'$. It is again a gerbe with band $\underline{\mathbb{T}}_{|X}$. We furthermore have the following relation of Dixmier–Douady classes.

$$d(H) + d(H') = d(H \otimes_X H').$$

6.1.10 The sheaf $\underline{\mathbb{T}}_{|X} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/X$ gives rise to the stack $\mathcal{B}\underline{\mathbb{T}}_{|X}$ (see 2.3.6). Then $X \times \mathcal{B}\underline{\mathbb{T}}_{|X} \to X$ is the trivial $\mathbb{T}$-banded gerbe over X. Let $H \to X$ be a gerbe with band $\underline{\mathbb{T}}_{|X}$.

Definition 6.6. A dual of the gerbe $H \to X$ is a pair $(H' \to X, \psi)$ of a gerbe $H' \to X$ and an isomorphism of gerbes $\psi\colon H \otimes_X H' \to X \times \mathcal{B}\underline{\mathbb{T}}_{|X}$.

Every gerbe $H \to X$ with band $\underline{\mathbb{T}}_{|X}$ admits a preferred dual $H^{\mathrm{op}} \to X$ which we call the opposite gerbe. The underlying stack of H^{op} is H, but we change its structure of a $\underline{\mathbb{T}}_{|X}$-banded gerbe using the inversion automorphism ${}^{-1}\colon \underline{\mathbb{T}}_{|X} \xrightarrow{\sim} \underline{\mathbb{T}}_{|X}$.

If $(H_0' \to X, \psi_0)$ and $(H_1' \to X, \psi)$ are two duals, then there exists a unique isomorphism class of isomorphisms $H_0' \to H_1'$ of gerbes such that the induced diagram

$$\begin{tikzcd}
H \otimes_X H_0' \arrow[rr] \arrow[dr, "\psi_0"'] & & H \otimes_X H_1' \arrow[dl, "\psi_1"] \\
& X \times \mathcal{B}\underline{\mathbb{T}}_{|X} &
\end{tikzcd}$$

can be filled with a two-isomorphism. Note that if $H' \to X$ is underlying gerbe of the dual of $H \to X$, then we have the relation of Dixmier–Douady classes

$$d(H) = -d(H').$$

Lemma 6.7. *Let $\hat{H} \to X$ and $H \to X$ be gerbes with band $\underline{\mathbb{T}}_{|X}$. There is a natural bijection between the sets of isomorphism classes of isomorphisms $\hat{H} \to H$ of gerbes over X and isomorphism classes of isomorphisms of $\underline{\mathbb{T}}_{|X}$-banded gerbes $\hat{H} \otimes H^{\mathrm{op}} \to X \times \mathcal{B}\underline{\mathbb{T}}_{|X}$.*

Proof. An isomorphism $\hat{H} \to H$ induces an isomorphism $\hat{H} \otimes_X H^{\mathrm{op}} \to H \otimes_X H^{\mathrm{op}} \xrightarrow{\psi} X \times \mathcal{B}\underline{\mathbb{T}}_{|X}$. On the other hand, an isomorphism $\hat{H} \otimes_X H^{\mathrm{op}} \to X \times \mathcal{B}\underline{\mathbb{T}}_{|X}$ presents $\hat{H} \to X$ as a dual of $H^{\mathrm{op}} \to X$. Since $\psi\colon H \otimes_X H^{\mathrm{op}} \to X \times \mathcal{B}\underline{\mathbb{T}}_{|X}$ presents H as a dual of H^{op} we have a preferred isomorphism class of isomorphisms $\hat{H} \to H$, too. □

6.1.11 In view of Lemma 6.7, in Definition 6.2 of a T-duality triple $((E, H), (\hat{E}, \hat{H}), u)$ we can consider u as an isomorphism

$$\hat{p}^* \hat{H}^{\mathrm{op}} \otimes_X p^* H \to E \times_B \hat{E} \times \mathcal{B}\underline{\mathbb{T}}_{|E \times_B \hat{E}}.$$

The condition 6.2 2. can be rephrased as follows. After restriction to a point $b \in B$ we can find isomorphisms

$$v\colon \underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{T}} \xrightarrow{\sim} H_b, \quad \hat{v}\colon \underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{T}} \xrightarrow{\sim} \hat{H}_b. \tag{50}$$

After a choice of $* \to \mathcal{B}\underline{\mathbb{T}}$ in order to define the map s below we obtain a map $r\colon \underline{\mathbb{T}}^n \times \underline{\mathbb{T}}^n \to \mathcal{B}\underline{\mathbb{T}}$ by the following diagram.

$$\begin{tikzcd}
\hat{H}_b^{\mathrm{op}} \times H_b \arrow[r] & \hat{H}_b^{\mathrm{op}} \otimes_{E_b \times \hat{E}_b} H_b \arrow[r, "u_b"] & E_b \times \hat{E}_b \times \mathcal{B}\underline{\mathbb{T}} \arrow[r, "\mathsf{pr}_{\mathcal{B}\underline{\mathbb{T}}}"] & \mathcal{B}\underline{\mathbb{T}} \\
(\underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{T}}^{\mathrm{op}}) \times (\underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{T}}) \arrow[u, "{(\hat{v},v)}"] & & & \\
\underline{\mathbb{T}}^n \times \underline{\mathbb{T}}^n \arrow[u, "s"] \arrow[uurrr, bend right, "r"'] & & &
\end{tikzcd}$$

The condition 6.2, 2. is now equivalent to the condition that we can choose the isomorphisms $v, \hat{v}$ in (50) such that

$$r^*(z) = \sum_{i=1}^{n} \mathsf{pr}_{1,i}^* x \cup \mathsf{pr}_{2,i}^* x,$$

where $z \in H^2(\mathcal{B}\underline{\mathbb{T}};\underline{\mathbb{Z}})$ and $x \in H^1(\underline{\mathbb{T}};\underline{\mathbb{Z}})$ are the canonical generators, and $\mathsf{pr}_{k,i},\colon \underline{\mathbb{T}}^n \times \underline{\mathbb{T}}^n \xrightarrow{\mathsf{pr}_k} \mathbb{T}^n \xrightarrow{\mathsf{pr}_i} \underline{\mathbb{T}}$, $k = 1, 2$, $i = 1, \dots, n$ are projections onto the factors.

6.1.12 Important topological invariants of T-duality triples are the Chern classes of the underlying $\mathbb{T}^n$-principal bundles. For a triple $t = ((E, H), (\hat{E}, \hat{H}), u)$ we define

$$c(t) := c(E), \quad \hat{c}(t) := c(\hat{E}).$$

These classes belong to $H^2(B; \underline{\mathbb{Z}}^n)$.

6.2 Torus bundles, torsors and gerbes

6.2.1 In this subsection we review various interpretations of the notion of a $\mathbb{T}^n$-principal bundle.

6.2.2 Let G be a topological group. Let us start with giving a precise definition of a G-principal bundle.

Definition 6.8. A G-principal bundle $\mathcal{E}$ over a space B consists of a map of spaces $\pi\colon E \to B$ which admits local sections together with a fibrewise right action $E \times G \to E$ such that the natural map

$$E \times G \to E \times_B E, \quad (e, t) \mapsto (e, et)$$

is an homeomorphism. An isomorphism of G-principal bundles $\mathcal{E} \to \mathcal{E}'$ is a G-equivariant map $E \to E'$ of spaces over B.

By $\mathsf{Prin}_B(G)$ we denote the category of G-principal bundles over B. The set $H^0(\mathsf{Prin}_B(G))$ of isomorphism classes in $\mathsf{Prin}_B(G)$ is in one-to-one correspondence with homotopy classes $[B, BG]$ of maps from B to the classifying space BG of G. If we fix a universal bundle $EG \to BG$, then the bijection

$$[B, BG] \xrightarrow{\sim} H^0(\mathsf{Prin}_B(G))$$

is given by

$$[f : B \to BG] \mapsto [B \times_{f,BG} EG \to B].$$

6.2.3 We now specialize to $G := \mathbb{T}^n$. The classifying space $B\mathbb{T}^n$ of $\mathbb{T}^n$ has the homotopy type of the Eilenberg–MacLane space $K(\mathbb{Z}^n, 2)$. We thus have a natural isomorphism

$$H^2(B; \mathbb{Z}^n) \overset{\text{Def.}}{\cong} [B, K(\mathbb{Z}^n, 2)] \cong [B, B\mathbb{T}^n] \cong H^0(\mathsf{Prin}_B(\mathbb{T}^n)).$$

So, $\mathbb{T}^n$-principal bundles are classified by the characteristic class $c(\mathcal{E}) \in H^2(B; \mathbb{Z}^n)$. Using the decomposition

$$H^2(B; \mathbb{Z}^n) \cong \underbrace{H^2(B; \mathbb{Z}) \oplus \dots H^2(B; \mathbb{Z})}_{n \text{ summands}}$$

we can write

$$c(\mathcal{E}) = (c_1(\mathcal{E}), \dots, c_n(\mathcal{E})).$$

Definition 6.9. The class $c(\mathcal{E})$ is called the Chern class of $\mathcal{E}$. The $c_i(\mathcal{E})$ are called the components of $c(\mathcal{E})$.

In fact, if $n = 1$, then $c(\mathcal{E})$ is the classical first Chern class of the $\mathbb{T}$-principal bundle $\mathcal{E}$.

6.2.4 Let $\mathbf{S}$ be a site and $F \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ be a sheaf of abelian groups.

Definition 6.10. An F-torsor $\mathcal{T}$ is a sheaf of sets $\mathcal{T} \in \mathsf{Sh}\mathbf{S}$ together with an action $\mathcal{T} \times F \to \mathcal{T}$ such that the natural map $\mathcal{T} \times F \to \mathcal{T} \times \mathcal{T}$ is an isomorphism of sheaves. An isomorphism of F-torsors $\mathcal{T} \to \mathcal{T}'$ is an isomorphism of sheaves which commutes with the action of F.

By $\mathsf{Tors}(F)$ we denote the category of F-torsors.

6.2.5 In 2.3.4 we have introduced the category $\mathsf{EXT}(\underline{\mathbb{Z}}, F)$ whose objects are extensions of sheaves of groups

$$\mathcal{W} : 0 \to F \to W \xrightarrow{w} \underline{\mathbb{Z}} \to 0, \tag{51}$$

and whose morphisms are isomorphisms of extensions. We have furthermore defined the equivalence of categories

$$U : \mathsf{EXT}(\underline{\mathbb{Z}}, F) \xrightarrow{\sim} \mathsf{Tors}(F)$$

given by $U(\mathcal{W}) := w^{-1}(1)$.

6.2.6 Consider an extension $\mathcal{W}$ as in (51) and apply $\mathsf{Ext}_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, \dots)$. We get the following piece of the long exact sequence

$$\begin{aligned} \dots \to \mathsf{Hom}_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, W) &\to \mathsf{Hom}_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) \\ &\xrightarrow{\delta_{\mathcal{W}}} \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, F) \to \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, W) \to \cdots . \end{aligned}$$

Let $1 \in \mathsf{Hom}_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}})$ be the identity and set

$$e(\mathcal{W}) := \delta_{\mathcal{W}}(1) \in \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, F).$$

Recall that $H^0(\mathsf{EXT}(\underline{\mathbb{Z}}, F))$ denotes the set of isomorphism classes of the category $\mathsf{EXT}(\underline{\mathbb{Z}}, F)$. The following lemma is well-known (see e.g. [Yon60]).

Lemma 6.11. *The map*

$$\mathsf{EXT}(\underline{\mathbb{Z}}, F) \ni \mathcal{W} \mapsto e(\mathcal{W}) \in \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, F)$$

induces a bijection

$$e \colon H^0(\mathsf{EXT}(\underline{\mathbb{Z}}, F)) \xrightarrow{\sim} \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, F).$$

In view of 6.2.5 we have natural bijections

$$H^0(\mathsf{Tors}(F)) \overset{U}{\cong} H^0(\mathsf{EXT}(\underline{\mathbb{Z}}, F)) \overset{e}{\cong} \mathsf{Ext}^1_{\mathsf{ShAb}\mathbf{S}}(\underline{\mathbb{Z}}, F). \tag{52}$$

6.2.7 Let G be an abelian topological group and consider a principal G-bundle $\mathcal{E}$ over B with underlying map $\pi \colon E \to B$. By $\mathcal{T}(\mathcal{E}) := \underline{E \to B} \in \mathsf{Sh}\mathbf{S}/B$ (see 3.2.3) we denote its sheaf of sections. The right action of G on E induces an action $\mathcal{T}(\mathcal{E}) \times \underline{G}_{|B} \to \mathcal{T}(\mathcal{E})$.

Lemma 6.12. *$\mathcal{T}(\mathcal{E})$ is a $\underline{G}_{|B}$-torsor.*

Proof. This follows from the following fact. If $X \to B$ and $Y \to B$ are two maps, then

$$\underline{X \times_B Y \to B} \cong \underline{X \to B} \times \underline{Y \to B}$$

in $\mathsf{Sh}\mathbf{S}/B$. We apply this to the isomorphism

$$E \times_B (B \times G) \cong E \times G \cong E \times_B E$$

of spaces over B in order to get the isomorphism

$$\mathcal{T}(\mathcal{E}) \times \underline{G}_{|B} \xrightarrow{\sim} \mathcal{T}(\mathcal{E}) \times \mathcal{T}(\mathcal{E}). \qquad \square$$

6.2.8 The construction $\mathcal{E} \mapsto \mathcal{T}(\mathcal{E})$ refines to a functor

$$\mathcal{T}\colon \mathsf{Prin}_B(G) \to \mathsf{Tors}(\underline{G}_{|B})$$

from the category of G-principal bundles $\mathsf{Prin}_B(G)$ over B to the category of $\underline{G}_{|B}$-torsors $\mathsf{Tors}(\underline{G}_{|B})$ over B.

Lemma 6.13. *The functor*

$$\mathcal{T}\colon \mathsf{Prin}_B(G) \to \mathsf{Tors}(\underline{G}_{|B})$$

is an equivalence of categories.

Proof. It is a consequence of the Yoneda Lemma that $\mathcal{T}$ is an isomorphism on the level of morphism sets. It remains to show that the underlying sheaf T of a $\underline{G}_{|B}$-torsor is representable by a G-principal bundle. Since T is locally isomorphic to $\underline{G}_{|B}$ this is true locally. The local representing objects can be glued to a global representing object. □

We can now prolong the chain of bijections (52) to

$$\begin{aligned}[B, BG] &\cong H^0(\mathsf{Prin}_B(G)) \cong H^0(\mathsf{Tors}(\underline{G}_{|B})) \\ &\cong H^0(\mathsf{EXT}(\underline{\mathbb{Z}}_{|B}, \underline{G}_{|B})) \cong \mathsf{Ext}^1_{\mathsf{ShAb}\,\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{G}_{|B}) \cong H^1(B; \underline{G}).\end{aligned} \tag{53}$$

6.2.9 Let $\mathbf{S}$ be some site and $F \in \mathbf{S}$ be a sheaf of abelian groups. By $\mathsf{Gerbe}(F)$ we denote the two-category of gerbes with band F over $\mathbf{S}$. It is well-known that isomorphism classes of gerbes with band F are classified by $\mathsf{Ext}^2_{\mathsf{ShAb}\,\mathbf{S}}(\underline{\mathbb{Z}}; F)$, i.e. there is a natural bijection

$$d\colon H^0(\mathsf{Gerbe}(F)) \xrightarrow{\sim} \mathsf{Ext}^2_{\mathsf{ShAb}\,\mathbf{S}}(\underline{\mathbb{Z}}; F).$$

6.2.10 Let $H \to G$ be a homomorphism of topological abelian groups with kernel $K := \ker(H \to G)$. Our main example is $\mathbb{R}^n \to \mathbb{T}^n$ with kernel $\mathbb{Z}^n$.

Definition 6.14. Let T be a space. An H-reduction of a G-principal bundle $\mathcal{E} = (E \to B)$ on T is a diagram

$$\begin{tikzcd}(F,\phi)\colon & F \arrow[r, "\phi"] \arrow[d] & E \arrow[d] \\ & T \arrow[r] & B\end{tikzcd},$$

where $F \to T$ is an H-principal bundle, and ϕ is H-equivariant, where H acts on E via $H \to G$. An isomorphism $(F, \phi) \to (F', \phi')$ of H-reductions over T is an isomorphism $f\colon F \to F'$ of H-principal bundles such that $\phi' \circ f = \phi$. Let $R^{\mathcal{E}}_H(T)$ denote the category of H-reductions of $\mathcal{E}$.

Observe that the group of automorphisms of every object in $R^{\mathcal{E}}_H(T)$ is isomorphic to $\mathsf{Map}(T, K)^\delta$ (the superscript δ indicates that we take the underlying set). For a map $u\colon T \to T'$ over B there is a natural pull-back functor $R^{\mathcal{E}}_H(u)\colon R^{\mathcal{E}}_H(T') \to R^{\mathcal{E}}_H(T)$.

Let $\mathbf{S}$ be a site as in 3.1.2 and assume that K, G, H, B belong to $\mathbf{S}$. In this case it is easy to check that

$$T \mapsto R^{\mathcal{E}}_H(T)$$

is a gerbe with band $\underline{K}_{|B}$ on $\mathbf{S}/B$.

6.2.11 Let $\pi : E \to B$ be a G-principal bundle. Note that the pull-back $\pi^* E \to E$ has a canonical section and is therefore trivialized. A trivialized G-bundle has a canonical H-reduction. In other words, there is a canonical map of stacks over B

$$\mathsf{can}\colon E \to R^{\mathcal{E}}_H. \tag{54}$$

Note that an object of $E(T)$ is a map $T \to E$; to this we assign the pull-back of the canonical H-reduction of $\pi^* E$.

6.2.12 The construction $\mathsf{Prin}_B(G) \ni \mathcal{E} \mapsto R^{\mathcal{E}}_H \in \mathsf{Gerbe}(\underline{K})$ is functorial in $\mathcal{E}$ and thus induces a natural map of sets of isomorphism classes

$$r\colon H^0(\mathsf{Prin}_B(G)) \to H^0(\mathsf{Gerbe}(\underline{K})).$$

Lemma 6.15. *If $H \to G$ is surjective and has local sections, and $H^1(B, \underline{H}) \cong H^2(B, \underline{H}) \cong 0$, then*

$$r\colon H^0(\mathsf{Prin}_B(G)) \to H^0(\mathsf{Gerbe}(\underline{K}))$$

is a bijection.

Proof. The exact sequence $0 \to K \to H \to G \to 0$ induces by 3.4 an exact sequence

$$0 \to \underline{K} \to \underline{H} \to \underline{G} \to 0$$

of sheaves. We consider the following segment of the associated long exact sequence in cohomology:

$$\cdots \to H^1(B; \underline{H}) \to H^1(B; \underline{G}) \stackrel{\delta}{\to} H^2(B; \underline{K}) \to H^2(B; \underline{H}) \to \cdots .$$

By our assumptions $\delta\colon H^1(B; \underline{G}) \to H^2(B; \underline{K})$ is an isomorphism. One can check that the following diagram commutes:

$$\begin{tikzcd} H^0(\mathsf{Prin}_B(G)) \arrow[r, "r"] \arrow[d] & H^0(\mathsf{Gerbe}(\underline{K}_{|B})) \arrow[d, "d"] \\ H^1(B; \underline{G}) \arrow[r, "\delta"] & H^2(B; \underline{K}). \end{tikzcd} \tag{55}$$

This implies the result since the vertical maps are isomorphisms. $\square$

Note that we can apply this lemma in our main example where $H = \mathbb{R}^n$ and $G = \mathbb{T}^n$. In this case the diagram (55) is the equality

$$c(E) = d(R^E_{\underline{\mathbb{Z}}^n_{|B}}) \tag{56}$$

in $H^2(B; \underline{\mathbb{Z}}^n)$.

6.3 Pairs and group stacks

6.3.1 Let $\mathcal{E}$ be a principal $\mathbb{T}^n$-bundle over B, or equivalently by (53), an extension $\mathcal{E} \in \mathsf{Sh_{Ab}}\mathbf{S}/B$

$$0 \to \underline{\mathbb{T}}^n_{|B} \to \mathcal{E} \to \underline{\mathbb{Z}}_{|B} \to 0 \tag{57}$$

of sheaves of abelian groups. Let $\tilde{c}(\mathcal{E}) \in \mathsf{Ext}^1_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}; \underline{\mathbb{T}}^n_{|B})$ be the class of this extension. Under the isomorphism

$$\mathsf{Ext}^1_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}; \underline{\mathbb{T}}^n_{|B}) \cong H^1(B; \underline{\mathbb{T}}^n) \cong H^2(B; \underline{\mathbb{Z}}^n)$$

it corresponds to the Chern class $c(\mathcal{E})$ of the principal $\mathbb{T}^n$-bundle introduced in 6.9.

6.3.2 We let $Q_{\mathcal{E}} = \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S})}(\mathcal{E}, \underline{\mathbb{T}}_{|B})$ (see Lemma 2.20 for the notation) denote the set of equivalence classes of Picard stacks $P \in \mathsf{PIC}(B)$ with isomorphisms

$$H^0(P) \xrightarrow{\cong} \mathcal{E}, \quad H^{-1}(P) \xrightarrow{\cong} \underline{\mathbb{T}}_B.$$

By Lemma 2.20 we have a bijection

$$\mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\mathcal{E}, \underline{\mathbb{T}}_{|B}) \cong Q_{\mathcal{E}}.$$

This bijection induces a group structure on $Q_{\mathcal{E}}$ which we will use in the discussion of long exact sequences below. We will not need a description of this group structure in terms of the Picard stacks themselves.

We apply $\mathsf{Ext}^*_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\dots, \underline{\mathbb{T}}_{|B})$ to the sequence (57) and get the following segment of a long exact sequence

$$\begin{aligned} &\mathsf{Ext}^1_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B}, \underline{\mathbb{T}}_{|B}) \xrightarrow{\alpha} \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B}) \to Q_{\mathcal{E}} \\ &\qquad \to \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B}, \underline{\mathbb{T}}_{|B}) \xrightarrow{\beta} \mathsf{Ext}^3_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B}). \end{aligned} \tag{58}$$

The maps α, β are given by the left Yoneda product with the class

$$\tilde{c}(\mathcal{E}) \in \mathsf{Ext}^1_{\mathsf{Sh_{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}; \underline{\mathbb{T}}^n_{|B}).$$

6.3.3 In this paragraph we identify the extension groups in the sequence (58) with sheaf cohomology. For a site $\mathbf{S}$ with final object $*$ and $X, Y \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}$ we have a local-global spectral sequence with second term

$$E_2^{p,q} \cong R^p\Gamma(*; \underline{\mathsf{Ext}}^q_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(X,Y))$$

which converges to $\mathsf{Ext}^{p+q}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(X,Y)$.

We apply this first to the sheaves $\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B$. The final object of the site $\mathbf{S}/B$ is $\mathsf{id}\colon B \to B$. We have $\underline{\mathsf{Ext}}^q_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B}) \cong 0$ for $q \geq 1$ so that this spectral sequence degenerates at the second page and gives

$$\begin{aligned}\mathsf{Ext}^p_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B}) &\cong H^p(B; \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B}))\\ &\cong H^p(B; \underline{\mathbb{T}}_{|B}) \cong H^p(B; \underline{\mathbb{T}}) \cong H^{p+1}(B; \underline{\mathbb{Z}}).\end{aligned}$$

Since the group $\mathbb{T}^n$ is admissible by Theorem 4.30, and by Corollary 3.14 we have $\underline{\mathsf{Ext}}^q_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B}, \underline{\mathbb{T}}_{|B}) \cong 0$ for $q = 1, 2$, we get

$$\begin{aligned}\mathsf{Ext}^p_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B}, \underline{\mathbb{T}}_{|B}) &\cong H^p(B; \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B}, \underline{\mathbb{T}}_{|B}))\\ &\cong H^p(B; \underline{\mathsf{Hom}}_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}}(\underline{\mathbb{T}}^n, \underline{\mathbb{T}})_{|B})\\ &\cong H^p(B; \underline{\mathbb{Z}}^n_{|B})\\ &\cong H^p(B; \underline{\mathbb{Z}}^n)\end{aligned}$$

for $p = 1, 2$. Therefore the sequence (58) has the form

$$H^1(B; \underline{\mathbb{Z}}^n) \xrightarrow{\alpha} H^3(B; \underline{\mathbb{Z}}) \to Q_{\mathcal{E}} \xrightarrow{\hat{c}} H^2(B; \underline{\mathbb{Z}}^n) \xrightarrow{\beta} H^4(B; \underline{\mathbb{Z}}). \tag{59}$$

In this picture the maps α, β are both given by the cup-product with the Chern class $c(\mathcal{E}) \in H^2(B; \underline{\mathbb{Z}}^n)$, i.e. $\alpha((x_i)) = \sum x_i \cup c_i(\mathcal{E})$.

6.3.4 Recall from 6.1.3 the decreasing filtration $(F^k H^*(E; \underline{\mathbb{Z}}))_{k \geq 0}$ and the spectral sequence associated to the decomposition of functors $\Gamma(E; \dots) = \Gamma(B, \dots) \circ p_*\colon \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/E \to \mathsf{Ab}$. This spectral sequence converges to $\mathsf{Gr}H^*(E; \underline{\mathbb{Z}})$. Its second page is given by $E_2^{i,j} \cong H^i(B; R^j p_* \underline{\mathbb{Z}})$. The edge sequence for $F^2H^3(E; \underline{\mathbb{Z}})$ has the form

$$\begin{aligned}&\ker(d_2^{0,2}\colon E_2^{0,2} \to E_2^{2,1}) \xrightarrow{d_3^{1,1}} \mathsf{coker}(d_2^{1,1}\colon E_2^{1,1} \to E_2^{3,0})\\ &\qquad \to F^2H^3(E; \underline{\mathbb{Z}}) \to (E_2^{2,1}/\mathsf{im}(d_2^{0,2}\colon E_2^{0,2} \to E_2^{2,1})) \xrightarrow{d_2^{2,1}} E_2^{4,0}.\end{aligned}$$

We now make this explicit. Since the fibre of p is an n-torus, $R^0 p_* \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}$, $R^1 p_* \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}^n$, $R^2 p_* \underline{\mathbb{Z}} \cong \Lambda^2 \underline{\mathbb{Z}}^n$. The differential d_2 can be expressed in terms of the Chern class $c(\mathcal{E})$. We get the following web of exact sequences, where K, A, B are defined as the appropriate kernels and cokernels.

$$
\begin{tikzcd}
 & & & K \arrow[d] \arrow[dddlll, bend right, "="'] & \\
 & H^1(B;\underline{\mathbb{Z}}^n) \arrow[d, "\alpha"] & & H^0(B;\Lambda^2\underline{\mathbb{Z}}^n) \arrow[d, "i_{c(\mathcal{E})}"] & \\
 & H^3(B;\underline{\mathbb{Z}}) \arrow[d] & & H^2(B;\underline{\mathbb{Z}}^n) \arrow[d] \arrow[dr, "\beta"] & \\
K \arrow[r] \arrow[ur, dotted, "s"] & A \arrow[r] \arrow[d] & F^2H^3(E;\underline{\mathbb{Z}}) \arrow[r] & B \arrow[r, "\bar{\beta}"] \arrow[d] & H^4(B;\underline{\mathbb{Z}}) \\
 & 0 & & 0 &
\end{tikzcd}
\tag{60}
$$

Since K as a subgroup of the free abelian group $H^0(B, \Lambda^2\underline{\mathbb{Z}}^n)$ is free we can choose a lift s as indicated.

6.3.5 We now define a map (the underlying pair map)

$$\mathrm{up}\colon Q_{\mathcal{E}} \to P(B)$$

as follows. Any Picard stack $P \in \mathsf{PIC}(\mathbf{S}/B)$ comes with a natural map of stacks $P \to H^0(P)$ (where on the right-hand side we consider a sheaf of sets as a stack). If $P \in Q_{\mathcal{E}}$, then $H^0(P) \cong \mathcal{E}$ has a natural map to $\underline{\mathbb{Z}}_{|B}$ (see (57)). We define the stack G by the pull-back in stacks on $\mathbf{S}/B$

$$
\begin{tikzcd}
G \arrow[r] \arrow[d] \arrow[dd, bend right=60] & P \arrow[d] \arrow[dd, bend left=60] \\
E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\underline{\{1\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B}.
\end{tikzcd}
$$

All squares are two-cartesian, and the outer square is the composition of the two inner squares. We have omitted to write the canonical two-isomorphisms. By construction E is a sheaf of $\underline{\mathbb{T}}^n$-torsors (see 6.2.5), i.e. by Lemma 6.13 a $\mathbb{T}^n$-principal bundle, and $G \to E$ is a gerbe with band $\underline{\mathbb{T}}$. We set

$$\mathrm{up}(P) := (E, G).$$

6.3.6 Recall Definition 6.4 of a dualizable pair.

Lemma 6.16. *If* $P \in Q_{\mathcal{E}}$, *then* $\mathrm{up}(P) \in P(B)$ *is dualizable.*

Proof. Let $(E, H) := \mathrm{up}(E)$. In view of Theorem 6.5 we must show that $d(H) \in F^2 H^3(E; \underline{\mathbb{Z}})$. For $k \in \mathbb{Z}$ we define $G(k) \to E(k)$ by the two-cartesian diagrams

$$\begin{tikzcd} G(k) \arrow[r] \arrow[d] \arrow[dd, bend right] & P \arrow[d] \arrow[dd, bend left] \\ E(k) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \underline{\{k\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B}. \end{tikzcd} \tag{61}$$

The group structure of $\mathcal{E}$ induces maps

$$\mu\colon E(k) \times_B E(m) \to E(k+m) \tag{62}$$

On fibres appropriately identified with $\mathbb{T}^n$, this map is the usual group structure on $\mathbb{T}^n$. Since P is a Picard stack these multiplications are covered by $\hat{\mu}\colon G(k) \times G(m) \to G(k+m)$. The isomorphism class of $G(k)$ therefore must satisfy

$$\mathsf{pr}^*_{E(k)} G(k) \otimes \mathsf{pr}^*_{E(m)} G(m) \cong \mu^* G(k+m) \in \mathsf{Gerbe}(E(k) \times_B E(m)). \tag{63}$$

We now write out this isomorphism in terms of Dixmier–Douady classes $d_k := d(G(k)) \in H^3(E(k); \underline{\mathbb{Z}})$.

We fix a generator of $H^1(\mathbb{T}; \mathbb{Z})$. This fixes a choice of generators of $x_i \in H^1(\mathbb{T}^n; \mathbb{Z})$, $i = 1, \dots, n$ via pullback along the coordinate projections. Let $a\colon \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T}^n$ be the group structure. Then we have

$$a^*(x_i) = \mathsf{pr}_1^* x_i + \mathsf{pr}_2^* x_i. \tag{64}$$

where $\mathsf{pr}_i\colon \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T}^n$, $i = 1, 2$, are the projections onto the factors.

Let us for simplicity assume that B is connected. Let $(E_r(k), d_r(k))$ be the Serre spectral sequence of the composition $E(k) \to B \to *$ (see 6.1.7). Then we can identify $E_2^{0,3}(k) \cong \Lambda^3_{\mathbb{Z}} H^1(\mathbb{T}^n; \mathbb{Z})$. The class d_k has a symbol in $E_2^{0,3}(k)$ which can be written as $\sum_{i<j<l} a_{i,j,l}(k) x_i \wedge x_j \wedge x_l$. Since the map μ (see 62) on fibres can be written in terms of the group structure we can use (64) in order to write out the symbol of $\mu^*(d_k)$. Equation (63) implies the identity

$$\begin{aligned} &\sum_{i,j,l} a_{i,j,l}(k) \mathsf{pr}_1^*(x_i \wedge x_j \wedge x_l) + \sum_{s,t,u} a_{s,t,u}(m) \mathsf{pr}_2^*(x_s \wedge x_t \wedge x_u) \\ &\quad = \sum_{a,b,c} a_{a,b,c}(k+m)(\mathsf{pr}_1^* x_a + \mathsf{pr}_2^* x_a) \wedge (\mathsf{pr}_1^* x_b + \mathsf{pr}_2^* x_b) \wedge (\mathsf{pr}_1^* x_c + \mathsf{pr}_2^* x_c). \end{aligned}$$

Because of the presence of mixed terms this is only possible if everything vanishes. This implies that $d_k \in F^1H^3(E(k);\underline{\mathbb{Z}})$. We write $\sum_{i,j} x_i \wedge x_j \otimes u_{i,j}(k)$ for its symbol in $E_2^{1,2}$, where $u_{i,j} \in H^1(B;\underline{\mathbb{Z}})$. As above equation (63) now implies

$$\begin{aligned}\sum_{i,j} \mathsf{pr}_1^*(x_i \wedge x_j) \otimes u_{i,j}(k) + \sum_{l,k} \mathsf{pr}_2^*(x_l \wedge x_r) \otimes u_{l,r}(m)\\ = \sum_{a,b} (\mathsf{pr}_1^* x_a + \mathsf{pr}_2^* x_a) \wedge (\mathsf{pr}_1^* x_b + \mathsf{pr}_2^* x_b) \otimes u_{a,b}(k+m).\end{aligned}$$

Again the presence of mixed terms implies that everything vanishes. This shows that $d_k \in F^2H^3(E(k);\underline{\mathbb{Z}})$. The assertion of the lemma is the case $k = 1$. □

6.3.7 Let us now combine (60) and (59) into a single diagram. We get the following web of horizontal and vertical exact sequences:

$$\begin{tikzcd}
 & H^1(B;\mathbb{Z}^n) \arrow[dl, equal] \arrow[d, "\alpha"] & & & \\
H^1(B;\underline{\mathbb{Z}}^n) \arrow[r] & H^3(B;\underline{\mathbb{Z}}) \arrow[r, "h"] \arrow[d] & Q_{\mathcal{E}} \arrow[r, "\hat{c}"] \arrow[d, "f"] & H^2(B;\underline{\mathbb{Z}}^n) \arrow[r, "\beta"] \arrow[d] & H^4(B;\underline{\mathbb{Z}}) \arrow[d, equal] \\
K \arrow[r] \arrow[ur, dotted, "s"] & A \arrow[r] \arrow[d] & F^2H^3(E;\underline{\mathbb{Z}}) \arrow[r] \arrow[d] & B \arrow[r, "\bar{\beta}"] \arrow[d] & H^4(B;\underline{\mathbb{Z}}) \\
 & 0 & 0 & 0. &
\end{tikzcd} \tag{65}$$

The map $f\colon Q_{\mathcal{E}} \to F^2H^3(E;\underline{\mathbb{Z}})$ associates to $P \in Q_{\mathcal{E}}$ the Dixmier–Douady class of the gerbe of the pair $\mathrm{up}(P) \in P(B)$ with underlying $\mathbb{T}^n$-bundle E. Here we use Lemma 6.16. Surjectivity of f follows by a diagram chase once we have shown the following lemma.

Lemma 6.17. *The diagram* (65) *commutes.*

Proof. We have to check that the left and the right squares

$$\begin{tikzcd}
H^3(B;\underline{\mathbb{Z}}) \arrow[r, "h"] \arrow[d] & Q_{\mathcal{E}} \arrow[d, "f"] \\
A \arrow[r] & F^2H^3(E;\underline{\mathbb{Z}})
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
Q_{\mathcal{E}} \arrow[r, "\hat{c}"] \arrow[d, "f"] & H^2(B;\underline{\mathbb{Z}}^n) \arrow[d] \\
F^2H^3(E;\underline{\mathbb{Z}}) \arrow[r, "e"] & B
\end{tikzcd} \tag{66}$$

commute. Let us start with the left square. Let $d \in H^3(B;\underline{\mathbb{Z}})$. Under the identification

$$H^3(B;\underline{\mathbb{Z}}) \cong H^2(B;\underline{\mathbb{T}}) \cong \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}/B}(\underline{\mathbb{Z}};\underline{\mathbb{T}})$$

it corresponds to a group stack $P \in \mathsf{PIC}(\mathbf{S}/B)$ with $H^0(P) \cong \underline{\mathbb{Z}}$ and $H^{-1}(P) \cong \underline{\mathbb{T}}$. The group stack $h(d) \in Q_{\mathcal{E}}$ is given by the pull-back (two-cartesian diagram)

$$\begin{tikzcd} h(d) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ P \arrow[r] & \underline{\mathbb{Z}}. \end{tikzcd}$$

In particular we see that the gerbe $H \to E$ of the pair $u(h(d)) =: (E, H)$ is given by a pull-back

$$\begin{tikzcd} H \arrow[r] \arrow[d] & E \arrow[d] \\ G \arrow[r] & B, \end{tikzcd}$$

where $G \in \mathsf{Gerbe}(B)$ is a gerbe with Dixmier–Douady class $d(G) = d$. The composition $f \circ h \colon H^3(B; \underline{\mathbb{Z}}) \to H^3(E; \underline{\mathbb{Z}})$ is thus given by the pull-back along the map $p \colon E \to B$, i.e. $p^* = f \circ h$. By construction the composition $H^3(B; \underline{\mathbb{Z}}) \to A \to F^2 H^3(E; \underline{\mathbb{Z}})$ is a certain factorization of p^*. This shows that the left square commutes.

Now we show that the right square in (66) commutes. We start with an explicit description of $\hat{c}$. Let $P \in Q_{\mathcal{E}}$. The principal $\mathbb{T}^n$-bundle in $G(0) \to E(0) \to B$ is trivial (see (61) for notation). Therefore the Serre spectral sequence $(E_r(0), d_r(0))$ degenerates at the second term. We already know by Lemma 6.16 that the Dixmier–Douady class of $G(0) \to E(0)$ satisfies $d_0 \in F^2 H^3(E(0); \underline{\mathbb{Z}})$. Its symbol can be written as $\sum_{i=1}^n x_i \otimes \hat{c}_i(P)$ for a uniquely determined sequence $\hat{c}_i(P) \in H^2(B; \underline{\mathbb{Z}})$. These classes constitute the components of the class $\hat{c}(P) \in H^2(B; \underline{\mathbb{Z}}^n)$.

We write the symbol of $f(P) = d_1$ as $\sum_i x_i \otimes a_i$ for a sequence $a_i \in H^2(B; \underline{\mathbb{Z}})$. As in the proof of Lemma 6.16 the equation (63) gives the identity

$$\sum_{i=1}^n \mathsf{pr}_2^* x_i \otimes a_i + \sum_{i=1}^n \mathsf{pr}_1^* x_i \otimes \hat{c}_i(P) \equiv \sum_{i=1}^n (\mathsf{pr}_1^* x_i + \mathsf{pr}_2^* x_i) \otimes a_i$$

modulo the image of a second differential $d_2^{0,2}(1)$. This relation is solved by $a_i := \hat{c}_i(P)$ and determines the image of the vector $a := (a_1, \ldots, a_n)$ under $H^2(B; \underline{\mathbb{Z}}^n) \to H^2(B; \underline{\mathbb{Z}}^n)/\mathsf{im}(d_2^{0,2}(1)) =: B$ uniquely. Note that $e \circ f(P)$ is also represented by the image of the vector a in B. This shows that the right square in (66) commutes. □

6.4 T-duality triples and group stacks

6.4.1 Let R_n be the classifying space of T-duality triples introduced in [BRS]. It carries a universal T-duality triple $t_{\text{univ}} := ((E_{\text{univ}}, H_{\text{univ}}), (\hat{E}_{\text{univ}}, \hat{H}_{\text{univ}}), u_{\text{univ}})$. Let $c_{\text{univ}}, \hat{c}_{\text{univ}} \in H^2(R_n; \mathbb{Z}^n)$ be the Chern classes of the bundles $E_{\text{univ}} \to R_n$, $\hat{E}_{\text{univ}} \to R_n$. They satisfy the relation $c_{\text{univ}} \cup \hat{c}_{\text{univ}} = 0$. Let $\mathcal{E}_{\text{univ}}$ be the extension of sheaves corresponding to $E_{\mathrm{univ}} \to R_n$ as in 6.3.1. In [BRS] we have shown that $H^3(R_n; \underline{\mathbb{Z}}) \cong 0$. The diagram (65) now implies that there is a unique Picard stack $P_{\mathrm{univ}} \in Q_{\mathcal{E}_{\mathrm{univ}}}$ with $\hat{c}(P_{\mathrm{univ}}) = \hat{c}_{\mathrm{univ}}$ and underlying pair $\mathrm{up}(P_{\mathrm{univ}}) \cong (E_{\mathrm{univ}}, H_{\mathrm{univ}})$ (see 6.3.5).

6.4.2 Let us fix a $\mathbb{T}^n$-principal bundle $E \to B$, or the corresponding extension of sheaves $\mathcal{E}$. Let us furthermore fix a class $h \in F^2H^3(E; \underline{\mathbb{Z}})$. In [BRS] we have introduced the set $\mathsf{Ext}(E, h)$ of extensions of (E, h) to a T-duality triple. The main theorem about this extension set is [BRS, Theorem 2.24].

Analogously, in the present paper we can consider the set of extensions of (E, h) to a Picard stack P with underlying $\mathbb{T}^n$-bundle $E \to B$ and $f(P) = h$, where $f : Q_{\mathcal{E}} \to F^2H^3(E; \underline{\mathbb{Z}})$ is as in (65). In symbols we can write $f^{-1}(h)$ for this set.

The main goal of the present section is to compare the sets $\mathsf{Ext}(E, h)$ and $f^{-1}(h) \subseteq Q_{\mathcal{E}}$. In the following paragraphs we construct maps between these sets.

6.4.3 We fix a $\mathbb{T}^n$-principal bundle $E \to B$ and let $\mathcal{E} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B$ be the corresponding extensions of sheaves. Let $\mathsf{Triple}_E(B)$ denote the set of isomorphism classes of triples t such that $c(t) = c(E)$. We first define a map

$$\Phi\colon \mathsf{Triple}_E(B) \to Q_{\mathcal{E}}.$$

Let $t \in \mathsf{Triple}_E(B)$ be a triple which is classified by a map $f_t : B \to R_n$. Pulling back the group stack $P_{\mathrm{univ}} \in \mathsf{PIC}(\mathbf{S}/R_n)$ we get an element $\Phi(t) := f_t^*(P_{\mathrm{univ}}) \in \mathsf{PIC}(\mathbf{S}/B)$. If $t \in \mathsf{Triple}_E(B)$, then we have $\Phi(t) \in Q_{\mathcal{E}}$. We further have

$$\hat{c}(\Phi(t)) = f_t^* \hat{c}(P_{\mathrm{univ}}) = f_t^* \hat{c}_{\mathrm{univ}} = \hat{c}(t). \tag{67}$$

6.4.4 In the next few paragraphs we describe a map

$$\Psi\colon Q_{\mathcal{E}} \to \mathsf{Triple}_E(B),$$

i.e. a construction of a T-duality triple $\Psi(P) \in \mathsf{Triple}_E(B)$ starting from a Picard stack $P \in Q_{\mathcal{E}}$.

6.4.5 Consider $P \in \mathsf{PIC}(\mathbf{S}/B)$. We have already constructed one pair (E, H). The dual $D(P) := \underline{\mathsf{HOM}}_{\mathsf{PIC}(\mathbf{S}/B)}(P, \mathcal{B}\underline{\mathbb{T}}_{|B})$ is a Picard stack with (see Corollary 5.10)

$$H^0(D(P)) \cong D(H^{-1}(P)) \cong D(\underline{\mathbb{T}}_{|B}) \cong \underline{\mathbb{Z}}_{|B},$$
$$H^{-1}(D(P)) \cong D(H^0(P)) \cong D(\mathcal{E}).$$

In view of the structure (57) of $\mathcal{E}$, the equalities $D(\underline{\mathbb{T}}^n_{|B}) \cong \underline{\mathbb{Z}}^n_{|B}$, $D(\underline{\mathbb{Z}}_{|B}) \cong \underline{\mathbb{T}}_{|B}$, and $\underline{\mathsf{Ext}}^1_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{T}}_{|B}, \underline{\mathbb{T}}_{|B}) \cong 0$ we have an exact sequence

$$0 \to \underline{\mathbb{T}}_{|B} \to D(\mathcal{E}) \to \underline{\mathbb{Z}}^n_{|B} \to 0. \tag{68}$$

Using the construction 2.5.12 we can form a quotient $\overline{D(P)}$ which fits into the sequence of maps of Picard stacks

$$\mathcal{B}\underline{\mathbb{T}}_{|B} \to D(P) \to \overline{D(P)},$$

where $H^0(\overline{D(P)}) \cong \underline{\mathbb{Z}}_{|B}$ and $H^{-1}(\overline{D(P)}) \cong \underline{\mathbb{Z}}^n_{|B}$. The fibre product

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \overline{D(P)} \arrow[d] \\ \{1\} \arrow[r] & \underline{\mathbb{Z}}_{|B} \end{tikzcd}$$

defines a gerbe $R \to B$ with band $\underline{\mathbb{Z}}^n_{|B}$.

6.4.6 By Lemma 6.15 there exists a unique isomorphism class $\widehat{E} \to B$ of a $\mathbb{T}^n$-bundle whose $\underline{\mathbb{Z}}^n_{|B}$-gerbe of $\mathbb{R}^n$-reductions $R^{\widehat{E}}_{\mathbb{Z}^n}$ is isomorphic to R. We fix such an isomorphism and obtain a canonical map of stacks can (see (54)) fitting into the diagram

$$\begin{tikzcd} \widehat{H}^{\mathrm{op}} \arrow[rr] \arrow[d] & & \widetilde{\widetilde{H}} \arrow[r] \arrow[d] & D(P) \arrow[d] \\ \widehat{E} \arrow[r, "\mathsf{can}"] \arrow[dr] & R^{\widehat{E}}_{\mathbb{Z}^n} \arrow[r, "\cong"] \arrow[d] & R \arrow[r] \arrow[d] \arrow[dl] & \overline{D(P)} \arrow[d] \\ & B & \{1\} \arrow[r] & \underline{\mathbb{Z}}_{|B}. \end{tikzcd} \tag{69}$$

The gerbes $\widehat{H}^{\mathrm{op}} \to \widehat{E}$ and $\widetilde{\widetilde{H}} \to R$ are defined such that the squares become two-cartesian (we omit to write the two-isomorphisms). In this way the Picard stack $P \in Q_{\mathcal{E}}$ defines the second pair $(\widehat{E}, \widehat{H})$ of the triple $\Psi(P) = ((E, H), (\widehat{E}, \widehat{H}), u)$ whose construction has to be completed by providing u.

Lemma 6.18. *We have the equality* $\hat{c}(P) = \hat{c}(\Psi(P))$ *in* $H^2(B; \underline{\mathbb{Z}}^n_{|B})$.

Proof. By the definition in 6.1.12 we have $\hat{c}(\Psi(P)) = c(\widehat{E})$. Furthermore, by (56) we have $c(\widehat{E}) = d(R^{\widehat{E}}_{\underline{\mathbb{Z}}^n_{|B}}) = d(R)$. By Lemma 5.14 we have $\phi(D(P)) = \mathcal{D}(\phi(P))$, where ϕ is the characteristic class (46), and

$$\mathcal{D} : Q_{\mathcal{E}} \cong \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}/B}(\mathcal{E}, \underline{\mathbb{T}}_{|B}) \xrightarrow{\sim} \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}/B}(D(\underline{\mathbb{T}}_{|B}), D(\mathcal{E}))$$

is as in Lemma 5.14. The map $\hat{c}\colon Q_{\mathcal{E}} \to H^2(B; \underline{\mathbb{Z}})$ by its definition fits into the diagram

$$\begin{tikzcd}
Q_{\mathcal{E}} \arrow[r, "\underline{\mathbb{T}}^n_{|B}\to\mathcal{E}"] \arrow[d, "D"] \arrow[ddr, dotted] \arrow[dddr, "\hat{c}"'] & \mathrm{Ext}^2_{\mathrm{ShAb}\,\mathbf{S}/B}(\underline{\mathbb{T}}^n_{|B};\underline{\mathbb{T}}_{|B}) \arrow[d, "\mathcal{D}"] \\
\mathrm{Ext}^2_{\mathrm{ShAb}\,\mathbf{S}/B}(D(\underline{\mathbb{T}}_{|B}), D(\mathcal{E})) \arrow[r, "D(\mathcal{E})\to D(\underline{\mathbb{T}}^n_{|B})", "a"'] & \mathrm{Ext}^2_{\mathrm{ShAb}\,\mathbf{S}/B}(D(\underline{\mathbb{T}}_{|B}); D(\underline{\mathbb{T}}^n_{|B})) \arrow[d, "\cong"] \\
 & \mathrm{Ext}^2_{\mathrm{ShAb}\,\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{Z}}^n_{|B}) \arrow[d, "\cong"] \\
 & H^2(B;\underline{\mathbb{Z}}^n_{|B}).
\end{tikzcd}$$

By construction the class $a(P) \in \mathrm{Ext}^2_{\mathrm{ShAb}\,\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{Z}}^n_{|B})$ classifies the Picard stack $\overline{D(\mathcal{E})}$. This implies that $\hat{c}(P)$ classifies the gerbe $R \to B$, i.e. $\hat{c}(P) = c(\widehat{E})$. □

6.4.7 It remains to construct the last entry

$$u\colon H \otimes_{E\times_B \widehat{E}} \widehat{H}^{\mathrm{op}} \to \mathcal{B}\underline{\mathbb{T}}_{|E\times_B\widehat{E}} \tag{70}$$

of the triple $\Psi(t) = ((E,H),(\widehat{E},\widehat{H}),u)$, where we use Picture 6.1.11. Note that $H^{-1}(P) \cong \underline{\mathbb{T}}_{|B}$. Construction 2.5.12 gives rise to a sequence of morphisms of Picard stacks

$$\mathcal{B}\underline{\mathbb{T}}_{|B} \to P \to \bar{P}$$

(we could write $\bar{P} = \mathcal{E}$).

We have an evaluation $\mathsf{ev}\colon P \times_B D(P) \to B\mathbb{T}_{|B}$. For a pair $(r,s) \in \mathbb{Z}\times\mathbb{Z}$ we consider the following diagram with two-cartesian squares.

$$\begin{tikzcd}
H_r \times_B \tilde{\widehat{H}}_s \arrow[r] \arrow[d] & P \times D(P) \arrow[r, "\mathsf{ev}"] \arrow[d] & \mathcal{B}\underline{\mathbb{T}}_{|B} \\
H_r \otimes_{E_r\times_B \widehat{R}_s} \tilde{\widehat{H}}_s \arrow[r] \arrow[d, "w"] \arrow[urr, dotted, "u_{r,s}"] & P \otimes_{\bar{P}\times\overline{D(P)}} D(P) \arrow[d] & \\
E_r \times_B R_s \arrow[r] \arrow[d] & \bar{P} \times \overline{D(P)} \arrow[d] & \\
\underline{\{r,s\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B} \times \underline{\mathbb{Z}}_{|B}. &
\end{tikzcd}$$

By an inspection of the definitions one checks that the natural factorization $u_{r,s}$ exists if $r = s$. Furthermore one checks that

$$(w, u_{r,s})\colon H_r \otimes_{E_r \times_B R_s} \tilde{\hat{H}}_s \to (E_r \times_B R_s) \times \mathcal{B}\underline{\mathbb{T}}_{|B}$$

is an isomorphism of gerbes with band $\underline{\mathbb{T}}_{|E_r \times_B R_s}$ if and only if $r = s = 1$. Both statements can be checked already when restricting to a point, and therefore become clear when considering the argument in the proof of Lemma 6.19.

We define the map (70) by

$$\begin{tikzcd}
H \otimes_{E \times_B \hat{E}} \hat{H}^{\mathrm{op}} \arrow[r] \arrow[d] \arrow[rr, bend left, "u"] & H_1 \otimes_{E_1 \times_B \hat{R}_1} \tilde{\hat{H}}_1 \arrow[r, "u_{1,1}"] \arrow[d] & \mathcal{B}\underline{\mathbb{T}}_{|B} \\
E \times_B \hat{E} \arrow[r, "\mathsf{(id,can)}"] & E_1 \times_B \hat{R}_1 &
\end{tikzcd}$$

(note that $R_1 = R$, $E_1 = E$ and $H_1 = H$).

Lemma 6.19. *The triple $\Psi(P) = ((E, H), (\hat{E}, \hat{H}), u)$ constructed above is a T-duality triple.*

Proof. It remains to show that the isomorphism of gerbes u satisfies the condition 6.2, 2 in the version of 6.1.11.

By naturality of the construction Ψ in the base B and the fact that condition 6.2, 2 can be checked at a single point $b \in B$, we can assume without loss of generality that B is a point. We can further assume that $\mathcal{E} = \underline{\mathbb{Z}} \times \underline{\mathbb{T}}^n$ and $P = \mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{Z}} \times \underline{\mathbb{T}}^n$. In this case

$$D(P) = D(\mathcal{B}\underline{\mathbb{T}}) \times D(\underline{\mathbb{Z}}) \times D(\underline{\mathbb{T}}^n) \cong \underline{\mathbb{Z}} \times \mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{Z}}^n.$$

We have

$$H \times \tilde{\hat{H}} \cong (\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n) \times (\mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{Z}}^n).$$

The restriction of the evaluation map $H \times \tilde{\hat{H}} \to H \otimes \tilde{\hat{H}} \to \mathcal{B}\underline{\mathbb{T}}$ is the composition

$$\begin{aligned}(\mathcal{B}\underline{\mathbb{T}} \times \underline{\mathbb{T}}^n) \times (\mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{Z}}^n) &\cong \mathcal{B}\underline{\mathbb{T}} \times (\mathcal{B}\underline{\mathbb{T}}) \times \underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{Z}}^n \\ &\xrightarrow{\mathsf{id} \times \mathsf{id} \times \mathsf{ev}} \mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{T}} \xrightarrow{\Sigma} \mathcal{B}\underline{\mathbb{T}}.\end{aligned}$$

We are interested in the contribution $\mathsf{ev}\colon \underline{\mathbb{T}}^n \times \mathcal{B}\underline{\mathbb{Z}}^n \to \mathcal{B}\mathbb{T}^n$.

It suffices to see that in the case $n = 1$ we have

$$\mathsf{ev}^*(z) = x \otimes y,$$

where $x \in H^1(\mathbb{T}; \underline{\mathbb{Z}})$, $y \in H^1(\mathcal{B}\mathbb{Z}; \underline{\mathbb{Z}})$, and $z \in H^2(\mathcal{B}T) \cong \mathbb{Z}$ are the canonical generators. In fact this implies via the Künneth formula that $\mathsf{ev}^*(z) = \sum_{i=1}^n x_i \otimes y_i$, where $x_i := p_i^*(x)$, $y_i := q_i^*(y)$ for the projections onto the components $p_i : \underline{\mathbb{T}}^n \to \underline{\mathbb{T}}$ and $q_i : \mathcal{B}\underline{\mathbb{Z}}^n \cong (\mathcal{B}\underline{\mathbb{Z}})^n \to \mathcal{B}\underline{\mathbb{Z}}$. Finally we use that $\mathsf{can}^*(y_i) = x_i$, where $\mathsf{can}\colon \underline{\mathbb{T}}^n \to \mathcal{B}\underline{\mathbb{Z}}^n$ is the canonical map (54) from a second copy of the torus to its gerbe of $\mathbb{R}^n$-reductions (after identification of this gerbe with $\mathcal{B}\underline{\mathbb{Z}}^n$). □

This finishes the construction of Ψ which started in 6.4.4.

6.4.8 In [BRS, 2.11] we have seen that the group $H^3(B;\underline{\mathbb{Z}})$ acts on $\mathsf{Triple}(B)$ preserving the subsets $\mathsf{Triple}_E(B) \subseteq \mathsf{Triple}(B)$ for every $\mathbb{T}^n$-bundle $E \to B$. We will recall the description of the action in the proof of Lemma 6.20 below. By (65) it also acts on $Q_{\mathcal{E}}$.

Lemma 6.20. *The map Ψ is $H^3(B;\underline{\mathbb{Z}})$-equivariant.*

The proof requires some preparations.

6.4.9 Note that we have a canonical isomorphism $\underline{\mathbb{T}} \cong D(\underline{\mathbb{Z}})$. In order to work with canonical identifications we are going to use $D(\underline{\mathbb{Z}})$ instead of $\underline{\mathbb{T}}$.

The isomorphisms classes of gerbes with band $D(\underline{\mathbb{Z}})$ over a space $B \in \mathbf{S}$ are classified by

$$H^2(B; D(\underline{\mathbb{Z}})_{|B}) \cong \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B}). \tag{71}$$

The latter group also classifies Picard stacks P with fixed isomorphisms $H^0(P) \cong \underline{\mathbb{Z}}_{|B}$ and $H^{-1}(P) \cong D(\underline{\mathbb{Z}}_{|B})$.

Given P the gerbe $G \to B$ can be reconstructed as a pull-back

$$\begin{array}{ccc} G & \longrightarrow & P \\ \downarrow & & \downarrow \\ \underline{\{1\}}_{|B} & \longrightarrow & \underline{\mathbb{Z}}_{|B}. \end{array}$$

If we want to stress the dependence of G on P we will write $G(P)$.

Recall that an object in $P(T)$ as a stack over $\mathbf{S}$ consists of a map $T \to B$ and an object of $P(T \to B)$. In a similar manner we interpret morphisms.

For example, the stack $\underline{\{1\}}_{|B}$ is the space B.

6.4.10 Let $P \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B})$ be a Picard stack P with a fixed isomorphisms $H^0(P) \cong \underline{\mathbb{Z}}_{|B}$ and $H^{-1}(P) \cong D(\underline{\mathbb{Z}}_{|B})$. Let $\phi\colon \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B}) \to \mathsf{Ext}^2_{\mathsf{Sh_{Ab}}\,\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B})$ be the characteristic class.

Lemma 6.21. $\phi(D(P)) \cong -\phi(P)$.

Proof. First of all note that we have canonical isomorphisms

$$H^0(D(P)) \cong D(H^{-1}(P) \cong D(D(\underline{\mathbb{Z}}_{|B})) \cong \underline{\mathbb{Z}}_{|B}$$

and

$$H^{-1}(D(P)) \cong D(H^0(P)) \cong D(\underline{\mathbb{Z}}_{|B}) \cong D(\underline{\mathbb{Z}})_{|B}.$$

Therefore we can consider $D(P) \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B})$ in a canonical way, and $\phi(P)$ and $\phi(D(P))$ belong to the same group.

Note that

$$d(G(P)) = \phi(P), \quad d(G(D(P))) = \phi(D(P))$$

under the isomorphism (71). It suffices to show that $d(G(P)) = -d(G(D(P)))$.

In fact, as in 6.4.7 we have the following factorization of the evaluation map:

$$\begin{tikzcd}
G(P) \otimes_{BG} (D(P)) \arrow[r] \arrow[d] & P \otimes_{\underline{\mathbb{Z}}_{|B}} D(P) \arrow[d] \arrow[rrr, bend left] & P \times_{\underline{\mathbb{Z}}_{|B}} D(P) \arrow[l] \arrow[r] \arrow[d] & P \times_B D(P) \arrow[r, "\mathsf{ev}"] \arrow[d] & \underline{\mathbb{T}}_{|B} \\
\underline{\{1\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B} \arrow[r, equal] & \underline{\mathbb{Z}}_{|B} \arrow[r, "\mathsf{diag}"] & \underline{\mathbb{Z}}_{|B} \times \underline{\mathbb{Z}}_{|B}. &
\end{tikzcd}$$

This represents $G(D(P))$ as the dual gerbe $G(P)^{\mathrm{op}}$ of $G(P)$ in the sense of Definition 6.6. The relation $d(G(P)) = -d(G(D(P)))$ follows. □

6.4.11 We now start the actual proof of Lemma 6.21.

In order to see that Ψ is $H^3(B;\underline{\mathbb{Z}})$-equivariant we will first describe the action of $H^3(B;\underline{\mathbb{Z}})$ on the sets of isomorphism classes of T-duality triples with fixed underlying $\mathbb{T}^n$-bundles E and $\hat{E}$ on the one hand, and on the set of isomorphism classes of Picard stacks $Q_{\mathcal{E}}$, on the other.

Consider $g \in H^3(B;\underline{\mathbb{Z}})$. It classifies the isomorphism class of a gerbe $G \to B$ with band $\underline{\mathbb{T}}_{|B}$. If t is represented by $((E,H),(\hat{E},\hat{H}),u)$, then $g+t$ is represented by

$$((E, H \otimes \pi^*G), (\hat{E}, \hat{H} \otimes \hat{\pi}^*G), u \otimes \mathsf{id}_{r^*G}),$$

where the maps $\pi, \hat{\pi}, r$ are as in (49).

Note the isomorphism $H^3(B;\underline{\mathbb{Z}}) \cong H^2(B;\underline{\mathbb{T}}) \cong \mathsf{Ext}^2_{\mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B}(\underline{\mathbb{Z}}_{|B}, \underline{\mathbb{T}}_{|B})$. Therefore the class g also classifies an isomorphism class of Picard stacks with $H^0(\tilde{G}) \cong \underline{\mathbb{Z}}_{|B}$ and $H^{-1}(\tilde{G}) \cong \underline{\mathbb{T}}_{|B}$. From $\tilde{G}$ we can derive the gerbe G by the pull-back

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \tilde{G} \arrow[d] \\
\underline{\{1\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B}.
\end{tikzcd}$$

6.4.12 Recall that we consider an extension $\mathcal{E} \in \mathsf{Sh}_{\mathsf{Ab}}\mathbf{S}/B$ of the form

$$0 \to \underline{\mathbb{T}}^n_{|B} \to \mathcal{E} \to \underline{\mathbb{Z}}_{|B} \to 0.$$

We consider a Picard stack $P \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\mathcal{E}, D(\underline{\mathbb{Z}})_{|B})$.

Let furthermore $\tilde{G} \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\underline{\mathbb{Z}}_{|B}, D(\underline{\mathbb{Z}})_{|B})$. Then we define

$$P \otimes_{\mathcal{E}} \tilde{G} \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\mathcal{E}, D(\underline{\mathbb{Z}})_{|B})$$

by the diagram

$$\begin{tikzcd}
\mathcal{B}\underline{\mathbb{T}}_{|B} \arrow[d] & \mathcal{B}\underline{\mathbb{T}}_{|B} \times_B \mathcal{B}\underline{\mathbb{T}}_{|B} \arrow[l, "+"'] \arrow[d] & \\
P \otimes_{\mathcal{E}} \tilde{G} & P \times_{\underline{\mathbb{Z}}_{|B}} \tilde{G} \arrow[l] \arrow[r] \arrow[d] & P \times_B \tilde{G} \arrow[d] \\
& \underline{\mathbb{Z}}_{|B} \arrow[r, "\mathrm{diag}"] & \underline{\mathbb{Z}}_{|B} \times_B \underline{\mathbb{Z}}_{|B} .
\end{tikzcd} \tag{72}$$

The right lower square is cartesian, and in the left upper square we take the fibre-wise quotient by the anti-diagonal action of $\mathcal{B}\underline{\mathbb{T}}_{|B}$.

6.4.13 We have $D(P) \in \mathsf{Ext}_{\mathsf{PIC}(\mathbf{S}/B)}(\underline{\mathbb{Z}}, D(\mathcal{E}))$, where

$$0 \to \mathcal{B}\underline{\mathbb{T}}_{|B} \to D(\mathcal{E}) \to D(\mathcal{E}) \to 0.$$

We define $\overline{D(P)}$ to be the quotient of $D(P)$ by $\mathcal{B}\underline{\mathbb{T}}_{|B}$ in the sense of 2.5.12 so that $D(P) \to \overline{D(P)}$ is a gerbe with band $\underline{\mathbb{T}}$.

We define $D(P) \otimes_{\overline{D(P)}} D(\tilde{G})$ by the diagram

$$\begin{tikzcd}
\mathcal{B}\underline{\mathbb{T}}_{|B} \arrow[d] & \mathcal{B}\underline{\mathbb{T}}_{|B} \times_B \mathcal{B}\underline{\mathbb{T}}_{|B} \arrow[l, "+"'] \arrow[d] & \\
D(P) \otimes_{\overline{D(P)}} D(\tilde{G}) & D(P) \times_{\underline{\mathbb{Z}}_{|B}} D(\tilde{G}) \arrow[l] \arrow[r] \arrow[d] & D(P) \times_B D(\tilde{G}) \arrow[d] \\
& \underline{\mathbb{Z}}_{|B} \arrow[r, "\mathrm{diag}"] & \underline{\mathbb{Z}}_{|B} \times_B \underline{\mathbb{Z}}_{|B} .
\end{tikzcd} \tag{73}$$

Again, the right lower square is cartesian, and in the left upper square we take the fibre-wise quotient by the anti-diagonal action of $\mathcal{B}\underline{\mathbb{T}}_{|B}$.

Proposition 6.22. *We have an equivalence of Picard stacks*

$$D(P \otimes_{\mathcal{E}} \tilde{G}) \cong D(P) \otimes_{\overline{D(P)}} D(\tilde{G}).$$

Proof. The diagram (73) defines $D(P) \otimes_{\overline{D(P)}} D(\tilde{G})$ by forming the pull-back to the diagonal and then taking the quotient of the anti-diagonal $\mathcal{B}\underline{\mathbb{T}}_{|B}$-action in the fibre. One can obtain this diagram by dualizing (72) and interchanging the order of pull-back and quotient. □

6.4.14 We can now finish the argument that Ψ is $H^3(B;\underline{\mathbb{Z}})$-equivariant. We let $\Psi(P) = ((E,H),(\hat{E},\hat{H}),u)$ and $\Psi(g+P) = ((E',H'),(\hat{E}',\hat{H}'),u')$. Note that $g + P = P \otimes_{\mathcal{E}} \tilde{G}$. An inspection of the construction of the first entry of Ψ shows that $E' \cong E$ and $H' \cong H \otimes_E \mathrm{pr}E \to BG$. Proposition 6.22 shows that $\overline{D(P \otimes_{\mathcal{E}} \tilde{G})} \cong \overline{D(P)}$. This implies that $\hat{E}' \cong \hat{E}$. Furthermore, if we restrict $D(P \otimes_{\mathcal{E}} \tilde{G})$ along $\underline{\{1\}}_{|B} \to \underline{\mathbb{Z}}_{|B}$ we get by Proposition 6.22 and the proof of Lemma 6.21 a diagram of stacks over B:

$$\begin{tikzcd}
\hat{H}^{\mathrm{op}} \otimes_{\hat{E}} G^{\mathrm{op}} \arrow[r] \arrow[d] & \hat{\tilde{H}} \otimes_{R^{\hat{E}}_{\mathbb{R}}} G^{\mathrm{op}} \arrow[r] \arrow[d] & D(P \otimes_{\mathcal{E}} \tilde{G}) \arrow[r, "\cong"] \arrow[d] & D(P) \otimes_{\overline{D(P)}} D(\tilde{G}) \arrow[d] \\
\hat{E} \arrow[r, "\mathsf{can}"] & R^{\hat{E}}_{\mathbb{R}} \arrow[r] \arrow[d] & \overline{D(P \otimes_{\mathcal{E}} \tilde{G})} \arrow[r, "\cong"] \arrow[d] & \overline{D(P)} \\
 & \underline{\{1\}}_{|B} \arrow[r] & \underline{\mathbb{Z}}_{|B}. &
\end{tikzcd}$$

The map u' is induced by the evaluation $(P \otimes_{\mathcal{E}} \tilde{G}) \times D(P \otimes_{\mathcal{E}} \tilde{G}) \to \mathcal{B}\underline{\mathbb{T}}$. With the given identifications using the "duality" between (73) and (72) we see that this evaluation the induced by the product of the evaluations

$$(P \times D(P)) \times (\tilde{G} \times D(\tilde{G})) \to \mathcal{B}\underline{\mathbb{T}} \times \mathcal{B}\underline{\mathbb{T}} \to \mathcal{B}\underline{\mathbb{T}}.$$

After restriction to $\underline{\{1\}}_{|B}$ we see that $u' = u \otimes v$, where $v \colon G \otimes G^{\mathrm{op}} \to \mathcal{B}\underline{\mathbb{T}}$ is the canonical pairing. This finishes the proof of the equivariance of Ψ. □

Theorem 6.23. *The maps Ψ and Φ are inverse to each other.*

Proof. We first show the assertion under the additional assumption that $H^3(B;\underline{\mathbb{Z}}) = 0$. In this case an element $P \in Q_{\mathcal{E}}$ is determined uniquely by the class $\hat{c}(P) \in H^2(B;\underline{\mathbb{Z}}^n)$. Similarly, a T-duality triple $t \in \mathsf{Triple}(B)$ with $c(t) = c(E)$ is uniquely determined by the class $\hat{c}(t) = c(\hat{E})$. Since for $P \in Q_{\mathcal{E}}$ we have $\hat{c}(\Phi \circ \Psi(P)) \overset{(67)}{=} \hat{c}(\Psi(P)) \overset{\text{Lemma 6.18}}{=} \hat{c}(P)$ and $\hat{c}(\Psi \circ \Phi(t)) \overset{\text{Lemma 6.18}}{=} \hat{c}(\Phi(t)) \overset{(67)}{=} \hat{c}(t)$ this implies that $\Phi \circ \Psi_{|Q_{\mathcal{E}}} = \mathsf{id}_{Q_{\mathcal{E}}}$ and $\Psi \circ \Phi_{|s^{-1}(E)} = \mathsf{id}_{|s^{-1}(E)}$, where $s \colon \mathsf{Triple} \to P$ is as in 6.1.7. Note that $H^3(R_n;\underline{\mathbb{Z}}) = 0$. Therefore

$$\Psi(\Phi(t_{\mathrm{univ}})) = t_{\mathrm{univ}}, \quad \Phi(\Psi(P_{\mathrm{univ}})) \cong P_{\mathrm{univ}}.$$

Now consider a general space $B \in \mathbf{S}$. We first show that $\Psi \circ \Phi = \mathsf{id}$. Let $t \in s^{-1}(E)$ be classified by the map $f_t \colon B \to R_n$, i.e. $f_t^* t_{\mathrm{univ}} = t$. Then we have $\Psi(\Phi(t)) = \Psi(f_t^* P_{\mathrm{univ}}) = f_t^* \Psi(P_{\mathrm{univ}}) = f_t^* t_{\mathrm{univ}} = t$, i.e.

$$\Psi \circ \Phi = \mathsf{id}. \tag{74}$$

We consider the group

$$\Gamma_{\mathcal{E}} := (\mathsf{im}(\alpha) + \mathsf{im}(s))/\mathsf{im}(\alpha) \subseteq H^3(B;\underline{\mathbb{Z}})/\mathsf{im}(\alpha).$$

This group is exactly the group $\ker(\pi^*)/C$ in the notation of [BRS, Theorem 2.24(3)]. It follows from (65) that the action of $H^3(B;\underline{\mathbb{Z}})$ on $Q_{\mathcal{E}}$ induces an action of $\Gamma_{\mathcal{E}}$ on $Q_{\mathcal{E}}$ which preserves the fibres of f. In [BRS, Theorem 2.24(3)] we have shown that it also acts on $\mathsf{Triple}_E(B)$ and preserves the subsets $\mathsf{Ext}(E,h)$.

Let us fix a Picard stack $P \in Q_{\mathcal{E}}$, and let $\hat{c} := \hat{c}(P)$ and $h \in H^3(E;\underline{\mathbb{Z}})$ be such that $f(P) \in \mathsf{Ext}(E,h)$. From (65) we see that the group $\Gamma_{\mathcal{E}}$ acts simply transitively on the set

$$A_{\mathcal{E},\hat{c}} := \{Q \in f^{-1}(h) \mid \hat{c}(Q) = \hat{c}\}.$$

By [BRS, Theorem 2.24(3)] it also acts simply transitively on the set

$$B_{\mathcal{E},\hat{c}} := \{t \in \mathsf{Ext}(E,h) \mid \hat{c}(t) = \hat{c}\}.$$

By Lemma 6.18 we have $\Psi(A_{\mathcal{E},\hat{c}}) \subseteq B_{\mathcal{E},\hat{c}}$. By Lemma 6.20 the map Ψ is $\Gamma_{\mathcal{E}}$-equivariant. Hence it must induce a bijection between $A_{\mathcal{E},\hat{c}}$ and $B_{\mathcal{E},\hat{c}}$. If we let $\hat{c}$ run over all possible choices (solutions of $\hat{c} \cup c(E) = 0$) we see that $\Psi\colon Q_{\mathcal{E}} \to \mathsf{Triple}_E(B)$ is a bijection. In view of (74) we now also get $\Phi \circ \Psi = \mathsf{id}$. □

Acknowledgment. We thank Tony Pantev for the crucial suggestion which initiated the work on this paper.

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Deformations of gerbes on smooth manifolds

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1 Introduction

In [5] we obtained a classification of formal deformations of a gerbe on a manifold (C^∞ or complex-analytic) in terms of Maurer–Cartan elements of the differential graded Lie algebra (DGLA) of Hochschild cochains twisted by the cohomology class of the gerbe. In the present paper we develop a different approach to the derivation of this classification in the setting of C^∞ manifolds, based on the differential-geometric approach of [4].

The main result of the present paper is the following theorem which we prove in Section 8.

Theorem 1. *Suppose that X is a C^∞ manifold and $\mathcal{S}$ is an algebroid stack on X which is a twisted form of $\mathcal{O}_X$. Then, there is an equivalence of 2-groupoid valued functors of commutative Artin $\mathbb{C}$-algebras*

$$\operatorname{Def}_X(\mathcal{S}) \cong \operatorname{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}).$$

Notations in the statement of Theorem 1 and the rest of the paper are as follows. We consider a paracompact C^∞-manifold X with the structure sheaf $\mathcal{O}_X$ of *complex valued* smooth functions. Let $\mathcal{S}$ be a twisted form of $\mathcal{O}_X$, as defined in Section 4.5. Twisted forms of $\mathcal{O}_X$ are in bijective correspondence with $\mathcal{O}_X^\times$-gerbes and are classified up to equivalence by $H^2(X;\mathcal{O}^\times) \cong H^3(X;\mathbb{Z})$.

One can formulate the formal deformation theory of algebroid stacks ([17], [16]) which leads to the 2-groupoid valued functor $\operatorname{Def}_X(\mathcal{S})$ of commutative Artin $\mathbb{C}$-algebras. We discuss deformations of algebroid stacks in Section 6. It is natural to expect that the deformation theory of algebroid pre-stacks is "controlled" by a suitably constructed differential graded Lie algebra (DGLA) well-defined up to isomorphism in the derived category of DGLA. The content of Theorem 1 can be stated as the existence of such a DGLA, namely $\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$, which "controls" the formal deformation theory of the algebroid stack $\mathcal{S}$ in the following sense.

To a nilpotent DGLA $\mathfrak{g}$ which satisfies $\mathfrak{g}^i = 0$ for $i < -1$ one can associate its Deligne 2-groupoid which we denote $\operatorname{MC}^2(\mathfrak{g})$, see [11], [10] and references therein. We review this construction in Section 3. Then Theorem 1 asserts equivalence of the 2-groupoids $\operatorname{Def}_X(\mathcal{S})$ and $\operatorname{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]})$.

*Supported by the Ellentuck Fund
†Partially supported by NSF grant DMS-0400342
‡Partially supported by NSF grant DMS-0605030

The DGLA $\mathfrak{g}(\mathcal{J}_X)_{[\mathcal{S}]}$ is defined as the $[\mathcal{S}]$-twist of the DGLA

$$\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X) := \Gamma(X; \mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X))[1]).$$

Here, $\mathcal{J}_X$ is the sheaf of infinite jets of functions on X, considered as a sheaf of topological $\mathcal{O}_X$-algebras with the canonical flat connection ∇^{can}. The shifted normalized Hochschild complex $\bar{C}^\bullet(\mathcal{J}_X)[1]$ is understood to comprise locally defined $\mathcal{O}_X$-linear continuous Hochschild cochains. It is a sheaf of DGLA under the Gerstenhaber bracket and the Hochschild differential δ. The canonical flat connection on $\mathcal{J}_X$ induces one, also denoted ∇^{can}, on $\bar{C}^\bullet(\mathcal{J}_X))[1]$. The flat connection ∇^{can} commutes with the differential δ and acts by derivations of the Gerstenhaber bracket. Therefore, the de Rham complex $\mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X))[1]) := \Omega^\bullet_X \otimes \bar{C}^\bullet(\mathcal{J}_X))[1])$ equipped with the differential $\nabla^{\mathrm{can}} + \delta$ and the Lie bracket induced by the Gerstenhaber bracket is a sheaf of DGLA on X giving rise to the DGLA $\mathfrak{g}(\mathcal{J}_X)$ of global sections.

The sheaf of abelian Lie algebras $\mathcal{J}_X/\mathcal{O}_X$ acts by derivations of degree -1 on the graded Lie algebra $\bar{C}^\bullet(\mathcal{J}_X)[1]$ via the adjoint action. Moreover, this action commutes with the Hochschild differential. Therefore, the (abelian) graded Lie algebra $\Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X$ acts by derivations on the graded Lie algebra $\Omega^\bullet_X \otimes \bar{C}^\bullet(\mathcal{J}_X))[1]$. We denote the action of the form $\omega \in \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X$ by ι_ω. Consider now the subsheaf of closed forms $(\Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}}$ which is by definition the kernel of ∇^{can}. $(\Omega^k_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}}$ acts by derivations of degree $k-1$ and this action commutes with the differential $\nabla^{\mathrm{can}} + \delta$. Therefore, for $\omega \in \Gamma(X; (\Omega^2 \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}})$ one can define the ω-twist $\mathfrak{g}(\mathcal{J}_X)_\omega$ as the DGLA with the same underlying graded Lie algebra structure as $\mathfrak{g}(\mathcal{J}_X)$ and the differential given by $\nabla^{\mathrm{can}} + \delta + \iota_\omega$. The isomorphism class of this DGLA depends only on the cohomology class of ω in $H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$.

More precisely, for $\beta \in \Gamma(X; \Omega^1_X \otimes \mathcal{J}_X/\mathcal{O}_X)$ the DGLA $\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_\omega$ and $\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{\omega+\nabla^{\mathrm{can}}\beta}$ are canonically isomorphic with the isomorphism depending only on the equivalence class $\beta + \mathrm{Im}(\nabla^{\mathrm{can}})$.

As we remarked before a twisted form $\mathcal{S}$ of $\mathcal{O}_X$ is determined up to equivalence by its class in $H^2(X; \mathcal{O}^\times)$. The composition $\mathcal{O}^\times \to \mathcal{O}^\times/\mathbb{C}^\times \xrightarrow{\log} \mathcal{O}/\mathbb{C} \xrightarrow{j^\infty} \mathsf{DR}(\mathcal{J}/\mathcal{O})$ induces the map $H^2(X; \mathcal{O}^\times) \to H^2(X; \mathsf{DR}(\mathcal{J}/\mathcal{O})) \cong H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$. We denote by $[\mathcal{S}] \in H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ the image of the class of $\mathcal{S}$. By the remarks above we have the well-defined up to a canonical isomorphism DGLA $\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}$.

The rest of this paper is organized as follows. In Section 2 we review some preliminary facts. In Section 3 we review the construction of Deligne 2-groupoid, its relation with the deformation theory and its cosimplicial analogues. In Section 4 we review the notion of algebroid stacks. Next we define matrix algebras associated with a descent datum in Section 5. In Section 6 we define the deformations of algebroid stacks and relate them to the cosimplicial DGLA of Hochschild cochains on matrix algebras. In Section 7 we establish quasiisomorphism of the DGLA controlling the deformations of twisted forms of $\mathcal{O}_X$ with a simpler cosimplicial DGLA. Finally, the proof the main result of this paper, Theorem 1, is given in Section 8

2 Preliminaries

2.1 Simplicial notions

2.1.1 The category of simplices. For $n = 0, 1, 2, \ldots$ we denote by $[n]$ the category with objects $0, \ldots, n$ freely generated by the graph

$$0 \to 1 \to \cdots \to n.$$

For $0 \leq p \leq q \leq n$ we denote by (pq) the unique morphism $p \to q$.

We denote by Δ the full subcategory of **Cat** with objects the categories $[n]$ for $n = 0, 1, 2, \ldots$.

For $0 \leq i \leq n+1$ we denote by $\partial_i = \partial_i^n : [n] \to [n+1]$ the i^{th} face map, i.e. the unique map whose image does not contain the object $i \in [n+1]$.

For $0 \leq i \leq n-1$ we denote by $s_i = s_i^n : [n] \to [n-1]$ the i^{th} degeneracy map, i.e. the unique surjective map such that $s_i(i) = s_i(i+1)$.

2.1.2 Simplicial and cosimplicial objects. Suppose that $\mathcal{C}$ is a category. By definition, *a simplicial object in* $\mathcal{C}$ (respectively, *a cosimplicial object in* $\mathcal{C}$) is a functor $\Delta^{\text{op}} \to \mathcal{C}$ (respectively, a functor $\Delta \to \mathcal{C}$). Morphisms of (co)simplicial objects are natural transformations of functors.

For a simplicial (respectively, cosimplicial) object F we denote the object $F([n]) \in \mathcal{C}$ by F_n (respectively, F^n).

2.2 Cosimplicial vector spaces. Let $V^\bullet$ be a cosimplicial vector space. We denote by $C^\bullet(V)$ the associated complex with component $C^n(V) = V^n$ and the differential $\partial^n : C^n(V) \to C^{n+1}(V)$ defined by $\partial^n = \sum_i (-1)^i \partial_i^n$, where ∂_i^n is the map induced by the i^{th} face map $[n] \to [n+1]$. We denote cohomology of this complex by $H^\bullet(V)$.

The complex $C^\bullet(V)$ contains a normalized subcomplex $\overline{C}^\bullet(V)$. Here $\overline{C}^n(V) = \{V \in V^n \mid s_i^n v = 0\}$, where $s_i^n : [n] \to [n-1]$ is the i^{th} degeneracy map. Recall that the inclusion $\overline{C}^\bullet(V) \to C^\bullet(V)$ is a quasiisomorphism.

Starting from a cosimplicial vector space $V^\bullet$ one can construct a new cosimplicial vector space $\widehat{V}^\bullet$ as follows. For every $\lambda : [n] \to \Delta$ set $\widehat{V}^\lambda = V^{\lambda(n)}$. Suppose given another simplex $\mu : [m] \to \Delta$ and morphism $\phi : [m] \to [n]$ such that $\mu = \lambda \circ \phi$, i.e., ϕ is a morphism of simplices $\mu \to \lambda$. The morphism $(0n)$ factors uniquely into $0 \to \phi(0) \to \phi(m) \to n$, which, under λ, gives the factorization of $\lambda(0n) : \lambda(0) \to \lambda(n)$ (in Δ) into

$$\lambda(0) \xrightarrow{f} \mu(0) \xrightarrow{g} \mu(m) \xrightarrow{h} \lambda(n), \tag{2.1}$$

where $g = \mu(0m)$. The map $\mu(m) \to \lambda(n)$ induces the map

$$\phi_* : \widehat{V}^\mu \to \widehat{V}^\lambda. \tag{2.2}$$

Set now $\widehat{V}^n = \prod_{[n] \xrightarrow{\lambda} \Delta} \widehat{V}^\lambda$. The maps (2.2) endow $\widehat{V}^\bullet$ with the structure of a cosimplicial vector space. We then have the following well-known result:

Lemma 2.1. $H^\bullet(V) \cong H^\bullet(\widehat{V})$.

Proof. We construct morphisms of complexes inducing the isomorphisms in cohomology. We will use the following notations. If $f \in \widehat{V}^n$ and $\lambda\colon [n] \to \Delta$ we will denote by $f(\lambda) \in \widehat{V}^\lambda$ its component in $\widehat{V}^\lambda$. For $\lambda\colon [n] \to \Delta$ we denote by $\lambda|_{[jl]}\colon [l-j] \to \Delta$ its truncation: $\lambda|_{[jl]}(i) = \lambda(i+j)$, $\lambda|_{[jl]}(i,k) = \lambda((i+j)\ (k+j))$. For $\lambda_1\colon [n_1] \to \Delta$ and $\lambda_2\colon [n_2] \to \Delta$ with $\lambda_1(n_1) = \lambda_2(0)$ define their concatenation $\Lambda = \lambda_1 * \lambda_2\colon [n_1+n_2] \to \Delta$ by the following formulas.

$$\Lambda(i) = \begin{cases} \lambda_1(i) & \text{if } i \le n_1, \\ \lambda_2(i-n_1) & \text{if } i \ge n_1, \end{cases}$$

$$\Lambda(ik) = \begin{cases} \lambda_1(ik) & \text{if } i,k \le n_1, \\ \lambda_2((i-n_1)\ (k-n_1)) & \text{if } i,k \ge n_1, \\ \lambda_2(0\ (k-n_1)) \circ \lambda(in_1) & \text{if } i \le n_1 \le k. \end{cases}$$

This operation is associative. Finally we will identify in our notations $\lambda\colon [1] \to \Delta$ with the morphism $\lambda(01)$ in Δ.

The morphism $C^\bullet(V) \to C^\bullet(\widehat{V})$ is constructed as follows. Let $\lambda\colon [n] \to \Delta$ be a simplex in Δ and define λ_k by $\lambda(k) = [\lambda_k]$, $k = 0, 1, \dots, n$. Let $\Upsilon(\lambda)\colon [n] \to \lambda(n)$ be a morphism in Δ defined by

$$(\Upsilon(\lambda))(k) = \lambda(kn)(\lambda_k). \tag{2.3}$$

Then define the map $\iota\colon V^\bullet \to \widehat{V}^\bullet$ by the formula

$$(\iota(v))(\lambda) = \Upsilon(\lambda)_* v \quad \text{for } v \in V^n.$$

This is a map of cosimplicial vector spaces, and therefore it induces a morphism of complexes.

The morphism $\pi\colon C^\bullet(\widehat{V}) \to C^\bullet(V)$ is defined by the formula

$$\pi(f) = (-1)^{\frac{n(n+1)}{2}} \sum_{0 \le i_k \le k+1} (-1)^{i_0+\cdots+i_{n-1}} f(\partial^0_{i_0} * \partial^1_{i_1} * \cdots * \partial^{n-1}_{i_{n-1}}) \quad \text{for } f \in \widehat{V}^n$$

when $n > 0$, and $\pi(f)$ is V^0 component of f if $n = 0$.

The morphism $\iota \circ \pi$ is homotopic to Id with the homotopy $h\colon C^\bullet(\widehat{V}) \to C^{\bullet-1}(\widehat{V})$ given by the formula

$$hf(\lambda) = \sum_{j=0}^{n-1} \sum_{0 \le i_k \le k+1} (-1)^{i_0+\cdots+i_{j-1}} f(\partial^0_{i_0} * \cdots * \partial^{j-1}_{i_{j-1}} * \Upsilon(\lambda|_{[0j]}) * \lambda|_{[j\ (n-1)]})$$

for $f \in \widehat{V}^n$ when $n > 0$, and $h(f) = 0$ if $n = 0$.

The composition $\pi \circ \iota\colon C^\bullet(V) \to C^\bullet(V)$ preserves the normalized subcomplex $\overline{C}^\bullet(V)$ and acts as the identity on it. Therefore $\pi \circ \iota$ induces the identity map on cohomology. It follows that π and ι are quasiisomorphisms inverse to each other. $\square$

2.3 Covers. A cover (open cover) of a space X is a collection $\mathcal{U}$ of open subsets of X such that $\bigcup_{U\in\mathcal{U}} U = X$.

2.3.1 The nerve of a cover. Let $N_0\mathcal{U} = \coprod_{U\in\mathcal{U}} U$. There is a canonical augmentation map
$$\epsilon_0\colon N_0\mathcal{U} \xrightarrow{\coprod_{U\in\mathcal{U}}(U\hookrightarrow X)} X.$$
Let
$$N_p\mathcal{U} = N_0\mathcal{U} \times_X \cdots \times_X N_0\mathcal{U}$$
be the $(p+1)$-fold fiber product.

The assignment $N\mathcal{U}\colon \Delta \ni [p] \mapsto N_p\mathcal{U}$ extends to a simplicial space called *the nerve of the cover* $\mathcal{U}$. The effect of the face map ∂_i^n (respectively, the degeneracy map s_i^n) will be denoted by $d^i = d_n^i$ (respectively, $\varsigma_i = \varsigma_i^n$) and is given by the projection *along* the i^{th} factor (respectively, the diagonal embedding on the i^{th} factor). Therefore for every morphism $f\colon [p] \to [q]$ in Δ we have a morphism $N_q\mathcal{U} \to N_p\mathcal{U}$ which we denote by f^*. We will denote by f_* the operation $(f^*)^*$ of pull-back along f^*; if $\mathcal{F}$ is a sheaf on $N_p\mathcal{U}$ then $f_*\mathcal{F}$ is a sheaf on $N_q\mathcal{U}$.

For $0 \le i \le n+1$ we denote by $\mathrm{pr}_i^n\colon N_n\mathcal{U} \to N_0\mathcal{U}$ the projection onto the i^{th} factor. For $0 \le j \le m$, $0 \le i_j \le n$ the map $\mathrm{pr}_{i_0} \times \cdots \times \mathrm{pr}_{i_m}\colon N_n\mathcal{U} \to (N_0\mathcal{U})^m$ can be factored uniquely as a composition of a map $N_n\mathcal{U} \to N_m\mathcal{U}$ and the canonical imbedding $N_m\mathcal{U} \to (N_0\mathcal{U})^m$. We denote this map $N_n\mathcal{U} \to N_m\mathcal{U}$ by $\mathrm{pr}^n_{i_0\ldots i_m}$.

The augmentation map ϵ_0 extends to a morphism $\epsilon\colon N\mathcal{U} \to X$ where the latter is regarded as a constant simplicial space. Its component of degree n $\epsilon_n\colon N_n\mathcal{U} \to X$ is given by the formula $\epsilon_n = \epsilon_0 \circ \mathrm{pr}_i^n$. Here $0 \le i \le n+1$ is arbitrary.

2.3.2 Čech complex. Let $\mathcal{F}$ be a sheaf of abelian groups on X. One defines a cosimplicial group $\check{C}^\bullet(\mathcal{U},\mathcal{F}) = \Gamma(N_\bullet\mathcal{U};\epsilon^*\mathcal{F})$, with the cosimplicial structure induced by the simplicial structure of $N\mathcal{U}$. The associated complex is the Čech complex of the cover $\mathcal{U}$ with coefficients in $\mathcal{F}$. The differential $\check{\partial}$ in this complex is given by $\sum(-1)^i(d^i)^*$.

2.3.3 Refinement. Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two covers of X. A morphism of covers $\rho\colon \mathcal{U} \to \mathcal{V}$ is a map of sets $\rho\colon \mathcal{U} \to \mathcal{V}$ with the property $U \subseteq \rho(U)$ for all $U \in \mathcal{U}$.

A morphism $\rho\colon \mathcal{U} \to \mathcal{V}$ induces the map $N\rho\colon N\mathcal{U} \to N\mathcal{V}$ of simplicial spaces which commutes with respective augmentations to X. The map $N_0\rho$ is determined by the commutativity of
$$\begin{tikzcd} U \arrow[r] \arrow[d] & N_0\mathcal{U} \arrow[d, "N_0\rho"] \\ \rho(U) \arrow[r] & N_0\mathcal{V}. \end{tikzcd}$$
It is clear that the map $N_0\rho$ commutes with the respective augmentations (i.e. is a map of spaces over X) and, consequently induces maps $N_n\rho = (N_0\rho)^{\times_X n+1}$ which commute with all structure maps.

2.3.4 The category of covers. Let $\mathrm{Cov}(X)_0$ denote the set of open covers of X. For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)_0$ we denote by $\mathrm{Cov}(X)_1(\mathcal{U}, \mathcal{V})$ the set of morphisms $\mathcal{U} \to \mathcal{V}$. Let $\mathrm{Cov}(X)$ denote the category with objects $\mathrm{Cov}(X)_0$ and morphisms $\mathrm{Cov}(X)_1$. The construction of 2.3.1 is a functor

$$N : \mathrm{Cov}(X) \to \mathrm{Top}^{\Delta^{\mathrm{op}}}/X.$$

3 Deligne 2-groupoid and its cosimplicial analogues

We begin this section by recalling the definition of Deligne 2-groupoid and its relation with the deformation theory. We then describe the cosimplicial analogues of Deligne 2-groupoids and establish some of their properties.

3.1 Deligne 2-groupoid. In this subsection we review the construction of Deligne 2-groupoid of a nilpotent differential graded algebra (DGLA). We follow [11], [10] and references therein.

Suppose that $\mathfrak{g}$ is a nilpotent DGLA such that $\mathfrak{g}^i = 0$ for $i < -1$.

A Maurer–Cartan element of $\mathfrak{g}$ is an element $\gamma \in \mathfrak{g}^1$ satisfying

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0. \tag{3.1}$$

We denote by $\mathrm{MC}^2(\mathfrak{g})_0$ the set of Maurer–Cartan elements of $\mathfrak{g}$.

The unipotent group $\exp \mathfrak{g}^0$ acts on the set of Maurer–Cartan elements of $\mathfrak{g}$ by the gauge equivalences. This action is given by the formula

$$(\exp X) \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\mathrm{ad}\, X)^i}{(i+1)!}(dX + [\gamma, X])$$

If $\exp X$ is a gauge equivalence between two Maurer–Cartan elements γ_1 and $\gamma_2 = (\exp X) \cdot \gamma_1$ then

$$d + \mathrm{ad}\,\gamma_2 = \mathrm{Ad} \exp X\, (d + \mathrm{ad}\,\gamma_1). \tag{3.2}$$

We denote by $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ the set of gauge equivalences between γ_1, γ_2. The composition

$$\mathrm{MC}^2(\mathfrak{g})_1(\gamma_2, \gamma_3) \times \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2) \to \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_3)$$

is given by the product in the group $\exp \mathfrak{g}^0$.

If $\gamma \in \mathrm{MC}^2(\mathfrak{g})_0$ we can define a Lie bracket $[\cdot, \cdot]_\gamma$ on $\mathfrak{g}^{-1}$ by

$$[a, b]_\gamma = [a, db + [\gamma, b]]. \tag{3.3}$$

With this bracket $\mathfrak{g}^{-1}$ becomes a nilpotent Lie algebra. We denote by $\exp_\gamma \mathfrak{g}^{-1}$ the corresponding unipotent group, and by $\exp_\gamma$ the corresponding exponential map $\mathfrak{g}^{-1} \to \exp_\gamma \mathfrak{g}^{-1}$. If γ_1, γ_2 are two Maurer–Cartan elements, then the group $\exp_\gamma \mathfrak{g}^{-1}$

acts on $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1,\gamma_2)$. Let $\exp_\gamma t \in \exp_\gamma \mathfrak{g}^{-1}$ and let $\exp X \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1,\gamma_2)$. Then

$$(\exp_\gamma t)\cdot(\exp X) = \exp(dt + [\gamma, t])\exp X \in \exp \mathfrak{g}^0.$$

Such an element $\exp_\gamma t$ is called a 2-morphism between $\exp X$ and $(\exp t)\cdot(\exp X)$. We denote by $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$ the set of 2-morphisms between $\exp X$ and $\exp Y$. This set is endowed with a vertical composition given by the product in the group $\exp_\gamma \mathfrak{g}^{-1}$.

Let $\gamma_1,\gamma_2,\gamma_3 \in \mathrm{MC}^2(\mathfrak{g})_0$. Let $\exp X_{12}, \exp Y_{12} \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1,\gamma_2)$ and $\exp X_{23}$, $\exp Y_{23} \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_2,\gamma_3)$. Then one defines the horizontal composition

$$\begin{aligned}\otimes\colon\ &\mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23}) \times \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12})\\ &\to \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23}\exp X_{12}, \exp X_{23}\exp Y_{12})\end{aligned}$$

as follows. Let

$$\exp_{\gamma_2} t_{12} \in \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12}), \quad \exp_{\gamma_3} t_{23} \in \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23}).$$

Then

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3}(e^{\operatorname{ad} X_{23}}(t_{12}))$$

To summarize, the data described above forms a 2-groupoid which we denote by $\mathrm{MC}^2(\mathfrak{g})$ as follows:

1. the set of objects is $\mathrm{MC}^2(\mathfrak{g})_0$,
2. the groupoid of morphisms $\mathrm{MC}^2(\mathfrak{g})(\gamma_1,\gamma_2)$, $\gamma_i \in \mathrm{MC}^2(\mathfrak{g})_0$ consists of
   - objects, i.e., 1-morphisms in $\mathrm{MC}^2(\mathfrak{g})$ are given by $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1,\gamma_2)$ – the gauge transformations between γ_1 and γ_2,
   - morphisms between $\exp X, \exp Y \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1,\gamma_2)$ which are given by $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$.

A morphism of nilpotent DGLA $\phi\colon \mathfrak{g} \to \mathfrak{h}$ induces a functor $\phi\colon \mathrm{MC}^2(\mathfrak{g}) \to \mathrm{MC}^2(\mathfrak{g})$.

We have the following important result ([12], [11] and references therein).

Theorem 3.1. *Suppose that $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of DGLA and let $\mathfrak{m}$ be a nilpotent commutative ring. Then the induced map $\phi\colon \mathrm{MC}^2(\mathfrak{g}\otimes\mathfrak{m}) \to \mathrm{MC}^2(\mathfrak{h}\otimes\mathfrak{m})$ is an equivalence of 2-groupoids.*

3.2 Deformations and Deligne 2-groupoid. Let k be an algebraically closed field of characteristic zero.

3.2.1 Hochschild cochains. Suppose that A is a k-vector space. The k-vector space $C^n(A)$ of Hochschild cochains of degree $n \geq 0$ is defined by

$$C^n(A) := \mathrm{Hom}_k(A^{\otimes n}, A).$$

The graded vector space $\mathfrak{g}(A) := C^\bullet(A)[1]$ has a canonical structure of a graded Lie algebra under the Gerstenhaber bracket denoted by $[\ ,\ ]$ below. Namely, $C^\bullet(A)[1]$ is canonically isomorphic to the (graded) Lie algebra of derivations of the free associative co-algebra generated by $A[1]$.

Suppose in addition that A is equipped with a bilinear operation $\mu\colon A \otimes A \to A$, i.e. $\mu \in C^2(A) = \mathfrak{g}^1(A)$. The condition $[\mu, \mu] = 0$ is equivalent to the associativity of μ.

Suppose that A is an associative k-algebra with the product μ. For $a \in \mathfrak{g}(A)$ let $\delta(a) = [\mu, a]$. Thus, δ is a derivation of the graded Lie algebra $\mathfrak{g}(A)$. The associativity of μ implies that $\delta^2 = 0$, i.e. δ defines a differential on $\mathfrak{g}(A)$ called the Hochschild differential.

For a unital algebra the subspace of *normalized cochains* $\bar{C}^n(A) \subset C^n(A)$ is defined by

$$\bar{C}^n(A) := \mathrm{Hom}_k((A/k \cdot 1)^{\otimes n}, A).$$

The subspace $\bar{C}^\bullet(A)[1]$ is closed under the Gerstenhaber bracket and the action of the Hochschild differential and the inclusion $\bar{C}^\bullet(A)[1] \hookrightarrow C^\bullet(A)[1]$ is a quasi-isomorphism of DGLA.

Suppose in addition that R is a commutative Artin k-algebra with the nilpotent maximal ideal $\mathfrak{m}_R$ The DGLA $\mathfrak{g}(A) \otimes_k \mathfrak{m}_R$ is nilpotent and satisfies $\mathfrak{g}^i(A) \otimes_k \mathfrak{m}_R = 0$ for $i < -1$. Therefore, the Deligne 2-groupoid $\mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$ is defined. Moreover, it is clear that the assignment $R \mapsto \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)$ extends to a functor on the category of commutative Artin algebras.

3.2.2 Star products. Suppose that A is an associative unital k-algebra. Let m denote the product on A.

Let R be a commutative Artin k-algebra with maximal ideal $\mathfrak{m}_R$. There is a canonical isomorphism $R/\mathfrak{m}_R \cong k$.

An (R-)*star product* on A is an associative R-bilinear product on $A \otimes_k R$ such that the canonical isomorphism of k-vector spaces $(A \otimes_k R) \otimes_R k \cong A$ is an isomorphism of algebras. Thus, a star product is an R-deformation of A.

The 2-category of R-star products on A, denoted $\mathrm{Def}(A)(R)$, is defined as the subcategory of the 2-category Alg^2_R of R-algebras (see 4.1.1) with

- Objects: R-star products on A,
- 1-morphisms $\phi\colon m_1 \to m_2$ between the star products μ_i those R-algebra homomorphisms $\phi\colon (A \otimes_k R, m_1) \to (A \otimes_k R, m_2)$ which reduce to the identity map modulo $\mathfrak{m}_R$, i.e. $\phi \otimes_R k = \mathrm{Id}_A$,

- 2-morphisms $b\colon \phi \to \psi$, where $\phi, \psi\colon m_1 \to m_2$ are two 1-morphisms, are elements $b \in 1 + A \otimes_k \mathfrak{m}_R \subset A \otimes_k R$ such that $m_2(\phi(a), b) = m_2(b, \psi(a))$ for all $a \in A \otimes_k R$.

It follows easily from the above definition and the nilpotency of $\mathfrak{m}_R$ that $\operatorname{Def}(A)(R)$ is a 2-groupoid.

Note that $\operatorname{Def}(A)(R)$ is non-empty: it contains the trivial deformation, i.e. the star product, still denoted m, which is the R-bilinear extension of the product on A.

It is clear that the assignment $R \mapsto \operatorname{Def}(A)(R)$ extends to a functor on the category of commutative Artin k-algebras.

3.2.3 Star products and the Deligne 2-groupoid. We continue in notations introduced above. In particular, we are considering an associative unital k-algebra A. The product $m \in C^2(A)$ determines a cochain, still denoted $m \in \mathfrak{g}^1(A) \otimes_k R$, hence the Hochschild differential $\delta = [m, \ ]$ in $\mathfrak{g}(A) \otimes_k R$ for any commutative Artin k-algebra R.

Suppose that m' is an R-star product on A. Since $\mu(m') := m' - m = 0 \mod \mathfrak{m}_R$ we have $\mu(m') \in \mathfrak{g}^1(A) \otimes_k \mathfrak{m}_R$. Moreover, the associativity of m' implies that $\mu(m')$ satisfies the Maurer–Cartan equation, i.e. $\mu(m') \in \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R)_0$.

It is easy to see that the assignment $m' \mapsto \mu(m')$ extends to a functor

$$\operatorname{Def}(A)(R) \to \mathrm{MC}^2(\mathfrak{g}(A) \otimes_k \mathfrak{m}_R). \tag{3.4}$$

The following proposition is well known (cf. [9], [11], [10]).

Proposition 3.2. *The functor* (3.4) *is an isomorphism of* 2*-groupoids.*

3.2.4 Star products on sheaves of algebras. The above considerations generalize to sheaves of algebras in a straightforward way.

Suppose that $\mathcal{A}$ is a sheaf of k-algebras on a space X. Let $m\colon \mathcal{A} \otimes_k \mathcal{A} \to \mathcal{A}$ denote the product.

An R-star product on $\mathcal{A}$ is a structure of a sheaf of an associative algebras on $\mathcal{A} \otimes_k R$ which reduces to μ modulo the maximal ideal $\mathfrak{m}_R$. The 2-category (groupoid) of R star products on $\mathcal{A}$, denoted $\operatorname{Def}(\mathcal{A})(R)$ is defined just as in the case of algebras; we leave the details to the reader.

The sheaf of Hochschild cochains of degree n is defined by

$$C^n(\mathcal{A}) := \underline{\operatorname{Hom}}(\mathcal{A}^{\otimes n}, \mathcal{A}).$$

We have the sheaf of DGLA $\mathfrak{g}(A) := C^\bullet(\mathcal{A})[1]$, and hence the nilpotent DGLA $\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R)$ for every commutative Artin k-algebra R concentrated in degrees ≥ -1. Therefore, the 2-groupoid $\mathrm{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R)$ is defined.

The canonical functor $\operatorname{Def}(\mathcal{A})(R) \to \mathrm{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R)$ defined just as in the case of algebras is an isomorphism of 2-groupoids.

3.3 $\mathfrak{G}$-stacks. Suppose that $\mathfrak{G}\colon [n] \to \mathfrak{G}^n$ is a cosimplicial DGLA. We assume that each $\mathfrak{G}^n$ is a nilpotent DGLA. We denote its component of degree i by $\mathfrak{G}^{n,i}$ and assume that $\mathfrak{G}^{n,i} = 0$ for $i < -1$.

Definition 3.3. A $\mathfrak{G}$-stack is a triple $\gamma = (\gamma^0, \gamma^1, \gamma^2)$, where

- $\gamma^0 \in \mathrm{MC}^2(\mathfrak{G}^0)_0$,
- $\gamma^1 \in \mathrm{MC}^2(\mathfrak{G}^1)_1(\partial_0^0\gamma^0, \partial_1^0\gamma^0)$,
  satisfying the condition
  $$s_0^1\gamma^1 = \mathrm{Id},$$
- $\gamma^2 \in \mathrm{MC}^2(\mathfrak{G}^2)_2(\partial_2^1(\gamma^1) \circ \partial_0^1(\gamma^1), \partial_1^1(\gamma^1))$

satisfying the conditions

$$\partial_2^2\gamma_2 \circ (\mathrm{Id} \otimes \partial_0^2\gamma_2) = \partial_1^2\gamma_2 \circ (\partial_3^2\gamma_2 \otimes \mathrm{Id}), \quad \text{and} \quad s_0^2\gamma^2 = s_1^2\gamma^2 = \mathrm{Id}. \tag{3.5}$$

Let $\mathrm{Stack}(\mathfrak{G})_0$ denote the set of $\mathfrak{G}$-stacks.

Definition 3.4. For $\gamma_1, \gamma_2 \in \mathrm{Stack}(\mathfrak{G})_0$ a 1-morphism $\gimel\colon \gamma_1 \to \gamma_2$ is a pair $\gimel = (\gimel^1, \gimel^2)$, where $\gimel^1 \in \mathrm{MC}^2(\mathfrak{G}^0)_1(\gamma_1^0, \gamma_2^0)$, $\gimel^2 \in \mathrm{MC}^2(\mathfrak{G}^1)_2(\gamma_2^1 \circ \partial_0^0(\gimel^1), \partial_1^0(\gimel^1) \circ \gamma_1^1)$, satisfying

$$\begin{gathered}(\mathrm{Id} \otimes \gamma_1^2) \circ (\partial_2^1\gimel^2 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \partial_0^1\gimel^2) = \partial_1^1\gimel^2 \circ (\gamma_2^2 \otimes \mathrm{Id}),\\ s_0^1\gimel^2 = \mathrm{Id}.\end{gathered} \tag{3.6}$$

Let $\mathrm{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ denote the set of 1-morphisms $\gamma_1 \to \gamma_2$.

Composition of 1-morphisms $\gimel\colon \gamma_1 \to \gamma_2$ and $\daleth\colon \gamma_2 \to \gamma_3$ is given by $(\daleth^1 \circ \gimel^1, (\gimel^2 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \daleth^2))$.

Definition 3.5. For $\gimel_1, \gimel_2 \in \mathrm{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ a 2-morphism $\phi\colon \gimel_1 \to \gimel_2$ is a 2-morphism $\phi \in \mathrm{MC}^2(\mathfrak{G}^0)_2(\gimel_1^1, \gimel_2^1)$ which satisfies

$$\gimel_2^2 \circ (\mathrm{Id} \otimes \partial_0^0\phi) = (\partial_1^0\phi \otimes \mathrm{Id}) \circ \gimel_1^2. \tag{3.7}$$

Let $\mathrm{Stack}(\mathfrak{G})_2(\gimel_1, \gimel_2)$ denote the set of 2-morphisms.

Compositions of 2-morphisms are given by the compositions in $\mathrm{MC}^2(\mathfrak{G}^0)_2$.

For $\gamma_1, \gamma_2 \in \mathrm{Stack}(\mathfrak{G})_0$, we have the groupoid $\mathrm{Stack}(\mathfrak{G})(\gamma_1, \gamma_2)$ with the set of objects $\mathrm{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ and the set of morphisms $\mathrm{Stack}(\mathfrak{G})_2(\gimel_1, \gimel_2)$ under vertical composition.

Every morphism θ of cosimplicial DGLA induces in an obvious manner a functor $\theta_*\colon \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{H} \otimes \mathfrak{m})$

We have the following cosimplicial analogue of Theorem 3.1:

Theorem 3.6. *Suppose that $\theta\colon \mathfrak{G} \to \mathfrak{H}$ is a quasi-isomorphism of cosimplicial DGLA and $\mathfrak{m}$ is a commutative nilpotent ring. Then the induced map $\theta_*\colon \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{H} \otimes \mathfrak{m})$ is an equivalence.*

Proof. The proof can be obtained by applying Theorem 3.1 repeatedly.

Let $\gamma_1,\ \gamma_2 \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$, and let $\gimel_1,\ \gimel_2$ be two 1-morphisms between γ_1 and γ_2. Note that $\theta_*\colon\ \mathrm{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_2(\gimel_1^1, \gimel_2^1) \to \mathrm{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_2(\theta_*\gimel_1^1, \theta_*\gimel_2^1)$ is a bijection by Theorem 3.1. Injectivity of the map $\theta_*\colon\ \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_2(\gimel_1, \gimel_2) \to \mathrm{Stack}(\mathfrak{H} \otimes \mathfrak{m})_2(\theta_*\gimel_1, \theta_*\gimel_2)$ follows immediately.

For the surjectivity, note that an element of $\mathrm{Stack}(\mathfrak{H}\otimes\mathfrak{m})_2(\theta_*\gimel_1, \theta_*\gimel_2)$ is necessarily given by $\theta_*\phi$ for some $\phi \in \mathrm{MC}^2(\mathfrak{G}^0\otimes\mathfrak{m})(\gimel_1^1, \gimel_2^1)_2$ and the following identity is satisfied in $\mathrm{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})(\theta_*(\gamma_2^1 \circ \partial_0^0\gimel_1^1), \theta_*(\partial_1^0\gimel_2^1 \circ \gamma_1^1))_2$:

$$\theta_*(\gimel_2^2 \circ (\partial_0^1\phi \otimes \mathrm{Id})) = \theta_*((\mathrm{Id} \otimes \partial_1^1\phi) \circ \gimel_1^2).$$

Since $\theta_*\colon\ \mathrm{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})(\gamma_2^1 \circ \partial_0^0\gimel_1^1, \partial_1^0\gimel_2^1 \circ \gamma_1^1)_2 \to \mathrm{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})(\theta_*(\gamma_2^1 \circ \partial_0^0\gimel_1^1), \theta_*(\partial_1^0\gimel_2^1 \circ \gamma_1^1))_2$ is bijective, and in particular injective, $\gimel_2^2 \circ (\partial_0^1\phi \otimes \mathrm{Id}) = (\mathrm{Id} \otimes \partial_1^1\phi) \circ \gimel_1^2$, and ϕ defines an element in $\mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_2(\gimel_1, \gimel_2)$.

Next, let $\gamma_1,\ \gamma_2 \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$, and let $\gimel$ be a 1-morphisms between $\theta_*\gamma_1$ and $\theta_*\gamma_2$. We show that there exists $\daleth \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ such that $\theta_*\daleth$ is isomorphic to $\gimel$. Indeed, by Theorem 3.1, there exists $\daleth^1 \in \mathrm{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ such that $\mathrm{MC}^2(\mathfrak{H}^0\otimes\mathfrak{m})_2(\theta_*\daleth^1, \gimel) \neq \emptyset$. Let $\phi \in \mathrm{MC}^2(\mathfrak{H}^0\otimes\mathfrak{m})_2(\theta_*\daleth^1, \gimel)$. Define $\psi \in \mathrm{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\theta_*(\gamma_2^1 \circ \partial_0^0\daleth^1), \theta_*(\partial_1^0\daleth^1 \circ \gamma_1^1))$ by $\psi = (\partial_1^0\phi \otimes \mathrm{Id})^{-1} \circ \gimel^2 \circ (\mathrm{Id} \otimes \partial_0^0\phi)$. It is easy to verify that the following identities holds:

$$(\mathrm{Id} \otimes \theta_*\gamma_1^2) \circ (\partial_2^1\psi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \partial_0^1\psi) = \partial_1^1\psi \circ (\theta_*\gamma_2^2 \otimes \mathrm{Id}),$$
$$s_0^1\psi = \mathrm{Id}.$$

By bijectivity of θ_* on MC^2 there exists a unique $\daleth^2 \in \mathrm{MC}^2(\mathfrak{H}^1\otimes\mathfrak{m})_2(\gamma_2^1\circ\partial_0^0\daleth^1, \partial_1^0\daleth^1 \circ \gamma_1^1)$ such that $\theta_*\daleth^2 = \psi$. Moreover, as before, injectivity of θ_* implies that the conditions (3.6) are satisfied. Therefore $\daleth = (\daleth^1, \daleth^2)$ defines a 1-morphism $\gamma_1 \to \gamma_2$ and ϕ is a 2-morphism $\theta_*\daleth \to \gimel$.

Now, let $\gamma \in \mathrm{Stack}(\mathfrak{H} \otimes \mathfrak{m})_0$. We construct $\lambda \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$ in such a way that $\mathrm{Stack}(\mathfrak{H} \otimes \mathfrak{m})_1(\theta_*\lambda, \gamma) \neq \emptyset$. By Theorem 3.1 there exists $\lambda^0 \in \mathrm{MC}^2(\mathfrak{G}^0 \otimes \mathfrak{m})_0$ such that $\mathrm{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_1(\theta_*\lambda^0, \gamma^0) \neq \emptyset$. Let $\gimel^1 \in \mathrm{MC}^2(\mathfrak{H}^0 \otimes \mathfrak{m})_1(\theta_*\lambda, \gamma)$. Applying Theorem 3.1 again we obtain that there exists $\mu \in \mathrm{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})_1(\partial_0^0\lambda^0, \partial_1^0\lambda^0)$ such that there exists $\phi \in \mathrm{MC}^2(\mathfrak{H}^1 \otimes \mathfrak{m})_2(\gamma^1 \circ \partial_0^0\gimel^1, \partial_1^0\gimel^1 \circ \theta_*\mu)$. Then $s_0^1\phi$ is then a 2-morphism $\gimel^1 \to \gimel^1 \circ (s_0^1\mu)$, which induces a 2-morphism $\psi\colon (s_0^1\mu)^{-1} \to \mathrm{Id}$.

As a next step set $\lambda^1 = \mu \circ (\partial_0^0(s_0^1\mu))^{-1} \in \mathrm{MC}^2(\mathfrak{G}^1 \otimes \mathfrak{m})_1(\partial_0^0\lambda^0, \partial_1^0\lambda^0)$, $\gimel^2 = \phi\otimes(\partial_0^0\psi)^{-1} \in \mathrm{MC}^2(\mathfrak{H}^1\otimes\mathfrak{m})_2(\gamma^1\circ\partial_0^0\gimel^1, \partial_1^0\gimel^1\circ\theta_*\lambda^1)$. It is easy to see that $s_0^1\lambda^1 = \mathrm{Id}$, $s_0^1\gimel^2 = \mathrm{Id}$.

We then conclude that there exists a unique λ^2 such that

$$(\mathrm{Id} \otimes \theta_*\lambda^2) \circ (\partial_2^1\gimel^2 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \partial_0^1\gimel^2) = \partial_1^1\gimel^2 \circ (\gamma^2 \otimes \mathrm{Id}).$$

Such a λ^2 necessarily satisfies the conditions

$$\partial_2^2\lambda^2 \circ (\mathrm{Id} \otimes \partial_0^2\lambda^2) = \partial_1^2\lambda^2 \circ (\partial_3^2\lambda^2 \otimes \mathrm{Id}),$$
$$s_0^2\lambda^2 = s_1^2\lambda^2 = \mathrm{Id}.$$

Therefore $\lambda = (\lambda^0, \lambda^1, \lambda^2) \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})$, and $\gimel = (\gimel^1, \gimel^2)$ defines a 1-morphism $\theta_*\lambda \to \gamma$. □

3.4 Acyclicity and strictness

Definition 3.7. A $\mathfrak{G}$-stack $(\gamma^0, \gamma^1, \gamma^2)$ is called *strict* if $\partial_0^0\gamma^0 = \partial_1^0\gamma^0$, $\gamma_1 = \mathrm{Id}$ and $\gamma_2 = \mathrm{Id}$.

Let $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0$ denote the subset of strict $\mathfrak{G}$-stacks.

Lemma 3.8. $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0 = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))_0$.

Definition 3.9. For elements γ_1, γ_2 in $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0$, a 1-morphism $\gimel = (\gimel^1, \gimel^2) \in \mathrm{Stack}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ is called *strict* if $\partial_0^0(\gimel^1) = \partial_1^0(\gimel^1)$ and $\gimel^2 = \mathrm{Id}$.

For $\gamma_1, \gamma_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0$ we denote by $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ the subset of strict morphisms.

Lemma 3.10. *For $\gamma_1, \gamma_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0$ one has*

$$\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_1(\gamma_1, \gamma_2) = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))_1.$$

For $\gamma_1, \gamma_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_0$ let $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})(\gamma_1, \gamma_2)$ denote the full subcategory of $\mathrm{Stack}(\mathfrak{G})(\gamma_1, \gamma_2)$ with objects $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_1(\gamma_1, \gamma_2)$.

Thus, we have the 2-groupoids $\mathrm{Stack}(\mathfrak{G})$ and $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})$ and an embedding of the latter into the former which is fully faithful on the respective groupoids of 1-morphisms.

Lemma 3.11. $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G}) = \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1))$.

Suppose that $\mathfrak{G}$ is a cosimplicial DGLA. For each n and i we have the vector space $\mathfrak{G}^{n,i}$, namely the degree i component of $\mathfrak{G}^n$. The assignment $n \mapsto \mathfrak{G}^{n,i}$ is a cosimplicial vector space $\mathfrak{G}^{\bullet,i}$.

We will consider the following acyclicity condition on the cosimplicial DGLA $\mathfrak{G}$:

$$H^p(\mathfrak{G}^{\bullet,i}) = 0 \quad \text{for all } i \in \mathbb{Z} \text{ and for } p \neq 0. \tag{3.8}$$

Theorem 3.12. *Suppose that $\mathfrak{G}$ is a cosimplicial DGLA which satisfies the condition* (3.8)*, and $\mathfrak{m}$ a commutative nilpotent ring. Then, the functor ι:* $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})$ *is an equivalence.*

Proof. As we already noted before, it is immediate from the definitions that if γ_1, $\gamma_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_0$, and $\gimel_1, \gimel_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$, then the mapping ι: $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_2(\gimel_1, \gimel_2) \to \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_2(\gimel_1, \gimel_2)$ is a bijection.

Let now $\gamma_1, \gamma_2 \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_0$, $\gimel = (\gimel^1, \gimel^2) \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$. We show that there exists $\daleth \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_1(\gamma_1, \gamma_2)$ and a 2-morphism $\phi\colon \gimel \to \daleth$. Let $\gimel^2 = \exp_{\partial_0^0\gamma_2^0} g$, where $g \in (\mathfrak{G}^1 \otimes \mathfrak{m})$. Then $\partial_0^1 g - \partial_1^1 g + \partial_2^1 g = 0 \mod \mathfrak{m}^2$. As a consequence of the acyclicity condition there exists $a \in (\mathfrak{G}^0 \otimes \mathfrak{m})$ such that $\partial_0^0 a - \partial_1^0 a = g \mod \mathfrak{m}^2$. Set $\phi_1 = \exp_{\gamma_2^0} a$. Define then $\daleth_1 = (\phi_1 \cdot \gimel^1, (\partial_1^0\phi_1 \otimes \mathrm{Id}) \circ$

$\gimel^2 \circ (\mathrm{Id} \otimes \partial_0^0\phi_1)^{-1}) \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$. Note that ϕ_1 defines a 2-morphism $\gimel \to \daleth_1$. Note also that $\daleth_1^2 \in \exp_{\partial_0^0\gamma_2^0}(\mathfrak{G} \otimes \mathfrak{m}^2)$. Proceeding inductively one constructs a sequence $\daleth_k \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_1(\gamma_1, \gamma_2)$ such that $\daleth_k^2 \in \exp(\mathfrak{G} \otimes \mathfrak{m}^{k+1})$ and 2-morphisms $\phi_k : \gimel \to \daleth_k$, $\phi_{k+1} = \phi_k \mod \mathfrak{m}^k$. Since $\mathfrak{m}$ is nilpotent, for k large enough we have $\daleth_k \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G})_1(\gamma_1, \gamma_2)$.

Assume now that $\gamma \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$. We will construct $\lambda \in \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G} \otimes \mathfrak{m})_0$ and a 1-morphism $\gimel : \gamma \to \lambda$.

We begin by constructing $\mu \in \mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m})_0$ such that $\mu^2 = \mathrm{Id}$ and a 1-morphism $\daleth : \gamma \to \lambda$. We have: $\gamma^2 = \exp_{\partial_1^1\partial_1^0\gamma^0} c$, $c \in (\mathfrak{G}^2 \otimes \mathfrak{m})$. In view of the equations (3.5) c satisfies the identities

$$\partial_0^2 c - \partial_1^2 c + \partial_2^2 c - \partial_3^2 c = 0 \mod \mathfrak{m}^2, \quad \text{and} \quad s_0^2 c = s_1^2 c = 0. \tag{3.9}$$

By the acyclicity of the normalized complex we can find $b \in \mathfrak{G}^1 \otimes \mathfrak{m}$ such that $\partial_0^1 b - \partial_1^1 b + \partial_2^1 b = c \mod \mathfrak{m}^2$, $s_0^1 b = 0$. Let $\phi_1 = \exp_{\partial_1^0\gamma^0}(-b)$. Define then $\mu_1 = (\mu_1^0, \mu_1^1, \mu_1^2)$ where $\mu_1^0 = \gamma^0$, $\mu_1^1 = \phi_1 \cdot \gamma^1$, and μ_1^2 is such that

$$(\mathrm{Id} \otimes \gamma^2) \circ (\partial_2^1\phi_1 \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \partial_0^1\phi_1) = \partial_1^1\phi_1 \circ (\mu_1^2 \otimes \mathrm{Id}).$$

Note that $\mu_1^2 = \mathrm{Id} \mod \mathfrak{m}^2$ and (Id, ϕ_1) is a 1-morphism $\gamma \to \mu_1$. As before we can construct a sequence μ_k such that $\mu_k^2 = \mathrm{Id} \mod \mathfrak{m}^{k+1}$, and 1-morphisms $(\mathrm{Id}, \phi_k) : \gamma \to \mu_k$, $\phi_{k+1} = \phi_k \mod \mathfrak{m}^k$. As before we conclude that this gives the desired construction of μ. The rest of the proof, i.e. the construction of λ is completely analogous. □

Corollary 3.13. *Suppose that $\mathfrak{G}$ is a cosimplicial DGLA which satisfies the condition* (3.8). *Then there is a canonical equivalence:*

$$\mathrm{Stack}(\mathfrak{G} \otimes \mathfrak{m}) \cong \mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1) \otimes \mathfrak{m}).$$

Proof. Combine Lemma 3.11 with Theorem 3.12. □

4 Algebroid stacks

In this section we review the notions of algebroid stack and twisted form. We also define the notion of descent datum and relate it with algebroid stacks.

4.1 Algebroids and algebroid stacks

4.1.1 Algebroids. For a category $\mathcal{C}$ we denote by $i\mathcal{C}$ the subcategory of isomorphisms in $\mathcal{C}$; equivalently, $i\mathcal{C}$ is the maximal subgroupoid in $\mathcal{C}$.

Suppose that R is a commutative k-algebra.

Definition 4.1. An *R-algebroid* is a *nonempty* R-linear category $\mathcal{C}$ such that the groupoid $i\mathcal{C}$ is connected

Let Algd_R denote the 2-category of R-algebroids (full 2-subcategory of the 2-category of R-linear categories).

Suppose that A is an R-algebra. The R-linear category with one object and morphisms A is an R-algebroid denoted A^+.

Suppose that $\mathcal{C}$ is an R-algebroid and L is an object of $\mathcal{C}$. Let $A = \mathrm{End}_{\mathcal{C}}(L)$. The functor $A^+ \to \mathcal{C}$ which sends the unique object of A^+ to L is an equivalence.

Let Alg^2_R denote the 2-category with

- objects: R-algebras,
- 1-morphisms: homomorphism of R-algebras,
- 2-morphisms $\phi \to \psi$, where $\phi, \psi \colon A \to B$ are two 1-morphisms: elements $b \in B$ such that $\phi(a) \cdot b = b \cdot \psi(a)$ for all $a \in A$.

It is clear that the 1- and the 2- morphisms in Alg^2_R as defined above induce 1- and 2-morphisms of the corresponding algebroids under the assignment $A \mapsto A^+$. The structure of a 2-category on Alg^2_R (i.e. composition of 1- and 2-morphisms) is determined by the requirement that the assignment $A \mapsto A^+$ extends to an embedding $(\bullet)^+ \colon \mathrm{Alg}^2_R \to \mathrm{Algd}_R$.

Suppose that $R \to S$ is a morphism of commutative k-algebras. The assignment $A \to A \otimes_R S$ extends to a functor $(\bullet) \otimes_R S \colon \mathrm{Alg}^2_R \to \mathrm{Alg}^2_S$.

4.1.2 Algebroid stacks. We refer the reader to [1] and [20] for basic definitions. We will use the notion of fibered category interchangeably with that of a pseudo-functor. A *prestack* $\mathcal{C}$ on a space X is a category fibered over the category of open subsets of X, equivalently, a pseudo-functor $U \mapsto \mathcal{C}(U)$, satisfying the following additional requirement. For an open subset U of X and two objects $A, B \in \mathcal{C}(U)$ we have the presheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, B)$ on U defined by $U \supseteq V \mapsto \mathrm{Hom}_{\mathcal{C}(V)}(A|_V, B|_B)$. The fibered category $\mathcal{C}$ is a prestack if for any U, $A, B \in \mathcal{C}(U)$, the presheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, B)$ is a sheaf. A prestack is a *stack* if, in addition, it satisfies the condition of effective descent for objects. For a prestack $\mathcal{C}$ we denote the associated stack by $\widetilde{\mathcal{C}}$.

Definition 4.2. A stack in R-linear categories $\mathcal{C}$ on X is an *R-algebroid stack* if it is locally nonempty and locally connected, i.e. satisfies

1. any point $x \in X$ has a neighborhood U such that $\mathcal{C}(U)$ is nonempty;
2. for any $U \subseteq X$, $x \in U$, $A, B \in \mathcal{C}(U)$ there exits a neighborhood $V \subseteq U$ of x and an isomorphism $A|_V \cong B|_V$.

Remark 4.3. Equivalently, the stack associated to the substack of isomorphisms $\widetilde{i\mathcal{C}}$ is a gerbe.

Example 4.4. Suppose that $\mathcal{A}$ is a sheaf of R-algebras on X. The assignment $X \supseteq U \mapsto \mathcal{A}(U)^+$ extends in an obvious way to a prestack in R-algebroids denoted $\mathcal{A}^+$.

The associated stack $\widetilde{\mathcal{A}^+}$ is canonically equivalent to the stack of locally free $\mathcal{A}^{\mathrm{op}}$-modules of rank one. The canonical morphism $\mathcal{A}^+ \to \widetilde{\mathcal{A}^+}$ sends the unique (locally defined) object of $\mathcal{A}^+$ to the free module of rank one.

1-morphisms and 2-morphisms of R-algebroid stacks are those of stacks in R-linear categories. We denote the 2-category of R-algebroid stacks by $\mathrm{AlgStack}_R(X)$.

4.2 Descent data

4.2.1 Convolution data

Definition 4.5. An *R-linear convolution datum* is a triple $(\mathcal{U}, \mathcal{A}_{01}, \mathcal{A}_{012})$ consisting of

- a cover $\mathcal{U} \in \mathrm{Cov}(X)$,
- a sheaf $\mathcal{A}_{01}$ of R-modules $\mathcal{A}_{01}$ on $N_1\mathcal{U}$,
- a morphism
$$\mathcal{A}_{012} \colon (\mathrm{pr}^2_{01})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^2_{12})^*\mathcal{A}_{01} \to (\mathrm{pr}^2_{02})^*\mathcal{A}_{01} \tag{4.1}$$
of R-modules

subject to the associativity condition expressed by the commutativity of the diagram

$$\begin{tikzcd}
(\mathrm{pr}^3_{01})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^3_{12})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^3_{23})^*\mathcal{A}_{01} \arrow[r, "(\mathrm{pr}^3_{012})^*(\mathcal{A}_{012})\otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes(\mathrm{pr}^3_{123})^*(\mathcal{A}_{012}"] & (\mathrm{pr}^3_{02})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^3_{23})^*\mathcal{A}_{01} \arrow[d, "(\mathrm{pr}^3_{023})^*(\mathcal{A}_{012})"] \\
(\mathrm{pr}^3_{01})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^3_{13})^*\mathcal{A}_{01} \arrow[r, "(\mathrm{pr}^3_{013})^*(\mathcal{A}_{012})"] & (\mathrm{pr}^3_{03})^*\mathcal{A}_{01}.
\end{tikzcd}$$

For a convolution datum $(\mathcal{U}, \mathcal{A}_{01}, \mathcal{A}_{012})$ we denote by $\underline{\mathcal{A}}$ the pair $(\mathcal{A}_{01}, \mathcal{A}_{012})$ and abbreviate the convolution datum by $(\mathcal{U}, \underline{\mathcal{A}})$.

For a convolution datum $(\mathcal{U}, \underline{\mathcal{A}})$ let

- $\mathcal{A} := (\mathrm{pr}^0_{00})^*\mathcal{A}_{01}$; $\mathcal{A}$ is a sheaf of R-modules on $N_0\mathcal{U}$,
- $\mathcal{A}^p_i := (\mathrm{pr}^p_i)^*\mathcal{A}$; thus for every p we get sheaves $\mathcal{A}^p_i$, $0 \le i \le p$ on $N_p\mathcal{U}$.

The identities $\mathrm{pr}^0_{01} \circ \mathrm{pr}^0_{000} = \mathrm{pr}^0_{12} \circ \mathrm{pr}^0_{000} = \mathrm{pr}_{02} \circ \mathrm{pr}^0_{000} = \mathrm{pr}^0_{00}$ imply that the pull-back of $\mathcal{A}_{012}$ to $N_0\mathcal{U}$ by pr^0_{000} gives the pairing

$$(\mathrm{pr}^0_{000})^*(\mathcal{A}_{012}) \colon \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}. \tag{4.2}$$

The associativity condition implies that the pairing (4.2) endows $\mathcal{A}$ with a structure of a sheaf of associative R-algebras on $N_0\mathcal{U}$.

The sheaf $\mathcal{A}_i^p$ is endowed with the associative R-algebra structure induced by that on $\mathcal{A}$. We denote by $\mathcal{A}_{ii}^p$ the $\mathcal{A}_i^p \otimes_R (\mathcal{A}_i^p)^{\mathrm{op}}$-module $\mathcal{A}_i^p$, with the module structure given by the left and right multiplication.

The identities $\mathrm{pr}_{01}^2 \circ \mathrm{pr}_{001}^1 = \mathrm{pr}_{00}^0 \circ \mathrm{pr}_0^1$, $\mathrm{pr}_{12}^2 \circ \mathrm{pr}_{001}^1 = \mathrm{pr}_{02}^2 \circ \mathrm{pr}_{001}^1 = \mathrm{Id}$ imply that the pull-back of $\mathcal{A}_{012}$ to $N_1\mathcal{U}$ by pr_{001}^1 gives the pairing

$$(\mathrm{pr}_{001}^1)^*(\mathcal{A}_{012})\colon \mathcal{A}_0^1 \otimes_R \mathcal{A}_{01} \to \mathcal{A}_{01}. \tag{4.3}$$

The associativity condition implies that the pairing (4.3) endows $\mathcal{A}_{01}$ with a structure of a $\mathcal{A}_0^1$-module. Similarly, the pull-back of $\mathcal{A}_{012}$ to $N_1\mathcal{U}$ by pr_{011}^1 endows $\mathcal{A}_{01}$ with a structure of a $(\mathcal{A}_1^1)^{\mathrm{op}}$-module. Together, the two module structures define a structure of a $\mathcal{A}_0^1 \otimes_R (\mathcal{A}_1^1)^{\mathrm{op}}$-module on $\mathcal{A}_{01}$.

The map (4.1) factors through the map

$$(\mathrm{pr}_{01}^2)^*\mathcal{A}_{01} \otimes_{\mathcal{A}_1^2} (\mathrm{pr}_{12}^2)^*\mathcal{A}_{01} \to (\mathrm{pr}_{02}^2)^*\mathcal{A}_{01}. \tag{4.4}$$

Definition 4.6. A *unit* for a convolution datum $\underline{\mathcal{A}}$ is a morphism of R-modules

$$\mathbf{1}\colon R \to \mathcal{A}$$

such that the compositions

$$\mathcal{A}_{01} \xrightarrow{\mathbf{1}\otimes\mathrm{Id}} \mathcal{A}_0^1 \otimes_R \mathcal{A}_{01} \xrightarrow{(\mathrm{pr}_{001}^1)^*(\mathcal{A}_{012})} \mathcal{A}_{01}$$

and

$$\mathcal{A}_{01} \xrightarrow{\mathrm{Id}\otimes\mathbf{1}} \mathcal{A}_{01} \otimes_R \mathcal{A}_1^1 \xrightarrow{(\mathrm{pr}_{011}^1)^*(\mathcal{A}_{012})} \mathcal{A}_{01}$$

are equal to the respective identity morphisms.

4.2.2 Descent data

Definition 4.7. A *descent datum* on X is an R-linear convolution datum $(\mathcal{U}, \underline{\mathcal{A}})$ on X with a unit which satisfies the following additional conditions:

1. $\mathcal{A}_{01}$ is locally free of rank one as a $\mathcal{A}_0^1$-module and as a $(\mathcal{A}_1^1)^{\mathrm{op}}$-module;
2. the map (4.4) is an isomorphism.

4.2.3 1-morphisms. Suppose given convolution data $(\mathcal{U}, \underline{\mathcal{A}})$ and $(\mathcal{U}, \underline{\mathcal{B}})$ as in Definition 4.5.

Definition 4.8. A 1-*morphism of convolution data*

$$\underline{\phi}\colon (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{B}})$$

is a morphism of R-modules $\phi_{01}\colon \mathcal{A}_{01} \to \mathcal{B}_{01}$ such that the diagram

$$\begin{array}{ccc} (\mathrm{pr}^2_{01})^*\mathcal{A}_{01} \otimes_R (\mathrm{pr}^2_{12})^*\mathcal{A}_{01} & \xrightarrow{\mathcal{A}_{012}} & (\mathrm{pr}^2_{02})^*\mathcal{A}_{01} \\ \Big\downarrow{\scriptstyle (\mathrm{pr}^2_{01})^*(\phi_{01})\otimes(\mathrm{pr}^2_{12})^*(\phi_{01})} & & \Big\downarrow{\scriptstyle (\mathrm{pr}^2_{02})^*(\phi_{01})} \\ (\mathrm{pr}^2_{01})^*\mathcal{B}_{01} \otimes_R (\mathrm{pr}^2_{12})^*\mathcal{B}_{01} & \xrightarrow{\mathcal{B}_{012}} & (\mathrm{pr}^2_{02})^*\mathcal{B}_{01} \end{array} \tag{4.5}$$

is commutative.

The 1-morphism $\underline{\phi}$ induces a morphism of R-algebras $\phi := (\mathrm{pr}^0_{00})^*(\phi_{01})\colon \mathcal{A} \to \mathcal{B}$ on $N_0\mathcal{U}$ as well as morphisms $\phi^p_i := (\mathrm{pr}^p_i)^*(\phi)\colon \mathcal{A}^p_i \to \mathcal{B}^p_i$ on $N_p\mathcal{U}$. The morphism ϕ_{01} is compatible with the morphism of algebras $\phi_0 \otimes \phi_1^{\mathrm{op}}\colon \mathcal{A}^1_0 \otimes_R (\mathcal{A}^1_1)^{\mathrm{op}} \to \mathcal{B}^1_0 \otimes_R (\mathcal{B}^1_1)^{\mathrm{op}}$ and the respective module structures.

A 1-*morphism of descent data* is a 1-morphism of the underlying convolution data which preserves respective units.

4.2.4 2-morphisms. Suppose that we are given descent data $(\mathcal{U}, \underline{\mathcal{A}})$ and $(\mathcal{U}, \underline{\mathcal{B}})$ as in 4.2.2 and two 1-morphisms

$$\underline{\phi}, \underline{\psi}\colon (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{B}})$$

as in 4.2.3.

A 2-morphism

$$\underline{b}\colon \underline{\phi} \to \underline{\psi}$$

is a section $b \in \Gamma(N_0\mathcal{U}; \mathcal{B})$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}_{01} & \xrightarrow{b\otimes\phi_{01}} & \mathcal{B}^1_0 \otimes_R \mathcal{B}_{01} \\ \Big\downarrow{\scriptstyle \psi_{01}\otimes b} & & \Big\downarrow \\ \mathcal{B}_{01} \otimes_R \mathcal{B}^1_1 & \longrightarrow & \mathcal{B}_{01} \end{array}$$

is commutative.

4.2.5 The 2-category of descent data. Fix a cover $\mathcal{U}$ of X.

Suppose that we are given descent data $(\mathcal{U}, \underline{\mathcal{A}})$, $(\mathcal{U}, \underline{\mathcal{B}})$, $(\mathcal{U}, \underline{\mathcal{C}})$ and 1-morphisms $\underline{\phi}\colon (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{B}})$ and $\underline{\psi}\colon (\mathcal{U}, \underline{\mathcal{B}}) \to (\mathcal{U}, \underline{\mathcal{C}})$. The map $\psi_{01} \circ \phi_{01}\colon \mathcal{A}_{01} \to \mathcal{C}_{01}$ is a 1-morphism of descent data $\underline{\psi} \circ \underline{\phi}\colon (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{C}})$, the *composition of* $\underline{\phi}$ *and* $\underline{\psi}$.

Let $\underline{\phi}^{(i)}\colon (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{B}})$, $i = 1, 2, 3$, be 1-morphisms and let $\underline{b}^{(j)}\colon \underline{\phi}^{(j)} \to \underline{\phi}^{(j+1)}$, $j = 1, 2$, be 2-morphisms. The section $b^{(2)} \cdot b^{(1)} \in \Gamma(N_0\mathcal{U}; \mathcal{B})$ defines a 2-morphism, denoted $\underline{b}^{(2)}\underline{b}^{(1)}\colon \underline{\phi}^{(1)} \to \underline{\phi}^{(3)}$, the *vertical composition of* $\underline{b}^{(1)}$ *and* $\underline{b}^{(2)}$.

Suppose that $\underline{\phi}^{(i)}: (\mathcal{U}, \underline{\mathcal{A}}) \to (\mathcal{U}, \underline{\mathcal{B}})$ and $\underline{\psi}^{(i)}: (\mathcal{U}, \underline{\mathcal{B}}) \to (\mathcal{U}, \underline{\mathcal{C}})$, $i = 1, 2$, are 1-morphisms and $\underline{b}: \underline{\phi}^{(1)} \to \underline{\phi}^{(2)}$ and $\underline{c}: \underline{\psi}^{(1)} \to \underline{\psi}^{(2)}$ are 2-morphisms. The section $c \cdot \psi^{(1)}(b) \in \Gamma(N_0\mathcal{U}; \mathcal{C})$ defines a 2-morphism, denoted $\underline{c} \otimes \underline{b}$, the *horizontal composition of* $\underline{b}$ *and* $\underline{c}$.

We leave it to the reader to check that with the compositions defined above descent data, 1-morphisms and 2-morphisms form a 2-category, denoted $\mathrm{Desc}_R(\mathcal{U})$.

4.2.6 Fibered category of descent data. Suppose that $\rho: \mathcal{V} \to \mathcal{U}$ is a morphism of covers and $(\mathcal{U}, \underline{\mathcal{A}})$ is a descent datum. Let $\mathcal{A}_{01}^{\rho} = (N_1\rho)^*\mathcal{A}_{01}, \mathcal{A}_{012}^{\rho} = (N_2\rho)^*(\mathcal{A}_{012})$. Then, $(\mathcal{V}, \underline{\mathcal{A}}^{\rho})$ is a descent datum. The assignment $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto (\mathcal{V}, \underline{\mathcal{A}}^{\rho})$ extends to a functor, denoted $\rho^*: \mathrm{Desc}_R(\mathcal{U}) \to \mathrm{Desc}_R(\mathcal{V})$.

The assignment $\mathrm{Cov}(X)^{\mathrm{op}} \ni \mathcal{U} \to \mathrm{Desc}_R(\mathcal{U})$, $\rho \to \rho^*$ is (pseudo-)functor. Let $\mathrm{Desc}_R(X)$ denote the corresponding 2-category fibered in R-linear 2-categories over $\mathrm{Cov}(X)$ with object pairs $(\mathcal{U}, \underline{\mathcal{A}})$ with $\mathcal{U} \in \mathrm{Cov}(X)$ and $(\mathcal{U}, \underline{\mathcal{A}}) \in \mathrm{Desc}_R(\mathcal{U})$; a morphism $(\mathcal{U}', \underline{\mathcal{A}}') \to (\mathcal{U}, \underline{\mathcal{A}})$ in $\mathrm{Desc}_R(X)$ is a pair $(\rho, \underline{\phi})$, where $\rho: \mathcal{U}' \to \mathcal{U}$ is a morphism in $\mathrm{Cov}(X)$ and $\underline{\phi}: (\mathcal{U}', \underline{\mathcal{A}}') \to \rho^*(\mathcal{U}, \underline{\mathcal{A}}) = (\mathcal{U}', \underline{\mathcal{A}}^{\rho})$.

4.3 Trivializations

4.3.1 Definition of a trivialization

Definition 4.9. A *trivialization* of an algebroid stack $\mathcal{C}$ on X is an object in $\mathcal{C}(X)$.

Suppose that $\mathcal{C}$ is an algebroid stack on X and $L \in \mathcal{C}(X)$ is a trivialization. The object L determines a morphism $\underline{\mathrm{End}}_{\mathcal{C}}(L)^+ \to \mathcal{C}$.

Lemma 4.10. *The induced morphism* $\widetilde{\underline{\mathrm{End}}_{\mathcal{C}}(L)^+} \to \mathcal{C}$ *is an equivalence.*

Remark 4.11. Suppose that $\mathcal{C}$ is an R-algebroid stack on X. Then, there exists a cover $\mathcal{U}$ of X such that the stack $\epsilon_0^*\mathcal{C}$ admits a trivialization.

4.3.2 The 2-category of trivializations. Let $\mathrm{Triv}_R(X)$ denote the 2-category with

- objects: the triples $(\mathcal{C}, \mathcal{U}, L)$ where $\mathcal{C}$ is an R-algebroid stack on X, $\mathcal{U}$ is an open cover of X such that $\epsilon_0^*\mathcal{C}(N_0\mathcal{U})$ is nonempty and L is a trivialization of $\epsilon_0^*\mathcal{C}$,
- 1-morphisms: $(\mathcal{C}', \mathcal{U}', L) \to (\mathcal{C}, \mathcal{U}, L)$ are pairs (F, ρ) where $\rho: \mathcal{U}' \to \mathcal{U}$ is a morphism of covers, $F: \mathcal{C}' \to \mathcal{C}$ is a functor such that $(N_0\rho)^*F(L') = L$,
- 2-morphisms $(F, \rho) \to (G, \rho)$, where $(F, \rho), (G, \rho): (\mathcal{C}', \mathcal{U}', L) \to (\mathcal{C}, \mathcal{U}, L)$: the morphisms of functors $F \to G$.

The assignment $(\mathcal{C}, \mathcal{U}, L) \mapsto \mathcal{C}$ extends in an obvious way to a functor $\mathrm{Triv}_R(X) \to \mathrm{AlgStack}_R(X)$.

The assignment $(\mathcal{C}, \mathcal{U}, L) \mapsto \mathcal{U}$ extends to a functor $\mathrm{Triv}_R(X) \to \mathrm{Cov}(X)$ making $\mathrm{Triv}_R(X)$ a fibered 2-category over $\mathrm{Cov}(X)$. For $\mathcal{U} \in \mathrm{Cov}(X)$ we denote the fiber over $\mathcal{U}$ by $\mathrm{Triv}_R(X)(U)$.

4.3.3 Algebroid stacks from descent data. Consider $(\mathcal{U}, \underline{\mathcal{A}}) \in \mathrm{Desc}_R(\mathcal{U})$.

The sheaf of algebras $\mathcal{A}$ on $N_0\mathcal{U}$ gives rise to the algebroid stack $\widetilde{\mathcal{A}^+}$. The sheaf $\mathcal{A}_{01}$ defines an equivalence

$$\phi_{01} := (\bullet) \otimes_{\mathcal{A}_0^1} \mathcal{A}_{01} \colon (\mathrm{pr}_0^1)^* \widetilde{\mathcal{A}^+} \to (\mathrm{pr}_1^1)^* \widetilde{\mathcal{A}^+}.$$

The convolution map $\mathcal{A}_{012}$ defines an isomorphism of functors

$$\phi_{012} \colon (\mathrm{pr}_{01}^2)^*(\phi_{01}) \circ (\mathrm{pr}_{12}^2)^*(\phi_{01}) \to (\mathrm{pr}_{02}^2)^*(\phi_{01}).$$

We leave it to the reader to verify that the triple $(\widetilde{\mathcal{A}^+}, \phi_{01}, \phi_{012})$ constitutes a descent datum for an algebroid stack on X which we denote by $\mathsf{St}(\mathcal{U}, \underline{\mathcal{A}})$.

By construction there is a canonical equivalence $\widetilde{\underline{\mathcal{A}}^+} \to \epsilon_0^* \mathsf{St}(\mathcal{U}, \underline{\mathcal{A}})$ which endows $\epsilon_0^* \mathsf{St}(\mathcal{U}, \underline{\mathcal{A}})$ with a canonical trivialization $\mathbb{1}$.

The assignment $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto (\mathsf{St}(\mathcal{U}, \underline{\mathcal{A}}), \mathcal{U}, \mathbb{1})$ extends to a cartesian functor

$$\mathsf{St} \colon \mathrm{Desc}_R(X) \to \mathrm{Triv}_R(X).$$

4.3.4 Descent data from trivializations. Consider $(\mathcal{C}, \mathcal{U}, L) \in \mathrm{Triv}_R(X)$. Since $\epsilon_0 \circ \mathrm{pr}_0^1 = \epsilon_0 \circ \mathrm{pr}_1^1 = \epsilon_1$ we have canonical identifications $(\mathrm{pr}_0^1)^* \epsilon_0^* \mathcal{C} \cong (\mathrm{pr}_1^1)^* \epsilon_0^* \mathcal{C} \cong \epsilon_1^* \mathcal{C}$. The object $L \in \epsilon_0^* \mathcal{C}(N_0\mathcal{U})$ gives rise to the objects $(\mathrm{pr}_0^1)^* L$ and $(\mathrm{pr}_1^1)^* L$ in $\epsilon_1^* \mathcal{C}(N_0\mathcal{U})$. Let $\mathcal{A}_{01} = \underline{\mathrm{Hom}}_{\epsilon_1^* \mathcal{C}}((\mathrm{pr}_1^1)^* L, (\mathrm{pr}_0^1)^* L)$. Thus, $\mathcal{A}_{01}$ is a sheaf of R-modules on $N_1\mathcal{U}$.

The object $L \in \epsilon_0^* \mathcal{C}(N_0\mathcal{U})$ gives rise to the objects $(\mathrm{pr}_0^2)^* L$, $(\mathrm{pr}_1^2)^* L$ and $(\mathrm{pr}_2^2)^* L$ in $\epsilon_2^* \mathcal{C}(N_2\mathcal{U})$. There are canonical isomorphisms

$$(\mathrm{pr}_{ij}^2)^* \mathcal{A}_{01} \cong \underline{\mathrm{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathrm{pr}_i^2)^* L, (\mathrm{pr}_j^2)^* L).$$

The composition of morphisms

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathrm{pr}_1^2)^* L, (\mathrm{pr}_0^2)^* L) \otimes_R \underline{\mathrm{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathrm{pr}_2^2)^* L, (\mathrm{pr}_1^2)^* L) \\ &\quad \to \underline{\mathrm{Hom}}_{\epsilon_2^* \mathcal{C}}((\mathrm{pr}_2^2)^* L, (\mathrm{pr}_0^2)^* L) \end{aligned}$$

gives rise to the map

$$\mathcal{A}_{012} \colon (\mathrm{pr}_{01}^2)^* \mathcal{A}_{01} \otimes_R (\mathrm{pr}_{12}^2)^* \mathcal{A}_{01} \to (\mathrm{pr}_{02}^2)^* \mathcal{A}_{02}.$$

Since $\mathrm{pr}_i^1 \circ \mathrm{pr}_{00}^0 = \mathrm{Id}$ there is a canonical isomorphism $\mathcal{A} := (\mathrm{pr}_{00}^0)^* \mathcal{A}_{01} \cong \underline{\mathrm{End}}(L)$ which supplies $\mathcal{A}$ with the unit section $\mathbf{1} \colon R \xrightarrow{1} \underline{\mathrm{End}}(L) \to \mathcal{A}$.

The pair $(\mathcal{U}, \underline{\mathcal{A}})$, together with the section $\mathbf{1}$ is a decent datum which we denote $\mathsf{dd}(\mathcal{C}, \mathcal{U}, L)$.

The assignment $(\mathcal{U}, \underline{\mathcal{A}}) \mapsto \mathsf{dd}(\mathcal{C}, \mathcal{U}, L)$ extends to a cartesian functor

$$\mathsf{dd}\colon \operatorname{Triv}_R(X) \to \operatorname{Desc}_R(X).$$

Lemma 4.12. *The functors* St *and* dd *are mutually quasi-inverse equivalences.*

4.4 Base change. For an R-linear category $\mathcal{C}$ and homomorphism of algebras $R \to S$ we denote by $\mathcal{C} \otimes_R S$ the category with the same objects as $\mathcal{C}$ and morphisms defined by $\operatorname{Hom}_{\mathcal{C}\otimes_R S}(A, B) = \operatorname{Hom}_{\mathcal{C}}(A, B) \otimes_R S$.

For an R-algebra A the categories $(A \otimes_R S)^+$ and $A^+ \otimes_R S$ are canonically isomorphic.

For a prestack $\mathcal{C}$ in R-linear categories we denote by $\mathcal{C} \otimes_R S$ the prestack associated to the fibered category $U \mapsto \mathcal{C}(U) \otimes_R S$.

For $U \subseteq X$, $A, B \in \mathcal{C}(U)$, there is an isomorphism of sheaves $\underline{\operatorname{Hom}}_{\mathcal{C}\otimes_R S}(A, B) = \underline{\operatorname{Hom}}_{\mathcal{C}}(A, B) \otimes_R S$.

Lemma 4.13. *Suppose that $\mathcal{A}$ is a sheaf of R-algebras and $\mathcal{C}$ is an R-algebroid stack.*

1. $(\widetilde{\mathcal{A}^+} \otimes_R S)^{\sim}$ *is an algebroid stack equivalent to* $\widetilde{(\mathcal{A} \otimes_R S)^+}$.
2. $\widetilde{\mathcal{C} \otimes_R S}$ *is an algebroid stack.*

Proof. Suppose that $\mathcal{A}$ is a sheaf of R-algebras. There is a canonical isomorphism of prestacks $(\mathcal{A} \otimes_R S)^+ \cong \mathcal{A}^+ \otimes_R S$ which induces the canonical equivalence $\widetilde{(\mathcal{A} \otimes_R S)^+} \cong \widetilde{\mathcal{A}^+ \otimes_R S}$.

The canonical functor $\mathcal{A}^+ \to \widetilde{\mathcal{A}^+}$ induces the functor $\mathcal{A}^+ \otimes_R S \to \widetilde{\mathcal{A}^+} \otimes_R S$, hence the functor $\widetilde{\mathcal{A}^+ \otimes_R S} \to (\widetilde{\mathcal{A}^+} \otimes_R S)^{\sim}$.

The map $\mathcal{A} \to \mathcal{A} \otimes_R S$ induces the functor $\widetilde{\mathcal{A}^+} \to \widetilde{(\mathcal{A} \otimes_R S)^+}$ which factors through the functor $\widetilde{\mathcal{A}^+} \otimes_R S \to \widetilde{(\mathcal{A} \otimes_R S)^+}$. From this we obtain the functor $(\widetilde{\mathcal{A}^+} \otimes_R S)^{\sim} \to \widetilde{(\mathcal{A} \otimes_R S)^+}$.

We leave it to the reader to check that the two constructions are mutually inverse equivalences. It follows that $(\widetilde{\mathcal{A}^+} \otimes_R S)^{\sim}$ is an algebroid stack equivalent to $\widetilde{(\mathcal{A} \otimes_R S)^+}$.

Suppose that $\mathcal{C}$ is an R-algebroid stack. Let $\mathcal{U}$ be a cover such that $\epsilon_0^*\mathcal{C}(N_0\mathcal{U})$ is nonempty. Let L be an object in $\epsilon_0^*\mathcal{C}(N_0\mathcal{U})$; put $\mathcal{A} := \underline{\operatorname{End}}_{\epsilon_0^*\mathcal{C}}(L)$. The equivalence $\widetilde{\mathcal{A}^+} \to \epsilon_0^*\mathcal{C}$ induces the equivalence $(\widetilde{\mathcal{A}^+} \otimes_R S)^{\sim} \to (\epsilon_0^*\mathcal{C} \otimes_R S)^{\sim}$. Since the former is an algebroid stack so is the latter. There is a canonical equivalence $(\epsilon_0^*\mathcal{C} \otimes_R S)^{\sim} \cong \epsilon_0^*(\widetilde{\mathcal{C} \otimes_R S})$; since the former is an algebroid stack so is the latter. Since the property of being an algebroid stack is local, the stack $\widetilde{\mathcal{C} \otimes_R S}$ is an algebroid stack. $\square$

4.5 Twisted forms. Suppose that $\mathcal{A}$ is a sheaf of R-algebras on X. We will call an R-algebroid stack locally equivalent to $\widetilde{\mathcal{A}^{\mathrm{op}+}}$ a *twisted form of* $\mathcal{A}$.

Suppose that $\mathcal{S}$ is twisted form of $\mathcal{O}_X$. Then, the substack $i\mathcal{S}$ is an $\mathcal{O}_X^\times$-gerbe. The assignment $\mathcal{S} \mapsto i\mathcal{S}$ extends to an equivalence between the 2-groupoid of twisted forms of $\mathcal{O}_X$ (a subcategory of $\mathrm{AlgStack}_{\mathbb{C}}(X)$) and the 2-groupoid of $\mathcal{O}_X^\times$-gerbes.

Let $\mathcal{S}$ be a twisted form of $\mathcal{O}_X$. Then for any $U \subseteq X$, $A \in \mathcal{S}(U)$ the canonical map $\mathcal{O}_U \to \underline{\mathrm{End}}_{\mathcal{S}}(A)$ is an isomorphism. Consequently, if $\mathcal{U}$ is a cover of X and L is a trivialization of $\epsilon_0^*\mathcal{S}$, then there is a canonical isomorphism of sheaves of algebras $\mathcal{O}_{N_0\mathcal{U}} \to \underline{\mathrm{End}}_{\epsilon_0^*\mathcal{S}}(L)$.

Conversely, suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a $\mathbb{C}$-descent datum. If the sheaf of algebras $\mathcal{A}$ is isomorphic to $\mathcal{O}_{N_0\mathcal{U}}$ then such an isomorphism is unique since the latter has no non-trivial automorphisms. Thus, we may and will identify $\mathcal{A}$ with $\mathcal{O}_{N_0\mathcal{U}}$. Hence, $\mathcal{A}_{01}$ is a line bundle on $N_1\mathcal{U}$ and the convolution map $\mathcal{A}_{012}$ is a morphism of line bundles. The stack which corresponds to $(\mathcal{U}, \underline{\mathcal{A}})$ (as in 4.3.3) is a twisted form of $\mathcal{O}_X$.

Isomorphism classes of twisted forms of $\mathcal{O}_X$ are classified by $H^2(X; \mathcal{O}_X^\times)$. We recall the construction presently. Suppose that the twisted form $\mathcal{S}$ of $\mathcal{O}_X$ is represented by the descent datum $(\mathcal{U}, \underline{\mathcal{A}})$. Assume in addition that the line bundle $\mathcal{A}_{01}$ on $N_1\mathcal{U}$ is trivialized. Then we can consider $\mathcal{A}_{012}$ as an element in $\Gamma(N_2\mathcal{U}; \mathcal{O}^\times)$. The associativity condition implies that $\mathcal{A}_{012}$ is a cocycle in $\check{C}^2(\mathcal{U}; \mathcal{O}^\times)$. The class of this cocycle in $\check{H}^2(\mathcal{U}; \mathcal{O}^\times)$ does not depend on the choice of trivializations of the line bundle $\mathcal{A}_{01}$ and yields a class in $H^2(X; \mathcal{O}_X^\times)$.

We can write this class using the de Rham complex for jets. We refer to Section 7.1 for the notations and a brief review.

The composition $\mathcal{O}^\times \to \mathcal{O}^\times/\mathbb{C}^\times \xrightarrow{\log} \mathcal{O}/\mathbb{C} \xrightarrow{j^\infty} \mathsf{DR}(\mathcal{J}/\mathcal{O})$ induces the map $H^2(X; \mathcal{O}^\times) \to H^2(X; \mathsf{DR}(\mathcal{J}/\mathcal{O})) \cong H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$. Here the latter isomorphism follows from the fact that the sheaf $\Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X$ is soft. We denote by $[\mathcal{S}] \in H^2(\Gamma(X; \Omega_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ the image of the class of $\mathcal{S}$ in $H^2(X; \mathcal{O}^\times)$. In Lemma 7.13 (see also Lemma 7.15) we will construct an explicit representative for $[\mathcal{S}]$.

5 DGLA of local cochains on matrix algebras

In this section we define matrix algebras from a descent datum and use them to construct a cosimplicial DGLA of local cochains. We also establish the acyclicity of this cosimplicial DGLA.

5.1 Definition of matrix algebras

5.1.1 Matrix entries. Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is an R-descent datum. Let $\mathcal{A}_{10} := \tau^*\mathcal{A}_{01}$, where $\tau = \mathrm{pr}^1_{10} \colon N_1\mathcal{U} \to N_1\mathcal{U}$ is the transposition of the factors. The pairings $(\mathrm{pr}^1_{100})^*(\mathcal{A}_{012}) \colon \mathcal{A}_{10} \otimes_R \mathcal{A}^1_0 \to \mathcal{A}_{10}$ and $(\mathrm{pr}^1_{110})^*(\mathcal{A}_{012}) \colon \mathcal{A}^1_1 \otimes_R \mathcal{A}_{10} \to \mathcal{A}_{10}$ of sheaves on $N_1\mathcal{U}$ endow $\mathcal{A}_{10}$ with a structure of a $\mathcal{A}^1_1 \otimes (\mathcal{A}^1_0)^{\mathrm{op}}$-module.

The identities $\mathrm{pr}^2_{01} \circ \mathrm{pr}^1_{010} = \mathrm{Id}$, $\mathrm{pr}^2_{12} \circ \mathrm{pr}^1_{010} = \tau$ and $\mathrm{pr}^2_{02} \circ \mathrm{pr}^1_{010} = \mathrm{pr}^0_{00} \circ \mathrm{pr}^1_0$ imply that the pull-back of $\mathcal{A}_{012}$ by pr^1_{010} gives the pairing

$$(\mathrm{pr}^1_{010})^*(\mathcal{A}_{012}) \colon \mathcal{A}_{01} \otimes_R \mathcal{A}_{10} \to \mathcal{A}^1_{00}. \tag{5.1}$$

which is a morphism of $\mathcal{A}^1_0 \otimes_K (\mathcal{A}^1_0)^{\mathrm{op}}$-modules. Similarly, we have the pairing

$$(\mathrm{pr}^1_{101})^*(\mathcal{A}_{012}) \colon \mathcal{A}_{10} \otimes_R \mathcal{A}_{01} \to \mathcal{A}^1_{11}. \tag{5.2}$$

The pairings (4.2), (4.3), (5.1) and (5.2), $\mathcal{A}_{ij} \otimes_R \mathcal{A}_{jk} \to \mathcal{A}_{ik}$ are morphisms of $\mathcal{A}^1_i \otimes_R (\mathcal{A}^1_k)^{\mathrm{op}}$-modules which, as a consequence of associativity, factor through maps

$$\mathcal{A}_{ij} \otimes_{\mathcal{A}^1_j} \mathcal{A}_{jk} \to \mathcal{A}_{ik} \tag{5.3}$$

induced by $\mathcal{A}_{ijk} = (\mathrm{pr}^1_{ijk})^*(\mathcal{A}_{012})$; here $i, j, k = 0, 1$. Define now for every $p \geq 0$ the sheaves $\mathcal{A}^p_{ij}$, $0 \leq i, j \leq p$, on $N_p\mathcal{U}$ by $\mathcal{A}^p_{ij} = (\mathrm{pr}^p_{ij})^*\mathcal{A}_{01}$. Define also $\mathcal{A}^p_{ijk} = (\mathrm{pr}^p_{ijk})^*(\mathcal{A}_{012})$. We immediately obtain for every p the morphisms

$$\mathcal{A}^p_{ijk} \colon \mathcal{A}^p_{ij} \otimes_{\mathcal{A}^p_j} \mathcal{A}^p_{jk} \to \mathcal{A}^p_{ik}. \tag{5.4}$$

5.1.2 Matrix algebras. Let $\mathsf{Mat}(\underline{\mathcal{A}})^0 = \mathcal{A}$; thus, $\mathsf{Mat}(\underline{\mathcal{A}})^0$ is a sheaf of algebras on $N_0\mathcal{U}$. For $p = 1, 2, \ldots$ let $\mathsf{Mat}(\underline{\mathcal{A}})^p$ denote the sheaf on $N_p\mathcal{U}$ defined by

$$\mathsf{Mat}(\underline{\mathcal{A}})^p = \bigoplus_{i,j=0}^{p} \mathcal{A}^p_{ij}.$$

The maps (5.4) define the pairing

$$\mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathsf{Mat}(\underline{\mathcal{A}})^p \to \mathsf{Mat}(\underline{\mathcal{A}})^p$$

which endows the sheaf $\mathsf{Mat}(\underline{\mathcal{A}})^p$ with a structure of an associative algebra by virtue of the associativity condition. The unit section 1 is given by $1 = \sum_{i=0}^{p} 1_{ii}$, where 1_{ii} is the image of the unit section of $\mathcal{A}^p_{ii}$.

5.1.3 Combinatorial restriction. The algebras $\mathsf{Mat}(\underline{\mathcal{A}})^p$, $p = 0, 1, \ldots$, do not form a cosimplicial sheaf of algebras on $N\mathcal{U}$ in the usual sense. They are, however, related by *combinatorial restriction* which we describe presently.

For a morphism $f \colon [p] \to [q]$ in Δ define a sheaf on $N_q\mathcal{U}$ by

$$f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q = \bigoplus_{i,j=0}^{p} \mathcal{A}^q_{f(i)f(j)}.$$

Note that $f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q$ inherits a structure of an algebra.

Recall from Section 2.3.1 that the morphism f induces the map $f^* \colon N_q\mathcal{U} \to N_p\mathcal{U}$ and that f_* denotes the pull-back along f^*. We will also use f_* to denote the canonical isomorphism of algebras

$$f_* \colon f_*\mathsf{Mat}(\underline{\mathcal{A}})^p \to f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q \tag{5.5}$$

induced by the isomorphisms $f_*\mathcal{A}^p_{ij} \cong \mathcal{A}^q_{f(i)f(j)}$.

5.1.4 Refinement. Suppose that $\rho\colon \mathcal{V} \to \mathcal{U}$ is a morphism of covers. For $p = 0, 1, \ldots$ there is a natural isomorphism

$$(N_p\rho)^*\mathsf{Mat}(\underline{\mathcal{A}})^p \cong \mathsf{Mat}(\underline{\mathcal{A}^\rho})^p$$

of sheaves of algebras on $N_p\mathcal{V}$. The above isomorphisms are obviously compatible with combinatorial restriction.

5.2 Local cochains on matrix algebras

5.2.1 Local cochains. A *sheaf of matrix algebras* is a sheaf of algebras $\mathcal{B}$ together with a decomposition

$$\mathcal{B} = \bigoplus_{i,j=0}^{p} \mathcal{B}_{ij}$$

as a sheaf of vector spaces which satisfies $\mathcal{B}_{ij} \cdot \mathcal{B}_{jk} = \mathcal{B}_{ik}$.

To a matrix algebra $\mathcal{B}$ one can associate to the DGLA of *local cochains* defined as follows. For $n = 0$ let $C^0(\mathcal{B})^{\text{loc}} = \bigoplus \mathcal{B}_{ii} \subset \mathcal{B} = C^0(\mathcal{B})$. For $n > 0$ let $C^n(\mathcal{B})^{\text{loc}}$ denote the subsheaf of $C^n(\mathcal{B})$ whose stalks consist of multilinear maps D such that for any collection of $s_{i_k j_k} \in \mathcal{B}_{i_k j_k}$

1. $D(s_{i_1 j_1} \otimes \cdots \otimes s_{i_n j_n}) = 0$ unless $j_k = i_{k+1}$ for all $k = 1, \ldots, n-1$,
2. $D(s_{i_0 i_1} \otimes s_{i_1 i_2} \otimes \cdots \otimes s_{i_{n-1} i_n}) \in \mathcal{B}_{i_0 i_n}$.

For $I = (i_0, \ldots, i_n) \in [p]^{\times n+1}$ let

$$C^I(\mathcal{B})^{\text{loc}} := \underline{\mathrm{Hom}}_k(\otimes_{j=0}^{n-1}\mathcal{B}_{i_j i_{j+1}}, \mathcal{B}_{i_0 i_n}).$$

The restriction maps along the embeddings $\otimes_{j=0}^{n-1}\mathcal{B}_{i_j i_{j+1}} \hookrightarrow \mathcal{B}^{\otimes n}$ induce an isomorphism $C^n(\mathcal{B})^{\text{loc}} \to \oplus_{I \in [p]^{\times n+1}} C^I(\mathcal{B})^{\text{loc}}$.

The sheaf $C^\bullet(\mathcal{B})^{\text{loc}}[1]$ is a subDGLA of $C^\bullet(\mathcal{B})[1]$ and the inclusion map is a quasi-isomorphism.

For a matrix algebra $\mathcal{B}$ on X we denote by $\mathrm{Def}(\mathcal{B})^{\text{loc}}(R)$ the subgroupoid of $\mathrm{Def}(\mathcal{B})(R)$ with objects R-star products which respect the decomposition given by $(\mathcal{B} \otimes_k R)_{ij} = \mathcal{B}_{ij} \otimes_k R$ and 1- and 2-morphisms defined accordingly. The composition

$$\mathrm{Def}(\mathcal{B})^{\text{loc}}(R) \to \mathrm{Def}(\mathcal{B})(R) \to \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})[1]) \otimes_k \mathfrak{m}_R)$$

takes values in $\mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\text{loc}}[1]) \otimes_k \mathfrak{m}_R)$ and establishes an isomorphism of 2-groupoids $\mathrm{Def}(\mathcal{B})^{\text{loc}}(R) \cong \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\text{loc}}[1]) \otimes_k \mathfrak{m}_R)$.

5.2.2 Combinatorial restriction of local cochains. Suppose given a matrix algebra $\mathcal{B} = \bigoplus_{i,j=0}^{q} \mathcal{B}_{ij}$ is a sheaf of matrix k-algebras.

The DGLA $C^\bullet(\mathcal{B})^{\mathrm{loc}}[1]$ has additional variance not exhibited by $C^\bullet(\mathcal{B})[1]$. Namely, for $f\colon [p] \to [q]$ – a morphism in Δ – there is a natural map of DGLA

$$f^\sharp\colon C^\bullet(\mathcal{B})^{\mathrm{loc}}[1] \to C^\bullet(f^\sharp\mathcal{B})^{\mathrm{loc}}[1] \tag{5.6}$$

defined as follows. Let $f_\sharp^{ij}\colon (f^\sharp\mathcal{B})_{ij} \to \mathcal{B}_{f(i)f(j)}$ denote the tautological isomorphism. For each multi-index $I = (i_0, \dots, i_n) \in [p]^{\times n+1}$ let

$$f_\sharp^I := \otimes_{j=0}^{n-1} f_\sharp^{i_j i_{j+1}}\colon \otimes_{j=0}^{n-1} (f^\sharp\mathcal{B})_{i_j i_{j+1}} \to \otimes_{i=0}^{n-1} \mathcal{B}_{f(i_j)f(i_{j+1})}.$$

Let $f_\sharp^n := \oplus_{I \in \Sigma_p^{\times n+1}} f_\sharp^I$. The map (5.6) is defined as restriction along $f_\sharp^n$.

Lemma 5.1. *The map* (5.6) *is a morphism of DGLA*

$$f^\sharp\colon C^\bullet(\mathcal{B})^{\mathrm{loc}}[1] \to C^\bullet(f^\sharp\mathcal{B})^{\mathrm{loc}}[1].$$

If follows that combinatorial restriction of local cochains induces the functor

$$\mathrm{MC}^2(f^\sharp)\colon\ \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R) \to \mathrm{MC}^2(\Gamma(X; C^\bullet(f^\sharp\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R).$$

Combinatorial restriction with respect to f induces the functor

$$f^\sharp\colon\ \mathrm{Def}(\mathcal{B})^{\mathrm{loc}}(R) \to \mathrm{Def}(f^\sharp\mathcal{B})^{\mathrm{loc}}(R).$$

It is clear that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Def}(\mathcal{B})^{\mathrm{loc}}(R) \arrow[r, "f^\sharp"] \arrow[d] & \mathrm{Def}(f^\sharp\mathcal{B})^{\mathrm{loc}}(R) \arrow[d] \\
\mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R) \arrow[r, "\mathrm{MC}^2(f^\sharp)"] & \mathrm{MC}^2(\Gamma(X; C^\bullet(f^\sharp\mathcal{B})^{\mathrm{loc}}[1]) \otimes_k \mathfrak{m}_R).
\end{tikzcd}$$

5.2.3 Cosimplicial DGLA from descent datum. Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a descent datum for a twisted sheaf of algebras as in 4.2.2. Then, for each $p = 0, 1, \dots$ we have the matrix algebra $\mathsf{Mat}(\underline{\mathcal{A}})^p$ as defined in 5.1, and therefore the DGLA of local cochains $C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]$ defined in 5.2.1. For each morphism $f\colon [p] \to [q]$ there is a morphism of DGLA

$$f^\sharp\colon C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1] \to C^\bullet(f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1]$$

and an isomorphism of DGLA

$$C^\bullet(f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1] \cong f_* C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]$$

induced by the isomorphism $f_*\colon f_*\mathsf{Mat}(\underline{\mathcal{A}})^p \to f^\sharp\mathsf{Mat}(\underline{\mathcal{A}})^q$ from the equation (5.5). These induce the morphisms of the DGLA of global sections

$$\begin{tikzcd}
\Gamma(N_q\mathcal{U}; C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}[1]) \arrow[dr, "f^\sharp"] & & \Gamma(N_p\mathcal{U}; C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]) \arrow[dl, "f_*"'] \\
& \Gamma(N_q\mathcal{U}; f_* C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}[1]). &
\end{tikzcd}$$

For $\lambda\colon [n] \to \Delta$ let

$$\mathfrak{G}(\underline{\mathcal{A}})^{\lambda} = \Gamma(N_{\lambda(n)}\mathcal{U}; \lambda(0n)_* C^{\bullet}(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{\mathrm{loc}}[1]).$$

Suppose given another simplex $\mu\colon [m] \to \Delta$ and morphism $\phi\colon [m] \to [n]$ such that $\mu = \lambda \circ \phi$ (i.e. ϕ is a morphism of simplices $\mu \to \lambda$). The morphism $(0n)$ factors uniquely into $0 \to \phi(0) \to \phi(m) \to n$, which, under λ, gives the factorization of $\lambda(0n)\colon \lambda(0) \to \lambda(n)$ (in Δ) into

$$\lambda(0) \xrightarrow{f} \mu(0) \xrightarrow{g} \mu(m) \xrightarrow{h} \lambda(n), \tag{5.7}$$

where $g = \mu(0m)$. The map

$$\phi_*\colon \mathfrak{G}(\underline{\mathcal{A}})^{\mu} \to \mathfrak{G}(\underline{\mathcal{A}})^{\lambda}$$

is the composition

$$\begin{aligned}
&\Gamma(N_{\mu(m)}\mathcal{U}; g_* C^{\bullet}(\mathsf{Mat}(\underline{\mathcal{A}})^{\mu(0)})^{\mathrm{loc}}[1]) \\
&\quad \xrightarrow{h_*} \Gamma(N_{\lambda(n)}\mathcal{U}; h_* g_* C^{\bullet}(\mathsf{Mat}(\underline{\mathcal{A}})^{\mu(0)})^{\mathrm{loc}}[1]) \\
&\qquad \xrightarrow{f^{\sharp}} \Gamma(N_{\lambda(n)}\mathcal{U}; h_* g_* f_* C^{\bullet}(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{\mathrm{loc}}[1]).
\end{aligned}$$

Suppose given yet another simplex, $\nu\colon [l] \to \Delta$, and a morphism of simplices $\psi\colon \nu \to \mu$, i.e. a morphism $\psi\colon [l] \to [m]$ such that $\nu = \mu \circ \psi$. Then, the composition $\phi_* \circ \psi_*\colon \mathfrak{G}(\underline{\mathcal{A}})^{\nu} \to \mathfrak{G}(\underline{\mathcal{A}})^{\lambda}$ coincides with the map $(\phi \circ \psi)_*$.

For $n = 0, 1, 2, \dots$ let

$$\mathfrak{G}(\underline{\mathcal{A}})^{n} = \prod_{[n]\xrightarrow{\lambda}\Delta} \mathfrak{G}(\underline{\mathcal{A}})^{\lambda}. \tag{5.8}$$

A morphism $\phi\colon [m] \to [n]$ in Δ induces the map of DGLA $\phi_*\colon \mathfrak{G}(\underline{\mathcal{A}})^{m} \to \mathfrak{G}(\underline{\mathcal{A}})^{n}$. The assignment $\Delta \ni [n] \mapsto \mathfrak{G}(\underline{\mathcal{A}})^{n}$, $\phi \mapsto \phi_*$ defines the cosimplicial DGLA denoted by $\mathfrak{G}(\underline{\mathcal{A}})$.

5.3 Acyclicity

Theorem 5.2. *The cosimplicial DGLA $\mathfrak{G}(\underline{\mathcal{A}})$ is acyclic, i.e. it satisfies the condition* (3.8).

The rest of the section is devoted to the proof of Theorem 5.2. We fix a degree of Hochschild cochains k.

For $\lambda\colon [n] \to \Delta$ let $\mathrm{c}^{\lambda} = \Gamma(N_{\lambda(n)}\mathcal{U}; \lambda(0n)_* C^{k}(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{\mathrm{loc}})$. For a morphism $\phi\colon \mu \to \lambda$ we have the map $\phi_*\colon \mathrm{c}^{\mu} \to \mathrm{c}^{\lambda}$ defined as in 5.2.3.

Let $(\mathfrak{C}^{\bullet}, \partial)$ denote the corresponding cochain complex whose definition we recall below. For $n = 0, 1, \dots$ let $\mathfrak{C}^{n} = \prod_{[n]\xrightarrow{\lambda}\Delta} \mathrm{c}^{\lambda}$. The differential $\partial^{n}\colon \mathfrak{C}^{n} \to \mathfrak{C}^{n+1}$ is defined by the formula $\partial^{n} = \sum_{i=0}^{n+1} (-1)^{i} (\partial_i^{n})_*$.

5.3.1 Decomposition of local cochains. As was noted in 5.2.1, for $n, q = 0, 1, \ldots$ there is a direct sum decomposition

$$C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}} = \bigoplus_{I \in [q]^{k+1}} C^I(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}. \tag{5.9}$$

In what follows we will interpret a multi-index $I = (i_0, \ldots, i_n) \in [q]^{k+1}$ as a map $I: \{0, \ldots, k\} \to [q]$. For I as above let $s(I) = |\operatorname{Im}(I)| - 1$. The map I factors uniquely into the composition

$$\{0, \ldots, k\} \xrightarrow{I'} [s(I)] \xrightarrow{m(I)} [q]$$

where the second map is a morphism in Δ (i.e. is order preserving). Then, the isomorphisms $m(I)_* \mathcal{A}_{I'(i)I'(j)} \cong \mathcal{A}_{I(i)I(j)}$ induce the isomorphism

$$m(I)_* C^{I'}(\mathsf{Mat}(\underline{\mathcal{A}})^{s(I)})^{\mathrm{loc}} \to C^I(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}.$$

Therefore, the decomposition (5.9) may be rewritten as follows:

$$C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}} = \bigoplus_e \bigoplus_I e_* C^I(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}} \tag{5.10}$$

where the summation is over *injective* (monotone) maps $e: [s(e)] \to [q]$ and *surjective* maps $I: \{0, \ldots, k\} \to [s(e)]$. Note that, for e, I as above, there is an isomorphism

$$e_* C^I(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}} \cong C^{e \circ I}(e^\sharp \mathsf{Mat}(\underline{\mathcal{A}})^{s(e)})^{\mathrm{loc}}.$$

5.3.2 Filtrations. Let $F^\bullet C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}}$ denote the *decreasing* filtration defined by

$$F^s C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}} = \bigoplus_{e: s(e) \geq s} \bigoplus_I e_* C^I(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}}$$

Note that $F^s C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}} = 0$ for $s > k$ and $Gr^s C^k(\mathsf{Mat}(\underline{\mathcal{A}})^q)^{\mathrm{loc}} = 0$ for $s < 0$.

The filtration $F^\bullet C^k(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)})^{\mathrm{loc}}$ induces the filtration $F^\bullet \mathrm{c}^\lambda$, hence the filtration $F^\bullet \mathfrak{C}^\bullet$ with $F^s \mathrm{c}^\lambda = \Gamma(N_{\lambda(n)} \mathcal{U}; \lambda(0n)_* F^s C^k(\mathsf{Mat}(\underline{A})^{\lambda(0)})^{\mathrm{loc}})$, $F^s \mathfrak{C}^n = \prod_{[n] \xrightarrow{\lambda} \Delta} F^s \mathrm{c}^\lambda$.

The following result then is an easy consequence of the definitions:

Lemma 5.3. *For $\phi: \mu \to \lambda$ the induced map $\phi_*: \mathrm{c}^\mu \to \mathrm{c}^\lambda$ preserves filtration.*

Corollary 5.4. *The differential ∂^n preserves the filtration.*

Proposition 5.5. *For any s the complex $Gr^s(\mathfrak{C}^\bullet, \partial)$ is acyclic in non-zero degrees.*

Proof. Use the following notation: for a simplex $\mu: [m] \to \Delta$ and an arrow $e: [s] \to [\mu(0)]$ the simplex $\mu^e: [m+1] \to \Delta$ is defined by $\mu^e(0) = [s]$, $\mu^e(01) = e$, and $\mu^e(i) = \mu(i-1)$, $\mu^e(i, j) = \mu(i-1, j-1)$ for $i > 0$.

For $D = (D_\lambda) \in F^s\mathfrak{C}^n$, $D_\lambda \in \mathrm{c}^\lambda$ and $\mu\colon [n-1] \to \Delta$ let

$$h^n(D)_\mu = \sum_{I,e} e_* D^I_{\mu^e}$$

where $e\colon [s] \to \mu(0)$ is an injective map, $I : \{0,\dots,k\} \to [s]$ is a surjection, and $D^I_{\mu^e}$ is the I-component of D_{μ^e} in the sense of the decomposition (5.9). Put $h^n(D) = (h^n(D)_\mu) \in \mathfrak{C}^{n-1}$. It is clear that $h^n(D) \in F^s\mathfrak{C}^{n-1}$. Thus the assignment $D \mapsto h^n(D)$ defines a filtered map $h^n : \mathfrak{C}^n \to \mathfrak{C}^{n-1}$ for all $n \geq 1$. Hence, for any s we have the map $Gr^s h^n : Gr^s\mathfrak{C}^n \to Gr^s\mathfrak{C}^{n-1}$.

Next, we calculate the effect of the face maps $\partial^{n-1}_i : [n-1] \to [n]$. First of all, note that, for $1 \leq i \leq n-1$, $(\partial^{n-1}_i)_* = \mathrm{Id}$. The effect of $(\partial^{n-1}_0)_*$ is the map on global sections induced by

$$\lambda(01)^\sharp : \lambda(1n)_* C^k(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(1)})^{\mathrm{loc}} \to \lambda(0n)_* C^k(\mathsf{Mat}(\underline{A})^{\lambda(0)})^{\mathrm{loc}}$$

and $(\partial^{n-1}_n)_* = \lambda(n-1,n)_*$. Thus, for $D = (D_\lambda) \in F^s\mathfrak{C}^n$, we have:

$$\begin{aligned}((\partial^{n-1}_i)_* h^n(D))_\lambda &= (\partial^{n-1}_i)_* \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_i)^e} \\ &= \begin{cases} \lambda(01)^\sharp \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_0)^e} & \text{if } i = 0, \\ \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_i)^e} & \text{if } 1 \leq i \leq n-1, \\ \lambda(n-1,n)_* \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_n)^e} & \text{if } i = n. \end{cases}\end{aligned}$$

On the other hand,

$$\begin{aligned}h^{n+1}((\partial^n_j)_* D)_\lambda &= \sum_{I,e} e_*((\partial^n_j)_* D)^I_{\lambda^e} \\ &= \begin{cases} \sum_{I,e} e_* e^\sharp D^I_\lambda & \text{if } j = 0, \\ \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_0)^{\lambda(01)\circ e}} & \text{if } j = 1, \\ \sum_{I,e} e_* D^I_{(\lambda\circ\partial^{n-1}_{j-1})^e} & \text{if } 2 \leq j \leq n, \\ \sum_{I,e} \lambda(n-1,n)_* e_* D^I_{(\lambda\circ\partial^{n-1}_n)^e} & \text{if } j = n+1. \end{cases}\end{aligned}$$

Note that $\sum_{I,e} e_* e^\sharp D^I_\lambda = D_\lambda \mod F^{s+1}\mathrm{c}^\lambda$, i.e. the map $h^{n+1} \circ \partial^n_0$ induces the identity map on $Gr^s\mathrm{c}^\lambda$ for all λ, hence the identity map on $Gr^s\mathfrak{C}^n$.

The identities $(\partial^{n-1}_i h^n(D))_\lambda = h^{n+1}((\partial^n_{i+1})_* D)_\lambda \mod F^s\mathrm{c}^\lambda$ hold for $0 \leq i \leq n$.

For $n = 0, 1, \dots$ let $\partial^n = \sum_i (-1)\partial^n_i$. The above identities show that, for $n > 0$, $h^{n+1} \circ \partial^n + \partial^{n-1} \circ h^n = \mathrm{Id}$. □

Proof of Theorem 5.2. Consider for a fixed k the complex of Hochschild degree k cochains $\mathfrak{C}^\bullet$. It is shown in Proposition 5.5 that this complex admits a finite filtration $F^\bullet\mathfrak{C}^\bullet$ such that $Gr\mathfrak{C}^\bullet$ is acyclic in positive degrees. Therefore $\mathfrak{C}^\bullet$is also acyclic in positive degree. As this holds for every k, the condition (3.8) is satisfied. □

6 Deformations of algebroid stacks

In this section we define a 2-groupoid of deformations of an algebroid stack. We also define 2-groupoids of deformations and star products of a descent datum and relate it with the 2-groupoid $\mathfrak{G}$-stacks of an appropriate cosimplicial DGLA.

6.1 Deformations of linear stacks

Definition 6.1. Let $\mathcal{B}$ be a prestack on X in R-linear categories. We say that $\mathcal{B}$ is *flat* if for any $U \subseteq X$, $A, B \in \mathcal{B}(U)$ the sheaf $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, B)$ is flat (as a sheaf of R-modules).

Suppose now that $\mathcal{C}$ is a stack in k-linear categories on X and R is a commutative Artin k-algebra. We denote by $\mathrm{Def}(\mathcal{C})(R)$ the 2-category with

- objects: pairs $(\mathcal{B}, \varpi)$, where $\mathcal{B}$ is a stack in R-linear categories flat over R and $\varpi\colon \widetilde{\mathcal{B} \otimes_R k} \to \mathcal{C}$ is an equivalence of stacks in k-linear categories;
- 1-morphisms: a 1-morphism $(\mathcal{B}^{(1)}, \varpi^{(1)}) \to (\mathcal{B}^{(2)}, \varpi^{(2)})$ is a pair (F, θ) where $F\colon \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$ is a R-linear functor and $\theta\colon \varpi^{(2)} \circ (F \otimes_R k) \to \varpi^{(1)}$ is an isomorphism of functors;
- 2-morphisms: a 2-morphism $(F', \theta') \to (F'', \theta'')$ is a morphism of R-linear functors $\kappa\colon F' \to F''$ such that $\theta'' \circ (\mathrm{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$.

The 2-category $\mathrm{Def}(\mathcal{C})(R)$ is a 2-groupoid.

Lemma 6.2. *Suppose that $\mathcal{B}$ is a flat R-linear stack on X such that $\widetilde{\mathcal{B} \otimes_R k}$ is an algebroid stack. Then, $\mathcal{B}$ is an algebroid stack.*

Proof. Let $x \in X$. Suppose that for any neighborhood U of x the category $\mathcal{B}(U)$ is empty. Then, the same is true about $\widetilde{\mathcal{B} \otimes_R k}(U)$ which contradicts the assumption that $\widetilde{\mathcal{B} \otimes_R k}$ is an algebroid stack. Therefore, $\mathcal{B}$ is locally nonempty.

Suppose that U is an open subset and A, B are two objects in $\mathcal{B}(U)$. Let $\overline{A}$ and $\overline{B}$ be their respective images in $\widetilde{\mathcal{B} \otimes_R k}(U)$. We have: $\mathrm{Hom}_{\widetilde{\mathcal{B}\otimes_R k}(U)}(\overline{A}, \overline{B}) = \Gamma(U; \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \otimes_R k)$. Replacing U by a subset we may assume that there is an isomorphism $\overline{\phi}\colon \overline{A} \to \overline{B}$.

The short exact sequence

$$0 \to \mathfrak{m}_R \to R \to k \to 0$$

gives rise to the sequence

$$0 \to \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \otimes_R \mathfrak{m}_R \to \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \otimes_R k \to 0$$

of sheaves on U which is exact due to flatness of $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, B)$. The surjectivity of the map $\underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \to \underline{\mathrm{Hom}}_{\mathcal{B}}(A, B) \otimes_R k$ implies that for any $x \in U$ there exists a neighborhood $x \in V \subseteq U$ and $\phi \in \Gamma(V; \underline{\mathrm{Hom}}_{\mathcal{B}}(A|_V, B|_V)) = \mathrm{Hom}_{\mathcal{B}(V)}(A|_V, B|_V)$ such that $\overline{\phi}|_V$ is the image of ϕ. Since $\overline{\phi}$ is an isomorphism and $\mathfrak{m}_R$ is nilpotent it follows that ϕ is an isomorphism. $\square$

6.2 Deformations of descent data. Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is an k-descent datum and R is a commutative Artin k-algebra.

We denote by $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ the 2-category with

- objects: *R-deformations of* $(\mathcal{U}, \underline{\mathcal{A}})$; such a gadget is a flat R-descent datum $(\mathcal{U}, \underline{\mathcal{B}})$ together with an isomorphism of k-descent data $\pi \colon (\mathcal{U}, \underline{\mathcal{B}} \otimes_R k)) \to (\mathcal{U}, \underline{\mathcal{A}})$;
- 1-morphisms: a 1-morphism of deformations $(\mathcal{U}, \underline{\mathcal{B}}^{(1)}, \pi^{(1)}) \to (\mathcal{U}, \underline{\mathcal{B}}^{(2)}, \pi^{(2)})$ is a pair $(\underline{\phi}, \underline{a})$, where $\underline{\phi} \colon (\mathcal{U}, \underline{\mathcal{B}}^{(1)}) \to (\mathcal{U}, \underline{\mathcal{B}}^{(2)})$ is a 1-morphism of R-descent data and $\underline{a} \colon \pi^{(2)} \circ (\underline{\phi} \otimes_R k) \to \pi^{(1)}$ is 2-isomorphism;
- 2-morphisms: a 2-morphism $(\underline{\phi}', \underline{a}') \to (\underline{\phi}'', \underline{a}'')$ is a 2-morphism $\underline{b} \colon \underline{\phi}' \to \underline{\phi}''$ such that $\underline{a}'' \circ (\mathrm{Id}_{\pi^{(2)}} \otimes (\underline{b} \otimes_R k)) = \underline{a}'$.

Suppose that $(\underline{\phi}, \underline{a})$ is a 1-morphism. It is immediate from the definition above that the morphism of k-descent data $\underline{\phi} \otimes_R k$ is an isomorphism. Since R is an extension of k by a nilpotent ideal the morphism $\underline{\phi}$ is an isomorphism. Similarly, any 2-morphism is an isomorphism, i.e. $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ is a 2-groupoid.

The assignment $R \mapsto \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ is fibered 2-category in 2-groupoids over the category of commutative Artin k-algebras ([3]).

6.2.1 Star products. An *(R-)star product* on $(\mathcal{U}, \underline{\mathcal{A}})$ is a deformation $(\mathcal{U}, \underline{\mathcal{B}}, \pi)$ of $(\mathcal{U}, \underline{\mathcal{A}})$ such that $\mathcal{B}_{01} = \mathcal{A}_{01} \otimes_k R$ and $\pi \colon \mathcal{B}_{01} \otimes_R k \to \mathcal{A}_{01}$ is the canonical isomorphism. In other words, a star product is a structure of an R-descent datum on $(\mathcal{U}, \underline{\mathcal{A}} \otimes_k R)$ such that the natural map

$$(\mathcal{U}, \underline{\mathcal{A}} \otimes_k R) \to (\mathcal{U}, \underline{\mathcal{A}})$$

is a morphism of such.

We denote by $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$ the full 2-subcategory of $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ whose objects are star products.

The assignment $R \mapsto \mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$ is fibered 2-category in 2-groupoids over the category of commutative Artin k-algebras ([3]) and the inclusions $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ extend to a morphism of fibered 2-categories.

Proposition 6.3. *Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a $\mathbb{C}$-linear descent datum with $\mathcal{A} = \mathcal{O}_{N_0\mathcal{U}}$. Then, the embedding* $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ *is an equivalence.*

6.2.2 Deformations and $\mathfrak{G}$-stacks. Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a k-descent datum and $(\mathcal{U}, \underline{\mathcal{B}})$ is an R-star product on $(\mathcal{U}, \underline{\mathcal{A}})$. Then, for every $p = 0, 1, \ldots$ the matrix algebra $\mathsf{Mat}(\underline{\mathcal{B}})^p$ is a flat R-deformation of the matrix algebra $\mathsf{Mat}(\underline{\mathcal{A}})^p$. The identification $\mathcal{B}_{01} = \mathcal{A}_{01} \otimes_k R$ gives rise to the identification $\mathsf{Mat}(\underline{\mathcal{B}})^p = \mathsf{Mat}(\underline{\mathcal{A}})^p \otimes_k R$ of the underlying sheaves of R-modules. Using this identification we obtain the Maurer–Cartan element $\mu^p \in \Gamma(N_p\mathcal{U}; C^2(\mathsf{Mat}(\underline{\mathcal{A}})^p)^{\mathrm{loc}} \otimes_k \mathfrak{m}_R)$. Moreover, the equation (5.5) implies that for a morphism $f : [p] \to [q]$ in Δ we have $f_* \mu^p = f^\sharp \mu^q$. Therefore the collection μ^p defines an element in $\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R)_0$. The considerations in 5.2.1 and 5.2.2 imply that this construction extends to an isomorphism of 2-groupoids

$$\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Stack}_{\mathrm{str}}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R). \tag{6.1}$$

Combining (6.1) with the embedding

$$\mathrm{Stack}_{\mathrm{str}}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \to \mathrm{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R) \tag{6.2}$$

we obtain the functor

$$\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R). \tag{6.3}$$

The naturality properties of (6.3) with respect to base change imply that (6.3) extends to morphism of functors on the category of commutative Artin algebras.

Combining this with the results of Theorems 5.2 and 3.12 implies the following:

Proposition 6.4. *The functor* (6.3) *is an equivalence.*

Proof. By Theorem 5.2 the DGLA $\mathfrak{G}(\underline{\mathcal{A}}) \otimes_k \mathfrak{m}_R$ satisfies the assumptions of Theorem 3.12. The latter says that the inclusion (6.2) is an equivalence. Since (6.1) is an isomorphism, the composition (6.3) is an equivalence as claimed. □

7 Jets

In this section we use constructions involving the infinite jets to simplify the cosimplicial DGLA governing the deformations of a descent datum.

7.1 Infinite jets of a vector bundle. Let M be a smooth manifold, and $\mathcal{E}$ a locally-free $\mathcal{O}_M$-module of finite rank.

Let $\pi_i : M \times M \to M$, $i = 1, 2$, denote the projection on the i^{th} factor. Denote by $\Delta_M : M \to M \times M$ the diagonal embedding and let $\Delta_M^* : \mathcal{O}_{M\times M} \to \mathcal{O}_M$ be the induced map. Let $\mathcal{I}_M := \ker(\Delta_M^*)$.

Let
$$\mathcal{J}^k(\mathcal{E}) := (\pi_1)_* \big(\mathcal{O}_{M\times M}/\mathcal{I}_M^{k+1} \otimes_{\pi_2^{-1}\mathcal{O}_M} \pi_2^{-1}\mathcal{E}\big),$$

$\mathcal{J}_M^k := \mathcal{J}^k(\mathcal{O}_M)$. It is clear from the above definition that $\mathcal{J}_M^k$ is, in a natural way, a sheaf of commutative algebras and $\mathcal{J}^k(\mathcal{E})$ is a sheaf of $\mathcal{J}_M^k$-modules. If moreover $\mathcal{E}$ is a sheaf of algebras, $\mathcal{J}^k(\mathcal{E})$ will canonically be a sheaf of algebras as well. We regard $\mathcal{J}^k(\mathcal{E})$ as $\mathcal{O}_M$-modules via the pull-back map $\pi_1^* : \mathcal{O}_M \to (\pi_1)_* \mathcal{O}_{M\times M}$

For $0 \le k \le l$ the inclusion $\mathcal{I}_M^{l+1} \to \mathcal{I}_M^{k+1}$ induces the surjective map $\mathcal{J}^l(\mathcal{E}) \to \mathcal{J}^k(\mathcal{E})$. The sheaves $\mathcal{J}^k(\mathcal{E})$, $k = 0, 1, \ldots$ together with the maps just defined form an inverse system. Define $\mathcal{J}(\mathcal{E}) := \varprojlim \mathcal{J}^k(\mathcal{E})$. Thus, $\mathcal{J}(\mathcal{E})$ carries a natural topology.

We denote by $p_{\mathcal{E}} : \mathcal{J}(\mathcal{E}) \to \mathcal{E}$ the canonical projection. In the case when $\mathcal{E} = \mathcal{O}_M$ we denote by p the corresponding projection $p : \mathcal{J}_M \to \mathcal{O}_M$. By $j^k : \mathcal{E} \to \mathcal{J}^k(\mathcal{E})$ we denote the map $e \mapsto 1 \otimes e$, and $j^\infty := \varprojlim j^k$. In the case $\mathcal{E} = \mathcal{O}_M$ we also have the canonical embedding $\mathcal{O}_M \to \mathcal{J}_M$ given by $f \mapsto f \cdot j^\infty(1)$.

Let

$$d_1 \colon \mathcal{O}_{M\times M} \otimes_{\pi_2^* \mathcal{O}_M} \pi_2^{-1}\mathcal{E} \to \pi_1^{-1}\Omega_M^1 \otimes_{\pi_1^{-1}\mathcal{O}_M} \mathcal{O}_{M\times M} \otimes_{\pi_2^{-1}\mathcal{O}_M} \pi_2^{-1}\mathcal{E}$$

denote the exterior derivative along the first factor. It satisfies

$$d_1(\mathcal{I}_M^{k+1} \otimes_{\pi_2^{-1}\mathcal{O}_M} \pi_2^{-1}\mathcal{E}) \subset \pi_1^{-1}\Omega_M^1 \otimes_{\pi_1^{-1}\mathcal{O}_M} \mathcal{I}_M^k \otimes_{\pi_2^{-1}\mathcal{O}_M} \pi_2^{-1}\mathcal{E}$$

for each k and, therefore, induces the map

$$d_1^{(k)} \colon \mathcal{J}^k(\mathcal{E}) \to \Omega^1_{M/\mathcal{P}} \otimes_{\mathcal{O}_M} \mathcal{J}^{k-1}(\mathcal{E}).$$

The maps $d_1^{(k)}$ for different values of k are compatible with the maps $\mathcal{J}^l(\mathcal{E}) \to \mathcal{J}^k(\mathcal{E})$ giving rise to the *canonical flat connection*

$$\nabla^{\mathrm{can}} \colon \mathcal{J}(\mathcal{E}) \to \Omega^1_M \otimes \mathcal{J}(\mathcal{E}).$$

Here and below we use notation $(\bullet) \otimes \mathcal{J}(\mathcal{E})$ for $\varprojlim (\bullet) \otimes_{\mathcal{O}_M} \mathcal{J}^k(\mathcal{E})$.

Since ∇^{can} is flat we obtain the complex of sheaves $\mathsf{DR}(\mathcal{J}(\mathcal{E})) = (\Omega^\bullet_M \otimes \mathcal{J}(\mathcal{E}), \nabla^{\mathrm{can}})$. When $\mathcal{E} = \mathcal{O}_M$ embedding $\mathcal{O}_M \to \mathcal{J}_M$ induces embedding of de Rham complex $\mathsf{DR}(\mathcal{O}) = (\Omega^\bullet_M, d)$ into $\mathsf{DR}(\mathcal{J})$. We denote the quotient by $\mathsf{DR}(\mathcal{J}/\mathcal{O})$. All the complexes above are complexes of soft sheaves. We have the following:

Proposition 7.1. *The (hyper)cohomology $H^i(M, \mathsf{DR}(\mathcal{J}(\mathcal{E}))) \cong H^i(\Gamma(M; \Omega_M \otimes \mathcal{J}(\mathcal{E})), \nabla^{\mathrm{can}})$ is 0 if $i > 0$. The map $j^\infty \colon \mathcal{E} \to \mathcal{J}(\mathcal{E})$ induces the isomorphism between $\Gamma(\mathcal{E})$ and $H^0(M, \mathsf{DR}(\mathcal{J}(\mathcal{E}))) \cong H^0(\Gamma(M; \Omega_M \otimes \mathcal{J}(\mathcal{E})), \nabla^{\mathrm{can}})$*

7.2 Jets of line bundles. Let, as before, M be a smooth manifold, $\mathcal{J}_M$ be the sheaf of infinite jets of smooth functions on M and $p \colon \mathcal{J}_M \to \mathcal{O}_M$ be the canonical projection. Set $\mathcal{J}_{M,0} = \ker p$. Note that $\mathcal{J}_{M,0}$ is a sheaf of $\mathcal{O}_M$ modules and therefore is soft.

Suppose now that $\mathcal{L}$ is a line bundle on M. Let $\underline{\mathrm{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ denote the sheaf of local $\mathcal{J}_M$-module isomorphisms $\mathcal{L} \otimes \mathcal{J}_M \to \mathcal{J}(\mathcal{L})$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{L} \otimes \mathcal{J}_M \arrow[rr] \arrow[dr, "\mathrm{Id}\otimes p"'] & & \mathcal{J}(\mathcal{L}) \arrow[dl, "p_{\mathcal{L}}"] \\
& \mathcal{L}. &
\end{tikzcd}$$

It is easy to see that the canonical map $\mathcal{J}_M \to \underline{\mathrm{End}}_{\mathcal{J}_M}(\mathcal{L} \otimes \mathcal{J}_M)$ is an isomorphism. For $\phi \in \mathcal{J}_{M,0}$ the exponential series $\exp(\phi)$ converges. The composition

$$\mathcal{J}_{M,0} \xrightarrow{\exp} \mathcal{J}_M \to \underline{\mathrm{End}}_{\mathcal{J}_M}(\mathcal{L} \otimes \mathcal{J}_M)$$

defines an isomorphism of sheaves of groups

$$\exp\colon \mathcal{J}_{M,0} \to \underline{\mathrm{Aut}}_0(\mathcal{L} \otimes \mathcal{J}_M),$$

where $\underline{\mathrm{Aut}}_0(\mathcal{L} \otimes \mathcal{J}_M)$ is the sheaf of groups of (locally defined) $\mathcal{J}_M$-linear automorphisms of $\mathcal{L} \otimes \mathcal{J}_M$ making the diagram

$$\begin{tikzcd} \mathcal{L} \otimes \mathcal{J}_M \arrow[rr] \arrow[dr, "\mathrm{Id}\otimes p"'] & & \mathcal{L} \otimes \mathcal{J}_M \arrow[dl, "\mathrm{Id}\otimes p"] \\ & \mathcal{L} & \end{tikzcd}$$

commutative.

Lemma 7.2. *The sheaf* $\underline{\mathrm{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ *is a torsor under the sheaf of groups* $\exp \mathcal{J}_{M,0}$.

Proof. Since $\mathcal{L}$ is locally trivial, both $\mathcal{J}(\mathcal{L})$ and $\mathcal{L} \otimes \mathcal{J}_M$ are locally isomorphic to $\mathcal{J}_M$. Therefore the sheaf $\underline{\mathrm{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ is locally non-empty, hence a torsor. □

Corollary 7.3. *The torsor* $\underline{\mathrm{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ *is trivial, that is,* $\mathrm{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L})) := \Gamma(M; \underline{\mathrm{Isom}}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))) \neq \varnothing$.

Proof. Since the sheaf of groups $\mathcal{J}_{M,0}$ is soft we have $H^1(M, \mathcal{J}_{M,0}) = 0$ (see [8], Lemme 22, cf. also [6], Proposition 4.1.7). Therefore every $\mathcal{J}_{M,0}$-torsor is trivial. □

Corollary 7.4. *The set* $\mathrm{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ *is an affine space with the underlying vector space* $\Gamma(M; \mathcal{J}_{M,0})$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles, and $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ an isomorphism. Then f induces a map $\mathrm{Isom}_0(\mathcal{L}_2 \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_2)) \to \mathrm{Isom}_0(\mathcal{L}_1 \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_1))$ which we denote by $\mathrm{Ad}\, f$:

$$\mathrm{Ad}\, f(\sigma) = (j^\infty(f))^{-1} \circ \sigma \circ (f \otimes \mathrm{Id}).$$

Let $\mathcal{L}$ be a line bundle on M and $f\colon N \to M$ is a smooth map. Then there is a pull-back map $f^*\colon \mathrm{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L})) \to \mathrm{Isom}_0(f^*\mathcal{L} \otimes \mathcal{J}_N, \mathcal{J}(f^*\mathcal{L}))$.

If $\mathcal{L}_1, \mathcal{L}_2$ are two line bundles, and $\sigma_i \in \mathrm{Isom}_0(\mathcal{L}_i \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_i))$, $i = 1, 2$. Then we denote by $\sigma_1 \otimes \sigma_2$ the induced element of $\mathrm{Isom}_0((\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}_1 \otimes \mathcal{L}_2))$.

For a line bundle $\mathcal{L}$ let $\mathcal{L}^*$ be its dual. Then given $\sigma \in \mathrm{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ there exists a unique a unique $\sigma^* \in \mathrm{Isom}_0(\mathcal{L}^* \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}^*))$ such that $\sigma^* \otimes \sigma = \mathrm{Id}$.

For any bundle E $\mathcal{J}(E)$ has a canonical flat connection which we denote by ∇^{can}. A choice of $\sigma \in \mathrm{Isom}_0(\mathcal{L} \otimes \mathcal{J}_M, \mathcal{J}(\mathcal{L}))$ induces the flat connection $\sigma^{-1} \circ \nabla^{\mathrm{can}}_{\mathcal{L}} \circ \sigma$ on $\mathcal{L} \otimes \mathcal{J}_M$.

Let ∇ be a connection on $\mathcal{L}$ with the curvature θ. It gives rise to the connection $\nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}}$ on $\mathcal{L} \otimes \mathcal{J}_M$.

Lemma 7.5. 1. *Choose σ, ∇ as above. Then the difference*

$$F(\sigma,\nabla)=\sigma^{-1}\circ\nabla^{\mathrm{can}}_{\mathcal{L}}\circ\sigma-(\nabla\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^{\mathrm{can}}) \tag{7.1}$$

is an element of $\in\Omega^1\otimes\underline{\mathrm{End}}_{\mathcal{J}_M}(\mathcal{L}\otimes\mathcal{J}_M)\cong\Omega^1\otimes\mathcal{J}_M$.

2. *Moreover, F satisfies*

$$\nabla^{\mathrm{can}}F(\sigma,\nabla)+\theta=0. \tag{7.2}$$

Proof. We leave the verification of the first claim to the reader. The flatness of $\sigma^{-1}\circ\nabla^{\mathrm{can}}_{\mathcal{L}}\circ\sigma$ implies the second claim. □

The following properties of our construction are immediate

Lemma 7.6. 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles, and $f\colon\mathcal{L}_1\to\mathcal{L}_2$ an isomorphism. Let ∇ be a connection on $\mathcal{L}_2$ and $\sigma\in\mathrm{Isom}_0(\mathcal{L}_2\otimes\mathcal{J}_M,\mathcal{J}(\mathcal{L}_2))$. Then*

$$F(\sigma,\nabla)=F(\mathrm{Ad}\,f(\sigma),\mathrm{Ad}\,f(\nabla)).$$

2. *Let $\mathcal{L}$ be a line bundle on M, ∇ a connection on $\mathcal{L}$ and $\sigma\in\mathrm{Isom}_0(\mathcal{L}\otimes\mathcal{J}_M,\mathcal{J}(\mathcal{L}))$. Let $f\colon N\to M$ be a smooth map. Then*

$$f^*F(\sigma,\nabla)=F(f^*\sigma,f^*\nabla).$$

3. *Let $\mathcal{L}$ be a line bundle on M, ∇ a connection on $\mathcal{L}$ and $\sigma\in\mathrm{Isom}_0(\mathcal{L}\otimes\mathcal{J}_M,\mathcal{J}(\mathcal{L}))$. Let $\phi\in\Gamma(M;\mathcal{J}_{M,0})$. Then*

$$F(\phi\cdot\sigma,\nabla)=F(\sigma,\nabla)+\nabla^{\mathrm{can}}\phi.$$

4. *Let $\mathcal{L}_1$, $\mathcal{L}_2$ be two line bundles with connections ∇_1 and ∇_2 respectively, and let $\sigma_i\in\mathrm{Isom}_0(\mathcal{L}_i\otimes\mathcal{J}_M,\mathcal{J}(\mathcal{L}_i))$, $i=1,2$. Then*

$$F(\sigma_1\otimes\sigma_2,\nabla_1\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla_2)=F(\sigma_1,\nabla_1)+F(\sigma_2,\nabla_2).$$

7.3 DGLAs of infinite jets. Suppose that $(\mathcal{U},\underline{\mathcal{A}})$ is a descent datum representing a twisted form of $\mathcal{O}_X$. Thus, we have the matrix algebra $\mathsf{Mat}(\underline{\mathcal{A}})$ and the cosimplicial DGLA $\mathfrak{G}(\underline{\mathcal{A}})$ of local $\mathbb{C}$-linear Hochschild cochains.

The descent datum $(\mathcal{U},\underline{\mathcal{A}})$ gives rise to the descent datum $(\mathcal{U},\mathcal{J}(\underline{\mathcal{A}}))$, $\mathcal{J}(\underline{\mathcal{A}})=(\mathcal{J}(\mathcal{A}),\mathcal{J}(\mathcal{A}_{01}),j^\infty(\mathcal{A}_{012}))$, representing a twisted form of $\mathcal{J}_X$, hence to the matrix algebra $\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))$ and the corresponding cosimplicial DGLA $\mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$ of local $\mathcal{O}$-*linear continuous* Hochschild cochains.

The canonical flat connection ∇^{can} on $\mathcal{J}(\mathcal{A})$ induces the flat connection, still denoted ∇^{can} on $\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$ for each p; the product on $\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$ is horizontal with respect to ∇^{can}. The flat connection ∇^{can} induces the flat connection, still denoted ∇^{can} on $C^\bullet(\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{\mathrm{loc}}[1]$ which acts by derivations of the Gerstenhaber bracket and commutes with the Hochschild differential δ. Therefore we have the sheaf of

DGLA $\mathsf{DR}(C^\bullet(\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{\mathrm{loc}}[1])$ with the underlying sheaf of graded Lie algebras $\Omega^\bullet_{N_p\mathcal{U}} \otimes C^\bullet(\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p)^{\mathrm{loc}}[1]$ and the differential $\nabla^{\mathrm{can}} + \delta$.

For $\lambda\colon [n] \to \Delta$ let

$$\mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda = \Gamma(N_{\lambda(n)}\mathcal{U}; \lambda(0n)_*\mathsf{DR}(C^\bullet(\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^{\lambda(0)})^{\mathrm{loc}}[1]))$$

be the DGLA of global sections. The "inclusion of horizontal sections" map induces the morphism of DGLA

$$j^\infty\colon \mathfrak{G}(\underline{\mathcal{A}})^\lambda \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda.$$

For $\phi\colon [m] \to [n]$ in Δ, $\mu = \lambda \circ \phi$ there is a morphism of DGLA

$$\phi_*\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\mu \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$$

making the diagram

$$\begin{array}{ccc}
\mathfrak{G}(\underline{\mathcal{A}})^\lambda & \xrightarrow{j^\infty} & \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda \\
{\scriptstyle \phi_*}\big\downarrow & & \big\downarrow{\scriptstyle \phi_*} \\
\mathfrak{G}(\underline{\mathcal{A}})^\mu & \xrightarrow{j^\infty} & \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\mu
\end{array}$$

commutative.

Let $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^n = \prod_{[n]\xrightarrow{\lambda}\Delta} \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^\lambda$. The cosimplicial DGLA $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))$ is defined by the assignment $\Delta \ni [n] \mapsto \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^n$, $\phi \mapsto \phi_*$.

Proposition 7.7. *The map $j^\infty\colon \mathfrak{G}(\underline{\mathcal{A}}) \to \mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$ extends to a quasiisomorphism of cosimplicial DGLA.*

$$j^\infty\colon \mathfrak{G}(\underline{\mathcal{A}}) \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}})).$$

The goal of this section is to construct a quasiisomorphism of the latter DGLA with the simpler DGLA.

The canonical flat connection ∇^{can} on $\mathcal{J}_X$ induces a flat connection on $\bar{C}^\bullet(\mathcal{J}_X)[1]$, the complex of *$\mathcal{O}$-linear continuous normalized* Hochschild cochains, still denoted ∇^{can} which acts by derivations of the Gerstenhaber bracket and commutes with the Hochschild differential δ. Therefore we have the sheaf of DGLA $\mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X)[1])$ with the underlying graded Lie algebra $\Omega^\bullet_X \otimes \bar{C}^\bullet(\mathcal{J}_X)[1]$ and the differential $\delta + \nabla^{\mathrm{can}}$.

Recall that the Hochschild differential δ is zero on $C^0(\mathcal{J}_X)$ due to commutativity of $\mathcal{J}_X$. It follows that the action of the sheaf of abelian Lie algebras $\mathcal{J}_X = C^0(\mathcal{J}_X)$ on $\bar{C}^\bullet(\mathcal{J}_X)[1]$ via the restriction of the adjoint action (by derivations of degree -1) commutes with the Hochschild differential δ. Since the cochains we consider are $\mathcal{O}_X$-linear, the subsheaf $\mathcal{O}_X = \mathcal{O}_X \cdot j^\infty(1) \subset \mathcal{J}_X$ acts trivially. Hence the action of $\mathcal{J}_X$ descends to an action of the quotient $\mathcal{J}_X/\mathcal{O}_X$. This action induces the action of the abelian graded Lie algebra $\Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X$ by derivations on the graded Lie algebra $\Omega^\bullet_X \otimes \bar{C}^\bullet(\mathcal{J}_X)[1]$. Further, the subsheaf $(\Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}} := \ker(\nabla^{\mathrm{can}})$ acts by derivation which commute with the differential $\delta + \nabla^{\mathrm{can}}$. For $\omega \in \Gamma(X; (\Omega^2_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}})$

we denote by $\mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X)[1])_\omega$ the sheaf of DGLA with the underlying graded Lie algebra $\Omega^\bullet_X \otimes \bar{C}^\bullet(\mathcal{J}_X)[1]$ and the differential $\delta + \nabla^{\mathrm{can}} + \iota_\omega$. Let

$$\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_\omega = \Gamma(X; \mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X)[1])_\omega),$$

be the corresponding DGLA of global sections.

Suppose now that $\mathcal{U}$ is a cover of X; let $\epsilon\colon N\mathcal{U} \to X$ denote the canonical map. For $\lambda\colon [n] \to \Delta$ let

$$\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^\lambda_\omega = \Gamma(N_{\lambda(n)}\mathcal{U}; \epsilon^*\mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X)[1])_\omega).$$

For $\mu\colon [m] \to \Delta$ and a morphism $\phi\colon [m] \to [n]$ in Δ such that $\mu = \lambda \circ \phi$ the map $\mu(m) \to \lambda(n)$ induces the map

$$\phi_*\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^\mu_\omega \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^\lambda_\omega.$$

For $n = 0, 1, \dots$ let $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^n_\omega = \prod_{[n]\xrightarrow{\lambda}\Delta} \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^\lambda_\omega$. The assignment $[n] \mapsto \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^n_\omega$ extends to a cosimplicial DGLA $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega$. If we need to explicitly indicate the cover we will also denote this DGLA by $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{U})$.

Lemma 7.8. *The cosimplicial DGLA $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega$ is acyclic, i.e. satisfies the condition* (3.8).

Proof. Consider the cosimplicial vector space $V^\bullet$ with

$$V^n = \Gamma(N_n\mathcal{U}; \epsilon^*\mathsf{DR}(\bar{C}^\bullet(\mathcal{J}_X)[1])_\omega)$$

and the cosimplicial structure induced by the simplicial structure of $N\mathcal{U}$. The cohomology of the complex $(V^\bullet, \partial)$ is the Čech cohomology of $\mathcal{U}$ with the coefficients in the soft sheaf of vector spaces $\Omega^\bullet \otimes \bar{C}^\bullet(\mathcal{J}_X)[1]$ and, therefore, vanishes in the positive degrees. $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega$ as a cosimplicial vector space can be identified with $\widehat{V}^\bullet$ in the notations of Lemma 2.1. Hence the result follows from Lemma 2.1. □

We leave the proof of the following lemma to the reader.

Lemma 7.9. *The map $\epsilon^*\colon \mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_\omega \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^0_\omega$ induces an isomorphism of DGLA*

$$\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_\omega \cong \ker(\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^0_\omega \rightrightarrows \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^1_\omega)$$

where the two maps on the right are $(\partial^0_0)_*$ and $(\partial^0_1)_*$.

Two previous lemmas together with Corollary 3.13 imply the following:

Proposition 7.10. *Let $\omega \in \Gamma(X; (\Omega^2_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}})$ and let $\mathfrak{m}$ be a commutative nilpotent ring. Then the map ϵ^* induces equivalence of groupoids:*

$$\mathrm{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \cong \mathrm{Stack}(\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega \otimes \mathfrak{m}).$$

For $\beta \in \Gamma(X; \Omega^1 \otimes \mathcal{J}_X/\mathcal{O}_X)$ there is a canonical isomorphism of cosimplicial DGLA $\exp(\iota_\beta)\colon \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_{\omega+\nabla^{\mathrm{can}}\beta} \to \mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega$. Therefore, $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega$ depends only on the class of ω in $H^2_{\mathrm{DR}}(\mathcal{J}_X/\mathcal{O}_X)$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 7.11. *Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a descent datum representing a twisted form $\mathcal{S}$ of $\mathcal{O}_X$. There exists a quasi-isomorphism of cosimplicial DGLA $\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_{[\mathcal{S}]} \to \mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}}))$.*

7.4 Quasiisomorphism.

Suppose that $(\mathcal{U}, \underline{\mathcal{A}})$ is a descent datum for a twisted form of $\mathcal{O}_X$. Thus, $\mathcal{A}$ is identified with $\mathcal{O}_{N_0\mathcal{U}}$ and $\mathcal{A}_{01}$ is a line bundle on $N_1\mathcal{U}$.

7.4.1 Multiplicative connections.

Let $\mathcal{C}^\mu(\mathcal{A}_{01})$ denote the set of connections ∇ on $\mathcal{A}_{01}$ which satisfy

1. $\mathrm{Ad}\,\mathcal{A}_{012}((\mathrm{pr}^2_{02})^*\nabla) = (\mathrm{pr}^2_{01})^*\nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes (\mathrm{pr}^2_{12})^*\nabla$,
2. $(\mathrm{pr}^0_{00})^*\nabla$ is the canonical flat connection on $\mathcal{O}_{N_0\mathcal{U}}$.

Let $\mathrm{Isom}^\mu_0(\mathcal{A}_{01}\otimes\mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ denote the subset of $\mathrm{Isom}_0(\mathcal{A}_{01}\otimes\mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ which consists of σ which satisfy

1. $\mathrm{Ad}\,\mathcal{A}_{012}((\mathrm{pr}^2_{02})^*\sigma) = (\mathrm{pr}^2_{01})^*\sigma \otimes (\mathrm{pr}^2_{12})^*\sigma$,
2. $(\mathrm{pr}^0_{00})^*\sigma = \mathrm{Id}$

Note that the vector space $\bar{Z}^1(\mathcal{U}; \Omega^1)$ of cocycles in the *normalized* Čech complex of the cover $\mathcal{U}$ with coefficients in the sheaf of 1-forms Ω^1 acts on the set $\mathcal{C}^\mu(\mathcal{A}_{01})$, with the action given by

$$\alpha \cdot \nabla = \nabla + \alpha. \tag{7.3}$$

Here $\nabla \in \mathcal{C}^\mu(\mathcal{A}_{01})$, $\alpha \in \bar{Z}^1(\mathcal{U}; \Omega^1) \subset \Omega^1(N_1\mathcal{U})$.

Similarly, the vector space $\bar{Z}^1(\mathcal{U}; \mathcal{J}_0)$ acts on the set $\mathrm{Isom}^\mu_0(\mathcal{A}_{01}\otimes\mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$, with the action given as in Corollary 7.4.

Note that since the sheaves involved are soft, cocycles coincide with coboundaries: $\bar{Z}^1(\mathcal{U}; \Omega^1) = \bar{B}^1(\mathcal{U}; \Omega^1)$, $\bar{Z}^1(\mathcal{U}; \mathcal{J}_0) = \bar{B}^1(\mathcal{U}; \mathcal{J}_0)$.

Proposition 7.12. *The set $\mathcal{C}^\mu(\mathcal{A}_{01})$ (respectively, $\mathrm{Isom}^\mu_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$) is an affine space with the underlying vector space being $\bar{Z}^1(\mathcal{U}; \Omega^1)$ (respectively, $\bar{Z}^1(\mathcal{U}; \mathcal{J}_0)$).*

Proof. Proofs of the both statements are completely analogous. Therefore we explain the proof of the statement concerning $\mathrm{Isom}^\mu_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ only.

We show first that $\mathrm{Isom}^\mu_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$ is nonempty. Choose an arbitrary $\sigma \in \mathrm{Isom}_0(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{ij}))$ such that $(\mathrm{pr}^0_{00})^*\sigma = \mathrm{Id}$. Then, by Corollary 7.4, there exists $c \in \Gamma(N_2\mathcal{U}; \mathcal{J}_0)$ such that $c\cdot(\mathrm{Ad}\,\mathcal{A}_{012}((d^1)^*\sigma_{02})) = (d^0)^*\sigma_{01}\otimes (d^2)^*\sigma_{12}$. It is easy to see that $c \in \bar{Z}^2(\mathcal{U}; \exp\mathcal{J}_0)$. Since the sheaf $\exp\mathcal{J}_0$ is soft, corresponding Čech cohomology is trivial. Therefore, there exists $\phi \in \bar{C}^1(\mathcal{U};\mathcal{J}_0)$ such that $c = \check{\partial}\phi$. Then, $\phi \cdot \sigma \in \mathrm{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$.

Suppose that $\sigma, \sigma' \in \mathrm{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$. By Corollary 7.4 $\sigma = \phi \cdot \sigma'$ for some uniquely defined $\phi \in \Gamma(N_2\mathcal{U};\mathcal{J}_0)$. It is easy to see that $\phi \in \bar{Z}^1(\mathcal{U};\mathcal{J}_0)$. □

We assume from now on that we have chosen $\sigma \in \mathrm{Isom}_0^\mu(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1\mathcal{U}}, \mathcal{J}(\mathcal{A}_{01}))$, $\nabla \in \mathcal{C}^\mu(\mathcal{A}_{01})$. Such a choice defines $\sigma_{ij}^p \in \mathrm{Isom}_0(\mathcal{A}_{ij} \otimes \mathcal{J}_{N_p\mathcal{U}}, \mathcal{J}(\mathcal{A}_{ij}))$ for every p and $0 \le i,j \le p$ by $\sigma_{ij}^p = (\mathrm{pr}_{ij}^p)^*\sigma$. This collection of σ_{ij}^p induces for every p algebra isomorphism $\sigma^p\colon \mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}} \to \mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p$. The following compatibility holds for these isomorphisms. Let $f\colon [p] \to [q]$ be a morphism in Δ. Then the following diagram commutes:

$$\begin{array}{ccc} f_*(\mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}}) & \xrightarrow{f_*} & f^\sharp\mathsf{Mat}(\underline{\mathcal{A}} \otimes \mathcal{J}_{N_q\mathcal{U}})^q \\ {\scriptstyle f_*(\sigma^p)}\downarrow & & \downarrow{\scriptstyle f^\sharp(\sigma^p)} \\ f_*\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^p & \xrightarrow{f_*} & f^\sharp\mathsf{Mat}(\mathcal{J}(\underline{\mathcal{A}}))^q. \end{array} \tag{7.4}$$

Similarly define the connections $\nabla_{ij}^p = (\mathrm{pr}_{ij}^p)^*\nabla$. For $p = 0,1,\ldots$ set $\nabla^p = \oplus_{i,j=0}^p \nabla_{ij}$; the connections ∇^p on $\mathsf{Mat}(\underline{\mathcal{A}})^p$ satisfy

$$f_*\nabla^p = (\mathrm{Ad}\, f_*)(f^\sharp\nabla^q). \tag{7.5}$$

Note that $F(\sigma,\nabla) \in \Gamma(N_1\mathcal{U};\Omega^1_{N_1\mathcal{U}} \otimes \mathcal{J}_{N_1\mathcal{U}})$ is a cocycle of degree 1 in $\check{C}^\bullet(\mathcal{U};\Omega^1_X \otimes \mathcal{J}_X)$. Vanishing of the corresponding Čech cohomology implies that there exists $F^0 \in \Gamma(N_0\mathcal{U};\Omega^1_{N_0\mathcal{U}} \otimes \mathcal{J}_{N_0\mathcal{U}})$ such that

$$(d^1)^*F^0 - (d^0)^*F^0 = F(\sigma,\nabla). \tag{7.6}$$

For $p = 0,1,\ldots$, $0 \le i \le p$, let $F_{ii}^p = (\mathrm{pr}_i^p)^*F^0$; put $F_{ij}^p = 0$ for $i \ne j$. Let $F^p \in \Gamma(N_p\mathcal{U};\Omega^1_{N_p\mathcal{U}} \otimes \mathsf{Mat}(\underline{\mathcal{A}})^p) \otimes \mathcal{J}_{N_p\mathcal{U}}$ denote the diagonal matrix with components F_{ij}^p. For $f\colon [p] \to [q]$ we have

$$f_*F^p = f^\sharp F^q. \tag{7.7}$$

Then, we obtain the following equality of connections on $\mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}}$:

$$(\sigma^p)^{-1} \circ \nabla^{\mathrm{can}} \circ \sigma^p = \nabla^p \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} + \mathrm{ad}\, F^p. \tag{7.8}$$

The matrices F^p also have the following property. Let $\nabla^{\mathrm{can}}F^p$ be the diagonal matrix with the entries $(\nabla^{\mathrm{can}}F^p)_{ii} = \nabla^{\mathrm{can}}F_{ii}^p$. Denote by $\overline{\nabla^{\mathrm{can}}F^p}$ the image of $\nabla^{\mathrm{can}}F^p$ under the natural map $\Gamma(N_p\mathcal{U};\Omega^2_{N_p\mathcal{U}} \otimes \mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}}) \to \Gamma(N_p\mathcal{U};\Omega^2_{N_p\mathcal{U}} \otimes \mathsf{Mat}(\underline{\mathcal{A}})^p \otimes (\mathcal{J}_{N_p\mathcal{U}}/\mathcal{O}_{N_p\mathcal{U}}))$. Recall the canonical map $\epsilon_p\colon N_p\mathcal{U} \to X$. Then, we have the following:

Lemma 7.13. *There exists a unique* $\omega \in \Gamma(X; (\Omega^2_X \otimes \mathcal{J}_X/\mathcal{O}_X)^{\mathrm{cl}})$ *such that*

$$\overline{\nabla^{\mathrm{can}} F^p} = -\epsilon_p^* \omega \otimes \mathrm{Id}_p,$$

where Id_p *denotes the* $(p+1) \times (p+1)$ *identity matrix.*

Proof. Using the definition of F^0 and formula (7.2) we obtain: $(d^1)^* \nabla^{\mathrm{can}} F^0 - (d^0)^* \nabla^{\mathrm{can}} F^0 = \nabla^{\mathrm{can}} F(\sigma, \nabla) \in \Omega^2(N_1 \mathcal{U})$. Hence $(d^1)^* \overline{\nabla^{\mathrm{can}} F^0} - (d^0)^* \overline{\nabla^{\mathrm{can}} F^0} = 0$, and there exists a unique $\omega \in \Gamma(X; \Omega^2_X \otimes (\mathcal{J}_X/\mathcal{O}_X))$ such that $\epsilon_0^* \omega = \overline{\nabla^{\mathrm{can}} F^0}$. Since $(\nabla^{\mathrm{can}})^2 = 0$ it follows that $\nabla^{\mathrm{can}} \omega = 0$. For any p we have: $(\overline{\nabla^{\mathrm{can}} F^p})_{ii} = \mathrm{pr}_i^* \overline{\nabla^{\mathrm{can}} F^0} = \epsilon_p^* \omega$, and the assertion of the lemma follows. □

Lemma 7.14. *The class of* ω *in* $H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ *does not depend on the choices made in its construction.*

Proof. The construction of ω is dependent on the choice of F^0, $\sigma \in \mathrm{Isom}_0^{\mu}(\mathcal{A}_{01} \otimes \mathcal{J}_{N_1 \mathcal{U}} \mathcal{J}(\mathcal{A}_{01}))$, and $\nabla \in \mathcal{C}^{\mu}(\mathcal{A}_{01})$ satisfying the equation (7.6). Assume that we make different choices: $\sigma' = (\check{\partial}\phi) \cdot \sigma$, $\nabla' = (\check{\partial}\alpha) \cdot \nabla$ and $(F^0)'$ satisfying $\check{\partial}(F^0)' = F(\sigma', \nabla')$. Here, $\phi \in \check{C}^0(\mathcal{U}; \mathcal{J}_0)$ and $\alpha \in \check{C}^0(\mathcal{U}; \Omega^1)$. We have: $F(\sigma', \nabla') = F(\sigma, \nabla) - \check{\partial}\alpha + \check{\partial}\nabla^{\mathrm{can}}\phi$. It follows that $\check{\partial}((F^0)' - F^0 - \nabla^{\mathrm{can}}\phi + \alpha) = 0$. Therefore $(F^0)' - F^0 - \nabla^{\mathrm{can}}\phi + \alpha = -\epsilon_0^* \beta$ for some $\beta \in \Gamma(X; \Omega^1_X \otimes \mathcal{J}_X)$. Hence if ω' is constructed using σ', ∇', $(F^0)'$ then $\omega' - \omega = \nabla^{\mathrm{can}} \bar{\beta}$ where $\bar{\beta}$ is the image of β under the natural projection $\Gamma(X; \Omega^1_X \otimes \mathcal{J}_X) \to \Gamma(X; \Omega^1_X \otimes (\mathcal{J}_X/\mathcal{O}_X))$. □

Let $\rho\colon \mathcal{V} \to \mathcal{U}$ be a refinement of the cover $\mathcal{U}$, and $(\mathcal{V}, \underline{\mathcal{A}}^{\rho})$ the corresponding descent datum. Choice of σ, ∇, F^0 on $\mathcal{U}$ induces the corresponding choice $(N\rho)^*\sigma$, $(N\rho)^*\nabla$, $(N\rho)^*(F^0)$ on $\mathcal{V}$. Let ω^{ρ} denotes the form constructed as in Lemma 7.13 using $(N\rho)^*\sigma$, $(N\rho)^*\nabla$, $(N\rho)^* F^0$. Then,

$$\omega^{\rho} = (N\rho)^* \omega.$$

The following result now follows easily and we leave the details to the reader.

Proposition 7.15. *The class of* ω *in* $H^2(\Gamma(X; \Omega^\bullet_X \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^{\mathrm{can}})$ *coincides with the image* $[\mathcal{S}]$ *of the class of the gerbe.*

7.5 Construction of the quasiisomorphism. For $\lambda\colon [n] \to \Delta$ let

$$\mathfrak{H}^{\lambda} := \Gamma(N_{\lambda(n)}\mathcal{U};\ \Omega^\bullet_{N_{\lambda(n)}\mathcal{U}} \otimes \lambda(0n)_* C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)} \otimes \mathcal{J}_{N_{\lambda(0)}\mathcal{U}})^{\mathrm{loc}}[1]),$$

considered as a graded Lie algebra. For $\phi\colon [m] \to [n]$, $\mu = \lambda \circ \phi$ there is a morphism of graded Lie algebras $\phi_*\colon \mathfrak{H}^{\mu} \to \mathfrak{H}^{\lambda}$. For $n = 0, 1, \ldots$ let $\mathfrak{H}^n := \prod_{[n] \xrightarrow{\lambda} \Delta} \mathfrak{H}^{\lambda}$. The assignment $\Delta \ni [n] \mapsto \mathfrak{H}^n$, $\phi \mapsto \phi_*$ defines a cosimplicial graded Lie algebra $\mathfrak{H}$.

For each $\lambda\colon [n] \to \Delta$ the map

$$\sigma_*^{\lambda} := \mathrm{Id} \otimes \lambda(0n)_*(\sigma^{\lambda(0)})\colon \mathfrak{H}^{\lambda} \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))^{\lambda}$$

is an isomorphism of graded Lie algebras. It follows from (7.4) that the maps σ_*^λ yield an isomorphism of cosimplicial graded Lie algebras

$$\sigma_* \colon \mathfrak{H} \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}})).$$

Moreover, the equation (7.8) shows that if we equip $\mathfrak{H}$ with the differential given on $\Gamma(N_{\lambda(n)}\mathcal{U};\ \Omega^\bullet_{N_{\lambda(n)}\mathcal{U}} \otimes \lambda(0n)_* C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^{\lambda(0)} \otimes \mathcal{J}_{N_{\lambda(0)}\mathcal{U}})^{\mathrm{loc}}[1])$ by

$$\begin{aligned}\delta + \lambda(0n)_*(\nabla^{\lambda(0)}) \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} + \mathrm{ad}\,\lambda(0n)_*(F^{\lambda(0)})& \\ = \delta + \lambda(0n)^\sharp(\nabla^{\lambda(n)}) \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} + \mathrm{ad}\,\lambda(0n)^\sharp(F^{\lambda(n)}))& \end{aligned} \tag{7.9}$$

then σ_* becomes an isomorphism of DGLA. Consider now an automorphism $\exp \iota_F$ of the cosimplicial graded Lie algebra $\mathfrak{H}$ given on $\mathfrak{H}^\lambda$ by $\exp \iota_{\lambda(0n)_* F_p}$. Note the fact that this morphism preserves the cosimplicial structure follows from the relation (7.7).

The following result is proved by the direct calculation; see [4], Lemma 16.

Lemma 7.16.

$$\begin{aligned}&\exp(\iota_{F_p}) \circ (\delta + \nabla^p \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} + \mathrm{ad}\, F^p) \circ \exp(-\iota_{F_p}) \\ &\quad = \delta + \nabla^p \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} - \iota_{\nabla F^p}.\end{aligned} \tag{7.10}$$

Therefore the morphism

$$\exp \iota_F \colon \mathfrak{H} \to \mathfrak{H} \tag{7.11}$$

conjugates the differential given by the formula (7.9) into the differential which on $\mathfrak{H}^\lambda$ is given by

$$\delta + \lambda(0n)_*(\nabla^{\lambda(0)}) \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} - \iota_{\lambda(0n)_*(\nabla^{\mathrm{can}} F^{\lambda(0)})}. \tag{7.12}$$

Consider the map

$$\mathrm{cotr} \colon \bar{C}^\bullet(\mathcal{J}_{N_p\mathcal{U}})[1] \to C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}})[1] \tag{7.13}$$

defined as follows:

$$\mathrm{cotr}(D)(a_1 \otimes j_1, \ldots, a_n \otimes j_n) = a_0 \ldots a_n D(j_1, \ldots, j_n). \tag{7.14}$$

The map cotr is a quasiisomorphism of DGLAs (cf. [15], section 1.5.6; see also [4] Proposition 14).

Lemma 7.17. *For every p the map*

$$\mathrm{Id} \otimes \mathrm{cotr} \colon \Omega^\bullet_{N_p\mathcal{U}} \otimes \bar{C}^\bullet(\mathcal{J}_{N_p\mathcal{U}})[1] \to \Omega^\bullet_{N_p\mathcal{U}} \otimes C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}})^p \otimes \mathcal{J}_{N_p\mathcal{U}})[1] \tag{7.15}$$

is a quasiisomorphism of DGLA, where the source and the target are equipped with the differentials $\delta + \nabla^{\mathrm{can}} + \iota_{\epsilon^*\omega}$ and $\delta + \nabla^p \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla^{\mathrm{can}} - \iota_{\nabla F^p}$ respectively.

Proof. It is easy to see that $\mathrm{Id}\otimes\mathrm{cotr}$ is a morphism of graded Lie algebras, which satisfies $(\nabla^p\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^{\mathrm{can}})\circ(\mathrm{Id}\otimes\mathrm{cotr}) = (\mathrm{Id}\otimes\mathrm{cotr})\circ\nabla^{\mathrm{can}}$ and $\delta\circ(\mathrm{Id}\otimes\mathrm{cotr}) = (\mathrm{Id}\otimes\mathrm{cotr})\circ\delta$. Since the domain of $(\mathrm{Id}\otimes\mathrm{cotr})$ is the normalized complex, in view of Lemma 7.13 we also have $\iota_{\nabla F^p}\circ(\mathrm{Id}\otimes\mathrm{cotr}) = -(\mathrm{Id}\otimes\mathrm{cotr})\circ\iota_{\epsilon^*\omega}$. This implies that $(\mathrm{Id}\otimes\mathrm{cotr})$ is a morphism of DGLA.

To see that this map is a quasiisomorphism, introduce filtration on $\Omega^\bullet_{N_p\mathcal{U}}$ by $F_i\Omega^\bullet_{N_p\mathcal{U}} = \Omega^{\geq -i}_{N_p\mathcal{U}}$ and consider the complexes $\overline{C}^\bullet(\mathcal{J}_{N_p\mathcal{U}})[1]$ and $C^\bullet(\mathsf{Mat}(\underline{\mathcal{A}}))^p\otimes\mathcal{J}_{N_p\mathcal{U}})[1]$ equipped with the trivial filtration. The map (7.15) is a morphism of filtered complexes with respect to the induced filtrations on the source and the target. The differentials induced on the associated graded complexes are δ (or, more precisely, $\mathrm{Id}\otimes\delta$) and the induced map of the associated graded objects is $\mathrm{Id}\otimes\mathrm{cotr}$ which is a quasi-isomorphism. Therefore, the map (7.15) is a quasiisomorphism as claimed. □

The map (7.15) therefore induces a morphism $\mathrm{Id}\otimes\mathrm{cotr}\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})^\lambda_\omega \to \mathfrak{H}^\lambda$ for every $\lambda\colon [n]\to\Delta$. These morphisms are clearly compatible with the cosimplicial structure and hence induce a quasiisomorphism of cosimplicial DGLAs

$$\mathrm{Id}\otimes\mathrm{cotr}\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega \to \mathfrak{H}$$

where the differential in the right hand side is given by (7.12).

We summarize our consideration in the following:

Theorem 7.18. *For a any choice of* $\sigma\in\mathrm{Isom}^\mu_0(\mathcal{A}_{01}\otimes\mathcal{J}_{N_1\mathcal{U}},\mathcal{J}(\mathcal{A}_{01}))$, $\nabla\in\mathcal{C}^\mu(\mathcal{A}_{01})$ *and* F^0 *as in* (7.6) *the composition* $\Phi_{\sigma,\nabla,F} := \sigma_*\circ\exp(\iota_F)\circ(\mathrm{Id}\otimes\mathrm{cotr})$

$$\Phi_{\sigma,\nabla,F}\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \tag{7.16}$$

is a quasiisomorphism of cosimplicial DGLAs.

Let $\rho\colon \mathcal{V}\to\mathcal{U}$ be a refinement of the cover $\mathcal{U}$ and let $(\mathcal{V},\underline{\mathcal{A}}^\rho)$ be the induced descent datum. We will denote the corresponding cosimplicial DGLAs by $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{U})$ and $\mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{V})$ respectively. Then the map $N\rho\colon N\mathcal{V}\to N\mathcal{U}$ induces a morphism of cosimplicial DGLAs

$$(N\rho)^*\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)) \tag{7.17}$$

and

$$(N\rho)^*\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{U}) \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{V}). \tag{7.18}$$

Notice also that the choice that the choice of data σ, ∇, F^0 on $N\mathcal{U}$ induces the corresponding data $(N\rho)^*\sigma$, $(N\rho)^*\nabla$, $(N\rho)^*F^0$ on $N\mathcal{V}$. This data allows one to construct using the equation (7.16) the map

$$\Phi_{(N\rho)^*\sigma,(N\rho)^*\nabla,(N\rho)^*F}\colon \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega \to \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)). \tag{7.19}$$

The following proposition is an easy consequence of the description of the map Φ, and we leave the proof to the reader.

Proposition 7.19. *The following diagram commutes:*

$$\begin{tikzcd} \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{U}) \arrow[r, "(N\rho)^*"] \arrow[d, "\Phi_{\sigma,\nabla,F}"] & \mathfrak{G}_{\mathsf{DR}}(\mathcal{J})_\omega(\mathcal{V}) \arrow[d, "\Phi_{(N\rho)^*\sigma,(N\rho)^*\nabla,(N\rho)^*F}"] \\ \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \arrow[r, "(N\rho)^*"] & \mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)). \end{tikzcd} \tag{7.20}$$

8 Proof of the main theorem

In this section we prove the main result of this paper. Recall the statement of the theorem from the introduction:

Theorem 1. *Suppose that X is a C^∞ manifold and $\mathcal{S}$ is an algebroid stack on X which is a twisted form of $\mathcal{O}_X$. Then, there is an equivalence of 2-groupoid valued functors of commutative Artin $\mathbb{C}$-algebras*

$$\mathrm{Def}_X(\mathcal{S}) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}).$$

Proof. Suppose $\mathcal{U}$ is a cover of X such that $\epsilon_0^*\mathcal{S}(N_0\mathcal{U})$ is nonempty. There is a descent datum $(\mathcal{U}, \underline{\mathcal{A}}) \in \mathrm{Desc}_{\mathbb{C}}(\mathcal{U})$ whose image under the functor $\mathrm{Desc}_{\mathbb{C}}(\mathcal{U}) \to \mathrm{AlgStack}_{\mathbb{C}}(X)$ is equivalent to $\mathcal{S}$.

The proof proceeds as follows.

Recall the 2-groupoids $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ and $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R)$ of deformations of and star-products on the descent datum $(\mathcal{U}, \underline{\mathcal{A}})$ defined in Section 6.2. Note that the composition $\mathrm{Desc}_R(\mathcal{U}) \to \mathrm{Triv}_R(X) \to \mathrm{AlgStack}_R(X)$ induces functors $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{S})(R)$ and $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{S})(R)$, the second one being the composition of the first one with the equivalence $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$. We are going to show that for a commutative Artin $\mathbb{C}$-algebra R there are equivalences

1. $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \cong \mathrm{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]})(R)$ and
2. the functor $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \cong \mathrm{Def}(\mathcal{S})(R)$ induced by the functor $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{S})(R)$ above.

Let $\mathcal{J}(\underline{\mathcal{A}}) = (\mathcal{J}(\mathcal{A}_{01}), j^\infty(\mathcal{A}_{012}))$. Then, $(\mathcal{U}, \mathcal{J}(\underline{\mathcal{A}}))$ is a descent datum for a twisted form of $\mathcal{J}_X$. Let R be a commutative Artin $\mathbb{C}$-algebra with maximal ideal $\mathfrak{m}_R$.

Then the first statement follows from the equivalences

$$\begin{aligned} \mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) &\cong \mathrm{Stack}(\mathfrak{G}(\underline{\mathcal{A}}) \otimes \mathfrak{m}_R) && (8.1)\\ &\cong \mathrm{Stack}(\mathfrak{G}_{\mathsf{DR}}(\mathcal{J}(\underline{\mathcal{A}}))) \otimes \mathfrak{m}_R) && (8.2)\\ &\cong \mathrm{Stack}(\mathfrak{G}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]} \otimes \mathfrak{m}_R) && (8.3)\\ &\cong \mathrm{MC}^2(\mathfrak{g}_{\mathsf{DR}}(\mathcal{J}_X)_{[\mathcal{S}]} \otimes \mathfrak{m}_R). && (8.4) \end{aligned}$$

Here the equivalence (8.1) is the subject of Proposition 6.4. The inclusion of horizontal sections is a quasi-isomorphism and the induced map in (8.2) is an equivalence by Theorem 3.6. In Theorem 7.18 we have constructed a quasi-isomorphism

$\mathfrak{G}(\mathcal{J}_X)_{[\mathcal{S}]} \to \mathfrak{G}(\mathcal{J}(\underline{\mathcal{A}}))$; the induced map (8.3) is an equivalence by another application of Theorem 3.6. Finally, the equivalence (8.4) is shown in Proposition 7.10

We now prove the second statement. We begin by considering the behavior of $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ under the refinement. Consider a refinement $\rho\colon \mathcal{V} \to \mathcal{U}$. Recall that by Proposition 7.10 the map ϵ^* induces equivalences $\mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m})$ and $\mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m})$. It is clear that the diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{MC}^2(\mathfrak{g}_{\mathrm{DR}}(\mathcal{J}_X)_\omega \otimes \mathfrak{m}) \arrow[dl] \arrow[dr] & \\ \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m}) \arrow[rr, "(N\rho)^*"] & & \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m}) \end{tikzcd}$$

commutes, and therefore $(N\rho)^*\colon \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{U}) \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J})_\omega(\mathcal{V}) \otimes \mathfrak{m})$ is an equivalence. Then Proposition 7.19 together with Theorem 3.6 implies that $(N\rho)^*\colon \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}})) \otimes \mathfrak{m}) \to \mathrm{Stack}(\mathfrak{G}_{\mathrm{DR}}(\mathcal{J}(\underline{\mathcal{A}}^\rho)) \otimes \mathfrak{m})$ is an equivalence. It follows that the functor $\rho^*\colon \mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R)$ is an equivalence. Note also that the diagram

$$\begin{tikzcd} \mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \arrow[r, "\rho^*"] \arrow[d] & \mathrm{Def}(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R) \arrow[d] \\ \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \arrow[r, "\rho^*"] & \mathrm{Def}'(\mathcal{V}, \underline{\mathcal{A}}^\rho)(R) \end{tikzcd}$$

is commutative with the top horizontal and both vertical maps being equivalences. Hence it follows that the bottom horizontal map is an equivalence.

Recall now that the embedding $\mathrm{Def}(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R)$ is an equivalence by Proposition 6.3. Therefore it is sufficient to show that the functor $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{S})(R)$ is an equivalence. Suppose that $\mathcal{C}$ is an R-deformation of $\mathcal{S}$. It follows from Lemma 6.2 that $\mathcal{C}$ is an algebroid stack. Therefore, there exists a cover $\mathcal{V}$ and an R-descent datum $(\mathcal{V}, \underline{\mathcal{B}})$ whose image under the functor $\mathrm{Desc}_R(\mathcal{V}) \to \mathrm{AlgStack}_R(X)$ is equivalent to $\mathcal{C}$. Replacing $\mathcal{V}$ by a common refinement of $\mathcal{U}$ and $\mathcal{V}$ if necessary we may assume that there is a morphism of covers $\rho\colon \mathcal{V} \to \mathcal{U}$. Clearly, $(\mathcal{V}, \underline{\mathcal{B}})$ is a deformation of $(\mathcal{V}, \underline{\mathcal{A}}^\rho)$. Since the functor $\rho^*\colon \mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}'(\mathcal{V}, \underline{\mathcal{A}}^\rho)$ is an equivalence there exists a deformation $(\mathcal{U}, \underline{\mathcal{B}}')$ such that $\rho^*(\mathcal{U}, \underline{\mathcal{B}}')$ is isomorphic to $(\mathcal{V}, \underline{\mathcal{B}})$. Let $\mathcal{C}'$ denote the image of $(\mathcal{U}, \underline{\mathcal{B}}')$ under the functor $\mathrm{Desc}_R(\mathcal{V}) \to \mathrm{AlgStack}_R(X)$. Since the images of $(\mathcal{U}, \underline{\mathcal{B}}')$ and $\rho^*(\mathcal{U}, \underline{\mathcal{B}}')$ in $\mathrm{AlgStack}_R(X)$ are equivalent it follows that $\mathcal{C}'$ is equivalent to $\mathcal{C}$. This shows that the functor $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{C})(R)$ is essentially surjective.

Suppose now that $(\mathcal{U}, \underline{\mathcal{B}}^{(i)})$, $i = 1, 2$, are R-deformations of $(\mathcal{U}, \underline{\mathcal{A}})$. Let $\mathcal{C}^{(i)}$ denote the image of $(\mathcal{U}, \underline{\mathcal{B}}^{(i)})$ in $\mathrm{Def}(\mathcal{S})(R)$. Suppose that $F\colon \mathcal{C}^{(1)} \to \mathcal{C}^{(2)}$ is a 1-morphism.

Let L denote the image of the canonical trivialization under the composition

$$\epsilon_0^*(F)\colon \widetilde{\mathcal{B}^{(1)+}} \cong \epsilon_0^*\mathcal{C}^{(1)} \xrightarrow{F} \epsilon_0^*\mathcal{C}^{(1)} \cong \widetilde{\mathcal{B}^{(2)+}}.$$

Thus, L is a $\mathcal{B}^{(1)} \otimes_R \mathcal{B}^{(2)op}$-module such that the line bundle $L \otimes_R \mathbb{C}$ is trivial. Therefore, L admits a non-vanishing global section. Moreover, there is an isomorphism $f\colon \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathrm{pr}_1^1)^*L \to (\mathrm{pr}_0^1)^*L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)}$ of $(\mathcal{B}^{(1)})_0^1 \otimes_R ((\mathcal{B}^{(2)})_1^1)^{\mathrm{op}}$-modules.

A choice of a non-vanishing global section of L gives rise to isomorphisms $\mathcal{B}_{01}^{(1)} \cong \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathrm{pr}_1^1)^*L$ and $\mathcal{B}_{01}^{(2)} \cong (\mathrm{pr}_0^1)^*L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)}$.

The composition

$$\mathcal{B}_{01}^{(1)} \cong \mathcal{B}_{01}^{(1)} \otimes_{(\mathcal{B}^{(1)})_1^1} (\mathrm{pr}_1^1)^*L \xrightarrow{f} (\mathrm{pr}_0^1)^*L \otimes_{(\mathcal{B}^{(2)})_0^1} \mathcal{B}_{01}^{(2)} \cong \mathcal{B}_{01}^{(2)}$$

defines a 1-morphism of deformations of $(\mathcal{U}, \underline{\mathcal{A}})$ such that the induced 1-morphism $\mathcal{C}^{(1)} \to \mathcal{C}^{(2)}$ is isomorphic to F. This shows that the functor $\mathrm{Def}'(\mathcal{U}, \underline{\mathcal{A}})(R) \to \mathrm{Def}(\mathcal{C})$ induces essentially surjective functors on groupoids of morphisms. By similar arguments left to the reader one shows that these are fully faithful.

This completes the proof of Theorem 1. □

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Torsion classes of finite type and spectra

Grigory Garkusha* and Mike Prest

1 Introduction

Non-commutative geometry comes in various flavours. One is based on abelian and triangulated categories, the latter being replacements of classical schemes. This is based on classical results of Gabriel and later extensions, in particular by Thomason. Precisely, Gabriel [6] proved that any noetherian scheme X can be reconstructed uniquely up to isomorphism from the abelian category, Qcoh X, of quasi-coherent sheaves over X. This reconstruction result has been generalized to quasi-compact schemes by Rosenberg in [15]. Based on Thomason's classification theorem, Balmer [3] reconstructs a noetherian scheme X from the triangulated category of perfect complexes $\mathcal{D}_{\mathrm{per}}(X)$. This result has been generalized to quasi-compact, quasi-separated schemes by Buan–Krause–Solberg [5].

In this paper we reconstruct affine and projective schemes from appropriate abelian categories. Our approach, similar to that used in [8], [9], is different from Rosenberg's [15] and less abstract. Moreover, some results of the paper are of independent interest.

Let Mod R (respectively QGr A) denote the category of R-modules (respectively graded A-modules modulo torsion modules) with R (respectively $A = \bigoplus_{n\geq 0} A_n$) a commutative ring (respectively a commutative graded ring). We first demonstrate the following result (cf. [8], [9]).

Theorem (Classification). *Let R (respectively A) be a commutative ring (respectively commutative graded ring which is finitely generated as an A_0-algebra). Then the maps*

$$V \mapsto \mathcal{S} = \{M \in \mathrm{Mod}\, R \mid \mathsf{supp}_R(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M\in\mathcal{S}} \mathsf{supp}_R(M)$$

and

$$V \mapsto \mathcal{S} = \{M \in \mathrm{QGr}\, A \mid \mathsf{supp}_A(M) \subseteq V\}, \quad \mathcal{S} \mapsto V = \bigcup_{M\in\mathcal{S}} \mathsf{supp}_A(M)$$

induce bijections between

1. *the set of all subsets $V \subseteq \mathsf{Spec}\, R$ (respectively $V \subseteq \mathsf{Proj}\, A$) of the form $V = \bigcup_{i\in\Omega} Y_i$ with $\mathsf{Spec}\, R \setminus Y_i$ (respectively $\mathsf{Proj}\, A \setminus Y_i$) quasi-compact and open for all $i \in \Omega$,*

*This paper was written during the visit of the first author to the University of Manchester supported by the MODNET Research Training Network in Model Theory. He would like to thank the University for the kind hospitality.

2. *the set of all torsion classes of finite type in* Mod R (*respectively tensor torsion classes of finite type in* QGr A).

This theorem says that $\mathsf{Spec}\, R$ and $\mathsf{Proj}\, A$ contain all the information about finite localizations in Mod R and QGr A respectively. The next result says that there is a 1-1 correspondence between the finite localizations in Mod R and the triangulated localizations in $\mathcal{D}_{\mathrm{per}}(R)$ (cf. [11], [8]).

Theorem. *Let R be a commutative ring. The map*

$$\mathcal{S} \mapsto \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid H_n(X) \in \mathcal{S} \text{ for all } n \in \mathbb{Z}\}$$

induces a bijection between

1. *the set of all torsion classes of finite type in* Mod R,
2. *the set of all thick subcategories of* $\mathcal{D}_{\mathrm{per}}(R)$.

Following Buan–Krause–Solberg [5] we consider the lattices $L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ and $L_{\mathrm{tor}}(\mathrm{QGr}\, A)$ of (tensor) torsion classes of finite type in Mod R and QGr A, as well as their prime ideal spectra $\mathsf{Spec}(\mathrm{Mod}\, R)$ and $\mathsf{Spec}(\mathrm{QGr}\, A)$. These spaces come naturally equipped with sheaves of rings $\mathcal{O}_{\mathrm{Mod}\, R}$ and $\mathcal{O}_{\mathrm{QGr}\, A}$. The following result says that the schemes $(\mathsf{Spec}\, R, \mathcal{O}_R)$ and $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$ are isomorphic to $(\mathsf{Spec}(\mathrm{Mod}\, R), \mathcal{O}_{\mathrm{Mod}\, R})$ and $(\mathsf{Spec}(\mathrm{QGr}\, A), \mathcal{O}_{\mathrm{QGr}\, A})$ respectively.

Theorem (Reconstruction). *Let R (respectively A) be a commutative ring (respectively commutative graded ring which is finitely generated as an A_0-algebra). Then there are natural isomorphisms of ringed spaces*

$$(\mathsf{Spec}\, R, \mathcal{O}_R) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{Mod}\, R), \mathcal{O}_{\mathrm{Mod}\, R})$$

and

$$(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{QGr}\, A), \mathcal{O}_{\mathrm{QGr}\, A}).$$

2 Torsion classes of finite type

We refer the reader to the appendix for necessary facts about localization and torsion classes in Grothendieck categories.

Proposition 2.1. *Assume that $\mathfrak{B}$ is a set of finitely generated ideals of a commutative ring R. The set of those ideals which contain a finite products of ideals belonging to $\mathfrak{B}$ is a Gabriel filter of finite type.*

Proof. See [17, VI.6.10]. □

Given a module M, we denote by $\mathsf{supp}_R(M) = \{P \in \mathsf{Spec}\, R \mid M_P \neq 0\}$. Here M_P denotes the localization of M at P, that is, the module of fractions $M[(R \setminus P)^{-1}]$.

Note that $V(I) = \{P \in \mathsf{Spec}\, R \mid I \leq P\}$ is equal to $\mathsf{supp}_R(R/I)$ for every ideal I and

$$\mathsf{supp}_R(M) = \bigcup_{x \in M} V(\mathsf{ann}_R(x)), \quad M \in \mathrm{Mod}\, R.$$

Recall from [10] that a topological space is *spectral* if it is T_0, quasi-compact, if the quasi-compact open subsets are closed under finite intersections and form an open basis, and if every non-empty irreducible closed subset has a generic point. Given a spectral topological space, X, Hochster [10] endows the underlying set with a new, "dual", topology, denoted X^*, by taking as open sets those of the form $Y = \bigcup_{i \in \Omega} Y_i$ where Y_i has quasi-compact open complement $X \setminus Y_i$ for all $i \in \Omega$. Then X^* is spectral and $(X^*)^* = X$ (see [10, Proposition 8]). The spaces, X, which we shall consider are not in general spectral; nevertheless we make the same definition and denote the space so obtained by X^*.

Given a commutative ring R, every closed subset of $\mathsf{Spec}\, R$ with quasi-compact complement has the form $V(I)$ for some finitely generated ideal, I, of R (see [2, Chapter 1, Ex. 17 (vii)]). Therefore a subset of $\mathsf{Spec}^*\, R$ is open if and only if it is of the form $\bigcup_\lambda V(I_\lambda)$ with each I_λ finitely generated. Notice that $V(I)$ with I a non-finitely generated ideal is not open in $\mathsf{Spec}^*\, R$ in general. For instance (see [18, 3.16.2]), let $R = \mathbb{C}[x_1, x_2, \dots]$ and $m = (x_1, x_2, \dots)$. It is clear that $V(m) = \{m\}$ is not open in $\mathsf{Spec}^*\, \mathbb{C}[x_1, x_2, \dots]$.

For definitions of terms used in the next result see the Appendix to this paper.

Theorem 2.2 (Classification). *Let R be a commutative ring. There are bijections between*

1. *the set of all open subsets $V \subseteq \mathsf{Spec}^*\, R$,*
2. *the set of all Gabriel filters $\mathfrak{F}$ of finite type,*
3. *the set of all torsion classes $\mathcal{S}$ of finite type in* Mod R.

These bijections are defined as follows:

$$V \mapsto \begin{cases} \mathfrak{F}_V = \{I \subset R \mid V(I) \subseteq V\}, \\ \mathcal{S}_V = \{M \in \mathrm{Mod}\, R \mid \mathsf{supp}_R(M) \subseteq V\}; \end{cases}$$

$$\mathfrak{F} \mapsto \begin{cases} V_{\mathfrak{F}} = \bigcup_{I \in \mathfrak{F}} V(I), \\ \mathcal{S}_{\mathfrak{F}} = \{M \in \mathrm{Mod}\, R \mid \mathsf{ann}_R(x) \in \mathfrak{F} \text{ for every } x \in M\}; \end{cases}$$

$$\mathcal{S} \mapsto \begin{cases} \mathfrak{F}_{\mathcal{S}} = \{I \subset R \mid R/I \in \mathcal{S}\}, \\ V_{\mathcal{S}} = \bigcup_{M \in \mathcal{S}} \mathsf{supp}_R(M). \end{cases}$$

Proof. The bijection between Gabriel filters of finite type and torsion classes of finite type is a consequence of a theorem of Gabriel (see, e.g., [7, 5.8]).

Let $\mathfrak{F}$ be a Gabriel filter of finite type. Then the set $\Lambda_{\mathfrak{F}}$ of finitely generated ideals I belonging to $\mathfrak{F}$ is a filter basis for $\mathfrak{F}$. Therefore $V_{\mathfrak{F}} = \bigcup_{I \in \Lambda_{\mathfrak{F}}} V(I)$ is open in $\mathsf{Spec}^* R$.

Now let V be an open subset of $\mathsf{Spec}^* R$. Let Λ denote the set of finitely generated ideals I such that $V(I) \subseteq V$. By definition of the topology $V = \bigcup_{I \in \Lambda} V(I)$ and $I_1 \cdots I_n \in \Lambda$ for any $I_1, \ldots, I_n \in \Lambda$. We denote by $\mathfrak{F}'_V$ the set of ideals $I \subset R$ such that $I \supseteq J$ for some $J \in \Lambda$. By Proposition 2.1 $\mathfrak{F}'_V$ is a Gabriel filter of finite type. Clearly, $\mathfrak{F}'_V \subset \mathfrak{F}_V = \{I \subset R \mid V(I) \subseteq V\}$. Suppose $I \in \mathfrak{F}_V \setminus \mathfrak{F}'_V$; by [17, VI.6.13–15] (cf. the proof of Theorem 6.4) there exists a prime ideal $P \in V(I)$ such that $P \notin \mathfrak{F}'_V$. But $V(I) \subseteq V$ and therefore $P \supset J$ for some $J \in \Lambda$, so $P \in \mathfrak{F}'_V$, a contradiction. Thus $\mathfrak{F}'_V = \mathfrak{F}_V$.

Clearly, $V = V_{\mathfrak{F}_V}$ for every open subset $V \subseteq \mathsf{Spec}^* R$. Let $\mathfrak{F}$ be a Gabriel filter of finite type and $I \in \mathfrak{F}$. Clearly $\mathfrak{F} \subset \mathfrak{F}_{V_{\mathfrak{F}}}$ and, as above, there is no ideal belonging to $\mathfrak{F}_{V_{\mathfrak{F}}} \setminus \mathfrak{F}$. We have shown the bijection between the sets of all Gabriel filters of finite type and all open subsets in $\mathsf{Spec}^* R$. The description of the bijection between the set of torsion classes of finite type and the set of open subsets in $\mathsf{Spec}^* R$ is now easily checked. $\square$

3 The fg-topology

Let $\mathsf{Inj}\, R$ denote the set of isomorphism classes of indecomposable injective modules. Given a finitely generated ideal I of R, we denote by $\mathcal{S}_I$ the torsion class of finite type corresponding to the Gabriel filter of finite type having $\{I^n\}_{n \geq 1}$ as a basis (see Proposition 2.1 and Theorem 2.2). Note that a module M has $\mathcal{S}_I$-torsion if and only if every element $x \in M$ is annihilated by some power $I^{n(x)}$ of the ideal I. Let us set

$$D^{\mathrm{fg}}(I) := \{E \in \mathsf{Inj}\, R \mid E \text{ is } \mathcal{S}_I\text{-torsion free}\}, \quad V^{\mathrm{fg}}(I) := \mathsf{Inj}\, R \setminus D^{\mathrm{fg}}(I)$$

("fg" referring to this topology being defined using only finitely generated ideals).

Let E be any indecomposable injective R-module. Set $P = P(E)$ to be the sum of annihilator ideals of non-zero elements, equivalently non-zero submodules, of E. Since E is uniform the set of annihilator ideals of non-zero elements of E is closed under finite sum. It is easy to check ([14, 9.2]) that $P(E)$ is a prime ideal and $P(E_P) = P$. Here E_P stands for the injective hull of R/P. There is an embedding

$$\alpha\colon \mathsf{Spec}\, R \to \mathsf{Inj}\, R, \quad P \mapsto E_P,$$

which need not be surjective. We shall identify $\mathsf{Spec}\, R$ with its image in $\mathsf{Inj}\, R$.

If P is a prime ideal of a commutative ring R its complement in R is a multiplicatively closed set S. Given a module M we denote the module of fractions $M[S^{-1}]$ by M_P. There is a corresponding Gabriel filter

$$\mathfrak{F}^P = \{I \mid P \notin V(I)\}.$$

Clearly, $\mathfrak{F}^P$ is of finite type. The $\mathfrak{F}^P$-torsion modules are characterized by the property that $M_P = 0$ (see [17, p. 151]).

More generally, let $\mathcal{P}$ be a subset of $\operatorname{Spec} R$. To $\mathcal{P}$ we associate a Gabriel filter

$$\mathfrak{F}^{\mathcal{P}} = \bigcap_{P \in \mathcal{P}} \mathfrak{F}^P = \{I \mid \mathcal{P} \cap V(I) = \emptyset\}.$$

The corresponding torsion class consists of all modules M with $M_P = 0$ for all $P \in \mathcal{P}$.

Given a family of injective R-modules $\mathcal{E}$, denote by $\mathfrak{F}_{\mathcal{E}}$ the Gabriel filter determined by $\mathcal{E}$. By definition, this corresponds to the localizing subcategory $\mathcal{S}_{\mathcal{E}} = \{M \in \operatorname{Mod} R \mid \operatorname{Hom}_R(M, E) = 0 \text{ for all } E \in \mathcal{E}\}$.

Proposition 3.1. *A Gabriel filter $\mathfrak{F}$ is of finite type if and only if it is of the form $\mathfrak{F}^{\mathcal{P}}$ with $\mathcal{P}$ a closed set in $\operatorname{Spec}^* R$. Moreover, $\mathfrak{F}^{\mathcal{P}}$ is determined by $\mathcal{E}_{\mathcal{P}} = \{E_P \mid P \in \mathcal{P}\}$ via $\mathfrak{F}^{\mathcal{P}} = \{I \mid \operatorname{Hom}_R(R/I, \mathcal{E}_{\mathcal{P}}) = 0\}$.*

Proof. This is a consequence of Theorem 2.2. □

Proposition 3.2. *Let $\mathcal{P}$ be the closure of P in $\operatorname{Spec}^* R$. Then $\mathcal{P} = \{Q \in \operatorname{Spec} R \mid Q \subseteq P\}$. Also $\mathfrak{F}^P = \mathfrak{F}^{\mathcal{P}}$.*

Proof. This is direct from the definition of the topology. □

Recall that for any ideal I of a ring, R, and $r \in R$ we have an isomorphism $R/(I : r) \cong (rR + I)/I$, where $(I : r) = \{s \in R \mid rs \in I\}$, induced by sending $1 + (I : r)$ to $r + I$.

Proposition 3.3. *Let E be an indecomposable injective module and let $P(E)$ be the prime ideal defined before. Let I be a finitely generated ideal of R. Then $E \in V^{\mathrm{fg}}(I)$ if and only if $E_{P(E)} \in V^{\mathrm{fg}}(I)$.*

Proof. Let I be such that $E = E(R/I)$. For each $r \in R \setminus I$ we have, by the remark just above, that the annihilator of $r + I \in E$ is $(I : r)$ and so, by definition of $P(E)$, we have $(I : r) \leq P(E)$. The natural projection $(rR + I)/I \cong R/(I : r) \to R/P(E)$ extends to a morphism from E to $E_{P(E)}$ which is non-zero on $r + I$. Forming the product of these morphisms as r varies over $R \setminus I$, we obtain a morphism from E to a product of copies of $E_{P(E)}$ which is monic on R/I and hence is monic. Therefore E is a direct summand of a product of copies of $E_{P(E)}$ and so $E \in V^{\mathrm{fg}}(J)$ implies $E_{P(E)} \in V^{\mathrm{fg}}(J)$, where J is a finitely generated ideal.

Now, $E_{P(E)} \in V^{\mathrm{fg}}(I)$, where I is a finitely generated ideal, means that there is a non-zero morphism $f\colon R/I^n \to E_{P(E)}$ for some n. Since $R/P(E)$ is essential in $E_{P(E)}$ the image of f has non-zero intersection with $R/P(E)$ so there is an ideal J, without loss of generality finitely generated, with $I^n < J \leq R$, J/I^n a cyclic module, and such that the restriction, f', of f to J/I^n is non-zero (and the image is contained in $R/P(E)$). Since J/I^n is a cyclic $\mathcal{S}_I$-torsion module, there is an epimorphism $g\colon R/I^m \to J/I^n$ for some m. By construction, $R/P(E) = \varinjlim R/I_\lambda$, where I_λ

ranges over the annihilators of non-zero elements of E. Since R/I^m is finitely presented, $0 \neq f'g$ factorises through one of the maps $R/I_\lambda \to R/P(E)$. In particular, there is a non-zero morphism $R/I^m \to E$ showing that $E \in V^{\mathrm{fg}}(I)$, as required. $\square$

Given a module M, we set

$$[M] := \{E \in \mathsf{Inj}\, R \mid \mathrm{Hom}_R(M, E) = 0\}, \quad (M) := \mathsf{Inj}\, R \setminus [M].$$

Remark 3.4. *For any finitely generated ideal I we have: $D^{\mathrm{fg}}(I) \cap \mathsf{Spec}\, R = D(I)$ and $V^{\mathrm{fg}}(I) \cap \mathsf{Spec}\, R = V(I)$. Moreover, $D^{\mathrm{fg}}(I) = [R/I]$.*

If I, J are finitely generated ideals, then $D(IJ) = D(I) \cap D(J)$. It follows from Proposition 3.3 and Remark 3.4 that $D^{\mathrm{fg}}(I) \cap D^{\mathrm{fg}}(J) = D^{\mathrm{fg}}(IJ)$. Thus the sets $D^{\mathrm{fg}}(I)$ with I running over finitely generated ideals form a basis for a topology on $\mathsf{Inj}\, R$ which we call the *fg-ideals topology*. This topological space will be denoted by $\mathsf{Inj}_{\mathrm{fg}}\, R$. Observe that if R is coherent then the fg-topology equals the Zariski topology on $\mathsf{Inj}\, R$ (see [14], [8]). The latter topological space is defined by taking the $[M]$ with M finitely presented as a basis of open sets.

Theorem 3.5. (cf. Prest [14, 9.6]) *Let R be a commutative ring, let E be an indecomposable injective module and let $P(E)$ be the prime ideal defined before. Then E and $E_{P(E)}$ are topologically indistinguishable in $\mathsf{Inj}_{\mathrm{fg}}\, R$.*

Proof. This follows from Proposition 3.3 and Remark 3.4. $\square$

Theorem 3.6. (cf. Garkusha–Prest [8, Theorem A]) *Let R be a commutative ring. The space $\mathsf{Spec}\, R$ is dense and a retract in $\mathsf{Inj}_{\mathrm{fg}}\, R$. A left inverse to the embedding $\mathsf{Spec}\, R \hookrightarrow \mathsf{Inj}_{\mathrm{fg}}\, R$ takes an indecomposable injective module E to the prime ideal $P(E)$. Moreover, $\mathsf{Inj}_{\mathrm{fg}}\, R$ is quasi-compact, the basic open subsets $D^{\mathrm{fg}}(I)$, with I finitely generated, are quasi-compact, the intersection of two quasi-compact open subsets is quasi-compact, and every non-empty irreducible closed subset has a generic point.*

Proof. For any finitely generated ideal I we have

$$D^{\mathrm{fg}}(I) \cap \mathsf{Spec}\, R = D(I)$$

(see Remark 3.4). From this relation and Theorem 3.5 it follows that $\mathsf{Spec}\, R$ is dense in $\mathsf{Inj}_{\mathrm{fg}}\, R$ and that $\alpha\colon \mathsf{Spec}\, R \to \mathsf{Inj}_{\mathrm{fg}}\, R$ is a continuous map.

One may check (see [14, 9.2]) that

$$\beta\colon \mathsf{Inj}_{\mathrm{fg}}\, R \to \mathsf{Spec}\, R, \quad E \mapsto P(E),$$

is left inverse to α. Remark 3.4 implies that β is continuous. Thus $\mathsf{Spec}\, R$ is a retract of $\mathsf{Inj}_{\mathrm{fg}}\, R$.

Let us show that each basic open set $D^{\mathrm{fg}}(I)$ is quasi-compact (in particular $\mathsf{Inj}_{\mathrm{fg}}\, R = D^{\mathrm{fg}}(R)$ is quasi-compact). Let $D^{\mathrm{fg}}(I) = \bigcup_{i \in \Omega} D^{\mathrm{fg}}(I_i)$ with each I_i finitely generated. It follows from Remark 3.4 that $D(I) = \bigcup_{i \in \Omega} D(I_i)$. Since I is finitely generated,

$D(I)$ is quasi-compact in $\mathsf{Spec}\, R$ by [2, Chapter 1, Ex. 17 (vii)]. We see that $D(I) = \bigcup_{i\in\Omega_0} D(I_i)$ for some finite subset $\Omega_0 \subset \Omega$.

Assume $E \in D^{\mathrm{fg}}(I) \setminus \bigcup_{i\in\Omega_0} D^{\mathrm{fg}}(I_i)$. It follows from Theorem 3.5 that $E_{P(E)} \in D^{\mathrm{fg}}(I)\setminus\bigcup_{i\in\Omega_0} D^{\mathrm{fg}}(I_i)$. But $E_{P(E)} \in D^{\mathrm{fg}}(I)\cap\mathsf{Spec}\, R = D(I) = \bigcup_{i\in\Omega_0} D(I_i)$, and hence it is in $D(I_{i_0}) = D^{\mathrm{fg}}(I_{i_0})\cap\mathsf{Spec}\, R$ for some $i_0 \in \Omega_0$, a contradiction. So $D^{\mathrm{fg}}(I)$ is quasi-compact. It also follows that the intersection $D^{\mathrm{fg}}(I) \cap D^{\mathrm{fg}}(J) = D^{\mathrm{fg}}(IJ)$ of two quasi-compact open subsets is quasi-compact. Furthermore, every quasi-compact open subset in $\mathsf{Inj}_{\mathrm{fg}}\, R$ must therefore have the form $D^{\mathrm{fg}}(I)$ with I finitely generated.

Finally, it follows from Remark 3.4 and Theorem 3.5 that a subset V of $\mathsf{Inj}_{\mathrm{fg}}\, R$ is Zariski-closed and irreducible if and only if there is a prime ideal Q of R such that $V = \{E \mid P(E) \geq Q\}$. This obviously implies that the point $E_Q \in V$ is generic. □

Notice that $\mathsf{Inj}_{\mathrm{fg}}\, R$ is not a spectral space in general, for it is not necessarily T_0.

Lemma 3.7. *Let the ring R be commutative. Then the maps*

$$\mathsf{Spec}^*\, R \supseteq V \overset{\phi}{\mapsto} \mathcal{Q}_V = \{E \in \mathsf{Inj}\, R \mid P(E) \in V\}$$

and

$$(\mathsf{Inj}_{\mathrm{fg}}\, R)^* \supseteq \mathcal{Q} \overset{\psi}{\mapsto} V_{\mathcal{Q}} = \{P(E) \in \mathsf{Spec}^*\, R \mid E \in \mathcal{Q}\} = \mathcal{Q} \cap \mathsf{Spec}^*\, R$$

induce a 1-1 *correspondence between the lattices of open sets of* $\mathsf{Spec}^*\, R$ *and those of* $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$.

Proof. First note that $E_P \in \mathcal{Q}_V$ for any $P \in V$ (see [14, 9.2]). Let us check that $\mathcal{Q}_V$ is an open set in $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$. Every closed subset of $\mathsf{Spec}\, R$ with quasi-compact complement has the form $V(I)$ for some finitely generated ideal, I, of R (see [2, Chapter 1, Exercise 17 (vii)]), so there are finitely generated ideals $I_\lambda \subseteq R$ such that $V = \bigcup_\lambda V(I_\lambda)$. Since the points E and $E_{P(E)}$ are, by Theorem 3.5, indistinguishable in $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$ we see that $\mathcal{Q}_V = \bigcup_\lambda V^{\mathrm{fg}}(I_\lambda)$, hence this set is open in $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$.

The same arguments imply that $V_{\mathcal{Q}}$ is open in $\mathsf{Spec}^*\, R$. It is now easy to see that $V_{\mathcal{Q}_V} = V$ and $\mathcal{Q}_{V_{\mathcal{Q}}} = \mathcal{Q}$. □

4 Torsion classes and thick subcategories

We shall write $L(\mathsf{Spec}^*\, R)$, $L((\mathsf{Inj}_{\mathrm{fg}}\, R)^*)$, $L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R))$, $L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ to denote:

- the lattice of all open subsets of $\mathsf{Spec}^*\, R$,
- the lattice of all open subsets of $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$,
- the lattice of all thick subcategories of $\mathcal{D}_{\mathrm{per}}(R)$,
- the lattice of all torsion classes of finite type in Mod R, ordered by inclusion.

(A thick subcategory is a triangulated subcategory closed under direct summands).

Given a perfect complex $X \in \mathcal{D}_{\mathrm{per}}(R)$ denote by $\mathsf{supp}(X) = \{P \in \mathsf{Spec}\, R \mid X \otimes_R^L R_P \neq 0\}$. It is easy to see that

$$\mathsf{supp}(X) = \bigcup_{n \in \mathbb{Z}} \mathsf{supp}_R(H_n(X)),$$

where $H_n(X)$ is the nth homology group of X.

Theorem 4.1 (Thomason [18]). *Let R be a commutative ring. The assignments*

$$\mathcal{T} \in L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R)) \overset{\mu}{\mapsto} \bigcup_{X \in \mathcal{T}} \mathsf{supp}(X)$$

and

$$V \in L(\mathsf{Spec}^* R) \overset{\nu}{\mapsto} \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid \mathsf{supp}(X) \subseteq V\}$$

are mutually inverse lattice isomorphisms.

Given a subcategory $\mathcal{X}$ in Mod R, we may consider the smallest torsion class of finite type in Mod R containing $\mathcal{X}$. This torsion class we denote by

$$\sqrt{\mathcal{X}} = \bigcap \{\mathcal{S} \subseteq \text{Mod } R \mid \mathcal{S} \supseteq \mathcal{X} \text{ is a torsion class of finite type}\}.$$

Theorem 4.2. (cf. Garkusha–Prest [8, Theorem C]) *Let R be a commutative ring. There are bijections between*

- *the set of all open subsets $Y \subseteq (\mathsf{Inj}_{\mathrm{fg}}\, R)^*$,*
- *the set of all torsion classes of finite type in* Mod R*,*
- *the set of all thick subcategories of $\mathcal{D}_{\mathrm{per}}(R)$.*

These bijections are defined as follows:

$$Y \mapsto \begin{cases} \mathcal{S} = \{M \mid (M) \subseteq Y\}, \\ \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid (H_n(X)) \subseteq Y \textit{ for all } n \in \mathbb{Z}\}; \end{cases}$$

$$\mathcal{S} \mapsto \begin{cases} Y = \bigcup_{M \in \mathcal{S}} (M), \\ \mathcal{T} = \{X \in \mathcal{D}_{\mathrm{per}}(R) \mid H_n(X) \in \mathcal{S} \textit{ for all } n \in \mathbb{Z}\}; \end{cases}$$

$$\mathcal{T} \mapsto \begin{cases} Y = \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} (H_n(X)), \\ \mathcal{S} = \sqrt{\{H_n(X) \mid X \in \mathcal{T}, n \in \mathbb{Z}\}}. \end{cases}$$

Proof. That $\mathcal{S}_Y = \{M \mid (M) \subseteq Y\}$ is a torsion class follows because it is defined as the class of modules having no non-zero morphism to a family of injective modules, $\mathcal{E} := \mathsf{Inj}\, R \setminus Y$. By Lemma 3.7, $\mathcal{E} \cap \mathsf{Spec}^* R = U$ is a closed set in $\mathsf{Spec}^* R$, that is $P(E) \in U$ for all $E \in \mathcal{E}$. $\mathcal{S}_Y$ is also determined by the family of injective modules $\{E_P\}_{P \in U}$. Indeed, any $E \in \mathcal{E}$ is a direct summand of some power of $E_{P(E)}$ by the proof of Proposition 3.3. Therefore $\mathrm{Hom}_R(M, E_{P(E)}) = 0$ implies $\mathrm{Hom}_R(M, E) = 0$. By Proposition 3.1 $\mathcal{S}_Y$ is of finite type. Conversely, given a torsion class of finite type $\mathcal{S}$, the set $Y_{\mathcal{S}} = \bigcup_{M \in \mathcal{S}} (M)$ is plainly open in $(\mathsf{Inj}_{\mathrm{fg}}\, R)^*$. Moreover, $\mathcal{S}_{Y_{\mathcal{S}}} = \mathcal{S}$ and $Y = Y_{\mathcal{S}_Y}$.

Consider the following diagram:

$$\begin{array}{ccc} L(\mathsf{Spec}^* R) & \underset{\mu}{\overset{\nu}{\rightleftarrows}} & L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R)) \\ \psi \uparrow\downarrow \phi & & \sigma \uparrow\downarrow \rho \\ L((\mathsf{Inj}_{\mathrm{fg}}\, R)^*) & \underset{\delta}{\overset{\zeta}{\rightleftarrows}} & L_{\mathrm{tor}}(\mathrm{Mod}\, R), \end{array}$$

where ϕ, ψ are as in Lemma 3.7, μ, ν are as in Theorem 4.1 and the remaining maps are the corresponding maps indicated in the formulation of the theorem. We have $\nu = \mu^{-1}$ by Theorem 4.1, $\phi = \psi^{-1}$ by Lemma 3.7, and $\zeta = \delta^{-1}$ by the above.

By construction,

$$\sigma\zeta\phi(V) = \left\{X \mid \textstyle\bigcup_{n \in \mathbb{Z}} \mathsf{supp}_R(H_n(X)) \subseteq V\right\} = \{X \mid \mathsf{supp}(X) \subseteq V\}$$

for all $V \in L(\mathsf{Spec}^* R)$. Thus $\sigma\zeta\phi = \nu$. Since ζ, ϕ, ν are bijections so is σ.

On the other hand,

$$\psi\delta\rho(\mathcal{T}) = \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} \mathsf{supp}_R(H_n(X)) = \bigcup_{X \in \mathcal{T}} \mathsf{supp}(X)$$

for any $\mathcal{T} \in L_{\mathrm{thick}}(\mathcal{D}_{\mathrm{per}}(R))$. We have used here the relation

$$\bigcup_{M \in \rho(\mathcal{T})} \mathsf{supp}_R(M) = \bigcup_{X \in \mathcal{T}, n \in \mathbb{Z}} \mathsf{supp}_R(H_n(X)).$$

One sees that $\psi\delta\rho = \mu$. Since δ, ψ, μ are bijections so is ρ. Obviously, $\sigma = \rho^{-1}$ and the diagram above yields the desired bijective correspondences. The theorem is proved. □

To conclude this section, we should mention the relation between torsion classes of finite type in Mod R and the Ziegler subspace topology on $\mathsf{Inj}\, R$ (we denote this space by $\mathsf{Inj}_{\mathrm{zg}}\, R$). The latter topology arises from Ziegler's work on the model theory of modules [20]. The points of the Ziegler spectrum of R are the isomorphism classes of indecomposable pure-injective R-modules and the closed subsets correspond to complete theories of modules. It is well known (see [14, 9.12]) that for every coherent ring R there is a 1-1 correspondence between the open (equivalently closed) subsets of $\mathrm{Inj}_{\mathrm{zg}}\, R$ and torsion classes of finite type in Mod R. However, this is not the case for general commutative rings.

The topology on $\mathrm{Inj}_{\mathrm{zg}}\, R$ can be defined as follows. Let $\mathcal{M}$ be the set of those modules M which are kernels of homomorphisms between finitely presented modules; that is $M = \mathrm{Ker}(K \xrightarrow{f} L)$ with K, L finitely presented. The sets (M) with $M \in \mathcal{M}$ form a basis of open sets for $\mathrm{Inj}_{\mathrm{zg}}\, R$. We claim that there is a ring R and a module $M \in \mathcal{M}$ such that the intersection $(M) \cap \mathrm{Spec}^* R$ is not open in $\mathrm{Spec}^* R$, and hence such that the open subset (M) cannot correspond to any torsion class of finite type on Mod R. Such a ring has been pointed out by G. Puninski.

Let V be a commutative valuation domain with value group isomorphic to $\Gamma = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}$, a $\mathbb{Z}$-indexed direct sum of copies of $\mathbb{Z}$. The order on Γ is defined as follows. $(a_n)_{n\in\mathbb{Z}} > (b_n)_{n\in\mathbb{Z}}$ if $a_i > b_i$ for some i and $a_k = b_k$ for every $k < i$. Then $J^2 = J$ where J is the Jacobson radical of V. Let r be an element with value $v(r) = (a_n)_{n\in\mathbb{Z}}$ where $a_0 = 1$ and $a_n = 0$ for all $n \neq 0$. Consider the ring $R = V/rJ$. Again $J(R)^2 = J(R)$. Denoting the image of r in R by r', note that $\mathrm{ann}_R(r') = J(R)$ which is not finitely generated, and so R is not coherent by Chase's Theorem (see [17, 1.13.3]). Note that R is a local ring and, as already observed, the simple module $R/J(R)$ is isomorphic to $r'R$. Therefore $R/J(R) = \mathrm{Ker}(R \to R/r'R)$. Thus $R/J(R) \in \mathcal{M}$. We have

$$(R/J(R)) \cap \mathrm{Spec}^* R = V(J(R)) = \{J(R)\}.$$

Suppose $V(J(R))$ is open in $\mathrm{Spec}^* R$; then $V(J(R)) = \bigcup_\lambda V(I_\lambda)$ with each I_λ finitely generated. Since $J(R)$ is the largest proper ideal each $V(I_\lambda)$, if non-empty, equals $\{J(R)\}$. Therefore $J(R) = \sqrt{I_\lambda}$ for some λ. But the prime radical of every finitely generated ideal in R is prime (since R is a valulation ring) and different from $J(R)$. To see the latter, we have, since I_λ is finitely generated, that all elements of I_λ have value $> (a'_n)_n$ for some $(a'_n)_n$ with $a'_n = 0$ for all $n \leq N$ for some fixed N. (Recall that the valuation v on R satisfies $v(r+s) \geq \min\{v(r), v(s)\}$ and $v(rs) = v(r) + v(s)$.) It follows that there is a prime ideal properly between I_λ and $J(R)$. This gives a contradiction, as required.

5 Graded rings and modules

In this section we recall some basic facts about graded rings and modules.

Definition. A (*positively*) *graded ring* is a ring A together with a direct sum decomposition $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ as abelian groups, such that $A_i A_j \subset A_{i+j}$ for $i, j \geq 0$. A *homogeneous element* of A is simply an element of one of the groups A_j, and a *homogeneous ideal* of A is an ideal that is generated by homogeneous elements. A *graded A-module* is an A-module M together with a direct sum decomposition $M = \bigoplus_{j\in\mathbb{Z}} M_j$ as abelian groups, such that $A_i M_j \subset M_{i+j}$ for $i \geq 0$, $j \in \mathbb{Z}$. One calls M_j the jth *homogeneous component of M*. The elements $x \in M_j$ are said to be *homogeneous* (*of degree* j).

Note that A_0 is a commutative ring with $1 \in A_0$, that all summands M_j are A_0-modules, and that $M = \bigoplus_{j\in\mathbb{Z}} M_j$ is a direct sum decomposition of M as an A_0-module.

Let A be a graded ring. The *category of graded A-modules*, denoted by $\mathrm{Gr}\,A$, has as objects the graded A-modules. A *morphism* of graded A-modules $f : M \to N$ is an A-module homomorphism satisfying $f(M_j) \subset N_j$ for all $j \in \mathbb{Z}$. An A-module homomorphism which is a morphism in $\mathrm{Gr}\,A$ will be called *homogeneous*.

Let M be a graded A-module and let N be a submodule of M. Say that N is a *graded submodule* if it is a graded module such that the inclusion map is a morphism in $\mathrm{Gr}\,A$. The graded submodules of A are called *graded ideals*. If d is an integer the *tail* $M_{\geq d}$ is the graded submodule of M having the same homogeneous components $(M_{\geq d})_j$ as M in degrees $j \geq d$ and zero for $j < d$. We also denote the ideal $A_{\geq 1}$ by A_+.

For $n \in \mathbb{Z}$, $\mathrm{Gr}\,A$ comes equipped with a shift functor $M \mapsto M(n)$ where $M(n)$ is defined by $M(n)_j = M_{n+j}$. Then $\mathrm{Gr}\,A$ is a Grothendieck category with generating family $\{A(n)\}_{n\in\mathbb{Z}}$. The tensor product for the category of all A-modules induces a tensor product on $\mathrm{Gr}\,A$: given two graded A-modules M, N and homogeneous elements $x \in M_i$, $y \in N_j$, set $\deg(x \otimes y) := i + j$. We define the *homomorphism A-module* $\mathcal{H}om_A(M, N)$ to be the graded A-module which is, in dimension $n \in \mathbb{Z}$, the group $\mathcal{H}om_A(M, N)_n$ of graded A-module homomorphisms of degree n, i.e.,

$$\mathcal{H}om_A(M, N)_n = \mathrm{Gr}\,A(M, N(n)).$$

We say that a graded A-module M is *finitely generated* if it is a quotient of a free graded module of finite rank $\bigoplus_{s=1}^{n} A(d_s)$ where $d_1, \ldots, d_s \in \mathbb{Z}$. Say that M is *finitely presented* if there is an exact sequence

$$\bigoplus_{t=1}^{m} A(c_t) \to \bigoplus_{s=1}^{n} A(d_s) \to M \to 0.$$

The full subcategory of finitely presented graded modules will be denoted by $\mathrm{gr}\,A$. Note that any graded A-module is a direct limit of finitely presented graded A-modules, and therefore $\mathrm{Gr}\,A$ is a locally finitely presented Grothendieck category.

Let E be any indecomposable injective graded A-module (we remind the reader that the corresponding ungraded module, $\bigoplus_n E_n$, need not be injective in the category of ungraded A-modules). Set $P = P(E)$ to be the sum of the annihilator ideals $\mathsf{ann}_A(x)$ of non-zero homogeneous elements $x \in E$. Observe that each ideal $\mathsf{ann}_A(x)$ is homogeneous. Since E is uniform the set of annihilator ideals of non-zero homogeneous elements of E is upwards closed so the only issue is whether the sum, $P(E)$, of them all is itself one of these annihilator ideals.

Given a prime homogeneous ideal P, we use the notation E_P to denote the injective hull, $E(A/P)$, of A/P. Notice that E_P is indecomposable. We also denote the set of isomorphism classes of indecomposable injective graded A-modules by $\mathsf{Inj}\,A$.

Lemma 5.1. *If $E \in \mathrm{Inj}\, A$ then $P(E)$ is a homogeneous prime ideal. If the module E has the form $E_P(n)$ for some prime homogeneous ideal P and integer n, then $P = P(E)$.*

Proof. The proof is similar to that of [14, 9.2]. □

It follows from the preceding lemma that the map

$$P \subset A \mapsto E_P \in \mathrm{Inj}\, A$$

from the set of homogeneous prime ideals to $\mathrm{Inj}\, A$ is injective.

A *tensor torsion class* in Gr A is a torsion class $\mathcal{S} \subset \mathrm{Gr}\, A$ such that for any $X \in \mathcal{S}$ and any $Y \in \mathrm{Gr}\, A$ the tensor product $X \otimes Y$ is in $\mathcal{S}$.

Lemma 5.2. *Let A be a graded ring. Then a torsion class $\mathcal{S}$ is a tensor torsion class of* Gr A *if and only if it is closed under shifts of objects, i.e. $X \in \mathcal{S}$ implies $X(n) \in \mathcal{S}$ for any $n \in \mathbb{Z}$.*

Proof. Suppose that $\mathcal{S}$ is a tensor torsion class of Gr A. Then it is closed under shifts of objects, because $X(n) \cong X \otimes A(n)$.

Assume the converse. Let $X \in \mathcal{S}$ and $Y \in \mathrm{Gr}\, A$. Then there is a surjection $\bigoplus_{i \in I} A(i) \overset{f}{\twoheadrightarrow} Y$. It follows that $1_X \otimes f \colon \bigoplus_{i \in I} X(i) \to X \otimes Y$ is a surjection. Since each $X(i)$ belongs to $\mathcal{S}$ then so does $X \otimes Y$. □

Lemma 5.3. *The map*

$$\mathcal{S} \mapsto \mathfrak{F}(\mathcal{S}) = \{\mathfrak{a} \subseteq A \mid A/\mathfrak{a} \in \mathcal{S}\}$$

establishes a bijection between the tensor torsion classes in Gr A *and the sets $\mathfrak{F}$ of homogeneous ideals satisfying the following axioms:*

- $T1$. *$A \in \mathfrak{F}$;*
- $T2$. *if $\mathfrak{a} \in \mathfrak{F}$ and if a is a homogeneous element of A then it holds that $(\mathfrak{a} : a) = \{x \in A \mid xa \in \mathfrak{a}\} \in \mathfrak{F}$;*
- $T3$. *if $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals of A such that $\mathfrak{a} \in \mathfrak{F}$ and $(\mathfrak{b} : a) \in \mathfrak{F}$ for every homogeneous element $a \in \mathfrak{a}$ then $\mathfrak{b} \in \mathfrak{F}$.*

*We shall refer to such filters as t-*filters. *Moreover, $\mathcal{S}$ is of finite type if and only if $\mathfrak{F}(\mathcal{S})$ has a basis of finitely generated ideals, that is every ideal in $\mathfrak{F}(\mathcal{S})$ contains a finitely generated ideal belonging to $\mathfrak{F}(\mathcal{S})$. In this case $\mathfrak{F}(\mathcal{S})$ will be referred to as a t-*filter of finite type.

Proof. It is enough to observe that there is a bijection between the Gabriel filters on the family $\{A(n)\}_{n \in \mathbb{Z}}$ of generators closed under the shift functor (i.e., if $\mathfrak{a}$ belongs to the Gabriel filter then so does $\mathfrak{a}(n)$ for all $n \in \mathbb{Z}$) and the t-filters. □

Proposition 5.4. *The following statements are true:*

1. *Let $\mathfrak{F}$ be a t-filter. If I, J belong to $\mathfrak{F}$, then $IJ \in \mathfrak{F}$.*

2. *Assume that $\mathfrak{B}$ is a set of homogeneous finitely generated ideals. The set $\mathfrak{B}'$ of finite products of ideals belonging to $\mathfrak{B}$ is a basis for a t-filter of finite type.*

Proof. 1. For any homogeneous element $a \in I$ we have $(IJ : a) \supset J$, so $IJ \in \mathfrak{F}$ by $T3$ and the fact that every homogeneous ideal containing an ideal from $\mathfrak{F}$ must belong to $\mathfrak{F}$.

2. We follow [17, VI.6.10]. We must check that the set $\mathfrak{F}$ of homogeneous ideals containing ideals in $\mathfrak{B}'$ is a t-filter of finite type. $T1$ is plainly satisfied. Let a be a homogeneous element in A and $I \in \mathfrak{F}$. There is an ideal $I' \in \mathfrak{B}'$ contained in I. Then $(I : a) \supset I'$ and therefore $(I : a) \in \mathfrak{F}$, hence $T2$ is satisfied as well.

Next we verify that $\mathfrak{F}$ satisfies $T3$. Suppose that I is a homogeneous ideal and there exists $J \in \mathfrak{F}$ such that $(I : a) \in \mathfrak{F}$ for every homogeneous element $a \in J$. We may assume that $J \in \mathfrak{B}'$. Let $a_1, \dots, a_n$ be generators of J. Then $(I : a_i) \in \mathfrak{F}$, $i \leq n$, and $(I : a_i) \supset J_i$ for some $J_i \in \mathfrak{B}'$. It follows that $a_i J_i \subset I$ for each i, and hence $JJ_1 \dots J_n \subset J(J_1 \cap \dots \cap J_n) \subset I$, so $I \in \mathfrak{F}$. $\square$

6 Torsion modules and the category QGr A

Let A be a commutative graded ring. In this section we introduce the category QGr A, which is analogous to the category of quasi-coherent sheaves on a projective variety. The non-commutative analog of the category QGr A plays a prominent role in "non-commutative projective geometry" (see, e.g., [1], [16], [19]).

Recall that the projective scheme $\mathsf{Proj}\, A$ is a topological space whose points are the homogeneous prime ideals not containing A_+. The topology of $\mathsf{Proj}\, A$ is defined by taking the closed sets to be the sets of the form $V(I) = \{P \in \mathsf{Proj}\, A \mid P \supseteq I\}$ for I a homogeneous ideal of A. We set $D(I) := \mathsf{Proj}\, A \setminus V(I)$.

In the remainder of this section the homogeneous ideal $A_+ \subset A$ is assumed to be finitely generated. This is equivalent to assuming that A is a finitely generated A_0-algebra. The space $\mathsf{Proj}\, A$ is spectral and the quasi-compact open sets are those of the form $D(I)$ with I finitely generated (see, e.g., [9, 5.1]). Let Tors A denote the tensor torsion class of finite type corresponding to the family of homogeneous finitely generated ideals $\{A_+^n\}_{n \geq 1}$ (see Proposition 5.4). We refer to the objects of Tors A as *torsion graded modules*.

Let QGr A = Gr A/ Tors A. Let Q denote the quotient functor Gr $A \to$ QGr A. We shall identify QGr A with the full subcategory of Tors-closed modules. The shift functor $M \mapsto M(n)$ defines a shift functor on QGr A for which we shall use the same notation. Observe that Q commutes with the shift functor. Finally we shall write $\mathcal{O} = Q(A)$. Note that QGr A is a locally finitely generated Grothendieck category with the family, $\{Q(M)\}_{M \in \mathrm{gr}\, A}$, of finitely generated generators (see [7, 5.8]).

The tensor product in Gr A induces a tensor product in QGr A, denoted by $\boxtimes$. More precisely, one sets

$$X \boxtimes Y := Q(X \otimes Y)$$

for any $X, Y \in$ QGr A.

Lemma 6.1. *Given $X, Y \in \mathrm{Gr}\, A$ there is a natural isomorphism in* $\mathrm{QGr}\, A$*: $Q(X) \boxtimes Q(Y) \cong Q(X \otimes Y)$. Moreover, the functor $- \boxtimes Y\colon \mathrm{QGr}\, A \to \mathrm{QGr}\, A$ is right exact and preserves direct limits.*

Proof. See [9, 4.2]. □

As a consequence of this lemma we get an isomorphism $X(d) \cong \mathcal{O}(d) \boxtimes X$ for any $X \in \mathrm{QGr}\, A$ and $d \in \mathbb{Z}$.

The notion of a tensor torsion class of $\mathrm{QGr}\, A$ (with respect to the tensor product $\boxtimes$) is defined analogously to that in $\mathrm{Gr}\, A$. The proof of the next lemma is like that of Lemma 5.2 (also use Lemma 6.1).

Lemma 6.2. *A torsion class $\mathcal{S}$ is a tensor torsion class of* $\mathrm{QGr}\, A$ *if and only if it is closed under shifts of objects, i.e. $X \in \mathcal{S}$ implies $X(n) \in \mathcal{S}$ for any $n \in \mathbb{Z}$.*

Given a prime ideal $P \in \mathsf{Proj}\, A$ and a graded module M, denote by M_P the homogeneous localization of M at P. If f is a homogeneous element of A, by M_f we denote the localization of M at the multiplicative set $S_f = \{f^n\}_{n \geq 0}$.

Lemma 6.3. *If T is a torsion module then $T_P = 0$ and $T_f = 0$ for any $P \in \mathsf{Proj}\, A$ and $f \in A_+$. As a consequence, $M_P \cong Q(M)_P$ and $M_f \cong Q(M)_f$ for any $M \in \mathrm{Gr}\, A$.*

Proof. See [9, 5.5]. □

Denote by $L_{\mathrm{tor}}(\mathrm{Gr}\, A, \mathrm{Tors}\, A)$ (respectively $L_{\mathrm{tor}}(\mathrm{QGr}\, A)$) the lattice of the tensor torsion classes of finite type in $\mathrm{Gr}\, A$ with torsion classes containing $\mathrm{Tors}\, A$ (respectively the tensor torsion classes of finite type in $\mathrm{QGr}\, A$) ordered by inclusion. The map

$$\ell\colon L_{\mathrm{tor}}(\mathrm{Gr}\, A, \mathrm{Tors}\, A) \to L_{\mathrm{tor}}(\mathrm{QGr}\, A), \quad \mathcal{S} \mapsto \mathcal{S}/\,\mathrm{Tors}\, A$$

is a lattice isomorphism, where $\mathcal{S}/\,\mathrm{Tors}\, A = \{Q(M) \mid M \in \mathcal{S}\}$ (see, e.g., [7, 1.7]). We shall consider the map ℓ as an identification.

Theorem 6.4 (Classification). *Let A be a commutative graded ring which is finitely generated as an A_0-algebra. Then the maps*

$$V \mapsto \mathcal{S} = \{M \in \mathrm{QGr}\, A \mid \mathsf{supp}_A(M) \subseteq V\} \quad \text{and} \quad \mathcal{S} \mapsto V = \bigcup_{M \in \mathcal{S}} \mathsf{supp}_A(M)$$

induce bijections between

1. *the set of all open subsets $V \subseteq \mathsf{Proj}^*\, A$,*
2. *the set of all tensor torsion classes of finite type in* $\mathrm{QGr}\, A$.

Proof. By Lemma 5.3 it is enough to show that the maps

$$V \mapsto \mathfrak{F}_V = \{I \in A \mid V(I) \subseteq V\} \quad \text{and} \quad \mathfrak{F} \mapsto V_{\mathfrak{F}} = \bigcup_{I \in \mathfrak{F}} V(I)$$

induce bijections between the set of all open subsets $V \subseteq \mathrm{Proj}^* A$ and the set of all t-filters of finite type containing $\{A_+^n\}_{n\geq 1}$.

Let $\mathfrak{F}$ be such a t-filter. Then the set $\Lambda_{\mathfrak{F}}$ of finitely generated graded ideals I belonging to $\mathfrak{F}$ is a basis for $\mathfrak{F}$. Clearly $V_{\mathfrak{F}} = \bigcup_{I\in\Lambda_{\mathfrak{F}}} V(I)$, so $V_{\mathfrak{F}}$ is open in $\mathrm{Proj}^* A$.

Now let V be an open subset of $\mathrm{Proj}^* A$. Let Λ be the set of finitely generated homogeneous ideals I such that $V(I) \subseteq V$. Then $V = \bigcup_{I\in\Lambda} V(I)$ and $I_1 \cdots I_n \in \Lambda$ for any $I_1, \dots, I_n \in \Lambda$. We denote by $\mathfrak{F}'_V$ the set of homogeneous ideals $I \subset A$ such that $I \supseteq J$ for some $J \in \Lambda$. By Proposition 5.4(2) $\mathfrak{F}'_V$ is a t-filter of finite type. Clearly, $\mathfrak{F}'_V \subset \mathfrak{F}_V$. Suppose $I \in \mathfrak{F}_V \setminus \mathfrak{F}'_V$.

We can use Zorn's lemma to find an ideal $J \supset I$ which is maximal with respect to $J \notin \mathfrak{F}'_V$ (we use the fact that $\mathfrak{F}'_V$ has a basis of finitely generated ideals). We claim that J is prime. Indeed, suppose $a, b \in A$ are two homogeneous elements not belonging to J. Then $J + aA$ and $J + bA$ must be members of $\mathfrak{F}'_V$, and also $(J + aA)(J + bA) \in \mathfrak{F}'_V$ by Proposition 5.4(1). But $(J + aA)(J + bA) \subset J + abA$, and therefore $ab \notin J$. We see that $J \in V(I) \subset V$, and hence $J \in V(I')$ for some $I' \in \Lambda$. But this implies $J \in \mathfrak{F}'_V$, a contradiction. Thus $\mathfrak{F}'_V = \mathfrak{F}_V$. Clearly, $V = V_{\mathfrak{F}_V}$ for every open subset $V \subseteq \mathrm{Proj}^* A$. Let $\mathfrak{F}$ be a t-filter of finite type and $I \in \mathfrak{F}$. Then $I \supset J$ for some $J \in \Lambda_{\mathfrak{F}}$, and hence $V(I) \subset V(J) \subset V_{\mathfrak{F}}$. It follows that $\mathfrak{F} \subset \mathfrak{F}_{V_{\mathfrak{F}}}$. As above, there is no ideal belonging to $\mathfrak{F}_{V_{\mathfrak{F}}} \setminus \mathfrak{F}$. We have shown the desired bijection between the sets of all t-filters of finite type and all open subsets in $\mathrm{Proj}^* A$. □

7 The prime spectrum of an ideal lattice

Inspired by recent work of Balmer [4], Buan, Krause, and Solberg [5] introduce the notion of an ideal lattice and study its prime ideal spectrum. Applications arise from abelian or triangulated tensor categories.

Definition (Buan, Krause, Solberg [5]). An *ideal lattice* is by definition a partially ordered set $L = (L, \leq)$, together with an associative multiplication $L \times L \to L$, such that the following holds.

(L1) The poset L is a *complete lattice*, that is,

$$\sup A = \bigvee_{a\in A} a \quad \text{and} \quad \inf A = \bigwedge_{a\in A} a$$

exist in L for every subset $A \subseteq L$.

(L2) The lattice L is *compactly generated*, that is, every element in L is the supremum of a set of compact elements. (An element $a \in L$ is *compact*, if for all $A \subseteq L$ with $a \leq \sup A$ there exists some finite $A' \subseteq A$ with $a \leq \sup A'$.)

(L3) We have for all $a, b, c \in L$

$$a(b \vee c) = ab \vee ac \quad \text{and} \quad (a \vee b)c = ac \vee bc.$$

(L4) The element $1 = \sup L$ is compact, and $1a = a = a1$ for all $a \in L$.

(L5) The product of two compact elements is again compact.

A *morphism* $\phi\colon L \to L'$ of ideal lattices is a map satisfying

$$\phi\Big(\bigvee_{a\in A} a\Big) = \bigvee_{a\in A} \phi(a) \quad \text{for } A \subseteq L,$$
$$\phi(1) = 1 \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b) \quad \text{for } a, b \in L.$$

Let L be an ideal lattice. Following [5] we define the spectrum of prime elements in L. An element $p \neq 1$ in L is *prime* if $ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. We denote by $\operatorname{Spec} L$ the set of prime elements in L and define for each $a \in L$

$$V(a) = \{p \in \operatorname{Spec} L \mid a \leq p\} \quad \text{and} \quad D(a) = \{p \in \operatorname{Spec} L \mid a \nleq p\}.$$

The subsets of $\operatorname{Spec} L$ of the form $V(a)$ are closed under forming arbitrary intersections and finite unions. More precisely,

$$V\Big(\bigvee_{i\in\Omega} a_i\Big) = \bigcap_{i\in\Omega} V(a_i) \quad \text{and} \quad V(ab) = V(a) \cup V(b).$$

Thus we obtain the *Zariski topology* on $\operatorname{Spec} L$ by declaring a subset of $\operatorname{Spec} L$ to be *closed* if it is of the form $V(a)$ for some $a \in L$. The set $\operatorname{Spec} L$ endowed with this topology is called the *prime spectrum* of L. Note that the sets of the form $D(a)$ with compact $a \in L$ form a basis of open sets. The prime spectrum $\operatorname{Spec} L$ of an ideal lattice L is spectral [5, 2.5].

There is a close relation between spectral spaces and ideal lattices. Given a topological space X, we denote by $L_{\text{open}}(X)$ the lattice of open subsets of X and consider the multiplication map

$$L_{\text{open}}(X) \times L_{\text{open}}(X) \to L_{\text{open}}(X), \quad (U, V) \mapsto UV = U \cap V.$$

The lattice $L_{\text{open}}(X)$ is complete.

The following result, which appears in [5], is part of the Stone Duality Theorem (see, for instance, [12]).

Proposition 7.1. *Let X be a spectral space. Then $L_{\text{open}}(X)$ is an ideal lattice. Moreover, the map*

$$X \to \operatorname{Spec} L_{\text{open}}(X), \quad x \mapsto X \setminus \overline{\{x\}},$$

is a homeomorphism.

We deduce from the classification of torsion classes of finite type (Theorems 2.2 and 6.4) the following.

Proposition 7.2. *Let R (respectively A) be a commutative ring (respectively graded commutative ring which is finitely generated as an A_0-algebra). Then* $L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ *and* $L_{\mathrm{tor}}(\mathrm{QGr}\, A)$ *are ideal lattices.*

Proof. The spaces $\mathsf{Spec}\, R$ and $\mathsf{Proj}\, A$ are spectral. Thus $\mathsf{Spec}^* R$ and $\mathsf{Proj}^* A$ are spectral, also $L_{\mathrm{open}}(\mathsf{Spec}^* R)$ and $L_{\mathrm{open}}(\mathsf{Proj}^* A)$ are ideal lattices by Proposition 7.1. By Theorems 2.2 and 6.4 we have isomorphisms $L_{\mathrm{open}}(\mathsf{Spec}^* R) \cong L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ and $L_{\mathrm{open}}(\mathsf{Proj}^* A) \cong L_{\mathrm{tor}}(\mathrm{QGr}\, A)$. Therefore $L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ and $L_{\mathrm{tor}}(\mathrm{QGr}\, A)$ are ideal lattices. □

Corollary 7.3. *Let R (respectively A) be a commutative ring (respectively the graded commutative ring which is finitely generated as an A_0-algebra). The points of* $\mathsf{Spec}\, L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ *(respectively* $\mathsf{Spec}\, L_{\mathrm{tor}}(\mathrm{QGr}\, A)$*) are the $\cap$-irreducible torsion classes of finite type in* $\mathrm{Mod}\, R$ *(respectively tensor torsion classes of finite type in* $\mathrm{QGr}\, A$*) and the maps*

$$f\colon \mathsf{Spec}^* R \to \mathsf{Spec}\, L_{\mathrm{tor}}(\mathrm{Mod}\, R), \quad P \mapsto \mathcal{S}_P = \{M \in \mathrm{Mod}\, R \mid M_P = 0\},$$
$$f\colon \mathsf{Proj}^* A \to \mathsf{Spec}\, L_{\mathrm{tor}}(\mathrm{QGr}\, A), \quad P \mapsto \mathcal{S}_P = \{M \in \mathrm{QGr}\, A \mid M_P = 0\}$$

are homeomorphisms of spaces.

Proof. This is a consequence of Theorems 2.2, 6.4 and Propositions 7.1, 7.2. □

8 Reconstructing affine and projective schemes

Let R (respectively A) be a commutative ring (respectively graded commutative ring which is finitely generated as an A_0-algebra). We shall write $\mathsf{Spec}(\mathrm{Mod}\, R) := \mathsf{Spec}^* L_{\mathrm{tor}}(\mathrm{Mod}\, R)$ (resp. $\mathsf{Spec}(\mathrm{QGr}\, A) := \mathsf{Spec}^* L_{\mathrm{tor}}(\mathrm{QGr}\, A)$) and $\mathsf{supp}(M) := \{\mathcal{P} \in \mathsf{Spec}(\mathrm{Mod}\, R) \mid M \notin \mathcal{P}\}$ (resp. $\mathsf{supp}(M) := \{\mathcal{P} \in \mathsf{Spec}(\mathrm{QGr}\, A) \mid M \notin \mathcal{P}\}$) for $M \in \mathrm{Mod}\, R$ (resp. $M \in \mathrm{QGr}\, A$). It follows from Corollary 7.3 that

$$\mathsf{supp}_R(M) = f^{-1}(\mathsf{supp}(M)) \quad (\text{resp. } \mathsf{supp}_A(M) = f^{-1}(\mathsf{supp}(M))).$$

Following [4], [5], we define a structure sheaf on $\mathsf{Spec}(\mathrm{Mod}\, R)$ ($\mathsf{Spec}(\mathrm{QGr}\, A)$) as follows. For an open subset $U \subseteq \mathsf{Spec}(\mathrm{Mod}\, R)$ ($U \subseteq \mathsf{Spec}(\mathrm{QGr}\, A)$), let

$$\mathcal{S}_U = \{M \in \mathrm{Mod}\, R\ (\mathrm{QGr}\, A) \mid \mathsf{supp}(M) \cap U = \emptyset\}$$

and observe that $\mathcal{S}_U = \{M \mid M_P = 0 \text{ for all } P \in f^{-1}(U)\}$ is a (tensor) torsion class. We obtain a presheaf of rings on $\mathsf{Spec}(\mathrm{Mod}\, R)$ ($\mathsf{Spec}(\mathrm{QGr}\, A)$) by

$$U \mapsto \mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_U}(R) \quad (\mathrm{End}_{\mathrm{QGr}\, A/\mathcal{S}_U}(\mathcal{O})).$$

If $V \subseteq U$ are open subsets, then the restriction map

$$\mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_U}(R) \to \mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_V}(R) \quad (\mathrm{End}_{\mathrm{QGr}\, A/\mathcal{S}_U}(\mathcal{O}) \to \mathrm{End}_{\mathrm{QGr}\, A/\mathcal{S}_V}(\mathcal{O}))$$

is induced by the quotient functor

$$\mathrm{Mod}\, R/\mathcal{S}_U \to \mathrm{Mod}\, R/\mathcal{S}_V \quad (\mathrm{QGr}\, A/\mathcal{S}_U \to \mathrm{QGr}\, A/\mathcal{S}_V).$$

The sheafification is called the *structure sheaf* of Mod R (QGr A) and is denoted by $\mathcal{O}_{\mathrm{Mod}\, R}$ ($\mathcal{O}_{\mathrm{QGr}\, A}$). This is a sheaf of commutative rings by [13, XI.2.4]. Next let $\mathcal{P} \in \mathsf{Spec}(\mathrm{Mod}\, R)$ and $P := f^{-1}(\mathcal{P})$. We have

$$\mathcal{O}_{\mathrm{Mod}\, R,\mathcal{P}} = \varinjlim_{\mathcal{P}\in U} \mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_U}(R) = \varinjlim_{f\notin P} \mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_{D(f)}}(R) \cong \varinjlim_{f\notin P} R_f = \mathcal{O}_{R,P}.$$

Similarly, for $\mathcal{P} \in \mathsf{Spec}(\mathrm{QGr}\, A)$ and $P := f^{-1}(\mathcal{P})$ we have

$$\mathcal{O}_{\mathrm{QGr}\, A,\mathcal{P}} \cong \mathcal{O}_{\mathsf{Proj}\, A,P}.$$

The next theorem says that the abelian category Mod R (QGr A) contains all the necessary information to reconstruct the affine (projective) scheme $(\mathsf{Spec}\, R, \mathcal{O}_R)$ (respectively $(\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A})$).

Theorem 8.1 (Reconstruction). *Let R (respectively A) be a ring (respectively graded ring which is finitely generated as an A_0-algebra). The maps of Corollary* 7.3 *induce isomorphisms of ringed spaces*

$$f\colon (\mathsf{Spec}\, R, \mathcal{O}_R) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{Mod}\, R), \mathcal{O}_{\mathrm{Mod}\, R})$$

and

$$f\colon (\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{QGr}\, A), \mathcal{O}_{\mathrm{QGr}\, A}).$$

Proof. The proof is similar to that of [5, 9.4]. Fix an open subset $U \subseteq \mathsf{Spec}(\mathrm{Mod}\, R)$ and consider the composition of the functors

$$F\colon \mathrm{Mod}\, R \xrightarrow{\widetilde{(-)}} \mathrm{Qcoh}\, \mathsf{Spec}\, R \xrightarrow{(-)|_{f^{-1}(U)}} \mathrm{Qcoh}\, f^{-1}(U).$$

Here, for any R-module M, we denote by $\widetilde{M}$ its associated sheaf. By definition, the stalk of $\widetilde{M}$ at a prime P equals the localized module M_P. We claim that F annihilates $\mathcal{S}_U$. In fact, $M \in \mathcal{S}_U$ implies $f^{-1}(\mathsf{supp}(M)) \cap f^{-1}(U) = \emptyset$ and therefore $\mathsf{supp}_R(M) \cap f^{-1}(U) = \emptyset$. Thus $M_P = 0$ for all $P \in f^{-1}(U)$ and therefore $F(M) = 0$. It follows that F factors through Mod $R/\mathcal{S}_U$ and induces a map $\mathrm{End}_{\mathrm{Mod}\, R/\mathcal{S}_U}(R) \to \mathcal{O}_R(f^{-1}(U))$ which extends to a map $\mathcal{O}_{\mathrm{Mod}\, R}(U) \to \mathcal{O}_R(f^{-1}(U))$. This yields the morphism of sheaves $f^\sharp\colon \mathcal{O}_{\mathrm{Mod}\, R} \to f_*\mathcal{O}_R$.

By the above $f^\sharp$ induces an isomorphism $f_P^\sharp\colon \mathcal{O}_{\mathrm{Mod}\, R, f(P)} \to \mathcal{O}_{R,P}$ at each point $P \in \mathsf{Spec}\, R$. We conclude that $f_P^\sharp$ is an isomorphism. It follows that f is an isomorphism of ringed spaces if the map $f\colon \mathsf{Spec}\, R \to \mathsf{Spec}(\mathrm{Mod}\, R)$ is a homeomorphism. This last condition is a consequence of Propositions 7.1 and 7.2. The same arguments apply to show that

$$f\colon (\mathsf{Proj}\, A, \mathcal{O}_{\mathsf{Proj}\, A}) \xrightarrow{\sim} (\mathsf{Spec}(\mathrm{QGr}\, A), \mathcal{O}_{\mathrm{QGr}\, A})$$

is an isomorphism of ringed spaces. □

9 Appendix

A subcategory $\mathcal{S}$ of a Grothendieck category $\mathcal{C}$ is said to be *Serre* if for any short exact sequence
$$0 \to X' \to X \to X'' \to 0$$
$X', X'' \in \mathcal{S}$ if and only if $X \in \mathcal{S}$. A Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ is said to be a *torsion class* if $\mathcal{S}$ is closed under taking coproducts. An object C of $\mathcal{C}$ is said to be *torsionfree* if $(\mathcal{S}, C) = 0$. The pair consisting of a torsion class and the corresponding class of torsionfree objects is referred to as a *torsion theory*. Given a torsion class $\mathcal{S}$ in $\mathcal{C}$ the *quotient category* $\mathcal{C}/\mathcal{S}$ is the full subcategory with objects those torsionfree objects $C \in \mathcal{C}$ satisfying $\mathrm{Ext}^1(T, C) = 0$ for every $T \in \mathcal{S}$. The inclusion functor $i : \mathcal{S} \to \mathcal{C}$ admits the right adjoint $t : \mathcal{C} \to \mathcal{S}$ which takes every object $X \in \mathcal{C}$ to the maximal subobject $t(X)$ of X belonging to $\mathcal{S}$. The functor t we call the *torsion functor*. Moreover, the inclusion functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ has a left adjoint, the *localization functor* $(-)_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}/\mathcal{S}$, which is also exact. Then,
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}/\mathcal{S}}(X_{\mathcal{S}}, Y)$$
for all $X \in \mathcal{C}$ and $Y \in \mathcal{C}/\mathcal{S}$. A torsion class $\mathcal{S}$ is *of finite type* if the functor $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ preserves directed sums. If $\mathcal{C}$ is a locally coherent Grothendieck category then $\mathcal{S}$ is of finite type if and only if $i : \mathcal{C}/\mathcal{S} \to \mathcal{C}$ preserves direct limits (see, e.g., [7]).

Let $\mathcal{C}$ be a Grothendieck category having a family of finitely generated projective generators $\mathcal{A} = \{P_i\}_{i \in I}$. Let $\mathfrak{F} = \bigcup_{i \in I} \mathfrak{F}^i$ be a family of subobjects, where each $\mathfrak{F}^i$ is a family of subobjects of P_i. We refer to $\mathfrak{F}$ as a *Gabriel filter* if the following axioms are satisfied:

$T1$. $P_i \in \mathfrak{F}^i$ for every $i \in I$;

$T2$. if $\mathfrak{a} \in \mathfrak{F}^i$ and $\mu \colon P_j \to P_i$ then $\{\mathfrak{a} : \mu\} = \mu^{-1}(\mathfrak{a}) \in \mathfrak{F}^j$;

$T3$. if $\mathfrak{a}$ and $\mathfrak{b}$ are subobjects of P_i such that $\mathfrak{a} \in \mathfrak{F}^i$ and $\{\mathfrak{b} : \mu\} \in \mathfrak{F}^j$ for any $\mu \colon P_j \to P_i$ with $\mathrm{Im}\,\mu \subset \mathfrak{a}$ then $\mathfrak{b} \in \mathfrak{F}^i$.

In particular each $\mathfrak{F}^i$ is a filter of subobjects of P_i. A Gabriel filter is *of finite type* if each of these filters has a cofinal set of finitely generated objects (that is, if for each i and each $\mathfrak{a} \in \mathfrak{F}_i$ there is a finitely generated $\mathfrak{b} \in \mathfrak{F}_i$ with $\mathfrak{a} \supseteq \mathfrak{b}$).

Note that if $\mathcal{A} = \{A\}$ is a ring and $\mathfrak{a}$ is a right ideal of A, then for every endomorphism $\mu \colon A \to A$
$$\mu^{-1}(\mathfrak{a}) = \{\mathfrak{a} : \mu(1)\} = \{a \in A \mid \mu(1)a \in \mathfrak{a}\}.$$
On the other hand, if $x \in A$, then $\{\mathfrak{a} : x\} = \mu^{-1}(\mathfrak{a})$, where $\mu \in \mathrm{End}\,A$ is such that $\mu(1) = x$.

It is well-known (see, e.g., [7]) that the map
$$\mathcal{S} \longmapsto \mathfrak{F}(\mathcal{S}) = \{\mathfrak{a} \subseteq P_i \mid i \in I,\ P_i/\mathfrak{a} \in \mathcal{S}\}$$
establishes a bijection between the Gabriel filters (respectively Gabriel filters of finite type) and the torsion classes on $\mathcal{C}$ (respectively torsion classes of finite type on $\mathcal{C}$).

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Parshin's conjecture revisited

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1 Introduction

Parshin's conjecture states that $K_i(X)_{\mathbb{Q}} = 0$ for $i > 0$ and X smooth and projective over a finite field $\mathbb{F}_q$. The purpose of this paper is to break up Parshin's conjecture into several independent statements, in the hope that each of them is easier to attack individually. If $CH_n(X, i)$ is Bloch's higher Chow group of cycles of relative dimension n, then in view of $K_i(X)_{\mathbb{Q}} \cong \bigoplus_n CH_n(X, i)_{\mathbb{Q}}$, Parshin's conjecture is equivalent to Conjecture $P(n)$ for all n, stating that $CH_n(X, i)_{\mathbb{Q}} = 0$ for $i > 0$, and all smooth and projective X. We show, assuming resolution of singularities, that Conjecture $P(n)$ is equivalent to the conjunction of three conjectures $A(n)$, $B(n)$ and $C(n)$, and give several equivalent versions of these conjectures. This is most conveniently formulated in terms of weight homology. We define $H_*^W(X, \mathbb{Q}(n))$ to be the homology of the complex $CH_n(W(X))_{\mathbb{Q}}$, where $W(X)$ is the weight complex defined by Gillet–Soulé [8] shifted by $2n$. Then, in a nutshell, Conjecture $A(n)$ states that for all schemes X over $\mathbb{F}_q$, the niveau spectral sequence of $CH_n(X, *)_{\mathbb{Q}}$ degenerates to one line, Conjecture $C(n)$ states that for all schemes X over $\mathbb{F}_q$ the niveau spectral sequence of $H_*^W(X, \mathbb{Q}(n))$ degenerates to one line, and Conjecture $B(n)$ states that, for all X over $\mathbb{F}_q$, the two lines are isomorphic. The conjunction of $A(n)$, $B(n)$, and $C(n)$ clearly implies $P(n)$, because $H_i^W(X, \mathbb{Q}(n)) = 0$ for $i \neq 2n$ and X smooth and projective over $\mathbb{F}_q$, and we show the converse. Note that in the above formulation, Parshin's conjecture implies statements on higher Chow groups not only for smooth and projective schemes, but gives a way to calculate $CH_n(X, i)_{\mathbb{Q}}$ for arbitrary X.

A more concrete version of $A(n)$ is that, for every smooth and projective scheme X of dimension d over $\mathbb{F}_q$, $CH_n(X, i)_{\mathbb{Q}} = 0$ for $i > d - n$. A reformulation of $B(n)$ is that for every smooth and projective scheme X of dimension $d > n + 1$, the following sequence is exact

$$0 \to K_{d-n}^M(k(X))_{\mathbb{Q}} \to \bigoplus_{x \in X^{(1)}} K_{d-n-1}^M(k(x))_{\mathbb{Q}} \to \bigoplus_{x \in X^{(2)}} K_{d-n-2}^M(k(x))_{\mathbb{Q}}$$

and that this sequence is exact at $K_1^M(k(X))_{\mathbb{Q}}$ if $d = \dim X = n + 1$.

In the second half of the paper, we focus on the case $n = 0$, because of its applications in [5]. A different version of weight homology has been studied by Jannsen [11], and he proved that Conjecture $C(0)$ holds under resolution of singularities. We use this to give two more versions of Conjecture $P(0)$. The first is that there is an isomorphism

*Supported in part by NSF grant No.0556263

from higher Chow groups with $\mathbb{Q}_l$-coefficients to l-adic cohomology, for all X and i,

$$CH_0(X,i)_{\mathbb{Q}_l} \oplus CH_0(X,i+1)_{\mathbb{Q}_l} \to H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l).$$

Conjecture $P(0)$ can also be recovered from, and implies a structure theorem for higher Chow groups of smooth affine schemes: For all smooth and affine schemes U of dimension d over $\mathbb{F}_q$, the groups $CH_0(U,i)$ are torsion for $i \neq d$, and the canonical map $CH_0(U,d)_{\mathbb{Q}_l} \to H_d(\bar{U}_{\text{et}}, \widehat{\mathbb{Q}}_l)^{\text{Gal}(\mathbb{F}_q)}$ is an isomorphism.

Finally, we reproduce an argument of Levine showing that if F is the absolute Frobenius, the push-forward F_* acts like q^n on $CH_n(X,i)$, and the pull-back F^* acts on motivic cohomology $H^i_M(X,\mathbb{Z}(n))$ like q^n for all n. As a corollary, Conjecture $P(0)$ follows from finite dimensionality of smooth and projective schemes over finite fields in the sense of Kimura [13].

Acknowledgements. This paper was inspired by the work of, and discussions with, U. Jannsen and S. Saito. We are indebted to the referee, whose careful reading helped to improve the exposition.

2 Parshin's conjecture

We fix a perfect field k of characteristic p, and consider the category of separated schemes of finite type over k. We recall some facts on Bloch's higher Chow groups [1], see [3] for a survey. Let $z_n(X,i)$ be the free abelian group generated by cycles of dimension $n+i$ on $X \times_k \Delta^i$ which meet all faces properly, and let $z_n(X,*)$ be the complex of abelian groups obtained by taking the alternating sum of intersection with face maps as differential. We define $CH_n(X,i)$ as the ith homology of this complex and motivic Borel–Moore homology to be

$$H_i^c(X,\mathbb{Z}(n)) = CH_n(X,i-2n).$$

For a proper map $f: X \to Y$ we have a push-forward $z_n(X,*) \to z_n(Y,*)$, for a flat, quasi-finite map $f: X \to Y$ we have a pull-back $z_n(X,*) \to z_n(Y,*)$, and a closed embedding $i: Z \to X$ with open complement $j: U \to X$ induces a localization sequence

$$\cdots \to H_i^c(Z,\mathbb{Z}(n)) \xrightarrow{i_*} H_i^c(X,\mathbb{Z}(n)) \xrightarrow{j^*} H_i^c(U,\mathbb{Z}(n)) \to \cdots.$$

If X is smooth of pure dimension d, then $H_i^c(X,\mathbb{Z}(n)) \cong H^{2d-i}(X,\mathbb{Z}(d-n))$, where the right-hand side is Voevodsky's motivic cohomology [15]. For a finitely generated field F over k, we define $H_i^c(F,\mathbb{Z}(n)) = \operatorname{colim}_U H_i^c(U,\mathbb{Z}(n))$, where the colimit runs through U of finite type over k with field of functions F. For the reader who is more familiar with motivic cohomology, we mention that Voevodsky's theorem implies that for a field F of transcendence degree d over k, we have

$$H_i^c(F,\mathbb{Z}(n)) \cong H^{2d-i}(F,\mathbb{Z}(d-n)) \cong \begin{cases} 0, & i < d+n, \\ K^M_{d-n}(F), & i = d+n. \end{cases}$$

The latter isomorphism is due to Nesterenko–Suslin and Totaro. It follows formally from localization that there are spectral sequences

$$E^1_{s,t}(X) = \bigoplus_{x \in X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}(n)) \Rightarrow H^c_{s+t}(X, \mathbb{Z}(n)). \tag{1}$$

Here $X_{(s)}$ denotes points of X of dimension s. Since $H^c_i(F, \mathbb{Z}(n)) = 0$ for $i < n + \operatorname{trdeg} F$, the spectral sequence is concentrated in the area $0 \le s \le \dim X$ and $t \ge n$. If we let

$$\widetilde{H}^c_i(X, \mathbb{Z}(n)) = E^2_{i+n,n}(X)$$

be the ith homology of the complex

$$0 \leftarrow \bigoplus_{x \in X_{(n)}} H^c_{2n}(k(x), \mathbb{Z}(n)) \leftarrow \cdots \leftarrow \bigoplus_{x \in X_{(s)}} H^c_{s+n}(k(x), \mathbb{Z}(n)) \leftarrow \cdots, \tag{2}$$

with $\bigoplus_{x \in X_{(s)}} H^c_{s+n}(k(x), \mathbb{Z}(n))$ in degree $s + n$, then we obtain a canonical and functorial map

$$\alpha \colon H^c_i(X, \mathbb{Z}(n)) \to \widetilde{H}^c_i(X, \mathbb{Z}(n)).$$

Note that the groups in (2) are Milnor-K-groups.

Parshin's conjecture states that for all smooth and projective X over $\mathbb{F}_q$, the groups $K_i(X)_{\mathbb{Q}}$ are torsion for $i > 0$. If Tate's conjecture holds and rational equivalence and homological equivalence agree up to torsion for all X, then Parshin's conjecture holds by [2]. Since $K_i(X)_{\mathbb{Q}} = \bigoplus_n CH_n(X, i)_{\mathbb{Q}}$, it follows that Parshin's conjecture is equivalent to the following conjecture for all n.

Conjecture $P(n)$. For all smooth and projective schemes X over the finite field $\mathbb{F}_q$, the groups $H^c_i(X, \mathbb{Q}(n))$ vanish for $i \neq 2n$.

We will refer to the following equivalent statements as Conjecture $A(n)$:

Proposition 2.1. *For a fixed integer n, the following statements are equivalent:*

a) *For all schemes $X/\mathbb{F}_q$ and all i, α induces an isomorphism $H^c_i(X, \mathbb{Q}(n)) \cong \widetilde{H}^c_i(X, \mathbb{Q}(n))$.*
b) *For all finitely generated fields $k/\mathbb{F}_q$ with $d := \operatorname{trdeg} k/\mathbb{F}_q$, and all $i \neq d + n$, we have $H^c_i(k, \mathbb{Q}(n)) = 0$.*
c) *For all smooth and projective X over $\mathbb{F}_q$ and all $i > \dim X + n$, we have $H^c_i(X, \mathbb{Q}(n)) = 0$.*
d) *For all smooth and affine schemes U over $\mathbb{F}_q$ and all $i > \dim U + n$, we have $H^c_i(U, \mathbb{Q}(n)) = 0$.*

Proof. a) $\Rightarrow$ c), d): The complex (2) is concentrated in degrees $[2n, d + n]$.

c) $\Rightarrow$ b): This is proved by induction on the transcendence degree. By de Jong's theorem, we find a smooth and proper model X of a finite extension of k. Looking at the niveau spectral sequence (1), we see that the induction hypothesis implies $H^c_i(X, \mathbb{Q}(n)) = 0$ for $i > d + n$ (see [2] for details).

d) $\Rightarrow$ b): follows by writing k as a colimit of smooth affine scheme schemes of dimension d.

b) $\Rightarrow$ a): The niveau spectral sequence collapses to the complex (2). $\square$

Using the Gersten resolution, the statement in the proposition implies that on a smooth X of dimension d, the motivic complex $\mathbb{Q}(d-n)$ is concentrated in degree $d-n$, and if $\mathcal{C}_n := \mathcal{H}^{d-n}(\mathbb{Q}(d-n)) = CH_n(-, d-n)_{\mathbb{Q}}$, then $CH_n(X,i) = H^{d-n-i}(X, \mathcal{C}_n)$.

Statement 2.1 d) is part of the following affine analog of $P(n)$:

Conjecture $L(n)$. For all smooth and affine schemes U of dimension d over the finite field $\mathbb{F}_q$, the group $H_i^c(U, \mathbb{Q}(n))$ vanishes unless $d \leq i \leq d+n$.

Since $H_i^c(U, \mathbb{Q}(n)) \cong H^{2d-i}(U, \mathbb{Q}(d-n))$, Conjecture $L(n)$ can be thought of as an analog of the affine Lefschetz theorem.

3 Weight homology

This section is inspired by Jannsen [11]. Throughout this section we assume resolution of singularities over the field k. Let C be the category with objects smooth projective varieties over a field k of characteristic 0, and $\mathrm{Hom}_C(X,Y) = \bigoplus_i CH^{\dim Y_i}(X \times Y_i)$, where Y_i runs through the connected components of Y. Let H be the homotopy category of bounded complexes over C. Gillet and Soulé define in [8], for every separated scheme of finite type, a weight complex $W(X) \in H$ satisfying the following properties [8, Theorem 2] (our notation differs from loc. cit. in variance):

a) $W(X)$ is represented by a bounded complex
$$M(X_0) \leftarrow M(X_1) \leftarrow \cdots \leftarrow M(X_k)$$
with $\dim X_i \leq \dim X - i$, where $M(X_i)$ placed in degree i.

b) $W(-)$ is covariant functorial for proper maps.

c) $W(-)$ is contravariant functorial for open embeddings.

d) If $T \to X$ is a closed embedding with open complement U, then there is a distinguished triangle
$$W(T) \xrightarrow{i_*} W(X) \xrightarrow{j^*} W(U).$$

e) If D is a divisor with normal crossings in a scheme X, with irreducible components Y_i, and if $Y^{(r)} = \coprod_{\#I=r} \cap_{i\in I} Y_i$, then $W(X-D)$ is represented by
$$M(X) \leftarrow M(Y^{(1)}) \leftarrow \cdots \leftarrow M(Y^{(\dim X)}).$$

The argument in loc. cit. only uses that resolution of singularities exists over k, and we assume from now on that k is such a field.

Given an additive covariant functor F from C to an abelian category, we define weight (Borel–Moore) homology $H_i^W(X,F)$ as the ith homology of the complex $F(W(X))$. Weight homology has the functorialities inherited from b) and c), and satisfies a localization sequence deduced from d). If K is a finitely generated field over k, then we define $H_i^W(K,F)$ to be colim $H_i^W(U,F)$, where the (filtered) limit runs through integral varieties having K as their function field. Similarly, a contravariant functor G from C to an abelian category gives rise to weight cohomology (with compact support) $H^i_W(X,G)$.

As a special case, we define the weight homology group $H_i^W(X,\mathbb{Z}(n))$ as the $i-2n$th homology of the homological complex of abelian groups $CH_n(W(X))$.

Lemma 3.1. *We have $H_i^W(X,\mathbb{Z}(n)) = 0$ for $i > \dim X + n$. In particular we have $H_i^W(K,\mathbb{Z}(n)) = 0$ for every finitely generated field K/k and every $i > \operatorname{trdeg}_k K + n$.*

Proof. This follows from the first property of weight complexes together with $CH_n(T) = 0$ for $n > \dim T$. □

It follows from Lemma 3.1 that the niveau spectral sequence

$$E^1_{s,t}(X) = \bigoplus_{x\in X_{(s)}} H^W_{s+t}(k(x),\mathbb{Z}(n)) \Rightarrow H^W_{s+t}(X,\mathbb{Z}(n)) \tag{3}$$

is concentrated on and below the line $t = n$. Let

$$\widetilde{H}_i^W(X,\mathbb{Z}(n)) = E^2_{i+n,n}(X)$$

be the ith homology of the complex

$$0 \leftarrow \bigoplus_{x\in X_{(n)}} H^W_{2n}(k(x),\mathbb{Z}(n)) \leftarrow \cdots \leftarrow \bigoplus_{x\in X_{(s)}} H^W_{s+n}(k(x),\mathbb{Z}(n)) \leftarrow \cdots, \tag{4}$$

where $\bigoplus_{x\in X_{(s)}} H^W_{s+n}(k(x),\mathbb{Z}(n))$ is placed in degree $s+n$. Then we obtain a canonical and natural map

$$\gamma\colon \widetilde{H}_i^W(X,\mathbb{Z}(n)) \to H_i^W(X,\mathbb{Z}(n)).$$

Consider the canonical map of covariant functors $\pi'\colon z_n(-,*) \to CH_n(-)$ on the category of smooth projective schemes over k, sending the cycle complex to its highest cohomology. Then by [11, Theorem 5.13, Remark 5.15], the set of associated homology functors extends to a homology theory on the category of all varieties over k. The argument of [11, Proposition 5.16] show that the extension of the associated homology functors for $z_n(-,*)$ are higher Chow groups $CH_n(-,i)$. The extension $CH_n(-)$ are by definition the functors $H_i^W(-,\mathbb{Z}(n))$. We obtain a functorial map

$$\pi\colon H_i^c(X,\mathbb{Z}(n)) \to H_i^W(X,\mathbb{Z}(n)).$$

Lemma 3.2. *For $i = 2n$, and for all schemes X, the map π is an isomorphism $H^c_{2n}(X,\mathbb{Z}(n)) \cong H^W_{2n}(X,\mathbb{Z}(n))$. In particular, $H^W_{2d}(K,\mathbb{Z}(d)) \cong \mathbb{Z}$ for all fields K of transcendence degree d over k.*

Proof. The statement is clear for X smooth and projective. We proceed by induction on the dimension of X. Using the localization sequence for both theories, we can assume that X is proper. Let $f: X' \to X$ be a resolution of singularities of X, Z be the closed subscheme (of lower dimension) where f is not an isomorphism, and $Z' = Z \times_X X'$. Then we conclude by comparing localization sequences

$$\begin{tikzcd}
H^c_{2n}(Z',\mathbb{Z}(n)) \arrow[r] \arrow[d, equal] & H^c_{2n}(Z,\mathbb{Z}(n)) \oplus H^c_{2n}(X',\mathbb{Z}(n)) \arrow[r] \arrow[d, equal] & H^c_{2n}(X,\mathbb{Z}(n)) \arrow[r] \arrow[d] & 0 \\
H^W_{2n}(Z',\mathbb{Z}(n)) \arrow[r] & H^W_{2n}(Z,\mathbb{Z}(n)) \oplus H^W_{2n}(X',\mathbb{Z}(n)) \arrow[r] & H^W_{2n}(X,\mathbb{Z}(n)) \arrow[r] & 0.
\end{tikzcd}$$

□

The map π for fields induces a map $\beta: \tilde{H}^c_i(X,\mathbb{Q}(n)) \to \tilde{H}^W_i(X,\mathbb{Q}(n))$, which fits into the (non-commutative) diagram

$$\begin{tikzcd}
H^c_i(X,\mathbb{Z}(n)) \arrow[r, "\pi"] \arrow[d, "\alpha"] & H^W_i(X,\mathbb{Z}(n)) \\
\tilde{H}^c_i(X,\mathbb{Z}(n)) \arrow[r, "\beta"] & \tilde{H}^W_i(X,\mathbb{Z}(n)). \arrow[u, "\gamma"']
\end{tikzcd}$$

We now return to the situation $k = \mathbb{F}_q$, and compare weight homology to higher Chow groups using their niveau spectral sequences. We saw that the niveau spectral sequence for higher Chow groups is concentrated above the line $t = n$, and that the niveau spectral sequence for weight homology is concentrated below the line $t = n$. Our aim is to show that Parshin's conjecture is equivalent to both being rationally concentrated on this line, and that the resulting complexes are isomorphic.

The following statements will be referred to as Conjecture $B(n)$:

Proposition 3.3. *For a fixed integer n, the following statements are equivalent:*

a) *The map β induces an isomorphism $\tilde{H}_i(X,\mathbb{Q}(n)) \cong \tilde{H}^W_i(X,\mathbb{Q}(n))$ for all schemes X and all i.*
b) *The map π induces an isomorphism $H^c_{d+n}(k,\mathbb{Q}(n)) \cong H^W_{d+n}(k,\mathbb{Q}(n))$ for all finitely generated fields $k/\mathbb{F}_q$, where $d = \operatorname{trdeg} k/\mathbb{F}_q$.*
c) *For every smooth and projective X over $\mathbb{F}_q$ the following holds: if $d = \dim X > n+1$, then $\tilde{H}^c_{d+n}(X,\mathbb{Q}(n)) = \tilde{H}^c_{d+n-1}(X,\mathbb{Q}(n)) = 0$, and if $\dim X = n+1$, then $\tilde{H}^c_{2n+1}(X,\mathbb{Q}(n)) = 0$.*

Note that assuming the conjecture $A(n)$, c) is equivalent to $H^c_{d+n}(X,\mathbb{Q}(n)) = H^c_{d+n-1}(X,\mathbb{Q}(n)) = 0$ and $H^c_{2n+1}(X,\mathbb{Q}(n)) = 0$ for all smooth and projective X of dimension $d > n+1$ and $d = n+1$, respectively, hence are part of Conjecture $P(n)$.

Proof. b) $\Rightarrow$ a) is trivial, and a) $\Rightarrow$ b) follows by a colimit argument because

$$\operatorname*{colim}_{U\subseteq X} \widetilde{H}_i^c(X,\mathbb{Q}(n)) \cong \begin{cases} H^c_{d+n}(k(X),\mathbb{Q}(n)), & i=d+n;\\ 0, & \text{otherwise.}\end{cases}$$

a) $\Rightarrow$ c) follows because for X smooth and projective of dimension d, the cohomology of the complex (2) tensored with $\mathbb{Q}$ equals $\widetilde{H}_i^c(X,\mathbb{Q}(n)) = \widetilde{H}_i^W(X,\mathbb{Q}(n))$ for $i=d+n$ and $i=d+n-1$ (or $i=2n+1$ in case $d=n+1$). An inspection of the niveau spectral sequence (3) shows that this is a subgroup of $H_i^W(X,\mathbb{Q}(n))=0$.

c) $\Rightarrow$ b): For $n>d$, both sides vanish, whereas for $d=n$, both sides are canonically isomorphic to $\mathbb{Q}$. For $n<d$, we proceed by induction on d. Choose a smooth and projective model X for k and compare the exact sequences (2) and (4)

$$\begin{array}{ccccc} A & \longleftarrow & \bigoplus_{x\in X_{(d-1)}} H^c_{d+n-1}(k(x),\mathbb{Q}(n)) & \longleftarrow & H^c_{d+n}(k,\mathbb{Q}(n)) \\ \| & & \| & & \downarrow \\ B & \longleftarrow & \bigoplus_{x\in X_{(d-1)}} H^W_{d+n-1}(k(x),\mathbb{Q}(n)) & \longleftarrow & H^W_{d+n}(k,\mathbb{Q}(n)). \end{array}$$

The terms on the left are $A=H^c_{2n}(X,\mathbb{Q}(n)) \cong B = H^W_{2n}(X,\mathbb{Q}(n))$ if $d=n+1$, and $A=\bigoplus_{x\in X_{(d-2)}} H^c_{d+n-2}(k(x),\mathbb{Q}(n)) \cong B = \bigoplus_{x\in X_{(d-2)}} H^W_{d+n-2}(k(x),\mathbb{Q}(n))$, if $d>n+1$. The upper sequence is exact by hypothesis, and an inspection of (3) shows that the lower sequence is exact because $H_i^W(X,\mathbb{Q}(n))=0$ for $i>2n$. $\square$

We refer to the following statements as Conjecture $C(n)$:

Proposition 3.4. *For a fixed integer n, the following statements are equivalent:*

a) *For all schemes X over $\mathbb{F}_q$ and for all i, the map γ induces an isomorphism $\widetilde{H}_i^W(X,\mathbb{Q}(n) \cong H_i^W(X,\mathbb{Q}(n))$.*

b) *For all finitely generated fields $k/\mathbb{F}_q$ and for all $i \neq \operatorname{trdeg} k/\mathbb{F}_q + n$, we have $H_i^W(k,\mathbb{Q}(n))=0$.*

c) *For all smooth and projective X, the map γ induces an isomorphism*

$$\widetilde{H}_i^W(X,\mathbb{Q}(n)) = \begin{cases} 0, & i>0;\\ CH_n(X)_{\mathbb{Q}}, & i=2n.\end{cases}$$

Proof. b) $\Rightarrow$ a) is trivial and a) $\Rightarrow$ b) follows by a colimit argument.

a) $\Rightarrow$ c) is trivial for $i>2n$, and Lemma 3.2 for $i=2n$.

c) $\Rightarrow$ b): This is proved like Proposition 2.1, by induction on the transcendence degree of k. Let X be a smooth and projective model of k. The induction hypothesis implies that the niveau spectral sequence (3) collapses to the horizontal line $t=n$ and the vertical line $s=d$. Since it converges to $H_i^W(X,\mathbb{Q}(n))$, which is zero for $i>2n$, we obtain isomorphisms $H^W_{d+n-i+1}(k(X),\mathbb{Q}(n)) \xrightarrow{d_i} \widetilde{H}^W_{d+n-i}(X,\mathbb{Q}(n))$ for

$1 < i < d - n$, and an exact sequence

$$H^W_{2n+1}(k(X), \mathbb{Q}(n)) \overset{d_{d-n}}{\hookrightarrow} \tilde{H}^W_{2n}(X, \mathbb{Q}(n)) \to H^W_{2n}(X, \mathbb{Q}(n)) \twoheadrightarrow H^W_{2n}(k(X), \mathbb{Q}(n)).$$

The claim follows because $\tilde{H}^W_i(X, \mathbb{Q}(n)) = H^W_i(X, \mathbb{Q}(n)) = 0$ for $i > 2n$, and because the maps $H^c_{2n}(X, \mathbb{Q}(n)) \xrightarrow{\pi} H^W_{2n}(X, \mathbb{Q}(n)) \xleftarrow{\gamma} \tilde{H}^W_{2n}(X, \mathbb{Q}(n))$ are isomorphisms by Lemma 3.2 and hypothesis. □

Theorem 3.5. *The conjunction of Conjectures $A(n)$, $B(n)$ and $C(n)$ is equivalent to $P(n)$.*

Proof. Given Conjectures $A(n)$, $B(n)$, and $C(n)$, we get

$$H^c_i(X, \mathbb{Q}(n)) = \tilde{H}^c_i(X, \mathbb{Q}(n)) \cong \tilde{H}^W_i(X, \mathbb{Q}(n)) \cong H^W_i(X, \mathbb{Q}(n))$$

for all X, and the latter vanishes for smooth and projective X and $i > 0$, hence Conjecture $P(n)$ follows. Conversely, Conjecture $P(n)$ implies Proposition 2.1 c), then 3.3 c), and finally 3.4 c) by using 2.1 a) and 3.3 a). □

Remark. Propositions 2.1, 3.4, 3.3 as well as Theorem 3.5 remain true if we restrict ourselves to schemes of dimension at most N, and to fields of transcendence degree at most N, for a fixed integer N.

Remark. Gillet announced that one can obtain a rational version of weight complexes by using de Jong's theorem on alterations instead of resolution of singularities. The same argument should then give the generalization [11, Theorem 5.13]. In this case, all arguments of this section hold true rationally, except the proof of c) ⇒ b) in Propositions 3.3 and 3.4, and the proof of $P(n) \Rightarrow A(n), B(n), C(n)$ in Theorem 3.5, which require that every finitely generated field over $\mathbb{F}_q$ has a smooth and projective model.

4 The case $n = 0$

Since $H^c_i(X, \mathbb{Q}(0)) = CH_0(X, i)_{\mathbb{Q}}$, we use higher Chow groups in this section.

Proposition 4.1. *We have $CH_0(X, i)_{\mathbb{Q}} \cong \tilde{H}^c_i(X, \mathbb{Q}(0))$ for $i \leq 2$, and the map $CH_0(X, 3)_{\mathbb{Q}} \to \tilde{H}^c_3(X, \mathbb{Q}(0))$ is surjective for all X. In particular, $A(0)$ holds in dimensions at most 2.*

Proof. Since $H^i(k(x), \mathbb{Q}(1)) = 0$ for $i \neq 1$ and $H^i(k(x), \mathbb{Q}(0)) = 0$ for $i \neq 0$, this follows from an inspection of the niveau spectral sequence. □

Jannsen [11] defines a variant of weight homology with coefficients A, $H^W_i(X, A)$, as the homology of the complex $\mathrm{Hom}(CH^0(W(X)), A)$. Note that $H^W_i(X, \mathbb{Q}) = H^W_i(X, \mathbb{Q}(0))$ because $\mathrm{Hom}(CH^0(X), \mathbb{Q}) \cong CH_0(X)_{\mathbb{Q}}$ for smooth and projective X, in a functorial way. Indeed, a map of connected, smooth and projective varieties induces the identity pull-back on CH^0 and the identity push-forward on CH_0.

Theorem 4.2 (Jannsen). *Under resolution of singularities, $H^W_a(k, A) = 0$ for $a \neq$ trdeg $k/\mathbb{F}_q$, hence $H^W_i(X, A)$ is the homology of the complex*

$$0 \leftarrow \bigoplus_{x\in X_{(0)}} H^W_0(k(x), A) \leftarrow \cdots \leftarrow \bigoplus_{x\in X_{(s)}} H^W_s(k(x), A) \leftarrow \cdots \tag{5}$$

for all schemes X. In particular, Conjecture $C(0)$ holds.

This is proved in [11, Proposition 5.4, Theorem 5.10]. The proof only works for $n = 0$, because it uses the bijectivity of $CH^0(Y) \to CH^0(X)$ for a map of connected smooth and projective schemes $X \to Y$. The second statement follows using the niveau spectral sequence, which exists because $H^W_i(X, A)$ satisfies the localization property by property (4) of weight complexes.

Let $\mathbb{Z}^c(0)$ be the complex of etale sheaves $z_0(-, *)$. For any prime l, consider l-adic cohomology

$$H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l) := \mathbb{Q} \otimes_{\mathbb{Z}} \lim H_i(X_{\text{et}}, \mathbb{Z}^c/l^r(0)).$$

In [4], we showed that for every positive integer m, and every scheme $f : X \to k$ over a perfect field, there is a quasi-isomorphism $\mathbb{Z}^c/m(0) \cong Rf^!\mathbb{Z}/m$. In particular, the above definition agrees with the usual definition of l-adic homology if $l \neq p = \text{char}\,\mathbb{F}_q$. If $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and $\widehat{G} = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, then there is a short exact sequence

$$0 \to H_{i+1}(\bar{X}_{\text{et}}, \widehat{\mathbb{Q}}_l)_{\hat{G}} \to H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l) \to H_i(\bar{X}_{\text{et}}, \widehat{\mathbb{Q}}_l)^{\hat{G}} \to 0$$

and for U affine and smooth, $H_i(U_{\text{et}}, \widehat{\mathbb{Q}}_l)$ vanishes for $i \neq d, d-1$ and $l \neq p$ by the affine Lefschetz theorem and a weight argument [12, Theorem 3a)]. The map from Zariski-hypercohomology of $\mathbb{Z}^c/m(0)$ to etale-hypercohomology of $\mathbb{Z}^c/m(0)$ induces a functorial map

$$CH_0(X, i)/m \to CH_0(X, i, \mathbb{Z}/m) \to H_i(X_{\text{et}}, \mathbb{Z}^c/m(0)),$$

hence in the limit a map

$$\omega\colon CH_0(X, i)_{\mathbb{Q}_l} \to H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l).$$

Similarly, the map $\bar{\omega}\colon CH_0(\bar{X}, i)_{\mathbb{Q}_l} \to H_i(\bar{X}_{\text{et}}, \widehat{\mathbb{Q}}_l)$ induces a map

$$\tau\colon CH_0(X, i+1)_{\mathbb{Q}_l} \xleftarrow{\sim} (CH_0(\bar{X}, i+1)_{\mathbb{Q}_l})_{\hat{G}} \to H_{i+1}(\bar{X}_{\text{et}}, \widehat{\mathbb{Q}}_l)_{\hat{G}} \to H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l)$$

(the left map is an isomorphism by a trace argument). The sum

$$\varphi^i_X\colon CH_0(X, i)_{\mathbb{Q}_l} \oplus CH_0(X, i+1)_{\mathbb{Q}_l} \to H_i(X_{\text{et}}, \widehat{\mathbb{Q}}_l) \tag{6}$$

is compatible with localization sequences, because all maps involved in the definition are. The following proposition shows that Parshin's conjecture can be recovered from and implies a structure theorem for higher Chow groups of smooth affine schemes; compare to Jannsen [11, Conjecture 12.4b)].

Proposition 4.3. *The following statements are equivalent:*

a) *Conjecture* $P(0)$.

b) *For all schemes* X*, and all* i*, the map* φ_X^i *is an isomorphism.*

c) *For all smooth and affine schemes* U *of dimension* d*, the groups* $CH_0(U,i)$ *are torsion for* $i \neq d$*, and the composition* $\omega\colon CH_0(U,d)_{\mathbb{Q}_l} \to H_d(U_{\text{et}},\widehat{\mathbb{Q}}_l) \to H_d(\bar{U}_{\text{et}},\widehat{\mathbb{Q}}_l)^{\hat{G}}$ *is an isomorphism.*

Proof. a) $\Rightarrow$ b): First consider the case that X is smooth and proper. Then Conjecture P(0) is equivalent to the vanishing of the left-hand side of (6) for $i \neq 0,-1$, whereas the right-hand side of (6) vanishes by the Weil-conjectures. On the other hand, φ_X^0 induces an isomorphism $CH_0(X)\otimes\mathbb{Q}_l \cong H^{2d}(X_{\text{et}},\widehat{\mathbb{Q}}_l(d))$ and φ_X^{-1} induces an isomorphism $CH_0(X)\otimes\mathbb{Q}_l \cong (CH_0(\bar{X})\otimes\mathbb{Q}_l)_{\hat{G}} \cong H^{2d}(\bar{X},\widehat{\mathbb{Q}}_l(d))_{\hat{G}}$. Indeed, both sides are isomorphic to $\mathbb{Q}_l$ if X is connected. Using localization and alterations, the statement for smooth and proper X implies the statement for all X.

b) $\Rightarrow$ a): The right-hand side of (6) is zero for $i \neq 0,-1$ by weight reasons for smooth and projective X, hence so is the left side.

b) $\Rightarrow$ c): This follows because $H_i(U_{\text{et}},\widehat{\mathbb{Q}}_l) = 0$ unless $i = d, d-1$ for smooth and affine U.

c) $\Rightarrow$ b): We first assume that X is smooth and affine. By hypothesis and the affine Lefschetz theorem, both sides of (6) vanish for $i \neq d, d-1$, and are isomorphic for $i = d$. For $i = d-1$, the vertical maps in the following diagram are isomorphisms by semi-simplicity,

$$\begin{array}{ccccc}
CH_0(X,d)_{\mathbb{Q}_l} \cong CH_0(\bar{X},d)_{\mathbb{Q}_l}^{\hat{G}} & \longrightarrow & H_d(\bar{X},\widehat{\mathbb{Q}}_l)^{\hat{G}} & & \\
\| & & \| & & \\
(CH_0(\bar{X},d)_{\mathbb{Q}_l})_{\hat{G}} & \longrightarrow & H_d(\bar{X},\widehat{\mathbb{Q}}_l)_{\hat{G}} & \xrightarrow{\sim} & H_{d-1}(X,\widehat{\mathbb{Q}}_l).
\end{array}$$

Hence the lower map φ_X^{d-1} is an isomorphism because the upper map is. Using localization, the statement for smooth and affine X implies the statement for all X. $\square$

Proposition 4.4. *Under resolution of singularities, the following are equivalent:*

a) *Conjecture* $P(0)$.

b) *For every smooth affine* U *of dimension* d *over* $\mathbb{F}_q$*, we have* $CH_0(U,i)_{\mathbb{Q}} \cong H_i^W(U,\mathbb{Q})$ *for all* i*, and these group vanish for* $i \neq d$.

c) *For every smooth affine* U *of dimension* d *over* $\mathbb{F}_q$*, the groups* $CH_0(U,i)_{\mathbb{Q}}$ *vanish for* $i > d$*, and* $CH_0(U,d)_{\mathbb{Q}} \cong H_d^W(U,\mathbb{Q})$.

Proof. a) $\Rightarrow$ b): It follows from the previous proposition that $CH_0(U,i)_{\mathbb{Q}} = 0$ for $i \neq d$. On the other hand, $P(0)$ for all X implies that $CH_0(X,i) \cong H_i^W(X,\mathbb{Q})$ for all i and X.

c) $\Rightarrow$ a): The statement implies Conjecture $A(0)$, and then Conjecture $B(0)$, version b), for all X. By Theorems 3.5 and 4.2, $P(0)$ follows. $\square$

Proposition 4.5. *Conjecture $P(0)$ for all smooth and projective X implies the following statements:*

a) (Affine Gersten) *For every smooth affine U of dimension d, the following sequence is exact:*

$$CH_0(U,d)_{\mathbb{Q}} \hookrightarrow \bigoplus_{x\in U^{(0)}} H^d(k(x),\mathbb{Q}(d)) \to \bigoplus_{x\in U^{(1)}} H^{d-1}(k(x),\mathbb{Q}(d-1)) \to \cdots .$$

b) *Let $X = X_d \supseteq X_{d-1} \supseteq \cdots X_1 \supseteq X_0$ be a filtration such that $U_i = X_i - X_{i-1}$ is smooth and affine of dimension i. Then $CH_0(X,i)_{\mathbb{Q}}$ is isomorphic to the ith homology of the complex*

$$0 \to CH_0(U_d,d)_{\mathbb{Q}} \to CH_0(U_{d-1},d-1)_{\mathbb{Q}} \to \cdots \to CH_0(U_0,0)_{\mathbb{Q}} \to 0.$$

The maps $CH_0(U_i,i)_{\mathbb{Q}} \to CH_0(X_{i-1},i-1)_{\mathbb{Q}} \to CH_0(U_{i-1},i-1)_{\mathbb{Q}}$ arise from the localization sequence.

Proof. a) follows because the spectral sequence (1) collapses, and b) by a diagram chase. □

Remark. If we fix a smooth scheme X of dimension d, and use cohomological notation, then by Proposition 2.1 and the Gersten resolution, we get that the rational motivic complex $\mathbb{Q}(d)$ is conjecturally concentrated in degree d, say $C_d = \mathcal{H}^d(\mathbb{Q}(d)) = CH_0(-,d)_{\mathbb{Q}} = H^W_d(-,\mathbb{Q})$. Then Conjecture $L(0)$ says that $H^i(U,C_d) = 0$ for $U \subseteq X$ affine and $i > 0$. This is analog to the mod p situation, where the motivic complex agrees with the logarithmic de Rham–Witt sheaf $\mathbb{Z}/p(n) \cong \nu^d[-d]$, and $H^i(U_{\text{et}},\nu^d) = 0$ for $U \subseteq X$ affine and $i > 0$. The latter can be proved by writing ν^d as the kernel of a map of coherent sheaves and using the vanishing of cohomology of coherent sheaves on affine schemes. This suggest that one might try to do the same for C_d.

4.1 Frobenius action. Let $F\colon X \to X$ be the Frobenius morphism induced by the qth power map on the structure sheaf.

Theorem 4.6. *The push-forward F_* acts like q^n on $CH_n(X,i)$, and the pull-back F^* acts on $H^i(X,\mathbb{Z}(n))$ as q^n for all n.*

The theorem is well known, but we could not find a proof in the literature. The proof of Soulé [14, Proposition 2] for Chow groups does not carry over to higher Chow groups, because the Frobenius does not act on the simplices Δ^n, hence a cycle $Z \subseteq \Delta^n \times X$ is not send to a multiple of itself by the Frobenius. We give an argument due to M. Levine.

Proof. Let DM^- be Voevodsky's derived category of bounded above complexes of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves. Then we have the isomorphisms

$$CH_n(X,i) \cong \mathrm{Hom}_{DM^-}(\mathbb{Z}(n)[2n+i], M_c(X)),$$
$$H^i(X,\mathbb{Z}(n)) \cong \mathrm{Hom}_{DM^-}(M(X), \mathbb{Z}(n)[i]).$$

The action of the Frobenius is given by composition with $F\colon M_c(X) \to M_c(X)$ and $F\colon M(X) \to M(X)$, respectively.

The Frobenius acts on the category DM^-, i.e. for every $\alpha \in \mathrm{Hom}_{DM^-}(X,Y)$ we have $F_Y \circ \alpha = \alpha \circ F_X$. This follows by considering composition of correspondences. Hence it suffices to calculate the action of the Frobenius on $\mathbb{Z}(n)$, i.e. show that $F = q^n \in \mathrm{Hom}_{DM^-}(\mathbb{Z}(n),\mathbb{Z}(n)) \cong \mathbb{Z}$. But $\mathrm{Hom}_{DM^-}(\mathbb{Z}(n),\mathbb{Z}(n))$ is a direct factor of $\mathrm{Hom}_{DM^-}(\mathbb{Z}(n),\mathbb{P}^n[-2n]) = CH_n(\mathbb{P}^n)$. The latter is the free abelian group generated by the generic point, and the Frobenius acts by q^n on it. □

Remark. 1) It would be interesting to write down an explicit chain homotopy between F_* and q^n on $z_n(X,*)$.

2) The proposition implies that the groups $CH_n(\mathbb{F}_q,i)$ are killed by q^n-1, and that $CH_n(X,i)$ is q-divisible for $n<0$.

Granted the theorem, the standard argument gives the following corollary, see also Jannsen [10, Theorem 12.5.7].

Corollary 4.7. *Assume that for all smooth and projective X of dimension d, the kernel of the map $CH^d(X\times X) \to \mathrm{End}_{\mathrm{hom}}(M(X))$ is nilpotent. Then Conjecture $P(0)$ holds.*

The hypothesis of the corollary is satisfied if X is finite dimensional in the sense of Kimura [13].

Proof. Using the existence of a zero-cycle c of degree 1, we see that the projector $\pi_{2d} = [X\times c]$ is defined. Let $\tilde{X}$ be the motive $\ker \pi_{2d} = X/\mathcal{L}^d$, where $\mathcal{L}$ is the Lefschetz motive. Consider the action of the geometric Frobenius $F \in \mathrm{End}_{\mathrm{rat}}(\tilde{X}) \subseteq CH^d(X\times X)$. Its image in the category of motives for homological equivalence is algebraic, and its minimal polynomial $P_{\tilde{X}}(T)$ has roots of absolute value $q^{\frac{j}{2}}$ for $0 \le j < 2d$. By hypothesis, $P_{\tilde{X}}(F)^a = 0$ in $\mathrm{End}_{\mathrm{rat}}(\tilde{X})$ for some integer a, but by the theorem, F^* acts like q^d on $CH^d(X,i)$. Hence $0 \neq P_{\tilde{X}}(q^d)^a = P_{\tilde{X}}(F^*)^a = 0$. □

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Axioms for the norm residue isomorphism

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Introduction

The purpose of this paper is to give an axiomatic framework for proving (by induction on n) that the norm residue map $K_n^M(k)/\ell \to H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ is an isomorphism, for any prime $\ell > 2$ and for any field k such that ℓ is invertible in k.

Fix ℓ and n. By a *Rost variety* for a sequence $\underline{a} = (a_1, \dots, a_n)$ of units in k, we shall mean a smooth projective variety X of dimension $d = \ell^{n-1} - 1$ satisfying the conditions of [8, 6.3] or [6, (0.1)]; the exact definition is given in Definition 1.1 below. One key requirement is that $\{a_1, \dots, a_n\}$ vanishes in $K_n^M(k(X))/\ell$. Rost constructed such a variety in his 1998 preprint [3]; the proof that it satisfies these properties was published in [1] and [6].

Theorem 0.1. *Let n and ℓ be such that the norm residues maps are isomorphisms for all $i < n$. Suppose that for every sequence $\underline{a}$ there is a direct summand M of the motive of a Rost variety X, satisfying the Axioms given in* 0.3 *below.*

Then the norm residue map $K_n^M(k)/\ell \to H^n_{\acute{e}t}(k, \mu_\ell^{\otimes n})$ is an isomorphism.

To state the axioms, let $\mathfrak{X}$ denote the simplicial Čech scheme $p \mapsto X^{p+1}$; see [10, 9.1]. By abuse of notation, we will regard $\mathfrak{X}$ as a cochain complex, and hence as an element of $\mathbf{DM}^{\text{eff}}$. By [10, 9.2], $\mathfrak{X} \otimes \mathfrak{X} \simeq \mathfrak{X}$ and $X \otimes \mathfrak{X} \simeq X$.

There is a duality isomorphism $X^* \otimes \mathbb{L}^d \cong X$, where X^* is the dual Chow motive of X and $\mathbb{L}$ is the Lefschetz motive $\mathbb{Z}(1)[2]$; see [2, 20.11]. If M is a summand of the motive of X, the motivic structure map $X \xrightarrow{p} \mathbb{Z}$ induces a relative map $y\colon M \to \mathfrak{X}$; its $\mathfrak{X}$*-dual* Dy is defined to be the composite

$$Dy\colon \mathfrak{X} \otimes \mathbb{L}^d \to \mathbb{L}^d \xrightarrow{p^* \otimes \mathbb{L}^d} X^* \otimes \mathbb{L}^d \to M^* \otimes \mathbb{L}^d. \tag{0.2}$$

Axioms 0.3. (a) M is a direct summand of X. We write y for $M \to X \to \mathfrak{X}$.

(b) The evident duality map $M^* \otimes \mathbb{L}^d \to M$ is an isomorphism.

(c) There is a motive D, related to y by two distinguished triangles:

$$D \otimes \mathbb{L}^b \to M \xrightarrow{y} \mathfrak{X} \to, \tag{0.4}$$

$$\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{Dy} M \to D \to . \tag{0.5}$$

Here $b = d/(\ell - 1) = 1 + \ell + \cdots + \ell^{n-2}$, and Dy is defined in (0.2) via 0.3 (b).

*Weibel's research was supported by NSA grant MSPF-04G-184, and by the Oswald Veblen Fund.

Given (0.4) and axiom 0.3 (b), triangle (0.5) is equivalent to the duality assertion that $D^* \otimes \mathbb{L}^{d-b} \cong D$; (0.5) is isomorphic to $\mathbb{L}^d$ times the dual of triangle (0.4).

Remark 0.6. In the language of Chow motives, any direct summand $M = (X, e)$ of X has a dual motive $M^* = (X, e^t, d)$ and a transpose summand $M' = (X, e^t)$ of X. Thus the summand $M' = M^* \otimes \mathbb{L}^d$ of X is *a priori* different from M. Axiom 0.3 (b) says that $M' \to X \to M$ is an isomorphism. (The differences between these languages is partly due to the fact that the category of Chow motives embeds contravariantly into $\mathbf{DM}^{\text{eff}}$.)

Although we only use $\mathfrak{X}$-duals fleetingly in Lemma 3.5 below, they fit into a larger context, which we shall now quickly describe. For this, we need to assume that k admits resolution of singularities, so that A^* exists by [2, 20.3].

$\mathbf{DM}_{\mathfrak{X}}$ and $\mathfrak{X}$-duals 0.7. Let $\mathbf{DM}_{\mathfrak{X}}$ denote the full subcategory of $\mathbf{DM}$ consisting of objects M isomorphic to $A \otimes \mathfrak{X}$ for a geometric motive A over k. For this, we assume that k admits resolution of singularities, so that A^* exists by [2, 20.3]. As pointed out in [11, 6.11], $\operatorname{Hom}(M, B) \cong \operatorname{Hom}(M, B \otimes \mathfrak{X})$ for all B in $\mathbf{DM}^{\text{eff}}$. Thus

$$\operatorname{Hom}(U \otimes \mathfrak{X}, A^* \otimes \mathfrak{X}) = \operatorname{Hom}(U \otimes \mathfrak{X}, A^*) = \operatorname{Hom}(U \otimes \mathfrak{X} \otimes A, \mathbb{Z}) = \operatorname{Hom}(U \otimes M, \mathfrak{X}).$$

That is, $\underline{\operatorname{Hom}}_{\mathfrak{X}}(M, \mathfrak{X}) = A^* \otimes \mathfrak{X}$ is an internal Hom object from M to $\mathfrak{X}$ in the tensor category $\mathbf{DM}_{\mathfrak{X}}$. As such, $\underline{\operatorname{Hom}}_{\mathfrak{X}}(M, \mathfrak{X})$ is unique up to canonical isomorphism (cf. [11, 8.1]). Any map $M_1 \to M_2$ induces a map $\underline{\operatorname{Hom}}_{\mathfrak{X}}(M_2, \mathfrak{X}) \to \underline{\operatorname{Hom}}_{\mathfrak{X}}(M_1, \mathfrak{X})$ via

$$\operatorname{Hom}(A_1 \otimes \mathfrak{X}, A_2 \otimes \mathfrak{X}) \cong \operatorname{Hom}(A_1 \otimes \mathfrak{X} \otimes A_2^*, \mathfrak{X}) \cong \operatorname{Hom}(A_2^* \otimes \mathfrak{X}, A_1^* \otimes \mathfrak{X}).$$

It is easy to check that $\underline{\operatorname{Hom}}_{\mathfrak{X}}(-, \mathfrak{X})$ is a contravariant functor, and that it is a duality in the sense that $\underline{\operatorname{Hom}}_{\mathfrak{X}}(\underline{\operatorname{Hom}}_{\mathfrak{X}}(M, \mathfrak{X}), \mathfrak{X}) \cong M$.

In this paper we consider the $\mathfrak{X}$-dual $DM(-) = \underline{\operatorname{Hom}}_{\mathfrak{X}}(-, \mathfrak{X}) \otimes \mathbb{L}^d$, which also satisfies $D^2(M) \cong M$. Our choice of the twist is such that $D(X) \cong X$, $D(M)$ is the transpose M' of Remark 0.6, and Dy is given by (0.2).

This paper is an attempt to clarify the ending of the Voevodsky–Rost program to prove that the norm residue map is an isomorphism, and specifically the proof as presented in Voevodsky's 2003 preprint [8]. One important feature of our Theorem 0.1 is that only published results (and [3]) are used.

When $\ell = 2$, triangle (0.4) was constructed in [10, 4.4] using $D = \mathfrak{X}$, and (0.5) is its $\mathfrak{X}$-dual. Axioms 0.3 (a)–(b) hold by a result of Rost (cited as [10, 4.3]).

When $\ell > 2$, there are two different programs for constructing M. Both start by using the norm residue symbol of $\underline{a}$ to construct a nonzero element μ of $H^{2b+1,b}(\mathfrak{X})$. The construction of μ is given in [10, p. 95] and [8, (5.2)] (see 2.5 below).

In Voevodsky's approach, the triangles (0.4) and (0.5) are constructed in (5.5) and (5.6) of [8] with $M = M_{\ell-1}$ and $D = M_{\ell-2}$, starting with $\mu \in H^{2b+1,b}(\mathfrak{X})$. The duality axiom 0.3 (b) for M and D is given by [8, 5.7].

Unfortunately, there is a gap in the proof of Lemma [8, 5.15], which asserts that Axiom 0.3 (a) holds, i. e., that $M_{\ell-1}$ is a summand of X. That proof depends via

Theorems 3.8 and 4.4 of [8] upon Lemmas 2.2 and 2.3 of [8], whose proofs are currently unavailable. Instead, [8, 5.11] proves that the canonical map $X \to \mathfrak{X}$ factors through the map $y\colon M \to \mathfrak{X}$. Since this paper was written, the gap in the proof was patched in [12].

In Rost's approach, μ defines an equivalence class of idempotent endomorphisms e of X and Rost defines M to be the Chow motive (X, e). By construction, his M satisfies Axioms 0.3 (a)–(b). Hopefully it also satisfies Axiom 0.3 (c), the existence of triangles (0.4) and (0.5).

Notation. The integer n and the prime $\ell > 2$ will be fixed. We will work over a fixed field k in which ℓ is invertible. The integer d will always be $\ell^{n-1} - 1$ and b will always be $d/(\ell - 1) = 1 + \cdots + \ell^{n-2}$.

We fix the sequence of units $\underline{a} = (a_1, \ldots, a_n)$, and X will always denote a d-dimensional Rost variety relative to $\underline{a}$, satisfying Axioms 0.3.

We will work in the triangulated category of motives $\mathbf{DM}^{\text{eff}}$ described in [2]. The Lefschetz motive is $\mathbb{L} = \mathbb{Z}(1)[2]$. Unless explicitly stated otherwise, motivic cohomology will always be taken with coefficients $\mathbb{Z}_{(\ell)}$. The notation $H^{p,q}_{\text{ét}}(-)$ refers to the étale motivic cohomology $H^p_{\text{ét}}(-, \mathbb{Z}_{(\ell)}(q))$ defined in [2, 10.1].

Finally, I would like to thank the Institute for Advanced Study for allowing me to present these results in a seminar during the Fall of 2006, and Pierre Deligne for his continued interest and encouragement.

1 Proof of Theorem 0.1

Recall [6, 1.20] that a ν_{n-1}*-variety* over a field k is a smooth projective variety X of dimension $d = \ell^{n-1} - 1$, with $\deg s_d(X) \not\equiv 0 \pmod{\ell^2}$. Here $s_d(X)$ is the characteristic class of the tangent bundle T_X corresponding to the symmetric polynomial $\sum t_j^d$ in the Chern roots t_j of T_X; see [9, 14.3]. We also recall that $H_{-1,-1}(Y)$ is the group $\operatorname{Hom}(\mathbb{Z}, Y(1)[1])$, and is covariant in Y; see [2, 14.17].

Definition 1.1. A *Rost variety* for a sequence $\underline{a} = (a_1, \ldots, a_n)$ of units in k is a ν_{n-1}-variety such that: $\{a_1, \ldots, a_n\}$ vanishes in $K^M_n(k(X))/\ell$; for each $i < n$ there is a ν_i-variety mapping to X; and the motivic homology sequence

$$H_{-1,-1}(X^2) \xrightarrow{\pi_0^* - \pi_1^*} H_{-1,-1}(X) \to H_{-1,-1}(k) \quad (= k^\times) \tag{1.2}$$

is exact. As mentioned in the Introduction, Rost varieties exist for every $\underline{a}$ by [1], [6].

Example 1.2.1. If $n = 1$ and $L = k(\sqrt[\ell]{a_1})$, then $\operatorname{Spec}(L)$ is a Rost variety for a_1, with the convention that $s_0(X) = [L : k] = \ell$. When $n = 2$, the Severi–Brauer variety corresponding to the degree ℓ algebra with symbol $\underline{a}$ is a Rost variety.

Proof of Theorem 0.1. By [10, 6.10], it suffices to prove the assertion

$$H90(n, \ell)\colon \quad H^{n+1,n}_{\text{ét}}(k) = 0.$$

As pointed out in the proof of [10, 7.4], it suffices to show that for every symbol $\underline{a} \in K_n^M(k)$ there is a field extension $k \subset F$ such that $\underline{a}$ vanishes in $K_n^M(F)/\ell$ and $H_{\text{ét}}^{n+1,n}(k) \to H_{\text{ét}}^{n+1,n}(F)$ is an injection. When $F = k(X)$, this is the bottom right arrow in the flowchart (1.3), which is implicit in [8, 6.11].

$$
\begin{tikzcd}
H^{2+2b\ell,1+b\ell}(\mathfrak{X}) & H^{2d+1,d+1}(D) \arrow[l, "\text{onto}"', "1.4"] \arrow[r, phantom, "\underset{3.6}{=}"] & 0 \\
H^{n+1,n}(\mathfrak{X}) \arrow[u, "\text{into}", "2.4"'] \arrow[r, "4.4", "\text{exact}"'] & H_{\text{ét}}^{n+1,n}(k) \arrow[r, "\text{sequence}"'] & H_{\text{ét}}^{n+1,n}(k(X)).
\end{tikzcd}
\tag{1.3}
$$

The other decorations in (1.3) refer to the properties we need and where they are established. Proposition 4.4 states that the bottom row of the flowchart (1.3) is exact. By Lemma 2.4 the group $H^{n+1,n}(\mathfrak{X})$ injects (by a cohomology operation) into $H^{2+2b\ell,1+b\ell}(\mathfrak{X})$, which is in turn a quotient of $H^{2d+1,d+1}(D)$ by 1.4. In Corollary 3.6, we prove that $H^{2d+1,d+1}(D) = 0$. All of this implies that $H_{\text{ét}}^{n+1,n}(k) \to H_{\text{ét}}^{n+1,n}(k(X))$ is an injection, finishing the proof of Theorem 0.1. $\square$

One part of the flowchart (1.3) is easy to establish.

Lemma 1.4. *The map* $s\colon \mathfrak{X} \to D \otimes \mathbb{L}^b[1]$ *in* (0.4) *induces a surjection:*

$$H^{2d+1,d+1}(D) \cong H^{2b\ell+2,b\ell+1}(D \otimes \mathbb{L}^b[1]) \xrightarrow{s} H^{2b\ell+2,b\ell+1}(\mathfrak{X}).$$

Proof. In the cohomology exact sequence arising from (0.4), the next term is $H^{2b\ell+2,b\ell+1}(M)$, which is a summand of $H^{2b\ell+2,b\ell+1}(X)$. Because $b\ell = d + b > d$, this group is zero by the Vanishing Theorem [2, 3.6] – which says that $H^{n,i}(X) = 0$ whenever $n > i + \dim(X)$. $\square$

The lower left map in flowchart (1.3) is just the motivic-to-étale map by the following basic result, which we quote from [10, 7.3].

Lemma 1.5. *The structure map* $\mathfrak{X} \to \operatorname{Spec}(k)$ *induces isomorphisms*

$$H_{\acute{e}t}^{*,*}(k) \cong H_{\acute{e}t}^{*,*}(\mathfrak{X}) \quad \textit{and} \quad H_{\acute{e}t}^{*,*}(k;\mathbb{Z}/\ell) \cong H_{\acute{e}t}^{*,*}(\mathfrak{X};\mathbb{Z}/\ell).$$

2 Motivic cohomology operations

In the course of the proof, we will need some facts about the motivic cohomology operations constructed in [9]. When $\ell > 2$, there are operations P^i on $H^{*,*}(-;\mathbb{Z}/\ell)$ of bidegree $(2i(\ell-1), i(\ell-1))$, for each $i \geq 0$ (with $P^0 = 1$), as well as the Bockstein operation β of bidegree $(1,0)$. These satisfy the Adem relations given in [5] for the usual topological cohomology operations. In fact, the subring of all (stable) motivic cohomology operations generated by the P^i and β is isomorphic to the topologists' Steenrod algebra $\mathcal{A}^*$, developed in [5].

In addition, the motivic operations satisfy the usual Cartan formula $P^n(xy) = \sum P^i(x)P^{n-i}(y)$, $P^i(x) = x^\ell$ if $x \in H^{2i,i}(Y;\mathbb{Z}/\ell)$, and $P^i = 0$ on $H^{p,q}(Y;\mathbb{Z}/\ell)$ when (p,q) is in the region $q \le i$, $p < i + q$. These are proven in [9, 9.7–9].

Still assuming $\ell > 2$, the dual to the usual Steenrod algebra $\mathcal{A}^*$ is a graded-commutative algebra on generators ξ_i in (even) degrees $(2\ell^i - 2, \ell^i - 1)$ and τ_i in (odd) degrees $(2\ell^i - 1, \ell^i - 1)$; see [9, 12.6]. The dual to τ_i is the motivic cohomology operation Q_i, which has bidegree $(2\ell^i - 1, \ell^i - 1)$. Because it is true in $\mathcal{A}^*$, the Q_i are derivations which form an exterior subalgebra of all (stable) motivic cohomology operations. They may be inductively defined by $Q_0 = \beta$ and $Q_{i+1} = [P^{\ell^i}, Q_i]$.

We now quickly establish those portions of [8] that we need, concerning these operations on the cohomology of $\mathfrak{X}$ and the unreduced simplicial suspension $\Sigma\mathfrak{X}$ of $\mathfrak{X}$ (the cofiber of $\mathfrak{X} \to \mathrm{cone}(\mathfrak{X})$). The proofs we give are due to Voevodsky, and only depend upon [10] and [9]. In particular, they do not depend upon the missing lemmas in *op. cit.*, or upon the Axioms 0.3.

Fix $q < n$. Because we have assumed that the norm residue map is an isomorphism in weight q, and hence that $H90(q,\ell)$ holds, it follows from 1.5 that $H^{p,q}(k;\mathbb{Z}/\ell) \cong H^{p,q}_{\text{ét}}(k;\mathbb{Z}/\ell) \cong H^{p,q}_{\text{ét}}(\mathfrak{X};\mathbb{Z}/\ell)$ for $p \le q$, and that $H^{q+1,q}_{\text{ét}}(\mathfrak{X};\mathbb{Z}/\ell) = 0$. As observed in [8, 6.6], it then follows from [10, 6.9] that

$$H^{p,q}(\Sigma\mathfrak{X};\mathbb{Z}/\ell) = 0 \text{ when } (p,q) \text{ is in the region } q < n,\ p \le 1 + q. \tag{2.1}$$

Since X is a $\nu_{\le n-1}$-variety, Theorem [10, 3.2] translates to:

Theorem 2.2. *If $i < n$, the sequences $\xrightarrow{Q_i} H^{*,*}(\Sigma\mathfrak{X};\mathbb{Z}/\ell) \xrightarrow{Q_i}$ are exact.*

Remark 2.2.1. A slight generalization is stated in [8, 4.3].

Example 2.2.2. For $(p,q) = (n - 2\ell + 3, n - \ell + 1)$ we have the exact sequence

$$H^{p,q}(\Sigma\mathfrak{X};\mathbb{Z}/\ell) \xrightarrow{Q_1} H^{n+2,n}(\Sigma\mathfrak{X};\mathbb{Z}/\ell) \xrightarrow{Q_1} H^{n+2\ell+1,n+\ell-1}(\Sigma\mathfrak{X};\mathbb{Z}/\ell).$$

Because $p \le q \le n$, the left group is zero by (2.1). Thus the right map Q_1 is an injection on $H^{n+2,n}(\Sigma\mathfrak{X};\mathbb{Z}/\ell)$.

Lemma 2.3. *The motivic cohomology groups $H^{*,*}(\Sigma\mathfrak{X})$ have exponent ℓ.*

Proof. We may assume, by the usual transfer argument, that k has no extensions of degree prime to ℓ. In this case, the variety X has a k'-point for the field $k' = k(\sqrt[\ell]{a_1})$ of degree ℓ over k, by [6, 1.23]. It follows that $\mathfrak{X} \otimes \mathrm{Spec}\, k' \cong M(\mathrm{Spec}\, k')$, and hence that $\Sigma\mathfrak{X} \otimes \mathrm{Spec}(k') \cong 0$. Since the composition $\Sigma\mathfrak{X} \to \Sigma\mathfrak{X} \otimes \mathrm{Spec}(k') \to \Sigma\mathfrak{X}$ is multiplication by ℓ (by the usual transfer argument), the result follows. $\square$

For any $p > q$ we have $H^{p,q}(\mathrm{Spec}\, k) = 0$ and hence $H^{p,q}(\mathfrak{X}) \cong H^{p+1,q}(\Sigma\mathfrak{X})$. As a consequence of Lemma 2.3 we have that $H^{p,q}(\mathfrak{X}) \to H^{p,q}(\mathfrak{X};\mathbb{Z}/\ell)$ is an injection. By [10, 7.2], the cohomology operations Q_i preserve integral classes, so they induce integral operations on $H^{p,q}(\mathfrak{X})$ in this range.

Consider the cohomology operation $Q = Q_{n-1} \dots Q_2 Q_1$.

Lemma 2.4. *The operation* $Q : H^{n+1,n}(\mathfrak{X}; \mathbb{Z}/\ell) \to H^{2+2b\ell,1+b\ell}(\mathfrak{X}; \mathbb{Z}/\ell)$ *is an injection, and induces an injection*

$$H^{n+1,n}(\mathfrak{X}) \hookrightarrow H^{2+2b\ell,1+b\ell}(\mathfrak{X}).$$

Proof (Voevodsky). By the above remarks, it suffices to show that the operation Q from $H^{n+2,n}(\Sigma\mathfrak{X}; \mathbb{Z}/\ell)$ to $H^{3+2b\ell,1+b\ell}(\Sigma\mathfrak{X}; \mathbb{Z}/\ell)$ is injective. As illustrated in Example 2.2.2, it is easy to see from 2.2 and (2.1) that each Q_i is injective on the group $H^{*,*}(\Sigma\mathfrak{X}; \mathbb{Z}/\ell)$ containing $Q_{i-1} \dots Q_1 H^{n+2,n}(\Sigma\mathfrak{X}; \mathbb{Z}/\ell)$, because the preceding term in 2.2 is zero. □

Remark 2.5. The same argument, given in [8, 6.7], shows that $Q' = Q_{n-2} \dots Q_0$ is an injection from $H^{n,n-1}(\mathfrak{X}; \mathbb{Z}/\ell)$ to $H^{2b+1,b}(\mathfrak{X}) \subset H^{2b+1,b}(\mathfrak{X}; \mathbb{Z}/\ell)$.

If $\{a_1, \dots, a_n\} \neq 0$ in $K_n^M(k)/\ell$, Voevodsky shows in [8, 6.5] that its norm residue symbol in $H^n_{\text{ét}}(k, \mu_\ell^{\otimes n})$ lifts to a nonzero element $\delta \in H^{n,n-1}(\mathfrak{X}; \mathbb{Z}/\ell)$. Using injectivity of Q', we get a nonzero symbol $\mu = Q'(\delta) \in H^{2b+1,b}(\mathfrak{X})$. This symbol is the starting point of the construction of the motive M in both the program of Voevodsky [8] and that of Rost [4].

3 Motivic homology

In this section, we prove Corollary 3.6, which depends upon Axiom 0.3 (b) and exactness of (1.2) via results about the motivic homology of $\mathfrak{X}$ and M.

We will make repeated use of the following basic lemma. In this section, the notation $H_{-p,-q}(Y)$ refers to the group $\operatorname{Hom}(\mathbb{Z}, Y(q)[p])$; see [2, 14.17].

Lemma 3.1. *For every smooth* (*simplicial*) Y *and* $p > q$, $\operatorname{Hom}(\mathbb{Z}, Y(q)[p]) = 0$.

Proof. We have $\operatorname{Hom}(\mathbb{Z}, Y(q)[p]) \cong H^{p-q}_{\text{zar}}(k, C_*(Y \times \mathbb{G}_m^q)) = H^{p-q} C_*(Y \times \mathbb{G}_m^q)(k)$; see [2, 14.16]. The chain complex $C_*(Y \times \mathbb{G}_m^q)$ is zero in positive cohomological degrees, so the H^{p-q} group vanishes. □

Lemma 3.2. *The structural map* $H_{-1,-1}(\mathfrak{X}) \to H_{-1,-1}(k) = k^\times$ *is injective.*

Proof (Voevodsky). By Lemma 3.1, $\operatorname{Hom}(\mathbb{Z}, X^p(1)[n]) = 0$ for all $n \geq 2$ and all p. Therefore the right half-plane homological spectral sequence

$$E^1_{p,q} = \operatorname{Hom}(\mathbb{Z}, X^{p+1}(1)[-q]) \Rightarrow \operatorname{Hom}(\mathbb{Z}, \mathfrak{X}(1)[p-q])$$

has no nonzero rows below $q = -1$, and the row $q = -1$ yields the exact sequence

$$0 \leftarrow H_{-1,-1}(\mathfrak{X}) \leftarrow H_{-1,-1}(X) \leftarrow H_{-1,-1}(X \times X).$$

Since (1.2) is exact, this implies the result. □

Corollary 3.3. *The structural map* $H_{-1,-1}(M) \to H_{-1,-1}(k) = k^\times$ *is injective.*

Proof. By (0.4) and Lemma 3.2, it suffices to show that $\mathrm{Hom}(\mathbb{Z}, D(b+1)[2b+1]) = 0$. By (0.5) this results from the vanishing of both $\mathrm{Hom}(\mathbb{Z}, M(b+1)[2b+1])$ and $\mathrm{Hom}(\mathbb{Z}, \mathfrak{X} \otimes \mathbb{L}^{b+d+1})$ which follows from Lemma 3.1. □

Lemma 3.4. $H^{0,1}(\mathfrak{X}) = H^{2,1}(\mathfrak{X}) = 0$ *and* $H^{1,1}(\mathfrak{X};\mathbb{Z}) \cong H^{1,1}(\mathrm{Spec}\, k;\mathbb{Z}) \cong k^\times$.

Proof. The spectral sequence $E_1^{p,q} = H^q(X^{p+1};\mathbb{Z}(1)) \Rightarrow H^{p+q,1}(\mathfrak{X};\mathbb{Z})$ degenerates, all rows vanishing except for $q = 1$ and $q = 2$, because $\mathbb{Z}(1) \cong \mathcal{O}^\times[-1]$; see [2, 4.2]. We compare this with the spectral sequence converging to $H^{p+q}(\mathfrak{X};\mathbb{G}_m)$; $H^q_{\mathrm{zar}}(Y, \mathcal{O}^\times) \to H^q_{\text{ét}}(-, \mathbb{G}_m)$ is an isomorphism for $q = 0, 1$ (and an injection for $q = 2$). Hence for $q \le 2$ we have $H^{q,1}(\mathfrak{X}) = H^{q,1}_{\text{ét}}(\mathfrak{X}) = H^{q,1}_{\text{ét}}(k) = H^{q-1}_{\text{ét}}(k, \mathbb{G}_m)$. □

Remark. The proof of 3.4 also shows that $H^{3,1}(\mathfrak{X})$ injects into $H^2_{\text{ét}}(k, \mathbb{G}_m) = \mathrm{Br}(k)$.

Lemma 3.5. $H^{2d+1,d+1}(D;\mathbb{Z})$ *is the kernel of* $H_{-1,-1}(M) \xrightarrow{y} H_{-1,-1}(k) = k^\times$.

Proof. From (0.5) we get an exact sequence with coefficients $\mathbb{Z}$:

$$H^{2d,d+1}(\mathfrak{X}\otimes\mathbb{L}^d) \to H^{2d+1,d+1}(D) \to H^{2d+1,d+1}(M) \xrightarrow{Dy} H^{2d+1,d+1}(\mathfrak{X}\otimes\mathbb{L}^d).$$

The first group is $H^{0,1}(\mathfrak{X})$, which is zero by 3.4, so it suffices to show that the map Dy identifies with the structural map y. This follows from Axiom 0.3 (b), because for any u in $H_{-1,-1}(M) = \mathrm{Hom}(\mathbb{Z}, M(1)[1])$, $\mathfrak{X}$ tensored with the composite

$$\mathfrak{X} \otimes \mathbb{L}^d \xrightarrow{Dy} M^* \otimes \mathbb{L}^d \xrightarrow{u^*} \mathbb{Z}(1)[1] \otimes \mathbb{L}^d = \mathbb{Z}(d+1)[2d+1]$$

is the $\mathfrak{X}$-dual of $\mathfrak{X}(-1)[-1] \xrightarrow{u} M \xrightarrow{y} \mathfrak{X}$. □

Corollary 3.6. $H^{2d+1,d+1}(D) = H^{2d+1,d+1}(D;\mathbb{Z}) \otimes \mathbb{Z}_{(\ell)} = 0$.

4 Between motivic and étale cohomology

Definition 4.1 ([10, p. 90]). Let $L(n)$ denote the truncation $\tau^{\le n+1}\mathbb{Z}^{\text{ét}}_{(\ell)}(n)$ of the complex in $\mathbf{DM}^{\mathrm{eff}}$ representing étale motivic cohomology; *i. e.*, $H^p(-, L(n)) \cong H^{p,n}_{\text{ét}}(-)$ for $p \le n+1$. Let $K(n)$ denote the mapping cone of the canonical map $\mathbb{Z}_{(\ell)}(n) \to L(n)$, and consider the triangle $\mathbb{Z}_{(\ell)}(n) \to L(n) \to K(n) \to \mathbb{Z}_{(\ell)}(n)[1]$.

Lemma 4.2. *The map* $H^{n+1}(\mathfrak{X}, K(n)) \xrightarrow{y} H^{n+1}(M, K(n))$ *is an injection.*

Proof. By triangle (0.4) we have an exact sequence:

$$H^n(D \otimes \mathbb{L}^b, K(n)) \to H^{n+1}(\mathfrak{X}, K(n)) \xrightarrow{y} H^{n+1}(M, K(n)).$$

We need to see that the left term vanishes. By (0.5), it is sufficient to show that $H^n(M \otimes \mathbb{L}^b, K(n))$ and $H^{n-1}(\mathfrak{X} \otimes \mathbb{L}^{b\ell}, K(n))$ vanish. This follows from [10, 6.12], which says that $H^*(Y(1), K(n)) = 0$ for every smooth Y, the assumption that M is a summand of X, and the consequent collapsing of the spectral sequence $E_1^{p,q} = H^q(X^p, K(n)) \Rightarrow H^{p+q}(\mathfrak{X}, K(n))$. □

Corollary 4.3. *The map $H^{n+1}(\mathfrak{X}, K(n)) \to H^{n+1}(k(X), K(n))$ is an injection.*

Proof. The map $H^{n+1}(X, K(n)) \to H^{n+1}(k(X), K(n))$ is an injection by [2, 11.1, 13.8, 13.10], or by [10, 7.4]. The corollary follows from Lemma 4.2, since $H^*(\mathfrak{X}, -) \to H^*(M, -)$ is a summand of $H^*(\mathfrak{X}, -) \to H^*(X, -)$. □

Proposition 4.4. *There is an exact sequence*

$$H^{n+1,n}(\mathfrak{X}) \to H^{n+1,n}_{\acute{e}t}(k) \to H^{n+1,n}_{\acute{e}t}(k(X)).$$

Proof. By 1.5, $\mathfrak{X} \to \mathrm{Spec}(k)$ is an isomorphism on étale motivic cohomology, so we have $H^{n+1,n}_{\text{ét}}(k) \cong H^{n+1}(\mathfrak{X}, L(n))$.

The map $\mathrm{Spec}\, k(X) \to \mathfrak{X}$ induces a commutative diagram with exact rows:

$$\begin{array}{ccccc}
H^{n+1,n}(\mathfrak{X}) & \longrightarrow & H^{n+1}(\mathfrak{X}, L(n)) & \longrightarrow & H^{n+1}(\mathfrak{X}, K(n)) \\
\downarrow & & \downarrow & & {\scriptstyle 4.3}\downarrow{\scriptstyle \text{into}} \\
0 = H^{n+1,n}(k(X)) & \longrightarrow & H^{n+1}(k(X), L(n)) & \longrightarrow & H^{n+1}(k(X), K(n)).
\end{array}$$

By 4.3, the right vertical map is an injection. The proposition now follows by a diagram chase. □

Acknowledgements. This paper is an attempt to clarify Voevodsky's preprint [8]. It is directly inspired by the work of Markus Rost, especially [4], who explicitly suggested the line of attack, based upon Axiom 0.3 (c), which is presented in this paper. I am greatly indebted to both of them.

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