4

Categorical Galois theory of commutative rings

Convention *In this chapter, all rings and algebras are commutative and have a unit.*

In this chapter, we propose an approach to Galois theory of rings originally inspired by the work of A.R. Magid [67], but put in a categorical presentation which will make transparent the generalization of chapter 5. In particular we make an extensive use of the Pierce spectrum functor and its adjoint, which relate the category of commutative rings with that of profinite topological spaces, and then naturally bring in profinite topological groupoids.

4.1 Stone duality

We shall now prove that the category of profinite topological spaces is dual to the category of boolean algebras. We first fix the terminology and recall some elementary facts.

Definition 4.1.1 Let B be a boolean algebra.

(i) A filter in B is a subset $F \subseteq B$ such that

(F1) $1 \in F$,

(F2) $x \in F$ and $y \in F \Rightarrow x \wedge y \in F$,

(F3) $x \in F$ and $x \leq y \Rightarrow y \in F$.

(ii) The filter F is proper when, moreover,

(F4) $0 \notin F$.

(iii) An ultrafilter is a maximal element of the poset of proper filters, ordered by inclusion.

The notion of *ideal* is dual to that of filter. More precisely,

Definition 4.1.2 Let B be a boolean algebra.

(i) An ideal in B is a subset $I \subseteq B$ such that

(I1) $0 \in I$,
(I2) $x \in I$ and $y \in I \Rightarrow x \vee y \in I$,
(I3) $x \in I$ and $x \geq y \Rightarrow y \in I$.

(ii) The ideal I is proper when, moreover,

(I4) $1 \notin I$.

(iii) An ideal is maximal when it is a maximal element in the poset of proper ideals, ordered by inclusion.

In a boolean algebra B, we shall write $\complement x$ for the complement of an element $x \in B$.

Proposition 4.1.3 *Let $F \subseteq B$ be a proper filter of a boolean algebra B. The following conditions are equivalent.*

(i) *F is an ultrafilter;*
(ii) *$\forall x \in B \quad x \in F$ or $\complement x \in F$;*
(iii) *$\forall x, y \in B \quad x \vee y \in F \Rightarrow x \in F$ or $y \in F$;*
(iv) *there exists a homomorphism of boolean algebras $f\colon B \longrightarrow \{0,1\}$ such that $F = f^{-1}(1)$.*

Proof (i) $\Rightarrow$ (ii) If $x \notin F$, then $G = \{z \mid x \vee z \in F\}$ is a proper filter which contains F, thus $G = F$ and $\complement x \in G = F$.

(ii) $\Rightarrow$ (iii) If $x \vee y \in F$ and $x \notin F$, one has $\complement x \in F$ and thus $y \wedge \complement x = (x \vee y) \wedge \complement x \in F$, from which $y \in F$.

(iii) $\Rightarrow$ (iv) Putting $f(x) = 1$ iff $x \in F$, f preserves $\wedge$, $\leq$, 1 because F is a filter and 0 because F is proper. Condition (iii) expresses the preservation of $\vee$. The preservation of $\complement$ follows at once from that of 0, 1, $\wedge$ and $\vee$.

(iv) $\Rightarrow$ (i) Since f preserves $\complement$, condition (ii) is satisfied. But then if G is a filter containing F, given $x \in G \setminus F$ one has $\complement x \in F$ and thus $0 = x \wedge \complement x \in G$, which proves $G = B$. □

Proposition 4.1.4 *In a boolean algebra B,*

(i) *every proper filter is contained in an ultrafilter,*
(ii) *every non-zero element belongs to an ultrafilter,*
(iii) *$x \not\leq y \Rightarrow \exists F$ ultrafilter, with $x \in F$, $y \notin F$,*
(iv) *every filter is the intersection of the ultrafilters which contain it.*

Proof Proper filters containing a given filter constitute an inductive poset, from which condition (i) holds by Zorn's lemma. Applying condition (i) to

$$\uparrow x = \{y \mid y \geq x\}$$

one gets condition (ii). Applying condition (ii) to $x \wedge \complement y$, one gets condition (iii); this implies condition (iv) for proper filters. Condition (iv) for the trivial filter B is obvious, since the intersection becomes that of the empty family and thus is equal to the whole space. □

Proposition 4.1.5 *Given a boolean algebra B, let us write $\mathsf{Spec}(B)$ for the set of its ultrafilters. For every filter H on B, consider*

$$\mathcal{O}_H = \left\{F \in \mathsf{Spec}(B) \middle| H \not\subseteq F\right\}.$$

The subsets $\mathcal{O}_H \subseteq \mathsf{Spec}(B)$ constitute a topology $\mathcal{T}$ on $\mathsf{Spec}(B)$. The map

$$\mathcal{O}\colon \mathsf{Filters}(B) \longrightarrow \mathcal{T}, \quad H \mapsto \mathcal{O}_H$$

is an isomorphism of posets.

Proof The map $\mathcal{O}$ is surjective by definition and injective by condition (iv) of proposition 4.1.4. It is a homomorphism of posets, since given filters $H \subseteq H'$, one has at once $\mathcal{O}_H \subseteq \mathcal{O}_{H'}$. Conversely $\mathcal{O}_H \subseteq \mathcal{O}_{H'}$, for arbitrary filters H and H', implies $H \subseteq H'$, again by condition (iv) of proposition 4.1.4. □

Corollary 4.1.6 *Given a boolean algebra B, the following conditions hold:*

(i) $\mathcal{O}_{\{1\}} = \emptyset$ *and* $\mathcal{O}_B = \mathsf{Spec}(B)$,
(ii) $\mathcal{O}_{H \cap H'} = \mathcal{O}_H \cap \mathcal{O}_{H'}$,
(iii) $\mathcal{O}_{\langle \bigcup_{i\in I} H_i\rangle} = \bigcup_{i\in I} \mathcal{O}_{H_i}$,

where H, H' and the H_i are filters in B, and $\langle \bigcup_{i\in I} H_i\rangle$ denotes the filter generated by the set theoretical union of the filters H_i.

Proof Any isomorphism of posets preserves the top and bottom elements, arbitrary infima and suprema. In $\mathcal{T}$, finite infima and arbitrary suprema are computed set theoretically. □

Definition 4.1.7 Given a boolean algebra B, the space $(\mathsf{Spec}(B), \mathcal{T})$ of proposition 4.1.5 is called the spectrum of B.

Lemma 4.1.8 *Let B be a boolean algebra. The subsets*

$$\forall b \in B \quad \mathcal{O}_b = \mathcal{O}_{\uparrow b} = \{F \in \mathsf{Spec}(B) \big| b \notin F\}$$

are both open and closed and constitute a base of open subsets for the topology of $\mathsf{Spec}(B)$. *We write $U_b = \mathcal{O}_{\complement b}$ for the complement of $\mathcal{O}_b$.*

Proof For every filter $H \subseteq B$, $H = \bigcup_{b \in H} \uparrow b$, from which $\mathcal{O}_H = \bigcup_{b \in H} \mathcal{O}_b$ by condition (iii) of corollary 4.1.6. Moreover condition (iii) of proposition 4.1.3 can be rephrased as $\mathcal{O}_b \cap \mathcal{O}_{b'} = \mathcal{O}_{b \vee b'}$, proving that the $\mathcal{O}_b$ are stable under finite intersections. Finally condition (ii) of proposition 4.1.3 indicates that $U_b = \mathcal{O}_{\complement b} = \complement \mathcal{O}_b$, proving that each open subset $\mathcal{O}_b$ is also closed. □

Lemma 4.1.9 *Let B be a boolean algebra. One has*

(i) $b \leq b' \Rightarrow \mathcal{O}_b \supseteq \mathcal{O}_{b'}$,
(ii) $b \neq b' \Rightarrow \mathcal{O}_b \neq \mathcal{O}_{b'}$,
(iii) $\mathcal{O}_{b \wedge b'} = \mathcal{O}_b \cup \mathcal{O}_{b'}$,
(iv) $\mathcal{O}_{b \vee b'} = \mathcal{O}_b \cap \mathcal{O}_{b'}$,
(v) $\mathcal{O}_1 = \emptyset, \quad \mathcal{O}_0 = B$,
(vi) $\mathcal{O}_{\complement b} = \complement \mathcal{O}_b$,

for all elements $b, b' \in B$.

Proof With the notation of 4.1.6 and by proposition 4.1.3, observe that

$$\langle \uparrow b \, \cup \, \uparrow b' \rangle = \uparrow (b \wedge b'), \qquad \uparrow b \, \cap \, \uparrow b' = \uparrow (b \vee b')$$

and apply proposition 4.1.5. □

Corollary 4.1.10 *Let B be a boolean algebra. Using the alternative notation $U_b = \mathcal{O}_{\complement b}$, one gets the corresponding covariant formulæ between basic open-closed subsets:*

(i) $b \leq b' \Rightarrow U_b \subseteq U_{b'}$;
(ii) $b \neq b' \Rightarrow U_b \neq U_{b'}$;
(iii) $U_{b \wedge b'} = U_b \cap U_{b'}$;
(iv) $U_{b \vee b'} = U_b \cup U_{b'}$;
(v) $U_1 = B, \quad U_0 = \emptyset$;
(vi) $U_{\complement b} = \complement U_b$;

for all elements $b, b' \in B$. □

Proposition 4.1.11 *The spectrum of a boolean algebra is a profinite space.*

Proof If $F \neq F'$ are distinct ultrafilters, the maximality of F implies at once $F \not\subseteq F'$. Choosing $b \in F \setminus F'$, we get $F' \in \mathcal{O}_b$ and $F \notin \mathcal{O}_b$. The open subset $\mathcal{O}_b$ is also closed by lemma 4.1.8. This proves that $\mathsf{Spec}(B)$ is totally disconnected (see definition 3.4.2).

By theorem 3.4.7, it remains to prove that $\mathsf{Spec}(B)$ is compact. For this apply lemma 4.1.8 and consider $\mathsf{Spec}(B) = \bigcup_{i \in I} \mathcal{O}_{b_i}$, for elements $b_i \in B$. We have thus

$$\forall F \in \mathsf{Spec}(B) \;\; \exists i \in I \;\; F \in \mathcal{O}_{b_i}$$

or, equivalently,

$$\forall F \in \mathsf{Spec}(B) \;\; \exists i \in I \;\; b_i \notin F.$$

Consider now

$$G = \{y \in B \mid \exists i_1, \dots, i_n \in I \;\; b_{i_1} \wedge \cdots \wedge b_{i_n} \leq y\}.$$

We argue by reduction *ad absurdum*. If we cannot extract a finite covering from the original one, for every finite sequence $b_{i_1}, \dots, b_{i_n}$, there exists a corresponding ultrafilter $F_{i_1,\dots,i_n}$ such that

$$F_{i_1,\dots,i_n} \notin \mathcal{O}_{b_{i_1}} \cup \cdots \cup \mathcal{O}_{b_{i_n}} = \mathcal{O}_{b_{i_1} \wedge \cdots \wedge b_{i_n}},$$

that is, $b_{i_1} \wedge \cdots \wedge b_{i_n} \in F_{i_1,\dots,i_n}$. In particular $b_{i_1} \wedge \cdots \wedge b_{i_n} \neq 0$, which proves that G, which is obviously a filter, is in fact a proper filter. By proposition 4.1.4, this proper filter G is contained in an ultrafilter F. But G, and thus F, contains all the elements b_i, $i \in I$, which is a contradiction. □

Corollary 4.1.12 *A subset $U \subseteq \mathsf{Spec}(B)$ of the spectrum of a boolean algebra is a clopen (= closed and open) iff it has the form U_b, for some element $b \in B$.*

Proof By lemma 4.1.8, all U_b are clopen. Conversely if U is a clopen, U is compact in $\mathsf{Spec}(B)$ by proposition 4.1.11. Moreover, by lemma 4.1.8, $U = \bigcup_{i \in I} U_{b_i}$, for elements $b_i \in B$. By compactness,

$$U = U_{b_1} \cup \cdots \cup U_{b_n} = U_{b_1 \vee \cdots \vee b_n}. \qquad □$$

Corollary 4.1.13 *Every boolean algebra B is isomorphic to the boolean algebra of clopens in $\mathsf{Spec}(B)$.* □

Corollary 4.1.14 *Every finite boolean algebra is isomorphic to the boolean algebra of subsets of a finite set.*

Proof The spectrum of a finite boolean algebra is compact Hausdorff and finite, thus discrete. Its clopens are all its subsets. □

Proposition 4.1.15 *Every profinite space is homeomorphic to the spectrum of the boolean algebra of its clopens.*

Proof Let X be a profinite space. Given $x \in X$, we put

$$F_x = \{U \subseteq X \mid U \text{ clopen}, \ x \in U\}.$$

Each F_x is trivially a proper filter in the boolean algebra $\mathsf{Clopen}(X)$ of clopens of X. Since moreover

$$\forall U \in \mathsf{Clopen}(X) \quad x \in U \text{ or } x \in \complement U,$$

the filter F_x is an ultrafilter (see proposition 4.1.3). We consider now the map

$$\varphi\colon X \longrightarrow \mathsf{Spec}\big(\mathsf{Clopen}(X)\big), \quad x \mapsto F_x$$

and we shall prove it is a homeomorphism.

If $x \neq y$, by theorem 3.4.7 there exists a clopen U with $x \in U$ and $y \in \complement U$. Thus $U \in F_x$ and $U \notin F_y$, from which $F_x \neq F_y$ and φ is injective.

If $\mathcal{F} \subseteq \mathsf{Clopen}(X)$ is an ultrafilter, consider

$$V = \bigcap\{U \subseteq X \mid U \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is a proper filter, a finite intersection of elements in $\mathcal{F}$ is never empty. By compactness of X, the subset V is thus non empty and we fix $x \in V$. The proper filter F_x contains $\mathcal{F}$, thus $\mathcal{F} = F_x = \varphi(x)$ by maximality of $\mathcal{F}$. This proves the surjectivity of φ.

Since φ is a bijection between compact Hausdorff spaces, it remains to prove its continuity. Applying lemma 4.1.8, we consider a basic open subset of $\mathsf{Spec}\big(\mathsf{Clopen}(X)\big)$, which has the form U_W for some $W \in \mathsf{Clopen}(X)$. It follows at once that

$$\varphi^{-1}(U_W) = \{x \in X \mid W \in F_x\} = \{x \in X \mid x \in W\} = W. \qquad \square$$

Theorem 4.1.16 (Stone duality) *The maps*

$$B \mapsto \mathsf{Spec}(B), \quad X \mapsto \mathsf{Clopen}(X)$$

extend to a contravariant equivalence between the categories of boolean algebras and profinite spaces.

Proof If $f\colon B \longrightarrow B'$ is a homomorphism of boolean algebras, composition with f

$$\mathsf{Hom}\big(B', \{0,1\}\big) \longrightarrow \mathsf{Hom}\big(B, \{0,1\}\big)$$

yields, by proposition 4.1.3, a map

$$\mathsf{Spec}(f)\colon \mathsf{Spec}(B') \longrightarrow \mathsf{Spec}(B).$$

This map is continous since given $b \in B$

$$\begin{aligned}
\mathsf{Spec}(f)^{-1}(\mathcal{O}_b) &= \big\{F \in \mathsf{Spec}(B') \big| b \notin \mathsf{Spec}(f)(F)\big\} \\
&= \big\{F \in \mathsf{Spec}(B') \big| b \notin f^{-1}(F)\big\} \\
&= \big\{F \in \mathsf{Spec}(B') \big| f(b) \notin F\big\} \\
&= \mathcal{O}_{f(b)}.
\end{aligned}$$

This yields at once a functor from the category Bool of boolean algebras to the category Prof of profinite spaces –

$$\mathsf{Spec}\colon \mathsf{Bool} \longrightarrow \mathsf{Prof}.$$

On the other hand every continuous map $g\colon X \longrightarrow X'$ between (profinite) topological spaces induces a homomorphism of boolean algebras

$$g^{-1}\colon \mathsf{Clopen}(X') \longrightarrow \mathsf{Clopen}(X)$$

which yields at once a functor

$$\mathsf{Clopen}\colon \mathsf{Prof} \longrightarrow \mathsf{Bool}.$$

Corollary 4.1.13 and proposition 4.1.15 imply already that Spec and Clopen are mutually inverse (up to isomorphisms) at the level of objects. Moreover given a homomorphism $f\colon B \longrightarrow B'$ of boolean algebras,

$$(\mathsf{Clopen} \circ \mathsf{Spec})(f)(\mathcal{O}_b) = \mathsf{Spec}(f)^{-1}(\mathcal{O}_b) = \mathcal{O}_{f(b)}$$

from which $\mathsf{Clopen} \circ \mathsf{Spec} \cong \mathsf{id}$ since $U_b = \mathcal{O}_{\mathsf{C}b}$. On the other hand, for a continuous map $g\colon X \longrightarrow X'$ between profinite spaces,

$$\begin{aligned}
(\mathsf{Spec} \circ \mathsf{Clopen})(g)(F_x) &= \big\{W \in \mathsf{Clopen}(X') \big| g^{-1}(W) \in F_x\big\} \\
&= \big\{W \in \mathsf{Clopen}(X') \big| x \in g^{-1}(W)\big\} \\
&= \big\{W \in \mathsf{Clopen}(X') \big| g(x) \in W\big\} \\
&= F_{g(x)},
\end{aligned}$$

which proves $\mathsf{Spec} \circ \mathsf{Clopen} \cong \mathsf{id}$. □

4.2 Pierce representation of a commutative ring

Let us recall that an element e of a ring R is idempotent when $e^2 = e$. We also recall that all our rings are commutative and have a unit.

Proposition 4.2.1 *The idempotents of a ring R, with the operations*

$$e \wedge e' = ee', \quad e \vee e' = e + e' - ee',$$

constitute a boolean algebra.

Proof If e, e' are idempotents,

$$\begin{aligned} (ee')^2 &= e^2 e'^2 = ee', \\ (e + e' - ee')^2 &= e^2 + e'^2 + e^2 e'^2 + 2ee' - 2e^2 e' - 2ee'^2 \\ &= e + e' + ee' + 2ee' - 2ee' - 2ee' \\ &= e + e' - ee' \end{aligned}$$

from which ee' and $e + e' - ee'$ are again idempotents.

Observe next that

$$e \wedge e' = e \Leftrightarrow ee' = e \Leftrightarrow e' = e + e' - ee' \Leftrightarrow e' = e \vee e'.$$

This shows the existence of a unique relation $\leq$ defined by

$$e \leq e' \text{ iff } e \wedge e' = e \text{ iff } e \vee e' = e'.$$

This yields a poset structure on the set of idempotents:

- $e \leq e$ because $ee = e$;
- $e \leq e' \leq e''$ yields $ee' = e$ and $e'e'' = e'$, from which $e = ee' = ee'e'' = ee''$ and thus $e \leq e''$;
- $e \leq e'$ and $e' \leq e$ yield $e = ee'$ and $e' = e'e$, from which $e = e'$.

It follows at once, by definition of the poset structure, that $\wedge$ and $\vee$ are the infimum and the supremum in the poset of idempotents. Clearly 0 is the bottom element and 1 is the top element. We prove next that every idempotent has a complement.

If e is idempotent,

$$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e$$

proving that $1 - e$ is idempotent as well. Moreover

$$e(1 - e) = e - e^2 = e - e = 0$$

proving $e \wedge (1-e) = 0$, while

$$e + (1-e) - e(1-e) = e + 1 - e - e + e^2 = e + 1 - e - e + e = 1$$

proving that $e \vee (1-e) = 1$. Thus $1-e$ is the complement of e.

The distributivity law

$$e \wedge (e' \vee e'') = (e \wedge e') \vee (e \wedge e'')$$

reduces to

$$e(e' + e'' - e'e'') = ee' + ee'' - e^2e'e''$$

which holds since $e = e^2$. Thus the idempotents of R constitute a boolean algebra. $\square$

Corollary 4.2.2 *If $I \lhd R$ is an ideal of the ring R, the idempotents of I constitute an ideal in the boolean algebra of idempotents of R.*

Proof Of course $0 \in I$. Write e, e' for idempotents in R. If $e, e' \in I$, then $e \vee e' = e + e' - ee' \in I$. If $e' \in I$ and $e \leq e'$, then $e = e \wedge e' = ee' \in I$. $\square$

Proposition 4.2.3 *The construction of proposition 4.2.1 extends to a functor defined on the category* Ring *of rings –*

$$\mathsf{Idemp}\colon \mathsf{Ring} \longrightarrow \mathsf{Bool}.$$

Proof A ring homomomorphism $f\colon R \longrightarrow R'$ maps an idempotent of R onto an idempotent of R' and preserves operations like $e \wedge e' = ee'$, $e \vee e' = e + e' - ee'$, $\complement e = 1 - e$. $\square$

Definition 4.2.4 The Pierce spectrum (or boolean spectrum) of a ring R is the spectrum of the boolean algebra of its idempotents.

The composite functor

$$\mathsf{Ring} \xrightarrow{\mathsf{Idemp}} \mathsf{Bool} \xrightarrow{\mathsf{Spec}} \mathsf{Prof}$$

will be written $\mathsf{Sp}\colon \mathsf{Ring} \longrightarrow \mathsf{Prof}$; it is a contravariant functor.

Lemma 4.2.5 *In the Pierce spectrum of a ring R, a partition of a clopen U_e into non-empty clopens has the form*

$$U_e = U_{e_1} \cup \cdots \cup U_{e_n}$$

where

(i) *each* $e_i \in R$ *is a non-zero idempotent,*

(ii) $e_1 + \cdots + e_n = e,$

(iii) $i \neq j \Rightarrow e_i e_j = 0$

and $U_{e'} = \{F \in \mathsf{Sp}(R) | e' \in F\}$ *is defined as in corollary 4.1.10. In particular, putting* $e = 1$*, this applies to* $\mathsf{Sp}(R) = U_1$.

Proof Since U_e is compact, every partition into non-empty clopens is necessarily finite. And each clopen has the form $U_{e'}$ for some idempotent $e' \in R$, by corollary 4.1.12. By corollary 4.1.10, a partition of U_e into clopens thus has the form

$$U_e = U_{e_1} \cup \cdots \cup U_{e_n}$$

with

$$\begin{aligned} e_1 \vee \cdots \vee e_n &= e, \\ e_i e_j &= 0 \quad \text{for } i \neq j. \end{aligned}$$

Notice next that $e_i \wedge e_j = e_i e_j = 0$ implies at once

$$e_i \vee e_j = e_i + e_j - e_i e_j = e_i + e_j.$$

It is straightforward to extend this formula by induction to an n term partition; this yields the required result. □

Definition 4.2.6 In a ring R, an ideal $I \triangleleft R$ is regular when, as an ideal, it is generated by its idempotent elements.

An ideal $I \triangleleft R$ is thus regular when every element $i \in I$ can be written

$$i = r_1 e_1 + \cdots + r_n e_n$$

with each $r_i \in R$, $e_i \in I$ and $e_i^2 = e_i$.

Lemma 4.2.7 *The following conditions are equivalent, for an ideal* $I \triangleleft R$*:*

(i) I *is regular;*

(ii) $\forall i \in I \;\; \exists e \in I \;\; e = e^2, \;\; i = ie;$

(iii) $\forall i_1, \ldots, i_n \in I \;\; \exists e \in I \;\; e = e^2, \;\; \forall k = 1, \ldots, n \; i_k = i_k e.$

Proof Trivially (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Let us prove (i) $\Rightarrow$ (iii). Each i_k can be written

$$i_k = r_1^{(k)} e_1^{(k)} + \cdots + r_{m_k}^{(k)} e_{m_k}^{(k)}$$

with each $r_i^{(j)} \in R$, $e_i^{(j)} \in I$ and $\left(e_i^{(j)}\right)^2 = e_i^{(j)}$. It suffices clearly to find $e \in I$ such that $e^2 = e$ and $e_i^{(j)} e = e_i^{(j)}$ for all possible indices i, j.

This reduces the problem to the following statement. Given idempotents $e_1, \ldots, e_m$ in I, there exists an idempotent $e \in I$ such that $e_i e = e_i$, that is $e_i \leq e$, for each index i. By corollary 4.2.2, it suffices to take $e = e_1 \vee \cdots \vee e_n$. □

Lemma 4.2.8 *In every ring,*

(i) *the ring itself is a regular ideal,*
(ii) *a finite non-empty intersection of regular ideals is again regular and coincides with the product of the ideals,*
(iii) *an arbitrary sum of regular ideals is again a regular ideal.*

Proof The ring itself is regular, since $r = r1$ for each $r \in R$ and $1 \in R$ is idempotent. This is in fact the case of an empty intersection of ideals, which is the ring itself.

Let us treat the case of a binary intersection, which will imply the case of a finite non-empty intersection. If I, J are regular ideals, one has at once $IJ \subseteq I \cap J$. Conversely if $r \in I \cap J$, by lemma 4.2.7 write $r = re = (re)e$ with $e = e^2 \in J$; this proves at once $r \in IJ$ and therefore, $I \cap J = IJ$. It remains to prove that IJ is regular. Indeed an element of IJ has the form

$$r = i_1 j_1 + \cdots + i_n j_n, \quad i_k \in I, \; j_k \in J.$$

By lemma 4.2.7, choose $e = e^2 \in I$ such that $i_k e = i_k$ for all indices k, and $e' = e'^2 \in J$ such that $j_k e' = j_k$ for all indices k. It follows immediately that $e \wedge e' = ee' \in IJ$ is an idempotent such that $ree' = r$.

Next choose a family $(I_k)_{k \in K}$ of regular ideals. An element $r \in I = \sum_{k \in K} I_k$ has the form $r = r_1 + \cdots + r_n$ with each $r_i \in I_{k_i}$. By lemma 4.2.7, for each index i choose $e_i = e_i^2 \in I_{k_i} \subseteq I$ such that $r_i e_i = r_i$. It remains to find $e = e^2 \in I$ such that $e_i e = e_i$, that is $e_i \leq e$, for each index i. By corollary 4.2.2, it suffices to choose $e = e_1 \vee \cdots \vee e_n$. □

Let us recall a classical definition.

Definition 4.2.9 A locale is a complete lattice in which the infinite distributivity law

$$a \wedge \left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I} (a \wedge b_i)$$

holds for all elements a, b_i and every set I of indices.

A typical example of a locale is the lattice of open subsets of a topological space: the distributivity law holds since finite meets and arbitrary joins of open subsets are just set theoretical intersections and unions.

Proposition 4.2.10 *The regular ideals of a ring R constitute a locale isomorphic to the locale of open subsets of the Pierce spectrum of R.*

Proof By proposition 4.1.5, the locale of open subsets of $\mathsf{Sp}(R)$ is isomorphic to the poset of filters in the boolean algebra of idempotents of R. But in every boolean algebra B the poset of filters is isomorphic to the poset of ideals: the correspondence between an ideal I and a filter F is simply given by

$$I = \{b \in B \mid \complement b \in F\}, \quad F = \{b \in B \mid \complement b \in I\}.$$

It remains to prove that the lattice of regular ideals of R is isomorphic to the lattice of ideals in the boolean algebra $\mathsf{Idemp}(R)$ of idempotents of R.

By corollary 4.2.2 we know already that the idempotents of a regular ideal I constitute an ideal $\mathsf{Idemp}(I)$ in the boolean algebra $\mathsf{Idemp}(R)$. By definition of a regular ideal, distinct regular ideals yield distinct ideals in $\mathsf{Idemp}(R)$, proving the injectivity of the correspondence. Moreover it is obvious that this correspondence preserves and respects the ordering. It remains to prove that every ideal J of $\mathsf{Idemp}(R)$ is the ideal of idempotents of a regular ideal $I \lhd R$. Of course, we define I as the ideal of R generated by all the elements of J, which implies at once that I is regular and $J \subseteq \mathsf{Idemp}(I)$. Conversely, if $e \in \mathsf{Idemp}(I)$, then e can be written $e = r_1e_1 + \cdots + r_ne_n$ with each $r_i \in R$ and $e_i \in J$. Since J is an ideal in $\mathsf{Idemp}(R)$, $e' = e_1 \vee \ldots \vee e_n \in J$ and $e_i = e_i \wedge e' = e_ie'$ for each index i. In particular $e = ee'$, that is $e \leq e'$ with $e' \in J$. Since J is an ideal, $e \in J$. $\square$

Corollary 4.2.11 *Let R be a ring. The Pierce spectrum of R is homeomorphic to the set of maximal regular ideals of R (that is, maximal elements in the poset of proper regular ideals), provided with the topology constituted of the subsets*

$$\mathcal{O}_I = \{M \mid I \not\subseteq M\}$$

for every regular ideal $I \lhd R$. We keep writing $\mathsf{Sp}(R)$ for the Pierce spectrum described in this way.

Proof By definition 4.1.7 and proposition 4.2.10. □

Given an idempotent $e \in R$, we shall generally write

$$\mathcal{O}_e = \mathcal{O}_{Re} = \{M \mid Re \not\subseteq M\} = \{M \mid e \notin M\}.$$

Definition 4.2.12 The Pierce structural space of a ring R is the disjoint union of all the quotients R/M, where M runs through all the maximal regular ideals of R; this set is provided with the final topology for all the maps

$$s_r^I \colon \mathcal{O}_I \longrightarrow \coprod_M R/M, \quad N \mapsto [r] \in R/N$$

where I runs through the regular ideals of R and r through the elements of R.

We recall another classical concept.

Definition 4.2.13 A map $f \colon X \longrightarrow Y$ between topological spaces is étale when, for every point $x \in X$, there exist open neighbourhoods $x \in U \subseteq X$ and $y \in V \subseteq Y$ such that f restricts as a homeomorphism $f \colon U \longrightarrow V$.

Of course, an étale map is both continuous and open.

Theorem 4.2.14 *Given a ring R, the projection*

$$p\colon \coprod_M R/M \longrightarrow \mathsf{Sp}(R), \quad [r] \in R/N \mapsto N$$

is étale, when the structural space $\coprod_M R/M$ is provided with its topology from definition 4.2.12.

Proof We coinsider first an element $r \in R$ and prove that

$$U_r = \{M \in \mathsf{Sp}(R) \big| r \in M\}$$

is open in $\mathsf{Sp}(R)$. For this consider

$$\begin{aligned} J &= \{e \in R \big| e = e^2, \ \forall N \in \mathsf{Sp}(R)\ r \notin N \Rightarrow e \in N\} \\ &= \bigcap \{\mathsf{Idemp}(N) \big| N \in \mathsf{Sp}(R), \ \ r \notin N\}. \end{aligned}$$

J is trivially an ideal in the boolean algebra $\mathsf{Idemp}(R)$. The proof of proposition 4.2.10 shows that $J = \mathsf{Idemp}(I)$ for a unique regular ideal $I \lhd R$. Observe moreover that given $N \in \mathsf{Sp}(R)$, one has immediately

$$N \in \mathcal{O}_I \quad \Leftrightarrow \quad I \not\subseteq N$$

$$\begin{aligned}
&\Leftrightarrow \quad \exists e \quad e = e^2, \quad e \in I, \quad e \notin N\\
&\Leftrightarrow \quad \exists e \quad e \in J, \quad e \notin N\\
&\Rightarrow \quad r \in N\\
&\Leftrightarrow \quad N \in U_r.
\end{aligned}$$

In fact, the implication in the previous formula is itself an equivalence. Indeed, if $r \in N$, then $r = re'$ with $e' = e'^2 \in N$ by lemma 4.2.7. If $M \in \mathsf{Sp}(R)$ and $r \notin M$, then $e' \notin M$ since $r = re'$. Thus $1 - e' \in M$ by maximality of M (propositions 4.1.3 and 4.2.10). This proves that $e = 1 - e' \in J$ with $e = 1 - e' \notin N$, since $e' \in N$. This concludes the proof that $U_r = \mathcal{O}_I$, thus U_r is open.

Now if s_r^I and $s_t^{I'}$ are two sections as in definition 4.2.12,

$$\begin{aligned}
(s_t^{I'})^{-1}\big(s_r^I(\mathcal{O}_I)\big) &= \big\{M \in \mathcal{O}_{I'} \big| s_t^{I'}(M) \in s_r^I(\mathcal{O}_I)\big\}\\
&= \big\{M \in \mathcal{O}_I \cap \mathcal{O}_{I'} \big| [t] = [r] \in R/M\big\}\\
&= \big\{M \in \mathcal{O}_{I \cap I'} \big| [t - r] = 0 \in R/M\big\}\\
&= \big\{M \in \mathcal{O}_{I \cap I'} \big| t - r \in M\big\}\\
&= \mathcal{O}_{I \cap I'} \cap U_{t-r}
\end{aligned}$$

and this is an open subset. By definition of a final topology, $s_r^I(\mathcal{O}_I)$ is thus an open subset of $\coprod_M R/M$.

Again by definition of a final topology, the continuity of p reduces to that of all composites $p \circ s_r^I$, which are the canonical inclusions of the open subsets $\mathcal{O}_I$ in $\mathsf{Sp}(R)$. As a consequence, for every element $[r] \in R/M$, we obtain reciprocal homeomorphisms

$$\begin{tikzcd} \mathsf{Sp}(R) \arrow[r, shift left, "s_r^R"'] & s_r^R\big(\mathsf{Sp}(R)\big) \arrow[l, shift left, "p"'] \end{tikzcd}$$

with moreover $s_r^R\big(\mathsf{Sp}(R)\big)$ a neighbourhood of $[r]$. □

Theorem 4.2.15 *Every ring R is isomorphic to the ring of continuous sections of the projection p,*

$$p\colon \coprod_M R/M \longrightarrow \mathsf{Sp}(R), \quad [r] \in R/N \mapsto N,$$

defined on the Pierce structural space of R. More generally, the ring of continous sections of p defined on the open subset $\mathcal{O}_e$, for an idempotent element $e \in R$, is isomorphic to the principal ideal Re.

Proof Let us fix an idempotent element $e \in R$; the first part of the statement is the special case $e = 1$. For every element $r \in R$ we have a continuous section

$$s_r^e = s_r^{Re} \colon \mathcal{O}_e \longrightarrow \coprod_M R/M, \quad N \mapsto [r] \in R/N.$$

Writing $\mathsf{Sec}_e(p)$ for the set of continuous sections of p on $\mathcal{O}_e$, we get a map

$$\varphi_e \colon R \longrightarrow \mathsf{Sec}_e(p), \quad r \mapsto s_r^e$$

It is obvious that the image of φ_e is a ring for the fibrewise operations and that φ_e, corestricted to its image $\varphi_e(R)$, is a ring homomorphism.

Now observe that $s_{1-e}^e = 0$ since

$$M \in \mathcal{O}_e \Leftrightarrow e \notin M \Leftrightarrow 1 - e \in M$$

by propositions 4.1.3 and 4.2.10. This implies the existence of a ring homomorphism factorization of φ_e as

$$R \longrightarrow\!\!\!\!\rightarrow R/R(1-e) \xrightarrow{\psi_e} \varphi_e(R), \quad \psi_e\big([r]\big) = \varphi_e(r).$$

This factorization ψ_e is injective because when $s_r^e = 0$, by propositions 4.1.4 and 4.2.10,

$$r \in \bigcap_{M \in \mathcal{O}_e} M = \bigcap_{e \notin M} M = \bigcap_{1-e \in M} M = \bigcap_{R(1-e) \subseteq M} M = R(1-e).$$

The factorization ψ_e is also surjective, by definition of its codomain.

Next we observe that $R/R(1-e) \cong Re$. Indeed we have a direct sum $R = Re \oplus R(1-e)$ since this sum contains $e + (1-e) = 1$ and $Re \cap R(1-e) = Re \cdot R(1-e) = 0$ (see lemma 4.2.8). Thus ψ_e yields an isomorphism between Re and $\varphi_e(R)$.

It remains to prove that $\varphi_e(R) = \mathsf{Sec}_e(p)$. For this choose a continuous section σ of p on $\mathcal{O}_e$. Given $M \in \mathcal{O}_e$, let us write $\sigma(M) = [r_M] \in R/M$. The proof of theorem 4.2.14 shows that $s_{r_M}^e(\mathcal{O}_e)$ is open, from which $\sigma^{-1}\big(s_{r_M}^e(\mathcal{O}_e)\big)$ is open. In other words

$$W_M = \big\{N \in \mathcal{O}_e \big| \sigma(N) = [r_M] \in R/N\big\} \subseteq \mathcal{O}_e$$

is open and contains M. This open subset W_M is a union of clopens of the form $\mathcal{O}_{e_i}$, with each e_i idempotent. In particular there exists an idempotent e_M such that $M \in \mathcal{O}_{e_M} \subseteq W_M \subseteq \mathcal{O}_e$. These various $\mathcal{O}_{e_M}$ constitute a covering of $\mathcal{O}_e$, since each $M \in \mathcal{O}_e$ belongs to $\mathcal{O}_{e_M}$. But $\mathcal{O}_e$ is closed and thus compact, from which we get finitely many $M_i \in \mathcal{O}_e$ such that the corresponding clopens $\mathcal{O}_{e_{M_i}}$ cover $\mathcal{O}_e$. A finite intersection of clopens of the form $\mathcal{O}_{e_{M_i}}$ is a clopen of the form $\mathcal{O}_{e'}$ for some idempotent e' (see lemma 4.1.9). Thus from the finite covering by clopens we get, by considering the various intersections, a finite partition $\mathcal{O}_e = \mathcal{O}_{e_1} \cup \cdots \cup \mathcal{O}_{e_n}$ with each e_i an idempotent. Now each $\mathcal{O}_{e_i}$ is by construction contained in some W_{M_i} and therefore $\sigma(N) = [r_{M_i}]$ for each $N \in \mathcal{O}_{e_i}$. For simplicity, write $r_i = r_{M_i}$, yielding $\sigma(N) = [r_i] \in R/N$ for each $N \in \mathcal{O}_{e_i}$.

We now put $r = r_1 e_1 + \cdots + r_n e_n$ and we shall prove that $\sigma = s_r^{\mathrm{e}}$. If $M \in \mathcal{O}_{e_i}$, then $e_i \notin M$ and thus $1 - e_i \in M$. The fact of working with a partition also yields $M \notin \mathcal{O}_{e_j}$ for $j \neq i$, thus $e_j \in M$. Therefore, in R/M,

$$\begin{aligned} s_r^{\mathrm{e}}(M) &= [r_1 e_1 + \cdots + r_n e_n] \\ &= [r_1][e_1] + \cdots + [r_n][e_n] \\ &= [r_i][e_i] \qquad \text{since } e_j \in M \text{ for } j \neq i \\ &= [r_i] \qquad \text{since } 1 - e_i \in M, \text{ thus } [e_i] = [1] \text{ in } R/M \\ &= \sigma(M). \end{aligned}$$

This concludes the proof. □

Proposition 4.2.16 *Given a maximal regular ideal M of a ring R, the quotient ring R/M admits only* 0 *and* 1 *as idempotents.*

Proof The ring R/M is the filtered colimit of the rings $R/\langle e_1, \ldots, e_n \rangle$, where $\langle e_1, \ldots, e_n \rangle$ is the ideal generated by the idempotents $e_1, \ldots, e_n$ of M. If $[r] \in R/M$ is idempotent, then r and r^2 are identified in R/M, thus are already identified in a quotient $R/\langle e_1, \ldots, e_n \rangle = R/\langle e_1 \vee \cdots \vee e_n \rangle$. But $R/\langle e_1 \vee \cdots \vee, e_n \rangle = R\big(1 - (e_1 \vee \cdots \vee e_n)\big)$ (see proof of theorem 4.2.15), thus $[r] = [e] \in R/M$, where $e = e^2 \in R\big(1 - (e_1 \vee \cdots \vee e_n)\big)$. If $e \in M$, then $[e] = 0$; otherwise $1 - e \in M$ and $[e] = 1$. □

4.3 The adjoint of the ‘spectrum’ functor

We recall once more that our rings and algebras are commutative and have a unit.

Lemma 4.3.1 *The category R-*Alg *of R-algebras on a ring R is isomorphic to the category $R/$*Ring *of rings under R. In other words, the dual of the category of R-algebras is the slice category* $(\mathsf{Ring})^{\mathsf{op}}/R$. *More generally, if S is an R-algebra, the dual of the category of S-algebras is the slice category* $(R\text{-}\mathsf{Alg})^{\mathsf{op}}/S$.

Proof An R-algebra A is a ring A provided with a ring homomorphism

$$\rho_A : R \longrightarrow A, \quad r \mapsto r1.$$

Conversely a ring A provided with such a ring homomorphism ρ_A is an R-algebra, with scalar multiplication defined by $ra = \rho_A(r)a$. The rest is obvious.

More generally, if S is an R-algebra and A is an S-algebra, the morphism

$$\rho_A : S \longrightarrow A, \quad s \mapsto s1$$

is now a homomorphism of R-algebras and, conversely, the existence of such a homomorphism ρ_A on an R-algebra A induces the structure of an S-algebra on A via the formula $sa = \rho_A(s)a$. □

Theorem 4.3.2 *Let R be a ring. The Pierce spectrum functor*

$$\mathsf{Sp}\colon (R\text{-}\mathsf{Alg})^{\mathsf{op}} \longrightarrow \mathsf{Prof}, \quad A \mapsto \mathsf{Sp}(A)$$

admits as right adjoint the functor

$$\mathcal{C}(-,R)\colon \mathsf{Prof} \longrightarrow (R\text{-}\mathsf{Alg})^{\mathsf{op}}, \quad X \mapsto \mathcal{C}(X,R)$$

where R is provided with the discrete topology and $\mathcal{C}(X,R)$ denotes the ring of continuous functions.

Proof The functor $\mathcal{C}(-,R)$ acts by composition on the arrows of Prof. To prove the adjunction, we shall construct the corresponding unit and counit and prove the required triangular identities. For the sake of clarity, we work in R-Alg instead of its dual.

Given a profinite space X, let us construct

$$\alpha_X : \mathsf{Sp}\big(\mathcal{C}(X,R)\big) \longrightarrow X.$$

By the Stone duality theorem (see 4.1.16), this reduces to constructing a homomorphism of boolean algebras

$$\widetilde{\alpha}_X : \mathsf{Clopen}(X) \longrightarrow \mathsf{Idemp}\big(\mathcal{C}(X,R)\big), \quad U \mapsto \widetilde{\alpha}_X(U) = f_U,$$

where we define

$$f_U : X \longrightarrow R, \quad f_U(x) = \begin{cases} 1 & \text{if} \quad x \in U, \\ 0 & \text{if} \quad x \notin U. \end{cases}$$

One has trivially

$$f_{U\cap V} = f_U \cdot f_V, \quad f_{U\cup V} = f_U + f_V - f_{U\cap V} = f_U + f_V - f_U \cdot f_V$$

from which it follows at once that $\widetilde{\alpha}_X$ is a homomorphism of boolean algebras.

The naturality of α is easy. If $h\colon X \longrightarrow Y$ is a morphism in Prof, for every $V \in \mathsf{Clopen}(Y)$ and $x \in X$,

$$f_{h^{-1}(U)}(x) = 1 \Leftrightarrow x \in h^{-1}(U) \Leftrightarrow h(x) \in U \Leftrightarrow f_U\big(h(x)\big) = 1$$

from which $\widetilde{\alpha}_X\big(h^{-1}(U)\big) = \mathcal{C}(h,-)\big(\widetilde{\alpha}_Y(U)\big)$. This expresses the naturality of $\widetilde{\alpha}$, thus of α.

Next for an R-algebra A, we construct

$$\beta_A \colon \mathcal{C}\big(\mathsf{Sp}(A), R\big) \longrightarrow A.$$

For every continuous map $f\colon \mathsf{Sp}(A) \longrightarrow R$, we shall prove the existence of a finite partition $\mathsf{Sp}(A) = U_{e_1} \cup \cdots \cup U_{e_n}$ of $\mathsf{Sp}(A)$ into clopens U_{e_i}, with $e_i \in A$ idempotent, and such that f is constant on each U_{e_i}, let us say, with value $r_i \in R$. We shall define $\beta_A(f) = \sum_{i=1}^{n} r_i e_i$ and we shall prove that this definition is independent of the choice of the partition.

To prove the existence of such a partition, observe we have a covering $\mathsf{Sp}(A) = \bigcup_{r\in R} f^{-1}(r)$. Clearly $r \neq r'$ implies that $f^{-1}(r)$ and $f^{-1}(r')$ are disjoint, thus the covering is a partition. Moreover since R is discrete, each $f^{-1}(r)$ is open and closed. And finally since $\mathsf{Sp}(A)$ is compact, we can extract a finite subpartition of $\mathsf{Sp}(A)$, that is, only finitely many $f^{-1}(r_i)$ are non empty. Let us write

$$\mathsf{Sp}(A) = f^{-1}(r_1) \cup \cdots \cup f^{-1}(r_n).$$

Since each $f^{-1}(r_i)$ is a clopen in the profinite space $\mathsf{Sp}(A)$, it has the form U_{e_i}, for an idempotent $e_i \in A$ (see corollary 4.1.12). And of course for each $a \in U_{e_i}$, we have $f(a) = r_i$.

Let us prove now that the definition $\beta_A(f) = r_1e_1 + \cdots + r_ne_n$ depends only on f, not on the choice of the partition. The fact that f is constant on each piece of the partition, with the r_i distinct, implies that every possible partition is necessarily a refinement of the partition we have just exhibited. But, with the previous notation, if $U_{e_i} = U_{e_1^i} \cup \cdots \cup U_{e_{k_i}^i}$, by lemma 4.2.5 we get $e_i = e_1^i + \cdots + e_{k_i}^i$ and obviously

$$r_ie_i = r_i(e_1^i + \cdots + e_{k_i}^i) = r_ie_1^i + \cdots + r_ie_{k_i}^i$$

which shows that the definition of $\beta_A(f)$ does not depend on the choice of the partition.

It is now easy to prove that β_A is a homomorphism of R-algebras. For example, given $f, g \in \mathcal{C}\big(\mathsf{Sp}(A), R\big)$, let us prove that $\beta_A(f+g) = \beta_A(f) + \beta_A(g)$. Using the formula $U_e \cap U_{e'} = U_{e\wedge e'} = U_{ee'}$ of corollary 4.1.10, one can choose a partition into clopens U_{e_i} such that both $f(a) = r_i$ and $g(a) = s_i$ hold for all elements $a \in U_{e_i}$. Then $(f+g)(a) = r_i + s_i$ for all $a \in U_{e_i}$. Therefore

$$\beta_A(f+g) = \sum_{i=1}^{n}(r_i + s_i)e_i = \sum_{i=1}^{n} r_i e_i + \sum_{i=1}^{n} s_i e_i = \beta_A(f) + \beta_A(g).$$

To prove the naturality of β, choose a morphism $k\colon A \longrightarrow B$ of R-algebras and $f \in \mathcal{C}\big(\mathsf{Sp}(A), R\big)$. With the above notation, the conditions

$$e_1 + \cdots + e_n = 1, \quad i \neq j \Rightarrow e_i e_j = 0$$

imply at once

$$k(e_1) + \cdots + k(e_n) = 1, \quad i \neq j \Rightarrow k(e_i)k(e_j) = 0.$$

But

$$\begin{aligned} U_{k(e)} &= \big\{F \in \mathsf{Sp}(B) \big| k(e) \in F\big\} \\ &= \big\{F \in \mathsf{Sp}(B) \big| e \in k^{-1}(F)\big\} \\ &= \mathsf{Sp}(k)^{-1}(U_e), \end{aligned}$$

proving that $\mathcal{C}\big(\mathsf{Sp}(k), \mathsf{id}\big)(f) = f \circ \mathsf{Sp}(k)$ maps $U_{k(e)}$ onto $f(e)$. Therefore

$$(k \circ \beta_A)(f) = k\left(\sum_{i=1}^{n} r_i e_i\right) = \sum_{i=1}^{n} r_i k(e_i) = \beta_B\Big(\mathcal{C}\big(\mathsf{Sp}(k), \mathsf{id}\big)\Big)(f).$$

We must now prove the triangular identities of the adjunction. The first identity is the commutativity of the left hand triangle below,

$$\begin{tikzcd} \mathsf{Sp}(A) \arrow[r, "\mathsf{Sp}(\beta_A)"] \arrow[dr, equal] & \mathsf{Sp}\Big(\mathcal{C}\big(\mathsf{Sp}(A), R\big)\Big) \arrow[d, "\alpha_{\mathsf{Sp}(A)}"] \\ & \mathsf{Sp}(A) \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Idemp}(A) \arrow[dr, equal] & \mathsf{Idemp}\Big(\mathcal{C}\big(\mathsf{Sp}(A), R\big)\Big) \arrow[l, "\mathsf{Idemp}(\beta_A)"'] \\ & \mathsf{Idemp}(A) \arrow[u, "\widetilde{\alpha}_{\mathsf{Sp}(A)}"'] \end{tikzcd}$$

which is equivalent to the commutativity of the right hand triangle, by identifying the clopens of $\mathsf{Sp}(A)$ with the idempotents of A. An idempotent $e \in A$ is mapped by $\widetilde{\alpha}_{\mathsf{Sp}(A)}$ onto

$$f_e\colon \mathsf{Sp}(A) \longrightarrow R, \qquad \begin{cases} f_e(e') = 1 & \text{if} \quad e' \in U_e, \\ f_e(e') = 0 & \text{if} \quad e' \notin U_e, \text{ i.e. } e' \in U_{1-e}, \end{cases}$$

from which $\beta_A(f_e) = 1e + 0(1-e) = e$.

The second triangular identity is the commutativity of

$$\begin{tikzcd}
\mathcal{C}(X,R) \arrow[r, "\mathcal{C}(\alpha_X,1)"] \arrow[dr, equal] & \mathcal{C}\Big(\mathsf{Sp}\big(\mathcal{C}(X,R)\big),R\Big) \arrow[d, "\beta_{\mathcal{C}(X,R)}"] \\
& \mathcal{C}(X,R)
\end{tikzcd}$$

Choose $g \in \mathcal{C}(X,R)$ and consider the composite

$$\mathsf{Sp}\big(\mathcal{C}(X,R)\big) \xrightarrow{\alpha_X} X \xrightarrow{g} R.$$

Write $X = \bigcup_{r \in R} g^{-1}(r)$, which is a partition into clopens, since R is discrete. By compactness, we extract a finite partition $X = g^{-1}(r_1) \cup \cdots \cup g^{-1}(r_n)$ and, obviously, $g = \sum_{i=1}^{n} r_i f_{g^{-1}(r_i)}$. Now the subsets $\alpha_X^{-1}\big(g^{-1}(r_i)\big)$ constitute a partition of $\mathsf{Sp}\big(\mathcal{C}(X,R)\big)$ into clopens and by lemma 4.2.5, $\alpha_X^{-1}\big(g^{-1}(r_i)\big) = U_{g_i}$ for some idempotent $g_i \in \mathcal{C}(X,R)$. By definition of α_X, we have $g_i = f_{g^{-1}(r_i)}$, from which $g = \sum_{i=1}^{n} r_i g_i$ and therefore

$$\beta_{\mathcal{C}(X,R)}(g \circ \alpha_X) = \sum_{i=1}^{n} r_i g_i = g. \qquad \square$$

Proposition 4.3.3 *In the conditions and with the notation of theorem 4.3.2, the following conditions are equivalent:*

(i) *the ring R has exactly two distinct idempotents, 0 and 1;*
(ii) *the functor $\mathcal{C}(-,R)\colon \mathsf{Prof} \longrightarrow (R\text{-}\mathsf{Alg})^{\mathsf{op}}$ is full and faithful.*

Proof Assume condition (i). Condition (ii) reduces to the fact that each morphism α_X is a homeomorphism. Since α_X is continuous between compact Hausdorff spaces, it suffices to prove its bijectivity. Observe at once that when X is empty, $\mathcal{C}(X,R)$ is the zero ring and $\mathsf{Sp}\big(\mathcal{C}(X,R)\big)$ is empty as well, from which α_X is a homeomorphism.

If X is not empty, an idempotent continuous map $f\colon X \longrightarrow R$, under assumption (i), takes only the values 0 and 1. Since R is discrete, f takes value 1 on a clopen of X and 0 on its complement. Therefore $\mathsf{Idemp}\big(\mathcal{C}(X,R)\big) \cong \mathsf{Clopen}(X)$ and $\widetilde{\alpha}_X$ is an isomorphism.

Conversely if α_X is an isomorphism for each X, choosing for X the singleton yields

$$\mathsf{Sp}(R) \cong \mathsf{Sp}\Big(\mathcal{C}\big(\{*\},R\big)\Big) \cong \{*\},$$

which proves that $\mathsf{Idemp}(R) = \{0,1\}$, with $0 \neq 1$. $\square$

$$\begin{tikzcd}
T \arrow[dr, dotted] \arrow[drr, dotted] \arrow[ddr, "t"'] & & & F(T) \arrow[drr, dotted] \arrow[ddr, "F(t)"'] & & \\
& Z \arrow[r, "\delta"] \arrow[d, "\gamma"'] & G(Y) \arrow[d, "G(\beta)"] & & & Y \arrow[d, "\beta"] \\
& A \arrow[r, "\eta_A"'] & (GF)(A) & & F(A) \arrow[r, equal] & F(A)
\end{tikzcd}$$

Diagram 4.1

The condition "R admits 0 and 1 as its only idempotents" is very strong, and will now be avoided via a localization process over $\mathsf{Sp}(R)$.

Lemma 4.3.4 *Consider an adjunction* $\mathcal{A} \underset{F}{\overset{G}{\leftrightarrows}} \mathcal{B}$, $F \dashv G$. *Assume that* $\mathcal{A}$ *has pullbacks. In those conditions, for every object* $A \in \mathcal{A}$*, the functor*

$$F_A\colon \mathcal{A}/A \longrightarrow \mathcal{B}/F(A), \quad (X,\alpha) \mapsto \big(F(X), F(\alpha)\big)$$

has a right adjoint functor

$$G_A\colon \mathcal{B}/F(A) \longrightarrow \mathcal{A}/A, \quad (Y,\beta) \mapsto (Z,\gamma)$$

where (Z,γ) *is defined by the pullback*

$$\begin{tikzcd}
Z \arrow[r, "\delta"] \arrow[d, "\gamma"'] & G(Y) \arrow[d, "G(\beta)"] \\
A \arrow[r, "\eta_A"'] & (GF)(A)
\end{tikzcd}$$

in which η *is the unit of the adjunction* $G \dashv F$.

Proof Considering diagram 4.1, we observe that

$$\begin{aligned}
\mathcal{A}/A\Big((T,t), G_A(Y,\beta)\Big) &\cong \mathcal{A}/GF(A)\Big(\big(T, \eta_A \circ t\big), \big(G(Y), G(\beta)\big)\Big) \\
&\cong \mathcal{B}/F(A)\Big(F_A(T,t), (Y,\beta)\Big).
\end{aligned}$$

This exhibits the required adjunction. □

Corollary 4.3.5 *Let R be a ring and S an R-algebra. When S has exactly two distinct idempotents 0 and 1, the isomorphism*

$$S \otimes_R \mathcal{C}(X,R) \cong \mathcal{C}(X,S)$$

holds for every profinite space X.

Proof By assumption, $\mathsf{Sp}(S)$ is a singleton, thus $\mathsf{Prof}/\mathsf{Sp}(S) \cong \mathsf{Prof}$. The adjunction of theorem 4.3.2

$$\begin{tikzcd} (R\text{-}\mathsf{Alg})^{\mathsf{op}} \arrow[r, shift right, "\mathsf{Sp}"'] & \mathsf{Prof} \arrow[l, shift right, "\mathcal{C}(-,R)"'] \end{tikzcd}$$

induces, via lemmas 4.3.1 and 4.3.4, an adjunction

$$\begin{tikzcd} (S\text{-}\mathsf{Alg})^{\mathsf{op}} \cong (R\text{-}\mathsf{Alg})^{\mathsf{op}}/S \arrow[r, shift right, "\mathsf{Sp}_S"'] & \mathsf{Prof}/\mathsf{Sp}(S) \cong \mathsf{Prof} \arrow[l, shift right, "\mathcal{C}(-,R)_S"'] \end{tikzcd}$$

where $\mathcal{C}(-,R)_S$ is defined via the following pushout diagram in R-Alg:

$$\begin{tikzcd} \mathcal{C}(X,R)_S \cong S \otimes_R \mathcal{C}(X,R) & \mathcal{C}(X,R) \arrow[l] \\ S \arrow[u] & \mathcal{C}(\mathsf{Sp}(S),R) \cong \mathcal{C}(\{*\},S) \cong R \arrow[l, "\beta_S"] \arrow[u] \end{tikzcd}$$

Applying theorem 4.3.2 to the ring S, we conclude that

$$S \otimes_R \mathcal{C}(X,R) \cong \mathcal{C}(X,S). \qquad \square$$

Theorem 4.3.6 *For every ring R, the right adjoint of the functor*

$$\mathsf{Sp}_R \colon (R\text{-}\mathsf{Alg})^{\mathsf{op}} \longrightarrow \mathsf{Prof}/\mathsf{Sp}(R), \quad A \mapsto \big(\mathsf{Sp}(A) \longrightarrow \mathsf{Sp}(R)\big)$$

is the functor

$$\mathcal{C}_R \colon \mathsf{Prof}/\mathsf{Sp}(R) \longrightarrow (R\text{-}\mathsf{Alg})^{\mathsf{op}}, \ (X,f) \mapsto \mathsf{Hom}\Big((X,f), \big(\textstyle\coprod_M R/M, p\big)\Big)$$

where $p\colon \coprod_M R/M \longrightarrow \mathsf{Sp}(R)$ is the projection of the Pierce structural space of the ring R (see theorem 4.2.14). This right adjoint is full and faithful.

Proof The case $R = \{0\}$ is trivial since $(R\text{-}\mathsf{Alg})^{\mathsf{op}} \cong \big\{\{0\}\big\}$ and $\mathsf{Prof}/\mathsf{Sp}(R) \cong \{\emptyset = \emptyset\}$. So we shall assume $0 \neq 1$ in R.

The set $\mathsf{Hom}\big((X,f),(\coprod_M R/M,p)\big)$ is provided pointwise with the structure of R-algebra, via the structure of R-algebra on each fibre R/M. Indeed, checking that

$$g, g' \text{ continuous} \Rightarrow g+g',\ gg' \text{ continuous}$$

is routine. An idempotent in this ring is a function $g\colon X \longrightarrow \coprod_M R/M$ such that $p \circ g = f$ and each $g(x) \in R/f(x)$ is idempotent, that is, equal to 0 or 1 by proposition 4.2.16. The subsets

$$s_0^{\mathsf{Sp}(R)}\big(\mathsf{Sp}(R)\big) = \big\{[0] \in R/M \big| M \in \mathsf{Sp}(R)\big\},$$
$$s_1^{\mathsf{Sp}(R)}\big(\mathsf{Sp}(R)\big) = \big\{[1] \in R/M \big| M \in \mathsf{Sp}(R)\big\}$$

are open (proof of 4.2.14) and disjoint (because $1 \notin M$).Therefore the idempotents of $\mathsf{Hom}\big((X,f),(\coprod_M R/M)\big)$ are given by

$$\mathsf{Hom}\Big((X,f),\big(\{0,1\} \times \mathsf{Sp}(R),p\big)\Big)$$

where $\{0,1\}$ is provided with the discrete topology and p is the projection onto the factor $\mathsf{Sp}(R)$. This is the same thing as $\mathcal{C}\big(X,\{0,1\}\big)$ or, equivalently, the boolean algebra of clopens in X. The Stone duality theorem (see 4.1.16) then yields

$$\mathsf{Sp}\bigg(\mathsf{Hom}\Big((X,f),\big(\textstyle\coprod_M R/M,p\big)\Big)\bigg) \cong X.$$

When the adjunction property is established, this will prove that the counit of the adjunction is an isomorphism, thus $\mathcal{C}_R$ is full and faithful.

To prove the adjointness property, we fix an R-algebra A; we must prove the existence of natural bijections

$$\mathsf{Hom}\bigg(\mathsf{Hom}\Big((X,f),\big(\textstyle\coprod_M R/M,p\big)\Big),A\bigg) \cong \mathsf{Hom}\Big(\big(\mathsf{Sp}(A),\alpha\big),(X,f)\Big)$$

where $\alpha\colon \mathsf{Sp}(A) \longrightarrow \mathsf{Sp}(R)$ is the canonical morphism corresponding, by the Stone duality, to the morphism

$$\mathsf{Idemp}(R) \longrightarrow \mathsf{Idemp}(A), \quad e \mapsto e \cdot 1$$

of boolean algebras.

Given $\varphi\colon \mathsf{Hom}\big((X,f),(\coprod_M R/M,p)\big) \longrightarrow A$ and a clopen $U \subseteq X$, we consider the continuous function

$$\chi_U\colon X \longrightarrow \coprod_M R/M, \quad \begin{cases} x \mapsto [1] \in R/f(x) & \text{if} \quad x \in U, \\ x \mapsto [0] \in R/f(x) & \text{if} \quad x \notin U. \end{cases}$$

The map χ_U is idempotent and $\varphi(\chi_U)$ is thus an idempotent of A. This yields a homomorphism

$$\varphi' \colon \mathsf{Clopen}(X) \longrightarrow \mathsf{Idemp}(A), \quad U \mapsto \varphi'(U) = \varphi(\chi_U);$$

this homomorphism corresponds by the Stone duality to a continuous map $\varphi'' \colon \mathsf{Sp}(A) \longrightarrow X$. To have a morphism $\varphi'' \colon (\mathsf{Sp}(A), \alpha) \longrightarrow (X, f)$, we must still prove the equality $f \circ \varphi'' = \alpha$. This reduces, for every idempotent $e \in R$, to the equality $\varphi'\big(f^{-1}(\mathcal{O}_e)\big) = e \cdot 1$. But

$$\chi_{f^{-1}(\mathcal{O}_e)}(x) = \begin{cases} [1] & \text{if } \ x \in f^{-1}(\mathcal{O}_e), \ \text{ i.e. } \ e \notin f(x), \\ [0] & \text{if } \ x \notin f^{-1}(\mathcal{O}_e), \ \text{ i.e. } \ e \in f(x). \end{cases}$$

But $e \notin f(x)$ implies $1-e \in f(x)$, thus $[1] = [e] \in R/f(x)$, while $e \in f(x)$ implies $[0] = [e] \in R/f(x)$. This proves that $\chi_{f^{-1}(\mathcal{O}_e)}(x) = [e] = e \cdot [1]$ for all x. Since φ is a homomorphism of R-algebras,

$$\varphi'\big(f^{-1}(\mathcal{O}_e)\big) = \varphi\big(\chi_{f^{-1}(\mathcal{O}_e)}\big) = e \cdot 1 = e.$$

It is obvious to observe that the correspondence $\varphi \mapsto \varphi''$ is natural.

To prove the injectivity of this correspondence, observe first that given $h \colon (X, f) \longrightarrow (\coprod_M R/M, p)$, the subsets $h^{-1}\big(s_r^{\mathsf{Sp}(R)}(\mathsf{Sp}(R))\big)$, for $r \in R$, constitute an open covering of X (see proof of 4.2.14). Since X has a base of clopens (see 3.4.9), there is a refinement of this covering constituted of clopens and, by compactness, we extract from it a finite covering. This thus yields clopens $U_1, \ldots, U_n \subseteq X$ and elements $r_1, \ldots, r_n \in R$ such that for each index i and each element $x \in U_i$, $h(x) = [r_i] \in R/f(x)$. This proves that $h = \sum_{i=1}^n r_i \cdot \chi_{U_i}$. Now choose $\varphi, \psi \colon \mathsf{Hom}\big((X, f), (\coprod_M R/M, p)\big) \rightrightarrows A$ such that $\varphi'' = \psi''$; proving $\varphi(\chi_U) = \psi(\chi_U)$ for each clopen $U \subseteq X$ will imply $\varphi(h) = \psi(h)$, by R-linearity of φ and ψ. But $\varphi(\chi_U) = \varphi'(U) = \psi'(U) = \psi(\chi_U)$, since $\varphi'' = \psi''$ implies $\varphi' = \psi'$ by the Stone duality.

To prove the surjectivity of the correspondence $\varphi \mapsto \varphi''$, let us consider a morphism $g \colon (\mathsf{Sp}(A), \alpha) \longrightarrow (X, f)$, which yields by the Stone duality a morphism $\overline{g} \colon \mathsf{Clopen}(X) \longrightarrow \mathsf{Idemp}(A)$ of boolean algebras, such that $\overline{g}\big(f^{-1}(\mathcal{O}_e)\big) = e \cdot 1$ for each idempotent $e \in R$. We have already observed that $h \colon (X, f) \longrightarrow (\coprod_M R/M)$ can be written $h = \sum_{i=1}^n r_i \chi_{U_i}$ with $r_i \in R$ and $X = U_1 \cup \cdots \cup U_n$ a partition into clopens such that $h(x) = [r_i] \in R/f(x)$ for $x \in U_i$. We define

$$\varphi \colon \mathsf{Hom}\Big((X, f), \big(\textstyle\coprod_M R/M, p\big)\Big) \longrightarrow A, \quad h \mapsto \varphi(h) = \textstyle\sum_{i=1}^n r_i \cdot \overline{g}(U_i).$$

If we can prove that this definition is independent of the decomposition $h = \sum_{i=1}^{n} r_i \chi_{U_i}$, the relation $\varphi(\chi_U) = \overline{g}(U)$ will yield $\varphi'' = g$ and prove the surjectivity.

So let us consider another decomposition $h = \sum_{j=1}^{m} s_j \chi_{V_j}$ as above. Putting $W_{ij} = U_i \cap V_j$, we obtain a new partition of X into clopens with the property

$$x \in W_{ij} \Rightarrow [r_i] = h(x) = [s_j] \in R/f(x).$$

On the other hand fixing the index i, the clopens W_{ij} for $j = 1, \dots, m$ constitute a finite partition of U_i into clopens, thus the elements $\overline{g}(W_{ij})$ constitute a partition of the idempotent $\overline{g}(U_i)$; by lemma 4.2.5, this implies $\overline{g}(U_i) = \sum_{j=1}^{m} \overline{g}(W_{ij})$. Consequently

$$r_i \overline{g}(U_i) = r_i \sum_{j=1}^{m} \overline{g}(W_{ij})$$

and analogously

$$s_j \overline{g}(V_j) = s_j \sum_{i=1}^{n} \overline{g}(W_{ij}).$$

It suffices now to prove that $r_i \overline{g}(W_{ij}) = s_j \overline{g}(W_{ij})$ for all indices i, j, to imply the independence with respect to the decomposition of h. But for $x \in W_{ij}$, the relation $[r_i] = [s_j] \in R/f(x)$ implies $r_i - s_j \in f(x)$. Thus it suffices to prove that given a clopen $U \subseteq X$ and an element $r \in R$

$$\big(\forall x \in U \quad r \in f(x)\big) \Rightarrow \big(r\overline{g}(U) = 0\big).$$

Let us prove this last fact. With the notation of theorem 4.2.14 and referring freely to its proof,

$$f(U) \subseteq \big\{M \in \mathsf{Sp}(R) \big| r \in M\big\} = U_r$$

with U_r open. By 3.4.9, U_r is a union of clopens, from which by finite intersections we deduce a partition $U_r = \bigcup_{i \in I} \mathcal{O}_{e_i}$ into clopens $\mathcal{O}_{e_i}$, with each $e_i \in R$ idempotent. Therefore the clopens $f^{-1}(\mathcal{O}_{e_i})$ cover the clopen $U \subset X$; but U is closed, thus compact, and therefore we can extract a finite covering by clopens

$$U \subseteq f^{-1}(\mathcal{O}_{e_{i_1}}) \cup \dots \cup f^{-1}(\mathcal{O}_{e_{i_k}}).$$

This implies

$$r \cdot \overline{g}(U) \quad = \quad r \cdot \overline{g}\left(U \cap \bigcup_{j=1}^{k} f^{-1}(\mathcal{O}_{e_{i_j}})\right)$$

$$\begin{aligned}
&= r\cdot\overline{g}\left(\bigcup_{j=1}^{k} U\cap f^{-1}(\mathcal{O}_{e_{i_j}})\right)\\
&= r\cdot\sum_{j=1}^{k}\Big(\overline{g}(U)\cdot\overline{g}\big(f^{-1}(\mathcal{O}_{e_{i_j}})\big)\Big)\\
&= \sum_{j=1}^{k} r\overline{g}(U)\cdot e_i
\end{aligned}$$

and it remains to prove that each $re_i = 0$ for each index i. But since $\mathcal{O}_{e_i} \subseteq U_r$,

$$\begin{aligned}
r \in \bigcap\big\{M\in\mathsf{Sp}(R)\big|M\in\mathcal{O}_{e_i}\big\} &= \bigcap\big\{M\in\mathsf{Sp}(R)\big|e_i\notin M\big\}\\
&= \bigcap\big\{M\in\mathsf{Sp}(R)\big|1-e_i\in M\big\}.
\end{aligned}$$

Applying proposition 4.2.10 and condition (iv) in 4.1.4, this proves $r\in R(1-e_i)$. But then $r = r(1-e_i)$, that is $re_i = 0$. □

Let us conclude this section with a last observation on the functor Sp.

Proposition 4.3.7 *For every ring R, the functor*

$$\mathsf{Sp}\colon (R\text{-}\mathsf{Alg})^{\mathsf{op}} \longrightarrow \mathsf{Prof}$$

preserves cofiltered limits.

Proof Applying the Stone duality, we must verify that the functor

$$\mathsf{Idemp}\colon R\text{-}\mathsf{Alg} \longrightarrow \mathsf{Bool}$$

preserves filtered colimits. But both categories are algebraic, thus their filtered colimits are computed as in the category of sets. Now the functor Idemp is the factorization, through the category of boolean algebras, of the representable functor

$$\mathsf{Hom}\big(R[X]/\langle X^2\rangle, -\big)\colon R\text{-}\mathsf{Alg} \longrightarrow \mathsf{Set}$$

with values in the category Set of sets. This representable functor preserves filtered colimits, because $R[X]/\langle X^2\rangle$ is finitely presentable as an R-algebra (see [1]). □

$$
\begin{tikzcd}
X \times_Y A \arrow[r, "p_A"] \arrow[d, "\mathrm{id}_X \times a"'] & A \arrow[d, "a"] \\
X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Diagram 4.2

4.4 Descent morphisms

This section presents a rough introduction to descent theory along pullbacks in a given category.

Definition 4.4.1 Let $\mathcal{C}$ be a category with pullbacks. A morphism $f: X \longrightarrow Y$ is an effective descent morphism when the functor "pullback along f"

$$f^*: \mathcal{C}/Y \longrightarrow \mathcal{C}/X$$

is monadic.

Let us recall the precise form of the monad. The functor f^* admits the left adjoint functor

$$\Sigma_f: \mathcal{C}/X \longrightarrow \mathcal{C}/Y, \quad (A, a) \mapsto (A, f \circ a)$$

of composition with f. This yields a composite functor

$$T = f^* \circ \Sigma_f: \mathcal{C}/X \longrightarrow \mathcal{C}/X, \quad (A, a) \mapsto \big(X \times_Y A, p_1 \circ (\mathrm{id}_X \times a)\big)$$

described by diagram 4.2, where both squares are pullbacks. We get at once natural transformations

$$\eta: \mathrm{id} \Longrightarrow T, \quad \mu: T \circ T \Longrightarrow T$$

defined by

$$\eta_{(A,a)} = \begin{pmatrix} a \\ \mathrm{id}_A \end{pmatrix}: A \longrightarrow X \times_Y A$$

and

$$\mu_{(A,a)} = p_1 \times \mathrm{id}_A \colon X \times_Y X \times_Y A \longrightarrow X \times_Y A,$$

and these are the unit and the multiplication of the monad $\mathbb{T} = (T, \varepsilon, \mu)$.

By the Beck criterion (see [66]), the monadicity of the functor f^* reduces to the following properties:

(i) the functor f^* reflects isomorphisms;
(ii) the functor f^* creates the coequalizers of those pairs (u, v) such that $\big(f^*(u), f^*(v)\big)$ has a split coequalizer.

For the sake of completeness, we also recall the precise meaning of this condition (ii). A split coequalizer of $\big(f^*(u), f^*(v)\big)$ consists in three arrows q, r, s,

$$\begin{tikzcd} f^*(U) \arrow[r, shift left, "f^*(u)"] \arrow[r, shift right, "f^*(v)"'] & f^*(V) \arrow[l, bend right=50, "s"'] \arrow[r, "q"] & Q \arrow[l, bend right=50, "r"'] \end{tikzcd}$$

such that

$$q \circ f^*(u) = q \circ f^*(v), \quad q \circ r = \mathrm{id}_Q, \quad f^*(u) \circ s = \mathrm{id}_{f^*(V)}, \quad f^*(v) \circ s = r \circ q;$$

this implies immediately that q is the coequalizer of $\big(f^*(u), f^*(v)\big)$ and that this coequalizer is preserved by every functor defined on $\mathcal{C}/X$. Condition (ii) requires that for every such pair (u, v), the coequalizer of (u, v) exists in $\mathcal{C}/Y$ and is preserved by f^*.

Our interest will be in descent morphisms in the dual of the category of rings (commutative, with unit). We must therefore study the comonadicity of the "pushout" functor in the category of rings. Let us recall that in the category of rings, the pushout is given by the tensor product

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ A \arrow[r] & S \otimes_R A \end{tikzcd}$$

Let us also recall a classical notion in commutative algebra.

Definition 4.4.2 A module M over a ring R is flat when the functor

$$M \otimes_R -: \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_R$$

preserves exact sequences. The module M is faithfully flat when it preserves and reflects exact sequences.

Proposition 4.4.3 *Let $f: R \longrightarrow S$ be a morphism of rings. When S is faithfully flat as an R-module, f is an effective descent morphism in the dual of the category of rings.*

Proof A morphism $g: A \longrightarrow B$ of R-modules is an isomorphism if and only if the sequence

$$0 \longrightarrow A \xrightarrow{\ g\ } B \longrightarrow 0$$

is exact. Since $S \otimes_R -: \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_R$ reflects exact sequences, it thus reflects isomorphisms. Since the forgetful functors $R\text{-}\mathsf{Alg} \longrightarrow \mathsf{Mod}_R$ and $S\text{-}\mathsf{Alg} \longrightarrow \mathsf{Mod}_S$ have "all properties" we need (preserve, reflect, create exact sequences), the functor $S \otimes_R -: R\text{-}\mathsf{Alg} \longrightarrow S\text{-}\mathsf{Alg}$, which is the restriction of the previous functor, reflects isomorphisms as well, since these are bijective homomorphisms.

Since R-Alg and S-Alg have equalizers computed as in Mod_R and Mod_S, the second condition of the Beck criterion is satisfied because $S \otimes_R -: \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_R$ preserves equalizers, by flatness of S. □

Proposition 4.4.4 *Let $f: R \longrightarrow S$ be a morphism of rings admitting an R-linear retraction $g: S \longrightarrow R$; that is, $g \circ f = \mathrm{id}_R$. Then f is an effective descent morphism in the dual of the category of rings.*

Proof If $u, v: A \rightrightarrows B$ are such that $S \otimes u = S \otimes v$, the consideration of the diagram

$$\begin{tikzcd}
R \otimes_R A \cong A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift right, "f \otimes A"'] & B \cong R \otimes_R B \arrow[d, shift right, "f \otimes B"'] \\
S \otimes_R A \arrow[r, shift left, "S \otimes u"] \arrow[r, shift right, "S \otimes v"'] \arrow[u, shift right, "g \otimes A"'] & S \otimes_R B \arrow[u, shift right, "g \otimes B"']
\end{tikzcd}$$

yields

$$(f \otimes B) \circ u = (S \otimes u) \circ (f \otimes A) = (S \otimes v) \circ (f \otimes A) = (f \otimes B) \circ v,$$

$$
\begin{tikzcd}
N \arrow[r, "n"'] \arrow[d, shift right, "\overline{f}"'] & A \cong R \otimes_R A \arrow[l, bend right, "\sigma"'] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift right, "f \otimes A"'] & B \cong R \otimes_R B \arrow[l, bend right, "\rho"'] \arrow[d, shift right, "f \otimes B"'] \\
K \arrow[r, "k"] \arrow[u, shift right, "\overline{g}"'] & S \otimes_R A \arrow[l, bend left, "s"] \arrow[r, shift left, "S \otimes u"] \arrow[r, shift right, "S \otimes v"'] \arrow[u, shift right, "g \otimes A"'] & S \otimes_R B \arrow[l, bend left, "r"] \arrow[u, shift right, "g \otimes B"']
\end{tikzcd}
$$

Diagram 4.3

from which $u = v$ since $f \otimes B$ admits the retraction $g \otimes B$ and therefore is injective. Thus the functor $S \otimes_R -\colon \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_R$ is faithful, and therefore reflects monomorphisms and epimorphisms. It thus reflects isomorphisms, since in Mod_R being an isomorphism reduces to being both a monomorphism and an epimorphism. And finally the functor $S \otimes_R -\colon R\text{-}\mathsf{Alg} \longrightarrow S\text{-}\mathsf{Alg}$ reflects isomorphisms, since in all those categories, isomorphisms are just bijective homomorphisms.

Let us check now the condition on split equalizers. We thus consider two morphisms $u, v\colon A \rightrightarrows B$ of R-algebras, such that the pair $(S \otimes u, S \otimes v)$ has a split equalizer in S-Alg. This yields a part of diagram 4.3 where

$$(S \otimes u) \circ k = (S \otimes v) \circ k, \quad s \circ k = \mathsf{id}, \quad r \circ (S \otimes u) = \mathsf{id}, \quad r \circ (S \otimes v) = k \circ s.$$

We put $n = \mathsf{Ker}\,(u, v)$ in R-Alg or Mod_R: this is the same map. We must prove that $(S \otimes_R -)$ preserves this equalizer. Considering both equalizers

$$n = \mathsf{Ker}\,(u, v), \quad k = \mathsf{Ker}\,(S \otimes u, S \otimes v)$$

we conclude that there exist morphisms $\overline{f}, \overline{g}$ inducing corresponding commutativities in the diagram in Mod_R. We put then

$$\sigma = \overline{g} \circ s \circ (f \otimes A), \quad \rho = (g \otimes A) \circ r \circ (f \otimes B).$$

It remains to observe that ρ and σ present n as the split equalizer of the pair (u, v), from which this coequalizer will be preserved by every functor and in particular by the functor $S \otimes_R -$. And indeed (omitting for short the composition symbols)

$$\begin{aligned}\sigma n &= \overline{g}s(f \otimes A)n = \overline{g}sk\overline{f} = \overline{g}\overline{f} = \mathsf{id},\\ \rho u &= (g \otimes A)r(f \otimes B)u = (g \otimes A)r(S \otimes u)(f \otimes A)\\ &= (g \otimes A)(f \otimes A) = \mathsf{id},\\ \rho v &= (g \otimes A)r(f \otimes B)v = (g \otimes A)r(S \otimes v)(f \otimes A)\\ &= (g \otimes A)ks(f \otimes A) = n\overline{g}s(f \otimes A) = n\sigma.\end{aligned}$$

This concludes the proof. □

Corollary 4.4.5 *Every morphism of fields $f\colon K \longrightarrow L$ is an effective descent morphism in the dual of the category of rings.*

Proof The field L is a K-vector space and the K-linear map f is injective, as a field homomorphism. Thus K is a sub-K-vector-space of L and has therefore a complementary sub-K-vector-space V. So $L \cong K \oplus V$ and the projection of the direct sum onto K is the expected retraction.

An alternative proof follows easily from proposition 4.4.3. □

The last result of this section exhibits an interesting class of effective descent morphisms; this is a special case of a more general result valid for exact categories (see [50]). This result will be useful in section 5.2 to apply Galois theory to the study of central extensions of groups.

Lemma 4.4.6 *In a category which is monadic over the category of sets, every regular epimorphism is an effective descent morphism.*

Proof Let $\mathbb{T}$ be a monad over Set and consider the corresponding category $\mathsf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras. In this category, a regular epimorphism $\sigma\colon (S,\xi) \longrightarrow (R,\zeta)$ is exactly a surjective morphism (see [8], volume 2). Via the axiom of choice, we fix a section $\tau\colon R \longrightarrow S$ of σ in the category of sets; thus $\sigma \circ \tau = \mathsf{id}_R$.

We shall use the Beck criterion to prove the statement. In $\mathsf{Set}^{\mathbb{T}}$, the functor σ^* admits as left adjoint the functor Σ_σ of composition with σ, which is one of the conditions of this criterion.

To prove that σ^* reflects isomorphisms, consider diagram 4.4, where the quadrilaterals containing σ are pullbacks. Since σ is surjective, so are σ_A and σ_B. If $\sigma^*(\gamma)$ is an isomorphism, then $\sigma_B \circ \sigma^*(\gamma) = \gamma \circ \sigma_A$ is surjective, from which γ is surjective. To prove that γ is also injective, consider $x, y \in A$ such that $\gamma(x) = \gamma(y)$. Since σ is surjective, we choose

$$\begin{tikzcd}
& (S,\xi)\times_{(R,\zeta)}(B,b) \arrow[rr, two heads, "\sigma_B"] \arrow[dd, "\beta'"'] & & (B,b) \arrow[dd, "\beta"] \\
(S,\xi)\times_{(R,\zeta)}(A,a) \arrow[ur, "\sigma^*(\gamma)"] \arrow[rr, two heads, "\sigma_A"] \arrow[dr, "\alpha'"'] & & (A,a) \arrow[ur, "\gamma"] \arrow[dr, "\alpha"] & \\
& (S,\xi) \arrow[rr, two heads, "\sigma"'] & & (R,\zeta)
\end{tikzcd}$$

Diagram 4.4

$$\begin{tikzcd}
& & (Q,\chi) \arrow[rr, "\theta"] \arrow[dl, "r"'] \arrow[dddl, "q'"] & & (C,c) \arrow[dddl, "\gamma"] \\
& (S,\xi)\times_{(R,\zeta)}(B,b) \arrow[ur, "p"'] \arrow[rr, "\sigma_B"] \arrow[dl, "t"'] \arrow[dd, "\beta'"'] & & (B,b) \arrow[ur, "w"] \arrow[dd, "\beta"] & \\
(S,\xi)\times_{(R,\zeta)}(A,a) \arrow[ur, shift left, "u'"] \arrow[ur, shift right, "v'"'] \arrow[rr, "\sigma_A"] \arrow[dr, "\alpha'"'] & & (A,a) \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] \arrow[dr, "\alpha"] & & \\
& (S,\xi) \arrow[rr, "\sigma"'] & & (R,\zeta) &
\end{tikzcd}$$

Diagram 4.5

$s \in S$ such that $\sigma(s) = \beta\gamma(x) = \beta\gamma(y)$. We have then

$$\sigma(s) = \beta\gamma(x) = \alpha(x), \quad \sigma(s) = \beta\gamma(y) = \alpha(y);$$

thus $(s,x) \in S \times_R A$ and $(s,y) \in S \times_R A$. But

$$\sigma^*(\gamma)(s,x) = \big(s,\gamma(x)\big) = \big(s,\gamma(y)\big) = \sigma^*(\gamma)(s,y)$$

and thus $(s,x) = (s,y)$ since $\sigma^*(\gamma)$ is an isomorphism. Finally $x = y$, which proves the injectivity of γ. Thus γ is an isomorphism.

To check the last condition in the Beck criterion, we refer to diagram 4.5. Consider two morphisms $u,v\colon \big((A,a),\alpha\big) \rightrightarrows \big((B,b),\beta\big)$ in

$$\begin{tikzcd}
 & & \overline{Q} \arrow[rr, "\tau_Q"] \arrow[dl, shift right, "\overline{r}"'] \arrow[dddl, "\overline{q}"] & & Q \arrow[rr, "\theta"] \arrow[dl, shift right, "r"'] \arrow[dddl, "q'"] & & C \arrow[dddl, "\gamma"] \\
 & B \arrow[rr, "\tau_B"] \arrow[ur, shift right, "\overline{p}"'] \arrow[dl, shift right, "\overline{t}"'] \arrow[dd, "\beta"] & & S \times_R B \arrow[rr, "\sigma_B"] \arrow[ur, shift right, "p"'] \arrow[dl, shift right, "t"'] \arrow[dd, "\beta'"] & & B \arrow[ur, "w"] \arrow[dd, "\beta"] & \\
A \arrow[rr, "\tau_A"] \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] \arrow[dr, "\alpha"'] & & S \times_R A \arrow[rr, "\sigma_A"] \arrow[ur, shift left, "u'"] \arrow[ur, shift right, "v'"'] \arrow[dr, "\alpha'"'] & & A \arrow[ur, shift left, "u"] \arrow[ur, shift right, "v"'] \arrow[dr, "\alpha"'] & & \\
 & R \arrow[rr, "\tau"'] \arrow[rrrr, bend right, "\mathrm{id}"'] & & S \arrow[rr, "\sigma"'] & & R &
\end{tikzcd}$$

Diagram 4.6

$\mathsf{Set}^{\mathbb{T}}/(R,\zeta)$ such that the pair $\big(\sigma^*(u),\sigma^*(v)\big)$ admits in $\mathsf{Set}^{\mathbb{T}}/(S,\xi)$ the coequalizer p which is split by morphisms r and t. To shorten notation, we write $u' = \sigma^*(u)$ and $v' = \sigma^*(v)$. Putting $w = \mathsf{Coker}\,(u,v)$ in $\mathsf{Set}^{\mathbb{T}}$, from $\beta \circ u = \alpha = \beta \circ v$ we get γ such that $\gamma \circ w = \beta$, yielding $w = \mathsf{Coker}\,(u,v)$ in $\mathsf{Set}^{\mathbb{T}}/(R,\zeta)$. We must prove that $p = \sigma^*(w)$. First, there is a factorization θ between the two coequalizer diagrams. An easy diagram chasing shows $\gamma \circ \theta \circ p = \sigma \circ q' \circ p$, from which $\gamma \circ \theta = \sigma \circ q'$, because p is a surjection. In the category of sets, let us further compute the pullbacks along the map $\tau\colon R \longrightarrow S$, as in diagram 4.6. The split coequalizer p is preserved by every functor, thus

$$\begin{tikzcd}
A \arrow[rr, shift left, "u"] \arrow[rr, shift right, "v"'] & & B \arrow[ll, bend right, "\overline{t}"'] \arrow[rr, "\overline{p}"'] & & \overline{Q} \arrow[ll, bend right, "\overline{r}"']
\end{tikzcd}$$

is a split coequalizer in Set. By the Beck criterion applied to the morphisms $u, v\colon (A,a) \rightrightarrows (B,b)$ and the forgetful functor $\mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}$, the coequalizer $w = \mathsf{Coker}\,(u,v)$ in $\mathsf{Set}^{\mathbb{T}}$ is also the coequalizer in Set, that is, $\theta \circ \tau_Q$ is a bijection. But since $\overline{p} \cong w$ is a split coequalizer of (u,v) in Set, the diagram

$$S \times_R A \overset{u'}{\underset{v'}{\rightrightarrows}} S \times_R B \xrightarrow{\sigma^*(w)} S \times_R C$$

is a coequalizer in Set. This proves that in Set, the factorization ψ in the diagram

$$\begin{tikzcd}
(S,\xi)\times_{(R,\zeta)}(B,b) \arrow[r, "p"] \arrow[dr, "\sigma^*(w)"'] & (Q,\chi) \arrow[d, dotted, "\psi"] \\
 & (S,\xi)\times_{(R,\zeta)}(C,c)
\end{tikzcd}$$

is bijective. Therefore ψ is an isomorphism in $\mathsf{Set}^{\mathbb{T}}$. □

4.5 Morphisms of Galois descent

This section generalizes, to the case of rings, the notion of "Galois extension of fields".

Definition 4.5.1 Let $\sigma\colon R \longrightarrow S$ be a morphism of rings. Write β for the unit of the adjunction

$$(S\text{-}\mathsf{Alg})^{\mathrm{op}} \underset{\mathsf{Sp}_S}{\overset{\mathcal{C}_S}{\rightleftarrows}} \mathsf{Prof}/\mathsf{Sp}(S)$$

described in theorem 4.3.6. For the sake of clarity, we express each β_A in the category S-Alg, not in its dual. An R-algebra A is split by σ when the morphism

$$\beta_{S\otimes_R A}\colon \mathcal{C}_S\mathsf{Sp}_S(S\otimes_R A) \longrightarrow S\otimes_R A$$

is an isomorphism.

Definition 4.5.2 A morphism $\sigma\colon R \longrightarrow S$ of rings is of (effective) Galois descent (also called 'normal') when

(i) σ is an effective descent morphism in the dual of the category of rings,
(ii) for every object $(X,\varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$, the R-algebra $\mathcal{C}_S(X,\varphi)$ is split by σ.

Let us mention that the second condition could be required only for $X = \{*\}$. Let us also mention that with every ring R can be associated a "separable closure" $\overline{R}$ of R, with the property that the inclusion $R \hookrightarrow \overline{R}$ is a Galois descent morphism. We shall not need this result in this book.

We prove now that the previous notions extend the situation we have studied in the special case of fields (see also 4.7.16).

Lemma 4.5.3

(i) *Every finite dimensional Galois extension of fields is a Galois descent morphism.*

(ii) *In these conditions, every finite dimensional K-algebra which is split by L in the sense of definition 2.3.1 is also split by σ.*

Proof We recall that a field admits only 0 and 1 as idempotents, thus its Pierce spectrum is a singleton. By corollary 4.4.5, σ is an effective descent morphism, and corollary 4.3.5 holds in the present case. We shall prove that for every finite dimensional K-algebra A which is split by L in the sense of 2.3.1 and every profinite space X, the canonical morphism

$$\mathcal{C}\Big(\mathsf{Sp}\big(A \otimes_K \mathcal{C}(X,L)\big), L\Big) \longrightarrow A \otimes_K \mathcal{C}(X,L)$$

is an isomorphism. Putting $A = L$ (see proposition 2.3.2) will yield that $\mathcal{C}(X,L)$ is split by σ. Putting $X = \{*\}$ will prove that A is split by σ.

By corollary 4.3.5, $\mathcal{C}(X,L) \cong L \otimes_K \mathcal{C}(X,K)$. Using proposition 2.3.2, we get

$$\begin{aligned} A \otimes_K \mathcal{C}(X,L) &\cong A \otimes_K L \otimes_K \mathcal{C}(X,K) \\ &\cong L^n \otimes_K \mathcal{C}(X,K) \\ &\cong \big(L \otimes_K \mathcal{C}(X,K)\big)^n \\ &\cong \mathcal{C}(X,L)^n. \end{aligned}$$

Let us recall that $\mathsf{Sp}\colon K\text{-}\mathsf{Alg} \longrightarrow \mathsf{Prof}$ transforms limits into colimits, because it has an adjoint functor. Therefore

$$\begin{aligned} \mathcal{C}\Big(\mathsf{Sp}\big(A \otimes_K \mathcal{C}(X,L)\big), L\Big) &\cong \mathcal{C}\Big(\mathsf{Sp}\big(\mathcal{C}(X,L)^n\big), L\Big) \\ &\cong \mathcal{C}\left(\coprod_{i=1}^{n} \mathsf{Sp}\,\mathcal{C}(X,L), L\right) \\ &\cong \mathcal{C}\left(\coprod_{i=1}^{n} X, L\right) \\ &\cong \prod_{i=1}^{n} \mathcal{C}(X,L) \\ &\cong A \otimes_K \mathcal{C}(X,L) \end{aligned}$$

where we have used proposition 4.3.3 for the isomorphism $\mathsf{Sp}\,\mathcal{C}(X,L)\cong X$. □

Proposition 4.5.4

(i) *Every Galois extension of fields $\sigma\colon K\longrightarrow L$ is a Galois descent morphism.*

(ii) *In these conditions, every K-algebra which is split by L in the sense of definition 2.3.1 is also split by σ.*

Proof As in the proof of 4.5.3, we recall that a field admits only 0 and 1 as idempotents, thus its Pierce spectrum is a singleton. By corollary 4.4.5, σ is an effective descent morphism; corollary 4.3.5 and proposition 4.3.3 hold in the present case. We shall prove that for every K-algebra A which is split by L in the sense of 2.3.1 and every profinite space X, the canonical morphism

$$\mathcal{C}\Big(\mathsf{Sp}\big(A\otimes_K \mathcal{C}(X,L)\big),L\Big)\longrightarrow A\otimes_K \mathcal{C}(X,L)$$

is an isomorphism. Putting $A=L$ (see proposition 2.3.2) will yield that $\mathcal{C}(X,L)$ is split by σ. Putting $X=\{*\}$ will prove that A is split by σ.

By proposition 3.1.5, every algebra A which is split by L in the sense of 2.3.1 is a filtered colimit of its finite dimensional subalgebras $B\subseteq A$, each of these being itself split, in the sense of 2.3.1, by a finite dimensional Galois extension $K\subseteq M_B\subseteq L$. By proposition 3.1.4, the extension L is itself the filtered colimit of its finite dimensional Galois subextensions $K\subseteq M\subseteq L$. We thus have

$$A\otimes_K L\cong\Big(\operatorname*{colim}_B B\Big)\otimes_K\Big(\operatorname*{colim}_M M\Big)\cong\operatorname*{colim}_{(B,M)} B\otimes_K M$$

where B and M are as above. We can equivalently compute the last filtered colimit on a cofinal subset of indices, for example, via proposition 3.1.5, by restricting our attention to those pairs (B,M) such that B is split by M in the sense of 2.3.1. Having made this choice, we get the following isomorphisms, using the same arguments as in the proof of lemma 4.5.3 and the fact that the functor $\mathsf{Sp}\colon K\text{-}\mathsf{Alg}\longrightarrow\mathsf{Prof}$ transforms filtered colimits into cofiltered limits (see proposition 4.3.7).

$$\begin{aligned}
&\mathcal{C}\big(\mathsf{Sp}\big(A\otimes_K\mathcal{C}(X,L)\big),L\big)\\
&\quad\cong L\otimes_K\mathcal{C}\Big(\mathsf{Sp}\big(A\otimes_K L\otimes_K\mathcal{C}(X,K)\big),K\Big)\\
&\quad\cong L\otimes_K\mathcal{C}\Big(\mathsf{Sp}\big((\operatorname{colim}_{(B,M)} B\otimes_K M)\otimes_K\mathcal{C}(X,K)\big),K\Big)
\end{aligned}$$

$$\begin{aligned}
&\cong L \otimes_K \mathcal{C}\Big(\mathsf{Sp}\big(\mathsf{colim}_{(B,M)} B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \\
&\cong L \otimes_K \mathcal{C}\Big(\mathsf{lim}_{(B,M)} \mathsf{Sp}\big(B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \\
&\cong L \otimes_K \mathsf{colim}_{(B,M)} \mathcal{C}\Big(\mathsf{Sp}\big(B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \quad (*) \\
&\cong \Big(\mathsf{colim}_{M'} M'\Big) \otimes_K \mathsf{colim}_{(B,M)} \mathcal{C}\Big(\mathsf{Sp}\big(B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \\
&\cong \mathsf{colim}_{(B,M,M')} M' \otimes_K \mathcal{C}\Big(\mathsf{Sp}\big(B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \\
&\cong \mathsf{colim}_{(B,M)} M \otimes_K \mathcal{C}\Big(\mathsf{Sp}\big(B \otimes_K M \otimes_K \mathcal{C}(X,K)\big), K\Big) \quad (**) \\
&\cong \mathsf{colim}_{(B,M)} \mathcal{C}\Big(\mathsf{Sp}\big(B \otimes_K \mathcal{C}(X,M)\big), M\Big) \\
&\cong \mathsf{colim}_{(B,M)} B \otimes_K \mathcal{C}(X,M) \quad \text{see 4.5.3} \\
&\cong \mathsf{colim}_{(B,M)} B \otimes_K M \otimes_K \mathcal{C}(X,K) \\
&\cong \big(\mathsf{colim}_B B\big) \otimes_K \big(\mathsf{colim}_M M\big) \otimes_K \mathcal{C}(X,K) \\
&\cong L \otimes_K A \otimes_K \mathcal{C}(X,K) \\
&\cong A \otimes_K \mathcal{C}(X,L).
\end{aligned}$$

The argument $(*)$ holds since the functor $\mathcal{C}(-,K)\colon \mathsf{Prof} \longrightarrow (K\text{-}\mathsf{Alg})^{\mathsf{op}}$ has the left adjoint Sp, thus preserves limits. The argument $(**)$ reduces to the computation of the colimit on a cofinal subset of indices. □

We conclude this section with a useful technical result, showing how the spectrum construction is naturally present in the theory developed in the previous chapters.

Proposition 4.5.5 *Let $\sigma\colon K \longrightarrow L$ be a Galois extension of fields. The functors, defined on the category of K-algebras split by L in the sense of definition 2.3.1,*

$$\mathsf{Split}_K(L) \longrightarrow \mathsf{Prof}, \qquad A \mapsto \mathsf{Hom}_K(A,L),$$
$$\mathsf{Split}_K(L) \longrightarrow \mathsf{Prof}, \qquad A \mapsto \mathsf{Sp}(A \otimes_K L)$$

(see lemma 3.5.3) are isomorphic.

Proof When the K-algebra A is finite dimensional, we know by proposition 3.1.5 that $\mathsf{Hom}_K(A,L) \cong \mathsf{Hom}_K(A,M)$ for some finite dimensional Galois extension $K \subseteq M \subseteq L$, and this isomorphism is clearly natural in A. Moreover, by theorem 2.3.3, $\#\mathsf{Hom}_K(A,M) = n = \dim A$ while $A \otimes_K L \cong L^n$, again with naturality properties, hidden in particular in the cardinality argument. A finite profinite space is discrete, thus

$\mathsf{Hom}_K(A, L)$ is the n point discrete space. On the other hand $\mathsf{Sp}(L^n)$ is the discrete n point space too, since L has only 0 and 1 as idempotents.

Now when A is infinite dimensional, by lemma 3.5.3

$$\mathsf{Hom}_K(A, L) \cong \lim_B \mathsf{Hom}_K(B, L)$$

where B runs through the finite dimensional subalgebras of A. On the other hand, applying proposition 4.3.7, we obtain

$$\begin{aligned}\mathsf{Sp}(A \otimes_K L) &\cong \mathsf{Sp}\big((\mathsf{colim}_B\, B) \otimes_K L\big) \cong \mathsf{Sp}\big(\mathsf{colim}_B(B \otimes_K L)\big)\\ &\cong \mathsf{lim}_B\, \mathsf{Sp}(B \otimes_K L) \cong \mathsf{lim}_B\, \mathsf{Hom}_K(B, L)\\ &\cong \mathsf{Hom}_K(\mathsf{colim}_B\, B, L) \cong \mathsf{Hom}_K(A, L).\end{aligned}$$

This concludes the proof. $\square$

4.6 Internal presheaves

In this section, "presheaf" will always mean "covariant presheaf". A covariant presheaf on a small category $\mathcal{C}$ is thus, classically, a functor $P: \mathcal{C} \longrightarrow \mathsf{Set}$. And a small category has a *set* of objects and a *set* of morphisms. It is well known that this situation can be generalized by replacing all *sets* by objects of an arbitrary category $\mathcal{X}$ with pullbacks. To avoid losing ourselves through heavy technical considerations, we recall only the spirit of the definitions and refer to section 7.1 or [8], volume 1, for the details.

An internal category in $\mathcal{X}$ consists in giving the following situation:

$$\begin{tikzcd} C_2 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n" description] \end{tikzcd}$$

where C_2 is defined by the following pullback:

$$\begin{tikzcd} C_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

One should think of

- C_0 as the "object of objects" of $\mathcal{C}$,

- C_1 as the "object of arrows" of $\mathcal{C}$,
- d_0 as the "domain morphism",
- d_1 as the "codomain morphism",
- n as the "identity morphism",
- C_2 as the "object of composable pairs",
- m as the "composition morphism".

It remains to impose on these data the diagrammatical transcription of the axioms of units and associativity.

The internal category is an internal groupoid when there exists an additional morphism $s\colon C_1 \longrightarrow C_1$ with axioms indicating that s formally inverts the arrows of $\mathcal{C}$.

We leave to the reader the definitions of an internal functor between two internal categories in $\mathcal{X}$ and of an internal natural transformation between internal functors: we shall not need these notions.

An internal covariant presheaf on the internal category $\mathcal{C}$ is a triple (P, p, π) where

(i) $p\colon P \longrightarrow C_0$ is a morphism of $\mathcal{X}$,
(ii) $\pi\colon P_1 \longrightarrow P$ is a morphism of $\mathcal{X}$, where P_1 is defined by the first square below, which is a pullback,

$$
\begin{tikzcd}
P_1 \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & P \arrow[d, "p"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P_1 \arrow[r, "\pi"] \arrow[d, "\alpha_1"'] & P \arrow[d, "p"] \\
C_1 \arrow[r, "d_1"'] & C_0
\end{tikzcd}
$$

while the second square above is commutative.

To get an intuition of this notion, consider the case $\mathcal{X} = \mathsf{Set}$. Giving $p\colon P \longrightarrow C_0$ is equivalent to giving the family $\big(p^{-1}(C)\big)_{C \in C_0}$; for each $C \in C_0$, we put $P(C) = p^{-1}(C)$. This defines the functor P on the objects. Next, we have by definition

$$P_1 = \big\{(f, a) \big| f \text{ arrow of } \mathcal{C},\quad a \in P(\text{domain of } f)\big\}.$$

Giving π as indicated in condition (ii) consists, for each pair (f, a) as above, in giving an element of $P(\text{codomain of } f)$, which we choose as $P(f)(a)$. This defines the functor P on the morphisms. It remains to impose axioms expressing respectively the compatibility of P with identities and composition.

Finally given two internal covariant presheaves (P,p,π), (P',p',π') on an internal category $\mathcal{C}$, an internal natural transformation

$$\gamma\colon (P,p,\pi) \Longrightarrow (P',p',\pi')$$

is a morphism $\alpha\colon P \longrightarrow P'$ such that $p' \circ \alpha = p$. In the case $\mathcal{X} = \mathsf{Set}$, the condition $p' \circ \alpha = p$ means precisely that γ splits as a family of maps

$$\gamma_C\colon P(C) = p^{-1}(C) \longrightarrow (p')^{-1}(C) = P'(C).$$

It remains to impose diagrammatically the naturality axiom.

It is straightforward to observe that internal covariant presheaves and internal natural transformations on an internal category $\mathcal{C}$ constitute a category $\mathcal{X}^{\mathcal{C}}$.

Proposition 4.6.1 *Let $\mathcal{X}$ be a category with pullbacks and $\mathcal{X}^{\mathcal{C}}$ the category of internal covariant presheaves on an internal category $\mathcal{C}$. The functor*

$$\mathcal{X}^{\mathcal{C}} \longrightarrow \mathcal{X}/C_0, \quad (P,p,\pi) \mapsto (P,p)$$

is monadic and the corresponding monad admits as functorial part the composite

$$\mathcal{X}/C_0 \xrightarrow{d_0^*} \mathcal{X}/C_1 \xrightarrow{\Sigma_{d_1}} \mathcal{X}/C_0$$

where d_0^* is pulling back along d_0 and Σ_{d_1} is composition with d_1.

Proof This is again a classical result (see [55]); therefore we shall only recall a sketch of the proof. One first writes $T = \Sigma_{d_1} \circ d_0^*$. In the case $\mathcal{X} = \mathsf{Set}$, an object of $\mathcal{X}/C_0$ is a family $(X_C)_{C \in C_0}$ of sets, and

$$T\Big((X_C)_{C \in C_0}\Big) = \left(\coprod_{d_1(f)=C} X_{d_0(f)}\right)_{C \in C_0}.$$

With this special case in mind, one observes that composition and identities in $\mathcal{C}$ induce at once the structure of a monad on T. An algebra for this monad is then a pair $\big((P,p),\pi\big)$ where $(P,p) \in \mathcal{X}/C_0$ and $\pi\colon T(P,p) \longrightarrow (P,p)$. Thus p and π make commutative the diagram:

$$\begin{tikzcd}
P & P_1 \arrow[l, "\pi"'] \arrow[r, "\alpha_1"] \arrow[d, "\alpha_1"'] \arrow[dr, phantom, "\text{p.b.}"] & P \arrow[d, "p"] \\
C_0 \arrow[from=u, "p"'] & C_1 \arrow[l, "d_1"] \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

This yields the same data as in the description of an internal covariant presheaf. It remains to observe that the axioms for being a T-algebra are equivalent to those for being an internal covariant presheaf, and analogously for the morphisms. $\square$

It is probably more interesting to really grasp the intuitive meaning of this result in the case of $\mathcal{X} = \mathsf{Set}$. In this case, writing $\mathcal{C}_0$ for the discrete category on the set C_0, it follows at once that the category Set/C_0 is isomorphic to the category of covariant presheaves $\mathsf{Set}^{\mathcal{C}_0}$. Indeed, in both cases, we recapture the category of C_0-indexed families of sets and maps. The functor

$$\mathsf{Set}^{\mathcal{C}} \longrightarrow \mathsf{Set}^{\mathcal{C}_0}$$

we are interested in is composition with the inclusion $i\colon \mathcal{C}_0 \longrightarrow \mathcal{C}$ and we are interested in its monadicity. This functor has a left adjoint, namely the Kan extension along i. This same functor also reflects isomorphisms (that is, natural transformations all of whose components are bijective) simply because i is bijective on the objects (a natural transformation α is an isomorphism when each α_C is an isomorphisms). Finally both categories have coequalizers computed pointwise, thus preserved by composition with i. By the Beck criterion (see [66]), we get the expected monadicity.

Example 4.6.2 Let $\sigma\colon S \longrightarrow R$ be a morphism in a category $\mathcal{X}$ with pullbacks. The diagram

$$\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[r, "(p_1, p_4)"] & S \times_R S \arrow[loop below, "\tau"] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & S \arrow[l, "\Delta"']
\end{tikzcd}$$

is an internal groupoid, with (p_0, p_1) the kernel pair of σ, Δ the diagonal of the pullback and τ the twisting isomorphism which interchanges factors.

Proof The proof is obvious. $\square$

In the previous example, the object $S \times_R S$ of morphisms is in fact a subobject of $S \times S$, where S is the object of arrows. Thus the internal category is in fact an internal preorder, thus a reflective and transitive relation. Being a groupoid expresses the symmetry of the relation. Thus the previous example is a reformulation of the well-known fact that the kernel pair of σ is an equivalence relation on S.

4.7 The Galois theorem for rings

Convention *Throughout this section, $\sigma\colon R \longrightarrow S$ indicates a Galois descent morphism of rings, in the sense of definition 4.5.2. This fact will generally not be recalled in the various statements of the present section.*

We shall now develop, in the special case of rings, the Galois theory of Janelidze. This is a special case of the more general theory of the same author, developed in section 5.1.

Lemma 4.7.1 *The following conditions are equivalent for an S-algebra A:*

(i) *A is split by $\mathsf{id}_S\colon S =\!=\!= S$;*
(ii) *the canonical morphism $\mathcal{C}_S\big(\mathsf{Sp}_S(A)\big) \longrightarrow A$ is an isomorphism.*

Proof One has $S \otimes_S A \cong A$ and thus $\mathcal{C}_S\big(\mathsf{Sp}_S(S \otimes_S A)\big) \cong \mathcal{C}_S\big(\mathsf{Sp}_S(A)\big)$, from which the result follows by 4.5.1. □

Corollary 4.7.2 *The following conditions are equivalent for an R-algebra A:*

(i) *A is split by $\sigma\colon R \longrightarrow S$;*
(ii) *the S-algebra $S \otimes_R A$ is split by $\mathsf{id}_S\colon S =\!=\!= S$.* □

Lemma 4.7.3 *The following conditions are equivalent for an S-algebra A:*

(i) *A is split by $\mathsf{id}_S\colon S =\!=\!= S$;*
(ii) *$A \cong \mathcal{C}_S(X, \varphi)$ for some $(X, \varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$.*

Proof Putting $(X, \varphi) = \mathsf{Sp}_S(A)$, lemma 4.7.1 yields (i) ⇒ (ii). Conversely, by lemma 4.7.1 again and theorem 4.3.6

$$A \cong \mathcal{C}_S(X, \varphi) \cong \mathcal{C}_S\big(\mathsf{Sp}_S\mathcal{C}_S(X, \varphi)\big) \cong \mathcal{C}_S\big(\mathsf{Sp}_S(A)\big). \qquad \square$$

Corollary 4.7.4 *The following conditions are equivalent for an R-algebra A:*

(i) *A is split by $\sigma\colon R \longrightarrow S$;*
(ii) *$S \otimes_R A \cong \mathcal{C}(X, \varphi)$ for some $(X, \varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$.* □

Lemma 4.7.5 *Let us write $\mathsf{Split}_S(S)$ for the category of S-algebras split by $\mathsf{id}_S\colon S = S$. The functors*

$$\big(\mathsf{Split}_S(S)\big)^{\mathsf{op}} \underset{\mathsf{Sp}_S}{\overset{\mathcal{C}_S}{\rightleftarrows}} \mathsf{Prof}/\mathsf{Sp}(S)$$

constitute an equivalence of categories.

Proof By lemma 4.7.3, $\mathcal{C}_S$ indeed takes values in $\mathsf{Split}_S(S)$. One has $\mathsf{Sp}_S \circ \mathcal{C}_S \cong \mathsf{id}$ by theorem 4.3.6 and $\mathcal{C}_S \circ \mathsf{Sp}_S \cong \mathsf{id}$ by lemma 4.7.1. □

Lemma 4.7.6 *If A is an S-algebra split by $\mathsf{id}_S\colon S = S$, then $S \otimes_R A$ is another S-algebra split by $\mathsf{id}_S\colon S = S$.*

Proof By lemma 4.7.3, $A \cong \mathcal{C}_S(X,\varphi)$ for some $(X,\varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$. Since σ is of Galois descent, $\mathcal{C}_S(X,\varphi)$ is an R-algebra split by σ, thus, by corollary 4.7.2, $S \otimes_R \mathcal{C}_S(X,\varphi)$ is an S-algebra split by id_S. □

Lemma 4.7.7 *The ring S is an S-algebra which is split by $\mathsf{id}_S\colon S = S$.*

Proof The algebra S is the initial object of the category S-Alg of S-algebras and $\mathsf{Sp}_S(S) = \big(\mathsf{Sp}(S) = \mathsf{Sp}(S)\big)$ is the final object of the category $\mathsf{Prof}/\mathsf{Sp}(S)$. The functor $\mathcal{C}_S\colon \mathsf{Prof}/\mathsf{Sp}(S) \longrightarrow (S\text{-}\mathsf{Alg})^{\mathsf{op}}$ preserves the terminal object, since it has a left adjoint Sp_S (see 4.5.1), thus $\mathcal{C}_S\big(\mathsf{Sp}_S(S)\big) \cong S$. One concludes the proof by lemma 4.7.1. □

Corollary 4.7.8 *The ring R is an R-algebra split by $\sigma\colon R \longrightarrow S$.* □

Lemma 4.7.9 *For every integer $n \in \mathbb{N}$, consider $\otimes_{i=1}^n S = S \otimes_R \cdots \otimes_R S$ provided with the structure of S-algebra induced by the multiplication on the first factor. These S-algebras $\otimes_{i=1}^n S$ are split by $\mathsf{id}_S\colon S = S$.*

Proof By lemma 4.7.7 and an iterated application of lemma 4.7.6. □

Let us generalize further some of the previous arguments. If A and B are S-algebras, $A \otimes_R B$ can be provided with two canonical structures of S-algebra:

$$\begin{aligned} s\left(\sum_i a_i \otimes b_i\right) &= \sum_i (sa_i) \otimes b_i, \\ s\left(\sum_i a_i \otimes b_i\right) &= \sum_i a_i \otimes (sb_i). \end{aligned}$$

Let us write $A \otimes_R^1 B$ and $A \otimes_R^2 B$ to distinguish those two structures of S-algebra. One has obviously $A \otimes_R^1 B \cong B \otimes_R^2 A$. Observe that $\otimes^1$ was already used in the proof of lemmas 4.7.1 and 4.7.6.

Lemma 4.7.10 *If A and B are S-algebras split by id_S, then so are $A \otimes_R^1 B$ and $A \otimes_R^2 B$.*

Proof It suffices to develop the proof for $A \otimes_R^1 B$. Obviously $A \otimes_R^1 B \cong A \otimes_S (S \otimes_R^1 B)$, with $S \otimes_R^1 B$ split by id_S, by lemma 4.7.6. So putting $C = S \otimes_R^1 B$, it suffices to prove that

$$(A \text{ and } C \text{ split by } \mathrm{id}_S) \Rightarrow (A \otimes_S C \text{ split by } \mathrm{id}_S).$$

This implication follows from lemma 4.7.3 and the fact that the functor

$$\mathcal{C}_S \colon \mathsf{Prof}/\mathsf{Sp}(S) \longrightarrow (S\text{-}\mathsf{Alg})^{\mathrm{op}}$$

preserves products, since it is a right adjoint. □

Lemma 4.7.11 *Let us write $\mathsf{Split}_R(\sigma)$ for the category of R-algebras split by σ. The functor*

$$S \otimes_R - \colon \mathsf{Split}_R(\sigma) \longrightarrow \mathsf{Split}_S(S)$$

is comonadic.

Proof By corollary 4.7.2, this functor is correctly defined. Observe that given $A \in \mathsf{Split}_S(S)$, lemma 4.7.10 implies $S \otimes_R A \in \mathsf{Split}_S(S)$ and thus $A \in \mathsf{Split}_R(\sigma)$ by corollary 4.7.2. Therefore the classical adjunction

$$R\text{-}\mathsf{Alg} \underset{S \otimes_R -}{\overset{U}{\rightleftarrows}} S\text{-}\mathsf{Alg}, \quad U(A) = A$$

restricts, via corollary 4.7.2, to an adjunction

$$\mathsf{Split}_R(\sigma) \underset{S \otimes_R -}{\overset{U}{\rightleftarrows}} \mathsf{Split}_S(S), \quad S \otimes_R - \dashv U.$$

This is one of the conditions of the Beck criterion.

Since σ is an effective descent morphism,

$$S \otimes_R - \colon R\text{-}\mathsf{Alg} \longrightarrow S\text{-}\mathsf{Alg}$$

is comonadic (see 4.4.1 and 4.5.2), thus reflects isomorphisms. Therefore its restriction to split algebras reflects isomorphisms as well.

Finally consider two morphisms $u, v \colon A \rightrightarrows B$ in $\mathsf{Split}_R(\sigma)$ whose images under $(S \otimes_R -)$ admit a split equalizer

$$
\begin{tikzcd}
K \arrow[r, "k"'] & S \otimes_R A \arrow[l, bend right, "r"'] \arrow[r, shift left, "S \otimes u"] \arrow[r, shift right, "S \otimes v"'] & S \otimes_R B \arrow[l, bend right, "t"']
\end{tikzcd}
$$

in $\mathsf{Split}_S(S)$. This split equalizer is preserved by every functor, thus in particular by the inclusion in S-Alg. The Beck criterion in the case of the functor

$$S \otimes_R -\colon R\text{-}\mathsf{Alg} \longrightarrow S\text{-}\mathsf{Alg}$$

implies thus the existence of an equalizer

$$
\begin{tikzcd}
N \arrow[r, "n"] & A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B
\end{tikzcd}
$$

in R-Alg which is preserved by $S \otimes_R -$, thus mapped onto k. In particular, $S \otimes_R N \cong K \in \mathsf{Split}_S(S)$, which implies by corollary 4.7.2 that $N \in \mathsf{Split}_R(\sigma)$. □

Corollary 4.7.12 *The functor*

$$\big(\mathsf{Split}_R(\sigma)\big)^{\mathsf{op}} \longrightarrow \mathsf{Prof}/\mathsf{Sp}(S), \quad A \mapsto \mathsf{Sp}_S(S \otimes_R A)$$

is monadic.

Proof This functor is the composite of functors in lemmas 4.7.11 and 4.7.5. □

For clarity, let us present on a single diagram the various functors we have already studied.

$$
\begin{tikzcd}
\mathsf{Split}_R(\sigma) \arrow[r, shift right, "S \otimes_R -"'] \arrow[d, tail] & \mathsf{Split}_S(S) \arrow[l, shift right, "U"'] \arrow[r, shift right, "\mathsf{Sp}_S"'] \arrow[d, tail] & \mathsf{Prof}/\mathsf{Sp}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \arrow[d, equal] \\
R\text{-}\mathsf{Alg} \arrow[r, shift right, "S \otimes_R -"'] & S\text{-}\mathsf{Alg} \arrow[l, shift right, "U"'] \arrow[r, shift right, "\mathsf{Sp}_S"'] & \mathsf{Prof}/\mathsf{Sp}(S) \arrow[l, shift right, "\mathcal{C}_S"']
\end{tikzcd}
$$

The next lemma is crucial, since it allows defining the Galois groupoid of the morphism $\sigma\colon R \longrightarrow S$.

Lemma 4.7.13 *Consider the cokernel pair of $\sigma\colon R \longrightarrow S$ in the category of R-algebras, viewed as a groupoid in the dual category (see example 4.6.2). The functor*

$$\mathsf{Sp}\colon (R\text{-}\mathsf{Alg})^{\mathsf{op}} \longrightarrow \mathsf{Prof}$$

transforms this groupoid into another groupoid $\mathsf{Gal}\,[\sigma]$ *in the category of profinite spaces.*

Proof The cogroupoid in the category of R-algebras (that is, the groupoid in the opposite category) is thus (see section 4.4)

$$\begin{tikzcd}(S \otimes_R S) \otimes_S (S \otimes_R S) & S \otimes_R S \arrow[l, "s_1 \otimes s_2"'] \arrow[loop below, "\tau"] \arrow[r, "\mu" description] & S \arrow[l, shift right=2, "s_1"'] \arrow[l, shift left=2, "s_2"]\end{tikzcd}$$

where

$$s_1(a) = a \otimes 1, \quad s_2(a) = 1 \otimes a, \quad \mu(a \otimes b) = ab, \quad \tau(a \otimes b) = b \otimes a.$$

Notice that S is canonically an S-algebra. The objects $S \otimes_R S$ and $(S \otimes_R S) \otimes_S (S \otimes_R S)$ can be provided with two distinct structures of S-algebra (see lemma 4.7.10). With the notation of 4.7.10, let us observe that the following morphisms are morphisms of S-algebras:

$$s_1 \colon S \longrightarrow S \otimes_R^1 S, \quad s_2 \colon S \longrightarrow S \otimes_R^2 S.$$

When we refer to $(S \otimes_R S) \otimes_S (S \otimes_R S)$ as an S-algebra, it will always be as the following pushout in S-Alg, which is by the way also a pushout in R-Alg:

$$\begin{tikzcd}S \arrow[r, "s_1"] \arrow[d, "s_2"'] & S \otimes_R^1 S \arrow[d] \\ S \otimes_R^2 S \arrow[r] & (S \otimes_R^2 S) \otimes_S (S \otimes_R^1 S)\end{tikzcd}$$

By lemma 4.7.9, the S-algebras S, $S \otimes_R^1 S$ and $S \otimes_R^2 S$ are split by $\mathsf{id}_S \colon S \!=\!=\! S$. On the other hand we have the classical isomorphism

$$(S \otimes_R^2 S) \otimes_S (S \otimes_R^1 S) \cong S \otimes_R S \otimes_R S$$

where the S-module structure on $S \otimes_R S \otimes_R S$ is now given by

$$s\left(\sum_i u_i \otimes v_i \otimes w_i\right) = \sum_i u_i \otimes (sv_i) \otimes w_i.$$

Again by lemma 4.7.9, $(S \otimes_R^2 S) \otimes_S (S \otimes_R^1 S)$ is thus also an S-algebra split by id_S.

This proves in particular that the pushout defining the S-algebra structure of $(S \otimes_R^2 S) \otimes_S (S \otimes_R^1 S)$ lies entirely in the category $\mathsf{Split}_S(S)$, thus is a pushout in this category. Applying lemma 4.7.5, we get therefore a pullback in $\mathsf{Prof}/\mathsf{Sp}(S)$

$$\begin{tikzcd}
\mathsf{Sp}_S\big((S \otimes_R^2 S) \otimes_S (S \otimes_R^1 S)\big) \arrow[r] \arrow[d] & \mathsf{Sp}_S(S \otimes_R^2 S) \arrow[d, "\mathsf{Sp}_S(s_2)"] \\
\mathsf{Sp}_S(S \otimes_R^1 S) \arrow[r, "\mathsf{Sp}_S(s_1)"'] & \mathsf{Sp}_S(S)
\end{tikzcd}$$

and therefrom the following pullback in Prof:

$$\begin{tikzcd}
\mathsf{Sp}\big((S \otimes_R S) \otimes_S (S \otimes_R S)\big) \arrow[r] \arrow[d] & \mathsf{Sp}(S \otimes_R S) \arrow[d, "\mathsf{Sp}(s_2)"] \\
\mathsf{Sp}(S \otimes_R S) \arrow[r, "\mathsf{Sp}(s_1)"'] & \mathsf{Sp}(S)
\end{tikzcd}$$

This shows that applying the functor Sp to the cokernel pair of σ, viewed as a groupoid in $(R\text{-}\mathsf{Alg})^{\mathsf{op}}$, yields the following situation in Prof:

$$\begin{tikzcd}
\mathsf{Sp}(S \otimes_R S) \times_{\mathsf{Sp}(S)} \mathsf{Sp}(S \otimes_R S) \arrow[r, "\mathsf{Sp}(s_1 \otimes s_2)"] & \mathsf{Sp}(S \otimes_R S) \arrow[loop below, "\mathsf{Sp}(\tau)"] \arrow[r, shift left=2, "\mathsf{Sp}(s_1)"] \arrow[r, shift right=2, "\mathsf{Sp}(s_2)"'] & \mathsf{Sp}(S) \arrow[l, "\mathsf{Sp}(\mu)"']
\end{tikzcd}$$

with the left hand object being the pullback of $\mathsf{Sp}(s_1)$, $\mathsf{Sp}(s_2)$.

It remains to prove that the axioms for a groupoid are satisfied in Prof. But the functor Sp, like every functor, preserves the commutativity of diagrams, which yields at once all axioms, with the exception maybe of the associativity of the composition, since this axiom involves a pullback. In fact, an argument perfectly analogous to the one we have just developed proves that we also have a three factor pullback in Prof

$$\begin{aligned}
&\mathsf{Sp}\big((S \otimes_R S) \otimes_S (S \otimes_R S) \otimes_S (S \otimes_R S)\big) \\
&\qquad\qquad \cong \mathsf{Sp}(S \otimes_R S) \times_{\mathsf{Sp}(S)} \mathsf{Sp}(S \otimes_R S) \times_{\mathsf{Sp}(S)} \mathsf{Sp}(S \otimes_R S),
\end{aligned}$$

so that the associativity of the composition reduces again to the preservation by the functor Sp of the diagram expressing that associativity for the cokernel pair of σ. $\square$

Lemma 4.7.13 thus allows the following definition.

Definition 4.7.14 Let $\sigma\colon R \longrightarrow S$ be a Galois descent morphism of rings. The Galois groupoid $\mathsf{Gal}\,[\sigma]$ of σ is the following internal groupoid in the category of profinite spaces:

$$\begin{tikzcd}[column sep=large]
\mathsf{Sp}(S \otimes_R S) \times_{\mathsf{Sp}(S)} \mathsf{Sp}(S \otimes_R S) \arrow[r, "\mathsf{Sp}(s_1 \otimes s_2)"] & \mathsf{Sp}(S \otimes_R S) \arrow[loop below, "\mathsf{Sp}(\tau)"] \arrow[r, shift left=3, "\mathsf{Sp}(s_1)"] \arrow[r, shift right=3, "\mathsf{Sp}(s_2)"'] & \mathsf{Sp}(S). \arrow[l, "\mathsf{Sp}(\mu)"']
\end{tikzcd}$$

Here now is the Galois theorem for rings (commutative, with a unit).

Theorem 4.7.15 (Galois theorem) *Let $\sigma\colon R \longrightarrow S$ be a Galois descent morphism of rings and $\mathsf{Gal}\,[\sigma]$ the corresponding Galois groupoid in the category of profinite spaces. There exists an equivalence of categories*

$$\big(\mathsf{Split}_R(\sigma)\big)^{\mathsf{op}} \approx \mathsf{Prof}^{\mathsf{Gal}\,[\sigma]}$$

between the dual of the category of R-algebras split by σ and the category of internal covariant presheaves on $\mathsf{Gal}\,[\sigma]$ in the category of profinite spaces.

Proof The category $\mathsf{Prof}^{\mathsf{Gal}\,[\sigma]}$ is the category of algebras for the monad on $\mathsf{Prof}/\mathsf{Sp}(S)$ described in proposition 4.6.1. We shall prove that the category $\big(\mathsf{Split}_R(\sigma)\big)^{\mathsf{op}}$ is also monadic on $\mathsf{Prof}/\mathsf{Sp}(S)$, for a monad which is isomorphic to that given by proposition 4.6.1. This will yield the expected result.

By corollary 4.7.12, the functor

$$\big(\mathsf{Split}_R(\sigma)\big)^{\mathsf{op}} \longrightarrow \mathsf{Prof}/\mathsf{Sp}(S), \quad A \mapsto \mathsf{Sp}_S(S \otimes_R A)$$

is indeed monadic. Its left adjoint is $U \circ \mathcal{C}_S$. Observe that the functorial part of the corresponding monad is

$$T\colon \mathsf{Prof}/\mathsf{Sp}(S) \longrightarrow \mathsf{Prof}/\mathsf{Sp}(S), \quad (X,\varphi) \mapsto \mathsf{Sp}_S\big(S \otimes_R \mathcal{C}_S(X,\varphi)\big).$$

Let us now compute explicitly the form of the functorial part of the monad given by proposition 4.6.1. Given $(X,\varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$, one computes first the pushout in S-Alg

$$\begin{tikzcd}
S \arrow[r, "s_1"] \arrow[d, "\alpha_{\mathcal{C}_S(X,\varphi)}"'] & S \otimes_R^1 S \arrow[d, "\mathsf{id}_S \otimes \alpha_{\mathcal{C}_S(X,\varphi)}"] \\
\mathcal{C}_S(X,\varphi) \arrow[r] & (S \otimes_R^1 S) \otimes_S \mathcal{C}_S(X,\varphi) \cong S \otimes_R \mathcal{C}_S(X,\varphi)
\end{tikzcd}$$

where

$$\alpha_{\mathcal{C}_S(X,\varphi)} \colon S \longrightarrow \mathcal{C}_S(X,\varphi), \quad s \mapsto s \cdot 1,$$

and the structure of S-module on $S \otimes_R \mathcal{C}_S(X,\varphi)$ is given by the action of S on $\mathcal{C}_S(X,\varphi)$ (see theorem 4.3.6), that is

$$S \otimes_R \mathcal{C}_S(X,\varphi) \longrightarrow \mathcal{C}_S(X,\varphi), \quad s \otimes a \mapsto \alpha_{\mathcal{C}_S(X,\varphi)}(s) \cdot a.$$

By lemma 4.7.10, this is also a pushout in $\mathsf{Split}_S(S)$. Applying lemma 4.7.5, we get the following pullback in $\mathsf{Prof}/\mathsf{Sp}(S)$, where the isomorphism follows from theorem 4.3.6.

$$\begin{tikzcd}
\mathsf{Sp}_S\big(S \otimes_R \mathcal{C}(X,\varphi)\big) \arrow[r] \arrow[d, "\mathsf{Sp}_S\left(\mathsf{id}_S \otimes \alpha_{\mathcal{C}_S(X,\varphi)}\right)"'] & \mathsf{Sp}_S\mathcal{C}_S(X,\varphi) \cong (X,\varphi) \arrow[d, "\mathsf{Sp}_S(\alpha_{\mathcal{C}_S(X,\varphi)})"] \\
\mathsf{Sp}_S(S \otimes_R S) \arrow[r, "\mathsf{Sp}_S(s_1)"'] & \mathsf{Sp}_S(S)
\end{tikzcd}$$

Since pullbacks in $\mathsf{Prof}/\mathsf{Sp}(S)$ are computed as in Prof, we get in fact the pullback of diagram 4.7 in Prof. The left vertical composite is precisely $\big(\Sigma_{\mathsf{Sp}(s_2)} \circ \mathsf{Sp}(s_1)^*\big)(X,\varphi)$.

Since the structure of S-module on $S \otimes_R \mathcal{C}_S(X,\varphi)$ is given by the action of S on $\mathcal{C}_S(X,\varphi)$, it is induced by the corresponding morphism

$$S \xrightarrow{\quad s_2 \quad} S \otimes_R S \xrightarrow{\mathsf{id}_S \otimes \alpha_{\mathcal{C}_S(X,\varphi)}} S \otimes_R \mathcal{C}_S(X,\varphi).$$

This shows that the left vertical composite in diagram 4.7 is also the object $T(X,\varphi) \in \mathsf{Prof}/\mathsf{Sp}(S)$, for the monad T described above. This shows already that the two monads T and $\Sigma_{\mathsf{Sp}(s_2)} \circ \mathsf{Sp}(s_1)^*$ coincide on objects. It is now routine, left to the reader, to verify that the two monads considered in this proof are in fact isomorphic. $\square$

Let us conclude this section by observing that theorem 4.7.15 extends the corresponding result for fields, namely, theorem 3.5.8.

$$
\begin{tikzcd}
\mathsf{Sp}(S \otimes_R \mathcal{C}(X,\varphi)) \arrow[r] \arrow[d, "\mathsf{Sp}(\mathsf{id}_S \otimes \alpha_{\mathcal{C}_S(X,\varphi)})"'] & X \arrow[d, "\varphi"] \\
\mathsf{Sp}_S(S \otimes^1_R S) \arrow[r, "\mathsf{Sp}(s_1)"'] \arrow[d, "\mathsf{Sp}(s_2)"'] & \mathsf{Sp}(S) \\
\mathsf{Sp}(S) &
\end{tikzcd}
$$

Diagram 4.7

Corollary 4.7.16 *Let $\sigma\colon K \longrightarrow L$ be a Galois extension of fields.*

(i) *The Galois groupoid $\mathsf{Gal}\,[\sigma]$ of σ, viewed as a Galois descent morphism of rings, coincides with the usual profinite Galois group $\mathsf{Gal}\,[L : K]$ of the field extension $[L : K]$.*

(ii) *The K-algebras split by L in the sense of definition 2.3.1 coincide with the K-algebras split by σ.*

(iii) *The internal covariant presheaves on the internal groupoid $\mathsf{Gal}\,[\sigma]$ coincide with the profinite $\mathsf{Gal}\,[L : K]$-spaces.*

(iv) *The equivalence of theorem 4.7.15 reduces to the equivalence of theorem 3.5.8.*

Proof In the case of the present corollary, $\mathsf{Sp}(K) \cong \{*\} \cong \mathsf{Sp}(L)$, and therefore $\mathsf{Prof}/\mathsf{Sp}(K) \cong \mathsf{Prof} \cong \mathsf{Prof}/\mathsf{Sp}(L)$. In particular the Galois groupoid $\mathsf{Gal}\,[\sigma]$ has a unique object $* \in \mathsf{Sp}(L)$ and is therefore an internal group in Prof, that is, a profinite group G. Obviously, the internal covariant presheaves on $\mathsf{Gal}\,[\sigma]$ are exactly the profinite G-sets. By proposition 4.5.5, this profinite group G is given by

$$G \cong \mathsf{Sp}(L \otimes_K L) \cong \mathsf{Aut}_K(L) \cong \mathsf{Gal}\,[L : K]$$

and the group structure of $\mathsf{Aut}_K(L)$ agrees with that of $\mathsf{Sp}(L \otimes_K L)$.

Theorems 3.5.8 and 4.7.15, together with propositions 4.5.5 and 4.5.4, yield the diagram

$$\begin{tikzcd}
\mathsf{Split}_K(L) \arrow[r, "\mathsf{Hom}_K(-,L)"] \arrow[d, tail] & \mathsf{Gal}\,[L:K]\text{-}\mathsf{Prof} \arrow[d, equal] \\
\mathsf{Split}_K(\sigma) \arrow[r, "\mathsf{Sp}(S \otimes_K -)"'] & \mathsf{Prof}^{\mathsf{Gal}\,[\sigma]}
\end{tikzcd}$$

In this commutative diagram of functors, the horizontal morphisms are equivalences. Since the right vertical morphism is an equality, the left vertical morphism is an equivalence as well. □