LOGARITHMIC GROWTH

5(a) Logarithmic growth

I explain the proof of a version of the Dolbeault lemma ($\bar\partial$-Poincaré lemma) with logarithmic singularities, published in the *Comptes Rendus* in a joint paper with D. H. Phong, and refined in a paper with Zucker. In what follows, let Δ denote the disk of radius $\frac{1}{2}$ around the origin in $\mathbb{C}$, $\Delta^* = \Delta \setminus \{0\}$ the punctured disk. The variable is denoted $z = re^{i\theta}$ for $z \neq 0$.

Lemma. *Let $N \in \mathbb{Z}$, $N \neq 1$, and let $g \in C^\infty(\Delta^*)$ be a function of logarithmic growth of order N:*

$$|g(z)| < C\frac{|\log\ r|^N}{r},\ C \in \mathbb{R}^+.$$

Then there is a solution $f \in C^\infty(\Delta^*)$ to the equation $\bar\partial\ f = g$ that satisfies

$$|f(z)| < C'|\log\ r|^{N+1},\ C' \in \mathbb{R}^+.$$

Proof. We begin as usual by writing the formula for the fundamental solution of the $\bar\partial$ equation:

$$f(z) = \frac{1}{2\pi i}\int\int_{\Delta^*}\frac{g(w)}{w-z}dwd\bar w,\ z \in \Delta^*.$$

We first show that this integral converges to a C^∞ solution to $\bar\partial\ f = g$. For any $w \in \Delta$ and $a > 0$, let $B(w,a)$ be the disk of radius a centered on w. Let $z_0 \in \Delta^*$, $a > 0$ such that the closure of $B(z_0, 2a)$ is contained in Δ^*. Then we can write $g = g_1 + g_2$ as a sum of two functions in $C^\infty(\Delta^*)$, where $supp(g_1) \subset B(z_0, 2a)$ and $g_2 \mid_{B(z_0,a)} \equiv 0$. In particular, the singularity at 0 is contained in the support of g_2.

Now setting $w = \rho e^{i\theta}$ we find

$$\begin{aligned}|\frac{1}{2\pi i}\int\int_{\Delta^*}\frac{g_2(w)}{w-z}dwd\bar w| &< \frac{1}{2\pi a}|\int\int_{\Delta^*} g_2(w)dwd\bar w| \\ &< \frac{C}{\pi a}\int_0^{2\pi}\int_0^{1/2}\frac{|\log\ \rho|^N}{\rho}\rho d\rho d\theta = O(\int_0^{1/2}|\log\ \rho|^N d\rho)\end{aligned}$$

and for any N, $|\log\ \rho|^N$ is integrable on $[0, \frac{1}{2}]$, so this integral converges. (There is a terrible misprint in my article with Phong! The integrability is by integration by parts: we have

$$\int_0^t |log\ \rho|^N d\rho = \rho\cdot|log\ \rho|^N\ |_0^t - N\int_0^t |log\ \rho|^{N-1}|d\rho$$

and we have it by induction for $N > 0$, and for $N < 0$ there is nothing to prove.)

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Thus

$$f_2(z) = \frac{1}{2\pi i}\int\int_{\Delta^*}\frac{g_2(w)}{w-z}dwd\bar{w} \in C^\infty(B(z_0,a)).$$

Moreover, we can differentiate under the integral sign and since the integrand is holomorphic in z, $\bar{\partial}\ f_2 = 0$ on $B(z_0,a)$.

Now $g_1 \in {C^\infty}_c(\Delta)$ so the usual arguments apply and we find that

$$f_1(z) = \frac{1}{2\pi i}\int\int_{\Delta}\frac{g_1(w)}{w-z}dwd\bar{w}\ \in C^\infty(B(z_0,2a))$$

and that $\bar{\partial}\ f_1 = g_1$ on $B(z_0,2a)$. Thus $f = f_1 + f_2$ is C^∞ in a neighborhood of z_0 and is a solution to $\bar{\partial}\ f = g$ on $B(z_0,a)$.

Now z_0 is arbitrary, so it remains to show that f has logarithmic growth. Choose $z \in \Delta^*$, $r = |z|$. We decompose Δ^* in three regions: $\Delta^* = D_1 \cup D_2 \cup D_3$ where

$$D_1 = B(0,\frac{r}{2}) \setminus \{0\},\ D_2 = B(z,\frac{r}{2}) \cap \Delta,\ D_3 = \Delta^* \setminus (D_1 \cup D_2).$$

We bound the integral on each of these three regions.

On D_1 we have $\frac{1}{|w-z|} \leq \frac{2}{r}$. Thus

$$\frac{1}{2\pi}|\int\int_{D_1}\frac{g(w)}{w-z}dwd\bar{w}| < \frac{4C}{2\pi r}\int\int_{D_1}|\log\ \rho|^N d\rho d\theta = C_1\frac{1}{r}\int_0^{r/2}|\log\rho|^N d\rho.$$

Integration by parts as above shows that this is $O(|log\frac{r}{2}|^N) = O(|log\frac{r}{2}|^N)$. (Again a terrible misprint!)

On D_2 we have the inequality $|g(w)| < C^*\frac{|log\frac{r}{2}|^N}{r}$. where C^* is independent of r. Thus

$$\begin{aligned}\frac{1}{2\pi}|\int\int_{D_2}\frac{g(w)}{w-z}dwd\bar{w}| &< \frac{C^*}{2\pi}\frac{|log\frac{r}{2}|^N}{r}\cdot|\int\int_{D_2}\frac{dwd\bar{w}}{|w-z|}\\ &\leq \frac{C^*}{2\pi}\frac{|log\frac{r}{2}|^N}{r}\cdot|\int\int_{B(0,r/2)}\frac{dud\bar{u}}{|u|}\ (u = w-z)\\ &= C^*|log\frac{r}{2}|^N\end{aligned}$$

by polar coordinates.

On D_3, finally, we have $|w-z| \geq |w/3| = \rho/3$. So

$$\begin{aligned}\frac{1}{2\pi}|\int\int_{D_3}\frac{g(w)}{w-z}dwd\bar{w}| &< \frac{6C}{2\pi}\int\int_{D_3}\frac{|\log\ \rho|^N}{\rho^2}\rho d\rho d\theta\\ &< \frac{3C}{\pi}\int\int_{D_3\cup D_2}\frac{|\log\ \rho|^N}{\rho}d\rho d\theta\\ &= 6C\int_{r/2}^1\frac{|\log\ \rho|^N}{\rho}d\rho = \frac{6C}{N+1}|\log\ r/2|^{N+1}.\end{aligned}$$

This completes the proof.

Let $A^0_N(\Delta^*)$ be the space of f satisfying the growth condition of degree N, $A^0_{si}(\Delta^*) = \cup_N A^0_N(\Delta^*)$ $A^0_{rd}(\Delta^*) = \cap_N A^0_N(\Delta^*)$, $A^1_? = A^0_? \cdot \frac{d\bar{z}}{\bar{z}}$. Then the above lemma implies that $\bar{\partial} : A^0_? \to A^1_?$ is surjective if $? = si$, $? = rd$. Moreover $\ker \bar{\partial} \cap A^0_{si}$ is the space of functions on Δ^* that are holomorphic and have logarithmic growth at 0; but then they have removable singularities, so in fact they extend holomorphically to Δ. Similarly, $\ker \bar{\partial} \cap A^0_{rd}$ is the space of holomorphic functions on Δ that vanish at 0.

In order to globalize and generalize to higher dimensions, we can improve the result.

Corollary. *Let $\theta = z\partial/\partial\, z$ and $\bar{\theta} = \bar{z}\bar{\partial}/\partial\bar{z}$ In the above Lemma, suppose $h = \bar{z}g(z)$ has the property that all its derivatives of the form $\theta^i\bar{\theta}^j h$ are bounded by some (respectively every) power of $|\log\rho|$ Then f has the same property.*

Indeed, letting $I(g) = \frac{1}{2\pi i}\int\int_{\Delta^*}\frac{g(w)}{w-z}dwd\bar{w}$ the main point is that $\frac{\partial I(g)}{\partial z} = I(\frac{\partial g}{\partial z})$, and then we argue by induction. This allows us to use the standard argument (see Griffiths-Harris, p. 25) to apply this lemma to complex algebraic varieties (even analytic varieties). Let M be a smooth compact complex algebraic variety of dimension n, $Z \subset M$ a divisor with (simple) normal crossings. This means that every point $z \in Z$ has a neighborhood $D \subset M$ such that $D \xrightarrow{\sim} \Delta^n$ with z as origin and

$$(*) \qquad (M \setminus Z) \cap D \xrightarrow{\sim} (\Delta^*)^r \times \Delta^{n-r}$$

for some integer r. In other words, up to complex analytic change of coordinates, $Z \cap D$ looks like a union of some of the coordinate hyperplanes. We define

$$A^{0,q}_{si}((\Delta^*)^r \times \Delta^{n-r}) = A^{0,q}_{si}(\Delta^*)^{\otimes\, r} \otimes A^{0,q}(\Delta)^{\otimes\, n-r}$$

and let $\mathcal{A}^{0,q}_{si}$ to be the sheaf of C^∞ $(0,q)$ forms on $M \setminus Z$ whose restriction to $(M\setminus Z)\cap D$ belongs to $A^{0,q}_{si}((\Delta^*)^r \times \Delta^{n-r})$ for some (equivalently any) holomorphic isomorphism (*) as above. We define $\mathcal{A}^{0,q}_{rd}$ similarly. Note that these define sheaves on M (not just on $M \setminus Z$ (a section on an open set U is in this sheaf if and only if its restriction to $U \setminus Z$ is).

Theorem (Harris-Phong). *Let $\mathcal{B}$ be a (holomorphic) vector bundle on M and let $\mathcal{A}^{0,q}_{si}(\mathcal{B}) = \mathcal{A}^{0,q}_{si} \otimes \mathcal{B}$, $\mathcal{A}^{0,q}_{rd}(\mathcal{B}) = \mathcal{A}^{0,q}_{rd} \otimes \mathcal{B}$. There are natural inclusions of sheaves*

$$\mathcal{B} \hookrightarrow \mathcal{A}^{0,0}_{si}(\mathcal{B}); \mathcal{B}(-Z) \hookrightarrow \mathcal{A}^{0,0}_{rd}(\mathcal{B})$$

where $\mathcal{B}(-Z) = \ker[\mathcal{B} \to \mathcal{B} \otimes \mathcal{O}_Z]$, and the complexes

$$0\mathcal{B} \to \mathcal{A}^{0,0}_{si}(\mathcal{B}) \to \mathcal{A}^{0,1}_{si}(\mathcal{B}) \to \ldots \to \mathcal{A}^{0,n}_{si}(\mathcal{B}) \to 0;$$

$$0\mathcal{B}(-Z) \to \mathcal{A}^{0,0}_{rd}(\mathcal{B}) \to \mathcal{A}^{0,1}_{si}(\mathcal{B}) \to \ldots \to \mathcal{A}^{0,n}_{si}(\mathcal{B}) \to 0;$$

are fine resolutions of $\mathcal{B}$ and $\mathcal{B}(-Z)$ respectively.

This follows by standard arguments from the 1-dimensional local case (and the existence of partitions of unity).

The Theorem was designed to apply to certain smooth projective compactifications M of non-compact Shimura varieties $S(G,X) = M \setminus Z$ (this should be ${}_{K_f}S(G,X)$ but I omit the subscript). The idea is that automorphic vector bundles $[W]$ on $S(G,X)$ have canonical extensions $[W]^{can}$ to M with very good properties; in particular, the growth conditions for forms in $\Gamma(M, \mathcal{A}^{0,q}_{si}([W]^{can})$ correspond, under the isomorphism with functions on $G(\mathbb{Q})\backslash G(\mathbf{A})$, to the growth conditions defining automorphic forms.