Problem Books in Mathematics

Walter K. Hayman
Eleanor F. Lingham

Research Problems in Function Theory

Fiftieth Anniversary Edition

Springer

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Preface

In 1967, the first author wrote 'Research Problems in Function Theory', which was published by the Athlone Press of the University of London. There had been earlier problem collections by Littlewood [678] and by Erdős [299] in the 1960s. Subsequent additions to the 1967 booklet were [504], [505], [36], [180], [86] and [156]. It was the idea of the second author to give an account of the progress that has been made on these problems in the intervening half-century, and for the addition of new problems to what is now known as 'Hayman's List'.

We are most grateful to the many mathematicians worldwide who helped to make this book possible by answering our queries and suggesting corrections, amendments, omissions and additions. Among these, we would like to single out the following persons and organisations:

- Alex Eremenko, who provided us with the updated information for most of Chaps. 1 and 2;
- The nine colleagues who have written prefaces for the chapters: A. Eremenko, P.J. Rippon, S.J. Gardiner, E. Crane, L.R. Sons, Ch. Pommerenke, D. Sixsmith, F. Holland and J.L. Rovnyak;
- Our friends who have spent a huge amount of time reading this work, including J.K. Langley, D.A. Brannan, Ch. Pommerenke and J. Becker;
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- The Mathematical Research Institute at Oberwolfach, which allowed us to make a start on writing this book, and the London Mathematical Society, which supported its completion;
- zbMATH for providing us with access to its database;
- Rémi Lodh and the Springer Press, who agreed to publish it.

The reader may find it helpful to know a little about how this book is structured. It is the amalgamation of the original edition, the additions and research which has occurred over the last few decades. This perhaps explains its idiosyncrasies, such as why the 'Miscellaneous' chapter is the seventh of nine, and why some of the more important or famous problems are buried in the middle of chapters. Also, as the language of mathematics has changed over the last half-century, we have adjusted chapter titles and problem statements accordingly, for example, ‘schlicht’ has been replaced by ‘univalent’ and ‘integral functions’ are now known as ‘entire functions’. Any reader who is greatly interested in a particular problem will find direction here, but is reminded of the value of also checking the original statement and the progress information in the subsequent additions. For this, the problem reference table at the end of this book will be useful.

Any science thrives on its problems, and we hope that this book will keep function theory flourishing for a while longer.

London, UK
Sheffield, UK
April 2019

Walter K. Hayman
Eleanor F. Lingham

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Chapter 1
Meromorphic Functions

1.1 Preface by A. Eremenko

According to Hayman [510], Hilbert once told Nevanlinna: "You have made a hole in the wall of Mathematics. Other mathematicians will fill it." Hayman continues: "If the hole means that many new problems were opened up, then this is indeed the case, and I am certain that Nevanlinna theory will continue to solve problems as it has done in the last 50 years."

This chapter is dedicated to the problems on meromorphic functions stated by various authors during the period 1967–1989 and collected by Hayman and his collaborators.

In this book a "meromorphic function" means a function meromorphic in the complex plane. Most problems are about transcendental meromorphic functions (having an essential singularity at infinity).

The theory of meromorphic functions was mostly created by Nevanlinna in the 1920s, and he wrote two influential books on it [756, 757].

These books, especially the second one, contained many unsolved problems, and the present collection mentions only a few of them. This chapter reflects very well the development of Nevanlinna theory in the the second half of the 20th century.

Here I will try to give a very brief overview of the most important problems and their solutions. Of course, this selection reflects my own taste.

We use the definitions and notation introduced in the beginning of the chapter, and add to this $n_1(r, f)$, the counting function of critical points of a meromorphic function f, including multiplicities, and the averaged counting function $N_1(r, f)$, see Update 1.33. The Second Fundamental Theorem of Nevanlinna says that

$$\sum_{j=1}^{q} m(r, a_j, f) + N_1(r, f) \leq 2T(r, f) + S(r, f), \tag{1.1}$$

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where $S(r, f) = o(T(r, f))$ outside an exceptional set of r of finite length. For functions of finite order, there is no exceptional set. (The fact that the exceptional set is in general really necessary was established by Hayman, see Update 1.22.)

This implies the defect relation:

$$\sum_a \delta(a, f) \leq 2.$$

Problem 1.1 asks whether anything else can be said about deficiencies in general, and the answer is "no": for every at most countable set of points a_j, and positive numbers δ_j whose sum is at most 2, there exists a meromorphic function f with $\delta(a_j, f) = \delta_j$ and $\delta(a, f) = 0$ when $a \notin \{a_j\}$ (see Update 1.1).

The situation is quite different for the class of meromorphic functions of finite order. Problems 1.3, 1.6, 1.14, 1.29, 1.33 address the question: what are the restrictions on deficiencies for functions in this class. An almost complete answer is now known. Assume that f is of finite order. First of all, for functions of finite order,

$$\sum_a \delta(a, f)^{1/3} < \infty.$$

Second, if f is of finite order and

$$\sum_a \delta(a, f) = 2,$$

then the number of deficient values is finite (see Updates 1.3, 1.33). Finally, if $\delta(b, f) = 1$ for some b, then

$$\sum_a \delta(a, f)^{1/3-\epsilon} < \infty,$$

where $\epsilon = 2^{-264}$. Of course, this value is not the best possible.

On the other hand, for every sequence of points a_j and numbers $\delta_j \in (0, 1)$ whose sum is strictly less than 2 and

$$\sum \delta_j^{1/3} < \infty$$

there exists a meromorphic function of finite order with deficient values a_j and deficiencies δ_j.

If we further restrict the class of functions by prescribing the order ρ, the results are much less complete, see Problems 1.5, 1.7–1.8, 1.13, 1.15, 1.17, 1.26. Most of these problems are still unsolved, except the Edrei spread conjecture 1.15 and Paley conjecture 1.17. The method invented by Albert Baernstein for solving the spread conjecture found a lot of applications outside the theory of meromorphic functions.

Another very popular line of research stems from the paper of Hayman [510]: Picard-type theorems for derivatives and differential polynomials.

Problems 1.18–1.20, 1.42 belong to this line. A remarkable application of these theorems is their use in the solution of an old question of Wiman about real zeros of derivatives of entire functions (Problem 2.64).

Let me also mention two recent major developments in Nevanlinna theory which are not reflected in this book. At the beginning of the theory, Nevanlinna had already asked whether the Second Fundamental Theorem can be generalised, if one uses "small functions" $a_j(z)$ instead of constants, where "small" means $T(r, a_j) = o(T(r, f))$. The weak form, without N_1, was proved by Charles Osgood [782] with some additional assumptions, and then by Osgood [783] and Steinmetz [921] in full generality (Steinmetz's proof is much simpler than Osgood's). The simple example $f(z) = \mathrm{e}^z + z$ shows that one cannot include the N_1 term as in (1.1). However, one can write a weaker form of (1.1) as

$$\sum_{j=1}^{q} \overline{N}(r, a_j) \geq (q-2)T(r, f) + S(r, f), \tag{1.2}$$

and in this form the Second Fundamental Theorem generalises to small functions, as was proved by Yamanoi [1001].

The next major development was also made by Yamanoi [1002]: he proved Gol'dberg's conjecture:

$$\overline{N}(r, f) \leq N(r, 1/f'') + S(r, f),$$

which implies the conjecture of Mues:

$$\sum_{a \neq \infty} \delta(a, f') \leq 1.$$

These two achievements, both made in the 21st century, show that the theory is very much alive.

1.2 Progress on Previous Problems

Notation We use the usual notations of Nevanlinna theory, see for example Nevanlinna [756, 757] and Hayman [493].

If $f(z)$ is meromorphic in $|z| < R$, and $0 < r < R$, we write

$$n(r, a) = n(r, a, f)$$

for the number of roots of the equation $f(z) = a$ in $|z| \leq r$, when *multiple roots are counted according to multiplicity*, and $\overline{n}(r, a)$ when *multiple roots are counted only once*. We also define

$$N(r,a) = \int_0^r \frac{[n(t,a) - n(0,a)]\,\mathrm{d}t}{t} + n(0,a)\log r,$$

$$\overline{N}(r,a) = \int_a^r \frac{[\overline{n}(t,a) - \overline{n}(0,a)]\,\mathrm{d}t}{t} + \overline{n}(0,a)\log r,$$

$$m(r,f) = m(r,\infty,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\mathrm{e}^{i\theta})|\,\mathrm{d}\theta,$$

where $\log^+ x = \max\big\{\log x, 0\big\}$,

$$m(r,a,f) = m\left(r,\infty,\frac{1}{f-a}\right), \qquad a \neq \infty,$$

and

$$T(r,f) = m(r,\infty,f) + N(r,\infty,f).$$

Then for every finite a, we have by the First Fundamental Theorem (see Hayman [493, p. 5]),

$$T(r,f) = m(r,a,f) + N(r,a,f) + O(1), \qquad \text{as } r \to R. \tag{1.3}$$

We further define the deficiency,

$$\delta(a,f) = \liminf_{r\to R} \frac{m(r,a,f)}{T(r,f)} = 1 - \limsup_{r\to R} \frac{N(r,a,f)}{T(r,f)},$$

the Valiron deficiency,

$$\varDelta(a,f) = \limsup_{r\to R} \frac{m(r,a,f)}{T(r,f)},$$

and further,

$$\varTheta(a,f) = 1 - \limsup_{r\to R} \frac{\overline{N}(r,a,f)}{T(r,f)}.$$

We then have the *"defect relation"* (see Hayman [493, p. 43]),

$$\sum_a \delta(a,f) \leq \sum_a \varTheta(a,f) \leq 2, \tag{1.4}$$

provided that either $R = \infty$ and $f(z)$ is not constant, or $R < +\infty$ and

$$\limsup_{r\to R} \frac{T(r,f)}{\log\big(1/(R-r)\big)} = +\infty.$$

If $R = +\infty$ we also define the *lower order* λ and *order* ρ,

$$\lambda = \liminf_{r \to R} \frac{\log T(r, f)}{\log r}, \qquad \rho = \limsup_{r \to R} \frac{\log T(r, f)}{\log r}.$$

If $\delta(a, f) > 0$ the value a is called *deficient*. It follows from (1.4) that there are at most countably many deficient values if the conditions for (1.4) are satisfied.

Problem 1.1 Is (1.4) all that is true in general? In other words, can we construct a meromorphic function $f(z)$ such that $f(z)$ has an arbitrary sequence a_n of deficient values and no others, and further that $\delta(a_n, f) = \delta_n$, where δ_n is an arbitrary sequence subject to $\sum \delta_n \leq 2$? (If $f(z)$ is an entire function $\delta(f, \infty) = 1$, so that $\sum_{a \neq \infty} \delta(a, f) \leq 1$. For a solution of the problem in this case, see Hayman [493, p. 80].)

Update 1.1 This problem has been completely settled by Drasin [255]. He constructs a meromorphic function $f(z)$ with arbitrary deficiencies and branching indices on a preassigned sequence a_n of complex numbers with $f(z)$ growing arbitrarily slowly, subject to having infinite order.

Problem 1.2 How big can the set of Valiron deficiencies be for functions in the plane? It is known that

$$N(r, a) = T(r, f) + O\big(T(r, f)^{\frac{1}{2}+\varepsilon}\big) \tag{1.5}$$

as $r \to \infty$, for all a outside a set of capacity zero (see Nevanlinna [757, pp. 260–264]).

In the case $R < +\infty$ this is more or less best possible, but in the plane we only know from an example of Valiron [963] that the corresponding set of a can be noncountably infinite. It is also not known whether (1.5) can be sharpened.

Update 1.2 Hyllengren [564] has shown that all values of a set E can have Valiron deficiency greater than a positive constant for a function of finite order in the plane if and only if there exists a sequence of complex numbers a_n and a positive k such that each point of E lies in infinitely many of the discs $\{z : |z - a_n| < \mathrm{e}^{-\mathrm{e}^{kn}}\}$.

Hayman [501] proved that all values of any F_σ set of capacity zero can be Valiron deficiencies for an entire function of infinite order, and a little more.

Problem 1.3 If $f(z)$ is meromorphic of finite order ρ and $\sum \delta(a, f) = 2$, it is conjectured that $\rho = n/2$, where n is an integer and $n \geq 2$, and all the deficiencies are rational. F. Nevanlinna [754] has proved this result on the condition that $f(z)$ has no multiple values, so that $n(r, a) = \overline{n}(r, a)$ for every a (see also R. Nevanlinna [755]).

Update 1.3 Weitsman [982] proved that the number of deficiencies is at most twice the order in this case. The conjecture was completely proved by Drasin [256]. Eremenko [314] gave a simpler proof of a stronger result, see Problem 1.33.

Problem 1.4 Let $f(z)$ be an entire function of finite order ρ, and let $n_1(r,a)$ denote the number of simple zeros of the equation $f(z)=a$. If

$$n_1(r,a)=O(r^c), \qquad n_1(r,b)=O(r^c), \qquad \text{as } r\to\infty,$$

where $a\neq b$, $c<\rho$, is it true that ρ is an integral multiple of $\frac{1}{2}$? More strongly, is this result true if $\Theta(a)=\frac{1}{2}=\Theta(b)$? (For a somewhat weaker result in this direction, see Gol'dberg and Tairova [416].)

Update 1.4 The answer is 'no', even in a very weak sense. Gol'dberg [405] has constructed an example of an entire function for which

$$n_1(r,a)=O((\log r)^{2+\varepsilon}), \qquad n_1(r,b)=O((\log r)^{2+\varepsilon}) \qquad \text{as } r\to\infty,$$

but the order is not a multiple of $\frac{1}{2}$.

Problem 1.5 Under what conditions can $\sum\delta(a,f)$ be nearly 2 for an entire function of finite order ρ? Pfluger [799] proved that if $\sum\delta(a,f)=2$, then (see Hayman [493, p. 115]) ρ is a positive integer q, the lower order λ is such that $\lambda=\rho$ and all the deficiencies are integral multiples of $1/q$. If further,

$$\sum\delta(a,f)>2-\varepsilon(\lambda),$$

where $\varepsilon(\lambda)$ is a positive quantity depending on λ, then Edrei and Fuchs [287, 288] proved that these results remain true 'nearly', in the sense that there exist 'large' deficiencies which are nearly positive integral multiplicities of $1/q$, and whose sum of deficiencies is 'nearly' 2. Can there also be a finite or infinite number of small deficiencies in this case?

Update 1.5 No progress on this problem has been reported to us. Hayman suspects that the answer is 'no'.

Problem 1.6 Arakelyan [41] has proved that, given $\rho>\frac{1}{2}$ and a countable set E, there exists an entire function $f(z)$ of order ρ for which all the points of E are deficient. Can E be the precise set of deficiencies of f in the sense that f has no other deficient values? It is also conjectured that if the a_n are deficient values for an entire function of finite order, then

$$\sum\big(\log[1/\delta(a_n,f)]\big)^{-1}<+\infty.$$

(N.U. Arakelyan)

Update 1.6 Eremenko [305] has proved the first conjecture. He also proved [313] that the second conjecture is false: given $\rho>1/2$ and a sequence of complex numbers (a_k), there is an entire function f of order ρ with the property $\delta(a_k,f)>c^k$, $k=1,2,\dots,$ for some $c\in(0,1)$. On the other hand, Lewis and Wu

[669] proved $\sum \delta(a_k, f)^\alpha < \infty$ for entire functions of finite order with an absolute constant $\alpha < 1/3 - 2^{-264}$. The exact rate of decrease of deficiencies of an entire function of finite order remains unknown.

Problem 1.7 If $f(z)$ is an entire function of finite order ρ which is not an integer, it is known that (see Pfluger [799] and Hayman [493, p. 104])

$$\sum \delta(a, f) \leq 2 - K(\rho),$$

where $K(\rho)$ is a positive quantity depending on ρ. What is the best possible value for $K(\rho)$? Edrei and Fuchs [288] conjectured (see also Hayman [493, p. 104]) that if q is the integral part of ρ, and if $q \geq 1$, then

$$K(\rho) = \frac{|\sin(\pi\rho)|}{q + |\sin(\pi\rho)|}, \qquad q \leq \rho < q + \frac{1}{2},$$

$$K(\rho) = \frac{|\sin(\pi\rho)|}{q + 1}, \qquad q + \frac{1}{2} \leq \rho < q + 1.$$

This result would be sharp.

If $\rho \leq \frac{1}{2}$, there are no deficient values, so that $K(\rho) = 1$. If $\frac{1}{2} < \rho < 1$, Pfluger [799] proved that $K(\rho) = \sin(\pi\rho)$. See also Hayman [493, p. 104].

Update 1.7 For Problems 1.7 and 1.8 a better lower bound was found by Miles and Shea [728], who also obtained the exact lower bound for any order ρ of

$$\limsup_{r\to\infty} \frac{N(r, 0) + N(r, \infty)}{m_2(r)},$$

where

$$m_2(r) = \left(\frac{1}{2\pi} \int_0^{2\pi} (\log|f(re^{i\theta})|)^2 \, d\theta\right)^{\frac{1}{2}}.$$

Hellerstein and Williamson [534] have solved the problems completely for entire functions with zeros on a ray.

Problem 1.8 Following the notation in Problem 1.7, if $f(z)$ is meromorphic in the plane of order ρ, it is conjectured by Pfluger [799] that, for $a \neq b$,

$$\limsup_{r\to\infty} \frac{N(r, a) + N(r, b)}{T(r, f)} \geq K(\rho).$$

This is known to be true for $0 < \rho \leq 1$. If equality holds in the above inequality, it is conjectured that $f(z)$ has regular growth, that is, $\rho = \lambda$.

Update 1.8 See Update 1.7.

Problem 1.9 If $f(z)$ is an entire function of finite order ρ which has a finite deficient value, find the best possible lower bound for the lower order λ of $f(z)$. (Edrei and Fuchs [288] showed that λ is positive.)

Gol'dberg [400] showed that for every $\rho > 1$, $\lambda \geq 1$ is possible.

Update 1.9 This has been settled by Gol'dberg [400].

Problem 1.10 If $f(z)$ is a meromorphic function of finite order with more than two deficient values, is it true that if $\sigma > 1$, then

$$\limsup_{r\to\infty} \frac{T(\sigma r)}{T(r)} < +\infty.$$

Update 1.10 No progress on this problem has been reported to us. (The update in [504] has been withdrawn.)

Problem 1.11 If $f(z)$ is a meromorphic function of finite order with at least one finite deficient value, does the conclusion of Problem 1.10 hold?

Update 1.11 Drasin writes that Kotman [629] has shown that the answer to this question is 'no'.

Problem 1.12 Edrei, Fuchs and Hellerstein [292] ask if $f(z)$ is an entire function of infinite order with real zeros, is $\delta(0, f) > 0$? More generally, is $\delta(0, f) = 1$?

Update 1.12 This has been disproved by Miles [725], who showed that $\delta(0, f) = 0$ is possible. However, Miles also showed that

$$\frac{N(r,0)}{T(r, f)} \to 0$$

as $r \to \infty$ outside a fairly small set in this case.

Problem 1.13 If $f(z)$ is an entire function of finite order ρ and lower order λ with real zeros, find the best possible bound $B = B(\rho, \lambda)$ such that $\delta(0, f) \geq B$. From Edrei, Fuchs and Hellerstein [292] it is known that $B > 0$ if $2 < \rho < \infty$, and it is conjectured that $B \to 1$ as $\rho \to +\infty$.

Update 1.13 An affirmative answer with the exact value of B was given by Hellerstein and Shea [533].

Problem 1.14 If $f(z)$ is a meromorphic function of finite order, then it is known (see Hayman [493, pp. 90, 98]) that $\sum \delta(a, f)^{\alpha}$ converges if $\alpha > \frac{1}{3}$, but may diverge if $\alpha < \frac{1}{3}$. What happens when $\alpha = \frac{1}{3}$?

Update 1.14 This has been completely settled by Weitsman [983], who proved that $\sum \left(\delta(a, f)\right)^{1/3}$ does indeed converge for any meromorphic function of finite order.

Problem 1.15 (*Edrei's spread conjecture*) If $f(z)$ is meromorphic in the plane and of lower order λ, and if $\delta = \delta(a, f) > 0$, is it true that, for a sequence $r = r_\nu \to \infty$, $f(z)$ is close to a on a part of the circle $|z| = r_\nu$ having angular measure at least

$$\frac{4}{\lambda} \sin^{-1} \sqrt{\left(\frac{\delta}{2}\right)} + o(1)?$$

(For a result in this direction, see Edrei [285].)

Update 1.15 This result was proved by Baernstein [54] by means of the function $T^*(r, \theta)$, where

$$T^*(r, \theta) = \sup_E \frac{1}{2\pi} \int_E \log |f(re^{i\varphi})| \, d\varphi + N(r, f),$$

and E runs over all sets of measure exactly 2θ. See also Baernstein [53, 61].

Problem 1.16 For any function $f(z)$ in the plane, let $n(r) = \sup_a n(r, a)$ be the maximum number of roots of the equation $f(z) = a$ in $|z| < r$, and

$$A(r) = \frac{1}{\pi} \int \int_{|z|<r} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} \, dx \, dy = \frac{1}{\pi} \int \int_{|a|<\infty} \frac{n(r, a) \, |da|^2}{(1 + |a|^2)^2}.$$

Then $\pi A(r)$ is the area, with due count of multiplicity, of the image on the Riemann sphere of the disc $|z| < r$ under f, and $A(r)$ is the average value of $n(r, a)$ as a moves over the Riemann sphere. It is known (see Hayman [493, p. 14]) that

$$1 \le \liminf_{r \to \infty} \frac{n(r)}{A(r)} \le e.$$

Can e be replaced by any smaller quantity, and in particular, by 1?

Update 1.16 Toppila [951] has shown that e cannot be replaced by 1. He has constructed an example of a meromorphic function for which

$$\liminf_{r \to \infty} \frac{n(r)}{A(r)} \ge \frac{80}{79}$$

for every sufficiently large r. The question remains open for entire functions. Among other examples, Toppila shows that for an entire function the following can occur:

$$\limsup_{r \to \infty} \frac{n(r, 0)}{A(7r/6)} \ge \frac{9}{5}$$

and

$$\limsup_{r \to \infty} \frac{n(r)}{A(Kr)} = \infty$$

for every K, $K \ge 1$.

Miles [727] gave a positive answer, by showing that for every meromorphic function

$$\liminf_{r\to\infty} \frac{\max_a n(r,a)}{A(r)} \le \mathrm{e} - 10^{-28}.$$

Problem 1.17 (*Paley's conjecture*) For any entire function $f(z)$ of finite order ρ in the plane, we have

$$1 \le \liminf_{r\to\infty} \frac{\log M(r,f)}{T(r,f)} \le C(\rho),$$

where $C(\rho)$ depends on ρ only. This follows very simply from Hayman [493, Theorem 1.6, p. 18]. Wahlund [975] has shown that the best possible value of $C(\rho)$ is $\pi\rho/\sin(\pi\rho)$ for $0 < \rho < \frac{1}{2}$, and it is conjectured that $C(\rho) = \pi\rho$ is the corresponding result for $\rho > \frac{1}{2}$.

Update 1.17 This inequality has been proved by Govorov [438] for entire functions, and by Petrenko [797] for meromorphic functions.

Problem 1.18 Suppose that $f(z)$ is meromorphic in the plane, and that $f(z)$ and $f^{(l)}(z)$ have no zeros, for some $l \ge 2$. Prove that $f(z) = \mathrm{e}^{az+b}$ or $(Az+B)^{-n}$.

The result is known if $f(z)$ has only a finite number of poles (see Clunie [209] and Hayman [493, p. 67]) or if $f(z)$ has finite order and $f \ne 0$, $f' \ne 0$, $f'' \ne 0$, and

$$\liminf_{r\to\infty} \frac{\log n(r,f)}{\log r} < +\infty,$$

(see Hayman [490]), or if none of the derivatives of $f(z)$ have any zeros and $f(z)$ has unrestricted growth (see Polya [804], Hayman [493, p. 63]).

Update 1.18 The conjecture was proved by Mues [745] if f has finite order and $ff'' \ne 0$ (instead of $ff'f'' \ne 0$). For $l > 2$, the conjecture was proved by Frank [352]. Since then Frank, Polloczek and Hennekemper [354] have obtained various extensions. Thus, if f and $f^{(l)}$ have only a finite number of zeros, and $l > 2$, then

$$f(z) = \frac{p_1}{p_2}\mathrm{e}^{p_3},$$

where p_1, p_2, p_3 are polynomials. However, the paper [352] contains a gap in the proof of the case $l = 2$.

The last case which remained unsolved, $l = 2$, was settled by Langley [649] who proved that the only meromorphic functions f, for which ff'' is zero-free, are $f(z) = \mathrm{e}^{az+b}$ and $f(z) = (az+b)^{-n}$.

Problem 1.19 Suppose that $f(z)$ is meromorphic in the plane and $f'(z)f(z)^n \ne 1$, where $n \ge 1$. Prove that $f(z)$ is constant. Hayman [490] has shown this to be true for $n \ge 3$.

Update 1.19 The case $n = 2$ has been settled by Mues [746]. The last case which remained unsolved, $n = 1$, was settled by Bergweiler and Eremenko [110]: for every non-constant meromorphic function f, the equation $f'(z)f(z) = c$ has solutions for every c, $c \neq 0, \infty$. This was first proved by Bergweiler and Eremenko for functions of finite order; then, Bergweiler and Eremenko [110], Chen and Fang [204], and Zalcman [1016] independently noticed that a general method of Pang [789] permits an extension to arbitrary meromorphic functions. The proof actually applies whenever $n \geq 1$.

There were many extensions and generalisations of this result of Bergweiler and Eremenko. The strongest result so far is due to Jianming Chang [201]: *Let f be a transcendental meromorphic function whose derivative is bounded on the set of zeros of f. Then the equation $f(z) = c$ has infinitely many solutions for every $c \in \mathbb{C} \setminus \{0\}$.*

Problem 1.20 If $f(z)$ is non-constant and meromorphic in the plane, and $n = 3$ or $n = 4$, prove that $\phi(z) = f'(z) - f(z)^n$ assumes all finite complex values. This is known to be true if $f(z)$ is an entire function, or if $n \geq 5$ in the case where $f(z)$ is meromorphic; see Hayman [490].

In connection to this, it would be most interesting to have general conditions under which a polynomial in $f(z)$ and its derivatives can fail to take some complex value. When $f(z)$ is a meromorphic rather than an entire function, rather little is known, see however Clunie [209, 210] and Hayman [493, Ch. 3].

Update 1.20 This question is closely related to Problem 1.19. Mues [746] proved that $\phi(z)$ may omit a finite non-zero value when $n = 3$ or 4. He also showed that ϕ must have infinitely many zeros for $n = 4$. The remaining case of zeros for $n = 3$ was settled by Bergweiler and Eremenko [110].

Problem 1.21 If $f(z)$ is non-constant in the plane, it is known (see Hayman [493, pp. 55–56]) that

$$\alpha_f = \limsup_{r\to\infty} \frac{T(r, f)}{T(r, f')} \geq \begin{cases} \frac{1}{2} & \text{if } f(z) \text{ is meromorphic,} \\ 1 & \text{if } f(z) \text{ is an entire function.} \end{cases}$$

These inequalities are sharp. It is not known whether

$$\beta_f = \liminf_{r\to\infty} \frac{T(r, f)}{T(r, f')}$$

can be greater than one, or even infinite. It is known that β_f is finite if $f(z)$ has finite order. Examples show that α_f may be infinite for entire functions of any order ρ, that is $0 \leq \rho \leq \infty$, and that given any positive constants K, ρ there exists an entire function of order at most ρ such that

$$\frac{T(r, f)}{T(r, f')} > K$$

on a set of r having positive lower logarithmic density. For this and related results, see Hayman [495].

Update 1.21 Let f be meromorphic in the plane. The relation between $T(r, f')$ and $T(r, f)$ constitutes an old problem of Nevanlinna theory. It is classical that

$$m(r, f') \leq m(r, f) + m\Big(r, \frac{f'}{f}\Big) \leq m(r, f) + O(\log T(r, f))$$

outside an exceptional set. In particular, if f is entire so that $m(r, f) = T(r, f)$, we deduce that $T(r, f') < (1 + o(1))T(r, f)$ outside an exceptional set.

The question of a corresponding result in the opposite direction had been open until fairly recently. Hayman [494] has shown that there exist entire functions of finite order ρ for which $T(r, f) > KT(r, f')$ on a set having positive lower logarithmic density, for every positive ρ and $K > 1$. Toppila [952] has given a simple example for which

$$\frac{T(r, f)}{T(r, f')} \geq 1 + \frac{7}{10^7}$$

for all sufficiently large r. In this example, he takes for f' the square of the sine product, having permuted the zeros in successive annuli to the positive or negative axis. The result is that f' is sometimes large on each half-axis, and so f, the integral of f', is always large on and near the real axis. Further, Langley [650] constructed an entire function of arbitrary order under $\rho > \frac{1}{2}$ for which $\beta_f > 1$, where

$$\beta_f = \limsup_{r\to\infty} \frac{T(r, f)}{T(r, f')}.$$

On the other hand, Hayman and Miles [518] have proved that $\beta_f \leq 3e$ if f is meromorphic, and $\beta_f \leq 2e$ if f is entire. Density estimates are also given to show that the previous examples are fairly sharp.

Problem 1.22 The defect relation (1.4) is a consequence of the inequality which is called the '*Second Fundamental Theorem*' (see Hayman [493, formula (2.9), p. 43]),

$$\sum_{\nu=1}^{k} \overline{N}(r, a_\nu, f) \geq \big(q - 2 + o(1)\big)T(r, f), \tag{1.6}$$

which holds for any distinct numbers a_ν and $q \geq 3$, as $r \to \infty$ outside a set E of finite measure, if $f(z)$ is meromorphic in the plane. The exceptional set E is known to be unnecessary if $f(z)$ has finite order. Does (1.6) also hold as $r \to \infty$ without restriction if $f(z)$ has infinite order?

Update 1.22 A negative answer to this question is provided by Hayman's examples [501] discussed in connection with Problem 1.2. These show that the Second Fundamental Theorem fails to hold on a certain sequence $r = r_\nu$.

Problem 1.23 Under what circumstances does $f(z_0 + z)$ have the same deficiencies as $f(z)$? It was shown by Dugué [265] that this need not be the case for meromorphic functions, and by Hayman [485] that it is not necessarily true for entire functions of infinite order. The case of functions of finite order remains open. Valiron [965] notes that a sufficient condition is

$$\frac{T(r+1, f)}{T(r, f)} \to 1, \qquad \text{as } r \to \infty,$$

and this is the case in particular if $\rho - \lambda < 1$. Since for entire functions of lower order λ, $\lambda \le \frac{1}{2}$ there are no deficiencies anyway, it follows that the result is true at any rate for entire functions of order $\rho < \frac{3}{2}$ and, since $\lambda \ge 0$ always, for meromorphic functions of order less than one.

Update 1.23 Gol'dberg and Ostrovskii [415] give examples of meromorphic functions of finite order for which the deficiency is not invariant under change of origin. See also Gol'dberg and Ostrovskii [415]; and Wittich [994, 996] for details.

Miles [726] provided a counterexample of an entire f of large finite order. Gol'dberg, Eremenko and Sodin [411] have constructed such f with preassigned order ρ, such that $5 < \rho < \infty$.

Problem 1.24 If f is meromorphic in the plane, can $n(r, a)$ be compared in general with its average value

$$A(r) = \frac{1}{\pi} \int\int_{|z|<r} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} \,\mathrm{d}x\,\mathrm{d}y$$

in the same sort of way that $N(r, a)$ can be compared with $T(r)$? In particular, is it true that

$$n(r, a) \sim A(r)$$

as $r \to \infty$, outside an exceptional set of r, independent of a, and possibly an exceptional set of a? (Compare Problem 1.16.)

(P. Erdős)

Update 1.24 Miles [723] has shown that

$$\lim_{r\to\infty,\, r\notin E} \frac{n(r, a)}{A(r)} = 1,$$

for all a not in A, a set of inner capacity zero, and all r not in E, a set of finite logarithmic measure.

Problem 1.25 In the opposite direction to Problem 1.24, does there exist a meromorphic function such that for every pair of distinct values a, b, we have

$$\limsup_{r\to\infty} \frac{n(r, a)}{n(r, b)} = \infty \qquad \text{and} \qquad \liminf_{r\to\infty} \frac{n(r, a)}{n(r, b)} = 0?$$

Note, of course, that either of the above limits for all distinct a, b implies the other.

(Compare the result (1.3) quoted in Problem 1.2, which shows that this certainly cannot occur for the N-function.) The above question can also be asked for entire functions.

(P. Erdős)

Update 1.25 Both Gol'dberg [406] and Toppila [951] have produced examples of entire functions for which

$$\limsup_{r\to\infty} \frac{n(r,a)}{n(r,b)} = \infty,$$

for every finite unequal pair (a, b). A corresponding example for meromorphic functions has also been given by Toppila [951].

Problem 1.26 The analogue of Problem 1.7 may be asked for meromorphic functions. The proposers conjecture that in this case

$$\sum \delta(a, f) \leq \max\{\Lambda_1(\rho), \Lambda_2(\rho)\},$$

where for $\rho \geq 1$, $q = [2\rho]$ we have

$$\Lambda_1(\rho) = 2 - \frac{2\sin\left(\frac{1}{2}\pi(2\rho - q)\right)}{q + 2\sin\left(\frac{1}{2}\pi(2\rho - q)\right)},$$
$$\Lambda_2(\rho) = 2 - \frac{2\cos\left(\frac{1}{2}\pi(2\rho - q)\right)}{q + 1}.$$

Drasin and Weitsman [259] shows that this result would be sharp. The correct bound is known for $0 \leq \rho \leq 1$.

(D. Drasin and A. Weitsman)

Update 1.26 No progress on this problem has been reported to us.

Problem 1.27 Let E be the set for which $m(r, a) \to \infty$ as $r \to \infty$. How large can E be if:

(a) f is entire and of order $\frac{1}{2}$ mean type,
(b) f is meromorphic of order ρ, where $0 \leq \rho \leq \frac{1}{2}$.

The proposers settled this problem in all other cases (see Updates 2.1(a) and 2.1(b) for more details).

(D. Drasin and A. Weitsman)

Update 1.27(b) Damodaran [232] proved the existence of meromorphic functions of growth $T(r, f) = O(\rho(r)(\log r)^3)$, where $\rho(r) \to \infty$ arbitrarily slowly, such that $m(r, a) \to \infty$ for all a in an arbitrarily prescribed set of capacity zero. Lewis [666] and Eremenko [303] independently improved this to $(\log r)^2$ in place of $(\log r)^3$. This is best possible, following from an old result of Tumura [958].

Problem 1.28 Are there upper bounds of any kind on the set of asymptotic values of a meromorphic function of finite order?

(D. Drasin and A. Weitsman)

Update 1.28 A negative answer to this question has been given by Eremenko [304], who has constructed meromorphic functions of positive and of zero order, having every value in the closed plane as an asymptotic value. This has been improved by Canton, Drasin and Granados [182], who proved that for every $\phi(r) \to +\infty$ and every analytic (Suslin) set A, there exists a meromorphic function f with the property $T(r, f) = O(\phi(r) \log^2 r)$ and whose set of asymptotic values coincides with A.

Problem 1.29 Under what circumstances does there exist a meromorphic function $f(z)$ of finite order ρ with preassigned deficiencies $\delta_n = \delta(a_n, f)$ at a preassigned sequence of complex numbers? Weitsman has solved this problem (see Update 1.14) by showing that it is necessary that

$$\sum \delta_n^{\frac{1}{3}} < \infty, \tag{1.7}$$

but the bound of the sum of the series depends on the largest term δ_1. On the other hand, Hayman [493, p. 98] showed that the condition

$$\sum \delta_n^{\frac{1}{3}} < A, \tag{1.8}$$

with $A = 9^{-\frac{1}{3}}$, is sufficient to yield a meromorphic function of order 1 mean type, such that $\delta(a_n, f) \geq \delta_n$. It may be that (1.8) is a sufficient condition with a constant $A = A_1(\rho)$, and, with a larger constant $A = A_2(\rho)$ is also a necessary condition. The problem may be a little easier if ρ is allowed to be arbitrary but finite.

Update 1.29 Eremenko [311] has constructed an example of a function of finite order, having preassigned deficiencies $\delta_n = \delta(a_n, f)$, subject to $0 < \delta_n < 1$, $\sum \delta_n < 2$ and $\sum \delta_n^{\frac{1}{3}} < \infty$, and no other conditions. In view of the results reported in Updates 1.3 and 1.33, this result is a complete solution of the Inverse Problem in the class of functions of (unspecified) finite order.

Problem 1.30 Can one establish an upper bound on the number of finite asymptotic values of a meromorphic function $f(z)$ in $\mathbb{C}$, taking into account both the order of f, and the angular measure of its tracts?

(W. Al-Katifi)

Update 1.30 No progress on this problem has been reported to us.

Problem 1.31 Let the function f be meromorphic in the plane, and not rational, and satisfy the condition

$$\frac{T(r, f)}{(\log r)^3} \to \infty, \qquad \text{as } r \to \infty, \tag{1.9}$$

where $T(r, f)$ is the Nevanlinna characteristic. A theorem of Yang Lo [1005] states that then there exists a direction $\theta_0 \in [0, 2\pi)$ such that for every positive ε, either f attains every finite value infinitely often in $D_\varepsilon = \{z : |\arg z - \theta_0| < \varepsilon\}$, or else $f^{(k)}$ attains every value, except possibly zero, infinitely often in D_ε for all positive integers k. Can the condition (1.9) be dropped completely? Or, possibly, can it be replaced by the 'more usual' condition

$$\frac{T(r, f)}{(\log r)^2} \to \infty, \qquad \text{as } r \to \infty\,?$$

One cannot expect any more from Yang Lo's method of finding θ_0 through the use of 'filling discs'. Rossi [849] has shown that (1.9) cannot be improved if θ_0 is sought in this way.

(D. Drasin; communicated by J. Rossi)

Update 1.31 Rossi writes that there is an incorrect paper of Zhu [1021] where he purports to use filling discs to solve this problem. However, Fenton and Rossi [340] remark that Zhu's approach is wrong, and point to the example in Rossi [849]. Some work on this problem has been produced by Sauer [878].

Problem 1.32 Let f be meromorphic in $\mathbb{C}$, and let f^{-1} denote any element of the inverse function that is analytic in a neighbourhood of a point w. A well-known theorem of Gross [446] states that f^{-1} may be continued analytically along almost all rays beginning at w. Is it possible to refine the exceptional set in this theorem?

(A. Eremenko)

Update 1.32 No progress on this problem has been reported to us.

Problem 1.33 Let f be a meromorphic function of finite order ρ. Does the condition

$$N(r, 1/f') + 2N(r, f) - N(r, f') = o(T(r, f)), \qquad \text{as } r \to \infty,$$

imply that 2ρ is an integer?

(A. Eremenko)

Update 1.33 This is a slightly more precise conjecture than Problem 2.25. Both problems are solved completely by the following theorem of Eremenko [314]: suppose that f is a meromorphic function of finite lower order λ, and that

$$N_1(r, f) := N(r, 1/f') + 2N(r, f) - N(r, f') = o(T(r, f)).$$

Then

(a) 2λ is an integer greater than or equal to 2.
(b) $T(r, f) = r^\lambda l(r)$, where l is a slowly varying function in the sense of Karamata.
(c) $\sum_a \delta(a, f) = 2$, all deficient values are asymptotic, and all deficiencies are multiples of $1/\lambda$.

Problem 1.34 Let $n_1(r, a, f)$ denote the number of simple zeros of $f(z) - a$ in $\{|z| \leq r\}$. Selberg [887] has shown that if:

(a) f is a meromorphic function of finite order ρ, and
(b) $n_1(r, a, f) = O(1)$, as $r \to \infty$ for four distinct values of a, then

ρ is an integral multiple of $\frac{1}{2}$ or $\frac{1}{3}$.

Does this conclusion remain true if (b) is replaced by:

(c) $n_1(r, a, f) = o(T(r, f))$, as $r \to \infty$, for four distinct values of a?

Gol'dberg [404] has constructed an entire function of arbitrary prescribed order which satisfies the condition (c) for two distinct values of a.

(A. Eremenko)

Update 1.34 The answer is 'no'. Künzi [646] has shown that ρ can be arbitrary, subject to $1 < \rho < \infty$, and Gol'dberg [404] has a counterexample for arbitrary positive ρ.

Problem 1.35 Determine the upper and lower estimates for the growth of entire and meromorphic solutions of algebraic ordinary differential equations (AODE). (This is a classical problem.)

For AODEs of first order, it is known that the meromorphic solutions f must have finite order (see Gol'dberg [399]) and that $(\log r)^2 = O(T(r, f))$; see Eremenko [307, 309]. (The latter two references contain a general account of first-order AODEs, including modern proofs of some classical results.) The order of entire solutions of first-order AODEs is an integral multiple of $\frac{1}{2}$; see Malmquist [704]. For AODEs of second order, it is known that the order of entire solutions is positive; see Zimogljad [1023]. There is no upper estimate valid for all entire or meromorphic solutions of AODEs of order greater than one, but there is an old conjecture that $\log |f(z)| \leq \exp_n(|z|)$ as $z \to \infty$ for entire solutions f of an AODE of order n.

(A. Eremenko)

Update 1.35 Steinmetz [919, 920] proved that every meromorphic solution of a homogeneous algebraic differential equation of second order has the form $f = (g_1/g_2)\exp(g_3)$, where the g_i are entire functions of finite order. Thus $T(r, f) = O(\exp(r^k))$ for some positive k. By Wiman–Valiron theory (see, for example [966]), it is known that 'most' algebraic differential equations do not have entire solutions of infinite order. A precise statement of this sort is contained in Hayman [512].

Problem 1.36 Let F be a polynomial in two variables, and let y be a meromorphic solution of the algebraic ordinary differential equation $F(y^{(n)}, y) = 0$. Is it true that y must be an elliptic function, or a rational function of exponentials, or a rational function? This is known in the following cases:

(a) $n = 1$: a classical result, probably due to Abel;
(b) $n = 2$: an old result of Picard [801], and independently, Bank and Kaufman [72];
(c) n is odd and y has at least one pole, Eremenko [308]; and

(d) the genus of the curve $F(x_1, x_2) = 0$ is at least equal to one, Eremenko [308].

(A. Eremenko)

Update 1.36 This has been solved by Eremenko, Liao and Tuen-Wai Ng [325], who prove (c) for all n, and give an example of an entire solution which is neither rational, nor a rational function of exponentials.

Problem 1.37 Find criteria for and/or give explicit methods for the construction of meromorphic functions f in $\mathbb{C}$ with the following properties:

(a) all poles of f are of odd multiplicity;
(b) all zeros of f are of even multiplicity.

(Here 'explicit methods' means that all computations must be practicable.) The background of this problem lies in the question of meromorphic solutions of the differential equation $y'' + A(z)y = 0$ in the whole plane.

(J. Winkler)

Update 1.37 We mention a result of Schmieder [884] which may be relevant: on every open Riemann surface, there exists an analytic function with prescribed divisors of zeros and critical points, subject to certain trivial restrictions.

Problem 1.38

(a) Let f be non-constant and meromorphic in the open unit disc $\mathbb{D}$, with $\alpha < +\infty$, where

$$\alpha = \limsup_{r \to 1} \frac{T(r, f)}{-\log(1-r)}, \tag{1.10}$$

and set

$$\Psi = (f)^{m_0}(f')^{m_1} \dots (f^{(k)})^{m_k}.$$

It is known that Ψ assumes all finite values, except possibly zero, infinitely often, provided that $m_0 \geq 3$ and $\alpha > 2/(m_0 - 2)$, (or $m_0 \geq 2$ and $\alpha > 2/(m_0 - 1)$, if f is analytic). For which smaller values of α does the same conclusion hold?

(b) Let f be non-constant and meromorphic in $\mathbb{D}$, with $\alpha < +\infty$ in (1.10); assume also that f has only finitely many zeros and poles in $\mathbb{D}$. Let l be a positive integer, and write $\Psi = \sum_{\nu=0}^{l} a_\nu f^{(\nu)}$, where the a_ν are functions in $\mathbb{D}$ for which $T(r, a_\nu) = o(T(r, f))$ as $r \to 1$ (for each ν). It is known that if Ψ is non-constant, then Ψ assumes every finite value, except possibly zero, infinitely often, provided that $\alpha > \frac{1}{2}l(l+1) + 1$. For which smaller values of α does the same conclusion hold?

(L.R. Sons)

Update 1.38(b) Gunsul [454] provides a condition that establishes the same conclusion for smaller values of α.

Problem 1.39 Let f be a function meromorphic in $\mathbb{D}$ for which $\alpha < +\infty$ in (1.10).

(a) Shea and Sons [893, Theorem 5] have shown that if $f(z) \neq 0, \infty$ and $f'(z) \neq 1$ in $\mathbb{D}$, then $\alpha \leq 2$. Is 2 best possible?
(b) Shea and Sons [893] have shown that, if $f(z) \neq 0$ and $f'(z) \neq 1$ in $\mathbb{D}$, then $\alpha \leq 7$. What is the best possible α in this case?

(L.R. Sons)

Update 1.39 No progress on this problem has been reported to us.

Problem 1.40 Let f be a function meromorphic in $\mathbb{D}$ of finite order ρ. Shea and Sons [893] have shown that

$$\sum_{a\neq\infty} \delta(a, f) \leq \delta(0, f')\big(1 + k(f)\big) + \frac{2}{\lambda}(\rho + 1),$$

where

$$k(f) = \limsup_{r\to 1} \frac{\overline{N}(r, \infty, f)}{T(r, f) + 1} \quad \text{and} \quad \lambda(f) = \liminf_{r\to\infty} \frac{T(r, f)}{\log(1/(1-r))}.$$

Can the factor 2 be eliminated? (If so, the result is then best possible.)

(L.R. Sons)

Update 1.40 No progress on this problem has been reported to us.

Problem 1.41 Let f be a function meromorphic in $\mathbb{D}$ for which $\alpha = +\infty$ in (1.10). Then it is known that

$$\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta(a, f) \leq 2.$$

Are there functions which have an 'arbitrary' assignment of deficiencies at an arbitrary sequence of complex numbers, subject only to these conditions?

For analytic functions, Girynk [394] has a result; whereas for arbitrary meromorphic functions, there is a result of Krutin [641].

(L.R. Sons)

Update 1.41 No progress on this problem has been reported to us.

Problem 1.42 Let f be meromorphic in $\mathbb{C}$, and suppose that the function

$$F(z) = f^{(k)}(z) + \sum_{j=0}^{k-2} a_j(z) f^{(j)}(z)$$

is non-constant, where $k \geq 3$ and the coefficients a_j are polynomials. Characterise those functions f for which f and F have no zeros.

The case where f is entire has been settled by Frank and Hellerstein [353]. If all the a_j are constant, then the problem has also been solved by Steinmetz [922] using results from Frank and Hellerstein [353]. It seems possible that if the a_j are not all constants, then the only solutions with infinitely many poles are of the form

$$f = (H')^{-\frac{1}{2}(k-1)} H^{-l}, \tag{1.11}$$

where l is a positive integer, and H''/H' is a polynomial.

(G. Frank and J.K. Langley)

Update 1.42 This was solved by Brüggemann [169], who proved the following: let a linear differential operator

$$L(f) = f^{(k)} + \sum_{j=0}^{k-2} a_j f^j$$

with polynomial coefficients a_j be given, with at least one non-constant a_j. Then the only meromorphic functions f with infinitely many poles, satisfying $f L(f) \neq 0$, are of the form (1.11). An extension to rational coefficients has been given by Langley [653].

Problem 1.43 Let f be a meromorphic function of lower order λ. Let

$$m_0(r, f) = \inf\{|f(z)| : |z| = r\}$$

and

$$M(r, f) = \sup\{|f(z)| : |z| = r\}$$

and suppose that

$$\log r = o(\log M(r, f)), \qquad \text{as } r \to \infty.$$

Gol'dberg and Ostrovskii [415] proved that if $0 < \lambda < \frac{1}{2}$, then

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f)} + \pi\lambda \sin(\pi\lambda) \limsup_{r\to\infty} \frac{N(r, f)}{\log M(r, f)} \geq \cos(\pi\lambda).$$

Does this inequality remain valid for $\frac{1}{2} \leq \lambda < 1$? See also Gol'dberg and Ostrovskii [415].

(A.A. Gol'dberg and I.V. Ostrovskii)

Update 1.43 No progress on this problem has been reported to us.

1.3 New Problems

Problem 1.44 Consider the differential equation $\dot{z} = \dfrac{\mathrm{d}z}{\mathrm{d}t} = f(z)$, where f is a meromorphic function in the plane: see [455, 752] for fundamental results on such flows. King and Needham [752] showed that if f has a pole at infinity of order at least 2 then there exists at least one trajectory $z(t)$ which tends to infinity in finite time, that is, $\lim_{t\to\beta-} z(t) = \infty$, where $\beta \in \mathbb{R}$ and $z'(t) = f(z(t))$. It is then natural to ask what happens when f is transcendental.

It turns out [652] that if f is a transcendental entire function then there are infinitely many trajectories tending to infinity in finite time, whereas a transcendental meromorphic function need not have any at all. On the other hand, such trajectories do exist [652] if the inverse of a meromorphic f has a logarithmic singularity over infinity: this means that there exist $M > 0$ and a component U_M of the set $\{z \in \mathbb{C} : M < |f(z)| \leq \infty\}$ such that $v = \log f$ maps U_M univalently onto the half-plane $\operatorname{Re} v > \log M$.

The problem is then to resolve what happens under the weaker hypothesis that the inverse of f has a direct singularity over infinity, which means that there exists a component U_M of $\{z \in \mathbb{C} : M < |f(z)| \leq \infty\}$ on which f has no poles. In this case the Wiman–Valiron theory [505], which played a key role in the proof for entire functions in [652], is available in the version from [116], but the difficulty seems to be to show that the trajectories arising from the method of [652] do not all tend to poles of f outside U_M.

(J.K. Langley)

Problem 1.45 Following Hinchliffe [548], call a plane domain D a *Rubel domain* if every unbounded analytic function f on D has a sequence (z_n) in D such that $\lim_{n\to\infty} f^{(k)}(z_n) = \infty$ for all $k \geq 0$. The name arises because Rubel [594, 856] asked whether the unit disc has this property, which was subsequently proved to be the case by Gordon [435].

Evidently a Rubel domain must be bounded (otherwise take $f(z) = z$), but it is easy to construct simply connected bounded domains on which $1/z$ is bounded but $\log z$ is not [856]. Hinchliffe [548] extended Gordon's method to show that a bounded quasidisc (that is, the image of the unit disc under a quasiconformal mapping which fixes infinity) is a Rubel domain, but nothing more appears to be known on this problem.

Determine necessary and/or sufficient conditions for a bounded plane domain to be a Rubel domain.

(J.K. Langley)

Problem 1.46 A transcendental entire function is called *pseudoprime* if it cannot be written as a composition $f \circ g$ of transcendental entire functions f and g. Suppose that the Maclaurin series of a transcendental entire function F has large gaps: for example, Hadamard gaps

$$F(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad \lambda_{n+1} > q\lambda_n,$$

for some $q > 1$. Must F be pseudoprime?

(J.K. Langley)

Problem 1.47 Let $k \geq 3$ and let $A_0, \ldots, A_{k-2}$ be entire functions. Let $f_1, \ldots, f_k$ be linearly independent solutions of

$$w^{(k)} + \sum_{j=0}^{k-2} A_j(z) w^{(j)} = 0$$

and let E be the product $E = f_1 \ldots f_k$. Must the order of growth of E be at least that of each A_j? Note that if $k = 2$ then this follows immediately from the Bank–Laine formula [73].

(J.K. Langley)

Chapter 2
Entire Functions

2.1 Preface by P.J. Rippon

The study of entire functions (formerly known as integral functions and here assumed to be transcendental, that is, not polynomial) is one of the oldest areas of research in function theory. Its early history goes back to Weierstrass and Hadamard for the establishment of representations of entire functions f by infinite products, the concepts of the *order* of f, $\rho(f)$, its *genus*, and the *exponent of convergence* of its zeros, and the relationships amongst these quantities. Moreover Picard's theorem (1879) that an entire function which omits two values must be constant, once described by Hadamard as 'mysterious and disconcerting', shows how complicated their behaviour must be, especially near ∞, and how different it is from that of polynomials.

Other differences from polynomials are seen in the behaviour of the *maximum modulus* and *minimum modulus* of an entire function:

$$M(r, f) = \max\{|f(z)| : |z| = r\} \quad \text{and} \quad m_0(r, f) = \min\{|f(z)| : |z| = r\}.$$

For a polynomial these two quantities have similar behaviour for large values of r but for entire functions they can be very different, and roughly speaking the smaller $m_0(r, f)$ is the greater $M(r, f)$ has to be. Several problems associated with the relationship between these two quantities appear in this chapter. (Note that $m_0(r, f)$ is sometimes denoted by $m(r, f)$ and sometimes by $L(r, f)$.)

Another difference from polynomials is that entire functions need not tend to ∞ along rays from 0; indeed there exist non-constant entire functions that are bounded on any ray to ∞. We say that a is an *asymptotic value* of f if there is a path Γ tending to ∞ such that

$$f(z) \to a \qquad \text{as } z \to \infty, z \in \Gamma.$$

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Three major results involving asymptotic values are as follows:

- a theorem of Iversen [566] states that any unbounded entire function f must have asymptotic value ∞;
- a theorem of Ahlfors [12] states that an entire function of order ρ can have at most 2ρ finite asymptotic values;
- a theorem of Hayman [492] implies that if f is entire and $\log M(r, f) = O((\log r)^2)$ as $r \to \infty$, then f tends to ∞ along almost every ray from 0 to ∞.

Many problems in this chapter concern developments of these three results.

There are also many problems in this chapter about entire functions that arise as solutions of ordinary differential equations, in particular concerning the order of these solutions, and the location and nature of their zeros.

In recent decades there has been a great increase in work on complex dynamics, the iteration of analytic functions f. This theory was initiated in the early 20th century by Fatou and Julia independently, continued in the 1950s, 60s and 70s mainly by Baker, and revived in the 1980s partly under the influence of easily accessible computer graphics. Here the objects of study are the Fatou set $F(f)$ (the open set where the family of iterates $\{f^n\}$ of f forms a normal family) and its complement the Julia set $J(f)$. (Note that these names and notations only became standard in the subject around 1990—earlier, for example, the Fatou set was called the domain of normality.)

A huge advance was made by Sullivan [930], using quasiconformal techniques to show that rational functions do not have wandering domains (the name for components of the Fatou set that are not eventually periodic under iteration of f). The theory of polynomial and rational dynamics has become especially highly developed, partly because the so-called MLC conjecture, which states that the Mandelbrot set (the parameter space for the quadratic family) is locally connected, remains stubbornly resistant.

There have also been major developments in the theory of transcendental dynamics, where Picard's theorem and the existence of finite asymptotic values lead to many differences. Entire functions can have wandering domains (early examples were given by Baker [69] and Herman [542]) but there are many classes of entire functions which do not have them, and it is of interest to find more such classes and to improve our understanding of the properties of wandering domains in general. Amongst the many open problems in transcendental dynamics is the so-called problem of 'bounded wandering', namely: Can there exist wandering domains in which the iterates $(f^n(z))$ form bounded sequences? Several versions of this problem are discussed in this chapter.

We draw the reader's attention to many other problems in complex dynamics raised in Douady [250, 251], Herman [544], Lyubich [690], Beardon [93], Milnor [739], and in the extensive survey by Bergweiler [106], which includes the well-known conjecture of Eremenko that all components of the escaping set

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

are unbounded. Note that updates on the problems and results in [106] can be found in an article on Bergweiler's website.

2.2 Progress on Previous Problems

Notation Let $f(z)$ be an entire function. We say that a is an *asymptotic value* of $f(z)$ if

$$f(z) \to a,$$

as $z \to \infty$ along a path Γ, called a corresponding *asymptotic path*. Some of the most interesting open problems concerning entire functions centre on these asymptotic values and paths. It follows from a famous result of Ahlfors [10] that an entire function of finite order ρ can have at most 2ρ distinct finite asymptotic values. On the other hand, by a theorem of Iversen [566], ∞ is an asymptotic value of every entire function. Some of the following problems are concerned with generalisations arising out of the above two theorems.

Throughout this section

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)|$$

denotes the *maximum modulus* of $f(z)$.

Problem 2.1 Suppose that $f(z)$ is an entire function of finite order. What can we say about the set E of values w such that

(a)

$$\lambda(r, f - w) = \min_{|z|=r} |f(z) - w| \to 0, \qquad \text{as } r \to \infty;$$

or

(b)

$$m\left(r, \frac{1}{f - w}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\theta}) - w} \right| d\theta \to \infty, \qquad \text{as } r \to \infty\,?$$

Clearly (b) implies (a). By a result of Arakelyan [41] (see also Problem 1.6), the set of deficient values, which is clearly contained in E, can include any countable set. Can E be non-countably infinite in case (b), or contain interior points in case (a)?

Update 2.1(a) This was settled by examples of Arakelyan [41]. Let E be a dense countable set in the plane, every value of which is deficient. Then clearly $\lambda(r, f - w) \to 0$ as $r \to \infty$, for every w in the plane.

Update 2.1(b) This was settled by Drasin and Weitsman [258]. The set of w for which $m(r, \frac{1}{f-w}) \to \infty$ as $r \to \infty$ must have capacity zero, and an arbitrary set of capacity zero may occur.

Problem 2.2 Produce a general method for constructing an entire function of finite order, and in fact, minimal growth, which tends to different asymptotic values $w_1, w_2, \dots, w_k$ as $z \to \infty$, along preassigned asymptotic paths $C_1, C_2, \dots, C_k$. (Known methods by Kennedy [592] and Al-Katifi [17] only seem to work if the w_ν are all equal, unless the C_ν are straight lines.)

Update 2.2 Such a construction has been given by Hayman [499].

Problem 2.3 If $\phi(z)$ is an entire function growing slowly compared with the function $f(z)$, we can consider $\phi(z)$ to be an asymptotic function of $f(z)$ if $f(z) - \phi(z) \to 0$ as $z \to \infty$ along a path Γ. Is it true that an entire function of order ρ can have at most 2ρ distinct asymptotic functions of order less than $\frac{1}{2}$?

If $f(z) - \phi_1(z) \to 0$ and $f(z) - \phi_1(z) \to 0$ along the same path Γ and $\phi_1(z)$, $\phi_2(z)$ have order less than $\frac{1}{2}$, then by Wiman's theorem $\phi_1(z) \equiv \phi_2(z)$. Also if, $\phi(z) = \phi_1(z) - \phi_2(z)$ and the minimum modulus of ϕ tends to zero, then ϕ has lower order at least $\frac{1}{2}$ mean type (see Hayman [511, p. 288]).

A positive result in this direction is due to Denjoy [244], but only when the paths are straight lines. The result when the $\phi_2(z)$ are polynomials is true (and is a trivial consequence of Ahlfors' theorem for asymptotic values [10]).

Update 2.3 An answer has been given by Somorjai [912] with $\frac{1}{30}$ instead of $\frac{1}{2}$, and Fenton [339] has obtained the same conclusion, if the orders are less than $\frac{1}{4}$. See also Hinkkanen and Rossi [549].

Problem 2.4 Suppose that $f(z)$ is a meromorphic function in the plane, and that for some θ, $0 \le \theta < 2\pi$, $f(z)$ assumes every value infinitely often, with at most two exceptions, in every angle $\theta - \varepsilon < \arg z < \theta + \varepsilon$, when ε is positive. Then the ray $\arg z = \theta$ is called a *Julia line*. Lehto [654] has shown that if $f(z)$ is an entire function, or if $f(z)$ is meromorphic and

$$\limsup_{r\to\infty} \frac{T(r, f)}{(\log r)^2} = +\infty,$$

(but not necessarily otherwise), at least one Julia line exists. What can we say about the exceptional values at different Julia lines? In particular, can an entire function $f(z)$ have one exceptional (finite) value a at one Julia line Γ_a, and a different exceptional value b at a different Julia line Γ_b?

(C. Rényi)

Update 2.4 An example has been given by Toppila [950] of a function having different exceptional values at each of n Julia lines. See also Gol'dberg [401].

Problem 2.5 What can we say about the set E of values a which an entire function $f(z)$ assumes infinitely often in every angle? Simple examples show that E may be the whole open plane, for example, if

$$f(z) = \sigma(z) = z \prod_{(m,n)\neq(0,0)} \left(1 - \frac{z}{z_{m,n}}\right) \exp\left\{\frac{z}{z_{m,n}} + \frac{1}{2}\left(\frac{z}{z_{m,n}}\right)^2\right\},$$

where $z_{m,n} = m + ni$, or the whole plane except one point, for example, if $f(z) = \mathrm{e}^{\sigma(z)}$. If $z_m = 2^m \mathrm{e}^{im}$, and

$$f(z) = \mathrm{e}^z \prod_{m=1}^{\infty} \left(1 - \frac{z}{z_m}\right),$$

then E consists of the value 0 only, since clearly $f(z) \to 0$ as $z \to \infty$, uniformly for $\pi/2 + \varepsilon < \arg z < 3\pi/2 - \varepsilon$, if ε is positive. Can E consist of exactly two values?

(C. Rényi)

Update 2.5 Gol'dberg [402] has answered several of the questions posed in this problem. In particular, he showed that given any countable set A, there exists an entire function $f(z)$ for which the set E satisfies $A \subset E \subset \overline{A}$, where $\overline{A}$ is the closure of A.

Problem 2.6 Let $f(z)$ be an entire function. Then Boas (unpublished) proved that there exists a path Γ_∞ such that, for every n,

$$\left|\frac{f(z)}{z^n}\right| \to \infty, \qquad \text{as } z \to \infty \text{ along } \Gamma. \tag{2.1}$$

Can we improve this result if something is known about the lower growth of $M(r, f)$? Hayman [491] has shown that there exist functions of infinite order and, in fact, growing arbitrarily rapidly, such that, on every path Γ on which $f(z) \to \infty$, we have

$$\log\log|f(z)| = O(\log|z|),$$

that is, $f(z)$ has finite order on Γ.

Update 2.6 Talpur [935] has shown that if f has order ρ and $\alpha < \rho < \frac{1}{2}$, $\varepsilon > 0$, then we can find a path Γ going to ∞ on which

$$\log|f(z)| > \log M(|z|^{(1-\rho/\alpha)/(1+\varepsilon)}) \cos(\pi\alpha).$$

Eremenko [306] has proved the following result: let f be an entire function of order ρ and lower order λ. Then there exists an asymptotic path Γ such that

$$\log|f(z)| > (A(\rho, \lambda) + o(1)) \log|z|, \qquad \text{as } z \to \infty, z \in \Gamma,$$

where $A(\rho, \lambda)$ is some explicitly written function, with the property $A(\rho, \lambda) > 0$, for $0 < \lambda \le \rho < \infty$. When $\lambda < 1/2$, we have $A(\rho, \lambda) \ge \lambda$. See also Update 2.8.

Problem 2.7 If $f(z)$ is of finite order, can anything be asserted about the length of Γ_∞, which is the path on which $f(z)$ tends to ∞, or the part of it in $|z| \le r$?

Update 2.7 Let $\ell(r)$ be the length of the arc of Γ_∞ to the first intersection with $|z| = r$. If

$$T(r, f) = O((\log r)^2) \qquad \text{as } r \to \infty,$$

then Hayman [492] showed that Γ_∞ can be taken to be a straight line.

Eremenko and Gol'dberg [410] have constructed examples for which $T(r, f)/(\log r)^2$ tends to ∞ arbitrarily slowly but $\ell(r) = O(r)$ fails to hold. An independent proof has been given by Toppila [953].

On the other hand, Chang [202] has proved that if f has finite order ρ, then for any positive ε,

$$\ell(r) = O(r^{(1+\frac{1}{2}\rho+\varepsilon)})$$

can always hold.

It is also possible that $f(z) \to a$ as $z \to \infty$ along a path Γ_a. In this case. Gol'dberg and Eremenko [410] have constructed examples with f having order arbitrarily close to $\frac{1}{2}$, while $\ell(r) \ne O(r)$. See also Update 2.10, and Lewis, Rossi and Weitsman [667].

Problem 2.8 Does (2.1) remain true if the number $n(r)$ of poles of $f(z)$ in $|z| < r$ satisfies $n(r) = O(r^k)$, where $k < \frac{1}{2} < \lambda$, and λ is the lower order of $f(z)$? Gol'dberg and Ostrovskii [415] have shown that (2.1) can be false if $\frac{1}{2} < k < \lambda$.

Update 2.8 The original version of this problem had order ρ instead of lower order λ. However, Gol'dberg pointed out to Hayman orally that this would give a negative answer.

One may ask for corresponding results if f is meromorphic with sufficiently few poles. If ∞ is Nevanlinna deficient, and $T(r, f) = O\big((\log r)^2\big)$, then Anderson and Clunie [37] showed that $f \to \infty$ along almost all straight lines. This result fails for functions of larger growth, according to an example of Hayman [507], even if the deficiency is one. He also proves that if

$$\limsup_{r\to\infty} \frac{r^{\frac{1}{2}}}{T(r, f)} \int_r^\infty \frac{N(t, f)\,\mathrm{d}t}{t^{\frac{3}{2}}} < 2,$$

then ∞ is an asymptotic value of f. This is true, for instance, if the order of the poles of f is smaller than $\frac{1}{2}$, and smaller than the lower order of f.

Problem 2.9 We ask the analogues of Problems 2.6–2.8 if, in addition, $f(z)$ has a finite Picard value, for example, $f(z) \ne 0$. In this case, if ∞ has deficiency one, in the sense of Nevanlinna (see the start of Chap. 1), (2.1) remains true for functions of finite order (see Edrei and Fuchs [288]), but not necessarily for functions of infinite order (see Gol'dberg and Ostrovskii [415]).

Update 2.9 If $f(z) \neq 0$, then for every K, the level set $|f(z)| = K$ contains a curve tending to infinity. Under this condition, Rossi and Weitsman [851] proved that there is an asymptotic curve Γ with the following properties:

$$\log|f(z)| > |z|^{1/2-\varepsilon(z)}, \qquad \text{for } z \in \Gamma, \text{ where } \varepsilon(z) \to 0, \text{ and} \tag{2.2}$$

$$\int_{\Gamma} (\log|f|)^{-(2+\alpha)}\,|\mathrm{d}z| < \infty, \qquad \text{for all positive } \alpha.$$

On the other hand, Barth, Brannan and Hayman [85] constructed a zero-free entire function, for which no asymptotic curve satisfies (2.2) with $\varepsilon = 0$. Furthermore, Brannan pointed out that, for their example, every asymptotic curve Γ satisfies

$$\int_{\Gamma} (\log|f(z)|)^{-2}|\mathrm{d}z| = \infty.$$

Problem 2.10 Huber [558] proved that, for every positive μ, there exists a path C_μ tending to infinity such that

$$\int_{C_\mu} |f(z)|^{-\mu}|\mathrm{d}z| < \infty, \tag{2.3}$$

provided that $f(z)$ does not reduce to a polynomial. Does there exist a C_∞ such that (2.3) holds for every positive μ with $C_\mu = C_\infty$?

Update 2.10 This question has been settled by Lewis, Rossi and Weitsman [667], who have proved that there is a path C_∞ such that (2.3) holds for every positive μ, with C_∞ instead of C_μ, thus answering the question affirmatively. Further, they prove that C_μ is asymptotic, that is, $f(z) \to \infty$ as $z \to \infty$ along C_μ; and they also obtain estimates for the length $\ell(r)$ in Problem 2.7. The case of finite order was dealt with by Chang [202].

Problem 2.11 If $f(z) = \sum a_n z^{\lambda_n}$ is an entire function, and $\sum(1/\lambda_n)$ converges, is it true that:

(a) $f(z)$ has no finite asymptotic value,

(b)
$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f)} = 1,$$

where $m_0(r, f) = \inf_{|z|=r} |f(z)|$ is the minimum modulus of $f(z)$?

(a) is known for $\lambda_n > n(\log n)^{1+\varepsilon}$ (see Kövari [631]); and (b) is known for $\lambda_n > n(\log n)^2$ (see Kövari [630]). It is also known that $f(z)$ has no finite radial asymptotic value if $\sum(1/\lambda_n)$ converges, and that here this hypothesis cannot be replaced by any weaker condition (see Macintyre [695]).

Update 2.11 Nazarov [750] proved that each of the following conditions

(a) $\lambda_{k+1} + \lambda_{k-1} \geq 2\lambda_k$ and $\sum(1/\lambda_k) < \infty$,
(b) $\sum(\log\log k)/\lambda_k < \infty$

imply

$$\limsup_{z\to\infty,\, z\in\Gamma} \frac{\log|f(z)|}{\log M(|z|, f)} = 1$$

for every curve Γ tending to infinity.

Problem 2.12 If the entire function $f(z)$ has finite order ρ, and the maximal density of non-zero coefficients is Δ, is it true that if $\rho\Delta < \frac{1}{2}$, $f(z)$ cannot have a finite deficient value? Edrei and Fuchs [288] have shown that if $\rho\Delta < 1$, $f(z)$ cannot have a finite deficient value with deficiency one.

Update 2.12 No progress on this problem has been reported to us.

Problem 2.12(a) Under the same conditions as in Problem 2.12, is it true that if $\rho\Delta < 1$, $f(z)$ cannot have a finite asymptotic value? This is known if $\rho\Delta < 1/\pi^2$; see Kövari [632]. Is it true that, if $m_0(r, f)$ is the minimum modulus, then

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f)} \geq \cos(\pi\rho\Delta)\,?$$

See Kövari [633].

Update 2.12(a) Fryntov [358] proved the following partial result: suppose that f is an entire function of lower order λ, with density of non-zero coefficients Δ. If $\lambda\Delta < 1/3$, and Γ is a curve which intersects each circle $|z| = r$ at most once, then

$$\limsup_{z\to\infty,\, z\in\Gamma} \frac{\log|f(z)|}{\log M(|z|, f)} \geq 2\cos(\pi\lambda\Delta) - 1.$$

Problem 2.13 If $f(z) = \sum a_n z^{\lambda_n}$ is an entire function, and $\lambda_n/n \to \infty$, is it true that $f(z)$ has

(a) no Picard value,
(b) no Borel exceptional value,
(c) no deficient value?

All this is known for functions of finite order (see Fuchs [360]). If the answer is 'no', are (b) and (c) true for $\sum(1/\lambda_n) < \infty$? Certainly by a theorem of Biernacki [124], (a) is true in this case.

(T. Kövari)

Update 2.13 Murai [748] has shown that $\sum(1/\lambda_n) < \infty$ does indeed imply (b) and (c), but that $\lambda_n/n \to \infty$ as $n \to \infty$ does not. He constructed an example to show that $\lambda_n/n \to \infty$ as $n \to \infty$ is consistent with $\delta(0, f) = 1$.

Only the question whether $\sum(1/\lambda_n) < \infty$ implies (a) remains open.

Problem 2.14

(a) Let $f(z) = \sum a_n z^n$ be entire and $m(r) = \max_n |a_n| r^n$. If $C > \frac{1}{2}$ then does there exist an entire f with
$$m(r)/M(r, f) \to C?$$

Any value of C such that $0 < C \leq \frac{1}{2}$ is possible.

(b) If $f(z) \neq 0$, then
$$\liminf_{r\to\infty} \frac{m(r)}{M(r, f)} = 0;$$
is
$$\lim_{r\to\infty} \frac{m(r)}{M(r, f)} = 0\,?$$

(c) Removing the condition that $f(z) \neq 0$, what is the exact upper bound β of
$$\beta_f = \liminf_{r\to\infty} \frac{m(r)}{M(r, f)}\,?$$

It is known that $\frac{4}{7} < \beta < 2/\pi$; see Clunie and Hayman [214].

Update 2.14(b) Davies [237] has shown that the upper limit can be positive.

Problem 2.15 (*Blumenthal's conjecture*) Let $w = f_1(z), f_2(z)$ be entire functions. Is it true that if
$$M(r, f_1) = M(r, f_2), \qquad 0 < r < \infty,$$
then $f_1(z), f_2(z)$ are equivalent, apart from rotations and reflections in the z and w planes? The corresponding problem for polynomials (of degree higher than about 6) is also open.

The functions $(1 - z)e^z$ and $e^{\frac{1}{2}z^2}$ have the same value of $M(r, f)$ for $0 < r < 2$, and e^{z-z^2} and $e^{z^2+\frac{1}{2}}$ for $r \geq \frac{1}{4}$.

Update 2.15 Hayman, Tyler and White [520] establish Blumenthal's conjecture for polynomials f and g with at most four non-zero terms (and so, in particular, for all quadratic and cubic polynomials). Short examples are also given to show that, in the general case, it is not sufficient only to consider arbitrarily large or arbitrarily small positive values of r. However, the following result is proved: if entire functions f and g are real on the real axis, $f(0)f'(0) \neq 0$ and $M(r, f) = M(r, g)$ in some range $0 < r < r_0$, then f and g are equivalent. Here we say that f and g are *equivalent* if
$$f(z) = e^{i\theta} g(ze^{i\phi})$$
for suitable θ and ϕ.

The corresponding problem for polynomials of degree higher than 3 remains open.

Problem 2.16 Let $\nu(r)$ be the number of points on $|z| = r$ such that $|f(z)| = M(r, f)$. Can we have

(a) $\limsup_{r\to\infty} \nu(r) = \infty$?
(b) $\liminf_{r\to\infty} \nu(r) = \infty$?

(P. Erdős)

Update 2.16 Herzog and Piranian [545] have shown that (a) is indeed possible, however the answer to (b) is still unknown. These authors also provided an example of a univalent function in $\mathbb{D}$ for which the analogue of (a) holds.

Problem 2.17 If $f(z)$ is a non-constant entire function and

$$b(r) = \left(r\frac{d}{dr}\right)^2 \log M(r, f),$$

then

$$\limsup_{r\to\infty} b(r) \geq A, \tag{2.4}$$

where A is an absolute constant, such that $0.18 < A \leq \frac{1}{4}$; see Hayman [497]. What is the best value of A? It seems fair to conjecture that the correct constant in (2.4) is in fact $\frac{1}{4}$.

Update 2.17 Kjellberg [609] proved that this conjecture is false, and that $0.24 < A < 0.248$. In this problem, and also in Problem 2.18, the monomials az^n should be excluded. Boĭčuk and Gol'dberg [147] have shown that the result is true if f has positive coefficients; and in fact

$$\limsup_{r\to\infty} b(r) \geq \frac{1}{4}A^2, \qquad \text{where } A = \limsup_{k\to\infty}(n_{k+1} - n_k).$$

Tyler (unpublished) has some numerical evidence that A is just under 0.247.

Problem 2.18 Consider the function $b(r)$ of Problem 2.17. Since $\log M(r, f)$ is an analytic function of r, except for isolated points, $b(r)$ exists except at isolated points where the right and left limits $b(r \pm 0)$ still exist, but may be different. It follows from Hadamard's convexity theorem (see Hayman [497]) that $b(r) \geq 0$. Is equality possible here for an entire function, or more generally, a function analytic on $|z| = r$, in the sense that

$$b(r + 0) = b(r - 0) = 0?$$

Clunie notes that if $f(z) = (z - 1)e^z$, then

$$M(r, f) = (r - 1)e^r, \qquad r > 2, \qquad \text{and} \qquad b(2 + 0) = 0.$$

Update 2.18 Let

$$b(r) = \left(\frac{\mathrm{d}}{\mathrm{d}\log r}\right)^2 \log M(r, f).$$

The quantity exists as a left or right limit everywhere and is non-negative. Examples had been constructed by Clunie previously to show that $b(r+0)$ or $b(r-0)$ may be zero. London [681] has given a positive answer to this question by constructing an example of the form $f(z) = (1-z)\mathrm{e}^{\alpha z+\beta z^2+\gamma z^3}$ for which $b(2+0) = b(2-0) = 0$. The powers z^n are the only functions for which $b(r) = 0$ for a whole interval $r_1 \le r \le r_2$ of values of r.

See also Update 2.17.

Problem 2.19 If $f(z)$ is an entire function of exponential type, that is, satisfying $|f(z)| \le M\mathrm{e}^{K|z|}$ for some constants M, K, and if, further, $|f(x)| \le A$ for negative x, and $|f(x)| \le B$ for positive x, what is the sharp bound for $|f(z)|$? If $A = B$, it is known that

$$|f(z)| \le A\mathrm{e}^{Ky}$$

is true and sharp.

Update 2.19 An analogous problem for subharmonic functions was solved by Gol'dberg and Levin [412]. This gives an upper estimate but it is not exact for entire functions. A sharp bound for $|f(x)|$ for entire functions was found by Eremenko [312].

Problem 2.20 If $f(z)$ is an entire function, the iterates $f_n(z)$, $n = 1, 2, \ldots$ are defined inductively by

$$f_{n+1}(z) = f(f_n(z)), \qquad f_1(z) = f(z).$$

A point z satisfying the equation $f_n(z) = z$, but such that $f_k(z) \ne z$ for $k < n$, is called a *fixed point of exact order* n. Prove that there always exist fixed points of exact order n if $f(z)$ is transcendental, and $n \ge 2$. For the case of polynomial or rational $f(z)$, see Baker [63]. For a proof that fixed points of exact order n exist, except for at most one value of n, see Baker [64].

Update 2.20 Bergweiler [105] proved that if f is a transcendental entire function, and $n \ge 2$, then f has infinitely many fixed points of exact order n. This also follows from another result by Bergweiler [108].

Problem 2.21 If, in the terminology of Problem 2.20, z_0 is a fixed point of exact order n for $f(z)$, the fixed point is called *repelling* if $|f_n'(z_0)| > 1$. It is a problem of Fatou (see [330, 331]) whether every entire transcendental function $f(z)$ has repelling fixed points. It is shown by Fatou (see [330, 331]) that for rational $f(z)$ (including polynomials), all fixed points of sufficiently high (exact) order are repelling.

Update 2.21 The existence of repelling fixed points was proved for the first time by Baker [67], who used Ahlfors' theory of covering surfaces [11] and showed that such points are dense in the non-normality set of f. Since then, the proof of this important result has been generalised to meromorphic functions, and ultimately evolved into an elementary half-page argument of Berteloot and Duval [117].

Problem 2.22 With the terminology of Problem 2.20, denote by $\mathcal{F}(f)$ the set of points where the sequence $\{f_n(z)\}$ is not normal. Fatou [333] asks if there is an entire function $f(z)$ for which $\mathcal{F}(f)$ is the whole plane, and, in particular, if this is the case for $f(z) = \mathrm{e}^z$. Since every point of $\mathcal{F}(f)$ is an accumulation point of fixed points of $f(z)$, this is equivalent to asking if the fixed points (of all orders) of e^z are dense in the plane.

Update 2.22 Baker [68] proved that if $f(z) = kz\mathrm{e}^z$, where k is a certain positive constant, then the set of non-normality, the Julia set, does indeed occupy the whole plane. Misiurewicz [741] proved that this is also the case for $f(z) = \mathrm{e}^z$, and many further examples are now known.

Problem 2.23 Baker [66] proved that if $f(z)$ is a transcendental entire function, then $\mathcal{F}(f)$ is not restricted to a straight line in the plane. This implies (see Problem 2.21) that, given a line l, there are fixed points (of sufficiently high order) not belonging to l. Is it already true that a transcendental $f(z)$ cannot have all its fixed points of order at most 2 on l? This is indeed true for $f(z)$ of order less than $\frac{1}{2}$.

Update 2.23 Bergweiler, Clunie and Langley [109] proved the conjecture by showing that for every transcendental entire function f and every line, infinitely many of the fixed points of every n-th iterate, $n \geq 2$, do not lie on this line. Bergweiler [108] improved this by showing that for every line, there are infinitely many repelling fixed points of each n-th iterate, $n \geq 2$, which do not lie on this line.

Problem 2.24 Can an entire function have all its zeros and ones on two distinct straight lines, having infinitely many on each line?

Edrei has proved (unpublished) that if l, m are intersecting straight lines, then it is impossible for all the zeros of an entire function $f(z)$ to lie on l, and all the ones on m.

(A. Edrei)

Update 2.24 Al-Katifi has noted orally that $f(z) = \sin(z^2)$ has this property on the real and imaginary axes. Ozawa (private communication to Hayman) made the same observation, and also proved some related uniqueness theorems.

When the lines in question are parallel, there are no entire functions with zeros on one line and ones on another, except one explicitly listed exceptional function. This was proved independently by Baker [65] and Kobayashi [611].

Bergweiler, Eremenko and Hinkkanen [113] proved a number of results on this subject, generalising Edrei's theorem.

Problem 2.25 If f and g are linearly independent entire functions of order ρ, which is not a positive multiple of $\frac{1}{2}$, can $fg' - gf'$ have order less than ρ? This is possible if $\rho = n/2$, $n \geq 2$. Clunie (unpublished) proved the result if $\rho < \frac{1}{3}$.

(A. Edrei)

Update 2.25 The original statement of this problem contained 'distinct' instead of 'linearly independent'. A negative answer follows from a stronger result by Eremenko [314] who writes that his solution of Problem 1.33 in [314] also contains a solution of Problem 2.25. See Update 1.33.

Problem 2.26 What is the least integer $k = k(N)$ such that every entire function $f(z)$ can be written as

$$f(z) = \sum_{\nu=1}^{k} [f_\nu(z)]^N,$$

where the $f_\nu(z)$ are entire functions? It is enough to solve the problem for $f(z) = z$ since then one can substitute $f(z)$ for z. The equation

$$z = \frac{1}{N^2} \sum_{\nu=1}^{N} \frac{(1 + \omega_\nu z)^N}{\omega_\nu},$$

where ω_ν are the distinct N-th roots of unity, shows that $k(N) \leq N$. On the other hand, for $N = 1, 2, 3$ we have $k(N) = N$.

To see, for example, that $k(3) \geq 3$, suppose that

$$z = f^3 + g^3 = (f + g)(f + \omega g)(f + \omega^2 g),$$

where $\omega = \exp(2\pi i/3)$. It follows that the meromorphic function $\phi(z) = f(z)/g(z)$ satisfies $\phi(z) \neq -1, -\omega, -\omega^2$, except possibly at $z = 0$. Thus, by Picard's Theorem, $\phi(z)$ must be rational, and so $\phi(z)$ assumes at least two of the three values $-1, -\omega, -\omega^2$. This gives a contradiction.

(H.A. Heilbronn)

Update 2.26 Let $\mathcal{P}$ be the class of polynomials, $\mathcal{E}$ that of entire functions, $\mathcal{R}$ that of rational functions and $\mathcal{M}$ that of meromorphic functions in the plane. An easy argument shows that it is enough to consider $f(z) = z$. Since a number of authors have obtained partial results, it is worth noting that all known lower bounds for k are immediate consequences of an old result of Cartan [197]. This yields

$$k(k-1) > N \text{ in } \mathcal{P}, \text{ if } N > 2; \qquad k(k-1) \geq N \text{ in } \mathcal{E};$$

$$k^2 - 1 > N \text{ in } \mathcal{R}; \qquad k^2 - 1 \geq N \text{ in } \mathcal{M}.$$

These results are all sharp for $k = 2$. (For the case $k = 2$, $N = 3$ in $\mathcal{M}$, see Gross and Osgood [445].) But for large k, no upper bounds substantially better than $k \leq N$ are known for any of the above classes. See also Gundersen and Hayman [452, Theorem 5.1].

For the related equation $1 = \sum_{\nu=1}^{k} f_\nu(z)^N$, where the terms on the right-hand side are assumed linearly independent, the above lower bounds for k remain valid, and are proved in the same way; but an example due to Molluzzo [743], quoted by Newman and Slater [761], shows that they give the correct order of magnitude $\sqrt{N}$ for k as a function of N.

It follows from this that the functional equation $f^n + g^n + h^n = 1$ cannot have non-constant meromorphic solutions for $n \geq 9$. Gundersen [449, 450] constructed examples of transcendental meromorphic solutions for $n = 5$ and $n = 6$. Thus, only the cases $n = 7$ and $n = 8$ remain unsolved.

Problem 2.27 Let $\phi_1, \ldots, \phi_n$ denote entire functions of the form

$$\phi(z) = \sum \mathrm{e}^{f_\nu(z)} / \sum \mathrm{e}^{g_\nu(z)}, \tag{2.5}$$

where $f_\nu(z)$, $g_\nu(z)$ are entire functions, and the indices run over the (finite) number of functions involved. In general, the right-hand side of (2.5) is not an entire function. Does there exist an entire function $f(z)$, not of the form $\phi(z)$, but satisfying an algebraic equation of the form $f^n + \phi_1 f^{n-1} + \ldots + \phi_n = 0$? The special cases $n = 2$, or when the $f_\nu(z)$ are linear polynomials, may be easier to settle.

As an example, Hayman notes that

$$f(z) = \frac{\sin \pi z^2}{\sin \pi z}$$

is not of the form $\sum \mathrm{e}^{f_\nu(z)}$, although it is a ratio of such functions.

Update 2.27 No progress on this problem has been reported to us.

Problem 2.28 A meromorphic function $f(z)$ in the plane is said to be of *bounded value distribution (b.v.d.)* if, for every positive r, there exists a fixed constant $C(r)$ such that the equation $f(z) = w$ never has more than $C(r)$ roots in any disc of radius r. (It is clearly enough to make the assumption for a single value of r.)

(a) If $f(z)$ is an entire function, prove that $C(r) = O(r)$ as $r \to \infty$, so that $f(z)$ has exponential type at most.
If a differential equation

$$y^{(n)} + f_1(z)y^{(n-1)} + \ldots + f_n(z)y = 0, \tag{2.6}$$

where the $f_n(z)$ are entire functions, has only b.v.d. solutions, Wittich [995] proves that the f_ν are all constants. The converse is also true.

(b) Is it sufficient to make the basic assumption, not for all values w, but for only three such values, to assure that $f(z)$ is of b.v.d.?

(P. Turán)

Update 2.28(a) A positive answer was provided by Hayman [502] when he showed that b.v.d. functions are precisely those whose derivatives have bounded index. (Let $f(z)$ be an entire function and for each z let $N(z)$ be the least integer such that

$$\sup_{0\leq j<\infty}\left|\frac{f^{(j)}(z)}{j!}\right| = \frac{|f^{(N)}(z)|}{N!}.$$

If $N(z)$ is bounded above for varying z, then $f(z)$ is said to be of *bounded index*, and the least upper bound N of $N(z)$ is called the *index* of $f(z)$.)

Update 2.28(b) A negative answer was provided by Gol'dberg [403]. Another such example is given by the sigma-function. If E is any bounded set, and $\varepsilon < \frac{1}{2}$, then, when $\omega \in E$, the equation $\sigma(z) = \omega$ has exactly one root in $|z - m - in| < \varepsilon$ for integers m, n with $m^2 + n^2 > r_0$ for r_0 depending on ε and E. But $\sigma(z)$ has order 2 and so cannot be a b.v.d. function.

Problem 2.29 Is it possible to give an analogous characterisation of the solutions of (2.6) in the case where the $f_\nu(z)$ are polynomials?

(P. Turán)

Update 2.29 Such a characterisation has been provided by Tijdeman [939, 940]. The coefficients $f_\nu(z)$ are polynomials if and only if there exist fixed numbers p and q such that each solution $g(z)$ of (2.6) is p-valent in any disc $\{z : |z - z_0| < 1/(1 + r^q)\}$ where $r = |z_0|$.

Problem 2.30 Let S_k, $k = 1, 2, \ldots$, be sets which have no finite limit points. Does there exist a sequence n_k and an entire function $f(z)$ such that whenever $z \in S_k$ we have $f^{(n_k)}(z) = 0$?

(P. Erdős)

Update 2.30 Functions satisfying the conditions of this problem have been constructed by Barth and Schneider [90].

Problem 2.31 Let A and B be two countable dense sets in the plane. Does there exist an entire function $f(z)$ such that $f(z) \in B$ if and only if $z \in A$? If the answer is negative, it would be desirable to have conditions on A and B when this is so.

(P. Erdős)

Update 2.31 Functions satisfying the conditions of this problem have been constructed by Barth and Schneider [89].

Problem 2.32 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function where $a_n \geq 0$ for $n \geq 0$, and set

$$p_n(z) = \frac{a_n z^n}{f(z)}.$$

Then

$$\sum_{n=0}^{\infty} p_n(z) = 1.$$

In addition, if $f(z) = \mathrm{e}^z = \sum_{n=0}^{\infty} z^n/n!$, we have

$$\int_0^{\infty} p_n(z)\,\mathrm{d}z = 1, \qquad n = 0, 1, 2, \ldots, \tag{2.7}$$

or, equivalently,

$$\int_0^{\infty} \frac{f(\rho z)}{f(z)}\,\mathrm{d}z = \frac{1}{1-\rho}, \qquad 0 < \rho < 1. \tag{2.8}$$

Does there exist a transcendental entire function $f(z)$, other than e^z, satisfying (2.7) or (2.8)?

(A. Rényi, St. Vincze)

Update 2.32 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a transcendental entire function with positive coefficients, and suppose that

$$\int_0^{\infty} \frac{a_n z^n}{f(z)}\,\mathrm{d}z = 1, \qquad n = 0, 1, 2, \ldots.$$

Rényi and Vincze had asked whether these conditions imply that $f(z) = c\mathrm{e}^z$, and this has been proved by Miles and Williamson [730]. Weaker results were proved earlier by Hall and Williamson [461] and Hayman and Vincze [521].

Notation Let $f(z)$ be an entire function of order ρ, and lower order λ, and let

$$m_0(r, f) = \inf_{|z|=r} |f(z)|, \qquad M(r, f) = \sup_{|z|=r} |f(z)|.$$

It is a classical result that

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f)} \geq C(\lambda).$$

Here $C(\lambda) = \cos(\pi\lambda)$ for $0 \leq \lambda \leq 1$, and Hayman [484] has shown that

$$-2.19 \log \lambda < C(\lambda) < -0.09 \log \lambda$$

when λ is large. We refer the reader to Barry [82] for a general account of the situation. For functions of infinite order the analogous result is

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f) \log\log\log M(r, f)} \geq C(\infty),$$

where

$$-2.19 < C(\infty) < -0.09.$$

Problem 2.33 Is it possible to obtain the exact value of $C(\infty)$ or the asymptotic behaviour of $\frac{C(\lambda)}{\log\lambda}$ as $\lambda \to \infty$? The question seems related to the number of zeros a function can have in a small disc centred on a point of $|z| = r$; see Hayman [484].

Update 2.33 No progress on this problem has been reported to us.

Problem 2.34 Is it possible to say something more precise about $C(\lambda)$ when λ is just greater than 1? In particular, is it true that $C(\lambda) = -1$ for such λ, or alternatively, is $C(\lambda)$ a strictly decreasing function of λ?

Update 2.34 Whenever $\rho > 1$, Fryntov [358] constructed an entire function f of order ρ with the property

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{\log M(r, f)} < -1.$$

Problem 2.35 If Γ is a continuum that recedes to ∞, it is known (see Hayman [484]) that as $z \to \infty$ on Γ,

$$\limsup_{r\to\infty} \frac{\log|f(z)|}{\log M(|z|)} \geq -A,$$

where A is an absolute constant. Is it true that $A = 1$? This is certainly the case if Γ is a ray through the origin; see Beurling [122]. If $A > 1$, is it possible to obtain a good numerical estimate for A?

Update 2.35 Hayman and Kjellberg [517] have given a positive answer by proving that, for any non-constant subharmonic function u, and $A > 1$, the set $\{z : u(z) + AB(z)\}$ where $B(z) = \max_{|\zeta|=|z|} u(\zeta)$ has no unbounded components. Furthermore, if the set $\{z : u(z) + B(|z|) < 0\}$ has an unbounded component, then: u has infinite lower order; or else, regular growth and mean or minimal type of order ρ, where $0 < \rho < \infty$; or else, u is linear.

Problem 2.36 Suppose that $0 < \rho < \alpha \leq 1$ where ρ is the order of an entire function f. Let E_α be the set of r for which $\log m_0(r, f) > \cos(\pi\alpha)\log M(r, f)$. Besicovitch [118] showed that the upper density of E_α is at least $1 - \rho/\alpha$, and Barry [83] proved the stronger result, that the same is true of the lower logarithmic density of E_α. Examples given by Hayman [500] show that Barry's theorem is sharp; in these examples, the logarithmic density exists, but the upper density is larger. This suggests that Besicovitch's theorem may be sharpened.

Update 2.36 No progress on this problem has been reported to us.

Problem 2.37 Let r_n be a sequence of Pólya peaks, as defined by Edrei [286], of order ρ. Then Edrei [286] showed that there exists a $K = K(\alpha, \rho)$ such that $\log m_0(r, f) > \cos(\pi\alpha) \log M(r, f)$ for some value r in the interval $r_n \le r \le K r_n$ and n sufficiently large. Is $K(\alpha, \rho)$ independent of α for fixed ρ? Can it be taken arbitrarily near 1?

(D. Drasin and A. Weitsman)

Update 2.37 This has been proved by Eremenko, Shea and Sodin [323] for the Pólya peaks of $N(r, 0, f)$ and they have given an example showing that the answer is negative for the Pólya peaks of $\log M(r, f)$ or $T(r, f)$.

Problem 2.38 It was shown by Kjellberg [608] that if $0 < \alpha < 1$ and

$$\log m_0(r, f) < \cos(\phi\alpha) \log M(r, f) + O(1), \qquad \text{as } r \to \infty,$$

then

$$\lim_{r\to\infty} \frac{\log M(r, f)}{r^{\alpha}} = \beta,$$

where $0 < \beta \le \infty$. If $\alpha = 1$, it was shown by Hayman [506] that unless $f(z) = Ae^{Bz}$, the corresponding result holds with $\beta = \infty$. Examples constructed by Hayman show that $m_0(r, f)M(r, f) \to \infty$ as $r \to \infty$ can occur for a function of order $1 + \varepsilon$ for every positive ε. The case of functions of order 1 and maximal type remains open.

Update 2.38 Drasin [257] constructed an entire function of order 1 with the property $m_0(r, f)M(r, f) \to 0$.

Problem 2.39 We can also compare $m_0(r, f)$ with the characteristic $T(r)$. We have

$$\limsup_{r\to\infty} \frac{\log m_0(r, f)}{T(r)} \ge D(\lambda)$$

and ask for the best constant $D(\lambda)$. In view of Petrenko's solution [797] of Problem 1.17, we certainly have $D(\lambda) \ge -\pi\lambda$ with $1 \le \lambda < \infty$. Also, Essén and Shea [327] show that $D(\lambda) \le \frac{\pi\lambda}{1+|\sin(\pi\lambda)|}$ for $1 < \lambda < \frac{3}{2}$, and $D(\lambda) \le \frac{-\pi\lambda}{2}$ for $\frac{3}{2} < \lambda < \infty$. Further, it follows from results of Valiron [964], and Edrei and Fuchs [290, 291], that

$$D(\lambda) = \begin{cases} \pi\lambda\cot(\pi\lambda), & \text{if } 0 \le \lambda < \frac{1}{2}, \\ \pi\lambda\cos(\pi\lambda), & \text{if } \frac{1}{2} \le \lambda < 1. \end{cases}$$

(D. Shea)

Update 2.39 No progress on this problem has been reported to us.

Problem 2.40 Let $f(z)$ be a non-constant entire function, and assume that for some constant c the plane measure of the set $E(c)$ where $|f(z)| > c$ is finite. What is the minimum growth rate of $f(z)$? Hayman conjectures that

$$\int_0^\infty \frac{r\,\mathrm{d}r}{\log\log M(r, f)} < \infty$$

is true and best possible. If $E(c)$ has finite measure, is the same true for $E(c')$ for $c' < c$?

(P. Erdős)

Update 2.40 Erdős had asked about the minimum growth of non-constant entire functions, bounded outside a set of finite area. Cámera [175] has established a conjecture of Hayman according to which

$$\int_0^\infty \frac{r\,\mathrm{d}r}{\log\log M(r, f)} < \infty$$

is true and best possible in the following sense: if $\phi(r)$ increases, and

$$\int_0^\infty \frac{r\,\mathrm{d}r}{\phi(r)} = \infty,$$

then there exists an entire $f(z)$ such that $\log\log M(r, f) < \phi(r)$, and f is bounded outside a set of finite area. Cámera has also established the analogous result for subharmonic functions $u(x)$ in $\mathbb{R}^m$. If $B(r) = \sup_{|x|=r} u(x)$, then

$$\int_0^\infty \left(\frac{r}{\log B(r)}\right)^{m-1} \mathrm{d}r < \infty$$

is true, and this is best possible in the same sense as above.

This was also proved independently by Hansen [473] and by Gol'dberg [407]. Gol'dberg also answered the second part of this problem by giving an example of a function f for which $A(c)$ is finite for some c, but not for all c, where $A(c)$ is the area of $E(c)$.

Problem 2.41 Suppose that $f(z)$ has finite order and that Γ is a locally rectifiable path on which $f(z) \to \infty$. Let $\ell(r)$ be the length of Γ in $|z| < r$. Find such a path for which $\ell(r)$ grows as slowly as possible, and estimate $\ell(r)$ in terms of $M(r, f)$. If $f(z)$ has zero order, or more generally, finite order, can a path be found for which $\ell(r) = O(r)$ as $r \to \infty$? If $\log M(r, f) = O\big((\log^2 r)\big)$ as $r \to \infty$, but under no weaker growth condition, it is shown by Hayman [492] and Piranian [802] that we may choose a ray through the origin for Γ.

If $f(z)$ has a finite asymptotic value a, the corresponding question may be asked for paths on which $f(z) \to a$.

(P. Erdős)

Update 2.41 This is a refined form of Problem 2.7. It was completely solved by Gol'dberg and Eremenko [410] who showed that for every function $\phi(r)$ tending to infinity there exists an entire function f such that $\ell(r) \neq O(r)$ for every asymptotic curve. Moreover, for every $\rho > 1/2$ there exists an entire function of order ρ with finite asymptotic value a such that $\ell(r) \neq O(r)$ for every asymptotic curve on which $f(z) \to a$.

Problem 2.42 Let $f(z)$ be an entire function (of sufficiently high order) with l, $l \geq 2$ different asymptotic values a_k, $k = 1, \ldots, l$. Suppose that γ_k is a path such that $f(z) \to a_k$ as $z \to \infty$, $z \in \gamma_k$. Let $n_1(r, a_k)$ be the number of zeros of $f(z) - a_k$ on γ_k, and in $|z| \leq r$. Can we find a function $f(z)$ such that

$$\frac{n_1(r, a_k)}{n(r, a_k)} \to b_k > 0, \qquad \text{as } r \to \infty,$$

for $k = 1, 2, \ldots, n$? Can we take $b_k = 1$?

(J. Winkler)

Update 2.42 Winkler has supplied the following additional information: the problem deals with the question of value distribution in the tongues which are excluded in Ahlfors' island theorems (see, for example, Hayman [493, Chap. 5]). The problem is connected with two previous papers of Winkler [992, 993]. A complete answer was obtained by Barsegyan [84], who has shown that $\sum b_k \leq 1$ for entire functions, and $\sum b_k \leq 2$ for meromorphic functions.

Problem 2.43 Let $f(z)$ be a transcendental entire function which permutes the integers, that is, gives an injective mapping of the integers onto themselves. Is it true that $f(z)$ is at least of order 1, type π? We can also ask the corresponding question for a function permuting the positive integers with the same conjectured answer. Note that $f(z) = z + \sin z$ satisfies both conditions and is of order 1, type π.

If $f(z)$ assumes integer values on the positive integers, then Hardy and Pólya proved (see [988, Theorem 11, p. 55]) that $f(z)$ is at least of order 1, type 2; and if $f(z)$ assumes integer values on all the integers, then Buck [171] has shown that $f(z)$ is at least of order 1, type $\log(\frac{3+\sqrt{5}}{2}) = 0.962\ldots$.

(L.A. Rubel)

Update 2.43 There was a mistake in the original statement of this problem, where log was omitted in the final sentence. Linden points out a contradiction with the original statement: that the function $f(z) = z + 4\cos(\pi(z-2)/3)$ gives (i) an injective mapping of the integers onto themselves, and (ii) an injective mapping of the positive integers onto themselves of type $\pi/3$. There is no contradiction with the corrected statement. Rubel adds that it would be interesting to find an entire function of reasonably small type that 'really' permutes the positive integers, that is, really scrambles them badly, as all the known examples are pretty tame.

Problem 2.44 For $f(z)$ entire of order ρ, and non-constant, let $\nu(r)$ be the number of points on $|z| = r$ where $|f(z)| = 1$. Is it true that

$$\limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \rho\,?$$

If one replaces $\nu(r)$ by the number of points on $|z| = r$ where $f(z)$ is real, then Hellerstein and Korevaar [531] have shown that the corresponding upper limit is always equal to ρ.

(J. Korevaar)

Update 2.44 The example $f(z) = \mathrm{e}^z$ shows this to be false ($\nu(r) = 2$). Miles and Townsend [729] consider the problem for meromorphic f of finding upper bounds for $\phi(r)$, the number of points on $|z| = r$ at which $f(z)$ is real. If there exists a meromorphic function F such that $F = u + iv$ and $f = (F - 1)(F + 1)$, then $u = 0 \Leftrightarrow |f| = 1$. Thus for meromorphic f the problems of obtaining upper bounds for $\phi(r)$ and for $\nu(r)$ are equivalent. Miles and Townsend [729] show that if f is meromorphic of order ρ and

$$\Phi(r) = \int_1^r \frac{\phi(t)}{t}\,\mathrm{d}t,$$

then

$$\limsup_{r\to\infty} \frac{\log \Phi(r)}{\log r} \le \rho.$$

They further show that

$$\limsup_{r\to\infty} \frac{\log \phi(r)}{\log r} \le \rho$$

outside an exceptional set, which may exist if f is meromorphic but not if f is entire; see Hellerstein and Korevaar [531].

Problem 2.45 Let $J_0(z)$ be the Bessel function of order zero. Is it true that the equation $J_0(z) = 1$ has at most one solution on each ray from the origin? An affirmative answer would show that the exceptional set in a theorem of Delsarte and Lions [242] is, in fact, void. Asymptotic estimates show that there can be at most a finite number of solutions on any ray, and yield even stronger information.

(L. Zalcman)

Update 2.45 No progress on this problem has been reported to us.

Problem 2.46 Let $\{f_\alpha(z)\}$ be a family of entire functions, and assume that for every z_0, there are only denumerably many distinct values of $f_\alpha(z_0)$. Then if $\mathfrak{c} = 2^{\aleph_0} > \aleph_1$, the family $\{f_\alpha(z)\}$ is itself denumerable. The above result was proved by Erdős [301]. If $\mathfrak{c} = \aleph_1$, he constructed a counterexample.

Suppose now that m is an infinite cardinal, $\aleph_0 < m < \mathfrak{c}$. Assume that for every z_0, there are at most m distinct values $f_\alpha(z_0)$. Does it then follow that the family $\{f_\alpha(z)\}$ has at most power m? If $m^+ < \mathfrak{c}$, where m^+ is the successor of m, it is easy to see that the answer is 'yes'. However, if $\mathfrak{c} = m^+$, the counterexample fails. It is possible the question is undecidable.

(P. Erdős)

Update 2.46 No progress on this problem has been reported to us.

Problem 2.47 Let E_ρ be the linear space of entire functions f such that $|f(z)| \le B\exp(A|z|^\rho)$ for some positive A and B. Let K_ρ be the family of functions $k(z)$ positive and continuous on $\mathbb{C}$, with $\exp(A|z|^\rho) = o(k(z))$ as $|z| \to \infty$, for all positive A. Let S be a subset of $\mathbb{C}$, and $\|\cdot\|_{k,S}$, $\|\cdot\|_k$ the semi-norms defined for f in E_ρ and k in K_ρ by

$$\|f\|_{k,S} = \sup_S \left\{\frac{|f(z)|}{k(z)}\right\},$$

$$\|f\|_k = \sup_{\mathbb{C}} \left\{\frac{|f(z)|}{k(z)}\right\}.$$

We say that S is a *sufficient set* for E_ρ if the topologies defined by the semi-norms $\{\|\cdot\|_k, k \in K_\rho\}$, $\{\|\cdot\|_{k,S}, k \in K_\rho\}$ coincide; see Ehrenpreis [294].

(a) Suppose that, whenever $f \in E_\rho$, f is bounded on S if and only if f is bounded on $\mathbb{C}$. Does it follow that S is a sufficient set for E_ρ?
(b) Suppose that S is a sufficient set for E_ρ. Then, if $f \in E_\rho$ and f is bounded on S, does it follow that f is bounded on $\mathbb{C}$, or (maybe) of small growth?

Some work of Oliver [779] suggests the latter, at least, is likely.

(D.A. Brannan)

Update 2.47 No progress on this problem has been reported to us.

Problem 2.48 If A and B are countable dense subsets of $\mathbb{R}$ and $\mathbb{C}$ respectively, does there necessarily exist a transcendental entire function that maps A onto B, and $\mathbb{R} \setminus A$ into $\mathbb{C} \setminus B$?

Suppose that E and F are countable dense subsets of $\mathbb{R}$, and that there exists an entire function f, monotonic on $\mathbb{R}$, that maps E onto F, and $\mathbb{R} \setminus E$ onto $\mathbb{R} \setminus F$. Find interesting conditions which imply that f is trivial. For example, if E and F are the real rationals, under what conditions is f necessarily linear with rational coefficients? One could also investigate this question in the case of real-valued harmonic or subharmonic functions in $\mathbb{R}^n$, $n \ge 2$.

If A and B are two countable dense subsets of $\mathbb{R}$, Barth and Schneider [88] have shown that there exists a transcendental entire function, monotonic on $\mathbb{R}$, that maps A onto B and $\mathbb{R} \setminus A$ onto $\mathbb{R} \setminus B$; also, if A and B are countable dense subsets of $\mathbb{C}$; see Barth and Schneider [89].

(K.F. Barth)

Update 2.48 No progress on this problem has been reported to us.

Problem 2.49 If $f(z)$ is a transcendental entire function, we define

$$M = \{z : |f(z)| = M(|z|, f)\}.$$

Tyler [961] has shown that M can have isolated points, and that, given any N, we can have $\nu(r) > N$ for infinitely many r, where $\nu(r)$ is the number of points in $M \cap \{|z| = r\}$. Herzog and Piranian [545] have shown that $\limsup_{r\to\infty} \nu(r)$ can be infinite; is it true that $\liminf_{r\to\infty} \nu(r) < \infty$ for all entire f?

(J.G. Clunie)

Update 2.49 This problem is related to Problem 2.16. No progress on this problem has been reported to us.

Problem 2.50 Characterise those entire functions having at least one continuous maximum modulus path going from 0 to ∞.

(W. Al-Katifi)

Update 2.50 No progress on this problem has been reported to us.

Problem 2.51 Suppose that an entire function f has exactly one curve Γ of maximum modulus (that is, Γ is connected, joins 0 to ∞, and f has no other maximum modulus points). What can be said about the minimum rate of growth of $M(r, f)$, if one is given information about the geometry of the curve Γ, for example, that Γ is a given infinitely-spiralling spiral? If Γ is a radial line, clearly nothing much can be said. (In a sense, this is the opposite of a Phragmén–Lindelöf principle.)

(D.A. Brannan)

Update 2.51 No progress on this problem has been reported to us.

Problem 2.52 What is the best function $g(\sigma)$, $\sigma \geq 0$, such that, for a non-constant entire function $f(z)$ with maximum and minimum modulus $M(r, f)$ and $m_0(r, f)$ respectively, the assumption

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{(\log r)^2} \leq \sigma$$

implies that

$$\limsup_{r\to\infty} \frac{m_0(r, f)}{M(r, f)} \geq g(\sigma)\ ?$$

(P.D. Barry)

Update 2.52 This is solved completely by Gol'dberg [408], who proved the following stronger result: let f be a meromorphic function of zero order, satisfying

$$\liminf_{r\to\infty} \frac{N(r, 0, f) + N(r, \infty, f)}{(\log r)^2} \leq \sigma < \infty,$$

then

$$\limsup_{r\to\infty} \frac{\min_\theta |f(re^{i\theta})|}{\max_\theta |f(re^{i\theta})|} \geq C(\sigma),$$

where

$$C(\sigma) = \left(\prod_{n=1}^{\infty} \frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^2, \qquad \text{and} \qquad q = \exp(-1/(4\sigma)),$$

and this estimate is best possible.

A different proof of Barry's original conjecture was given by Fenton [338].

Problem 2.53 For entire or, more generally, meromorphic functions f and g, let '$f \leq g$' mean that, for any sequence $\{z_n\}_1^\infty$ for which $|f(z_n)| \to \infty$, then $|g(z_n)| \to \infty$. For entire functions, it can be proved that, if $f' \leq f$, then f is of exponential type; what can be said if $f'' \leq f$? Does $f'' \leq f$ imply that f is normal (that is, that $|f'(z)|(1 + |f(z)|^2)^{-1}$ is bounded)? (An analogue in the case of the unit disc $\mathbb{D}$ has been proved by Pommerenke [819]). In the above ordering, does there exist $f \wedge g$ and $f \vee g$ for any two entire functions f and g? That is, given f and g, does there exist an h such that $h \leq f$ and $h \leq g$, and so that if $k \leq f$ and $k \leq g$, then $k \leq h$? Similarly, for $f \vee g$. Finally, if f is meromorphic and $f' \leq f$, does this imply a growth restriction on f, for example, is the order of f at most two?

Note: Any constant c satisfies $c \leq f$ for all entire f, and any non-constant polynomial p satisfies $f \leq p$ for any entire f. Observe also that e^{cz} are all equivalent if $c > 0$.

(L.A. Rubel and J.M. Anderson)

Update 2.53 No progress on this problem has been reported to us.

Problem 2.54 Let E be a closed set in $\mathbb{C}$, with the following properties: (1) there exists a transcendental entire function $f(z)$ that is bounded on E; and (2), there exists a transcendental entire function $g(z)$ that is bounded away from 0 on the complement of E. For each such set E, must there exist one transcendental entire function that is simultaneously bounded on E and bounded away from 0 on the complement of E?

(L.A. Rubel)

Update 2.54 No progress on this problem has been reported to us.

Problem 2.55 Let $f_i(z)$, $i = 1, 2, 3$, be non-constant entire functions of one complex variable, and

$$V = \{z : z = (z_1, z_2, z_3) \in \mathbb{C}^3,\ f_1(z_1) + f_2(z_2) + f_3(z_3) = 0\}.$$

If F is an entire function of $z = (z_1, z_2, z_3)$ that is bounded on V, is F necessarily constant on V?

(J.M. Anderson and L.A. Rubel)

Update 2.55 Rubel points out that there has been progress in one very special case by Demailly [243].

Problem 2.56 Prove or disprove the conjecture that an entire function f of n complex variables is an L-atom (where this is defined in a way analogous to the definition for $n = 1$; see Problem 5.55 with $\mathbb{D}$ replaced by $\mathbb{C}$) if and only if there are entire functions $f_2, f_3, \ldots, f_n$ of the n variables such that $(f, f_2, f_3, \ldots, f_n)$ is an analytic automorphism of $\mathbb{C}^n$, that is, an injective biholomorphic map of $\mathbb{C}^n$ onto $\mathbb{C}^n$. Rubel can prove the result in the case $n = 1$.

(L.A. Rubel)

Update 2.56 No progress on this problem has been reported to us.

Problem 2.57 If f is an entire function such that $\log M(r, f) = O\big((\log r)^2\big)$ as $r \to \infty$, then Hayman [492] has shown that $\log|f(re^{i\theta})| \sim \log M(r, f)$, as $r \to \infty$, for $re^{i\theta}$ outside an exceptional set E of circles subtending angles at the origin, whose sum is finite. In particular,

$$\log|f(re^{i\theta})| \sim \log M(r, f), \text{ as } r \to \infty, \text{ for almost every } \theta.$$

Using this result, Anderson and Clunie [37] showed that if f is meromorphic and $T(r, f) = O\big((\log r)^2\big)$ as $r \to \infty$, then a deficient value (there is at most one) must be asymptotic and, moreover, if $\delta(a, f) > 0$, then

$$f(re^{i\theta}) \to a \text{ as } r \to \infty \text{ for almost every } \theta.$$

Now consider a new class of entire functions I_α, $\alpha > 1$, defined by

(a) $\log M(r, f) = O\big((\log r)^{1+\alpha}\big)$, as $r \to \infty$,
(b) $(\log r)^\alpha = o(\log M(r, f))$ as $r \to \infty$.

(Hayman's functions are I_1.) Do the results of Hayman, and the corresponding results of Anderson and Clunie, still hold? In other words, do these results depend on smallness of growth, or only on smoothness of growth?

(J.M. Anderson)

Update 2.57 This problem has been solved by Anderson [34] as follows: let $f(z)$ be meromorphic in $\mathbb{C}$ and suppose that, for some a, $\delta(a, f) > 0$. Then under the hypothesis

$$T(r, f) \sim T(2r, f) \qquad \text{as } r \to \infty, \tag{2.9}$$

$f(z)$ has asymptotic value a. More precisely, there is an asymptotic path Γ such that

$$\frac{1}{T(r, f)} \log\left|\frac{1}{f(z) - a}\right| \geq \delta, \qquad \text{as } z \to \infty \text{ along } \Gamma, \qquad |z| = r,$$

and the length $\ell(r)$ of $\Gamma \cap \{|z| < r\}$ satisfies $\ell(r) = r\{1 + o(1)\}$ as $r \to \infty$. By (2.9) $f(z)$ is forced to be of order zero, but there are examples (see Gol'dberg and Eremenko [410]) of entire functions $g(z)$ satisfying

$$T(r, g) = O(\psi(r))(\log r)^2 \qquad \text{as } r \to \infty,$$

where $\psi(r)$ is any assigned real-valued function tending to ∞ with r such that no asymptotic path satisfies $\ell(r) = O(r)$.

Problem 2.58 Suppose that f is entire with a non-zero Picard exceptional value α. Then f has α as an asymptotic value. It can be shown that $f \to \alpha$ along a level curve $|f| = |\alpha|$, if f is of finite order. This follows readily from the fact that arg f is monotone along such a curve, together with an application of the Denjoy–Carleman–Ahlfors theorem. We call such a level curve a *natural asymptotic path* for α. Does there exist an entire function of infinite order, with a non-zero Picard exceptional value α, having no natural asymptotic paths?

(S. Hellerstein)

Update 2.58 An example of such a function has been given by Eremenko [310] who constructed a suitable Riemann surface.

Problem 2.59 (*A width conjecture*) Given a power series $\sum_{k=0}^{\infty} a_k z^k$, suppose that there is a non-negative ρ such that all of the partial sums $S_n(z) = \sum_{k=0}^{n} a_k z^k$, $n = 1, 2, 3, \ldots$, are non-zero in the region

$$S\rho = \{z = x + iy : |y| < Kx^{1-(\rho/2)}, x > 0\}.$$

We conjecture that $f(z)$ must be entire of order at most ρ. When $\rho = 0$, S_ρ is a sector with vertex at $z = 0$, and the conjecture is a consequence of results of Carlson [194] which were later generalised by Rosenbloom [846], Ganelius [370], and Korevaar and McCoy [625]. Remark: If $f(z) = e^z$, which is of order 1, Saff and Varga [873] have shown that the partial sums $\sum_{k=0}^{n} z^k/k!$ are in fact zero-free in the parabolic region

$$\{z = z + iy : y^2 \leq 4(x+1), x > -1\}.$$

(E.B. Saff and R.S. Varga)

Update 2.59 No progress on this problem has been reported to us.

Problem 2.60 Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a non-vanishing entire function, and let $S_n(z) = \sum_{k=0}^{n} a_k z^k$. Given a positive ε, must there exist a z_0 and an n such that $S_n(z_0) = 0$ and $|f(z_0)| < \varepsilon$? (The Hurwitz theorem shows that the result is true if f has a zero.)

(D.J. Newman and A. Abian)

Update 2.60 No progress on this problem has been reported to us.

Problem 2.61 Let Γ be a rectifiable curve. Suppose f is a continuous function on the plane satisfying

$$\int_{\sigma(\Gamma)} f(z)\,dz = 0, \qquad \text{for all rigid motions } \sigma.$$

Does this imply that f is an entire function? The answer is 'yes' for some choices of Γ, and 'no' for others. For example, the answer is 'no' for the circle ($f(x+iy) = \sin(ax)$ is a counterexample for a suitable choice of a); 'yes' for an ellipse; 'yes' for any polygonal Jordan curve; and 'yes' for the boundary of any convex set with at least one corner. Prove that the circle is the only closed rectifiable Jordan curve (or the only curve among the class of curves which are boundaries of bounded convex sets) for which the answer is 'no'. This problem is related to the 'Pompeiu Problem' discussed by Brown, Schreiber and Taylor [167].

(L. Brown)

Update 2.61 Significant progress has been made on this by Ebenfelt [282, 283]

Problem 2.62 Let f denote a rational or entire function of a complex variable, and f^n, $n = 1, 2, \ldots$, the n-th iterate of f, so that $f^1 = f$, $f^{n+1} = f \circ f^n = f^n \circ f$. Provided that f is not rational of degree 0 or 1, the set C of those points where $\{f^n\}$ forms a normal family is a proper open subset of the plane, and is invariant under the map $z \mapsto f(z)$. A component G of C is a wandering domain of f if $f^k(G) \cap f^n(G) = \emptyset$ for all $\{k, n \mid k \geq 1, n \geq 1, k \neq n\}$.

Jakobson has asked whether it is possible for a rational function f to have a wandering domain. Baker [69] gave a transcendental entire function which does have such domains.

(I.N. Baker)

Update 2.62 Sullivan [930] has proved that rational functions have no wandering domains, so the answer to Jakobson's question is 'no'.

Problem 2.63 Let f be a rational function and C be as in Problem 2.62. We say that g is a *limit function* for f if g is defined in some component G of C and is the limit of some subsequence of (f^n) in G. In the simplest examples, the number of limit functions is finite, which implies that each has a constant α, say, such that $f^k(\alpha) = \alpha$ for some positive integer k. If, in addition, $|(f^k)'(\alpha)| < 1$ for each of the limit functions, we say that the function f belongs to the class N.

(a) Does there exist a rational function f which has an infinity of constant limit functions?
(b) Is the property of belonging to N 'generic' in some sense for rational functions?

An account of the older established results can be found in Fatou [329–331] and a sketch from a more modern point of view is given by Guckenheimer [447].

(I.N. Baker)

Update 2.63 Sullivan [930] proved that the answer to the first question is 'no'.

On the other hand, Eremenko and Lyubich [320] have constructed an example of an entire function, iterates of which have an infinite set of constant limit functions in a component of normality, a Fatou component.

Problem 2.64 Let $f(z)$ be a real entire function, that is, $f(z)$ is real for real z. It has been shown by Hellerstein and Williamson [535] that if f, f', f'' have only real zeros, then f is in the Laguerre–Pólya class, that is, $f(z) = \mathrm{e}^{-az^2} g(z)$, $a \geq 0$, g a real entire function of genus 0 or 1, with only real zeros, thus affirming a conjecture of Pólya. Wiman made the stronger conjecture that the above is true with no assumption on the zeros of f'. Even the simplest case, $f(z) = \exp(q)$, where q is a real polynomial, has not been settled; see Problem 4.28. Wiman's conjecture has been proved by Levin and Ostrovskii [660] for f of infinite order, growing sufficiently fast.

(S. Hellerstein)

Update 2.64 Wiman's conjecture has been proved by Bergweiler, Eremenko and Langley [114]. For functions of finite order, the conjecture was proved previously by Sheil-Small [897]. Both works use the methods developed by Levin and Ostrovskii [660].

Problem 2.65 Since the knowledge of the zeros of an entire function f leaves an unknown factor, e^h say, in the Hadamard product for f, one can ask if f is determined by the zeros of f, and of its first few derivatives. (Further details on Hadamard products can be found in Sheil-Small [895, p. 515] and Titchmarsh [942, Sect. 4.6].) Does there exist an integer $k \geq 2$ such that, if f and g are entire, and $f^{(n)}/g^{(n)}$ is entire and non-vanishing for $0 \leq n \leq k$, then f/g is constant, unless

$$f(z) = \mathrm{e}^{az+b}, g(z) = \mathrm{e}^{cz+d} \text{ or } f(z) = A(\mathrm{e}^{az} - b), g(z) = B(\mathrm{e}^{-az} - b^{-1})\,?$$

The proposer has shown (unpublished) that $k = 2$ will do in certain cases: for example, when f and g have finite order. The example

$$f(z) = (\mathrm{e}^{2z} - 1)\exp(-i\mathrm{e}^z), \qquad g(z) = (1 - \mathrm{e}^{-2z})\exp(i\mathrm{e}^{-z})$$

shows that one sometimes needs $k = 3$. One can ask a similar question for meromorphic functions, with the additional possibility that

$$f(z) = A(\mathrm{e}^{h(z)} - 1)^{-1}, \qquad g(z) = B(1 - \mathrm{e}^{-h(z)}),$$

for any non-constant entire function h.

(A. Hinkkanen)

Update 2.65 Köhler [613] proved that the answer is 'yes' for meromorphic functions and $k = 6$. Namely, if f and g are meromorphic function such that $f^{(n)}/g^{(n)}$ are entire and without zeros for $0 \leq n \leq 6$, then f and g satisfy one of the four relations suggested by Hinkkanen. If one makes additional assumptions about the growth of f and g, one needs fewer derivatives to achieve the same conclusion [613, 944]. See also Langley [651] for related results.

Yang [1007] shows that the answer to Hinkkanen's problem is positive for meromorphic functions of finite order if $n = 1$, and an additional condition holds. Yang

[1007] also generalises this result to meromorphic functions of hyper-order less than one. Examples show that the order restriction is sharp. See also Tohge [944].

Problem 2.66 Given a countable number of entire functions, one can find an entire function growing faster than any of these. Without making any assumption about the Continuum Hypothesis, can one associate with every countable ordinal number α an entire function f_α such that

(a) if $\alpha < \beta$, then $M(r, f_\alpha)/M(r, f_\beta) \to 0$ as $r \to \infty$,

and

(b) if f is an entire function, then there exists a γ such that $M(r, f)/M(r, f_\gamma) \to 0$ as $r \to \infty$?

See also Problem 7.62.

(A. Hinkkanen)

Update 2.66 In terms of the cardinality A of the smallest set of entire functions which exhaust all orders of growth, Hechler [524] has proved that A can be anything between $\aleph_1$ and $\mathfrak{c}$, having the same general properties as $\mathfrak{c}$, that is, $A \neq \sum_{i \in I} M_i$, where $\#I < A$ and $\#M_i < A$, for $i \in I$. The answer is independent of the axioms of set theory.

Problem 2.67 Let f be an entire function, and let D be a component of the set in $\mathbb{C}$ where the family of iterates $\{f_n\}$ is normal. Can this family have an infinite bounded set of constant limit functions? Eremenko and Lyubich [320] have shown that the set of constant limit functions may be infinite.

(A. Eremenko)

Update 2.67 This is related to Problem 2.63. See Update 2.87 for details on the equivalence of Problem 2.77 and Problem 2.87, and the fact that Problem 2.67 is a special case of these problems. There has only been partial progress on these problems.

Problem 2.68 Let f be an entire function satisfying the condition

$$\log M(r, f) \leq (1 + o(1))r^\rho, \qquad \text{as } r \to \infty.$$

Suppose that there exists a curve Γ tending to ∞ such that on Γ

$$\log |f(z)| \leq (\alpha + o(1))r^\rho, \qquad \text{as } r = |z| \to \infty,$$

for some α in $[-1, 1)$; and denote by $E(r, \varepsilon)$ the angular measure of the set

$$\{re^{i\theta} : \log |f(re^{i\theta})| \leq (1 - \varepsilon)r^\rho\}.$$

Eremenko conjectures that

$$\limsup_{\varepsilon\to 0,\, r\to\infty} E(r,\varepsilon) \geq \frac{2}{\rho} \arccos \alpha.$$

Jaenisch [570] has proved some related results.

(A. Eremenko)

Update 2.68 No progress on this problem has been reported to us.

Problem 2.69 Hayman [495] has shown that

$$\liminf_{r\to\infty} \frac{T(r,f)}{T(r,f')} \leq 1 \tag{2.10}$$

for transcendental entire functions f of lower order zero. Toppila [952] has shown that there exists an entire function of order one which does not satisfy (2.10). Does there exist a constant $d > 0$ such that (2.10) holds for all transcendental entire functions f of order less than d?

(S. Toppila)

Update 2.69 This problem is a continuation of Problem 1.21. See Update 1.21.

Problem 2.70 (*The Bank–Laine conjecture*) Let H be an entire function, let f_1, f_2 be linearly independent solutions of the differential equation $w'' + Hw = 0$, and let $E = f_1 f_2$. Clearly f_1, f_2 and E are entire. Also, it is well known that if H is a polynomial of degree n, then the orders of f_1 and f_2 are $\rho(f_1) = \rho(f_2) = \frac{1}{2}(n+2)$. Furthermore, the exponent of convergence of the zeros of E is $\lambda(E) = \frac{1}{2}(n+2)$, provided that $n > 1$. If H is transcendental then

$$\rho(f_1) = \rho(f_2) = +\infty$$

and one might hope that, by analogy with the previous remarks, $\lambda(E) = +\infty$.

However, examples of Bank and Laine [73] showed that this is not necessarily the case if $\rho(H)$ is a positive integer or $+\infty$. They also asked whether $\lambda(E) = +\infty$ if $\rho(H)$ is non-integral and finite and they showed that this is indeed the case if $\rho(H) < \frac{1}{2}$, a result improved to $\rho(H) \leq \frac{1}{2}$ by Rossi [848]. What happens in general?

(S. Hellerstein and J. Rossi)

Update 2.70 This question became known as the Bank–Laine conjecture. A negative answer was given by Bergweiler and Eremenko [111, 112]: for every $\rho > \frac{1}{2}$ there exists an entire H for which $\rho(H) = \rho$, but $\lambda(E) < +\infty$.

Problem 2.71 It is shown by Hellerstein and Rossi [532], and Gundersen [448] that if f_1 and f_2 are two linearly independent solutions to the differential equation $w'' + Hw = 0$, where H is a polynomial and f_1 and f_2 have only finitely many non-real zeros, then H is a non-negative constant. It is also shown that, if $H(z) = az + b$, for $a, b \in \mathbb{R}$, then the differential equation admits a solution with only real zeros (and infinitely many of them). Furthermore, as pointed out by Gundersen [448], Titchmarsh [941, pp. 172–173] showed that, if $H(z) = z^4 - \beta$ for special choices of β, then the differential equation also admits solutions with only real zeros (and infinitely many of them).

Characterise all non-constant polynomials H such that the differential equation admits a solution with only real zeros (and infinitely many of them).

(S. Hellerstein and J. Rossi)

Update 2.71 Gundersen [451] showed that for given $a > 0$ and $b \geq 0$, there exists an infinite sequence of real numbers when $\lambda_k \to +\infty$ such that the differential equation

$$f'' + (az^4 + bz^2 - \lambda_k)f = 0$$

has solutions with infinitely many zeros, and all these zeros, except finitely many of them, are real. Rossi and Wang [850] proved that if an equation $f'' + Pf = 0$, where P is a polynomial, has a solution with infinitely many zeros, all of them real, then the number of real zeros of P must be less than than $1 + \frac{1}{2}\deg P$, counted with multiplicities.

Eremenko and Merenkov [321] proved that for every d there exist polynomials P of degree d such that some solution of the equation $f'' + Pf = 0$ has only real zeros. The zero set of such f can be infinite if and only if $d \not\equiv 2 \pmod 4$. They show that for every non-negative integer d, there exist differential equations $w'' + Pw = 0$, where P is a polynomial of degree d, such that some non-trivial solution w has only real roots. For polynomials of degree 3, Eremenko and Gabrielov [316] give a curve Γ_0, which parametrises all such equations; and a curve Γ_n which parametrises all such equations having solutions with exactly $2n$ non-real zeros.

Problem 2.72 Let $\{f_1, \ldots, f_n\}$ be a fundamental system for the differential equation

$$L_n(w) \equiv w^{(n)} + a_{n-1}(z)w^{(n-1)} + \ldots + a_0(z) = 0, \qquad (2.11)$$

where a_o, ..., a_{n-1} are polynomials. Frank [351] has proved there exists such a fundamental set such that each of $f_1, \ldots, f_n$ has finitely many zeros if and only if (2.11) can be transformed into a differential equation with constant coefficients by a transformation of the form $w(z) = \exp(q(z)u(z))$, where q is a suitable polynomial. Does the same result hold if each function $f_1, \ldots, f_n$ is assumed to have only finitely many non-real zeros? In view of Hellerstein and Rossi [532] and Gundersen [448], we may assume that $n \geq 3$.

(S. Hellerstein and J. Rossi)

Update 2.72 Brüggemann [168] and Steinmetz [923] have independently given a positive answer. In fact, each of them proved a stronger result than conjectured.

Problem 2.73 Let $F(z, a, b)$ be an entire function of three complex variables, and suppose that F is not of the form

$$F(z, a, b) = G(z, H(a, b)) \qquad (2.12)$$

for any entire functions G and H of two complex variables. Can

$$\{F(z, a, b) : a, b \in \mathbb{C}\}$$

constitute a normal family of entire functions of z? (Put rather loosely, does there exist a two-parameter normal family of entire functions?) Notice that if F does have the form (2.12) then $F_b F_{a,z} = F_a F_{b,z}$ where subscripts denote partial differentiation. The purpose of ruling out the form (2.12) is to ensure that there are two honest parameters of the family. Otherwise we could have, say,

$$F(z, a, b) = z + 5a^2 + \sin b,$$

which is really a one-parameter family in disguise.

(L.A. Rubel)

Update 2.73 No progress on this problem has been reported to us.

The next two problems concern the class of real entire functions of finite order with only real zeros. We may partition this class as follows: for each non-negative integer p, denote by V_{2p} the class of entire functions f of the form

$$f(z) = g(z)\exp(-az^{2p+2})$$

where $a \geq 0$, and $g(z)$ is a constant multiple of a real entire function of genus at most $2p+1$ with only real zeros. Then write $U_0 = V_0$, and $U_{2p} = V_{2p} - V_{2p-2}$ for $p \in \mathbb{N}$.

Problem 2.74 Suppose that $f(z) = 1 + a_1 z + a_2 z^2 + \ldots \in U_{2p}$. If $p = 0$ (so that $f \in U_0$) and if f is not a polynomial, it is well known that f cannot have two consecutive Taylor coefficients equal to zero.

If $p > 0$, can an analogous assertion be made? Is it true, for example, that the Taylor series of f in U_{2p} cannot have $2p+2$ consecutive coefficients equal to zero?

(J. Williamson)

Update 2.74 No progress on this problem has been reported to us.

Problem 2.75 Suppose that f is entire of proximate order $\rho(r)$ and that f has a representation as a Dirichlet series

$$f(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \qquad 0 \leq \lambda_1 < \lambda_2 < \ldots \to \infty, \qquad a_n > 0.$$

Can one give a complete characterisation of the indicator

$$h(\theta, f) = \limsup_{r\to\infty} r^{-\rho(r)} \log|f(re^{i\theta})|$$

of such functions? If f has a representation

$$f(z) = \int_0^\infty e^{iz}\, dF(t),$$

where F is positive and increasing, Gol'dberg and Ostrovskii [413] gave such a characterisation.

(A.A. Gol'dberg and I.V. Ostrovskii)

Update 2.75 Gol'dberg and Ostrovskii [414] solved the problem under the following additional assumption on the sequence of the exponents: there exists an entire function $L(\lambda)$ of exponential type, such that $L(\lambda_n) = 0$ for $n = 1, 2, \ldots,$ and

$$\lim_{n\to\infty} \lambda_n^{-1} \log(1/|L'(\lambda_n)|) < \infty.$$

If these conditions are satisfied, the sequence (λ_n) is said to have *finite index of concentration*. If the sequence of exponents of a Dirichlet series f has finite index of concentration, the only possible indicators are $h(f, \theta) = a(\cos^+ \theta)^\rho$, where $a > 0$. Gol'dberg and Ostrovskii [414] also obtained other results, with weaker conditions on the sequence (λ_n), and studied the lower indicators.

Problem 2.76 Let Ω be a component of the normal set of an entire function (under iteration). Is $\dim(\partial\Omega) > 1$? Or is $\partial\Omega$ a circle/line?

(D.H. Hamilton)

Update 2.76 Bishop [127] has constructed transcendental entire functions with Julia sets of dimension 1. In particular, the boundaries of the components of the Fatou set have dimension 1 but they are not circles or lines.

Problem 2.77 Let Ω be a component of the normal set of an entire function f (under iteration). Do there exist such an f and such an Ω with the following properties:

(a) $f^n(\Omega)$ is uniformly bounded, for $n = 0, 1, 2, \ldots$;
(b) $f^n(\Omega) \cap f^m(\Omega) = \emptyset$ for $n \neq m$.

(I.N. Baker, M.R. Herman and I. Kra; communicated by D.H. Hamilton)

Update 2.77 We note that 'uniformly' has been added to the original statement of (a), as otherwise there are examples of such f.

See Update 2.87 for details on the equivalence of Problems 2.77 and 2.87, and the fact that Problem 2.67 is a special case of these problems. There has only been partial progress on these problems.

In the following problems we denote by R_d the class of rational functions of degree d on the Riemann sphere $\hat{\mathbb{C}}$, and we suppose that $d \geq 2$.

Problem 2.78 (*Fatou's conjecture*) Show that the subset U of functions g in R_d, such that all the critical points of g are in the basins of attraction of periodic sinks, is dense in R_d.

The property '$g \in U$' is also sometimes called Axiom A. See also Fatou [332].

(P. Fatou; communicated by M.R. Herman)

Update 2.78 An analogue of the Fatou conjecture for the real quadratic family

$$\{z \mapsto z^2 + c : c \in \mathbb{R}\}$$

(instead of the family R_d of all rational functions of degree d at least 2) has been established by Graczyk and Świątek [440], and by Lyubich [689].

The next three problems were suggested in order to give a positive answer to the Fatou conjecture. See, for example, Mañé [691].

Problem 2.79

(a) Show that if a function g in R_d has the property that its Julia set $J(g) \neq \hat{\mathbb{C}}$, then g does not leave invariant a non-trivial Beltrami form on $J(g)$. Here a *Beltrami form* μ means that $\mu \in L^\infty(\hat{\mathbb{C}})$ and $\|\mu\|_{L^\infty} < 1$; and *trivial* on $J(g)$ means that $\mu(x) = 0$ for almost all x in $J(g)$.
(b) More generally, is the Lebesgue measure of $J(g)$ in $\hat{\mathbb{C}}$ equal to zero? (This is the analogous conjecture to the Ahlfors conjecture for finitely-generated Kleinian groups.)

Negative answers to both (a) and (b) have been proved (by McMullen (no citation given) and Eremenko and Lyubich [320]) for the class of transcendental entire functions g. Douady has conjectured that the answer is negative, and that a counterexample is the function $P_\lambda(z) = \lambda(z + z^2)$ for some λ of modulus one. See also Douady [250], Lyubich [688] and Mañé [691].

(D. Sullivan; communicated by M.R. Herman)

Update 2.79(a) This has been established for the real quadratic family by McMullen [713], and extended to the family $\{z \mapsto z^d + c, c \in \mathbb{R}\}$ by Levin and van Strien [661].

Update 2.79(b) Buff and Chéritat [173] showed that the answer is 'no' in general, and that a counterexample is given by $\lambda(z + z^2)$ for suitable λ, as conjectured by Douady.

Problem 2.80 Let the function g in R_d have the property that its Julia set $J(g) = \hat{\mathbb{C}}$. Is the dimension k of the space of Beltrami forms on $\hat{\mathbb{C}}$, invariant under g, at most one? Also, are certain of the Lattès examples the only rational functions such that $k \neq 0$? For further information, see [691].

(D. Sullivan; slightly modified and communicated by M.R. Herman)

Update 2.80 No progress on this problem has been reported to us.

Problem 2.81 Let the function g in R_d have the property that its Julia set $J(g) = \hat{\mathbb{C}}$. Is g ergodic for Lebesgue measure? In other words, if $B \subset \hat{\mathbb{C}}$ is a Borel-invariant set under g (that is, $g^{-1}(B) = B$), does it follow that either B or $\hat{\mathbb{C}} \setminus B$ has Lebesgue measure zero? For further information, see [691].

(D. Sullivan; communicated by M.R. Herman)

Update 2.81 No progress on this problem has been reported to us.

Problem 2.82 Let L_d denote the class of those functions g in R_d such that every critical point of g is preperiodic but not periodic. Show that if the function g in R_d has the property that $J(g) = \hat{\mathbb{C}}$, then g belongs to the closure of L_d in R_d.

(M.R. Herman)

Update 2.82 No progress on this problem has been reported to us.

Problem 2.83 Let a function f in R_d have the property that

$$f(z) = \lambda_\alpha z + O(z^2) \qquad \text{as } z \to 0,$$

where $\lambda_\alpha = \mathrm{e}^{2\pi i\alpha}$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume also that f is linearisable at 0, and denote by S its Siegel singular disc.

(a) Is α necessarily a Brjuno number? In other words, is it true that

$$\sum_{n=0}^{\infty} (\log q_{n+1})/q_n < +\infty,$$

where $\{p_n/q_n\}_{n=0}^{\infty}$ are the convergents of the continued fraction expansion of α? Also, what is the situation here when f is a non-linear entire function?

(b) Is f necessarily injective on the boundary ∂S of S in $\hat{\mathbb{C}}$?

(c) Is it true that f has no periodic points on ∂S?

Both (b) and (c) are open even under the additional hypothesis that f has no critical point on ∂S. A positive answer to (b), under this additional hypothesis, would imply that when α satisfies a diophantine condition (that is, there exist β, γ, with $\beta \geq 2$ and $\gamma > 0$ such that, for every number p/q in $\mathbb{Q}$, we have $|\alpha - (p/q)| \geq \gamma q^{-\beta}$), then f has a critical point on ∂S.

For further information, see Herman [543].

(J.-C. Yoccoz and M.R. Herman)

Update 2.83(a) Yoccoz [1011] has proved that the answer is 'yes' if f is a polynomial of degree 2.

Problem 2.84 Does there exist a number λ of modulus one, that is not a root of unity, such that the positive orbit of $-\frac{1}{2}$ under $P_\lambda(z) = \lambda(z + z^2)$ is dense in $J(P_\lambda)$?

(M.R. Herman)

Update 2.84 No progress on this problem has been reported to us.

Problem 2.85 Suppose that λ is of modulus one and not a root of unity, and suppose that

$$P_\lambda(z) = \lambda(z + z^2) \qquad \text{and that} \qquad h_\lambda(z) = z + O(z^2)$$

is the unique formal power series such that $P_\lambda(h_\lambda(z)) = h_\lambda(\lambda z)$. Denote by $R(\lambda)$ the radius of convergence of h_λ.

(a) Calculate (or, at least, estimate up to $\pm 10^{-10}$) the value of $m = \sup_\lambda R(\lambda)$.
(b) Prove that m is realised by a function h_λ where $\lambda = e^{2\pi i\alpha}$ and α is a real algebraic number of degree 2.
(c) If $R(\lambda) = 0$ is the following true: for every positive ε, the function P_λ has a repelling periodic cycle included in $\{|z| < \varepsilon\}$? This property is known to hold for a dense G_δ-set of numbers λ of modulus one; see Cremer [228].

(M.R. Herman and J.-C. Yoccoz)

Update 2.85 No progress on this problem has been reported to us.

Problem 2.86 Let the function $f(z)$, $f(z) = \lambda(e^z - 1)$ with $|\lambda| = 1$, have a Siegel singular disc S_λ that contains zero.

(a) Prove that there exists some number λ, where $|\lambda| = 1$, such that S_λ is bounded in $\mathbb{C}$.
(b) If S_λ is unbounded in $\mathbb{C}$, does $-\lambda$ belong to ∂S_λ?

(M.R. Herman, I.N. Baker and P.J. Rippon)

Update 2.86(a) Rempe-Gillen writes that this can be solved by a method of Ghys which is mentioned by Douady [251] and found in detail in Rempe-Gillen [836].

Update 2.86(b) A positive answer is given by Rempe-Gillen [837] with a relatively simple proof.

Problem 2.87 Does there exist a non-linear entire function g with wandering domain W such that $\bigcup_{n\geq 0} g^n(W)$ is bounded in $\mathbb{C}$?

It has been conjectured by Lyubich [688] that if g and W exist, then $g^n(W)$ cannot converge to a fixed point of g as $n \to \infty$.

(M.R. Herman; A. Eremenko and M.Yu. Lyubich)

Update 2.87 The first question is the same as Problem 2.77 (as restated here), and is sometimes called the problem of 'bounded wandering'. Problem 2.67 is a special case of this question. All these problems remain unsolved.

Bergweiler writes that in principle it is conceivable that there exists a function satisfying Problem 2.67 but not satisfying Problems 2.77 and 2.87. In other words, the constant limit functions in the wandering domain form a bounded set, but the union of the wandering domains does not.

In relation to Problem 2.67, several authors have given families of entire functions each having the property that in any wandering domain the limit functions of the iterates must include ∞; see Zheng [1020], Osborne [781], and Mihaljević-Brandt and Rempe-Gillen [721].

Nicks [765] has shown that there does exist a quasiregular mapping g of the plane, with an essential singularity at infinity, which has a wandering domain W such that $\bigcup_{n\geq 0} g^n(W)$ is a bounded set.

Finally, the result that iterates in a wandering domain cannot converge to fixed point is contained in the work of Pérez-Marco [794, 795].

Problem 2.88 Let B denote the boundary of the Mandelbrot set (or, equivalently, the topological bifurcation set of the family $z \mapsto z^2 + c$, $c \in \mathbb{C}$).

(a) Is B locally connected?
(b) Prove that B has Hausdorff dimension 2.
(c) Does B have Lebesgue measure zero?

For further information, see Douady [250] and Lyubich [688].

(A. Douady and J.H. Hubbard; N. Sibony; M. Rees; M.R. Herman)

Update 2.88

(a) It is shown by Hubbard [557] and Lyubich [690] that B is locally connected in the neighbourhoods of certain points.
(b) Shishikura [902, 903] proved that the boundary of the Mandelbrot set B has Hausdorff dimension 2.

Problem 2.89 Let the function f_0 in R_d have an invariant Herman singular ring A_f of rotation number α, where α satisfies a diophantine condition. Denote by $H_{d,\alpha}$ the class of all functions f_1 in R_d such that f_1 can be joined to f_0 by a continuous path f_t, $0 \leq t \leq 1$, in R_d, where each f_t has a Herman singular ring A_{f_t} of rotation α, and the annuli A_f vary continuously with f (in the sense of Carathéodory).

(a) Is $H_{d,\alpha}$ locally closed in R_d?
(b) Is the boundary of $H_{d,\alpha}$ in its closure in R_d a topological manifold?

Both (a) and (b) are related to the investigation of rational functions with an invariant Herman singular ring when the moduli of their invariant rings tend to zero; see Herman [542].

(M.R. Herman)

Update 2.89 No progress on this problem has been reported to us.

Problem 2.90 Does there exist a number α in $\mathbb{R} \setminus \mathbb{Q}$ that does not satisfy a diophantine condition such that every $\mathbb{R}$-analytic orientation-preserving diffeomorphism of the circle with rotation number α is $\mathbb{R}$-analytically conjugate to a rotation?

For related questions, see Douady [251]. If α satisfies a diophantine condition, the global analytical conjugacy theorem has been proved. See Herman [541] and Yoccoz [1010].

(M.R. Herman)

Update 2.90 No progress on this problem has been reported to us.

2.3 New Problems

Problem 2.91 Let $k(x) \geq 0 \in L^1(-\infty, \infty)$ and let

$$\int_{-\infty}^{\infty} k(x)\,\mathrm{d}x = 1.$$

Let f be a continuous function such that when x is large: $f(x)$ is positive,

$$k * f(x) = \int_{-\infty}^{\infty} k(x-y) f(y)\,\mathrm{d}y$$

exists, and

$$k * f(x) = (1 + o(1)) f(x), \qquad \text{as } x \to \infty. \tag{2.13}$$

The problem is to show that the hypothesis (8.5) is sufficient to show that f must have the form

$$f(x) = \mathrm{e}^{\lambda x} L(x), \tag{2.14}$$

where λ is any scalar and

$$\int_{-\infty}^{\infty} k(x)\mathrm{e}^{\lambda x}\,\mathrm{d}x = 1$$

and L is slowly-varying in the sense of Karamata: that is, L is measurable and

$$L(x+h)/L(x) \to 1, \qquad \text{as } x \to \infty, \tag{2.15}$$

for every fixed h.

If f is increasing (or weakly decreasing) and the Laplace transform of k exists on an interval about λ this has been proved by Drasin [254], and there are extensions to kernels of varying signs by Jordan [583]. However, the hypothesis (8.5) is very strong, in the sense that the smoothing of f by k should force f to have sufficient regularity that no supplemental regularity of conditions on f might be needed.

This problem arose from a Nevanlinna theory paper of Edrei and Fuchs [289], where they showed that the condition (8.6) (where L is subject to (8.7)) is best possible.

(D. Drasin)

Problem 2.92 Let f be an entire function of order ρ, with $\frac{1}{2} \leq \rho \leq \infty$. Denote by $n(r, a)$ the number of zeros of $f(z) - a$ in the disc $|z| \leq r$. It was shown by Bergweiler [107] that if $a, b \in \mathbb{C}$ are distinct, then

$$\limsup_{r\to\infty} \frac{n(r,a) + n(r,b)}{\log M(r, f)} \geq \frac{1}{\pi}. \tag{2.16}$$

An immediate consequence is that

$$\limsup_{r\to\infty} \frac{n(r,a)}{\log M(r,f)} \geq \frac{1}{2\pi} \tag{2.17}$$

for all $a \in \mathbb{C}$ with at most one exception.

Can the constant $1/2\pi$ in (2.17) be replaced by $1/\pi$? This would be best possible.

For $f(z) = e^z$ and $a = 0$ we have equality in (2.16). Can the constant $1/\pi$ in (2.16) be improved for large ρ? In particular, does (2.16) hold with $1/\pi$ replaced by $2/\pi$ if $\rho = \infty$?

(W. Bergweiler)

Problem 2.93 Let f be a transcendental entire function of order less than 1. Can f have (an orbit of) unbounded wandering domains?

This problem arises from the following question, which has its origins in the work of Baker [70], and which is still open: *Can a transcendental entire function of order less than* $1/2$*, or even of order at most* $1/2$ *minimal type, have unbounded Fatou components?*

Zheng [1019] showed that f has no unbounded periodic Fatou components whenever it has order at most $1/2$ minimal type, and this is sharp by examples of Baker [70].

Nicks, Rippon and Stallard [766] showed that if f is real and has only real zeros, then order at most $1/2$ minimal type implies that f has no unbounded wandering domains, and also that order less than 1 implies that f has no orbits of unbounded wandering domains.

(G.M. Stallard)

Problem 2.94 Let U be a component of the Fatou set of a transcendental entire function which lies in the escaping set $I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$.

(a) Must there exist at least one point on ∂U which is escaping?
(b) If U is a bounded escaping wandering domain of f, must *all* points on ∂U be escaping?

It is known that if U is an escaping wandering domain, then almost all points of ∂U (in the sense of harmonic measure) must be escaping; see Rippon and Stallard [839]. This answers question (a) for escaping wandering domains and gives a context to question (b). Question (a) is thus reduced to considering Baker domains (periodic Fatou components in $I(f)$). The answer is known to be 'yes' for some types of Baker domain but in general the question is open; see Barański, Fagella, Jarque and Karpińska [74].

(P.J. Rippon)

Chapter 3
Subharmonic and Harmonic Functions

3.1 Preface by S.J. Gardiner

Among the answers that have been found to many of the questions in this chapter are:

(A) the striking resolution, by Hansen and Nadirashvili, of two well-known problems of Littlewood concerning bounded continuous functions which possess a one-radius mean value property (Problem 3.8);
(B) the verification of the boundary Harnack principle, which is now a widely used tool in potential theory (Problem 3.17);
(C) Fuglede's treatment of asymptotic paths for subharmonic functions, which illustrates the utility of fine topology methods in addressing classical questions (Problem 3.2).

An enticing question that has remained open for more than half a century is Problem 3.10 from Lipman Bers. This asks whether a non-constant harmonic function u on the unit ball of $\mathbb{R}^3$ that is smooth up to the boundary can vanish, along with its normal derivative, on a set of positive surface area measure. Bourgain and Wolff have shown that this can indeed happen if u is merely required to be C^1 on the closed ball. However, the answer is still unknown when the partial derivatives of all orders are continuous up to the boundary.

Another interesting problem which has long resisted resolution concerns null quadrature domains. An open set $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), where $\overline{\Omega} \neq \mathbb{R}^n$, is called a *null quadrature domain* (*nqd*) for harmonic functions if $\int_\Omega u \, \mathrm{d}m = 0$ for all integrable harmonic functions u on Ω, where m denotes n-dimensional Lebesgue measure. Sakai [874] showed that, when $n = 2$, the nqd's are precisely the open sets which have one of the following forms:

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9_3

(i) the exterior of an ellipse;
(ii) the non-convex component of the complement of a parabola;
(iii) a half-plane;
(iv) the complement of a strip.

He subsequently asked in Problem 3.28 for the corresponding characterisation when $n \geq 3$. An initial step towards this goal was made by Friedman and Sakai [356], who showed that the complement of an nqd must be an ellipsoid if it is compact.

Karp and Margulis [587] conjectured that the nqd's in higher dimensions are precisely the open sets which have one of the following forms:

(1) the exterior of an ellipsoid;
(2) the non-convex component of the complement of an elliptical paraboloid;
(3) a cylinder over a set of the form (1) or (2) above;
(4) a half-space;
(5) the complement of a strip.

Some further partial results may be found in [585–587]. For example, for nqd's Ω, it is known that:

(a) if $\partial\Omega$ is bounded in the directions of the x_{n-1} and x_n axes, then Ω is either the exterior of an ellipsoid or a cylinder over the exterior of an ellipsoid (see [587, Theorem 4.13]);
(b) the complement of Ω is a convex set with analytic boundary (see [586, Sect. 4]).

However, the full classification of null quadrature domains in higher dimensions remains an open problem.

3.2 Progress on Previous Problems

Notation A function $u(z)$ in a domain D of the plane is said to be *subharmonic* in D if:

(a) $u(z)$ is upper semi-continuous in D,
(b) $-\infty \leq u(z_0) < +\infty$, and $u(z) \not\equiv -\infty$ in D,
(c) for every z_0 in D, and all sufficiently small r (depending on z_0), we have

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + r\mathrm{e}^{i\theta})\,\mathrm{d}\theta.$$

In space of higher dimensions, subharmonic functions are defined analogously. If $u(z)$ and $-u(z)$ are subharmonic, then $u(z)$ is *harmonic*. If $f(z)$ is analytic in a domain D, and $f(z) \not\equiv 0$, then

$$u(z) = \log|f(z)|$$

is subharmonic in D, and many properties of the modulus of analytic functions suggest corresponding properties of subharmonic functions in the plane.

Sometimes, analogous properties of subharmonic functions in space also hold. We recall, from the introduction to Chap. 2, Iversen's theorem [566], that if $f(z)$ is a non-constant entire function, then there exists a path Γ such that $f(z) \to \infty$ as $z \to \infty$ along Γ. The analogue, that a subharmonic function $u(z)$ in the plane, which is not constant, tends to $+\infty$ along a path Γ was proved by Talpur [934].

Problems 2.6 and 2.7 have immediate analogues for subharmonic functions, and an analogue of Problem 2.8 can be formulated without too much difficulty. If $f(z)$ is analytic and non-zero, then $u(z) = \log|f(z)|$ is harmonic. Thus, Problem 2.9 in its first formulation reduces to a problem on harmonic functions $u(z)$. It should be said that Huber [558] proved his result (2.3) for subharmonic functions (see Problem 2.10) that for every positive λ there exists a path C_λ such that

$$\int_{C_\lambda} |f(z)|^{-\lambda}\, |\mathrm{d}z| < \infty.$$

That is, with $|f(z)|$ replaced by $\exp(u(z))$ in Problem 2.10, where $u(z)$ is a subharmonic function such that

$$\frac{B(r,u)}{\log r} \to +\infty,$$

where, throughout this section,

$$A(r,u) = \inf_{|z|=r} u(z), \qquad B(r,u) = \sup_{|z|=r} u(z).$$

Unless the contrary is explicitly stated, we shall assume that the functions $u(z)$ are defined in a plane domain.

Problem 3.1 If $u(z)$ is harmonic in the plane, and not a polynomial, does there exist a path Γ_n for every positive integer n such that

$$\frac{u(z)}{|z|^n} \to +\infty \tag{3.1}$$

as $z \to \infty$ along Γ_n. Does there exist a path Γ_∞ such that (3.1) holds for every fixed n, as $z \to \infty$ along Γ_∞?

We note that the result (see Problem 2.6) of Boas, that for every transcendental entire function there exists a path Γ_∞ such that for every n, $\left|\frac{f(z)}{z^n}\right| \to \infty$ as $z \to \infty$ along Γ, can be applied to $f(z) = \mathrm{e}^{u+iv}$, where v is the harmonic conjugate of u, but this only yields

$$\frac{u(z)}{\log|z|} \to +\infty.$$

Update 3.1 Barth, Brannan and Hayman [85] have shown that, for every α with $\alpha < \frac{1}{2}$, there exists a path for every α, with $\alpha < \frac{1}{2}$ such that

$$\frac{u(z)}{|z|^{\alpha}} \to +\infty,$$

and a little more. On the other hand, there need not exist such a path if $\alpha \geq \frac{1}{2}$.

Problem 3.2 If $u(x)$ is harmonic and not constant in space of 3 or more dimensions, is it true that there exists a path Γ such that $u(x) \to +\infty$ as $x \to \infty$ along Γ?

The corresponding result for subharmonic functions is certainly false, since if

$$r = \left(\sum_{\nu=1}^{n} x_{\nu}^{2}\right)^{\frac{1}{2}}$$

is the distance of $x = (x_1, x_2, \ldots, x_n)$ from the origin, then $u(x) = \max(-1, -r^{2-n})$ is subharmonic and bounded in space of n dimensions, when $n > 2$. On the other hand, a bounded harmonic function in space is constant.

Update 3.2 This problem has been largely settled. If $u(x)$ is subharmonic and has finite least upper bound M, then Hayman [515, Chap. 4] showed that $u(x) \to M$ as $x \to \infty$ along almost all rays through the origin. If $u(x)$ is not bounded above, and in particular, if $u(x)$ is harmonic, there always exists a path Γ such that

$$u(x) \to +\infty, \qquad \text{as } x \to \infty \text{ along } \Gamma. \tag{3.2}$$

This is a result of Fuglede [362] which completes earlier results of Talpur and Hayman [933]. Fuglede uses a deep theorem about Brownian motion by Nguyen and Watanabe [762]. The path Γ is locally a Brownian motion and so highly irregular. Carleson [191] has proved the existence of a polygonal path using classical analytic arguments, and Fuglede [363] subsequently gave an elegant alternative proof of this fact based on the fine topology.

Problem 3.3 Suppose that $u(z)$ is subharmonic and $u(z) < 0$ in the half-plane $|\theta| < \pi/2$ where $z = re^{i\theta}$. Suppose also that

$$A(r) = \inf_{|\theta|<\pi/2} u(re^{i\theta}) \leq -K, \qquad 0 < r < \infty.$$

Is it true then that

$$u(r) \leq -\frac{1}{2}K, \qquad 0 < r < \infty\,?$$

The result $u(r) \leq -K/3$ is true, and is due to Hall [462].

Update 3.3 The answer is ‘no’. Hayman [511, Chap. 7] has constructed an example of a function satisfying Hall’s conditions and such that $u(r) > -\frac{1}{2}K$ on the whole positive axis. This leaves open the question as to what is the best constant α such that $u(r) < -\alpha K$ on the whole positive axis.

Problem 3.4 Consider the class of functions subharmonic in the unit disc $\mathbb{D}$, and satisfying $u(z) \leq 0$ there. Suppose also that $A(r, u) \leq -1$, for r lying on a set E consisting of a finite number of straight line segments. Then it is known (see, for example, R. Nevanlinna [757, p. 100]) that $B(r, u)$ is maximal when $u(z) = u_E(z)$ where $u_E(z)$ is harmonic in $\mathbb{D}$ except on a set E of the positive real axis, and $u(z)$ assumes boundary values 0 on $|z| = 1$, and -1 on E. This is the solution to the so-called Carleman–Milloux problem [757]. For $0 < r < 1, 0 < K < 1$, let $C(r, K)$ be the set of all θ such that $u(re^{i\theta}) < -K$.

Is it true that $C(r, K)$ has minimal length only if $u(z) = u_E(z)$? The special case where E consists of the whole interval $[0, 1]$ is of particular interest.

(T. Kövari)

Update 3.4 No progress on this problem has been reported to us.

Problem 3.5 Suppose that $u(z)$ is positive and subharmonic in $\mathbb{D}$ and that there exists a series of arcs γ_n tending to the arc $\alpha \leq \theta \leq \beta$ of $|z| = 1$ such that

$$u(z) \leq M, \qquad z \text{ on } \gamma_n, \qquad n = 1, 2, \ldots. \tag{3.3}$$

If in addition,

$$\int_0^1 (1 - r)u(re^{i\theta})\, d\theta < +\infty, \tag{3.4}$$

for a set E of θ which is dense in the interval (α, β), then MacLane [698] proved that $u(re^{i\theta})$ is uniformly bounded for

$$\alpha + \delta \leq \theta \leq \beta - \delta, \qquad 0 \leq r < 1,$$

and any fixed positive δ. These conclusions thus hold in particular, if

$$\int_0^1 (1 - r)B(r, u)\, dr < +\infty. \tag{3.5}$$

Can the growth conditions (3.4) and (3.5) be weakened without weakening the conclusions?

Update 3.5 Hornblower [553, 554] has shown that (3.5) can indeed be replaced by the much weaker condition

$$\int_0^1 \log^+ B(r)\, dr < \infty,$$

and that this is more or less best possible. However, it is still an open question as to whether condition (3.4) is sharp.

Problem 3.6 It follows from a result of Wolf [997] that if

$$u(re^{i\theta}) \le f(\theta), \qquad 0 < r < +\infty,$$

where

$$\int_0^{2\pi} \log^+ f(\theta)\, d\theta < +\infty,$$

then $u(z)$ is bounded above, and so is constant. What is the 3-dimensional analogue of this result?

Update 3.6 We note that the original statement of Problem 3.6 had $\log^+ \log^+ f(\theta)$ instead of $\log^+ f(\theta)$. Wolf [997] proved that if u is subharmonic in the plane, and satisfies

$$u(re^{i\theta}) \le f(\theta), \qquad 0 < r < \infty,$$

where

$$\int_0^{2\pi} \log^+ f(\theta)\, d\theta < \infty,$$

then u is bounded above, and so constant. Yoshida [1012] established the higher-dimensional analogue of this result.

Problem 3.7 Problem 1.17 can be reformulated for subharmonic functions if we replace $\log M(r, f)$ by a general subharmonic function $u(z)$. The same positive theorems hold, and the same conclusions are conjectured in the general case.

Update 3.7 The result, generalising Govorov's theorem [438] to subharmonic functions in space, has been obtained by Dahlberg [230].

Problem 3.8 If $\omega(\zeta)$ is continuous and bounded in $\mathbb{D}$ and each point of ζ of $\mathbb{D}$ is the centre of at least one circle C_ζ lying in $\mathbb{D}$ such that $\omega(\zeta)$ is equal to the average of the values of $\omega(z)$ on C_ζ, is it true that $\omega(z)$ is harmonic in $\mathbb{D}$? The result is true if $\omega(z)$ is continuous in $|z| \le 1$, and is certainly false if this condition is removed. The corresponding problem with C_ζ replaced by its interior D_ζ is also open.

(J.E. Littlewood)

Update 3.8 Hansen and Nadirashvili [474] have shown that the conclusion is true for area mean values, and [475] that it is false for circular mean values. See also the papers of Fenton, including [337].

Problem 3.9 If D is a convex domain in space of 3 or more dimensions, can we assert any inequalities for the Green's function $g(P, Q)$ of D which generalise the results for 2 dimensions, that follow from the classical inequalities for convex univalent functions?

Gabriel [364] proved that the level surfaces $G(P, Q) = \lambda > 0$ are convex, but the proof is long. It would be interesting to find a simpler proof, and also to obtain definite inequalities for the curvatures. It may be conjectured that half-space gives the extreme case.

(G.E.H. Reuter)

Update 3.9 Alternative proofs of a more general result may be found in Rosay and Rudin [845] and in some of the references therein. See also Lewis [665].

In general, there are many problems for harmonic functions in 3 dimensions where the proofs of the corresponding two-dimensional results depend on conjugacy arguments and analytic functions, and so do not readily extend. An example is the following.

Problem 3.10 Suppose that $u(X)$ is harmonic on the unit ball $|X| < 1$, and remains continuous with partial derivatives of all orders on $|X| = 1$, where X is a point (x_1, x_2, x_3) in space, and

$$|X|^2 = x_1^2 + x_2^2 + x_3^2.$$

Suppose further that there is a set E of positive area on $|X| = 1$ such that both u and its normal derivative vanish on E. Is it true that $u \equiv 0$?

Here the two-dimensional analogue is almost trivial, since if u is harmonic in $\mathbb{D}$ and u and its partial derivatives remain continuous on the unit circle $\mathbb{T}$, we may consider

$$f(z) = z\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right), \qquad \text{where } z = x + iy.$$

If $u(z)$ and its normal derivatives both vanish on a set E, then $f(z)$ vanishes at all limit points of E. Now the Poisson–Jensen formula shows at once that $\log|f(z)| = -\infty$, that is $f(z) = 0$, identically in $\mathbb{D}$, provided that the closure of E has positive 1-dimensional measure.

(L. Bers)

Update 3.10 Bourgain and Wolff [153] have shown that the conclusion fails if one merely requires u to be continuously differentiable up to the boundary. The complete question remains open.

Problem 3.11 If $u(x)$ is a homogeneous harmonic polynomial of degree n in $\mathbb{R}^m$, what are the upper and lower bounds of

$$-\frac{A(r, u)}{B(r, u)},$$

where $A(r, u) = \inf_{|x|=r} u(x)$ and $B(r, u) = \sup_{|x|=r} u(x)$? If n is odd, it is evident that $A(r, u) = -B(r, u)$, but if $u(x) = x_1^2 + x_2^2 - 2x_3^2$ in $\mathbb{R}^3$, then $B(r, u) = r^2$ and $A(r, u) = -2r^2$. For transcendental harmonic functions such that $u(0) = 0$, we can prove that

$$-A(r, u) \leq \frac{(R + r)R^{m-2}}{(R - r)^{m-1}} B(r, u), \qquad 0 < r < R,$$

by Poisson's formula and this leads to

$$- A(r, u) < B(r)(\log B(r))^{m-1+\varepsilon} \tag{3.6}$$

outside a set of r of finite logarithmic measure. However (3.6) is unlikely to be sharp. We note that for $m = 2$, it follows from a classical result of Wiman that for any harmonic function u,

$$A(r) \sim -B(r)$$

as $r \to \infty$ outside a set of finite logarithmic measure.

Update 3.11 Armitage [44] has solved this problem by obtaining the least upper bound of

$$\frac{\sup_{|x|=1} u(x)}{\inf_{|x|=1} u(x)},$$

where $u(x)$ is a homogeneous harmonic polynomial of degree n in $\mathbb{R}^m$, $m \geq 3$. He shows that the ratio is bounded by a constant depending only on m.

Problem 3.12 Consider a domain of infinite connectivity in $\mathbb{R}^3$ whose complement E lies in the plane $P : x_3 = 0$. Suppose further than any disc of positive radius R in P contains a subset of E having area at least ε, where ε, R are fixed positive constants. If u is positive and harmonic in D, continuous in $\mathbb{R}^3$, and zero on E, is it true that

$$u = cx_3 + \phi(x), \qquad x_3 > 0,$$

where c is a constant and $\phi(x)$ is uniformly bounded? One can also ask the analogue of this result for $\mathbb{R}^m$ when $m > 3$. It is true in $\mathbb{R}^2$ (but Kjellberg cannot remember who proved it).

(B. Kjellberg)

Update 3.12 A fairly complete answer has been given by Benedicks [103].

Problem 3.13 Let $u(x)$ be subharmonic in $\mathbb{R}^m$. One can define the quantities $n(r, 0)$, $N(r, 0)$, $T(r)$ as in Nevanlinna theory in the plane, taking the analogue of the case $u(z) = \log |f(z)|$ where $f(z)$ is an entire function; see for example Hayman [492]. Define

$$\delta(u) = 1 - \limsup_{r\to\infty} \frac{N(r, u)}{T(r)}.$$

If the order ρ of u is less than 1, it is possible to obtain the sharp upper bound for $\delta(u)$ in terms of ρ and m. The bound is attained when $u(x)$ has all its mass on a ray (see Hayman and Kennedy [515, Chap. IV]). One can ask the corresponding question for $\rho > 1$.

One can ask whether a lower bound $A(\rho)$ can be obtained for $\delta(u)$ if $\rho > 1$ and all the mass of $u(x)$ lies on a ray, or more generally, on some suitable lower-dimensional subspace S of $\mathbb{R}^m$ and $\rho > \rho_0(S)$. For fixed S we may conjecture, by analogy with the $m = 2$ case, that $A(\rho) \to 1$ as $\rho \to \infty$. This is proved by Hellerstein and Shea [533] in the $m = 2$ case.

(D. Shea)

Update 3.13 Rao and Shea [833] have extended from 2 to m dimensions the work of Hellerstein and Shea [533] on the deficiency of a subharmonic function $u(x)$ with all its mass concentrated on a ray. More precisely, write

$$N(r, u) = \int_{|x|=r} u(x)\,\mathrm{d}\sigma(x), \qquad m_2(r) = \int_{|x|=r} u(x)^2\,\mathrm{d}\sigma(x),$$

where σ denotes surface measure on $|x| = r$ normalised to have total value one. Then for $m = 2, 3, 4$ they obtain the sharp lower bound for

$$\limsup_{r\to\infty} \frac{m_2(r)}{N(r)}$$

as a function of the order of u. It turns out again that the limit is attained when all the Riesz mass lies on a ray. This leads to nice bounds for the deficiencies of subharmonic functions in space in terms of the order.

Problem 3.14 Let there be given an integrable function F on the unit circumference $\mathbb{T}$ and a point z_0 in $\mathbb{D}$. The problem is to maximise $u(z_0)$, where u runs through all functions which are subharmonic in $\mathbb{D}$, equal to F on $\mathbb{T}$, and which satisfy

$$\inf u(r\mathrm{e}^{i\theta}) \le 0, \qquad 0 < r < 1.$$

(A. Baernstein)

Update 3.14 No progress on this problem has been reported to us.

Problem 3.15 Let D be a doubly-connected domain with boundary curves α and β and let z_0, z_1 be points of D. Let A, B be given real numbers. The problem is to maximise $u(z_0)$, where u runs through all functions which are subharmonic in D, take the values A and B at α and β respectively and are non-positive on some curve connecting z_1 to α.

(A. Baernstein)

Update 3.15 No progress on this problem has been reported to us.

Problem 3.16 A compact set E in $\mathbb{R}^n$, $n \ge 3$, is said to be *thin* at P_0 if

$$\int_0^1 \frac{c(P_0, r)}{r^{n-1}}\,\mathrm{d}r < \infty, \tag{3.7}$$

where $c(P_0, r) = \mathrm{cap}\big[E \cap \{P : |P - P_0| \le r\}\big]$; this is the integrated form of the Wiener criterion. It follows from Kellogg's theorem [590] that the points of E where

(3.7) holds form a polar set. Can one give a direct proof of this fact which shows perhaps that (3.7) is best possible?

(P.J. Rippon)

Update 3.16 No progress on this problem has been reported to us.

Problem 3.17 Let D, D' be Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with D' contained in D and $\partial D' \cap \partial D$ lying compactly in the interior of a set Γ in ∂D. For any fixed P_0 in D', let $H(P_0)$ denote the family of positive harmonic functions h on D that vanish continuously on Γ and satisfy $h(P_0) = 1$. Is there a constant C such that for all h_1, h_2 in $H(P_0)$, we have $h_1(P) \leq Ch_2(P)$ for all P in D'?

The problem was first considered by Kemper [591], but the proof he gives contains an error.

(P.J. Rippon)

Update 3.17 Suppose D is a Lipschitz domain, P_0 is a point in D, E is a relatively open set on ∂D, and S is a subdomain of D satisfying $\partial S \cap \partial D \subseteq E$. Dahlberg [231] and Wu [999] proved independently that there is a constant C such that whenever u_1 and u_2 are two positive harmonic functions in D vanishing on E, and $u_1(P_0) = u_2(P_0)$, then $u_1(P) \leq CU_2(P)$ for all P in S. Ancona [31] proved the above result for solutions of certain elliptic operators instead of harmonic functions.

Problem 3.18 It is known that the set E of least capacity C and given volume is a ball. If E displays some measure of asymmetry (for instance, if every ball with the same volume as E in space contains a minimum proportion δ in the complement of E), can one obtain a lower bound for the capacity of E which exceeds C by some positive function of δ?

(L.E. Fraenkel; communicated by W.K. Hayman)

Update 3.18 Hall, Hayman and Weitsman [460] obtain the following for general sets if $n = 2$, and for convex sets if $n \geq 3$:

$$\operatorname{cap} E \geq C(1 + k\delta^{n+1}),$$

where k is a constant depending on the dimension n.

Problem 3.19 Let C_0 be a tangential path in $\mathbb{D}$ which ends at $z = 1$, and let C_θ be any rotation of C_0. Littlewood [673] showed that there exists a function $u(z)$, harmonic and satisfying $0 < u(z) < 1$ in $\mathbb{D}$, such that

$$\lim_{|z|\to 1,\, z\in C_\theta} u(z)$$

does not exist for almost all θ, $0 \leq \theta \leq 2\pi$. Surprisingly, it seems to be unknown whether there exists a $v(z)$, positive and harmonic in $\mathbb{D}$, such that

$$\lim_{|z|\to 1,\, z\in C_\theta} v(z)$$

does not exist for all θ, $0 \leq \theta \leq 2\pi$. The corresponding result is known for bounded analytic functions in $\mathbb{D}$. See, for example, Collingwood and Lohwater [222, Chap. 2].

(K.F. Barth)

Update 3.19 Aikawa [15] proves a theorem which automatically answers Barth's question in the affirmative. Further, he notes that the question can be solved more easily, and does in fact find a positive unbounded harmonic function with the required property.

Problem 3.20 Suppose that you have a continuous real function $u(x)$ on $\mathbb{R}^n$, and you want to know whether a homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and a harmonic function v on $\mathbb{R}^n$ exist such that

$$v(x) = u(\phi(x)).$$

Is it necessary and sufficient that there should exist mappings $\mu_2, \ldots, \mu_n$ such that

$$U = (u, \mu_2, \ldots, \mu_n)$$

is a light open mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$? For example, the $n = 2$ case is a result of Stoilow [989].

(L.A. Rubel; communicated by D.A. Brannan)

Update 3.20 No progress on this problem has been reported to us.

Problem 3.21 Let α be a continuum in the closure of the unit disc $\mathbb{D}$, and let $\omega(z) = \omega(z; \mathbb{D}; \alpha)$ be the harmonic measure at z of α with respect to $\mathbb{D}$. Is it true that $\omega(0) \geq \frac{1}{\pi} \arcsin \frac{1}{2}d$, where d is the diameter of α?

(B. Rodin)

Update 3.21 FitzGerald, Rodin and Warschawski [348] show that the answer is positive.

Problem 3.22 Let D be a domain containing the origin whose 'outer boundary' is $\mathbb{T}$ and whose 'inner boundary' is a closed set E in $\mathbb{D}$. If every radius of the unit disc meets E, determine the supremum of the harmonic measure at 0 of $\mathbb{T}$ with respect to D.

(W.H.J. Fuchs)

Update 3.22 A special case of this is listed as an open problem by Betsakos [119]. This problem is to find a continuum in the closed unit disc which meets every radius and whose harmonic measure at the origin is minimal. Jenkins [576] characterises the class of continua in which such an extremal continuum must occur; and Marshall and Sundberg [706] determine the essentially unique extremal continuum. A simplified derivation of this solution using the method of the extremal metric is given by Jenkins [577]. Marshall and Sundberg [706] showed how close to 1 the above supremum can be in a special case. Further related results have been obtained by Marshall and Sundberg [707] and Solynin [911].

Problem 3.23 Determine whether or not there exists a function $g(r)$, defined for $r \geq 0$ with $g(r) \to 0$ as $r \to \infty$, such that the following holds: if u is any Green potential in $\mathbb{D}$ satisfying $u(0) = 1$, then for every non-negative r the set

$$E_r = \{z : z \in \mathbb{D}, u(z) > r\}$$

can be covered by a family of discs $\{D(a_k; r_k)\}$ (with centres a_k and radii r_k), depending on r, such that $\sum_k r_k \leq g(r)$. One can ask the same question with 'Green potential' replaced by 'positive harmonic function'.

Results of this type are known for ordinary logarithmic potentials (see Cartan's lemma [453]) and the Riesz potentials (in higher dimensions).

(R. Zeinstra)

Update 3.23 Eiderman points out that an old result of Govorov [437] shows this to be the case with

$$g(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 28, \\ \frac{27}{r-1} & \text{if } r > 28. \end{cases}$$

Problem 3.24 For which positive p does there exist a function $u \not\equiv 0$ harmonic on $\mathbb{R}^3$ and vanishing on the cone $x_1^2 + x_2^2 = px_3^2$?

(H.S. Shapiro)

Update 3.24 Armitage [45] characterised the circular cones in $\mathbb{R}^n$ on which a non-trivial entire harmonic function can vanish. His result was subsequently generalised by Agranovsky and Krasnov [2] who gave a complete characterisation of all quadratic harmonic divisors in $\mathbb{R}^n$.

Problem 3.25 Is there a harmonic polynomial $P(x_1, x_2, x_3) \not\equiv 0$ that is divisible by $x_1^4 + x_2^4 + x_3^4$?

(H.S. Shapiro)

Update 3.25 The answer is 'no' since by a theorem of Brelot and Choquet [158], any divisor of a harmonic polynomial in $\mathbb{R}^n$ assumes positive and negative values.

Problem 3.26 Given n, $n \geq 4$, find a continuous function f on $(0, 1)$ such that the following statement is true: if u is a subharmonic function in the unit ball B of $\mathbb{R}^n$ with $u(0) > 0$ and $0 \leq u < 1$ in B, then there exists a path γ from the origin to ∂B with $u > 0$ on γ and

$$\text{length of } \gamma \leq f(u(0)).$$

Such an f exists when $n = 2$ and when $n = 3$; see Davis and Lewis [238]. In the particular case $n = 2$, it has been shown by Lewis, Weitsman and Rossi [667] that one can take

$$f(t) = c_1 t^{-c_2}, \qquad 0 < t < 1,$$

where c_1, c_2 are absolute constants. What is the smallest exponent c_2 for which such an f exists (for the case $n = 2$)?

(J.L. Lewis)

Update 3.26 For $n = 2$, Wu [1000] has obtained such a result with

$$f(t) = C\left(1 + \log \frac{1}{t}\right).$$

Lewis has reported that Ancona pointed out to him that the same f will do for general n. This follows from a result of Brelot and Choquet [157].

Problem 3.27 Let D be an unbounded domain in $\mathbb{R}^n$, $n \geq 2$. Is there a positive continuous function $\varepsilon(|x|)$ such that if u is harmonic in D and $|u(x)| < \varepsilon(|x|)$, then $u \equiv 0$?

For $n = 2$, the answer is 'yes'. The answer is also 'yes' if we restrict our attention to positive harmonic functions. For fine domains and finely harmonic functions, it follows from an example of Lyons [686, 687] that the answer is 'no'.

(P.M. Gauthier and W. Hengartner)

Update 3.27 A positive answer has been given by Armitage, Bagby and Gauthier [47]. See also Armitage [46] and Armitage and Goldstein [48].

Problem 3.28 Determine all domains Ω in $\mathbb{R}^n$, $n \geq 2$, satisfying the identity $\int_\Omega h(x)\,\mathrm{d}x = 0$ for every function h harmonic and integrable on Ω.

In the case $n = 2$, the answer is given by Sakai [874].

(M. Sakai)

Update 3.28 Partial results on this problem may be found in Karp [585] and Karp and Margulis [586].

Problem 3.29 It is known that the Newtonian potential of a uniform mass distribution spread over an ellipsoid K in $\mathbb{R}^n$, $n \geq 2$, is a quadratic function of the coordinates of $x = (x_1, \ldots, x_n)$ for x in K.

Nikliborc [767] and Dive [247, 248] independently proved that for $n = 2$ and $n = 3$, the ellipsoid is the only body with this property. Prove this converse assertion for $n > 3$ (preferably by a new method, since Nikliborc and Dive both use methods involving highly non-trivial calculations).

(H.S. Shapiro)

Update 3.29 D. Khavinson writes that Givental [395] has shown that the potential of any polynomial density of degree m is a polynomial of degree $m + 2$ inside an ellipsoid. The converse stated in Problem 3.29 has been proven by DiBenedetto and Friedman [245] in all dimensions. A shorter proof was subsequently found by Karp [584] based on Brouwer's fixed point theorem. See Khavinson and Lundberg [596] for further details.

Problem 3.30 Let $K(z, z')$ denote the kernel of the double layer potential occurring in Fredholm's theory where $z, z' \in \Gamma$, Γ being a smooth Jordan curve. (Recall that $K(z, z') = \frac{\cos\phi}{|z-z'|}$, where ϕ is the angle between the inward normal to Γ at z and the line (z, z').)

When Γ is a circle, the function $z \mapsto K(z, z')$ is a constant (that is, the same for each choice of z') and consequently the integral operator

$$T_\Gamma : f \mapsto \int_\Gamma f(z) K(z, z') \, \mathrm{d}s_z$$

is of rank one (as an operator from $C(\Gamma)$ to $C(\Gamma)$). Are there any other Γ for which the rank of T_Γ is finite?

(H.S. Shapiro)

Update 3.30 Shapiro [891, Theorem 7.6] has shown that if the rank of T_Γ is finite, then Γ is a circle. D. Khavinson writes that in the plane the kernel of T_Γ is either trivial or infinite-dimensional and it is an algebra [284, Theorem 1.2, Corollary 1.3, Remark 1.4]. Also, it is conjectured there that the only closed curves for which the kernel of T_Γ is nontrivial are rational lemniscates, that is, curves of the form:

$$\Gamma = \{|r(z)| = 1, \ r \text{ is a rational function all of whose poles are outside } \Gamma$$
$$\text{while all of its zeros are inside } \Gamma\}.$$

This problem in higher dimensions is discussed in [601, Sect. 8.4]. No examples of domains where T_Γ would have a nontrivial kernel are known although, as conjectured in [601], such domains probably should exist.

Problem 3.31 Let D be an unbounded domain in $\mathbb{R}^n$, $n \geq 2$. Denote points in $\mathbb{R}^n$ by $x = (x_1, x_2, \ldots, x_n)$, and by $|x|$ denote the Euclidean norm of x. The following result was given by Essén [326]:

Theorem 1 *Assume that the least harmonic majorant Ψ of $|x_1|$ in D is such that $\Psi(x) = O(|x|)$ as $x \to \infty$ in D. If $|x|$ has a harmonic majorant in D, then $|x_1| \log^+ |x_1|$ has a harmonic majorant in D.*

In the plane $|x|^p$ (if p is positive) has a harmonic majorant in D if and only if $F \in H^p$, where $F : \{|x| < 1\} \to D$ is a universal covering map with $F(0) = 0$. There are similar statements for $|x_1| \log^+ |x_1|$ and $\operatorname{Re} F \in L \log L$. Thus when $n = 2$, Theorem 1 is closely related to the following:

Theorem 2 *Suppose that $n = 2$, and that $F \in H^1(\mathbb{D})$. Then Re $F \in L \log L$ if and only if*

$$\int_{-\infty}^{\infty} N(1, iv, F) \log^+ |v| \, \mathrm{d}v < \infty, \tag{3.8}$$

where $N(1, \omega, F)$ is the Nevanlinna counting function (see [328, Theorem 1]).

Theorem 2 is an extension of a well-known result of Zygmund. Note also that in the case $n = 2$ there are functions F in $H^1(\mathbb{D})$ such that $\operatorname{Re} F \notin L \log L$; see [328, Sect. 6].

(a) In the case $n = 2$, what is the relation between the condition on Ψ in Theorem 1 and condition (3.8) in Theorem 2?
(b) Suppose now that $n \geq 2$. The assumption on Ψ in Theorem 1 was introduced in the proof for purely technical reasons. Is this the correct condition needed in Theorem 1 (we assume always that $|x|$ has a harmonic majorant in D)? Does there exist a domain D such that $|x|$ has a harmonic majorant in D while $|x_1| \log^+ |x_1|$ does not have a harmonic majorant in D?

(M. Essén)

Update 3.31 Brossard and Chevalier [160] gave an example of a domain D such that $|x|$ has a harmonic majorant in D while $|x_1| \log^+ |x_1|$ does not.

Problem 3.32 Let Ω be an open ball in $\mathbb{R}^n$, $n \geq 2$. It is shown by Armitage [43] that $V \in L^p(\Omega)$ for any positive superharmonic function V on Ω and any p in $(0, n/(n-1))$. Now suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ for which the interior cones have half-angle at least α. For what values of p do we have $V \in L^p(\Omega)$ for every positive superharmonic function V on Ω?

(S.J. Gardiner)

Update 3.32 Aikawa [16] has given a sharp upper bound on p for every positive superharmonic function to be in $L^p(\Omega)$ when Ω is such a Lipschitz domain.

Problem 3.33 Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and suppose that $y \in \partial\Omega$. Denote the open ball of centre y and radius r by $B(r)$. A point y is said to be *B-regular* for Ω if, for each resolutive function f on $\partial\Omega$ that is bounded in $B(R) \cap \partial\Omega$ for some positive R, the Perron–Wiener–Brelot solution H_f^Ω of the Dirichlet problem is bounded in $B(r) \cap \Omega$ for some positive r. Also, a point y is said to be *lB-regular* for Ω if there exists a positive null sequence $\{r_m\}_1^\infty$ such that y is B-regular for $B(r_m) \cap \Omega$ for all m. If y is lB-regular for Ω then it is B-regular for Ω; see Sadi [871]. Is the converse true?

(D.H. Armitage and A. Sadi)

Update 3.33 No progress on this problem has been reported to us.

Problem 3.34 Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with the property that there exists an α in $(0, \pi]$ such that for every point y in $\partial\Omega$ there is an open truncated cone with vertex y and angle α contained in Ω. Is there some positive number $p = p(\alpha)$ such that every positive superharmonic function in Ω belongs to $L^p(\Omega)$? If so, can such a p be characterised in terms of α?

(D.H. Armitage)

Update 3.34 This problem is answered by Aikawa [16].

Problem 3.35 For $r_1 < r_2$ we will call the set $\{x \in \mathbb{R}^n : n \geq 3, r_1 < \|x\| < r_2\}$ an *annulus* and its closure a *closed annulus*. Let Ω be a non-empty subset of $\mathbb{R}^n$, $n \geq 3$, such that $\lambda(\overline{\Omega}) < +\infty$, where λ denotes n-dimensional Lebesgue measure. Then if, for each point $x \in \mathbb{R}^n \setminus \overline{\Omega}$, we have that

$$\frac{1}{\lambda(\overline{\Omega})}\int_{\overline{\Omega}} \|x-y\|^{2-n}\,\mathrm{d}\lambda(y)$$

equals the mean-value of the function $y \mapsto \|x-y\|^{2-n}$ over the unit sphere of $\mathbb{R}^n$, it can be shown that $\overline{\Omega}$ is a closed annulus.

If, throughout the hypotheses, we replace $\overline{\Omega}$ by Ω, can we conclude that Ω is an annulus?

For details of closely-related work see [419, 516].

(D.H. Armitage and M. Goldstein)

Update 3.35 An affirmative answer has been given by Gardiner and Sjödin [377].

3.3 New Problems

Problem 3.36 This problem has its genesis as the so-called $K(\rho)$ problem arising from R. Nevanlinna's theory of meromorphic functions: if f is meromorphic in the plane and of finite order ρ, how large can the Nevanlinna deficiency sum $\sum_a \delta(a, f)$ be? The sum can attain its maximum, 2, only for isolated values of ρ, and the sharp bounds for general ρ are known only when $\rho \leq 1$.

It is possible that the solution for the special case of entire functions might be more tractable, and this problem has already been introduced as Problem 1.7 in the context of meromorphic functions. However, for entire functions, there is an intrinsic way to frame it as a problem in subharmonic functions, the connection being that

$$u(z) = \log|f(z)| \tag{3.9}$$

is subharmonic when f is entire, although additional regularity properties are present in this special case. As the notation suggests, this is a two-dimensional problem.

The most ideal subharmonic functions are harmonic, which means that the Riesz measure Δu vanishes. In our situation, we ask: given that u has order ρ, how small can Δu be? The Nevanlinna characteristic counting-function and order of a subharmonic function are defined in Hayman and Kennedy [515, Chap. 3].

To state it concretely we ask: if

$$\limsup_{r\to\infty} \frac{\log T(r,u)}{\log r} = \rho < \infty$$

and $\rho \notin \mathbb{Z}$, find the best bound for

$$K^*(\rho) = \limsup_{r\to\infty} \frac{\log N(r,u)}{\log r}.$$

When u has the special form (3.9), Cartan's identity (see Hayman [493, p. 8]) provides a more transparent way to view $T(r)$, which we may phrase intrinsically in terms of u. Let Γ be the preimage of the level-set $\{u = 0\}$ and for $r > 0$, let $\Gamma(r)$ be the length of $\Gamma \cap \{|z| < r\}$. Then

$$T(r, f) = \frac{1}{2\pi} \int_0^r \frac{\Gamma(t)}{t}\, \mathrm{d}t + \mathrm{C},$$

where C is a constant.

(D. Drasin)

Chapter 4
Polynomials

4.1 Preface by E. Crane

Polynomials occupy a central position in many branches of mathematics and its applications. There are many different ways of thinking about a complex polynomial f of degree n in one variable. It is characterised by being a holomorphic self-mapping of the Riemann sphere that is topologically a branched covering of degree n such that ∞ maps to ∞ with local degree n. Of course it also has concrete representations in terms of coefficients, roots, and critical points:

$$\begin{aligned} f &= a_n \prod_{i=1}^{n} (z - z_i) \\ &= a_n z^n + \cdots + a_1 z + a_0 \\ &= n a_n \int_0^z \prod_{j=1}^{n-1} (w - \zeta_i)\, \mathrm{d}w. \end{aligned}$$

A polynomial of degree n is also uniquely determined by its values at $n + 1$ distinct points, explicitly by Lagrange's interpolation formula. We might also represent f in terms of another useful basis for the vector space of polynomials, for example one of the various famous sequences of orthogonal polynomials. The Hermite and Chebyshev polynomials make an appearance in this collection of problems. Many important problems and theorems about polynomials concern the transition from one representation to another. Any one of these representations might be the natural data of a mathematical problem. The wide variety of applications and the variety of representations of polynomials lead to diverse types of questions that are natural to ask about these apparently simple objects. It is interesting to guess at the fields of enquiry that motivated each of the problems listed in this eclectic chapter!

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There are now several good textbooks devoted to the analytic and geometric properties of polynomials in one variable: two graduate level textbooks by Borwein and Erdélyi [145] and by Sheil-Small [899], and an impressively comprehensive monograph by Rahman and Schmeisser [831]. When one is faced with a geometric inequality about some class of polynomials it is well worth checking these references for clues.

Polynomials are used as approximations to holomorphic functions in many situations. For example, Mergelyan's theorem states that if $K \subset \mathbb{C}$ is compact and $\mathbb{C} \setminus K$ is connected, and f is a continuous complex-valued function on K that is holomorphic on the interior of K, then polynomials approximate f uniformly. Hilbert showed that for a polynomial f every such K is approximated in the Hausdorff topology by sublevel sets of the form

$$E_f^{(n)} = \{z \in \mathbb{C} \,:\, |f(z)| \leq 1\}.$$

The relationship between this sublevel set and the polynomial f is the subject of Problems 4.7–4.11. These can be thought of as questions about how rapidly sublevel sets of polynomials approximate a fixed compact set.

Many questions about polynomials in one complex variable can be interpreted in terms of planar electrostatics. This is because $\log|f|$ is the logarithmic potential generated by unit point charges located at the roots of f. The function $\frac{1}{n}\log|f|$ vanishes on the boundary of $E_f^{(n)}$, is harmonic on its complement and asymptotic to $\log|z|$ as $z \to \infty$: that is, it is the Green's function of $E_f^{(n)}$ with respect to ∞. The harmonic measure of $E_f^{(n)}$ with respect to ∞ is the pullback of the uniform measure on the unit circle under f. The logarithmic capacity of $E_f^{(n)}$, a linearly scaling measure of its size, is $a_n^{-1/n}$, where a_n is the leading coefficient of f. Thus in posing geometric problems about $E_f^{(n)}$ or about its boundary, the *lemniscate* $\{z \,:\, |f(z)| = 1\}$, it is often convenient to restrict attention to monic polynomials. The logarithmic capacity of a compact subset K of the plane is equal to its transfinite diameter, which is defined to be $\lim_{n\to\infty} d_n(K)$, where $d_n(K)$ is the maximum value of the geometric mean of the pairwise distances between n distinct points of K. This can be rephrased in terms of maximizing the absolute value of the discriminant of a monic polynomial constrained to have its roots in K.

Several of the problems concern polynomials whose coefficients are constrained, for example to be real, to lie on the unit circle or to be bounded. Of course such constraints occur naturally in many applications. Even in applications where the polynomials are real, the locations of the complex roots can be key, for example when a polynomial appears as the denominator in the Laplace transform of a solution of a differential equation.

In Problem 4.27 we may assume the polynomial is monic, in which case its coefficients are rational. One cannot expect to solve problems about classes of polynomials whose coefficients are constrained to be integral, or rational, using only the tools of function theory. However, a negative answer to Problem 4.27 turns out to be a simple consequence of Faltings' famous proof in 1983 of Mordell's conjecture that there are only finitely many rational points on any smooth algebraic curve over the rationals having genus at least 2.

In the case of Problems 4.13–4.17 the coefficients lie in the set $\{-1, 1\}$, so one might make the same objection. Nevertheless there are some fascinating problems with a definitely geometric rather than number-theoretic flavour concerning this class. In relation to Problem 4.15 it is interesting to note that the set of all the roots of all 2^n polynomials of degree n with coefficients in $\{-1, 1\}$ has a beautiful appearance reminiscent of a fractal. It is discussed by John Baez in his blog [62], which contains many beautiful computer-generated illustrations. The same aspect of the closely related class of polynomials with coefficients in $\{0, 1\}$ is discussed by Odlyzko and Poonen [771]. The prolific problem setter and solver Paul Erdős had a fascination with this class of polynomials; perhaps he had the chance to see the images in [771] and to try to explain their features.

Beautiful images of subsets of the plane generated by polynomials also arise in the field of holomorphic dynamics. Here polynomials form a class of maps with particularly good properties. Under iteration, any nonlinear polynomial f has a superattracting fixed point at infinity whose basin of attraction is the complement of the filled Julia set of f. The Julia set $J(f)$ is the boundary of this basin, and the measure of maximal dynamical entropy coincides with the harmonic measure of the filled Julia set. The orbits of the critical points of f are of crucial importance in understanding the dynamics, and inequalities from geometric function theory can give useful constraints on the locations of the critical points and critical values. Distortion estimates are very useful in understanding the geometry of the Julia set and of the components of its complement, the Fatou set. Of course Hayman's original collection of problems predates the resurgence in holomorphic dynamics that occurred in the 1980s, which may explain why this chapter contains no problems relating to iteration of polynomials.

Stephen Smale made his well-known mean value conjecture about the critical points and critical values of polynomials when he was studying the computational complexity of root-finding by Newton's iteration. The conjecture says that if f is a polynomial of degree $n \geq 2$ and $f'(z) \neq 0$ then there exists at least one critical point ζ (a root of f') such that

$$\left| \frac{f(z) - f(\zeta)}{z - \zeta} \right| \leq |f'(z)|.$$

It is further conjectured that the RHS can be replaced by $\frac{n-1}{n}|f'(z)|$. Smale proved the inequality with $4|f'(z)|$ on the right-hand side, using Koebe's 1/4-theorem. Smale's mean value conjecture was formulated after Hayman's original list of problems was compiled, but it has attracted a lot of attention, and deserves to be mentioned here. Another well-studied conjecture that asserts something about the location of at least one critical point is Sendov's conjecture. This does appear in the list, as Problem 4.5, and is mentioned again in Problem 4.30.

In common with several of the extremal problems posed in this chapter, such as Problems 4.2, 4.3, 4.5 and 4.26, 4.31, the problem of finding the sharp constant in Smale's mean value conjecture has the feature that for each fixed value of the degree n, the problem can in principle be solved algorithmically because it is a problem of semi-algebraic geometry; this follows from Tarski's elimination theorem. However, the difficulty of this approach grows very rapidly with the degree! When promising guesses for the extremal polynomials are known, if these are shown to be locally extremal, with quantitative bounds, then the problem may also be amenable to computer solution using verifiable interval arithmetic, again for small fixed values of the degree. But such approaches do not promise to give any insight into the case of general degree.

Specific inequalities concerning trigonometric polynomials, or equivalently about the restriction of a complex polynomial to the unit circle, often arise in harmonic analysis, additive combinatorics and analytic number theory. In this collection Problems 4.3, 4.19, 4.20, 4.25 and 4.26 are of this type.

One of the joys (or frustrations) of attacking geometric problems about polynomials is that there are so many different tools that one can try to apply!

4.2 Progress on Previous Problems

Problem 4.1 Let $\{z_n\}$, $1 \le n < \infty$, be an infinite sequence such that $|z_n| = 1$. Define

$$A_n = \max_{|z|=1} \prod_{i=1}^{n} |z - z_i|.$$

Is it true that $\limsup_{n\to\infty} A_n = \infty$, and if so, how quickly must A_n tend to infinity?

We may define z_n inductively as follows: $z_1 = 1$, $z_2 = -1$ and if z_ν has already been defined for $1 \le \nu \le 2^k$, then we define for $1 \le p \le 2^k$,

$$z_{p+2^k} = z_p \exp\left(\frac{\pi i}{2^k}\right).$$

With this definition we easily see that $A_n \le n + 1$ with equality if, and only if, $n = 2^k - 1$ for some integer k. Is this example extreme?

(P. Erdős)

Update 4.1 Wagner [974] has proved that

$$\limsup_{n\to\infty} \frac{\log A_n}{\log\log n} > 0$$

thus answering the first question positively. Linden [671] proved that A_n can be less than $n^{1-\delta}$ for all large n, where δ is a positive constant, thus giving a negative answer to the second question. A follow-up question by Erdős asks about the behaviour of

$$\left(\prod_{n=1}^{N} A_n\right)^{1/N}.$$

Problem 4.2 Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ be a polynomial, all of whose zeros are on $|z| = 1$. If

$$A = \max_{0 \le k \le n} |a_k|, \qquad M = \max_{|z|=1} |p(z)|,$$

is $M \ge 2A$?

Update 4.2 The original problem was misstated as $M \le 2A$. The inequality $|a_\nu| \le \frac{1}{2}M$ has been proved by Saff and Sheil-Small [872] except when n is even and $\nu = \frac{1}{2}n$. Sheil-Small points out that Kristiansen [637] has proved that $M \ge 2A$ in general.

Problem 4.3 Let $P_N(z)$ be a polynomial with N terms, satisfying $|P_N(z)| \le 1$ on $|z| = 1$. How large can $P_n(z)$ be if $P_n(z)$ is a partial sum of $P_N(z)$?

(P. Erdős)

Update 4.3 Kuznetsova and Tkachev [647] have proved some related results.

Problem 4.4 Is there a function $f(k)$ of the positive integer k so that the square of every polynomial having at least $f(k)$ terms has at least k terms? Erdős proved that at any rate $f(k) > k^{1+c}$ for a positive constant c.

(P. Erdős)

Update 4.4 No progress on this problem has been reported to us.

Problem 4.5 (*The Sendov conjecture*) Let $P(z)$ be a polynomial whose zeros $z_1, z_2, \ldots, z_n$ lie in $|z| \le 1$. Is it true that $P'(z)$ always has a zero in $|z - z_1| \le 1$?

(B. Sendov; communicated by L. Ilieff)

Update 4.5 This problem was originally called the Ilieff conjecture as it had been erroneously attributed. With over eighty papers published on the Sendov conjecture, it would be a difficult task to list them all here, but we note that it was proved by Meir and Sharma [715] for $n \le 5$, and obtained for general n if $|z_1| = 1$ by Rubinstein [858].

Gundersen writes that [599] contains a useful survey, which draws attention to some highlights. Brown [163] proves the conjecture for polynomials of degree $n = 6$ with respect to a zero of the polynomial of modulus not exceeding $\frac{63}{64}$; and an asymptotic proof of a version of the conjecture is obtained by Dégot [241]. Kasmalkar [588] strengthens a theorem by Chijiwa on this conjecture, and offers results which sharpen some of the estimates. Sendov [888] announces a stronger conjecture; proves it for polynomials of degree $n = 3$; and also announces a number of other conjectures, including a variation of Smale's mean value conjecture [909].

The Sendov conjecture was proved for $n = 6$ and $n = 7$ by Borcea and he proposed *Borcea's Variance Conjectures on the Critical Points of Polynomials*; see [599]. It was proved for $n \le 7$ by Brown [164], and for $n \le 8$ by Brown and Xiang [166]. Dégot [241] has proved that for each a, there exists an integer N such that the disc $|z - a| \le 1$ contains a critical point of p' when the degree of p is larger than N. This result is obtained by studying the geometry of the zeros and critical points of a polynomial which would eventually contradict Sendov's conjecture.

Further relevant works include a result by Bojanov, Rahman and Szynal [135], who prove the validity of the conjecture in an asymptotic sense without any additional restrictions; and an article by Schmeisser [883] on Sendov's conjecture and Smale's conjecture.

Problem 4.6 If $H_\nu(z)$ is the ν-th Hermite polynomial, so that

$$H_\nu(z)\mathrm{e}^{-z^2} = (-1)^\nu \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\nu \mathrm{e}^{-z^2},$$

is it true that the equation

$$1 + H_1(z) + aH_n(z) + bH_m(z) = 0,$$

where $2 \le n < m$, and a, b are complex, has at least one zero in the strip $|\mathrm{Im}\, z| \le c$, where c is an absolute constant? (This is true if $b = 0$; see Makai and Turán [700].)

(P. Turán)

Update 4.6 No progress on this problem has been reported to us, although progress on Problem 4.5 was erroneously listed under this number in [36].

In the following five problems we use the notation

$$E_f^{(n)} = \{z \in \mathbb{C} : |f(z)| \le 1\},$$

where n is the degree of the polynomial f.

Problem 4.7 Let $f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ be a polynomial of degree n. Cartan [196] proved that the set $E_f^{(n)}$ can always be covered by discs, the sum of whose radii is at most 2e. It seems likely that 2e can be replaced by 2. If $E_f^{(n)}$ is connected, this was proved by Pommerenke [809], who also proved the general result with 2.59 instead of 2.

Update 4.7 Mo [742] attempts to prove this conjecture with a sharp constant $2^{1-1/n}$ for each fixed n, instead of 2. Unfortunately there are gaps in the argument which so far have proved impossible to fill. Eremenko and Hayman [324] proved that the boundary of $E_f^{(n)}$ has length less than $9.173n$. See also Borwein [144] and Kuznetsova and Tkachev [647].

Problem 4.8 Assume that $E_f^{(n)}$ is connected. Is it true that

$$\max_{z \in E_f^{(n)}} |f'(z)| \leq 2^{1/n-1} n^2 \,? \tag{4.1}$$

Pommerenke [808] proved this with $\frac{1}{2}\mathrm{e}n^2$ instead of $\frac{1}{2}n^2$.

Update 4.8 We note that the original statement of this problem contained

$$\max_{z \in E_f^{(n)}} |f'(z)| \leq \frac{1}{2} n^2$$

instead of (4.1). However, Eremenko writes that this was not true, as it is violated by Chebyshev's polynomials. The correct inequality (4.1) is best possible, which has been proved by Eremenko and Lempert [319]. There is also a generalisation of this inequality by Eremenko [315], and Dubinin [263, 264] has obtained a sharp upper bound of the absolute value of the derivative of a polynomial at any point on the plane.

Problem 4.9 Is it true that to every positive c, there exists an $A(c)$ independent of n such that $E_f^{(n)}$ can have at most $A(c)$ components of diameter greater than $1 + c^2$?

(P. Erdős)

Update 4.9 Erdős wrote that Pommerenke [811] showed that to every positive ε and integer k there is an integer n_0 such that for $n > n_0$ there is a polynomial $f_n(z)$ of degree n for which $E_f^{(n)}$ has at least k components of diameter greater than $4 - \varepsilon$, thus showing that the original conjecture is false.

One could try to estimate the number of components of $E_f^{(n)}$ having diameter greater than $1 + c$. Is it $o(n)$ as $n \to \infty$? This seems certain, but it could be $o(n^{\varepsilon})$ as $n \to \infty$.

Problem 4.10 Is it true that the length of the curve $|f_n(z)| = 1$ is maximal for $f_n(z) = z^n - 1$?

(P. Erdős)

Update 4.10 Eremenko and Hayman [324] proved this for $n = 2$. They also proved that the set $L(f_n) = \{z : |f_n(z)| = 1\}$ is connected for extremal polynomials. Further progress was made by Fryntov and Nazarov [359] who proved that the length of $L(f_n)$ is at most $(2 + o(1))n$, which is the optimal asymptotics. They also proved that the polynomial $z^n - 1$ gives a local maximum for this problem.

Problem 4.11 If $|z_i| \leq 1$, estimate from below the area of $E_f^{(n)}$. Erdős, Herzog and Piranian [302] prove that: given positive ε, the area of $E_f^{(n)}$ can be made less that ε, if $n > n_0(\varepsilon)$.

Update 4.11 No progress on this problem has been reported to us.

Problem 4.12 If $-1 \leq z_1 \leq z_2 \leq \ldots \leq z_n \leq 1$, where the z_i are the zeros of $f(z)$, it is known that the diameter of the set on the real line for which $|f(x)| \leq 1$ is at most 3. Is it true that the measure of this set is at most $2\sqrt{2}$ (see also Problem 4.7)?

Update 4.12 This conjecture has been proved by Elbert [295, 296].

Problem 4.13 It is known (see Clunie [208]) that there exists a polynomial $P(z)$

$$P(z) = \sum_{k=1}^{n} \varepsilon_k z^k, \qquad \varepsilon_k = \mp 1,$$

for which

$$\max_{|z|=1} |P(z)| < C_1 \sqrt{n}. \tag{4.2}$$

Is it necessarily true that $C_1 > 1 + A$ if (4.2) holds, where A is a positive absolute constant?

Update 4.13 No progress on this problem has been reported to us.

Problem 4.14 Does there exist a polynomial of the type in Problem 4.13, for which

$$\min_{|z|=1} |P(z)| > C_2 \sqrt{n} \tag{4.3}$$

for every n? More generally, does there exist such a polynomial satisfying both (4.2) and (4.3)?

Update 4.14 An affirmative answer has been given by Beller and Newman [102] for the case when $|\varepsilon_k| \le 1$, and by Körner [628] for $|\varepsilon_k| = 1$. The case $\varepsilon_k = \pm 1$ remains open.

Problem 4.15 If $\varepsilon_k = \mp 1$, is it true that, for large n, all but $o(2^n)$ polynomials $P(z) = \sum_{k=1}^{n} \varepsilon_k z^k$ have just $n/2 + o(n)$ roots in $\mathbb{D}$?

Update 4.15 No progress on this problem has been reported to us.

Problem 4.16 Is it true that for all but $o(2^n)$ polynomials $P(z)$

$$\min_{|z|=1} |P(z)| < 1,$$

or, if not, what is the corresponding correct result?

Update 4.16 This has been solved by Konyagin [615], where he shows the stronger result that all but $o(2^n)$ polynomials satisfy $\min |P(z)| \le n^{\{-\frac{1}{2}+\epsilon\}}$ for any positive ϵ. Konyagin and Schlag [616] show that the exponent $-\frac{1}{2}$ is optimal.

Problem 4.17 It is shown by Salem and Zygmund [875] that there exist positive constants c_3, c_4 such that for every positive ε, we have

$$(c_3 - \varepsilon)\big(n \log n\big)^{\frac{1}{2}} < \max_{|z|=1} |P(z)| < (c_4 + \varepsilon)\big(n \log n\big)^{\frac{1}{2}}$$

apart from $o(2^n)$ polynomials $P(z)$. Is this result true with $c_3 = c_4$ and if so, what is the common value?

Update 4.17 Halász [457] has proved this conjecture with $c_3 = c_4 = 1$ together with the analogous result for trigonometric polynomials.

Problem 4.18 If f is any polynomial or rational function of degree n, find the least upper bound $\phi(n)$ of

$$\frac{1}{r}\int_0^r dt \int_{-\pi}^{\pi} \frac{|f'(re^{i\theta})|}{1+|f|^2}\,\mathrm{d}\theta$$

for varying r and f. It is known only that $\phi(n) = O(n^{\frac{1}{2}})$ and $\phi(n) \neq O(\log n)^{\frac{1}{2}-\varepsilon}$ as $n \to \infty$. The upper bound for varying f of

$$\int_0^{\infty}\int_{-\pi}^{\pi} \frac{|f'|}{1+|f|^2}\,\mathrm{d}r\,\mathrm{d}\theta$$

should not be much larger than $\phi(n)$, but there is no proof that it is not $+\infty$. The special cases $n = 2, 3, \ldots$ would be of interest. See Littlewood [676].

Update 4.18 If f is a polynomial or rational function, Littlewood asked for bounds depending on n only, where n is the degree or order of f, of the functionals

$$J_1(f) = \int_0^1 r\,\mathrm{d}r \int_{-\pi}^{\pi} \frac{|f'(re^{i\theta})|}{1+|f|^2}\,\mathrm{d}\theta, \qquad J_2(f) = \int_0^{\infty} \mathrm{d}r \int_{-\pi}^{\pi} \frac{|f'(te^{i\theta})|}{1+|f|^2}\,\mathrm{d}\theta.$$

Hayman [508] has shown that the sharp bounds $\psi_1(n)$, $\psi_2(n)$ for $J_1(f)$ and $J_2(f)$ and rational functions f of degree n have orders $n^{1/2}$ and n, respectively.

The corresponding questions for polynomials are more difficult and more interesting. If $\phi_1(n)$, $\phi_2(n)$ are the bounds, then Hayman [508] has proved

$$\phi_1(n) > A\log n, \qquad \phi_2(n) < A(n\log n)^{1/2},$$

thus slightly improving previous results by Littlewood [676] and Chen and Liu [206] respectively. If f is a polynomial of degree n, let $\phi(n)$ be the supremum of

$$\frac{1}{r}\int_0^r \mathrm{d}t \int_{-\pi}^{\pi} \frac{|f'(re^{i\theta})}{1+|f|^2}\,\mathrm{d}\theta,$$

for varying r and f. Littlewood conjectured that

$$\phi(n) = O(n^{\frac{1}{2}-\eta})$$

for some positive η, and this has been proved by Lewis and Wu [669] by the same method which they used for their attack on Problem 1.6. It depends on some harmonic measure estimates of Bourgain [149]. Earlier Eremenko and Sodin [322] had shown that

$$\phi(n) = o(n^{\frac{1}{2}}).$$

A consequence of these results is that an entire function of finite order has 'almost all roots of almost all equations $f(z) = a$' in a set of zero density in the plane.

D. Beliaev writes that Problems 4.18, 6.5, 6.7, and 6.8 are essentially equivalent, and the current best estimates for the optimal exponent in all of these problems are: lower bound 0.23 [100, 101], upper bound 0.46 [528]. See Problem 6.5 for details of the equivalence of Problems 6.5, 6.7 and part of 6.8. To see how Problem 4.18 is also essentially equivalent we need a proof of Binder and Jones that was never published.

Problem 4.19 Littlewood conjectured that if $n_1, n_2, \ldots, n_k$ are distinct positive integers then

$$\int_0^{2\pi} \left| \sum_{i=1}^{k} \cos(n_1 x) \right| \mathrm{d}x > c \log k. \tag{4.4}$$

The best result known in this direction is due to Davenport [235] who proved (4.4) with $\left(\log k / \log\log k\right)^{\frac{1}{4}}$ instead of $\log k$, thus sharpening an earlier result of Cohen [219].

Update 4.19 This was solved independently by Konyagin [614] and by McGehee, Pigno and Smith [711].

Problem 4.20 Let $f_n(\theta)$ be a trigonometric polynomial of degree n, all of whose roots are real. Is it true that

$$\int_0^{\pi} \left| f_n(\theta) \right| \mathrm{d}\theta \le 4\,?$$

See Erdős [299].

(P. Erdős)

Update 4.20 This problem was originally misattributed to F.R. Keogh. It has been settled affirmatively by Saff and Sheil-Small [872]. They have also obtained sharp bounds for

$$\int_{-\pi}^{\pi} \left| f_n(r\mathrm{e}^{i\theta}) \right|^p \mathrm{d}\theta$$

whenever $f(z) = z^n + \ldots$ is a polynomial of degree n, all of whose zeros lie on $|z| = 1$, and $p > 0$.

Problem 4.21 If $a_k = \mp 1$, $k = 0, \ldots, n$ and

$$b_k = a_n a_{n-k} + a_{n-1} a_{n-k-1} + \ldots + a_k a_0,$$

is it true that

$$\sum_{k=1}^{n} |b_k|^2 > A n^2,$$

where A is an absolute constant?

If $p(z) = a_0 + a_1 z + \ldots + a_n z^n$, then

$$|p(z)|^2 = n + 2\sum_{k=1}^{n} b_k \cos(k\theta),$$

so that the truth of the above inequality would imply

$$\frac{1}{2\pi}\int_0^{2\pi} \left|p(e^{i\theta})\right|^4 d\theta \geq n^2(1 + A).$$

(F.R. Keogh)

Update 4.21 No progress on this problem has been reported to us.

Problem 4.22 Using the notation of Problem 4.7, if $|z_i| \leq 1$, Clunie and Netanyahu (personal communication) showed that a path exists joining the origin to $|z| = 1$ in $E_f^{(n)}$. What is the shortest length $L_f^{(n)}$ of such a path? Presumably $L_f^{(n)}$ tends to infinity with n, but not too fast.

(P. Erdős)

Update 4.22 No progress on this problem has been reported to us.

Problem 4.23 Some of the Problems 4.7–4.12 extend naturally to higher dimensions. Let x_i be a set of n points in $\mathbb{R}^m$ and let $E_n^{(m)}$ be the set of points for which

$$\prod_{i=1}^{n} |x - x_i| \leq 1.$$

When is the maximum volume of $E_n^{(m)}$ attained and how large can it be? Piranian observed that the ball is not extreme for $m = 3$, $n = 2$. If $E_n^{(m)}$ is connected, can it be covered by a ball of radius 2? For $m = 2$, this was proved by Pommerenke [807].

(P. Erdős)

Update 4.23 No progress on this problem has been reported to us.

Problem 4.24 Let

$$P(z) = \sum_{k=0}^{n} a_k z^k$$

be a self-inversive polynomial, that is, if ζ is a zero of $P(\zeta)$ with multiplicity m, then $1/\zeta$ is also a zero with multiplicity m. Is it true that $w = P(z)$ maps $\mathbb{D}$ onto a domain containing a disc of radius $A = \max_{0 \leq k \leq n} |a_k|$?

(T. Sheil-Small)

Update 4.24 No progress on this problem has been reported to us.

Problem 4.25 Determine

$$\inf \int_{-\pi}^{\pi} \left|1 - e^{i\theta}\right|^{2\lambda} \left|P(e^{i\theta})\right|^2 d\theta, \qquad \lambda > 0,$$

where $P(z)$ ranges over all polynomials with integer coefficients and leading coefficient unity. (The solution would have number-theoretic applications.)

(W.H.J. Fuchs)

Update 4.25 No progress on this problem has been reported to us.

Problem 4.26 Let P_n denote the class of polynomials $p(z)$, $p(0) = 1$, of degree at most n and of positive real part in $\mathbb{D}$. Find

$$\max_{p \in P_n} \int_0^{2\pi} |p(e^{i\theta})|^2 d\theta.$$

(F. Holland)

Update 4.26 Goldstein and McDonald [420] have proved that $\Lambda_n \leq n + 1$ and calculated Λ_2 to Λ_5, where

$$\Lambda_n = \max_{p \in P_n} \sum_0^n |a_\nu|^2.$$

Their research suggests that

$$\frac{\Lambda_n}{n} \to \Lambda, \qquad \text{as } n \to \infty,$$

where $\frac{2}{3} \leq \Lambda \leq 1$.

Problem 4.27 Let $p(x)$ be a real polynomial of degree n in the real variable x such that $p(x) = 0$ has n distinct (real) rational roots. Does there necessarily exist a (real) non-zero number t such that $p(x) - t = 0$ has n distinct (real) rational roots? (Rubel can prove this for $n = 1, 2, 3$.)

(L.A. Rubel)

Update 4.27 Crane (personal communication to Lingham) has given a simple proof using Faltings' theorem showing that there is either a counterexample in degree 6 or in degree 12, but it is not constructive. It seems likely that the answer to Rubel's problem is negative in every degree $n \geq 4$, and it is even possible that for each degree $n \geq 12$, every polynomial satisfying the hypotheses of the problem is a counterexample.

Problem 4.28 Suppose that p is a non-linear polynomial with real coefficients. Show that $p^2(z) + p'(z)$ has non-real zeros. We conjecture that the lower bound for the number of non-real zeros is $\deg(p) - 1$. If p itself has only real zeros, this is proved by Pólya and Szegö [806]. For the origin of this problem, see Problem 2.64.

(S. Hellerstein)

Update 4.28 This has been solved by Sheil-Small [897]. An alternative solution was proposed by Eremenko (unpublished), which is reproduced in a paper by Bergweiler, Eremenko and Langley [115].

Problem 4.29 Yang [1004] claims to prove the following: let $p(z)$, $q(z)$ be monic polynomials such that (i) $p(z) = 0 \Longleftrightarrow q(z) = 0$, and (ii) $p'(z) = 0 \Longleftrightarrow q'(z) = 0$. Then there exist positive integers m, n such that $p(z)^m \equiv q(z)^n$. As pointed out by Rubinstein (personal communication), Yang's proof is incorrect since the inequalities [1004, p. 597] are wrong. The problem then is to settle the Yang conjecture. If we let $\{z_1, z_2, \ldots, z_\nu\}$ be the distinct points at which p (and therefore q) has zeros, then the conjecture is easily established if $\nu \leq 5$, and also in the case when ν is arbitrary and all the points z_i are collinear.

(E.B. Saff)

Update 4.29 A counterexample has been given by Roitman [843].

Problem 4.30 Let $\mathcal{P}$ denote the set of all polynomials of the form

$$p(z) = \prod_{\nu=1}^{n}(z - \zeta_\nu),$$

where $n \geq 2$ and $|\zeta_\nu| \leq 1, \nu = 1, 2, \ldots, n$. The Sendov conjecture (see Problem 4.5) states: if $p(z) \in \mathcal{P}$ then each disc

$$\{z : |z - \zeta_\nu| \leq 1\}, \qquad \nu = 1, 2, \ldots, n,$$

contains at least one zero of $p'(z)$. Schmeisser [882] proved this conjecture for certain subsets of $\mathcal{P}$. In all but two of these special cases, the proof shows that the following stronger result is true:

If ζ is an arbitrary point of the convex hull of the zeros of $p(z)$, then the disc $\{z : |z - \zeta| \leq 1$ contains at least one zero of $p'(z)\}$.

We ask:

(a) Is this stronger result true for all $p(z)$ in $\mathcal{P}_1$, where $\mathcal{P}_1$ is the subset of all polynomials in $\mathcal{P}$ which vanish at 0?
(b) Is this stronger result true for all $p(z)$ in $\mathcal{P}_2$, where $\mathcal{P}_2$ is the subset of all polynomials in $\mathcal{P}$ which are of the form $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$, $a_\nu \leq 0, \nu = 0, 1, \ldots, n-1$?

(G. Schmeisser)

Update 4.30 The relationship between this problem and Problem 4.5 is that in the latter, the discs to contain a critical point are centred at the zeros, while in this problem they are centred at any point of the convex hull of the zeros (including the zeros themselves as a special case). No progress on this problem has been reported to us.

Problem 4.31 Erdős and Newman conjectured that if

$$f(z) = \sum_{k=0}^{n} a_k z^k, \qquad |a_k| = 1, \qquad 0 \le k \le n, \tag{4.5}$$

then there is an absolute constant c such that

$$\max_{|z|=1} |f(z)| > (1+c)n^{1/2}. \tag{4.6}$$

The weaker form of our conjecture stated that (4.6) holds if we assume $a_k = \pm 1$ (Problem 4.13). A stronger form would be that (4.6) holds even if $f(z) = \sum_{k=1}^{n} a_{n_k} z^{n_k}$, n_k natural numbers, $|a_{n_k}| = 1$. However, (4.6) was disproved by Kahane (no citation given). In fact, he showed that, given a positive ε, there are polynomials of the form (4.5) for which

$$\max_{|z|=1} |f(z)| < n^{1/2} + O(n^{(3/10)+\varepsilon}) \qquad \text{as } n \to \infty. \tag{4.7}$$

Show that $n^{(3/10)+\varepsilon}$ cannot be replaced by n^{ε} in (4.7). Is there an n-th degree polynomial of the form (4.5) for which

$$\min_{|z|=1} |f(z)| > (1-\varepsilon)n^{1/2} \tag{4.8}$$

for every positive ε if $n > n_0(\varepsilon)$? Perhaps there is an n-th degree polynomial of the form (4.5) for which, for all z, $|z| = 1$,

$$(1-\varepsilon)n^{1/2} < |f(z)| < (1+\varepsilon)n^{1/2}. \tag{4.9}$$

In fact, (4.9) could possibly hold with $n^{1/2} + O(1)$ on the left and right. It would be worthwhile to determine if (4.6) holds for $a_k = \pm 1$ and for $a_{n_k} = \pm 1$.

(P. Erdős and D. Newman)

Update 4.31 No progress on this problem has been reported to us.

4.3 New Problems

Problem 4.32 The effect on the critical points of a real polynomial under motion of the real roots (also known as *root dragging*) has been studied; see, for example, Anderson [32], Boelkins, From and Kolins [133], and Frayer and Swenson [355]. Nothing seems to be known about root dragging for arbitrary complex polynomials, beyond that the critical points are continuous as a function of the zeros. One observes that when the roots of $z^n - 1$ are varied randomly by a distance at most some small positive ε, the critical points 'explode' away from zero by a distance much more than ε. The conjecture is that if the zeros of a complex polynomial of degree n are each varied by at most some positive ε, the corresponding change in the critical points is at most $\sim\varepsilon^{1/(n-1)}$.

(T. Richards)

Problem 4.33 If all of the zeros of a complex polynomial p of degree n are dragged by the same amount in the same direction (precomposing $p(z)$ by $z - c$ for example, for some constant c), then the critical points will also be translated in the same way. What if all zeros of p are dragged in the same direction, increasing real parts, constant imaginary parts for example, but by differing amounts? What can be said about the effect on the critical points? Is there a bound on the number of critical points which will have decreasing real part as a result?

(T. Richards)

Problem 4.34 Give a geometric interpretation of the effect of root dragging for a complex polynomial.

(T. Richards)

Problem 4.35 Let $h(z) := f(z) + g(\overline{z})$ be a harmonic polynomial, where f and g are analytic polynomials of degrees n and m respectively, $n > m$. How many zeros can h have?

It is known that if $N = \#\{z : h(z) = 0\}$, then $N \leq n^2$; see Wilmshurst [991]. If $m = 1$, then $N \leq 3n - 2$ (see Khavinson and Świątek [602]) and this is sharp; see Geyer [393]. Not much is known when $m = 2$; see [595]. See Khavinson and Neumann [598] for the history of the problem, more references, and the relation to problems of gravitational lensing in astrophysics.

(D. Khavinson)

Problem 4.36 Does there exist an increasing function $f : [1/2, 1] \to (0, 1]$ with the following property: if K is a compact convex plane set and U is an open neighbourhood of K, and if p_n is a sequence of polynomials whose degrees d_n tend to infinity with n, such that each p_n has at least $(t - o(1))d_n$ zeros in K, then p_n' has at least $(f(t) - o(1))(d_n - 1)$ zeros in U (in both cases, counting multiplicities)? A result and an example of Totik [954] show that $f(1) = 1$ and that t needs to be at least $1/2$.

(E. Crane)

Chapter 5
Functions in the Unit Disc

5.1 Preface by L.R. Sons

If you are looking for a problem to captivate you, this chapter should meet your needs. While some problems from the "old" lists have been solved or partially resolved, there still is a treasure trove of problems for those willing to be bounded by the unit disc. Among them some interesting coefficient problems and some problems about symmetrisation are looking for solutions. Will you be snared by a conjecture of Littlewood?

Do you like to construct functions with peculiar properties—if so, problems posed by K.F. Barth and J.G. Clunie may spark a fire for you. Concerning the behaviour at boundary points for an analytic function in the unit disc, how much can "dense" be improved in the well-known theorem of Plessner?

And just what IS Bloch's constant? Nothing pretty it seems. And Landau's constant is also still waiting to be determined. Further, if f is univalent in the unit disc, up for grabs is the schlicht Bloch's constant and analogous considerations for other more specialised functions f.

Perhaps your attention can be drawn to describing coefficient multipliers from H^p to H^p for $0 < p < 1$ or from S to S where S is the class of univalent functions in the unit disc.

Then there's a problem which should be understood by those having completed a good first course in function theory: Determine the Laurent coefficient bodies for analytic functions taking values of modulus at most 1 in a given annulus $A(r) = \{z : r < |z| < 1\}$.

As a child in an old-fashioned candy store is caught up with the array of types of candies, wanting to sample many, so the interested researcher can be caught up here with an array of problems to try to solve. And the question arises, which one will you work on first?

For those (like me) fascinated by analytic functions f in the disc defined by power series about zero with Hadamard gaps, see if such functions f must have a radial limit,

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9_5

what kinds of subsets S of the disc on which f is bounded imply f is bounded in the whole disc, and does the unbounded nature of the function's Nevanlinna characteristic imply Nevanlinna deficiency zero for each complex number?

ALL these and many more problems are ready for the researcher to attack in order to provide solutions, to make progress towards solutions, and to enjoy the journey with the mathematics involved. Happy reading and researching!!!

5.2 Progress on Previous Problems

Notation In this chapter, $\mathbb{D}$ denotes the open unit disc $|z| < 1$, $\mathbb{T}$ denotes the circumference $|z| = 1$ and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$. There are a number of problems concerned with the growth of the coefficients, means and maximum modulus of functions in $\mathbb{D}$ omitting certain values. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an analytic function in $\mathbb{D}$ with coefficients a_n, and suppose that

$$f(z) \neq w_k,$$

where w_k is complex, $r_k = |w_k|$ is monotonic increasing and $r_k \to \infty$, as $k \to \infty$.

Write

$$M(r, f) = \sup_{|z|=r} |f(z)|,$$

$$I_\lambda(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\lambda d\theta.$$

While the theory for $M(r, f)$ is well-worked out (see for example Cartwright [198] and Hayman [481, 482]) very little that is not trivial is known about $I_\lambda(r, f)$ and the coefficients a_n. If

$$r_{k+1} \leq C r_k,$$

it is known (see Littlewood [673]) that

$$M(r, f) = O(1 - r)^{-A(C)}$$

where $A(C)$ depends on C only. If

$$\frac{r_{n+1}}{r_n} \to 1, \tag{5.1}$$

Cartwright [198] proved that

$$M(r, f) = O(1-r)^{-2-\varepsilon}, \quad \text{as } r \to 1, \tag{5.2}$$

for every positive ε. If

$$\sum \left(\log \frac{r_{n+1}}{r_n} \right)^2 < +\infty, \tag{5.3}$$

then Hayman [482] proved that

$$M(r, f) = O(1-r)^{-2}. \tag{5.4}$$

These two results are essentially best possible.

Problem 5.1 Is it true that (5.1) implies

$$I_1(r, f) = O(1-r)^{-1-\varepsilon}$$

and

$$|a_n| = O(n^{1+\varepsilon}) \, ?$$

Update 5.1 For Problems 5.1–5.3, substantial progress has been made by Pommerenke [821]. He has shown that if $f(z)$ satisfies

$$\left| \frac{f'(z)}{f(z)} \right| \le \frac{\alpha}{1-|z|}, \quad r_0 < |z| < 1,$$

then

$$I_1(r, f) = O((1-r)^{-\lambda(\alpha)}), \quad r \to 1,$$

and hence

$$a_n = O(n^{\lambda(\alpha)}),$$

where $\lambda(\alpha) = \frac{1}{2}(\sqrt{1+4\alpha^2} - 1)$. In particular, if f is weakly univalent (see Problem 5.3) we may take $\alpha = 2 + \varepsilon$, $\lambda(\alpha) = \frac{1}{2}(\sqrt{17} - 1) + \varepsilon = 1.562\ldots + \varepsilon$.

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\mathbb{D}$ and $f(z) \neq w_n$ there, where w_n is a sequence of complex values such that

$$w_n \to \infty, \quad \left| \frac{w_{n+1}}{w_n} \right| \to 1, \quad \text{as } n \to \infty.$$

Then Cartwright [198] has shown that

$$M(r, f) = O(1-r)^{-2-\varepsilon}, \qquad \text{as } r \to \infty$$

for every positive ε. Baernstein and Rochberg [60] have obtained the analogous results for the means,

$$I_\lambda(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\lambda d\theta,$$

namely

$$I_\lambda(r, f) = O(1-r)^{1-2\lambda-\varepsilon}, \qquad \lambda \geq \frac{1}{2},$$
$$I_\lambda(r, f) = O(1-r)^{-\varepsilon}, \qquad \lambda < \frac{1}{2},$$

as $r \to 1$. From the case $\lambda = 1$, they deduce that

$$a_n = O(n^{1+\varepsilon}).$$

Baernstein and Rochberg [60] also obtain the analogous result when the equations $f(z) = w_n$ have at most p roots for some positive integer p.

It is natural to ask whether, under stronger hypotheses on the w_n, it is possible to get rid of the ε in the above results. Hansen [472], Baernstein [56], and Hayman and Weitsman [522] each did so on the hypothesis that $f(z)$ fails to take some value on every circle $|w| = R$ (or every circle apart from a set of R of finite logarithmic measure). However, it ought to be possible to obtain the same conclusion under weaker hypotheses.

Problem 5.2 Is it true that (5.3) implies that

$$I_1(r, f) = O(1-r)^{-1} \tag{5.5}$$

and

$$|a_n| = O(n)\ ? \tag{5.6}$$

Update 5.2 See Update 5.1.

Problem 5.3 An even stronger hypothesis than (5.3) is that $f(z)$ is *weakly univalent* (see Hayman [483]) that is, for every r with $0 < r < \infty$, either $f(z)$ assumes every value on $|w| = r$ exactly once, or there exists a complex $w = w_r$ such that $|w_r| = r$ and $f(z) \neq w_r$. Even with this assumption, nothing stronger than the results

$$I_\lambda(r, f) = O(1-r)^{-2},$$

$$|a_n| = O(n^2)$$

are known (which are trivial consequences of (5.4)). It would be interesting to obtain some sharpening of these results even if it is not possible to deduce the full strength of (5.5), (5.6).

Update 5.3 See Update 5.1.

Problem 5.4 If the sequence w_n satisfies

$$\arg w_n = O\left(|w_n|^{-\frac{1}{2}}\right) \tag{5.7}$$

and

$$|w_{n+1} - w_n| = O(|w_n|^{\frac{1}{2}}) \tag{5.8}$$

then it is known (see Hayman [486]) that (5.5) and hence (5.6) hold. It is interesting to ask whether the method will yield the same conclusions under somewhat weaker hypotheses, such as for instance, replacing the index $\frac{1}{2}$ in (5.7) and (5.8) by a smaller positive number.

Update 5.4 If the sequence w_n satisfies (5.7) and $|w_{n+1} - w_n| = O\left(|w_n|^{\lambda}\right)$ where $\lambda < 1$ then Higginson (unpublished) has shown that this is indeed possible. Hayman [486] had previously obtained the corresponding result when $\lambda = \frac{1}{2}$. Higginson [546] has also shown that if

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots$$

is *weakly* p*-valent* in $\mathbb{D}$, that is, if $f(z)$ either assumes, for every R, all values on $|w| = R$ exactly p times, or assumes at least one such value less than p times, then

$$\limsup_{n\to\infty} \frac{|a_n|}{n^{2p-1}} < \frac{1}{\Gamma(2p)} \tag{5.9}$$

unless $f(z) \equiv z^p(1 - z\mathrm{e}^{i\theta})^{-2p}$. The sharp bounds for the means of this class of functions had previously been obtained by Baernstein [56]. In contrast to the p-valent or mean p-valent case, it is not true that the upper limit in (5.9) exists as a limit in general. It does so if and only if there is exactly one radius of greatest growth.

Problem 5.5 If $f(z) = u + iv$ assumes only values in the right half-plane, then subordination shows that

$$a_n = O(1). \tag{5.10}$$

It is of interest to ask what other hypotheses on the values assumed by $f(z)$ result in (5.10). Let $d(r)$ be the radius of the largest disc whose centre lies on $|w| = r$, and every value of whose interior is assumed by $f(z)$. If $d(r) \le d$, then it is shown by Hayman [486] that (5.10) holds. It is fair to ask whether this conclusion still holds if $d(r) \to \infty$ sufficiently slowly.

Update 5.5 For Problems 5.5–5.7, let D be a domain and let $w = f(z) = \sum_0^\infty a_n z^n$ be a function in the unit disc $\mathbb{D}$ with values in D. It was asked under what conditions on D we can conclude that

(a) $a_n = o(1)$

and

(b) $a_n = O(1)$.

Fernández [342] has shown that if the complement of D has capacity zero, then (a) is never true, and (b) holds if and only if D is a *Bloch domain*, that is, D does not contain arbitrarily large discs. Pommerenke [820] has shown that if the complement of D has positive capacity and D is a Bloch domain, then (a) holds.

Problem 5.6 It is known that there exist functions which fail to take any of the values $2\pi ik$, $-\infty < k < +\infty$, and which do not satisfy (5.10), and in fact $|a_n| \le \log\log n$ (see Littlewood [675, p. 205], and Hayman [486]). However (5.10) holds if $f(z)$ omits all but a finite interval of the imaginary axis (again by subordination). This suggests that (5.10) might still hold if the omitted values w_n cluster near ∞ sufficiently close to the imaginary axis.

Update 5.6 See Update 5.5.

Problem 5.7 If c_k is a sequence of positive numbers such that

$$\sum c_k = S < +\infty,$$

and n_k is an arbitrary sequence of positive integers, then

$$f(z) = \sum_{n=0}^{\infty} c_k z^{n_k} = \sum_{n=0}^{\infty} a_n z^n$$

is bounded in $\mathbb{D}$, and so takes no values outside a fixed disc. This shows that no conditions on the omitted values w can imply more than

$$a_n = o(1). \tag{5.11}$$

Clearly (5.11) holds if $f(z)$ is bounded, since then

$$I_2(r, f) = \Big(\sum_{n=0}^{\infty} |a_n|^2\Big)^{\frac{1}{2}} < +\infty.$$

It would be of interest to obtain a non-trivial condition on the values omitted by $f(z)$, which would imply (5.11). Such a condition might be $d(r) \to 0$, where $d(r)$ is defined as in Problem 5.5.

Update 5.7 The work of Hansen [472] and of Hayman and Weitsman [522] also leads to conditions on a set of omitted values such that the corresponding function satisfies

$$a_n \to 0. \tag{5.12}$$

However all our conditions require the set of omitted values to have positive capacity. The biggest functions omitting a set E of complex values, from the point of view of the means and maximum modulus, are the functions F which map $\mathbb{D}$ onto the infinite covering surface over the complement of E. This led Pommerenke to ask (see Problem 5.33) whether these functions $F(z)$ satisfy (5.12). Patterson, Pommerenke and Hayman [519] have been able to give an affirmative answer to this question, by showing that (5.12) holds for $F(z)$ if E is a lattice or if E is thick at ∞ in the sense that the distance $d(w)$ of any point in the w-plane from E tends to zero as $w \to \infty$. This answers Problem 5.7 for the functions $F(z)$. However the case of the subordinate functions $f(z)$ remains open. See Update 5.5 also.

Problem 5.8 Suppose that $f(z) = z + a_2z^2 + \ldots$ is analytic in $\mathbb{D}$. Then $f(z)$ maps some sub-domain of $\mathbb{D}$ univalently into a disc of radius at least B, where B is Bloch's constant. What is the value of B? The best results known are $B \geq \sqrt{3}/4 > 0.433$, due to Ahlfors [12], and $B < 0.472$ due to Ahlfors and Grunsky [14]. The upper bound is conjectured to be the right one. Heins [529] has shown that $B > \sqrt{3}/4$.

Update 5.8 Turán [960] has shown that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is univalent in $\mathbb{D}$ with $\sum_{p}^{2p} |a_n|^2 \geq 1$ for some p, then the corresponding Bloch constant is at least $(32e^2)^{-1}$. An analogous result holds with $2p$ replaced by Cp, where C is any constant greater than 1, but not if C is allowed to tend to ∞ with p, however slowly. This latter result is due to Petruska [798]. Bonk [139] has proved that $B > \sqrt{3}/4 + 10^{-14}$. The best current estimate is $B \geq \sqrt{3}/4 + 2 \times 10^{-4}$ by Chen and Gauthier [205].

Problem 5.9 With the hypotheses of Problem 5.8, it follows that $f(z)$ assumes all values in some disc of radius L, $L \geq B$. What is the value of L? The best lower bound for the Landau constant L is $L \geq \frac{1}{2}$, due to Ahlfors [12].

Update 5.9 An upper bound for L is $L < 0.54326$, which has been obtained by Rademacher [830]. Rademacher mentions that this bound has also been found, but not published, by R.M. Robinson.

Problem 5.10 If, in addition, $f(z)$ is univalent in $\mathbb{D}$, the conclusions of Problems 5.8 and 5.9 follow with a constant S, $S \geq L$, known as the *schlicht Bloch's constant*. What is the value of S? We may also ask the same question when, in addition, $f(z)$ is starlike, thus obtaining a still larger constant S_1. If $f(z)$ is convex, the correct value of the constant is $\pi/4$, attained for $f(z) = \frac{1}{2}\log\frac{1+z}{1-z}$, which maps $\mathbb{D}$ onto the strip $|\mathrm{Im}\, z| < \pi/4$.

Update 5.10 No progress on this problem has been reported to us.

Various conditions on functions $f(z)$ in the plane result in these functions being constant. This is the case, for instance, if $f(z)$ is analytic and bounded (Liouville's theorem) or more generally, if $f(z)$ is meromorphic and fails to take three fixed values (Picard's theorem), or if $f(z)$ is analytic, $f(z) \neq 0$, $f'(z) \neq 1$ (see Hayman

[490]). Classes of functions satisfying one of the above conditions in $\mathbb{D}$ form normal families, the results being due to Montel [744] and Miranda [740], respectively.

This suggests investigating other conditions for functions in the unit disc, where the corresponding conditions for functions in the plane lead to constants.

The interested reader is directed to a more recent result in this area, namely Zalcman's Lemma [1015], as stated by Schwick [885]: Let $\mathcal{F}$ be a family of meromorphic functions on $\mathbb{D}$ which are not normal at 0. Then there exist sequences f_n in $\mathcal{F}$, z_n, ρ_n, and a non-constant function f meromorphic in the plane such that

$$f_n(z_n + \rho_n z) \to f(z),$$

locally and uniformly (in the spherical sense) in $\mathbb{C}$, where $z_n \to 0$ and $\rho_n \to 0$.

By means of this lemma, a positive answer to several of the questions in this subsection can be given.

Do the following classes form normal families, possibly after suitable normalisations?

Problem 5.11 $f(z)$ meromorphic in $\mathbb{D}$, $f(z) \neq 0$, $f^{(l)}(z) \neq 1$, where $l \geq 1$.

Update 5.11 This has been shown to be the case by Ku Yung Hsing [643].

Problem 5.12 $f(z)$ analytic in $\mathbb{D}$, $f'(z)f(z)^n \neq 1$, where $n \geq 1$.

Update 5.12 This has been answered affirmatively for $n \geq 2$ by Yang and Zhang [1006]. The case $n = 1$ has also been answered affirmatively by Oshkin [784].

Problem 5.13 $f(z)$ meromorphic in $\mathbb{D}$, $f'(z)f(z)^n \neq 1$, for $n \geq 3$.

Update 5.13 This is solved for $n \geq 2$ by Pang [789]; and for $n \geq 1$ this follows from combining the results of Pang [790] and Bergweiler and Eremenko [110]. See also Wang and Fang [979].

Problem 5.14 $f' - f^n \neq a$, where a is some complex number, and $n \geq 5$ if f is meromorphic, $n \geq 3$ if f is entire.

The corresponding results for functions in the plane are proved by Hayman [490], except for the case $n = 1$ of Problem 5.12, which is a result of Clunie [210].

Update 5.14 Eremenko writes that this problem with $n \geq 3$ is equivalent to Problem 5.13 with $n \geq 1$.

Notation Let D be a domain in the plane. The *circularly symmetrised domain* D^* of D is defined as follows: For every r, with $0 < r < \infty$, the intersection of D^* and the circle $|z| = r$ is

(i) the whole circle $|z| = r$, if D contains the whole circle $|z| = r$;
(ii) null if D does not meet $|z| = r$;
(iii) the arc $|\arg z| < \frac{1}{2}\ell(r)$ if neither (a) nor (b) holds, but D meets $|z| = r$ in a set of arcs of total length $r\ell(r)$.

In addition D^* contains the origin if and only if D does.

Suppose now that a_0 is a point of D on the positive real axis. Let

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n$$

be analytic in $\mathbb{D}$, and assume only values lying in D, and let

$$f^*(z) = a_0 + \sum_{n=1}^{\infty} a_n^* z^n$$

map $\mathbb{D}$ onto the infinite covering surface over D^*. The following facts are known:

(a) If D^* is simply connected, so that $f^*(z)$ is univalent, then $|a_1| \leq |a_1^*|$. From this, it is not difficult to deduce that:

(b) If $m_0(r, f) = \inf_{|z|=r} |f(z)|$, and $M(r, f) = \sup_{|z|<r} |f(z)|$, then

$$\mu(r, f^*) \leq m_0(r, f) \leq M(r, f) \leq M(r, f^*),$$

and, in special cases, rather more (see Hayman [513, Chap. 4]).

This leads to the following questions:

Problem 5.15 Is it possible to remove the restriction that D^* is simply connected in (a) and (b) above? It might be possible to start with the case when D and D^* are both doubly-connected.

Update 5.15 Another way to phrase this question is to ask whether the Poincaré metric is decreased by symmetrisation. This result is known to be true if the symmetrised domain is simply connected. Lai [648] and Hempel [536] have independently proved that this is indeed the case when the complement of the domain D consists of the origin and one other point. This leads to sharp forms of the theorems of Schottky and Landau. The result has also been proved by Weitsman [984] when the domain is already symmetrical about some ray.

Weitsman [985] has proved in general that the Poincaré metric of a domain D is decreased by symmetrisation. It follows that the maximum modulus of a function f mapping $\mathbb{D}$ into D is dominated by that of a function F mapping $\mathbb{D}$ onto the universal cover surface over D^*. The corresponding conclusion for the means of f and F is still open, unless D^* is simply connected, in which case it has been proved by Baernstein [56].

Problem 5.16 Do corresponding results to Problem 5.15(b) apply to the means

$$I_\lambda(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^\lambda d\theta\right\}^{1/\lambda}, \quad 0 < \lambda < \infty,$$

or the Nevanlinna characteristic

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta \,?$$

This is known to be the case when D is already symmetrical (but possibly multiply-connected) so that $D = D^*$, as a consequence of the theory of subordination (see, for example, Littlewood [675, Theorem 210, p. 164]).

Update 5.16 These results have been proved by Baernstein [54] if D (and so D^*) is simply-connected. See also Baernstein [55].

Problem 5.17 Let $D = D_0$ be a domain, $g(z, a_0)$ be the Green's function of D with respect to a point a_0 on the positive real axis, and let D_λ be the part of D where $g > \lambda$ for $0 < \lambda < \infty$. Is it true, at least in some simple cases, that $(D_\lambda)^* \subset (D^*_\lambda)$? The cases where D^* consists of the plane or the unit disc cut along the negative real axis are of particular interest. A positive answer to this problem for simply-connected domains D would lead to a positive answer of Problem 5.16 for the same class of domains using a formula of Hardy–Stein–Spencer; see Hayman [513, Chap. 3].

With the general notation of the introduction above, when can we assert that $|a_n| \le |a^*_n|$ for general n? This is true, for instance, when D^* is the plane cut along the negative real axis, so that we obtain the Littlewood conjecture that $|a_n| \le 4|a_0|n$ for non-zero univalent f. In fact, $|a_1| \le 4|a_0|$ in this case, by symmetrisation and subordination, and $|a_n| \le n|a_1|$ by de Branges' proof of the Bieberbach conjecture (see Update 6.1).

If D is convex and $D = D^*$, the exact bound for $|a_n|$ is $|a_1|$, but $|a^*_n| \le |a_1|$ in general; see Hayman [513, Theorem 1.7].

Update 5.17 In the original formulation of this question, Littlewood's conjecture was left open. There has been no further progress on this problem reported to us.

Problem 5.18 Let $f(z) = \lambda + a_1 z + \ldots$ be analytic in $\mathbb{D}$, where $0 < \lambda < 1$. Find the best constant $B(\lambda)$ such that if

$$F(r) = \lambda + |a_1|r + |a_2|r^2 + \ldots$$

then

$$F[B(\lambda)] \le 1$$

for all f. It is known (see Bombieri [136]) that:

$$B(\lambda) = \begin{cases} (1+2\lambda)^{-1} & \text{if } \frac{1}{2} \le \lambda < 1, \\ 1/\sqrt{2} & \text{if } \lambda = 0, \end{cases}$$

and that

$$B(\lambda) > [(1-\lambda)/2]^{\frac{1}{2}}, \quad \text{if } 0 < \lambda < \frac{1}{2}.$$

Update 5.18 There has been lots of activity around this problem stemming from a theorem of Bohr [134] which states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on the unit disc such that $|f(z)| \leq 1$ for each z in the disc, then $\sum_{n=0}^{\infty} |a_n z^n| \leq 1$ when $|z| \leq \frac{1}{3}$, and moreover the radius $\frac{1}{3}$ is best possible. See Boas and Khavinson [131]; Bénéteau, Dahlner and Khavinson [104]; Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip [240]; and many more references found on MathSciNet.

Problem 5.19 A function meromorphic in $\mathbb{D}$ which has no asymptotic value assumes every value infinitely often in the disc. Every point of the circumference $\mathbb{T}$ is a Picard point for such a function, that is, a point such that all except perhaps two values are taken in every neighbourhood. Functions with no exceptional values in the global sense are known. Can locally exceptional values occur? (See Cartwright and Collingwood [199, 221].)

(E.F. Collingwood)

Update 5.19 A strongly positive answer has been given by Eremenko (unpublished) who noted that one can construct a function for which the only limit point of poles and zeros is the point $z = 1$, by slightly modifying an example of Barth and Schneider [90].

Problem 5.20 Plessner [803] proves after Privaloff that if f is analytic in $\mathbb{D}$, almost all points P of the boundary are of two kinds, either

(a) f tends to a finite limit as $z \to P$ in any Stolz angle S, lying in $\mathbb{D}$, or
(b) as $z \to P$ in any S, f takes (infinitely) often all values of a dense set.

Can 'dense set' be replaced by anything bigger here; for example, the complement of a set of measure zero?

(E.F. Collingwood)

Update 5.20 Nicolau writes that Problem 5.57 is a strong version of this problem.

Problem 5.21 Corresponding to each function f analytic in $\mathbb{D}$, and each value w with $|w| < 1$, write

$$f_w(z) = f\left(\frac{z - w}{1 - wz}\right) = \sum_{n=0}^{\infty} a_n^{(w)} z^n,$$

$$\|f_w\| = \sum_{n=0}^{\infty} |a_n^{(w)}|,$$

and let W_f be the set of all values w with $|w| < 1$ for which

$$\|f_w\| < \infty.$$

Since $a_n^{(w)}$ is a continuous function of w, it follows that W_f is a set of type F_σ. What more can be said? For example, if W_f is everywhere dense in D (or uncountable, or of positive measure), is W_f the unit disc? It is known that W_f may be a proper non-empty subset of D. Is W_f either empty or all of D if f is univalent?

Update 5.21 No progress on this problem has been reported to us.

Problem 5.22 Let H^p be the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in $\mathbb{D}$, and such that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^p \, d\theta$$

remains bounded as $r \to 1$. We define H^∞ to be the class of bounded functions in $\mathbb{D}$. For $0 < p < 1$ describe the coefficient multipliers from H^p to H^p. That is, for each such p describe the sequences λ_n such that

$$\sum \lambda_n a_n z^n \in H^p \quad \text{whenever } \sum a_n z^n \in H^p.$$

(P.L. Duren)

Update 5.22 No progress on this problem has been reported to us.

Problem 5.23 Describe similarly the coefficient multipliers from S to S, where S is the class of functions $\sum_{n=1}^{\infty} a_n z^n$ univalent in $\mathbb{D}$, either

(a) with the normalisation $a_1 = 1$, or
(b) generally.
(c) What are the multipliers of the space of close-to-convex functions into itself?
(d) What are the multipliers of S into the class C of convex functions?
(e) What are the multipliers from the class N of functions of bounded characteristic into itself? The analogous problem for the class N^+ may be more tractable. (The definition of N^+ is too lengthy for this work, but the reader is directed to Duren [273, p. 25] for more details.)

Ruscheweyh and Sheil-Small in solving Problem 6.9 have shown that (λ_n) is a multiplier sequence from C into itself if and only if $\sum \lambda_n z^n \in C$. In general, one can obtain only some sufficient conditions. Thus in most cases $f(z) = \sum a_n z^n$ belongs to a class A if a_n is sufficiently small, and conversely if $f \in A$, then a_n cannot be too big. For example, $\sum_2^{\infty} n|a_n| \le 1$ is a sufficient condition for $f(z)$ in S, and $|a_n| \le n\sqrt{7/6}$ is a necessary condition; see FitzGerald [346]. Similarly if $\sum_1^{\infty} |a_n| < \infty$, then $f(z)$ is continuous in $\overline{\mathbb{D}}$, and so belongs to H^p for every positive p and to N; whilst if $f \in N$, then $|a_n| \le \exp(cn^{\frac{1}{2}})$ for some constant c. Again, if $f(z)$ belongs to one of the above classes, then so does $\frac{1}{t} f(tz)$ for $0 < t < 1$, so that the sequence (t^{n-1}) is a multiplier sequence. In other cases, negative results are known. Thus Frostman [357] showed that (n) is not a multiplier sequence from N to N, and Duren [273] showed that $\left(\frac{1}{n+1}\right)$ is not such a sequence either.

(P.L. Duren, except for (e), which is due to A.L. Shields)

Update 5.23(a) Sheil-Small observes that $\sum_{n=1}^{\infty} b_n a_n z^n$ is close-to-convex for every function $\sum_{n=1}^{\infty} a_n z^n$ which is close-to-convex if and only if $\sum_{n=1}^{\infty} b_n z^n$ is convex in $\mathbb{D}$. The sufficiency was shown by Ruscheweyh and Sheil-Small [867, 868] while Sheil-Small notes that the necessity is a simple application of the duality principle.

Problem 5.24 Is the intersection of two finitely generated ideals in H^∞ finitely generated?

(L.A. Rubel)

Update 5.24 For $H^\infty(\mathbb{D})$ an affirmative answer was given by McVoy and Rubel [714]. This was extended to any finitely connected domain by Barnard (no citation given). Amar (no citation given) has given a negative answer in $H^\infty(D^n)$ or $H^\infty(B^n)$ where D^n is the polydisc and B^n is the unit ball, for $n \geq 3$. The case $n = 2$ is still open.

Problem 5.25 Let W^+ be the Banach algebra of power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ absolutely convergent in $|z| \leq 1$, with $\|f\| = \sum_{n=0}^{\infty} |a_n|$. Which functions generate W^+? More precisely, for which functions f is it true that the polynomials in f are dense in W^+?

It is clear that a necessary condition is that f be univalent in $\overline{\mathbb{D}}$. Newman [760] has shown that if in addition, $f' \in H^1$ then f generates W^+. Hedberg [525] and Lisin [672] have shown (independently) that if f is univalent and $\sum n|a_n|^2(\log n)^{1+\varepsilon} < \infty$ for some positive ε, then f generates W^+. Neither Newman's condition nor Hedberg–Lisin's condition implies the other. Is univalence enough?

(L. Zalcman)

Update 5.25 No progress on this problem has been reported to us.

Problem 5.26 Let B be the *Bergman space of square integrable functions* in $\mathbb{D}$, that is, those functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for which $\sum_{n=1}^{\infty} n^{-1}|a_n|^2 < \infty$. A subspace S is said to be *invariant* if $zf \in S$ whenever $f \in S$. What are the invariant subspaces of B? The corresponding problem for H^2 was solved by Beurling [121] and uses the inner-outer factorisation of H^p functions, a tool unavailable in the present context.

(L. Zalcman)

Update 5.26 D. Khavinson writes that a lot of progress has been achieved, starting with the development of the theory of contractive divisors which originated with Hedenmalm [526]. An equivalent form of Beurling's theorem holds for zero invariant subspaces in all Bergman spaces A^p, $p \geq 1$, where a *zero invariant subspace* N is defined as

$$\{f \in A^p : f(z_k) = 0, k = 1, 2, 3..., z_k \text{ is a zero set for the Bergman space } A^p\},$$

which is equivalent to

$$\{f : f \text{ analytic in } \mathbb{D} \text{ with } |f| \in L^p(\mathrm{d}A), \mathrm{d}A \text{ is the area measure}\}.$$

See [267, 268] for more details. A general extension of Beurling's theorem for all invariant subspaces, not only those of index 1 (like zero subspaces), is found in [23] with further details in [270, 527]. Yet, due to the complexity of Bergman spaces, this extended Beurling's theorem does not possess the simplicity and elegance of its Hardy space predecessor.

Problem 5.27 (*The corona conjecture*) Let D be an arbitrary domain in the plane that supports non-constant bounded analytic functions. Suppose that $f_1(z), \dots, f_n(z)$ are bounded and analytic in D and satisfy

$$\sum_{\nu=1}^{n} |f_\nu(z)| \geq \delta > 0$$

in D. Can one find bounded analytic functions $g_\nu(z)$ in D such that

$$\sum_{\nu=1}^{n} f_\nu(z) g_\nu(z) \equiv 1$$

in D?

When D is a disc, Carleson [190] proved that the answer is 'yes', and the result extends to finitely connected domains. The result is also known to be true for certain infinitely connected domains (see Behrens [99], Gamelin [367]), but false for general Riemann surfaces of infinite genus (B. Cole, unpublished). Presumably the answer for the general plane domain is negative. Proofs of all positive results depend on Carleson's theorem.

(L. Zalcman)

Update 5.27 A positive answer for certain classes of domains has been given by Cole [368, Chap. 4], Carleson [192] and Jones [580]. It has also been proved for domains D whose complements lie on the real axis, by Garnett and Jones [381].

Problem 5.28 Let f be continuous in $\overline{\mathbb{D}}$ and analytic in $\mathbb{D}$. Let

$$\begin{aligned} \omega(f, \delta) &= \sup |f(z) - f(w)|, \quad \text{for } |z - w| \leq \delta \text{ and } z, w \in \mathbb{D}, \\ \tilde{\omega}(f, \delta) &= \sup |f(z) - f(w)|, \quad \text{for } |z - w| \leq \delta \text{ and } |z| = |w| = 1. \end{aligned}$$

Is it true that

$$\lim_{\delta \to 0} \frac{\omega(f, \delta)}{\tilde{\omega}(f, \delta)} = 1?$$

Rubel, Shields and Taylor (unpublished) have shown that there is an absolute constant C such that

$$\omega(f, \delta) \leq C \tilde{\omega}(f, \delta),$$

but that one may not take $C = 1$.

(L.A. Rubel, A.L. Shields)

Update 5.28 A negative answer was given by Rubel, Shields and Taylor [857].

Problem 5.29 A G_δ *set* is a subset of a topological space that is a countable intersection of open sets. Let F be a G_δ set of measure zero on $|z| = 1$. Then does there exist an f in H^∞, $f \neq 0$, such that $f = 0$ on F, and every point of the unit circle is a Fatou point of f?

(L.A. Rubel)

Update 5.29 An affirmative solution has been given by Danielyan [233]. Danielyan writes that [234, Theorem 2] improves the solution further, by showing that at the points of the set F the desirable bounded analytic function not only has radial (and angular), but also unrestricted, vanishing limits. This is implied by the second conclusion of [234, Theorem 2] in the following way: Rubel's Problem 5.29 is about extending Fatou's interpolation theorem [617, p. 30] to the case when the closed set is replaced by the G_δ set. Fatou's interpolation function in fact belongs to the disc algebra, while in the case of Problem 5.29, the interpolation function can be a bounded analytic function for which the radial limits exist everywhere. The second conclusion of [234, Theorem 2] means that the resulting interpolation function is continuous at each point of the set F, which makes [234, Theorem 2] a better analogue of Fatou's original theorem than the main result of [233].

Problem 5.30 Let $\mathcal{B}$ be the space of Bloch functions, that is, the space of functions analytic in $\mathbb{D}$ with

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

Let $\mathcal{B}_S$ be the space of functions of the form

$$f(z) = \log g'(z), \quad g \in S, \tag{5.13}$$

where S is the class of functions as in Problem 5.23(a). Let $\mathcal{B}_Q$ be the space of functions g in S that have a quasiconformal extension to the closed plane; see Anderson, Clunie and Pommerenke [38].

(a) Is $\mathcal{B}_S$ connected in the norm topology?
(b) Is $\mathcal{B}_Q$ dense in $\mathcal{B}_S$ in the norm topology?

(L. Bers)

Update 5.30(b) Becker points out that the answer is 'no'. If $f = \log g' \in \mathcal{B}_S$, then define a new norm by $\|f\| = \sup\left\{(1 - |z|^2)^2|S_g(z)| : |z| < 1\right\}$ where S_g is the Schwarzian of g. It is easy to show that

$$\|f_n - f\|_B \to 0 \qquad \Longrightarrow \qquad \|f_n - f\| \to 0 \tag{5.14}$$

(compare with Becker [95, Lemma 6.1]). Gehring [389] disproved the Bers conjecture, which means that $\mathcal{B}_Q$ is not dense in $\mathcal{B}_S$ with respect to $\|\cdot\|$. By (5.14) it follows immediately that $\mathcal{B}_Q$ is not dense in $\mathcal{B}_S$ with respect to $\|\cdot\|_{\mathcal{B}}$. Baernstein notes that it is an interesting open problem to determine geometrically the closure of $\mathcal{B}_Q$.

Problem 5.31 It was shown by Becker [94] that

$$\{f : \|f\|_{\mathcal{B}} < 1\} \subset \mathcal{B}_Q.$$

Is the radius 1 best possible? Is it true that for f in $\mathcal{B}_S$,

$$\limsup_{|z|\to 1} (1-|z|^2)|f'(z)| < 1 \qquad \Longrightarrow \qquad f \in \mathcal{B}.$$

(Ch. Pommerenke)

Update 5.31 Campbell [179] showed that the answer to the second part is 'no'. The reason is that one of the assumptions has been omitted, namely that the image domain of the function $g(f = \log g')$ must be a Jordan domain. However, if the image domain of the functions $g(f = \log g')$ is a Jordan domain, then even the condition

$$\limsup_{|z|\to 1} \left|(1-|z|^2)zf'(z) - c\right| < 1,$$

where $|c| < 1$, is enough to imply that f is in $\mathcal{B}_Q$. Becker's proof of this latter statement consists in noting that

$$g(z,t) = g(\mathrm{e}^{-t}z) + (1-c)^{-1}(\mathrm{e}^{t} - \mathrm{e}^{-t})zg'(\mathrm{e}^{-t}z)$$

is a subordination chain for small t, hence univalent for small t with $g(z,t)$ giving a quasiconformal extension of $g(z,0)$ onto a larger disc.

Problem 5.32 Suppose that $f_n \in \mathcal{B}_S$. What does $\|f_n - f\|_{\mathcal{B}} \to 0$ as $n \to \infty$ mean geometrically for the functions g_n related to f_n by (5.13)?

(Ch. Pommerenke)

Update 5.32 No progress on this problem has been reported to us.

Problem 5.33 Let L be a regular triangular lattice in the plane. Let $f(z)$ map $\mathbb{D}$ onto the universal covering surface over the complement of L. Is it true that the coefficients a_n of f tend to 0 as $n \to \infty$?

(Ch. Pommerenke)

Update 5.33 See Update 5.7. An affirmative answer was given by Hayman, Patterson and Pommerenke [519]. They proved that $a_n = O\big((\log n)^{-1/2}\big)$ in this case.

Problem 5.34 It was proved by Hall that every Bloch function has (possibly infinite) angular limits on an uncountably dense subset of $|z| = 1$. Do there always exist angular limits on a set of positive measure relative to some fixed Hausdorff measure, such as logarithmic measure, for example?

(J.E. McMillan, Ch. Pommerenke)

Update 5.34 Let Λ_ϕ be the Hausdorff measure corresponding to the function

$$\phi(t) = t\sqrt{\log t^{-1} \log\log\log t^{-1}}.$$

Makarov [703] has proved the following result: Let f be a function in the Bloch space which has no radial limit at almost every point. Let E be the set of points in the unit circle where f has infinite radial limit. Then $\Lambda_\phi(E) > 0$.

Problem 5.35 Let Γ be any discontinuous group of Möbius transformations of $\mathbb{D}$. Does there always exist a meromorphic function automorphic with respect to Γ and normal, that is, such that

$$(1 - |z|^2)\frac{|f'(z)|}{1 + |f(z)|^2}$$

is bounded in $\mathbb{D}$?

(Ch. Pommerenke)

Update 5.35 Pommerenke [822] has shown that there does exist a character-automorphic normal function f, that is, one for which $|f|$ is invariant under Γ. The original question for a normal f which is invariant under Γ remains open.

Problem 5.36 Let (n_k) be a sequence of positive integers such that

$$n_{k+1} > \lambda n_k, \quad \text{where } \lambda > 1, \tag{5.15}$$

and suppose that

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \tag{5.16}$$

is analytic in $\mathbb{D}$. Is it true that if $\sum_{k=0}^{\infty} |a_k| = \infty$, then $f(z)$ assumes every finite value

(a) at least once;
(b) infinitely often;
(c) in every angle $\alpha < \arg z < \beta$ of $|z| < 1$?

See Weiss and Weiss [981].

(J.P. Kahane)

Update 5.36 Murai [747] has proved that $f(\mathbb{D}) \neq \mathbb{C}$ implies $\sum_{k=0}^{\infty} |a_k| < \infty$. Hence (a) is true. Sons points out that the best known result is due to I-Lok Chang [200], who showed that $\sum_{k=0}^{\infty} |a_k|^{2+\varepsilon} = \infty$ for some positive ε implies infinitely many zeros in any sector.

Problem 5.37 Suppose that $f(z)$ is a function as in (5.16) and define

$$\mu = \limsup_{r\to 1} \frac{\log\log M(r, f)}{-\log(1 - r)},$$

where $M(r, f)$ is the maximum modulus of $f(z)$ on $|z| = r$. We do not now assume (5.15), but let $N^0(t)$ be the number of n_k not greater than t. If $N(r, a)$ is the function of Nevanlinna theory (see Chap. 1) it is known that

$$\limsup_{r \to 1} \frac{N(r, 0)}{\log M(r, f)} = 1$$

provided that either

(a) $\mu > 0$ and
$$\liminf_{k \to \infty} \frac{\log(n_{k+1} - n_k)}{\log n_k} > \frac{1}{2}\Big(\frac{2 + \mu}{1 + \mu}\Big),$$
(this is implicit in Wiman; see Sunyer and Balaguer [931] for details), or

(b) $\mu > \frac{1-\beta}{\beta}$ and $N^0(t) = O(t^{1-\beta})$ as $t \to \infty$, where $0 < \beta < 1$; see Sons [913].

If $0 < \mu < \frac{1-\beta}{\beta}$ with $N^0(t) = O(t^{1-\beta})$ as $t \to \infty$ we ask (a), (b) and (c) of the preceding problem, at least for those cases not covered above. In particular we may consider the cases $n_k = [k^\alpha]$, where $1 < \alpha < \frac{3}{2}$.

(L.R. Sons)

Update 5.37 The best result on this problem so far is by Nicholls and Sons [763].

For the rest of this chapter, and for Chap. 6, S denotes the class of analytic functions f univalent in $\mathbb{D}$ with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

In the next three questions $\prec$ means 'is subordinate to'. If f and g are analytic functions in $\mathbb{D}$, then g is *subordinate to* f if $g = f \circ \phi$ where ϕ is analytic in $\mathbb{D}$ and satisfies $\phi(\mathbb{D}) \subset \mathbb{D}$, $\phi(0) = 0$. See also Pommerenke [823, Chap. 2].

Problem 5.38 Tao-Shing Shah [889] has shown that if $g(z) \prec f(z)$ in $\mathbb{D}$, $g'(0)/f'(0)$ is real, and f is in S, then

$$|g(z)| \le |f(z)| \text{ for } |z| \le \frac{1}{2}(3 - \sqrt{5}), \tag{5.17}$$

$$|g'(z)| \le |f'(z)| \text{ for } |z| \le 3 - \sqrt{8}. \tag{5.18}$$

Both constants are best possible. Shah's proofs are technically very involved and it would be nice to have simpler proofs. Goluzin [422] gave simpler proofs but with worse constants in each case. His methods appear to be incapable of yielding (5.17) and (5.18).

(P.L. Duren)

Update 5.38 Shah's [889] majorisation results were generalised by Campbell [176–178] as well as similar questions by Biernacki, MacGregor and Lewandowski. The proper setting for such questions is not univalent function theory but locally univalent functions of finite order. In particular, Shah's sharp results of $|g(z)| \le |f(z)|$ in $(3-\sqrt{5})/2$ and $|g'(z)| \le |f'(z)|$ in $3-\sqrt{8}$ hold for all functions in U_2, the *universal linear invariant family* of order 2, which contains S as a proper subclass, and which contains functions of infinite valence.

Problem 5.39 Goluzin [422] has shown that if $g(z) \prec f(z)$ in $\mathbb{D}$, then

$$M_2(r, g') \le M_2(r, f'), \quad 0 \le r \le \frac{1}{2}.$$

Here, for $p > 0$,

$$M_p(r, h) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|^p \, d\theta\Big)^{1/p}.$$

The result is not necessarily true if $r > \frac{1}{2}$, as $f(z) = z$, $g(z) = z^2$ shows, though it follows from a theorem of Littlewood that, for all p,

$$M_p(r, g) \le M_p(r, f), \quad 0 < r < 1.$$

Find the largest number r_p, $0 < r_p < 1$, independent of f and g such that

$$M_p(r, g') \le M_p(r, f'), \quad 0 < r < r_p,$$

if $g \prec f$.

Note: If $g(z) \prec f(z)$ in $\mathbb{D}$, then $g(z) = f(\phi(z))$ so that

$$|g'(z)| \le |f'(\phi(z))|, \quad |z| \le \sqrt{2} - 1,$$

(see Carathéodory [183, p. 19]). Thus if $h(z) = f'(\phi(z))$ then $h \prec f'$ so that, by Littlewood's theorem, $M_p(r, g') \le M_p(r, h) \le M_p(r, f')$ for $p > 0$, $r \le \sqrt{2} - 1$. This improves what one gets if one applied (5.18) to the p-th means since $\sqrt{2} - 1 > 3 - \sqrt{8}$; but it may not be best possible.

(P.L. Duren)

Update 5.39 No progress on this problem has been reported to us.

Problem 5.40 Suppose that $f(z) = \sum_0^\infty a_n z^n$ and that $F(z)$ is analytic in $\mathbb{D}$, with $f \prec F$. What non-trivial conditions on F imply that

$$a_n \to 0, \quad \text{as } n \to \infty\,? \tag{5.19}$$

In particular, is (5.19) implied by

$$F \text{ is a Bloch function, } \textit{and} \tag{5.20}$$

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta = o\Big(\frac{1}{1-r}\Big)^2, \quad \text{as } r \to 1^- ? \tag{5.21}$$

Is it true that (5.21) by itself is preserved under subordination?

It is known that (5.19) holds if

$$(1-r)|F'(re^{i\theta})| \to 0$$

uniformly in θ as $r \to 1$ and that (5.19) holds if both (5.20) is satisfied, and given any positive ε, F can be written in the form

$$F(z) = F_1(z) + F_2(z)$$

where $(1-|z|^2)|F_1'(z)| \leq \varepsilon$ and F_2 has bounded characteristic in $\mathbb{D}$.

(W.K. Hayman)

Update 5.40 The problem asked whether various conditions on a function F might imply that (5.19) above holds when f is subordinate to F. The examples of Fernández [342] show that the answer is 'no'. In particular, the condition

$$\int_0^{2\pi} |F'(re^{i\theta})|^2 \, d\theta = \frac{o(1)}{(1-r)^2} \tag{5.22}$$

is not preserved under subordination, even if F is a Bloch function. For instance, if D is a domain such that the distance $d(w)$ of w from the complement of D tends to zero as $w \to \infty$, then (5.19) and (5.22) hold for the functions F mapping $\mathbb{D}$ into the universal covering surface over D, by a theorem of Hayman, Patterson and Pommerenke [519]. However by Fernández's examples, neither (5.19) nor (5.22) hold for functions f analytic in $\mathbb{D}$ and with values in D, if the complement of D is countable or, more generally, has capacity zero.

Problem 5.41 Let k be a positive integer, and define

$$\Phi_k = \{\phi : \phi \in C[0, 2\pi], \phi \uparrow, \phi(0) = 0, \phi(2\pi) = 2k\pi\}.$$

Let $e^{i\phi(t)}$ with $\phi \in \Phi_k$ have the Fourier expansion

$$\sum_{n=-\infty}^{\infty} C_n e^{int}.$$

(a) Shapiro conjectures that

$$\sum_{n=0}^{k} |C_n|^2 \geq \delta_k, \tag{5.23}$$

where δ_k is a positive constant depending only on k. (For $k = 1$ this is known, and it can be proved by combining results of Heinz (no citation given) and Radó (no citation given). Inequalities of this type arise in studying the distortion of harmonic mappings.)

(b) Conceivably (5.23) holds even with $\delta_k \geq \delta > 0$ for some absolute constant δ. In any case Shapiro conjectures also that, given only that $\phi \in C(0, 2\pi)$ and $\phi \uparrow$ then

$$\sum_{n=0}^{\infty} |C_n|^2 \geq \delta > 0.$$

(c) Is it true that, for $\phi \in \Phi_k$,

$$\sum_{n=1}^{k} |C_n|^2 \geq \sum_{n=-k}^{-1} |C_n|^2\ ?$$

(This is known to be true for $k = 1$).

(H.S. Shapiro)

Update 5.41 Sheil-Small [896] has proved this conjecture. The case $k = 2$ had been previously settled by Hall [458]. Hall [459] also proved that $\delta_k \to 0$ as $k \to \infty$, and gave an example for which

$$\sum_{n=1}^{k} |C_n|^2 \leq \frac{1}{k^2} \sum_{n=-k}^{-1} |C_n|^2, \tag{5.24}$$

thus answering in the negative two other questions of Shapiro. He conjectured that the factor k^{-2} in (5.24) is extreme.

Problem 5.42 Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\mathbb{D}$, with

$$\sum_{n=0}^{\infty} |a_n| = 1, \quad |f(z)| \geq \delta > 0 \text{ in } \mathbb{D}, \quad \text{and} \quad \frac{1}{f(z)} = \sum_{n=0}^{\infty} b_n z^n.$$

The following facts are known about $M = \sum_{n=0}^{\infty} |b_n|$:

(a) $M < +\infty$.

(b) If $\delta < \frac{1}{2}$, M cannot be bounded in terms of δ; Katznelson (no citation given).

(c) If $\delta > 2^{-\frac{1}{2}}$, M can be bounded in terms of δ; Katznelson (no citation given), Newman (no citation given).

What is the infimum of those δ such that M can be bounded in terms of δ? (A likely guess is $\frac{1}{2}$.)

(H.S. Shapiro)

Update 5.42 No progress on this problem has been reported to us.

Problem 5.43 Determine the Laurent coefficient bodies for analytic functions taking values of modulus at most unity in a given annulus

$$A_r = \{z : r < |z| < 1\}.$$

Determine the extremal functions.

(M. Heins)

Update 5.43 D. Khavinson writes that the progress achieved on studying Carathéodory bodies, as they are usually called in arbitrary finitely connected domains (not only annuli), can be found in [345, 604, 605].

Problem 5.44 (*Brannan's conjecture*) Suppose that $0 < \alpha < 1$ and

$$\frac{(1+xz)^{\alpha}}{1-z} = \sum_{n=0}^{\infty} A_n(x)z^n, \quad A_0(x) = 1, \quad |x| = 1.$$

Is it true that

$$|A_{2n+1}(x)| \leq |A_{2n+1}(1)|, \quad n \geq 2, \quad |x| = 1?$$

The above is true for $n = 1$; the corresponding result is false for all $A_{2m}(x)$ with $m \geq 1$; see Brannan [155]. More generally, one can ask the same question for the coefficients of

$$(1+xz)^{\alpha}(1-z)^{-\beta}, \quad |x| = 1, \quad \alpha > 0, \quad \beta > 0.$$

Here it is not even known if $|A_3(x)| \leq A_3(1)$.

(D.A. Brannan)

Update 5.44 These two conjectures are both known as 'Brannan's conjecture'. Brannan [155] stated the conjecture and showed that it holds if:

(i) $\beta > \alpha > 0$ and $\alpha + \beta \geq 2$, or
(ii) $\alpha = \beta \geq 1$, or
(iii) $\alpha \geq 1$ and $\beta = 1$.

Aharonov and Friedland [4] had previously shown that the conjecture holds if $\alpha \geq 1$ and $\beta > 1$. The following special cases have since been proved:

(a) $n = 2$, $\beta = 1$, by Milcetich [722].
(b) $n = 5$, $\alpha = \beta$ by Silverman and Silvia [904]. They point out that the conjecture is related to estimates of Gegenbauer polynomials and has applications for coefficient estimates for univalent functions in the unit disc which are starlike with respect to the boundary point. Their proofs make use of some well-known properties of Gaussian hypergeometric functions.
(c) $n = 7$, $\beta = 1$ by Barnard, Pearce and Wheeler [80].
(d) Odd n with $n \le 51$, $\beta = 1$ by Jayatilake [574], whose proofs make use of the program Mathematica.
(e) $\alpha = \beta$, with positive α and β, by Ruscheweyh and Salinas [866], whose proofs make use of some well-known properties of Gaussian hypergeometric functions.
(f) In a fresh approach Barnard, Jayatilake and Solynin [76] have expressed the coefficients in

$$\frac{(1 + xz)^{\alpha}}{(1 - z)^{\beta}} = \sum_{n=0}^{\infty} A_n(\alpha, \beta, z)x^n, \quad \alpha > 0, \quad \beta > 0,$$

as analytic functions of z, and developed integral representations for them. Hence they proved, for odd integers $n \ge 1$, $0 < \alpha < 1$, $\beta = 1$ and $z = re^{i\theta}$, $0 < \theta < 2\pi$, that

$$\left|A_n(\alpha, 1, z)\right| < \left|A_n(\alpha, 1, r)\right| \quad \text{for } 0 < r \le \frac{1}{2},$$

and

$$\mathrm{Re}[A_n(\alpha, 1, z)] < A_n(\alpha, 1, r) \quad \text{for } 0 < r \le 1.$$

More recently, in a preprint on arXiv, Szász [932] has asserted that he can prove the conjecture in the cases when $|\arg x| \le \frac{2\pi}{3}$, $\beta = 1$, or $\frac{1}{3} \le \alpha < 1$, $\beta = 1$.

Problem 5.45 If A is any analytic subset of the Riemann sphere it was shown by Kierst [606] that A is (exactly) the set of asymptotic values of a function meromorphic in $\mathbb{D}$. If $\infty \in A$, Kierst also proved that A is the set of asymptotic values of a function f analytic in $\mathbb{D}$. However, there exist analytic sets which are not the set of asymptotic values of a function analytic in $\mathbb{D}$. Ryan [869] has characterised those subsets of the Riemann sphere which are the set of asymptotic values of a function analytic in $\mathbb{D}$. Can one find a simpler characterisation?

(K.F. Barth)

Update 5.45 No progress on this problem has been reported to us.

Problem 5.46 A non-constant function f analytic in $\mathbb{D}$ is said to be in the *MacLane class* $\mathcal{A}$ if the set of points of the unit circle $\mathbb{T}$ at which f has an asymptotic value is a dense subset of $\mathbb{T}$; see MacLane [698]. A function f is said to have an *arc tract* if there exists a sequence $\{\gamma_n\}$ of arcs, γ_n in $\mathbb{D}$, and a non-degenerate subarc γ of $\mathbb{T}$ such that $\gamma_n \to \gamma$ (in the obvious fashion) and $\min\{|f(z)| : z \in \gamma_n\} \to \infty$

as $n \to \infty$. Does there exist a function $f \in \mathcal{A}$ with an arc tract and with non-zero derivative; see MacLane [699, p. 281]?

(K.F. Barth)

Update 5.46 No progress on this problem has been reported to us.

Problem 5.47 Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n}$$

be analytic in $\mathbb{D}$ and have Hadamard gaps, that is, $\lambda_{n+1}/\lambda_n \geq q > 1$. Need f have any radial limits (finite or infinite)? If not, need f have any asymptotic value on a path ending at a single point? For the case $q \geq 3$, see MacLane [698].

(J.M. Anderson and R. Hornblower)

Update 5.47 Murai [749] has shown that for $q > 1$, a Hadamard function has the asymptotic value ∞ if the coefficients are unbounded. It is still not known whether every such function has any radial limits whatsoever.

There are several partial results: Gnuschke and Pommerenke [396, Theorem 1]) have shown that if $q > 1$ and f is unbounded, and if furthermore

$$|a_k|/(|a_1| + \ldots + |a_k|) \to 0, \quad \text{as } k \to \infty,$$

then $\operatorname{Re} f$ has the angular limit $+\infty$ on a set of positive Hausdorff dimension. This improves a (proof and) result of Csordas, Lohwater and Ramsey [229]. Gnuschke and Pommerenke [396, Theorem 3] have also proved that there exists a Hadamard function f (with $q = \frac{33}{32}$) such that $\operatorname{Re} f$ and $\operatorname{Im} f$ oscillate between $-\infty$ and $+\infty$ on every radius. This does not exclude the possibility that f spirals to ∞ on some radii. Furthermore, there is a result of Hawkes [480] that, for instance, $f(z) = \sum z^{2^k}$ has the radial limit ∞ on a set of Hausdorff dimension one.

Problem 5.48 Let

$$f(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}$$

be analytic in $\mathbb{D}$ and have Hadamard gaps. Characterise sets S in $\mathbb{D}$ with the property that if f is bounded on S then $f(z)$ is bounded for $z \in \mathbb{D}$.

(K.G. Binmore)

Update 5.48 No progress on this problem has been reported to us.

Problem 5.49 What kind of gaps can the Taylor expansion of a non-constant automorphic function have? For example, can it have Hadamard gaps? (Presumably the sharp answer would depend on the group concerned.) This is closely related to a theorem of Rényi (no citation given) that a non-constant periodic entire function cannot have more than half of its coefficients zero.

(L.A. Rubel)

Update 5.49 Nicholls and Sons [764] have shown that $f(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ cannot have

$$n_k^{-1}(n_{k+1} - n_k)\log n_k \to \infty.$$

Furthermore, if f is automorphic and of the above form, and

$$N_0(t) \equiv \max\{k : n_k < t\},$$

then $N_0(t) \neq o(\log\log t)$ as $t \to \infty$.

Hwang [563] has proved that, if f is automorphic with respect to any group containing a parabolic element, then f does not have Hadamard gaps. The proof that this holds for any Fuchsian group appears to be false.

Problem 5.50 A function f analytic in $\mathbb{D}$ is said to be *annular* if there exists a sequence $\{J_n\}$ of Jordan curves in $\mathbb{D}$ such that

(a) J_n lies in the inside of J_{n+1},
(b) for each positive ε there exists a number $N(\varepsilon)$ such that for if $n > N(\varepsilon)$, J_n lies in the domain $\{z : 1 - \varepsilon < |z| < 1\}$,
(c) $\min\{|f(z)| : z \in J_n\} \to \infty$ as $n \to \infty$.

Let $\mathbb{T}$ denote the unit circle and let $Z'(f)$ denote the set of limit points of the zeros of an annular function f. Write

$$S(f) = \{a : a \in \mathbb{C} \text{ and } Z'(f - a) \neq T\},$$

and let $|S(f)|$ denote the cardinality of this set. Can $|S(f)| = \aleph_0$? The cases $|S(f)| = 1$ (Barth and Schneider [88]) and $|S(f)| = 2$ (Osada (no citation given)) are known, however neither construction can be easily adapted to the general case.

(K.F. Barth and D.D. Bonar)

Update 5.50 Barth observes that Carroll [195] has constructed a strongly annular function f such that the set of singular values of f is countably infinite. An analytic function f in $\mathbb{D}$ is said to be *strongly annular* if there exists a sequence $\{r_n\}$, $0 < r_n < 1$, such that the minimum modulus of f on the circle $|z| = r_n$ tends to infinity as $n \to \infty$.

Problem 5.51 It can be shown that there exists a Blaschke product $B(z)$ with $B(0) = 0$ such that

$$\frac{1 + B(z)}{1 - B(z)}$$

is a Bloch function. Give an explicit construction for such a product.

(F. Holland)

Update 5.51 Let ϕ be a positive continuous function defined on the unit interval with $\phi(0) = 0$. Aleksandrov, Anderson and Nicolau [21] proved that there exists an inner function I such that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)|I'(z)|}{\phi(1-|I(z)|^2)} = 0.$$

Moreover, I is constructed explicitly. This is related to the existence of singular measures which are symmetric and satisfy a Zygmund-type condition.

Problem 5.52 Let F be a Fuchsian group in $\mathbb{D}$, and let B be a set on $\mathbb{T}$ such that $B \cap \gamma(B) = \emptyset$ for $\gamma \in F$, $\gamma \neq I$. (This is the case, for instance, if

$$B = (\partial F \cap \mathbb{T}) \setminus C \quad (C \text{ countable}),$$

where F is a normal fundamental domain.) If cap $B > 0$, does it follow that F is of convergence type? (Conversely, it is known that, for every group F of convergence type, there exists such a set B with cap $B > 0$.)

(Ch. Pommerenke)

Update 5.52 No progress on this problem has been reported to us.

Problem 5.53 The ratio R of two Blaschke products $B(z, a_n)$, $B(z, b_n)$ is of bounded characteristic, but need not be a normal meromorphic function. That is, if a_n is 'close' to b_n for infinitely many n we can arrange that the spherical derivative of R is too large for R to be normal. When is R a normal function? Cima and Colwell [207] have shown that if $\{a_n\}$ and $\{b_n\}$ are both interpolating sequences, then R is normal if and only if $\{a_n\} \cup \{b_n\}$ is also an interpolating sequence. What happens if the sequences are not interpolating sequences?

(J.M. Anderson)

Update 5.53 No progress on this problem has been reported to us.

Problem 5.54 Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is convergent in $\mathbb{D}$, that $|z_0| = 1$, and that neither of the series

$$\sum_{n=0}^{\infty} (\operatorname{Re} a_n) z_0^n, \quad \sum_{n=0}^{\infty} (\operatorname{Im} a_n) z_0^n,$$

is absolutely convergent. Let S be the set of complex values assumed by the series

$$\sum_{n=0}^{\infty} \varepsilon_n a_n z_0^n,$$

where $\varepsilon = \pm 1$, and is allowed to vary over all possible choices. Is it true that $S = \mathbb{C}$?

(A.C. Offord; communicated by J.G. Clunie)

Update 5.54 Partial results have been obtained by Jakob and Offord [571]. In particular, if $R_n = \sum_n^\infty |a_n|^2$ is such that

$$\sum n^{-1}(\log n)^{-1/2} R_n^{1/2}$$

is convergent, then almost all the functions $\sum_0^\infty \pm a_n z^n$ will be bounded. Related results are due to Salem and Zygmund [875]. Notice that $\sum |a_n|^2 < \infty$ is in itself not sufficient to secure boundedness.

Problem 5.55 Let the function $h(z)$ be analytic in $\mathbb{D}$, $G_h = \{h(\mathbb{D})\}$, and K_h be a compact subset of G_h. The function $h(z)$ will be said to have the *L-property* ('$h(z_n)$ leaves the range of h') on the sequence $\{z_n\}_1^\infty$ with $\lim |z_n| = 1$ if only finitely many points of $\{h(z_n)\}_1^\infty$ lie in K_h for every K_h. We will define two functions f, g analytic in $\mathbb{D}$ to be an *ordered L-pair* if, on any sequence for which f has the L-property, g also has the L-property. A non-constant function $\alpha(z)$ analytic in $\mathbb{D}$ will be called an *L-atom* if, corresponding to any function f such that f, α is an ordered L-pair, there exists a function ϕ_f analytic on G_α such that $f = \phi_f \circ \alpha$. Prove or disprove the conjecture that α is an L-atom if and only if it is univalent. What happens if $\mathbb{D}$ is replaced by more general domains?

(L.A. Rubel)

Update 5.55 No progress on this problem has been reported to us.

Problem 5.56 By a *first-order property* we shall mean a ring-theoretic property in the first-order predicate calculus. For functions f analytic in $\mathbb{D}$, are the following first-order properties:

(a) f is constant?
(b) f is bounded?
(c) f is admissible, that is, $T(r, f) \neq O(\log(1-r)^{-1})$?

What is the situation in more general domains?

(L.A. Rubel)

Update 5.56 No progress on this problem has been reported to us.

Problem 5.57 Suppose that f is analytic in $\mathbb{D}$. Plessner's theorem [803] asserts that at almost all points $e^{i\theta}$ on the unit circle, either f has an angular limit, or else the image $f(S)$ of every Stolz angle S with vertex at $e^{i\theta}$ is dense in $\mathbb{C}$. How much can 'dense' be improved? In particular, is it true that at almost all $e^{i\theta}$ either f has an angular limit, or else $f(S)$ is all of $\mathbb{C}$, except perhaps for a set of zero logarithmic capacity? This result would be best possible, since if E is any closed set of capacity zero, then the universal covering map of the disc onto $\mathbb{C} \setminus E$ has angular limits almost nowhere.

(A. Baernstein)

Update 5.57 Nicolau writes that this problem is a strong version of Problem 5.20.

Problem 5.58 Suppose that f is univalent and zero-free in $\mathbb{D}$. Baernstein [57] has shown that for each $p \in (0, \frac{1}{2})$, f admits a factorisation $f = B_p(F_p)^{1/p}$, where $B_p \in H^\infty$, $1/B_p \in H^\infty$, and $\operatorname{Re} F_p > 0$. Is it possible to pass to the limit $p = \frac{1}{2}$, and thus factor f into a bounded function times a function subordinate to a map onto a slit plane?

(A. Baernstein)

Update 5.58 While this conjecture is still open, Wolff [998] has shown that a certain natural stronger conjecture is false; on the other hand, the factorisation given in Problem 5.58 is possible when f is a monotone slit mapping. A *monotone slit mapping* is a function $f(z)$ which is analytic and univalent in $\mathbb{D}$ and for which $f(0) = 0$ and $f'(0) = 1$, whose image domain is the complement of a path $\Gamma(t)$ on $[0, \infty)$ for which $|\Gamma(t_1)| < |\Gamma(t_2)|$ if $t_1 < t_2$.

Problem 5.59 (*Subordination and extreme point problem*) Let $g(z) = \sum_{n=0}^{\infty} B_n z^n$ be analytic in $\mathbb{D}$. Denote by S_g the family of functions $f(z)$ subordinate to g. Find general conditions on g so that the only extreme points of the closed convex hull of S_g are the functions $g(ze^{it})$, $0 \le t < 2\pi$. This is known for certain functions g, see Clunie [218], for example:

(a) $g(z) = [(1 + cz)/(1 - z)]^\alpha$, where $|c| \le 1$, $\alpha \ge 1$;
(b) $g(z) = \exp[(1 + z)/(1 - z)]$.

Sheil-Small can prove it for functions of the form $g(z) = h((1 + z)/(1 - z))$, where $h(w)$ is a univalent quadratic polynomial in $\operatorname{Re} w > 0$. One might expect that the conclusion would hold for functions g with positive coefficients increasing in a suitably regular manner.

(T. Sheil-Small)

Update 5.59 No progress on this problem has been reported to us.

Problem 5.60 (*Hadamard convolution*) Suppose that $\alpha \ge 1$, $\beta \ge 1$ and that ϕ is analytic in $\mathbb{D}$, and satisfies

$$\phi(z) * \frac{(1 + xz)^\alpha}{(1 - z)^\beta} \ne 0, \quad |x| = 1, |z| < 1.$$

Is it true that

$$\phi(z) * \frac{(1 + xz)^{\alpha-1}}{(1 - z)^\beta} \ne 0, \quad |x| = 1, |z| < 1\,?$$

This is true when α is a natural number. For the proof and further background, see Sheil-Small [895]. *Hadamard convolution* is another name for *Hadamard product*. Further details on this can be found in Sheil-Small [895, p. 515] and Titchmarsh [942, Sect. 4.6].

(T. Sheil-Small)

Update 5.60 No progress on this problem has been reported to us.

Problem 5.61 Let $w(z)$ be analytic in $\mathbb{D}$ with $w(0) = 0$. If $|w(z) + zw'(z)| < 1$, for $|z| < 1$, then a simple application of Schwarz's lemma shows that $|w(z)| < 1$, for $|z| < 1$. Miller and Mocanu [738] showed that

$$|w(z) + zw'(z) + z^2 w''(z)| < 1, \quad |z| < 1,$$

implies that $|w(z)| < 1$, for $|z| < 1$. Is it true that

$$|w + zw' + z^2 w'' + \ldots + z^n w^{(n)}| < 1 \qquad \Longrightarrow \qquad |w(z)| < 1,$$

for $n = 1, 2, 3, \ldots$?

(S. Miller)

Update 5.61 This has been proved to be the case by Goldstein, Hall, Sheil-Small and Smith [418].

Problem 5.62 Let u be a continuous real-valued function on the unit circle $\mathbb{T}$. Give a necessary and sufficient condition on u such that u is the real part of a function f in the disc algebra $A(\overline{\mathbb{D}})$.
Remarks:

(a) A solution would have applications in the algebraic ideal theory of $A(\overline{\mathbb{D}})$.
(b) An answer to the analogous problem for $L^p(T)$, $H^p(\mathbb{D})$ is the Burkholder–Gundy–Silverstein theorem; see Peterson [796, p. 13] and Garnett [379, Chap. III].

(M. von Renteln)

Update 5.62 No progress on this problem has been reported to us.

Problem 5.63 One of the many equivalent norms on BMOA on $\mathbb{D}$ is defined by

$$\|f\|_h = \inf_q \sup_{z \in \mathbb{D}} |f(z) + \overline{q(z)}|,$$

the infimum being taken over all functions analytic in $\mathbb{D}$.

Given f in BMOA, does there exist a q such that

$$|f(z) + \overline{q(z)}| \equiv \|f\|_h \text{ a.e. on } |z| = 1?$$

The answer is 'yes' when f is a rational function.

(J.A. Hempel)

Update 5.63 No progress on this problem has been reported to us.

Problem 5.64 Let f be analytic in $\mathbb{D}$ with

$$|f(z)| = O((1 - |z|)^{-k}), \quad k \geq 0. \tag{5.25}$$

Then f induces a distribution on $C^\infty(T)$, as follows: for ϕ in $C^\infty(T)$,

$$\lim_{r\to 1}\left(\frac{1}{2\pi}\int f(re^{i\theta})\phi(e^{-i\theta})\,d\theta\right) = \Lambda_f(\phi).$$

What can be said about the order of the distributions satisfying (5.25) and

$$\lim_{|z|\to 1}|f(z)|(1-|z|)^k = 0\,?$$

(J.A. Cima)

Update 5.64 No progress on this problem has been reported to us.

Problem 5.65 Does there exist a non-constant function f in the disc algebra such that $f(e^{i\theta}) \in f(\mathbb{D})$ for almost all θ? Caution: the Rudin–Carleson theorem (see Bishop [128]) allows the construction of a good candidate, but it is not immediately clear whether it works.

(K. Stephenson)

Update 5.65 Stegenga and Stephenson (unpublished) have shown this to be the case for almost all f in a category sense. A solution has also been obtained by Gol'dberg [409] using a simple construction and without the use of the Rudin–Carleson theorem.

Problem 5.66 Let B be an infinite Blaschke product in $\mathbb{D}$. Does there exist a positive δ, depending on B, such that, for every w, $|w| < \delta$, the set $B^{-1}(\{w\})$ is infinite? Stephenson has obtained some related results.

(K. Stephenson)

Update 5.66 Stephenson (unpublished) has answered this question negatively by constructing an inner function f such that $f^{-1}(w)$ is finite on a dense subset of $|w| < 1$. Then

$$f_\alpha = \frac{f-\alpha}{1-\overline{\alpha}f}$$

is a Blaschke product for almost all α.

Problem 5.67 Let the function f in $\mathbb{D}$ be given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \ge \lambda > 1, \quad k \ge 0,$$

with

$$m_0(r, f) \equiv \max_{k\ge 0}|a_k|r^{n_k} \to \infty, \quad \text{as } r \to 1^-.$$

Define

$$E = \left\{\theta : \liminf_{r\to 1^-} \frac{|f(re^{i\theta})|}{m_0(r, f)} > 0\right\}.$$

Is it true that E has measure 0?

This has been proved for the case when $\mu(\frac{1}{2}(1+r))/m_0(r, f) \le$ constant.

(D. Gnuschke and Ch. Pommerenke)

Update 5.67 No progress on this problem has been reported to us.

Problem 5.68 Let the function f where

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad |z| \le 1,$$

be a Bloch function with positive real part in $\mathbb{D}$. Determine the rate of growth (as $N \to \infty$) of the sequence $\left\{\sum_{n=1}^{N} |a_n|^2\right\}$.

(F. Holland)

Update 5.68 No progress on this problem has been reported to us.

Problem 5.69 Let the function f where

$$f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, \quad |z| \le 1,$$

be a Bloch function with positive real part in $\mathbb{D}$ and such that each $a_n \ge 0$. Does it follow that $\sum_{n=1}^{\infty} a_n^2 < \infty$?

An equivalent formulation of the problem is the following: let μ be a probability measure in Zygmund's class Λ_* on the circle, and let

$$\hat{\mu}(n) = \int_0^{2\pi} e^{-inx}\, d\mu(x) \ge 0, \quad n \in \mathbb{Z}.$$

Is it true that $\hat{\mu} \in l_2$? A counterexample, if one exists, cannot be constructed using Riesz products; see Duren [271] and Holland and Twomey [552].

An affirmative answer would mean that if $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ is a Bloch function in $\mathbb{D}$ with positive real part, then $\sum_{n=1}^{\infty} |a_n|^4 < \infty$. Even if this last inequality is false, perhaps it is still true in the general case that there exists a $p > 4$, such that $\sum_{n=1}^{\infty} |a_n|^p < \infty$.

(F. Holland)

Update 5.69 No progress on this problem has been reported to us.

Problem 5.70 Barth and Clunie [87] have constructed a bounded analytic function in $\mathbb{D}$ with a level set component of infinite length; this component is highly branched. Can one construct a bounded analytic function with an unbranched level set component of infinite length?

(K.F. Barth and J.G. Clunie)

Update 5.70 No progress on this problem has been reported to us.

Problem 5.71 Suppose that

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}, \quad n_{k+1}/n_k \geq q > 1,$$

is an analytic function in $\mathbb{D}$ with Hadamard gaps such that $T(r, f) \to \infty$ as $r \to 1$. Does $\delta(w, f) = 0$ hold for every (finite) complex number w?

(T. Murai)

Update 5.71 No progress on this problem has been reported to us.

Problem 5.72 Let the function f have the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of radius of convergence 1; let E be the set of singular points of f on $\mathbb{T}$, and suppose that

$$\sup_{\xi \in E} \sup_{N \geq 0} \left| \sum_{n=0}^{N} a_n \xi \right| < \infty.$$

It has been shown by Allan, O'Farrell and Ransford [30] that if E has measure zero, then $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$ at each point z in $\mathbb{T} \setminus E$. Does the conclusion remain true if E has positive measure? (Nothing appears to be known either way.)

(T.J. Ransford)

Update 5.72 No progress on this problem has been reported to us.

Problem 5.73 Let $0 < \alpha < 1$ and let R_α denote the set of all Riesz potentials $p(x)$ of finite positive Borel measures μ on $\mathbb{R}$:

$$p(x) = \int_{-\infty}^{\infty} |x-t|^{-\alpha} \, \mathrm{d}\mu(t).$$

Characterise those non-negative measurable functions $f(x)$ on $\mathbb{R}$ that are dominated by some function p in R_α.

(B. Korenblum)

Update 5.73 No progress on this problem has been reported to us.

Problem 5.74 Characterise those non-negative measurable functions f on the unit circle that are dominated almost everywhere by moduli of the boundary values of an analytic function in the unit disc with positive real part. (Problem 5.73 might be considered a step towards solving this problem.)

(B. Korenblum)

Update 5.74 No progress on this problem has been reported to us.

Problem 5.75 Does there exist a bounded analytic function in $\mathbb{D}$ such that the image of every radius has infinite length? See, for example, Anderson [33] and Rudin [859].

(W. Rudin; communicated by K.F. Barth)

Update 5.75 This has been solved in the negative by Bourgain [151] who showed that 'good' radii (that is, those that have images of finite length) always exist. Moreover, the set of good radii (viewed as a subset of the unit circle) has Hausdorff dimension 1. The related conjecture by Anderson [33] has been proved by Jones and Müller [581].

Problem 5.76 Suppose γ is an arc that lies in $\mathbb{D}$ except for one endpoint at $z = 1$, and that fails to be non-tangential to $\mathbb{T}$. Define

$$\gamma_\theta = \mathrm{e}^{i\theta}\gamma, \quad \theta \in [0, 2\pi). \tag{5.26}$$

Does there exist a function g in H^∞ such that

$$\lim_{z \to \mathrm{e}^{i\theta},\, z \in \gamma_\theta} g(z)$$

exists for no value of θ? (See Rudin [861].)

If γ is tangential, the answer is 'yes', see Collingwood and Lohwater [222, p. 43].

(W. Rudin; communicated by K.F. Barth)

Update 5.76 In the original statement of this problem, it was erroneously assumed that γ is a non-tangential arc. In that form, the question did not make much sense, since Fatou's theorem says that the angular limit exists for almost all points on $\mathbb{T}$; see Collingwood and Lohwater [222, p. 21]. In addition, there was a typing error in (5.26). Dyakonov suggests the new formulation. A discussion of the difference between *'tangential'* boundary paths and *'not nontangential'* boundary paths can be found in Rudin [861, Part V].

Problem 5.77 In general, the radial behaviour of the derivative of a bounded analytic function in $\mathbb{D}$ can be fairly arbitrary, even under stronger conditions than boundedness.

Is it true that, given any measurable function $m(\theta)$ on $[0, 2\pi)$, there exists a function $f(z)$, continuous on $\overline{\mathbb{D}}$ and univalent on $\mathbb{D}$, such that

$$\lim_{r \to 1} f'(r\mathrm{e}^{i\theta}) = m(\theta)$$

for almost all θ? (In terms of known results, the conjecture seems quite plausible. Ortel and Schneider [780] showed that the conjecture is true under slightly strengthened hypotheses; and Lohwater, Piranian and Rudin [680] showed that the conjecture is true with a slightly weakened conclusion.)

(W.J. Schneider)

Update 5.77 No progress on this problem has been reported to us.

Problem 5.78 Let H^1 denote Hausdorff one-dimensional measure on $\mathbb{C}$, and let $g : \mathbb{T} \to [-\infty, \infty]$ denote an arbitrary Borel function on the boundary of the unit circle $\mathbb{T}$. Does there exist a corresponding function f, analytic in $\mathbb{D}$ and with bounded Taylor coefficients, such that

$$\lim_{r \to 1^-} f(rz) = g(z)$$

for H^1-almost all z in $\mathbb{T}$?

For work on related questions, see Ortel and Schneider [780].

(M. Ortel and W.J. Schneider)

Update 5.78 No progress on this problem has been reported to us.

Problem 5.79

(a) Let f be a non-constant analytic function in $\mathbb{D}$, m be a positive integer, and define $\psi = (f)^m f'$. Then it is shown by Sons [914, Theorem 2] that, when f and ψ both belong to MacLane's class A [698], either

 (i) f has finite asymptotic values on a dense subset of $\mathbb{T}$, or
 (ii) ψ assumes every finite value infinitely often.

What replacements can be found for (i) to give another correct theorem? Can ψ be taken to be of the form

$$\psi = (f)^{m_0}(f')^{m_1} \dots (f^{(k)})^{m_k},$$

whenever $k, m_0, m_1, \dots, m_k \in \mathbb{N} \cup \{0\}$?

(b) It is known that the conclusion in (a) is true when ψ is replaced by

$$\psi = f^{(l)} + \sum_{\nu=0}^{i-1} a_\nu f^{(\nu)},$$

where $l \in \mathbb{N}$ and the a_ν are analytic functions in $\{|z| < 1 + r\}$ for some positive r.

What replacements can be found in this case for (i) to give another correct theorem? Can the a_ν be taken as functions analytic in $\mathbb{D}$ with $T(r, a_\nu) = o(T(r, f))$ as $r \to 1$?

For both parts, compare Problem 1.38.

(L.R. Sons)

Update 5.79 No progress on this problem has been reported to us.

5.3 New Problems

The following three problems are related to Problem 5.29, and involve G_δ sets which we recall are subsets of a topological space that are a countable intersection of open sets. These problems can be found as Problems 3, 4 and 5 respectively in Danielyan [234].

Problem 5.80 Describe all G_δ sets F on $\mathbb{T}$ such that whenever f in $C(F)$ coincides almost everywhere on F with the radial limits of a function g in H^∞, then the radial limits of g exist on F and coincide with f at each point of F.

(A.A. Danielyan)

Problem 5.81 Describe all G_δ sets F on $\mathbb{T}$ such that whenever f in $C(\mathbb{T})$ coincides almost everywhere on F with the radial limits of a function g in H^∞, then the radial limits of g exist on F and coincide with f at each point of F.

(A.A. Danielyan)

Problem 5.82 Problem 5.29 and the results in Update 5.29 extend Fatou's interpolation theorem for the case of G_δ sets, which we recall are subsets of a topological space that are a countable intersection of open sets. As is well known, Fatou's interpolation theorem has been the basis for another well known interpolation theorem, the Rudin–Carleson theorem (see Bishop [128]). The following questions ask the possibility of an analogue of the Rudin–Carleson theorem for G_δ sets:

(a) Let F be a G_δ set of measure zero on $\mathbb{T}$ and let either f be in $C(\mathbb{T})$ or, more generally, f be in $C(F)$. Then does there exist a function g in H^∞ such that the radial limits of g exist everywhere on $\mathbb{T}$ and coincide with f on F?
(b) In addition to the requirement of part (a), is it possible that g has unrestricted limits at each point of F?

(A.A. Danielyan)

Problem 5.83 (*The Krzyż conjecture*) Let $\mathcal{B}_*$ be the class of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D}$ such that $|f(z)| < 1$ and $f(z)$ is non-zero for z in $\mathbb{D}$. Krzyż conjectured [642] that for each $n \geq 1$ and each f in $\mathcal{B}_*$,

$$|a_n| \leq \frac{2}{e}.$$

Equality holds for

$$f_n(z) = \mathrm{e}^{(z^n-1)/(z^n+1)} = \frac{1}{e} + \frac{2}{e} z^2 + \cdots.$$

See Martín, Sawyer, Uriarte-Tuero and Vukotić [708].

(J.G. Krzyż)

Chapter 6
Univalent and Multivalent Functions

6.1 Preface by Ch. Pommerenke

Let S denote the family of all function $f(z) = z + a_2 z^2 + \ldots$ that are analytic and univalent in the unit disc $\mathbb{D}$. In [123] Bieberbach proved that $|a_2| \leq 2$ and asked whether $|a_n| \leq n$ was true for all n. This was the birth of the Bieberbach conjecture.

Complex analysts became interested in the family S and in particular in the coefficients of functions f in S. There were many interesting results, for instance the asymptotic Bieberbach conjecture by Walter Hayman [487].

If $f \in S$ *and unless* $f(z) = z/(1 - z\mathrm{e}^{i\theta})^2$,

$$\frac{|a_n|}{n} \to \alpha, \quad \text{where } \alpha < 1.$$

One could say that the coefficients were "in" (that is, "fashionable"). Of course, complex analysts were also working on many other problems not connected in any way with the Bieberbach conjecture.

In 1984 FitzGerald and I were in LaJolla. He showed me a manuscript of de Branges where he claimed to have proved the Bieberbach conjecture. I told FitzGerald that, as a referee, I had seen several wrong proofs by various authors. He said "Okay, let's find the error in this paper!"

So we read the manuscript together. "Ha, here is the error!" I exclaimed. But I was wrong. We continued reading. My friend said "I do not believe this here", and it turned out that he was wrong. Finally we came to the end of the manuscript without finding any error. Twice we reread the paper. Three days later we had to admit that de Branges had finally proved the Bieberbach conjecture.

But he had proved more, namely the Milin conjecture, which implies the Robertson conjecture, which in turn implies the Bieberbach conjecture. De Branges also used a recent result about Jacobi polynomials. Can one say that the time was ripe for a proof of the Bieberbach conjecture?

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
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Afterwards there was less interest in the coefficients of univalent functions, just as the interest of many group theorists had shifted after all finite simple groups had been found.

In all parts of science there are questions that are "in" for some time and later attract less research interest. They move into books. Fractals even reached the popular literature. The interest had waned after Sullivan had solved the outstanding problems about the iteration of polynomials.

The open problem about univalent functions that I find most interesting is the BCJK conjecture, see Problem 6.96 below. It is a search to describe the precise shape of a certain curve $y = B(x)$ for $-\infty < x < +\infty$.

The most famous conjecture is the Riemann hypothesis, namely that the analytic continuation to $0 < \operatorname{Re} z < 1$ of the Riemann zeta function has no zeros. This would have very important consequences in number theory.

I want to conclude with some philosophy about human knowledge and mathematical proofs. The German philosopher Nicolai Hartmann once said: There are three kingdoms of knowledge, namely:

(a) Things that we already know.
(b) Things that we can in principle know in the future.
(c) Things that we can never know.

Applied to mathematics this would say about theorems and their proofs that there are three cases:

(a) Theorems that have been proved.
(b) Conjectures that possibly can be proved or disproved in the future.
(c) Conjectures that we can never prove or disprove.

Why should case (c) occur? The reason is that every proof can only have finite length. Perhaps the Riemann hypothesis is true but it has no proof of finite length. Who knows?

6.2 Progress on Previous Problems

Notation A function $w = f(z)$ analytic or meromorphic in a domain D is said to be *schlicht* or *univalent* in D if $f(z)$ assumes different values at different points of D. Such a function maps D univalently onto a domain Δ in the w plane. Among the classes of univalent functions, the following play a particularly important role: we denote by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{6.1}$$

analytic and univalent in the unit disc $\mathbb{D}$ and by Σ the class of functions

$$F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \tag{6.2}$$

analytic and univalent for $1 < |z| < \infty$. In the definition (6.2) of Σ the sum often starts with $n = 0$.

An account of various elementary extremal problems for the class S is given by Hayman [513, Chaps. 1, 6]. It is known that for many problems in S, the extremal functions are the Koebe functions

$$w = f_\theta(z) = \frac{z}{(1 - z\mathrm{e}^{i\theta})^2} = z + \sum_{n=2}^{\infty} n z^n \mathrm{e}^{i(n-1)\theta},$$

which map $\mathbb{D}$ onto the plane cut along a ray going from $w = -\frac{1}{4}\mathrm{e}^{-i\theta}$ to ∞ in a straight line away from the origin. The corresponding functions of Σ are

$$w = F_\theta(z) = f_\theta(z^{-2})^{-\frac{1}{2}} = z - \frac{\mathrm{e}^{i\theta}}{z},$$

which map $|z| > 1$ onto the plane cut along the segment from $-2i\mathrm{e}^{i(\theta/2)}$ to $2i\mathrm{e}^{i(\theta/2)}$.

Many of the most interesting problems for S and Σ relate to the size of the coefficients a_n and b_n. Pre-eminent among these is the famous Bieberbach conjecture.

Problem 6.1 (*The Bieberbach conjecture*) Is it true that $|a_n| \le n$ for f in S with equality only for $f(z) \equiv f_\theta(z)$? The result is known to be true for $n = 2$ (see Bieberbach [123]), $n = 3$ (see Löwner [683]), and $n = 4$ (see Garabedian and Schiffer [372], Charzynski and Schiffer [203]).

Update 6.1 De Branges' [239] proof of the Bieberbach conjecture has transformed this field. Suppose that S is the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

univalent in $\mathbb{D}$. We also define

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} \gamma_k z^k$$

and

$$\left(\frac{f(z)}{z}\right)^p = \sum_{n=0}^{\infty} a_{n,p} z^n, \qquad p > 0.$$

Then de Branges proved Milin's conjecture (see [239, 731]) that

$$\sum_{n=1}^{N}\sum_{k=1}^{n} k|\gamma_k|^2 \le \sum_{n=1}^{N}\sum_{k=1}^{n}\frac{1}{k}$$

(Problem 6.42), and this in turn implies Robertson's conjecture (see Sheil-Small [894])

$$\sum_{k=0}^{n} |a_{k,\frac{1}{2}}|^2 \le n+1$$

(Problem 6.39), which implies the Bieberbach conjecture $|a_n| \le n$ (Problem 6.1), as well as the corresponding result for functions subordinate to f, and a positive answer to Problems 6.2 and 6.85.

Among earlier proofs, the result for $n = 6$ was proved independently by Pederson [791] and Ozawa [788], and for $n = 5$ by Pederson and Schiffer [792]. Among later proofs of the full Bieberbach conjecture, that of FitzGerald and Pommerenke [347] may be mentioned.

Problem 6.2 Define $A_n = \sup_{f\in S} |a_n|$. It is shown by Hayman [489] that

$$\frac{A_n}{n} \to K_0, \qquad \text{as } n \to \infty.$$

Is it true that $K_0 = 1$? The best known result so far is $K_0 < 1.243$ due to Milin [732].

Update 6.2 See Update 6.1.

With regard to earlier weaker results, FitzGerald [346] had proved that

$$|a_n| \le Kn,$$

where $K = (7/6)^{\frac{1}{2}} = 1.080\ldots$. Horowitz [556] sharpened this to $K < 1.0657$, and FitzGerald has remarked that the best his method might be expected to yield is $K < (7/6)^{\frac{1}{4}} = 1.039\ldots$. Further, Hamilton [465] proved that $A_{n+1} - A_n \to K_0$, and also [466] that $K_0 = 1$ is implied by, and so is equivalent to, Littlewood's conjecture $|a_n| \le 4dn$, where d is the distance from the origin to the complement of the image of $\mathbb{D}$ by $f(z)$. By de Branges' theorem (see Update 6.1) $A_n = n$ for all n, so that $K_0 = 1$, and Littlewood's conjecture holds.

Problem 6.3 If $f(z)$ is in S it is shown by Bombieri [137] that there exist constants c_n such that for $f(z)$ in S

$$|\mathrm{Re}\,(n - a_n)| \le c_n \mathrm{Re}\,(2 - a_2).$$

What is it the exact size of the constants c_n? Is it true that there exists d_n such that

$$\big|n - |a_n|\big| \le d_n\big(2 - |a_2|\big)\,?$$

(E. Bombieri)

Update 6.3 Roth writes that the value of c_2 is now known, see Greiner and Roth [441]; and that there has been interest in these numbers, particularly the 'local' versions of them, see for example Leung [659] and Aharanov and Bshouty [3].

Problem 6.4 Suppose that $F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n} \in \Sigma$ and that

$$G(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$$

is the inverse function of $F(z)$. What are the exact bounds for $|b_n|$ and $|c_n|$? The following results are known

$|b_1| = |c_1|$ (Bieberbach [123]).
$|b_2| = |c_2| \leq \frac{2}{3}$ (Schiffer [881], Goluzin [421]).
$|b_3| \leq \frac{1}{2} + e^{-6}$ (Garabedian and Schiffer [373]).
$|c_3| \leq 1$ (Springer [916]).

These are sharp.

Update 6.4 Springer [916] conjectured that

$$|c_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!}, \qquad n = 1, 2, \ldots.$$

Kubota [645] has proved this conjecture for $n = 3, 4, 5$, and has also [644] obtained the sharp upper bounds of $|b_4|$ and $|b_5|$ among all F with real coefficients.

Problem 6.5 If it proves too difficult to obtain sharp bounds for all of the coefficients in Problem 6.4, we ask for the orders of magnitude. An area principle shows that

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

and hence

$$b_n = o(n^{-\frac{1}{2}}).$$

Clunie and Pommerenke [216] have shown that

$$|b_n| = O(n^{-\frac{1}{2}-\frac{1}{300}}).$$

In the opposite direction, examples due to Clunie [208] show that

$$|b_n| > n^{-1+\delta}$$

is possible for indefinitely many n and a fixed $F(z)$ in Σ, where δ is an absolute constant.

Update 6.5 Pommerenke [818] has proved that

$$|b_n| \geq n^{0.17-1}$$

can hold for infinitely many n.

As stated above, Clunie and Pommerenke [216] proved that

$$|b_n| = O\left(n^{-\frac{1}{2}-\delta}\right)$$

where $\delta = 1/300$. A corresponding improvement to the coefficient estimates for bounded functions in S follows from this method.

See also Update 4.18.

Problem 6.6 What are the orders of magnitude of the c_n in Problem 6.4? Springer [916] obtained the estimate

$$|c_n| \leq \frac{2^n}{n}$$

and also showed that, given a positive ε,

$$|c_{2n-1}| > (1-\varepsilon)2^{2n-2}e/(\pi n^3)^{\frac{1}{2}}$$

is possible for all sufficiently large n.

Update 6.6 No progress on this problem has been reported to us.

Problem 6.7 If $f(z)$ is in S and is bounded, that is, f satisfies $|f(z)| < M$ for $z \in \mathbb{D}$, we again ask for the order of magnitude of the coefficients a_n. Since the area of the image of $\mathbb{D}$ by $f(z)$ is at most πM^2, we deduce that $\sum n|a_n|^2 \leq M^2$, so that again

$$|a_n| = o(n^{-\frac{1}{2}}), \qquad \text{as } n \to \infty.$$

Here also Clunie and Pommerenke [216] have shown that

$$|a_n| = O(n^{-\frac{1}{2}-\frac{1}{300}}).$$

Examples in the opposite direction due to Littlewood [674] show again that for a sufficiently small positive δ, we can have

$$|a_n| > n^{-1+\delta}$$

for infinitely many n and a fixed $f(z)$. The problems for this class of functions seem very analogous to the corresponding problems for Σ; see Problem 6.5.

Update 6.7 Carleson and Jones [193] established that this problem is equivalent to Problem 6.5; they also have shown that, as conjectured above, the bounds for the coefficient $|a_n|$ in Σ and those for bounded functions in S are of the form $n^{-\gamma+o(1)}$, where γ is an absolute constant, which they conjecture to be $\frac{3}{4}$. See also Updates 4.18 and 6.8.

Problem 6.8 We write

$$I_\lambda(r, f) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\lambda \,d\theta\right\}^{1/\lambda}.$$

What are the exact bounds for $I_\lambda(r, f)$ and $I_\lambda(r, f')$ when f is in S or f is in Σ? If $f(z)$ is in S, it is known that for fixed λ, the orders of magnitude of $I_\lambda(r, f)$ and $I_\lambda(r, f')$ are maximal when $f(z)$ is the Koebe function. For the best results in this direction, see Bazilevič [92].

If f is in S and $|f| < M$, or if f is in Σ, it is almost trivial from the area principle that

$$I_1(r, f') = o(1-r)^{-\frac{1}{2}}, \qquad \text{as } r \to 1.$$

Clunie and Pommerenke [216] have improved this to

$$I_1(r, f') = O(1-r)^{-\frac{1}{2}+\frac{1}{300}}.$$

A function $f(z)$ in S is said to be *convex* if the image of $\mathbb{D}$ by $w = f(z)$ is a convex domain D in the w-plane, that is, for any two points w_1, w_2 in D, the straight line segment w_1, w_2 also lies in D. A function $F(z)$ in Σ is said to be *convex* if the complement of the image of $|z| > 1$ by $F(z)$ is convex.

Update 6.8 Baernstein [55] has shown that if $\phi(R)$ is any convex function of $\log R$ then for each r, with $0 < r < 1$,

$$\int_0^{2\pi} \phi\big(|f(re^{i\theta})|\big)\,d\theta$$

is maximised in the class S by the Koebe function. Thus he completely settled this maximising integral means problem for class S. The problem for class Σ remains open. Carleson and Jones [193] proved that Problems 6.5, 6.7 and 6.8 for class Σ are equivalent. See also Updates 4.18 and 6.30.

Problem 6.9 (*Schoenberg's conjecture*) If $f(z) = \sum_{n=1}^\infty a_n z^n$ and $g(z) = \sum_{n=1}^\infty b_n z^n$ are convex and f, g belong to S, is it true that

$$f * g(z) = \sum_{n=1}^\infty a_n b_n z^n$$

is also a convex function in S? See Pólya and Schoenberg [805].

Update 6.9 Ruscheweyh and Sheil-Small [867, 868] have proved this result.

Problem 6.10 If $F(z)$, $G(z)$ are convex functions in Σ, it is known that for $0 < \lambda < 1$,

$$H(z) = \lambda F(z) + (1 - \lambda)G(z) \in \Sigma,$$

see Pommerenke [813]. Is it true that $H(z)$ is also convex?

(Ch. Pommerenke)

Update 6.10 Contrary to previous updates, no progress on this problem has been reported to us.

Problem 6.11 If $f(z)$, $g(z)$ are convex functions in S, is it true that for $0 < \lambda < 1$, $\lambda f + (1 - \lambda)g$ is starlike and univalent? A function $w = f(z)$ in S is said to be *starlike* if the image domain D is starlike with respect to the origin O, that is, if for any point P in D the straight line segment OP lies in D. It is known that $f(z)$ is convex if and only if $zf'(z)$ is starlike; see for example Hayman [513, Chap. 1].

Update 6.11 A counterexample has been given by MacGregor [693], who has also found the largest disc in which $\lambda f + (1 - \lambda)g$ is starlike.

Problem 6.12 If $f(z) = z + \sum_{k=2}^{\infty} a_{n_k} z^{n_k} \in S$, and

$$\liminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1,$$

then Pommerenke [814] has proved that

$$a_n = o\left(\frac{1}{n}\right) \tag{6.3}$$

and this is sharp. If

$$\liminf_{k\to\infty} (n_{k+1} - n_k) > 4,$$

then Hayman [496] has shown that

$$a_n = o(n^{-\frac{1}{2}}). \tag{6.4}$$

Are there intermediate gap conditions which allow us to interpolate between (6.3) and (6.4)?

Update 6.12 No progress on this problem has been reported to us.

Problem 6.13 Suppose that $f(z)$ is in S and that positive integers k, m, n are given. It is known that there exist complex numbers $c_0, c_1, \ldots, c_m$, depending on k, m, n and f, such that $c_0 = 1$, $|c_m| \geq 4^{-m}$ and

$$|c_0 a_{n+j} + c_1 a_{n+j+1} + \ldots + c_m a_{n+j+m}| \leq K n^{\alpha_m}, \qquad 0 \leq j \leq k, \tag{6.5}$$

where K is a constant depending on k, m and f, and $\alpha_m = 8/\sqrt{m - \frac{1}{2}}$. What are the best possible values for the α_m? See Pommerenke [816]. It is known that $\alpha_1 = 0$, and it may be conjectured that $\alpha_2 = -\frac{1}{3}$, $\alpha_3 = -\frac{1}{2} + \varepsilon$, and $\alpha_m < -\frac{1}{2}$ for $m > 4$.

If, in addition, $f(z)$ is starlike, then Pommerenke has shown that (6.5) holds with $\alpha_m = -1 + \frac{2}{m+1}$. The result is sharp.

Update 6.13 Pommerenke [818] has made some progress with this, showing that for large m,

$$\alpha_m < -\frac{1}{2} + \frac{2}{m^2}.$$

Problem 6.14 For $f(z)$ in S, set

$$A_n^{(k)} = \begin{vmatrix} a_n, & a_{n+1}, & \dots, & a_{n+k-1} \\ \dots & \dots & \dots & \dots \\ a_{n+k-1}, & a_{n+k}, & \dots, & a_{n+2k-2} \end{vmatrix}.$$

Using (6.5), Pommerenke [816] proved that if k is fixed

$$|A_n^{(k)}|^{1/k} = O(n^{j_k}), \qquad \text{as } n \to \infty, \tag{6.6}$$

with $j_k = -\frac{1}{2} + 16/\sqrt{k}$. Here the conjecture is that $j_k < -\frac{1}{2}$ for large k. It can be shown that for $k = 2$, (6.6) is false with $j_2 = 0$, and holds with $j_2 = \frac{1}{4}$; see Hayman [498].

If $f(z)$ is in addition starlike, then Pommerenke [816] showed that the best possible values of j_k are $j_k = -1 + 2/k$.

Update 6.14 Pommerenke [817] has proved that for $k = 2, 3, \dots$,

$$A_n^{(k)} = O\big(n^{\frac{3}{2} - \frac{1}{2}k - k\beta}\big), \qquad \text{as } n \to \infty$$

for some constant β, where $\beta > \frac{1}{4000}$.

Problem 6.15 If $f(z)$ is in S, write

$$f_\alpha(z) = \int_0^z f'(\zeta)^\alpha \, \mathrm{d}\zeta.$$

For what values of α is it true that $f_\alpha(z) \in S$? The result is known to hold for $\alpha \le (\sqrt{5} - 2)/3$ (see Duren, Shapiro and Shields [277]), but not for $\alpha > \frac{1}{3}$; see Royster [854].

If for z in $\mathbb{D}$, $f(z)$ is analytic and $|f''(z)/f'(z)| \le c/(1 - |z|^2)$, then $f(z)$ is univalent if

$$c \le 2(\sqrt{5} - 2), \tag{6.7}$$

see Duren, Shapiro and Shields [277]; but not if $c > 2$; see Hille [547]. What is the best value of c?

(P.L. Duren)

Update 6.15 Becker [94] has shown that the right-hand side of (6.7) could be improved to 1. Becker and Pommerenke [97] showed that this is best possible.

Problem 6.16 Let S^* be the class of all starlike functions $f(z)$ in S. Marx [709] conjectured that for each fixed z_0, $|z_0| < 1$, the set of all numbers $f'(z_0)$ for f in S^* coincides with the set of all numbers $k'(z)$, $|z| \leq |z_0|$, where

$$k(z) = \frac{z}{(1-z)^2}$$

is the Koebe function. This is known to be true for $|z| \leq 0.736$; see Duren [272].

(P.L. Duren)

Update 6.16 This is a version of the Marx conjecture [709]. A counterexample has been given by Hummel [562].

Problem 6.17 If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in S, then

$$A = \pi \sum_{n=1}^{\infty} n|a_n|^2$$

is the area of the image domain. What is the minimum value of A when a_2 is given?

Clearly $A \geq \pi(1 + 2|a_2|^2)$ always, but this bound is not sharp if $|a_2| > \frac{1}{2}$, since in this case $f(z) = z + a_2 z^2$ is not univalent in $\mathbb{D}$.

(H.S. Shapiro)

Update 6.17 This is solved by Aharonov, Shapiro and Solynin [8, 9].

Problem 6.18 If $F(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$ in Σ, then

$$A(F) = \pi - \pi \sum_{n=1}^{\infty} n|b_n|^2$$

is the area of the set of values not assumed by $F(z)$ in $|z| > 1$. If $F_n(z) \in \Sigma$, and

$$F_n(z) \to F(z), \qquad \text{as } n \to \infty,$$

for $|z| > 1$, under what additional hypotheses is it true that

$$A(F_n) \to A(F), \qquad \text{as } n \to \infty\,? \tag{6.8}$$

It is suggested that (6.8) might be true under some hypotheses on $(|z|^2 - 1)^2\{F(z), z\}$, where

$$\{F(z), z\} = \left(\frac{F''}{F'}\right)' - \frac{1}{2}\left(\frac{F''}{F'}\right)^2$$

is the Schwarzian derivative of $F(z)$.

(L. Bers)

Update 6.18 We note that there were errors in the original statement of this problem. No progress on this problem has been reported to us.

Problem 6.19 If $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $\sum_{n=1}^{\infty} |a_n| < +\infty$, can $f(z)$ map the unit circle $\mathbb{T}$ onto a curve of positive two-dimensional measure if

(a) $f(z)$ is in S,
(b) more generally, $f'(z) \neq 0$ in $\mathbb{D}$?

Update 6.19 No progress on this problem has been reported to us.

Problem 6.20 Let C be a closed curve inside the unit circle $\mathbb{T}$. Under what conditions on C does there exist a univalent function f in $\mathbb{D}$ such that $f(C)$ and $f(\mathbb{T})$ are both convex?

Update 6.20 No progress on this problem has been reported to us.

Problem 6.21 A function $f(z)$ analytic in $\mathbb{D}$ is said to be *typically real* if $f(z)$ is real, when and only when z is real; see Rogosinski [841].

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is typically real in $\mathbb{D}$, then

$$f(z) = \frac{z}{1 - z^2} P(z),$$

where $P(0) = 1, \operatorname{Re} P(z) > 0$ in $\mathbb{D}$. What other conditions must $P(z)$ satisfy to make $f(z)$ univalent in $\mathbb{D}$?

Update 6.21 No progress on this problem has been reported to us.

Problem 6.22 If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent and starlike of order $\frac{1}{2}$ in $\mathbb{D}$, that is,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \frac{1}{2},$$

find the radius of the largest disc $|z| < r$ in which $f(z)$ is convex. In other words, when is

$$\min_{|z|=r} \operatorname{Re} \left\{ \frac{P+1}{2} + z \frac{P'(z)}{P+1} \right\} > 0,$$

where $\operatorname{Re} P(z) > 0$ in $\mathbb{D}$ and $P(0) = 1$?

Update 6.22 No progress on this problem has been reported to us.

Notation Let $f(z)$ be analytic in a domain D and let $n(w)$ be the number of roots of the equation $f(z) = w$ in D. Then $f(w)$ is said to be *mean p-valent* in D if

$$\int_0^{2\pi}\int_0^R n(\rho e^{i\phi})\rho \, d\rho \, d\phi \le \pi p R^2, \qquad 0 < R < \infty. \tag{6.9}$$

Frequently, it is possible to prove analogous results to those for univalent functions for the wider class of mean p-valent functions. Counterexamples for mean p-valent functions are in general much easier to find. We briefly discuss how some of the preceding problems could be modified for mean p-valent functions. In connection to this, the following counterexample is valuable; see Pommerenke [812]:

If

$$f(z) = 1 + \sum_{m=1}^{\infty} a_m z^m, \tag{6.10}$$

where

(a) $\sum_{m=1}^{\infty} |a_m| < \varepsilon$,
(b) $\sum_{m=1}^{\infty} m|a_m|^2 < \varepsilon$,

then $f(z)$ is bounded and mean p-valent in $\mathbb{D}$ with $p < \varepsilon/(1-\varepsilon)^2$. In fact

$$n(\rho e^{i\phi}) = 0, \qquad \rho < 1 - \varepsilon,$$

and

$$\int_0^{2\pi}\int_0^R n(\rho e^{i\phi}) \, d\rho \, d\phi \le \pi \sum_{n=1}^{\infty} n|a_n|^2 < \pi\varepsilon < \pi \frac{\varepsilon}{(1-\varepsilon)^2} R^2,$$

if $R > 1 - \varepsilon$.

On the other hand, (a) and (b) permit isolated relatively large coefficients so that no boundedness or gap-condition for mean p-valent functions can imply more than $|a_n| = o(n^{-\frac{1}{2}})$ for this class. In particular, the results of Pommerenke and Clunie quoted in Problems 6.5, 6.7, 6.8 and 6.12 have no analogue for mean p-valent functions, though (6.4) holds for the wider class provided that

$$\limsup(n_{k+1} - n_k) > 4p,$$

and this condition is sharp; see Hayman [496].

We discuss modifications of some of the other problems for p-valent functions, and denote these by inserting a $'$ after the problem number. Since a polynomial of degree p is automatically mean p-valent for any positive integer p, it is clear that we cannot in general expect to limit the growth of mean p-valent functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

unless we have information about the coefficients a_0 to a_p. We write

$$\mu_p = \max_{0 \le \nu \le p} |a_\nu|,$$

and denote by $S(p)$ the class of functions $f(z)$ analytic and mean p-valent in $\mathbb{D}$, and such that $\mu_p = 1$. It is known (see for example Hayman [513, Theorem 3.5]) that, if $f(z)$ is in $S(p)$ with $p > \frac{1}{4}$,

$$|a_n| < A(p)n^{2p-1}.$$

The counterexample (6.10) shows that this is false for $p < \frac{1}{4}$. It was also shown by Spencer (personal communication to Hayman) that $|a_3| > 3$ is possible for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $S(1)$, so that the analogue of Bieberbach's conjecture fails for $n = 3$. However, Spencer [915] proved that $|a_2| \le 2$ in this case.

Problem 6.2′ If $A_n^{(p)} = \sup_{f \in S_p} |a_n|$ is it true that

$$\frac{A_n^{(p)}}{n^{2p-1}} \to K_p, \qquad \text{as } n \to \infty,$$

and if so, what is K_p? It is known that for a fixed f in $S(p)$, the limit

$$\alpha_f = \lim_{n \to \infty} \frac{|a_n|}{n^{2p-1}}$$

exists if $p > \frac{1}{4}$; see Hayman [487] and Eke [915].

Update 6.2′ No progress on this problem has been reported to us.

Problem 6.7′ Here the counterexample (6.10) shows that $|a_n| = o(n^{-\frac{1}{2}})$ is best possible for a bounded function $f(z)$ in $S(p)$.

Problem 6.8′ If we ask the analogous problems to those of Problem 6.8 for the class $S(p)$, the correct orders of magnitude are again known in many cases, but not the exact bounds. If f is in $S(p)$ and f is bounded, then

$$I_1(r, f') = o(1 - r)^{-\frac{1}{2}}$$

and this is sharp for the class, by our introductory example.

Problem 6.13′ The results of Pommerenke [816] were proved in fact for mean p-valent functions, and if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is mean p-valent with $p > \frac{1}{4}$, then (6.5) holds with $\alpha_m = -\frac{1}{2} + 8p^{\frac{3}{2}}/\sqrt{m}$. This result is probably far from the best possible, though clearly $\alpha_m \ge -\frac{1}{2}$ in all cases. Lucas [684] has shown that $\alpha_1 = 2p - 2$ if $p \ge 1$, and $\alpha_1 \le 2p - 2\sqrt{p}$ for $\frac{1}{4} < p < 1$. It is fair to conjecture that the correct value of α_1 is $p - 1$ for $\frac{1}{2} < p < 1$ and $-\frac{1}{2}$ if $p < \frac{1}{2}$. It might also be conjectured that $\alpha_m = -\frac{1}{2}$ for $m + 1 > 4p$.

The functions

$$f(z) = \left(\frac{1+z^{m+1}}{1-z^{m+1}}\right)^{2p/(m+1)} = 1 + \sum_{n=2}^{\infty} a_n z^n$$

provide counterexamples. These functions are mean p-valent, and $a_n = 0$ except when $(m+1)$ divides n, and for the remaining values of n we have, as $n \to \infty$,

$$|a_n| \sim C n^{2p/(m+1)-1}, \qquad \text{where } C \text{ is a constant.}$$

Thus (6.5) cannot hold with $\alpha_m < 2p/(m+1) - 1$.

Update 6.13′ Some work on this problem has been done by Noonan and Thomas [769].

Problem 6.14′ Here again the main conclusions extend to mean p-valent functions. In this case (6.6) holds with $j_k = -\frac{1}{2} + 16(p^3/k)^{\frac{1}{2}}$. This is still unlikely to be best possible.

Update 6.14′ Some work on this problem has been done by Noonan and Thomas [769].

Problem 6.23 A related problem concerns upper bounds for $|a_{n+1}| - |a_n|$ when $f(z)$ is mean p-valent. Lucas [684] has proved that

$$\big||a_{n+1}| - |a_n|\big| = O(n^{j_d}),$$

where $j_d = 2p - 2$, if $p \geq 1$; $j_d \leq 2p - 2\sqrt{p}$ if $\frac{1}{4} < p < 1$; and $j_d = -\frac{1}{2}$, if $p < \frac{1}{4}$. The result for $\frac{1}{4} < p < 1$ is probably not sharp. A similar question may be asked for symmetric mean p-valent functions of the type

$$f(z) = \sum_{n=0}^{\infty} a_n z^{an+b}.$$

The coefficients of such functions behave rather like those of functions $\sum a_n z^n$ which are mean p-valent. In particular, if $f(z) = z + \sum a_n z^{2n+1}$ is mean univalent, then Lucas [684] proved that

$$\big||a_{n+1}| - |a_n|\big| = O(n^{1-\sqrt{2}}).$$

Update 6.23 No progress on this problem has been reported to us.

Problem 6.24 If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(1)$, prove that on $|z| = r$,

$$|f(z)| \leq \frac{r}{(1-r)^2}.$$

It is shown by Garabedian and Royden [371] that $f(z)$ assumes in $\mathbb{D}$ each value w such that $|w| < \frac{1}{4}$ and hence that

$$|f(z)| \geq \frac{r}{(1+r)^2}, \qquad |z| = r.$$

The question of sharp bounds for $|f'(z)|$ and $|f'(z)|/|f(z)|$ is also open. The corresponding results for S are elementary; see for example Hayman [513, Chap. 1].

Update 6.24 No progress on this problem has been reported to us.

Notation If $f(z)$ is analytic in a domain D, and $n(w)$ is the number of roots of the equation $f(z) = w$ in D, then $f(z)$ is called *p-valent* in D if p is an integer and $n(w) \leq p$ always. If p is any positive number and

$$\frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\phi})\,\mathrm{d}\phi \leq p, \qquad 0 < R < \infty, \tag{6.11}$$

then $f(z)$ is called *circumferentially mean p-valent*. Clearly (6.11) implies (6.9), that is, circumferentially mean p-valence implies mean p-valence.

Problem 6.25 Suppose that p is an integer and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is p-valent in $\mathbb{D}$. It is conjectured by Goodman [424] that

$$|a_n| \leq \sum_{k=1}^{p} |a_k| D(p,k,n),$$

where

$$D(p,k,n) = \frac{2kn\prod_{\alpha=1}^{p}(n^2-\alpha^2)}{(p+k)!(p-k)!(n^2-k^2)}, \qquad 1 \leq k \leq p < n.$$

This result, containing the Bieberbach conjecture as a special case, is likely to be extremely difficult. The inequality if true would be sharp in all cases. No counterexamples are known and the conjecture has been proved only if $a_k = 0$ for $k = 1, 2, \ldots, (p-1)$ and $n = p+1$, see Spencer [915]; and $n = p+2$, see Jenkins [575, p. 160]. The conjecture is true and sharp for certain classes of p-valent functions, namely those which are typically real of order p; see Goodman and Robertson [433].

We recall that the conjecture is definitely false for areally mean p-valent functions, if $p = 1$ and $n = 3$, by the example of Spencer [915], though for circumferentially mean p-valent functions it remains true in this case; see Jenkins [575, p. 160].

Update 6.25 No progress on this problem has been reported to us.

Problem 6.26 Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is circumferentially mean p-valent and $f(z) \neq 0$ in $\mathbb{D}$. (This latter condition is a consequence of mean p-valency if $p < 1$.) It is conjectured that in this case, at least if $p \geq 1$, we have

$$|a_n| \leq A_{n,p} \tag{6.12}$$

where

$$F(z) = a_0 \left(\frac{1+z}{1-z} \right)^{2p} = \sum_{n=0}^{\infty} A_{n,p} z^{2n}. \tag{6.13}$$

The conjecture (6.12) is known to be true for $n = 1$ and all p; and for $n = 2, 3$ if $p = 1$, by the results quoted in Problem 6.25. It is certainly false for small positive p, and large n, since it would imply $a_n = O(n^{\varepsilon - 1})$ for every positive ε as $n \to \infty$, for a bounded univalent function. For $g(z)$ bounded and univalent and a positive ε, then if K is a sufficiently large positive constant, $g(z) + K$ is circumferentially mean ε-valent.

On the other hand, if $p > \frac{1}{4}$ and $f(z)$ is fixed, (6.12) holds for all sufficiently large n; see Hayman [513, Theorem 5.10]. In the special case when $p = 1$ and $f(z)$ is univalent, (6.12) reduces to the Littlewood conjecture $|a_n| \leq 4|a_0|n$. This conjecture is somewhat weaker than the Bieberbach conjecture. It was shown by Nehari [753] (see also Bombieri [138]) that we have at any rate $|a_n| \leq 4K_0|a_0|n$ if $f(z)$ is univalent, where K_0 is the constant of Problem 6.2. Thus, Littlewood's conjecture holds since $K_0 = 1$, as was pointed out in Update 6.2.

It also seems likely that for $\lambda p > 1$

$$I_\lambda(r, f) \leq I_\lambda(r, F)$$

if $f(z)$ satisfies the above hypotheses, and $F(z)$ is given by (6.13).

Update 6.26 See Update 6.2. Nothing beyond this is known.

Problem 6.27 Suppose that

$$g(z) = z + b_0 + b_1 z^{-1} + \dots$$

is univalent in $|z| > 1$. Is it true that for each positive ε we have

$$n|b_n| = O(n^\varepsilon) \left\{ \max_{0 < |\nu - n| < \frac{n}{2}} \nu(|b_\nu| + 1) \right\}, \qquad \text{as } n \to \infty\,?$$

This is suggested by some results of Clunie and Pommerenke.

(J. Clunie, Ch. Pommerenke)

Update 6.27 There were errors in the original statement of this problem. No progress on this problem has been reported to us.

Problem 6.28 Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is in S and that $P(z) = \sum_{k=0}^{n} b_k z^k$ is a polynomial of degree at most n. Is it true that

$$\max_{|z|=1} |P(z) * f(z)| \le n \max_{|z|=1} |P(z)|?$$

Here $P * f = \sum_{k=0}^{n} a_k b_k z^k$. The above result would imply Rogosinski's generalised Bieberbach conjecture but is weaker than Robertson's conjecture; see Sheil-Small [894].

(T. Sheil-Small)

Update 6.28 No progress on this problem has been reported to us.

Problem 6.29 With the above notation, $f(z)$ is in S if and only if for each pair of numbers ξ_1, ξ_2 satisfying $|\xi_1| \le 1$, $|\xi_2| \le 1$, we have

$$f(z) * \frac{z}{(1-\xi_1 z)(1-\xi_2 z)} \ne 0, \qquad 0 < |z| < 1.$$

On the other hand it is true that if $F(z) * f(z) \ne 0$, $0 < |z| < 1$ whenever f is in S, then $F(z)$ is starlike. What is the complete class of starlike functions having this property? $F(z) = z + z^n/n$ has the property since the Bieberbach conjecture holds.

(T. Sheil-Small)

Update 6.29 No progress on this problem has been reported to us.

Problem 6.30 If f is in S, Baernstein has shown that

$$\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |k(re^{i\theta})|^p \, d\theta, \qquad 0 < r < 1, \qquad 0 < p < \infty,$$

where $k(z)$ is the Koebe function.

Does the corresponding inequality hold for integral means of the derivatives at least for certain values of p? The best we can hope for is that it holds for $p \ge \frac{1}{3}$ because $k'(z) \in H^p$ for $p < \frac{1}{3}$, and there exist functions f in S for which $f'(z)$ does not belong to any H^p. (An example is due to Lohwater, Piranian and Rudin [680]). For close-to-convex functions it was proved by MacGregor [692] that the result holds for $p \ge 1$, and in fact the corresponding inequality holds for derivatives of all orders.

(A. Baernstein)

Update 6.30 Leung [657] proved that

$$\int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |k'(re^{i\theta})|^p \, d\theta, \qquad 0 < r < 1, \qquad 0 < p < \infty,$$

for all functions in a certain subclass of the Bazilevič functions which includes the close-to-convex functions. For $p < 1$ this was previously unknown even for starlike functions. Feng and MacGregor [336] have shown that in the full class S the inequality is correct in order of magnitude as $r \to 1$ if $p > 2/5$. The asymptotic problem therefore remained open for $1/3 \le p \le 2/5$, but Makarov [702] has given a counterexample if $p - \frac{1}{3}$ is sufficiently small. This is related to Problem 6.8 and to the Brennan conjecture; see Problem 6.96.

Problem 6.31 Duren [274] has shown that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is in S and if

$$(1-r)^2 f(r) = \lambda + O\big((1-r)^{\delta}\big), \qquad \text{as } r \to 1-,$$

for some λ, δ where $\lambda \neq 0$, $\delta > 0$, then

$$\frac{a_n}{n} = \lambda + O\Big(\frac{1}{\log n}\Big), \qquad \text{as } n \to \infty.$$

To what extent can this estimate be improved?

(P.L. Duren)

Update 6.31 No progress on this problem has been reported to us.

Problem 6.32 Let S_α, $0 < \alpha \le 1$, be the subclass of S of functions f such that $\mathbb{C} \setminus f(\mathbb{D})$ is a single piecewise analytic slit from some finite point ω_0 to ∞ that makes an angle at most $\alpha\pi/2$ with the radii vectors. What can be said about the Taylor coefficients of functions in S_α? If $f(-1) = \infty$ and $f(\mathrm{e}^{i\phi_f}) = \omega_0$, $-\pi < \phi < \pi$, find $\sup_{f \in S_\alpha} |\phi_f|$.

(K.W. Lucas)

Update 6.32 This has been solved by Prokhorov [827], who showed that

$$\sup_{f \in S_\alpha} |\phi_f| = \frac{1+\alpha}{1-\alpha}, \qquad 0 \le \alpha < 1.$$

Problem 6.33 The same questions as in Problem 6.32 can be asked under the alternative hypothesis that $\mathbb{C} \setminus f(\mathbb{D})$ is a single piecewise analytic slit from some finite point (ω_0, say) to ∞ lying in an infinite sector with opening $\alpha\pi$, $0 < \alpha \le 2$, and vertex ω_0.

(K.W. Lucas and D.A. Brannan)

Update 6.33 No progress on this problem has been reported to us.

Problem 6.34 A function $f(z) = z + a_2 z^2 + \ldots$ analytic in $\mathbb{D}$ is said to belong to *Ruscheweyh's class* M if the $*$ Hadamard convolution of f with every normalised convex function is univalent. All close-to-convex functions lie in M. Suppose that $g(z) = z + b_2 z^2 + \ldots$ is analytic in $\mathbb{D}$ and satisfies the condition

$$\operatorname{Re}\left\{\frac{\phi * (gF)}{\phi * g}\right\} > 0, \qquad |z| < 1, \tag{6.14}$$

for all normalised convex functions ϕ, and all normalised functions F of positive real part in $\mathbb{D}$. If g is starlike, then (6.14) certainly holds; is (6.14) true for any other g, or (maybe) for some significant larger family of g?

If g satisfies (6.14) and the condition

$$\operatorname{Re}\left(\frac{zf'}{g}\right) > 0, \qquad |z| < 1,$$

then $f \in M$; does this classify M?

(T. Sheil-Small)

Update 6.34 Sheil-Small observes that the suggested classification for M is invalid. The duality method shows that if $g(z) = z + \dots$ is analytic and satisfies

$$\phi(z) * \frac{1 + xz}{1 - yz} g(z) \neq 0, \qquad 0 < |z| < 1,$$

for $|x| = |y| = 1$ and every convex function ϕ, then g is starlike. Duality also shows that M is a linear invariant family. Its geometrical classification remains unknown.

Problem 6.35 Let $\mathbb{O}$ be a subset of $\mathbb{D} = \{|\omega| < 1\}$. Find a characterisation of those $\mathbb{O}$ that are of the form $(\mathbb{C} \setminus f(\mathbb{D})) \cap \mathbb{D}$ for some f in S. What is the maximum area of $\mathbb{O}$?

Given that $\omega_1, \omega_2 \in \mathbb{D}$, how does one tell whether there exists such an f with $\omega_1, \omega_2 \in \mathbb{O}_f$? The same question could be asked for $\omega_1, \omega_2, \omega_3$ etc.

(A.W. Goodman)

Update 6.35 This problem was originally wrongly attributed to D.A. Brannan. Barnard and Suffridge [35] have observed that an extremal domain $g(\mathbb{D})$, g in S, for this problem must be circularly symmetric and have as its boundary (up to rotation) the negative reals up to -1, an arc λ of the unit circle which is symmetric about -1, and a Jordan curve γ symmetric about the real axis, lying in the unit disc $\mathbb{D}$ and connecting the endpoints of λ. If γ is assumed to be piecewise twice continuously differentiable, then it can be proved that $|zg'(z)|$ is constant on $g^{-1}(\gamma)$. See also Barnard, Pearce, and Solynin [79] for a summary of progress.

Problem 6.36 Suppose that f is in S and define

$$f_p(z) = [f(z)]^p = z^p + \sum_{n=p+1}^{\infty} a_{n,p} z^n.$$

What can be said about bounds for $a_{n,p}$? If $|a_{n,1}| \leq Kn$ for all n and fixed K, then, for integral p,

$$|a_{n,\,p}| \leq K^p \frac{2p(2p+1)\dots(n+p-1)}{(n-p)!},$$

but it might be easier to obtain bounds for f_p than for f.

(W.K. Hayman)

Update 6.36 It was observed independently by Aharonov (unpublished), Grinshpan [444] and Hayman and Hummel [514], that de Branges' theorem yields the sharp bounds

$$|a_{n,\,p}| \le \frac{\Gamma(2p+n)}{\Gamma(n+1)\Gamma(2p)} = b_{n,\,p}, \qquad n = 0, 1, 2, \dots, \qquad p \ge 1.$$

The result holds trivially for all p if $n = 0$ or 1, but, as Grinshpan [444] has shown, it fails for $p < 1$ whenever $n \ge 2$. However, Hayman and Hummel [514] showed that at least for $\frac{1}{4} \le p < 1$, if

$$A_{n,\,p} = \sup |a_{n,\,p}|,$$

then

$$\frac{A_{n,\,p}}{b_{n,\,p}} \to K_p, \qquad \text{as } n \to \infty.$$

Also $K_p > 1$ if $p < 0.499$. We conjecture that $K_p = 1$ if and only if $p \ge \frac{1}{2}$.

Problem 6.37 Suppose that $f(z) = z + c_3 z^3 + c_5 z^5 + \dots$ is an odd univalent function in $\mathbb{D}$, and let $d_n = |c_{2n+1}| - |c_{2n-1}|$. It is known that $d_n \to 0$, the best known estimate being $d_n = O(n^{1-\sqrt{2}})$; can this be improved to

$$d_n = O(n^{-\frac{1}{2}})\,? \tag{6.15}$$

(Nothing better is possible, as is shown by the fourth-root transform of the Koebe-function.)

Milin [734–736] proves that $d_n \le K(\alpha)n^{-\frac{1}{2}}$ for functions f such that $g(z) = [f(z^{\frac{1}{2}})]^2 = z + a_2 z^2 + \dots$ is univalent in $\mathbb{D}$ and has positive Hayman number $\alpha = \lim_{n\to\infty} n^{-1}|a_n|$; but $K(\alpha) \to \infty$ as $\alpha \to 0$. Levin [662] showed that $d_n = O(n^{-\frac{1}{2}} \log n)$ if c_n vanishes for $n \not\equiv 1 \pmod 4$.

(P.L. Duren)

Update 6.37 Elhosh [297] proved (6.15) for close-to-convex functions. For odd univalent functions Ye [1008] has shown that $d_n = O(n^{-\frac{1}{2}} \log n)$, thus proving this conjecture.

Problem 6.38 With the notation of Problem 6.37, is it true that

$$\sum_{n=1}^{\infty} n^{-\beta} d_n^2 < \infty,$$

where $\beta = (\sqrt{2} - 1)^2$?

(K.W. Lucas)

Update 6.38 Milin [733] has proved that for every positive β,

$$\sum_{n=1}^{\infty} n^{-2\beta} d_n^2 < 50 \sum_{n=1}^{\infty} \frac{1}{n^{1+2\beta}}.$$

Problem 6.39 Suppose f is in S and define $h(z) = \{f(z^2)\}^{\frac{1}{2}} = z + c_3 z^3 + c_5 z^5 + \ldots$ Robertson's conjecture (see Sheil-Small [894]) asserts that

$$1 + |c_3|^2 + |c_5|^2 + \ldots + |c_{2n-1}|^2 \leq n.$$

This is known to be true if f is starlike; is it true if f has real coefficients, or if f is close-to-convex?

(P.L. Duren)

Update 6.39 See Update 6.1.

Problem 6.40 If $f(z)$ is in S and if the a_n are real, then

$$1 + a_3 + \ldots + a_{2n-1} \geq a_n^2, \qquad n \geq 1. \tag{6.16}$$

The Bieberbach conjecture for such functions follows easily; see for example FitzGerald [346]. It was pointed out by Clunie and Robertson that (6.16) holds for normalised typically-real functions; the inequality is clear from the representation formula for these functions.

The Bieberbach conjecture for S would follow analogously if we could prove that if f is in S,

$$1 + |a_3| + \ldots + |a_{2n-1}| \geq |a_n|^2, \qquad n \geq 1. \tag{6.17}$$

Bshouty (unpublished) has shown that if f is in S, then there exists an $N(f)$ such that (6.17) holds for $n > N(f)$.

(C.H. FitzGerald)

Update 6.40 No progress on this problem has been reported to us.

Problem 6.41 Let $K(\alpha)$ and $S^*(\alpha)$ be those subsets of S consisting of the class of functions convex in $\mathbb{D}$ of order α, that is,

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geq \alpha, \qquad |z| < 1,$$

and starlike of order α in $\mathbb{D}$ that is,

$$\operatorname{Re}\left[z\frac{f'z)}{f(z)}\right] \geq \alpha, \qquad |z| < 1,$$

respectively.

(a) Prove that (see Goel [398])

$$\min_{f \in K(\alpha)} \min_{|z|=r} \left| \frac{zf'(z)}{f(z)} \right| = \min_{f \in K(\alpha)} \min_{|z|=r} \left[\operatorname{Re} \frac{zf'(z)}{f(z)} \right].$$

(b) Show that the functions

$$(2\alpha - 1)^{-1}[1 - (1 - z)^{2\alpha - 1}], \qquad \alpha \neq \frac{1}{2} \qquad \text{and} \qquad -\log(1 - z)$$

are starlike of order

$$4^{\alpha}(2\alpha - 1)[4 - 4.2^{\alpha}]^{-1} \qquad \text{and} \qquad (\log 4)^{-1}$$

respectively; see MacGregor [694].

Either (a) or (b), combined with the work of Jack [569], would solve the following problem of Keogh: find

$$\max_{f \in K(\alpha)} \{\beta : f \in S^*(\beta)\}.$$

(D. Benjamin)

Update 6.41 Sheil-Small observes that Feng and Wilken [990] have established Jack's assertion [569], which therefore completes the solution of the order of starlikeness of a function convex of order α. Fournier [350] gives a new proof of Jack's assertion, and discusses an extension for polynomials.

Problem 6.42 If f is in S, write

$$\log[f(z)/z] = 2 \sum_{k=1}^{\infty} \gamma_k z^k.$$

If f is starlike then $|\gamma_k| \leq 1/k$; this is false in general, even in order of magnitude. Milin [276, p. 151] has shown that

$$\sum_{k=1}^{n} k|\gamma_k|^2 \leq \sum_{k=1}^{n} \frac{1}{k} + \delta,$$

where $\delta < 0.312$, and that δ cannot be reduced to 0. Milin conjectured that

$$\sum_{n=1}^{N} \sum_{k=1}^{n} k|\gamma_k|^2 \leq \sum_{n=1}^{N} \sum_{k=1}^{n} \frac{1}{k}, \tag{6.18}$$

which would imply Robertson's conjecture (see Problem 6.39). Inequality (6.18) is known to be true for $N = 1, 2, 3$; see Grinshpan [442]. Is it true in general if f has real coefficients, or if f is close-to-convex?

(P.L. Duren)

Update 6.42 See Update 6.1.

Problem 6.43 Using the notation of Problem 6.42, it is well-known that

$$\left|\sum_{k=1}^{\infty} k\gamma_k z^k\right| = O\left(\frac{1}{1-r}\right), \qquad r \to 1-,$$

for $|z| = r < 1$; is it true that

$$\sum_{k=1}^{\infty} k|\gamma_k|r^k = O\left(\frac{1}{1-r}\right), \qquad r \to 1- ?$$

(D. Aharonov)

Update 6.43 Hayman [509] has constructed a function f in S for which

$$\sum_{k=1}^{\infty} k^2|\gamma_k|^2 r^k \neq o\left(\frac{1}{1-r}\log\frac{1}{1-r}\right),$$

but the general problem remains open.

Problem 6.44 Let f, g be formal power series

$$\sum_{n=0}^{\infty} a_n z^n, \qquad \sum_{n=0}^{\infty} b_n z^n$$

respectively, and define

$$(f \otimes g)(z) = \sum_{n=1}^{\infty} a_n b_n n^{-1} z^n.$$

Let S_R denote the class of functions in S with real coefficients. Prove (or disprove) that $f, g \in S_R$ implies that $f \otimes g \in S_R$. (Robertson (no citation given) has proved the corresponding result for typically-real functions.)

(J.G. Krzyż)

Update 6.44 Bshouty [170] has shown that the result is false by producing a counterexample using a deep result of Jenkins; see [823, Corollary 4.8 and Example 4.5, p. 120].

Problem 6.45 Let $S^*(\alpha)$ be the *class of* α*-strongly-starlike functions* f, that is, those f in S for which

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, \qquad \text{for } |z| < 1,$$

where $0 < \alpha < 1$.

(a) Prove (or disprove) that $S^*(\alpha)$ is closed under $\otimes$ (see Problem 6.44).
(b) Prove (or disprove) that, if $f \in S^*(\alpha)$ and $g \in S^*(\beta)$, then $f \otimes g \in S^*(\gamma)$ where $\gamma = \gamma(\alpha, \beta) < 1$.

One could ask the same question for different convolutions in place of $\otimes$.

(J.G. Krzyż)

Update 6.45 Barnard and Kellogg [77] have observed that $S^*(\alpha)$ is closed under the Hadamard convolution $*$ and that $\gamma(\alpha, \beta) = \min[\alpha, \beta]$ involves the Ruscheweyh–Sheil-Small theorem [868]. Suppose that $f, g \in S^*(\alpha)$ and define

$$\phi = g * \log\frac{1}{1-z}.$$

Then ϕ is convex and f starlike, so that

$$\left|\arg\left(\frac{\phi * (zf')}{\phi * f}\right)\right| < \frac{\alpha\pi}{2},$$

as required.

Problem 6.46 Suppose that f is in S and is starlike. Is it true that

$$\big||a_{n+1}| - |a_n|\big| \leq 1\,? \tag{6.19}$$

This is certainly true if $\lim_{r\to 1}(1-r)M(r, f) > 0$ (Brannan, not published). Sheil-Small (no citation given) has obtained an upper bound 2 in (6.19). Notice that in addition to the 'obvious' extremal functions $z(1-z^2)^{-1}$ and $z(1-z)^{-2}$ we have $z(1+z+z^2)^{-1}$.

(J.G. Clunie)

Update 6.46 Leung [656] has solved this problem by proving that

$$\big||a_{n+1}| - |a_n|\big| \leq 1, \qquad n = 1, 2, \ldots,$$

for all starlike functions f in S. By a similar method, Leung [658] has also proved a conjecture of Robertson that

$$\big|n|a_n| - m|a_m|\big| \leq |n^2 - m^2|, \qquad n, m = 1, 2, \ldots,$$

for all close-to-convex functions f in S. For the full class S, Grinshpan [443] gives

$$-2.97 \le |a_{n+1}| - |a_n| \le 3.61, \qquad n = 1, 2, \ldots.$$

Duren [275] has shown that

$$\big||a_{n+1}| - |a_n|\big| \le \mathrm{e}^{\delta}\alpha^{-1/2} < 1.37\alpha^{-1/2}, \qquad n = 1, 2, \ldots,$$

for all f in S with positive Hayman index α, where δ is Milin's constant, $\delta < 0.312$. Hamilton [463] has proved Leung's results independently.

Problem 6.47 If f is in S and f' is also univalent in $\mathbb{D}$, what can be said about $\max |a_n|$, $n \ge 2$? The function $z(1-z)^{-1}$ shows that $\max |a_n| \ge 1$ whenever $n \ge 2$.

(J.G. Clunie)

Update 6.47 Define $S' = \{f : f \in S \text{ and } f' \text{ is univalent}\}$. It was noted by Barnard [75] that if $f(z) = \sum a_n z^n$ is in S' and $A_2 = \max_{f \in S'} |a_2|$, then $|a_n| \le 2A_2(n-1)/n$, whenever the Bieberbach conjecture holds for $(f'-1)/2A_2$. It was conjectured that $A_2 = 2/(\pi - 2) \approx 1.75$. However, it has been shown by Barnard and Suffridge [81] that the function F defined by

$$F'(z) = \frac{1+z}{(1-z)^2} \exp\left\{\frac{1}{i\pi}\int_0^z \log\left(\frac{1+iw}{1-iw}\right)\frac{\mathrm{d}w}{w}\right\} = 1 + 2B_2 z + \ldots$$

is in S' with $B_2 = \frac{3}{2} + \frac{1}{\pi} \approx 1.82$. F has the property that it takes the left half of the unit circle onto a slit along the negative reals such that $-\infty < F(i) \le F(\mathrm{e}^{i\theta}) \le F(-1)$ for $\pi/2 \le \theta \le 3\pi/2$; while F' takes the right half of the unit circle onto a slit along the negative reals such that $-\infty \le F'(\mathrm{e}^{i\theta}) \le F'(i) < 0$ for $-\pi/2 \le \theta \le \pi/2$. If $C' = \{g \in S' : g \text{ and } g' \text{ are convex}\}$, then Barnard and Suffridge [81] have shown that if $g(z) = \sum b_n z^n$ is in C' then $|b_n| \le 4n/3$. This is sharp since the function defined by $G'(z) = 1 + 4z/3(1-z)$ is in C'.

Problem 6.48 Suppose that f is in S. The coefficient problem, except in certain cases, remains open for each of the following subclasses of univalent functions (we limit ourselves to one-parameter families):

(a) $B(\alpha)$, the class of Basilevič functions comprising functions f such that

$$f(z) = \left[\int_0^z p(t)s^{\alpha}(t)t^{-1}\,\mathrm{d}t\right]^{1/\alpha},$$

where $\alpha > 0$, $p(t) = 1 + p_1 t + \ldots$ is analytic and of positive real part in $|t| < 1$ and $s(t) = t + s_2 t^2 + \ldots$ is starlike in $|t| < 1$.

(b) $M(\alpha)$, the class of Mocanu–Reade functions comprising functions f such that

$$\mathrm{Re}\left[\alpha\left(1 + \frac{zf''}{f'}\right) + (1-\alpha)\frac{zf'}{f}\right] > 0,$$

where $0 < \alpha < 1$.

(c) $S^*(\alpha)$, the class of strongly-starlike functions, comprising functions f such that

$$\left|\arg\left(\frac{zf'}{f}\right)\right| < \frac{\alpha\pi}{2},$$

where $0 < \alpha < 1$.

(D.A. Brannan)

Update 6.48(a) Thomas writes that in recent collaboration with Nak Eun Cho and Young Jae Sim (no citation given), partial answers to the problem of finding bounds for the difference of coefficients for Basilevič functions have been obtained.

Update 6.48(b) This has been solved completely. A full treatment can be found in Thomas, Tuneski and Allu [938].

Update 6.48(c) Sharp bounds are now known for $n = 2$, 3 and 4; see Ali [28] and Ali and Singh [29].

Problem 6.49 What are the extreme points of the following classes of functions:

(a) Basilevič functions (see Problem 6.48)?
(b) $S^*(\alpha)$ (see Problem 6.48)?
(c) Close-to-convex functions of order α, $0 < \alpha < 1$?
(d) Functions of boundary rotation $k\pi$, $2 < k < 4$?

(D.A. Brannan)

Update 6.49 No progress on this problem has been reported to us.

Problem 6.50 If $0 \le \alpha \le 1$, and $f(z), g(z) \in \Sigma$, and if we define $F(z)$ by

$$F(z) = f(z)^{1-\alpha} g(z)^{\alpha}, \qquad |z| > 1; \tag{6.20}$$

$$= z + \sum_{n=0}^{\infty} A_n z^{-n}, \qquad |z| \text{ sufficiently large}; \tag{6.21}$$

is it true that

$$\sum_{n=1}^{\infty} n|A_n|^2 \le 1\,?$$

See Thomas [936].

(D.K. Thomas)

Update 6.50 The original statement of this problem contained errors which have now been corrected. No progress on this problem has been reported to us.

Problem 6.51 Let D be a domain in $\mathbb{C}$ (containing the origin) of connectivity n, and let $S(D)$ be the class of analytic univalent functions in D with $f(0) = 0$, $f'(0) = 1$. Find the functions f in $S(D)$ that minimise

$$\int_D\int |f'(z)|^2 \, \mathrm{d}\sigma_z.$$

(D. Aharonov)

Update 6.51 No progress on this problem has been reported to us.

Problem 6.52 Suppose that $f(z)$ is analytic in $\mathbb{D}$ and has the whole complex plane as its range. Does there necessarily exist a bounded univalent function $g(z)$ in $\mathbb{D}$ such that $f(z) + g(z)$ has the whole complex plane as its range?

(L.A. Rubel)

Update 6.52 No progress on this problem has been reported to us.

Problem 6.53 Hengartner and Schober [537] and Goodman and Saff [434] have shown that if $f(z) = z + a_2z^2 + \ldots$ maps $\mathbb{D}$ univalently onto a domain G_1 that is convex in the direction of the imaginary axis (CIA), then $G_r = \{f(|z| < r)\}$ is not necessarily CIA for all r, $r < 1$, or even for r bigger than some constant.

(a) Find $\sup\{r : G_r$ is necessarily CIA$\}$. (A result of Goodman and Saff [434] suggests that this is $\sqrt{2} - 1$.)
(b) Find reasonable sufficient conditions on G_1 (or equivalently, on f) that imply that G_r is in CIA for all r in $(0, 1)$.
(c) Suppose that we slit G_1 along the real axis and let G', G'' be those two components of the resulting family of domains that have 0 on their boundaries. If $G_1 = G' \cup G''$, does it necessarily follow that G_r is CIA for all r in $(0, 1)$?
(d) If G_1 is CIA but G_{r_0} is not CIA, is it true that G_r is not CIA for $r_0 < r < 1$?

(D.A. Brannan)

Update 6.53 Brown [162] proves a weaker form of the Goodman–Saff conjecture and answers Problem 6.53(d) by Brannan negatively. Prokhorov [828] and Ruscheweyh and Salinas [865] independently verify the Goodman–Saff conjecture using different methods. A more general problem was solved by Prokhorov and Szynal [829].

Problem 6.54 Let D be a Jordan domain with boundary C; $\{F_n(z)\}_1^\infty$ be the sequence of Faber polynomials for D; and $S(D)$ be the class of univalent functions

$$f(z) = F_1(z) + \sum_{n=2}^{\infty} a_n F_n(z)$$

in D. Does the coefficient region of at least one a_n have the same shape as C, or is it at least in the subclass of starlike functions? Royster [852] has shown that if f is starlike in an ellipse, then there exists a direction θ_f and sequences

$$\{\lambda_n\}_2^\infty, \qquad \{\mu_n\}_2^\infty, \qquad \lambda_n > \mu_n > 0,$$

such that each coefficient a_n lies in an ellipse of centre 0, major axis λ_n, minor axis μ_n, inclined at an angle θ_f to the real axis. (The λ_n, μ_n are known.)

(K.W. Lucas)

Update 6.54 No progress on this problem has been reported to us. See Problem 7.2 for information on Faber polynomials.

Problem 6.55 Let $f(z)$ be a normalised bounded starlike function in $\mathbb{D}$, and set

$$f(\xi) = \lim_{r \to 1-} f(r\xi),$$

where $\xi \in E \subset \mathbb{T}$. Is it true that $f(E) = \{f(\xi) : \xi \in E\}$ has zero linear (that is, one-dimensional Hausdorff) measure if cap $E = 0$, or at least if E has zero logarithmic measure? (It is known that meas$\{\arg f(\xi) : \xi \in E\} = 0$ if cap $E = 0$.)

(Ch. Pommerenke)

Update 6.55 No progress on this problem has been reported to us.

Problem 6.56 Let $S_R(q)$ be the class of normalised univalent functions in $\mathbb{D}$ with real coefficients that admit a quasiconformal extension to the whole plane with complex dilatation bounded (in modulus) almost everywhere by q, $q < 1$. Following the notation in Problem 6.44, prove (or disprove) that, if $f \in S_R(p)$ and $g \in S_R(q)$, then $f \otimes g \in S_R(r)$.

(J.G. Krzyż)

Update 6.56 No progress on this problem has been reported to us.

Problem 6.57 It is known that functions f in $\Sigma(q)$, that is, those f analytic and univalent in $\mathbb{D}$, normalised by $f(z) = z + b_0 + b_1 z^{-1} + \ldots$ and having a quasiconformal extension to the whole plane with $|f_{\bar{z}}/f_z| \le q < 1$ almost everywhere, satisfy

$$\int\int_{|\zeta|>1} |u(z,\zeta)|^2 \, \mathrm{d}\sigma_\zeta \le \frac{\pi q^2}{(|z|^2 - 1)^2}, \tag{6.22}$$

where

$$u(z,\zeta) = \frac{f'(z) f'(\zeta)}{f(z) - f(\zeta)} - \frac{1}{(z-\zeta)^2}.$$

Prove (or disprove) that (6.22) is also sufficient for f to have a quasiconformal extension to the whole plane, possibly for q sufficiently small.

(J.G. Krzyż)

Update 6.57 This has been answered affirmatively by Zuravlev [1022].

Problem 6.58 Following the notation in Problem 6.57, the well-known Golusin inequality for functions f in $\Sigma(q)$ (defined in Problem 6.57) is:

$$\left|\log \frac{f'(z)f'(\zeta)(z-\zeta)^2}{[f(z)-f(\zeta)]^2}\right| \leq q \log \frac{|z\overline{\xi}-1|^2}{(|z|^2-1)(|\zeta|^2-1)}. \tag{6.23}$$

Prove (or disprove) that (6.23) is also sufficient for f to have a quasiconformal extension to the whole plane, possibly for q sufficiently small.

(J.G. Krzyż)

Update 6.58 No progress on this problem has been reported to us.

Problem 6.59 Let D be a plane domain containing ∞. Let there be given a continuous assignment of numbers (thought of as angles) to the components of $\mathbb{C} \setminus D$. Consider conformal mappings of D onto the complement in the extended plane of straight line segments. Show that one of these mappings is such that the straight line segments make angles with the positive real axis equal to the corresponding preassigned angles. In other words, under the mapping, each component of $\mathbb{C} \setminus D$ is associated with a slit in its preassigned direction.

(B. Rodin; communicated by C. FitzGerald)

Update 6.59 FitzGerald and Weening [349] show that Rodin's continuity assumption is sufficient to imply the existence of a rectilinear slit map achieving any angle assignment with finite range.

Problem 6.60 Let C be a closed Jordan curve. Then if $f(z) = z + a_2z^2 + \ldots$ and $g(z) = z^{-1} + b_0 + b_1z + \ldots$ map $\mathbb{D}$ onto the inside and outside of C respectively, the area principle shows that C is the unit circumference $\mathbb{T}$. If we remove the normalisation on g and replace g by $g_1(z) = b_{-1}z^{-1} + b_0 + b_1z + \ldots$, what can be said about the connection between f and g_1? For example, what about the asymptotic behaviour of their coefficients, or their Lipschitz behaviour on $\mathbb{T}$, or about prime ends (if we make C slightly non-Jordan)?

(D.A. Brannan)

Update 6.60 Lesley [655] has obtained best possible results on the Lipschitz behaviour.

Problem 6.61 Let D_1, D_2 be Jordan domains bounded by rectifiable curves C_1, C_2 of equal length. Suppose that an isometric sewing of C_1 and C_2 is everywhere conformally admissible, thus generating a Riemann surface equivalent to a sphere S. Is the curve C on S which corresponds to C_1 (and C_2) necessarily rectifiable?

(A. Huber)

Update 6.61 No progress on this problem has been reported to us.

Problem 6.62 Let D_1 and D_2 be bounded Jordan domains, bounded by curves C_1 and C_2 of bounded boundary rotation (in the sense of Paatero; see for example Noonan

[768]); then C_1 and C_2 are rectifiable, and we shall assume that they have the same length. It is known that in this case, every isometric sewing is conformally admissible and generates a Riemann surface which is equivalent to a sphere S. It follows from results of Aleksandrov (no citation given) and Reshetnjak (no citation given) that the curve C on S which corresponds to C_1 (and C_2) is of bounded boundary rotation. Can one find a function-theoretic proof of this?

(A. Huber)

Update 6.62 No progress on this problem has been reported to us.

Problem 6.63 Let α be a homeomorphic mapping of $(0, \infty)$ onto $(\alpha(0), \infty)$, $\alpha(0) \geq 0$, such that $x \to \alpha(x) + i$ defines a conformal sewing of the half-strip

$$H = \{z : \operatorname{Re} z > 0, 0 \leq \operatorname{Im}(z) \leq 1\}.$$

If α is hyperbolic (that is, if the generated Riemann surface has a hyperbolic end at ∞), does there exist a positive continuous function $\varepsilon : (0, \infty) \to \mathbb{R}$ with the property that each conformal sewing $x \to \beta(x) + i$ of H satisfying the inequality

$$|\beta(x) - \alpha(x)| < \varepsilon(x), \qquad 0 < x < \infty,$$

is also hyperbolic?

(C. Constantinescu; communicated by A. Huber)

Update 6.63 No progress on this problem has been reported to us.

Problem 6.64 Let α be real and suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ with $f(z)f'(z)/z \neq 0$. We say f is in M_α, the *class of alpha-convex functions*, if

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'}{f} + \alpha\left(1 + \frac{zf''}{f'}\right)\right] > 0$$

for $|z| < 1$. Note that $M_0 = S^*$, the class of starlike functions and $M_1 = K$, the class of convex functions. It is known that $M_\alpha \subset S^*$ for all α.

Clunie and Keogh [215] have shown the following:

(a) If $\sum_{n=2}^{\infty} n|a_n| < 1$ then $f \in M_0$.
(b) If $\sum_{n=2}^{\infty} n^2|a_n| < 1$ then $f \in M_1$.

What generalisation of these conditions implies that f belongs to M_α?

(S. Miller)

Update 6.64 No progress on this problem has been reported to us.

Problem 6.65 Given M, $1 < M < \infty$, let $S^*(M)$ be the class of starlike univalent functions f in $\mathbb{D}$ with $f(0) = 0$, $f'(0) = 1$, and $|f(z)| \leq M$ for $|z| < 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, find $\sup\{|a_3| : f \in S^*(M)\}$ for $\mathrm{e} < M < 5$. Barnard and Lewis [78] have found the supremum for other values of M.

(J.L. Lewis)

Update 6.65 No progress on this problem has been reported to us.

Problem 6.66 Describe the extreme points of the class Σ_0 consisting of all functions g in Σ with constant term $b_0 = 0$. Springer (see Pommerenke [823, p. 183]) showed that every g in Σ_0 whose omitted set has measure zero is an extreme point. Is this condition also necessary?

(P.L. Duren)

Update 6.66 Hamilton [464] answers this question positively.

Problem 6.67 Let f be univalent in $\mathbb{D}$ and let $f(\mathbb{D})$ be a Jordan domain. Does the condition

$$\limsup_{|z|\to 1}(1-|z|^2)|f''(z)/f'(z)| < 2 \tag{6.24}$$

imply that f has a quasiconformal extension over the unit circle?

It is known that, for $|c| < 1$, the condition

$$\limsup_{|z|\to 1}\left|(1-|z|^2)zf''(z)/f'(z) - c\right| < 1$$

is sufficient for f to have a quasiconformal extension. It is also known that (6.24) is sufficient if f satisfies in addition

$$\limsup_{|z|\to 1}(1-|z|^2)^2|S_f(z)| < 2,$$

where S_f denotes the Schwarzian derivative. It might also be asked whether the last condition alone is already sufficient. In both cases, the constant 2 on the right-hand side would be best possible.

(J. Becker)

Update 6.67 The first question has been answered positively by Gehring and Pommerenke [391].

Problem 6.68 Let Σ be the class of univalent functions in $\{|z| > 1\}$ with the usual normalisation $f(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$. Let S_f denote the Schwarzian derivative of f. Does

$$\sup_{|z|>1}(|z|^2-1)^2\left|S_{f_n}(z) - S_f(z)\right| \to 0, \qquad \text{as } n \to \infty,$$

imply that

$$\sup_{|z|>1}(|z|^2-1)\left|\frac{f_n''(z)}{f_n'(z)} - \frac{f''(z)}{f'(z)}\right| \to 0, \qquad \text{as } n \to \infty,$$

if $f, f_n \in \Sigma, n = 1, 2, \ldots$?

This is known to be true if f, f_n have quasiconformal extensions onto the plane.

(J. Becker)

Update 6.68 No progress on this problem has been reported to us.

Problem 6.69 Let B be the Banach space of analytic functions ϕ in $\{|z| > 1\}$ with finite norm

$$\|\phi\| := \sup_{|z|>1} (|z|^2 - 1)|z\phi(z)|.$$

Let S and T be the subsets defined by $S := \{f''/f' : f \in \Sigma\}$ and $T := \{f''/f' : f \in \sum, f$ has a quasiconformal extension to $\mathbb{C}\}$.

It is known that T is topologically equivalent to the universal Teichmüller space. From results of Ahlfors and Gehring (no citations given), it follows that T is a subdomain in B and that $S \setminus \overline{T} \neq \emptyset$, where $\overline{T}$ denotes the closure of T.

Is it also true (analogous to another result of Gehring (no citation given)) that $T = S^0$, where S^0 denotes the interior of S?

This would follow if the answer to Problem 6.68 were affirmative. The problem is closely related to a characterisation of quasicircles given by Gehring.

(J. Becker)

Update 6.69 This has been proved by Astala and Gehring [50].

Problem 6.70 Is every extreme point of S a support point? Is every support point an extreme point?

(P.L. Duren)

Update 6.70 Hamilton [468] has constructed examples of extreme points which are not support points and a different construction has been given by Duren and Leung [269]. The converse problem still seems to be open.

Problem 6.71 For each function f in S, it can be shown that

$$\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right)$$

as $r \to 1$. Hayman [509] has constructed an example showing that 'O' cannot be replaced by 'o'. For certain subclasses of S, such as the starlike functions and the functions with positive Hayman index (including all support points of S), the estimate can be improved to $O(1/(1-r))$. Can the same improvement be made for the extreme points of S?; for close-to-convex functions?

(P.L. Duren)

Update 6.71 No progress on this problem has been reported to us.

Problem 6.72 Let Γ be the analytic arc omitted by a support point of S. Must Γ have monotonic argument? Must the angle between the radius and tangent vectors be monotonic on Γ? (The second property implies the first. Brown [161] has shown that the support points associated with point-evaluation functions have both properties.)

(P.L. Duren)

Update 6.72 No progress on this problem has been reported to us.

Problem 6.73 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in S. Is it true that

$$\limsup_{n\to\infty} \big||a_{n+1}| - |a_n|\big| \leq 1?$$

Hamilton [463] has proved that this is true for odd functions in S, functions of maximal growth in S, and spiral-like functions.

(D.H. Hamilton)

Update 6.73 Hamilton [466] has proved more precisely that $\big||a_{n+1}| - |a_n|\big| \leq 1$, $n > n_0(f)$, with equality only for

$$f(z) = \frac{z}{(1 - z\mathrm{e}^{-\theta_1})(1 - z\mathrm{e}^{i\theta_2})}$$

when $(\theta_2 - \theta_1)$ is a rational multiple of π.

Problem 6.74 Suppose $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is univalent and bounded by M in $\mathbb{D}$. Find

$$\sup_t \max_{0\leq t\leq 2\pi} |s_n(\mathrm{e}^{it})|,$$

where

$$s_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

(F. Holland)

Update 6.74 No progress on this problem has been reported to us.

Problem 6.75 Let $\mathcal{P}_n$ be the class of polynomials

$$\mathrm{P}_n(z) = z + a_2 z^2 + \ldots + a_n z^n$$

univalent in $\mathbb{D}$, and let

$$A_m(n) = \max_{\mathcal{P}_n} |a_m|.$$

If n is fixed, is it true that as m increases, the quantity $A_m(n)$ increases strictly up to some index n_0, and then decreases strictly? Again, for fixed (but arbitrary) n determine the least n_1 such that

$$A_m(n) \leq 1, \qquad n_1 \leq m \leq n.$$

(D.A. Brannan)

Update 6.75 No progress on this problem has been reported to us.

Problem 6.76 Let $\mathcal{V}_n$ denote the class of polynomials

$$P_n(z) = z + a_2 z^2 + \ldots + a_n z^n$$

analytic and bi-univalent in $\mathbb{D}$ (that is, P_n and P_n^{-1} are both univalent in the unit disc). Determine $\max_{\mathcal{V}_n} |a_2|$ and $\max_{\mathcal{V}_n} |a_n|$.

(D.A. Brannan)

Update 6.76 No progress on this problem has been reported to us.

Problem 6.77 Let $\mathcal{P}_n$ be the class of polynomials

$$P_n(z) = z + a_2 z^2 + \ldots + a_n z^n$$

univalent in $\mathbb{D}$. Determine

$$\max_{P \in \mathcal{P}_n} \int_0^{2\pi} |P(e^{it})|^q \, dt, \qquad 0 < q < \infty.$$

(F. Holland)

Update 6.77 No progress on this problem has been reported to us.

Problem 6.78 Suppose that f is in S. Consider the region $\mathbb{D}(f)$ on the Riemann sphere which is the stereographic projection of the image of the unit disc under f. We can associate with each f in S the spherical area of $\mathbb{D}(f)$. What is

$$\min_{f \in S} \{\text{area of } \mathbb{D}(f)\},$$

and what is the extremal function?

(Y. Avci)

Update 6.78 No progress on this problem has been reported to us.

Problem 6.79 Let $S_k(\infty)$ denote the class of all analytic and univalent functions $f(z) = z + a_2 z^2 + \ldots$ defined in $\mathbb{D}$ which admit a k-quasiconformal extension to the whole plane, $0 < k < 1$, with $f(\infty) = \infty$. Prove or disprove:

$$f(z) \in S_k(\infty) \qquad \Longrightarrow \qquad \frac{f(rz)}{r} \in S_k(\infty), \qquad 0 < r < 1.$$

(D. Bshouty)

Update 6.79 This has been proved by Krushkal [638].

Problem 6.80 If f is univalent analytic in $\mathbb{D}$, then it is well known (see Pommerenke [823, p. 262]) that both f and its first derivative f' must be normal, while the higher derivatives $f^{(n)}$ need not be normal if $n \geq 2$. Setting $f^{(-1)}(z) = \int_0^z f(t)\,\mathrm{d}t$ and $f^{(-n-1)}(z) = \int_0^z f^{(-n)}(t)\,\mathrm{d}t$, it is easy to verify that $f^{(-2)}$ is Bloch; while if $n \geq 3$, $f^{(-n)}$ is bounded (hence Bloch, and therefore normal). Thus, if f is univalent analytic in $\mathbb{D}$, then the functions $f^{(n)}$ must be normal for $n = 1, 0, -2, -3, \ldots$, and need not be normal for $n = 2, 3, \ldots$. Must $f^{(-1)}(z) = \int_0^z f(t)\,\mathrm{d}t$ be normal if $f(z)$ is univalent analytic in $\mathbb{D}$?

(D. Campbell)

Update 6.80 No progress on this problem has been reported to us.

Problem 6.81 Let G be the set of functions analytic and *not* univalent in $\mathbb{D}$. Set, for f in G,

$$M_f = \sup\{|f'(z)| : |z| < 1\}, \qquad m_f = \inf\{|f'(z)| : |z| < 1\},$$

and put

$$\gamma = \inf\left\{\frac{M_f}{m_f} : f \in G\right\}.$$

Find γ.

John [578] has proved that $\gamma \geq \mathrm{e}^{\pi/2} \approx 4.7$ and Yamashita [1003] has shown that $\gamma \leq \mathrm{e}^{\pi} \approx 23.1$.

(G.M. Rassias, J.M. Rassias and T.M. Rassias)

Update 6.81 No progress on this problem has been reported to us.

Problem 6.82 Let σ denote the class of bi-univalent functions, namely the class of functions $f(z) = z + a_2 z^2 + \ldots$ analytic and univalent in $\mathbb{D}$ such that their inverses f^{-1} have an analytic continuation to $\mathbb{D}$, being also univalent in $\mathbb{D}$. Brannan conjectures that

$$a_2^* \equiv \sup_{\sigma} |a_2| = \sqrt{2}.$$

It is known that $a_2^* < 1.51$ (see Lewin [664]) and that $a_2^* > \frac{4}{3} + 0.02$; see Styer and Wright [929].

Prove or disprove the statement that if $f \in \sigma$, then

$$d_n \equiv \big||a_{n+1}| - |a_n|\big| \leq A$$

for some absolute constant A with $0 < A < 1$.

Alternatively, what is $\sup_\sigma d_n$ for $n \geq 2$?

(D.A. Brannan)

Update 6.82 Brannan's conjecture is verified for some classes by Sivasubramanian, Sivakumar, Kanas, and Kim [905].

Problem 6.83 Let S be the usual class of normalised univalent functions in $\mathbb{D}$. Characterise those sequences $\{z_n\}$ of points in $\mathbb{D}$ for which $f(z_n) = g(z_n)$ for two different functions f, g in S.

Notice that a necessary condition is that $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$ because $(f - g) \in H^p$ for all $p < \frac{1}{2}$.

(P.L. Duren)

Update 6.83 Overholt [786] has provided a partial solution to this problem.

Problem 6.84 If $f(z)$ is in S, write

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n$$

and

$$f(z^p)^{1/p} = z + \sum_{n=1}^{\infty} c_n^{(p)} z^{pn+1}, \qquad p = 1, 2, \ldots.$$

Szegö's conjecture asserts that $c_n^{(p)} = O(n^{2/p-1})$ as $n \to \infty$. This has been proved for $p = 1, 2$ and 3, but Pommerenke [818] has shown that it is false for $p \geq 12$; his example also has $\gamma_n \neq O(1/n)$. Milin [737] has shown that, if a function f in S has the property that $\gamma_n = O(1/n)$, then $c_n^{(p)} = O(n^{2/p-1})$ for every p. Is the converse true?

(P.L. Duren)

Update 6.84 Szegö's conjecture has been proved by Baernstein [59] for $p = 4$. The cases $5 \leq p \leq 11$ remain open.

Problem 6.85 Each function f in S that maximises $\mathrm{Re}\{L(g) : g \in S\}$ for some continuous linear functional L must map the unit disc onto the complement of an arc Γ that is asymptotic at infinity to the half-line

$$w = \frac{L(f^3)}{3L(f^2)} - L(f^2)t, \qquad t \geq 0.$$

For the coefficient functional $\Lambda_n(f) = a_n$, show that the asymptotic half-line is radial; that is, that $\Lambda_n(f^3)/[\Lambda_n(f^2)]^2$ is real. (This is true for $n = 2, 3, 4, 5, 6$.)

(P.L. Duren)

Update 6.85 See Update 6.1.

Problem 6.86 Sundberg notes that it is well-known fact (see Hayman [513, Chap. 1]) that, for each fixed z_0 in $\mathbb{D}$,

$$\left| z_0 \frac{f''(z_0)}{f'(z_0)} - \frac{2\rho^2}{1-\rho^2} \right| \leq \frac{4\rho}{1-\rho^2},$$

and asks if this can be improved, if we have that f is real on the real axis, or equivalently, if f has real coefficients?

(C. Sundberg; communicated by P.L. Duren)

Update 6.86 This problem has been restated. No progress on this problem has been reported to us.

Problem 6.87 Let L_1, L_2 be two complex-valued continuous linear functionals on $H(\mathbb{D})$, the space of all analytic functions on the unit disc $\mathbb{D}$ that are not constant on the set S of all normalised univalent functions on $\mathbb{D}$. Assume, in addition, that $L_1 \neq tL_2$ for any positive t. If a function f in S maximises both $\mathrm{Re}\,\{L_1\}$ and $\mathrm{Re}\,\{L_2\}$ in S, must f be a rotation of the Koebe function?

(P.L. Duren)

Update 6.87 Bakhtin [71] has answered this positively for certain classes of coefficient functionals.

Problem 6.88 Let the function $f = z + a_2 z^2 + \dots$ in S map $\mathbb{D}$ onto a domain with finite area A. Then Bieberbach's inequality $|a_2| \leq 2$ can be sharpened to the following:

$$|a_2| \leq 2 - cA^{-1/2} \tag{6.25}$$

where c is an absolute constant. What is the best value of c? Aharonov and Shapiro [5] have shown that (6.25) holds for some c, and have a conjecture concerning the sharp constant c and the extremal function for (6.25). They also conjecture that $|a_2| \leq 2 - c_1 l^{-1}$ where l is the length of $\partial f(\mathbb{D})$. For some background information, see Aharonov and Shapiro [5, 6] and Aharonov, Shapiro and Solynin [8, 9].

(H.S. Shapiro)

Update 6.88 No progress on this problem has been reported to us.

Problem 6.89 Let $S^*(\frac{1}{2})$ denote the class of functions g analytic in $\mathbb{D}$ and such that $\mathrm{Re}\,(zg'/g) > \frac{1}{2}$ in $\mathbb{D}$. Is it true that, if $f \in S^*(\frac{1}{2})$, $r \in (0, 1)$ and $\theta \in [0, 2\pi)$, then

$$\frac{1}{|f(re^{-i\theta})|} \int_0^r |f'(te^{i\theta})|\, dt \leq \frac{\arcsin r}{r}\ ? \tag{6.26}$$

(The left-hand side of (6.26) is the ratio of the length of the image of a radius and the distance between the endpoints of that image (as the crow flies).) Inequality (6.26) is true when $f(z) = z/(1 - z)$. In addition, the left-hand side of (6.26) never exceeds $\pi/2$ for any r or f.

Let z_1, z_2 lie in $\mathbb{D}$. Then, in the smaller class of convex functions f, what can be said about

$$\frac{1}{|f(z_1) - f(z_2)|} \int_{z_1}^{z_2} |f'(t)|\, |dt|?$$

(R.R. Hall)

Update 6.89 No progress on this problem has been reported to us.

Problem 6.90 Let E be a set of positive logarithmic capacity on the unit circle $\mathbb{T}$. Is E necessarily a set of uniqueness for functions univalent in $\mathbb{D}$?

Carleson [188] has shown that this is false for functions f analytic in $\mathbb{D}$ for which

$$\int\int_{|z|<1} |f'(z)|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

Beurling [120] has shown that univalent functions cannot have constant boundary values on a set of positive capacity.

(D.H. Hamilton)

Update 6.90 Progress reported in a previous update [156] has not been published.

Problem 6.91 Let Ω be an arbitrary domain in $\mathbb{C}$. Does there necessarily exist a set E in $\partial\Omega$ of full harmonic measure, with the following property: For each z in E, there exist circular arcs C_r in Ω, of radius r (where r is small) and centred at z, for which

$$\lim_{R\to 0}\left(\frac{1}{\pi R^2}\int_0^R \theta(C_r) r \, \mathrm{d}r\right) = \frac{1}{2},$$

where $\theta(\cdot)$ denotes angular measure?

For simply-connected domains Ω this is a theorem of McMillan [712]. For general domains Ω at least there is a sequence $\{r_n\}$ decreasing to 0 for which $\theta(C_{r_n})$ approaches $\pi/4$.

(D. Stegenga and K. Stephenson)

Update 6.91 No progress on this problem has been reported to us.

Problem 6.92 Let $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and suppose that $E \subset \mathbb{R}^2_+$, and let $f : \mathbb{R}^2_+ \to B^2$ be analytic and conformal, where $B^2 = \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 < 1\}$. Assume also that

$$\lim_{x\to 0,\, x\in E} f(x) = \alpha.$$

Lindelöf's theorem shows that α is an angular limit of f at 0 if E is a curve terminating at 0. Vuorinen [972] has shown that the same conclusion holds under much weaker hypotheses on E. For instance, the condition that

$$\liminf_{r\to 0} m(A \cap (0, r))/r > 0,$$

where $A = \{|x| : x \in E\}$, is sufficient; see Vuorinen [972, pp. 169, 176]. This problem concerns a converse result.

Denote by $\mathcal{F}$ the class of all analytic and conformal maps of $\mathbb{R}^2_+$ into B^2. Let K be a subset of $\mathbb{R}^2_+$ with the property that whenever

$$\lim_{x\to 0,\, x\in E} f(x) = \alpha \qquad \text{and} \qquad f \in \mathcal{F},$$

then necessarily f has an angular limit α at 0. Since f is conformal it follows (see Vuorinen [972, p. 176]) that

$$\lim_{x\to 0,\, x\in K_1} f(x) = \alpha,$$

where $K_1 = \{(x, y) \in \mathbb{R}^2_+ : \rho((x, y), K) < l\}$ and ρ is the Poincaré metric of $\mathbb{R}^2_+$. What can be said about the thickness of K_1 at 0? Is it true that

$$\limsup_{r\to 0} m\big(B' \cap (0, r)\big)/r > 0, \tag{6.27}$$

where $B' = \{(x^2 + y^2)^{1/2} : (x, y) \in K_1\}$? It is easy to see that (6.27) must hold if K is contained in an angle with vertex at 0 whose closure lies inside $\mathbb{R}^2_+$.

(M. Vuorinen)

Update 6.92 No progress on this problem has been reported to us.

Problem 6.93 Let the function $f(z) = z + a_2 z^2 + \ldots$ map $\mathbb{D}$ univalently onto a domain Ω, and let $F : \Omega \to \mathbb{D}$ denote the inverse of f. Is it true that

$$\int_{\Omega\cap\mathbb{R}} |F'(x)|^p \, \mathrm{d}x < \infty, \qquad \text{for } 1 \le p < 2\,?$$

Hayman and Wu [523] (in the so-called Hayman–Wu theorem) and Garnett, Gehring and Jones [380] have shown that the answer is 'yes' for $p = 1$. For $p = 2$, it is possible that $\int_{\Omega\cap\mathbb{R}} |F'(x)|^2 \, \mathrm{d}x = \infty$; for example, when $\Omega = \{w : |w| < R, R > 1\} \setminus L$ where, for a suitably chosen R_1, $L = \{(u, 0) : -R \le u \le -R_1 < 0\}$.

(A. Baernstein)

Update 6.93 No progress on this problem has been reported to us.

Problem 6.94 (*Brennan problem*) Let D be a simply-connected domain in $\mathbb{C}$ with at least two boundary points, and let the function ϕ map D analytically and conformally onto $\mathbb{D}$. For which values of p is it true that

$$\int\int_D |\phi'|^p \, \mathrm{d}x \, \mathrm{d}y < \infty\,? \tag{6.28}$$

If D is starlike or close-to-convex, (6.28) holds for $\frac{4}{3} < p < 4$ and this is sharp. More generally, it is known that there is a universal constant τ, independent of D, with $0 < \tau < 1$ such that (6.28) holds whenever $\frac{4}{3} < p < 3 + \tau$. Is $\frac{4}{3} < p < 4$ the correct range for general types of D? (For background material and additional information, see Brennan [159].)

(J.E. Brennan)

Update 6.94 Pommerenke [825] proved that (6.28) is true for $\frac{4}{3} < p < 4$.

Problem 6.95 Determine an intrinsic characterisation for the class $\mathcal{H}$ of functions h analytic in $\mathbb{D}$ that admit a decomposition of the form $2h = f + f^{-1}$ for some function f in the class S of normalised univalent functions in $\mathbb{D}$. (Here f^{-1} denotes the function inverse to f and it is assumed that f^{-1} has an analytic continuation to $\mathbb{D}$.) Notice that the functions $z \mapsto z$ and $z \mapsto z/(1-z^2)$ both belong to $\mathcal{H}$. In particular, does the function $z \mapsto z + a\sin(2\pi z)$ belong to $\mathcal{H}$ for any non-zero constant a?

(F. Holland)

Update 6.95 No progress on this problem has been reported to us.

Problem 6.96 For $-\infty < p < +\infty$ let

$$B(p) := \sup\{\beta_f(p) : f \text{ conformal map of } \mathbb{D} \text{ into } \mathbb{D}\},$$

where

$$\beta_f(p) = \limsup_{r\to 1}\left(\int_{|\zeta|=r} |f'(r\zeta)|^p\,|\mathrm{d}\zeta|\right)/\log\frac{1}{1-r}.$$

The BCJK-conjecture states that

$$B(p) = \begin{cases} -p-1, & \text{for } p \le -2,\\ p^2/4, & \text{for } -2 \le p \le 2,\\ p-1, & \text{for } p \ge 2.\end{cases}$$

The claim that $B(p) = |p| - 1$ is the famous Brennan conjecture. Carleson and Jones [193] proved that $B(p) = p - 1 + O((p-2)^2)$ as $p \to 2$, $p > 2$. Based on extensive computer experiments Kraetzer conjectured that $B(p) = p^2/4$ for $|p| \le 2$. See Garnett and Marshall [382, p. 305].

(Ch. Pommerenke)

Update 6.96 No progress on this problem has been reported to us.

Problem 6.97 Goodman [425] conjectured that if $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is p-valent in $\mathbb{D}$, then for each $n > p$, we have

$$|a_n| \le \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}|a_k|.$$

If the bound is true, then it is sharp in all of the variables $p, n, a_1, \ldots, a_p$.

The conjecture has been proved for large subclasses [433, 679, 840] but is still open in general. The simplest case, namely for all 2-valent functions, is still open, but Watson [980] has made important contributions towards proving this inequality.

(A.W. Goodman)

Update 6.97 See Update 6.25.

Problem 6.98 The coefficients of a p-valent function are bounded by some function of its zeros. In particular, let the function

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$$

be p-valent in $\mathbb{D}$ and have s zeros $\beta_k, k = 1, 2, \ldots, s$, where $0 < |\beta_k| < 1$. Goodman [427] conjectured that under these hypotheses, $|a_n| \leq |A_n|$ where A_n is defined by the identity

$$F(z) = \frac{z^q}{(1-z)^{2q+2s}} \left(\frac{1+z}{1-z}\right)^{2t} \prod_{k=1}^{s} \left(1 + \frac{z}{|\beta_k|}\right)\left(1 + |\beta_k| z\right) = z + \sum_{n=2}^{\infty} A_n z^n,$$

where $t = p - q - s \geq 0$.

The conjecture has been proved if $t = 0$ and $f(z)$ is p-valent starlike with respect to the origin [426]. However, it is still open in general.

(A.W. Goodman)

Update 6.98 No progress on this problem has been reported to us.

Problem 6.99 A function $f(z) = z + a_2 z^2 + \ldots$ is said to belong to the *class* $CV(R_1, R_2)$ if it is univalent and convex in $\mathbb{D}$, and if on $f(\{|z| = 1\})$ the curvature ρ satisfies the inequalities $R_1 \leq \rho \leq R_2$. So far little progress has been made on the study of this class of functions, see Goodman [430, 431]; we do not even know the sharp bound for $|a_2|$ in $CV(R_1, R_2)$.

(A.W. Goodman)

Update 6.99 No progress on this problem has been reported to us.

Problem 6.100 Given two functions f, g in the (usual) class S, we can form the new functions (arithmetic and geometric mean functions)

$$F(z) = \alpha f(z) + \beta g(z) \qquad \text{and} \qquad G(z) = z\left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{g(z)}{z}\right)^{\beta},$$

where $\alpha, \beta \in (0, 1)$ and $\alpha + \beta = 1$. Goodman [428] has shown that if

$$0.042 \simeq \frac{1}{1 + \mathrm{e}^{\pi}} < \alpha, \qquad \beta < \frac{\mathrm{e}^{\pi}}{1 + \mathrm{e}^{\pi}} \simeq 0.988,$$

then there are functions f, g in S such that F and G have valence infinity in $\mathbb{D}$. What can be said about the 'fringes' of the interval $(0, 1)$? Is there some bound on the valence of F and G that is a function of α for $0 < \alpha \leq 1/(1 + \mathrm{e}^{\pi})$?

(A.W. Goodman)

Update 6.100 No progress on this problem has been reported to us.

Problem 6.101 Let K be a closed set of points in $\mathbb{C}$, and let $F(K)$ denote the family of functions f of the form

$$f(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k},$$

where $A_k > 0$ and $a_k \in K$, $k = 1, 2, \ldots, n$. Find a maximal domain of p-valence for the class $F(K)$.

For $p = 1$, this problem was completely solved by Distler [246]; however, for $p > 1$, we do not even have a good conjecture. (See also Goodman [432].)

(A.W. Goodman)

Update 6.101 No progress on this problem has been reported to us.

Problem 6.102 Let $\{v_n\}_1^\infty$ be a sequence of positive integers (which may include ∞); the sequence is called a *valence sequence* if there is a function $f(z)$, analytic in $\mathbb{D}$, such that $f^{(n)}(z)$ has valence v_n in $\mathbb{D}$, for $n = 1, 2, \ldots$.

Find interesting necessary conditions for $\{v_n\}$ to be a valence sequence. Also, find sufficient conditions for $\{v_n\}$ to be a valence sequence. (For some results of this type, see Goodman [429].)

(A.W. Goodman)

Update 6.102 No progress on this problem has been reported to us.

The next five problems concern harmonic univalent functions, that is, functions that can be written in the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} \overline{z}^n$$

and are univalent. For a good introduction to these, see Duren [276] and Clunie and Sheil-Small [217]. We will denote by S the (usual) class of normalised analytic univalent functions in $\mathbb{D}$, by S_H the class of normalised harmonic univalent functions in $\mathbb{D}$ (with $a_0 = 0$ and $a_1 = 1$), and by S_H^0 the subclass of S_H for which $a_{-1} = 0$.

Problem 6.103 The function

$$k(z) = 2\mathrm{Re}\left[\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right] = \sum_{n=1}^{\infty} \frac{1}{3}(2n^2+1) r^n (\mathrm{e}^{in\theta} + \mathrm{e}^{-in\theta}),$$

where $z = r\mathrm{e}^{i\theta}$, lies in the closure of S_H. Prove that k is extremal for the coefficient bounds in S_H.

Clunie and Sheil-Small [217] have shown that $|a_n| < \frac{1}{3}(2n^2+1)$ for functions in S_H with real coefficients, and that $|a_n| \le \frac{1}{3}(2n^2+1)$ for functions in S_H for which the domain $f(\mathbb{D})$ is close-to-convex.

(T. Sheil-Small)

Update 6.103 No progress on this problem has been reported to us.

Problem 6.104 Clunie and Sheil-Small [217] have shown that for functions f in S_H^0, $\{|w| < \frac{1}{16}\} \subset f(\mathbb{D})$. Prove that the correct value d of the Koebe constant for the class S_H^0 is $\frac{1}{6}$.

Note that the function

$$k_0(z) = \mathrm{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i\mathrm{Im}\left(\frac{z}{(1-z)^2}\right)$$

belongs to S_H^0 and maps $\mathbb{D}$ onto the plane cut along the real axis from $-\frac{1}{6}$ to $-\infty$, and that $\frac{1}{6}$ is the correct constant for those functions f in S_H^0 for which $f(\mathbb{D})$ is close-to-convex.

Also, determine the number

$$\alpha = \sup\{|a_2| : f \in S_H\}.$$

The best known estimate for α is $\alpha < 57.05$; see Sheil-Small [898]. It is also known that $d \geq 1/(2\alpha)$.

(T. Sheil-Small)

Update 6.104 No progress on this problem has been reported to us.

Problem 6.105 What are the convolution multipliers $\phi^* : K_H \to K_H$, where K_H is the subclass of functions f in S_H with convex images $f(\mathbb{D})$?

A particularly interesting case is the radius of convexity problem: for which values of r in $(0, 1)$ is the function $z \mapsto f(rz)$ convex in $\mathbb{D}$, when f is convex in $\mathbb{D}$? (It is known that $r \leq \sqrt{(2)} - 1$; see Clunie and Sheil-Small [217].)

(T. Sheil-Small)

Update 6.105 No progress on this problem has been reported to us.

Problem 6.106 Let J be a Jordan curve in $\mathbb{C}$ bounding a domain D. Suppose that $f : \mathrm{e}^{it} \mapsto f(\mathrm{e}^{it})$ is a sense-preserving homeomorphism of the unit circle $\mathbb{T}$ onto J, and that the harmonic extension of f to $\mathbb{D}$ satisfies the relation $f(\mathbb{D}) \subset D$. Prove that f is a homeomorphism of $\mathbb{D}$ onto D.

This is known to be true if J is convex when the hypothesis $f(\mathbb{D}) \subset D$ is automatically satisfied because of the positivity of the Poisson kernel by the Kneser–Radó–Choquet theorem; see Duren [266]. The result is also known to be true when $\partial f/\partial t$ is continuous and non-zero (on $\mathbb{T}$), and when the co-analytic and analytic parts of f have continuous derivatives on $\overline{\mathbb{D}}$.

(T. Sheil-Small)

Update 6.106 No progress on this problem has been reported to us.

Problem 6.107 Prove that, for f in S_H^0,

$$\big||a_n| - |a_{-n}|\big| \le n, \qquad n = 2, 3, 4, \ldots. \tag{6.29}$$

(This is a generalisation of the Bieberbach conjecture for S.)

It is known that (6.29) holds in the following cases:

(i) when f has real coefficients, see Clunie and Sheil-Small [217];
(ii) when $f(\mathbb{D})$ is starlike with respect to the origin, see Sheil-Small [898];
(iii) when $f(\mathbb{D})$ is convex in one direction, see Sheil-Small [898].

(T. Sheil-Small)

Update 6.107 No progress on this problem has been reported to us.

Problem 6.108 Let f be analytic univalent in $\mathbb{D}$, and consider

$$I_\lambda(r, f') = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f'(re^{i\theta})\right|^\lambda \,\mathrm{d}\theta\right)^{1/\lambda},$$

where λ is positive.

(a) What is the maximal order of magnitude of $I_\lambda(r, f')$ as $r \to 1$, where $0 \le \lambda \le \frac{2}{5}$? If $\lambda > \frac{2}{5}$, it is known that $I_\lambda(r, f') = O\big(1/(1-r)^{3-1/\lambda}\big)$ with equality when f is the Koebe function. The case $\lambda > \frac{1}{2}$ follows easily from classical facts, while the case $\frac{2}{5} \le \lambda \le \frac{1}{2}$ is due to Feng and MacGregor [336]. (It was formerly conjectured that the Koebe function would still be extremal for $\lambda > \frac{1}{3}$; but an example of Makarov [702] has shown that this is not the case for $\lambda \le \frac{1}{3} + \varepsilon$, for some positive ε.)

(b) Now normalise the function f to belong to the usual class S. For $\lambda > \frac{2}{5}$, we know that

$$I_\lambda(r, f') \le C_\lambda I_\lambda(r, k'),$$

where k is the Koebe function. For which λ is the best constant C_λ equal to 1? (It follows from de Branges' theorem [239] that $C_\lambda = 1$ for $\lambda = 2, 4, 6, \ldots$. Presumably $C_\lambda = 1$ for $\lambda \ge 2$ but the proposer knows of no proof of this.)

(A. Baernstein)

Update 6.108 Related questions have become of great interest, see Problem 6.96. No progress on this problem has been reported to us.

Problem 6.109 Let f be analytic univalent in $\mathbb{D}$, and consider

$$I_{-\lambda}(r, f') = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f'(re^{i\theta})\right|^{-\lambda} \,\mathrm{d}\theta\right)^{1/\lambda},$$

where λ is positive. Except for the elementary case $\lambda = \infty$, the maximal order of magnitude as $r \to 1$ is not known for any positive λ. A particularly interesting case is when $\lambda = 2$. It seems possible that

$$I_{-2}(r, f') = O\big(I_{-2}(r, k')\big) = O\big((1-r)^{-1/2}\big) \tag{6.30}$$

might be true, where k is the Koebe function. The equation (6.30) is slightly stronger than an earlier conjecture of Brennan, see Problem 6.94. The best known bounds are due to Pommerenke [826]. See Becker and Pommerenke [98] for related information.

(A. Baernstein)

Update 6.109 No progress on this problem has been reported to us.

Problem 6.110 Let D be a simply-connected domain in the finite plane whose complement contains n disjoint closed balls with centres on the interval [0, 1] and common radius ε, $\varepsilon < 1/n$, (such domains are the so-called 'ball and chain domains'). Let z_0 in D be a point at a distance at least 1 from each ball, and let $w(z_0, D)$ denote the harmonic measure at z_0 of the union of the balls, relative to D. Is it true that, for every positive δ, there is an estimate

$$w(z_0, D) \leq C_\delta (n\varepsilon)^{\frac{1}{2}-\delta}, \tag{6.31}$$

where C_δ depends only on δ.

An affirmative answer would imply the L^p extension, for $1 < p < 2$, of the Hayman–Wu theorem [523] mentioned in Problem 6.93. In the case $n = 1$, the Beurling–Nevanlinna projection theorem shows that (6.31) is true with exponent $\frac{1}{2}$; however, for large n there are examples that show that this is then false.

(A. Baernstein)

Update 6.110 No progress on this problem has been reported to us.

Problem 6.111 Let A denote the class of functions $f(z) = z + a_2 z^2 + \ldots$ analytic in $\mathbb{D}$. For $\delta \geq 0$ and $T = \{T_k\}_2^\infty$ a sequence of non-negative real numbers, define a *T-δ-neighbourhood* of f in A by

$$TN_\delta(f) = \left\{g : g(z) = z + b_2 z^2 + \ldots \in A, \sum_{k=2}^{\infty} T_k |a_k - b_k| \leq \delta\right\}.$$

When $T = \{k\}_2^\infty$, we call $TN_\delta(f) = N_\delta(f)$ a *δ-neighbourhood* of f. These neighbourhoods were introduced by Ruscheweyh [864] who used them to generalise the result [25] that $N_1(z) \subset St$, the class of normalised starlike functions in $\mathbb{D}$.

Now let $K[A, B]$ denote the class of univalent functions

$$\{f : f \in S, 1 + zf''(z)/f'(z) \prec (1 + Az)/(1 + Bz), z \in \mathbb{D}\},$$

where $-1 \le B < A \le 1$, introduced by Janowski [572]. See also Pommerenke [823, Chap. 2] for discussion on $\prec$, the *subordination*. The proposers have shown [900] that if $f \in K[A, B]$, and either

$$(i) \quad -\frac{1}{4}(\sqrt{3}+2) \le B < A \le 1 \quad \text{or} \quad (ii) \quad -1 \le -A \le B < A \le 1,$$

then $N_\delta(f) \subset St$, where

$$\delta = \begin{cases} (1-B)^{(A-B)/B}, & B \ne 0, \\ \mathrm{e}^{-A}, & B = 0, \end{cases}$$

and that this value of δ is best possible. Is this conclusion still valid without hypotheses (i) and (ii)?

(T. Sheil-Small and E.M. Silvia)

Update 6.111 No progress on this problem has been reported to us.

Problem 6.112 If f in A and δ is positive, define a $\Sigma_\delta(f)$ *neighbourhood* of f to be $\left\{g : g \in A, \left|(g'(z) - f'(z)) - \frac{1}{z}(g(z) - f(z))\right| + \left|(g'(z) - f'(z)) + \frac{1}{z}(g(z) - f(z))\right| < 2\delta\right\}$. Clearly, $N_\delta(f) \subset \Sigma_\delta(f)$. Sheil-Small and Silvia [900] have shown that $\Sigma_1(z) \subset St$, and $\Sigma_{\frac{1}{4}}(f) \subset St$ for convex functions f, with the notation in Problem 6.111.

Given a normal family $\mathcal{F}$ in A, the *dual*, $\mathcal{F}^*$, of $\mathcal{F}$ is the set

$$\left\{f : f \in A, f * g \ne 0 \text{ for all } g \in \mathcal{F}, 0 < |z| < 1\right\},$$

where $*$ denotes the Hadamard product; see Ruscheweyh [863]. The Bieberbach conjecture (see de Branges [239]) is equivalent to the statement that $N_1(z) \subset S^*$, where S^* is the dual of S. It seems reasonable therefore to ask whether $\sum_1(z) \subset S^*$. (Recall that $S^* \subset St$; see Ruscheweyh [862].)

(T. Sheil-Small and E.M. Silvia)

Update 6.112 No progress on this problem has been reported to us.

Problem 6.113 Following the notation of Problems 6.111 and 6.112, it is known [900] that, if $|x| \le \rho \le 1$ and $\gamma = 1/(1+\rho)^2$, then

$$N_\gamma\left(\frac{z}{1-xz}\right) \subset S^*.$$

Is it true that

$$\Sigma_\gamma\left(\frac{z}{1-xz}\right) \subset S^* ?$$

Notice that since not all convex functions belong to S^* (see Sheil-Small and Silvia [900]) it follows that we cannot replace in the above $N_\gamma(z/(1-xz))$ by $N_\gamma(g)$, where g is an arbitrary convex function.

(T. Sheil-Small and E.M. Silvia)

Update 6.113 No progress on this problem has been reported to us.

Problem 6.114 Let Γ be a regular curve and f an analytic and conformal function in $\mathbb{D}$. Does $f^{-1}(\Gamma)$ necessarily have finite length? A curve Γ is said to be *regular* if the intersection of Γ with a disc of radius r has one-dimensional measure at most Cr, where C is a constant independent of r.

(J.L. Fernández and D.H. Hamilton)

Update 6.114 No progress on this problem has been reported to us.

Problem 6.115 Let Γ be a rectifiable curve, and let E be a subset of Γ having zero length. If D is any simply-connected domain and $z \in D$, is it true that the harmonic measure satisfies the equation $\omega(z, E, D) = 0$?

(B. Øksendal; R. Kaufman and J.-M.G. Wu; communicated by D.H. Hamilton)

Update 6.115 D. Khavinson writes that this problem is unclear in that, is Γ part of the boundary of D? He writes that it is well known that if Γ is a closed, Jordan, rectifiable curve with $\Gamma = \partial D$, then the statement is true.

Problem 6.116 Let D be a domain in $\mathbb{C}$ containing the origin 0. For $t > 0$, let Ω_t be the component of $D \cap \{|z| \le t\}$ containing 0. In the usual notation for harmonic measure, define the function $\omega = \omega_D : [0, \infty) \to [0, 1]$ by the formula

$$\omega(t) = \omega\big(0, \partial\Omega_t \cap \{|z| = t\}, \Omega_t\big).$$

What can be said about ω and about its relations with D? For instance:

(a) What are necessary and sufficient conditions on a function to be ω_D for some D?
(b) If $\omega_{D_1} \equiv \omega_{D_2}$, are D_1 and D_2 essentially the same? (That is, will D_1 be a rotation or reflection of D_2? Or will D_1 and D_2 differ only on sets of capacity zero?) What happens if $\omega_{D_1} \equiv \omega_{D_2}$ only on some subinterval of $[0, 1]$?
(c) Given a function ω_D, can one 'reconstruct' D?
(d) Can one infer such properties as connectivity of D from the behaviour of ω_D? One might start by dealing with domains D that are circularly symmetric.

(K. Stephenson)

Update 6.116 Snipes and Ward [910] and Walden and Ward [976] provide results which answer the analogues of Stephenson's questions for their function $h(r)$. Barton and Ward [91] establish several sets of sufficient conditions on a function f, for f to arise as a harmonic measure distribution function.

Problem 6.117 Let G be a domain in $\mathbb{C}$ that contains the origin 0 and is *symmetric with respect to the real and imaginary axes*, that is, if a point z in G then the line segment with endpoints z and $\bar{z}$ also lies in G. (In particular, it follows that G is simply-connected.) For $t > 0$, let G_t and G_t' be the components of the domains

$$\{z : z \in G, \operatorname{Im} z < t\} \qquad \text{and} \qquad \{z : z \in G, \operatorname{Re} z < t\}$$

containing 0. Let ω_t and ω'_t denote the harmonic measure at 0 of the sets

$$\partial G_t \cap \{\operatorname{Im} z = t\} \qquad \text{and} \qquad \partial G'_t \cap \{\operatorname{Re} z = t\}$$

with respect to the domains G_t and G'_t, respectively.

Now let D_1 and D_2 be two domains in $\mathbb{C}$ of this type, and use for each the notation just described. Then:

(a) If $\omega_t(D_1) \equiv \omega_t(D_2)$ for each $t > 0$, is it true that $D_1 = D_2$?
(b) If $\omega'_t(D_1) \equiv \omega'_t(D_2)$ for each $t > 0$, is it true that $D_1 = D_2$?
(c) If both (a) and (b) fail, is it true that $D_1 = D_2$ if $\omega_t(D_1) \equiv \omega_t(D_2)$ for each $t > 0$ and $\omega'_t(D_1) \equiv \omega'_t(D_2)$ for each $t > 0$?

If the answer to (a) or (b) is 'yes', in that case, how few t's does one need for the conclusion to hold? (For example, infinitely many t's such that $\{\operatorname{Im} z = t\}$ or $\{\operatorname{Re} z = t\}$ meet the domains? Or do the t's need to be dense?) If the answer to (a), (b) or (c) is 'yes', in that case can one replace 'axially-symmetric' by 'simply-connected', or perhaps drop this requirement completely?

(D.A. Brannan)

Update 6.117 The original statement of this problem contained errors which have now been fixed. No progress on this problem has been reported to us.

6.3 New Problems

Problem 6.118 In relation to the Bieberbach conjecture (Problem 6.1), what are the sharp bounds for $|\gamma_n|$, for $n \geq 3$ and f in S? The bounds for $n = 1$ and $n = 2$ follow at once from the Fekete–Szegö inequality. The *Fekete–Szegö inequality* states that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

is a univalent analytic function on $\mathbb{D}$ and $0 \leq \lambda < 1$, then

$$|a_3 - \lambda a_2^2| \leq 1 + 2\mathrm{e}^{-2\lambda/(1-\lambda)}.$$

(D.K. Thomas)

Problem 6.119 In relation to the Bieberbach conjecture (Problem 6.1), find the sharp upper bound for $|\gamma_3|$, when f is close-to-convex. Sharp bounds $|\gamma_1| \leq 1$, and $|\gamma_2| \leq 11/18$ follow at once from the Fekete–Szegö inequality for close-to-convex functions; see Keogh and Merkes [593]. It is conjectured by Ali and Vasudevarao [27] that the sharp bound is

$$|\gamma_3| \leq \frac{1}{243}(28 + 19\sqrt{19}) = 0.4560\ldots$$

See also Ali and Vasudevarao [26], and Thomas [937].

(D.K. Thomas)

Problem 6.120 In relation to the Bieberbach conjecture (Problem 6.1), Ye [1009] has shown that when f is close-to-convex

$$\gamma_n = O\left(\frac{\log n}{n}\right).$$

Can the log factor be removed?

(D.K. Thomas)

Problem 6.121 As in Problem 6.14, for $f(z)$ in S, set

$$A_n^{(k)} = \begin{vmatrix} a_n, & a_{n+1}, & \ldots, & a_{n+k-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+k-1}, & a_{n+k}, & \ldots, & a_{n+2k-2} \end{vmatrix}.$$

Janteng, Halim and Darus [573] have shown that $A_2^{(2)} \leq 1$ when f is starlike. Is this true when f is close-to-convex? Răducanu and Zaprawa [855] have achieved the best bound to date, $1.242\ldots$.

(D.K. Thomas)

Problem 6.122 Suppose that f is in S. As in Problem 6.46:

(a) When f is close-to-convex, is it true that

$$\big||a_{n+1}| - |a_n|\big| \leq 1\,?$$

Koepf [612] has shown that this is true for $n = 2$.

(b) When f is convex, what are the sharp bounds for $\big||a_{n+1}| - |a_n|\big|$? See Li and Sugawa [670].

(D.K. Thomas)

Problem 6.123 Let $\mathcal{P}$ denote the class of functions satisfying Re $p(z) > 0$ in $\mathbb{D}$. For p in $\mathcal{P}$, is it true that with $z = re^{i\theta}$,

$$\int_0^{2\pi} \left|\frac{zp'(z)}{p(z)}\right|^2 \mathrm{d}\theta = O\left(\frac{1}{1-r}\right)?$$

If not, what is the correct rate of growth?

(D.K. Thomas)

Problem 6.124 A pioneering proof of the Nitsche conjecture [568] on univalent harmonic mappings between two annuli has opened a door into interesting problems related to the question of characterising doubly-connected domains (Ω, Ω^*) that admit a univalent harmonic mapping from Ω onto Ω^*.

The *affine modulus*, $\mathrm{Mod}_@\, \Omega$, of a doubly connected domain $\Omega \subset \mathbb{C}$ is defined by

$$\mathrm{Mod}_@\, \Omega = \sup\{\mathrm{Mod}\, \phi(\Omega); \quad \phi : \mathbb{C} \xrightarrow{\text{onto}} \mathbb{C} \text{ affine}\}.$$

The above problem was tackled by Iwaniec, Kovalev and Onninen [567] from different directions, and culminated with the following conjecture:

If $h : \Omega \to \Omega^*$ *is a harmonic homeomorphism between doubly connected domains, and* $\mathrm{Mod}\, \Omega \neq \infty$, *then*

$$\mathrm{Mod}_@\, \Omega^* \geq \log\cosh(\mathrm{Mod}\, \Omega).$$

Let

$$\mathcal{G}(s) = \{z : |z| > 1\} \setminus [s, \infty) \quad \text{for } s > 1$$

be a Grötzsch ring which is a doubly connected domain. Iwaniec, Kovalev and Onninen [567] proposed the following:
Grötzsch–Nitsche Problem: for which $1 < s, t < \infty$ does there exist a harmonic homeomorphism $h : \mathcal{G}(s) \xrightarrow{\text{onto}} \mathcal{G}(t)$?

(Communicated by A. Vasudevarao)

Problem 6.125 Let A denote the class of analytic functions in $\mathbb{D}$, and think of A as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. A function f in A that is univalent in $\mathbb{D}$ is called *exponentially convex* if $e^{f(z)}$ maps $\mathbb{D}$ onto a convex domain. Let $\mathcal{A}$ denote the family of functions f in $\mathcal{H}$ normalized by $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be *in the family* $\mathcal{S}$ if it is univalent in $\mathbb{D}$. For α in $\mathbb{C} \setminus \{0\}$, the family $\mathcal{E}(\alpha)$ of α-exponential functions was introduced by Arango, Mejía and Ruscheweyh [42]. A function f is said to be *in* $\mathcal{E}(\alpha)$ if $F(\mathbb{D})$ is a convex domain, where $F(z) = e^{\alpha f(z)}$. Set $\mathcal{F}_\alpha = \mathcal{A} \cap \mathcal{E}(\alpha)$. We now recall a number of basic properties of the class $\mathcal{F}_\alpha$ obtained in [42]: for α in $\mathbb{C} \setminus \{0\}$, a function f is in $\mathcal{F}_\alpha$ if and only if $\mathrm{Re}\, P_f(z) > 0$ in $\mathbb{D}$, where

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)} + \alpha z f'(z).$$

By [42, Theorem 2] we have that if $f \in \mathcal{E}(\alpha)$, then $f(\mathbb{D})$ is convex in the $\overline{\alpha}-$direction (and therefore close-to-convex). It is not necessarily starlike univalent.

Because each f in $\mathcal{F}_\alpha$ can be represented in the form

$$e^{\alpha f(z)} = 1 + \alpha g(z), \quad g \in \mathcal{A},$$

as shown in [42], g belongs to the class $\mathcal{C}(\alpha)$ of (normalised) convex univalent functions with $-1/\alpha \notin g(\mathbb{D})$. In [42, Theorem 3], for α in $\mathbb{C} \setminus \{0\}$ we have

$$\mathcal{E}(\alpha) = \left\{ \frac{1}{\alpha} \log(1 + \alpha g) \colon g \in \mathcal{C}(\alpha) \right\}.$$

In [42, Theorem 4], it is also observed that $\mathcal{E}(\alpha) = \emptyset$ if $|\alpha| > 2$ and for $|\alpha| = 2$, $\mathcal{E}(\alpha)$ consists of the functions

$$f(z) = \frac{1}{\alpha} \log \frac{2 + \alpha z}{2 - \alpha z}.$$

The region of variability problem for exponentially convex functions has been studied by Saminathan, Vasudevarao and Vuorinen [876], and they proposed the following open problems on exponentially convex univalent functions:

(a) For f in $\mathcal{E}(\alpha)$, what are the sharp lower and upper bounds of $|f(z)|$ and $|f'(z)|$ for $z \in \mathbb{D}$?

(a) Let f in $\mathcal{E}(\alpha)$ be given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \quad z \in \mathbb{D}.$$

What are the sharp coefficient bounds for $|a_n|$ for $n \geq 2$?

(Communicated by A. Vasudevarao)

Problem 6.126 Let L be a line or a circle in a simply connected domain Ω in the complex plane and let f be a conformal map from Ω to the unit disc. The Hayman–Wu theorem [523] asserts the existence of a universal constant C such that $l(f(L)) \leq C$, where l denotes length. The smallest such constant is called the *Hayman–Wu constant*, denoted by Φ. It has been proved that $\pi^2 \leq \Phi \leq 4\pi$; see Øyma [787] and Rohde [842]. What is the value of Φ?

(A. Nicolau)

Problem 6.127 (*The Zalcman conjecture*) Let S be the class of normalised univalent functions

$$f(z) = z + a_2 z^2 + \ldots$$

on the unit disc $\mathbb{D}$. In the late 1960s, Zalcman conjectured that

$$|a_{2n-1} - a_n^2| \leq (n-1)^2$$

for each f in S and $n \geq 2$. This remained oral tradition until the publication of Brown and Tsao [165]. The Zalcman conjecture has been verified for many special classes of univalent functions and is now known to hold nonrestrictedly for $n = 2, 3, 4, 5, 6$; see Krushkal [639, 640]. The case $n = 2$ is classical.

The principal interest of this question is that an affirmative solution easily implies $|a_n| \leq n$ for all f in S, and thus would provide an alternative proof of the Bieberbach conjecture (Problem 6.1). The Zalcman conjecture seems to be the simplest unproved coefficient conjecture for S with this property.

(L. Zalcman)

Chapter 7
Miscellaneous

7.1 Preface by D. Sixsmith

So it falls to me to introduce the "miscellaneous" section. How does one do this, and make it sound exciting, and interesting—worth reading on? After all, here are the bits and pieces. The odds and sods. The various and the sundry. A potpourri of questions that don't quite fit here and don't quite fit there. A mishmash of scraps and sundries, of leftovers and loose ends. Surely this is the least attractive chapter in which to spend time.

Yet I'd like to suggest that there may be more here than meets the eye. The Spanish biologist Ramón Margalef wrote[1]

> *The scientist must feel more attraction for the new and unusual, than for what can be stored in the pigeonholes we have prepared.*

Well, this is exactly what this chapter offers. We are away from the usual pigeonholes, and here are the questions that may lead to "the new and the unusual".

Let's look quickly at three examples. Problem 7.54 is a question originally raised to solve a problem in dynamics which, itself, is now resolved.[2] But that does not detract from a lovely question, which is simple to state, easy to understand, but hard to see how to answer. Perhaps a solution to this question may lead to some new ideas in dynamics?

Personally, I am very fond of Problem 7.73, which is due to Fuchs. Suppose that $D_1, D_2 \subset \{|z| < R\}$ are domains, with hyperbolic metrics $\lambda_1|\mathrm{d}z|$ and $\lambda_2|\mathrm{d}z|$ respectively. Let $\lambda|\mathrm{d}z|$ denote the hyperbolic metric in $D_1 \cap D_2$. What is the least number $A = A(R)$ such that

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[1]The quote is taken from Margalef, R., 1974. Ecología. Barcelona: Ediciones Omega, S.A. 951 pp. It reads in Spanish: "El cientfico ha de sentirse más atraído por lo nuevo e insólito, que por lo que se puede archivar en los huequecitos que tenemos preparados." R. Margalef (1974 p. 882)

[2]Private communication with Phil Rippon.

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9_7

$$\lambda(z) < A(\lambda_1(z) + \lambda_2(z))\,?$$

Once again, this question is easy to state, and easy to understand, but apparently hard to make progress on. It also gives rise to many subsidiary questions: is the lower bound for A attained? If so, what can be said about the class of domains for which it is attained? Is the question easier if D_1, D_2 are simply connected? Is it the case that A is independent of R? Can anything similar be said about the corresponding *lower* bound on $\lambda(z)$?

Finally, I would like to mention Problem 7.78, which is due to Rubel. Roughly speaking, this says the following: if I place infinitely many electrons in the plane, is there necessarily an equilibrium point? (Of course, this statement can be made more precise, and becomes a question in complex analysis.) An elegant problem, with an unexpected physical implication. Once again, this question then leads to others: If the answer is "yes", then is there a configuration with only finitely many equilibrium points? What happens in more than 2 (real) dimensions? Does the question have a different answer if we use "electrons" with a different potential? Is there a version of this problem with "electrons" arranged, say, in the open unit disc?

So. Here are the problems that refused to be pigeonholed. Off you go, and good luck.

7.2 Progress on Previous Problems

Problem 7.1 Let E be a compact plane set of logarithmic capacity $d(E) = 1$ and define

$$d_n(E)^{n(n-1)/2} = \max_{w_\nu \in E} \prod_{1\le\mu<\nu\le n} |w_\mu - w_\nu|.$$

It is known that $d_n(E)$ decreases with n and $d_n(E) \to d(E)$ as $n \to \infty$, so that $d_n(E) \ge 1$ is trivial. It is shown by Pommerenke [815] that $d_n(E) \ge n^{2/(n-1)}$ if E is connected. Is this inequality true in general?

Further, is it true that

$$d_n(E)^{(n-1)/2} \le Kn$$

if E is connected, where K is some absolute constant? It is known that

$$d_n(E)^{(n-1)/2} \le \left(\frac{4}{e}\log n + 4\right)n$$

in this case.

(Ch. Pommerenke)

Update 7.1 No progress on this problem has been reported to us.

Problem 7.2 Let $f(z)$ be analytic in a simply-connected domain D. The *Faber polynomials* P_m of a Laurent series

$$f(z) = z^{-1} + a_0 + a_1 z + \cdots$$

are the polynomials such that

$$P_n(f) - z^{-n}$$

vanishes at $z = 0$. It is known that $f(z)$ can be expanded in a series of Faber polynomials

$$f(z) = \sum_{n=0}^{\infty} a_n P_n(z).$$

(Faber polynomials can be used to calculate the number of lattice paths from a point $(r, 0)$ to a point (a, b) that remain below the line $y = cx$, where c is a constant.) Find the domain of variability V of a_n, that is, the set V of all points $f(a_n)$ where $f(z)$ runs through all functions analytic in D and having positive real part there. It is known that if D is a circle, V is a circle; and if D is an ellipse, then V is an ellipse; see Royster [853].

(W.C. Royster)

Update 7.2 No progress on this problem has been reported to us.

Problem 7.3 Let z_i, $1 \le i \le n$, be a finite sequence of complex numbers such that $|z_i| \le 1$. Set

$$S_k = \sum_{i=1}^{n} z_i^k.$$

Can we have

$$\max_{2 \le k \le n+1} |S_k| < A^{-n}, \tag{7.1}$$

where A is an absolute constant greater than one? If we assume $z_1 = 1$, $|z_i| \le 1$, $2 \le i \le n$, then (7.1) can be satisfied. See Turán [959].

(P. Erdős)

Update 7.3 No progress on this problem has been reported to us.

Problem 7.4 If $z_1 = 1$, and the z_i are arbitrary complex numbers for $2 \le i \le n$, then Atkinson [51] proved that

$$\max_{1 \le k \le n} |S_k| > c$$

with $c = \frac{1}{3}$. What is the best value for the constant c?

Update 7.4 No progress on this problem has been reported to us.

Problem 7.5 Let $z_i, 1 \le i \le n$, be n complex numbers such that $|z_i| \ge 1$. Then there exists an absolute constant c such that the number of sums

$$\sum_{i=1}^{n} \varepsilon_i z_i, \qquad \varepsilon_i = \mp 1, \tag{7.2}$$

which fall into the interior of an arbitrary circle of radius 1 is less than $(c2^n \log n)/n^{\frac{1}{2}}$; see Littlewood and Offord [678]. If the z_i are real, Erdős [300] proves that the number of sums (7.2) which fall into the interior of any interval of length 2 is at most $\binom{n}{[\frac{n}{2}]}$ and this is sharp. He asks whether this estimate remains true for complex z_i, where his proof gives only $c2^n/\sqrt{n}$; or more generally, for vectors of a Hilbert space of norm at least one, where he can prove that the number of sums (7.2) falling into an arbitrary unit sphere is $o(2^n)$.

Update 7.5 The plane case has been settled independently by Katona [589] and Kleitman [610]. The general case was also settled by Kleitman [610].

Problem 7.6 We consider the range of the random function

$$F(z) = \sum_{n=0}^{\infty} \pm a_n z^n,$$

(F chosen at random in the natural way) defined in $\mathbb{D}$, where $\sum |a_n|^2 = \infty$. Is the image of $w = F(z)$ with probability one:

(a) everywhere dense in the plane?
(b) the whole plane?
(c) does it contain any given point with probability one?

If $a_n = n^\lambda$, (b) holds if $\lambda > \frac{1}{2}$, and (a) holds if $-\frac{1}{2} < \lambda < +\frac{1}{2}$.

(J.P. Kahane)

Update 7.6 Partial results were obtained by Offord [772]. This problem has been solved by Nazarov, Sodin and Nishry [751] who proved that (b) always holds.

Problem 7.7 Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathbb{D}$,

$$f_n(z) = \sum_{k=0}^{n} a_k z^k,$$

and let Γ_n be the modulus of the largest zero of $f_n(z)$. Then Kakeya (no citation given) showed that $\liminf_{n\to\infty} \Gamma_n \le A$, with $A = 2$. What is the best possible value of A? Clunie and Erdős [212] have shown that $\sqrt{2} < A < 2$.

Update 7.7 This has been settled by Buckholtz [172]. He has found a method by which the constant in question can be calculated with any desired accuracy. Sixsmith writes that the statement of this problem only makes sense if f is assumed to be analytic in $\mathbb{D}$ with radius of convergence equal to one.

Problem 7.8 Is it possible to express each K-quasiconformal map in 3-space as the composition of two quasiconformal maps with maximal dilatation less than K? The corresponding plane result is true.

(F.W. Gehring)

Update 7.8 A reviewer directs the interested reader to the following papers of Hamilton [469, 470].

Problem 7.9 Suppose that f is a plane K-quasiconformal mapping of the unit disc $\mathbb{D}$ onto itself. Show that there exists a finite constant $b = b(K)$ such that

$$m(f(E)) \leq b\{m(E)\}^{1/K}$$

for each measurable set E in $\mathbb{D}$. Here m denotes plane Lebesgue measure. Such an inequality is known (see Gehring and Reich [392]) with the exponent $\frac{1}{K}$ replaced by a constant $a = a(K)$.

(F.W. Gehring)

Update 7.9 This has been proved by Astala [49]. His proof has been substantially simplified by Eremenko and Hamilton [317].

Problem 7.10 It was proved by Boyarskiĭ [154] that the partial derivatives of a plane K-quasiconformal mapping are locally L^p-integrable for $2 \leq p < 2 + c$, where $c = c(K) > 0$. Show that this is true with $c = \frac{2}{K-1}$.

The example $f(z) = |z|^{\frac{1}{K}-1} \cdot z$ shows that such a result would be sharp.

(F.W. Gehring)

Update 7.10 Gehring points out that the proof published by Okabe [774], who claims to solve this problem, contains an error. Details of this are found in Gehring's review of Okabe's article for *Mathematical Reviews*. Also see the follow-up papers by Okabe [775, 776].

Problem 7.11 Show that each quasiconformal mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ has a quasiconformal extension to $\mathbb{R}^{n+1}$. This has been established by Ahlfors when $n = 2$ and by Carleson when $n = 3$.

(F.W. Gehring)

Update 7.11 This has been established in all cases by Tukia and Väisälä [957].

Problem 7.12 Suppose that f is an n-dimensional K-quasianalytic function. Show that the partial derivatives of f are locally L^p-integrable for $n \leq p \leq n + c$, where $c = c(K, n) > 0$. This was shown by Gehring [385] to be true if f is injective.

(F.W. Gehring)

Update 7.12 This has been solved by Elcrat and Meyers [720] modulo a slight problem observed by Gehring, which was straightened out by Stredulinsky [928].

Problem 7.13 One part of Nevanlinna theory is devoted to the following problem: How does the geometric structure of a simply connected Riemann covering surface of the sphere influence the value distribution of the meromorphic function generating the surface? It is suggested that one should also consider halfsheets among the constituent pieces of such surfaces, which have an infinite number of branch points on their boundary.

A typical example is the covering surface generated by $e^z - z$. It contains a right half-plane with a second order branch point at $2\pi i n$ for each integer n.

(F. Huckemann)

Update 7.13 No progress on this problem has been reported to us.

Problem 7.14 Let γ be a C^1 curve in the plane and let f be a continuous function on γ. Put

$$F(z) = \int_\gamma \frac{f(t)}{t-z}\,\mathrm{d}t, \qquad z \notin \gamma.$$

Does F have non-tangential boundary values almost everywhere on γ? This is a very old question which has been studied quite extensively, especially by Russian mathematicians. It is known that the answer is 'yes' if slightly greater smoothness is assumed for γ or f; see for example Havin [479] for an outstanding contribution to this subject.

Suppose now that γ is a Jordan curve and let ϕ be a conformal map from $\mathbb{D}$ onto the inside of γ. Does $F \circ \phi$ belong to H^p for some p, $p < 1$, or perhaps to the class N of functions with bounded Nevanlinna characteristic?

(A. Baernstein)

Update 7.14 Calderón [174] has proved that Cauchy integrals of measures on Lip^1 curves have boundary values almost everywhere. From this it is possible to extend the result to rectifiable curves. The second part of the problem, about possibly pulling back to an H^p function, is still open. For non-rectifiable γ one needs to consider $F(z) = \int_\gamma (t-z)^{-1} f(t)\,\mathrm{d}\mu(t)$ for measures μ since $F(z) = \int_\gamma (t-z)^{-1} f(t)\,\mathrm{d}t$ does not make sense.

D. Khavinson writes that if the measure $\mathrm{d}\mu = f\mathrm{d}\omega$, where $\mathrm{d}\omega$ is the harmonic measure on γ, then the Cauchy integral of $\mathrm{d}\mu$ is in the Smirnov class N^+; see [603, Lemma 2]. Further progress on the second part of the problem is also found in Björn [129].

Problem 7.15 Let D be a domain in the extended complex plane. A finite point z_0 on the boundary ∂D of D is called *angular* (relative to D) if there exists a positive ε such that every component domain of $D \cap \{|z - z_0| < \varepsilon\}$ which has z_0 as a boundary point is contained in an angle less than π with vertex at z_0. Angularity at infinity is similarly defined.

Let $A = A(D)$ be the set of angular points of ∂D relative to D. Obviously if A is not empty then ∂D has positive capacity. The set $A(D)$ can have positive linear measure. For example, let C be a Cantor set on $|z| = 1$ and let D consist of the open unit disc from which have been deleted all points rz with z in C and $\frac{1}{2} \le r < 1$. Then $A(D) = C$ which can, of course, have positive linear measure.

Yet the following holds for arbitrary domains: $A(D)$ is either empty or its harmonic measure relative to any point of D is zero. This result follows easily from an unpublished theorem on Brownian paths $\omega(t)$ in the complex plane, which says that almost all such paths have the property that for every real t_0 and every positive ε the set of numbers $\frac{\omega(t)-\omega(t_0)}{|\omega(t)-\omega(t_0)|}$ with $t_0 < t < t_0 + \varepsilon$ fills at least an open arc of length π on the unit circle. This theorem is not easy and it would be desirable to give a direct proof of the above result on $A(D)$.

Moreover, the Brownian paths approach will certainly not yield a similar result for the set $B_\alpha(D)$ of ∂D, whose points are defined by replacing the angles less than π with translates of $\{x + iy : 0 < x < |y|^\alpha\}$ for a given α with $\frac{1}{2} < \alpha < 1$. (For $\alpha = \frac{1}{2}$ the result is false, as can be seen by taking D to be a disc.)

Problem: For which α in $(\frac{1}{2}, 1)$ is the harmonic measure of the set $B_\alpha(D)$ always zero? (Of course we may assume that the capacity of ∂D is positive.)

A much more difficult problem would be to characterise the monotone functions $f(y)$ having the property that the set obtained on replacing the angles by translates of $\{x + iy : 0 < x < f(|y|)\}$ necessarily has harmonic measure zero.

Similar questions can be asked for Riemann surfaces and for n-dimensional space.

(A. Dvoretzky)

Update 7.15 No progress on this problem has been reported to us.

Problem 7.16 Let γ be a Jordan arc and $\mathrm{d}\mu$ a measure on γ. Does the Laplace transform

$$f(z) = \int_\gamma \mathrm{e}^{z\zeta}\, \mathrm{d}\mu(\zeta) \tag{7.3}$$

always have 'asymptotically regular' growth as $|z| \to \infty$. The answer might be 'no'. However it could be true anyway that the zeros of $f(z)$ have 'measurable distribution' in the sense of Pfluger [800].

(J. Korevaar)

Update 7.16 On regularity of growth and zero distribution of Laplace transforms $f(z) = \int_\gamma \mathrm{e}^{z\zeta}\, \mathrm{d}\mu(\zeta)$ along arcs, Dixon and Korevaar [249] have obtained some results for arcs of bounded slope. A negative answer has been given by Wiegerinck (personal communication to Hayman) using Borel transforms, to the question whether the Laplace transform (7.3) always has asymptotically regular growth as $|z| \to \infty$.

Problem 7.17 Let $f(z)$ be analytic and bounded for $\mathrm{Re}\,[z] > 0$. Suppose that $|\alpha| < \frac{1}{2}\pi$ and that (r_n) is a sequence of positive integers with $\sum \frac{1}{r_n} = \infty$. Show that the exponential type of f on the sequence $z_n = r_n \mathrm{e}^{i\alpha}$ is equal to the type of f on the ray $z = r\mathrm{e}^{i\alpha}$. (Proofs by Boas [132] and by Levinson [663] for $\alpha = 0$ do not seem to work for $|\alpha| > \frac{1}{4}\pi$.)

(J. Korevaar)

Update 7.17 This has been proved by Korevaar and Zeinstra [627]. See also Zeinstra [1017].

Problem 7.18 Let Γ be a closed Jordan curve containing $z = 0$ in its interior. Wermer [986] showed that when Γ has infinite length, the powers z^n, $n \neq 0$ span all of $C(\Gamma)$. One can indicate conditions on Γ under which the powers $z^n, n \neq n_1, \ldots, n_k$, form a spanning set; see Korevaar and Pfluger [626]. Under what conditions on Γ can one omit an infinite set of powers, and still have a spanning set?

(J. Korevaar)

Update 7.18 This problem has been altered to include 'closed' on the recommendation of a reviewer, as otherwise there is confusion about where the origin is an interior point of the arc. No progress on this problem has been reported to us.

Problem 7.19 For what sets Ω of lattice points (m_k, n_k) do the monomials $x^{m_k} y^{n_k}$ span L^2 or C_0 on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$? It is conjectured that the condition $\sum \frac{1}{m_k n_k} = \infty$ is sufficient for sets Ω in an angle $\varepsilon x \leq y \leq x/\varepsilon$, $\varepsilon > 0$. Hellerstein [530] has shown that the condition is not necessary.

(J. Korevaar)

Update 7.19 No progress on this problem has been reported to us.

Problem 7.20 (*Two-constants theorem for the polydisc*) Let $F(z_1, z_2)$ be defined for $|z_1| \leq 1$, $|z_2| \leq 2$ except when $z_1 = z_2$ and $|z_1| = |z_2| = 1$. Suppose that F is plurisubharmonic in $|z_1| < 1, |z_2| < 1$ and that $F(z_1, z_2) \leq \log \frac{1}{|z_1 - z_2|}$, whenever the left-hand side is defined. Further, suppose that $F(z_1, z_2) \leq 0$ for $\{|z_1| = |z_2| = 1$, $z_1 \neq z_2\}$. Does it follow that $F(z_1, z_2) \leq 0$ for $|z_1| < 1$, $|z_2| < 1$?

(L.A. Rubel, A.L. Shields)

Update 7.20 No progress on this problem has been reported to us.

Problem 7.21 Suppose that $|z_k| = 1$, $1 \leq k < \infty$. Put

$$A_l = \limsup_{m \to \infty} \Big| \sum_{k=1}^{m} z_k^l \Big|.$$

It is easy to see that there is a sequence z_k for which $A_l < cl$ for all l, and Clunie [211] proved that $A_l > cl^{\frac{1}{2}}$ for infinitely many l. Is there a sequence for which $A_l = o(l)$ as $l \to \infty$?

(P. Erdős)

Update 7.21 No progress on this problem has been reported to us.

Problem 7.22 Suppose that A and B are disjoint linked Jordan curves in $\mathbb{R}^3$ which lie at a distance 1 from each other. Show that the length of A is at least 2π. The corresponding result with a positive absolute constant instead of 2π is due to Gehring [386].

(F.W. Gehring)

Update 7.22 This problem was solved by several people. Osserman's survey article [785, p. 1226] contains details of its solution and some extensions. Cantarella, Fu, Kusner, Sullivan and Wrinkle [181] state that this was soon solved by Ortel, but since his elegant solution was never published, they reproduce it with his permission.

Problem 7.23 The expression $u(z, \zeta)$ of Problem 6.57 is closely related to the Schwarzian derivative $\{f(z), z\}$, for example, it is invariant under compositions with Möbius transformations and

$$\lim_{\zeta \to z} u(z, \zeta) = \frac{1}{6}\{f(z), z\}.$$

Kühnau (no citation given) showed that if $f(z)$ is analytic in $\mathbb{D}$ and has a quasiconformal extension to the whole plane with

$$|f_{\bar{z}}/f_z| \le q < 1, \qquad \text{almost everywhere,}$$

then

$$|u(z, \zeta)| \le q(1 - |z|^2)^{-1}(1 - |\zeta|^2)^{-1}. \tag{7.4}$$

Show that (7.4) is also sufficient for f to have a quasiconformal extension to the whole plane (possibly with $\frac{1}{3} \le q < 1$).

(J.G. Krzyż)

Update 7.23 No progress on this problem has been reported to us.

Problem 7.24 Let 0, a, b, where $a > 0$, $b = |b|e^{i\beta}$, $-\pi < \beta \le \pi$ be distinct points of $\mathbb{D}$; and let $\mathcal{K}$ be the family of continua K with the properties that $\{a, b\} \subset K \subset \mathbb{D} \setminus \{0\}$ and $\mathbb{D} \setminus K$ is connected. On any K in $\mathcal{K}$ there is a continuous function $\arg z$; we thus subdivide $\mathcal{K}$ into homotopy classes $\{\mathcal{K}_n\}$ according to the value of $V(K)$, where

$$V(K) = \arg b - \arg a \in \{\beta + 2n\pi, n \in \mathbb{Z}\}.$$

Let us call $K \in \mathcal{K}$ a *natural continuum* if K is a trajectory of a quadratic differential σ with the following properties:

(i) $0, a, b$ are simple poles of σ, and there is no other pole of σ in $\overline{\mathbb{D}}$;
(ii) σ is real on $\partial\mathbb{D}$.

There are many problems that then arise, for example:

(a) Do all homotopy classes contain natural continua? (There are many in $\mathcal{K}_n$ when $|\beta + 2n\pi| < 2\pi$.)
(b) Find all natural continua in $\mathcal{K}$.
(c) How does the modulus of $\mathbb{D} \setminus K$ vary when K runs through the natural continua in $\mathcal{K}_n$?

(F. Huckemann)

Update 7.24(a) An affirmative answer has been given by Hamilton [467].

Problem 7.25 Let K be a compact set of positive measure in $\mathbb{C}$. Does there necessarily exist a non-constant analytic function in $\mathbb{C} \setminus K$ with $f(\infty) = 0$ such that

$$\frac{f(z) - f(\zeta)}{z - \zeta} \neq \pm 1$$

for any z, ζ in $\mathbb{C} \setminus K$? Conceivably, the hypotheses even imply the existence of non-linear analytic functions f with $|f(z) - f(\zeta)|/|z - \zeta|$ *bounded*, but this is a well-known unsolved problem. Of course one can pose more general problems, such as requiring that the difference quotient omit all values in some pre-assigned plane set. (These problems arise in the use of variational methods.)

(D. Aharonov and H.S. Shapiro)

Update 7.25 No progress on this problem has been reported to us.

Problem 7.26 Is there a homeomorphism of the open unit ball in $\mathbb{R}^3$ onto $\mathbb{R}^3$ whose coordinate functions are harmonic? In other words, do there exist u_1, u_2, u_3 harmonic in $|\mathbf{x}| < 1$, $\mathbf{x} = (x_1, x_2, x_3)$, such that

$$(x_1, x_2, x_3) \to (u_1, u_2, u_3)$$

is a homeomorphism of $|\mathbf{x}| < 1$ onto all of $\mathbb{R}^3$? The analogous problem in $\mathbb{R}^2$ is answered negatively; the result is due to T. Radó (see [266, Sect. 2.4]) and is an important lemma in the theory of minimal surfaces.

(H.S. Shapiro)

Update 7.26 No progress on this problem has been reported to us.

Problem 7.27 For a domain D in $\mathbb{C}$, define

$$\rho(x, y) = \sup\{|f(x) - f(y)| : x, y \in D;\ f \text{ analytic in } D;\ |f'| \leq 1 \text{ in } D\}.$$

If D is convex, then $\rho(x, y) = |x - y|$, but not otherwise. Clearly $\rho(x, y) \leq L(x, y)$, where L is the infimum of the lengths of paths in G that join x to y. What can be said about ρ for general D in terms of the geometry of D?

(L.A. Rubel)

Update 7.27 No progress on this problem has been reported to us.

Problem 7.28 Suppose that $f(z)$ is continuous in a domain D in $\mathbb{C}$, and that either

(i) $\int_{|\zeta - z| = r} f(\zeta)\, \mathrm{d}\zeta = 0$ for all z in D and $0 < r \leq r(z)$,

or (weaker),

(ii) $\lim_{r \to 0} \left(r^{-2} \int_{|\zeta - z| = r} f(\zeta)\, \mathrm{d}\zeta \right) = 0$ for all z in D.

Does it follow that $f(z)$ is analytic in D? (See, for example, Zalcman [1013, 1014].)

(D. Gaier and L. Zalcman)

Update 7.28 No progress on this problem has been reported to us.

Problem 7.29 Let $f(z)$ be continuous on $\{|z| \le 1\}$, and let α be a fixed number with $0 < \alpha \le 1$. If, for each z in $\mathbb{D}$, the area integral

$$\int_{|\zeta - z| < \alpha(1-|z|)} f(\zeta)\,\mathrm{d}\zeta = 0,$$

is f necessarily analytic in $\mathbb{D}$? What happens if we are given that f is continuous only in $\mathbb{D}$?

(L. Zalcman)

Update 7.29 No progress on this problem has been reported to us.

Problem 7.30 Let $u(z)$ be a real bounded continuous function on $\mathbb{D}$, and suppose that to each z in $\mathbb{D}$ there corresponds a number $r(z)$ with $0 < r(z) < 1 - |z|$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} u(z + r(z)\mathrm{e}^{i\theta})\,\mathrm{d}\theta = u(z). \tag{7.5}$$

Must $u(z)$ be harmonic on $\mathbb{D}$? Volterra (no citation given) showed that this is true in the case that $u(z)$ is given to be continuous on $\mathbb{D}$; the case in which (7.5) is replaced by an areal-mean-value (and the continuity condition on $u(z)$ is relaxed) has been studied by Veech [968, 969] and others.

(L. Zalcman)

Update 7.30 This is the same as Problem 3.8. A complete answer was obtained by Hansen and Nadirashvili [474, 475]. See Update 3.8.

Problem 7.31 Suppose that

$$a_1 > 0, \qquad 0 \le a_n \le n, \qquad n \ge 1, \qquad b_m = \sum_{\nu=1}^{m} a_\nu, \qquad c_n = \sum_{\nu=1}^{n} b_\nu.$$

Then $\sum (c_n/a_n)^\alpha < \infty$ if $\alpha < -\frac{1}{2}$. For what other functions $f(t)$ is it true that $\sum f(c_n/a_n) < \infty$? Is it true, for instance, that (under some smoothness condition on f) $\sum f(c_n/a_n)$ converges with $\sum f(n^2)$?

The analogous result for c_n/b_n was obtained by Borwein [143], that if $xf(x)$ is positive and non-increasing for $x \ge a > 0$ and $\sum f(n) < \infty$, then $\sum f(c_n/b_n) < \infty$.

(W.K. Hayman)

Update 7.31 No progress on this problem has been reported to us.

Problem 7.32 Let $\mu(t)$ be a continuous monotonic increasing function of t for $t \in [0, 1]$ and let $\omega_1(h, \mu)$, $\omega_2(h, \mu)$ denote the modulus of continuity and the modulus of smoothness of μ, respectively. It is known that, if

$$\omega_1(h) = O(h), \qquad h \to 0,$$

or

$$\omega_2(h) = O(h(\log 1/h)^{-c}), \qquad h \to 0,$$

where $c > \frac{1}{2}$, then $\mu(t)$ is absolutely continuous (with respect to Lebesgue measure). It is also known that each of these conditions is essentially best possible. Are they simultaneously best possible? More precisely, is it true that given any function $\phi(t)$ such that $\phi(t)$ increases to ∞ as t decreases to 0, there is a continuous, monotonic increasing *singular* function $\mu(t)$ such that

$$\omega_1(h) = O(h\phi(h)), \qquad h \to 0$$

and

$$\omega_2(h) = O(h(\log 1/h)^{-\frac{1}{2}}), \qquad h \to 0\,?$$

(J.M. Anderson)

Update 7.32 An affirmative answer has been given by Anderson, Fernández and Shields [40].

Problem 7.33 Let $P(\theta) = \sum_{n=1}^{N} \mathrm{e}^{i\lambda_n\theta}$ be a finite Dirichlet series where $\lambda_m \neq \lambda_n$ for $m \neq n$. What can be said about $\mu \equiv \inf |P(\theta)|$?

A trivial argument shows that $\mu \leq (N-1)^{1/2}$. In fact,

$$|P|^2 = N + 2\sum_{m\neq n} \cos(\lambda_m - \lambda_n)\theta.$$

If $w(\theta) = |P|^2 - N \geq -c$, then write

$$h(\theta) = c + w(\theta) = 2\sum b_n \cos(\delta_n\theta),$$

where the b_n are positive integers, if $\delta_n > 0$ and $h(\theta) \geq 0$. Then

$$b_n = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} h(t)\cos(\delta_n t)\,\mathrm{d}t \leq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} h(t)\,\mathrm{d}t = c.$$

Thus $c \geq 1$. The problem arises in prediction theory, where $\mu \leq (N-1)^{1/2}$ is adequate. If the λ_n are rational, then the problem reduces to a problem on polynomials with coefficients 0 and 1.

(S. Rudolfer and W.K. Hayman)

Update 7.33 Goddard [397] explicitly calculates the minimum of μ when $N = 4$, and in the process, discovers some examples of Newman polynomials with few terms, but large minimum modulus, where a *Newman polynomial* is a sum of powers of z, with constant term 1.

Problem 7.34 Suppose that $\beta_j \in \mathbb{R}^+$ and $\zeta_j \in \mathbb{C}$ and for suitable small z define

$$f(z) = \prod_{j=1}^{n} (1 - \zeta_j z)^{\beta_j} = 1 + a_1 z + a_2 z^2 + \dots. \tag{7.6}$$

Suppose further that the a_k are all real. Then there exists N,

$$N = N(\beta_1, \beta_2, \beta_3, \dots, \beta_n) < \infty,$$

independent of the ζ_j, such that $\min(a_1, a_2, \dots, a_N) \leq 0$. Find a sharp, or good, upper bound for N as a function of the β's.

This would be significant for Turán's power sum method. To see that $N < \infty$: β_j's are all integers, then f is a polynomial and $N = \beta_1 + \beta_2 + \beta_3 + \dots + \beta_n + 1$. (The sort of estimate wanted in the general case.) If not, assume that $|\zeta_1| \leq |\zeta_2| \leq \dots \leq |\zeta_n| \leq 1$, and let $m = \max\{j : \beta_j \notin \mathbb{Z}\}$. The radius of convergence of (7.6) is $|\zeta_m|^{-1}$ and if the a_k were all positive (or even greater than or equal to 0). Then by Pringsheim's theorem (see Titchmarsh [942]), $|\zeta_m|^{-1}$ would be a singularity of f. Since f only has the singularities $\{\zeta_j^{-1} : \beta_j \notin \mathbb{Z}\}$ we deduce that $\zeta_m > 0$. We multiply through by $(1 - \zeta_m z)^{-\beta_m}$, which has positive coefficients, and repeat the argument. After at most n steps, we obtain a contradiction. This gives a finite N for any particular set of ζ_j and the uniformity is straightforward.

(R.R. Hall)

Update 7.34 No progress on this problem has been reported to us.

Problem 7.35 According to Fefferman's theorem (see [334, 335]), a real function u on the unit circle which has bounded mean oscillation can be decomposed as $u = b_1 + \tilde{b_2}$, where b_1 and b_2 are bounded functions and $\tilde{b_2}$ denotes the conjugate of b_2. Given u, what is the smallest possible $\|b_2\|_\infty$? This problem is discussed by Baernstein [58, Sect. 10].

An affirmative answer to Baernstein [58, Sect. 10, Conjecture 2] would prove the conjecture about factoring non-zero univalent functions that Baernstein made (see Problem 5.58).

(A. Baernstein)

Update 7.35 No progress on this problem has been reported to us. It has been suggested that there should be some constraints on b_1 and b_2. The Baernstein conjecture [58, Sect. 10, Conjecture 2] cites Jones' method [579] of achieving constructively the decomposition $u = b_1 + \tilde{b_2}$.

Problem 7.36 This problem is equivalent to Problem 7.9 due to Gehring and Reich about best bounds for area distribution under quasiconformal mapping. Let E denote a measurable subset of the unit disc $\mathbb{D}$, m denote 2-dimensional measure, and define

$$f_E(z) = -\frac{1}{\pi} \int\int_E \frac{\mathrm{d}m(w)}{(w-z)^2}, \qquad z \in \mathbb{C} \setminus E.$$

Thus, f_E is the 2-dimensional Hilbert transform of the characteristic function of E. It follows from the Calderón–Zygmund theory of singular integrals that there are constants a and b such that

$$\int\int_{\mathbb{D}\setminus E} |f_E| \,\mathrm{d}m \leq am(E) \log \frac{\pi}{m(E)} + bm(E)$$

for every E. The problem is to find the smallest possible a (for which there exists some b such that the inequality holds for every E in $\mathbb{D}$). Consideration of $E = \{z : |z| < \delta\}$ for small δ shows that $a = 1$ would be best possible, and this is conjectured by Gehring and Reich.

An analogous sharp inequality for sets E in $[-1, 1]$ and the 1-dimensional Hilbert transforms

$$f_E(x) = -\frac{1}{\pi} \int_E \frac{\mathrm{d}t}{x-t}$$

is known, and can be proved either by a subordination argument or by the use of a theorem of Stein and Weiss.

(A. Baernstein)

Update 7.36 This has been solved by Eremenko and Hamilton [317].

Problem 7.37 Let T denote the class of all rational functions g of the form

$$g(z) = \sum_{j=1}^{n} \frac{\lambda_j}{(z-z_j)^2},$$

where the constants λ_j satisfy $\lambda_j > 0$ and $\sum_{j=1}^{n} \lambda_j = 1$. Prove (or disprove): There is a constant C with the property that for each g in T we can find a set $S = S(g)$ with $m(S) = \pi$ such that

$$\int\int_{\Delta(R)\setminus S} |g| \,\mathrm{d}x \,\mathrm{d}y \leq 2\pi \log R + C$$

for every R in $(1, \infty)$. Here $\Delta(R) = \{z : |z| < R\}$. This assertion, if true, would imply that the inequality of Problem 7.36 holds with $a = 1$, and thus solve the area problem of Gehring and Reich. Problems of this sort have been considered by Fuchs and MacIntyre [696].

(A. Baernstein)

Update 7.37 D. Khavinson writes that a similar Fuchs and MacIntyre problem for logarithmic derivatives of a polynomial was solved by Anderson and Eiderman [39].

Problem 7.38 The Hankel matrices of a function f having a Taylor expansion

$$f(z) = a_0 + a_1 z + \dots$$

are defined by

$$H_p^{(n)} = (a_{ij}); \qquad a_{ij} = a_{n+i+j-2}; \qquad 1 \le i,\ j \le p+1.$$

If f belongs to the Pick–Nevanlinna class (Det $H_p^{(n)} \ge 0$, all n, p), then all the poles of f are simple and they lie on the positive real axis. Denote by $\varepsilon_j^{(n)}$,

$$\varepsilon_1^{(n)} \ge \varepsilon_2^{(n)} \ge \dots \ge \varepsilon_p^{(n)} \ge 0,$$

the eigenvalues of $H_p^{(n)}$. Then

$$\limsup_{n\to\infty} (\varepsilon_j^{(n)})^{1/n} = \frac{1}{\lambda_j},$$

where λ_j is the jth pole, where the poles are numbered in order of increasing modulus. What can be said about the eigenvalues under less restrictive conditions?

(R. Bouteiller)

Update 7.38 No progress on this problem has been reported to us.

Problem 7.39 (*Subadditivity problem for analytic capacity*) The *analytic capacity* of a compact set K in $\mathbb{C}$ is defined by

$$\gamma(K) = \sup\{\lim_{|z|\to\infty} |zf(z)| : f \in A(K)\},$$

where $A(K)$ is the set of functions which are analytic outside K, vanish at infinity and for which $|f(z)| \le 1$ for z in $\mathbb{C} \setminus K$. Prove or disprove the existence of a constant M such that

$$\gamma(K_1 \cup K_2) \le M\{\gamma(K_1) + \gamma(K_2)\}$$

for all compact sets K_1, K_2 in $\mathbb{C}$. Even the case where $\gamma(K_2) = 0$ is open. For background, see Garnett [378].

(J. Korevaar)

Update 7.39 This has been proved by Tolsa [946] and is the same as Problem 7.75(c).

Problem 7.40 Define $D(z,r)=\{w:|w-z|\le r\}$. Given a sequence $\{D(z_j,r)\}_{j=1}^N$ of disjoint closed balls all contained in $|z|\le\frac{1}{2}$, put

$$\Omega=\mathbb{D}\setminus\cup_{j=1}^N D(z_j,r).$$

Let h be the function in $|z|\le 1$ satisfying: h is harmonic in Ω, $h=1$ on $|z|=1$, and $h=0$ on $\cup_{j=1}^N D(z_j,r)$. Does there exist a positive δ such that whenever $r\le\delta$ and $N\ge[1/r]^{2-\delta}$, then

$$\int_0^{2\pi} h(z_j+2re^{i\theta})\,d\theta\le r^{2+\delta}$$

holds for at least one $D(z_j,r)$? Here $[1/r]$ denotes the greatest integer less than or equal to $1/r$. An affirmative answer should imply a weak version of Arakelian's conjecture for entire functions (Problem 1.6).

(J.L. Lewis)

Update 7.40 Lewis and Wu [669] have proved a similar conjecture for harmonic measure which provides the key to their solution of Problems 1.6 and 4.18.

Problem 7.41 Let $\{z_\nu\}$ be a sequence of distinct points in $\mathbb{D}$ such that $\sum(1-|z_\nu|)<\infty$. Let B be the Blaschke product corresponding to this sequence. If $0<t<\infty$, define $W_t=\{z\in\mathbb{C}:|B(z)|<t\}$. Denote the space of all bounded analytic functions in W_t by $H^\infty(W_t)$, and put $S=\{z_\nu\}_{\nu=1}^\infty$. Is

$$H^\infty(\mathbb{D})|_S=H^\infty(W_t)|_S \tag{7.7}$$

for all t? $H^\infty(W_t)|_S$ denotes the restrictions to S of the functions in $H^\infty(W_t)$.

We note the following:

(i) For an interpolating sequence $H^\infty(\mathbb{D})|_S=l^\infty$ this is true, and can easily be deduced from Earl's proof [280] of Carleson's interpolation theorem.
(ii) For *any* Blaschke sequence, the result is true when $0<t<1$ for then the result is contained in Carleson's original proof of the corona theorem for H^∞.

(A. Stray)

Update 7.41 Stray [926] has proved (7.7).

Problem 7.42 Define the *Harnack function* $H_{z_0}(z)$ for a Green domain D relative to z_0 in D to be the supremum of all positive harmonic functions h on D which satisfy $h(z_0)\le 1$. If $K_\zeta(z)$ is the Martin kernel for D relative to z_0, $K_\zeta(z_0)=1$ for all ζ in the Martin boundary Δ_1, then

$$H_{z_0}(z)=\sup\{H_\zeta(z):z\in D,\zeta\in\Delta_1\}.$$

In particular, for the unit disc $\mathbb{D}$, the boundary Δ_1 may be identified with the unit circle $\mathbb{T}$ and, if $z_0=0$, then $K_\zeta(z)$ is the Poisson kernel with pole at ζ in $\mathbb{T}$. In this case, $K_\zeta = H_0$ along the radius to ζ. Using the Riemann mapping functions, this property may be described for simply connected D as follows: for each ζ in Δ_1, K_ζ touches H_{z_0} along a Green line for D issuing from z_0.

Does this property continue to hold for multiply connected D?

(M. Arsove and G. Johnson)

Update 7.42 No progress on this problem has been reported to us.

Problem 7.43 A bounded simply-connected domain D is said to be *conformally rigid* if there is some positive ε such that, if f is a conformal self-map of D satisfying $|f(z) - z| < \varepsilon$, then $f(z) \equiv z$. Clearly, if each prime end of D is a singleton, then D is *not* conformally rigid. Show the converse.

(P.M. Gauthier)

Update 7.43 Gaier [366] has shown that the converse is false.

Problem 7.44 Let U be an open set in the plane and let λ_a^U be the harmonic measure at the point a with respect to U. Then Øksendal [778] showed that λ_a^U is singular with respect to area measure. Is it also true that λ_a^U is singular with respect to β-dimensional Hausdorff measure for all $\beta > 1$?

The same question can be asked for the Keldysh measure μ_a^K at a in K with respect to a compact set K. Øksendal has shown that μ_a^K is singular with respect to area measure.

(B. Øksendal)

Update 7.44 This was proved by Makarov [701] in the case where U is simply connected, and by Jones and Wolff [582] in the general case.

Problem 7.45 Let D be the unit disc cut along p radial slits from the outer boundary, all of the same given length less than one. Let u be the harmonic measure of $\{|z| = 1\}$ in D. Find the configuration of slits which makes $u(0)$ minimal, when p is fixed.

(A.A. Gonchar; communicated by M. Essén)

Update 7.45 Dubinin [260] has proved that this occurs when the slits are symmetrically distributed. Baernstein in some of his talks asked about the harmonic measure problem when the removed slits are not connected. Is harmonic measure minimised when the modified slits are equally spaced on the circle?

Problem 7.46 (*A 'Universal' Phrágmen–Lindelöf theorem*) Let D be an arbitrary unbounded plane domain. Suppose that $f(z)$ is analytic on D and continuous on $\overline{D}$. If $|f(z)| \le 1$ on ∂D and $f(z) = o(|z|)$ at ∞, show that $|f(z)| \le 1$ throughout D. The $o(|z|)$ would then be the 'right' condition since that is what is needed for the case $D = \{z : |z| > 1\}$.

(D.J. Newman)

Update 7.46 Fuchs [361] has shown that $|f(z)| \le 1$ throughout D. A slightly more general result, where $z^\alpha f(z)$ is one-valued for some real α, was obtained by Gehring, Hayman and Hinkkanen [390] and was applied to problems involving moduli of continuity.

Problem 7.47 Let K be a compact subset of $\mathbb{C}$ and let x_0 in K be a *non-peak point* for $R(K)$, the uniform limits on K of rational functions with poles outside K. Will there always exist a continuous curve Γ in K terminating at x_0?

It is easy to see that if $\sum_{n=1}^{\infty} 2^n M_1(A_n(x_0)\backslash K) < \infty$, where $A_n(x_0) = \{z : 2^{-n-1} \leq |z - x_0| \leq 2^{-n}\}$, and M_1 denotes 1-dimensional Hausdorff content, then Γ can be chosen to be a straight line segment. In this case, Øksendal [777] showed that the integrated Brownian motion starting at x_0 stays inside K for a positive period of time, almost surely. There is an example of a compact set K and a non-peak point x_0 in K such that no straight line segment terminating at x_0 is included in K.

(B. Øksendal)

Update 7.47 No progress on this problem has been reported to us.

Problem 7.48 A domain D in $\mathbb{R}^n$ is said to be *linearly accessible* if each point in the complement of D can be joined to ∞ by a ray which does not meet D. Let $g(\,\cdot\,, x_0)$ be the Green's function for D with pole at x_0 in D.
Is $\{x : g(x, x_0) > t\}$ linearly accessible for $0 < t < \infty$ if D is linearly accessible? This conclusion is valid in $\mathbb{R}^2$.

(J.L. Lewis)

Update 7.48 No progress on this problem has been reported to us.

Problem 7.49 Let $x = (x_1, x_2, \ldots, x_n)$ be a point in Euclidean n-space, $n \geq 3$, with $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$. A function u on $\mathbb{R}^n$ is said to be *homogeneous of degree* m if $u(\lambda x) = \lambda^m u(x)$ for all positive λ. If u has continuous second partial derivatives, put $\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ and $\nabla\cdot = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$. Fix p, $1 < p < \infty$, $p \neq 2$. Prove there are no homogeneous polynomials u of degree $m \geq 2$, with real coefficients, such that if $x \in \mathbb{R}^n$

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-4} \left((p-2) \sum_{i,j=1}^{n} u_{x_i} u_{x_i} u_{x_i x_j} + |\nabla u|^2 \Delta u \right) \equiv 0. \quad (7.8)$$

In $\mathbb{R}^2$ there are no polynomial solutions for $m \geq 2$, whenever $1 < p < \infty$, $p \neq 2$. Proof of the above would imply that if f is any solution to (7.8) on a domain D in $\mathbb{R}^n$, then f is real analytic in D if and only if ∇f does not vanish in D.

(J.L. Lewis)

Update 7.49 The original statement of this problem has been amended by its proposer. It is known that there are no polynomials of degree $m = 2, 3, 4$ in $\mathbb{R}^n$, $n \geq 3$, and none for $m = 5$ in $\mathbb{R}^3$ which are solutions to (7.8) whenever $1 < p < \infty$, $p \neq 2$. For $m = 3$, see Tkachev [943], and for $m = 4, 5$, see Lewis and Vogel [668].

Problem 7.50 Let U be a connected open set in $\mathbb{R}^n$. Brelot and Choquet [157] showed that the set of points on the boundary of U which are accessible from the interior by (finite length) rectifiable paths supports harmonic measure. It is natural, in view of polygonal path connectedness of finely open sets, to ask if the same is true for finely open sets and the Keldysh measure.

(T. Lyons and B. Øksendal)

Update 7.50 No progress on this problem has been reported to us.

Problem 7.51 A domain $D \subset \mathbb{C}^n$ is said to be *pseudoconvex* if there exists a continuous plurisubharmonic function φ on D such that the set $\{z \in D \mid \varphi(z) < x\}$ is a relatively compact subset of D for all real numbers x. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, with smooth boundary. Denote by $A^\infty(D)$ the class of functions analytic in D, continuous on $\overline{D}$, all of whose derivatives are continuous on $\overline{D}$. Let E be a closed subset of the boundary of D which is not a set of uniqueness for $A^\infty(D)$, that is, there exists a function f, $f \not\equiv 0$, which belongs to $A^\infty(D)$ such that f vanishes exactly on E and all of the derivatives of f vanish on E. Is every closed subset of E a set of non-uniqueness for $A^\infty(D)$? This is true in the case of the unit disc in $\mathbb{C}$.

(A.-M. Chollet)

Update 7.51 No progress on this problem has been reported to us.

Problem 7.52 Let K be a compact subset of $\mathbb{C}^n$ and let $P_0(K)$ be the set of all polynomials on K. The P*-hull* of K, the *polynomial convex hull* of K, is defined by

$$P\text{-hull } K = \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p(z)| \text{ for all } p \text{ in } P_0(K)\}.$$

Let $P(K)$ be the uniform closure of $P_0(K)$ in $C(K)$, the continuous functions on K. Let Šilov Bd $(P(K))$ denote the Šilov boundary of the uniform algebra $P(K)$; that is, the smallest closed subset of the structure space of a commutative Banach algebra where an analogue of the maximum modulus principle holds; see Alexander and Wermer [24, Chap. 9]. Determine all compact subsets K in $\mathbb{C}^n$, $n > 1$ such that

$$\text{Šilov Bd } (P(K)) = \text{Boundary } (P\text{-hull } K).$$

For $n = 1$, every compact set K in $\mathbb{C}$ has this property. For $n > 2$, examples of K are compact convex sets and closed spheres.

(S. Kilambi)

Update 7.52 No progress on this problem has been reported to us.

Problem 7.53

(a) (One-dimensional version). Let E be a compact set in $\mathbb{R}$ and for each x in E, let a positive δ_x be given. Define $I_x = (x - \delta_x, x + \delta_x)$. For what values of c can one always find a disjoint collection of such intervals, $\{I_{x_j}\}$ say, such that $\sum_j |I_{x_j}| \geq c|E|$? It is known that this is possible for $c = \frac{1}{2}$, but is impossible in general for $c > \frac{2}{3}$.

(b) (n-dimensional version). Let E be a compact set in $\mathbb{R}^n$; and let K be an open bounded symmetric convex set in $\mathbb{R}^n$, where we say that a convex body K is *symmetric* if it is centrally symmetric with respect to the origin, that is, a point x lies in K if and only if its antipode, $-x$, also lies in K. Symmetric convex bodies are in a one-to-one correspondence with the unit balls of norms on $\mathbb{R}^n$. For each x in E, let a positive δ_x be given; and define $K_x = x + \delta_x K$ to be the dilation of K by a factor δ_x, centred at x. For what values of c can one always find a disjoint collection of such sets, $\{K_x\}$ say, such that $\sum_j |K_{x_j}| \geq c|E|$? If $c(K)$ denotes the best value, it is known that $2^{-n} \leq c(K) < 1$.

The facts in (a) and (b), together with some sketchy information on $c(Q_n)$ (where Q_n is the n-cube) and $c(S_n)$ (where S_n is the n-sphere), are given in Walker [977], but there is no information on the correct asymptotic behaviour of $c(Q_n)$ or $c(S_n)$. The problem has applications to the best constants in results concerning the Hardy–Littlewood maximal function.

(P.L. Walker)

Update 7.53 No progress on this problem has been reported to us.

Problem 7.54 Define $\phi_t(z) = \mathrm{e}^{tz} - 1$, $\phi_t^1 = \phi_t$ and $\phi_t^{n+1} = \phi_t \circ \phi_t^n$ for $n \geq 1$; it follows that

$$\phi_t^1(-1) = \mathrm{e}^{-t} - 1 = -t + \frac{t^2}{2!} - \dots,$$

$$\phi_t^2(-1) = \mathrm{e}^{t(\mathrm{e}^{-t}-1)} - 1 = -t^2 + \dots \text{ and so on.}$$

Are the coefficients in these formal power series for $\{\phi_t^n(-1)\}_{n=1}^{\infty}$ uniformly bounded by 1 in modulus?

(P.J. Rippon)

Update 7.54 No progress on this problem has been reported to us.

Problem 7.55 It is known that any quasiconformal homeomorphism f of $B^n = \{x : x \in \mathbb{R}^n, |x| < 1\}$ onto a Jordan domain D in $\mathbb{R}^n$ can be extended to a homeomorphism of $\overline{B^n}$ onto $\overline{D}$. If ∂D is rectifiable (in the sense that $\Lambda^{n-1}(\partial D) < \infty$), is $f|_{\partial B^n}$ absolutely continuous (in the sense that $\Lambda^{n-1}\big(f(E)\big) = 0$ for every set E in ∂B^n with $\Lambda^{n-1}(E) = 0$), where Λ^{n-1} is $(n-1)$-dimensional Hausdorff measure? One can also ask the analogous question about f^{-1}.

When $n = 2$, the answer to both questions is 'yes' for conformal mappings, but 'no' for quasiconformal mappings. When $n = 3$, Gehring [387] has proved that, if in addition the function f has a quasiconformal extension to $\mathbb{R}^n$, then $f|_{\partial B^n}$ is absolutely continuous; but, even in this special case, it is not known if $f^{-1}|_{\partial D}$ is absolutely continuous.

(A. Baernstein)

Update 7.55 No progress on this problem has been reported to us.

Problem 7.56 Let Γ be a closed Jordan curve in the extended plane, and suppose that $\infty \in \Gamma$. Let f_1, f_2 map the upper and lower half-planes, respectively, onto the two different domains in $\mathbb{C}$ bounded by Γ, with $f_1(\infty) = f_2(\infty) = \infty$. Then, if $h = f_2^{-1} \circ f_1$ is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, it is known that Γ is a quasicircle if and only if h is quasisymmetric (that is, there exists a constant c such that

$$\frac{1}{c} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq c$$

for all x, t in $\mathbb{R}$). Can one characterise the function h for general Jordan curves Γ? In particular, can every function $h : \mathbb{R} \to \mathbb{R}$ be generated in this fashion?

(L. Bers; communicated by A. Baernstein)

Update 7.56 Counterexamples to the latter question have been given by Oikawa [773] and Huber [559].

Problem 7.57 If $n \geq 2$, define

$$\mathbb{R}^n_+ = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\big\}$$

and

$$\mathbb{R}^n_- = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < 0\big\};$$

let E, F be non-empty compact subsets of $\mathbb{R}^n_+$, $\mathbb{R}^n_-$ (respectively), and let F^* denote the symmetric image of F in $\partial\mathbb{R}^n_+$. Denote by $\Delta(E, F)$, $\Delta(E, F^*)$ the families of all curves in $\overline{\mathbb{R}^n}$ joining E and F, E and F^* (respectively). Is it true that

$$M\big(\Delta(E, F)\big) \leq M\big(\Delta(E, F^*)\big), \tag{7.9}$$

where M denotes the n-modulus of a curve family? For further details on the n-modulus of a curve family, see Väisälä [962].

It is easy to show that strict inequality holds in (7.9) if E and F are balls. Also, if $\Delta(E, F)$ is obtained from $\Delta(E, F^*)$ as a result of symmetrisation, then (7.9) holds. Note too that it follows from the symmetry principle for the modulus that $\frac{1}{2}M\big(\Delta(E, F)\big) \leq M\big(\Delta(E, F^*)\big)$, at least if $E \cap F^* = \emptyset$. For further details on the symmetry principle, see for example Gehring [388, Lemma 1].

(M. Vuorinen)

Update 7.57 This has been solved by Dubinin [261, 262].

Problem 7.58 Let E be a compact set on $[0, 1]$; and let E have positive conformal 2-capacity, that is, $M\big(\Delta(E, \partial B^2(2); \mathbb{R}^2)\big) > 0$ where $\Delta(E, \partial B^2(2); \mathbb{R}^2)$ is the family of all curves joining E to $\partial B^2(2)$ and $M(\Delta)$ is the 2-modulus of Δ. Is it true that $M\big(\Delta(E, F; \mathbb{R}^2)\big) = \infty$, where $F = \mathbb{R} \setminus E$?

(A.A. Gonchar; communicated by M. Vuorinen)

Update 7.58 No progress on this problem has been reported to us. It has been suggested that some of the terms in the statement of this problem are unclear, but no more detail has been found.

The following three problems are about polynomials in n variables. We write $z = (z_1, \ldots, z_n)$ and $D = (D_1, \ldots, D_n)$, where D_i denotes $\frac{\partial}{\partial z_i}$. Also $\mathcal{E}$ denotes *the set of entire functions* in $\mathbb{C}^n$.

Problem 7.59 Let (P, Q) denote a pair of polynomials with the following property:

$$\text{The map } f \mapsto P(D)(Qf) \text{ carries } \mathcal{E} \text{ bijectively onto } \mathcal{E}. \tag{7.10}$$

If (P, Q) has the property (7.10), is it necessarily true that (Q, P) also has the property (7.10)? (There is little theoretical ground so far to support such a conjecture, but in all examples the proposer has been able to check, it is true.)

Note that it is fairly easy to show that, under the hypotheses above, the map $F \mapsto Q(D)(PF)$ carries $\tilde{\mathcal{E}}$ bijectively onto $\tilde{\mathcal{E}}$, where

$$\tilde{\mathcal{E}} = \{F : F \in \mathcal{E},\ F \text{ is of exponential type}\}.$$

(H.S. Shapiro)

Update 7.59 Meril and Struppa [719] have shown that the answer to the first question is 'no' in general.

Problem 7.60 Let P be a polynomial of degree m in which the coefficient of z_1^m is non-zero, and define $Q(z) = z_1^m$.

(a) Does the pair (P, Q) have the property (7.10)?
(b) Does the pair (Q, P) have the property (7.10)?

The proposer can prove the conjectures in the case when

$$Q(z) = z_1^m + \big(\text{polynomial in } (z_2, \ldots, z_n)\big).$$

Note that if (a) were true, then it would follow that the non-characteristic Cauchy problem with entire data on a hyperplane has a unique entire solution. (The uniqueness follows from classical results; only the entirety is in question.)

(H.S. Shapiro)

Update 7.60 Meril and Struppa [719] have shown that the answer to (a) and (b) is 'yes'.

Problem 7.61 Is it true that for any polynomial P with complex coefficients, the mapping $f \mapsto P^*(D)(Pf)$, where $P^*(z) = \overline{P(\overline{z})}$, is a bijection of $\mathcal{E}$ onto $\mathcal{E}$?

The proposer can prove that this is true when P is a homogeneous polynomial; and Newman told the proposer that he could prove the injectivity half of the conjecture, but the proposer has seen no details of the proof. The conjecture would follow if one could show that the partial differential equation $P^*(D)(Pf) = z^\alpha$ where $P^*(z) = \overline{P(\overline{z})}$, has a solution f that is entire and of exponential type for every multi-index α. Here $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $z^\alpha = (z_1^{\alpha_1}, z_2^{\alpha_2}, \ldots, z_n^{\alpha_n})$.

(H.S. Shapiro)

Update 7.61 No progress on this problem has been reported to us.

Problem 7.62 Given a countable number of convergent series with positive decreasing terms, one can find such a series converging more slowly than any of these. Without making any assumption about the Continuum Hypothesis, can one associate with every countable ordinal number α a convergent series $\sum_{n=1}^{\infty} x_{n,\alpha}$ with $0 \le x_{n+1,\alpha} \le x_{n,\alpha}$ such that:

(a) if $\alpha < \beta$, then $x_{n,\alpha}/x_{n,\beta} \to 0$ as $n \to \infty$,
 and
(b) if $x_n > 0$ and $\sum_{n=1}^{\infty} x_n < \infty$, then there exists an α such that $x_n/x_{n,\alpha} \to 0$ as $n \to \infty$.

(See also Problem 2.66.)

(A. Hinkkanen)

Update 7.62 No progress on this problem has been reported to us.

Problem 7.63 If $n \ge 2$ and $\alpha > 0$, define

$$\widehat{T_R^\alpha f}(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)_+^\alpha \hat{f}(\xi), \qquad a_+ = \max\{a, 0\},$$

where $R > 0$, f is in the Schwartz space of smooth rapidly decaying functions $\mathcal{S}(\mathbb{R}^n)$, $\hat{f}$ signifies the Fourier transform of f, and $(\cdot)_+$ signifies the positive part, that is, $a_+ = \max\{a, 0\}$. Is it true, whenever $f \in L^{2n/(n+1)}(\mathbb{R}^n)$ and α is positive, that $T_R^\alpha f(x) \to f(x)$ almost everywhere as $R \to \infty$?

When $n = 2$ and $\{R_j\}$ is a lacunary sequence tending to ∞, Carbery [184], Córdoba and López-Melero [224] and also Igari [565] have shown that the answer is 'yes'. When $n \ge 3$, the more 'elementary' problem of norm convergence remains unsolved. See also Carbery [185] and Córdoba [223].

(A. Carbery)

Update 7.63 Carbery, Rubio de Francia and Vega [186] have answered the almost everywhere convergence problem in the affirmative in all dimensions.

Problem 7.64 Let Γ be a Fuchsian group in $\mathbb{D}$, and set $i(z) \equiv z$. Is it true that

$$\sum_{\gamma \in \Gamma} |\gamma'(0)| \ge \prod_{\gamma \in \Gamma,\, \gamma \ne i} |\gamma(0)|^2\ ?$$

Note that this is an equivalent formulation of a problem connecting the Bergman kernel function and the capacity function of a Riemann surface.

(N. Suita; communicated by Ch. Pommerenke)

Update 7.64 This has been solved by Błocki [130].

Problem 7.65 Let $\mathcal{W}$ be a hyperbolic Riemann surface, G_ω the Green's function with pole ω in $\mathcal{W}$, and Γ be the fundamental group $\Gamma = \{[\gamma_n]\}$. Let $\tilde{\Gamma}$ be the subgroup of equivalence classes $[\gamma]$ such that, for every ω in $\mathcal{W}$, the harmonic conjugate G^*_ω of G_ω changes by an integral multiple of 2π on a representative path in $[\gamma]$. Certainly $[\Gamma, \Gamma] \lhd \tilde{\Gamma} \lhd \Gamma$, where $\lhd$ denotes *is a normal subgroup of*. Is it true that, if $[\Gamma, \Gamma] \neq \tilde{\Gamma}$, then $\mathcal{W}$ is necessarily of the form $\mathcal{V} \setminus A$, where $\mathcal{V}$ is a hyperbolic surface and A is a non-empty relatively closed subset of zero logarithmic capacity?

The conjecture is true if $\tilde{\Gamma} = \Gamma$ and $\mathcal{V}$ is $\mathbb{D}$; and it is false if $[\Gamma, \Gamma] = \tilde{\Gamma}$. (The conjecture arises from work in Stephenson [925].)

(K. Stephenson)

Update 7.65 No progress on this problem has been reported to us.

Problem 7.66 A continuous mapping $f : B^n \to \mathbb{R}^n$, where $B^n = \{x : x \in \mathbb{R}^n, |x| < 1\}$, and $n \geq 2$, is said to be *proper* if $f^{-1}(K)$ is a proper subset of B^n whenever K is a proper subset of $f(B^n)$. Define

$$B_f = \{z : z \in B^n, f \text{ is not a local homeomorphism at } z\}.$$

Is it true that, if

(i) $n \geq 3$,
(ii) $f : B^n \to \mathbb{R}^n$ is proper and quasiregular,
(iii) B_f is compact

then f is necessarily injective?

Note that the mapping $f(z) = z^2$, where $z \in B^2$, shows that the conjecture is false when $n = 2$. The conjecture is known to be true in the special case when $f(B^n) = B^n$, $n \geq 3$. The conjecture is a special case of a more general open problem; see Vuorinen [973, Problem 4, p. 193].

(M. Vuorinen)

Update 7.66 No progress on this problem has been reported to us.

Problem 7.67 Let V be the zero set of some analytic function in a strictly pseudoconvex domain Ω in $\mathbb{C}^2$ (see Problem 7.51 for the definition of pseudoconvex). If V has finite area inside Ω, is it necessarily true that V is the zero set of some bounded analytic function on Ω?

Berndtsson (no citation given) has shown that the answer is 'yes' when Ω is the ball; and easy examples show that the answer is 'no' for strictly pseudoconvex domains in $\mathbb{C}^n$ when $n > 2$. Skoda [906, 907] and independently Henkin [538] have shown (under some cohomology condition on Ω), for all n, that V is the zero set of a function of Nevanlinna class, that is, that it is the zero set of a function of bounded characteristic, only if V satisfies the Blaschke condition.

(R. Zeinstra)

Update 7.67 No progress on this problem has been reported to us.

Problem 7.68 For a Hadamard gap sequence $\{n_k\}_{k=1}^{\infty}$, $n_{k+1}/n_k \geq q > 1$, is it true that the measure in $[0, 2\pi)$ of the set of those points x for which

$$\liminf_{m\to\infty} \left| \sum_{k=1}^{m} \cos(n_k x) - \xi \right| = 0, \qquad \text{for all } \xi \in \mathbb{R},$$

equals 2π?

(T. Murai)

Update 7.68 No progress on this problem has been reported to us.

Problem 7.69 Given an associative algebra A with identity 1 and countable basis, then for a finite-dimensional subspace V spanned by the vectors $\{e_j\}_{j=1}^{k}$ we have the differential operator

$$\sum_{j=1}^{k} e_j \frac{\partial}{\partial x_j}$$

acting on differentiable functions defined over domains in V and taking their values in A. We call a function $f : U \to A$, where U is a subset of V, a *left analytic function* if

$$\sum_{j=1}^{k} e_j \frac{\partial f}{\partial x_j}(x) = 0$$

for all x in U. Ryan [870] has shown that there is a generalised Cauchy integral formula

$$f(x_0) = \int_{\partial M} G(x, x_0) \sum_{j=1}^{k} (-1)^j e_j \, \mathrm{d}\hat{x}_j \, f(x),$$

with real-analytic kernel $G(x, x_0)$, where M is an arbitrary, compact real n-dimensional manifold lying in U and x_0 in $\mathring{M}$, if and only if there are elements $\{p_j\}_{j=1}^{k}$ in A satisfying the relation

$$p_j e_l + p_l e_j = 2\delta_{jl}.$$

Here δ_{jl} is the Kronecker δ-function, with $\delta_{jl} = 1$ if $j = l$; $\delta_{jl} = 0$ otherwise.

(a) Is the result still valid if we only assume $G(x, x_0)$ to be a C^1 function?
(b) What analogous result holds if we assume the algebra to be non-associative?

(J. Ryan)

Update 7.69 No progress on this problem has been reported to us.

Problem 7.70 By means of arguments due to Ahlfors (see, for example [13]) any Möbius transformation in $\mathbb{R}^n$ can be written in the form $(ax+b)(cx+d)^{-1}$, where $x \in \mathbb{R}^n$ and a, b, c, d are elements of a Clifford algebra A_n that satisfies certain constraints. It can be shown that the linear differential equations whose solution spaces are conformally invariant are of the type

$$D^k f_k((ax+b)(cx+d)^{-1}) = 0, \qquad k \in \mathbb{N},$$

where D is the Euclidean Dirac operator, and the associated conformal weight is

$$J_k(cx+d) = \begin{cases} (cx+d) * |cx+d|^{-n-1+k}, & \text{for } k = 2p-1, \\ |cx+d|^{-n+k}, & \text{for } k = 2p, \end{cases}$$

where $*$ is the involution described in Ahlfors [13].

(a) What are the non-linear differential equations whose solution spaces are conformally invariant?
(b) Can their conformal weights also be expressed in terms of $cx+d$, and what relationship do these solutions have to the linear conformally-invariant differential equations?

(J. Ryan)

Update 7.70 No progress on this problem has been reported to us.

Problem 7.71 Given a domain of holomorphy Ω in $\mathbb{C}^n$, $n \geq 2$, what conditions are required on Ω to admit an analytic function $p : \Omega \to \mathbb{C}$ which cannot be analytically extended beyond the boundary of Ω, and satisfies the complex version

$$\sum_{k=1}^{n} \frac{\partial^2}{\partial z_j^2} p(z) = 0$$

of Laplace's equation?

(J. Ryan)

Update 7.71 D. Khavinson writes that no such domains exist, not only for the complex Laplace equation, but for all holomorphic linear PDEs with entire coefficients. The reason is that, in view of [597, Theorems 4.1 and 19.2], every point on the boundary $\partial\Omega$ of such a domain must be characteristic with respect to the differential operator in question. The latter points have $2n-1$ dimensional measure zero on $\partial\Omega$. More precisely, they belong to a set of real codimension 2 on the surface $\partial\Omega$.

Problem 7.72 Let N denote the class of complex-valued L^∞-functions v on the unit disc $\mathbb{D}$ such that $\int_{\mathbb{D}} v\phi \, dx \, dy = 0$ whenever ϕ is analytic in $\mathbb{D}$ with $\int_{\mathbb{D}} |\phi(x+iy)| \, dx \, dy < \infty$. The Cauchy principal value of

$$(Bv)(z) = \frac{-1}{\pi} \int_{\mathbb{D}} \frac{v(\zeta)}{(z-\zeta)^2} \, d\xi \, d\eta, \qquad \zeta = \xi + i\eta,$$

defines the Beurling transform Bv of v.

Is it true that $Bv \in L^\infty$ and, furthermore, that for some positive finite constant C independent of v,

$$\|Bv\|_\infty \le C\|v\|_\infty, \tag{7.11}$$

whenever $v \in N$. A weaker question is whether this holds for all v in $N \cap P$, where P is the class of all polynomials in z and $\overline{z}$. (The inequality (7.11) is true at least for certain subclasses of $N \cap P$.)

(A. Hinkkanen)

Update 7.72 This problem has been resolved by Volberg; see Bañuelos and Janakiraman [52] for a full account and related details.

Problem 7.73 Let D_1, D_2 be domains in $\{|z| < R\}$, and let $\lambda_1(z)\,|dz|$ and $\lambda_2(z)\,|dz|$ be their hyperbolic metrics. What is the least number A, $A = A(R)$, such that the hyperbolic metric $\lambda(z)\,|dz|$ of $D_1 \cap D_2$ satisfies the inequality

$$\lambda(z) < A(\lambda_1(z) + \lambda_2(z))\,?$$

(W.H.J. Fuchs)

Update 7.73 No progress on this problem has been reported to us.

Problem 7.74 In their famous Acta paper, Hardy and Littlewood [476] introduced the celebrated Hardy–Littlewood maximal function in connection with complex function theory. Since then it has proved an invaluable tool in real analysis. Here we ask some questions about the dependence of constants on dimension. Let B be a convex compact symmetric body in $\mathbb{R}^n$, normalised to have Euclidean volume 1. For an arbitrary measurable function f, let the Hardy–Littlewood maximal functions be

$$Mf(x) = \sup_{t>0} \left(\frac{1}{t^n} \int_{tB} |f(x+y)| \, dy \right)$$

and

$$\tilde{M}f(x) = \sup_{k \in Z} \left(\frac{1}{2^{kn}} \int_{2^k B} |f(x+y)| \, dy \right).$$

(a) If B is the Euclidean ball in $\mathbb{R}^n$, does there exist a constant C such that

$$\operatorname{meas}\{x : \tilde{M}f(x) > \lambda\} \le C\lambda^{-1}\|f\|_1$$

for all positive λ, with C independent of n?

(b) If so, what is the answer to the same question for Mf?

(c) In the case when $n = 1$, a conjecture of F. Soria and the proposer is that the best constant in the inequality meas$\{x : Mf(x) > \lambda\} \leq C\lambda^{-1}\|f\|_1$ is $C = \frac{3}{2}$. Prove this.

(d) If $1 < p \leq \frac{3}{2}$, can the best constant in the inequality

$$\|Mf\|_p \leq C_p\|f\|_p$$

be taken to be independent of n and the body B? Even if $B = [-\frac{1}{2}, \frac{1}{2}]^n$, can the constant be chosen to be independent of n?

The following relevant facts are known: For (a) and (b), the best known constants have been found by Stein and Stromberg [918]. For (c), F. Soria and the proposer have shown that the answer is 'yes' if $\frac{3}{2} < p \leq \infty$; for B with suitably-curved boundary, the answer is 'yes' for $1 < p \leq \infty$; for the sphere, see Stein [917].

(A. Carbery)

Update 7.74(b) Melas [717, 718] investigates the exact value of the best possible constant C for the weak-type $(1, 1)$ inequality for the one-dimensional centred Hardy–Littlewood maximal operator. In connection with this problem, the sharp bound for the weak-type $(1, 1)$ inequality for the n-dimensional Hardy operator is obtained by Zhao, Fu and Lu [1018].

Update 7.74(c) Melas [716] studies the centred Hardy–Littlewood maximal operator acting on positive linear combinations of Dirac deltas, and uses this to obtain improvements in both the lower and upper bounds for the best constant C in the $L^1 \to$ weak L^1 inequality for this operator. A counterexample is given by Aldaz [18], and Aldaz [19] also shows that the general conjecture fails whenever $n \geq 2$, and also asymptotically, that is

$$\liminf c_n > \lim\left(\frac{1+2^{1/n}}{2}\right)^n = \sqrt{2}.$$

Update 7.74(d) Bourgain [152] has answered the second part of (d) in the affirmative.

Problem 7.75 The *analytic capacity* γ of a compact set E in $\mathbb{C}$ is defined by

$$\gamma(E) = \sup\left\{\lim_{|z|\to\infty}|zf(z)| : f \in A(E)\right\},$$

where $A(E)$ is the set of functions which are analytic outside E, vanish at infinity and for which $|f(z)| \leq 1$ for $z \in \mathbb{C} \setminus E$. A related concept is *continuous analytic capacity* $\alpha(E)$, which is defined as for $\gamma(E)$, but the functions f are additionally required to be defined and continuous in the whole complex plane.

Prove or disprove the following statements about analytic capacity γ:

(a) If E is a compact subset of $\mathbb{C}$ and ϕ is a C^1-diffeomorphism of $\mathbb{C}$ onto $\mathbb{C}$, then $\gamma(E) = 0$ if and only if $\gamma(\phi(E)) = 0$. The statement is false if ϕ is a homeomorphism or a quasiconformal mapping.
(b) If E is a compact subset of $\mathbb{C}$ and $\phi \in \mathrm{GL}(2, \mathbb{R})$, then $\gamma(E) = 0$ if and only if $\gamma(\phi(E)) = 0$.
(c) If E, F are compact subsets of $\mathbb{C}$, then there exists a positive constant K (independent of the choice of E and F) such that

$$\gamma(E \cup F) \leq K(\gamma(E) + \gamma(F)).$$

Perhaps one can take $K = 1$? See Davie [236] for related results.
(d) If K is a compact subset of $\mathbb{C}$ with $\gamma(K) = 0$, then

$$\gamma(E \setminus K) = \gamma(E)$$

for all compact subsets E of $\mathbb{C}$. Here $\gamma(E \setminus K)$ means the inner capacity

$$\sup\{\gamma(L) : L \subseteq E \setminus K, L \text{ compact}\}.$$

An interesting special case would be that when K is the 'corner quarters square Cantor set' (also known as the *Garnett set*). Tolsa [949] gives a construction of this set.

(A.G. O'Farrell)

Update 7.75(a) Tolsa [948] has shown that it is true that the property of having zero capacity is invariant under C^1 homeomorphisms. Tolsa writes that this is true for any bilipschitz map; and more generally, that for any compact set E, the analytic capacity of E is comparable to the analytic capacity of $f(E)$, with the comparability constant depending only on the bilipschitz constant.

Update 7.75(c) This is the same as Problem 7.39, which has been proved by Tolsa [946]. If E is a subset of $\mathbb{R}$ then $\gamma(E) = \frac{1}{4}\text{length}(E)$; see Pommerenke [810].

Problem 7.76 Prove or disprove the following statement: if K is a compact subset of $\mathbb{C}$ whose continuous analytic capacity $\alpha(K)$ is zero, then

$$\alpha(E \setminus K) = \alpha(E)$$

for all compact subsets E of $\mathbb{C}$.

An interesting special case is when K is a C^1-arc. The case when K is a $C^{1+\varepsilon}$-arc has already been settled.

(A.G. O'Farrell)

Update 7.76 Note that continuous analytic capacity is different from the analytic capacity used in Problem 7.75. Tolsa [947] has shown that $\alpha(E)$ and $\alpha(E \setminus K)$ are comparable by the semiadditivity of continuous analytic capacity.

Problem 7.77 We will say that $g(x)$ is a *rearrangement* of $f(x)$ if

$$m\{x : g(x) < y\} = m\{x : f(x) < y\}, \qquad \text{for all } y \in \mathbb{R},$$

where m is Lebesgue measure and f and g are defined on some finite interval I. What are those functions $f(x)$ for which $f'(x)$ is a rearrangement $f(x)$?

Obvious examples of such functions are $f(x) = ke^x$ on any interval and $f(x) = k \sin x$ on $[0, \frac{1}{2}\pi]$. What others are there? How about other 'differential rearrangements' than $f'(x) \sim f(x)$?

(L.A. Rubel)

Update 7.77 No progress on this problem has been reported to us.

Problem 7.78 Does there exist a sequence $\{z_n\}_1^\infty$ of distinct complex numbers such that

$$\sum \frac{1}{|z_n|} < \infty \qquad \text{and} \qquad \sum \frac{1}{z - z_n} \neq 0,$$

for all z in $\mathbb{C}$?

This has the following physical interpretation: if we imagine electrons (really unit-charged wires perpendicular to the complex plane) placed at each point z_n, then these generate a logarithmic potential given by $\sum \log |z - z_n|$. The gradient of this potential is $\sum 1/(z - z_n)$. Thus the question is whether such a field must always have an equilibrium point – that is, a point where a free electron (or wire), once placed there, would remain there.

Of course, the corresponding problem could also be asked for $\mathbb{R}^n$, $n \geq 3$.

(L.A. Rubel)

Update 7.78 For $n = 2$ this was solved by Clunie, Eremenko and Rossi [213]. They also have partial results for $n \geq 3$. Further generalisations were obtained by Eremenko, Langley and Rossi [318]. D. Khavinson writes that Fefferman and Stein [365] is also relevant, and that this problem is closely related to Maxwell's conjecture, see [599]. See also Killian [607].

Problem 7.79 Let f be analytic on a domain D in $\mathbb{C}$. We will say that a point z_0 in D is a *MacLane point* of f if there exists some neighbourhood N of z_0 such that the restrictions to N of the successive derivatives of f,

$$\{f^{(n)}|_N : n \in \mathbb{N}\},$$

form a normal family of functions on N. Let $M(f)$ denote the set of MacLane points of f.

What can be said about the set $M(f)$, besides the fact that it is open? Must $M(f)$ be connected? If D is simply-connected, must $M(f)$ be simply-connected? Or can $M(f)$ be an arbitrarily prescribed open subset of D?

Similar questions can be asked about functions meromorphic in D. Perhaps it is more natural, also, to ask such questions about$\{f^{(n)}(z)/n!\}$ rather than simply $\{f^{(n)}(z)\}$?

See Edrei and MacLane [293] and MacLane [697].

(L.A. Rubel)

Update 7.79 No progress on this problem has been reported to us.

Problem 7.80 An *inner function* is an H^∞ function that has unit modulus almost everywhere on the unit circumference $\mathbb{T}$. Let f be an inner function with $f(0) = 0$; then f induces an ergodic (Lebesgue-) measure-preserving map of the circle onto itself. What is the entropy $h(f)$ of f? It is conjectured that $h(f) < \infty$ if and only if f' belongs to the Nevanlinna class; and that, in that case, then

$$h(f) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f'(\mathrm{e}^{i\theta}) \right| \mathrm{d}\theta.$$

See Fernández [343] and Pommerenke [824].

(J.L. Fernández)

Update 7.80 This question has been solved in the affirmative by Craizer [227].

Problem 7.81 Let I denote the class of all inner functions. Then I, as a subset of H^∞, enjoys some of the properties that the collection of unimodular functions have as a subset of $L^\infty_{\mathbb{R}}$ (the real-valued functions in $L^\infty(\Pi)$, see Shapiro [890]).

Is it true that if $\{\Lambda_n\} \subset (H^\infty)^*$ and if for each function ϕ in I one has $|\Lambda_n(\phi)| \le C(\phi)$, then

$$\sup_n \|\Lambda_n\|_{(H^\infty)^*} < \infty\,?$$

This is known to hold if $\Lambda_n \in L^1 \setminus H^1_0 (\subset (H^\infty)^*)$, see [344, 560, 890]. The corresponding real-variable result is also known, see [890].

(J.L. Fernández)

Update 7.81 No progress on this problem has been reported to us.

Problem 7.82 We suppose that E is a subset of $\mathbb{C}$, and that the function $F : E \times \mathbb{D} \to \mathbb{C}$ satisfies the following conditions:

(a) F is injective on E, for z in $\mathbb{D}$;
(b) F is analytic for z in $\mathbb{D}$, for each w in E;
(c) $F(z) = z$, when $w = 0$.

Does there exist a function $G : \mathbb{C} \times \mathbb{D} \to \mathbb{C}$ that satisfies conditions (a), (b) and (c), and for which $G = F$ on E?

(D. Sullivan, W. Thurston, H. Royden; communicated by D.H. Hamilton)

Update 7.82 Slodkowski [908] has proved that the answer is ‘yes’.

Problem 7.83 Let G be a finitely-generated Fuchsian group of the first kind, which means that the limit set is the whole boundary.

(a) If $F : \mathbb{T} \to \mathbb{T}$ is a G-invariant quasisymmetric function, is F totally singular?
(b) Is the Teichmüller space T_G dense in the space S_G (Schwarzians of G-invariant univalent functions)?

(O. Lehto; L. Bers; I. Kra; communicated by D.H. Hamilton)

Update 7.83 No progress on this problem has been reported to us.

7.3 New Problems

Problem 7.84 Let U be an open set in $\mathbb{R}^n$, $n > 2$, and λ_a^U be the harmonic measure at the point a with respect to U. Then what is the sharp upper bound for the dimension of harmonic measure in U?

An example of Wolff [153] shows that the dimension of harmonic measure can be greater than $n - 1$. Bourgain [150] has shown that it is at most $n - \varepsilon$ for some positive ε depending on n.

This problem is a higher-dimensional analogue of Problem 7.44, which was solved by Jones and Wolff [582]. Their solution shows that the sharp upper bound in the plane is 1.

(Communicated by X. Tolsa)

Chapter 8
Spaces of Functions

8.1 Preface by F. Holland

The problems in this chapter were posed in the second half of the last century, and are reflective of research topics in classical analysis under investigation at that time. For the most part, these topics had their origins in the pioneering work of Hardy and Littlewood who had revolutionised mathematical analysis in the UK when they began their collaboration in the early part of the century, and were discussed at various function theory workshops and symposia organised at different times and venues in England since the 1960s. They focused mainly on aspects related to, for example, Lusin's conjecture on the pointwise convergence of Fourier series, Bieberbach's conjecture on the coefficients of normalised univalent functions and off-shoots of Hilbert's double series theorem, which emerged prior to 1920. They occupied the attention of many attendees at these instructional conferences, where short courses were delivered by leading experts, and ad-hoc lectures by others in attendance. Even though Lusin's conjecture was settled by Carleson in 1966, and Bieberbach's was disposed of by de Branges in 1984, interest in both conjectures hasn't completely waned, and problems related to them are still under active investigation, especially Lusin's.

The twenty six problems listed here were contributed by participants at one of these conferences, only a handful of which have been attacked, let alone solved. Indeed only four, Problems 8.10, 8.11, 8.15, 8.16, appear to have been completely solved. But the fact that so many are still open—some 50 or more years after they were first stated—shouldn't discourage people from attempting to solve them, because the history of mathematics is replete with examples of intriguing problems that have baffled people for thousands of years. Foremost among these, perhaps, are the three famous geometric problems from antiquity: the problems of doubling the cube, squaring the circle, and trisecting an angle. To satisfy the standards of rigour

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of the day, imposed by the Greeks since Plato's time (fifth century BC), these had to be solved using plane methods, that is, by using an unmarked straightedge and compass in a finite number of steps. While they stood open for millennia, efforts to resolve these ancient problems led to the development of new and important results which had a profound effect on the growth of mathematics. The study of π and its rational approximations began around this time. Arguably, too, efforts to square the circle led ultimately to modern theories of integration, while it is said that efforts to double the cube and trisect an angle led to the discovery of conic sections, and exotic curves such as lunes and spirals. Indeed, the latter were actually employed by Hippocrates of Chios (c.470–c.410 BC) (not to be confused with the man after whom the Hippocratic oath is named) and Archimedes (c.287–c.212 BC) to square the circle. But such curves could not be constructed using plane methods, and so didn't provide acceptable solutions.

As we now know, the search for solutions of these three problems persisted down through the succeeding generations, and, until the beginning of the nineteenth century, defied the best efforts of many strong mathematicians—as well as numerous enthusiastic amateurs—to solve them. The first breakthrough came in 1837 when Wantzel (1814–1848) proved conclusively that one can't double a cube or trisect an arbitrary angle by using a straightedge and compass any finite number of times, by reducing both problems to the solution of cubic equations. Later on, Lindemann (1852–1939) put paid to the problem of squaring the circle by plane methods when he proved in 1882 that π is not the root of any polynomial with integer coefficients. Thus, more than two thousand years were to pass before these ancient problems were satisfactorily put to bed.

Of course, readers will be well able to cite their own favourite examples of other simply stated problems that remained open for a very long time after they were first formulated, and even know ones that are still open and carry monetary rewards for their solution. But they will also know that efforts to deal with many significant problems led to the discovery of major mathematical achievements, and the growth of the subject generally. So, the lesson of history for potential problem solvers of mathematical problems is that, while their efforts may not always settle the problems they set out to tackle, there is a fair chance that they will gain enjoyment and personal satisfaction, discover new results—and possibly earn a little momentary fame by their work!

The spaces mentioned in the chapter heading refer mainly to Banach spaces of analytic functions defined on the open unit disc whose growth is restricted in one way or another. For the most part, the spaces mentioned are either subalgebras of the disc algebra or one of the familiar ones that are named after Bergman, Besov and Hardy, respectively. The problems themselves are concerned with algebraic, geometric and or analytic properties of members of one of the spaces, or with linear transformations acting on or between them. People seriously interested in trying their hand at solving them should therefore probably have a rudimentary knowledge of basic complex analysis of one complex variable, and be acquainted with elementary operator theory. But not all the problems appear to require a deep knowledge of any of these areas, and there may be scope for somebody with a general mathematical background in modern analysis to tackle them ab initio.

It strikes me that a small number of problems fall into this category and require little or no special knowledge to understand them, and may therefore be attractive to a novice problem solver with little or no particular preparation who has the courage to tackle them. For instance, Problem 8.1 comes without references to prior work, and doesn't fall into any of the standard spaces mentioned; it is virgin territory for an ambitious researcher in search of a challenging project rather than a single problem. A potential solver of it might profit by studying coefficient multiplier problems that have arisen in Fourier series and Hardy spaces. Of the other problems on which no progress has been reported since they were published, several have reasonably succinct statements, are quickly and easily grasped and may fall to a persistent worker. A particularly tempting one is Problem 8.7 which is described as slightly "weird" by its poser, H.S. Shapiro, who also contributed Problem 8.12, another tantalising problem with a conjectured response. Problem 8.24 is also not unduly encumbered with background material, and offers an interesting challenge for a non-specialist.

On the other hand, Problem 8.14 is an example of a problem that may require an acquaintance with the study of Hankel operators and their companions, Toeplitz operators, which grew out of Hilbert's double series theorem. It asks for a characterisation of those functions f, belonging to L^∞ of the unit circle, for which the numbers

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{int}\,\mathrm{d}t,\ \ n = 0, 1, 2\ldots,$$

are the moments of a positive Borel measure on $(-\infty,\infty)$. It is motivated by the fact that Hilbert's matrix $[\frac{1}{n+m+1}]$ generates a positive Hankel operator on H^2, and the observation that

$$\frac{1}{n+1} = \frac{1}{2\pi}\int_0^{2\pi} i(t+\pi)\mathrm{e}^{i(1+n)t}\,\mathrm{d}t = \int_0^1 t^n\,\mathrm{d}t,\ \ n = 0, 1, 2, \ldots,$$

so that $i(t+\pi)\mathrm{e}^{it}$ is one such function with the required property. Nehari's 1957 paper, *On bounded bilinear forms*, and Widom's 1966 paper, *Hankel matrices*, provide the link between bounded Hankel operators and positive ones.

An account of Peller's solution of Problem 8.15—one of the few problems in the chapter that has been solved—is given in his 2003 Springer Monograph *Hankel operators and their applications*, a book that contains a wealth of related information.

To conclude: this chapter contains an interesting variety of unsolved problems that are deserving of solution, and attractive enough to whet the appetite of an open-minded mathematical tourist in search of a worthwhile challenge!

8.2 Progress on Previous Problems

Problem 8.1 Let $L = \{L_k\}_0^\infty$ be a non-negative increasing sequence such that

$$\sum_{k=0}^{\infty} L_k r^k < \infty, \qquad 0 < r < 1.$$

If $f(z)$ is analytic in $\mathbb{D}$, we will say that f is in $\mathcal{P}_L$ if there exists a constant A such that, for each non-negative integer n and each polynomial P_n of degree n,

$$\|P_n * f\|_\infty \leq AL_n\|P\|_\infty,$$

where $*$ denotes the Hadamard product. (We note that *Hadamard product* is another name for *Hadamard convolution*. Further details on this can be found in Sheil-Small [895, p. 515] and Titchmarsh [942, Sect. 4.6].) The infimum of such A for a given f in $\mathcal{P}_L$ defines a norm on $\mathcal{P}_L$ which then becomes a Banach space. A variety of linear operators (such as subordination) have the property that they are norm-decreasing on $\mathcal{P}_L$. This enables one to obtain sharp coefficient inequalities for subordinate functions etc., once a function has been shown to lie in $\mathcal{P}_L$. The spaces $\mathcal{P}_L$ are 'large-growth' spaces; for example, the case $L_k = 1, k \geq 0$, is the space of Cauchy–Stieltjes transforms. Convex sequences $\{L_k\}_0^\infty$ (such as $L_k = k$) have the property that

$$\sum_{k=0}^{\infty} L_k z^k$$

lies in the unit ball of $\mathcal{P}_L$.

Can one relate known spaces to these spaces, for example H^p, for $0 < p < 1$?

(T. Sheil-Small)

Update 8.1 No progress on this problem has been reported to us.

Problem 8.2 Given functions $f_1, \ldots, f_N$ in H^∞, let $I = I(f_1, \ldots, f_N)$ be the ideal of H^∞ generated by $f_1, \ldots, f_N$ and let $J = J(f_1, \ldots, f_N)$ denote the set of all g in H^∞ for which there exists a non-negative constant $C = C(g)$ for which

$$|g(z)| \leq C[|f_1(z)| + \ldots + |f_N(z)|], \qquad |z| < 1.$$

J is an ideal of H^∞ which contains I. The corona theorem states that $I = H^\infty$ if and only if $J = H^\infty$; in general $I \not\subseteq J$, and one seeks further relations between I and J when these are proper ideals. In particular, does there exist a positive absolute constant κ such that, if $g \in H^\infty$ and

$$|g(z)|^\kappa \leq C[|f_1(z)| + \ldots + |f_N(z)|], \qquad |z| < 1,$$

then necessarily $g \in I$? (If so, we must have $\kappa \geq 2$.) As a special case, is it true that $J^2 \subset I$? This is true in appropriate algebras of functions defined in terms of faster rates of growth as $|z| \to 1$.

(J.J. Kelleher)

Update 8.2 Von Renteln points out that the problem is improperly stated. The problem should ask for a constant k, $k \geq 2$, such that $|g(z)| \leq C\big(|f_1(z)| + \ldots + |f_N(z)|\big)$, $|z| < 1$, implies $g^k \in I(f_1, \ldots, f_N)$. Wolff [594] proved the result with $k = 3$, and Treil [955] showed that $k = 3$ is best possible. Nikolski writes that Problems 8.2, 8.20, 9.3 and 9.15 are almost identical.

Problem 8.3 For a bounded plane domain D, denote by $N(D)$ the class of functions analytic on D of bounded characteristic also known as the Nevanlinna class (see Duren [273, p. 16]), that is, all quotients of functions in $H^\infty(D)$ with nonvanishing denominator. Let $f_1, \ldots, f_N$ in $N(D)$ have no common zeros in D. Find necessary and sufficient conditions on $f_1, \ldots, f_N$ in order that they generate the full ring $N(D)$.

Equivalently, if $g_1, \ldots, g_N \in H^\infty(D)$, when does the ideal generated by $g_1, \ldots, g_N$ in $H^\infty(D)$, or in $N(D)$, contain a non-vanishing function?

For $D = \mathbb{D}$, for example, the zeros of $g_1, \ldots, g_N$ should not get too close together as one approaches ∂D, and one would like to obtain a corona-type theorem for this problem, that is, a lower estimate for

$$|g_1(z)| + \ldots |g_N(z)|$$

in D.

(J.J. Kelleher)

Update 8.3 For the unit disc, this problem was solved by Dahlberg, Kelleher and Taylor (not published). Earlier partial results are due to Mantel [705] and von Renteln [970].

Problem 8.4 Let D be a simply-connected domain in the complex plane $\mathbb{C}$ and let $A(D)$ be the ring of functions $f : D \to \mathbb{C}$ analytic in D. Bers has shown that (for domains of arbitrary connectivity) the algebraic structure of $A(D)$ determines the conformal structure of D, see Kra [635].

Can $A(D)$ be the direct sum of two non-trivial subrings of itself? (This would represent a generalisation of Taylor's theorem.) Is there a more general result for multiply-connected domains D?

(J.J. Kelleher)

Update 8.4 No progress on this problem has been reported to us.

Problem 8.5 Let D be a domain in $\mathbb{C}$, and let $H(D)$ be the ring of functions analytic on D. It is known that, for two domains D_1 and D_2, $H(D_1)$ is isomorphic to $H(D_2)$ if and only if D_1 and D_2 are conformally (or anticonformally) equivalent. What can be said under only the hypothesis that $H(D_1)$ and $H(D_2)$ are elementarily equivalent in the sense of model theory? For a large class of domains the corresponding problem has been solved for $H_{\mathbb{C}}(D)$, the algebra of functions analytic on D, by Henson and Rubel [539, 540].

(L.A. Rubel)

Update 8.5 Some real progress has been made by Becker, Hansen and Rubel [96]. The emphasis is on $H_{\mathbb{C}}(D)$. The problem as stated is open when D_1 and D_2 are just annuli.

Problem 8.6 Let A^p, $p > 0$, be the Bergman space of functions $f(z)$ analytic in $\mathbb{D}$ such that

$$\|f\|_p = \left(\int\int_{|z|<1} |f(r\mathrm{e}^{i\theta})|^p\, r\,\mathrm{d}r\,\mathrm{d}\theta\right)^{1/p} < \infty;$$

clearly $H^p \subset A^p$. Horowitz [555] has shown that if $f \in A^p$ and f has zeros $\{z_k\}$ in $\mathbb{D}$ then

$$\prod_{k=1}^{n} |z_k|^{-1} = O\left(n^{(1/p)+\varepsilon}\right) \qquad \text{as } n \to \infty, \tag{8.1}$$

for all positive ε; in (8.1) ε cannot be replaced by 0.

Recall that $\{z_k\}$ is a zero set for H^p if and only if $\sum(1 - |z_k|) < \infty$. Characterise the zero sets for A^p, or at least find some non-trivial converse to (8.1).

(P.L. Duren)

Update 8.6 For a finite set F on $|z| = 1$ construct the domain

$$D_F = \{|z| < 1\} \setminus \cup_\nu R(I_\nu),$$

where I_ν are the complementary arcs of F and

$$R(I) = \left\{z = r\mathrm{e}^{i\theta} : 1 - \frac{|I|}{2\pi} \le r < 1, \mathrm{e}^{i\theta} \in \overline{I}\right\},$$

$|I|$ being the angular length of an arc I.

Set further

$$\kappa(F) = \frac{1}{2\pi}\sum_\nu \frac{|I_\nu|}{2\pi} \log\left(\frac{2\pi}{|I_\nu|} + 1\right).$$

Then there exists an absolute constant $\lambda > 1$ such that the two conditions:

(a)

$$\sup_F \left\{\sum_{z_k \in D_F} (1 - |z_k|) - \lambda p^{-1}\kappa(F)\right\} < \infty,$$

(b)

$$\sup_F \left\{\sum_{z_k \in D_F} (1 - |z_k|) - \lambda^{-1} p^{-1}\kappa(F)\right\} < \infty,$$

are respectively necessary and sufficient for z_k to be a zero set for A^p (sup is taken over all finite F). From this it follows that:

(c)

$$\sup_F \left\{ \kappa(F)^{-1} \sum_{z_k \in D_F} (1 - |z_k|) \right\} < \infty$$

is necessary and sufficient for $\{z_k\}$ to be a zero set for some A^p, $p > 0$.

This is a modified version of a theorem of Korenblum [618].

A reviewer comments that in some sense this problem has been solved by Luecking [685].

Problem 8.7 We use the notation of Problem 8.6. Let $\{z_n\}_1^\infty$ be a sequence of points in $\mathbb{D}$ such that the kernel functions $k_n(z) = (1 - z_n z)^{-2}$ do not span the space A^2, and let $\{f_j\}_1^\infty$ be finite linear combinations of the k_n which converge (in norm) to some function f in A^2. Prove that, if the sequence $\{f_j\}$ converges uniformly to 0 on some disc $\varDelta$ in $\{|z| > 1\}$, then $f \equiv 0$.

Similar problems can of course be stated for the functions $(1 - z_n z)^{-1}$, and for spaces other than A^2. This slightly 'weird' problem arises in the theory of generalised analytic continuation; it is known to be true if the closure of $\{z_n\}_1^\infty$ does not contain all of $\{|z| = 1\}$; it is also known that the analogous result for H^2 is true.

(H.S. Shapiro)

Update 8.7 No progress on this problem has been reported to us. A reviewer recommends Ross and Shapiro [847] as a good source on this subject.

Problem 8.8 Suppose that F is a relatively-closed subset of $\mathbb{D}$, and define

$$\|f\|_F = \sup\{|f(z)|; z \in F\}$$

for functions f in the Bergman space A^2.

Describe geometrically the set

$$\{z : |z| < 1, |f(z)| \le \|f\|_F \text{ for all } f \text{ in } A^2\}.$$

(A. Stray)

Update 8.8 No progress on this problem has been reported to us.

Problem 8.9 Determine

$$\|\varLambda\| = \sup \left| \int\int_{|z|<1} f(z)\phi(z)\, d\sigma_z \right|$$

over those f in A^1 with $\|f\|_1 \le 1$, where

$$\phi(z) = \operatorname{sgn}(\operatorname{Re} z).$$

It is shown by Reich and Strebel [835] that $\|\Lambda\| < 1$, and that there exists an extremal function for the problem. See also Harrington and Ortel [478]. Problems of this type are of interest in connection with a theorem of Hamilton [471] in quasiconformal mapping.

(K. Strebel; communicated by M. Ortel)

Update 8.9 Reich [834] proved that $\|\Lambda\| < 1$. Another proof is due to Harrington and Ortel [477], who also showed [478] that the supremum is actually attained. A different (analytic) proof that $\|\Lambda\| < 1$ is given by Anderson [35].

Problem 8.10 If f and $1/f$ belong to the Bergman space A^2, does it follow that $\mathcal{P}f$ is dense in A^2? Here $\mathcal{P}f$ denotes the set of all polynomial multiples of f, that is,

$$\mathcal{P}f = \{pf : p \text{ a polynomial}\}.$$

More generally, if $f \in A^2$ and $|f(z)| \geq c(1-|z|)^a$ where $a, c > 0$, then does it follow that $\mathcal{P}f$ is dense in A^2? (For partial results in this direction, see [7].)

(A.L. Shields)

Update 8.10 This has been solved by Borichev and Hedenmalm [140]. See also [141, 142].

Problem 8.11 Let g be a function in the space D of functions analytic in $|z| < 1$ with finite Dirichlet integral, that is,

$$\int\int_{|z|<1} |h'(z)|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

If $\mathcal{P}g$, as defined in Problem 8.10, is dense in D and if, for some f in D, $|f(z)| \geq |g(z)|$ whenever $|z| < 1$, is it necessarily true that $\mathcal{P}f$ is dense in D?

The analogous result is true in H^2 and in A^2. Shields [901] solved the special case $g(z) \equiv 1$.

(A.L. Shields)

Update 8.11 This has been confirmed independently by Richter and Sundberg [838] and by Aleman [22].

Problem 8.12 Let A denote the set of functions continuous on $|z| = 1$ which extend continuously to analytic functions on $|z| < 1$ (the disc algebra). Let $\tilde{A}$ denote the set of functions of the form $f \circ \phi$, where f ranges over A and ϕ ranges over the set of sense-preserving homeomorphisms of $|z| = 1$. Find a 'good' characterisation of $\tilde{A}$.

Is there a function in $\tilde{A}$ which coincides with

$$\sum_{n=1}^{\infty} 2^{-n} z^{-2^n}$$

on some subset of $|z| = 1$ having positive measure? The proposer conjectures not. (This problem is related to generalised analytic continuation.)

(H.S. Shapiro)

Update 8.12 No progress on this problem has been reported to us.

Problem 8.13 Let M be a positive real number. The *Zygmund class* A^* is the class of continuous 2π-periodic functions f with the property that for all x and all positive h the inequality

$$|f(x+h) - 2f(x) + f(x-h)| \leq Mh$$

holds. Does there exist a singular measure in Zygmund's class A^* all of whose Fourier–Stieltjes coefficients are non-negative?

(F. Holland)

Update 8.13 No progress on this problem has been reported to us.

Problem 8.14 A *Hankel operator* on a Hilbert space is one whose matrix with respect to an orthonormal basis is a (possibly infinite) Hankel matrix $(A_{i,j})_{i,j\geq 1}$, where $A_{i,j}$ depends only on $i + j$. Which functions in L^∞ on the unit circle generate positive Hankel operators?

(F. Holland)

Update 8.14 No progress on this problem has been reported to us.

Problem 8.15 A bounded linear operator A over a separable Hilbert space H is said to be in the *trace class* if for some (and hence all) orthonormal bases $\{e_k\}_k$ of H, the sum of positive terms

$$\operatorname{Tr}|A| := \sum_k \langle (A^*A)^{1/2} e_k, e_k \rangle$$

is finite. Characterise the Hankel operators on the Hardy space H^2 on the circle that are of trace class.

(F. Holland)

Update 8.15 This was solved by Peller [793], where he characterised the Hankel operators that belong to the Schatten–von Neumann classes.

Problem 8.16 Miles [724] and Rudin [860] have shown that in $\mathbb{C}^n$ a function analytic in the unit polydisc and in the Hardy class H^1 may not be expressible as the product of two functions in H^2, if $n \geq 3$. Is this result also true for $n = 2$?

(J.G. Clunie)

Update 8.16 Rudin notes that the problem was settled for $n = 2$, and hence for $n \geq 2$, by Rosay [844]. The analogous problem (for $n > 1$), if the polydisc U^n is replaced by the unit ball B^n in $\mathbb{C}^n$, has been solved by Gowda [439]. Further results related to this problem have been established by weak Ferguson–Lacey factorisation [341, p. 145].

Problem 8.17 In the ring of bounded analytic functions on the unit ball or the polydisc in n variables, is the intersection of two finitely-generated ideals again finitely generated? This was proved for $n = 1$ by McVoy and Rubel [714].

(L.A. Rubel)

Update 8.17 No progress on this problem has been reported to us.

Problem 8.18 For which simply-connected domains D (with 0 in D) is it true that there is a constant $K = K(D)$ such that

$$\int\int_D |f|^2\,\mathrm{d}x\,\mathrm{d}y \le K\int\int_D |f'|^2\,\mathrm{d}x\,\mathrm{d}y \tag{8.2}$$

for all functions f analytic on D with $f(0) = 0$? This inequality (8.2) is known as the *analytic Poincaré inequality*.

Courant and Hilbert [226] have given a Jordan domain for which (8.2) is false; and Hummel [561] has given an example of a spiral domain D for which $\int\int_D |f|^2\,\mathrm{d}x\,\mathrm{d}y = \infty$ and $\int\int_D |f'|^2\,\mathrm{d}x\,\mathrm{d}y < \infty$.

(D.H. Hamilton)

Update 8.18 No progress on this problem has been reported to us.

Problem 8.19 Let μ, $\mu \ge 1$, be a singular measure on the boundary of the unit disc $\mathbb{D}$; and let S_μ be the corresponding singular inner function

$$S_\mu(z) = \exp\left(\int_0^{2\pi} \frac{z + \mathrm{e}^{i\theta}}{z - \mathrm{e}^{i\theta}}\,\mathrm{d}\mu(\theta)\right), \qquad z \in \mathbb{D}.$$

The function S_μ is said to be *discrete* if μ is discrete, and *continuous* if μ has no discrete part. If S_μ is discrete, does there exist some positive δ such that $\|S_\mu - S_\nu\|_\infty > \delta$ for all continuous functions S_ν?

The answer is 'no' if 'discrete' and 'continuous' are interchanged [924].

(K. Stephenson)

Update 8.19 No progress on this problem has been reported to us.

Problem 8.20 Let $f_1, \ldots, f_n$ and g be H^∞ functions on $\mathbb{D}$. If $|g(z)| \le |f_1(z)| + \ldots + |f_n(z)|$, are there necessarily functions $g_1, \ldots, g_n$ in H^∞ such that

$$g^2 = f_1 g_1 + \ldots + f_n g_n? \tag{8.3}$$

Wolff [379, p. 329] has shown that (8.3) holds with g^2 replaced by g^3, and Rao [379, 832] has shown that (8.3) is false with g^2 replaced by g. For general background on the problem, see Garnett [379, Chap. VIII].

(J.B. Garnett)

Update 8.20 Dyakanov writes that this problem, and Problem 9.15 which is essentially the same, have been solved in the negative by Treil [955]. Woytaszczyk writes that a paper by Tolokonnikov [945] is related to this problem. Lazarev writes that there is a result of Bourgain [148] which is stronger than what is stated in this problem, and also suggests the following articles [252, 253]. Nikolski writes that Problems 8.2, 8.20, 9.3 and 9.15 are almost identical.

Problem 8.21 Let w, $w \geq 0$, be an integrable function on the circle or on the line. Then w is said to belong to the *class* A_2 if

$$\sup_I \left(\frac{1}{|I|}\int_I w\,dx\right)\left(\frac{1}{|I|}\int_I \frac{1}{w}\,\mathrm{d}x\right) < \infty$$

over all (appropriate) intervals I ; and w is said to satisfy the *Helson–Szegö condition* if $\log w = u + \tilde{v}$, where $u \in L^\infty$, $\|v\|_\infty < \frac{1}{2}\pi$ and $\tilde{v}$ is the conjugate function of v. It is known that the Helson–Szegö condition is equivalent to A_2, since both conditions are necessary and sufficient for

$$\int_I |\tilde{f}|^2 w\,dx \leq \text{ const. } \int_I |f|^2 w\,\mathrm{d}x. \tag{8.4}$$

The problem is to prove that the Helson–Szegö condition and A_2 are equivalent without using (8.4). For information on the case $\|v\|_\infty < \pi$, see Coifman, Jones and Rubio de Francia [220].

(J.B. Garnett and P.W. Jones)

Update 8.21 No progress on this problem has been reported to us.

Problem 8.22 Do there exist inner functions in any strictly pseudoconvex domain of $\mathbb{C}^n$, $n \geq 2$? See Problem 7.51 for a definition of pseudoconvex. Alexandrov (no citation) and independently Løw (no citation given) and Hakim and Sibony [456] have shown the existence of inner functions in the case of the ball.

(R. Zeinstra)

Update 8.22 There is a positive answer by Løw [682] if the domain is smoothly bounded. Also, Aleksandrov [20] has constructed such inner functions if the domain has a C^2 boundary.

Problem 8.23 In Pompeiu's formula, $f(z)$ and $\int_\Gamma f(\zeta)/(\zeta - z)\,\mathrm{d}\zeta$ make sense for merely continuous functions, but

$$\int\int_\Omega \frac{\partial f}{\partial \overline{\zeta}} \frac{1}{\zeta - z}\,\mathrm{d}\zeta\,\mathrm{d}\eta$$

needs at least some weak differentiability of f. It would be useful to extend the validity of the formula, for example to cover the case of functions f in Lip α, with positive α. This prompts the following question: For which Borel sets Ω is the inequality

$$\left|\left\langle \frac{\partial \chi_\Omega}{\partial \bar{z}}, f \right\rangle\right| \le C_\Omega \|f\|_{\mathrm{Lip}\,\alpha}$$

valid? Putting it another way, for which Borel sets Ω does χ_Ω act on the Besov space $B^{\alpha-1}_{\infty,\infty}$? An interesting special case would be when Ω is a 'Swiss cheese'.

(A.G. O'Farrell)

Update 8.23 No progress on this problem has been reported to us.

Problem 8.24 Let the sequence $\{M_k\}_0^\infty$ of positive numbers be such that

$$M_0 = 1 \qquad \text{and} \qquad \frac{M_{k+j}}{M_k M_j} \ge \binom{k+j}{j}.$$

Assume also the non-quasianalyticity condition that $\sum_k \left(M_k/M_{k+1}\right) < \infty$.

Consider those functions f on a compact subset X of $\mathbb{C}$ that are limits of rational functions with poles off X. In the norm

$$g \mapsto \sum_{k=0}^{\infty} \frac{1}{M_k} \sup_X |g^{(k)}|$$

these functions form a Banach algebra B. Is X the maximal ideal space of B? (The answer is 'yes' if X is the unit disc or the unit interval.)

(A.G. O'Farrell)

Update 8.24 No progress on this problem has been reported to us.

Problem 8.25 Let $\psi : S^1 \to S^1$ be a direction-reversing homeomorphism, and let A_ψ denote the set of functions $f : S^1 \to \mathbb{C}$ such that both f and $f \circ \psi$ belong to the disc algebra. When does A_ψ contain only constant functions?

(A.G. O'Farrell)

Update 8.25 No progress on this problem has been reported to us.

Problem 8.26 Let $\psi : S^1 \to S^1$ be a homeomorphism. When is it true that $\mathrm{Re}\, A = \mathrm{Re}\, A \circ \psi$? That is, when is each function f in the real part of the disc algebra also of the form $g \circ \psi$, for some function g in the disc algebra?

O'Connell [770] has shown that it is necessary that ψ is absolutely continuous, and sufficient that ψ is $C^{1+\varepsilon}$.

(A.G. O'Farrell)

Update 8.26 No progress on this problem has been reported to us.

8.3 New Problems

Problem 8.27 Suppose that $k(x)$ is non-negative, is a member of $L^1(-\infty, \infty)$ and satisfies

$$\int_{-\infty}^{\infty} k(x)\,\mathrm{d}x = 1.$$

Let $f(x)$ be a continuous function on the real line, positive when x is large and suppose that

$$k * f(x) = \int_{-\infty}^{\infty} k(x-y) f(y)\,\mathrm{d}y$$

exists, and that

$$k * f(x) = (1 + o(1)) f(x), \qquad \text{as } x \to \infty. \tag{8.5}$$

This result, its further developments and applications have been cited and used in more general situations, where it is referred to as the *Drasin–Shea theorem*, see [125, Sect. 5.2] and [622, Chap. IV.13].

The problem is to show that the hypothesis (8.5) is sufficient to show that f must have the form

$$f(x) = \mathrm{e}^{\lambda x} L(x), \tag{8.6}$$

where λ is a scalar,

$$\int_{-\infty}^{\infty} k(x)\mathrm{e}^{\lambda x}\,\mathrm{d}x = 1,$$

and L is slowly-varying in the sense of Karamata: that is, L is measurable and

$$\frac{L(x+h)}{L(x)} \to 1, \qquad \text{as } x \to +\infty, \tag{8.7}$$

for every fixed h.

If f is increasing (or weakly decreasing) and the Laplace transform of k exists on an interval about λ this has been proved by Drasin [254], and there are extensions to kernels of varying signs by Jordan [583]. However, the hypothesis (8.5) is very strong, in the sense that the smoothing of f by k should force f to have sufficient regularity so that no supplementary regularity conditions on f would be needed.

This problem arose from a Nevanlinna theory paper of Edrei and Fuchs [289] where they showed that the condition (8.6) (where L is subject to (8.7)) is best possible.

(D. Drasin)

Chapter 9
Interpolation and Approximation

9.1 Preface by J.L. Rovnyak

Open problems stimulate research and provide entry points to seemingly overwhelming subjects. Two open problems of the mid-20th century, the characterisation of interpolating sequences and the corona problem, initiated major new directions in interpolation and approximation.

Let B be the set of bounded analytic functions on the unit disc $\mathbb{D}$. Call a sequence $\{z_n\}$ in $\mathbb{D}$ *interpolating* for B if for every bounded sequence $\{w_n\}$ of complex numbers there exists an f in B such that $f(z_n) = w_n$ for all $n = 1, 2, \ldots$. R.C. Buck conjectured that a sequence is interpolating for B if it approaches the boundary sufficiently fast. It is natural to ask, more generally, what are all such sequences? Existence and partial results were shown by Buck, A. Gleason, Hayman [488], and Newman [758]. The final result is due to Carleson [189]: *A sequence $\{z_n\}$ in $\mathbb{D}$ is interpolating for B if and only if*

$$\prod_{\{k\,:\,k \neq j\}} \left| \frac{z_k - z_j}{1 - z_k \bar{z}_j} \right| \geq \delta, \qquad j = 1, 2, \ldots,$$

for some $\delta > 0$. Shapiro and Shields [892] and Carleson [190] added different proofs.

The corona problem originates in functional analysis. It dates to the 1940s and is attributed to S. Kakutani (see Duren [273, p. 218]). Here we view B as a commutative Banach algebra in the supremum norm. The maximal ideal space $\mathcal{M}$ of B is conveniently defined as the set of nonzero homomorphisms from B into the complex numbers. Then $\mathcal{M}$ is a compact Hausdorff space in the weak*-topology of the dual space of B (see Hoffman [550, Chap. 10]). We can view $\mathbb{D}$ as a subset of $\mathcal{M}$ by identifying any point a of $\mathbb{D}$ with evaluation at a. The corona problem asks if $\mathbb{D}$ is dense in $\mathcal{M}$. The alternative is that there are unknown points of $\mathcal{M}$ beyond the closure of $\mathbb{D}$ that form a solar-like corona about the disc. By the 1950s, quite a lot was known about the structure of $\mathcal{M}$ and set down in I.J. Schark [880]; according to [*MathSciNet*, MR0125442], the actual authors of this paper are I. Kaplansky, J. Wermer,

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9_9

S. Kakutani, R.C. Buck, H. Royden, A. Gleason, R. Arens, and K. Hoffman. The corona problem, however, remained elusive. Enter classical complex analysis. By the definition of the weak*-topology, the corona problem has an equivalent formulation, which is commonly also called the corona problem: *Given any finite number of functions* $f_1, \ldots, f_n$ *in* B *such that*

$$|f_1| + \cdots + |f_n| \geq \delta > 0 \quad \text{on} \quad \mathbb{D},$$

can one find $g_1, \ldots, g_n$ *in* B *such that*

$$f_1 g_1 + \cdots + f_n g_n = 1 \quad \text{on} \quad \mathbb{D}\,?$$

The answer is affirmative, which means that there is no corona. This was proved by Carleson [190], who credits key input from unpublished work of D.J. Newman. Precursors of this famous result appeared in Carleson [187, Theorem 6] and Newman [759, Theorem 3] (see also the review of W. Rudin [*MathsSciNet*, MR0106290]).

The solutions to Buck's problem and the corona problem motivated many subsequent developments that greatly expanded the scope of the original investigations. We append a small sample of this activity and related areas, with the caveat that the list is highly incomplete.

History: Douglas, Krantz, Sawyer, Treil, and Wick [252]
Monographs: Duren [273], Garnett [379], Sawyer [879]
Wolff's proof: Gamelin [369]
Interpolation and sampling: Seip [886]
Multivariable theory: Costea, Sawyer, and Wick [225], Krantz [636]
Operator corona theorem: Treil and Wick [956]
Generalised interpolation: Sarason [877]
Hilbert space methods and multipliers: Agler and McCarthy [1].

The historical roots of interpolation and approximation are older, and the scope of the subject is much broader than indicated in this brief introduction. The topics treated here owe much to mathematicians of the early 20th century, including C. Carathéodory, L. Fejér, R. Nevanlinna, G. Pick, I. Schur, and O. Toeplitz. Some sense of the scope of the area as a whole, and the vastness of the classical literature, can be gleaned from the treatise of Walsh [978].

9.2 Progress on Previous Problems

Problem 9.1 A sequence $\{z_n\}_1^\infty$ in the unit disc $\mathbb{D}$ is *interpolating* for bounded analytic (harmonic) functions if, for each bounded sequence $\{\alpha_n\}_1^\infty$, there exists a function $u(z)$ bounded and analytic (harmonic) in $\mathbb{D}$ with $u(z_n) = \alpha_n$, $n \geq 1$. It is known that a sequence is interpolating for bounded harmonic functions if and only if it is interpolating for bounded analytic functions [190, 379, 967]; however all such proofs require knowledge of conditions implying that a sequence interpolates for bounded analytic functions. Find a simple proof of this equivalence which does not rely on knowledge of such conditions.

(L. Zalcman)

Update 9.1 No progress on this problem has been reported to us.

Problem 9.2 Suppose that $f(z) \in H^\infty$, and let $\{z_n\}$ be a *Blaschke sequence*, that is, a sequence satisfying the condition

$$\sum_{n=1}^{\infty}(1 - |z_n|) < \infty.$$

(a) Does there always exist a (constant multiple of a) Blaschke product $B(z)$ of norm not necessarily equal to 1 such that

$$B(z_n) = f(z_n) \tag{9.1}$$

for $n \geq 1$? This is certainly the case if the Blaschke sequence is uniformly separated, that is,

$$\inf_n \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \overline{z_m} z_n} \right| > 0.$$

See, for example, Earl [280].

(b) Is the unique H^∞ function of minimal norm assuming the values $\{f(z_n)\}$ at $\{z_n\}$ a constant multiple of a Blaschke product? The answer is 'yes' if the sequence $\{z_n\}$ is finite; see Earl [281]. What happens if we know, in addition, that $f(z_n) \to 0$ as $n \to \infty$? The answer is again 'yes' if $z_n \to 1$ non-tangentially as $n \to \infty$.

(c) One can ask questions (a) and (b) for sequences $\{z_n\}$ that are weakly separated, that is,

$$\inf_{m \neq n} \left| \frac{z_m - z_n}{1 - \overline{z_m} z_n} \right| > 0.$$

(d) Also, one can ask questions (a) and (b) for inner functions instead of Blaschke products.

(J.P. Earl and A. Stray)

Update 9.2 Stray [927] has proved that if there exist two distinct solutions $B(z)$ of (9.1) which have minimum supremum norm, then one of them is a Blaschke product. Stray [926] has also shown that if $\{z_n\}$ is an interpolation sequence and $f(z_n) \to 0$, then the solution of (9.1) with minimum norm is unique and is a Blaschke product.

Problem 9.3

(a) Suppose that $f, f_1, f_2, \ldots, f_n \in H^\infty$, and

$$|f| \leq |f_1| + |f_2| + \ldots + |f_n|.$$

Do there necessarily exist $h_1, h_2, \ldots, h_n$ in H^∞ such that

$$f = f_1 h_1 + f_2 h_2 + \ldots + f_n h_n\,?$$

(If $|f_1| + |f_2| + \ldots + |f_n| \geq \delta > 0$, this is the corona theorem.)

(b) Suppose that $f_1, f_2 \in H^\infty$. Do there necessarily exist f in H^∞ and a positive δ such that

$$\delta(|f_1| + |f_2|) \leq |f| \leq |f_1| + |f_2|\,?$$

If the answer is 'yes', is f necessarily or possibly of the form $h_1 f_1 + h_2 f_2$ for some $h_1, h_2 \in H^\infty$? (Notice that (b) would imply (a).)

(J.P. Earl)

Update 9.3 Nikolski writes that Problems 8.2, 8.20, 9.3 and 9.15 are almost identical.

Update 9.3(a) The problem was originally posed by Rubel in Birtel's collections [126, Problem 12, p. 347]. A counterexample ($N = 2$, f_1 and f_2 Blaschke products) was given by Rao [832]. An easier counterexample is the following: define $F_1(z) = 1 - z$, $F_2(z) = \exp[-(1+z)/(1-z)]$ and $f_1 = F_1^2$, $f_2 = F_2^2$, $f = F_1 F_2$. Then $|f| \leq |f_1| + |f_2|$. Since F_1 and F_2 are relatively prime, the existence of h_1, h_2 in H^∞ with $f = f_1 h_1 + f_2 h_2$ is equivalent to the existence of g_1, g_2 in H^∞ with $1 = F_1 g_1 + F_2 g_2$. But this is impossible, since the right-hand side of the equation tends to zero on the real axis.

In the case $n = 2$, a description of the functions f_1 and f_2 for which the answer is positive has been given by Gorkin, Mortini and Nicolau [436], who have shown that any f in H^∞ satisfying

$$|f| \leq C(f)(|f_1| + |f_2|)$$

is in the ideal generated by f_1 and f_2 if and only if

$$\inf_{z \in \mathbb{D}} |f_1(z)| + |f_2(z)| + (1 - |z|)(|f_1'(z)| + |f_2'(z)|) > 0.$$

Update 9.3(b) The following is a counterexample: take $N = 2$, $f_1 = F_1$ and $f_2 = F_2$, where F_1 and F_2 are defined as in Update 9.3(a). Assume that there exists an f in H^∞ and a positive δ such that $\delta(|f_1| + |f_2|) \leq |f| \leq |f_1 + |f_2|$ holds. The left-hand side of the inequality implies $f \neq 0$, $\delta|f_1| \leq |f|$ and $\delta|f_2| \leq |f|$, that is, f is a divisor of f_1 and of f_2 (in the algebra H^∞). Since f_1 is an outer function, so is f, and since f_2 is an inner function, so is f (up to invertible elements). That is, f has the factorisation $f = gh$, where g is invertible in H^∞ and h is both outer and inner. Therefore h is a non-zero constant function. This implies $|f| \geq \varepsilon$ for some positive ε, but this is a contradiction to $|f| \leq |f_1| + |f_2|$ since the right-hand side of the inequality tends to zero on the real axis.

Problem 9.4 For each pair of functions f, g in H^∞, does there necessarily exist another pair of functions a, b in H^∞ such that

$$af + gb \neq 0, \qquad |z| < 1\,? \tag{9.2}$$

It is easy to see that a necessary condition for this is that $\log \big| |f| - |g| \big|$ has a harmonic minorant. Is this condition also sufficient for (9.2)? This problem is closely related to Problem 9.3.

(B.A. Taylor; communicated by L.A. Rubel)

Update 9.4 This is a special case of Problem 8.3, and a positive solution has been given by Dahlberg, Kelleher and Taylor (not published).

Problem 9.5 Let K_1, K_2, K_3 be disjoint closed sets in the extended complex plane, and C_1, C_2, C_3 constants. Let $\rho_n(f)$ be the best rational approximation to the function f which equals C_i on K_i, $i = 1, 2, 3$; that is,

$$\rho_n(f) = \inf_{g \in R_n} \max_{z \in \cup_i K_i} |f(z) - g(z)|,$$

where R_n is the class of rational functions f order at most n. Find a geometric characterisation of $\lim_{n\to\infty} \rho_n^{1/n}$. For the case of two sets, see Gonchar [423].

(T. Ganelius)

Update 9.5 No progress on this problem has been reported to us.

Problem 9.6 Let D be an open subset of the extended complex plane with non-empty boundary ∂D, and let F be a relatively-closed subset of D. Let f be a function given on F, and $\{f_n\}_1^\infty$ a sequence of functions analytic on D such that $f_n \to f$ uniformly on F. If $E \subset (\partial F \cap \partial D)$ and if f extends continuously to $F \cup E$, can each f_n be extended continuously to $F \cup E$? The answer is 'yes' if D is the unit disc, or if E is compact.

(A. Stray)

Update 9.6 No progress on this problem has been reported to us.

Problem 9.7 Let us call a closed set E in $\mathbb{C}$ a *weak Arakelian set* if, corresponding to each function $f(z)$ continuous on E and analytic in the interior of E, there exists an entire function $g(z)$ such that, for any sequence $\{z_n\}_1^\infty$ in E, $|f(z_n)| \to \infty$ if and only if $|g(z_n)| \to \infty$. Find a geometric characterisation of the weak Arakelian sets.

(L.A. Rubel)

Update 9.7 Goldstein [417] and Gauthier, Hengartner and Stray [384] have obtained necessary conditions and sufficient conditions for a closed subset of $\mathbb{C}$ to be a weak Arakelian set, but have not obtained conditions which are both necessary and sufficient.

Problem 9.8 Let γ be a Jordan arc in $\mathbb{C}^n$, $n \geq 2$, such that the projections γ_j on the complex coordinate planes $j = 1, \ldots, n$ have area zero. Then $R(\gamma) = C(\gamma)$. Is it true that $P(\gamma) = C(\gamma)$? See Korevaar [621] and Wermer [987].

(J. Korevaar)

Update 9.8 No progress on this problem has been reported to us.

Problem 9.9 An arc γ is said to be of *locally limited rotation* if for some rectifiable subarc, all oriented chordal directions fall within an angle less that π. Does the condition $\sum 1/p_n < \infty$ for positive integers p_n guarantee that the sequence of powers $\{z^{p_n}\}$ fails to span $C(\gamma)$ for every Jordan arc γ? Korevaar and Dixon [623] have shown that for arcs of locally limited rotation (for example, C^1 arcs), the condition

$$p_n \geq nL(n), \quad \text{where } L(n) \text{ is positive increasing and } \quad \sum 1/nL(n) < \infty$$

ensures a non-spanning sequence $\{z^{p_n}\}$.

(J. Korevaar)

Update 9.9 No progress on this problem has been reported to us.

Problem 9.10 Let F be a closed subset of $\mathbb{R}^n$, $n \geq 2$. Call F a set of harmonic approximation if every function continuous on F and harmonic in the interior of F can be uniformly approximated there by a harmonic function in $\mathbb{R}^n$. Give necessary and sufficient conditions for F to be a set of harmonic approximation. If F is nowhere dense, Šaginjan [971] has done this. If F is the closure of its interior, Gauthier, Ow and Goldstein [383] have given necessary conditions and sufficient conditions when $n = 2$, but not necessary and sufficient conditions. This has applications to Rubel's Problem 9.7.

(M. Goldstein)

Update 9.10 This has been solved by Gardiner [374]. An exposition of such results may be found in Gardiner [375].

Problem 9.11 Let D be a planar domain. A sequence $\{z_j\}$ of points in D is said to be an *interpolating sequence* if whenever $\{\alpha_j\} \in \ell^\infty$ there is a function F in $H^\infty(D)$ such that $F(z_j) = \alpha_j$ for all j.

Suppose that the sequence $\{z_j\}$ has the property that for each j there is a function F_j in $H^\infty(D)$ such that $F_j(z_k) = 0$ if $k \neq j$, $F_j(z_j) = 1$ and $\|F_j\|_\infty \leq C$. Is $\{z_j\}$ necessarily an interpolating sequence?

(P.W. Jones)

Update 9.11 No progress on this problem has been reported to us.

Problem 9.12 Let Γ be a Jordan curve of logarithmic capacity 1 in $\mathbb{C}$, and let ϕ be a conformal map from the exterior of Γ to the exterior of the unit circle such that $\phi(\infty) = \infty$. We consider charge distributions on Γ consisting of n point charges $1/n$ at nth order Fekete points $z_1, \dots, z_n$ on Γ, $n \in \mathbb{N}$, where *Fekete points* are points that provide excellent polynomial interpolation; see Bos, Levenberg and Waldron [146]. If Γ is smooth enough, the corresponding potentials

$$\frac{1}{n}\sum_{k=1}^{n} \log |z - z_k|$$

give approximations to $\log |\phi(z)|$ (outside Γ) and to 0 (inside Γ) which are $O(1/n)$ away from Γ; see [619, 620, 624, 634].

Prove a similar result for the case where Γ is a square. It does hold in the degenerate case $\Gamma = [-2, 2]$.

(J. Korevaar)

Update 9.12 No progress on this problem has been reported to us.

Problem 9.13 Let K be a compact subset of $\mathbb{R}^n$, $n \geq 3$. If $\phi \in \mathcal{D}$, let $D(\phi)$ be a least-diameter disc containing spt ϕ; define $d(\phi) = \text{diam (spt } \phi)$, and

$$\|\phi\|_* = \|\phi\|_\infty + d(\phi) \cdot \|\nabla\phi\|_\infty.$$

Are the following conditions equivalent for continuous functions $f : \mathbb{R}^n \to \mathbb{R}$?

(i) There exists a sequence $\{f_n\}_1^\infty$ of functions harmonic near K, such that $f_n \to f$ uniformly on K.
(ii) There exists a function η such that $\eta(\delta)$ decreases to 0 as δ decreases to 0, for which

$$\left| \int_{\mathbb{R}^n} f \Delta\phi \, \mathrm{d}x \right| \leq \eta(d(\phi)) \|\phi\|_* C(D(\phi) - X).$$

Here C denotes the harmonic capacity of $\mathbb{R}^n$ obtained from the kernel r^{-n+2}.

Note that (i) implies (ii), and that (ii) implies (i) if f is a C^2-function. The condition (ii) is formally analogous to one that occurs in rational approximation theory.

(A.G. O'Farrell)

Update 9.13 The $n = 3$ case has been solved in the affirmative by Mazalov [710], where he showed that the two conditions are equivalent.

Problem 9.14 Let f be continuous on a compact subset K of $\mathbb{C}$. If there exists a sequence $\{g_n\}_1^\infty$ of functions analytic near K for which $g_n \to f^2$ uniformly on K, does there necessarily exist a sequence $\{h_n\}_1^\infty$ of functions analytic near K for which $h_n \to f$ uniformly on K?

Paramanov has proved this under the stronger hypothesis that $f \in \text{Lip}\,(\frac{1}{2})$; it is also true under the hypothesis that $f \in W^{1,p}$, $p > 2$.

(A.G. O'Farrell)

Update 9.14 No progress on this problem has been reported to us.

Problem 9.15 Suppose that $f_1, f_2 \in H^\infty(\mathbb{D}) = H^\infty$; and let the function g in H^∞ satisfy the inequality

$$|g(z)| \le |f_1(z)| + |f_2(z)|.$$

Do there necessarily exist functions g_1 and g_2 in H^∞ such that

$$g^2 = f_1 g_1 + f_2 g_2\,?$$

In other words, is it true that $g^2 \in I(f_1, f_2)$ (the ideal generated by f_1 and f_2)?

Wolff has proved that $g^3 \in I(f_1, f_2)$. Also Rao has given an example of a function g, $g \notin I(f_1, f_2)$. For related results by Tolokonnikov, see [594, p. 399].

(J.B. Garnett)

Update 9.15 Dyakanov writes that this problem and Problem 8.20 are essentially the same, and have been solved in the negative by Treil [955]. Nikolski writes that Problems 8.2, 8.20, 9.3 and 9.15 are almost identical.

Problem 9.16 Let Γ be a curve of the form

$$\{x + iA(x) : -\infty < x < \infty\}$$

with

$$|A(x_1) - A(x_2)| \le M|x_1 - x_2|.$$

Let E be a compact subset of Γ, $\Lambda_1(E) > 0$, where Λ_1 is one-dimensional Hausdorff measure or, equivalently, arc length; and let

$$\Omega = \mathbb{C}^* \setminus E, \qquad \text{where } \mathbb{C}^* = \mathbb{C} \cup \{\infty\}.$$

Prove the corona theorem for Ω.

(J.B. Garnett)

Update 9.16 No progress on this problem has been reported to us.

Problem 9.17 Let K denote the $\frac{1}{3}$-Cantor set on $\mathbb{R}$; define $E = K \times K$, and $\Omega = \mathbb{C}^* \setminus E$. Prove the corona theorem for Ω.

(J.B. Garnett)

Update 9.17 No progress on this problem has been reported to us.

9.3 New Problems

Problem 9.18 Let Ω be a domain in $\mathbb{C}$ containing 0, with compact complement K. Given a holomorphic function f on Ω, we denote the Taylor polynomial

$\sum_0^N (f^{(n)}(0)/n!)z^n$ by $S_N(f)(z)$. Then f is said to have a *universal Taylor series* about 0 (written $f \in \mathcal{U}(\Omega, 0)$) if, for any polynomial p, there is a subsequence $(S_{N_k}(f))$ that converges uniformly to p on K. It is known that $\mathcal{U}(\Omega, 0) \neq \emptyset$ if K is connected. However, the situation is more delicate when K is the disjoint union of a closed disc $\overline{D}(\xi, r)$ and a singleton $\{\xi_0\}$. It is known that $\mathcal{U}(\Omega, 0) \neq \emptyset$ if $|\xi_0| \geq |\xi| + r$, and that $\mathcal{U}(\Omega, 0) = \emptyset$ if $|\xi_0| < \sqrt{|\xi|^2 + r^2}$. (See Gardiner [376] for references.) What happens if

$$\sqrt{|\xi|^2 + r^2} \leq |\xi_0| < |\xi| + r\ ?$$

(S.J. Gardiner)

Problem 9.19 Khavinson, Pérez-González and Shapiro [600] have shown that the following is true:
Let f be in $C(\mathbb{T})$. Assume that for some H^1-analytic function G we have

$$\frac{1}{2n} \int_{\mathbb{T}} |f(\mathrm{e}^{i\theta}) - G(\mathrm{e}^{i\theta})|\, \mathrm{d}\theta < \epsilon.$$

Then there is a function g in the disc algebra such that

$$\|g\|_{C(\mathbb{T})} \leq \|f\|_{C(\mathbb{T})}$$

and

$$\frac{1}{2n} \int_{\mathbb{T}} |f(\mathrm{e}^{i\theta}) - g(\mathrm{e}^{i\theta})|\, \mathrm{d}\theta < C\epsilon \log \frac{1}{\epsilon},$$

where C is a constant not depending on f. Moreover, the estimate cannot be strengthened to $O(\epsilon)$.

This can be viewed as the quantitative version of the celebrated Hoffman–Wermer approximation theorem [551].

Is the estimate $O(\epsilon \log \frac{1}{\epsilon})$ sharp asymptotically? What happens if we replace the unit disc with a finitely connected domain, or a finite Riemann surface with a (smooth) boundary?

The same question in the context of the Bergman space $A^1(\mathbb{D})$ with $f \in C(\overline{\mathbb{D}})$ is completely open still. In the Hardy space H^1-context, Totik (2019, private communication to D. Khavinson) has shown that $O(\epsilon \log \frac{1}{\epsilon})$ is sharp for the disc.

(D. Khavinson)

Problem 9.20 Let $1 \leq p \leq \infty$ and let B be an interpolating Blaschke product on the unit disc. The star-invariant subspace K_B^p of the Hardy space H^p is the set of those $f \in H^p$ for which the function $\overline{z f} B$ (defined almost everywhere on the unit circle) is in H^p. Which sequences of complex numbers $\{w_n\}$ arise as traces of the functions from K_B^1 on the zero sequence, say $\{a_n\}$, of B?

The proposer conjectures that the trace space is characterised by the conditions

$$\sum_n |w_n|(1-|a_n|) < \infty$$

and

$$\sum_n |\widetilde{w}_n|(1-|a_n|) < \infty,$$

where

$$\widetilde{w}_n := \sum_j \frac{w_j}{B'(a_j)\cdot(1-a_j\overline{a}_n)}.$$

In fact, the necessity of the two conditions is easy to verify, see Dyakonov [279]. The sufficiency seems plausible in light of a result from Dyakonov [278] that solves a similar problem for K_B^∞. In the intermediate case of K_B^p with $1 < p < \infty$, the problem becomes trivial due to the M. Riesz projection theorem; see [278, 279] for a more detailed discussion.

(K.M. Dyakonov)

Problem 9.21 An *interpolating Blaschke product* is a Blaschke product whose zero set is an interpolating sequence for the algebra H^∞ of bounded analytic functions in the unit disc. The following open problem is mentioned by Garnett [379]:

(a) Can every Blaschke product be uniformly approximated in the unit disc by an interpolating Blaschke product?

The following weak version is also open:

(b) Can every Blaschke product be uniformly approximated in the unit disc by a bounded analytic function whose zero set forms an interpolating sequence for H^∞?

(Communicated by A. Nicolau)

Appendix
Tables

See Tables A.1, A.2 and A.3.

Table A.1 Abbreviations for reference documents

A	W.K. Hayman, Research problems in function theory, Athlone Press, London, 1967
B	J.G. Clunie and W.K. Hayman (eds.), Symposium on complex analysis, Canterbury, 1973, LMS Lect. Notes Ser. 12, Cambridge Univ. Press, 1974, 143–180
C	J.M. Anderson, K.F. Barth, D.A. Brannan and W.K. Hayman, Research problems in complex analysis, Bull. London Math. Soc. 9 (1977), 129–162
D, E, F, G, H	D.M. Campbell, J.G. Clunie and W.K. Hayman, Aspects of contemporary complex analysis (eds: D.A. Brannan and J.G. Clunie, Acad. Press 1980), 527–572
I, J	K.F. Barth, D.A. Brannan and W.K. Hayman, Research problems in complex analysis, Bull. London Math. Soc. 16 (1984), 490–517
K, L	D.A. Brannan and W.K. Hayman, Research problems in complex analysis, Bull. LMS, 21 (1989), 1–35
M	A. Eremenko, Progress in Entire and Meromorphic Functions, https://www.math.purdue.edu/~eremenko/haym.html
N	W.K. Hayman and E.F. Lingham, Research problems in function theory (Fiftieth Anniversary Edition), Springer (2019).

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9

Table A.2 List of problems proposed

Topic	A	B	C	H	I	L	N
1	1–23	24–29	30	–	31–36	37–43	44–47
2	1–32	33–46	47–57	58–64	65–68	69–90	91–94
3	1–10	11–15	16–18	19–20	21–30	31–35	36
4	1–21	22–24	25–27	28–31	–	–	32–36
5	1–21	22–37	38–58	59–66	67–70	71–79	80–83
6	1–26	27–31	32–63	64–82	83–95	96–117	118–127
7	1–8	9–22	23–32	33–52	53–67	68–83	84
8	–	–	1–17	–	18–22	23–26	27
9	–	–	1–7	8–10	11	12–17	18–21

Table A.3 Progress on problems

Problem	1	2	3	4	5	6	7	8	9
1	B	B	C, G	D	B, C	B, K			
2	B	B	B, F	B, D, F		B, C, J, K		D	K
3	B, J	D, J	B		B, C			D	D
4	B, F	B, F			C, G	C			D
5		B, F	B	B, D, K	K	B, E, K	B	D	
6	E, K, M	C, M	D	C, D	K		B	D	
7	B	D, J	B	J	C, G, J, K	E, K	B, E		D
8	B	B, C, E, G			B	B, E			
9	B	E, M		B	E	B		D	
10	B, K	D, J	B, K			B	D		
11	B, K	M	D		J	B	J		
12	D	M	D	B	B, J	E	D		
13	B	J	C, G			B, E			
14	B	C, G		C, D	B, E	B, E	D		
15	B				D, K	B, E			
16	C, M	B, C			B	B	D, J	D	
17	B	B, E, F, J	D	B, F			J		
18	B, C, G, M	C		D, K			D, F		
19	D, M	J, K, M			J				

(continued)

Table A.3 (continued)

Problem	1	2	3	4	5	6	7	8	9
20	D, E, M	M		B, F			D	K	
21	C, K	B, M							
22	B	B, J					D	K	
23	B, J, K	M	K		D	E			
24	C	C			D		J		
25	C, G	B, K, M	K						
26		J, M	K	J		J			
27	D		K		J, K				
28	C, D, J	B, E, F			D, F				
29	K	B, F		J					
30		B			D, F	C, D, K	D		
31		B			D	F			
32		C, G, K							
33	M				C, D, G				
34	K	M				D			
35	M	M			C	D			
36					D	K			
37	M	J			D	J			
38		F, M			D	J			
39						K			
40		C, J			K		K		
41		J			K	D	K		
42	M	D, J				K			
43		D				D	J		
44		C, G				D			
45						D	K		
46						D	J		
47					J	D			
49					D, J				
50					D				
52		M							
54					D				
55		D							
56							K		
57		D				D			
58		J			G, J				
59							K		

(continued)

Table A.3 (continued)

Problem	1	2	3	4	5	6	7	8	9
60						D	K		
61					J				
62		J							
63		J							
64		M							
65		M			J				
66		K			K	J			
67						J			
69		M				J			
70		M				K			
71		M							
72		M							
73						J			
75		M							
78		M							
79		M				K			
83		M							
84						K			
86		M							
87		M				K			
88		M							
90						K			
94						K			

References

1. J. Agler and J.E. McCarthy. *Pick interpolation and Hilbert function spaces*, volume 44 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
2. M.L. Agranovsky and Y. Krasnov. Quadratic divisors of harmonic polynomials in $\mathbb{R}^n$. *J. Anal. Math.*, 82:379–395, 2000; **3.24**.
3. D. Aharonov and D. Bshouty. A problem of Bombieri on univalent functions. *Comput. Methods Funct. Theory*, 16(4):677–688, 2016; **6.3**.
4. D. Aharonov and S. Friedland. On an inequality connected with the coefficient conjecture for functions of bounded boundary rotation. *Ann. Acad. Sci. Fenn., Ser. A I*, 524:14, 1973; **5.44**.
5. D. Aharonov and H.S. Shapiro. A minimal area problem in conformal mapping (Abstract), pages 1–5. London Math. Soc. Lecture Note Ser., No. 12, 1974; **6.88**.
6. D. Aharonov and H.S. Shapiro. A minimal area problem in conformal mapping (Preliminary Report). 1978; **6.88**.
7. D. Aharonov, H.S. Shapiro, and A.L. Shields. Weakly invertible elements in the space of square-summable holomorphic functions. *J. London Math. Soc. (2)*, 9:183–192, 1974/75; **8.10**.
8. D. Aharonov, H.S. Shapiro, and A.Y. Solynin. A minimal area problem in conformal mapping. *J. Anal. Math.*, 78:157–176, 1999; **6.17, 6.88**.
9. D. Aharonov, H.S. Shapiro, and A.Y. Solynin. A minimal area problem in conformal mapping. II. *J. Anal. Math.*, 83:259–288, 2001; **6.17, 6.88**.
10. L.V. Ahlfors. Untersuchungen zur Theorie der konformen Abbildung und der ganzen Funktionen. *Acta Soc. Sci. Fenn.*, Nova Ser. I, no. 9, 1930; **2.3**.
11. L.V. Ahlfors. Über die Kreise die von einer Riemannschen Fläche schlicht überdeckt werden. *Comment. Math. Helv.*, 5(1):28–38, 1933; **2.21**.
12. L.V. Ahlfors. An extension of Schwarz's lemma. *Trans. Amer. Math. Soc.*, 43(3):359–364, 1938; **5.8, 5.9**.
13. L.V. Ahlfors. Clifford numbers and Möbius transformations in $\mathbb{R}^n$. In *Clifford algebras and their applications in mathematical physics (Canterbury, 1985)*, volume 183 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–175. Reidel, Dordrecht, 1986; **7.70**.
14. L.V. Ahlfors and H. Grunsky. Über die Blochsche Konstante. *Math. Z.*, 42(1):671–673, 1937; **5.8**.

W. K. Hayman and E. F. Lingham, *Research Problems in Function Theory*, Problem Books in Mathematics,
https://doi.org/10.1007/978-3-030-25165-9

15. H. Aikawa. Harmonic functions having no tangential limits. *Proc. Amer. Math. Soc.*, 108(2):457–464, 1990; **3.19**.
16. H. Aikawa. Integrability of superharmonic functions and subharmonic functions. *Proc. Amer. Math. Soc.*, 120(1):109–117, 1994; **3.32, 3.34**.
17. W. Al-Katifi. On the asymptotic values and paths of certain integral and meromorphic functions. *Proc. London Math. Soc. (3)*, 16:599–634, 1966; **2.2**.
18. J.M. Aldaz. Remarks on the Hardy–Littlewood maximal function. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(1):1–9, 1998; **7.74**.
19. J.M. Aldaz. A remark on the centered n-dimensional Hardy–Littlewood maximal function. *Czechoslovak Math. J.*, 50(125)(1):103–112, 2000; **7.74**.
20. A.B. Aleksandrov. Inner functions on compact spaces. *Funktsional. Anal. i Prilozhen.*, 18(2):1–13, 1984; **8.22**.
21. A.B. Aleksandrov, J.M. Anderson, and A. Nicolau. Inner functions, Bloch spaces and symmetric measures. *Proc. Lond. Math. Soc. (3)*, 79(2):318–352, 1999; **5.51**.
22. A. Aleman. Hilbert spaces of analytic functions between the Hardy and the Dirichlet space. *Proc. Amer. Math. Soc.*, 115(1):97–104, 1992; **8.11**.
23. A. Aleman, S. Richter, and C. Sundberg. Beurling's theorem for the Bergman space. *Acta Math.*, 177(2):275–310, 1996; **5.26**.
24. H. Alexander and J. Wermer. *Several complex variables and Banach algebras*, volume 35. New York, NY: Springer, 3rd edition, 1998; **7.52**.
25. J.W. Alexander. Functions which map the interior of the unit circle upon simple regions. *Ann. of Math. (2)*, 17(1):12–22, 1915; **6.111**.
26. M.F. Ali and A. Vasudevarao. Logarithmic coefficients of some close-to-convex functions. *Bull. Aust. Math. Soc.*, 95(2):228–237, 2017; **6.119**.
27. M.F. Ali and A. Vasudevarao. On logarithmic coefficients of some close-to-convex functions. *Proc. Amer. Math. Soc.*, 146(3):1131–1142, 2018; **6.119**.
28. R.M. Ali. Coefficients of the inverse of strongly starlike functions. *Bull. Malays. Math. Sci. Soc. (2)*, 26(1):63–71, 2003; **6.48**.
29. R.M. Ali and V. Singh. On the fourth and fifth coefficients of strongly starlike functions. *Result. Math.*, 29(3-4):197–202, 1996; **6.48**.
30. G.R. Allan, A.G. O'Farrell, and T.J. Ransford. A Tauberian theorem arising in operator theory. *Bull. London Math. Soc.*, 19(6):537–545, 1987; **5.72**.
31. A. Ancona. Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine Lipschitzien. *Ann. Inst. Fourier (Grenoble)*, 28(4):169–213, x, 1978; **3.17**.
32. B. Anderson. Polynomial root dragging. *Amer. Math. Mon.*, 100(9):864–866, 1993; **4.32**.
33. J.M. Anderson. Category theorems for certain Banach spaces of analytic functions. *J. Reine Angew. Math.*, 249:83–91, 1971; **5.75**.
34. J.M. Anderson. Asymptotic values of meromorphic functions of smooth growth. *Glasgow Math. J.*, 20(2):155–162, 1979; **2.57**.
35. J.M. Anderson. The extremum problem for analytic functions with finite area integral. *Comment. Math. Helv.*, 55(1):87–96, 1980; **8.9**.
36. J.M. Anderson, K.F. Barth, and D.A. Brannan. Research problems in complex analysis. *Bull. London Math. Soc.*, 9:129–162, 1977.
37. J.M. Anderson and J. Clunie. Slowly growing meromorphic functions. *Comment. Math. Helv.*, 40:267–280, 1966; **2.8, 2.57**.
38. J.M. Anderson, J. Clunie, and Ch. Pommerenke. On Bloch functions and normal functions. *J. Reine Angew. Math.*, 270:12–37, 1974; **5.30**.
39. J.M. Anderson and V.Y. Eiderman. Cauchy transforms of point masses: the logarithmic derivative of polynomials. *Ann. Math. (2)*, 163(3):1057–1076, 2006; **7.29**.
40. J.M. Anderson, J.L. Fernández, and A.L. Shields. Inner functions and cyclic vectors in the Bloch space. *Trans. Amer. Math. Soc.*, 323(1):429–448, 1991; **7.32**.
41. N.U. Arakeljan. Entire functions of finite order with an infinite set of deficient values. *Dokl. Akad. Nauk SSSR*, 170:999–1002, 1966; **1.6, 2.1**.

42. J.H. Arango, D. Mejía, and S. Ruscheweyh. Exponentially convex univalent functions. *Complex Variables, Theory Appl.*, 33(1-4):33–50, 1997; **6.125**.
43. D.H. Armitage. On the global integrability of superharmonic functions in balls. *J. London Math. Soc. (2)*, 4:365–373, 1971; **3.32**.
44. D.H. Armitage. Spherical extrema of harmonic polynomials. *J. London Math. Soc. (2)*, 19(3):451–456, 1979; **3.11**.
45. D.H. Armitage. Cones on which entire harmonic functions can vanish. *Proc. R. Ir. Acad., Sect. A*, 92(1):107–110, 1992; **3.24**.
46. D.H. Armitage. On solutions of elliptic equations that decay rapidly on paths. *Proc. Amer. Math. Soc.*, 123(8):2421–2422, 1995; **3.27**.
47. D.H. Armitage, T. Bagby, and P.M. Gauthier. Note on the decay of solutions of elliptic equations. *Bull. London Math. Soc.*, 17(6):554–556, 1985; **3.27**.
48. D.H. Armitage and M. Goldstein. Radial limiting behaviour of harmonic functions in cones. *Complex Variables Theory Appl.*, 22(3-4):267–276, 1993; **3.27**.
49. K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994; **7.9**.
50. K. Astala and F.W. Gehring. Injectivity, the BMO norm and the universal Teichmüller space. *J. Analyse Math.*, 46:16–57, 1986; **6.69**.
51. F.V. Atkinson. On sums of powers of complex numbers. *Acta Math. Acad. Sci. Hungar.*, 12:185–188, 1961; **7.4**.
52. R. Bañuelos and P. Janakiraman. L^p-bounds for the Beurling–Ahlfors transform. *Trans. Amer. Math. Soc.*, 360(7):3603–3612, 2008; **7.72**.
53. A. Baernstein, II. Proof of Edrei's spread conjecture. *Bull. Amer. Math. Soc.*, 78:277–278, 1972; **1.15**.
54. A. Baernstein, II. Proof of Edrei's spread conjecture. *Proc. London Math. Soc. (3)*, 26:418–434, 1973; **1.15, 5.16**.
55. A. Baernstein, II. Some extremal problems for univalent functions, harmonic measures, and subharmonic functions. Pages 11–15. London Math. Soc. Lecture Note Ser., No. 12, 1974; **5.16, 6.8**.
56. A. Baernstein, II. Integral means, univalent functions and circular symmetrization. *Acta Math.*, 133:139–169, 1974; **5.4, 5.15**.
57. A. Baernstein, II. Univalence and bounded mean oscillation. *Michigan Math. J.*, 23(3):217–223, 1976; **5.58**.
58. A. Baernstein, II. Analytic functions of bounded mean oscillation. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 3–36. Academic Press, London-New York, 1980; **7.35**.
59. A. Baernstein, II. Coefficients of univalent functions with restricted maximum modulus. *Complex Variables Theory Appl.*, 5(2-4):225–236, 1986; **6.84**.
60. A. Baernstein, II and R. Rochberg. Means and coefficients of functions which omit a sequence of values. *Math. Proc. Cambridge Philos. Soc.*, 81(1):47–57, 1977; **5.1**.
61. A. Baernstein II. *Symmetrization in Analysis*. New Mathematical Monographs. Cambridge University Press, 2019; **1.15**.
62. J. Baez. The beauty of roots. *Blog, available on* http://math.ucr.edu/home/baez/roots/.
63. I.N. Baker. Some entire functions with fixpoints of every order. *J. Austral. Math. Soc.*, 1:203–209, 1959/1961; **2.20**.
64. I.N. Baker. Fixpoints of polynomials and rational functions. *J. London Math. Soc.*, 39:615–622, 1964; **2.20**.
65. I.N. Baker. Entire functions with linearly distributed values. *Mathematische Zeitschrift*, 86(4):263–267, Aug 1964; **2.24**.
66. I.N. Baker. The distribution of fixpoints of entire functions. *Proc. London Math. Soc. (3)*, 16:493–506, 1966; **2.23**.
67. I.N. Baker. Repulsive fixpoints of entire functions. *Math. Z.*, 104:252–256, 1968; **2.21**.
68. I.N. Baker. Limit functions and sets of non-normality in iteration theory. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 467:11, 1970; **2.22**.

69. I.N. Baker. An entire function which has wandering domains. *J. Austral. Math. Soc. Ser. A*, 22(2):173–176, 1976; **2.62**.
70. I.N. Baker. The iteration of polynomials and transcendental entire functions. *J. Aust. Math. Soc., Ser. A*, 30:483–495, 1981; **2.93**.
71. A.K. Bakhtin. Extrema of linear functionals. *Akad. Nauk Ukrain. SSR Inst. Mat. Preprint*, (25):8, 1986; **6.87**.
72. S.B. Bank and R.P. Kaufman. On Briot–Bouquet differential equations and a question of Einar Hille. *Math. Z.*, 177(4):549–559, 1981; **1.36**.
73. S.B. Bank and I. Laine. On the oscillation theory of $f'' + Af = 0$ where A is entire. *Trans. Amer. Math. Soc.*, 273(1):351–363, 1982; **1.47, 2.70**.
74. K. Barański, N. Fagella, X. Jarque, and B. Karpińska. Escaping points in the boundaries of Baker domains. *J. Anal. Math.*, 137(2):679–706, 2019; **2.93, 2.94**.
75. R.W. Barnard. On the coefficient bounds on f in S when f' is univalent. *Notices Amer. Math. Soc.*, 26:A–69, 1979; **6.47**.
76. R.W. Barnard, U.C. Jayatilake, and A.Y. Solynin. Brannan's conjecture and trigonometric sums. *Proc. Amer. Math. Soc.*, 143(5):2117–2128, 2015; **5.44**.
77. R.W. Barnard and C. Kellogg. Applications of convolution operators to problems in univalent function theory. *Michigan Math. J.*, 27(1):81–94, 1980; **6.45**.
78. R.W. Barnard and J.L. Lewis. Subordination theorems for some classes of starlike functions. *Pacific J. Math.*, 56(2):333–366, 1975; **6.65**.
79. R.W. Barnard, K. Pearce, and A.Y. Solynin. Iceberg-type problems: estimating hidden parts of a continuum from the visible parts. *Math. Nachr.*, 285(17-18):2042–2058, 2012; **6.35**.
80. R.W. Barnard, K. Pearce, and W. Wheeler. On a coefficient conjecture of Brannan. *Complex Variables, Theory Appl.*, 33(1-4):51–61, 1997; **5.44**.
81. R.W. Barnard and T.J. Suffridge. On the simultaneous univalence of f and f'. *Michigan Math. J.*, 30(1):9–16, 1983; **6.35**, **6.47**.
82. P.D. Barry. The minimum modulus of small integral and subharmonic functions. *Proc. Lond. Math. Soc. (3)*, 12:445–495, 1962.
83. P.D. Barry. On a theorem of Besicovitch. *Quart. J. Math. Oxford Ser. (2)*, 14:293–302, 1963; **2.36**.
84. G.A. Barsegyan. On the relation between the behaviour of asymptotic values and a-points of meromorphic functions (Russian). *Akad. Nauk Armyan. SSR Dokl.*, 18(2):124–133, 1983; **2.42**.
85. K.F. Barth, D.A. Brannan, and W.K. Hayman. The growth of plane harmonic functions along an asymptotic path. *Proc. London Math. Soc. (3)*, 37(2):363–384, 1978; **2.9, 3.1**.
86. K.F. Barth, D.A. Brannan, and W.K. Hayman. Research problems in complex analysis. *Bull. London Math. Soc.*, 16(5):490–517, 1984.
87. K.F. Barth and J.G. Clunie. A bounded analytic function in the unit disk with a level set component of infinite length. *Proc. Amer. Math. Soc.*, 85(4):562–566, 1982; **5.70**.
88. K.F. Barth and W.J. Schneider. Entire functions mapping countable dense subsets of the reals onto each other monotonically. *J. London Math. Soc. (2)*, 2:620–626, 1970; **2.48, 5.50**.
89. K.F. Barth and W.J. Schneider. Entire functions mapping arbitrary countable dense sets and their complements onto each other. *J. London Math. Soc. (2)*, 4:482–488, 1971/72; **2.31, 2.48**.
90. K.F. Barth and W.J. Schneider. On a problem of Erdös concerning the zeros of the derivatives of an entire function. *Proc. Amer. Math. Soc.*, 32:229–232, 1972; **2.30, 5.19**.
91. A. Barton and L.A. Ward. A new class of harmonic measure distribution functions. *J. Geom. Anal.*, 24(4):2035–2071, 2014; **6.116**.
92. I.E. Bazilevič. On distortion theorems and coefficients of univalent functions. *Mat. Sbornik N.S.*, 28(70):147–164, 1951; **6.8**.
93. A.F. Beardon. *Iteration of rational functions. Complex analytic dynamical systems*, volume 132. New York, NY: Springer, paperback edition, 2000.
94. J. Becker. Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. *J. Reine Angew. Math.*, 255:23–43, 1972; **5.31, 6.15**.

95. J. Becker. Conformal mappings with quasiconformal extensions. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 37–77. Academic Press, London-New York, 1980; **5.30**.
96. J. Becker, C.W. Henson, and L.A. Rubel. First-order conformal invariants. *Ann. of Math. (2)*, 112(1):123–178, 1980; **8.5**.
97. J. Becker and Ch. Pommerenke. Schlichtheitskriterien und Jordangebiete. *J. Reine Angew. Math.*, 354:74–94, 1984; **6.15**.
98. J. Becker and Ch. Pommerenke. On the Hausdorff dimension of quasicircles. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 12(2):329–333, 1987; **6.109**.
99. M. Behrens. The corona conjecture for a class of infinitely connected domains. *Bull. Amer. Math. Soc.*, 76:387–391, 1970; **5.27**.
100. D. Beliaev. Integral means spectrum of random conformal snowflakes. *Nonlinearity*, 21(7):1435–1442, 2008; **4.18**.
101. D. Beliaev and S. Smirnov. Random conformal snowflakes. *Ann. of Math. (2)*, 172(1):597–615, 2010; **4.18**.
102. E. Beller and D.J. Newman. The minimum modulus of polynomials. *Proc. Amer. Math. Soc.*, 45:463–465, 1974; **4.14**.
103. M. Benedicks. Positive harmonic functions vanishing on the boundary of certain domains in $\mathbb{R}^n$. *Ark. Mat.*, 18(1):53–72, 1980; **3.12**.
104. C. Bénéteau, A. Dahlner, and D. Khavinson. Remarks on the Bohr phenomenon. *Comput. Methods Funct. Theory*, 4(1):1–19, 2004; **5.18**.
105. W. Bergweiler. Periodic points of entire functions: proof of a conjecture of Baker. *Complex Variables Theory Appl.*, 17(1-2):57–72, 1991; **2.30**.
106. W. Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc., New Ser.*, 29(2):151–188, 1993.
107. W. Bergweiler. A quantitative version of Picard's theorem. *Ark. Mat.*, 34(2):225–229, 1996; **2.92**.
108. W. Bergweiler. Non-real periodic points of entire functions. *Canad. Math. Bull.*, 40(3):271–275, 1997; **2.20, 2.23**.
109. W. Bergweiler, J. Clunie, and J. Langley. Proof of a conjecture of Baker concerning the distribution of fixpoints. *Bull. London Math. Soc.*, 27(2):148–154, 1995; **2.23**.
110. W. Bergweiler and A. Eremenko. On the singularities of the inverse to a meromorphic function of finite order. *Rev. Mat. Iberoamericana*, 11(2):355–373, 1995; **1.19, 1.20, 5.13**.
111. W. Bergweiler and A. Eremenko. Quasiconformal surgery and linear differential equations. *J. Analyse Math. To appear* arXiv:1510.05731, October 2015; **2.70**.
112. W. Bergweiler and A. Eremenko. On the Bank–Laine conjecture. *J. Eur. Math. Soc. (JEMS)*, 19(6):1899–1909, 2017; **2.70**.
113. W. Bergweiler, A. Eremenko, and A. Hinkkanen. Entire functions with two radially distributed values. *Math. Proc. Cambridge Philos. Soc.*, 165(1):93–108, 2018; **2.24**.
114. W. Bergweiler, A. Eremenko, and J.K. Langley. Real entire functions of infinite order and a conjecture of Wiman. *Geom. Funct. Anal.*, 13(5):975–991, 2003; **2.64**.
115. W. Bergweiler, A. Eremenko, and J.K. Langley. Zeros of differential polynomials in real meromorphic functions. *Proc. Edinb. Math. Soc., II. Ser.*, 48(2):279–293, 2005; **4.28**.
116. W. Bergweiler, P.J. Rippon, and G.M. Stallard. Dynamics of meromorphic functions with direct or logarithmic singularities. *Proc. Lond. Math. Soc. (3)*, 97(2):368–400, 2008.
117. F. Berteloot and J. Duval. Une démonstration directe de la densité des cycles répulsifs dans l'ensemble de Julia. In *Complex analysis and geometry (Paris, 1997)*, volume 188 of *Progr. Math.*, pages 221–222. Birkhäuser, Basel, 2000; **2.21**.
118. A.S. Besicovitch. On integral functions of order <1. *Math. Ann.*, 97(1):677–695, 1927; **2.36**.
119. D. Betsakos. Geometric theorems and problems for harmonic measure. *Rocky Mountain J. Math.*, 31(3):773–795, 2001; **3.22**.
120. A. Beurling. Ensembles exceptionnels. *Acta Math.*, 72:1–13, 1940; **6.90**.
121. A. Beurling. On two problems concerning linear transformations in Hilbert space. *Acta Math.*, 81:17, 1948; **5.26**.

122. A. Beurling. Some theorems on boundedness of analytic functions. *Duke Math. J.*, 16:355–359, 1949; **2.35**.
123. L. Bieberbach. Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *Berl. Ber.*, 1916:940–955, 1916; **6.1, 6.4**.
124. M. Biernacki. Sur les équations algébriques contenant des paramètres arbitraires. *Bull. del. Acad. Polon. Sci.*, Cl. III:562, 1927; **2.13, 5.38**.
125. N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular variation. Paperback ed.*, volume 27. Cambridge etc.: Cambridge University Press, 1989; **8.27**.
126. F.T. Birtel, editor. *Function algebras*. Proceedings of the International Symposium on Function Algebras, Tulane University, Chicago. Scott, Foresman, 1966; **9.3**.
127. C.J. Bishop. A transcendental Julia set of dimension 1. *Inventiones mathematicae*, 212(2):407–460, May 2018; **2.76**.
128. E. Bishop. A general Rudin–Carleson theorem. *Proc. Amer. Math. Soc.*, 13:140–143, 1962; **5.65**.
129. A. Björn. *Removable singularities for Hardy spaces of analytic functions.* Linköping: Linköping Univ., Department of Mathematics 1995, 1994; **7.14**.
130. Z. Błocki. Suita conjecture and the Ohsawa–Takegoshi extension theorem. *Invent. Math.*, 193(1):149–158, 2013; **7.64**.
131. H.P. Boas and D. Khavinson. Bohr's power series theorem in several variables. *Proc. Amer. Math. Soc.*, 125(10):2975–2979, 1997; **5.18**.
132. R.P. Boas, Jr. Asymptotic properties of functions of exponential type. *Duke Math. J.*, 20:433–448, 1953; **7.17**.
133. M. Boelkins, J. From, and S. Kolins. Polynomial root squeezing. *Math. Mag.*, 81(1):39–44, 2008; **4.32**.
134. H. Bohr. A theorem concerning power series. *Proc. Lond. Math. Soc. (2)*, 13:1–5, 1914; **5.18**.
135. B.D. Bojanov, Q.I. Rahman, and J. Szynal. On a conjecture of Sendov about the critical points of a polynomial. *Math. Z.*, 190:281–285, 1985; **4.5**.
136. E.L. Bombieri. Sopra un teorema di H. Bohr e G. Ricci sulle funzioni maggioranti delle serie di potenze. *Boll. Un. Mat. Ital. (3)*, 17:276–282, 1962; **5.18**.
137. E.L. Bombieri. *Sull'integrazione approssimatat dell'equazione differenziale di Loewner e le sue applicazioni alla theoria delle funzioni univalenti*. Università di Milano, 1963; **6.3**.
138. E.L. Bombieri. On functions which are regular and univalent in a half-plane. *Proc. London Math. Soc. (3)*, 14a:47–50, 1965; **6.26**.
139. M. Bonk. On Bloch's constant. *Proc. Amer. Math. Soc.*, 110(4):889–894, 1990; **5.8**.
140. A. Borichev and H. Hedenmalm. Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces. *J. Amer. Math. Soc.*, 10(4):761–796, 1997; **8.10**.
141. A. Borichev, H. Hedenmalm, and A. Volberg. Large Bergman spaces: Invertibility, cyclicity, and subspaces of arbitrary index. *J. Funct. Anal.*, 207(1):111–160, 2004; **8.10**.
142. A.A. Borichev and P.J.H. Hedenmalm. Cyclicity in Bergman-type spaces. *Int. Math. Res. Not.*, 1995(5):253–262, 1995; **8.10**.
143. D. Borwein. On a class of convergent series of positive terms. *J. London Math. Soc.*, 40:587–588, 1965; **7.31**.
144. P. Borwein. The arc length of the lemniscate $\{|p(z)| = 1\}$. *Proc. Amer. Math. Soc.*, 123(3):797–799, 1995; **4.7**.
145. P. Borwein and T. Erdélyi. *Polynomials and polynomial inequalities.* New York, NY: Springer-Verlag, 1995.
146. L. Bos, N. Levenberg, and S. Waldron. On the spacing of Fekete points for a sphere, ball or simplex. *Indag. Math., New Ser.*, 19(2):163–176, 2008; **9.12**.
147. V.S. Boĭčuk and A.A. Gol'dberg. On the three lines theorem. *Mat. Zametki*, 15:45–53, 1974; **2.17**.
148. J. Bourgain. On finitely generated closed ideals in $H^\infty(D)$. *Ann. Inst. Fourier*, 35(4):163–174, 1985; **8.20**.

149. J. Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56(2):222–230, 1986; **4.18**.
150. J. Bourgain. On the Hausdorff dimension of harmonic measure in higher dimension. *Invent. Math.*, 87:477–483, 1987; **7.84**.
151. J. Bourgain. On the radial variation of bounded analytic functions on the disc. *Duke Math. J.*, 69(3):671–682, 1993; **5.75**.
152. J. Bourgain. On the Hardy–Littlewood maximal function for the cube. *Isr. J. Math.*, 203:275–293, 2014; **7.74**.
153. J. Bourgain and T. Wolff. A remark on gradients of harmonic functions in dimension ≥ 3. *Colloq. Math.*, 60/61(1):253–260, 1990; **3.10, 7.84**.
154. B.V. Boyarskiĭ. Homeomorphic solutions of Beltrami systems. *Dokl. Akad. Nauk SSSR (N.S.)*, 102:661–664, 1955; **7.10**.
155. D.A. Brannan. On coefficient problems for certain power series. Pages 17–27. London Math. Soc. Lecture Note Ser., No. 12, 1974; **5.44**.
156. D.A. Brannan and W.K. Hayman. Research problems in complex analysis. *Bull. London Math. Soc.*, 21(1):1–35, 1989.
157. M. Brelot and G. Choquet. Espaces et lignes de Green. *Ann. Inst. Fourier Grenoble*, 3:199–263 (1952), 1951; **3.26, 7.5**.
158. M. Brelot and G. Choquet. Polynômes harmoniques et polyharmoniques. In *Second colloque sur les équations aux dérivées partielles, Bruxelles, 1954*, pages 45–66. G. Thone, Liège; Masson & Cie, Paris, 1955; **3.25**.
159. J.E. Brennan. The integrability of the derivative in conformal mapping. *J. London Math. Soc. (2)*, 18(2):261–272, 1978; **6.94**.
160. J. Brossard and L. Chevalier. Majoration harmonique et mouvement brownien. *Ill. J. Math.*, 37(1):33–48, 1993; **3.31**.
161. J.E. Brown. A method for investigating geometric properties of support points and applications. *Trans. Amer. Math. Soc.*, 287(1):285–291, 1985; **6.72**.
162. J.E. Brown. Level sets for functions convex in one direction. *Proc. Amer. Math. Soc.*, 100(3):442–446, 1987; **6.53**.
163. J.E. Brown. On the Sendov conjecture for sixth degree polynomials. *Proc. Amer. Math. Soc.*, 113(4):939–946, 1991; **4.5**.
164. J.E. Brown. A proof of the Sendov conjecture for polynomials of degree seven. *Complex Variables, Theory Appl.*, 33(1-4):75–95, 1997; **4.5**.
165. J.E. Brown and A. Tsao. On the Zalcman conjecture for starlike and typically real functions. *Math. Z.*, 191:467–474, 1986; **6.127**.
166. J.E. Brown and G. Xiang. Proof of the Sendov conjecture for polynomials of degree at most eight. *J. Math. Anal. Appl.*, 232(2):272–292, 1999; **4.5**.
167. L. Brown, B. Schreiber, and B.A. Taylor. Spectral synthesis and the Pompeiu problem. *Annales de l'Institut Fourier*, 23(3):125–154, 1973; **2.61**.
168. F. Brüggemann. On solutions of linear differential equations with real zeros; proof of a conjecture of Hellerstein and Rossi. *Proc. Amer. Math. Soc.*, 113(2):371–379, 1991; **2.72**.
169. F. Brüggemann. Proof of a conjecture of Frank and Langley concerning zeros of meromorphic functions and linear differential polynomials. *Analysis*, 12(1-2):5–30, 1992; **1.42**.
170. D. Bshouty. A note on Hadamard products of univalent functions. *Proc. Amer. Math. Soc.*, 80(2):271–272, 1980; **6.44**.
171. R.C. Buck. Integral valued entire functions. *Duke Math. J.*, 15:879–891, 1948; **2.43**.
172. J.D. Buckholtz. Zeros of partial sums of power series. II. *Michigan Math. J.*, 17:5–14, 1970; **7.7**.
173. X. Buff and A. Chéritat. Quadratic Julia sets with positive area. *Ann. Math. (2)*, 176(2):673–746, 2012; **2.79**.
174. A.-P. Calderón. Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.*, 74(4):1324–1327, 1977; **7.14**.
175. G. Cámera. Doctoral thesis. *University of London*, 1977; **2.40**.

176. D.M. Campbell. Majorization-subordination theorems for locally univalent functions. *Bull. Amer. Math. Soc.*, 78:535–538, 1972; **5.38**.
177. D.M. Campbell. Majorization-subordination theorems for locally univalent functions. II. *Canad. J. Math.*, 25:420–425, 1973; **5.38**.
178. D.M. Campbell. Majorization-subordination theorems for locally univalent functions. III. *Trans. Amer. Math. Soc.*, 198:297–306, 1974; **5.38**.
179. D.M. Campbell. The limiting behaviour of $zf''(z)/f'(z)$ and two conjectures on univalent functions. *Notices Amer. Math. Soc.*, 22:A–120, 1975; **5.31**.
180. D.M. Campbell, J.G. Clunie, and W.K. Hayman. Research problems in complex analysis. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 527–572. Academic Press, London-New York, 1980.
181. J. Cantarella, J.H.G. Fu, R. Kusner, J.M. Sullivan, and N.C. Wrinkle. Criticality for the Gehring link problem. *Geom. Topol.*, 10:2055–2116, 2006; **7.22**.
182. A. Cantón, D. Drasin, and A. Granados. Asymptotic values of meromorphic functions of finite order. *Indiana University Mathematics Journal*, 59(3):1057–1095, 2010; **1.28**.
183. C. Carathéodory. *Theory of functions of a complex variable. Vol. 2*. Chelsea Publishing Company, New York, 1954; **5.39**. Translated by F. Steinhardt.
184. A. Carbery. The boundedness of the maximal Bochner–Riesz operator on $L^4(R^2)$. *Duke Math. J.*, 50:409–416, 1983; **7.63**.
185. A. Carbery. A weighted inequality for the maximal Bochner–Riesz operator on $\mathbb{R}^2$. *Trans. Amer. Math. Soc.*, 287:673–680, 1985; **7.63**.
186. A. Carbery, J.L. Rubio de Francia, and L. Vega. Almost everywhere summability of Fourier integrals. *J. London Math. Soc. (2)*, 38(3):513–524, 1988; **7.63**.
187. L. Carleson. On bounded analytic functions and closure problems. *Ark. Mat.*, 2:283–291, 1952.
188. L. Carleson. Sets of uniqueness for functions regular in the unit circle. *Acta Math.*, 87:325–345, 1952; **6.90**.
189. L. Carleson. An interpolation problem for bounded analytic functions. *Amer. J. Math.*, 80:921–930, 1958.
190. L. Carleson. Interpolations by bounded analytic functions and the corona problem. *Ann. of Math. (2)*, 76:547–559, 1962; **5.27, 9.1**.
191. L. Carleson. Asymptotic paths for subharmonic functions in $\mathbb{R}^n$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 2:35–39, 1976; **3.2**.
192. L. Carleson. On H^∞ in multiply connected domains. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 349–372. Wadsworth, Belmont, CA, 1983; **5.27**.
193. L. Carleson and P.W. Jones. On coefficient problems for univalent functions and conformal dimension. *Duke Math. J.*, 66(2):169–206, 1992; **6.7, 6.8, 6.96**.
194. F. Carlson. Sur les fonctions entières. *Ark. Mat. Astr. Fys.*, 35A(14):18, 1948; **2.59**.
195. F.W. Carroll. A strongly annular function with countably many singular values. *Math. Scand.*, 44(2):330–334, 1979; **5.50**.
196. H. Cartan. Sur les systèmes de fonctions holomorphes à variétés linéaires lacunaires et leurs applications. *Ann. Sci. École Norm. Sup. (3)*, 45:255–346, 1928; **4.7**.
197. H. Cartan. Sur les zéros des combinaisons linéaires de p fonctions holomorphes données. *Mathematica, Cluj*, 7:5–31, 1933; **2.26**.
198. M.L. Cartwright. Some inequalities in the theory of functions. *Math. Ann.*, 111(1):98–118, 1935; **5.1**.
199. M.L. Cartwright and E.F. Collingwood. The radial limits of functions meromorphic in a circular disc. *Math. Z.*, 76:404–410, 1961; **5.19**.
200. I.L. Chang. On the zeros of power series with Hadamard gaps-distribution in sectors. *Trans. Amer. Math. Soc.*, 178:393–400, 1973; **5.36**.
201. J. Chang. On meromorphic functions whose first derivatives have finitely many zeros. *Bull. London Math. Soc.*, 44(4):703–715, 2012; **1.19**.

202. K.H. Chang. Asymptotic values of entire and meromorphic functions. *Sci. Sinica*, 20(6):720–739, 1977; **2.7, 2.10**.
203. Z. Charzyński and M. Schiffer. A new proof of the Bieberbach conjecture for the fourth coefficient. *Arch. Rational Mech. Anal.*, 5:187–193 (1960), 1960; **6.1**.
204. H.H. Chen and M.L. Fang. The value distribution of $f^n f'$. *Sci. China Ser. A*, 38(7):789–798, 1995; **1.19**.
205. H.H. Chen and P.M. Gauthier. On Bloch's constant. *J. Anal. Math.*, 69:275–291, 1996; **5.8**.
206. Y.M. Chen and M.C. Liu. On Littlewoods's conjectural inequalities. *J. London Math. Soc. (2)*, 1:385–397, 1969; **4.18**.
207. J.A. Cima and P. Colwell. Blaschke quotients and normality. *Proc. Amer. Math. Soc.*, 19:796–798, 1968; **5.53**.
208. J. Clunie. On schlicht functions. *Ann. of Math. (2)*, 69:511–519, 1959; **4.13, 6.5**.
209. J. Clunie. On integral and meromorphic functions. *J. London Math. Soc.*, 37:17–27, 1962; **1.18, 1.20**.
210. J. Clunie. On a result of Hayman. *J. London Math. Soc.*, 42:389–392, 1967; **1.20, 5.14**.
211. J. Clunie. On a problem of Erdős. *J. London Math. Soc.*, 42:133–136, 1967; **7.21**.
212. J. Clunie and P. Erdős. On the partial sums of power series. *Proc. Roy. Irish Acad. Sect. A*, 65:113–123 (1967), 1967; **7.7**.
213. J. Clunie, A. Eremenko, and J. Rossi. On equilibrium points of logarithmic and Newtonian potentials. *J. Lond. Math. Soc., II. Ser.*, 47(2):309–320, 1993; **7.78**.
214. J. Clunie and W.K. Hayman. The maximum term of a power series. *J. Analyse Math.*, 12:143–186, 1964; **2.14**.
215. J. Clunie and F.R. Keogh. On starlike and convex schlicht functions. *J. London Math. Soc.*, 35:229–233, 1960; **6.64**.
216. J. Clunie and Ch. Pommerenke. On the coefficients of univalent functions. *Michigan Math. J.*, 14:71–78, 1967; **6.5, 6.7, 6.8**.
217. J. Clunie and T. Sheil-Small. Harmonic univalent functions. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9:3–25, 1984; **6.103, 6.104, 6.105, 6.107**.
218. J.G. Clunie. Some remarks on extreme points in function theory. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 137–146. Academic Press, London-New York, 1980; **5.59**.
219. P.J. Cohen. On a conjecture of Littlewood and idempotent measures. *Amer. J. Math.*, 82:191–212, 1960; **4.19**.
220. R. Coifman, P.W. Jones, and J.L. Rubio de Francia. Constructive decomposition of BMO functions and factorization of A_p weights. *Proc. Amer. Math. Soc.*, 87(4):675–676, 1983; **8.21**.
221. E.F. Collingwood and M.L. Cartwright. Boundary theorems for a function meromorphic in the unit circle. *Acta Math.*, 87:83–146, 1952; **5.19**.
222. E.F. Collingwood and A.J. Lohwater. *The theory of cluster sets*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 56. Cambridge University Press, Cambridge, 1966; **3.19, 5.76**.
223. A. Córdoba. A radial multiplier and a related Kakeya maximal function. *Bull. Amer. Math. Soc.*, 81:428–430, 1975; **7.63**.
224. A. Córdoba and B. López-Melero. Spherical summation: a problem of E.M. Stein. *Ann. Inst. Fourier (Grenoble)*, 31(3):x, 147–152, 1981; **7.63**.
225. Ş. Costea, E.T. Sawyer, and B.D. Wick. The corona theorem for the Drury–Arveson Hardy space and other holomorphic Besov–Sobolev spaces on the unit ball in $\mathbb{C}^n$. *Anal. PDE*, 4(4):499–550, 2011.
226. R. Courant and D. Hilbert. *Methoden der Mathematischen Physik. Vols. I, II*. Interscience Publishers, Inc., N.Y., 1943; **8.18**.
227. M. Craizer. Entropy of inner functions. *Isr. J. Math.*, 74(2-3):129–168, 1991; **7.80**.
228. H. Cremer. Zur Zentrumproblem. *Math. Ann.*, 98:151–163, 1928; **2.85**.
229. G. Csordas, A.J. Lohwater, and T. Ramsey. Lacunary series and the boundary behavior of Bloch functions. *Michigan Math. J.*, 29(3):281–288, 1982; **5.41**.

230. B.E.J. Dahlberg. Mean values of subharmonic functions. *Ark. Mat.*, 10:293–309, 1972; **3.7**.
231. B.E.J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977; **3.17**.
232. M. Damodaran. On the distribution of values of meromorphic functions of slow growth. pages 17–21. Lecture Notes in Math., Vol. 599, 1977; **1.27**.
233. A.A. Danielyan. Rubel's problem on bounded analytic functions. *Ann. Acad. Sci. Fenn. Math.*, 41(2):813–816, 2016; **5.29**.
234. A.A. Danielyan. Interpolation by bounded analytic functions and related questions. In *New Trends in Approximation Theory*, volume 81 of *Fields Inst. Commun.*, pages 215–224. Toronto: The Fields Institute for Research in the Mathematical Sciences; New York, NY: Springer,, 2018; **5.29, 5.80, 5.81, 5.82**.
235. H. Davenport. On a theorem of P. J. Cohen. *Mathematika*, 7:93–97, 1960; **4.19**.
236. A.M. Davie. Analytic capacity and approximation problems. *Trans. Amer. Math. Soc.*, 171:409–444, 1972; **7.75**.
237. S.T. Davies. On the maximum term of an entire function without zeros. *J. London Math. Soc. (2)*, 18(2):253–260, 1978; **2.14**.
238. B. Davis and J.L. Lewis. Paths for subharmonic functions. *Proc. London Math. Soc. (3)*, 48(3):401–427, 1984; **3.26**.
239. L. de Branges. A proof of the Bieberbach conjecture. *Acta Math.*, 154(1-2):137–152, 1985; **6.1, 6.108,6.112**.
240. A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes, and K. Seip. The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive. *Ann. Math. (2)*, 174(1):485–497, 2011; **5.18**.
241. J. Dégot. Sendov conjecture for high degree polynomials. *Proc. Amer. Math. Soc.*, 142(4):1337–1349, 2014; **4.5**.
242. J. Delsarte and J.-L. Lions. Moyennes généralisées. *Comment. Math. Helv.*, 33:59–69, 1959; **2.45**.
243. J.-P. Demailly. Fonctions holomorphes à croissance polynomiale sur la surface d'équation $e^x + e^y = 1$. *Bull. Sci. Math. (2)*, 103(2):179–191, 1979; **2.55**.
244. A. Denjoy. L'allure asymptotique des fonctions entières d'ordre fini. *C. R. Acad. Sci. Paris*, 242:213–218, 1956; **2.3**.
245. E. DiBenedetto and A. Friedman. Bubble growth in porous media. *Indiana Univ. Math. J.*, 35:573–606, 1986; **3.29**.
246. R.J. Distler. The domain of univalence of certain classes of meromorphic functions. *Proc. Amer. Math. Soc.*, 15:923–928, 1964; **6.101**.
247. P. Dive. Sur l'attraction des ellipsoides homogènes. *C.R. Acad. Sci. Paris*, 192:1443–1446, 1931; **3.29**.
248. P. Dive. Sur une propriété exclusive des homonoïdes ellipsoidaux. *C.R. Acad. Sci. Paris*, 193:141–142, 1931; **3.29**.
249. M. Dixon and J. Korevaar. Nonspanning sets of powers on curves: analyticity theorem. *Duke Math. J.*, 45(3):543–559, 1978; **7.16**.
250. A. Douady. Systèmes dynamiques holomorphes. In *Bourbaki seminar, Vol. 1982/83*, volume 105 of *Astérisque*, pages 39–63. Soc. Math. France, Paris, 1983; **2.79, 2.88**.
251. A. Douady. Disques de Siegel et anneaux de Herman. *Astérisque*, (152-153):4, 151–172 (1988), 1987; **2.86, 2.90**. Séminaire Bourbaki, Vol. 1986/87.
252. R.G. Douglas, S.G. Krantz, E.T. Sawyer, S. Treil, and B.D. Wick. A history of the Corona problem. In *The Corona problem. Connections between operator theory, function theory, and geometry*, pages 1–29. Toronto: The Fields Institute for Research in the Mathematical Sciences; New York, NY: Springer, 2014; **8.20**.
253. R.G. Douglas, S.G. Krantz, E.T. Sawyer, S. Treil, and B.D. Wick, editors. *The Corona problem. Connections between operator theory, function theory, and geometry*, volume 72. Toronto: The Fields Institute for Research in the Mathematical Sciences; New York, NY: Springer, 2014; **8.20**.

254. D. Drasin. Tauberian theorems and slowly varying functions. *Trans. Amer. Math. Soc.*, 133:333–356, 1968; **8.27**.
255. D. Drasin. A meromorphic function with assigned Nevanlinna deficiencies. *Symp. Complex Analysis*, Canterbury 1973, 31-41 (1974), 1974; **1.1**.
256. D. Drasin. Proof of a conjecture of F. Nevanlinna concerning functions which have deficiency sum two. *Acta Mathematica*, 158(1):1–94, Jul 1987; **1.3**.
257. D. Drasin. The minimum modulus of subharmonic functions of order one and a method of Fryntov. *J. London Math. Soc. (2)*, 54(2):239–250, 1996; **2.38**.
258. D. Drasin and A. Weitsman. The growth of the Nevanlinna proximity function and the logarithmic potential. *Indiana Univ. Math. J.*, 20:699–715, 1971; **2.1**.
259. D. Drasin and A. Weitsman. Meromorphic functions with large sums of deficiencies. *Adv. Math.*, 15:93–126, 1975; **1.26**.
260. V.N. Dubinin. Change of harmonic measure in symmetrization. *Mat. Sb. (N.S.)*, 124(166)(2):272–279, 1984; **7.45**.
261. V.N. Dubinin. Transformation of condensers in n-dimensional space. *J. Math. Sci., New York*, 70(6):2085–2096, 1991; **7.57**.
262. V.N. Dubinin. Capacities and geometric transformations of subsets in n-space. *Geom. Funct. Anal.*, 3(4):342–369, 1993; **7.57**.
263. V.N. Dubinin. Markov-type inequality and a lower bound for the moduli of critical values of polynomials. *Dokl. Math.*, 88(1):449–450, 2013; **4.8**.
264. V.N. Dubinin. On one extremal problem for complex polynomials with constraints on critical values. *Sib. Math. J.*, 55(1):63–71, 2014; **4.8**.
265. D. Dugué. Le défaut au sens de M. Nevanlinna dépend de l'origine choisie. *C. R. Acad. Sci. Paris*, 225:555–556, 1947; **1.23**.
266. P. Duren. *Harmonic mappings in the plane*, volume 156. Cambridge: Cambridge University Press, 2004; **6.106, 7.26**.
267. P. Duren, D. Khavinson, H.S. Shapiro, and C. Sundberg. Contractive zero-divisors in Bergman spaces. *Pac. J. Math.*, 157(1):37–56, 1993; **5.26**.
268. P. Duren, D. Khavinson, H.S. Shapiro, and C. Sundberg. Invariant subspaces in Bergman spaces and the biharmonic equation. *Mich. Math. J.*, 41(2):247–259, 1994; **5.26**.
269. P. Duren and Y.-J. Leung. Generalized support points of the set of univalent functions. *J. Analyse Math.*, 46:94–108, 1986; **6.70**.
270. P. Duren and A. Schuster. *Bergman spaces*, volume 100. Providence, RI: American Mathematical Society (AMS), 2004; **5.26**.
271. P.L. Duren. Smoothness of functions generated by Riesz products. *Proc. Amer. Math. Soc.*, 16:1263–1268, 1965; **5.69**.
272. P.L. Duren. On the Marx conjecture for starlike functions. *Trans. Amer. Math. Soc.*, 118:331–337, 1965; **6.16**.
273. P.L. Duren. *Theory of H^p spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970; **5.23, 8.3**.
274. P.L. Duren. Estimation of coefficients of univalent functions by a Tauberian remainder theorem. *J. London Math. Soc. (2)*, 8:279–282, 1974; **6.31**.
275. P.L. Duren. Successive coefficients of univalent functions. *J. London Math. Soc. (2)*, 19(3):448–450, 1979; **6.46**.
276. P.L. Duren. *Univalent functions*, volume 259 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1983; **6.42**.
277. P.L. Duren, H.S. Shapiro, and A.L. Shields. Singular measures and domains not of Smirnov type. *Duke Math. J.*, 33:247–254, 1966; **6.15**.
278. K.M. Dyakonov. A free interpolation problem for a subspace of H^∞. *Bull. London Math. Soc.*, 50(3):477–486, 2018; **9.20**.
279. K.M. Dyakonov. Interpolating by functions from model subspaces in H^1. *Integral Equations Oper. Theory*, 90(4):7, 2018; **9.20**.

280. J.P. Earl. On the interpolation of bounded sequences by bounded functions. *J. London Math. Soc. (2)*, 2:544–548, 1970; **7.41, 9.2**.
281. J.P. Earl. A note on bounded interpolation in the unit disc. *J. London Math. Soc. (2)*, 13(3):419–423, 1976; **9.2**.
282. P. Ebenfelt. Some results on the Pompeiu problem. *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, 18(2):323–341, 1993; **2.61**.
283. P. Ebenfelt. Propagation of singularities from singular and infinite points in certain complex-analytic Cauchy problems and an application to the Pompeiu problem. *Duke Math. J.*, 73(3):561–582, 1994; **2.61**.
284. P. Ebenfelt, D. Khavinson, and H.S. Shapiro. An inverse problem for the double layer potential. *Comput. Methods Funct. Theory*, 1(2):387–401, 2001; **3.30**.
285. A. Edrei. Sums of deficiencies of meromorphic functions. *J. Analyse Math.*, 14:79–107, 1965; **1.15**.
286. A. Edrei. A local form of the Phragmén–Lindelöf indicator. *Mathematika*, 17:149–172, 1970; **2.37**.
287. A. Edrei and W.H.J. Fuchs. On the growth of meromorphic functions with several deficient values. *Trans. Amer. Math. Soc.*, 93:292–328, 1959; **1.5**.
288. A. Edrei and W.H.J. Fuchs. Valeurs déficientes et valeurs asymptotiques des fonctions méromorphes. *Comment. Math. Helv.*, 33:258–295, 1959; **1.5, 1.7, 1.9, 2.9, 2.12**.
289. A. Edrei and W.H.J. Fuchs. Tauberian theorems for a class of meromorphic functions with negative zeros and positive poles. *Sovremen. Probl. Teor. Analit. Funktsij, Mezhdunarod. Konf. Teor. Analit. Funktsij*, Erevan 1965, 339-358 (1966), 1966; **8.27**.
290. A. Edrei and W.H.J. Fuchs. Asymptotic behavior of meromorphic functions with extremal spread. I. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 2:67–111, 1976; **2.39**.
291. A. Edrei and W.H.J. Fuchs. Asymptotic behavior of meromorphic functions with extremal spread. II. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 3(1):141–168, 1977; **2.39**.
292. A. Edrei, W.H.J. Fuchs, and S. Hellerstein. Radial distribution and deficiencies of the values of a meromorphic function. *Pacific J. Math.*, 11:135–151, 1961; **1.12**.
293. A. Edrei and G.R. MacLane. On the zeros of the derivatives of an entire function. *Proc. Amer. Math. Soc.*, 8:702–706, 1957; **7.79**.
294. L. Ehrenpreis. *Fourier analysis in several complex variables*. Pure and Applied Mathematics, Vol. XVII. Wiley-Interscience Publishers: A Division of J. Wiley & Sons, New York-London-Sydney, 1970; **2.47**.
295. Á. Elbert. A contribution to the problem of the convergence of the Fourier series of L^2 integrable functions. *Studia Sci. Math. Hungar.*, 1:147–151, 1966; **4.12**.
296. Á. Elbert. Über eine Vermutung von Erdös betreffs Polynome. I. *Studia Sci. Math. Hungar.*, 1:119–128, 1966; **4.12**.
297. M.M. Elhosh. On successive coefficients of close-to-convex functions. *Proc. Roy. Soc. Edinburgh Sect. A*, 96(1-2):47–49, 1984; **6.37**.
298. P. Erdős. Some unsolved problems. *Publ. Math. Inst. Hung. Acad. Sci., Ser. A*, 6:221–254, 1961.
299. P. Erdős. Note on some elementary properties of polynomials. *Bull. Amer. Math. Soc.*, 46:954–958, 1940; **4.20**.
300. P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945; **7.5**.
301. P. Erdős. An interpolation problem associated with the continuum hypothesis. *Michigan Math. J.*, 11:9–10, 1964; **2.46**.
302. P. Erdős, F. Herzog, and G. Piranian. Metric properties of polynomials. *J. Analyse Math.*, 6:125–148, 1958; **4.11**.
303. A. Eremenko. The growth of the Nevanlinna proximity function. *Sibirsk. Mat. Zh.*, 19(3):571–576, 717, 1978; **1.27**.
304. A. Eremenko. The set of asymptotic values of a finite order meromorphic function. *Mat. Zametki*, 24(6):779–783, 893, 1978; **1.28**.

305. A. Eremenko. Growth of entire and subharmonic functions on asymptotic curves. *Siberian Mathematical Journal*, 21(5):673–683, Sep 1980; **1.6**.
306. A. Eremenko. The growth of entire and subharmonic functions on asymptotic curves. *Sibirsk. Mat. Zh.*, 21(5):39–51, 189, 1980; **2.6**.
307. A. Eremenko. Meromorphic solutions of algebraic differential equations. *Uspekhi Mat. Nauk*, 37(4(226)):53–82, 240, 1982; **1.35**.
308. A. Eremenko. Meromorphic solutions of equations of Briot–Bouquet type. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (38):48–56, 127, 1982; **1.36**.
309. A. Eremenko. Meromorphic solutions of first-order algebraic differential equations. *Functional Analysis and Its Applications*, 18(3):246–248, Jul 1984; **1.35**.
310. A. Eremenko. On the natural asymptotic curves of meromorphic functions. *Complex Variables Theory Appl.*, 4(4):305–309, 1985; **2.58**.
311. A. Eremenko. Inverse problem of the theory of distribution of values for finite-order meromorphic functions. *Sibirsk. Mat. Zh.*, 27(3):87–102, 223, 1986; **1.29**.
312. A. Eremenko. Entire functions bounded on the real axis. *Dokl. Akad. Nauk SSSR*, 300(3):544–546, 1988; **2.19**.
313. A. Eremenko. A counterexample to the Arakelyan conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):159–164, 1992; **1.6**.
314. A. Eremenko. Meromorphic functions with small ramification. *Indiana Univ. Math. J.*, 42(4):1193–1218, 1993; **1.3, 1.33, 2.25**.
315. A. Eremenko. A Markov-type inequality for arbitrary plane continua. *Proc. Amer. Math. Soc.*, 135(5):1505–1510, 2007; **4.8**.
316. A. Eremenko and A. Gabrielov. Singular perturbation of polynomial potentials with applications to PT-symmetric families. *Mosc. Math. J.*, 11(3):473–503, 629–630, 2011; **2.71**.
317. A. Eremenko and D.H. Hamilton. On the area distortion by quasiconformal mappings. *Proc. Amer. Math. Soc.*, 123(9):2793–2797, 1995; **7.9, 7.36**.
318. A. Eremenko, J. Langley, and J. Rossi. On the zeros of meromorphic functions of the form $f(z) = \sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$. *J. Anal. Math.*, 62:271–286, 1994; **7.78**.
319. A. Eremenko and L. Lempert. An extremal problem for polynomials. *Proc. Amer. Math. Soc.*, 122(1):191–193, 1994; **4.8**.
320. A. Eremenko and M.Yu. Lyubich. Examples of entire functions with pathological dynamics. *J. London Math. Soc. (2)*, 36(3):458–468, 1987; **2.63, 2.67, 2.79**.
321. A. Eremenko and S. Merenkov. Nevanlinna functions with real zeros. *Illinois J. Math.*, 49(4):1093–1110, 2005; **2.71**.
322. A. Eremenko and M.L. Sodin. A proof of the conditional Littlewood theorem on the distribution of the values of entire functions. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(2):421–428, 448, 1987; **4.18**.
323. A. Eremenko, M.L. Sodin, and D.F. Shea. The minimum of the modulus of an entire function on a sequence of Pólya peaks. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (45):26–40, i, 1986; **2.37**.
324. A.E. Eremenko and W.K. Hayman. On the length of lemniscates. In *Paul Erdős and his mathematics I. Based on the conference, Budapest, Hungary, July 4–11, 1999*, pages 241–242. Berlin: Springer; Budapest: János Bolyai Mathematical Society, 2002; **4.7, 4.10**.
325. A.E. Eremenko, L. Liao, and T.W. Ng. Meromorphic solutions of higher order Briot–Bouquet differential equations. *Math. Proc. Camb. Philos. Soc.*, 146(1):197–206, 2009; **1.36**.
326. M. Essén. A generalization of the M. Riesz theorem on conjugate functions and the Zygmund $L \log L$-theorem to $\mathbb{R}^d$, $d \geq 2$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 15(1):83–105, 1990; **3.31**.
327. M. Essén and D.F. Shea. Applications of Denjoy integral inequalities to growth problems for subharmonic and meromorphic functions. Pages 59–68. London Math. Soc. Lecture Note Ser., No. 12, 1974; **2.39**.
328. M. Essén, D.F. Shea, and C.S. Stanton. A value-distribution criterion for the class $L \log L$, and some related questions. *Ann. Inst. Fourier (Grenoble)*, 35(4):127–150, 1985; **3.31**.
329. P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919; **2.63**.

330. P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 48:208–314, 1920; **2.21, 2.63**.
331. P. Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 48:33–94, 1920; **2.21, 2.63**.
332. P. Fatou. Sur les frontières de certains domaines. *Bull. Soc. Math. France*, 51:16–22, 1923; **2.78**.
333. P. Fatou. Sur l'itération des fonctions transcendantes entières. *Acta Math.*, 47(4):337–370, 1926; **2.22**.
334. C.L. Fefferman. Characterizations of bounded mean oscillation. *Bull. Amer. Math. Soc.*, 77:587–588, 1971; **7.35**.
335. C.L. Fefferman and E.M. Stein. H^p spaces of several variables. *Acta Math.*, 129:137–193, 1972; **7.35**.
336. J. Feng and T.H. MacGregor. Estimates on integral means of the derivatives of univalent functions. *J. Anal. Math.*, 29:203–231, 1976; **6.30, 6.108**.
337. P.C. Fenton. Functions having the restricted mean value property. *J. Lond. Math. Soc., II. Ser.*, 14:451–458, 1976; **3.8**.
338. P.C. Fenton. The minimum of small entire functions. *Proc. Amer. Math. Soc.*, 81(4):557–561, 1981; **2.52**.
339. P.C. Fenton. Entire functions having asymptotic functions. *Bull. Austral. Math. Soc.*, 27(3):321–328, 1983; **2.3**.
340. P.C. Fenton and J. Rossi. Cercles de remplissage for entire functions. *Bull. London Math. Soc.*, 31(1):59–66, 1999; **1.31**.
341. S.H. Ferguson and M.T. Lacey. A characterization of product BMO by commutators. *Acta Math.*, 189(2):143–160, 2002; **8.16**.
342. J.L. Fernández. On the growth and coefficients of analytic functions. *Ann. of Math. (2)*, 120(3):505–516, 1984; **5.5, 5.40**.
343. J.L. Fernández. A note on entropy and inner functions. *Israel J. Math.*, 53(2):158–162, 1986; **7.80**.
344. J.L. Fernández. A boundedness theorem for L^1/H^1_0. *Michigan Math. J.*, 35(2):227–231, 1988; **7.81**.
345. S.D. Fisher and D. Khavinson. Extreme Pick–Nevanlinna interpolants. *Can. J. Math.*, 51(5):977–995, 1999; **5.43**.
346. C.H. FitzGerald. Quadratic inequalities and coefficient estimates for schlicht functions. *Arch. Rational Mech. Anal.*, 46:356–368, 1972; **5.23, 6.2, 6.40**.
347. C.H. FitzGerald and Ch. Pommerenke. The de Branges theorem on univalent functions. *Trans. Amer. Math. Soc.*, 290(2):683–690, 1985; **6.1**.
348. C.H. FitzGerald, B. Rodin, and S.E. Warschawski. Estimates of the harmonic measure of a continuum in the unit disk. *Trans. Amer. Math. Soc.*, 287(2):681–685, 1985; **3.21**.
349. C.H. FitzGerald and F. Weening. Existence and uniqueness of rectilinear slit maps. *Trans. Amer. Math. Soc.*, 352(2):485–513, 2000; **6.59**.
350. R. Fournier. On a new proof and an extension of Jack's lemma. *J. Appl. Anal.*, 23(1):21–24, 2017; **6.41**.
351. G. Frank. Picardsche Ausnahmewerte bei Lösungen linearer Differentialgleichungen. *Manuscripta Math.*, 2:181–190, 1970; **2.72**.
352. G. Frank. Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen. *Math. Z.*, 149(1):29–36, 1976; **1.18**.
353. G. Frank and S. Hellerstein. On the meromorphic solutions of nonhomogeneous linear differential equations with polynomial coefficients. *Proc. London Math. Soc. (3)*, 53(3):407–428, 1986; **1.42**.
354. G. Frank, W. Hennekemper, and G. Polloczek. Über die Nullstellen meromorpher Funktionen und deren Ableitungen. *Math. Ann.*, 225(2):145–154, 1977; **1.18**.
355. C. Frayer and J.A. Swenson. Polynomial root motion. *Amer. Math. Mon.*, 117(7):641–646, 2010; **4.32**.

356. A. Friedman and M. Sakai. A characterization of null quadrature in $\mathbb{R}^N$. *Indiana Univ. Math. J.*, 35:607–610, 1986.
357. O. Frostman. Sur les produits de Blaschke. *Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund]*, 12(15):169–182, 1942; **5.23**.
358. A. Fryntov. On behavior of gap series on curves and a $\cos \pi\lambda$-type theorem. *Complex Variables Theory Appl.*, 37(1-4):195–209, 1998; **2.12a, 2.34**.
359. A. Fryntov and F. Nazarov. New estimates for the length of the Erdős–Herzog–Piranian lemniscate. In *Linear and complex analysis. Dedicated to V. P. Havin on the occasion of his 75th birthday*, pages 49–60. Providence, RI: American Mathematical Society (AMS), 2009; **4.10**.
360. W.H.J. Fuchs. Proof of a conjecture of G. Pólya concerning gap series. *Illinois J. Math.*, 7:661–667, 1963; **2.13**.
361. W.H.J. Fuchs. A Phragmen–Lindelöf theorem conjectured by D. J. Newman. *Trans. Amer. Math. Soc.*, 267:285–293, 1981; **7.46**.
362. B. Fuglede. Asymptotic paths for subharmonic functions. *Math. Ann.*, 213:261–274, 1975; **3.2**.
363. B. Fuglede. Asymptotic paths for subharmonic functions and polygonal connectedness of fine domains. Semin. Theorie du potentiel, Paris, No. 5, Lect. Notes Math. 814, 97-116 (1980), 1980; **3.2**.
364. R.M. Gabriel. An extended principle of the maximum for harmonic functions in 3-dimensions. *J. London Math. Soc.*, 30:388–401, 1955; **3.9**.
365. A. Gabrielov, D. Novikov, and B. Shapiro. Mystery of point charges. *Proc. Lond. Math. Soc. (3)*, 95(2):443–472, 2007; **7.78**.
366. D. Gaier. Über Räume konformer Selbstabbildungen ebener Gebiete. *Math. Z.*, 187:227–257, 1984; **7.43**.
367. T.W. Gamelin. Localization of the corona problem. *Pacific J. Math.*, 34:73–81, 1970; **5.27**.
368. T.W. Gamelin. *Uniform algebras and Jensen measures*, volume 32. Cambridge University Press, Cambridge. London Mathematical Society, London, 1978; **5.27**.
369. T.W. Gamelin. Wolff's proof of the corona theorem. *Israel J. Math.*, 37(1-2):113–119, 1980.
370. T. Ganelius. The zeros of the partial sums of power series. *Duke Math. J.*, 30:533–540, 1963; **2.59**.
371. P.R. Garabedian and H.L. Royden. The one-quarter theorem for mean univalent functions. *Ann. of Math. (2)*, 59:316–324, 1954; **6.24**.
372. P.R. Garabedian and M. Schiffer. A proof of the Bieberbach conjecture for the fourth coefficient. *J. Rational Mech. Anal.*, 4:427–465, 1955; **6.1**.
373. P.R. Garabedian and M. Schiffer. A coefficient inequality for schlicht functions. *Ann. of Math. (2)*, 61:116–136, 1955; **6.4**.
374. S.J. Gardiner. Superharmonic extension and harmonic approximation. *Ann. Inst. Fourier*, 44(1):65–91, 1994; **9.10**.
375. S.J. Gardiner. *Harmonic approximation*, volume 221. Cambridge: Univ. Press, 1995; **9.10**.
376. S.J. Gardiner. Existence of universal Taylor series for nonsimply connected domains. *Constr. Approx.*, 35(2):245–257, 2012; **9.18**.
377. S.J. Gardiner and T. Sjödin. A characterization of annular domains by quadrature identities. *Bull. London Math. Soc.*, 51:436–442, 2019; **3.35**.
378. J.B. Garnett. *Analytic capacity and measure*. Lecture Notes in Mathematics, Vol. 297. Springer-Verlag, Berlin-New York, 1972; **7.39**.
379. J.B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981; **8.20, 9.1, 9.21**.
380. J.B. Garnett, F.W. Gehring, and P.W. Jones. Conformally invariant length sums. *Indiana Univ. Math. J.*, 32(6):809–829, 1983; **6.93**.
381. J.B. Garnett and P.W. Jones. The corona theorem for Denjoy domains. *Acta Math.*, 155(1-2):27–40, 1985; **5.27**.

382. J.B. Garnett and D.E. Marshall. *Harmonic measure*, volume 2. Cambridge: Cambridge University Press, paperback reprint of the hardback edition 2005 edition, 2008; **6.96**.
383. P.M. Gauthier, M. Goldstein, and W.H. Ow. Uniform approximation on unbounded sets by harmonic functions with logarithmic singularities. *Trans. Amer. Math. Soc.*, 261(1):169–183, 1980; **9.10**.
384. P.M. Gauthier, W. Hengartner, and A. Stray. A problem of Rubel concerning approximation on unbounded sets by entire functions. *Rocky Mountain J. Math.*, 19(1):127–136, 03 1989; **9.7**.
385. F.W. Gehring. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *Acta Math.*, 130:265–277, 1973; **7.12**.
386. F.W. Gehring. The Hausdorff measure of sets which link in Euclidean space. Contribut. to Analysis, Collect. of Papers dedicated to Lipman Bers, 159-167 (1974), 1974; **7.22**.
387. F.W. Gehring. Lower dimensional absolute continuity properties of quasiconformal mappings. *Math. Proc. Cambridge Philos. Soc.*, 78:81–93, 1975; **7.55**.
388. F.W. Gehring. A remark on domains quasiconformally equivalent to a ball. *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, 2:147–155, 1976; **7.57**.
389. F.W. Gehring. Spirals and the universal Teichmüller space. *Acta Math.*, 141(1-2):99–113, 1978; **5.30**.
390. F.W. Gehring, W.K. Hayman, and A. Hinkkanen. Analytic functions satisfying Hölder conditions on the boundary. *J. Approx. Theory*, 35(3):243–249, 1982; **7.46**.
391. F.W. Gehring and Ch. Pommerenke. On the Nehari univalence criterion and quasicircles. *Comment. Math. Helv.*, 59(2):226–242, 1984; **6.67**.
392. F.W. Gehring and E. Reich. Area distortion under quasiconformal mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 388:15, 1966; **7.9**.
393. L. Geyer. Sharp bounds for the valence of certain harmonic polynomials. *Proc. Amer. Math. Soc.*, 136(2):549–555, 2008; **4.35**.
394. M.A. Girnyk. On the inverse problem of the theory of the distribution of values for functions that are analytic in the unit disc. *Ukrain. Mat. Ž.*, 29(1):32–39, 142, 1977; **1.41**.
395. A.B. Givental. Polynomiality of electrostatic potentials. *Uspekhi Mat. Nauk*, 39(5):253–254, 1984; **3.29**.
396. D. Gnuschke and Ch. Pommerenke. On the radial limits of functions with Hadamard gaps. *Mich. Math. J.*, 32:21–31, 1985; **5.47**.
397. B. Goddard. Finite exponential series and Newman polynomials. *Proc. Amer. Math. Soc.*, 116(2):313–320, 1992; **7.33**.
398. R.M. Goel. Functions starlike and convex of order α. *J. London Math. Soc. (2)*, 9:128–130, 1974/75; **6.41**.
399. A.A. Gol'dberg. On one-valued integrals of differential equations of the first order. *Ukrain. Mat. Ž.*, 8:254–261, 1956; **1.35**.
400. A.A. Gol'dberg. On the possible value of the lower order of an entire function with a finite deficient value. *Dokl. Akad. Nauk SSSR*, 159:968–970, 1964; **1.9**.
401. A.A. Gol'dberg. The distribution of the values of an entire function with respect to arguments. *Acta Math. Acad. Sci. Hungar.*, 19:191–199, 1968; **2.4**.
402. A.A. Gol'dberg. Über die Werteverteilung einer ganzen Funktion nach dem Argument. *Acta Math. Acad. Sci. Hung.*, 19:191–199, 1968; **2.5**.
403. A.A. Gol'dberg. The representation of a meromorphic function in the form of a quotient of entire functions. *Izv. Vysš. Učebn. Zaved. Matematika*, (10(125)):13–17, 1972; **2.28**.
404. A.A. Gol'dberg. The branched values of entire functions. *Sibirsk. Mat. Ž.*, 14:862–866, 911, 1973; **1.34**.
405. A.A. Gol'dberg. On ramified values of entire functions. *Sib. Math. J.*, 14:599–602, 1974; **1.4**.
406. A.A. Gol'dberg. Counting functions of sequences of a-points for entire functions. *Sibirsk. Mat. Ž.*, 19(1):28–36, 236, 1978; **1.25**.
407. A.A. Gol'dberg. Sets on which the modulus of an entire function has a lower bound. *Sibirsk. Mat. Zh.*, 20(3):512–518, 691, 1979; **2.40**.

408. A.A. Gol'dberg. The minimum modulus of a meromorphic function of slow growth. *Mat. Zametki*, 25(6):835–844, 956, 1979; **2.52**.
409. A.A. Gol'dberg. Analytic functions mapping a disk on a disk. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (6):24–25, 1984; **5.65**.
410. A.A. Gol'dberg and A. Eremenko. On asymptotic curves of entire functions of finite order (Russian). *Mat. Sb. (N. S.)*, 79, 109 (151)(No. 4):555–581, 1982; **2.7, 2.41, 2.57**.
411. A.A. Gol'dberg, A. Eremenko, and M.L. Sodin. Exceptional values in the sense of R. Nevanlinna and in the sense of V.P. Petrenko. I. *Teor. Funktsiĭ Funktsional. Anal. i Prilozhen.*, (47):41–51, 1987; **1.23**.
412. A.A. Gol'dberg and B.Ja. Levin. Entire functions which are bounded on the real axis. *Dokl. Akad. Nauk SSSR*, 157:19–21, 1964; **2.19**.
413. A.A. Gol'dberg and I.V. Ostrovskii. Indicators of entire absolutely monotone functions of finite order. *Sibirsk. Mat. Zh.*, 27(6):33–49, 1986; **2.75**.
414. A.A. Gol'dberg and I.V. Ostrovskii. Indicators of finite-order entire functions that can be represented by Dirichlet series. *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, (1):14–16, 85, 1990; **2.75**.
415. A.A. Gol'dberg and I.V. Ostrovskii. *Value distribution of meromorphic functions*, volume 236 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2008; **1.23, 1.43, 2.8, 2.9**. Translated from the 1970 Russian original by M. Ostrovskii.
416. A.A. Gol'dberg and V.G. Tairova. On integral functions with two finite completely multiple values (Russian). *Zap. Mech.-Mat-Fak. Harkov. Gos. Univ.*, XXIX:67–78, 1963; **1.4**.
417. M. Goldstein. An example of an Arakelian glove which is a weak Arakelian set. *Illinois J. Math.*, 27(1):138–144, 03 1983; **9.7**.
418. M. Goldstein, R.R. Hall, T. Sheil-Small, and H.L. Smith. Convexity preservation of inverse Euler operators and a problem of S. Miller. *Bull. London Math. Soc.*, 14(6):537–541, 1982; **5.61**.
419. M. Goldstein, W. Haussmann, and L. Rogge. On the mean value property of harmonic functions and best harmonic L^1-approximation. *Trans. Amer. Math. Soc.*, 305(2):505–515, 1988; **3.35**.
420. M. Goldstein and J.N. McDonald. An extremal problem for nonnegative trigonometric polynomials. *J. London Math. Soc. (2)*, 29(1):81–88, 1984; **4.26**.
421. G.M. Goluzin. Einige Koeffizientenabschätzungen für schlichte Funktionen. *Rec. Math. [Mat. Sbornik] N. S.*, 3(2):321–30, 1938; **6.4**.
422. G.M. Goluzin. On majorants of subordinate analytic functions. I. *Mat. Sbornik N.S.*, 29(71):209–224, 1951; **5.38, 5.39**.
423. A.A. Gončar. Generalized analytic continuation. *Mat. Sb. (N.S.)*, 76 (118):135–146, 1968; **9.5**.
424. A.W. Goodman. On some determinants related to p-valent functions. *Trans. Amer. Math. Soc.*, 63:175–192, 1948; **6.25**.
425. A.W. Goodman. *On some determinants related to p-valent functions*. ProQuest LLC, Ann Arbor, MI, 1948; **6.97**. Doctoral Thesis–Columbia University.
426. A.W. Goodman. On the Schwarz–Christoffel transformation and p-valent functions. *Trans. Amer. Math. Soc.*, 68:204–223, 1950; **6.98**.
427. A.W. Goodman. Typically-real functions with assigned zeros. *Proc. Amer. Math. Soc.*, 2:349–357, 1951; **6.98**.
428. A.W. Goodman. The valence of certain means. *J. Analyse Math.*, 22:355–361, 1969; **6.100**.
429. A.W. Goodman. Valence sequences. *Proc. Amer. Math. Soc.*, 79(3):422–426, 1980; **6.102**.
430. A.W. Goodman. Convex functions of bounded type. *Proc. Amer. Math. Soc.*, 92(4):541–546, 1984; **6.99**.
431. A.W. Goodman. More on convex functions of bounded type. *Proc. Amer. Math. Soc.*, 97(2):303–306, 1986; **6.99**.
432. A.W. Goodman. The complete multivalence of Čakalov–Distler sums. *Complex Variables Theory Appl.*, 11(2):87–93, 1989; **6.101**.

433. A.W. Goodman and M.S. Robertson. A class of multivalent functions. *Trans. Amer. Math. Soc.*, 70:127–136, 1951; **6.25, 6.97**.
434. A.W. Goodman and E.B. Saff. On univalent functions convex in one direction. *Proc. Amer. Math. Soc.*, 73:183–187, 1979; **6.53**.
435. A.Y. Gordon. Strong unboundedness of unbounded analytic functions. *Proc. Amer. Math. Soc.*, 122(2):525–529, 1994; **1.45**.
436. P. Gorkin, R. Mortini, and A. Nicolau. The generalized corona theorem. *Math. Ann.*, 301(1):135–154, 1995; **9.3**.
437. N.V. Govorov. The estimation from below of a function that is subharmonic in a disk. *Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp.*, 6:130–150, 1968; **3.23**.
438. N.V. Govorov. The Paley conjecture. *Funkcional. Anal. i Priložen.*, 3(2):41–45, 1969; **1.17**.
439. S.M. Gowda. Nonfactorization theorems in weighted Bergman and Hardy spaces on the unit ball of $\mathbb{C}^n (n>1)$. *Trans. Amer. Math. Soc.*, 277:203–212, 1983; **8.16**.
440. J. Graczyk and G. Światek. Generic hyperbolicity in the logistic family. *Ann. of Math. (2)*, 146(1):1–52, 1997; **2.78**.
441. R. Greiner and O. Roth. On support points of univalent functions and a disproof of a conjecture of Bombieri. *Proc. Amer. Math. Soc.*, 129(12):3657–3664, 2001; **6.3**.
442. A.Z. Grinshpan. Logarithmic coefficients of functions of class S. *Sibirsk. Mat. Ž.*, 13:1145–1157, 1199, 1972; **6.42**.
443. A.Z. Grinshpan. Improved bounds for the difference of the moduli of adjacent coefficients of univalent functions, in “Some questions in the modern theory of functions” (in Russian). *Sib. Inst. Mat. Novosibirsk*, pages 41–45, 1976; **6.46**.
444. A.Z. Grinshpan. On the power stability for the Bieberbach inequality. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 125:58–64, 1983; **6.36**. Analytic number theory and the theory of functions, 5.
445. F. Gross and C.F. Osgood. On the functional equation $f^n + g^n = h^n$ and a new approach to a certain class of more general functional equations. *Indian J. Math.*, 23(1-3):17–39, 1981; **2.26**.
446. W. Groß. Über die Singularitäten analytischer Funktionen. *Monatsh. Math. Phys.*, 29(1):3–47, 1918; **1.32**.
447. J. Guckenheimer. Endomorphisms of the Riemann sphere. In *Proceedings Symposia in Pure Mathematics*, volume 15, pages 95–123. Amer. Math. Soc., Providence, RI, 1970; **2.63**.
448. G.G. Gundersen. On the real zeros of solutions of $f'' + A(z)f = 0$ where $A(z)$ is entire. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 11(2):275–294, 1986; **2.71, 2.72**.
449. G.G. Gundersen. Meromorphic solutions of $f^6 + g^6 + h^6 \equiv 1$. *Analysis (Munich)*, 18(3):285–290, 1998; **2.26**.
450. G.G. Gundersen. Meromorphic solutions of $f^5 + g^5 + h^5 \equiv 1$. *Complex Variables Theory Appl.*, 43(3-4):293–298, 2001; **2.26**.
451. G.G. Gundersen. Solutions of $f'' + P(z)f = 0$ that have almost all real zeros. *Ann. Acad. Sci. Fenn. Math.*, 26(2):483–488, 2001; **2.71**.
452. G.G. Gundersen and W.K. Hayman. The strength of Cartan’s version of Nevanlinna theory. *Bull. London Math. Soc.*, 36(4):433–454, 2004; **2.26**.
453. R.C. Gunning and H. Rossi. *Analytic functions of several complex variables*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965; **3.23**.
454. P.A. Gunsul. Value distribution for a class of small functions in the unit disk. *Int. J. Math. Math. Sci.*, 2011:24, 2011; **1.38**.
455. O. Hajek. Notes on meromorphic dynamical systems. I-III. *Czech. Math. J.*, 16:14–27, 28–35, 36–40, 1966; **1.44**.
456. M. Hakim and N. Sibony. Quelques conditions pour l’existence de fonctions pics dans des domaines pseudoconvexes. *Duke Math. J.*, 44(2):399–406, 1977; **8.22**.
457. G. Halász. On a result of Salem and Zygmund concerning random polynomials. *Studia Sci. Math. Hungar.*, 8:369–377, 1973; **4.17**.
458. R.R. Hall. On a conjecture of Shapiro about trigonometric series. *J. London Math. Soc. (2)*, 25(3):407–415, 1982; **5.41**.

459. R.R. Hall. On some theorems of Hurwitz and Sheil-Small. *Math. Proc. Cambridge Philos. Soc.*, 100(2):365–370, 1986; **5.41**.
460. R.R. Hall, W.K. Hayman, and A.W. Weitsman. On asymmetry and capacity. *J. Anal. Math.*, 56:87–123, 1991; **3.18**.
461. R.R. Hall and J.H. Williamson. On a certain functional equation. *J. London Math. Soc. (2)*, 12(2):133–136, 1975/76; **2.32**.
462. T. Hall. Sur la mesure harmonique de certains ensembles. *Ark. Mat. Astr. Fys.*, 25(28):8 pp., 1937; **3.3**.
463. D.H. Hamilton. Doctoral thesis. *University of London*, 1980; **6.46, 6.73**.
464. D.H. Hamilton. The extreme points of Σ. *Proc. Amer. Math. Soc.*, 85(3):393–396, 1982; **6.66**.
465. D.H. Hamilton. The dispersion of the coefficients of univalent functions. *Trans. Amer. Math. Soc.*, 276(1):323–333, 1983; **6.2**.
466. D.H. Hamilton. A sharp form of the Ahlfors' distortion theorem, with applications. *Trans. Amer. Math. Soc.*, 282(2):799–806, 1984; **6.2, 6.73**.
467. D.H. Hamilton. Covering theorems for univalent functions. *Math. Z.*, 191:327–349, 1986; **7.24**.
468. D.H. Hamilton. Extremal boundary problems. *Proc. London Math. Soc. (3)*, 56(1):101–113, 1988; **6.70**.
469. D.H. Hamilton. Mapping Theorems. *arXiv Mathematics e-prints*, page math/0509703, Sep 2005; **7.8**.
470. D.H. Hamilton. QC Riemann mapping theorem in space. In *Complex analysis and dynamical systems III. Proceedings of the 3rd conference in honor of the retirement of Dov Aharonov, Lev Aizenberg, Samuel Krushkal, and Uri Srebro, Nahariya, Israel, January 2–6, 2006*, pages 131–149. Providence, RI: American Mathematical Society (AMS); Ramat Gan: Bar-Ilan University, 2008; **7.8**.
471. R.S. Hamilton. Extremal quasiconformal mappings with prescribed boundary values. *Trans. Amer. Math. Soc.*, 138:399–406, 1969; **8.9**.
472. L.J. Hansen. Hardy classes and ranges of functions. *Michigan Math. J.*, 17:235–248, 1970; **5.7**.
473. L.J. Hansen. On the growth of entire functions bounded on large sets. *Canad. J. Math.*, 29(6):1287–1291, 1977; **2.40**.
474. W. Hansen and N. Nadirashvili. A converse to the mean value theorem for harmonic functions. *Acta Math.*, 171(2):139–163, 1993; **3.8, 7.30**.
475. W. Hansen and N. Nadirashvili. Littlewood's one circle problem. *J. Lond. Math. Soc., II. Ser.*, 50(2):349–360, 1994; **3.8, 7.30**.
476. G.H. Hardy and J.E. Littlewood. A maximal theorem with function-theoretic applications. *Acta Math.*, 54(1):81–116, 1930; **7.74**.
477. A. Harrington and M. Ortel. The dilatation of an extremal quasi-conformal mapping. *Duke Math. J.*, 43(3):533–544, 1976; **8.9**.
478. A. Harrington and M. Ortel. Two extremal problems. *Trans. Amer. Math. Soc.*, 221(1):159–167, 1976; **8.9**.
479. V.P. Havin. Boundary properties of integrals of Cauchy type and of conjugate harmonic functions in regions with rectifiable boundary. *Mat. Sb. (N.S.)*, 68 (110):499–517, 1965; **7.14**.
480. J. Hawkes. Probabilistic behaviour of some lacunary series. *Z. Wahrsch. Verw. Gebiete*, 53(1):21–33, 1980; **5.47**.
481. W.K. Hayman. Some inequalities in the theory of functions. *Proc. Camb. Philos. Soc.*, 44:159–178, 1948.
482. W.K. Hayman. Inequalities in the theory of functions. *Proc. Lond. Math. Soc. (2)*, 51:450–473, 1949.
483. W.K. Hayman. Some applications of the transfinite diameter to the theory of functions. *J. Analyse Math.*, 1:155–179, 1951; **5.3**.

484. W.K. Hayman. The minimum modulus of large integral functions. *Proc. London Math. Soc. (3)*, 2:469–512, 1952; **2.33, 2.35**.
485. W.K. Hayman. An integral function with a defective value that is neither asymptotic nor invariant under change of origin. *J. London Math. Soc.*, 28:369–376, 1953; **1.23**.
486. W.K. Hayman. Uniformly normal families. In *Lectures on functions of a complex variable*, pages 199–212. The University of Michigan Press, Ann Arbor, 1955; **5.4, 5.5, 5.6**.
487. W.K. Hayman. The asymptotic behaviour of p-valent functions. *Proc. London Math. Soc. (3)*, 5:257–284, 1955; **6.2**$'$.
488. W.K. Hayman. Interpolation by bounded functions. *Ann. Inst. Fourier. Grenoble*, 8:277–290, 1958.
489. W.K. Hayman. Bounds for the large coefficients of univalent functions. *Ann. Acad. Sci. Fenn. Ser. A.I, no.* 250/13:13, 1958; **6.2**.
490. W.K. Hayman. Picard values of meromorphic functions and their derivatives. *Ann. of Math. (2)*, 70:9–42, 1959; **1.18, 1.19, 1.20, 5.14**.
491. W.K. Hayman. On the growth of integral functions of asymptotic paths. *J. Indian Math. Soc. (N.S.)*, 24:251–264 (1961), 1960; **2.6**.
492. W.K. Hayman. Slowly growing integral and subharmonic functions. *Comment. Math. Helv.*, 34:75–84, 1960; **2.7, 2.41, 2.57, 3.13**.
493. W.K. Hayman. *Meromorphic functions.* Oxford University Press, Oxford, 1964.
494. W.K. Hayman. On the characteristic of functions meromorphic in the unit disk and of their integrals. *Acta Math.*, 112:181–214, 1964; **1.21**.
495. W.K. Hayman. On the characteristic of functions meromorphic in the plane and of their integrals. *Proc. London Math. Soc. (3)*, 14a:93–128, 1965; **1.21, 2.68**.
496. W.K. Hayman. Mean p-valent functions with gaps. *Colloq. Math.*, 16:1–21, 1967; **6.12**.
497. W.K. Hayman. Note on Hadamard's convexity theorem. In *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966)*, pages 210–213. Amer. Math. Soc., Providence, R.I., 1968; **2.17, 2.18**.
498. W.K. Hayman. On the second Hankel determinant of mean univalent functions. *Proc. London Math. Soc. (3)*, 18:77–94, 1968; **6.14**.
499. W.K. Hayman. On integral functions with distinct asymptotic values. *Proc. Cambridge Philos. Soc.*, 66:301–315, 1969; **2.2**.
500. W.K. Hayman. Some examples related to the cos $\pi\rho$ theorem. In *Mathematical Essays Dedicated to A. J. Macintyre*, pages 149–170. Ohio Univ. Press, Athens, Ohio, 1970; **2.36**.
501. W.K. Hayman. On the Valiron deficiencies of integral functions of infinite order. *Arkiv för Matematik*, 10(1):163–172, Dec 1972; **1.2, 1.22**.
502. W.K. Hayman. Differential inequalities and local valency. *Pacific J. Math.*, 44:117–137, 1973; **2.28**.
503. W.K. Hayman. Research problems in function theory: new problems. Pages 155–180. London Math. Soc. Lecture Note Ser., No. 12, 1974.
504. W.K. Hayman. Research problems in function theory: progress on the previous problems. pages 143–154. London Math. Soc. Lecture Note Ser., No. 12, 1974.
505. W.K. Hayman. The local growth of power series: a survey of the Wiman–Valiron method. *Can. Math. Bull.*, 17:317–358, 1974; **1.44**.
506. W.K. Hayman. The minimum modulus of integral functions of order one. *J. Analyse Math.*, 28:171–212, 1975; **2.38**.
507. W.K. Hayman. On Iversen's theorem for meromorphic functions with few poles. *Acta Math.*, 141(1-2):115–145, 1978; **2.8**.
508. W.K. Hayman. On a conjecture of Littlewood. *J. Analyse Math.*, 36:75–95 (1980), 1979; **4.18**.
509. W.K. Hayman. The logarithmic derivative of multivalent functions. *Michigan Math. J.*, 27(2):149–179, 1980; **6.43, 6.71**.
510. W.K. Hayman. Some achievements of Nevanlinna theory. *Ann. Acad. Sci. Fenn., Ser. A I, Math.*, 7:65–71, 1982.
511. W.K. Hayman. *Subharmonic functions. Volume 2.* London: Academic Press, Inc., 1989.

512. W.K. Hayman. The growth of solutions of algebraic differential equations. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 7(2):67–73, 1996; **1.35**.
513. W.K. Hayman. *Multivalent functions. Paperback reprint of the 2nd hardback edition 1994*, volume 110. Cambridge: Cambridge University Press, 2008.
514. W.K. Hayman and J.A. Hummel. Coefficients of powers of univalent functions. *Complex Variables Theory Appl.*, 7(1-3):51–70, 1986; **6.36**.
515. W.K. Hayman and P.B. Kennedy. *Subharmonic functions. Vol. I*. Academic Press, London-New York, 1976. London Mathematical Society Monographs, No. 9.
516. W.K. Hayman, D. Kershaw, and T.J. Lyons. The best harmonic approximant to a continuous function. In *Anniversary volume on approximation theory and functional analysis (Oberwolfach, 1983)*, volume 65 of *Internat. Schriftenreihe Numer. Math.*, pages 317–327. Birkhäuser, Basel, 1984; **3.35**.
517. W.K. Hayman and B. Kjellberg. On the minimum of a subharmonic function on a connected set. In *Studies in pure mathematics*, pages 291–322. Birkhäuser, Basel, 1983; **2.35**.
518. W.K. Hayman and J. Miles. On the growth of a meromorphic function and its derivatives. *Complex Variables Theory Appl.*, 12(1-4):245–260, 1989; **1.21**.
519. W.K. Hayman, S.J. Patterson, and Ch. Pommerenke. On the coefficients of certain automorphic functions. *Math. Proc. Cambridge Philos. Soc.*, 82(3):357–367, 1977; **5.7, 5.33, 5.40**.
520. W.K. Hayman, T.F. Tyler, and D.J. White. The Blumenthal conjecture. In *Complex analysis and dynamical systems V*, volume 591 of *Contemp. Math.*, pages 149–157. Amer. Math. Soc., Providence, RI, 2013; **2.15**.
521. W.K. Hayman and I. Vincze. A problem on entire functions. Complex analysis and its applications, Collect. Artic., Steklov Math. Inst., Moscow 1978, 591-594 (1978), 1978; **2.32**.
522. W.K. Hayman and A. Weitsman. On the coefficients and means of functions omitting values. *Math. Proc. Cambridge Philos. Soc.*, 77:119–137, 1975; **5.7**.
523. W.K. Hayman and J.M.G. Wu. Level sets of univalent functions. *Comment. Math. Helv.*, 56(3):366–403, 1981; **6.93, 6.110, 6.126**.
524. S.H. Hechler. On the existence of certain cofinal subsets of ${}^{\omega}\omega$. Axiom. Set Theor., Proc. Symp. Los Angeles 1967, 155-173 (1974), 1974; **2.66**.
525. L.I. Hedberg. Weighted mean square approximation in plane regions, and generators of an algebra of analytic functions. *Ark. Mat.*, 5:541–552 (1965), 1965; **5.25**.
526. H. Hedenmalm. A factorization theorem for square area-integrable analytic functions. *J. Reine Angew. Math.*, 422:45–68, 1991; **5.26**.
527. H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman spaces*, volume 199. New York, NY: Springer, 2000; **5.26**.
528. H. Hedenmalm and S. Shimorin. Weighted Bergman spaces and the integral means spectrum of conformal mappings. *Duke Math. J.*, 127(2):341–393, 2005; **4.18**.
529. M. Heins. A class of conformal metrics. *Bull. Amer. Math. Soc.*, 67:475–478, 1961; **5.8**.
530. S. Hellerstein. Some analytic varieties in the polydisc and the Müntz–Szasz problem in several variables. *Trans. Amer. Math. Soc.*, 158:285–292, 1971; **7.19**.
531. S. Hellerstein and J. Korevaar. The real values of an entire function. *Bull. Amer. Math. Soc.*, 70:608–610, 1964; **2.44**.
532. S. Hellerstein and J. Rossi. Zeros of meromorphic solutions of second order linear differential equations. *Math. Z.*, 192(4):603–612, 1986; **2.71, 2.72**.
533. S. Hellerstein and D.F. Shea. An extremal problem concerning entire functions with radially distributed zeros. Pages 81–87. London Math. Soc. Lecture Note Ser., No. 12, 1974; **1.13, 3.13**.
534. S. Hellerstein and J. Williamson. Entire functions with negative zeros and a problem of R. Nevanlinna. *J. Analyse Math.*, 22:233–267, 1969; **1.7**.
535. S. Hellerstein and J. Williamson. Derivatives of entire functions and a question of Pólya. *Trans. Amer. Math. Soc.*, 227:227–249, 1977; **2.64**.

536. J.A. Hempel. Precise bounds in the theorems of Landau and Schottky. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 421–424. Academic Press, London-New York, 1980; **5.15**.
537. W. Hengartner and G. Schober. A remark on level curves for domains convex in one direction. *Appl. Anal.*, 3:101–106, 1973; **6.53**.
538. G.M. Henkin. Solutions with bounds for the equations of H. Lewy and Poincaré–Lelong. Construction of functions of Nevanlinna class with given zeros in a strongly pseudoconvex domain. *Dokl. Akad. Nauk SSSR*, 224(4):771–774, 1975; **7.67**.
539. C.W. Henson and L.A. Rubel. Some applications of Nevanlinna theory to mathematical logic: Identities of exponential functions. *Trans. Amer. Math. Soc.*, 282:1–32, 1984; **8.5**.
540. C.W. Henson and L.A. Rubel. Correction to "Some applications of Nevanlinna theory to mathematical logic: identities of exponential functions". *Trans. Amer. Math. Soc.*, 294:381, 1986; **8.5**.
541. M.-R. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Inst. Hautes Études Sci. Publ. Math.*, (49):5–233, 1979; **2.90**.
542. M.-R. Herman. Exemples de fractions rationnelles ayant une orbite dense sur la sphère de Riemann. *Bull. Soc. Math. France*, 112(1):93–142, 1984; **2.89**.
543. M.-R. Herman. Are there critical points on the boundaries of singular domains? *Comm. Math. Phys.*, 99(4):593–612, 1985; **2.83**.
544. M.-R. Herman. Recent results and some open questions on Siegel's linearization theorem of germs of complex analytic diffeomorphisms of $\mathbb{C}^n$ near a fixed point. In *VIIIth international congress on mathematical physics (Marseille, 1986)*, pages 138–184. World Sci. Publishing, Singapore, 1987.
545. F. Herzog and G. Piranian. The counting function for points of maximum modulus. In *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966)*, pages 240–243. Amer. Math. Soc., Providence, R. I., 1968; **2.16, 2.49**.
546. C.G. Higginson. The asymptotic Bieberbach conjecture for weakly p-valent functions. *Proc. London Math. Soc. (3)*, 35(2):291–312, 1977; **5.4**.
547. E. Hille. Remarks on a paper by Zeev Nehari. *Bull. Amer. Math. Soc.*, 55:552–553, 1949; **6.15**.
548. J.D. Hinchliffe. Unbounded analytic functions on plane domains. *Mathematika*, 50(1-2):207–214, 2003; **1.45**.
549. A. Hinkkanen and J. Rossi. Entire functions with asymptotic functions. *Math. Scand.*, 77(1):153–160, 1995; **2.3**.
550. K. Hoffman. *Banach spaces of analytic functions*. Prentice-Hall Series in Modern Analysis. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1962.
551. K. Hoffman and J. Wermer. A characterization of $C(X)$. *Pac. J. Math.*, 12:941–944, 1962; **9.19**.
552. F. Holland and J.B. Twomey. Fourier–Stieltjes series of measures in Zygmund's class Λ_*. *Proc. Roy. Irish Acad. Sect. A*, 76(26):289–299, 1976; **5.69**.
553. R. Hornblower. A growth condition for the MacLane class $\mathcal{A}$. *Proc. London Math. Soc. (3)*, 23:371–384, 1971; **3.5**.
554. R.J.M. Hornblower. Subharmonic analogues of MacLane's classes. *Ann. Polon. Math.*, 26:135–146, 1972; **3.5**.
555. C. Horowitz. Zeros of functions in the Bergman spaces. *Duke Math. J.*, 41:693–710, 1974; **8.6**.
556. D. Horowitz. A further refinement for coefficient estimates of univalent functions. *Proc. Amer. Math. Soc.*, 71(2):217–221, 1978; **6.2**.
557. J.H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 467–511. Publish or Perish, Houston, TX, 1993; **2.88**.
558. A. Huber. On subharmonic functions and differential geometry in the large. *Comment. Math. Helv.*, 32:13–72, 1957; **2.10**.

559. A. Huber. Isometrische und Konforme Verheftung. *Comment. Math. Helv.*, 51(3):319–331, 1976; **7.56**.
560. S. Hui. An extension of a theorem of J. Fernández. *Bull. London Math. Soc.*, 20(1):34–36, 1988; **7.81**.
561. J.A. Hummel. Counterexamples to the Poincaré inequality. *Proc. Amer. Math. Soc.*, 8:207–210, 1957; **8.18**.
562. J.A. Hummel. The Marx conjecture for starlike functions. *Michigan Math. J.*, 19:257–266, 1972; **6.16**.
563. J.S. Hwang. A problem on automorphic functions and gap series. *Bull. Inst. Math. Acad. Sinica*, 11(3):401–406, 1983; **5.49**.
564. A. Hyllengren. Valiron deficient values for meromorphic functions in the plane. *Acta Mathematica*, 124(1):1–8, Jul 1970; **1.2**.
565. S. Igari. Decomposition theorem and lacunary convergence of Riesz–Bochner means of Fourier transforms of two variables. *Tohoku Math. J. (2)*, 33:413–419, 1981; **7.63**.
566. F. Iversen. Recherches sur les fonctions inverses des fonctions méromorphes. Thèse présenté à la faculté des sciences de l'Université de Helsingfors (1914), 1914.
567. T. Iwaniec, L.V. Kovalev, and J. Onninen. The harmonic mapping problem and affine capacity. *Proc. R. Soc. Edinb., Sect. A, Math.*, 141(5):1017–1030, 2011; **6.124**.
568. T. Iwaniec, L.V. Kovalev, and J. Onninen. The Nitsche conjecture. *J. Amer. Math. Soc.*, 24(2):345–373, 2011; **6.124**.
569. I.S. Jack. Functions starlike and convex of order α. *J. London Math. Soc. (2)*, 3:469–474, 1971; **6.41**.
570. S. Jaenisch. Abschätzungen subharmonischer und ganzer Funktionen in der Umgebung einer Kurve. *Mitt. Math. Sem. Giessen Heft*, 65:iv+50, 1965; **2.68**.
571. M. Jakob and A.C. Offord. The distribution of the values of a random power series in the unit disk. *Proc. Roy. Soc. Edinburgh Sect. A*, 94(3-4):251–263, 1983; **5.54**.
572. W. Janowski. Some extremal problems for certain families of analytic functions. I. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 21:17–25, 1973; **6.111**.
573. A. Janteng, S.A. Halim, and M. Darus. Hankel determinant for starlike and convex functions. *Int. J. Math. Anal., Ruse*, 1(13-16):619–625, 2007; **6.121**.
574. U.C. Jayatilake. Brannan's conjecture for initial coefficients. *Complex Var. Elliptic Equ.*, 58(5):685–694, 2013; **5.44**.
575. J.A. Jenkins. *Univalent functions and conformal mapping*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958; **6.25**.
576. J.A. Jenkins. Some estimates for harmonic measures. In *Complex analysis, I (College Park, Md., 1985–86)*, volume 1275 of *Lecture Notes in Math.*, pages 210–214. Springer, Berlin, 1987; **3.22**.
577. J.A. Jenkins. Some estimates for harmonic measures. III. *Proc. Amer. Math. Soc.*, 119(1):199–201, 1993; **3.22**.
578. F. John. A criterion for univalency brought up to date. *Comm. Pure Appl. Math.*, 29(3):293–295, 1976; **6.81**.
579. P.W. Jones. Constructions for BMO($\mathbb{R}$) and $A_p(\mathbb{R}^n)$. Harmonic analysis in Euclidean spaces, Part 1, Williamstown/ Massachusetts 1978, Proc. Symp. Pure Math., Vol. 35, 417-419 (1979), 1979; **7.35**.
580. P.W. Jones. Estimates for the corona problem. *J. Funct. Anal.*, 39(2):162–181, 1980; **5.27**.
581. P.W. Jones and P.F.X. Müller. Radial variation of Bloch functions. *Math. Res. Lett.*, 4(2-3):395–400, 1997; **5.75**.
582. P.W. Jones and T.H. Wolff. Hausdorff dimension of harmonic measures in the plane. *Acta Math.*, 161(1-2):131–144, 1988; **7.44, 7.84**.
583. G.S. Jordan. Regularly varying functions and convolutions with real kernels. *Trans. Amer. Math. Soc.*, 194:177–194, 1974; **8.27**.
584. L. Karp. On the Newtonian potential of ellipsoids. *Complex Variables, Theory Appl.*, 25(4):367–371, 1994; **3.29**.

585. L. Karp. On null quadrature domains. *Comput. Methods Funct. Theory*, 8(1):57–72, 2008; **3.28**.
586. L. Karp and A. Margulis. Null quadrature domains and a free boundary problem for the Laplacian. *Indiana Univ. Math. J.*, 61(2):859–882, 2012; **3.28**.
587. L. Karp and A.S. Margulis. Newtonian potential theory for unbounded sources and applications to free boundary problems. *J. Anal. Math.*, 70:1–63, 1996.
588. I.G. Kasmalkar. On the Sendov conjecture for a root close to the unit circle. *Aust. J. Math. Anal. Appl.*, 11(1):Art. 4, 34, 2014; **4.5**.
589. G. Katona. On a conjecture of Erdös and a stronger form of Sperner's theorem. *Studia Sci. Math. Hungar*, 1:59–63, 1966; **7.5**.
590. O.D. Kellogg. On the derivatives of harmonic functions on the boundary. *Trans. Amer. Math. Soc.*, 33(2):486–510, 1931; **3.16**.
591. J.T. Kemper. A boundary Harnack principle for Lipschitz domains and the principle of positive singularities. *Comm. Pure Appl. Math.*, 25:247–255, 1972; **3.17**.
592. P.B. Kennedy. A class of integral functions bounded on certain curves. *Proc. London Math. Soc. (3)*, 6:518–547, 1956; **2.2**.
593. F.R. Keogh and E.P. Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, 20:8–12, 1969; **6.119**.
594. V.P. Khavin, S.V. Khrushchev, and N.K. Nikol'skij, editors. *Linear and complex analysis. Problem book. 199 research problems*, volume 1043. Springer, Cham, 1984; **1.45, 8.2, 9.15**.
595. D. Khavinson, S.-Y. Lee, and A. Saez. Zeros of harmonic polynomials, critical lemniscates, and caustics. *Complex Anal. Synerg.*, 4:20, 2018; **4.35**.
596. D. Khavinson and E. Lundberg. A tale of ellipsoids in potential theory. *Notices Amer. Math. Soc.*, 61(2):148–156, 2014; **3.29**.
597. D. Khavinson and E. Lundberg. *Linear holomorphic partial differential equations and classical potential theory*, volume 232. Providence, RI: American Mathematical Society (AMS), 2018; **7.71**.
598. D. Khavinson and G. Neumann. From the fundamental theorem of algebra to astrophysics: a "harmonious" path. *Notices Amer. Math. Soc.*, 55(6):666–675, 2008; **4.35**.
599. D. Khavinson, R. Pereira, M. Putinar, E.B. Saff, and S. Shimorin. Borcea's variance conjectures on the critical points of polynomials. In *Notions of positivity and the geometry of polynomials*, Trends Math., pages 283–309. Birkhäuser/Springer Basel AG, Basel, 2011; **4.5, 7.78**.
600. D. Khavinson, F. Pérez-González, and H.S. Shapiro. Approximation in L^1-norm by elements of a uniform algebra. *Constr. Approx.*, 14(3):401–410, 1998; **9.19**.
601. D. Khavinson, M. Putinar, and H.S. Shapiro. Poincaré's variational problem in potential theory. *Arch. Ration. Mech. Anal.*, 185(1):143–184, 2007; **3.30**.
602. D. Khavinson and G. Świątek. On the number of zeros of certain harmonic polynomials. *Proc. Amer. Math. Soc.*, 131(2):409–414, 2003; **4.35**.
603. S.Y. Khavinson. Removable singularities of analytic functions of the V. I. Smirnov class. *Some problems in modern function theory (Proc. Conf. Modern Problems of Geometric Theory of Functions, Inst. Math., Acad. Sci. USSR, Novosibirsk, 1976) (Russian)*, pages 160–166, 1976; **7.14**.
604. S.Y. Khavinson. Foundations of the theory of extremal problems for bounded analytic functions and various generalizations of them. *Transl., Ser. 2, Amer. Math. Soc.*, 129:1–61, 1986; **5.43**.
605. S.Y. Khavinson. Foundations of the theory of extremal problems for bounded analytic functions with additional conditions. *Transl., Ser. 2, Amer. Math. Soc.*, 129:63–114, 1986; **5.43**.
606. S. Kierst. Sur l'ensemble des valeurs asymptotiques d'une fonction méromorphe dans le cercle-unité. *Fund. Math.*, 27:226–233, 1936; **5.45**.
607. K. Killian. A remark on Maxwell's conjecture for planar charges. *Complex Var. Elliptic Equ.*, 54(12):1073–1078, 2009; **7.78**.
608. B. Kjellberg. A theorem on the minimum modulus of entire functions. *Math. Scand.*, 12:5–11, 1963; **2.38**.

609. B. Kjellberg. The convexity theorm of Hadamard–Hayman. In *Proceedings of the Symposium in Mathematics, Royal Institute of Technology, Stockholm, (June 1973)*, pages 87–114. 1973; **2.17**.
610. D.J. Kleitman. On a lemma of Littlewood and Offord on the distribution of certain sums. *Math. Z.*, 90:251–259, 1965; **7.5**.
611. T. Kobayashi. An entire function with linearly distributed values. *Kodai Math. J.*, 2(1):54–81, 1979; **2.24**.
612. W. Koepf. On the Fekete–Szegö problem for close-to-convex functions. *Proc. Amer. Math. Soc.*, 101:89–95, 1987; **6.122**.
613. L. Köhler. Meromorphic functions sharing zeros and poles and also some of their derivatives sharing zeros. *Complex Variables Theory Appl.*, 11(1):39–48, 1989; **2.65**.
614. S.V. Konyagin. On the problem of Littlewood. *Math. USSR, Izv.*, 18:205–225, 1982; **4.19**.
615. S.V. Konyagin. Minimum of the absolute value of random trigonometric polynomials with coefficients ± 1. *Math. Notes*, 56(3):931–947, 1994; **4.16**.
616. S.V. Konyagin and W. Schlag. Lower bounds for the absolute value of random polynomials on a neighborhood of the unit circle. *Trans. Amer. Math. Soc.*, 351(12):4963–4980, 1999; **4.16**.
617. P. Koosis. *Introduction to H_p spaces*, volume 115. Cambridge: Cambridge University Press, paperback reprint of the 2nd hardback edition 1998 edition, 2008; **5.29**.
618. B. Korenblum. An extension of the Nevanlinna theory. *Acta Math.*, 135(3-4):187–219, 1975; **8.6**.
619. J. Korevaar. Equilibrium distributions of electrons on roundish plane conductors. I. *Nederl. Akad. Wet., Proc., Ser. A*, 77:423–437, 1974; **9.12**.
620. J. Korevaar. Equilibrium distributions of electrons on roundish plane conductors. II. *Nederl. Akad. Wet., Proc., Ser. A*, 77:438–456, 1974; **9.12**.
621. J. Korevaar. Polynomial and rational approximation in the complex domain. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 251–292. Academic Press, London-New York, 1980; **9.8**.
622. J. Korevaar. *Tauberian theory. A century of developments*, volume 329. Berlin: Springer, 2004; **8.27**.
623. J. Korevaar and M. Dixon. Nonspanning sets of exponentials on curves. *Acta Math. Acad. Sci. Hung.*, 33:89–100, 1979; **9.9**.
624. J. Korevaar and R.A. Kortram. Equilibrium distributions of electrons on smooth plane conductors. *Nederl. Akad. Wetensch. Indag. Math.*, 45(2):203–219, 1983; **9.12**.
625. J. Korevaar and T.L. McCoy. Power series whose partial sums have few zeros in an angle. *J. Math. Anal. Appl.*, 8:461–473, 1964; **2.59**.
626. J. Korevaar and P. Pfluger. Spanning sets of powers on wild Jordan curves. *Nederl. Akad. Wet., Proc., Ser. A*, 77:293–305, 1974; **7.18**.
627. J. Korevaar and R. Zeinstra. Transformées de Laplace pour les courbes à pente bornée et un résultat correspondant du type Müntz–Szasz. (Laplace transforms along curves of bounded slope and a corresponding Müntz–Szász type result). *C. R. Acad. Sci., Paris, Sér. I*, 301:695–698, 1985; **7.17**.
628. T.W. Körner. On a polynomial of Byrnes. *Bull. London Math. Soc.*, 12(3):219–224, 1980; **4.14**.
629. L.J. Kotman. An entire function with irregular growth and more than one deficient value. In *Complex analysis Joensuu 1978 (Proc. Colloq., Univ. Joensuu, Joensuu, 1978)*, volume 747 of *Lecture Notes in Math.*, pages 219–229. Springer, Berlin, 1979; **1.11**.
630. T. Kövari. A gap-theorem for entire functions of infinite order. *Michigan Math. J.*, 12:133–140, 1965; **2.11**.
631. T. Kövari. On the asymptotic paths of entire functions with gap power series. *J. Analyse Math.*, 15:281–286, 1965; **2.11**.
632. T. Kövari. Asymptotic values of entire functions of finite order with density conditions. *Acta Sci. Math. (Szeged)*, 26:233–237, 1965; **2.12**.

633. T. Kövari. On the growth of entire functions of finite order with density conditions. *Quart. J. Math. Oxford Ser. (2)*, 17:22–30, 1966; **2.12a**.
634. T. Kövari and Ch. Pommerenke. On the distribution of Fekete points. *Mathematika*, 15:70–75, 1968; **9.12**.
635. I. Kra. Automorphic forms and Kleinian groups. Mathematics Lecture Note Series. Reading, Mass.: W. A. Benjamin, Inc., Advanced Book Program. XIV, 1972; **8.4**.
636. S.G. Krantz. The corona problem in several complex variables. In *The corona problem*, volume 72 of *Fields Inst. Commun.*, pages 107–126. Springer, New York, 2014.
637. G.K. Kristiansen. Some inequalities for algebraic and trigonometric polynomials. *J. London Math. Soc. (2)*, 20(2):300–314, 1979; **4.2**.
638. S.L. Krushkal. The interpolating family of a univalent analytic function. *Sibirsk. Mat. Zh.*, 28(5):88–94, 1987; **6.79**.
639. S.L. Krushkal. Proof of the Zalcman conjecture for initial coefficients. *Georgian Math. J.*, 17(4):663–681, 2010; **6.127**.
640. S.L. Krushkal. Erratum to: "Proof of the Zalcman conjecture for initial coefficients". *Georgian Math. J.*, 19(4):777, 2012; **6.127**.
641. V.I. Krutin. The size of the Nevanlinna deficiencies of functions meromorphic in $|z|<1$. *Izv. Akad. Nauk Armjan. SSR Ser. Mat.*, 8(5):347–358, 425, 1973; **1.41**.
642. J. Krzyż. Coefficient problem for bounded nonvanishing functions in 'Proceedings of the Fourth Conference on Analytic Functions'. *Ann. Pol. Math.*, 20:314–316, 1968; **5.83**.
643. Y.H. Ku. Sur les familles normales de fonctions méromorphes. *Sci. Sinica*, 21(4):431–445, 1978; **5.11**.
644. Y. Kubota. A coefficient inequality for certain meromorphic univalent functions. *Kōdai Math. Sem. Rep.*, 26:85–94, 1974/75; **6.4**.
645. Y. Kubota. Coefficients of meromorphic univalent functions. *Kōdai Math. Sem. Rep.*, 28(2–3):253–261, 1976/77; **6.4**.
646. H.P. Künzi. Konstruktion Riemannscher Flächen mit vorgegebener Ordnung der erzeugenden Funktionen. *Math. Ann.*, 128:471–474, 1955; **1.34**.
647. O.S. Kuznetsova and V.G. Tkachev. Length functions of lemniscates. *Manuscr. Math.*, 112(4):519–538, 2003; **4.3, 4.7**.
648. W.T. Lai. The exact value of Hayman's constant in Landau's theorem. *Sci. Sinica*, 22(2):129–134, 1979; **5.15**.
649. J.K. Langley. Proof of a conjecture of Hayman concerning f and f''. *J. London Math. Soc. (2)*, 48(3):500–514, 1993; **1.18**.
650. J.K. Langley. On the deficiencies of composite entire functions. *Proc. Edinburgh Math. Soc. (2)*, 36(1):151–164, 1993; **1.21**.
651. J.K. Langley. Two results related to a question of Hinkkanen. *Kodai Math. J.*, 19(1):52–61, 1996; **2.65**.
652. J.K. Langley. Trajectories escaping to infinity in finite time. *Proc. Amer. Math. Soc.*, 145(5):2107–2117, 2017; **1.44**.
653. J.K. Langley. Linear differential polynomials in zero-free meromorphic functions. *Ann. Acad. Sci. Fenn., Math.*, 43(2):693–735, 2018; **1.42**.
654. O. Lehto. The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity. *Comment. Math. Helv.*, 33:196–205, 1959; **2.4**.
655. F.D. Lesley. On interior and conformal mappings of the disk. *J. London Math. Soc. (2)*, 20(1):67–78, 1979; **6.60**.
656. Y.-J. Leung. Successive coefficients of starlike functions. *Bull. London Math. Soc.*, 10(2):193–196, 1978; **6.46**.
657. Y.-J. Leung. Integral means of the derivatives of some univalent functions. *Bull. London Math. Soc.*, 11(3):289–294, 1979; **6.30**.
658. Y.-J. Leung. Robertson's conjecture on the coefficients of close-to-convex functions. *Proc. Amer. Math. Soc.*, 76(1):89–94, 1979; **6.46**.
659. Y.-J. Leung. On the Bombieri numbers for the class S. *J. Anal.*, 24(2):229–250, 2016; **6.3**.

660. B.Ja. Levin and I.V. Ostrovskii. The dependence of the growth of an entire function on the distribution of zeros of its derivatives. *Sibirsk. Mat. Ž.*, 1:427–455, 1960; **2.64**.
661. G. Levin and S. van Strien. Local connectivity of the Julia set of real polynomials. *Ann. of Math. (2)*, 147(3):471–541, 1998; **2.79**.
662. V. Levin. Ein Beitrag zum Koeffizientenproblem der schlichten Funktionen. *Math. Z.*, 38(1):306–311, 1934; **6.37**.
663. N. Levinson. On the growth of analytic functions. *Trans. Amer. Math. Soc.*, 43(2):240–257, 1938; **7.17**.
664. M. Lewin. On a coefficient problem for bi-univalent functions. *Proc. Amer. Math. Soc.*, 18:63–68, 1967; **6.82**.
665. J.L. Lewis. Capacitary functions in convex rings. *Arch. Ration. Mech. Anal.*, 66:201–224, 1977; **3.9**.
666. J.L. Lewis. Note on the Nevanlinna proximity function. *Proc. Amer. Math. Soc.*, 69:129–134, 1978; **1.27**.
667. J.L. Lewis, J. Rossi, and A. Weitsman. On the growth of subharmonic functions along paths. *Ark. Mat.*, 22(1):109–119, 1984; **2.7, 2.10, 3.26**.
668. J.L. Lewis and A. Vogel. On p Laplace polynomial solutions. *J. Anal.*, 24(1):143–166, 2016; **7.49**.
669. J.L. Lewis and J.M.G. Wu. On conjectures of Arakelyan and Littlewood. *J. Analyse Math.*, 50:259–283, 1988; **1.6, 4.18, 7.40**.
670. M. Li and T. Sugawa. A note on successive coefficients of convex functions. *Comput. Methods Funct. Theory*, 17(2):179–193, 2017; **6.122**.
671. C.N. Linden. The modulus of polynomials with zeros on the unit circle. *Bull. London Math. Soc.*, 9(1):65–69, 1977; **4.1**.
672. F.S. Lisin. A problem of mean square approximation related to the study of generators in the algebra l^1. *Mat. Zametki*, 3:703–706, 1968; **5.25**.
673. J.E. Littlewood. Mathematical Notes (4): On a Theorem of Fatou. *J. London Math. Soc.*, S1-2(3):172, 1927; **3.19**.
674. J.E. Littlewood. On the coefficients of schlicht functions. *Quart. J. Math.*, Oxford(9):14–20, 1938; **6.7**.
675. J.E. Littlewood. *Lectures on the Theory of Functions*. Oxford University Press, 1944; **5.6, 5.16**.
676. J.E. Littlewood. On some conjectural inequalities, with applications to the theory of integral functions. *J. London Math. Soc.*, 27:387–393, 1952; **4.18**.
677. J.E. Littlewood. Some problems in real and complex analysis. Mimeographed (University of Wisconsin), 1965.
678. J.E. Littlewood and A.C. Offord. On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.*, 12(54):277–286, 1943; **7.5**.
679. A.E. Livingston. The coefficients of multivalent close-to-convex functions. *Proc. Amer. Math. Soc.*, 21:545–552, 1969; **6.97**.
680. A.J. Lohwater, G. Piranian, and W. Rudin. The derivative of a schlicht function. *Math. Scand.*, 3:103–106, 1955; **5.77, 6.30**.
681. R.R. London. A note on Hadamard's three circles theorem. *Bull. London Math. Soc.*, 9(2):182–185, 1977; **2.18**.
682. E. Løw. *Inner functions and boundary values in $H^\infty(\Omega)$ and $A(\Omega)$ in smoothly bounded pseudoconvex domains*. ProQuest LLC, Ann Arbor, MI, 1983; **8.22**. Doctoral Thesis–Princeton University.
683. K. Löwner. Untersuchungen über schlichte konforme Abbildungen des Einheitskreises. i. *Mathematische Annalen*, 89:103–121, 1923; **6.1**.
684. K.W. Lucas. A two-point modulus bound for areally mean p-valent functions. *J. Lond. Math. Soc.*, 43:487–494, 1968; **6.13′, 6.23**.
685. D.H. Luecking. Zero sequences for Bergman spaces. *Complex Variables, Theory Appl.*, 30(4):345–362, 1996; **8.6**.
686. T.J. Lyons. Finely holomorphic functions. *J. Funct. Anal.*, 37(1):1–18, 1980; **3.27**.

687. T.J. Lyons. A theorem in fine potential theory and applications to finely holomorphic functions. *J. Funct. Anal.*, 37(1):19–26, 1980; **3.27**.
688. M.Yu. Lyubich. The dynamics of rational transforms: the topological picture. *Russian Mathematical Surveys*, 41(4):43–117, August 1986; **2.79, 2.87, 2.88**.
689. M.Yu. Lyubich. Dynamics of quadratic polynomials. I, II. *Acta Math.*, 178(2):185–247, 247–297, 1997; **2.78**.
690. M.Yu. Lyubich. Dynamics of quadratic polynomials. III. Parapuzzle and SBR measures. *Astérisque*, (261):xii–xiii, 173–200, 2000; **2.88**. Géométrie complexe et systèmes dynamiques (Orsay, 1995).
691. R. Mañé. On the instability of Herman rings. *Invent. Math.*, 81(3):459–471, 1985; **2.79, 2.80, 2.81**.
692. T.H. MacGregor. Certain integrals of univalent and convex functions. *Math. Z.*, 103:48–54, 1968; **6.30**.
693. T.H. MacGregor. The univalence of a linear combination of convex mappings. *J. London Math. Soc.*, 44:210–212, 1969; **6.11**.
694. T.H. MacGregor. A subordination for convex functions of order α. *J. London Math. Soc. (2)*, 9:530–536, 1974/75; **6.41**.
695. A.J. Macintyre. Asymptotic paths of integral functions with gap power series. *Proc. London Math. Soc. (3)*, 2:286–296, 1952; **2.11**.
696. A.J. Macintyre and W.H.J. Fuchs. Inequalities for the logarithmic derivatives of a polynomial. *J. London Math. Soc.*, 15:162–168, 1940; **7.37**.
697. G.R. MacLane. Sequences of derivatives and normal families. *J. Analyse Math.*, 2:72–87, 1952; **7.79**.
698. G.R. MacLane. Asymptotic values of holomorphic functions. *Rice Univ. Studies*, 49(1):83, 1963; **3.5, 5.46, 5.47, 5.79**.
699. G.R. MacLane. Exceptional values of $f^{(n)}(z)$, asymptotic values of $f(z)$, and linearly accessible asymptotic values. In *Mathematical Essays Dedicated to A. J. Macintyre*, pages 271–288. Ohio Univ. Press, Athens, Ohio, 1970; **5.46**.
700. E. Makai and P. Turán. Hermite expansion and distribution of zeros of polynomials. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 8:157–163, 1963; **4.6**.
701. N.G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. Lond. Math. Soc. (3)*, 51:369–384, 1985; **7.44**.
702. N.G. Makarov. A note on integral means of the derivative in conformal mapping. *Proc. Amer. Math. Soc.*, 96(2):233–235, 1986; **6.30, 6.108**.
703. N.G. Makarov. On the radial behavior of Bloch functions. *Sov. Math., Dokl.*, 40(3):505–508, 1990; **5.34**.
704. J. Malmquist. Sur les fonctions à un nombre fini de branches satisfaisant à une équation différentielle du premier ordre. *Acta Math.*, 42(1):317–325, 1920; **1.35**.
705. J.H. Mantel. Conditions for generating a nonvanishing bounded analytic function. *Proc. Amer. Math. Soc.*, 66(1):62–64, 1977; **8.3**.
706. D.E. Marshall and C. Sundberg. Harmonic measure and radial projection. *Trans. Amer. Math. Soc.*, 316(1):81–95, 1989; **3.22**.
707. D.E. Marshall and C. Sundberg. Harmonic measure of curves in the disk. *J. Anal. Math.*, 70:175–224, 1996; **3.22**.
708. M.J. Martín, E.T. Sawyer, I. Uriarte-Tuero, and D. Vukotić. The Krzyż conjecture revisited. *Adv. Math.*, 273:716–745, 2015; **5.83**.
709. A. Marx. Untersuchungen über schlichte Abbildungen. *Math. Ann.*, 107(1):40–67, 1933; **6.16**.
710. M.Y. Mazalov. Criterion of uniform approximability by harmonic functions on compact sets in $\mathbb{R}^3$. *Proc. Steklov Inst. Math.*, 279:110–154, 2012; **9.13**.
711. O.C. McGehee, L. Pigno, and B. Smith. Hardy's inequality and the L^1 norm of exponential sums. *Ann. Math. (2)*, 113:613–618, 1981; **4.19**.
712. J.E. McMillan. On the boundary correspondence under conformal mapping. *Duke Math. J.*, 37:725–739, 1970; **6.91**.

713. C.T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1994; **2.79**.
714. W.S. McVoy and L.A. Rubel. Coherence of some rings of functions. *J. Functional Analysis*, 21(1):76–87, 1976; **5.24, 8.17**.
715. A. Meir and A. Sharma. On Ilyeff's conjecture. *Pacific J. Math.*, 31:459–467, 1969; **4.5**.
716. A.D. Melas. On the centered Hardy–Littlewood maximal operator. *Trans. Amer. Math. Soc.*, 354(8):3263–3273, 2002; **7.74**.
717. A.D. Melas. The best constant for the centered Hardy–Littlewood maximal inequality. *Ann. of Math. (2)*, 157(2):647–688, 2003; **7.74**.
718. A.D. Melas. On a covering problem related to the centered Hardy–Littlewood maximal inequality. *Ark. Mat.*, 41(2):341–361, 2003; **7.74**.
719. A. Meril and D.C. Struppa. Equivalence of Cauchy problems for entire and exponential type functions. *Bull. London Math. Soc.*, 17(5):469–473, 1985; **7.59, 7.60**.
720. N.G. Meyers and A. Elcrat. Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. *Duke Math. J.*, 42:121–136, 1975; **7.12**.
721. H. Mihaljević-Brandt and L. Rempe-Gillen. Absence of wandering domains for some real entire functions with bounded singular sets. *Math. Ann.*, 357(4):1577–1604, 2013; **2.87**.
722. J.G. Milcetich. On a coefficient conjecture of Brannan. *J. Math. Anal. Appl.*, 139(2):515–522, 1989; **5.44**.
723. J. Miles. On the counting function for the a-values of a meromorphic function. *Trans. Amer. Math. Soc.*, 147:203–222, 1970; **1.24**.
724. J. Miles. A factorization theorem in $H^1(U^3)$. *Proc. Amer. Math. Soc.*, 52:319–322, 1975; **8.16**.
725. J. Miles. On entire functions of infinite order with radially distributed zeros. *Pacific J. Math.*, 81(1):131–157, 1979; **1.12**.
726. J. Miles. Some examples of the dependence of the Nevanlinna deficiency upon the choice of origin. *Proc. London Math. Soc. (3)*, 47(1):145–176, 1983; **1.23**.
727. J. Miles. On a theorem of Hayman and Stewart. *Complex Variables, Theory and Application: An International Journal*, 37(1-4):425–455, 1998; **1.16**.
728. J. Miles and D.F. Shea. An extremal problem in value-distribution theory. *Quart. J. Math. Oxford Ser. (2)*, 24:377–383, 1973; **1.7**.
729. J. Miles and D. Townsend. Imaginary values of meromorphic functions. *Indiana Univ. Math. J.*, 27(3):491–503, 1978; **2.44**.
730. J. Miles and J. Williamson. A characterization of the exponential function. *J. London Math. Soc. (2)*, 33(1):110–116, 1986; **2.32**.
731. I.M. Milin. The area method in the theory of univalent functions. *Dokl. Akad. Nauk SSSR*, 154:264–267, 1964; **6.1**.
732. I.M. Milin. A bound for the coefficients of schlicht functions. *Dokl. Akad. Nauk SSSR*, 160:769–771, 1965; **6.2**.
733. I.M. Milin. Adjacent coefficients of univalent functions. *Dokl. Akad. Nauk SSSR*, 180:1294–1297, 1968; **6.38**.
734. I.M. Milin. *Odnolistnye funktsii i ortonormirovannye sistemy*. Izdat. "Nauka", Moscow, 1971; **6.37**.
735. I.M. Milin. Errata: "*Univalent functions and orthonormal systems*" (Trans. Math. Monographs, Vol. 49, Amer. Math. Soc., Providence, R.I., 1977). *American Mathematical Society, Providence, R.I.*, page 1, 1977; **6.37**.
736. I.M. Milin. *Univalent functions and orthonormal systems*. American Mathematical Society, Providence, R. I., 1977; **6.37**. Translated from the Russian, Translations of Mathematical Monographs, Vol. 49.
737. I.M. Milin. A problem for coefficients of p-multiply symmetric univalent functions. *Mat. Zametki*, 38(1):66–73, 169, 1985; **6.84**.
738. S.S. Miller and P.T. Mocanu. Second-order differential inequalities in the complex plane. *J. Math. Anal. Appl.*, 65(2):289–305, 1978; **5.61**.

739. J. Milnor. *Dynamics in one complex variable. 3rd ed.* Princeton, NJ: Princeton University Press, 2006.
740. C. Miranda. Sur un nouveau critère de normalité pour les familles de fonctions holomorphes. *Bull. Soc. Math. France*, 63:185–196, 1935.
741. M. Misiurewicz. On iterates of e^z. *Ergodic Theory Dynamical Systems*, 1(1):103–106, 1981; **2.22**.
742. G. Mo. An improvement upon Cartan's theorem. *Acta Math. Sin.*, 25:287–296, 1982; **4.7**.
743. J. Molluzzo. Doctoral thesis. *Yeshiva University*, 1972; **2.26**.
744. P. Montel. Leçons sur les familles normales de fonctions analytiques et leur applications. *Paris*, 1927.
745. E. Mues. Über eine Vermutung von Hayman. *Math. Z.*, 119:11–20, 1971; **1.18**.
746. E. Mues. Über ein Problem von Hayman. *Math. Z.*, 164(3):239–259, 1979; **1.19, 1.20**.
747. T. Murai. The value-distribution of lacunary series and a conjecture of Paley. *Ann. Inst. Fourier*, 31(1):135–156, 1981; **5.36**.
748. T. Murai. The deficiency of entire functions with Fejér gaps. *Ann. Inst. Fourier (Grenoble)*, 33(3):39–58, 1983; **2.13**.
749. T. Murai. The boundary behaviour of Hadamard lacunary series. *Nagoya Math. J.*, 89:65–76, 1983; **5.47**.
750. F. Nazarov. Growth of entire functions with sparse spectra. *Unpublished, posted on* http://users.math.msu.edu/users/fedja/prepr.html; **2.11**.
751. F. Nazarov, A. Nishry, and M. Sodin. Log-integrability of Rademacher Fourier series, with applications to random analytic functions. *St. Petersbg. Math. J.*, 25(3):467–494, 2014; **7.6**.
752. D.J. Needham and A.C. King. On meromorphic complex differential equations. *Dyn. Stab. Syst.*, 9(2):99–122, 1994; **1.44**.
753. Z. Nehari. On the coefficients of univalent functions. *Proc. Amer. Math. Soc.*, 8:291–293, 1957; **6.26**.
754. F. Nevanlinna. Über eine Klasse meromorpher Funktionen. *C. R. 7e Congr. Math. Scand., Oslo*, pages 81–3, 1929; **1.3**.
755. R. Nevanlinna. Über Riemannsche flächen mit endlich vielen Windungspunkten. *Acta Mathematica*, 58(1):295–373, Dec 1932; **1.3**.
756. R. Nevanlinna. *Le théorème de Picard–Borel et la théorie des fonctions méromorphes*. Chelsea Publishing Co., New York, 1974. Reprinting of the 1929 original.
757. R. Nevanlinna. *Eindeutige analytische Funktionen*. Springer-Verlag, Berlin-New York, 1974; **1.2, 3.4**. Zweite Auflage, Reprint, Die Grundlehren der mathematischen Wissenschaften, Band 46.
758. D.J. Newman. Interpolation in H^∞. *Trans. Amer. Math. Soc.*, 92:501–507, 1959.
759. D.J. Newman. Some remarks on the maximal ideal structure of H^∞. *Ann. of Math. (2)*, 70:438–445, 1959.
760. D.J. Newman. Generators in l_1. *Trans. Amer. Math. Soc.*, 113:393–396, 1964; **5.25**.
761. D.J. Newman and M. Slater. Waring's problem for the ring of polynomials. *J. Number Theory*, 11(4):477–487, 1979; **2.26**.
762. X.L. Nguyen and T. Watanabe. A characterization of fine domains for a certain class of Markov processes with applications to Brelot harmonic spaces. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 21:167–178, 1972; **3.2**.
763. P.J. Nicholls and L.R. Sons. Minimum modulus and zeros of functions in the unit disc. *Proc. Lond. Math. Soc. (3)*, 31:99–113, 1975; **5.37**.
764. P.J. Nicholls and L.R. Sons. Automorphic functions with gap power series. *Illinois J. Math.*, 25(3):383–389, 1981; **5.49**.
765. D.A. Nicks. Wandering domains in quasiregular dynamics. *Proc. Amer. Math. Soc.*, 141(4):1385–1392, 2013; **2.87**.
766. D.A. Nicks, P.J. Rippon, and G.M. Stallard. Baker's conjecture for functions with real zeros. *Proc. Lond. Math. Soc. (3)*, 117(1):100–124, 2018; **2.93**.
767. W. Nikliborc. Eine Bemerkung über die Volumpotentiale I. *Math. Z.*, 35(1):625–631, 1932; **3.29**.

768. J.W. Noonan. Boundary behavior of functions with bounded boundary rotation. *Journal of Mathematical Analysis and Applications*, 38(3):721 – 734, 1972; **6.62**.
769. J.W. Noonan and D.K. Thomas. On the Hankel determinants of areally mean p-valent functions. *Proc. London Math. Soc. (3)*, 25:503–524, 1972; **6.13′, 6.14′**.
770. J.M.F. O'Connell. Real parts of uniform algebras. *Pacific J. Math.*, 46(1):235–247, 1973; **8.26**.
771. A.M. Odlyzko and B. Poonen. Zeros of polynomials with 0, 1 coefficients. *Enseign. Math. (2)*, 39(3-4):317–348, 1993.
772. A.C. Offord. The distribution of the values of a random function in the unit disk. *Studia Math.*, 41:71–106, 1972; **7.6**.
773. K. Oikawa. Welding of polygons and the type of Riemann surfaces. *Kōdai Math. Sem. Rep.*, 13:37–52, 1961; **7.56**.
774. K. Okabe. On some open problems in the theory of quasiconformal mappings. I. The L^p-integrability of the partial derivatives of a quasiconformal mapping. *J. Hokkaido Univ. Ed. Sect. II A*, 28(2):65–69, 1977/78; **7.10**.
775. K. Okabe. On some open problems in the theory of quasiconformal mappings. II. Area distortion under quasiconformal mappings. *J. Hokkaido Univ. Ed. Sect. II A*, 30(1):5–12, 1979/80; **7.10**.
776. K. Okabe. On some open problems in the theory of quasiconformal mappings. III. Extension of quasiconformal mappings from n to $(n + 1)$ dimensions. *J. Hokkaido Univ. Ed. Sect. II A*, 30(2):109–122, 1979/80; **7.10**.
777. B. Øksendal. A Wiener test for integrals of Brownian motion and the existence of smooth curves in nowhere dense sets. *J. Funct. Anal.*, 36(1):72–87, 1980; **7.47**.
778. B. Øksendal. Brownian motion and sets of harmonic measure zero. *Pacific J. Math.*, 95(1):179–192, 1981; **7.44**.
779. P. Oliver. Doctoral thesis. *London University*, 1975; **2.47**.
780. M. Ortel and W. Schneider. Radial limits of functions of slow growth in the unit disk. *Math. Scand.*, 56(2):287–310, 1985; **5.77, 5.78**.
781. J.W. Osborne. The structure of spider's web fast escaping sets. *Bull. Lond. Math. Soc.*, 44(3):503–519, 2012; **2.87**.
782. C.F. Osgood. A number theoretic-differential equations approach to generalizing Nevanlinna theory. *Indian J. Math.*, 23:1–15, 1981.
783. C.F. Osgood. Sometimes effective Thue–Siegel–Roth–Schmidt–Nevanlinna bounds, or better. *J. Number Theory*, 21:347–389, 1985.
784. I.B. Oshkin. On a condition for the normality of families of holomorphic functions. *Uspekhi Mat. Nauk*, 37(2(224)):221–222, 1982; **5.12**.
785. R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84:1182–1238, 1978; **7.22**.
786. M. Overholt. Sets of uniqueness for univalent functions. *Canad. Math. Bull.*, 43(1):105–107, 2000; **6.83**.
787. K. Øyma. The Hayman–Wu constant. *Proc. Amer. Math. Soc.*, 119(1):337–338, 1993; **6.126**.
788. M. Ozawa. On the Bieberbach conjecture for the sixth coefficient. *Kōdai Math. Sem. Rep.*, 21:97–128, 1969; **6.1**.
789. X.C. Pang. Bloch's principle and normal criterion. *Sci. China Ser. A*, 32(7):782–791, 1989; **1.19, 5.13**.
790. X.C. Pang. On normal criterion of meromorphic functions. *Sci. China, Ser. A*, 33(5):521–527, 1990; **5.13**.
791. R.N. Pederson. A proof of the Bieberbach conjecture for the sixth coefficient. *Arch. Rational Mech. Anal.*, 31:331–351, 1968/1969; **6.1**.
792. R.N. Pederson and M. Schiffer. A proof of the Bieberbach conjecture for the fifth coefficient. *Arch. Rational Mech. Anal.*, 45:161–193, 1972; **6.1**.
793. V.V. Peller. Smooth Hankel operators and their applications (the ideals S_p Besov classes, and random processes). *Sov. Math., Dokl.*, 21:683–688, 1980; **8.15**.

794. R. Pérez-Marco. Sur une question de Dulac et Fatou. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(8):1045–1048, 1995; **2.87**.
795. R. Pérez-Marco. Fixed points and circle maps. *Acta Math.*, 179(2):243–294, 1997; **2.87**.
796. K.E. Petersen. *Brownian motion, Hardy spaces and bounded mean oscillation*, volume 28. Cambridge University Press, Cambridge. London Mathematical Society, London, 1977; **5.62**.
797. V.P. Petrenko. Investigation of the structure of the set of positive deviations of meromorphic functions. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 33:1330–1348, 1969; **1.17, 2.39**.
798. G. Petruska. A contribution to Bloch's theorem. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 12:39–42, 1969; **5.8**.
799. A. Pfluger. Zur Defektrelation ganzer Funktionen endlicher Ordnung. *Commentarii Mathematici Helvetici*, 19(1):91–104, Dec 1946; **1.5, 1.7, 1.8**.
800. A. Pfluger. Über ganze Funktionen ganzer Ordnung. *Comment. Math. Helv.*, 18:177–203, 1946; **7.16**.
801. E. Picard. Sur une propriété de certaines fonctions analogues aux fonctions algébriques. *C. R. Acad. Sci., Paris*, 89:1106–1108, 1880; **1.36**.
802. G. Piranian. An entire function of restricted growth. *Comment. Math. Helv.*, 33:322–324, 1959; **2.41**.
803. A. Plessner. Über das Verhalten analytischer Funktionen am Rande ihres Definitionsbereiches. *J. Reine Angew. Math.*, 158:219–227, 1927; **5.20, 5.57**.
804. G. Pólya. Über die Nullstellen sukzessiver Derivierten. *Math. Z.*, 12(1):36–60, 1922; **1.18**.
805. G. Pólya and I.J. Schoenberg. Remarks on de la Vallée Poussin means and convex conformal maps of the circle. *Pacific J. Math.*, 8:295–334, 1958; **6.9**.
806. G. Pólya and G. Szegő. *Problems and theorems in analysis II. Theory of functions, zeros, polynomials, determinants, number theory, geometry.* Berlin: Springer, reprint of the English translation by C. E. Billigheimer 1976 edition, 1998; **4.28**.
807. Ch. Pommerenke. On some problems by Erdös, Herzog and Piranian. *Michigan Math. J.*, 6:221–225, 1959; **4.23**.
808. Ch. Pommerenke. On the derivative of a polynomial. *Michigan Math. J*, 6:373–375, 1959; **4.8**.
809. Ch. Pommerenke. Einige Sätze über die Kapazität ebener Mengen. *Math. Ann.*, 141:143–152, 1960; **4.7**.
810. Ch. Pommerenke. Über die analytische Kapazität. *Arch. Math.*, 11:270–277, 1960; **7.75**.
811. Ch. Pommerenke. On metric properties of complex polynomials. *Michigan Math. J.*, 8:97–115, 1961; **4.9**.
812. Ch. Pommerenke. Über die Mittelwerte und Koeffizienten multivalenter Funktionen. *Math. Ann.*, 145:285–296, 1961/1962.
813. Ch. Pommerenke. Über einige Klassen meromorpher schlichter Funktionen. *Math. Z.*, 78:263–284, 1962; **6.10**.
814. Ch. Pommerenke. Lacunary power series and univalent functions. *Michigan Math. J.*, 11:219–223, 1964; **6.12**.
815. Ch. Pommerenke. Über die Faberschen Polynome schlichter Funktionen. *Math. Z.*, 85:197–208, 1964; **7.1**.
816. Ch. Pommerenke. On the coefficients and Hankel determinants of univalent functions. *J. London Math. Soc.*, 41:111–122, 1966; **6.13, 6.14, 6.13′**.
817. Ch. Pommerenke. On the Hankel determinants of univalent functions. *Mathematika*, 14:108–112, 1967; **6.14**.
818. Ch. Pommerenke. Relations between the coefficients of a univalent function. *Invent. Math.*, 3:1–15, 1967; **6.5, 6.13, 6.84**.
819. Ch. Pommerenke. Normal functions. Proc. NRL Conf. Classical Function Theory, Washington, D.C., 77-93, 1970; **2.53**.
820. Ch. Pommerenke. On Bloch functions. *J. London Math. Soc. (2)*, 2:689–695, 1970; **5.5**.
821. Ch. Pommerenke. On the growth of the coefficients of analytic functions. *J. London Math. Soc. (2)*, 5:624–628, 1972; **5.1**.

822. Ch. Pommerenke. On normal and automorphic functions. *Michigan Math. J.*, 21:193–202, 1974; **5.35**.
823. Ch. Pommerenke. Univalent functions. Studia Mathematica/Mathematische Lehrbücher. Band XXV. Göttingen: Vandenhoeck & Ruprecht, 1975; **6.44, 6.66, 6.80, 6.111**.
824. Ch. Pommerenke. On ergodic properties of inner functions. *Math. Ann.*, 256(1):43–50, 1981; **7.80**.
825. Ch. Pommerenke. On the integral means of the derivative of a univalent function. II. *Bull. London Math. Soc.*, 17(6):565–570, 1985; **6.94**.
826. Ch. Pommerenke. The growth of the derivative of a univalent function. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, volume 21 of *Math. Surveys Monogr.*, pages 143–152. Amer. Math. Soc., Providence, RI, 1986; **6.109**.
827. D.V. Prokhorov. The ratio of measures of boundary sets under a mapping of the disc onto a plane with slits. *Sov. Math.*, 29(4):66–69, 1985; **6.32**.
828. D.V. Prokhorov. Level curves of functions convex in the direction of an axis. *Math. Notes*, 44(3-4):767–769, 1988; **6.53**.
829. D.V. Prokhorov and J. Szynal. Directional convexity of level lines for functions convex in a given direction. *Proc. Amer. Math. Soc.*, 131(5):1453–1457, 2003; **6.53**.
830. H. Rademacher. On the Bloch–Landau constant. *Amer. J. Math.*, 65:387–390, 1943; **5.9**.
831. Q.I. Rahman and G. Schmeisser. *Analytic theory of polynomials.* Oxford: Oxford University Press, 2002.
832. R.K.V. Rajeswara. On a generalized corona problem. *J. Anal. Math.*, 18:277–278, 1967; **8.20, 9.3**.
833. N.V. Rao and D.F. Shea. Growth problems for subharmonic functions of finite order in space. *Trans. Amer. Math. Soc.*, 230:347–370, 1977; **3.13**.
834. E. Reich. An extremum problem for analytic functions with area norm. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 2:429–445, 1976; **8.9**.
835. E. Reich and K. Strebel. Extremal quasiconformal mappings with given boundary values. Contribut. to Analysis, Collect. of Papers dedicated to Lipman Bers 375-391 (1974), 1974; **8.9**.
836. L. Rempe. Doctoral thesis. *Ch.-Albrechts-Universität zu Kiel*, 2003; **2.86**.
837. L. Rempe. On a question of Herman, Baker and Rippon concerning Siegel disks. *Bull. London Math. Soc.*, 36(4):516–518, 2004; **2.86**.
838. S. Richter and C. Sundberg. A formula for the local Dirichlet integral. *Mich. Math. J.*, 38(3):355–379, 1991; **8.11**.
839. P.J. Rippon and G.M. Stallard. Boundaries of escaping Fatou components. *Proc. Amer. Math. Soc.*, 139(8):2807–2820, 2011; **2.93, 2.94**.
840. M.S. Robertson. A coefficient problem for functions regular in an annulus. *Canadian J. Math.*, 4:407–423, 1952; **6.97**.
841. W. Rogosinski. Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen. *Math. Z.*, 35(1):93–121, 1932; **6.21**.
842. S. Rohde. On the theorem of Hayman and Wu. *Proc. Amer. Math. Soc.*, 130(2):387–394, 2002; **6.126**.
843. M. Roitman. On roots of polynomials and of their derivatives. *J. London Math. Soc. (2)*, 27(2):248–256, 1983; **4.29**.
844. J.-P. Rosay. Sur la non-factorisation des éléments de l'espace de Hardy $H^1(U^2)$. *Illinois J. Math.*, 19:479–482, 1975; **8.16**.
845. J.-P. Rosay and W. Rudin. A maximum principle for sums of subharmonic functions, and the convexity of level sets. *Mich. Math. J.*, 36(1):95–111, 1989; **3.9**.
846. P.C. Rosenbloom. On sequences of polynomials, especially sections of power series. *Stanford University*, Doctoral Thesis, 1949; **2.59**.
847. W.T. Ross and H.S. Shapiro. *Generalized analytic continuation*, volume 25. Providence, RI: American Mathematical Society (AMS), 2002; **8.7**.
848. J. Rossi. Second order differential equations with transcendental coefficients. *Proc. Amer. Math. Soc.*, 97(1):61–66, 1986; **2.70**.

849. J. Rossi. A sharp result concerning cercles de remplissage. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(1):179–185, 1995; **1.31**.
850. J. Rossi and S. Wang. The radial oscillation of solutions to ODE's in the complex domain. *Proc. Edinburgh Math. Soc. (2)*, 39(3):473–483, 1996; **2.71**.
851. J. Rossi and A. Weitsman. The growth of entire and harmonic functions along asymptotic paths. *Comment. Math. Helv.*, 60(1):1–16, 1985; **2.9**.
852. W.C. Royster. Coefficient problems for functions regular in an ellipse. *Duke Math. J.*, 26:361–371, 1959; **6.54**.
853. W.C. Royster. A Poisson integral formula for the ellipse and some applications. *Proc. Amer. Math. Soc.*, 15:661–670, 1964; **7.2**.
854. W.C. Royster. On the univalence of a certain integral. *Michigan Math. J.*, 12:385–387, 1965; **6.15**.
855. D. Răducanu and P. Zaprawa. Second Hankel determinant for close-to-convex functions. *C. R., Math., Acad. Sci. Paris*, 355(10):1063–1071, 2017; **6.121**.
856. L.A. Rubel. Unbounded analytic functions and their derivatives on plane domains. *Bull. Inst. Math., Acad. Sin.*, 12:363–377, 1984; **1.45**.
857. L.A. Rubel, A.L. Shields, and B.A. Taylor. Mergelyan sets and the modulus of continuity of analytic functions. *J. Approximation Theory*, 15(1):23–40, 1975; **5.28**.
858. Z. Rubinstein. On a problem of Ilyeff. *Pacific J. Math.*, 26:159–161, 1968; **4.5**.
859. W. Rudin. The radial variation of analytic functions. *Duke Math. J.*, 22:235–242, 1955; **5.75**.
860. W. Rudin. Function theory in polydiscs. Mathematics Lecture Note Series. New York-Amsterdam: W.A. Benjamin, Inc., 188 p. (1969), 1969; **8.16**.
861. W. Rudin. Inner function images of radii. *Math. Proc. Cambridge Philos. Soc.*, 85(2):357–360, 1979; **5.76**.
862. S. Ruscheweyh. Über die Faltung schlichter Funktionen. *Math. Z.*, 128:85–92, 1972; **6.112**.
863. S. Ruscheweyh. Duality for Hadamard products with applications to extremal problems for functions regular in the unit disc. *Trans. Amer. Math. Soc.*, 210:63–74, 1975; **6.112**.
864. S. Ruscheweyh. Neighborhoods of univalent functions. *Proc. Amer. Math. Soc.*, 81(4):521–527, 1981; **6.111**.
865. S. Ruscheweyh and L.C. Salinas. On the preservation of direction-convexity and the Goodman–Saff conjecture. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 14(1):63–73, 1989; **6.53**.
866. S. Ruscheweyh and L.C. Salinas. On Brannan's coefficient conjecture and applications. *Glasg. Math. J.*, 49(1):45–52, 2007; **5.44**.
867. S. Ruscheweyh and T. Sheil-Small. Corrigendum: "Hadamard products of schlicht functions and the Pólya–Schoenberg conjecture" (Comment. Math. Helv. **48** (1973), 119–135). *Comment. Math. Helv.*, 48:194, 1973; **5.23, 6.9**.
868. S. Ruscheweyh and T. Sheil-Small. Hadamard products of schlicht functions and the Pólya–Schoenberg conjecture. *Comment. Math. Helv.*, 48:119–135, 1973; **5.23, 6.9, 6.45**.
869. F.B. Ryan. The set of asymptotic values of a bounded holomorphic function. *Duke Math. J.*, 33:477–484, 1966; **5.45**.
870. J. Ryan. Hypercomplex algebras, hypercomplex analysis and conformal invariance. *Compositio Math.*, 61(1):61–80, 1987; **7.69**.
871. A. Sadi. Some types of regularity for the Dirichlet problem. *Nagoya Math. J.*, 126:103–124, 1992; **3.33**.
872. E.B. Saff and T. Sheil-Small. Coefficient and integral mean estimates for algebraic and trigonometric polynomials with restricted zeros. *J. London Math. Soc. (2)*, 9:16–22, 1974/75; **4.2, 4.20**.
873. E.B. Saff and R.S. Varga. Zero-free parabolic regions for sequences of polynomials. *SIAM J. Math. Anal.*, 7(3):344–357, 1976; **2.59**.
874. M. Sakai. Null quadrature domains. *J. Analyse Math.*, 40:144–154 (1982), 1981; **3.28**.
875. R. Salem and A. Zygmund. Some properties of trigonometric series whose terms have random signs. *Acta Math.*, 91:245–301, 1954; **4.17, 5.54**.
876. P. Saminathan, A. Vasudevarao, and M. Vuorinen. Region of variability for exponentially convex univalent functions. *Complex Anal. Oper. Theory*, 5(3):955–966, 2011; **6.125**.

877. D. Sarason. Generalized interpolation in H^∞. *Trans. Amer. Math. Soc.*, 127:179–203, 1967.
878. A. Sauer. How to detect Hayman directions. *Comput. Methods Funct. Theory*, 9(1):57–64, 2009; **1.31**.
879. E.T. Sawyer. *Function theory: interpolation and corona problems*, volume 25 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2009.
880. I.J. Schark. Maximal ideals in an algebra of bounded analytic functions. *J. Math. Mech.*, 10:735–746, 1961. "I. J. Schark" is a pseudonym for I. Kaplansky, J. Wermer, S. Kakutani, R. C. Buck, H. Royden, A. Gleason, R. Arens and K. Hoffman.
881. M. Schiffer. Sur un problème d'extrémum de la représentation conforme. *Bull. Soc. Math. France*, 66:48–55, 1938; **6.4**.
882. G. Schmeisser. On Ilieff's conjecture. *Math. Z.*, 156(2):165–173, 1977; **4.30**.
883. G. Schmeisser. The conjectures of Sendov and Smale. In *Approximation theory. A volume dedicated to Blagovest Sendov*, pages 353–369. Sofia: DARBA, 2002; **4.5**.
884. G. Schmieder. Funktionen mit vorgeschriebenen Null- und Verzweigungsstellen auf Riemannschen Flächen. *Arch. Math. (Basel)*, 37(1):72–77, 1981; **1.37**.
885. W. Schwick. A note on Zalcman's lemma. *New Zealand J. Math.*, 29(1):71–72, 2000.
886. K. Seip. *Interpolation and sampling in spaces of analytic functions*, volume 33 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2004.
887. H.L. Selberg. Über einige Eigen schaften bei der Werteverteilung der meromorphen Funktionen endlicher Ordnung. *Avh. Norske Vid.-Akad. Oslo 1(N.S)*, No. 7:1–17, 1928; **1.34**.
888. B. Sendov. New conjectures in the Hausdorff geometry of polynomials. *East J. Approx.*, 16(2):179–192, 2010; **4.5**.
889. T. Shah. Goluzin's number $(3 - \sqrt{5})/2$ is the radius of superiority in subordination. *Sci. Record (N.S.)*, 1:219–222, 1957; **5.38**.
890. H.S. Shapiro. A uniform boundedness principle concerning inner functions. *J. Analyse Math.*, 50:183–188, 1988; **7.81**.
891. H.S. Shapiro. *The Schwarz function and its generalization to higher dimensions.* New York: John Wiley & Sons Ltd., 1992; **3.30**.
892. H.S. Shapiro and A.L. Shields. On some interpolation problems for analytic functions. *Amer. J. Math.*, 83:513–532, 1961.
893. D.F. Shea and L.R. Sons. Value distribution theory for meromorphic functions of slow growth in the disk. *Houston J. Math.*, 12(2):249–266, 1986; **1.39, 1.40**.
894. T. Sheil-Small. On the convolution of analytic functions. *J. Reine Angew. Math.*, 258:137–152, 1973; **6.1, 6.28, 6.39**.
895. T. Sheil-Small. Applications of the Hadamard product. In *Aspects of contemporary complex analysis (Proc. NATO Adv. Study Inst., Univ. Durham, Durham, 1979)*, pages 515–523. Academic Press, London-New York, 1980; **2.65, 5.60, 8.1**.
896. T. Sheil-Small. On the Fourier series of a finitely described convex curve and a conjecture of H. S. Shapiro. *Math. Proc. Cambridge Philos. Soc.*, 98(3):513–527, 1985; **5.41**.
897. T. Sheil-Small. On the zeros of the derivatives of real entire functions and Wiman's conjecture. *Ann. Math. (2)*, 129(1):179–193, 1989; **2.64, 4.28**.
898. T. Sheil-Small. Constants for planar harmonic mappings. *J. London Math. Soc. (2)*, 42(2):237–248, 1990; **6.104, 6.107**.
899. T. Sheil-Small. *Complex polynomials*, volume 75. Cambridge: Cambridge University Press, 2002.
900. T. Sheil-Small and E.M. Silvia. Neighborhoods of analytic functions. *J. Analyse Math.*, 52:210–240, 1989; **6.111, 6.112, 6.113**.
901. A.L. Shields. Cyclic vectors in some spaces of analytic functions. *Proc. Roy. Irish Acad. Sect. A*, 74:293–296, 1974; **8.11**. Spectral Theory Symposium (Trinity College, Dublin, 1974).
902. M. Shishikura. The boundary of the Mandelbrot set has Hausdorff dimension two. *Astérisque*, (222):7, 389–405, 1994; **2.88**. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).

903. M. Shishikura. The Hausdorff dimension of the boundary of the Mandelbrot set and Julia sets. *Ann. of Math. (2)*, 147(2):225–267, 1998; **2.88**.
904. H. Silverman and E.M. Silvia. Subclasses of univalent functions starlike with respect to a boundary point. *Houston J. Math.*, 16(2):289–300, 1990; **5.44**.
905. S. Sivasubramanian, R. Sivakumar, S. Kanas, and S.-A. Kim. Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent functions. *Ann. Polon. Math.*, 113(3):295–304, 2015; **6.82**.
906. H. Skoda. Zéros des fonctions de la classe de Nevanlinna dans les ouverts strictement pseudoconvexes. *C. R. Acad. Sci. Paris Sér. A-B*, 280(24):Aii, A1677–A1680, 1975; **7.67**.
907. H. Skoda. Valeurs au bord pour les solutions de l'opérateur d'', et caractérisation des zéros des fonctions de la classe de Nevanlinna. *Bull. Soc. Math. France*, 104(3):225–299, 1976; **7.67**.
908. Z. Slodkowski. Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.*, 111(2):347–355, 1991; **7.82**.
909. S. Smale. The fundamental theorem of algebra and complexity theory. *Bull. Amer. Math. Soc., New Ser.*, 4:1–36, 1981; **4.5**.
910. M.A. Snipes and L.A. Ward. Realizing step functions as harmonic measure distributions of planar domains. *Ann. Acad. Sci. Fenn. Math.*, 30(2):353–360, 2005; **6.116**.
911. A.Yu. Solynin. Extremal problems on conformal moduli and estimates for harmonic measures. *J. Anal. Math.*, 74:1–49, 1998; **3.22**.
912. G. Somorjai. On asymptotic functions. *J. London Math. Soc. (2)*, 21(2):297–303, 1980; **2.3**.
913. L.R. Sons. Value distribution and power series with moderate gaps. *Michigan Math. J.*, 13:425–433, 1966; **5.37**.
914. L.R. Sons. Value distribution for unbounded functions in the unit disk. *Complex Variables Theory Appl.*, 7(4):337–341, 1987; **5.79**.
915. D.C. Spencer. On mean one-valent functions. *Ann. of Math. (2)*, 42:614–633, 1941; **6.2′, 6.25**.
916. G. Springer. The coefficient problem for schlicht mappings of the exterior of the unit circle. *Trans. Amer. Math. Soc.*, 70:421–450, 1951; **6.4, 6.6**.
917. E.M. Stein. Some results in harmonic analysis in $\mathbb{R}^n$, for $n \to \infty$. *Bull. Amer. Math. Soc. (N.S.)*, 9(1):71–73, 1983; **7.74**.
918. E.M. Stein and J.-O. Strömberg. Behavior of maximal functions in $\mathbb{R}^n$ for large n. *Ark. Mat.*, 21(2):259–269, 1983; **7.74**.
919. N. Steinmetz. Über das Anwachsen der Lösungen homogener algebraischer Differentialgleichungen zweiter Ordnung. *Manuscripta Math.*, 32(3-4):303–308, 1980; **1.35**.
920. N. Steinmetz. Über die eindeutigen Lösungen einer homogenen algebraischen Differentialgleichung zweiter Ordnung. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 7(2):177–188, 1982;**1.35**.
921. N. Steinmetz. Eine Verallgemeinerung des zweiten Nevanlinnaschen Hauptsatzes. *J. Reine Angew. Math.*, 368:134–141, 1986.
922. N. Steinmetz. On the zeros of $(f^{(p)} + a_{p-1}f^{(p-1)} + \cdots + a_0 f)f$. *Analysis*, 7(3-4):375–389, 1987; **1.42**.
923. N. Steinmetz. Linear differential equations with exceptional fundamental sets. II. *Proc. Amer. Math. Soc.*, 117(2):355–358, 1993; **2.72**.
924. K. Stephenson. Omitted values of singular inner functions. *Michigan Math. J.*, 25(1):91–100, 1978; **8.19**.
925. K. Stephenson. Analytic functions and hypergroups of function pairs. *Indiana Univ. Math. J.*, 31(6):843–884, 1982; **7.65**.
926. A. Stray. Minimal interpolation by Blaschke products. *J. London Math. Soc. (2)*, 32(3):488–496, 1985; **7.41, 9.2**.
927. A. Stray. Minimal interpolation by Blaschke products. II. *Bull. London Math. Soc.*, 20(4):329–332, 1988; **9.2**.
928. E.W. Stredulinsky. Higher integrability from reverse Hölder inequalities. *Indiana Univ. Math. J.*, 29(3):407–413, 1980; **7.12**.

929. D. Styer and D.J. Wright. Results on bi-univalent functions. *Proc. Amer. Math. Soc.*, 82(2):243–248, 1981; **6.82**.
930. D. Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou–Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985; **2.62, 2.63**.
931. F. Sunyer Balaguer. Sur la substitution d'une valeur exceptionnelle par une propriété lacunaire. *Acta Math.*, 87:17–31, 1952; **5.37**.
932. R. Szász. On the Brannan's conjecture. arXiv:1710.09153, 2017; **5.44**.
933. M.N.M. Talpur. Doctoral thesis. *London University*, 1967; **3.2**.
934. M.N.M. Talpur. A subharmonic analogue of Iversen's theorem. *Proc. Lond. Math. Soc. (3)*, 31:129–148, 1975.
935. M.N.M. Talpur. On the growth of subharmonic functions on asymptotic paths. *Proc. London Math. Soc. (3)*, 32(2):193–198, 1976; **2.6**.
936. D.K. Thomas. On the coefficients of meromorphic univalent functions. *Proc. Amer. Math. Soc.*, 47:161–166, 1975; **6.50**.
937. D.K. Thomas. On the logarithmic coefficients of close to convex functions. *Proc. Amer. Math. Soc.*, 144(4):1681–1687, 2016; **6.119**.
938. D.K. Thomas, N. Tuneski, and V. Allu. *Univalent functions. A primer*, volume 69. Berlin: De Gruyter, 2018; **6.48**.
939. R. Tijdeman. *On the distribution of the values of certain functions*. Doctoral dissertation, University of Amsterdam. 1969; **2.29**.
940. R. Tijdeman. On the number of zeros of general exponential polynomials. *Nederl. Akad. Wet., Proc., Ser. A*, 74:1–7, 1971; **2.29**.
941. E.C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations. Part I*. Second Edition. Clarendon Press, Oxford, 1962; **2.71**.
942. E.C. Titchmarsh. The theory of functions. 2nd ed. London: Oxford University Press. X, 454 p., 1975; **2.65, 5.60, 7.34, 8.1**.
943. V.G. Tkachev. On the non-vanishing property for real analytic solutions of the p-Laplace equation. *Proc. Amer. Math. Soc.*, 144(6):2375–2382, 2016; **7.49**.
944. K. Tohge. On a problem of Hinkkanen about Hadamard products. *Kodai Math. J.*, 13(1):101–120, 1990; **2.65**.
945. V.A. Tolokonnikov. Estimates in the Carleson corona theorem, ideals of the algebra H^∞, a problem of Sz.-Nagy. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 113:178–198, 267, 1981; **8.20**. Investigations on linear operators and the theory of functions, XI.
946. X. Tolsa. Painlevé's problem and the semiadditivity of analytic capacity. *Acta Math.*, 190(1):105–149, 2003; **7.39, 7.75**.
947. X. Tolsa. The semiadditivity of continuous analytic capacity and the inner boundary conjecture. *Amer. J. Math.*, 126(3):523–567, 2004; **7.76**.
948. X. Tolsa. Bilipschitz maps, analytic capacity, and the Cauchy integral. *Ann. Math. (2)*, 162(3):1243–1304, 2005; **7.75**.
949. X. Tolsa. *Analytic capacity, the Cauchy transform, and non-homogeneous Calderón–Zygmund theory*, volume 307. Cham: Birkhäuser/Springer, 2014; **7.75**.
950. S. Toppila. Some remarks on exceptional values at Julia lines. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 456:20, 1970; **2.4**.
951. S. Toppila. On the counting function for the a-values of a meromorphic function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 2:565–572, 1976; **1.16, 1.25**.
952. S. Toppila. On Nevanlinna's characteristic functions of entire functions and their derivatives. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 3(1):131–134, 1977; **1.21, 2.69**.
953. S. Toppila. On the length of asymptotic paths of entire functions of order zero. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 5(1):13–15, 1980; **2.7**.
954. V. Totik. The Gauss–Lucas theorem in an asymptotic sense. *Bull. London Math. Soc.*, 48(5):848–854, 2016; **4.36**.
955. S. Treil. Estimates in the corona theorem and ideals of H^∞: a problem of T. Wolff. *J. Anal. Math.*, 87:481–495, 2002; **8.2, 8.20, 9.15**.

956. S. Treil and B.D. Wick. Analytic projections, corona problem and geometry of holomorphic vector bundles. *J. Amer. Math. Soc.*, 22(1):55–76, 2009.
957. P. Tukia and J. Väisälä. Quasiconformal extension from dimension n to $n+1$. *Ann. of Math. (2)*, 115(2):331–348, 1982; **7.11**.
958. Y. Tumura. Recherches sur la distribution des valeurs des fonctions analytiques. *Jap. J. Math.*, 18:797–876, 1943; **1.27**.
959. P. Turán. *Eine neue Methode in der Analysis und deren Anwendungen*. Akadémiai Kiadó, Budapest, 1953; **7.3**.
960. P. Turán. Problem 153. *Mat. Lapok*, 17:215, 1966; **5.8**.
961. T.F. Tyler. Maximum curves and isolated points of entire functions. *Proc. Amer. Math. Soc.*, 128(9):2561–2568, 2000; **2.49**.
962. J. Väisälä. *Lectures on n-dimensional quasiconformal mappings*, volume 229. Berlin-Heidelberg-New York: Springer-Verlag, 1971; **7.57**.
963. G. Valiron. Sur les valeurs asymptotiques de quelques fonctions méromorphes. *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 49(3):415–421, Oct 1925; **1.2**.
964. G. Valiron. Sur une classe de fonctions entières admettant deux directions de Borel d'ordre divergent. *Compositio Math.*, 1:193–206, 1935; **2.39**.
965. G. Valiron. Valeurs exceptionnelles et valeurs déficientes des fonctions méromorphes. *C. R. Acad. Sci. Paris*, 225:556–558, 1947; **1.23**.
966. G. Valiron. *Fonctions analytiques*. Presses Universitaires de France, Paris, 1954; **1.35**.
967. N.T. Varopoulos. BMO functions and the $\overline{\partial}$-equation. *Pacific J. Math.*, 71(1):221–273, 1977; **9.1**.
968. W.A. Veech. A converse to Gauss' theorem. *Bull. Amer. Math. Soc.*, 78:444–446, 1972; **7.30**.
969. W.A. Veech. A zero-one law for a class of random walks and a converse to Gauss' mean value theorem. *Ann. of Math. (2)*, 97:189–216, 1973; **7.30**.
970. M. von Renteln. Ideals in the Nevanlinna class N. *Mitt. Math. Sem. Giessen*, (Heft 123):57–65, 1977; **8.3**.
971. A.A. Šaginjan. The uniform and tangent harmonic approximation of continuous functions on arbitrary sets. *Mat. Zametki*, 9:131–142, 1971; **9.10**.
972. M. Vuorinen. On the existence of angular limits of n-dimensional quasiconformal mappings. *Ark. Mat.*, 18(2):157–180, 1980; **6.92**.
973. M. Vuorinen. Queries: No. 249. *Notices Amer. Math. Soc.*, 28(7):607, 1981; **7.66**.
974. G. Wagner. On a problem of Erdös in Diophantine approximation. *Bull. London Math. Soc.*, 12(2):81–88, 1980; **4.1**.
975. A. Wahlund. Über einen Zusammenhang zwischen dem Maximalbetrage der ganzen Funktion und seiner unteren Grenze nach dem Jensensche Theoreme. *Ark. Mat.*, 21A(23):34pp, 1929; **1.17**.
976. B.L. Walden and L.A. Ward. Asymptotic behaviour of distributions of harmonic measure for planar domains. *Complex Variables Theory Appl.*, 46(2):157–177, 2001; **6.116**.
977. P.L. Walker. On rearranging maximal functions in $\mathbb{R}^n$. *Proc. Edinburgh Math. Soc. (2)*, 19(4):363–369, 1974/75; **7.53**.
978. J.L. Walsh. *Interpolation and approximation by rational functions in the complex domain*. Fourth edition. American Mathematical Society Colloquium Publications, Vol. XX. American Mathematical Society, Providence, R.I., 1965.
979. Y. Wang and M. Fang. Picard values and normal families of meromorphic functions with multiple zeros. *Acta Math. Sin., New Ser.*, 14(1):17–26, 1998; **5.13**.
980. M. Watson. On functions that are bivalent in the unit circle. *J. Analyse Math.*, 17:383–409, 1966; **6.97**.
981. M. Weiss and G. Weiss. On the Picard property of lacunary power series. *Studia Math.*, 22:221–245, 1962/1963; **5.36**.
982. A. Weitsman. Meromorphic functions with maximal deficiency sum and a conjecture of F. Nevanlinna. *Acta Mathematica*, 123(1):115, Dec 1969; **1.3**.
983. A. Weitsman. A theorem on Nevanlinna deficiencies. *Acta Math.*, 128(1-2):41–52, 1972; **1.14**.

984. A. Weitsman. A symmetry property of the Poincaré metric. *Bull. London Math. Soc.*, 11(3):295–299, 1979; **5.15**.
985. A. Weitsman. Symmetrization and the Poincaré metric. *Ann. of Math. (2)*, 124(1):159–169, 1986; **5.15**.
986. J. Wermer. Non-rectifiable simple closed curve. *Amer. Math. Monthly*, 64:372, 1957; **7.18**.
987. J. Wermer. *Banach algebras and several complex variables*. Springer-Verlag, New York-Heidelberg, second edition, 1976; **9.8**. Graduate Texts in Mathematics, No. 35.
988. J.M. Whittaker. Interpolatory function theory. (Cambridge Tracts in Math. a. Math. Phys. 33) London: Cambridge Univ. Press. 107 p. (1935), 1935; **2.43**.
989. G.T. Whyburn. *Topological analysis*. Second, revised edition. Princeton Mathematical Series, No. 23. Princeton University Press, Princeton, N.J., 1964; **3.20**.
990. D.R. Wilken and J. Feng. A remark on convex and starlike functions. *J. London Math. Soc. (2)*, 21(2):287–290, 1980; **6.41**.
991. A.S. Wilmshurst. The valence of harmonic polynomials. *Proc. Amer. Math. Soc.*, 126(7):2077–2081, 1998; **4.35**.
992. J. Winkler. Zum Verzweigungsindex der a-Stellen ganzer und meromorpher Funktionen. *Math. Z.*, 113:353–362, 1970; **2.42**.
993. J. Winkler. Über den Verzweigungsindex bei ganzen Funktionen. *Manuscripta Math.*, 4:135–148, 1971; **2.42**.
994. H. Wittich. *Neuere Untersuchungen über eindeutige analytische Funktionen*. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 8. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955; **1.23**.
995. H. Wittich. Zur Theorie linearer Differentialgleichungen im Komplexen. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 379:19, 1966; **2.28**.
996. H. Wittich. *Neuere Untersuchungen über eindeutige analytische Funktionen*. Zweite, korrigierte Auflage. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 8. Springer-Verlag, Berlin-New York, 1968; **1.23**.
997. F. Wolf. An extension of the Phragmén–Lindelöf theorem. *J. London Math. Soc.*, 14:208–216, 1939; **3.6**.
998. T.H. Wolff. Counterexamples to two variants of the Helson–Szegö theorem. *J. Anal. Math.*, 88:41–62, 2002; **5.58**.
999. J.M.G. Wu. Comparisons of kernel functions, boundary Harnack principle and relative Fatou theorem on Lipschitz domains. *Ann. Inst. Fourier (Grenoble)*, 28(4):147–167, vi, 1978; **3.17**.
1000. J.M.G. Wu. Length of paths for subharmonic functions. *J. London Math. Soc. (2)*, 32(3):497–505, 1985; **3.26**.
1001. K. Yamanoi. On the truncated small function theorem in Nevanlinna theory. *Int. J. Math.*, 17(4):417–440, 2006.
1002. K. Yamanoi. Zeros of higher derivatives of meromorphic functions in the complex plane. *Proc. Lond. Math. Soc. (3)*, 106(4):703–780, 2013.
1003. S. Yamashita. On the John constant. *Math. Z.*, 161(2):185–188, 1978; **6.81**.
1004. C.C. Yang. A problem on polynomials. *Rev. Roumaine Math. Pures Appl.*, 22(5):595–598, 1977; **4.29**.
1005. L. Yang. Meromorphic functions and their derivatives. *J. London Math. Soc. (2)*, 25(2):288–296, 1982; **1.31**.
1006. L. Yang and G.H. Zhang. Deficient values and asymptotic values of entire functions. *Sci. Sinica*, (Special Issue II on Math.):190–203, 1979; **5.12**.
1007. L.-Z. Yang. Meromorphic functions and also their first two derivatives have the same zeros. *Ann. Acad. Sci. Fenn. Math.*, 30(1):205–218, 2005; **2.65**.
1008. Z. Ye. On successive coefficients of odd univalent functions. *Proc. Amer. Math. Soc.*, 133(11):3355–3360, 2005; **6.37**.
1009. Z. Ye. The logarithmic coefficients of close-to-convex functions. *Bull. Inst. Math., Acad. Sin. (N.S.)*, 3(3):445–452, 2008; **6.120**.
1010. J.-C. Yoccoz. Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne. *Ann. Sci. École Norm. Sup. (4)*, 17(3):333–359, 1984; **2.90**.

1011. J.-C. Yoccoz. Théorème de Siegel, nombres de Bruno et polynômes quadratiques. *Astérisque*, (231):3–88, 1995; **2.83**. Petits diviseurs en dimension 1.
1012. H. Yoshida. A boundedness criterion for subharmonic functions. *J. Lond. Math. Soc., II. Ser.*, 24:148–160, 1981; **3.6**.
1013. L. Zalcman. Analyticity and the Pompeiu problem. *Arch. Rational Mech. Anal.*, 47:237–254, 1972; **7.28**.
1014. L. Zalcman. Mean values and differential equations. *Israel J. Math.*, 14:339–352, 1973; **7.28**.
1015. L. Zalcman. A heuristic principle in complex function theory. *Amer. Math. Monthly*, 82(8):813–817, 1975.
1016. L. Zalcman. A tale of three theorems. *Amer. Math. Monthly*, 123(7):643–656, 2016; **1.19**.
1017. R. Zeinstra. Zeros and regular growth of Laplace transforms along curves. *J. Reine Angew. Math.*, 424:1–15, 1992; **7.17**.
1018. F. Zhao, Z. Fu, and S. Lu. Endpoint estimates for n-dimensional Hardy operators and their commutators. *Sci. China Math.*, 55(10):1977–1990, 2012; **7.74**.
1019. J. Zheng. Unbounded domains of normality of entire functions of small growth. *Math. Proc. Camb. Philos. Soc.*, 128(2):355–361, 2000; **2.93**.
1020. J.-H. Zheng. Singularities and wandering domains in iteration of meromorphic functions. *Illinois J. Math.*, 44(3):520–530, 2000; **2.87**.
1021. J.H. Zhu. Hayman direction of meromorphic functions. *Kodai Math. J.*, 18(1):37–43, 1995; **1.31**.
1022. I.V. Zhuravlev. Some sufficient conditions for the quasi-conformal extension of analytic functions. *Sov. Math., Dokl.*, 19:1549–1552, 1978; **6.57**.
1023. V.V. Zimogljad. The order of growth of entire transcendental solutions of second order algebraic differential equations. *Mat. Sb. (N.S.)*, 85(127):286–302, 1971; **1.35**.