TORI AND CANONICAL MODELS OF SHIMURA VARIETIES

Tori and canonical models

We study Shimura data when the group G is a torus.

Tori. Suppose $G = T$ is a torus. The set X is then a single homomorphism h_X. There is a perfect pairing

$$<,>: X^*(T)_{\overline{\mathbb{Q}}} \otimes X_*(T)_{\overline{\mathbb{Q}}} \to \mathbb{Z}$$

where $X_*(T) = Hom(\mathbb{G}_m, T)$, $X^*(T) = Hom(T, \mathbb{G}_m)$. The duality is defined as follows: if $\mu \in X_*(T)$ and $\chi \in X^*(T)$ then $\chi \circ \mu \in Hom(\mathbb{G}_m, \mathbb{G}_m)$ is a character of the form $t \mapsto t^r$, and we let $r =< \chi, \mu >$. This duality is compatible with the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action on both sides.

We introduce the cocharacter $\mu : \mathbb{G}_m \to \mathbb{S}_{\mathbb{C}}$ with the property that, for any $h : \mathbb{S} \to GL(V)$, $h \circ \mu(z)$ acts as z^{-p} on $V^{p,q}$; equivalently, if $e_1 = (1,0)$ and $e_2 = (0,1)$ are a basis of $X^*(\mathbb{S})_{\mathbb{C}}$, then $< e_1, \mu >= 1$, $< e_2, \mu >= 0$. Let $\mu_h = h \circ \mu$; then we can reconstruct h from μ_h via $h(z) = \mu_h(z) \cdot \mu_h(\bar{z})$. In this way a Shimura datum (T, h_X) is determined by the cocharacter $\mu_X = \mu_{h_X} \in X_*(T)_{\mathbb{C}}$. Moreover, any μ_X will do, since the two hypotheses are vacuous.

Some μ_X are especially interesting, however. Suppose $\mathcal{K}$ is a CM field, i.e. a totally imaginary quadratic extension of a totally real field F. Say $d = [F : \mathbb{Q}]$. Let $T = R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_m$. A basis of $X^*(T)_{\overline{\mathbb{Q}}} = X^*(T)_{\mathbb{C}}$ is given by $\Sigma_{\mathcal{K}} = Hom(\mathcal{K}, \overline{\mathbb{Q}}) = Hom(\mathcal{K}, \mathbb{C})$. Indeed,

$$T(\overline{\mathbb{Q}}) = (\mathcal{K} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times = (\oplus_{\sigma \in \Sigma_{\mathcal{K}}} \overline{\mathbb{Q}}_\sigma)^\times,$$

where $\overline{\mathbb{Q}}_\sigma$ is just $\overline{\mathbb{Q}}$ indexed by σ. Projection on each factor comes from an algebraic character which we denote σ. Let $i_\sigma, \sigma \in \Sigma_{\mathcal{K}}$ denote the dual basis of $X_*(T)_{\overline{\mathbb{Q}}}$.

Exercise. *Show that these are genuine algebraic characters, and that $X^*(T)_{\overline{\mathbb{Q}}}$ is canonically isomorphic to $\mathbb{Z}^{Hom(\mathcal{K},\overline{\mathbb{Q}})}$ with the canonical action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the latter.*

A *CM type* for $\mathcal{K}$ is a set $\Phi \subset \Sigma_{\mathcal{K}}$ of d elements such that $\Sigma_{\mathcal{K}} = \Phi \coprod c\Phi$. Equivalently, restriction to F defines a bijection $\Phi \leftrightarrow \Sigma_F = Hom(F, \overline{\mathbb{Q}}) = Hom(F, \mathbb{R})$. To a CM type Σ we can associate a cocharacter $\mu_\Phi = \sum_{\sigma \in \Phi} i_\sigma$. This defines a unique $\mathbb{R}$-homomorphism $h_\Phi : \mathbb{S} \to T_{\mathbb{R}}$, such that $\mathfrak{h}_\Phi \cdot \mu = \mu_\Phi$. Concretely, the CM type Φ defines an identification

$$T(\mathbb{R}) = \mathcal{K} \otimes_{\mathbb{Q}} \mathbb{R})^\times \xrightarrow{\sim} (\mathbb{C}^\times)^\Phi$$

and the map $h_\Phi : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \to (\mathbb{C}^\times)^\Phi$ is the diagonal map $z \mapsto (z, \dots, z)$. When $\mathcal{K}$ is imaginary quadratic and Φ is a single embedding then this notation is consistent with our previous notation in that case.

Note that h_Φ takes values in the subtorus $H_{\mathcal{K}} \subset T$ defined by the Cartesian diagram

$$\begin{array}{ccc} H_{CK} & \longrightarrow & R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}} \\ \downarrow & & \downarrow{\scriptstyle N_{\mathcal{K}/F}} \\ \mathbb{G}_{m,\mathbb{Q}} & \xrightarrow{\iota} & R_{\mathcal{K}/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}} \end{array}$$

Here $N_{\mathcal{K}/F}$ is the map on tori defined by the obvious map on characters

$$\sigma \mapsto \sum_{\sigma' \mid_F = \sigma} \sigma'$$

and ι is the inclusion defined by the map $\sigma \mapsto Id$ on characters, where Id is the identity character of $\mathbb{G}_m$. Concretely, $H_{\mathcal{K}}(\mathbb{Q})$ is the subgroup of elements of $y \in \mathcal{K}^\times$ such that $yy^c \in \mathbb{Q}^\times$; it is the Serre torus attached to $\mathcal{K}$. Thus there is a Shimura datum $(H_{\mathcal{K}}, h_\Phi)$.

The simplest non-trivial Shimura variety is $(\mathbb{G}_m, N)$, where $N(z) = z\bar{z} \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$.

The reflex field

To each $h \in X$ we have assigned a homomorphism $\mu = \mu_h : \mathbb{G}_m \rightarrow G_{\mathbb{C}}$. Up to conjugation (by $G(\mathbb{R})$), the homomorphism μ depends only on X. Thus *a fortiori* the $G(\mathbb{C})$-conjugacy class $M(X)$ of μ depends only on X.

Let $T \subset G$ be a maximal torus; for example, we can let T be a maximal torus of K_h for some $h \in X$. Here we view G, T, K_h as abstract algebraic groups, over $\mathbb{C}$, for instance. It is a theorem that any two maximal tori of G are conjugate under $G(\mathbb{C})$; so are any two Borel subgroups. It follows that, if $\mu \in M(X)$ then, up to conjugation, we can assume $\mu : \mathbb{G}_m \rightarrow T$, i.e. $\mu \in X_*(T)_{\mathbb{C}}$. It is also a theorem that G has a maximal torus (many maximal tori, in fact) defined over $\mathbb{Q}$; however, it is not the case that any two maximal tori are conjugate under $G(\mathbb{Q})$. For instance, if $G = GL(2)$, then the diagonal matrices form a maximal torus isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$; but if $L/\mathbb{Q}$ is any quadratic extension then, as we have already seen, choice of a $\mathbb{Q}$-isomorphism $L \xrightarrow{\sim} \mathbb{Q}^2$ determines an injective homomorphism $R_{L/\mathbb{Q}}\mathbb{G}_{m,L} \rightarrow GL(2)$ whose image is a maximal torus. For distinct L, these tori cannot be conjugate under $G(\mathbb{Q})$, because they are not even isomorphic as algebraic groups over $\mathbb{Q}$; but they do become conjugate (and in particular isomorphic) over $\overline{\mathbb{Q}}$.

So we can assume T defined over $\mathbb{Q}$. Then the conjugacy class of μ equals the G-conjugacy class of some $\mu_0 \in X_*(T)$. This can be determined more precisely using the Dynkin diagram; a choice of Borel subgroup B containing T – it is not generally possible to find B defined over $\mathbb{Q}$ – determines a unique set of positive roots $\Phi^+ \in X^*(T)_{\mathbb{C}}$, and there is a *unique* conjugate μ_0 of μ in $X_*(T)_{\mathbb{C}}$ such that $< \alpha, \mu_0 >= \alpha \circ \mu_0 \geq 0$ for all $\alpha \in \Phi^+$. (This follows from the theory of Weyl chambers.) Now since T is defined over $\mathbb{Q}$, every element of $X_*(T)_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$, and in particular this positive μ_0 is defined over some number field E. It follows that the conjugacy class $M(X)$ is defined over a number field $E(G,X)$, and it is not difficult to see that $E(G,X) \subset E = E(T, \mu_0)$, the field of definition of μ_0 for the chosen rational maximal torus T. The field $E(G,X)$ is called the *reflex field* of the Shimura datum (G,X).

The main theorem of the subject is the existence and uniqueness of a *canonical model* of the Shimura variety $Sh(G,X) = \varprojlim_K {}_KSh(G,X)$ over the reflex field $E(G,X)$ with respect to which the action of $G(\mathbf{A}_f)$ is rational. More precisely, one first defines canonical models when G is a torus, explicitly, via class field theory. The defining property of canonical models is that, when $(H,Y)\rightarrow(G,X)$ is a morphism of Shimura data, the natural map $Sh(H,Y)\rightarrow Sh(G,X)$ is rational over $E(H,Y)$. Deligne, following Shimura, showed that this property, applied to varying (H,Y) with H a torus, suffices to determine the canonical model of $Sh(G,X)$ uniquely. The existence for most classical groups was established by Shimura using the Shimura-Taniyama reciprocity law for abelian varieties with complex multiplication. The general case involves different considerations, namely Margulis' characterization of arithmetic subgroups of real groups.

Examples of reflex fields.

The Siegel modular variety. Consider the pair $(G,X) = (GSp(2n), \mathfrak{S}_n^{\pm})$. Let $V = \mathbb{Q}^{2n}$ be the standard symplectic representation space of $GSp(2n)$. The elements $h \in X$ correspond to Hodge structures of type $(0,-1)+(-1,0)$ on $V_{\mathbb{R}}$. I leave it as an (important) exercise to verify that these Hodge structures are *symplectic*, in the sense that the standard symplectic pairing on V

$$<,>: V \otimes V \rightarrow \mathbb{Q}$$

defines a morphism of Hodge structures where $\mathbb{Q}$ is endowed with the pure Hodge structure of of type $(-1,-1)$. (Hint: One calculates the map $\nu \circ h$ for any $h \in X$, where $\nu : G\rightarrow GL(1)$ is the similitude map). It follows that, for any $h \in X$, the subspaces $V_h^{0,-1}$ and $V_h^{-1,0}$, defined as in the case $n = 1$, are Lagrangian with respect to $<,>$ – i.e.

$$< V_h^{0,-1}, V_h^{0,-1} >=< V_h^{-1,0}, V_h^{-1,0} >= 0$$

and hence that $<,>: V_h^{0,-1} \otimes V_h^{-1,0} \rightarrow \mathbb{C}$ is a perfect pairing.

The homomorphism μ_h determines a splitting $V_{\mathbb{C}} = V_h^{0,-1} \oplus V_h^{-1,0}$ where $V_h^{0,-1}$, resp. $V_h^{-1,0}$, is the eigenspace of $\mu_h(t)$ on which $t \in \mathbb{C}^{\times}$ acts trivially, resp. as t. On the other hand, any splitting of V as a sum of Lagrangian subspaces $V = W_0 \oplus W_1$ defines a homomorphism $\mu : \mathbb{G}_m \rightarrow G$ such that W_i becomes the t^i-eigenspace of $\mu(t)$. Moreover, any two Lagrangian splittings of V are conjugate under G. Thus the $G(\mathbb{C})$-conjugacy class of μ_h, $h \in X$, can be identified with the set of Lagrangian splittings of V. Obviously there is a Lagrangian splitting defined over $\mathbb{Q}$, and the action of G on the set of such splittings is $\mathbb{Q}$-rational. Thus $E(G,X) = \mathbb{Q}$. By a similar argument, we see that the reflex field in the Hilbert-Siegel modular case is $\mathbb{Q}$.

Unitary Shimura varieties. The unitary case is more complicated, and depends on the signatures. We revert to the notation previously used, and consider the set $\Sigma_{\mathcal{K}}$ of $\overline{\mathbb{Q}}$ embeddings of $\mathcal{K}$. We define a function $P : \Sigma_{\mathcal{K}} \rightarrow \mathbb{Z}$ as follows: for $\sigma \in \Phi$, set $P(\sigma) = p_{\sigma}$; for $\tau = c\sigma \in c\Phi$, let $P(\tau) = q_{\sigma}$. The Galois group $\Gamma = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of $\mathbb{Z}$-valued functions on $\Sigma_{\mathcal{K}}$.

Exercise. *The reflex field $E(G,X)$ in the unitary case is the fixed field of the stablizer in Γ of the signature function P.*

Our goal for the remainder of this section is to define the canonical model when $G=T$ is a torus, and to determine some of its elementary properties.

The case of tori

We consider a Shimura datum (T,h) with T a torus. The conjugacy class of $\mu=\mu_h$ is just the point $\mu_h \in X_*(T)_{\overline{\mathbb{Q}}}$. The Galois group $\Gamma = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $X_*(T)_{\overline{\mathbb{Q}}} = Hom(X^*(T)_{\overline{\mathbb{Q}}}, \mathbb{Z})$. Let $\Gamma_\mu \subset \Gamma$ denote the stablizer of μ_h. Then $E(T,h)$ is the fixed field of Γ_μ in $\overline{\mathbb{Q}}$.

The reflex norm.

We write $X^*(T)$ for $X^*(T)_{\overline{\mathbb{Q}}}$. Let L be a number field, and let $\Gamma_L = Gal(\overline{\mathbb{Q}}/L)$. Recall that the map $T \rightarrow X^*(T)_{\overline{\mathbb{Q}}}$ defines an anti-equivalence of categories

$$\{ \text{ tori over } L\} \leftrightarrow Mod(\Gamma_L) = \{\mathbb{Z}[\Gamma_L]- \text{ modules free of finite type over } \mathbb{Z}\}.$$

In this way we obtain functorial operations on tori. The restriction map $Mod(\Gamma) \rightarrow Mod(\Gamma_L)$ defines a map (base change)

$$\mathbb{Q}-tori \rightarrow L-tori; \ T \mapsto T_L$$

where $X^*(T_L) = X^*(T)$ viewed as a Γ_L-module. In the opposite direction, we have induction

$$\begin{aligned} I_{L/\mathbb{Q}} : Mod(\Gamma_L) \rightarrow Mod(\Gamma); I_{L/\mathbb{Q}}(M) &= Hom_{\mathbb{Z}[\Gamma_L]}(\mathbb{Z}[\Gamma], M) \\ &= \{f : \Gamma \rightarrow M \mid f(hg) = h^{-1}f(g), g \in \Gamma, h \in \Gamma_L\}. \end{aligned}$$

Here we have to be careful about the sign of the action: if $f \in I_{L/\mathbb{Q}}(M)$ and $g \in \Gamma$ then $g \cdot f(g') = f(g'g)$. If $M = X_*(S)$, for some L-torus S, then the $\mathbb{Z}[\Gamma]$-module $I_{L/\mathbb{Q}}(M)$ is just $R_{L/\mathbb{Q}}S$.

Combining these two operations, we can define $R_{L/\mathbb{Q}}(T_L)$ for any $\mathbb{Q}$-torus T, so that $X^*(R_{L/\mathbb{Q}}(T_L)) = I_{L/\mathbb{Q}}X^*(T)$. On the other hand, for any $M \in Mod(\Gamma)$ and $N \in Mod(\Gamma_L)$ there is a canonical isomorphism (Frobenius reciprocity)

$$Hom_{\mathbb{Z}[\Gamma_L]}(M,N) \overset{R}{\xrightarrow{\sim}} Hom_{\mathbb{Z}[\Gamma]}(M, I_{L/\mathbb{Q}}N); R(\phi)(m)(g) = \phi(g \cdot m).$$

Applied to $M = N = X^*(T)$, the image $R(id)$ of the identity map is a homomorphism $X^*(T) \rightarrow X^*(R_{L/\mathbb{Q}}(T_L))$ which by duality defines a homomorphism

$$N_{L/\mathbb{Q}} : R_{L/\mathbb{Q}}(T_L) \rightarrow T.$$

Exercise. *Show that when $T = \mathbb{G}_m$, $N_{L/\mathbb{Q}}$ is just the usual norm map.*

It will be easier to work with the functor $X_*(T) = Hom(\mathbb{G}_m, T)$, which also defines a covariant equivalence of categories

$$\{ \text{ tori over } L\} \leftrightarrow Mod(\Gamma_L) = \{\mathbb{Z}[\Gamma_L]- \text{ modules free of finite type over } \mathbb{Z}\}.$$

as above. Restriction of scalars then corresponds to the functor (coinduction):

$$C_{L/\mathbb{Q}} : Mod(\Gamma_L) \to Mod(\Gamma); C_{L/\mathbb{Q}}(M) = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[G_L]} M).$$

When $N \in Mod(\Gamma)$, $M \in Mod(\Gamma_L)$, Frobenius reciprocity then takes the form of an isomorphism

$$Hom_{\mathbb{Z}[\Gamma_L]}(M,N) \overset{R}{\underset{\sim}{\longrightarrow}} Hom_{\mathbb{Z}[\Gamma]}(C_{L/\mathbb{Q}}M, N)$$

defined by linearity.

We return to the reflex field $E(T,h)$, which we denote E. This is the field of definition of the cocharacter $\mu = \mu_h$, which means precisely that there is a character $\mu : \mathbb{G}_{m,E} \to T_E$. Applying the functorial restriction of scalars map, we obtain a homomorphism of $\mathbb{Q}$-tori

$$R_{E/\mathbb{Q}}(\mu) : R_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to R_{E/\mathbb{Q}}T_E.$$

Composing with the norm map, we obtain a homomorphism of $\mathbb{Q}$-tori, the *reflex norm*

$$r(T,h) = N_{E/\mathbb{Q}} \circ R_{E/\mathbb{Q}}(\mu) : R_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to T.$$

This can be applied to points over $\mathbb{Q}$-algebras, and we obtain a continuous map $E^\times_{\mathbf{A}} \to T(\mathbf{A})$ which sends $E^\times$ to $T(\mathbb{Q})$ and $E^\times(\mathbb{R})$ to $T(\mathbb{R})$, hence defines a map

$$r(T,h) : E^\times_{\mathbf{A}}/\overline{E^\times}E^\times(\mathbb{R})^0 \to T(\mathbf{A})/\overline{T(\mathbb{Q})}T(\mathbb{R}) = Sh(T,h)(\mathbb{C}).$$

Note that in this case $Sh(T,h)(\mathbb{C})$ is a profinite group, the quotient indicated on the right-hand side. (The finiteness of the quotients ${}_KSh(T,h)$ is ultimately a consequence of the finiteness of class number, which we do not pursue here.) Our goal is to define a canonical model of $Sh(T,h)$ over $E(T,h)$, i.e. a profinite scheme over $E(T,h)$ whose $\overline{\mathbb{Q}}$-valued points are given canonically by $Sh(T,h)(\mathbb{C})$. But a (pro)-finite scheme over E is just a (pro)-finite set of points with (continuous) action of Γ_E. We decree that the action of Γ_E on $Sh(T,h)(\mathbb{C})$ factor through Γ^{ab}_E. By class field theory, there is a canonical isomorphism

$$rec : \Gamma^{ab}_E \overset{\sim}{\longrightarrow} E^\times_{\mathbf{A}}/\overline{E^\times}E^\times(\mathbb{R})^0,$$

the normalization taking a uniformizer at v (modulo an appropriate compact open subgroup) on the right to geometric Frobenius at v (on finite abelian extensions of E unramified at v).

Definition. *The canonical model of $Sh(T,h)(\mathbb{C})$ is the model over $E = E(T,h)$ such that all points of $Sh(T,h)(\mathbb{C})$ are defined over the maximal abelian extension E^{ab} of E, and such that*

$$\gamma(x) = (r(T,h) \circ rec(\gamma) \cdot x), \gamma \in \Gamma^{ab}_E, x \in Sh(T,h)(\mathbb{C}).$$

Such a model obviously exists and is unique. We now work out the meaning of this definition in the essential case.

Reciprocity for CM types. We now let $(\mathcal{K}, \Phi)$ be a CM type, and consider the Shimura datum $(T, h) = (H_{\mathcal{K}}, h_\Phi)$. The reflex field $E(T, h)$ is the fixed field of

$$\Gamma_E = \{g \in \Gamma \mid g \cdot \Phi = \Phi\}$$

with respect to the natural action, via composition, of Γ on $Hom(\mathcal{K}, \overline{\mathbb{Q}})$. In order to calculate the map $r(T, h)$, we need to describe in more detail the structure of CM fields. This is somewhat intricate inasmuch as $\mathcal{K}/\mathbb{Q}$ is not necessarily a Galois extension.

We may write $\mathcal{K} = F(\sqrt{\alpha})$ where F is the totally real subfield and $\alpha \in F$; then the fact that $\mathcal{K}$ is totally imaginary implies $\sigma(\alpha) < 0$ for every $\sigma \in \Sigma_F$. The embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ defines a natural complex conjugation, denoted c, in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. For any $\tau \in \Sigma_{\mathcal{K}}$, $\tau(\sqrt{\alpha})$ is purely imaginary, hence $c(\tau(\sqrt{\alpha})) = -\tau(\sqrt{\alpha})$. It follows that c generates $Gal(\tau(\mathcal{K})/\tau(F))$ for every $\tau \in \Sigma_{\mathcal{K}}$; in particular $g\mathcal{K} \subset \overline{\mathbb{Q}}$ is a CM field for every $g \in \Gamma$. Let $\tilde{\mathcal{K}}$ (resp. $\tilde{F}$) be the Galois closure of $\mathcal{K}$ (resp. F) in $\overline{\mathbb{Q}}$, and let $\tilde{\mathcal{K}}^+$ be the fixed field of c in $\tilde{\mathcal{K}}$.

Lemma. *$\tilde{\mathcal{K}}$ and $E(T, h)$ are CM fields.*

Proof. We show that the composite of two CM fields is a CM field. Say $K_1 = F_1(\sqrt{\alpha})$, $K_2 = F_2(\sqrt{\beta})$, with F_i totally real and α, β totally negative. Let $F = F_1 \cdot F_2$, a totally real field. Then

$$K_1 \cdot K_2 = F(\sqrt{\alpha}, \sqrt{\beta}) = F(\sqrt{\alpha \cdot \beta}, \sqrt{\alpha})$$

which is obviously a totally imaginary quadratic extension of the totally real field $F(\sqrt{\alpha \cdot \beta})$.

By the above, this suffices to prove that $\tilde{\mathcal{K}}$ is a CM field. Hence every subfield of $\tilde{\mathcal{K}}$ is either totally real or a CM field. This applies in particular to $E(T, h)$, and it is left as an exercise to show that $E(T, h)$ cannot be totally real.

For any number field L we write $T(L) = R_{L/\mathbb{Q}}\mathbb{G}_{m,L}$. We write $E = E(T, h)$. The map $r(T, h)$ can be viewed as a hommorphism from $T(L)$ to $T(\mathcal{K})$, taking values in $H_{\mathcal{K}}$. In making this homomorphism explicit, it will be easier to ignore $H_{\mathcal{K}}$. Let Id denote the identity (trivial) module of rank 1. At the level of cocharacters, $r(T, h)$ then factors:

$$X_*(T(E)) \rightarrow X_*(R_{E/\mathbb{Q}}T(\mathcal{K})_E) \rightarrow X_*(T(\mathcal{K})) = \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id.$$

or more compactly

$$\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} Id \rightarrow \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} \otimes \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id \rightarrow \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id.$$

Exercise. *Show that*

(i) the first map $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} Id \rightarrow \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} \otimes \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id$ is given by

$$\epsilon \otimes 1 \mapsto \sum_{\sigma \in \Phi} \epsilon \otimes \sigma \otimes 1.$$

(ii) the norm map $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} \otimes \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id \rightarrow \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_{\mathcal{K}}]} Id$ is given by

$$\gamma \otimes \gamma' \otimes 1 \mapsto \gamma \cdot \gamma' \otimes 1.$$

The module $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}[\Gamma_E]} Id$ has as basis the elements of Γ/Γ_E, naturally in bijection with the embeddings ϵ of E in $\overline{\mathbb{Q}}$ (or $\mathbb{C}$). Letting $\Phi(\epsilon) = \epsilon \cdot \Phi \subset \Gamma$, the conclusion is that, on the indicated basis of cocharacters,

$$X_*(r(T, h))(\epsilon) = \sum_{\sigma' \in \Phi(\epsilon)} \sigma' \otimes 1.$$

Exercise. *Show that on $\overline{\mathbb{Q}}$-points, the corresponding map*

$$T(E)(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^{\times})^{\Sigma_E} \to T(\mathcal{K})(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^{\times})^{\Sigma_{\mathcal{K}}}$$

is given by

$$(z_\epsilon)_{\epsilon\in\Sigma_E} \mapsto (y_\gamma)_{\gamma\in\Sigma_{\mathcal{K}}}$$

where

$$y_\gamma = \prod_{\epsilon^{-1}\in\Phi\cdot\gamma^{-1}} z_\epsilon.$$

Note that the product in the final formula makes sense for for elements $\epsilon \in \Gamma/\Gamma_E$, or equivalently $\epsilon^{-1} \in \Gamma_E\backslash\Gamma$, since both $\Gamma_E\backslash\Gamma$ and $\Phi \cdot \gamma^{-1}$ (for any γ) are subsets of Γ left-invariant under Γ_E; similarly, $\gamma^{-1} \in \Gamma_{\mathcal{K}}\backslash\Gamma$ whereas $\Phi \subset \Gamma/\Gamma_{\mathcal{K}}$, so the product $\Phi \cdot \gamma^{-1}$ is well-defined. This is the origin of the complicated formulas in Shimura-Taniyama.

This can be written in a different way. We can choose $\gamma = 1$, the initial (given) embedding of $\mathcal{K}$ in $\overline{\mathbb{Q}}$. Then we can write $y = r(T,h)(z)$ the above formula is gien by

$$r(T,h)(z) = \prod_{\epsilon\in\Phi^{-1}} z_\epsilon.$$

One verifies that this is independent of the choice of $\gamma = 1$ in the appropriate sense (as γ varies, the different components y_γ are the different embeddings of the same element of $\mathcal{K}$, identified with y_1. Of course $\Phi^{-1} \subset \Gamma_{\mathcal{K}}\backslash\Gamma$ is right-invariant under Γ_E, hence can be interpreted as a subset of $G/\Gamma_E = \Sigma_E$. The set Ψ of restrictions to E of the elements of Φ^{-1}, pulled back to Γ, is (exercise) a CM type of E, called the reflex (or dual) type. The formula for y defines what Lang calls the *type norm.*

Definition of a canonical model

In the following definitions, (G,X) is a Shimura datum and E is an extension of $E(G,X)$ contained in $\mathbb{C}$.

Definition. *A model of $Sh(G,X)$ over E is a variety (scheme) M over E (Spec(E)) with a continuous E-rational action of $G(\mathbf{A}_f)$ and a $G(\mathbf{A}_f)$-equivariant isomorphism $M \times_E \mathbb{C} \xrightarrow{\sim} Sh(G,X)$.*

This can of course be expressed in terms of the finite level quotients ${}_KSh(G,X)$: for each K there is a scheme ${}_KSh$, and there are morphisms connecting them for different K such that the isomorphisms after base change to $\mathbb{C}$ fit into commutative diagrams. This is made explicit in Deligne's Bourbaki article.

Exercise (difficult). *Reformulate this definition in terms of the connected components of ${}_KSh(G,X)$.*

When $G = T$ is a torus, the construction above provides a canonical model of $M(T,h)$ which is the model considered in the following definition.

Definition. *A model $Sh_E(G,X)$ of $Sh(G,X)$ over E is called weakly canonical if, for every injective morphism $u : (T,h) \to (G,X)$ of Shimura data, the corresponding morphism $Sh(T,h) \to Sh(G,X)$ is defined over the composite $E(T,h) \cdot E \subset \mathbb{C}$. The model is canonical if $E = E(G,X)$.*

Fix a level subgroup $K \subset G(\mathbf{A}_f)$. The image of $Sh(T,h)$ in ${}_K Sh(G,X)$ is a finite set, since $K \subset T(\mathbf{A}_f)$ is open. The hypothesis is that all points in the image are defined over $E(T,h) \cdot E$, and that the action of the corresponding subgroup of Γ on this finite set is given by the reciprocity map $r(T,h)$.

Our goal for the remainder of the course is to prove the following facts, not necessarily in this order:

(1) When $(G,X) = (GSp(2n), \mathfrak{S}_n^{\pm})$, there is a canonical identification of $Sh(G,X)$ with the moduli space of principally polarized abelian varieties with level structure N for all N.
(2) The reciprocity law for the canonical model in the Siegel case is exactly the Shimura-Taniyama reciprocity law for abelian varieties with complex multiplication.
(3) One wants to prove the Shimura-Taniyama reciprocity law.
(4) The following theorem

Theorem. *Let $(H,Y) \subset (G,X)$ be an embedding of Shimura data, and suppose $Sh(G,X)$ has a canonical model. Then $Sh(H,Y)$ has a canonical model.*

This provides a substantial list of examples, including the Hilbert-Siegel and unitary cases considered above.