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The Novikov conjecture for groups with finite asymptotic dimension

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Dedicated to Professor Ronald G. Douglas on the occasion of his sixtieth birthday

1. Introduction

In this paper we shall prove the coarse Baum-Connes conjecture for proper metric spaces with finite asymptotic dimension. Combining this result with a certain descent principle we obtain the following application to the Novikov conjecture on homotopy invariance of higher signatures.

THEOREM 1.1. *Let Γ be a finitely generated group whose classifying space $B\Gamma$ has the homotopy type of a finite* CW*-complex. If Γ has finite asymptotic dimension as a metric space with a word-length metric, then the Novikov conjecture holds for Γ.*

Recall that the asymptotic dimension is a coarse geometric analogue of the covering dimension in topology (page 28, [14]). More precisely, the asymptotic dimension for a metric space is the smallest integer n such that for any $r > 0$, there exists a uniformly bounded cover $C = \{U_i\}_{i \in I}$ of the metric space for which the r-multiplicity of C is at most $n+1$; i.e., no ball of radius r in the metric space intersects more than $n+1$ members of C [14]. The class of finitely generated discrete groups with finite asymptotic dimension is hereditary in the sense that if a finitely generated group has finite asymptotic dimension as metric space with a word-length metric, then its finitely generated subgroups also have finite asymptotic dimension as metric spaces with word-length metrics (cf. Section 6). This, together with a result of Gromov in [14], implies that finitely generated subgroups of Gromov's hyperbolic groups have finite asymptotic dimension. Currently no example of a finitely generated group with infinite asymptotic dimension and finite classifying space is known. It should also be noted that two different definitions of asymptotic dimension

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are introduced by Gromov and it is not known if the two definitions are equal (see page 28–32 of [14] for more details).

The validity of the Novikov conjecture has been established, by a variety of techniques, for many groups [3], [5], [7], [8], [9], [10], [23], [24], [27], most notably for fundamental groups of complete manifolds with nonpositive sectional curvature, closed discrete subgroups of Lie groups with finite connected components, and Gromov's hyperbolic groups. Our approach to the Novikov conjecture is coarse geometric in spirit and is based on the descent principle that the coarse Baum-Connes conjecture for a finitely generated group Γ (as a metric space with a word length metric) implies the strong Novikov conjecture for Γ if the classifying space $B\Gamma$ has the homotopy type of a finite CW-complex [33] (as pointed out by the referee, this descent principle is a C^*-algebra version of various descent principles previously known to topologists [4], [5], [11]). Our main new tool is controlled operator K-theory. Controlled operator K-theory is a refinement of ordinary K-theory, incorporating additional norm and propagation control.

Our result on the coarse Baum-Connes conjecture also implies the Gromov-Lawson-Rosenberg conjecture on the nonexistence of Riemannian metrics with positive scalar curvature for compact $K(\pi, 1)$-manifolds when the fundamental group π has finite asymptotic dimension as a metric space with a word-length metric and Gromov's zero-in-the-spectrum conjecture for uniformly contractible Riemannian manifolds with finite asymptotic dimension. Recall that Gromov's zero-in-the-spectrum conjecture says that the spectrum of the Laplacian operator acting on the space of L^2-forms of a uniformly contractible Riemannian manifold with bounded geometry contains zero.

We shall also construct a proper metric space with infinite asymptotic dimension for which the coarse Baum-Connes conjecture fails. This indicates that our result on the coarse Baum-Connes conjecture is best possible in some sense.

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2. The descent principle

In this section we shall briefly discuss the coarse Baum-Connes conjecture and the descent principle. We remark that C^*-algebras in this paper are complex C^*-algebras and K-theory is 2-periodic complex topological K-theory.

Let X be a proper metric space. Recall that a metric space is called proper if every closed ball in the metric space is compact. An X-module is a separable Hilbert space equipped with a $*$-representation of $C_0(X)$, the algebra of all complex-valued continuous functions on X which vanish at infinity. An X-module is called nondegenerate if the $*$-representation of $C_0(X)$ is nondegenerate. An X-module is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

Definition 2.1. Let H_X and H_Y be X and Y-modules, respectively. The *support of a bounded operator* from H_X to H_Y is defined to be the *complement* (in $X \times Y$) *of the set of all points* $(x, y) \in X \times Y$ for which there exist functions $f \in C_0(X), g \in C_0(Y)$ such that $gTf = 0$, and $f(x) \neq 0$, $g(y) \neq 0$.

Definition 2.2. Let H_X be a X-module; let T be a bounded linear operator acting on H_X.
(1) The *propagation* of T is defined to be: $\sup\{d(x,y)\colon (x,y) \in \mathrm{Supp}(T)\}$;
(2) T is said to be *locally compact* if fT and Tf are compact for all $f \in C_0(X)$.

Definition 2.3 ([31]). Let H_X be a standard nondegenerate X-module. $C^*(X)$ is defined to be the C^*-*algebra* generated by all locally compact operators acting on H_X with finite propagations.

$C^*(X)$ does not depend on the choice of the standard nondegenerate X-module H_X up to unnatural isomorphism and its K-theory groups are independent of the choice of H_X up to natural isomorphism [31].

A Borel map f from a proper metric space X to another metric space Y is called coarse if (1) f is proper, i.e., the inverse image of any bounded set is bounded; (2) for every $R > 0$, there exists $R' > 0$ such that $d(f(x), f(y)) \leq R'$ for all $x, y \in X$ satisfying $d(x, y) \leq R$.

LEMMA 2.4. *Let f be as above; let H_X and H_Y be respectively standard nondegenerate X and Y-modules. For any $\varepsilon > 0$, there exists an isometry V_f from H_X to H_Y such that*

$$\mathrm{Supp}(V_f) \subseteq \{(x,y) \in X \times Y\colon\ d(f(x),y) \leq \varepsilon\}.$$

Proof. The $*$-representation of $C_0(Y)$ on H_Y can be extended to a $*$-representation of the algebra of all bounded Borel functions on Y. There exists a Borel cover $\{Y_i\}_i$ of Y such that (1) $Y_i \cap Y_j = \emptyset$ if $i \neq j$; (2) diameter$(Y_i) \leq \varepsilon/2$ for all i; (3) each Y_i has nonempty interior.

(3), together with the standardness of H_Y, implies that $\chi_{Y_i} H$ is infinite dimensional, where χ_{Y_i} is the characteristic function of Y_i. This implies that there exists an isometry V_f from H_X to H_Y such that V_f maps $\chi_{f^{-1}(Y_i)} H_X$ to $\chi_{Y_i} H_Y$, where $\chi_{f^{-1}(Y_i)}$ is the characteristic function of $f^{-1}(Y_i)$ and the $*$-representation of $C_0(X)$ on H_X is extended to a $*$-representation of the algebra of all bounded Borel functions on X. It is not difficult to see that V_f satisfies the required conditions. □

V_f gives rise to a homomorphism $\mathrm{ad}(V_f)$ from $C^*(X)^+$ to $C^*(Y)^+$ defined by:

$$\mathrm{ad}(V_f)(a+cI) = V_f a V_f^* + cI$$

for all $a \in C^*(X)$ and $c \in \mathbb{C}$, where $C^*(X)^+$ and $C^*(Y)^+$ are obtained from $C^*(X)$ and $C^*(Y)$ by adjoining identities. Also $\mathrm{ad}(V_f)$ induces a homomorphism $\mathrm{ad}(V_f)_*$ from $K_i(C^*(X))$ to $K_i(C^*(Y))$.

The following lemma says that $\mathrm{ad}(V_f)_*$ does not depend on the choice of V_f.

LEMMA 2.5 ([19]). *Let f, H_X and H_Y be as in Lemma 2.4. If V_1 and V_2 are two isometries from H_X to H_Y such that for some $\varepsilon > 0$*

$$\mathrm{Supp}(V_i) \subseteq \{(x,y) \in X \times Y\colon\ d(f(x),y) \leq \varepsilon\}$$

for $i = 1,2$, then

$$\mathrm{ad}(V_1)_* = \mathrm{ad}(V_2)_*\colon K_i(C^*(X)) \to K_i(C^*(Y)).$$

See Section 4 of [19] for a proof.

Recall that the K-homology groups $K_i(X) = KK_i(C_0(X), \mathbb{C})$ $(i = 0, 1)$ are generated by certain cycles modulo a certain equivalence relation [23]:

(1) A cycle for $K_0(X)$ is a pair (H_X, F), where H_X is an X-module and F is a bounded linear operator acting on H_X such that $F^*F - I$ and $FF^* - I$ are locally compact, and $\phi F - F\phi$ is compact for all $\phi \in C_0(X)$;

(2) A cycle for $K_1(X)$ is a pair (H_X, F), where H_X is an X-module and F is a self-adjoint operator acting on H_X such that $F^2 - I$ is locally compact, and $\phi F - F\phi$ is compact for all $\phi \in C_0(X)$.

In the above description of cycles for $K_i(X)$, H_X can always be chosen to be standard and nondegenerate. See [18], [20] for more details of the above description of K-homology.

Next we shall define the index map from $K_i(X)$ to $K_i(C^*(X))$. Let (H_X, F) be a cycle for $K_0(X)$ such that H_X is a standard nondegenerate X-module. Let $\{U_i\}_i$ be a locally finite and uniformly bounded open cover of X and $\{\phi_i\}_i$ be a continuous partition of unity subordinate to the open cover. Define

$$F' = \sum_i \phi_i^{\frac{1}{2}} F \phi_i^{\frac{1}{2}},$$

where the infinite sum converges in strong topology.

LEMMA 2.6. *Let F and F' be as above.*

(1) *F' has finite propagation;*

(2) $\|F'\| \leq 4\|F\|$.

Proof. (1) is obvious. To prove (2), we write $F = \sum_{m=1}^{4} c_m T_m$ such that c_m is a complex number satisfying $|c_i| \leq 1$ for each m, and T_m is a nonnegative operator acting on H_X satisfying $\|T_m\| \leq \|F\|$ for each m. It is enough to prove that

$$\left\| \sum_{i=1}^{n} \phi_i^{\frac{1}{2}} T_m \phi_i^{\frac{1}{2}} \right\| \leq \|T_m\|$$

for all n and m. But this follows from the fact that $\sum_{i=1}^{n} \phi_i^{\frac{1}{2}} T_m \phi_i^{\frac{1}{2}}$ is nonnegative and the following estimation:

$$\begin{aligned} \left\langle \left(\sum_{i=1}^{n} \phi_i^{\frac{1}{2}} T_m \phi_i^{\frac{1}{2}} \right) h, h \right\rangle &= \sum_{i=1}^{n} \langle \phi_i^{\frac{1}{2}} T_m \phi_i^{\frac{1}{2}} h, h \rangle = \sum_{i=1}^{n} \langle T_m \phi_i^{\frac{1}{2}} h, \phi_i^{\frac{1}{2}} h \rangle \\ &\leq \sum_{i=1}^{n} \|T_m\| \langle \phi_i^{\frac{1}{2}} h, \phi_i^{\frac{1}{2}} h \rangle = \sum_{i=1}^{n} \|T_m\| \langle \phi_i h, h \rangle = \|T_m\| \langle h, h \rangle \end{aligned}$$

for all $h \in H_X$. □

It is easy to verify that (H_X, F') is equivalent to (H_X, F) in $K_0(X)$. By the above lemma it is not difficult to see that F' is a multiplier of $C^*(X)$ and F' is a unitary modulo $C^*(X)$. Hence F' gives rise to an element $[F']$ in $K_0(C^*(X))$. We define the index of (H_X, F) to be $[F']$. Similarly we can define the index map from $K_1(X)$ to $K_1(C^*(X))$.

Let C be a locally finite and uniformly bounded cover for X. The nerve space N_C associated to C is defined to be the simplicial complex whose set of vertices equals C and where a finite subset $\{U_0, \ldots, U_n\} \subseteq C$ spans an n-simplex in N_C if and only if $\cap_{i=0}^{n} U_i \neq \emptyset$. Endow N_C with the spherical metric. Recall that the spherical metric on the simplicial complex N_C is defined as follows. Identify every simplex $\{U_0, \ldots, U_n\}$ in N_C with $S_+^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \colon x_i \geq 0, \sum_{i=0}^{n} x_i^2 = 1\}$ by:

$$\sum_{i=0}^{n} t_i U_i \to \left(t_0 / \left(\sum_{i=0}^{n} t_i^2 \right)^{1/2}, \ldots, t_n / \left(\sum_{i=0}^{n} t_i^2 \right)^{1/2} \right),$$

where $t_i \geq 0$, $\sum_{i=0}^{n} t_i = 1$. The standard spherical metric on the simplex $\{U_0, \ldots, U_n\}$ is defined to be the metric induced from the standard Riemannian metric on the n^{th} unit sphere S^n. The spherical metric on N_C is defined to be the maximal metric whose restriction to each simplex is the standard spherical metric. If x and y lie in different connected components of N_C, then we define $d(x, y) = \infty$. Use of the spherical metric is necessary to avoid certain pathological phenomena when n goes to infinity.

A sequence of locally finite and uniformly bounded covers $\{C_k\}_{k=1}^{\infty}$ of X is called an anti-Čech system of X [31] if there exists a sequence of positive numbers $R_k \to \infty$ such that for each k,

(1) every set $U \in C_k$ has diameter less than or equal to R_k;
(2) any set of diameter R_k in X is contained in some member of C_{k+1}.

An anti-Čech system always exists [31].

By the property of the anti-Čech system, for every pair $k_2 > k_1$, there exists a simplicial map $i_{k_1 k_2}$ from $N_{C_{k_1}}$ to $N_{C_{k_2}}$ such that $i_{k_1 k_2}$ maps a simplex $\{U_0, \ldots, U_n\}$ in $N_{C_{k_1}}$ to a simplex $\{U'_0, \ldots, U'_n\}$ in $N_{C_{k_2}}$ satisfying $U_i \subseteq U'_i$ for all $0 \le i \le n$. Also, $i_{k_1 k_2}$ gives rise to the following inductive systems of groups:

$$\begin{aligned} (i_{k_1 k_2})_*\colon &\quad K_i(N_{C_{k_1}}) \to K_i(N_{C_{k_2}}); \\ \mathrm{ad}(V_{i_{k_1 k_2}})_*\colon &\quad K_i(C^*(N_{C_{k_1}})) \to K_i(C^*(N_{C_{k_2}})). \end{aligned}$$

The following conjecture is called the coarse Baum-Connes conjecture.

CONJECTURE 2.7 ([31]). *The index map induces an isomorphism from* $\lim_{k\to\infty} K_i(N_{C_k})$ *to* $K_i(C^*(X)) = \lim_{k\to\infty} K_i(C^*(N_{C_k}))$.

It is not difficult to see that the coarse Baum-Connes conjecture for X does not depend on the choice of the anti-Čech system.

The following descent principle [33] is a C^*-algebra version of various descent principles previously known to topologists [4], [5], [11].

THEOREM 2.8 ([33]). *Let* Γ *be a finitely generated group. If the classifying space* $B\Gamma$ *of* Γ *has the homotopy type of a finite* CW*-complex, then the coarse Baum-Connes conjecture for* Γ *(as a metric space with a word-length metric) implies the strong Novikov conjecture for* Γ.

Recall that the word-length metric on a finitely generated group Γ is defined as follows. By choosing a finite generating set S for Γ, for any $\gamma \in \Gamma$, we can define its length $l_S(\gamma)$ to be the smallest integer n such that there exists $\{s_i\}_{i=1}^n$ for which $\gamma = s_1 \ldots s_n$ and either s_i or $s_i^{-1} \in S$ for all i. The word-length metric d_S on Γ is defined by:

$$d_S(\gamma_1, \gamma_2) = l_S(\gamma_1^{-1}\gamma_2)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. It is not difficult to see that, for any two finite generating sets S_1, S_2 of Γ, (Γ, d_{S_1}) is quasi-isometric to (Γ, d_{S_2}).

3. Localization and the obstruction group

In this section we shall recall the localization technique introduced in [39] and introduce an obstruction group to the coarse Baum-Connes conjecture.

Definition 3.1 ([39]). (1) Let X be a proper metric space. The *localization algebra* $C_L^*(X)$ is defined to be the C^*-algebra generated by all bounded and uniformly norm-continuous functions f from $[0,\infty)$ to $C^*(X)$ such that

$$\text{propagation}(f(t)) \to 0 \quad \text{as } t \to \infty;$$

(2) Letting f be an element in $C_L^*(X)$, we define its *propagation* to be

$$\sup_{t\in[0,\infty)} \text{propagation}(f(t)).$$

Next we shall define a local index map from the K-homology group $K_i(X)$ to the K-theory group $K_i(C_L^*(X))$. For each positive integer n, there exists a locally finite open cover $\{U_{n,i}\}_i$ for X such that diameter$(U_{n,i}) < 1/n$ for all i. Let $\{\phi_{n,i}\}_i$ be a continuous partition of unity subordinate to $\{U_{n,i}\}_i$. Let (H_X, F) be a cycle for $K_0(X)$ such that H_X is a standard nondegenerate X-module. Define a family of operators $F(t)$ $(t \in [0,\infty))$ acting on H_X by:

$$F(t) = \sum_i \left((n-t)\phi_{n,i}^{\frac{1}{2}} F \phi_{n,i}^{\frac{1}{2}} + (t-n+1)\phi_{n+1,i}^{\frac{1}{2}} F \phi_{n+1,i}^{\frac{1}{2}}\right)$$

for all positive integers n and $t \in [n-1, n]$, where the infinite sum converges in strong topology. Lemma 2.6 implies that $F(t)$ is a bounded and uniformly norm-continuous function from $[0,\infty)$ to the C^*-algebra of all bounded operators acting on H_X. Notice that

$$\text{propagation}(F(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Using the above facts it is not difficult to see that $F(t)$ is a multiplier of $C_L^*(X)$ and $F(t)$ is a unitary modulo $C_L^*(X)$. Hence $F(t)$ gives rise to an element $[F(t)]$ in $K_0(C_L^*(X))$. We define the local index of the cycle (H_X, F) to be $[F(t)]$. Similarly we can define the local index map from $K_1(X)$ to $K_1(C_L^*(X))$.

Let e be the evaluation homomorphism from $C_L^*(X)$ to $C^*(X)$ defined by:

$$e(f) = f(0)$$

for every $f \in C_L^*(X)$.

We have the following commuting diagram:

$$\begin{tikzcd} & K_*(X) \arrow[dl, "\mathrm{Ind}_L"'] \arrow[dr, "\mathrm{Ind}"] & \\ K_*(C_L^*(X)) \arrow[rr, "e_*"] & & K_*(C^*(X)), \end{tikzcd}$$

where Ind and Ind_L are respectively the index and local index map.

We define the dimension of a simplicial complex X to be the smallest integer n such that the dimension of each simplex of X is at most n.

THEOREM 3.2 ([39]). *Let X be a simplicial complex endowed with the spherical metric. If X is of finite dimension, then the local index map from $K_i(X)$ to $K_i(C_L^*(X))$ is an isomorphism.*

This theorem can be proved by induction on the i-skeleton $X^{(i)}$ (as a metric subspace of X) and a Mayer-Vietoris sequence argument (cf. [39]).

(We would like to correct a misleading notational error in the proof of the above theorem in [39] (Proof of Proposition 3.7, page 314, [39]). Let $F(x,t)$, $t_{i,j}$ and ε_i be as in the proof of Proposition 3.7 in [39]. Let $V_{k,i}$ be an isometry from H_X to H_X such that $\mathrm{Supp}(V_{k,i}) \subseteq \{(x,y) \in X \times X\colon d(F(x,t_{k,i}),y) \leq \varepsilon_i\}$, and $V_{k,i} = I$ if $F(x,t_{k,i}) = x$ for all $x \in X$. Define a family of isometries $V_k(t)$ $(t \in [0,\infty))$ from $H_X \oplus H_X$ to $H_X \oplus H_X$ by: $V_k(t) = R(t-i+1)$ $(V_{k,i-1} \oplus V_{k,i})R^*(t-i+1)$ for all $i-1 \leq t \leq i$ and $i \geq 1$, where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Finally, for any unitary u representing an element in $K_1(C_L^*(X))$, we define $a = \oplus_{k\geq 0}\mathrm{Ad}(V_k)(u)(u^{-1} \oplus I)$, $b = \oplus_{k\geq 0}\mathrm{Ad}(V_{k+1})(u)(u^{-1} \oplus I)$, $c = I \oplus_{k>1} \mathrm{Ad}(V_k)(u)(u^{-1} \oplus I)$, where $\mathrm{Ad}(V_k)(g+cI) = V_k(t)(g(t) \oplus 0)V_k^*(t) + cI$ for any $g \in C_L^*(X)$ and $c \in \mathbb{C}$. With the above correction, the main argument in the proof of Proposition 3.7 in [39] remains the same.)

Let f be a proper map from a proper metric space X to another proper metric space Y. Assume that f is uniformly continuous, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ for $x, y \in X$ satisfying $d(x,y) < \delta$. Let $\{\varepsilon_k\}_k$ be a sequence of positive number such that $\varepsilon_k \to 0$ as $k \to \infty$. By Lemma 2.4, for each $k > 0$, there exists an isometry V_k from a standard nondegenerate X-module H_X to a standard nondegenerate Y-module H_Y satisfying

$$\mathrm{Supp}(V_k) \subseteq \{(x,y) \in X \times Y\colon d(f(x),y) \leq \varepsilon_k\}.$$

Define a family of isometries $V_f(t)$ $(t \in [0,\infty))$ from $H_X \oplus H_X$ to $H_Y \oplus H_Y$ by:

$$V_f(t) = R(t-k+1)(V_k \oplus V_{k+1})R^*(t-k+1)$$

for all $k-1 \leq t \leq k$, where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

$V_f(t)$ induces a homomorphism $\mathrm{Ad}(V_f)$ from $C_L^*(X)^+$ to $C_L^*(Y)^+ \otimes M_2(\mathbb{C})$ defined as follows:

$$\mathrm{Ad}(V_f)(u+cI) = V_f(t)(u(t) \oplus 0)V_f^*(t) + cI$$

for any $u \in C_L^*(X)$ and $c \in \mathbb{C}$, where $C_L^*(X)^+$ and $C_L^*(Y)^+$ are obtained from $C_L^*(X)$ and $C_L^*(X)$ by adjoining identities. Notice that $\mathrm{Ad}(V_f)(u+cI)$ is uniformly norm-continuous in t although $V_f(t)$ is not norm-continuous.

We have the following useful lemma:

LEMMA 3.3. *Let X, Y, f, $\{\varepsilon_k\}_k$ and* $\mathrm{Ad}(V_f)$ *be as above.* $\mathrm{Ad}(V_f)$ *is a homomorphism from $C_L^*(X)^+$ to $C_L^*(Y)^+ \otimes M_2(\mathbb{C})$ satisfying*

$$\text{propagation}(\mathrm{Ad}(V_f)(u)) \leq \text{propagation}(u) + 2\sup_k(\varepsilon_k).$$

Definition 3.4. Let X be a proper metric space. $C_{L,0}^*(X)$ is defined to be the C^*-algebra generated by all bounded and uniformly norm-continuous functions f from $[0,\infty)$ to $C^*(X)$ such that *propagation*$(f(t)) \to 0$ as $t \to \infty$, $f(0) = 0$.

We have the following short exact sequence:

$$0 \to C_{L,0}^*(X) \to C_L^*(X) \to C^*(X) \to 0.$$

This and Theorem 3.2, imply the following:

THEOREM 3.5. *Let X be a proper metric space. Assume that there exists an anti-Čech system $\{C_k\}_k$ for X such that N_{C_k} is finite dimensional for all k. Then the coarse Baum-Connes conjecture holds for X if and only if*

$$\lim_{k\to\infty} K_i(C_{L,0}^*(N_{C_k})) = 0$$

for $i = 0,1$.

In the above theorem, the inductive system of groups is given by:

$$\mathrm{Ad}(V_{i_{k_1k_2}})_*\colon\ K_i(C_{L,0}^*(N_{C_{k_1}})) \to K_i(C_{L,0}^*(N_{C_{k_2}})),$$

where $i_{k_1k_2}$ is as in Conjecture 2.7, and $\mathrm{Ad}(V_{i_{k_1k_2}})$ is as in Lemma 3.3.

Recall that X is said to have (locally) bounded geometry if there exists a subspace Γ such that (1) there exists $c > 0$ for which $d(x,\Gamma) \leq c$ for all $x \in X$; (2) for any $r > 0$, there exists a natural number $n(r)$ such that the number of elements in $B_\Gamma(\gamma, r) = \{\gamma' \in \Gamma\colon\ d(\gamma',\gamma) \leq r\}$ is at most $n(r)$ for every $\gamma \in \Gamma$. If X has (locally) bounded geometry, then there exists an anti-Čech system $\{C_k\}_k$ for X such that N_{C_k} is finite dimensional for all k.

For obvious reason $\lim_{k\to\infty} K_i(C_{L,0}^*(N_{C_k}))$ is called the obstruction group to the coarse Baum-Connes conjecture.

4. Controlled obstructions: $QP_{\delta,r,s,k}(X)$, $QU_{\delta,r,s,k}(X)$

In this section we shall introduce and study $QP_{\delta,r,s,k}(X)$ and $QU_{\delta,r,s,k}(X)$, which can be respectively considered as controlled versions of $K_0(C_{L,0}^*(X) \otimes C_0((0,1)^k))$ and $K_1(C_{L,0}^*(X) \otimes C_0((0,1)^k))$.

Let A be a C^*-algebra and δ be a positive number. An element p in A is called a δ-quasi-projection if

$$p^* = p, \quad \|p^2 - p\| < \delta.$$

Similarly an element u in A is called a δ-quasi-unitary if

$$\|u^*u - I\| < \delta, \quad \|uu^* - I\| < \delta.$$

Let X be a proper metric space; let $C^*_{L,0}(X)^+$ be the C^*-algebra obtained from $C^*_{L,0}(X)$ by adjoining an identity. Let δ, r and s be positive numbers, k and n be nonnegative integers. Define $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous functions from $[0,1]^k$ to $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that:

(1) $f(t)$ is a δ-quasi-projection and propagation $(f(t)) \leq r$ for all $t \in [0,1]^k$;

(2) f is piecewise smooth in t_i and $\|\frac{\partial f}{\partial t_i}(t)\| \leq s$ for all $t = (t_1, \ldots, t_k) \in [0,1]^k$;

(3) $\|f(t) - p_m\| < \delta$ for all $t \in \mathrm{bd}([0,1]^k)$, the boundary of $[0,1]^k$ in $\mathbb{R}^k$, where $p_m = I \oplus \cdots \oplus I \oplus 0 \oplus \cdots \oplus 0$ with m identities;

(4) $\pi(f(t)) = p_m$, where π is the canonical homomorphism from $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C})$.

Definition 4.1. $QP_{\delta,r,s,k}(X)$ is defined to be the direct limit of $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ under the embedding: $p \to p \oplus 0$.

Similarly we define $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ to be the set of all continuous functions from $[0,1]^k$ to $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ such that:

(1) $f(t)$ is a δ-quasi-unitary and propagation$(f(t)) \leq r$ for all $t \in [0,1]^k$;

(2) f is piecewise smooth in t_i and $\|\frac{\partial f}{\partial t_i}(t)\| \leq s$ for all $t \in [0,1]^k$;

(3) $\|f(t) - I\| < \delta$ for all $t \in \mathrm{bd}([0,1]^k)$;

(4) $\pi(f(t)) = I$, where π is the canonical homomorphism from $C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C})$ to $M_n(\mathbb{C})$.

Definition 4.2. $QU_{\delta,r,s,k}(X)$ is defined to be the direct limit of $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ under the embedding: $u \to u \oplus I$.

Definition 4.3. Let p and q be two elements in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$. Now p is said to be (δ, r, s)-equivalent to q if there exists a piecewise smooth homotopy $a(t')$ ($t' \in [0,1]$) in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that (1) $a(0) = p$ and $a(1) = q$; (2) $\|a'(t')\| \leq s$.

The above concept can be used to define the concept of (δ, r, s)-equivalence between elements in $QP_{\delta,r,s,k}(X)$.

Notice that if $\delta < \frac{1}{100}$, then there exist universal constants c_0 and s_0 such that (1) any $p \in QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ is $(10\delta, r, c_0 s + s_0)$-equivalent to some q in $QP_{10\delta,r,c_0 s+s_0,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ satisfying $q(t) = \pi(q(t)) =$

p_m for some m and all $t \in \mathrm{bd}([0,1]^k)$; (2) if p and q are two elements in $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that p is (δ, r, s)-equivalent to q, $p(t) = \pi(p(t))$ and $q(t) = \pi(q(t))$ for all $t \in \mathrm{bd}([0,1]^k)$, then there exists a homotopy $a(t')$ ($t' \in [0,1]$) in $QP_{10\delta,r,c_0 s+s_0,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ satisfying $a(0) = p, a(1) = q$, and $(a(t'))(t) = \pi((a(t'))(t))$ for all $t' \in [0,1]$ and $t \in \mathrm{bd}([0,1]^k)$.

By the following lemma, $QP_{\delta,r,s,k}(X)$ can be considered as a controlled version of $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$.

LEMMA 4.4. *Let* $0 < \delta < 1/100$; *let* f *be a continuous function on* $\mathbb{R}$ *satisfying* $f(x) = 1$ *for all* $x \in [1/2,3/2]$, *and* $f(x) = 0$ *for all* $x \in [-1/5,1/5]$.

(1) *For any* $p \in QP_{\delta,r,s,k}(X)$, $f(p)$ *is a projection and defines an element* $[f(p)]$ *in* $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$;

(2) *Let* p *and* q *be elements in* $QP_{\delta,r,s,k}(X)$. *If* p *is* (δ,r,s)*-equivalent to* q, *then* $[f(p)] = [f(q)]$ *in* $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$;

(3) *Every element in* $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ *can be represented as* $[f(p_1)] - [f(p_2)]$, *where* $p_1,p_2 \in QP_{\delta,r,s,k}(X)$ *for some* $r > 0$ *and* $s > 0$.

Proof. (1) and (2) are straightforward. To prove (3), note that every element in $K_0(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ can be represented as $[q_1] - [q_2]$, where q_1 and q_2 are projections in $(C^*_{L,0}(X) \otimes C_0((0,1)^k))^+ \otimes M_n(\mathbb{C})$. By an approximation argument, there exist p_1 and p_2 in $QP_{\delta/10,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ for some $r > 0$ and $s > 0$ such that $\|p_i - q_i\| < \delta/10$ for $i = 1,2$. Let $p_i(t') = t'p_i + (1-t')q_i$ for $i = 1,2$, and $t' \in [0,1]$. It is not difficult to see

$$(p_i(t'))^* = p_i(t'), \quad \|(p_i(t'))^2 - p_i(t')\| < \delta$$

for $i = 1,2$, and $t' \in [0,1]$. It follows that $f(p_i(t'))$ is a homotopy of projections for $i = 1,2$ satisfying $f(p_i(0)) = q_i, \quad f(p_i(1)) = f(p_i)$. This implies (3). □

Similarly we can define the concept of (δ, r, s)-equivalence between elements in $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ and $QU_{\delta,r,s,k}(X)$. It is also not difficult to verify that if $\delta < \frac{1}{100}$, then there exist universal constants c_1 and s_1 such that (1) any $u \in QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ is $(10\delta, r, c_1 s + s_1)$-equivalent to some v in $QU_{10\delta,r,c_1 s+s_1,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ satisfying $v(t) = \pi(v(t)) = I$ for all $t \in \mathrm{bd}([0,1]^k)$; (2) if u and v are two elements in $QU_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ such that u is (δ, r, s)-equivalent to v, $u(t) = \pi(u(t))$ and $v(t) = \pi(v(t))$ for all $t \in \mathrm{bd}([0,1]^k)$, then there exists a homotopy $a(t')$ ($t' \in [0,1]$) in $QU_{10\delta,r,c_1 s+s_1,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ satisfying $a(0) = u, a(1) = v$, and $(a(t'))(t) = \pi((a(t'))(t))$ for all $t' \in [0,1]$ and $t \in \mathrm{bd}([0,1]^k)$.

By the above discussions and the following lemma, $QU_{\delta,r,s,k}(X)$ can be considered as a controlled version of $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$.

LEMMA 4.5. *Let* $0 < \delta < 1/100$.

(1) *Every* $u \in QU_{\delta,r,s,k}(X)$ *satisfying* $u(t) = \pi(u(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$ *defines an element* $[u]$ *in* $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$;

(2) *Let* u *and* v *be elements in* $QU_{\delta,r,s,k}(X)$ *satisfying* $u(t) = \pi(u(t)) = I$ *and* $v(t) = \pi(v(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$. *If* u *is* (δ,r,s)*-equivalent to* v, *then* $[u] = [v]$ *in* $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$;

(3) *Every element in* $K_1(C^*_{L,0}(X) \otimes C_0((0,1)^k))$ *can be represented as* $[u]$, *where* $u \in QU_{\delta,r,s,k}(X)$ *for some* $r > 0$ *and* $s > 0$ *and* $u(t) = \pi(u(t)) = I$ *for all* $t \in \mathrm{bd}([0,1]^k)$.

The proof of the above lemma is similar to that of Lemma 4.4 and is therefore omitted.

LEMMA 4.6. *Let* $0 < \delta < 1/100$. *If* p *and* q *are two* (δ,r,s)*-equivalent elements in* $QP_{\delta,r,s,k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$, *then there exists* $u \in QU_{\delta,c_1(\delta,s)r,c_2(s),k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ *such that* $\|p - u^*qu\| < c_3(s)\delta$, *where* $c_1(\delta,s)$ *depends only on* δ *and* s, $c_2(s)$ *and* $c_3(s)$ *depend only on* s.

Proof. Let $a(t)$ be the homotopy in Definition 4.3. There exists a partition:

$$0 = t'_0 < t'_1 < \cdots < t'_{m_s} = 1$$

such that

$$\|a(t'_{i+1}) - a(t'_i)\| < 1/100,$$

where m_s depends only on s. Consider

$$u_i = [(2a(t'_{i+1}) - I)(2a(t'_i) - I) + I]/2.$$

We have

$$I - u_i = (2a(t'_{i+1}) - I)(a(t'_{i+1}) - a(t'_i)) + 2(a(t'_{i+1}) - a^2(t'_{i+1})).$$

This implies that

$$\|I - u_i\| < 1/10.$$

It follows that

$$\text{(A)} \qquad \|I - u_i^*u_i\| < 3/10.$$

We also have

$$\begin{aligned}\text{(B)} \quad & \|a(t'_{i+1})u_i - u_i a(t'_i)\| \\ & \le \|(a^2(t'_{i+1}) - a(t'_{i+1}))a(t'_i)\| + \|a(t'_{i+1})(a^2(t'_i) - a(t'_i))\| \\ & \quad + \|a^2(t'_{i+1}) - a(t'_{i+1})\| + \|a^2(t'_i) - a(t'_i)\| < 6\delta.\end{aligned}$$

Let $P_l(x)$ be the l^{th} Taylor polynomial for $\frac{1}{\sqrt{1-x}}$ at 0. Choose l_0 such that

$$\left|P_{l_0}(x) - \frac{1}{\sqrt{1-x}}\right| < \frac{\delta}{10\, 2^{10m_s}}$$

for all $x \in [0, 3/10]$. Let

$$w_i = u_i P_{l_0}(I - u_i^* u_i).$$

Define

$$u = w_{m_s - 1} \cdots w_0.$$

Notice that $P_l'(x)$ has nonnegative coefficients and the sequence $\{P_l'(x)\}_l$ is uniformly bounded on $[0, 3/10]$. This, together with the chain rule, implies that

$$\left\| \frac{\partial u}{\partial t_i} \right\| \le c_2(s)$$

for some $c_2(s)$ depending only on s. By the choice of l_0 it is not difficult to verify that u is a δ-quasi-unitary. (A), (B) and the self-adjointness of $a(t)$ imply

$$\|u_i^* u_i a(t_i') - a(t_i') u_i^* u_i\| < 20\delta.$$

Hence

$$\begin{aligned}
&\|a(t_i')(I - u_i^* u_i)^n - (I - u_i^* u_i)^n a(t_i')\| \\
&\quad \le \|(a(t_i')u_i^* u_i - u_i u_i^* a(t_i'))(I - u_i^* u_i)^{n-1}\| \\
&\qquad + \|(I - u_i^* u_i)(a(t_i')u_i^* u_i - u_i u_i^* a(t_i'))(I - u_i^* u_i)^{n-2}\| \\
&\qquad \cdots + \|(I - u_i^* u_i)^{n-1}(a(t_i')u_i^* u_i - u_i u_i^* a(t_i'))\| \\
&\quad \le 20\delta n (3/10)^{n-1}.
\end{aligned}$$

This, together with the definition of w_i, implies that

$$\|a(t_{i+1}')w_i - w_i a(t_i')\| < b\delta,$$

where b is a universal constant. Now it is not difficult to see that u satisfies the desired properties. □

For any proper metric space X, let $GQP_{\delta,r,s,k}(X)$ be the set of formal difference $p - q$, where $p, q \in QP_{\delta,r,s,k}(X)$, and $\pi(p) = \pi(q)$. Two elements $p - q$ and $p' - q'$ in $GQP_{\delta,r,s,k}(X)$ are said to be (δ, r, s)-equivalent if $p \oplus q'$ and $p' \oplus q$ are (δ, r, s)-equivalent. It is not difficult to see that there exists a universal constant s_0 such that any element $p - q$ in $GQP_{\delta,r,s,k}(X)$ is $(10\delta, r, s + s_0)$-equivalent to an element $p' - p_m$ for some $p' \in QP_{\delta,r,s,k}(X)$ and some nonnegative integer m, where $p_m = I \oplus \cdots \oplus I \oplus 0 \oplus \cdots \oplus 0$ with m identities.

An element $p - q$ in $GQP_{\delta,r,s,k}(X)$ is said to be (δ, r, s)-equivalent to 0 if $p \oplus (I \oplus 0)$ is (δ, r, s)-equivalent to $q \oplus (I \oplus 0)$.

For any $u \in QU_{\delta,r,s,k}(X)$, let $z_t(u)$ be the homotopy connecting $I \oplus I$ to $u \oplus u^*$ obtained by combining the linear homotopy connecting $I \oplus I$ to $uu^* \oplus I$ with the rotation homotopy connecting $uu^* \oplus I$ to $u \oplus u^*$:

$$(u \oplus I)R(t)(u^* \oplus I)R^*(t),$$

where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Let

$$e_t(u) = z_t(u)(I \oplus 0)z_t^*(u).$$

We define a map θ from $QU_{\delta,r,s,k}(X)$ to $GQP_{100\delta,100r,100(s+1),k+1}(X)$ by:

$$\theta(u) = e_t(u) - (I \oplus 0),$$

where t is incorporated as the $(k+1)^{\text{th}}$ suspension parameter.

The following result can be considered as a controlled version of a classical result in topological K-theory.

LEMMA 4.7. *θ: $QU_{\delta,r,s,k}(X) \to GQP_{100\delta,100r,100s,k+1}(X)$ is an asymptotic isomorphism in the following sense:*

(1) *For any $0 < \delta < 1/100, r > 0, s > 0$, there exist $0 < \delta_1 < \delta, 0 < r_1 < r$ and $s_1 > 0$, such that if two elements u and v in $QU_{\delta_1,r_1,s,k}(X)$ are (δ_1,r_1,s)-equivalent, then $\theta(u)$ and $\theta(v)$ are (δ,r,s_1)-equivalent, where δ_1 depends only on δ; r_1 depends only on r; and s_1 depends only on s;*

(2) *For any $0 < \delta < 1/100, r > 0, s > 0$, there exist $0 < \delta_2 < \delta, 0 < r_2 < r$ and $s_2 > 0$ for which if u and v are two elements in $QU_{\delta_2,r_2,s,k}(X)$ such that $\theta(u)$ and $\theta(v)$ are (δ_2,r_2,s)-equivalent, then u and v are (δ,r,s_2)-equivalent, where δ_2 depends only on δ and s; r_2 depends only on δ,r and s; and s_2 depends only on s;*

(3) *For any $0 < \delta < 1/100, r > 0, s > 0$, there exist $0 < \delta_3 < \delta, 0 < r_3 < r$ and $s_3 > 0$, such that for any $p - p_m \in GQP_{\delta_3,r_3,s,k+1}(X)$, there exists $u \in QU_{\delta,r,s_3,k}(X)$ for which $\theta(u)$ is (δ,r,s_3)-equivalent to $p - p_m$, where δ_3 depends only on δ and s; r_3 depends only on δ, r and s; and s_3 depends only on s.*

Proof. (1) Let $w(t)$ be the homotopy realizing the (δ_1, r_1, s)-equivalence between u and v. Let $a(t)$ be the homotopy connecting I to v^*u obtained by combining the linear homotopy between I and u^*u with the homotopy $w(t)^*u$; let $b(t)$ be the homotopy connecting I to vu^* obtained by combining the linear homotopy between I and uu^* with the homotopy $w(t)u^*$. Define

$$x_t = z_t(v)(a(t) \oplus b(t))z_t^*(u),$$

where z_t is as in the definition of the map θ. We have

$$x_0 = I, \quad \|x_1 - I\| < c\delta_1,$$

$$\|x_t e_t(u) x_t^* - e_t(v)\| < c\delta_1,$$

where c is a universal constant. Now (1) follows from the above conditions by choice of appropriate δ_1, r_1 and s_1.

(2) Choose

$$\delta_2 < \delta/10^{10}(1+10^{10}c_3(s)),$$

where $c_3(s)$ is as in Lemma 4.6. By Lemma 4.6 there exists x in $QU_{\delta_2,c_1(\delta_2,s)r_2,c_2(s),k+1}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$ (for some n) such that

$$\|xe_t(u\oplus I)x^* - e_t(v\oplus I)\| < c_3(s)\delta_2.$$

It follows that

(A) $\|z_t^*(v\oplus I)x_tz_t(u\oplus I)(I\oplus 0) - (I\oplus 0)z_t^*(v\oplus I)x_tz_t(u\oplus I)\| < 10^3c_3(s)\delta_2,$

where x is identified with a piecewise smooth family of elements x_t ($t\in[0,1]$) in $QU_{\delta_2,c_1(\delta_2,s)r_2,c_2(s),k}(C^*_{L,0}(X)^+ \otimes M_n(\mathbb{C}))$. Write

$$z_t^*(v\oplus I)x_tz_t(u\oplus I) = \begin{pmatrix} c_t & g_t \\ h_t & d_t \end{pmatrix}. \tag{B}$$

(A) implies that

$$\|g_t\| < 1000c_3(s)\delta_2, \quad \|h_t\| < 1000c_3(s)\delta_2. \tag{C}$$

(B), together with the properties of x_t and the definitions of $z_t(v\oplus I)$ and $z_t(u\oplus I)$, implies that

$$\|c_0 - I\| < 100\delta_2, \quad \|c_1 - (v\oplus I)^*(u\oplus I)\| < 10^{10}\delta_2. \tag{D}$$

Now, by choice of appropriate r_2 and s_2, (2) follows from the existence of the homotopy c_t satisfying (D).

(3) Choose

$$\delta_3 < \delta/10^{10}(1+10c_3(s)),$$

where $c_3(s)$ is as in Lemma 4.6. Without loss of generality, we can assume $p(0)=p(1)=p_m=I\oplus 0$, where p is identified with a piecewise smooth family of $p(t)\in QP_{\delta_3,r_3,s,k}(X)$ ($t\in[0,1]$). By the proof of Lemma 4.6 there exists a homotopy $w(t)$ in $QU_{\delta_3,c_1(\delta_3,s)r_3,c_2(s),k}(X)$ such that

$$w(0)=I, \quad \|w(t)(I\oplus 0)w(t)^* - p(t)\| < c_3(s)\delta_3.$$

It follows that

$$\|w(1)(I\oplus 0) - (I\oplus 0)w(1)\| < 10c_3(s)\delta_3.$$

Hence we can write

$$w(1) = \begin{pmatrix} u & g \\ h & v \end{pmatrix},$$

where $\|g\| < 10c_3(s)\delta_3$, $\|h\| < 10c_3(s)\delta_3$.

Define

$$y_t = (w(t)\oplus I)(I\oplus z_t^*(v)w^*(t))(z_t^*(u)\oplus I).$$

We have

(A) $$y_0 = I \oplus I, \quad \|y_1 - (I \oplus I)\| < 10^5 c_3(s)\delta_3,$$

(B) $$\|y_t(e_t(u) \oplus 0)y_t^* - (p(t) \oplus 0)\| < 10^5 c_3(s)\delta_3.$$

Now, by choice of appropriate r_3 and s_3, (3) follows from the existence of y_t satisfying (A) and (B). □

Let X and Y be two proper metric spaces; let f and g be two proper Lipschitz maps from X to Y. A continuous homotopy $F(t,x)$ ($t \in [0,1]$) between f and g is said to be strongly Lipschitz if:

(1) $F(t,x)$ is a proper map from X to Y for each t;

(2) $d(F(t,x), F(t,y)) \leq Cd(x,y)$ for all $x, y \in X$ and $t \in [0,1]$, where C is a constant (called the Lipschitz constant of F);

(3) F is equicontinuous in t, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(F(t_1,x), F(t_2,x)) < \varepsilon$ for all $x \in X$ if $|t_1 - t_2| < \delta$;

(4) $F(0,x) = f(x), F(1,x) = g(x)$ for all $x \in X$.

Note that the above condition of strong Lipschitz homotopy is stronger than the condition of Lipschitz homotopy introduced by Gromov (see page 25 of [14] or [38]).

Also, X is said to be strongly Lipschitz homotopy equivalent to Y if there exist proper Lipschitz maps $f : X \to Y$ and $f_1 : Y \to X$ such that $f_1 f$ and $f f_1$ are strongly Lipschitz homotopic to id_X and id_Y, respectively.

LEMMA 4.8. *Let f and g be two proper Lipschitz maps from X to Y. Assume that f is strongly Lipschitz homotopic to g. There exists $S_0 > 0, C_0 > 0$ such that for any $u \in QU_{\delta,r,s,k}(X)$, there exists a homotopy $w(t')$ ($t' \in [0,1]$) in $QU_{C_0\delta, C_0 r, C_0(s+1),k}(C^*_{L,0}(Y)^+ \otimes M_n(\mathbb{C}))$ for some n for which*

$$w(0) = \mathrm{Ad}(V_f)(u) \oplus I,$$

$$w(1) = \mathrm{Ad}(V_g)(u) \oplus I,$$

and $\|w'(t')\| \leq S_0$, where $\mathrm{Ad}(V_f)$ and $\mathrm{Ad}(V_g)$ are as in Lemma 3.3, S_0 and C_0 depend only on the Lipschitz constant C of the strong Lipschitz homotopy F between f and g.

Proof. Choose $\{t_{i,j}\}_{i\geq 0, j\geq 0} \subseteq [0,1]$ satisfying

(1) $t_{0,j} = 0, \quad t_{i,j+1} \leq t_{i,j}, \quad t_{i+1,j} \geq t_{i,j}$;

(2) there exists a sequence $N_j \to \infty$ such that $t_{i,j} = 1$ for all $i \geq N_j$, and $N_{j+1} \geq N_j$ for all j;

(3) $d(F(t_{i+1,j},x), F(t_{i,j},x)) < \varepsilon_j = r/(j+1)$, $d(F(t_{i,j+1},x), F(t_{i,j},x)) < \varepsilon_j$ for all $x \in X$.

For example, we can take

$$t_{i,j} = \begin{cases} i/(Nj+N) & \text{if } i \le Nj+N, \\ 1 & \text{if } i \ge Nj+N, \end{cases}$$

where N is some large positive number.

Let $f_{i,j}(x) = F(t_{i,j}, x)$ for all $x \in X$. By Lemma 2.4 there exists an isometry $V_{f_{i,j}}$ from H_X to H_Y such that

$$\mathrm{Supp}(V_{f_{i,j}}) \subseteq \{(x,y) \in X \times X\colon\ d(f_{i,j}(x), y) < r/(1+i+j)\}.$$

For each $i > 0$, define a family of isometries $V_i(t)$ $(t \in [0,\infty))$ from $H_X \oplus H_X$ to $H_Y \oplus H_Y$ by:

$$V_i(t) = R(t-j)(V_{f_{i,j}} \oplus V_{f_{i,j+1}})R^*(t-j)$$

for all $t \in [j, j+1]$, where

$$R(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

Consider

$$\begin{aligned} u_0(t) &= \mathrm{Ad}(V_f)(u) = V_f(t)(u(t) \oplus I)V_f^*(t) + (I - V_f(t)V_f^*(t)); \\ u_\infty(t) &= \mathrm{Ad}(V_g)(u) = V_g(t)(u(t) \oplus I)V_g^*(t) + (I - V_g(t)V_g^*(t)); \\ u_i(t) &= \mathrm{Ad}(V_i)(u) = V_i(t)(u(t) \oplus I)V_i^*(t) + (I - V_i(t)V_i^*(t)) \end{aligned}$$

for $i > 0$. Notice that $u_i(t)$ is uniformly continuous in t although $V_i(t)$ is not continuous in t.

For each i, define n_i to be the largest integer j satisfying $i \ge N_j$ if $\{j\colon\ i \ge N_j\} \ne \emptyset$, and define n_i to be 0 otherwise. We can choose $V_{f_{i,j}}$ in such a way that $u_i(t) = u_\infty(t)$ when $t \le n_i$.

Define

$$w_i(t) = \begin{cases} u_i(t)(u_\infty(t))^* & \text{if } t \ge n_i; \\ (n_i - t)I + (t - n_i + 1)u_i(t)(u_\infty(t))^* & \text{if } n_i - 1 \le t \le n_i; \\ I & \text{if } 0 \le t \le n_i - 1. \end{cases}$$

Consider

$$\begin{aligned} a &= \oplus_{i=0}^\infty (w_i \oplus I); \\ b &= \oplus_{i=0}^\infty (w_{i+1} \oplus I); \\ c &= (I \oplus I) \oplus_{i=1}^\infty (w_i \oplus I), \end{aligned}$$

where a, b and c act on the standard and nondegenerate Y-module $H_Y^\infty = \oplus_{i=0}^\infty ((H_Y \oplus H_Y) \oplus (H_Y \oplus H_Y))$.

By (2), (3) and the construction of a, b and c, we know that a, b and c are elements in $QU_{C_1\delta, C_1 r, C_1(s+1), k}(Y)$ for some constant C_1 depending only on C.

Let

$$V_{i,i+1}(t') = R(t')(V_i \oplus V_{i+1})R^*(t')$$

for all $t' \in [0,1]$.

Define

$$u_{i,i+1}(t') = V_{i,i+1}(t')((u \oplus I) \oplus I)V^*_{i,i+1}(t') + (I - V_{i,i+1}(t')V^*_{i,i+1}(t'))$$

for $t' \in [0,1]$. Notice that

$$\begin{aligned} u_{i,i+1}(0) &= (V_i(u \oplus I)V_i^* + (I - V_iV_i^*)) \oplus I; \\ u_{i,i+1}(1) &= (V_{i+1}(u \oplus I)V^*_{i+1} + (I - V_{i+1}V^*_{i+1})) \oplus I. \end{aligned}$$

Using $u_{i,i+1}(t')$ we can construct a homotopy $v_1(t')$ ($t' \in [0,1]$) in $QU_{C_2\delta, C_2 r, C_2 s, k}(Y)$ for some $C_2 \geq C_1$ such that

$$v_1(0) = a, v_1(1) = b;$$

$$\|v_1'(t')\| \leq s_0$$

for all t', where s_0 is a universal constant.

We can also construct a homotopy $v_2(t')$ ($t' \in [0,1]$) in $QU_{C_3\delta, C_3 r, C_3 s, k}(Y)$ for some $C_3 \geq C_1$ such that

$$v_2(0) = b, v_2(1) = c;$$

$$\|v_2'(t')\| \leq s_1$$

for all t', where s_1 is a universal constant.

Finally we define $w(t')$ to be the homotopy obtained by combining the following homotopies:

(1) the linear homotopy between

$$(u_0 \oplus I) \oplus_{i=1}^{\infty} (I \oplus I) \text{ and } c^*a((u_\infty \oplus I) \oplus_{i=1}^{\infty} (I \oplus I));$$

(2) $v_2^*(1-t')a((u_\infty \oplus I)\oplus_{i=1}^{\infty}(I \oplus I)$;

(3) $v_1^*(1-t')a((u_\infty \oplus I)\oplus_{i=1}^{\infty}(I \oplus I))$;

(4) the linear homotopy between

$$a^*a((u_\infty \oplus I)\oplus_{i=1}^{\infty}(I \oplus I)) \text{ and } (u_\infty \oplus I)\oplus_{i=1}^{\infty}(I \oplus I).$$

It is not difficult to see that $w(t')$ is the desired homotopy connecting $\mathrm{Ad}(V_f)(u)\oplus_{i=1}^{\infty} I$ to $\mathrm{Ad}(V_g)(u)\oplus_{i=1}^{\infty} I$. □

Combining Lemma 4.7 with Lemma 4.6, we have the following result.

LEMMA 4.9. *Let X, Y, f and g be as in Lemma 4.8. For any $0 < \delta < 1/100$, $r > 0$, $s > 0$, there exist $0 < \delta_1 < \delta$, $0 < r_1 < r$ and $s_1 > 0$ such that for* *any* $p \in QP_{\delta_1,r_1,s,k}(X)$ $(k > 1)$, *there exists a homotopy* $w(t')$ $(t' \in [0,1])$ *in* $QP_{\delta,r,s_1,k}(C^*_{L,0}(Y)^+ \otimes M_n(\mathbb{C}))$ *for some* n *satisfying*

$$w(0) = \mathrm{Ad}(V_f)(p \oplus 0) \oplus (I \oplus 0),$$

$$w(1) = \mathrm{Ad}(V_g)(p \oplus 0) \oplus (I \oplus 0),$$

and $\|w'(t')\| \leq s_1$, *where* δ_1 *depends only on* δ, s, C (C *is as in Lemma* 4.8); r_1 *depends only on* δ, r, s, C; *and* s_1 *depends only on* s *and* C.

5. Controlled cutting and pasting

In this section we shall prove a controlled cutting and pasting result for $QU_{\delta,r,s,k}(X)$.

Definition 5.1. Let X be a proper metric space and X_i $(i = 1,2)$ be metric subspaces. The triple $(X; X_1, X_2)$ is said to satisfy the strong excision condition if

(1) $X = X_1 \cup X_2$, X_i is a Borel subset and the interior of X_i is dense in X_i for $i = 1,2$;

(2) there exists $r_0 > 0, c_0 > 0$ such that (i) for any $r \leq r_0$, $\mathrm{bd}_r(X_1) \cap \mathrm{bd}_r(X_2) = \mathrm{bd}_r(X_1 \cap X_2)$, where $\mathrm{bd}_r(A)$ is defined to be $\{x \in X\colon\ d(x, A) \leq r\}$ for any $A \subseteq X$; (ii) for each $X' = X_1, X_2, X_1 \cap X_2$, and any $r \leq r_0$, $\mathrm{bd}_r(X')$ is strongly Lipschitz homotopy equivalent to X' with c_0 as the Lipschitz constant of the strong Lipschitz homotopies realizing the strong Lipschitz homotopy equivalence.

Let $(X; X_1, X_2)$ be as in Definition 5.1. We shall first construct a boundary map ∂ from $QU_{\delta,r,s,k}(X)$ to $GQP_{N_0\delta,N(\delta)r,N_0 s,k}(X_1 \cap X_2)$, where $0 < \delta < 1/100$, $N(\delta)$ depends only on δ, N_0 is a universal constant, and $r < r_0/(1 + N(\delta))$ (r_0 is as in Definition 5.1). Our construction is modeled after the standard construction of the boundary map in K-theory (cf. [2] and [26]). The modification is necessary in order to control the propagations.

We shall first make a few remarks about notation. Let Y be a metric subspace of X such that Y is Borel and the interior of Y is dense in Y. Let H_X be as in Definition 2.3. Recall that the $*$-representation of $C_0(X)$ on H_X can be extended to a $*$-representation of the algebra of all bounded Borel functions. Hence we can define $H_Y = \chi_Y H_X$, where χ_Y is the characteristic function of Y. Define a Y-module structure on H_Y by: $fh = f'h$ for any $f \in C_0(Y), h \in H_Y \subseteq H_X$, where f' is any bounded Borel function on X whose restriction to Y is equal to f. Note that $f'h$ is independent of the choice of f'. Since the interior of Y is dense in Y, H_Y is a standard nondegenerate Y-module. For technical convenience we shall choose H_Y as the standard nondegenerate Y-module in the definitions of $C^*(Y)$ and $C^*_{L,0}(Y)$ throughout this section. Each operator T acting on H_Y defines an operator (still denoted by T throughout this section) on H_X by: $Tx = T(\chi_Y x)$ for all $x \in H_X$. We shall also identify the adjoined identity in $C^*_{L,0}(Y)^+$ as the identity operator acting on H_Y.

Let $0 < \delta < 1/100$. For any $u \in QU_{\delta,r,s,k}(X)$, we can define $u_{X_1} = \chi_{X_1} u \chi_{X_1}$, where χ_{X_1} is the characteristic function of X_1. Define

$$w_1 = \begin{pmatrix} I & u_{X_1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -u^*_{X_1} & I \end{pmatrix} \begin{pmatrix} I & u_{X_1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The above formula has its origin in [26]. Let $P_l(x)$ be the l^{th} Taylor polynomial for $1/(10\sqrt{1-x})$. Choose l_0 to be the smallest integer such that

$$|P_{l_0}(x) - 1/(10\sqrt{1-x})| < \delta/100$$

for all $x \in [0, 99/100]$. Notice that

$$0 \leq I - w_1 w_1^*/100 \leq 99/100.$$

Let

$$w_u = w_1 P_{l_0}(I - w_1^* w_1/100).$$

Define

$$\partial_0(u) = \chi_{\mathrm{bd}_{10l_0r}(X_1)\cap \mathrm{bd}_{10l_0r}(X_2)} w_u (I \oplus 0) w_u^* \chi_{\mathrm{bd}_{10l_0r}(X_1)\cap \mathrm{bd}_{10l_0r}(X_2)},$$

where $\chi_{\mathrm{bd}_{10l_0r}(X_1)\cap \mathrm{bd}_{10l_0r}(X_2)}$ is the characteristic function of $\mathrm{bd}_{10l_0r}(X_1) \cap \mathrm{bd}_{10l_0r}(X_2)$.

Let $N(\delta) = 100 l_0$. Notice that $P_l'(x)$ has nonnegative coefficients and $\{P_l'(x)\}_l$ is uniformly bounded on $[0, 99/100]$. This, together with the definition of $\partial_0(u)$, implies that there exists a universal constant N_0 such that $\partial_0(u) \in QP_{N_0\delta, N(\delta)r, N_0 s, k}(\mathrm{bd}_{N(\delta)r/10} X_1 \cap \mathrm{bd}_{N(\delta)r/10} X_2)$.

Assume that $r < r_0/(1 + N(\delta))$, where r_0 is as in Definition 5.1. Let f be the proper strong Lipschitz map from $\mathrm{bd}_{N(\delta)r/10}(X_1) \cap \mathrm{bd}_{N(\delta)r/10}(X_2)$ to $X_1 \cap X_2$ realizing the strong Lipschitz homotopy equivalence in Definition 5.1; let $V_f(t)$ be the family of isometries as in Lemma 3.3, where $\{\varepsilon_k\}_k$ (in Lemma 3.3) is chosen in such a way that $\sup_k \varepsilon_k < r/10$.

We define the boundary of u by:

$$\partial(u) = \mathrm{Ad}(V_f)(\partial_0(u)) - (I \oplus 0).$$

Consider the following sequence:

$$QU_{\delta,r,s,k}(X_1) \oplus QU_{\delta,r,s,k}(X_2) \xrightarrow{j} QU_{\delta,r,s,k}(X) \xrightarrow{\partial} GQP_{N_0\delta, N(\delta)r, N_0 s, k}(X_1 \cap X_2),$$

where $j(u_1 \oplus u_2) = (u_1 + \chi_{X - X_1}) \oplus (u_2 + \chi_{X-X_2})$, $r < r_0/(1 + N(\delta))$.

LEMMA 5.2. *Let* $(X;X_1,X_2)$ *be as in Definition* 5.1. *The above sequence is asymptotically exact in the following sense:*

(1) *For any* $0 < \delta < 1/100, r > 0, s > 0$, *there exists* $0 < \delta_1 < \delta, 0 < r_1 < \min\{r, r_0/(1+N(\delta_1))\}, s_1 > 0$, *such that* $\partial j(u_1 \oplus u_2)$ *is* (δ,r,s_1)*-equivalent to* 0 *for any* $u_i \in QU_{\delta_1,r_1,s,k}(X_i)$ $(i = 1,2)$, *where* δ_1 *depends only on* δ *and* s; r_1 *depends only on* δ, r *and* s; *and* s_1 *depends only on* s;

(2) *For any* $0 < \delta < 1/100, r > 0, s > 0$, *there exist* $0 < \delta_2 < \delta, 0 < r_2 < \min\{r, r_0/(1+N(\delta_2))\}, s_2 > 0$, *such that if* u *is an element in* $QU_{\delta_2,r_2,s,k}(X)$ *for which* $\partial(u)$ *is* (δ_2,r_2,s)*-equivalent to* 0 *in* $GQP_{\delta_2,r_2,s,k}(X_1 \cap X_2)$, *then there exists* $u_i \in QU_{\delta,r,s_2,k}(X_i)$ $(i = 1,2)$ *such that* $j(u_1 \oplus u_2)$ *is* (δ,r,s_2)*-equivalent to* u, *where* δ_2 *depends only on* δ, s, r_0 *and* c_0 (c_0 *is as in the Definition* 5.1); r_2 *depends only on* δ, r, s, r_0 *and* c_0; *and* s_2 *depends only on* s, r_0 *and* c_0.

Proof. (1) follows from the definition of the boundary map.

(2) By Lemmas 4.6, 4.9 and the definition of boundary map, for any $0 < \delta' < \delta$ and $0 < r_2' < \min\{r, r_0/(1+N(\delta_2))\}$, there exist $0 < \delta_2 < \delta'$ (δ_2 depends only on δ', s and c_0) and $0 < r_2 < r_2'$ (r_2 depends only on δ', r_2', s and c_0) such that, for any u in $QU_{\delta_2,r_2,s,k}(X)$ whose boundary $\partial(u)$ is (δ_2,r_2,s)-equivalent to 0, there exists $y \in QU_{\delta',c_1(\delta',s)r_2',c_2(s),k}(\mathrm{bd}_{N(\delta_2)r_2/10}(X_1) \cap \mathrm{bd}_{N(\delta_2)r_2/10}(X_2))$ for which

$$\|xw(I \oplus 0)w^*x^* - (I \oplus 0)\| < c_3(s)\delta',$$

where $x = y + \chi_{X - \mathrm{bd}_{N(\delta_2)r_2/10}(X_1) \cap \mathrm{bd}_{N(\delta_2)r_2/10}(X_2)}$, $w = w_{u \oplus I}$ and $N(\delta_2)$ are as in the definition of the boundary $\partial(u \oplus I)$; $c_1(\delta',s)$ depends only on δ', s and c_0; $c_2(s)$ and $c_3(s)$ depend only on s and c_0.

This implies that

$$\text{(A)} \qquad \|xw(I \oplus 0) - (I \oplus 0)xw\| < 10c_3(s)\delta'.$$

Hence we have

$$\text{(B)} \qquad xw = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \|b\| \le 10c_3(s)\delta', \quad \|c\| \le 10c_3(s)\delta'.$$

Define

$$v_1 = a\chi_{\mathrm{bd}_{N(\delta_2)r_2/10}(X_1)}.$$

(A) and (B) imply that v_1 is a δ''-quasi-unitary acting on $H_{\mathrm{bd}_{N(\delta_2)r_2/10}(X_1)}$, where $\delta'' = 100\delta'(1+N_0)(1+c_3(s)+c_3^2(s))$ (N_0 is as in the definition of $\partial(u \oplus I)$). (B), together with the definition of w, implies that there exists a constant $N_1(\delta',s)$ (depending only on δ', s and c_0) such that

$$\text{(C)} \qquad \|\chi_{X - \mathrm{bd}_{N_1(\delta',s)r_2}(X_2)}(v_1^*(u \oplus I) - I)\| < c_4(s)\delta',$$

$$\text{(D)} \qquad \|(v_1^*(u \oplus I) - I)\chi_{X - \mathrm{bd}_{N_1(\delta',s)r_2}(X_2)}\| < c_4(s)\delta',$$

where $c_4(s)$ depends only on s. We choose

$$\delta' < \delta/10^{10}(1+N_0)(1+C_0+c_3(s)+c_3^2(s)+c_4(s)),$$

where C_0 is as in Lemma 4.8 (depending only on the Lipschitz constant c_0 in our case). Define

$$v_2 = \chi_{\mathrm{bd}_{N_1(\delta',s)r_2}(X_2)} v_1^*(u\oplus I)\chi_{\mathrm{bd}_{N_1(\delta',s)r_2}(X_2)}.$$

(C) and (D), together with the choice of δ', imply that v_2 is a $(\delta/(1+C_0))$-quasi-unitary acting on $H_{\mathrm{bd}_{N_1(\delta',s)r_2}(X_2)}$.

We have

$$\mathrm{propagation}(v_i) \leq r_2'(1+N(\delta_2)+c_1(\delta',s)).$$

We require

$$0 < r_2' < r_0/100(1+N_1(\delta',s)+N(\delta_2)),$$

where r_0 is as in Definition 5.1. Let f_1 be the proper strong Lipschitz map from $\mathrm{bd}_{N(\delta_2)r_2/10}(X_1)$ to X_1 realizing the strong Lipschitz homotopy equivalence in Definition 5.1; let f_2 be the proper strong Lipschitz map from $\mathrm{bd}_{N_1(\delta',s)r_2}(X_2)$ to X_2 realizing the strong Lipschitz homotopy equivalence in Definition 5.1. Define $u_i = \mathrm{Ad}(V_{f_i})(v_i)$ for $i=1,2$, where $\mathrm{Ad}(V_{f_i})$ is as in Lemma 3.3, and $\{\varepsilon_k\}$ (in Lemma 3.3) is chosen in such a way that $\sup_k \varepsilon_k < r_2'$.

(C) and (D) imply that $(v_1+\chi_{X-\mathrm{bd}_{N(\delta_2)r_2/10}(X_1)})\oplus(v_2+\chi_{X-\mathrm{bd}_{N_1(\delta',s)r_2}(X_2)})$ is $(\delta/(1+C_0), r_3, s_3)$-equivalent to $u\oplus I$, where $r_3 = c_5(\delta',s)r_2'$ for some $c_5(\delta',s)$ depending only on δ', s and c_0; and s_3 depends only on s and c_0. This, together with the construction of u_1 and u_2 and Lemma 4.8, implies that u_1 and u_2 satisfy the desired properties if we choose appropriate r_2' and s_2, where r_2' depends only on δ, r, s, r_0 and c_0; and s_2 depends only on s, r_0 and c_0. □

Combining Lemma 5.2 with Lemma 4.7 we have the following asymptotically exact sequence for QU when $k>1$:

$$QU_{\delta,r,s,k}(X_1)\oplus QU_{\delta,r,s,k}(X_2) \to QU_{\delta,r,s,k}(X) \to QU_{\delta,r,s,k-1}(X_1\cap X_2).$$

6. Spaces with finite asymptotic dimension

In this section we shall prove that the class of finitely generated groups with finite asymptotic dimension (as metric spaces with word length metrics) is hereditary. We shall also prove a localization result which will play an important role in the proof of our main result.

Definition 6.1 ([14]). The *asymptotic dimension of a metric space* is the smallest integer m such that for any $r>0$, there exists a uniformly bounded cover $C = \{U_i\}_{i \in I}$ of the metric space for which the r-multiplicity of C is at most $m+1$; i.e., no ball of radius r in the metric space intersects more than $m+1$ members of C.

Notice that the concept of asymptotic dimension is a coarse geometric analogue of the covering dimension in topology. It is not difficult to verify that the asymptotic dimension is invariant under quasi-isometry. In particular, this implies that if a finitely generated group is endowed with a word-length metric, then its asymptotic dimension does not depend on the choice of the word-length metric.

PROPOSITION 6.2. *Let Γ be a finitely generated group and Γ_1 be a finitely generated subgroup of Γ. If Γ has finite asymptotic dimension as a metric space with a word-length metric, then Γ_1 also has finite asymptotic dimension as a metric space with a word-length metric.*

Proof. For any $r > 0$, let $C = \{U_i\}_{i \in I}$ be the cover for Γ as in Definition 6.1. Define a cover C_1 for Γ_1 by: $C_1 = \{U_i \cap \Gamma_1\}_{i \in I}$. Choose finite generating sets G and G_1 for Γ and Γ_1 such that $G_1 \subseteq G$, G and G_1 are closed under the inverse operation. Let d and d_1 be respectively the word length metrics on Γ and Γ_1 associated to G and G_1. We have $d(x,y) \leq d_1(x,y)$ for all $x, y \in \Gamma_1$. This implies that the r-multiplicity of C_1 is at most $m+1$.

Next we shall prove that C_1 is uniformly bounded with respect to d_1. By assumption there exists $R > 0$ such that the diameter of $U_i \cap \Gamma_1$ is at most R with respect to d for all i. This implies that for any $x \in U_i \cap \Gamma_1$, we can write $x = x_0 g_1 \dots g_k$ for $k \leq R$, where $g_l \in G$ for all $0 \leq l \leq k$, and x_0 is a fixed element in $U_i \cap \Gamma_1$. Let A_R be the set of all elements in Γ_1 which can be written as $h_1 \dots h_j$, where $h_1, \dots, h_j \in G$, and $j \leq R$. It is not difficult to see that A_R is a finite set. Therefore there exists $R_1 > 0$ such that $d_1(A_R, \mathrm{id}) \leq R_1$. This implies that the diameter of $U_i \cap \Gamma_1$ is at most R_1 with respect to d_1 for all i. □

Gromov proved that hyperbolic groups have finite asymptotic dimension as metric spaces with word-length metrics (see page 31–32 in [14]). This result, together with Lemma 6.2, implies that finitely generated subgroups of hyperbolic groups have finite asymptotic dimension as metric spaces with word-length metrics.

Let X be a proper metric space with finite asymptotic dimension m. By the definition of asymptotic dimension there exists a sequence of covers C_k of X for which there exists a sequence of positive numbers $R_k \to \infty$ such that (1) $R_{k+1} > 4R_k$ for all k; (2) *diameter*$(U) < R_k/4$ for all $U \in C_k$; (3) the R_k-multiplicity of C_{k+1} is at most $m+1$; i.e., no ball with radius R_k intersects more than $m+1$ members of C_{k+1}.

Let $C'_k = \{B(U, R_k) \mid U \in C_{k+1}\}$, where $B(U, R_k) = \{x \in X \mid d(x, U) < R_k\}$. (1) and (2) imply that $\{C'_k\}_k$ is an anti-Čech system for X.

Fix a positive integer n_0. For each $n > n_0$, let $r_n = \pi(\frac{R_n}{8R_{n_0+1}} - 1)$. By property (1) of the sequence $\{R_k\}$, there exists $n_1 > n_0$ such that $r_n > \pi/2$ if $n > n_1$ and there exists a sequence of nonnegative smooth functions $\{\chi_n\}_{n>n_1}$ on $[0, \infty)$ for which (1) $\chi_n(t) = 1$ for all $0 \leq t \leq \pi/2$, and $\chi_n(t) = 0$ for all $t \geq r_n$; (2) there exists a sequence of positive numbers $\varepsilon_n \to 0$ satisfying $|\chi'_n(t)| < \varepsilon_n \leq 1$ for all $n > n_1$.

For each $U \in C_{n+1}$ $(n > n_1)$, define

$$U' = \{V \in N_{C'_{n_0}} \mid V \in C'_{n_0}, V \cap U \neq \emptyset\},$$

where $V \in N_{C'_{n_0}}$ is the vertex of $N_{C'_{n_0}}$ corresponding to $V \in C'_{n_0}$. We define a map G_n from $N_{C'_{n_0}}$ to $N_{C'_n}$ by:

$$G_n(x) = \sum_{U \in C_{n+1}} \frac{\chi_n(d(x, U'))}{\sum_{V \in C_{n+1}} \chi_n(d(x, V'))} B(U, R_n)$$

for all $x \in N_{C'_{n_0}}$. Lemma 6.4 (1) shows that $G_n(x)$ is indeed in $N_{C'_n}$.

Let $n > n_1$. We can choose the map $i_{n_0 n}$ from $N_{C'_{n_0}}$ to $N_{C'_n}$ in Conjecture 2.7 in such a way that, for each $V \in C_{n_0+1}$,

$$i_{n_0 n}(B(V, R_{n_0})) = B(U, R_n)$$

for some $U \in C_{n+1}$ satisfying $U \cap V \neq \emptyset$.

Define

$$F(t, x) = tG_n(x) + (1 - t) i_{n_0 n}(x)$$

for all $t \in [0, 1]$, and $x \in N_{C'_{n_0}}$.

LEMMA 6.3. *Let X be a proper metric space with finite asymptotic dimension m, and G_n, F and $i_{n_0 n}$ be as above.*

(1) G_n is a proper Lipschitz map from $N_{C'_{n_0}}$ to $N_{C'_n}$ with a Lipschitz constant depending only on m;

(2) $F(t,x)$ is a strong Lipschitz homotopy between G_n and $i_{n_0 n}$ with a Lipschitz constant depending only on m;

(3) For any $\varepsilon > 0$, $R > 0$, there exists $K > 0$ such that $d(G_n(x), G_n(y)) < \varepsilon$ if $n > K$, $d(x,y) \leq R$, where $N_{C'_0}$ and $N_{C'_n}$ are endowed with the spherical metrics.

Proof. (1) By property (1) of χ_n,

$$\text{(A)} \qquad \sum_{V \in C_{n+1}} \chi_n(d(x, V')) \geq 1$$

for all $x \in N_{C'_{n_0}}$.

Let U be an element in C_{n+1} such that $\chi_n(d(x, U')) \neq 0$ for some $x \in N_{C'_{n_0}}$. By property (1) of χ_n,

$$d(x, U') < r_n. \tag{B}$$

Let

$$x = \sum_i t_i B(V_i, R_{n_0}),$$

where $t_i > 0$, $\sum_i t_i = 1$, $V_i \in C_{n_0+1}$. Combining (B) with the definition of the spherical metric on $N_{C'_{n_0}}$, we have

$$d(V_i, U) < (r_n + \pi/2)2R_{n_0+1}2/\pi + 2R_{n_0+1} = R_n/2, \tag{C}$$

where $d(V_i, U)$ is the distance between two subsets V_i and U of X. (C) implies that

$$\cap_{U:\chi_n(d(x,U'))\neq 0} B(U, R_n) \neq \emptyset.$$

Hence $G_n(x)$ is in $N_{C'_n}$ for all $x \in N_{C'_{n_0}}$. The inequality (C), together with the condition on the R_n-multiplicity of C_{n+1}, implies that there are at most $m+1$ number of nonzero terms in $\sum_{V \in C_{n+1}} \chi_n(d(x, V'))$. Now (1) follows from the above fact, (A) and property (2) of χ_n.

(2) We shall first prove that, for each $x \in N_{C'_{n_0}}$, $G_n(x)$ and $i_{n_0 n}(x)$ live on a common simplex of $N_{C'_n}$.

Let

$$x = \sum_i t_i B(V_i, R_{n_0}),$$

where $t_i > 0$, $\sum_i t_i = 1$, and $V_i \in C_{n_0+1}$. By the choice of $i_{n_0 n}$,

$$i_{n_0 n}(x) = \sum_i t_i B(U_i, R_n),$$

where $U_i \in C_{n+1}, U_i \cap V_i \neq \emptyset$.

$$\chi_n(d(x, U_i')) = 1$$

for all i. Hence both $i_{n_0 n}(x)$ and $G_n(x)$ live on the simplex with vertices $\{B(U, R_n) \mid U \in C_{n+1}, \chi_n(d(x, U')) \neq 0\}$, which is a simplex in $N_{C'_n}$ by the argument in the proof of (1).

Now it is not difficult to see (2).

(3) follows from the properties of χ_n, G_n and an argument similar to that in the proof of (1). □

7. Main result and applications

In this section we shall prove the following result and discuss its applications to topology, geometry and analysis.

THEOREM 7.1. *The coarse Baum-Connes conjecture holds for proper metric spaces with finite asymptotic dimension.*

The above theorem greatly improves a result in [40], where, using a construction of a Chern character, it is proved that the indices of Dirac type operators on a uniformly contractible Riemannian manifold M are nonzero if the asymptotic dimension of M is equal to the dimension of M.

Combining Theorem 7.1 with the descent principle (Theorem 2.8), we obtain the following corollary.

COROLLARY 7.2. *Let Γ be a finitely generated group whose classifying space has the homotopy type of a finite* CW*-complex. If Γ has finite asymptotic dimension (as a metric space with a word-length metric), then the Novikov conjecture holds for Γ.*

The class of finitely generated groups with finite asymptotic dimension is extremely large and is hereditary (cf. Lemma 6.2). A result of Gromov in [14] and Lemma 6.2 imply that finitely generated subgroups of hyperbolic groups have finite asymptotic dimension. Currently no example of a finitely generated group with infinite asymptotic dimension and finite classifying space is known.

Combining Theorem 7.1 with Propositions 4.33 and 4.46 in [31], Theorem 3.2 in [38] and the Lichnerowicz argument [25], we have the following application to the positive scalar curvature problem.

COROLLARY 7.3. *A uniformly contractible Riemannian manifold with finite asymptotic dimension cannot have uniform positive scalar curvature.*

In particular, this corollary implies the Gromov-Lawson-Rosenberg conjecture for compact $K(\pi, 1)$-manifolds when the fundamental group π has finite asymptotic dimension as a metric space with a word-length metric. Recall that the Gromov-Lawson-Rosenberg conjecture says that there is no Riemannian metric with positive scalar curvature on compact $K(\pi, 1)$-manifolds.

Combining Theorem 7.1 with Propositions 4.33 and 4.46 in [31], and Theorem 3.2 in [38], we have the following application to the spectrum of the Laplacian.

COROLLARY 7.4. *Gromov's zero-in-the-spectrum conjecture holds for uniformly contractible Riemannian manifolds with finite asymptotic dimension.*

Recall that Gromov's zero-in-the-spectrum conjecture says that the spectrum of the Laplacian operator acting on the space of L^2-forms of a uniformly contractible Riemannian manifold with bounded geometry contains zero ([13], [14]).

Recall that the dimension of a simplicial complex X is the smallest integer m such that each simplex of X has dimension at most m.

PROPOSITION 7.5. *Let X be a simplicial complex with finite dimension m and endowed with the spherical metric. For any $k > m+1$, $0 < \delta < 1/100$, $r > 0$, $s \geq 0$, there exist $0 < \delta_1 \leq \delta$, $0 < r_1 \leq r$, $s_1 \geq s$ such that every element u in $QU_{\delta_1, r_1, s, k}(X)$ is (δ, r, s_1)-equivalent to I, where δ_1 depends only on δ, s, k and m; r_1 depends only on δ, r, s, k and m; and s_1 depends only on s, k, and m.*

Proof. Let $X^{(n)}$ be the n-skeleton of X and be endowed with the subspace metric induced from the spherical metric on X. We shall prove our proposition for $X^{(n)}$ by induction on n.

When $n = 0$, we choose r_1 to be $\min\{r, \pi/10\}$. By the choice of r_1, u has propagation 0. Without loss of generality we can assume that $u(t)$ is piecewise smooth with respect to $t \in [0, \infty)$ and $\|\frac{du}{dt}\|$ is at most $1/c$ for some constant $c > 0$. For each $t_0 \in [0, \infty)$, we define

$$u_{t_0}(t) = \begin{cases} I & \text{if } 0 \leq t \leq ct_0; \\ u(ct - ct_0) & \text{if } ct_0 \leq t < \infty. \end{cases}$$

Let H_X be as in Definition 2.3. Define

$$H_X^\infty = (\oplus_{k=0}^\infty H_X) \oplus H_X.$$

Notice that H_X^∞ is a standard nondegenerate X-module. Consider

$$w_1(t') = (\oplus_{k=0}^\infty u_k \oplus I)(I \oplus_{k=1}^\infty u_{k-t'}^{-1} \oplus I),$$

where $t' \in [0, 1]$ and $w_1(t')$ acts on H_X^∞. Notice that

$$w_1(0) = u \oplus_{k=1}^\infty I \oplus I.$$

It is easy to see that there exists a smooth homotopy $v(t')$ ($t' \in [0, 1]$) such that

$$\begin{aligned} v(0) &= I \oplus_{k=1}^\infty u_{k-1}^{-1} \oplus I; \\ v(1) &= \oplus_{k=0}^\infty u_k^{-1} \oplus I; \\ &\|v'(t')\| \leq 100 \end{aligned}$$

for all $t' \in [0, 1]$. Define

$$w(t') = \begin{cases} w_1(2t') & \text{if } 0 \leq t' \leq 1/2; \\ (\oplus_{k=0}^\infty u_k \oplus I) v(2t' - 1) & \text{if } 1/2 \leq t' \leq 1. \end{cases}$$

It is not difficult to see that $w(t')$ realizes the (δ, r, s_1)-equivalence between $u \oplus I$ and I if we choose appropriate δ_1 and s_1, where δ_1 depends only on δ, and s_1 depends only on s.

Assume by induction that the proposition holds when $n = l - 1$. Next we shall prove the proposition when $n = l$. For each simplex $\triangle$ of dimension l in X, we define

$$\triangle_1 = \{x \in \triangle \mid d(x, c(\triangle)) \leq 1/100\}, \quad \triangle_2 = \{x \in \triangle \mid d(x, c(\triangle)) \geq 1/100\},$$

where $c(\triangle)$ is the center of $\triangle$. Let

$X_1 = \cup_{\triangle:\text{ simplex of dimension } l \text{ in } X}\triangle_1, \qquad X_2 = \cup_{\triangle:\text{ simplex of dimension } l \text{ in } X}\triangle_2.$

Notice that

(1) X_1 is strongly Lipschitz homotopy equivalent to

$$\{c(\triangle) \mid \triangle\colon\ l-\text{dimensional simplex in } X\};$$

(2) X_2 is strongly Lipschitz homotopy equivalent to $X^{(l-1)}$;

(3) $X^{(l)} = X_1 \cup X_2$, and $X_1 \cap X_2$ is the disjoint union of the boundaries of all l-dimensional $\triangle_1$ in $X^{(l)}$.

(1) and (2), together with Lemma 4.8 and the induction hypothesis, imply that our proposition holds for X_1 and X_2. By Lemmas 4.8, 4.7 and 5.2, we also know that our proposition holds for $X_1 \cap X_2$. It is not difficult to verify that $(X^{(l)}; X_1, X_2)$ satisfies the strong excision condition in Definition 5.1. Now we can complete the induction process by using the asymptotically exact sequence for QU (cf. Lemmas 5.2 and 4.7). □

Proof of Theorem 7.1. Let X be a proper metric space with asymptotic dimension m. By Theorem 3.5 it is enough to prove that

$$\lim_{n\to\infty} K_i(C^*_{L,0}(N_{C'_n})) = 0,$$

where C'_n is as in Section 6. By Lemma 4.5 any element in $K_i(C^*_{L,0}(N_{C'_{n_0}}))$ can be represented as an element $u \in QU_{\delta_1,r,s,k}(N_{C'_{n_0}})$ for some r, s and $k > m+1$, where δ_1 is as in Proposition 7.5 for some $0 < \delta < 1/100$. Let

$$u_n = \mathrm{Ad}(V_{G_n})(u),$$

where G_n is as in Lemma 6.3, $\mathrm{Ad}(V_{G_n})$ is as in Lemma 3.3, and $\{\varepsilon_k\}_k$ (in Lemma 3.3) is chosen in such a way that $\sup_k \varepsilon_k < r_1/10$, where r_1 is as in Proposition 7.5. By Lemma 6.3, there exists $K > 0$ such that u_n has propagation at most r_1 for $n > K$. Our assumption on the asymptotic dimension of X implies that the dimension of $N_{C'_n}$ is at most m for all n. By Proposition 7.5 we know that u_n is (δ, r, s_1)-equivalent to I in $QU_{\delta,r,s,k}(N_{C'_n})$ for $n > K$, where s_1 is as in Proposition 7.5. By Lemmas 4.8 and 6.3, u_n is equivalent to $\mathrm{Ad}(V_{i_{n_0 n}})(u)$ in $K_i(C^*_{L,0}(N_{C'_n}))$. Hence $[u] = 0$ in $\lim_{n\to\infty} K_i(C^*_{L,0}(N_{C'_n}))$. □

8. A counterexample to the coarse Baum-Connes conjecture

In this section we shall construct a proper metric space with infinite asymptotic dimension for which the coarse Baum-Connes conjecture fails. This indicates that our result on the coarse Baum-Connes conjecture is best possible in some sense.

Let X be the disjoint union of S^{2n} ($n = 1, 2, 3, \ldots$), where S^{2n} is the sphere of dimension $2n$. Endow a metric d on X such that

(1) $d|_{S^{2n}} = nd_n$, where $d|_{S^{2n}}$ is the restriction of d to S^{2n}, and d_n is the standard Riemannian metric on the sphere S^{2n} with radius 1;

(2) $d(S^{2n}, S^{2n'}) > \max\{n, n'\}$ if $n \neq n'$.

PROPOSITION 8.1. *Let X be the proper metric space defined as above.*

(1) X is a counterexample to the coarse Baum-Connes conjecture;

(2) X has infinite asymptotic dimension.

Proof. Let D_n be the Dirac operator on S^{2n}. Define

$$D = \oplus_{n=1}^{\infty} D_n.$$

D gives rise to a K-homology class $[D]$ in $K_0(X)$. Let k_n be the scalar curvature for S^{2n} with the Riemannian metric nd_n. We have

$$k_n = \frac{2n(2n-1)}{2}\frac{1}{n^2} \geq 1.$$

This, together with the Lichnerowicz formula, implies that D is invertible. It follows that the index of D, $\mathrm{Ind}(D)$, is zero in $K_0(C^*(X))$.

Let $\{C_k\}_{k=1}^{\infty}$ be an anti-Čech system for X such that each C_k is an open cover for X. Let $\{\phi_U\}_{U \in C_1}$ be a continuous partition of unity subordinate to C_1. Define a proper continuous map ϕ from X to N_{C_1} by:

$$\phi(x) = \sum_{U \in C_1} \phi_U(x) U.$$

By the properties of the metric d on X, it is not difficult to see that $\phi_*([D]) \neq 0$ in $\lim_{k\to\infty} K_0(N_{C_k})$. This, together with the fact that $\mathrm{Ind}(D) = 0$, implies that X is a counterexample to the coarse Baum-Connes conjecture.

The proof for (2) is straightforward and is therefore omitted. □

Finally we remark that, based on the above example, one can construct a counterexample to the coarse Baum-Connes conjecture which is a connected finite-dimensional Riemannian manifold and has infinite asymptotic dimension.

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