AUTOMORPHIC VECTOR BUNDLES

2(b) Automorphic vector bundles

Let G be any connected semisimple Lie group, $K \subset G(\mathbb{R})$ a maximal compact subgroup, (τ, W) a finite-dimensional complex representation of K, $\Gamma \subset G(\mathbb{R})$ a torsion-free discrete arithmetic subgroup. We let $X = G(\mathbb{R})/K$ be the symmetric space attached to G and define the homogeneous vector bundle

$$[W] = \Gamma\backslash(G \times W)/K \to X_\Gamma = \Gamma\backslash X.$$

Here Γ acts trivially on W but K acts on $G \times W$ on the right by $(g, w)k = (gk, \tau(k)^{-1}w)$. The global sections of this vector bundle are then invariant functions from $G(\mathbb{R})$ to W:

$$f(\gamma g k) = \tau(k)^{-1} f(g). \tag{1}$$

Indeed, this is consistent:

$$f(gk_1k_2) = \tau(k_1k_2)^{-1}f(g) = \tau(k_2)^{-1}[\tau(k_1)^{-1}f(g)] = \tau(k_2)^{-1}f(gk_1).$$

The invariance condition (1) is one of the conditions defining automorphic forms, the other two being the moderate growth condition and annihilation by an ideal of finite codimension in the center of $U(\mathfrak{g})$.

We will mostly be assuming X has a $G(\mathbb{R})$-invariant complex structure, so that X_Γ is a complex analytic variety. Then it is well known, and we will admit, that X_Γ has a projective embedding, and so is in fact a complex algebraic variety. The construction of $[W]$ then shows that it has (at least) an analytic structure and (in fact) an algebraic structure. This goes as follows. We have seen that the Borel embedding $\beta : X \hookrightarrow \hat{X}$ is a complex open immersion that commutes with the $G(\mathbb{R})$ action. Now (τ, K) extends to an algebraic representation of the algebraic group K_h, the complexification of K, in the notation of the previous lecture. This in turn extends to the stabilizer P_h of $\beta(h)$ in G, which means that $(G(\mathbb{R}) \times W)/K$ extends to the $G(\mathbb{C})$-equivariant vector bundle $\hat{W} = (G(\mathbb{C}) \times W)/P_h$ over $\hat{X}$. This is obviously a complex analytic vector bundle – even algebraic over the flag variety $\hat{X}$. The pullback and quotient by Γ preserves the analytic structure.

More generally, we can let (G, X) be a Shimura datum and define

$$[W] = G(\mathbb{Q})\backslash G(\mathbf{A}) \times W/(K_h \cap G(\mathbb{R})) \times K \to {}_K Sh(G, X).$$

Let $g \in G(\mathbf{A}_f)$

We are interested in the coherent cohomology (in the analytic or Zariski topology) $H^q(X_\Gamma, [W])$ or $H^q({}_K Sh(G, X), [W])$. Let $d = \dim(X)$. For reasons to be explained later, this is only the right object when G^{der} has $\mathbb{Q}$-rank 0 (so the Shimura

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variety is projective), but the following discussion is valid in complete generality. To distinguish the holomorphic vector bundle $[W]$ from the associated locally free sheaf, I will temporarily label the latter $\mathcal{O}[W]$. Then as on any analytic variety one can compute its cohomology by the Dolbeault resolution:

$$0 \to \mathcal{O}[W] \to \mathcal{A}^{0,0}[W] \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}[W] \xrightarrow{\bar{\partial}} \ldots \to \mathcal{A}^{0,d-1}[W] \xrightarrow{\bar{\partial}} \mathcal{A}^{0,d}[W] \to 0.$$

Here $\mathcal{A}^{0,q}[W] = \mathcal{A}^{0,q} \otimes \mathcal{O}[W]$, where $\mathcal{A}^{0,q}$ is the sheaf of local C^∞ antiholomorphic q-forms, and $\bar{\partial}$ is the antiholomorphic derivative, given locally on functions by $f \mapsto \sum_i \frac{\partial f}{d\bar{z}_i} \otimes d\bar{z}_i$ and extended antisymmetrically to antiholomorphic forms. The exactness of the Dolbeault resolution is the $\bar{\partial}$-Poincaré lemma, and so, letting $A^{0,q}[W] = \Gamma({}_K Sh(G,X), \mathcal{A}^{0,q}[W])$, we have the Dolbeault isomorphism

$$H^q({}_K Sh(G,X),[W]) \xrightarrow{\sim} (\ker \bar{\partial} : A^{0,q}[W] \to A^{0,q+1}[W])/(im\bar{\partial} : A^{0,q-1}[W] \to A^{0,q}[W]).$$

The existence of a canonical invariant hermitian metric on $Sh(G,X)$ allows us to compute this more readily in terms of Hodge theory. Choose a $K_h \cap G^{der}(\mathbb{R})$-invariant hermitian metric on W. This extends by the action of $G(\mathbb{R})$ uniquely to a hermitian metric on $(G \times W)/K$ invariant under $G^{der}(\mathbb{R})$, and hence under any sufficiently small arithmetic subgroup Γ. There is also an invariant volume form (which pulls back to Haar measure on $G^{der}(\mathbb{R})$. There is something to be done about the action of the non-compact part of the center of $G(\mathbb{R})$ but this can be ignored in what follows. This metric defines an L_2-norm on each $A^{0,q}[W]$) by

$$||\omega||^2 = \int_{G(\mathbb{Q})\backslash G^{der}(\mathbb{R}) \times G(\mathbf{A}_f)/[K_h \cap G^{der}(\mathbb{R}) \times K]} \omega \wedge \star\omega.$$

We write $\mathcal{H}^{0,q}[W]$ for the Hilbert space completion of $A^{0,q}[W]$. Then (since X_Γ is a complete manifold, whether or not it is compact) there is a (densely defined) dual map $\bar{\partial}^* : \mathcal{H}^{0,q}[W] \to \mathcal{H}^{0,q-1}[W]$ and the Laplacian

$$\Box^q = \bar{\partial}^* \circ \bar{\partial} + \bar{\partial} \circ \bar{\partial}^* : \mathcal{H}^{0,q}[W] \to \mathcal{H}^{0,q}[W]$$

is an elliptic PDE; its kernel $H^{0,q}[W]$ (*harmonic forms*) is contained in the C^∞ space $A^{0,q}[W]$, is finite-dimensional, and there is a canonical isomorphism for each q.

$$H^{0,q}[W] \xrightarrow{\sim} H^q({}_K Sh(G,X),[W]).$$

What is important for us is that $\Box^q$ is a $G(\mathbb{R})$-invariant differential operator on X, and in fact comes from an element of $Z(\mathfrak{g})$. Since $H^{0,q}[W]$ is a finite-dimensional space, an ideal of $Z(\mathfrak{g})$ of finite codimension (in fact, codimension 1) annihilates $H^{0,q}[W]$ and therefore it consists of a space of automorphic forms. It is better, and more consistent with the adelic approach automorphic forms, to reinterpret the above calculation purely in terms of the representation theory of Lie groups. The point is that an automorphic representation has an archimedean and a non-archimedean adelic part, and the cohomology calculations only concern the archimedean part. Ultimately the determination of harmonic forms is translated into an algebraic problem in the representation theory of $U(\mathfrak{g})$.