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Bachir Bekka, Pierre de la Harpe  
and Alain Valette

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# Kazhdan's Property (T)

BACHIR BEKKA, PIERRE DE LA HARPE AND  
ALAIN VALETTE



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# Symbols

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$\mathcal{U}(\mathcal{H})$	unitary group of a Hilbert space $\mathcal{H}$ , page 27
$1_G$	unit representation of a topological group $G$ , page 28
$\kappa(G, Q, \pi)$	Kazhdan constant, page 29
$\kappa(G, Q)$	Kazhdan constant, page 30
$\mathcal{H}^G$	subspace of $G$ -invariant vectors in $\mathcal{H}$ , page 31
$\subset$	sign for subrepresentation, page 32
$\delta_x$	Dirac measure at a point $x$ , page 37
$F_k$	non-abelian free group on $k$ generators, page 39
$SL_n(\mathbf{K})$	special linear group, page 40
$\mathcal{L}(\mathcal{H})$	algebra of bounded operators on a Hilbert space, page 40
$E_{ij}(x)$	elementary matrix, page 44
$Sp_{2n}(\mathbf{K})$	symplectic group, page 50
$S^{2*}(\mathbf{K}^2)$	space of symmetric bilinear forms on $\mathbf{K}^2$ , page 51
$S^2(\mathbf{K}^2)$	second symmetric power of $\mathbf{K}^2$ , page 51
$\text{rank}_{\mathbf{K}} \mathbb{G}$	$\mathbf{K}$ -rank of an algebraic group $\mathbb{G}$ , page 58
$\mathbb{G}(\mathbf{K})$	group of $\mathbf{K}$ -rational points of an algebraic group $\mathbb{G}$ , page 58
$HS(\mathcal{H})$	space of Hilbert–Schmidt operators on a Hilbert space $\mathcal{H}$ , page 65
$\tilde{G}$	universal covering group of $G$ , page 67
$M_{n,m}(\mathbf{K})$	space of $(n \times m)$ matrices over $\mathbf{K}$ , page 68
$M_n(\mathbf{K})$	algebra of $(n \times n)$ matrices over $\mathbf{K}$ , page 68
$H_{2n+1}(\mathbf{K})$	Heisenberg group, page 69
$\mathcal{O}(\mathcal{H}^0)$	orthogonal group of a real Hilbert space $\mathcal{H}^0$ , page 74
$Z^1(G, \pi)$	space of 1-cocycles with coefficients in a representation $\pi$ , page 76
$B^1(G, \pi)$	space of 1-coboundaries in $Z^1(G, \pi)$ , page 76
$H^1(G, \pi)$	1-cohomology group with coefficients in a representation $\pi$ , page 76

$\text{Hom}(G, H)$	space of continuous homomorphisms from $G$ to $H$ , page 77
$X = (V, \mathbb{E})$	graph with vertex set $V$ and edge set $\mathbb{E}$ , page 81
$\mathcal{H}_{\mathbf{R}}$	complex Hilbert space $\mathcal{H}$ viewed as a real Hilbert space, page 88
$\pi_{\mathbf{R}}$	unitary representation $\pi$ viewed as an orthogonal representation, page 88
$O(n, 1)$	orthogonal group of the form $-x_{n+1}^2 + \sum_{i=1}^n x_i^2$ over $\mathbf{R}^{n+1}$ , page 93
$\mathbf{H}^n(\mathbf{R})$	real hyperbolic space, page 93
$U(n, 1)$	orthogonal group of the form $- z_{n+1} ^2 + \sum_{i=1}^n  z_i ^2$ on $\mathbf{C}^{n+1}$ , page 100
$Sp(n, 1)$	orthogonal group of the form $- z_{n+1} ^2 + \sum_{i=1}^n  z_i ^2$ on $\mathbf{H}^{n+1}$ , page 100
$\mathbf{H}^n(\mathbf{C})$	complex hyperbolic space, page 100
$\mathbf{H}^n(\mathbf{H})$	quaternionic hyperbolic space, page 100
$\Gamma \wr H$	wreath product of the groups $\Gamma$ and $H$ , page 105
$\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$	a diffeomorphism group of the circle, page 107
$\text{Homeo}_+(\mathbf{R})$	group of orientation preserving homeomorphisms of $\mathbf{R}$ , page 116
$C_c(G//K)$	convolution algebra of continuous $K$ -bi-invariant functions on $G$ with compact support, page 152
$M(G//K)$	algebra of $K$ -bi-invariant compactly supported probability measures on $G$ , page 154
$R[t]$	ring of polynomials over a ring $R$ , page 201
$R[[t^{-1}]]$	ring of formal power series in $t^{-1}$ over $R$ , page 201
$G^X$	group of continuous mappings from $X$ to $G$ , page 210
$LG$	loop group of $G$ , page 210
$\Omega_{\mathcal{H}}^k(X)$	Hilbert spaces associated to a random walk, page 219
$\mathcal{E}_{\pi}^k(X)$	Hilbert spaces associated to $G$ -equivariant random walk, page 227
$\partial A$	boundary of a subset $A$ of a graph, page 254
$h(\mathcal{G})$	expanding constant of a graph $\mathcal{G}$ , page 254
$C(G)_{1,+}$	continuous non-negative on $G$ of integral one, page 262
$L_0^2(\Omega)$	space of square-integrable functions on $\Omega$ with zero mean, page 265
$\langle \xi, \eta \rangle$	inner product of two vectors, page 289
$\pi_1 \simeq \pi_2$	equivalent representations, page 290
$\rho \subset \pi$	representation $\rho$ contained in $\pi$ , page 291
$\widehat{G}$	unitary dual (dual group, in case $G$ is abelian), page 292
$\mathbf{S}^1$	circle group, page 293

$\pi(G)'$	commutant of $\pi(G)$ , page 296
$\mathcal{B}(X)$	$\sigma$ -algebra of Borel subsets in a topological space $X$ , page 299
$C_c(X)$	space of complex-valued functions with compact support on a topological space $X$ , page 300
$\check{f}$	$\check{f}(x) = f(x^{-1})$ , page 303
${}_af$	${}_af(x) = f(ax)$ for a function $f$ on a group $G$ and $a \in G$ , page 305
$f_a$	$f_a(x) = f(xa)$ for a function $f$ on a group $G$ and $a \in G$ , page 305
$\lambda_G$	left regular representation of a locally compact group $G$ , page 305
$\lambda_{G/H}$	quasi-regular representation of a locally compact group $G$ on $L^2(G/H)$ , page 331
<b>P</b>	Poincaré half-plane, page 331
<b>H</b>	Hamiltonian quaternions, page 337
$C(X)$	space of complex-valued continuous functions on a topological space $X$ , page 342
$f * g$	convolution product of two functions $f$ and $g$ on a locally compact group, page 352
$\tilde{f}$	$\tilde{f}(x) = f(x^{-1})$ , page 353
$C_0(X)$	space of complex-valued continuous functions vanishing at infinity on a topological space $X$ , page 353
$\mathcal{P}(G)$	convex cone of functions of positive type on a topological group $G$ , page 357
$\mathcal{P}_1(G)$	convex set of normalised functions of positive type on $G$ , page 357
$\text{ext}(\mathcal{P}_1(G))$	extreme points of $\mathcal{P}_1(G)$ , page 358
$L^\infty(G)$	space of complex-valued locally measurable functions on $G$ which are bounded locally almost everywhere, page 359
$f^*$	$f^*(x) = \Delta_G(x^{-1})f(x^{-1})$ , page 360
$L^1(G)$	Banach $*$ -algebra of absolutely integrable complex-valued functions on a locally compact group, page 360
$\mathcal{P}_{\leq 1}(G)$	convex set of $\varphi \in \mathcal{P}(G)$ with $\varphi(e) \leq 1$ , page 360
$\Delta(A)$	character space of a Banach algebra $A$ , page 370
$M(G)$	Banach- $*$ -algebra of finite complex regular measures on $G$ , page 370
$d\mu^*(x)$	defined for a measure $\mu$ on a group by $d\mu^*(x) = \overline{d\mu(x^{-1})}$ , page 370
$\hat{\mu}$	Fourier transform of a measure $\mu \in M(G)$ , page 370

$C^b(X)$	space of complex-valued bounded continuous functions on a topological space $X$ , page 370
$\widehat{f}$	Fourier transform of a function $f \in L^1(G)$ , page 371
$M_{\leq 1}(G)$	space of positive Borel measures $\mu$ on $G$ with $\mu(G) \leq 1$ , page 372
$\text{Proj}(\mathcal{H})$	set of orthogonal projections on subspaces of a Hilbert space $\mathcal{H}$ , page 373
$\mathbf{Q}_p$	field of $p$ -adic numbers, page 377
$\mathbf{Z}_p$	ring of $p$ -adic integers, page 378
$k((X))$	field of Laurent series, page 378
$\text{Ind}_H^G \sigma$	induced representation, page 388
$\prec$	sign for weak containment of unitary representations, page 396
$\pi_{\text{univ}}$	universal representation of a locally compact group, page 412
$C^*(G)$	maximal $C^*$ -algebra of a locally compact group, page 412
$C_{\text{red}}^*(G)$	reduced $C^*$ -algebra of a locally compact group, page 413



# Introduction

---

In the mid-1960s, D. Kazhdan defined his *Property (T)* for locally compact groups and used it as a tool to demonstrate that a large class of lattices are finitely generated. Recall that a lattice  $\Gamma$  in a locally compact group  $G$  is a discrete subgroup such that the quotient space  $G/\Gamma$  carries a  $G$ -invariant probability measure; arithmetic and geometry provide many examples of countable groups which are lattices in semisimple groups, such as the special linear groups  $SL_n(\mathbf{R})$ , the symplectic groups  $Sp_{2n}(\mathbf{R})$ , and various orthogonal or unitary groups. Property (T) was defined in terms of unitary representations, using only a limited representation theoretic background. Later developments showed that it plays an important role in many different subjects.

Chapter 1 begins with the original definition of Kazhdan:

A topological group  $G$  has *Property (T)* if there exist a compact subset  $Q$  and a real number  $\varepsilon > 0$  such that, whenever  $\pi$  is a continuous unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$  for which there exists a vector  $\xi \in \mathcal{H}$  of norm 1 with  $\sup_{q \in Q} \|\pi(q)\xi - \xi\| < \varepsilon$ , then there exists an invariant vector, namely a vector  $\eta \neq 0$  in  $\mathcal{H}$  such that  $\pi(g)\eta = \eta$  for all  $g \in G$ .

We discuss some of its first implications, the trivial examples (which are the compact groups), and the following three main ingredients of Kazhdan's proof of the finite generation of lattices:

- (i) A locally compact group with Property (T) is compactly generated, and in particular a discrete group with Property (T) is finitely generated.
- (ii) For a local field  $\mathbf{K}$  (in particular for  $\mathbf{K} = \mathbf{R}$ ), the groups  $SL_n(\mathbf{K})$ ,  $n \geq 3$ , and  $Sp_{2n}(\mathbf{K})$ ,  $n \geq 2$ , have Property (T). This carries over to other groups  $\mathbb{G}(\mathbf{K})$  of  $\mathbf{K}$ -rational points of appropriate simple algebraic groups, and in particular to simple real Lie groups of real rank at least two.

(iii) A lattice  $\Gamma$  in a locally compact group  $G$  has Property (T) if and only if  $G$  has it.

Chapter 2 concentrates on a property which was shown in the late 1970s to be equivalent to Kazhdan's property for a large class of groups:

A topological group  $G$  has *Property (FH)* if any continuous action of  $G$  by affine isometries on a Hilbert space has a fixed point.

We have kept the discussions in Chapters 1 and 2 mostly independent of each other, so that either can be chosen as an introduction to our subject.

If  $\pi$  is a representation of a group  $G$ , let  $H^1(G, \pi)$  denote the first cohomology space of  $\pi$ . It is straightforward to translate Property (FH) as a vanishing property:  $H^1(G, \pi) = 0$  for all unitary representations  $\pi$  of  $G$ . There are strong consequences on several types of actions: for a group with Property (FH), any isometric action on a tree has a fixed point or a fixed edge (this is *Property (FA)* of Serre), any isometric action on a real or complex hyperbolic space has a fixed point, and any action on the circle which is orientation preserving and smooth enough factors through a finite cyclic group (a result of Navas and Reznikov). There is also a reformulation of Property (FH) in terms of scalar-valued functions on the group: any function which is conditionally of negative type is bounded.

In the last section, we prove the Delorme–Guichardet theorem: for  $\sigma$ -compact locally compact groups, Properties (T) and (FH) are equivalent.

Chapter 3 is devoted to reduced cohomology spaces  $\overline{H}^1(G, \pi)$ , which are the Hausdorff spaces associated to the cohomology spaces  $H^1(G, \pi)$  for the appropriate topology;  $\overline{H}^1$  is from several points of view a “better” functor than  $H^1$ . For a compactly generated locally compact group  $G$ , Shalom has shown that several vanishing properties are equivalent, including:

- (i) Property (T) or (FH), namely the vanishing of the cohomology space  $H^1(G, \pi)$  for every unitary representation  $\pi$  of  $G$ ;
- (ii) the vanishing of the *reduced* cohomology space  $\overline{H}^1(G, \pi) = 0$  for every unitary representation  $\pi$ .

Shalom's result implies that a countable group with Property (T) is always the quotient of a finitely presented group with Property (T). This answers a natural question, since Property (T) implies finite generation (Kazhdan's observation) but not finite presentation (as shown by examples discovered later).

There is also a section on Kostant's result according to which the isometry group  $Sp(n, 1)$  of a quaternionic hyperbolic space ( $n \geq 2$ ) has Property (T). The



method of Section 3.3 uses properties of harmonic mappings and rests on ideas of Gromov.

To find examples of non-compact groups with Property (T), the only methods known from the time of Kazhdan's paper until about 30 years later have been to use at some point the theory of Lie groups or of algebraic groups, and the resulting fact that groups like  $SL_n(\mathbf{K})$ ,  $n \geq 3$ , and  $Sp_{2n}(\mathbf{K})$ ,  $n \geq 2$ , have Property (T). Chapter 4 focuses on another method, due to Shalom, that shows again Property (T) for  $SL_n(\mathbf{Z})$ , and as a bonus shows it also for  $SL_n(R)$  for other rings  $R$  ( $n \geq 3$ ).

The new notion entering the scene is that of bounded generation,<sup>1</sup> of which the relevance for Property (T) was pointed out by Colin de Verdière and Shalom. On the one hand, the method can be used to estimate various *Kazhdan constants*, namely to obtain quantitative sharpenings of the qualitative notion of Property (T). On the other hand, the groups to which the method applies need not be locally compact. For example, for  $n \geq 3$ , the loop group  $LSL_n(\mathbf{C})$  consisting of all continuous functions from the circle to  $SL_n(\mathbf{C})$  has Property (T).

Chapter 5 is an account of the so-called *spectral criterion*. More precisely, given a group  $\Gamma$  generated by a finite set  $S$ , there is a finite graph  $\mathcal{G}(S)$  attached to the situation; if this graph is connected and if its smallest non-zero eigenvalue is strictly larger than  $1/2$ , then  $\Gamma$  has Property (T); moreover, the method gives an estimate of the Kazhdan constant for  $S$ . The spectral criterion is due to Zuk (1996) and Ballmann-Swiatkowski (1997); it relies on some fundamental work by Garland and Borel (1973), and is strongly used in more recent work of Gromov (2003) and others concerning random groups.

Chapter 6 is a small sample of applications of Property (T). We indicate a construction, due to Margulis, of finite graphs with good expanding properties. Then we discuss some applications to ergodic theory: estimates of spectral gaps for operators associated to appropriate actions, the importance for the so-called strongly ergodic actions (Schmidt and Connes–Weiss), and invariance of Property (T) by “measure equivalence” (work of Furman and Popa). The final section of Chapter 6 is about the Banach–Ruziewicz problem, which asks whether the normalised Lebesgue measure on the unit sphere  $\mathbf{S}^{n-1}$  of  $\mathbf{R}^n$  is the unique rotation-invariant *finitely additive* measure defined on all Lebesgue-measurable sets; the answer, which is positive for all  $n \geq 3$ , follows when  $n \geq 5$  from the fact that the special orthogonal group  $SO(n)$  contains a dense subgroup which has Property (T).

<sup>1</sup> A group  $\Gamma$  is boundedly generated if there exist a finite family  $C_1, \dots, C_k$  of cyclic subgroups in  $\Gamma$  and an integer  $N \geq 1$  such that any  $\gamma \in \Gamma$  is a product of at most  $N$  elements from the union  $\cup_{1 \leq j \leq k} C_j$ .

Despite their importance, applications to operator algebras in general and the work of Popa in particular are almost not discussed within this book.

Chapter 7 is a short collection of open problems which, at the time of writing, are standard in the subject.

A significant part of the theory of Property (T) uses the theory of unitary representations in a non-technical way. Accordingly, we use freely in this book the “soft parts” of representation theory, with as little formalism as possible. The reader who wishes to rely now and then on a more systematic exposition will find one in the appendix which appears as the second half of this book.

We discuss there some of the basic notions: generalities about unitary representations, invariant measures on homogeneous spaces, functions of positive type and functions conditionally of negative type, unitary representations of abelian groups, unitary induction, and weak containment. Moreover, we have one chapter on amenable groups, a notion which goes back to the time (1929) when von Neumann wrote up his view of the Hausdorff–Banach–Tarski paradox (itself from the period 1914–1924); amenability and Property (T) are two fundamental properties in our subject, and the second cannot be fully appreciated without some understanding of the first.

The size of the present book has grown to proportions that we did not plan! There are several much shorter introductions to the subject which can be recommended: the original Bourbaki seminar [DelKi–68], Chapter 7 of [Zimm–84a], Chapter III of [Margu–91], Chapter 3 of [Lubot–94], Chapter 5 of [Spatz–95], the Bourbaki seminar by one of us [Valet–04], and a book in preparation [LubZu] on *Property* ( $\tau$ ), which is a variant of Property (T). Despite its length, the present book is far from complete; the list of references should help the reader to appreciate the amount of material that we do not discuss.

## Historical introduction

For readers who already have some knowledge of Property (T), here is our personal view on the history of this notion.

### The first 25 years

#### First appearance of Property (T)

The subject of this book began precisely with a three page paper [Kazhd–67]. Kazhdan’s insight was the key to many unexpected discoveries. Indeed, in the mathematical literature, there are very few papers with such a rich offspring. Property (T) is now a basic notion in domains as diverse as group

theory, differential geometry, ergodic theory, potential theory, operator algebras, combinatorics, computer science, and the theory of algorithms.

On the one hand, Kazhdan defines a locally compact group  $G$  to have *Property (T)*, now also called the *Kazhdan Property*, if the unit representation<sup>2</sup> is isolated in the appropriate space of unitary representations of  $G$ . It is straightforward to show that a group  $G$  with this property is compactly generated and that its largest Hausdorff abelian quotient  $G/[\overline{G}, \overline{G}]$  is compact; in particular, a countable group  $\Gamma$  with Property (T) is finitely generated and its first homology group  $H_1(\Gamma, \mathbf{Z}) = \Gamma/[\Gamma, \Gamma]$  is finite. On the other hand, Kazhdan shows that, besides compact groups, groups having Property (T) include  $SL_n(\mathbf{K})$ ,  $n \geq 3$ , and  $Sp_{2n}(\mathbf{K})$ ,  $n \geq 2$ , for any local field  $\mathbf{K}$ . This implies in particular that a simple real Lie group  $G$  with large real rank and with finite centre has Property (T); in Kazhdan's paper, "large" real rank  $l$  means  $l \geq 3$ , but shortly afterwards<sup>3</sup> it was checked that  $l \geq 2$  is sufficient. Moreover, a lattice  $\Gamma$  in a locally compact group  $G$  has Property (T) if and only if  $G$  has Property (T).

One spectacular consequence of these results and observations can be phrased as follows. Let  $M = \Gamma \backslash G/K$  be a locally symmetric Riemannian manifold of finite volume, where  $G$  is a connected semisimple Lie group with finite centre, with all factors of real ranks at least 2, and where  $K$  is a maximal compact subgroup of  $G$ . Then:

- (i) the fundamental group  $\Gamma = \pi_1(M)$  is finitely generated;
- (ii) the first Betti number  $b_1(M) = \dim_{\mathbf{R}} \text{Hom}(\Gamma, \mathbf{R})$  is zero.

Statement (i) "gives a positive answer to part of a hypothesis of Siegel on the finiteness of the number of sides of a fundamental polygon" (the quotation is from [Kazhd-67]). Here is what *we* understand by the "hypothesis of Siegel": there exists a convenient fundamental domain for the action of  $\Gamma$  on  $G$ , namely a Borel subset  $\Omega \subset G$  such that  $(\gamma\Omega)_{\gamma \in \Gamma}$  is a partition of  $G$ , such that each element of  $G$  has a neighbourhood contained in a finite union of translates  $\gamma\Omega$ , and more importantly such that the set  $S = \{\gamma \in \Gamma : \gamma\overline{\Omega} \cap \overline{\Omega} \neq \emptyset\}$  is finite; it follows that  $S$  generates  $\Gamma$  (Section 9 in [Siege-43]).

Before Kazhdan's paper, some results of vanishing cohomology had been obtained in [CalVe-60] and [Matsu-62]. Soon after Kazhdan's paper, it was also established<sup>4</sup> that any lattice  $\Gamma$  in a semisimple Lie group  $G$  is finitely

<sup>2</sup> The unit representation is also called the *trivial representation*, and "T" holds for "trivial".

<sup>3</sup> Indeed, only  $SL_n(\mathbf{K})$  appears in Kazhdan's paper. Similar considerations hold for  $Sp_{2n}(\mathbf{K})$ , as was shown independently by [DeKi-68], [Vaser-68], and [Wang-69].

<sup>4</sup> Existence of nice fundamental domains for lattices was shown separately in the real rank one case [GarRa-70] and in the higher rank case [Margu-69]. See Chapter XIII in [Raghu-72], and an appendix of Margulis to the Russian translation of this book (the appendix has also appeared in English [Margu-84]).

generated, without any restriction on the ranks of the factors of  $G$ , and moreover the “Selberg conjecture” was proven.<sup>5</sup>

When  $M$  is not compact, it is often difficult to establish (i), and more generally finite generation for lattices in semisimple algebraic groups over local fields, by any other method than Property (T).<sup>6</sup>

Note however that some simple Lie groups of real rank one, more precisely the groups locally isomorphic to  $SO(n, 1)$  and  $SU(n, 1)$ , do *not* have Property (T). Let us try and reconstruct the way this fact was realised.

For  $SO(n, 1)$ , spherical functions of positive type have been determined independently by Vilenkin (see [Vilen–68]) and Takahashi [Takah–63]. (Particular cases have been known earlier: [Bargm–47] for  $SL_2(\mathbf{R})$ , [GelNa–47] and [Haris–47] for  $SL_2(\mathbf{C})$ .) As a consequence, it is clear that the unit representation is not isolated in the unitary dual of  $SO(n, 1)$ , even if this is not explicitly stated by Takahashi and Vilenkin. In 1969, S.P. Wang writes in our terms that  $SO(n, 1)$  does not have the Kazhdan Property (Theorem 4.9 in [Wang–69]).

As far as we know, it is Kostant [Kosta–69] who first worked out the spherical irreducible representations of *all* simple Lie groups of real rank one (see below), and in particular who has first shown that  $SU(n, 1)$  does not have Property (T). This can be found again in several later publications, among which we would like to mention [FarHa–74].

For the related problem to find representations  $\pi$  of  $G = SO(n, 1)$  or  $G = SU(n, 1)$  with non-vanishing cohomology  $H^1(G, \pi)$ , see [VeGeG–73], [VeGeG–74], [Delor–77], and [Guic–77b]. The interest in non-vanishing spaces  $H^1(G, \pi)$ , and more generally  $H^j(G, \pi)$  for  $j \geq 1$ , comes also from the following decomposition. Let  $G$  be a Lie group with finitely many connected

<sup>5</sup> Kazhdan and Margulis have shown that, if  $G$  is a connected linear semisimple Lie group without compact factor, there exists a neighbourhood  $W$  of  $e$  in  $G$  such that any discrete subgroup  $\Gamma$  in  $G$  has a conjugate  $g\Gamma g^{-1}$  disjoint from  $W \setminus \{e\}$ . It follows that the volume of  $G/\Gamma$  is bounded below by that of  $W$ . By ingenious arguments, it also follows that, if  $\Gamma$  is moreover a lattice such that  $G/\Gamma$  is *not* compact, then  $\Gamma$  contains unipotent elements distinct from  $e$  (Selberg conjecture). See [KazMa–68], [Bore–69a], and Chapter XI in [Raghu–72].

<sup>6</sup> For lattices in real Lie groups, there is a proof by Gromov using the “Margulis Lemma” and Morse theory [BaGrS–85]; see [Gelan–04] for a simple account. Of course, it is classical that arithmetic lattices are finitely generated, indeed finitely presented (Theorem 6.12 in [BorHa–62]), and Margulis has shown that lattices in semisimple Lie groups of rank at least two are arithmetic; but Margulis’ proof uses finite generation from the start. For lattices in algebraic groups defined over fields of characteristic zero, there is an approach to superrigidity and arithmeticity of lattices which does not use finite generation [Venka–93]; but this is not available in finite characteristic. Indeed, in characteristic  $p$ , lattices in rank one groups need not be finitely generated: see §II.2.5 in [Serre–77], as well as [Lubot–89] and [Lubot–91]. (However, still in characteristic  $p$ , irreducible lattices in products of at least two rank one groups are always finitely generated [Raghu–89].) To sum up, the use of the Kazhdan Property to prove finite generation of lattices is very efficient in most cases, and is currently unavoidable in some cases.

components and let  $K$  be a maximal compact subgroup; the quotient  $G/K$  is contractible, indeed homeomorphic to a Euclidean space (the Cartan–Iwasawa–Malcev–Mostow theorem). Let  $\Gamma$  be a torsion free cocompact lattice in  $G$ , so that  $M = \Gamma \backslash G/K$  is both a closed manifold and an Eilenberg–McLane space  $K(\Gamma, 1)$ . There are integers  $m(\Gamma, \pi) \geq 0$  such that

$$H^*(\Gamma, \mathbf{C}) = H^*(M, \mathbf{C}) = \bigoplus_{\pi \in \hat{G}} m(\Gamma, \pi) H^*(G, \pi)$$

where the summation can be restricted to those representations  $\pi$  in the unitary dual  $\hat{G}$  of  $G$  such that  $H^*(G, \pi) \neq 0$ . See [BorWa–80], Chapters VI and VII.

### Property (T) for the groups $Sp(n, 1)$ and $F_{4(-20)}$

Shortly after Kazhdan’s paper, Kostant made an analysis of the spherical irreducible representations of simple Lie groups; his results were announced in [Kosta–69] and the detailed paper was published later [Kosta–75]. His aim was to demonstrate the irreducibility of a (not necessarily unitary) representation of  $G$  which is either in the so-called principal series, or in the complementary series (when the latter exists). Only in the very last line (of both the announcement and the detailed paper), Kostant relates his work to that of Kazhdan, establishing<sup>7</sup> that the rank one groups  $Sp(n, 1)$ ,  $n \geq 2$ , and the rank one real form  $F_{4(-20)}$  of the simple complex Lie group of type  $F_4$ , have Property (T), thus completing the hard work for the classification of simple real Lie groups with Property (T). As a consequence, the semisimple Lie groups having Property (T) are precisely the Lie groups<sup>8</sup> locally isomorphic to products of simple Lie groups with Lie algebras *not* of type  $\mathfrak{so}(n, 1)$  or  $\mathfrak{su}(n, 1)$ .

Let  $\mathbb{G}$  be a connected linear algebraic group defined over a local field  $\mathbf{K}$ ; set  $G = \mathbb{G}(\mathbf{K})$ . For  $\mathbb{G}$  semisimple and  $\mathbf{K}$  a non-archimedean field, the situation is much simpler than if  $\mathbf{K}$  is  $\mathbf{R}$  or  $\mathbf{C}$ , since then  $G$  has Property (T) if and

<sup>7</sup> Now, we know several proofs that the groups  $Sp(n, 1)$ ,  $n \geq 2$ , and  $F_{4(-20)}$  have Property (T): the “cohomological proof” of Borel and Wallach (see Corollary 5.3 of Chapter V in [BorWa–80], and [HarVa–89]), a proof using harmonic analysis on groups of Heisenberg type (see [CowHa–89], the indication in [Cowli–90], and Theorem 1.16 in [Valet–94]), as well as proofs of Gromov, Korevaar–Schoen, Mok, Pansu, and others using Bochner’s formula of differential geometry and properties of harmonic mappings defined on Riemannian symmetric spaces (see [KorSc–93], [Mok–95], [Pansu–95], [Pansu–98], and [Gromo–03, Item 3.7.D’]). The last proofs are part of a theory of “geometric superrigidity”, of which the first goal had been to put into a differential geometric setting the superrigidity theorem of Margulis; see, among others, [Corle–92], [GroSc–92], and [MoSiY–93].

<sup>8</sup> Strictly speaking, this was only clear in 1969 for groups with finite centre. It holds in the general case by Lemma 1.7 in [Wang–82], or by results of Serre first published in Sections 2.c and 3.d of [HarVa–89]. Serre’s results on Property (T) for central extensions follow also simply from results of Shalom, as in §3.3 of [Valet–04].

only if it has no simple factor of  $\mathbf{K}$ -rank one.<sup>9</sup> In the general case, S.P. Wang has determined when  $G$  has Property (T) in terms of a Levi decomposition of  $G$  (assuming that such a decomposition exists, this being always the case in characteristic zero); see [Wang–82], as well as [Shal–99b] and [Cornu–06d, Corollary 3.2.6].

### Construction of expanding graphs and Property (T) for pairs

The first application of the Kazhdan Property outside group theory was the explicit construction by Margulis of remarkable families of finite graphs. In particular, for any degree  $k \geq 3$ , there are constructions of families of finite  $k$ -regular graphs which are *expanders*; this means that there exists a so-called *isoperimetric constant*  $\varepsilon > 0$  such that, in each graph of the family, any non-empty subset  $A$  of the set  $V$  of vertices is connected to the complementary set by at least  $\varepsilon \min\{\#A, \#(V \setminus A)\}$  edges. While the existence of such graphs is easily established on probabilistic grounds, explicit constructions require other methods.

A basic idea of [Margu–73] is that, if an infinite group  $\Gamma$  generated by a finite set  $S$  has Property (T) and is residually finite, then the finite quotients of  $\Gamma$  have Cayley graphs with respect to  $S$  which provide a family of the desired kind. Margulis’ construction is explicit for the graphs, but does not provide explicit estimates for the isoperimetric constants. Constructions given together with lower bounds for these constants were given later, for example in [GabGa–81]; see also the discussion below on Kazhdan constants.<sup>10</sup>

Rather than Property (T) for one group, Margulis used there a formulation for a pair consisting of a group and a subgroup. This *Property (T) for pairs*, also called *relative Property (T)*, was already important in Kazhdan’s paper, even though a name for it appears only in [Margu–82]. It has since become a basic notion, among other reasons for its role in operator algebras, as recognized by Popa. Recent progress involves defining Property (T) for a pair consisting of a group and a subset [Cornu–06d].

There is more on Property (T) for pairs and for semidirect products in [Ferno–06], [Shal–99b], [Valet–94], and [Valet–05].

<sup>9</sup> If  $\mathbb{G}$  is connected, simple, and of  $\mathbf{K}$ -rank one, then  $G$  acts properly on its Bruhat–Tits building [BruTi–72], which is a tree, and it follows that  $G$  does not have Property (T).

<sup>10</sup> More recently, there has been important work on finding more expanding families of graphs, sometimes with optimal or almost optimal constants. We wish to mention the so-called Ramanujan graphs, first constructed by Lubotzky–Phillips–Sarnak and Margulis (see the expositions of [Valet–97] and [DaSaV–03]), results of J. Friedman [Fried–91] based on random techniques, and the zig-zag construction of [ReVaW–02], [AILuW–01]. Most of these constructions are related to some weak form of Property (T).

### Group cohomology, affine isometric actions, and Property (FH)

Kazhdan's approach to Property (T) was expressed in terms of weak containment of unitary representations. There is an alternative approach involving group cohomology and affine isometric actions.

In the 1970s, cohomology of groups was a very active subject with (among many others) an influential paper by Serre. In particular (Item 2.3 in [Serre-71]), he conjectured that  $H^i(\Gamma, \mathbf{R}) = 0$ ,  $i \in \{1, \dots, l-1\}$ , for a cocompact discrete subgroup  $\Gamma$  in an appropriate linear algebraic group  $G$  "of rank  $l$ ". This conjecture was partially solved by Garland in an important paper [Garla-73] which will again play a role in the later history of Property (T). Shortly after, S.P. Wang [Wang-74] showed that  $H^1(G, \pi) = 0$  for a separable locally compact group  $G$  with Property (T), where  $\pi$  indicates here that the coefficient module of the cohomology is a finite-dimensional Hilbert space on which  $G$  acts by a unitary representation.

In 1977, Delorme showed that, for a topological group  $G$ , Property (T) implies that  $H^1(G, \pi) = 0$  for all unitary representations  $\pi$  of  $G$  [Delor-77]. Previously, for a group  $G$  which is locally compact and  $\sigma$ -compact, Guichardet had shown that the converse holds (see [Guic-72a], even if the expression "Property (T)" does not appear there, and [Guic-77a]). Delorme's motivation to study 1-cohomology was the construction of unitary representations by continuous tensor products.

A topological group  $G$  is said to have *Property (FH)* if every continuous action of  $G$  by affine isometries on a Hilbert space has a fixed point. It is straightforward to check that this property is equivalent to the vanishing of  $H^1(G, \pi)$  for all unitary representations  $\pi$  of  $G$ , but this formulation was not standard before Serre used it in talks (unpublished). Today, we formulate the result of Delorme and Guichardet like this: a  $\sigma$ -compact<sup>11</sup> locally compact group has Property (T) if and only if it has Property (FH).

Recall that there is a *Property (FA)* for groups, the property of having fixed points for all actions by automorphisms on trees. It was first studied in [Serre-74] (see also §1.6 in [Serre-77]); it is implied by Property (FH) [Watat-82].

We make one more remark, in order to resist the temptation of oversimplifying history. Delorme and Guichardet also showed that a  $\sigma$ -compact locally compact group  $G$  has Property (T) if and only if all real-valued continuous

<sup>11</sup> The hypothesis of  $\sigma$ -compactness is necessary, since there exist discrete groups with Property (FH) which are not countable, and therefore which are not finitely generated, and even less with Property (T). An example is the group of all permutations of an infinite set, with the discrete topology [Cornu-06c].

functions on  $G$  which are conditionally of negative type are bounded.<sup>12</sup> On the one hand, this was independently rediscovered and proved by other methods in [AkeWa–81], a paper in which the authors seek to understand the unitary dual of a group in terms of functions of conditionally negative type, and also a paper written under the strong influence of Haagerup’s work on reduced  $C^*$ -algebras of non-abelian free groups [Haage–78]. On the other hand, a weak form of the same result on Property (T) and functions of negative type appeared earlier in [FarHa–74], itself motivated by the wish to understand which are the invariant metrics on a space of the form  $G/K$  which are induced by embeddings in Hilbert spaces.

Whatever the early history has been, there has been a growing interest in the point of view of Property (FH) which is now considered as basic. Recently, it has been shown that, for locally compact groups, Property (T) implies the property of having fixed points for affine isometric actions on various Banach spaces, for example<sup>13</sup> on spaces of the form  $L^p(\mu)$  with  $1 < p \leq 2$  [BaFGM].

### Normal subgroups in lattices

Let  $G$  be a connected semisimple Lie group with finite centre, without compact factor, and of real rank at least 2. Let  $\Gamma$  be an irreducible lattice in  $G$  and let  $N$  be a normal subgroup of  $\Gamma$ . Margulis has shown that

- either  $N$  is of finite index in  $\Gamma$ ;
- or  $N$  is central in  $G$ , and in particular finite.

Property (T) is a crucial ingredient of Margulis’ proof. Indeed, for non-central  $N$ , the proof of the finiteness of the quotient cannot rely on any size estimate since  $\#(\Gamma/N)$  can be arbitrarily large (think of congruence subgroups in  $SL_3(\mathbf{Z})$ ). The strategy is to show that  $\Gamma/N$  is amenable and has Property (T), and it follows that  $\Gamma/N$  is a finite group.<sup>14</sup> (In the special case where all factors of  $G$  have real ranks at least 2, Property (T) for  $\Gamma/N$  is straightforward.)

<sup>12</sup> This reformulation is particularly well suited to Coxeter groups. Let  $(W, S)$  be a Coxeter system. The  $S$ -word length  $\ell_S : W \rightarrow \mathbf{R}_+$  is conditionally of negative type [BoJaS–88]. It follows that  $W$  cannot have Property (T) as soon as  $W$  is infinite.

<sup>13</sup> This cannot hold for a Banach space of the form  $C(X)$ ; see Remark 8.c of Chapter 4 in [HarVa–89]. More on this in Exercises 1.8.20 and 2.14.12.

<sup>14</sup> Though stated here for real semisimple Lie groups, this result holds for  $G$  of rank at least 2 in a much larger setting. See [Margu–78] and [Margu–79], as well as Theorems IV.4.9 (page 167) and IX.5.6 (page 325) in [Margu–91]. The Normal Subgroup Theorem and the strategy for its proof have been extended to irreducible lattices in products  $\text{Aut}(T_1) \times \text{Aut}(T_2)$  of automorphism groups of trees [BurMo–00] and in a larger family of products of two locally compact groups. One of the consequences is that the result holds for many Kac–Moody groups [BadSh–06].



### Property (T) versus amenability

Compact groups can be characterised as the locally compact groups which are amenable and which have Property (T). Otherwise, it is a straightforward but basic fact that, for non-compact locally compact groups, Property (T) and amenability are two extreme and opposite properties. The first is a strong rigidity property and the second quite a soft property. Some groups have properties “in between” these two, such as free groups,  $SL_2(\mathbf{Z})$ , fundamental groups of compact Riemannian manifolds of constant negative curvature, and “many” Gromov-hyperbolic groups.

A group has the *Haagerup Property* if there exists a unitary representation  $\pi$  of  $G$  which contains  $1_G$  weakly and which is such that all its coefficients vanish at infinity; it is a weakening of amenability [CCJJV–01]. It is still true that the Haagerup Property and Property (T) are opposite and that some groups are “in between”, such as  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ ; more on this in [Cornu–06a].

### The Ruziewicz problem

The Ruziewicz problem asks whether Lebesgue measure is the only finitely-additive measure on the unit sphere  $\mathbf{S}^{n-1}$  of the Euclidean space  $\mathbf{R}^n$  which is defined on all Lebesgue measurable sets and which is invariant by rotations. After Banach’s negative answer for  $n = 2$  (1923), the problem was long open for  $n \geq 3$ . There are analogous uniqueness problems concerning Lebesgue measure viewed as a finitely-additive measure invariant by isometries and defined on all Lebesgue measurable sets on various spaces, such as  $\mathbf{R}^n$  (the measure being normalised on the unit cube) and the  $n$ -torus  $\mathbf{T}^n$ .

Rosenblatt [Rosen–81] showed that, if the answer to the Ruziewicz question is negative for  $\mathbf{S}^{n-1}$ , then there exist non-trivial nets of measurable subsets of the sphere with “asymptotically invariant properties” (see below); there are related results in [DeJR–79] and [LosRi–81]. Very shortly afterwards, Sullivan [Sulli–81] and Margulis [Margu–80] independently constructed dense subgroups with Property (T) in the special orthogonal group  $SO(n)$  for  $n \geq 5$ . This shows that spheres of dimension at least 4 cannot have asymptotically invariant nets of measurable subsets, so that Lebesgue measure on such a sphere is the unique finitely additive  $SO(n)$ -invariant measure.

The Ruziewicz problem was solved for  $\mathbf{R}^n$ ,  $n \geq 3$ , in [Margu–82], and later for  $\mathbf{S}^2$ ,  $\mathbf{S}^3$  by Drinfeld in [Drinf–84]. See also [Sarna–90], [Lubot–94], and [GaJaS–99]. A reformulation of part of Drinfeld’s result is that, for  $n \geq 3$ , the compact group  $SO(n)$  has the *strong Property (T)*, namely that there exists for  $SO(n)$  a pair  $(Q, \varepsilon)$  as in the definition of Property (T) with moreover  $Q$  finite; for more on this property, see [Shal–99b] and [Bekka–03].

### Fundamental groups of $II_1$ factors

Let  $N$  be a von Neumann algebra which is a factor of type  $II_1$ . Set  $\tilde{N} = N \otimes \mathcal{L}(\ell^2)$ , where  $\mathcal{L}(\ell^2)$  is “the” factor of type  $I_\infty$ , and let  $\tilde{\mathcal{P}}$  denote the set of projections in  $\tilde{N}$ . Murray and von Neumann [ROIV] defined a dimension function  $D : \tilde{\mathcal{P}} \rightarrow [0, \infty]$  which classifies projections up to conjugacy by unitary operators. For  $e, f \in \tilde{\mathcal{P}}$ , the factors  $e\tilde{N}e$  and  $f\tilde{N}f$  are isomorphic whenever  $D(e) = D(f)$ . The *fundamental group* of  $N$  is

$$\mathcal{F}(N) = \left\{ t \in ]0, \infty[ \mid \begin{array}{l} \text{there exists } e \in \tilde{\mathcal{P}} \text{ with } D(e) = t \\ \text{such that } e\tilde{N}e \text{ is isomorphic to } N \end{array} \right\}.$$

Though Murray and von Neumann conjectured in 1943 the “great general significance” of the fundamental group, they could not compute any example besides a few for which  $\mathcal{F}(N) = \mathbf{R}_+^*$ .

Now let  $\Gamma$  be a countable group which is icc, namely an infinite group in which all conjugacy classes distinct from  $\{1\}$  are infinite, so that the von Neumann algebra  $L(\Gamma)$  of  $\Gamma$  is a factor of type  $II_1$ . In the case when  $\Gamma$  has Property (T), Connes [Conne–80] discovered two remarkable properties of  $L(\Gamma)$ . The first is that the subgroup  $\text{Int}(L(\Gamma))$  of inner automorphisms of  $L(\Gamma)$  is open in  $\text{Aut}(L(\Gamma))$ , for the topology of the pointwise norm convergence in the predual of  $L(\Gamma)$ ; as  $\text{Aut}(L(\Gamma))$  is a Polish group, this implies that the quotient group  $\text{Aut}(L(\Gamma))/\text{Int}(L(\Gamma))$  is countable.<sup>15</sup> The second is that the fundamental group  $\mathcal{F}(L(\Gamma))$  is countable;<sup>16</sup> to deduce the second property from the first, Connes associates to every element in the group  $\mathcal{F}(L(\Gamma))$  a non-inner automorphism of  $L(\Gamma \times \Gamma)$ .

The 1980 paper of Connes was followed by a definition of Property (T), first for factors of type  $II_1$  (in the introduction of [Conne–82]) and then for finite von Neumann algebras [ConJo–85]. The definition is tailored for the factor  $L(\Gamma)$  of an icc group  $\Gamma$  to have Property (T) if and only if the group  $\Gamma$  has Property (T).<sup>17</sup> The property was later defined for a pair  $A \subset B$  consisting of a von Neumann subalgebra  $A$  of a von Neumann algebra  $B$  ([Popa–06a], but see also [PetPo–05]).

<sup>15</sup> For a long time, it has been an open problem to know whether there exists a factor  $N$  such that the outer automorphism group  $\text{Out}(N)$  is finite. A recent construction from [IoPeP], using Property (T) in a crucial way, shows that this group can be reduced to one element.

<sup>16</sup> In contrast, if  $\Gamma$  is an amenable icc group, then  $L(\Gamma)$  is hyperfinite,  $\text{Int}(L(\Gamma))$  is proper and dense in  $\text{Aut}(L(\Gamma))$ , and  $\mathcal{F}(L(\Gamma)) = \mathbf{R}_+^*$ . Remember also that, if a group  $\Gamma$  is not inner amenable, then  $\text{Int}(L(\Gamma))$  is closed in  $\text{Aut}(L(\Gamma))$ , by [Effro–75] and [Conne–74].

<sup>17</sup> This carries over to groups with finitely many finite conjugacy classes, but it is shown in [Jolis–93] that a more subtle statement is necessary for an arbitrary countable group – unless the definition for non-factorial algebras is slightly modified, as in [Bekk–06].

Since 1980, our understanding of fundamental groups has progressed significantly (even if several basic questions remain open). Some progress has been obtained concerning a related notion of fundamental group in ergodic theory; see e.g. [GefGo–89]. As an application of his free probability theory, Voiculescu has shown that  $\mathcal{F}(L(F_\infty))$  contains  $\mathbf{Q}_+^*$ , where  $F_\infty$  denotes the free group on a countable infinite number of generators [Voic–90a]; Radulescu proved that in fact  $\mathcal{F}(L(F_\infty)) = \mathbf{R}_+^*$  [Radul–92]. Building on Gaboriau’s theory of  $L^2$ -Betti numbers for measured equivalence relations, Popa first produced explicit factors of type  $II_1$  with fundamental group reduced to one element, such as  $\mathcal{F}(L(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2)) = \{1\}$  (see [Popa–06a] and [Popa–04]; for more examples, see [Valet–05]), and then has shown that any countable subgroup of  $\mathbf{R}_+^*$  is the fundamental group of some factor of type  $II_1$  [Popa–06b].

As for locally compact groups, Property (T) and amenability are for von Neumann algebras two extreme and opposite properties, respectively “rigid” and “soft”.

The role of amenability for von Neumann algebras has long been recognised, a landmark being Connes’ classification of injective factors [Conne–76]. A systematic exploration of Property (T) for operator algebras was conducted, from the 1980s onwards: see, e.g. [Popa], [Anant–87], and [Bekk–06]. More generally, together with amenability, Property (T) is now a standard concept both in functional analysis<sup>18</sup> and in harmonic analysis.<sup>19</sup>

### Property (T) in ergodic theory

Let  $\Gamma$  be a countable group acting on a standard non-atomic probability space  $(\Omega, \mathcal{B}, \mu)$  in a measure-preserving and ergodic way. A sequence  $(B_n)_n$  of Borel subsets of  $\Omega$  is *asymptotically invariant* if  $\lim_{n \rightarrow \infty} \mu(B_n \Delta \gamma B_n) = 0$  for all  $\gamma \in \Gamma$  (where  $\Delta$  denotes a symmetric difference); the action of  $\Gamma$  is *strongly ergodic* if every such sequence is trivial, namely such that  $\lim_{n \rightarrow \infty} \mu(B_n) (1 - \mu(B_n)) = 0$ . An amenable group does not have any strongly ergodic action;

<sup>18</sup> It has been used for constructing particular operators in Hilbert spaces [Voic–90b], in relation with subfactors [PimPo–86], with tensor products of group  $C^*$ -algebras [Kirch–94], with  $KK$ -theory of  $C^*$ -algebras [Skand–88], and to produce new examples related to the Baum–Connes conjecture [HiLaS–02].

<sup>19</sup> The existence for a compact group  $G$  of a dense subgroup which has Property (T) for its discrete topology has consequences on several well-established problems. For example, it implies automatic continuity for  $G$ -invariant linear forms on various spaces of functions on  $G$ ; see [ChLaR–85], [Krawc–90], [Rosen–85], and [Willi–88]. Any simply connected compact Lie group  $G$  which is not locally isomorphic to  $SO(3)$  has dense subgroups with Property (T) [Margu–80]; when  $G = SO(n)$ ,  $n \geq 5$ , this is an important ingredient used by Margulis and Sullivan to solve the Ruziewicz problem for  $S^{n-1}$ , as already mentioned. On the contrary, any Kazhdan subgroup of  $SO(3)$  or  $SO(4)$  is finite [Zimm–84c].

moreover, two measure-preserving ergodic actions of infinite amenable groups are orbit-equivalent [OrnWei–80].<sup>20</sup>

Schmidt showed how this is connected with our subject; it is a straightforward consequence of Proposition 2.10 in [Schmi–80] that, for an infinite group with Property (T), ergodicity implies strong ergodicity (see also [Schmi–81]). Conversely, Connes and Weiss have shown that, if every ergodic action of a countable group  $\Gamma$  is strongly ergodic, then  $\Gamma$  has Property (T) [ConWe–80]. (The measure-preserving hypothesis is crucial, since any countable infinite group has non-singular actions which are ergodic and not strongly ergodic [Schmi–85].)

As we already mentioned, the Ornstein–Weiss theorem establishes that two measure-preserving ergodic actions of infinite amenable groups are orbit-equivalent. Indeed, this characterises amenability, since it follows from [ConWe–80] and [Schmi–85] that non-amenable groups without Property (T) have at least two non-orbit-equivalent measure preserving ergodic actions. Moreover every infinite Property (T) group admits uncountably many pairwise non-orbit-equivalent measure preserving ergodic actions [Hjort–05]. These results of Ornstein–Weiss, Schmidt–Connes–Weiss, and Hjorth illustrate once more that amenability and Property (T) are two opposite properties.

There is another dynamical characterisation of Property (T) for countable groups, in terms of measure-preserving actions on measure spaces with infinite measures [RobSt–98].

Property (T) is an important ingredient for establishing that the classification of subgroups of  $\mathbf{Q}^{n+1}$  is “strictly harder” than the corresponding problem for  $\mathbf{Q}^n$ , for all  $n \geq 1$ , in the sense of descriptive set theory. See [Thoma–02], [Thoma–03], and the review in [Thoma–01].

Consider, as in topological dynamics, a countable group  $\Gamma$  acting by homeomorphisms on a compact metric space  $X$  and the compact convex set  $M(X)^\Gamma$  of  $\Gamma$ -invariant probability measures on  $X$ . There is a characterisation of Property (T) for  $\Gamma$  in terms of the geometry of the convex sets  $M(X)^\Gamma$  [GlaWe–97].

Property (T) was defined by Zimmer for free measure-preserving actions [Zimme–81] and (almost equivalently) by Moore for measured equivalence relations [Moore–82]; it is an invariant of *orbit equivalence*. Property (T) is moreover an invariant of *measured equivalence*, a notion defined in [Gromo–93, Item 0.5.E]. The later invariance appears in [Furma–99a] and [Furma–99b]; it is also a consequence of Theorems 4.1.7 and 4.1.9 in the preprint [Popa]. Ergodic

<sup>20</sup> See also [CoFeW–81], as well as the discussions in [Kaima–97] and [KecMi–04]. In this context, orbit equivalence was proved earlier for cyclic groups in [Dye–59] and for a class of groups containing those of polynomial growth in [Dye–63].

theory is inseparable from the theory of operator algebras, as shown for example by the work of Popa already mentioned.

### Property (T) in differential geometry

Several of the results discussed above have geometric consequences. For example, let  $X = \mathbf{H}^n(\mathbf{R})$  or  $X = \mathbf{H}^n(\mathbf{C})$  be a real or complex hyperbolic space. If  $d(x, y)$  denotes the Riemannian distance between two points of  $X$ , the kernel  $\Phi : X \times X \longrightarrow \mathbf{R}$  defined by  $\Phi(x, y) = \log \cosh d(x, y)$  is conditionally of negative type [FarHa-74], which means that there exists a continuous mapping  $\eta$  from  $X$  to a real Hilbert space such that  $\Phi(x, y) = \|\eta(y) - \eta(x)\|^2$ . Obviously, the kernel  $\Phi$  is also unbounded.

The result of Delorme and Guichardet characterising Property (T) in terms of functions conditionally of negative type implies that any isometric action of a group with Property (T) on  $X$  has a fixed point. In other words, besides Property (FA) concerning fixed points for actions on trees, Kazhdan groups have two further fixed-point properties, sometimes written  $\text{FHyp}_{\mathbf{R}}$  and  $\text{FHyp}_{\mathbf{C}}$ . (This contrasts with the situation for hyperbolic spaces over the quaternions and for the hyperbolic plane over the octonions, by the result of Kostant discussed above.)

More generally, actions of groups with Property (T) on manifolds with geometric structures show strong rigidity properties. This is in particular the case for volume-preserving actions [Zimm-84b], for actions on Lorentz manifolds [Zimme-86], and on Riemannian manifolds (see the infinitesimal rigidity established in [LubZi-89]). Rather than quoting a few of the many existing theorems from the late 1980s and the 1990s, let us state as a sample the following result from [FisMa-05].

Let  $\Gamma$  be a group with Property (T) and let  $M$  be a compact Riemannian manifold. Let  $\rho$  be an isometric action of  $\Gamma$  on  $M$  and let  $\rho'$  be a smooth action of  $\Gamma$  on  $M$  which is sufficiently  $C^\infty$ -close to  $\rho$ . Then the actions  $\rho$  and  $\rho'$  are conjugate by a  $C^\infty$ -diffeomorphism.

In the late 1980s, two of the authors of the present book wrote up (in French) a set of notes on Property (T) [HarVa-89], which were later updated [Vale-94]. The continuing growth of the subject has motivated us to write the present account. We will now describe some results obtained essentially in the last 15 years.

### Later progress on Property (T)

#### Further examples of groups with Property (T)

At the beginning of the theory, when some group was known to have Property (T), it was either for straightforward reasons (compact groups), or because

the group was essentially an algebraic group of the appropriate kind or a lattice in such a group. In particular, there were only countably many *known* examples of countable groups with Property (T).

Gromov has announced that a non-elementary hyperbolic group  $\Gamma$  has uncountably many pairwise non-isomorphic quotient groups with all elements of finite order; see Corollary 5.5.E of [Grom87], as well as [Iva01–98] and [Ozawa04]. If moreover  $\Gamma$  has Property (T), for example if  $\Gamma$  is a cocompact lattice in  $Sp(n, 1)$  for some  $n \geq 2$ , all its quotients have Property (T). Let us digress to mention that this gave new examples (even if not the first ones) of non-amenable groups without free subgroups.<sup>21</sup> In particular, there exist groups with Property (T) which are not finitely presented; this answers a question of [Kazhd67]. A first concrete example is  $SL_3(\mathbb{F}_p[X])$ , a group which appears in [Behr79]; it is isomorphic to the lattice  $SL_3(\mathbb{F}_p[X^{-1}])$  in the locally compact group  $SL_3(\mathbb{F}_p((X)))$ . A second example, shown to us by Cornuier, is  $Sp_4(\mathbb{Z}[1/p]) \ltimes \mathbb{Z}[1/p]^4$ ; see Theorem 3.4.2. There are also examples of finitely presented groups with Property (T) which are non-Hopfian [Cornu07].

For a topological space, there is a natural topology on the space of its closed subspaces, which has been considered by Vietoris (1922), Michael (1951), Fell (1962), and others; see for example [Engel89], as well as Section 4.F and 12.C in [Kechr95]. For a locally compact group (possibly discrete), the Vietoris–Fell topology induces a topology on the space of closed subgroups that we like to call the *Chabauty topology*; the original reference is [Chaba50], and there is an account from another point of view in [Bou–Int2]. As a nice example, it is known that the space of closed subgroups of  $\mathbb{R}^2$  is a 4-sphere [HubPo79].

In particular, for an integer  $m \geq 1$ , the space  $\mathcal{G}_m$  of normal subgroups of the non-abelian free group  $F_m$  on a set  $S_m$  of  $m$  generators, with the Chabauty topology, is a totally disconnected compact metric space. There is a natural bijection between the set of normal subgroups of  $F_m$  and the set of marked groups with  $m$  generators, namely the set of groups given together with an ordered set of  $m$  generators (up to isomorphisms of marked groups). Thus  $\mathcal{G}_m$  is also known as the *space of marked groups with  $m$  generators*.

Here is a sample genericity result, from [Champ91] (see also [Ol’sh92], [Champ00], and [Ghys04]). For  $m \geq 2$ , let  $\mathcal{H}$  be the subset of  $\mathcal{G}_m$  defined by torsion-free non-elementary hyperbolic groups. Then the closure  $\overline{\mathcal{H}}$  contains a

<sup>21</sup> This answers a *question* which can reasonably be attributed to von Neumann. Twisting the question, several people – but not von Neumann! – have imagined a conjecture that groups without non-abelian free subgroups should be amenable. But the conjecture was wrong, as these and other examples show.

dense  $G_\delta$  consisting of pairs  $(\Gamma, S)$  with  $\Gamma$  an infinite group having Property (T) and  $S$  a generating set. Here is another result: in  $\mathcal{G}_m$ , the subset of Kazhdan groups is open (Theorem 6.7 in [Shal–00a]).

A completely different family of examples of groups with Property (T) is provided by many of the Kac–Moody groups as shown in [DymJa–00] and [DymJa–02]. More information on Kac–Moody lattices can be found in [CarGa–99], [Rémy–99], [Rémy–02] and [BadSh–06].

### Random groups

A *model for random groups* is a family  $(\mathcal{X}_t)_t$ , where each  $\mathcal{X}_t$  is a finite set of presentations of groups and where the parameter  $t$  is either a positive integer or a positive real number. Given a Property (P) of countable groups and a value  $t$  of the parameter, let  $\mathbb{A}(P, t)$  denote the quotient of the number of groups with Property (P) appearing in  $\mathcal{X}_t$  by the total number of elements in  $\mathcal{X}_t$ . Say Property (P) is *generic* for the model if, for any  $\varepsilon > 0$ , there exists  $t_0$  such that  $\mathbb{A}(P, t) \geq 1 - \varepsilon$  for all  $t \geq t_0$ . (This is a rather unsatisfactory definition since it does not capture what makes a model “good” or “interesting”).

There are several models for random groups currently in use, for which Property (T) has been shown to be generic. Historically, the first models appeared in [Gromo–87, Item 0.2.A]; another model is proposed in the last chapter of [Gromo–93], where Gromov asks whether it makes Property (T) generic. Positive results are established in [Champ–91], [Ol’sh–92], [Champ–95], [Gromo–00] (see in particular page 158), [Zuk–03], [Olliv–04], and [Ghys–04]. There is a review in [Olliv–05].

### Finite presentations with Property (T), examples beyond locally compact groups, and other new examples

Other types of arguments independent of Lie theory, including analysis of particular presentations, can be used to study groups with Property (T). Here is a typical criterion.

Let  $\Gamma$  be the fundamental group of a finite simplicial 2-complex  $X$  with the two following properties: (i) each vertex and each edge is contained in a triangle, and (ii) the link of each vertex is connected. For each vertex  $v$  of  $X$ , let  $\lambda_v$  denote the smallest positive eigenvalue of the combinatorial Laplacian on the link of  $v$ . Here is the result: if  $\lambda_v + \lambda_w > 1$  for each pair  $(v, w)$  of adjacent vertices of  $X$ , then  $\Gamma$  has Property (T). See [Zuk–96], [BalSw–97], [Pansu–98], [Wang–98], and [Zuk–03]. Proofs are strongly inspired by [Garla–73] and [Borel–73]; see also [Matsu–62] and [Picho–03]. Garland’s paper has been an inspiration for much more work, including [DymJa–00] and [DymJa–02]. There are related spectral criteria for groups acting on  $\tilde{A}_2$ -buildings [CaMIS–93].



It was realised by Colin de Verdière [ColVe–98] and Shalom [Shal–99a] that Property (T) is connected with the property of *bounded generation*. For example, if  $n \geq 3$ , any element in  $SL_n(\mathbf{Z})$  can be written as a product of at most  $\frac{1}{2}(3n^2 - n) + 36$  elementary matrices (see [CarKe–83], [CarKe–84], and [AdiMe–92]), and this was used in [Shal–99a] to show Property (T) without reference to Lie theory.

As a by-product of this work of Shalom, it became clear that Kazhdan’s definition of Property (T) makes sense for topological groups which are not locally compact. There are natural examples, such as the *loop group* of all continuous functions from a circle to  $SL_n(\mathbf{C})$ ,  $n \geq 3$ , [Shal–99a]; for other groups of mappings with Property (T), see [Cornu–06e]. The unitary group of an infinite-dimensional Hilbert space, with the strong topology, also has Property (T) [Bekka–03].

Later, using among other things ideas of algebraic K-theory, Shalom proved Property (T) for other kinds of groups, including  $SL_n(\mathbf{Z}[X_1, \dots, X_m])$  when  $n \geq m + 3$  [Shal–ICM]. Slightly earlier, these groups had been shown to have Property  $(\tau)$  for  $n \geq 3$  [KasNi–06].

Ershov has shown that there are groups satisfying the “Golod–Shafarevich condition” with respect to some prime  $p$  which also have Property (T). This provides the first proof that these groups are not amenable, and indeed the first known examples of residually finite torsion groups which are not amenable [Ershov].

### Kazhdan constants

Kazhdan’s definition has a quantitative reformulation. Let  $G$  be a topological group. If  $G$  has Property (T), easy arguments show that  $G$  has a *Kazhdan pair*  $(Q, \varepsilon)$ ; this consists of a compact subset  $Q$  of  $G$ , called a *Kazhdan subset*, and a positive real number  $\varepsilon$  such that a unitary representation  $\pi$  of  $G$  in a Hilbert space  $\mathcal{H}$  has a non-zero fixed vector whenever there exists  $\xi \in \mathcal{H}$  with  $\sup_{q \in Q} \|\pi(q)\xi - \xi\| < \varepsilon \|\xi\|$ . Conversely, the existence of a Kazhdan pair clearly implies Property (T). If  $G$  is moreover compactly generated, any compact generating set  $Q$  of  $G$  is a Kazhdan subset.

Note that the compact set  $Q$  entering a Kazhdan pair need not be generating. However, if a countable group  $\Gamma$  has a Kazhdan pair  $(Q, \varepsilon)$ , then the finite set  $Q$  necessarily generates  $\Gamma$ . See Section 1.3.

Given  $G$  with Property (T) and a Kazhdan subset  $Q$  of  $G$  (for example a generating compact subset), the problem of estimating a *Kazhdan constant*  $\varepsilon > 0$  making  $(Q, \varepsilon)$  a Kazhdan pair has received considerable attention. The problem was raised by Serre, and in [HarVa–89]. The first computations of Kazhdan constants for some representations of  $SL_3(\mathbf{Z})$  are due to Burger: see the appendix to [HarVa–89], and [Burge–91].



A large number of questions have been addressed concerning these Kazhdan constants: for lattices in algebraic groups ([BeMa–00a], [CaMIS–93], [Kassa], [Neuh–03b], [Neuh–03c], [Oh–02], [Shal–00c]), for special kinds of presentations [Zuk–03], for semidirect products [Cheri–95]. Or in relation to the spectrum of some Laplacian [BeChJ–98], with random walks [PakZu–02], and with expanding graphs (see Section 10.3 in [Lubot–94]). There is a kind of converse: properties of some finite graphs discovered by Gabber and Galil can be used to prove a weak form of Property (T) for the group  $SL_3(\mathbf{Z})$ , as shown in Sections 3.5–3.8 of [ColVe–98].

The case of a compact group  $G$  is particular. When  $Q = G$ , the smallest Kazhdan constant is  $\sqrt{2n/(n-1)}$  if  $G$  is finite of order  $n$  and  $\sqrt{2}$  if  $G$  is infinite [DeuVa–95]. Otherwise, there are estimates for symmetric groups in [BacHa–94] and for related groups in [Bagno–04]; in the first of these, it is shown that, for a given finite group and a given generating subset  $Q$ , the smallest constant  $\varepsilon'$  such that  $(Q, \varepsilon')$  is a Kazhdan pair for irreducible representations can be strictly larger than the smallest constant such that  $(Q, \varepsilon)$  is a Kazhdan pair for all representations. For some examples of proper closed subsets  $Q$  in compact groups  $G$ , estimates (sometimes optimal) for  $\varepsilon$  can be found in [Valet–94] and [Neuh–03a].

Let  $\Gamma$  be a countable group with Property (T) and let  $Q$  be a finite subset. For a unitary representation  $\pi$  of  $\Gamma$  in a Hilbert space  $\mathcal{H}$ , define

$$\kappa(\Gamma, Q, \pi) = \inf_{\xi \in \mathcal{H}, \|\xi\|=1} \max_{q \in Q} \|\pi(q)\xi - \xi\|.$$

Let  $\kappa(\Gamma, Q)$  be the infimum of the numbers  $\kappa(\Gamma, Q, \pi)$  over all unitary representations  $\pi$  of  $\Gamma$  without non-zero invariant vectors, namely the maximum of the numbers  $\varepsilon > 0$  which are Kazhdan constants for  $Q$ . The *uniform Kazhdan constant* of  $\Gamma$  is the infimum  $\kappa(\Gamma) = \inf_Q \kappa(\Gamma, Q)$  taken over all finite generating subsets  $Q$  of  $\Gamma$ . We know that  $\kappa(\Gamma) = 0$  for any Gromov-hyperbolic group with Property (T) [Osin–02] and for any dense subgroup of a connected topological group which has a unitary representation without non-zero invariant vectors [GelZu–02]. For example,  $\kappa(SL_3(\mathbf{Z}[1/p])) = 0$ , but it is not known if  $\kappa(SL_3(\mathbf{Z}))$  is zero.

On the other hand, for the left regular representation  $\lambda_\Gamma$  of a finitely generated group  $\Gamma$ , a strict inequality  $\inf_Q \kappa(\Gamma, Q, \lambda_\Gamma) > 0$  implies that  $\Gamma$  is a group of uniformly exponential growth. When  $\Gamma$  is a non-elementary residually finite Gromov hyperbolic group, we know that  $\inf_Q \kappa(\Gamma, Q, \lambda_\Gamma) > 0$  (Theorem 8.4 in [Shal–00c]).

Consider now a family  $(\Gamma_i)_i$  of groups which have Property (T), for example of finite groups. Recent work addresses the question of the existence of

generating subsets  $Q_i$  of  $\Gamma_i$  such that  $\inf_i \kappa(\Gamma_i, Q_i) > 0$ ; this is of interest in computer science, combinatorics, and group theory. See [ALLuW-01], recent work of Bourgain and Gamburd [BouGa-06], as well as the discussion in [Shal-ICM].

### Reduced 1-cohomology

Let  $G$  be a second countable locally compact group. If  $G$  has Property (T), then  $G$  is compactly generated and the reduced cohomology group  $\overline{H}^1(G, \pi)$  is reduced to zero for any unitary representation  $\pi$  of  $G$ . Shalom [Shal-00a] has shown the converse: if  $G$  is compactly generated and if  $\overline{H}^1(G, \pi) = 0$  for all  $\pi$ , then  $G$  has Property (T) [Shal-00a, Theorem 6.1]; since  $\overline{H}^1$  is a quotient of  $H^1$ , this is a strengthening of the Guichardet part of the Delorme–Guichardet theorem.

Together with a result of [VerKa-82] which asserts that irreducible representations with non-trivial  $H^1$  are not Hausdorff separated<sup>22</sup> from the unit representation in the Fell topology, this implies that a second countable locally compact group  $G$  has Property (T) if and only if it has the following three properties:

- (i)  $G$  is compactly generated;
- (ii) the only continuous homomorphism from  $G$  to the additive group  $\mathbf{R}$  is zero;
- (iii) the only irreducible unitary representation  $\pi$  of  $G$  which is not Hausdorff separated from the unit representation  $1_G$  is  $1_G$  itself.

(Theorem 6.2 in [Shal-00a].) In particular, this establishes a conjecture from [VerKa-82]: for  $G$  compactly generated, Property (T) is equivalent to the vanishing of  $H^1(G, \pi)$  for all *irreducible* representations  $\pi$  of  $G$ .

One more consequence is that any countable group with Property (T) is a quotient of a finitely presented group with Property (T). This can be viewed as the definitive answer to the question written as “Hypothesis 1” in [Kazhd-67] (see the paper in its original Russian version, since the English translation does not make sense at this point).

### Product replacement algorithm

The object of computational group theory is to discover and prove new results on finite groups. In this kind of work, it is crucial to generate random group elements. The “product replacement algorithm” is used widely and successfully in practice (for example in the packages GAP and MAGMA), but poorly understood in theory. Given a finite group  $G$  which can be generated by  $d$  elements

<sup>22</sup> Two representations are not Hausdorff separated if, in the unitary dual of the group with the Fell topology, any neighbourhood of one intersects any neighbourhood of the other. For an account of examples of representations with  $H^1(G, \pi) \neq 0$ , see [Louve-01].

and an integer  $k \geq d$ , the algorithm provides  $k$ -tuples of elements generating  $G$  by applying randomly one of  $4k(k-1)$  operations similar to some of the Nielsen transformations of classical combinatorial group theory. The algorithm can be seen as a random walk on a graph  $\Gamma_k(G)$  whose vertices are generating  $k$ -tuples of elements of  $G$ . There is also a natural action of an index two subgroup  $A^+(k)$  of the automorphism group  $\text{Aut}(F_k)$  of the free group on  $k$  generators.

Lubotzky and Pak have observed that, if the group  $\text{Aut}(F_k)$  could be shown to have Property (T) for  $k$  large enough, the effect of the product replacement algorithm would be much better understood. It is enough to know that appropriate quotients of  $A^+(k)$  have Property (T) for understanding how the algorithm works on restricted classes of finite groups; for example  $SL_k(\mathbf{Z})$  for finite abelian groups, or more generally groups described in [LubPa-01] for finite nilpotent groups. The approach of [LubPa-01] is closely related to some computations of Kazhdan constants due to Shalom.

### Action of Kazhdan groups on manifolds of dimensions $\leq 2$

Countable groups with Property (T) have very restricted actions on manifolds of low dimensions. It is a particular case of a theorem of Thurston [Thurs-74] that, for a non-trivial finitely generated subgroup  $\Gamma$  of the orientation-preserving diffeomorphism group of the interval, we have  $H^1(\Gamma, \mathbf{R}) \neq 0$ ; in particular, such a group cannot have Property (T).

The case of the other connected manifold of dimension one is covered by Navas' theorem, also announced by Reznikov [Rezni-01]: any homomorphism from a discrete Kazhdan group to the group of orientation-preserving  $C^{1+\alpha}$ -diffeomorphisms of the circle,  $\alpha > \frac{1}{2}$ , has finite image [Nava-02a]. Related results have been proved for particular groups (lattices in appropriate groups) acting on the circle by homeomorphisms ([BurMo-99], [Ghys-99], [Rezni-00], [Witte-94]). Navas has also a result about subgroups of Neretin groups, which are some groups in the  $p$ -adic world analogous to the diffeomorphism group of the circle [Nava-02b].

Infinite Kazhdan groups can act faithfully on compact surfaces, as shown by natural actions of  $SL_3(\mathbf{Z})$  on the sphere and of  $PSL_3(\mathbf{Z})$  on the real projective plane. However, no such action can preserve a Riemannian metric [Zimm-84c]. This is the only result that we quote here out of many others showing that actions of a Kazhdan group on a manifold of dimension small enough (this depends on the group) are extremely rigid.

Let us finally mention the fact that infinite groups with Property (T) do not appear as fundamental groups of compact manifolds of dimensions at most

3 satisfying Thurston's geometrization conjecture [Fujiw-99]. The case of mapping class groups of surfaces is open (see [Taher-00] for small genus).

### Variations

A topological group  $G$  has Property (T) if the unit representation  $1_G$  is isolated in the unitary dual  $\widehat{G}$ , for the Fell topology. If  $\mathcal{R}$  is a subspace of  $\widehat{G}$ , Lubotzky and Zimmer have defined  $G$  to have *Property (T;  $\mathcal{R}$ )* if  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$  [LubZi-89]. Let  $\Gamma$  be a countable group, let  $\mathcal{N}$  be a family of finite index normal subgroups in  $\Gamma$ , and let  $\mathcal{R}(\mathcal{N})$  denote the space of irreducible unitary representations which factor through  $\Gamma/N$  for some  $N \in \mathcal{N}$ ; then  $\Gamma$  has *Property ( $\tau$ ) with respect to  $\mathcal{N}$*  if it has Property (T;  $\mathcal{R}(\mathcal{N})$ ), and simply *Property ( $\tau$ )* in the particular case  $\mathcal{N}$  is the family of *all* finite index normal subgroups.

Being weaker than Property (T), Property ( $\tau$ ) applies to more groups. For example,  $SL_2(\mathcal{O})$  has Property ( $\tau$ ) for  $\mathcal{O}$  the ring of integers in a number field  $\mathbf{K}$  such that  $r_1 + r_2 \geq 2$ , where  $r_1$  is the number of real places and  $r_2$  the number of complex places of  $\mathbf{K}$  (more on this below). Also, irreducible lattices in direct products  $G_1 \times G_2$  have Property ( $\tau$ ) if  $G_1, G_2$  are separable locally compact groups with  $G_1$  having Property (T) and  $G_2$  being minimally almost periodic<sup>23</sup> (see [LubZi-89] and [BekLo-97]). This applies for example to irreducible lattices in  $O(n-1, 2) \times O(n, 1)$ .

Property ( $\tau$ ) for a countable group  $\Gamma$  implies some of the useful consequences of Property (T), such as the finiteness of the abelian quotients  $\Gamma_0/[\Gamma_0, \Gamma_0]$  for the subgroups  $\Gamma_0$  of finite index in  $\Gamma$ . However, Property ( $\tau$ ) does not imply finite generation; indeed, if  $\mathcal{P}$  is a non-empty set of primes, the group  $SL_2(\mathbf{Z}[1/\mathcal{P}])$  has Property ( $\tau$ ) [LubZu], but it is not finitely generated when  $\mathcal{P}$  is infinite, and it is residually finite as long as  $\mathcal{P}$  is a proper set of primes. It is important that Property ( $\tau$ ) is sufficient for several applications, for example to produce a family of expanders defined in terms of the finite quotients of an infinite residually finite finitely generated group with that property.

In case  $\Gamma$  is the fundamental group of a closed manifold  $M$ , Property ( $\tau$ ) has reformulations which involve the geometry and the potential theory of the finite coverings of  $M$ . In particular, Property ( $\tau$ ) for  $\pi_1(M)$  is equivalent to  $\inf \lambda_1(\widetilde{M}) > 0$ , where the infimum is taken over all finite coverings  $\widetilde{M}$  of  $M$  and where  $\lambda_1(\widetilde{M})$  denotes the smallest non-zero eigenvalue of the Riemannian Laplacian acting on  $L^2$ -functions on  $\widetilde{M}$ ; this is a result from [Brook-86] which has been influential (see [LubZi-89], Theorem 4.3.2 in [Lubot-94], [Lubot-97], [Leuzi-03], and [LubZu]).

<sup>23</sup> Recall that a topological group  $G$  is *minimally almost periodic* if any finite dimensional unitary representation of  $G$  is a multiple of the identity.

Recent work indicates a strong connection between Property  $(\tau)$  and expanders on the one hand, and some of the major problems of three-dimensional hyperbolic geometry on the other [Lacke–06].

The group  $SL_2(\mathbf{Z})$  does not have Property  $(\tau)$ , since it has a subgroup of finite index which surjects onto  $\mathbf{Z}$ ; this carries over to  $SL_2(\mathcal{O}_{\sqrt{-d}})$  for  $\mathcal{O}_{\sqrt{-d}}$  the ring of integers in an imaginary quadratic number field (Theorem 6 in [Serr–70b]). However,  $SL_2(\mathcal{O})$  has Property  $(\tau)$  with respect to the family of all *congruence subgroups* for  $\mathcal{O}$  the ring of integers in any number field  $\mathbf{K}$ . For  $SL_2(\mathbf{Z})$ , this is a qualitative consequence of Selberg’s “ $\frac{3}{16}$ -Theorem” about the spectrum of the Laplacian on the quotient of the Poincaré half plane by a congruence subgroup of  $SL_2(\mathbf{Z})$ ; for other cases, the standard references are [Sarna–83] and [GelJa–78]. If, moreover  $r_1 + r_2 \geq 2$ , any finite index subgroup in  $SL_2(\mathcal{O})$  is a congruence subgroup [Serr–70b], and it follows that  $SL_2(\mathcal{O})$  has the full Property  $(\tau)$ . For  $S$ -arithmetic subgroups of groups other than  $SL_2$ , see [Cloze–03].

*Property (T;  $\mathcal{FD}$ )* refers to the space  $\mathcal{FD}$  of irreducible finite-dimensional unitary representations. We do not know any example of a group which has Property  $(\tau)$  and which has not Property  $(T; \mathcal{FD})$ .

Another interesting case is the space  $\mathcal{R}_\infty$  of irreducible unitary representations with coefficients vanishing at infinity;  $SL_n(\mathbf{Q})$  has Property  $(T; \mathcal{R}_\infty)$  for  $n \geq 3$ .

There are other kinds of variations. One is the *strong Property (T)*, already mentioned in our account of the Ruziewicz problem.

Another is *Property (TT)*, which is also stronger than Property (T), and holds for a simple Lie group  $G$  with finite centre if and only if  $G$  has real rank at least 2 (this carries over to other local fields). A group  $G$  has Property (TT) if all the orbits of all its rough actions on Hilbert spaces are bounded. A continuous mapping  $\alpha : G \rightarrow \text{Isom}(\mathcal{H}) = \mathcal{U}(\mathcal{H}) \ltimes \mathcal{H}$  is a *rough action* if  $\sup_{g,h \in G} \sup_{\xi \in \mathcal{H}} \|\alpha(gh)\xi - \alpha(g)\alpha(h)\xi\| < \infty$ . Property (TT) has a reformulation in terms of *bounded cohomology*. See [BurMo–99] and [Monod–01].

In his review MR 1431309 (98a: 22003), Lubotzky observed that [Vale–94] was not up-to-date, and added kindly: “This is not the author’s fault; the area is so active, with some additional beautiful works appearing in the last three years. One hopes that an additional update will appear soon.” No doubt the present book will also be immediately out-of-date.

This book owes a lot to the collaboration and interest of a large number of colleagues and correspondents. We would like to single out Yves de Cornulier and Yehuda Shalom for the uncountable number of constructive and critical remarks they have made to us over the years. We are also most grateful

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# PART I

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## Kazhdan's Property (T)





# 1

## Definitions, first consequences, and basic examples

In this chapter, we first give the definition of Property (T) with minimal technicalities. We then reformulate it in terms of Fell's topology and establish easy consequences, such as compact generation for locally compact groups with Property (T). The main result in this chapter concerns the basic examples: if  $\mathbf{K}$  is a local field,  $SL_n(\mathbf{K})$  for  $n \geq 3$  and  $Sp_{2n}(\mathbf{K})$  for  $n \geq 2$ , as well as lattices in such groups, have Property (T). This is then generalised to the group of  $\mathbf{K}$ -rational points of an algebraic group with  $\mathbf{K}$ -rank at least 2. We end the chapter by discussing how Property (T) behaves with respect to lattices, short exact sequences, and coverings.

### 1.1 First definition of Property (T)

Kazhdan's Property (T) for a topological group involves its unitary representations in Hilbert spaces. In this chapter, *Hilbert spaces are always complex*; the inner product of two vectors  $\xi, \eta$  in such a space  $\mathcal{H}$  is written  $\langle \xi, \eta \rangle$ , is linear in  $\xi$  and antilinear in  $\eta$ .

The *unitary group*  $\mathcal{U}(\mathcal{H})$  of  $\mathcal{H}$  is the group of all invertible bounded linear operators  $U : \mathcal{H} \rightarrow \mathcal{H}$  which are unitary, namely such that, for all  $\xi, \eta \in \mathcal{H}$ ,

$$\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle,$$

or equivalently such that  $U^*U = UU^* = I$ , where  $U^*$  denotes the *adjoint* of  $U$  and  $I$  the identity operator on  $\mathcal{H}$ .

Let  $G$  be a topological group. A *unitary representation* of  $G$  in  $\mathcal{H}$  is a group homomorphism  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$  which is *strongly continuous*, that is, such that the mapping

$$G \rightarrow \mathcal{H}, \quad g \mapsto \pi(g)\xi$$

is continuous for every  $\xi \in \mathcal{H}$ . We often write  $(\pi, \mathcal{H})$  for such a representation, and  $\mathcal{H}_\pi$  instead of  $\mathcal{H}$  whenever useful.

Basic examples of unitary representations are associated to actions of groups on measure spaces (see Section A.6). In particular, we denote by  $1_G$  the *unit representation* of  $G$  in  $\mathbb{C}$  (associated to the action of  $G$  on a one-point space). In case of a locally compact group, we denote by  $\lambda_G$  the *left regular representation* of  $G$  on the space  $L^2(G)$  of complex-valued functions on  $G$  which are square-integrable with respect to a left Haar measure.

**Definition 1.1.1** Let  $(\pi, \mathcal{H})$  be a unitary representation of a topological group  $G$ .

- (i) For a subset  $Q$  of  $G$  and real number  $\varepsilon > 0$ , a vector  $\xi$  in  $\mathcal{H}$  is  $(Q, \varepsilon)$ -invariant if

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon \|\xi\|.$$

- (ii) The representation  $(\pi, \mathcal{H})$  *almost has invariant vectors* if it has  $(Q, \varepsilon)$ -invariants vectors for every *compact* subset  $Q$  of  $G$  and every  $\varepsilon > 0$ . If this holds, we write  $1_G < \pi$ .
- (iii) The representation  $(\pi, \mathcal{H})$  has *non-zero invariant vectors* if there exists  $\xi \neq 0$  in  $\mathcal{H}$  such that  $\pi(g)\xi = \xi$  for all  $g \in G$ . If this holds, we write  $1_G \subset \pi$ .

**Remark 1.1.2** (i) The symbol  $1_G < \pi$  is that of *weak containment* for representations, for which we refer to Appendix F; see in particular Corollary F.1.5. The symbol  $1_G \subset \pi$  is that indicating a *subrepresentation*, as in Definition A.1.5. It is straightforward that  $1_G \subset \pi$  implies  $1_G < \pi$ . Recall from Corollary F.2.9 that, if  $\pi$  is finite dimensional, then  $1_G < \pi$  is equivalent to  $1_G \subset \pi$ .

(ii) If  $Q' \subset Q$  and  $\varepsilon' \geq \varepsilon$ , then every  $(Q, \varepsilon)$ -invariant vector is  $(Q', \varepsilon')$ -invariant.

(iii) A vector  $\xi \in \mathcal{H}$  is  $(Q, \varepsilon)$ -invariant if and only if  $\xi$  is  $(\overline{Q \cup Q^{-1}}, \varepsilon)$ -invariant. This follows from the fact that  $\|\pi(g)\xi - \xi\| = \|\pi(g^{-1})\xi - \xi\|$  for every  $\xi \in \mathcal{H}$  and  $g \in G$ .

(iv) For a subset  $Q$  of  $G$  and an integer  $n \geq 1$ , denote by  $Q^n$  the set of all  $g \in G$  of the form  $g = q_1 \cdots q_n$  with  $q_1, \dots, q_n \in Q$ . Then every  $(Q, \varepsilon/n)$ -invariant vector is  $((Q \cup Q^{-1})^n, \varepsilon)$ -invariant (compare with the proof of Proposition F.1.7).

(v) Assume that  $G$  is a compactly generated locally compact group, and let  $Q$  be a compact generating subset of  $G$ . Then a unitary representation  $\pi$  of  $G$

almost has invariant vectors if and only if  $\pi$  has a  $(Q, \varepsilon)$ -invariant vector for every  $\varepsilon > 0$  (this follows from (iv)).

(vi) The regular representation  $\lambda_{\mathbf{R}}$  of  $\mathbf{R}$  by translations on  $L^2(\mathbf{R})$  almost has invariant vectors. Indeed, let  $Q$  be any compact subset of  $\mathbf{R}$  and let  $\varepsilon > 0$ ; consider a compact interval  $[a, b]$  with  $a < b$ , the characteristic function  $\chi$  of  $[a, b]$ , and  $\xi = (b - a)^{-1/2} \chi \in L^2(\mathbf{R})$ . Then  $\|\xi\| = 1$  and

$$\|\lambda_{\mathbf{R}}(t)\xi - \xi\|^2 = \frac{2|t|}{b - a} < \varepsilon^2$$

for all  $t \in Q$  as long as  $b - a$  is large enough. Similar arguments show that  $1_G \prec \lambda_G$  for any locally compact abelian group  $G$  of the form  $\mathbf{Z}^m \oplus \mathbf{R}^n$ . More generally, for a locally compact group  $G$ , the weak containment  $1_G \prec \lambda_G$  holds if and only if  $G$  is amenable (Theorem G.3.2). Recall that, in particular, compact extensions of solvable groups are amenable.

(vii) If  $G$  is a locally compact group, the following three properties are equivalent (Proposition A.5.1):  $G$  is compact,  $G$  has finite Haar measure, and  $1_G \subset \lambda_G$ . It follows that, for an amenable and non-compact locally compact group  $G$ , we have  $1_G \prec \lambda_G$  and  $1_G \not\subset \lambda_G$ .

**Definition 1.1.3** Let  $G$  be a topological group. A subset  $Q$  of  $G$  is a *Kazhdan set* if there exists  $\varepsilon > 0$  with the following property: every unitary representation  $(\pi, \mathcal{H})$  of  $G$  which has a  $(Q, \varepsilon)$ -invariant vector also has a non-zero invariant vector.

In this case,  $\varepsilon > 0$  is called a *Kazhdan constant* for  $G$  and  $Q$ , and  $(Q, \varepsilon)$  is called a *Kazhdan pair* for  $G$ .

The group  $G$  has *Kazhdan's Property (T)*, or is a *Kazhdan group*, if  $G$  has a compact Kazhdan set.

In other words,  $G$  has Kazhdan's Property (T) if there exists a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$  such that, whenever a unitary representation  $\pi$  of  $G$  has a  $(Q, \varepsilon)$ -invariant vector, then  $\pi$  has a non-zero invariant vector.

**Remark 1.1.4** Let  $G$  be a topological group  $G$ . For a compact subset  $Q$  and a unitary representation  $(\pi, \mathcal{H})$  of  $G$ , we define the *Kazhdan constant associated to  $Q$  and  $\pi$*  as the following non-negative constant:

$$\kappa(G, Q, \pi) = \inf_{x \in Q} \{\max \|\pi(x)\xi - \xi\| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

It is clear that  $1_G \prec \pi$ , if and only if  $\kappa(G, Q, \pi) = 0$  for all  $Q$ 's.

We also define the constant

$$\kappa(G, Q) = \inf_{\pi} \kappa(G, Q, \pi),$$

where  $\pi$  runs over all equivalence classes of unitary representations of  $G$  without non-zero invariant vectors. The group  $G$  has Property (T) if and only if there exists a compact subset  $Q$  such that  $\kappa(G, Q) > 0$ . If this is the case,  $\kappa(G, Q)$  is the optimal constant  $\varepsilon$  such that  $(Q, \varepsilon)$  is a Kazhdan pair.

As the next proposition shows, compact groups are the first – and obvious – examples of groups with Property (T). Other examples of groups with Property (T) will be given in Section 1.4.

**Proposition 1.1.5** *Let  $G$  be a topological group. The pair  $(G, \sqrt{2})$  is a Kazhdan pair; that is, if a unitary representation  $(\pi, \mathcal{H})$  of  $G$  has a unit vector  $\xi$  such that*

$$\sup_{x \in G} \|\pi(x)\xi - \xi\| < \sqrt{2},$$

*then  $\pi$  has a non-zero invariant vector. In particular, every compact group has Property (T).*

**Proof** Let  $\mathcal{C}$  be the closed convex hull of the subset  $\pi(G)\xi$  of  $\mathcal{H}$ . Let  $\eta_0$  be the unique element in  $\mathcal{C}$  with minimal norm, that is,

$$\|\eta_0\| = \min\{\|\eta\| : \eta \in \mathcal{C}\}.$$

As  $\mathcal{C}$  is  $G$ -invariant,  $\eta_0$  is  $G$ -invariant.

We claim that  $\eta_0 \neq 0$ . Indeed, set

$$\varepsilon = \sqrt{2} - \sup_{x \in G} \|\pi(x)\xi - \xi\| > 0.$$

For every  $x \in G$ , we have

$$2 - 2\operatorname{Re}\langle \pi(x)\xi, \xi \rangle = \|\pi(x)\xi - \xi\|^2 \leq (\sqrt{2} - \varepsilon)^2.$$

Hence,

$$\operatorname{Re}\langle \pi(x)\xi, \xi \rangle \geq \frac{2 - (\sqrt{2} - \varepsilon)^2}{2} = \frac{\varepsilon(2\sqrt{2} - \varepsilon)}{2} > 0.$$

This implies that

$$\operatorname{Re}\langle \eta, \xi \rangle \geq \frac{\varepsilon(2\sqrt{2} - \varepsilon)}{2}$$

for all  $\eta \in \mathcal{C}$ . In particular,  $\eta_0 \neq 0$ . ■

The following result is now straightforward.

**Theorem 1.1.6** *For a locally compact group  $G$ , the following properties are equivalent:*

- (i)  $G$  is amenable and has Property (T);
- (ii)  $G$  is compact.

**Proof** If  $G$  is a compact group, then  $G$  has Property (T) by the previous proposition, and is amenable (see Example G.1.5).

Conversely, assume that the locally compact group  $G$  is amenable and has Property (T). Since  $G$  is amenable,  $\lambda_G$  almost has invariant vectors (Remark 1.1.2.vi). Hence,  $\lambda_G$  has a non-zero invariant vector. This implies that  $G$  is compact (Remark 1.1.2.vii). ■

**Example 1.1.7** The groups  $\mathbf{R}^n$  and  $\mathbf{Z}^n$  do not have Property (T).

**Remark 1.1.8** The previous theorem fails for topological groups which are not locally compact. Indeed, as mentioned in Remark G.3.7, the unitary group  $\mathcal{U}(\mathcal{H})$  of an infinite dimensional separable Hilbert space  $\mathcal{H}$ , endowed with the weak operator topology, is amenable. On the other hand, it is shown in [Bekka–03] that  $\mathcal{U}(\mathcal{H})$  has Property (T).

It is a useful fact that invariant vectors are known to exist near “almost-invariant vectors”. Here is a more precise formulation.

**Proposition 1.1.9** *Let  $G$  be a topological group, let  $(Q, \varepsilon)$  be a Kazhdan pair for  $G$ , and let  $\delta > 0$ . Then, for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  and every  $(Q, \delta\varepsilon)$ -invariant vector  $\xi$ , we have*

$$\|\xi - P\xi\| \leq \delta\|\xi\|,$$

where  $P : \mathcal{H} \rightarrow \mathcal{H}^G$  is the orthogonal projection on the subspace  $\mathcal{H}^G$  of all  $G$ -invariant vectors in  $\mathcal{H}$ .

**Proof** Write  $\xi = \xi' + \xi''$ , where  $\xi' = P\xi \in \mathcal{H}^G$  and  $\xi'' = \xi - \xi' \in (\mathcal{H}^G)^\perp$ . Let  $t \in \mathbf{R}$  with  $0 < t < 1$ . Since  $(\mathcal{H}^G)^\perp$  contains no non-zero invariant vectors and since  $(Q, \varepsilon)$  is a Kazhdan pair, we have

$$\|\pi(x)\xi'' - \xi''\| \geq t\varepsilon\|\xi''\|,$$

for some  $x \in Q$ . On the other hand, we have

$$\|\pi(x)\xi'' - \xi''\| = \|\pi(x)\xi - \xi\| < \delta\varepsilon\|\xi\|.$$

Hence,

$$\|\xi - \xi'\| = \|\xi''\| < \frac{\delta}{t}\|\xi\|$$

for all  $0 < t < 1$  and the claim follows. ■

**Remark 1.1.10** The proof of the previous proposition is shorter when  $Q$  is compact, as we can take  $t = 1$  in this case. Moreover, we obtain the slightly better estimate  $\|\xi - P\xi\| < \delta\|\xi\|$ .

## 1.2 Property (T) in terms of Fell's topology

Property (T) can be characterised in terms of Fell's topology. For this, we will freely use facts established in the appendix.

**Proposition 1.2.1** *Let  $G$  be a topological group. The following statements are equivalent:*

- (i)  $G$  has Kazhdan's Property (T);
- (ii) whenever a unitary representation  $(\pi, \mathcal{H})$  of  $G$  weakly contains  $1_G$ , it contains  $1_G$  (in symbols:  $1_G \prec \pi$  implies  $1_G \subset \pi$ ).

**Proof** It is obvious from the definitions that (i) implies (ii).

To show the converse, assume that  $G$  does not have Property (T). Let  $I$  be the set of all pairs  $(Q, \varepsilon)$ , where  $Q$  is a compact subset of  $G$  and  $\varepsilon > 0$ . Then, for every  $(Q, \varepsilon) \in I$ , there exists a unitary representation  $(\pi_{Q, \varepsilon}, \mathcal{H}_{Q, \varepsilon})$  of  $G$  without non-zero invariant vectors and with a unit vector  $\xi_{Q, \varepsilon} \in \mathcal{H}_{Q, \varepsilon}$  which is  $(Q, \varepsilon)$ -invariant.

Set

$$\pi = \bigoplus_{(Q, \varepsilon) \in I} \pi_{Q, \varepsilon}.$$

It is clear that  $1_G \prec \pi$ . On the other hand,  $1_G$  is not contained in  $\pi$ . Indeed, let

$$\xi = \bigoplus_{(Q, \varepsilon) \in I} \xi_{Q, \varepsilon}$$

be an invariant vector for  $\pi$ . Then every  $\xi_{Q,\varepsilon}$  is an invariant vector for  $\pi_{Q,\varepsilon}$ . Hence,  $\xi_{Q,\varepsilon} = 0$  for all  $(Q, \varepsilon) \in I$ . Therefore,  $\xi = 0$ . This shows that (ii) implies (i). ■

**Remark 1.2.2** Let  $G$  be a locally compact group which is generated by a compact set  $Q$ ; assume that  $G$  has Property (T). Then there exists  $\varepsilon > 0$  such that  $(Q, \varepsilon)$  is a Kazhdan set.

Indeed, assume that this not true. Then, for every  $\varepsilon > 0$ , there exists a unitary representation  $\pi_\varepsilon$  of  $G$  without non-zero invariant vectors and with a  $(Q, \varepsilon)$ -invariant vector. Set  $\pi = \bigoplus_\varepsilon \pi_\varepsilon$ . Then, as in the proof above,  $\pi$  has  $(Q, \varepsilon)$ -invariant vectors for every  $\varepsilon > 0$ . Since  $Q$  generates  $G$ , we conclude that  $1_G < \pi$  (Proposition F.1.7). Hence,  $1_G \subset \pi$ , by the previous proposition. This is a contradiction.

Recall that *Fell's topology* is defined on every set of equivalence classes of unitary representations of a topological group (see Definition F.2.1). Recall also that a point  $x_0$  in a topological space  $X$  is *isolated* if  $\{x_0\}$  is open in  $X$ .

**Proposition 1.2.3** *Let  $G$  be a topological group. The following statements are equivalent:*

- (i)  $G$  has Kazhdan's Property (T);
- (ii)  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$ , for every set  $\mathcal{R}$  of equivalence classes of unitary representations of  $G$  without non-zero invariant vectors.

**Proof** To show that (i) implies (ii), assume that  $G$  has Property (T). Suppose, by contradiction, that there exists a set  $\mathcal{R}$  of equivalence classes of unitary representations of  $G$  without non-zero invariant vectors such that  $1_G$  is not isolated in  $\mathcal{R} \cup \{1_G\}$ . Hence, we can find a net  $(\pi_i)_{i \in I}$  in  $\mathcal{R}$  which converges to  $1_G$ . Then  $1_G < \bigoplus_{i \in I} \pi_i$  (Proposition F.2.2). By the previous proposition,  $1_G$  is contained in  $\bigoplus_{i \in I} \pi_i$ . Therefore,  $1_G$  is contained in  $\pi_i$  for some  $i \in I$ . This is a contradiction. Hence, (i) implies (ii).

Assume that  $G$  does not have Property (T). By the previous proposition, there exists a unitary representation  $(\pi, \mathcal{H})$  of  $G$  such that  $1_G$  is weakly contained but not contained in  $\pi$ . Let  $\mathcal{R} = \{\pi\}$ . Then  $1_G$  is not isolated in  $\mathcal{R} \cup \{1_G\}$ . This shows that (ii) implies (i). ■

Theorem 1.2.5 below shows that, in case  $G$  is locally compact, Property (T) is equivalent to the isolation of  $1_G$  in the *unitary dual*  $\widehat{G}$  of  $G$ , the set of equivalence classes of irreducible unitary representations of  $G$ . The following lemma is proved in [Wang–75, Corollary 1.10], by a different method and under the much weaker assumption that the Hilbert space of  $\pi_0$  is separable.

**Lemma 1.2.4** *Let  $G$  be a locally compact group, and  $\pi_0$  be a finite dimensional irreducible unitary representation of  $G$ . The following properties are equivalent:*

- (i) *whenever  $\pi_0$  is weakly contained in a unitary representation  $\pi$  of  $G$ , then  $\pi_0$  is contained in  $\pi$ ;*
- (ii)  *$\pi_0$  is isolated in  $\widehat{G}$ .*

**Proof** Assume that  $\pi_0$  is not isolated in  $\widehat{G}$ . Then there exists a net  $(\pi_i)_{i \in I}$  in  $\widehat{G} \setminus \{\pi_0\}$  which converges to  $\pi_0$ . Let  $\pi = \bigoplus_{i \in I} \pi_i$ . Then  $\pi_0 \prec \pi$  and  $\pi_0$  is not contained in  $\pi$ . This shows that (i) implies (ii).

To show the converse, assume that there exists a unitary representation  $\pi$  of  $G$  such that  $\pi_0$  is weakly contained but not contained in  $\pi$ . We claim that  $\pi_0$  is not isolated in  $\widehat{G}$ .

Fix a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$ . Let  $\varphi_0$  be a normalised function of positive type associated to  $\pi_0$ . Since  $\pi_0 \prec \pi$  and since  $\pi_0$  is irreducible, there exists a normalised function of positive type  $\varphi$  associated to  $\pi$  such that

$$(*) \quad \sup_{x \in Q} |\varphi(x) - \varphi_0(x)| \leq \varepsilon/2$$

(see Proposition F.1.4). By Theorem C.5.5, there exists a net  $(\varphi_i)_{i \in I}$  of convex combinations of normalised functions of positive type associated to *irreducible* unitary representations of  $G$  such that

$$\lim_i \varphi_i = \varphi$$

in the weak\* topology on  $L^\infty(G)$ .

For each  $i \in I$ , we can write

$$\varphi_i = a_i \psi_i + b_i \psi'_i,$$

where  $\psi_i$  is a convex combination of normalised functions of positive type associated to unitary representations in  $\widehat{G} \setminus \{\pi_0\}$ , where  $\psi'_i$  is a convex combination of normalised functions of positive type associated to  $\pi_0$ , and where  $a_i, b_i$  are real numbers with  $0 \leq a_i, b_i \leq 1$  and  $a_i + b_i = 1$ .

Upon passing to a subnet, we can assume that

$$\lim_i \psi_i = \psi \quad \text{and} \quad \lim_i \psi'_i = \psi'$$

in the weak\* topology for some functions of positive type  $\psi$  and  $\psi'$  on  $G$ , and that  $\lim_i a_i = a$  and  $\lim_i b_i = b$  for some real numbers  $a, b$  with  $0 \leq a, b \leq 1$



and  $a + b = 1$ . We then have

$$\varphi = a\psi + b\psi'.$$

We claim that either  $b = 0$  or  $\psi' = 0$ . Indeed, assume by contradiction that  $b \neq 0$  and  $\psi'(e) \neq 0$ . Then  $\psi'(e) = 1$ , since  $\varphi(e) = 1$  and  $\psi(e) \leq 1$ ,  $\psi'(e) \leq 1$ . It follows from Raikov's Theorem C.5.6 that  $\lim_i \psi'_i = \psi'$  uniformly on compact subsets of  $G$ . Since  $\pi_0$  is finite dimensional, Lemma F.2.8 shows that  $\psi'$  is a sum of functions of positive type associated to  $\pi_0$ . It follows that  $\pi_0$  is contained in  $\pi$ , since  $\pi_0$  is irreducible (Proposition C.5.1) and this is a contradiction.

Therefore, we have  $\lim_i a_i \psi_i = \varphi$  and  $\lim_i a_i = 1$ . Thus,  $\lim_i \psi_i = \varphi$  in the weak\* topology; by Raikov's Theorem, this holds also uniformly on compact subsets of  $G$ . Hence, using (\*), there exists  $i \in I$  such that

$$\sup_{x \in Q} |\psi_i(x) - \varphi_0(x)| \leq \varepsilon.$$

This shows that  $\varphi_0$  can be approximated, uniformly on compact subsets of  $G$ , by convex combinations of normalised functions of positive type associated to unitary representations in  $\widehat{G} \setminus \{\pi_0\}$ . By a standard fact (see [Conwa–87, Theorem 7.8]), it follows that  $\varphi_0$  can be approximated, uniformly on compact subsets of  $G$ , by functions of positive type associated to unitary representations in  $\widehat{G} \setminus \{\pi_0\}$  (compare with the proof of Proposition F.1.4). Hence,  $\pi_0$  is not isolated in  $\widehat{G}$ . ■

**Theorem 1.2.5** *Let  $G$  be a locally compact group. The following statements are equivalent:*

- (i)  $G$  has Kazhdan's Property (T);
- (ii)  $1_G$  is isolated in  $\widehat{G}$ ;
- (iii) every finite dimensional irreducible unitary representation of  $G$  is isolated in  $\widehat{G}$ ;
- (iv) some finite dimensional irreducible unitary representation of  $G$  is isolated in  $\widehat{G}$ .

**Proof** For the equivalence of (i) and (ii), see Lemma 1.2.4 and Proposition 1.2.1.

To show that (i) implies (iii), assume that  $G$  has Property (T), and let  $\pi_0$  be a finite dimensional irreducible unitary representation of  $G$ . Let  $\pi$  be a unitary representation of  $G$  with  $\pi_0 < \pi$ . Then

$$\pi_0 \otimes \overline{\pi}_0 < \pi \otimes \overline{\pi}_0$$

(continuity of the tensor product, see Proposition F.3.2). Since  $\pi_0$  is finite dimensional,  $1_G$  is contained in  $\pi_0 \otimes \bar{\pi}_0$  (Proposition A.1.12). Hence,  $1_G < \pi \otimes \bar{\pi}_0$ . Therefore,  $1_G$  is contained in  $\pi \otimes \bar{\pi}_0$ , by Proposition 1.2.1. Since  $\pi_0$  is irreducible, this implies that  $\pi_0$  is contained in  $\pi$  (Corollary A.1.13). Hence, by Lemma 1.2.4,  $\pi_0$  is isolated in  $\widehat{G}$ .

It is obvious that (iii) implies (iv). It remains to show that (iv) implies (ii). Let  $\pi_0$  be a finite dimensional irreducible unitary representation of  $G$  which is isolated in  $\widehat{G}$ . The finite dimensional unitary representation  $\pi_0 \otimes \bar{\pi}_0$  decomposes as a direct sum of irreducible unitary representations:

$$\pi_0 \otimes \bar{\pi}_0 = \pi_1 \oplus \cdots \oplus \pi_n.$$

By Corollary F.2.9,  $\{\pi_k\}$  is a closed point in  $\widehat{G}$  for every  $k \in \{1, \dots, n\}$ .

Assume, by contradiction, that  $1_G$  is not isolated in  $\widehat{G}$ . Then there exists a net  $(\sigma_i)_{i \in I}$  in  $\widehat{G} \setminus \{\pi_1, \dots, \pi_n\}$  such that  $\lim_i \sigma_i = 1_G$ . We then have  $1_G < \bigoplus_{i \in I} \sigma_i$  (Proposition F.2.2) and therefore

$$\pi_0 < \bigoplus_{i \in I} \sigma_i \otimes \pi_0.$$

Hence, by Lemma 1.2.4,  $\pi_0$  is contained in  $\bigoplus_{i \in I} \sigma_i \otimes \pi_0$ . Since  $\pi_0$  is irreducible, there exists  $i \in I$  such that  $\pi_0$  is contained in  $\sigma_i \otimes \pi_0$ . Then  $\pi_0 \otimes \bar{\pi}_0$  is contained in  $\sigma_i \otimes \pi_0 \otimes \bar{\pi}_0$ . It follows that  $1_G$  is contained in  $\sigma_i \otimes \pi_0 \otimes \bar{\pi}_0$  (Proposition A.1.12). Since  $\sigma_i$  is irreducible,  $\sigma_i$  is contained in  $\pi_0 \otimes \bar{\pi}_0$ , by Proposition A.1.12 again. Since  $\pi_0 \otimes \bar{\pi}_0 = \pi_1 \oplus \cdots \oplus \pi_n$ , it follows that  $\sigma_i$  is unitarily equivalent to one of the  $\pi_k$ 's. This is a contradiction to the choice of  $(\sigma_i)_{i \in I}$ . ■

**Remark 1.2.6** The equivalence between (i) and (ii) in the previous theorem is due to Kazhdan [Kazhd–67] and the equivalence between (i) and (iii) to S.P. Wang [Wang–75, Theorem 2.1].

### 1.3 Compact generation and other consequences

The first spectacular application of Property (T) is the following result, due to Kazhdan.

**Theorem 1.3.1** *Let  $G$  be a locally compact group with Property (T). Then  $G$  is compactly generated. In particular, a discrete group  $\Gamma$  with Property (T) is finitely generated.*

**Proof** Let  $\mathcal{C}$  be the set of all open and compactly generated subgroups of  $G$ . Since  $G$  is locally compact, we have

$$G = \bigcup_{H \in \mathcal{C}} H.$$

Indeed, every element in  $G$  has a compact neighbourhood and the subgroup generated by a set containing an open non-empty subset is open.

For every  $H \in \mathcal{C}$ , observe that  $G/H$  is discrete (since  $H$  is open) and denote by  $\lambda_{G/H}$  the *quasi-regular representation* of  $G$  on  $\ell^2(G/H)$ . Let  $\delta_H \in \ell^2(G/H)$  be the Dirac function at the point  $H \in G/H$ . Observe that  $\delta_H$  is  $H$ -invariant. Let

$$\pi = \bigoplus_{H \in \mathcal{C}} \lambda_{G/H}$$

be the direct sum of the representations  $\lambda_{G/H}$ . Then  $\pi$  almost has invariant vectors. Indeed, let  $Q$  be a compact subset of  $G$ . Let  $K \in \mathcal{C}$  be the subgroup generated by a relatively compact open neighbourhood of  $Q$ . Since  $Q \subset K$ , we have

$$\|\pi(x)\delta_K - \delta_K\| = 0 \quad \text{for all } x \in Q,$$

where  $\delta_K$  is viewed in the obvious way as a unit vector in  $\bigoplus_{H \in \mathcal{C}} \ell^2(G/H)$ . Since  $G$  has Property (T), there exists a non-zero  $G$ -invariant vector

$$\xi = \bigoplus_{H \in \mathcal{C}} \xi_H \in \bigoplus_{H \in \mathcal{C}} \ell^2(G/H).$$

Let  $H \in \mathcal{C}$  be such that  $\xi_H \neq 0$ . Then  $\xi_H$  is a non-zero  $G$ -invariant vector in  $\ell^2(G/H)$ . This implies that  $G/H$  is finite. Since  $H$  is compactly generated,  $G$  is compactly generated. ■

For a generalisation of the previous theorem, see Exercise 1.8.16.ii. Concerning the question of finite presentability of groups with Property (T), see Section 3.4.

**Proposition 1.3.2** *Let  $G$  be a locally compact group which has Property (T) and let  $Q$  be a subset of  $G$ .*

- (i) *Assume that  $Q$  is a generating set for  $G$ . Then  $Q$  is a Kazhdan set.*
- (ii) *Assume that  $Q$  is a Kazhdan set and has a non-empty interior. Then  $Q$  is a generating set for  $G$ .*

In particular, a subset  $Q$  of a discrete group  $\Gamma$  with Property (T) is a Kazhdan set if and only if  $Q$  is a generating set for  $\Gamma$ .

**Proof** (i) Let  $(R, \varepsilon)$  be a Kazhdan pair for  $G$ , where  $R$  is compact. Since  $\tilde{Q}$  is a generating set, there exists an integer  $n \geq 1$  such that  $R \subset (\tilde{Q})^n$ , where  $\tilde{Q} = Q \cup Q^{-1} \cup \{e\}$  (see the proof of Proposition F.1.7). It follows that  $((\tilde{Q})^n, \varepsilon)$  is a Kazhdan pair for  $G$ . Hence,  $(Q, \varepsilon/n)$  is a Kazhdan pair, by Remark 1.1.2.iv.

(ii) Denote by  $H$  the subgroup of  $G$  generated by  $Q$ . Since  $Q$  has non-empty interior,  $H$  is open in  $G$ . Let  $\lambda_{G/H}$  the quasi-regular representation of  $G$  on  $\ell^2(G/H)$ . Denote by  $P$  the orthogonal projection of  $\ell^2(G/H)$  onto its subspace of constant functions and by  $\xi \in \ell^2(G/H)$  the Dirac function at the point  $H \in G/H$ . By definition of  $H$  and  $\lambda_{G/H}$ , we have  $\lambda_{G/H}(x)\xi = \xi$  for all  $x \in Q$ . It follows from Proposition 1.1.9 that  $\xi = P\xi$ , so that  $H = G$ . ■

**Remark 1.3.3** The following example shows that, in a non-discrete locally compact group with Property (T), a Kazhdan set *need not* be a generating set.

Consider an integer  $n \geq 3$  and let  $Q$  be the subset of  $SL_n(\mathbf{R})$  consisting of the two matrices

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

By Theorem A in [Shal-00c], it is known that  $Q$  is a Kazhdan set for  $SL_n(\mathbf{R})$ . Other examples of Kazhdan sets which are not generating in  $SL_n(\mathbf{R})$  and  $SL_n(\mathbf{Q}_p)$  are given in [Oh-02].

**Theorem 1.3.4** Let  $G_1$  and  $G_2$  be topological groups, and let  $\varphi : G_1 \rightarrow G_2$  be a continuous homomorphism with dense image. If  $G_1$  has Property (T), then  $G_2$  has Property (T).

In particular, Property (T) is inherited by quotients: if  $G_1$  has Property (T), then so does  $G_1/N$  for every closed normal subgroup  $N$  of  $G_1$ .

**Proof** Let  $(Q_1, \varepsilon)$  be a Kazhdan pair for  $G_1$ , with  $Q_1$  compact. Then  $Q_2 = \varphi(Q_1)$  is a compact subset of  $G_2$ , and we claim that  $(Q_2, \varepsilon)$  is a Kazhdan pair for  $G_2$ . Indeed, let  $\pi$  be a unitary representation of  $G_2$  with a  $(Q_2, \varepsilon)$ -invariant vector  $\xi$ . Then  $\pi \circ \varphi$  is a unitary representation of  $G_1$ , and  $\xi$  is  $(Q_1, \varepsilon)$ -invariant for  $\pi \circ \varphi$ . Hence, there exists a non-zero vector  $\eta$  which is invariant under  $\pi \circ \varphi(G_1) = \pi(\varphi(G_1))$ . Since  $\varphi(G_1)$  is dense in  $G_2$  and since  $\pi$  is strongly continuous,  $\eta$  is invariant under  $\pi(G_2)$ . ■

**Corollary 1.3.5** *Let  $G_1$  be a topological group with Property (T), and let  $G_2$  be a locally compact amenable group. Every continuous homomorphism  $\varphi : G_1 \rightarrow G_2$  has a relatively compact image.*

*In particular, every continuous homomorphism  $\varphi : G_1 \rightarrow \mathbf{R}^n$  or  $\varphi : G_1 \rightarrow \mathbf{Z}^n$  is constant.*

**Proof** Let  $H$  be the closure of  $\varphi(G_1)$  in  $G_2$ . Then  $H$  is an amenable locally compact group. Moreover,  $H$  has Property (T), by the previous theorem. Hence,  $H$  is compact (Theorem 1.1.6).

The last statement follows, since  $\{0\}$  is the unique compact subgroup of  $\mathbf{R}^n$  or  $\mathbf{Z}^n$ . ■

**Corollary 1.3.6** *Let  $G$  be a locally compact group with Property (T). Then:*

- (i) *the Hausdorff abelianised group  $G/[\overline{G}, \overline{G}]$  is compact;*
- (ii)  *$G$  is unimodular.*

*In particular, if  $\Gamma$  is a discrete group with Property (T), then its abelianisation  $\Gamma/[\Gamma, \Gamma]$  is finite.*

**Proof** (i) and (ii) follow from the previous corollary. ■

We now give examples of non-amenable groups *without* Property (T).

**Example 1.3.7** (i) For  $k \geq 2$ , let  $\Gamma = F_k$  be the non-abelian free group on  $k$  generators. As  $\Gamma/[\Gamma, \Gamma] \cong \mathbf{Z}^k$ , it follows from the previous corollary that  $\Gamma$  does not have Property (T).

(ii) Let  $\Gamma$  be the fundamental group of an orientable closed surface of genus  $g \geq 1$ . Then  $\Gamma$  does not have Property (T), since  $\Gamma/[\Gamma, \Gamma] \cong \mathbf{Z}^{2g}$ . Similarly, the fundamental group  $\Gamma$  of a non-orientable closed surface of genus  $g \geq 2$  does not have Property (T), since  $\Gamma/[\Gamma, \Gamma] \cong \mathbf{Z}^{g-1} \oplus (\mathbf{Z}/2\mathbf{Z})$ ; see [Masse–67], Chapter 4, Proposition 5.1.

(iii) Let us show that the group  $G = SL_2(\mathbf{R})$  does not have Property (T). Recall that the free group  $\Gamma = F_2$  embeds as a lattice in  $G$  (see Example B.2.5). Let  $\Delta = [\Gamma, \Gamma]$ . We claim that the quasi-regular representation  $\lambda_{G/\Delta}$  on  $L^2(G/\Delta)$  almost has invariant vectors and has no non-zero invariant ones.

Since the group  $\Gamma/\Delta$  is abelian (and hence amenable),  $1_\Gamma \prec \lambda_{\Gamma/\Delta}$ . Therefore

$$\lambda_{G/\Gamma} = \text{Ind}_\Gamma^G 1_\Gamma \prec \text{Ind}_\Gamma^G \lambda_{\Gamma/\Delta},$$

by continuity of induction (Theorem F.3.5). Since  $\text{Ind}_\Gamma^G \lambda_{\Gamma/\Delta} = \text{Ind}_\Gamma^G (\text{Ind}_\Delta^\Gamma 1_\Delta)$  is equivalent to  $\text{Ind}_\Delta^G 1_\Delta = \lambda_{G/\Delta}$  (induction by stages, see Theorem E.2.4), it follows that  $\lambda_{G/\Gamma} \prec \lambda_{G/\Delta}$ . As  $\Gamma$  is a lattice in  $G$ , the unit representation  $1_G$  is contained in  $\lambda_{G/\Gamma}$ . Hence,  $1_G \prec \lambda_{G/\Delta}$ .

On the other hand, assume by contradiction that  $\lambda_{G/\Delta}$  has a non-zero invariant vector. This implies that  $G/\Delta$  has a finite invariant measure (Theorem E.3.1). As  $\Gamma$  is a lattice,  $\Gamma/\Delta$  is finite. This is a contradiction, since  $\Gamma/\Delta \cong \mathbf{Z}^2$ .

We will give other proofs of the fact that  $SL_2(\mathbf{R})$  does not have Property (T); see Example 1.7.4 and Remark 2.12.8.

(iv) For any  $n \geq 2$ , the discrete group  $SL_n(\mathbf{Q})$  does not have Property (T). Indeed,  $SL_n(\mathbf{Q})$  is not finitely generated, since every finite subset  $\{x_1, \dots, x_m\}$  of  $SL_n(\mathbf{Q})$  is contained in  $SL_n(\mathbf{Z}[1/N])$ , where  $N$  is a common multiple of the denominators of the matrix coefficients of  $x_1, \dots, x_m$ . The claim follows from Theorem 1.3.1.

## 1.4 Property (T) for $SL_n(\mathbf{K})$ , $n \geq 3$

Let  $\mathbf{K}$  be a *local field*. Recall that this is a non-discrete locally compact field, and that its topology is defined by an absolute value (see Section D.4, for more details).

We proceed to show that Property (T) holds for the *special linear group*  $SL_n(\mathbf{K})$  when  $n \geq 3$ , for the symplectic group  $Sp_{2n}(\mathbf{K})$  when  $n \geq 2$  (Section 1.5), and more generally for higher rank simple algebraic groups over  $\mathbf{K}$  (Section 1.6). The group  $SL_2(\mathbf{K})$  does not have Property (T); see Examples 1.3.7 and 1.7.4.

### Some general facts

We collect the common ingredients used in the proofs of Property (T) for  $SL_n(\mathbf{K})$  and  $Sp_{2n}(\mathbf{K})$ .

**Lemma 1.4.1** *Let  $G$  be a topological group, and let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  with  $1_G \prec \pi$ . Then there exists a linear functional  $\varphi$  on the algebra  $\mathcal{L}(\mathcal{H})$  of bounded operators on  $\mathcal{H}$  with the following properties:*

- (i)  $\varphi(I) = 1$ ;
- (ii)  $\varphi$  is positive, that is,  $\varphi(T^*T) \geq 0$  for all  $T \in \mathcal{L}(\mathcal{H})$ ;
- (iii)  $\varphi(\pi(x)T) = \varphi(T\pi(x)) = \varphi(T)$  for all  $x \in G$  and  $T \in \mathcal{L}(\mathcal{H})$ .

**Proof** Since  $1_G \prec \pi$ , there is a net of unit vectors  $(\xi_i)_{i \in I}$  in  $\mathcal{H}$  such that

$$(*) \quad \lim_i \|\pi(x)\xi_i - \xi_i\| = 0,$$

for all  $x \in G$ . For each  $T \in \mathcal{L}(\mathcal{H})$ , let  $D_T$  be the closed disc in  $\mathbf{C}$  of radius  $\|T\|$ , and consider the product space

$$X = \prod_{T \in \mathcal{L}(\mathcal{H})} D_T,$$

endowed with the product topology. By Tychonoff's Theorem,  $X$  is compact. Since  $(\langle T\xi_i, \xi_i \rangle)_{T \in \mathcal{L}(\mathcal{H})}$  is an element of  $X$  for all  $i \in I$ , there exists a subnet  $(\xi_j)_{j \in J}$  such that, for all  $T \in \mathcal{L}(\mathcal{H})$ , the limit

$$\varphi(T) = \lim_j \langle T\xi_j, \xi_j \rangle$$

exists. It is clear that  $T \mapsto \varphi(T)$  is a positive linear functional on  $\mathcal{L}(\mathcal{H})$  with  $\varphi(I) = 1$ . Moreover, for every  $x \in G$  and  $T \in \mathcal{L}(\mathcal{H})$ , we have

$$|\langle T\pi(x)\xi_j, \xi_j \rangle - \langle T\xi_j, \xi_j \rangle| = |\langle T(\pi(x)\xi_j - \xi_j), \xi_j \rangle| \leq \|T\| \|\pi(x)\xi_j - \xi_j\|$$

and

$$|\langle \pi(x)T\xi_j, \xi_j \rangle - \langle T\xi_j, \xi_j \rangle| = |\langle T\xi_j, \pi(x^{-1})\xi_j - \xi_j \rangle| \leq \|T\| \|\pi(x)\xi_j - \xi_j\|.$$

Hence, using  $(*)$ , we obtain

$$\varphi(T\pi(x)) = \lim_j \langle T\pi(x)\xi_j, \xi_j \rangle = \lim_j \langle T\xi_j, \xi_j \rangle = \varphi(T)$$

and, similarly,  $\varphi(\pi(x)T) = \varphi(T)$ . ■

**Remark 1.4.2** The topology of  $G$  plays no role in the above proof, so that the result is true under the assumption that  $1_G$  is weakly contained in  $\pi$  when both representations are viewed as representations of  $G$  endowed with the discrete topology.

It is useful to introduce Property (T) for pairs.

**Definition 1.4.3** Let  $G$  be a topological group, and let  $H$  be a closed subgroup. The pair  $(G, H)$  has *Property (T)* if, whenever a unitary representation  $(\pi, \mathcal{H})$  of  $G$  almost has invariant vectors, it actually has a non-zero  $\pi(H)$ -invariant vector.

Examples of pairs with Property (T) appear in Corollaries 1.4.13 and 1.5.2, and in Theorem 4.2.2.

**Remark 1.4.4** (i) Let  $(G, H)$  be a pair with Property (T). A straightforward modification of the proof of Proposition 1.2.1 shows that there exist a compact subset  $Q$  of  $G$  and a real number  $\varepsilon > 0$  with the following property: whenever  $(\pi, \mathcal{H})$  is a unitary representation of  $G$  which has a  $(Q, \varepsilon)$ -invariant vector, then  $(\pi, \mathcal{H})$  has a non-zero  $\pi(H)$ -invariant vector. Such a pair  $(Q, \varepsilon)$  is called a *Kazhdan pair* for  $(G, H)$ .

(ii) A topological group  $G$  has Property (T) if and only if the pair  $(G, G)$  has Property (T).

(iii) Let  $H$  be a closed subgroup of a topological group  $G$ . If  $H$  has Property (T), then the pair  $(G, H)$  has Property (T).

(iv) Let  $G_1 \subset G_2 \subset G_3 \subset G_4$  be a nested sequence of topological groups, each one being closed in the next one. If  $(G_3, G_2)$  has Property (T), then  $(G_4, G_1)$  has Property (T).

(v) If a pair  $(G, H)$  has Property (T), neither  $G$  nor  $H$  need be compactly generated. Indeed, let  $\Gamma$  be a subgroup of  $SL_3(\mathbf{Z})$  which is not finitely generated. As will be seen in Example 1.7.4,  $SL_3(\mathbf{Z})$  has Property (T). It is a particular case of the previous remark (with  $G_2 = G_3 = SL_3(\mathbf{Z})$ ) that the pair  $(SL_3(\mathbf{Z}) \times \mathbf{Q}, \Gamma \times \{0\})$  has Property (T). See however Exercise 1.8.16.

(vi) Let  $G$  be a locally compact group and  $H$  a closed subgroup of  $G$ . Property (T) for the pair  $(G, H)$  can be characterised in terms of *irreducible* unitary representations of  $G$  in the following way:  $(G, H)$  has Property (T) if and only if there exists a neighbourhood  $V$  of  $1_G$  in  $\widehat{G}$  such that  $\pi|_H$  contains  $1_H$  for every  $\pi \in V$  (see Exercise 1.8.15).

Observe that, if a topological group  $G$  acts continuously by automorphisms on another topological group  $X$ , then it acts continuously on the unitary dual  $\widehat{X}$  by

$$(g\pi)(x) = \pi(g^{-1}x), \quad g \in G, \pi \in \widehat{X}, x \in X.$$

The following theorem is a crucial step in our proof of Property (T). It was established in [Shal–99b, Theorem 5.5], with a different proof. For the notion of an *invariant mean*, we refer to Section G.1.

**Theorem 1.4.5** *Let  $G$  be a locally compact group and  $N$  an abelian closed normal subgroup. Assume that the Dirac measure at the unit character  $1_N$  of  $N$  is the unique mean on the Borel subsets of  $\widehat{N}$  which is invariant under the action of  $G$  on  $\widehat{N}$  dual to the conjugation action. Then the pair  $(G, N)$  has Property (T).*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  almost having invariant vectors. We have to prove that  $\mathcal{H}$  contains a non-zero  $N$ -invariant vector. Let



$\mathcal{B}(\widehat{N})$  be the  $\sigma$ -algebra of the Borel subsets of  $\widehat{N}$ , and let

$$E : \mathcal{B}(\widehat{N}) \rightarrow \mathcal{L}(\mathcal{H}), \quad B \mapsto E(B)$$

be the projection valued measure on  $\widehat{N}$  associated to the unitary representation  $\pi|_N$  of the abelian group  $N$  (see Theorem D.3.1). Thus,

$$\pi(x) = \int_{\widehat{N}} \chi(x) dE(\chi), \quad x \in N.$$

For every  $g \in G$ , we have

$$\pi(g)\pi(x)\pi(g^{-1}) = \pi(gxg^{-1}) = \int_{\widehat{N}} (g^{-1}\chi)(x) dE(\chi), \quad x \in N.$$

Hence,

$$B \mapsto E(gB) \quad \text{and} \quad B \mapsto \pi(g)E(B)\pi(g^{-1})$$

are projection valued measures associated to the unitary representation

$$x \mapsto \pi(gxg^{-1})$$

of  $N$ . By uniqueness, it follows that

$$(*) \quad \pi(g)E(B)\pi(g)^{-1} = E(gB), \quad \text{for all } B \in \mathcal{B}(\widehat{N}), g \in G.$$

By Lemma 1.4.1, there exists a positive linear functional  $\varphi$  on  $\mathcal{L}(\mathcal{H})$  such that  $\varphi(I) = 1$ , and such that  $\varphi(\pi(g)T) = \varphi(T\pi(g)) = \varphi(T)$  for all  $g \in G$  and  $T \in \mathcal{L}(\mathcal{H})$ . In particular,

$$\varphi(T) = \varphi(\pi(g)T\pi(g^{-1})), \quad \text{for all } g \in G, T \in \mathcal{L}(\mathcal{H}).$$

Define  $m : \mathcal{B}(\widehat{N}) \mapsto \mathbf{R}$  by

$$m(B) = \varphi(E(B)), \quad B \in \mathcal{B}(\widehat{N}).$$

Then  $m$  is a mean on  $\mathcal{B}(\widehat{N})$ . Moreover,  $m$  is  $G$ -invariant, by (\*). Hence,  $m$  is the Dirac measure at the unit character  $1_N$  of  $N$ . In particular,  $E(\{1_N\}) \neq 0$ . Let  $\xi$  be a non-zero vector in the range of the projection  $E(\{1_N\})$ . Then  $\pi(x)\xi = \xi$  for all  $x \in N$ , that is,  $\xi$  is  $N$ -invariant. ■

### Some facts about unitary representations of $SL_2(\mathbf{K})$

Let  $\mathbf{K}$  be a field. We consider the following subgroups of  $SL_2(\mathbf{K})$ :

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{K} \right\}$$

$$N^- = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} : x \in \mathbf{K} \right\}$$

and

$$A = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbf{K}^* \right\}.$$

The group  $SL_2(\mathbf{K})$  is generated by  $N \cup N^-$ . More generally, the following lemma is true. For  $n \in \mathbf{N}$  and  $1 \leq i, j \leq n$ ,  $i \neq j$ , denote by  $\Delta_{ij}$  the  $(n \times n)$ -matrix with 1 as  $(i, j)$ -entry and 0 elsewhere. For  $x \in \mathbf{K}$ , the *elementary matrix*  $E_{ij}(x) \in SL_n(\mathbf{K})$  is defined by

$$E_{ij}(x) = I_n + x\Delta_{ij}.$$

**Lemma 1.4.6** *For any field  $\mathbf{K}$ , the group  $SL_n(\mathbf{K})$  is generated by the set of all elementary matrices  $E_{ij}(x)$  for  $x \in \mathbf{K}$  and  $1 \leq i, j \leq n$ ,  $i \neq j$ .*

**Proof** See Exercise 1.8.2 or [Jacob–85, (6.5), Lemma 1]. ■

The next lemmas will be constantly used in the sequel. They show that vectors which are invariant under some subgroups are necessarily invariant under appropriate larger subgroups. The first lemma is a simple but crucial computation.

**Lemma 1.4.7** *Let  $\mathbf{K}$  be a non-discrete topological field. Let  $(\lambda_i)_i$  be a net in  $\mathbf{K}$  with  $\lambda_i \neq 0$  and  $\lim_i \lambda_i = 0$ . Then, for  $a_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}$ ,  $h \in N$  and  $h^- \in N^-$ , we have*

$$\lim_i a_i h a_i^{-1} = \lim_i a_i^{-1} h^- a_i = I.$$

**Proof** This follows from the formulae

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 x \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda^2 x & 1 \end{pmatrix}. \quad \blacksquare$$

Our next lemma goes back to F. Mautner [Mautn–57, Lemma 7].

**Lemma 1.4.8 (Mautner’s Lemma)** *Let  $G$  be a topological group, and let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Let  $x \in G$  and assume that there exists a net  $(y_i)_i$  in  $G$  such that  $\lim_i y_i x y_i^{-1} = e$ . If  $\xi$  is a vector in  $\mathcal{H}$  which is fixed by  $y_i$  for all  $i$ , then  $\xi$  is fixed by  $x$ .*

**Proof** Since  $\xi$  is fixed by the unitary operators  $\pi(y_i)$ , we have

$$\begin{aligned} \|\pi(x)\xi - \xi\| &= \|\pi(x)\pi(y_i^{-1})\xi - \pi(y_i^{-1})\xi\| \\ &= \|\pi(y_i)\pi(x)\pi(y_i^{-1})\xi - \xi\|. \end{aligned}$$

As  $\lim_i y_i x y_i^{-1} = e$  and as  $\pi$  is strongly continuous,  $\pi(x)\xi = \xi$ .  $\blacksquare$

The next lemma will be used to establish invariance of vectors under copies of  $SL_2(\mathbf{K})$ .

**Lemma 1.4.9** *Let  $\mathbf{K}$  be a non-discrete topological field, and let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_2(\mathbf{K})$ . Let  $\xi$  be a vector in  $\mathcal{H}$  which is invariant under the subgroup  $N$ . Then  $\xi$  is invariant under  $SL_2(\mathbf{K})$ .*

**Proof** It suffices to show that  $\xi$  is invariant under  $A$ . Indeed, it will then follow from Lemma 1.4.7 and from Mautner’s Lemma that  $\xi$  is invariant under  $N^-$  and, hence, under  $SL_2(\mathbf{K})$  since  $N \cup N^-$  generates  $SL_2(\mathbf{K})$ .

To show the  $A$ -invariance of  $\xi$ , consider the function of positive type  $\varphi$  on  $SL_2(\mathbf{K})$  defined by

$$\varphi(x) = \langle \pi(x)\xi, \xi \rangle, \quad x \in SL_2(\mathbf{K}).$$

Since  $\xi$  is  $N$ -invariant,  $\varphi$  is  $N$ -bi-invariant, that is,  $\varphi$  is constant on every double coset  $NgN$  for  $g \in SL_2(\mathbf{K})$ . Since  $\mathbf{K}$  is non-discrete, there exists a net  $(\lambda_i)_i$  in  $\mathbf{K}$

with  $\lambda_i \neq 0$  and such that  $\lim_i \lambda_i = 0$ . Set

$$g_i = \begin{pmatrix} 0 & -\lambda_i^{-1} \\ \lambda_i & 0 \end{pmatrix} \in SL_2(\mathbf{K}).$$

Then, for every  $a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in A$ , we have

$$\begin{pmatrix} 1 & \lambda\lambda_i^{-1} \\ 0 & 1 \end{pmatrix} g_i \begin{pmatrix} 1 & \lambda^{-1}\lambda_i^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ \lambda_i & \lambda^{-1} \end{pmatrix}.$$

Hence, since  $\varphi$  is continuous and  $N$ -bi-invariant,

$$\varphi(a) = \lim_i \varphi \begin{pmatrix} \lambda & 0 \\ \lambda_i & \lambda^{-1} \end{pmatrix} = \lim_i \varphi(g_i)$$

for all  $a \in A$ , and the latter is independent of  $a$ . Therefore, for all  $a \in A$ ,

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle = \|\xi\|^2.$$

By the equality case of the Cauchy–Schwarz inequality,

$$\pi(a)\xi = \xi, \quad \text{for all } a \in A$$

(Exercise 1.8.3). ■

**Remark 1.4.10** (i) The previous lemma is not true for non-unitary representations of  $SL_2(\mathbf{K})$ : consider, for instance, the standard representation of  $SL_2(\mathbf{K})$  on  $\mathbf{K}^2$ . The vector  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is  $N$ -invariant and is not  $SL_2(\mathbf{K})$ -invariant.

(ii) The lemma always fails if  $\mathbf{K}$  is discrete: for the quasi-regular representation  $\lambda_{SL_2(\mathbf{K})/N}$  of  $SL_2(\mathbf{K})$  on  $\ell^2(SL_2(\mathbf{K})/N)$ , the Dirac function at  $N$  is  $N$ -invariant and is not  $SL_2(\mathbf{K})$ -invariant.

Let  $n \geq 2$  and  $m < n$ . Choosing  $m$  distinct vectors  $e_{i_1}, \dots, e_{i_m}$  from the standard basis  $\{e_1, \dots, e_n\}$  of  $\mathbf{K}^n$ , we have an embedding of  $SL_{n-m}(\mathbf{K})$  in  $SL_n(\mathbf{K})$  as the subgroup of matrices in  $SL_n(\mathbf{K})$  which fix  $e_{i_1}, \dots, e_{i_m}$  and leave invariant the linear span of the remaining  $n - m$  vectors  $e_i$ . We refer to these embeddings as to the *standard embeddings* of  $SL_{n-m}(\mathbf{K})$ .

**Proposition 1.4.11** *Let  $\mathbf{K}$  be a topological non-discrete field, and let  $n \geq 2$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_n(\mathbf{K})$ . If  $\xi \in \mathcal{H}$  is invariant*

under  $SL_2(\mathbf{K})$ , for any one of the standard embeddings of  $SL_2(\mathbf{K})$ , then  $\xi$  is invariant under  $SL_n(\mathbf{K})$ .

**Proof** We can of course assume that  $n \geq 3$ . Using induction on  $n$ , it suffices to show that, if  $\xi \in \mathcal{H}$  is invariant under  $SL_{n-1}(\mathbf{K})$ , for any one of the standard embeddings of  $SL_{n-1}(\mathbf{K})$  in  $SL_n(\mathbf{K})$ , then  $\xi$  is invariant under  $SL_n(\mathbf{K})$ .

Without loss of generality, we can also assume that  $SL_{n-1}(\mathbf{K})$  is embedded as the subgroup of all matrices in  $SL_n(\mathbf{K})$  which fix  $e_n$  and leave invariant the linear span of  $\{e_1, \dots, e_{n-1}\}$ . Using Lemma 1.4.6, it suffices to show that  $\xi$  is fixed by every elementary matrix  $E_{in}(x)$  and  $E_{ni}(x)$  for  $1 \leq i \leq n-1$  and  $x \in \mathbf{K}$ .

Fix  $i \in \{1, \dots, n-1\}$  and  $x \in \mathbf{K}$ . Choose a net  $(\lambda_\alpha)_\alpha$  in  $\mathbf{K}$  with  $\lambda_\alpha \neq 0$  and  $\lim_\alpha \lambda_\alpha = 0$ . Let  $g_\alpha$  be a diagonal matrix in  $SL_{n-1}(\mathbf{K})$  with  $g_\alpha(e_i)$ . Then

$$g_\alpha E_{in}(x) g_\alpha^{-1} = E_{in}(\lambda_\alpha x) \quad \text{and} \quad g_\alpha^{-1} E_{ni}(x) g_\alpha = E_{ni}(\lambda_\alpha x).$$

As  $\lim_\alpha E_{in}(\lambda_\alpha x) = \lim_\alpha E_{ni}(\lambda_\alpha x) = I_n$ , the claim follows from Mautner's Lemma 1.4.8. ■

### Proof of Property (T) for $SL_n(\mathbf{K})$ , $n \geq 3$

For  $n \geq 3$ , we consider the subgroups

$$G = \left\{ \begin{pmatrix} A & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} : A \in SL_2(\mathbf{K}), x \in \mathbf{K}^2 \right\} \cong SL_2(\mathbf{K}) \ltimes \mathbf{K}^2$$

$$N = \left\{ \begin{pmatrix} I_2 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} : x \in \mathbf{K}^2 \right\} \cong \mathbf{K}^2$$

of  $SL_n(\mathbf{K})$ , where  $SL_2(\mathbf{K}) \ltimes \mathbf{K}^2$  denotes the semidirect product for the natural action of  $SL_2(\mathbf{K})$  on  $\mathbf{K}^2$ . The study of the pair  $(SL_2(\mathbf{K}) \ltimes \mathbf{K}^2, \mathbf{K}^2)$  is a cornerstone of the subject of Property (T).

Let  $\mathbf{K}$  be a local field. The dual group  $\widehat{\mathbf{K}^2}$  of  $\mathbf{K}^2$  can be identified with  $\mathbf{K}^2$  as follows (see Corollary D.4.6). Fix a unitary character  $\chi$  of the additive group of  $\mathbf{K}$  distinct from the unit character. The mapping

$$\mathbf{K}^2 \rightarrow \widehat{\mathbf{K}^2}, \quad x \mapsto \chi_x$$

is a topological group isomorphism, where  $\chi_x$  is defined by  $\chi_x(y) = \chi(\langle x, y \rangle)$  and  $\langle x, y \rangle = x_1 y_1 + x_2 y_2$  for  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  in  $\mathbf{K}^2$ . Under this identification, the dual action of a matrix  $g \in SL_2(\mathbf{K})$  on  $\widehat{\mathbf{K}^2}$  corresponds to the

inverse transpose of the standard action on  $\mathbf{K}^2$ , that is, to the action defined by  $x \mapsto {}^t g^{-1}x$  for  $x \in \mathbf{K}^2$ .

We first prove that the pair  $(G, N)$  has Property (T). In view of Theorem 1.4.5, this will follow from the following proposition.

**Proposition 1.4.12** *Let  $\mathbf{K}$  be a local field. The Dirac measure at 0 is the unique  $SL_2(\mathbf{K})$ -invariant mean on the Borel subsets of  $\mathbf{K}^2$ .*

**Proof** Let  $m$  be an  $SL_2(\mathbf{K})$ -invariant mean on the  $\sigma$ -algebra  $\mathcal{B}(\mathbf{K}^2)$  of the Borel sets of  $\mathbf{K}^2$ . Let  $|\cdot|$  be an absolute value defining the topology on  $\mathbf{K}$ . Consider the following subset of  $\mathbf{K}^2 \setminus \{0\}$  (see Figure 1.1):

$$\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{K}^2 \setminus \{0\} : |y| \geq |x| \right\}.$$

Choose a sequence  $(\lambda)_n$  in  $\mathbf{K}$  with  $|\lambda_{n+1}| > |\lambda_n| + 2$  for all  $n \in \mathbf{N}$ , and set

$$g_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbf{K}).$$

If  $\Omega_n = g_n \Omega$ , then

$$\Omega_n \subset \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{K}^2 \setminus \{0\} : \frac{|x|}{|\lambda_n| + 1} \leq |y| \leq \frac{|x|}{|\lambda_n| - 1} \right\}.$$

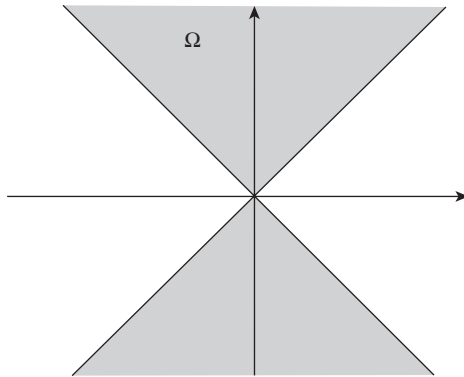


Figure 1.1 The subset  $\Omega$  of  $\mathbf{K}^2$

Indeed, for  $\begin{pmatrix} x \\ y \end{pmatrix} \in \Omega$ ,

$$g_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \lambda_n y \\ y \end{pmatrix}$$

and

$$|x + \lambda_n y| \geq |\lambda_n y| - |x| \geq (|\lambda_n| - 1)|y|,$$

$$|x + \lambda_n y| \leq |x| + |\lambda_n y| \leq (|\lambda_n| + 1)|y|.$$

Since

$$\frac{1}{|\lambda_n| - 1} < \frac{1}{|\lambda_m| + 1} \quad \text{for } n > m,$$

the sets  $\Omega_n$  are pairwise disjoint. Hence,

$$\sum_{i=1}^n m(\Omega_i) \leq m(\mathbf{K}^2 \setminus \{0\}) \leq 1$$

for all  $n \in \mathbf{N}$ . As  $m(\Omega_i) = m(g_i \Omega) = m(\Omega)$ , it follows that  $m(\Omega) = 0$ . If

$$\Omega' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{K}^2 \setminus \{0\} : |x| \geq |y| \right\},$$

then  $m(\Omega') = m(\Omega) = 0$ . Since  $\Omega \cup \Omega' = \mathbf{K}^2 \setminus \{0\}$ , we have  $m(\mathbf{K}^2 \setminus \{0\}) = 0$ . Therefore,  $m$  is the Dirac measure at 0. ■

**Corollary 1.4.13** *The pair  $(SL_2(\mathbf{K}) \ltimes \mathbf{K}^2, \mathbf{K}^2)$  has Property (T), for every local field  $\mathbf{K}$ .*

**Remark 1.4.14** The semidirect product  $SL_2(\mathbf{K}) \ltimes \mathbf{K}^2$  does not have Property (T). Indeed,  $SL_2(\mathbf{K})$  is a quotient of  $SL_2(\mathbf{K}) \ltimes \mathbf{K}^2$  and does not have Property (T); see Example 1.3.7 for the case  $\mathbf{K} = \mathbf{R}$  and Example 1.7.4 for the other cases.

We are now ready to show that  $SL_n(\mathbf{K})$  has Property (T) for  $n \geq 3$ .

**Theorem 1.4.15** *Let  $\mathbf{K}$  be a local field. The group  $SL_n(\mathbf{K})$  has Property (T) for any integer  $n \geq 3$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_n(\mathbf{K})$  almost having invariant vectors. Let  $G \cong SL_2(\mathbf{K}) \ltimes \mathbf{K}^2$  and  $N \cong \mathbf{K}^2$  be the subgroups of  $SL_n(\mathbf{K})$

introduced above. By the previous corollary, the pair  $(G, N)$  has Property (T). Hence, there exists a non-zero  $N$ -invariant vector  $\xi \in \mathcal{H}$ . By Lemma 1.4.9,  $\xi$  is invariant under the following copy of  $SL_2(\mathbf{K})$

$$\begin{pmatrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}$$

inside  $SL_n(\mathbf{K})$ . It follows from Proposition 1.4.11 that  $\xi$  is invariant under the whole group  $SL_n(\mathbf{K})$ . ■

Other examples of groups with Property (T) are provided by the following corollary and by Exercises 1.8.6–1.8.10.

**Corollary 1.4.16** *Let  $\mathbf{K}$  be a local field. The semidirect product  $G = SL_n(\mathbf{K}) \ltimes \mathbf{K}^n$  has Property (T) for  $n \geq 3$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  almost having invariant vectors. Since  $SL_n(\mathbf{K})$  has Property (T), there exists a non-zero vector  $\xi \in \mathcal{H}$  which is  $SL_n(\mathbf{K})$ -invariant. For every  $x \in \mathbf{K}^n$ , there exists a sequence  $A_i \in SL_n(\mathbf{K})$  with  $\lim_i A_i x = 0$ . It follows from Mautner's Lemma 1.4.8 that  $\xi$  is invariant under  $\mathbf{K}^n$ . Hence,  $\xi$  is invariant under  $SL_n(\mathbf{K}) \ltimes \mathbf{K}^n$ . ■

## 1.5 Property (T) for $Sp_{2n}(\mathbf{K})$ , $n \geq 2$

In this section, we prove that the symplectic group  $Sp_{2n}(\mathbf{K})$  has Property (T) for  $n \geq 2$ . The strategy of the proof is similar to that for  $SL_n(\mathbf{K})$ : we show that an appropriate subgroup of  $Sp_{2n}(\mathbf{K})$  gives rise to a pair which has Property (T).

Recall that  $Sp_{2n}(\mathbf{K})$  is the closed subgroup of  $GL_{2n}(\mathbf{K})$  consisting of all matrices  $g$  with  ${}^t g J g = J$ , where  ${}^t g$  is the transpose of  $g$ ,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and  $I_n$  is the  $n \times n$  identity matrix. Observe that  $Sp_2(\mathbf{K}) = SL_2(\mathbf{K})$ ; see Exercise 1.8.1. Writing matrices in  $GL_{2n}(\mathbf{K})$  as blocks

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$



of  $(n \times n)$  matrices, we have:  $g \in Sp_{2n}(\mathbf{K})$  if and only if

$${}^tAC - {}^tCA = {}^tBD - {}^tDB = 0$$

$${}^tAD - {}^tCB = I.$$

Let  $S^{2*}(\mathbf{K}^2)$  be the vector space of all symmetric bilinear forms on  $\mathbf{K}^2$ . The group  $GL_2(\mathbf{K})$  acts on  $S^{2*}(\mathbf{K}^2)$  by  $\beta \mapsto {}^g\beta$ , where

$${}^g\beta(x, y) = \beta({}^t gx, {}^t gy)$$

for  $g \in GL_2(\mathbf{K})$  and  $x, y \in \mathbf{K}^2$ . Each  $\beta \in S^{2*}(\mathbf{K}^2)$  is of the form  $\beta(x, y) = \langle X_\beta x, y \rangle$  for a unique symmetric  $(2 \times 2)$  matrix  $X_\beta$  with coefficients in  $\mathbf{K}$ , where  $\langle x, y \rangle = x_1 y_1 + x_2 y_2$  is the standard symmetric bilinear form on  $\mathbf{K}^2$ . The matrix corresponding to  ${}^g\beta$  is then  $gX_\beta {}^t g$ .

Consider the subgroups

$$G_2 = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I_{n-2} & 0 & 0 \\ 0 & 0 & {}^t A^{-1} & 0 \\ 0 & 0 & 0 & I_{n-2} \end{pmatrix} : A \in SL_2(\mathbf{K}) \right\} \cong SL_2(\mathbf{K})$$

$$N_2 = \left\{ \begin{pmatrix} I_2 & 0 & B & 0 \\ 0 & I_{n-2} & 0 & 0 \\ 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & I_{n-2} \end{pmatrix} : B \in M_2(\mathbf{K}), {}^t B = B \right\} \cong S^{2*}(\mathbf{K}^2)$$

of  $Sp_{2n}(\mathbf{K})$ . Since

$$\begin{pmatrix} A & 0 \\ 0 & {}^t A \end{pmatrix} \begin{pmatrix} I_2 & B \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} I_2 & AB {}^t A \\ 0 & I_2 \end{pmatrix},$$

the subgroup  $G = G_2 N_2$  is isomorphic to  $SL_2(\mathbf{K}) \ltimes S^{2*}(\mathbf{K}^2)$  for the action described above. Our next goal (Corollary 1.5.2) is to show that the pair  $(G, N_2)$  has Property (T).

Let  $S^2(\mathbf{K}^2)$  be the second symmetric tensor power of  $\mathbf{K}^2$ . Denote by  $\rho$  the natural action of  $SL_2(\mathbf{K})$  on  $S^2(\mathbf{K}^2)$ :

$$\rho(g) \sum_i x_i \otimes y_i = g x_i \otimes g y_i, \quad g \in SL_2(\mathbf{K}), \quad \sum_i x_i \otimes y_i \in S^2(\mathbf{K}^2).$$

We identify the dual vector space of  $S^{2*}(\mathbf{K}^2)$  with  $S^2(\mathbf{K}^2)$  by means of the duality formula

$$\left( \beta, \sum_i x_i \otimes y_i \right) \mapsto \sum_i \beta(x_i, y_i), \quad \beta \in S^{2*}(\mathbf{K}^2), \quad \sum_i x_i \otimes y_i \in S^2(\mathbf{K}^2).$$

Under this identification, the action of  $SL_2(\mathbf{K})$  on the dual of  $S^{2*}(\mathbf{K}^2)$  corresponds to the inverse transpose action of the natural action  $\rho$  on  $S^2(\mathbf{K}^2)$ .

The dual group of  $S^{2*}(\mathbf{K}^2)$  will be identified with  $S^2(\mathbf{K}^2)$  as follows (see Corollary D.4.6). Fix a non-trivial unitary character  $\chi$  of the additive group of  $\mathbf{K}$ . Then, for any  $X \in S^2(\mathbf{K}^2)$ , the formula

$$\chi_X(Y) = \chi(X(Y)), \quad \text{for all } Y \in S^{2*}(\mathbf{K}^2)$$

defines a character on  $S^2(\mathbf{K}^{2*})$  and the mapping  $X \rightarrow \chi_X$  is an isomorphism between  $S^2(\mathbf{K}^2)$  and the dual group of  $S^{2*}(\mathbf{K}^2)$ .

To show that the pair  $(G, N_2)$  has Property (T), it suffices, by Theorem 1.4.5, to prove the following proposition.

**Proposition 1.5.1** *Let  $\mathbf{K}$  be a local field. The Dirac measure  $\delta_0$  at 0 is the unique mean on the Borel subsets of  $S^2(\mathbf{K}^2)$  which is invariant under the natural action of  $SL_2(\mathbf{K})$  on  $S^2(\mathbf{K}^2)$ .*

**Proof** Let  $\{e_1, e_2\}$  be the standard basis of  $\mathbf{K}^2$ . We identify  $S^2(\mathbf{K}^2)$  with  $\mathbf{K}^3$  by means of the basis  $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2\}$ . The matrix of  $\rho(g)$  for  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $SL_2(\mathbf{K})$  is

$$\rho(g) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

In particular, for  $u_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ , we have

$$\rho(u_b) = \begin{pmatrix} 1 & b & b^2 \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{pmatrix},$$

for all  $b \in \mathbf{K}$ . Hence,

$$\rho(u_b) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + by + b^2 z \\ y + 2bz \\ z \end{pmatrix}, \quad \text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3.$$

We have to consider two cases, depending on whether the characteristic  $\text{char}(\mathbf{K})$  of the field  $\mathbf{K}$  is 2 or not.

• Assume first that  $\text{char}(\mathbf{K}) \neq 2$ . For every  $t \in \mathbf{R}$  with  $t > 0$ , consider the Borel subset

$$\Omega_t = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3 : |z| > t |y| \right\}$$

of  $\mathbf{K}^3$ , where  $|\cdot|$  denotes the absolute value of  $\mathbf{K}$ . For every  $b \in \mathbf{K}$  and every  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Omega_t$ , we have

$$(*) \quad \left( |2b| - \frac{1}{t} \right) |z| < |y + 2bz| < \left( |2b| + \frac{1}{t} \right) |z|.$$

Let  $m$  be an  $SL_2(\mathbf{K})$ -invariant mean on  $\mathcal{B}(\mathbf{K}^3)$ . Choose a sequence  $(b_j)_j$  in  $\mathbf{K}$  such that

$$|b_{j+1}| \geq |b_j| + \frac{2}{|2|t}, \quad \text{for all } j \in \mathbf{N}.$$

Then, by  $(*)$ , we have

$$\rho(u_{b_j})\Omega_t \cap \rho(u_{b_k})\Omega_t = \emptyset \quad \text{for all } j \neq k.$$

Since  $m(\mathbf{K}^3) < \infty$ , this implies that

$$m(\Omega_t) = 0, \quad \text{for all } t > 0.$$

We claim now that  $\mathbf{K}^3 \setminus \{0\}$  is contained in four sets of the form  $\rho(g)\Omega_t$ . The claim implies that  $m(\mathbf{K}^3 \setminus \{0\}) = 0$ , as needed to prove the proposition in case  $\text{char}(\mathbf{K}) \neq 2$ . For this, consider the matrices  $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  and

$r = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -1 & 1 \end{pmatrix}$  in  $SL_2(\mathbf{K})$ . We have

$$\rho(\omega) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix} \quad \text{and} \quad \rho(r) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{4}(x+y+z) \\ -x+z \\ x-y+z \end{pmatrix},$$

for all  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3$ . Let  $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3 \setminus \{0\}$ . We split the discussion in several cases.

If  $|z| > |y|$ , then  $v \in \Omega_1$ . If  $|x| > |y|$ , then  $\rho(\omega)v \in \Omega_1$ . If  $x = z = 0$  and  $y \neq 0$ , then  $\rho(r)v \in \Omega_1$ .

In the other cases, we have  $|y| \geq |z|$ ,  $|y| \geq |x|$ , and  $(x, z) \neq (0, 0)$ , so that

$$\begin{aligned} |x - y + z| + \left| \frac{1}{4}(x + y + z) \right| &\geq \frac{1}{|4|} (|x + y + z| + |y - x - z|) \\ &\geq \frac{1}{|2|} |y| \geq \frac{1}{2|2|} (|x| + |z|) \\ &> 2t_0 |-x + z|, \end{aligned}$$

for  $0 < t_0 < 1/(4|2|)$ . It follows that either  $|x - y + z| > t_0 |-x + z|$  and then  $\rho(r)v \in \Omega_{t_0}$ , or  $\left| \frac{1}{4}(x + y + z) \right| > t_0 |-x + z|$  and then  $\rho(\omega^{-1}r)v \in \Omega_{t_0}$ .

Thus

$$\mathbf{K}^3 \setminus \{0\} = \Omega_1 \cup \rho(\omega)\Omega_1 \cup \rho(r^{-1})\Omega_{t_0} \cup \rho(r^{-1}\omega)\Omega_{t_0},$$

as claimed.

• Assume now that  $\text{char}(\mathbf{K}) = 2$ ; in particular, the absolute value on  $\mathbf{K}$  is non-archimedean. With the notation above, we have

$$\rho(u_b) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + by + b^2z \\ y \\ z \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3.$$

Consider the Borel subset

$$\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3 \setminus \{0\} : |z| \geq \max\{|x|, |y|\} \right\}$$

of  $\mathbf{K}^3 \setminus \{0\}$ . For every  $b \in \mathbf{K}$  with  $|b| > 1$  and every  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Omega$ , we have  $|b^2 z| > \max\{|x|, |by|\}$ ; therefore we have also  $|b^2 z| > |x + by|$  and

$$(**) \quad \left| x + by + b^2 z \right| = \left| b^2 z \right|.$$

Let  $m$  be an  $SL_2(\mathbf{K})$ -invariant mean on  $\mathcal{B}(\mathbf{K}^3)$ . Choose a sequence  $(b_j)_j$  in  $\mathbf{K}$  such that

$$|b_{j+1}| > |b_j| > 1, \quad \text{for all } j \in \mathbf{N}.$$

Then, by (\*\*),

$$\rho(u_{b_j})\Omega \cap \rho(u_{b_k})\Omega = \emptyset \quad \text{for all } j \neq k.$$

This implies

$$m(\Omega) = 0,$$

and also  $m(\rho(\omega)\Omega) = 0$ .

Any non-zero vector in  $\mathbf{K}^3$  is either in  $\Omega$ , or in  $\rho(\omega)\Omega$ , or in the Borel subset

$$\Omega' = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbf{K}^3 : |y| > \max\{|x|, |z|\} \right\}$$

of  $\mathbf{K}^3 \setminus \{0\}$ . Since it is clear that  $\rho(u_1)(\Omega')$  is contained in  $\rho(\omega)\Omega$ , we have  $m(\Omega') = 0$ . Therefore

$$m(\mathbf{K}^3 \setminus \{0\}) \leq m(\Omega) + m(\rho(\omega)\Omega) + m(\Omega') = 0,$$

and this ends the proof. ■

**Corollary 1.5.2** *The pair  $(SL_2(\mathbf{K}) \ltimes S^{2*}(\mathbf{K}^2), S^{2*}(\mathbf{K}^2))$  has Property (T), for any local field  $\mathbf{K}$ .*

We are now ready to prove Property (T) for  $Sp_{2n}(\mathbf{K})$ .

**Theorem 1.5.3** *Let  $\mathbf{K}$  be a local field. The group  $Sp_{2n}(\mathbf{K})$  has Property (T), for any integer  $n \geq 2$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $Sp_{2n}(\mathbf{K})$  almost having invariant vectors. Let  $G = G_2 \ltimes N_2 \cong SL_2(\mathbf{K}) \ltimes S^{2*}(\mathbf{K}^2)$  and  $N_2 \cong S^{2*}(\mathbf{K}^2)$  be the subgroups of  $Sp_{2n}(\mathbf{K})$  introduced above. By the previous corollary, there exists a non-zero  $N_2$ -invariant vector  $\xi \in \mathcal{H}$ . Consider the following copy of  $SL_2(\mathbf{K})$  inside  $Sp_{2n}(\mathbf{K})$  :

$$H = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & I_{n-1} & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} : a, b, c, d \in \mathbf{K}, ad - bc = 1 \right\} \cong SL_2(\mathbf{K}).$$

Since  $\xi$  is invariant under the subgroup

$$\left\{ \begin{pmatrix} I_n & b & 0 \\ 0 & 0 & 0 \\ 0 & I_n \end{pmatrix} : b \in \mathbf{K} \right\}$$

of  $H$ , Lemma 1.4.9 shows that  $\xi$  is invariant under  $H$ . In particular,  $\xi$  is fixed by the subgroup

$$\Lambda = \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & I_{n-1} & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix} : \lambda \in \mathbf{K}^* \right\} \cong \mathbf{K}^*.$$

Considering a sequence  $(\lambda_k)_k$  in  $\mathbf{K}^*$  with  $\lim_k \lambda_k = 0$  or  $\lim_k \lambda_k^{-1} = 0$  and using Mautner's Lemma 1.4.8, we see that  $\xi$  is invariant under the subgroup

$$G_2 = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I_{n-2} & 0 & 0 \\ 0 & 0 & {}^t A^{-1} & 0 \\ 0 & 0 & 0 & I_{n-2} \end{pmatrix} : A \in SL_2(\mathbf{K}) \right\} \cong SL_2(\mathbf{K})$$

introduced above. Hence, by Proposition 1.4.11,  $\xi$  is invariant under the subgroup

$$G_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in SL_n(\mathbf{K}) \right\} \cong SL_n(\mathbf{K})$$

and, therefore, under the subgroup

$$\Lambda G_n = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in GL_n(\mathbf{K}) \right\}.$$

By Mautner's Lemma again,  $\xi$  is invariant under the subgroups

$$N_n = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} : B \in M_n(\mathbf{K}), {}^t B = B \right\}$$

$$N_n^- = \left\{ \begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix} : B \in M_n(\mathbf{K}), {}^t B = B \right\}$$

(Exercise 1.8.5). Now,  $\Lambda G_n \cup N_n \cup N_n^-$  generates  $Sp_{2n}(\mathbf{K})$ ; see [O'Mea-78, (2.2)]. Hence,  $\xi$  is invariant under  $Sp_{2n}(\mathbf{K})$ . ■

**Remark 1.5.4** The two representations of  $SL_2(\mathbf{K})$  on  $S^2(\mathbf{K}^2)$  and  $S^{2*}(\mathbf{K}^2)$  are easily seen to be equivalent in the case  $\text{char}(\mathbf{K}) \neq 2$  (Exercise 1.8.4). This fails if  $\text{char}(\mathbf{K}) = 2$ . In fact, the following result holds in this case.

**Proposition 1.5.5** *The pair  $(SL_2(\mathbf{K}) \ltimes S^2(\mathbf{K}^2), S^2(\mathbf{K}^2))$  does not have Property (T) if  $\mathbf{K}$  is a local field with  $\text{char}(\mathbf{K}) = 2$ .*

**Proof** Set  $N = S^2(\mathbf{K}^2)$ . Recall from the discussion before Proposition 1.5.1 that we can identify the dual group of  $N$  with  $S^{2*}(\mathbf{K}^2)$  in a  $SL_2(\mathbf{K})$ -equivariant way. We claim that  $SL_2(\mathbf{K})$  has non-zero fixed points in  $S^{2*}(\mathbf{K}^2)$ . Indeed, since  $\text{char}(\mathbf{K}) = 2$ ,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{K}).$$

Hence, for each  $t \in \mathbf{K}$ , the symmetric bilinear form

$$\beta_t : \mathbf{K}^2 \times \mathbf{K}^2 \rightarrow \mathbf{K}, \quad (x, y) \mapsto t(x_1 y_2 + x_2 y_1)$$

defined by the matrix  $\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$  is fixed under  $SL_2(\mathbf{K})$ . The character  $\lambda_t$  of  $N$  corresponding to  $\beta_t$  extends to a character  $\tilde{\lambda}_t$  of  $G = SL_2(\mathbf{K}) \ltimes N$  defined by

$$\tilde{\lambda}_t(A, X) = \lambda_t(X), \quad \text{for all } A \in SL_2(\mathbf{K}), X \in N.$$

Now,  $\lim_{t \rightarrow 0} \tilde{\lambda}_t = 1_G$  uniformly on compact subsets of  $G$  and  $\lambda_t \neq 1_N$  for  $t \neq 0$ . Hence,  $(G, N)$  does not have Property (T). ■

## 1.6 Property (T) for higher rank algebraic groups

Let  $\mathbf{K}$  be a local field. In this section, we indicate how Property (T) carries over from  $SL_3(\mathbf{K})$  and  $Sp_4(\mathbf{K})$  to other groups.

An algebraic group  $\mathbb{G}$  over  $\mathbf{K}$  is said to be *almost  $\mathbf{K}$ -simple* if the only proper algebraic normal subgroups of  $\mathbb{G}$  defined over  $\mathbf{K}$  are finite. The  $\mathbf{K}$ -rank of such a group, denoted  $\text{rank}_{\mathbf{K}} \mathbb{G}$ , is the dimension of a maximal  $\mathbf{K}$ -split torus. For instance, the  $\mathbf{K}$ -rank of  $SL_n$  is  $n - 1$  and the  $\mathbf{K}$ -rank of  $Sp_{2n}$  is  $n$ , independently of  $\mathbf{K}$ . As another example, let  $SO(n, 1)$  be the group of matrices in  $GL_{n+1}$  which preserve the form  $x_1^2 + \cdots + x_n^2 - x_{n+1}^2$ . Then  $\text{rank}_{\mathbb{C}} SO(n, 1) = [(n + 1)/2]$  and  $\text{rank}_{\mathbf{R}} SO(n, 1) = 1$ .

The set  $\mathbb{G}(\mathbf{K})$  of  $\mathbf{K}$ -rational points in  $\mathbb{G}$  is a locally compact group. This is a compact group if and only if  $\text{rank}_{\mathbf{K}}(\mathbb{G}) = 0$ .

Let  $G$  be a simple real Lie group with trivial centre. Let  $\mathbb{G}$  be the connected component of the automorphism group of the complexified Lie algebra of  $G$ . This is an almost  $\mathbf{R}$ -simple (in fact, an  $\mathbf{R}$ -simple) algebraic group  $\mathbb{G}$  over  $\mathbf{R}$ , and  $G$  is isomorphic, as a Lie group, to the connected component of  $\mathbb{G}(\mathbf{R})$  (see [Zimm–84a, Proposition 3.1.6]). The  $\mathbf{R}$ -rank of  $\mathbb{G}$  coincides with the dimension of  $A$  in an *Iwasawa decomposition*  $G = KAN$  of  $G$  (such a decomposition exists for any semisimple Lie group, see [Walla–73, Theorem 7.4.3]). It also coincides with the rank of the associated Riemannian symmetric space  $G/K$ , that is, the maximal dimension of a flat totally geodesic subspace of  $G/K$ .

The following result was proved under the assumption  $\text{rank}_{\mathbf{K}}(\mathbb{G}) \geq 3$  in [Kazhd–67], and under the assumption  $\text{rank}_{\mathbf{K}}(\mathbb{G}) \geq 2$  by Delaroché and Kirillov [DelKi–68], Vaserstein [Vaser–68], and Wang [Wang–69], independently.

**Theorem 1.6.1** *Let  $\mathbf{K}$  be a local field, and let  $\mathbb{G}$  be a connected, almost  $\mathbf{K}$ -simple algebraic group over  $\mathbf{K}$  with  $\text{rank}_{\mathbf{K}}(\mathbb{G}) \geq 2$ . Then  $G = \mathbb{G}(\mathbf{K})$  has Property (T).*

The proof is based on the following reduction to the cases  $SL_3$  and  $Sp_4$ .

**Lemma 1.6.2** *Under the assumption of the previous theorem,  $\mathbb{G}$  contains an almost  $\mathbf{K}$ -simple algebraic group  $\mathbf{H}$  over  $\mathbf{K}$  whose simply connected covering is isomorphic over  $\mathbf{K}$  to either  $SL_3$  or  $Sp_4$ .*

The lemma follows from the fact that, due to the rank assumption, the root system of  $\mathbb{G}$  with respect to a maximal  $\mathbf{K}$ -split torus in  $\mathbb{G}$  contains a subsystem of type  $A_2$  or  $C_2$ . For more details, see [Margu–91, Chapter I, (1.6.2)].

We give the proof of the theorem above in the case  $\mathbf{K} = \mathbf{R}$ , that is, in the case where  $G$  is a simple real Lie group. The general case is similar in spirit, but is technically more involved; see [Margu–91, Chapter III, (5.3)].



**Proof of Theorem 1.6.1 for  $K = \mathbf{R}$**  Let  $\mathbf{H}$  be as in the previous lemma, let  $\tilde{\mathbf{H}}$  be its simply connected covering, and  $\varphi : \tilde{\mathbf{H}} \rightarrow \mathbf{H}$  the canonical homomorphism. Set  $H = \mathbf{H}(\mathbf{R})$ . The group  $\varphi(\tilde{\mathbf{H}}(\mathbf{R}))$  has Property (T), since  $\tilde{\mathbf{H}}(\mathbf{R})$  has Property (T). As  $\varphi(\tilde{\mathbf{H}}(\mathbf{R}))$  is a normal subgroup of finite index in  $H$ , it follows from Proposition 1.7.6 in the next section that  $H$  has Property (T). Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Since  $\text{rank}_{\mathbf{R}}(\mathbf{H}) > 0$ , we can find an element  $a = \exp X \in H$ ,  $a \neq e$ , such that  $\text{ad}X \in \text{End}(\mathfrak{g})$  is diagonalisable over  $\mathbf{R}$ . Let

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbf{R}} \mathfrak{g}^{\lambda}$$

be the eigenspace decomposition of  $\mathfrak{g}$  under  $\text{ad}X$ . Set

$$\mathfrak{g}^{+} = \bigoplus_{\lambda > 0} \mathfrak{g}^{\lambda}, \quad \mathfrak{g}^{-} = \bigoplus_{\lambda < 0} \mathfrak{g}^{\lambda}$$

and let  $\mathfrak{g}^0$  be the kernel of  $\text{ad}X$ .

We claim that  $\mathfrak{g}$  is generated as a Lie algebra by  $\mathfrak{g}^{+} \cup \mathfrak{g}^{-}$ . Indeed, Let  $\mathfrak{n}$  be the subalgebra generated by this set. Since, for all  $\lambda, \mu \in \mathbf{R}$ ,

$$[\mathfrak{g}^{\lambda}, \mathfrak{g}^{\mu}] \subset \mathfrak{g}^{\lambda+\mu},$$

we have  $[\mathfrak{g}^0, \mathfrak{n}] \subset \mathfrak{n}$ . This shows that  $\mathfrak{n}$  is an ideal in  $\mathfrak{g}$ . Hence,  $\mathfrak{n} = \{0\}$  or  $\mathfrak{n} = \mathfrak{g}$ , by simplicity of  $\mathfrak{g}$ . The first case cannot occur since, otherwise,  $\mathfrak{g} = \mathfrak{g}^0 = \text{Ker}(\text{ad}X)$ . This would mean that  $X$  is in the centre of  $\mathfrak{g}$ , that is,  $X = 0$  and  $a = e$ .

For every  $Y \in \mathfrak{g}^{\lambda}$ , we have

$$a \exp Ya^{-1} = \exp((\text{Ada})Y) = \exp(e^{\text{ad}X} Y) = \exp(e^{\lambda} Y)$$

and therefore  $a^n \exp Ya^{-n} = \exp(e^{\lambda n} Y)$  for all  $n \in \mathbf{Z}$ . Hence,

$$\lim_{n \rightarrow +\infty} a^{-n} \exp Ya^n = e \quad \text{if } Y \in \mathfrak{g}^{+}$$

and

$$\lim_{n \rightarrow +\infty} a^n \exp Ya^{-n} = e \quad \text{if } Y \in \mathfrak{g}^{-}.$$

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  almost having invariant vectors. Since  $H$  has Property (T), there exists a non-zero  $\pi(H)$ -invariant vector  $\xi \in \mathcal{H}$ . The relations above and Mautner's Lemma 1.4.8 applied to  $a$  and  $a^{-1}$  show that

$\xi$  is invariant under  $\pi(\exp Y)$  for every  $Y \in \mathfrak{g}^+ \cup \mathfrak{g}^-$ . Hence,  $\xi$  is invariant under  $\pi(G)$ , as  $\mathfrak{g}$  is generated by  $\mathfrak{g}^+ \cup \mathfrak{g}^-$ . This shows that  $G$  has Property (T). ■

**Remark 1.6.3** Let  $\mathbb{G}$  be a connected, almost  $\mathbf{K}$ -simple algebraic group over the local field  $\mathbf{K}$ , and set  $G = \mathbb{G}(\mathbf{K})$ . If  $\text{rank}_{\mathbf{K}}(\mathbb{G}) = 0$ , then  $G$  is compact and hence has Property (T). We assume now that  $\text{rank}_{\mathbf{K}}(\mathbb{G}) = 1$  and we anticipate on several results shown below.

If  $\mathbf{K}$  is non-archimedean,  $G$  does not have Property (T). This follows from Theorem 2.3.6 and from the fact that  $G$  acts properly on a tree which is the Bruhat–Tits building of  $\mathbb{G}$  over  $\mathbf{K}$  [BruTi–72]. This also follows from Theorem 1.7.1 and from the fact that  $G$  has lattices which are isomorphic to a non-abelian free group (see Section 2 in [Lubot–91]).

If  $\mathbf{K} = \mathbf{C}$ , then  $G$  is locally isomorphic to  $SL_2(\mathbf{C})$ , and Example 1.7.4 shows that  $G$  does not have Property (T).

If  $\mathbf{K} = \mathbf{R}$ , Theorem 3.5.4 shows that  $G$  does not have Property (T) if and only if  $G$  is locally isomorphic to one of the groups  $SO(n, 1)$ ,  $SU(n, 1)$ .

## 1.7 Hereditary properties

We investigate Property (T) for lattices, extensions, and in particular coverings.

### Property (T) is inherited by lattices

An important feature of Property (T) is that it is inherited by *lattices*, a fact discovered by Kazhdan. This is one of the main methods for proving that certain discrete groups have Property (T); see footnote 6 in the historical introduction. The following more general result is true.

**Theorem 1.7.1** *Let  $G$  be a locally compact group, and let  $H$  be a closed subgroup of  $G$  such that  $G/H$  has a finite invariant regular Borel measure. The following are equivalent:*

- (i)  $G$  has Property (T);
- (ii)  $H$  has Property (T).

*In particular, if  $\Gamma$  is a lattice in  $G$ , then  $\Gamma$  has Property (T) if and only if  $G$  has Property (T).*

**Proof** Assume that  $G$  has Property (T), and let  $\sigma$  be a unitary representation of  $H$  such that  $1_H \prec \sigma$ . Then, by continuity of induction (Theorem F.3.5),

$$\text{Ind}_H^G 1_H \prec \text{Ind}_H^G \sigma.$$

Since  $G/H$  has a finite invariant measure,  $1_G$  is contained in  $\text{Ind}_H^G 1_H$ . Hence,

$$1_G \prec \text{Ind}_H^G \sigma.$$

As  $G$  has Property (T), it follows that  $1_G$  is contained in  $\text{Ind}_H^G \sigma$ . By Theorem E.3.1, this implies that  $1_H$  is contained in  $\sigma$ . Hence,  $H$  has Property (T).

To show the converse, assume that  $H$  has Property (T). Let  $(Q, \varepsilon)$  be a Kazhdan pair for  $H$  where  $Q$  is compact and  $\varepsilon < 1$ . Let  $\mu$  be an invariant probability measure on  $G/H$ . Choose a compact subset  $\tilde{Q}$  of  $G$  with  $Q \subset \tilde{Q}$  and

$$(*) \quad \mu(p(\tilde{Q})) > \frac{\varepsilon + 9}{10},$$

where  $p : G \rightarrow G/H$  denotes the canonical projection (see Lemma B.1.1). We claim that  $(\tilde{Q}, \varepsilon/4)$  is a Kazhdan pair for  $G$ .

Indeed, let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  with a  $(\tilde{Q}, \varepsilon/4)$ -invariant unit vector  $\xi$ . Then, restricting  $\pi$  to  $H$  and taking  $\delta = 1/4$  in Proposition 1.1.9, we see that

$$(**) \quad \|\xi - \xi'\| \leq 1/4,$$

where  $\xi'$  is the orthogonal projection of  $\xi$  on  $\mathcal{H}^H$ . The mapping

$$G/H \rightarrow \mathcal{H}, xH \mapsto \pi(x)\xi'$$

is well defined, continuous and bounded. Let  $\eta \in \mathcal{H}$  be defined by the  $\mathcal{H}$ -valued integral

$$\eta = \int_{G/H} \pi(x)\xi' d\mu(xH).$$

By  $G$ -invariance of  $\mu$ , we have, for all  $g \in G$ ,

$$\pi(g)\eta = \int_{G/H} \pi(gx)\xi' d\mu(xH) = \int_{G/H} \pi(x)\xi' d\mu(xH) = \eta.$$

Hence,  $\eta$  is  $G$ -invariant. It remains to show that  $\eta \neq 0$ .

Since  $\xi$  is a unit vector, observe that

$$(***) \quad \frac{3}{4} \leq \|\xi'\| \leq \frac{5}{4},$$

by (\*\*). For every  $g \in \tilde{Q}$ , we have

$$\begin{aligned} \|\pi(g)\xi' - \xi'\| &\leq \|\pi(g)(\xi - \xi') - (\xi - \xi')\| + \|\pi(g)\xi - \xi\| \\ &\leq \frac{1}{2} + \frac{\varepsilon}{4}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\eta - \xi'\| &= \left\| \int_{G/H} (\pi(x)\xi' - \xi') d\mu(xH) \right\| \\ &\leq \int_{p(\tilde{Q})} \|\pi(x)\xi' - \xi'\| d\mu(xH) + 2\|\xi'\|(1 - \mu(p(\tilde{Q}))) \\ &\leq \frac{1}{2} + \frac{\varepsilon}{4} + \frac{5}{2}(1 - \mu(p(\tilde{Q}))) \\ &< \frac{1}{2} + \frac{\varepsilon}{4} + \frac{1 - \varepsilon}{4} = \frac{3}{4}. \end{aligned}$$

Combining this inequality with (\*\*), we obtain  $\|\eta\| > 0$ , so that  $\eta \neq 0$  as claimed. ■

The following corollary is a consequence of Theorem 1.7.1 and Corollary 1.3.6.

**Corollary 1.7.2** *Let  $\Gamma$  be a discrete group with Property (T), and let  $\Lambda$  be a subgroup of  $\Gamma$  of finite index. Then the abelianisation  $\Lambda/[\Lambda, \Lambda]$  of  $\Lambda$  is finite.*

**Remark 1.7.3** The previous corollary has the following interpretation in geometry. Let  $X$  be a connected manifold. Assume that the fundamental group  $\Gamma = \pi_1(X)$  has Property (T). Let  $Y \rightarrow X$  be a finite covering of  $X$ . Then the first Betti number  $\beta_1(Y)$  of  $Y$  is 0.

Indeed,  $\beta_1(Y)$  is the rank of the abelian group  $H_1(Y, \mathbf{Z})$ , the fundamental group  $\Lambda$  of  $Y$  is a subgroup of finite index of  $\Gamma$ , and  $H^1(Y, \mathbf{Z})$  is isomorphic to the abelianisation of  $\Lambda$ .

**Example 1.7.4** (i) We can now give examples of infinite discrete groups with Property (T):  $SL_n(\mathbf{Z})$  and  $SL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$  have Property (T) for  $n \geq 3$ . Indeed, the first group is a lattice in  $SL_n(\mathbf{R})$  and the second one is a lattice in  $SL_n(\mathbf{R}) \ltimes \mathbf{R}^n$ , and both Lie groups have Property (T), by Theorem 1.4.15 and Corollary 1.4.16.

(ii) The free group  $F_2$  embeds as a lattice in  $SL_2(\mathbf{R})$  and does not have Property (T); see Example B.2.5 and Example 1.3.7. It follows that  $SL_2(\mathbf{R})$  does not have Property (T), a fact we already know from Example 1.3.7.

(iii) The free group  $F_2$  embeds as a lattice in  $SL_2(\mathbf{K})$ , for any non-archimedean local field  $\mathbf{K}$  (see [Serre–77, Chapitre II, 1.5, Théorème 4 and Exemples]). This shows that  $SL_2(\mathbf{K})$  does not have Property (T).

(iv) The group  $SL_2(\mathbf{C})$  contains lattices with infinite abelianisations. Any torsion-free subgroup of finite index in  $SL_2(\mathbf{Z}[\sqrt{-d}])$  for  $d \in \mathbf{N}$  is such a lattice (see [Serr–70b, Théorème 6]). Hence,  $SL_2(\mathbf{C})$  does not have Property (T).

### Behaviour under short exact sequences

We have already seen that Property (T) is inherited by quotient groups (Theorem 1.3.4). Observe that Property (T) is not inherited by closed normal subgroups. Indeed, for any local field  $\mathbf{K}$ , the semi-direct product  $SL_n(\mathbf{K}) \ltimes \mathbf{K}^n$  has Property (T) for  $n \geq 3$  (Corollary 1.4.16), but  $\mathbf{K}^n$  does not have Property (T).

**Lemma 1.7.5** *Let  $G$  be a topological group,  $N$  a closed normal subgroup of  $G$ , and  $p : G \rightarrow G/N$  the canonical projection. Let  $(Q_1, \varepsilon_1)$  and  $(Q_2, \varepsilon_2)$  be Kazhdan pairs for  $N$  and  $G/N$ , respectively. Let  $Q$  be a subset of  $G$  with  $Q_1 \subset Q$  and  $Q_2 \subset p(Q)$ . Set  $\varepsilon = \min\{\varepsilon_1/2, \varepsilon_2/2\}$ . Then  $(Q, \varepsilon)$  is a Kazhdan pair for  $G$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  with a  $(Q, \varepsilon)$ -invariant unit vector  $\xi$ . We claim that  $G$  has  $(Q, \varepsilon_2)$ -invariant vectors in the subspace  $\mathcal{H}^N$  of all  $N$ -fixed vectors.

Since  $(Q_1, \varepsilon_1)$  is a Kazhdan pair for  $N$ , taking  $\delta = 1/2$  in Proposition 1.1.9, we have  $\|\xi - P\xi\| \leq 1/2$ , where  $P : \mathcal{H} \rightarrow \mathcal{H}^N$  is the orthogonal projection on  $\mathcal{H}^N$ . In particular,  $\|P\xi\| \geq 1/2$ . Observe that  $\mathcal{H}^N$  is  $G$ -invariant, as  $N$  is normal. Therefore,  $P$  intertwines  $\pi$  with itself, and we have, for all  $x \in Q$ ,

$$\|\pi(x)P\xi - P\xi\| \leq \|\pi(x)\xi - \xi\| < \varepsilon_2/2 \leq \varepsilon_2\|P\xi\|.$$

This proves the claim.

Now, the restriction of  $\pi$  to  $\mathcal{H}^N$  factorises to a unitary representation  $\rho$  of  $G/N$ . By what we have seen above,  $\rho$  has a  $(Q_2, \varepsilon_2)$ -invariant vector, since  $Q_2 \subset p(Q)$ . As  $(Q_2, \varepsilon_2)$  is a Kazhdan pair for  $G/N$ , there exists a non-zero vector in  $\mathcal{H}^N$  which is invariant under  $\rho(G/N)$  and hence under  $\pi(G)$ . Therefore,  $(Q, \varepsilon)$  is a Kazhdan pair for  $G$ . ■

**Proposition 1.7.6** *Let  $G$  be a locally compact group, and let  $N$  be a closed normal subgroup of  $G$ . If  $N$  and  $G/N$  have Property (T), then  $G$  has Property (T).*

**Proof** Let  $Q_1$  and  $Q_2$  be compact Kazhdan sets for  $N$  and  $G/N$ , respectively. Since  $G$  is locally compact, there exists a compact subset  $Q'$  of  $G$  with  $p(Q') = Q_2$  (Lemma B.1.1). Then, by the previous lemma, the compact subset  $Q = Q_1 \cup Q'$  is a Kazhdan set for  $G$ . ■

**Remark 1.7.7** A stronger result is true:  $G$  has Property (T) if and only if both the quotient  $G/N$  and the pair  $(G, N)$  have Property (T); the proof is left as Exercise 1.8.12.

**Proposition 1.7.8** *Let  $G_1$  and  $G_2$  be topological groups. The direct product  $G_1 \times G_2$  has Property (T) if and only if  $G_1$  and  $G_2$  have Property (T).*

**Proof** The “only if” part is clear since Property (T) is inherited by quotients (Theorem 1.3.4). If  $Q_1$  and  $Q_2$  are compact Kazhdan sets for  $G_1$  and  $G_2$ , then, by the previous lemma, the compact set  $Q = (Q_1 \times \{e\}) \cup (\{e\} \times Q_2)$  is a Kazhdan set for  $G$ . This shows the “if” part of the proposition. ■

**Remark 1.7.9** Let  $G$  be a topological group, and let  $N$  a closed normal subgroup. As the proof shows, the conclusion of Proposition 1.7.6 is true if the pair  $(G, N)$  satisfies the following condition:

(\*) For every compact subset  $Q_2$  of  $G/N$ , there exists a compact subset  $Q'$  of  $G$  with  $p(Q') = Q_2$ .

As we have seen,  $(G, N)$  satisfies Condition (\*) for any normal subgroup  $N$ , if  $G$  is locally compact. By [Bou–Top2, §2, Proposition 18], this is also the case if  $G$  is a complete metrizable topological group.

However, Condition (\*) is not always satisfied. Indeed, given any topological group  $H$ , there exist a complete topological group  $G$ , in which all compact subsets are finite, and a closed normal subgroup  $N$  of  $G$  such that  $H$  is topologically isomorphic to  $G/N$ . The special case of this statement for  $H$  abelian is Proposition 1.1 of [RoeDi–81]; the general case is from [Pestov]; we are grateful to V. Pestov for his indications on this point.

## Covering groups

Let  $G$  be a topological group and  $C$  a closed subgroup contained in the centre of  $G$ . If  $C$  is compact and  $G/C$  has Property (T), then  $G$  has Property (T), by Remark 1.7.9. We will investigate what happens when  $C$  is not compact. Our analysis is based on the following lemma, which is due to J-P. Serre.

**Lemma 1.7.10** *Let  $G$  be a locally compact group and  $C$  a closed subgroup contained in the centre of  $G$ . Let  $\mathcal{R}$  denote the set of irreducible unitary representations  $\pi$  of  $G$  with  $2 \leq \dim \pi \leq +\infty$ .*

*If  $G/C$  has Property (T), then  $1_G$  is isolated in  $\mathcal{R} \cup \{1_G\}$ .*

**Proof** Assume, by contradiction, that  $1_G$  is not isolated in  $\mathcal{R} \cup \{1_G\}$ . Then there is a net  $(\pi_i)_{i \in I}$  in  $\mathcal{R}$  converging to  $1_G$ . Denote by  $\mathcal{H}_i$  the Hilbert space of  $\pi_i$ , by  $\overline{\mathcal{H}}_i$  its conjugate space, and by  $\overline{\pi}_i$  its conjugate representation (see Definition A.1.10). We can find a unit vector  $\xi_i \in \mathcal{H}_i$  such that

$$(*) \quad \lim_i \|\pi_i(g)\xi_i - \xi_i\| = 0,$$

uniformly on compact subsets of  $G$ . Since  $\pi_i$  is irreducible and  $C$  is contained in the centre of  $G$ , it follows from Schur's Lemma (Theorem A.2.2) that the representation  $\pi_i \otimes \overline{\pi}_i$  of  $G$  on  $\mathcal{H}_i \otimes \overline{\mathcal{H}}_i$  factorises through  $G/C$ . Denote by  $\overline{\xi}_i$  the vector  $\xi_i$  viewed in  $\overline{\mathcal{H}}_i$ . We have  $\|\xi_i \otimes \overline{\xi}_i\| = 1$  and

$$\begin{aligned} \|(\pi_i \otimes \overline{\pi}_i)(g)(\xi_i \otimes \overline{\xi}_i) - \xi_i \otimes \overline{\xi}_i\|^2 &= \|\pi_i(g)\xi_i \otimes \overline{\pi_i(g)\xi_i} - \xi_i \otimes \overline{\xi}_i\|^2 \\ &= 2 - \langle \pi_i(g)\xi_i, \xi_i \rangle \langle \overline{\pi_i(g)\xi_i}, \overline{\xi}_i \rangle \\ &\quad - \langle \xi_i, \pi_i(g)\xi_i \rangle \langle \overline{\xi}_i, \overline{\pi_i(g)\xi_i} \rangle \\ &= 2(1 - |\langle \pi_i(g)\xi_i, \xi_i \rangle|)(1 + |\langle \pi_i(g)\xi_i, \xi_i \rangle|) \\ &\leq 4(1 - \operatorname{Re}\langle \pi_i(g)\xi_i, \xi_i \rangle) \\ &= 2\|\pi_i(g)\xi_i - \xi_i\|^2. \end{aligned}$$

Together with (\*), this shows that

$$(**) \quad \lim_i \|(\pi_i \otimes \overline{\pi}_i)(g)\xi_i \otimes \overline{\xi}_i - \xi_i \otimes \overline{\xi}_i\| = 0,$$

uniformly on compact subsets of  $G$  and, since  $G$  is locally compact, uniformly on compact subsets of  $G/C$  (see Lemma B.1.1). Since  $G/C$  has Property (T), it follows that, for  $i$  large enough,  $\pi_i \otimes \overline{\pi}_i$  contains  $1_{G/C}$ . Therefore,  $\pi_i$  is finite dimensional for  $i$  large enough (Corollary A.1.13). Without loss of generality, we can assume that  $\pi_i$  is finite dimensional, say of dimension  $n_i$ , for all  $i \in I$ .

Recall that we can realise  $\pi_i \otimes \overline{\pi}_i$  in the space  $HS(\mathcal{H}_i)$  of Hilbert–Schmidt operators on  $\mathcal{H}_i$  by

$$(\pi_i \otimes \overline{\pi}_i)(g)T = \pi_i(g)T\pi_i(g^{-1}), \quad \text{for all } g \in G, T \in HS(\mathcal{H}_i)$$

(see remark after Definition A.1.11). Observe that, by Schur's Lemma (Theorem A.2.2), the space of  $G$ -invariant vectors in  $HS(\mathcal{H}_i)$  consists of the

scalar multiples of  $I_{n_i}$ . The operator corresponding to  $\xi_i \otimes \overline{\xi_i}$  is the orthogonal projection  $P_i$  on the line  $\mathbb{C}\xi_i$ . We have, by (\*\*),

$$(***) \quad \lim_i \|\pi_i(g)P_i\pi_i(g^{-1}) - P_i\| = 0,$$

uniformly on compact subsets of  $G/C$ . Let  $HS(\mathcal{H}_i)^0$  be the orthogonal complement of the space of  $G$ -invariant vectors in  $HS(\mathcal{H}_i)$ , that is,  $HS(\mathcal{H}_i)^0 = (CI_{n_i})^\perp$ . Set

$$P_i^0 = P_i - \frac{1}{n_i}I_{n_i}.$$

As  $\text{Trace}(P_i) = 1$ , we have  $P_i^0 \in HS(\mathcal{H}_i)^0$  and therefore

$$\|P_i^0\|^2 = 1 - \frac{1}{n_i}.$$

Since, by assumption,  $n_i \geq 2$ , we have

$$\|P_i^0\| \geq 1/2, \quad \text{for all } i \in I.$$

For  $Q_i = \|P_i^0\|^{-1}P_i^0$ , we have  $Q_i \in HS(\mathcal{H}_i)^0$ ,  $\|Q_i\| = 1$  and, by (\*\*\*),

$$\lim_i \|\pi_i(g)Q_i\pi_i(g^{-1}) - Q_i\| = 0,$$

uniformly on compact subsets of  $G/C$ . Hence the net  $((\pi_i \otimes \overline{\pi_i})^0)_i$  converges to  $1_{G/C}$ , where  $(\pi_i \otimes \overline{\pi_i})^0$  denotes the restriction of  $\pi_i \otimes \overline{\pi_i}$  to  $HS(\mathcal{H}_i)^0$ . This is a contradiction, since  $G/C$  has Property (T). ■

The previous lemma has the following consequence, which is also due to J.-P. Serre and for which we will give another proof in Corollary 3.5.3.

**Theorem 1.7.11** *Let  $G$  be a locally compact group and  $C$  a closed subgroup contained in the centre of  $G$ . Assume that  $G/C$  has Property (T) and that  $G/\overline{[G, G]}$  is compact. Then  $G$  has Property (T).*

**Proof** Assume, by contradiction, that  $G$  does not have Property (T). Then there exists a net  $(\pi_i)_{i \in I}$  in  $\widehat{G} \setminus \{1_G\}$  converging to  $1_G$ . Since  $G/C$  has Property (T), we can assume, by the previous lemma, that  $\dim \pi_i = 1$  for all  $i \in I$ . Thus, all  $\pi_i$ 's factorise through the abelianisation  $G/\overline{[G, G]}$  of  $G$ . On the other hand, since  $G/\overline{[G, G]}$  is compact, its unitary dual is discrete (see Example F.2.5.ii). This is a contradiction. ■



**Remark 1.7.12** (i) The assumptions on  $G/C$  and  $G/\overline{[G, G]}$  in the previous theorem are necessary, by Theorem 1.3.4 and Corollary 1.3.6. Observe that the additive group  $\mathbf{R}$ , which does not have Property (T), is the universal covering of the circle group, which is compact and therefore has Property (T).

(ii) The previous theorem was proved in [Wang–82, Lemma 1.8] in the special case where  $G$  is minimally almost periodic, that is, every finite dimensional unitary representation of  $G$  is a multiple of  $1_G$ . Examples of such groups are the non-compact connected simple Lie groups.

(iii) For an extension of the previous theorem to a pair of discrete groups with Property (T), see [NiPoS–07].

**Example 1.7.13** (i) Let  $G$  be a connected Lie group, with universal covering group  $\tilde{G}$ . Assume that  $G$  has Property (T) and that  $\tilde{G}/[\tilde{G}, \tilde{G}]$  is compact. Then  $\tilde{G}$  has Property (T). Indeed, it is well known that  $G \cong \tilde{G}/C$  for a (discrete) subgroup  $C$  of the centre of  $\tilde{G}$ .

(ii) We can give examples of groups with Property (T) which have non-compact centres. The fundamental group of  $G = Sp_{2n}(\mathbf{R})$  is isomorphic to  $\mathbf{Z}$ ; hence the universal covering  $\tilde{G}$  is perfect (that is,  $[\tilde{G}, \tilde{G}] = \tilde{G}$ ), with centre isomorphic to  $\mathbf{Z}$  and therefore non-compact. For  $n \geq 2$ , Theorem 1.7.11 implies that  $\tilde{G}$  has Property (T).

(iii) We can also give an example of a discrete group with Property (T) and with an infinite centre:  $\Gamma = Sp_{2n}(\mathbf{Z})$  is known to be a lattice in  $G = Sp_{2n}(\mathbf{R})$ . The inverse image  $\tilde{\Gamma}$  of  $\Gamma$  in  $\tilde{G}$  is a lattice containing the centre of  $\tilde{G}$ . Hence,  $\tilde{\Gamma}$  has an infinite centre and has Property (T) for  $n \geq 2$ .

(iv) Let  $G$  be a connected simple real Lie group of real rank at least 2, let  $K$  be a maximal compact subgroup of  $G$ , and let  $\Gamma$  be a lattice in  $G$ . Assume that the symmetric space  $G/K$  is hermitian. Then Examples (ii) and (iii) carry over: the groups  $\tilde{G}$  and  $\tilde{\Gamma}$  have infinite centres and have Property (T); for all this, see Remark 3.5.5.

## 1.8 Exercises

**Exercise 1.8.1** Let  $\mathbf{K}$  be a field. Verify that  $Sp_2(\mathbf{K}) = SL_2(\mathbf{K})$ .

**Exercise 1.8.2** Let  $\mathbf{K}$  be a field. Prove that  $SL_n(\mathbf{K})$  is generated by the set of all elementary matrices  $E_{ij}(x)$ ,  $x \in \mathbf{K}$ ,  $1 \leq i, j \leq n$ ,  $i \neq j$  (Lemma 1.4.6).

**Exercise 1.8.3** Let  $\xi$  and  $\eta$  be vectors in Hilbert space with  $\|\xi\| = \|\eta\|$ . Prove that  $\xi = \eta$  if and only if  $\operatorname{Re}\langle \xi, \eta \rangle = \|\xi\|^2$ .

**Exercise 1.8.4** Let  $\mathbf{K}$  be a field with  $\text{char}(\mathbf{K}) \neq 2$ . Show that the two natural representations of  $SL_2(\mathbf{K})$  on  $S^2(\mathbf{K}^2)$  and  $S^{2*}(\mathbf{K}^2)$  are equivalent.

**Exercise 1.8.5** Let  $\mathbf{K}$  be a topological non-discrete field. For  $n \geq 2$ , let  $G = H \ltimes N \subset SL_{2n}(\mathbf{K})$  be the semidirect product of the subgroup

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in GL_n(\mathbf{K}) \right\} \cong GL_n(\mathbf{K})$$

with the subgroup

$$N = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} : B \in M_n(\mathbf{K}), {}^t B = B \right\}.$$

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Show that, if  $\xi \in \mathcal{H}$  is invariant under  $H$ , then  $\xi$  is invariant under  $G$ .

[Hint: Use Mautner's Lemma 1.4.8.]

**Exercise 1.8.6** Let  $\mathbf{K}$  be a local field. Prove that the semidirect product

$$G = \left\{ \begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} : A \in SL_n(\mathbf{K}), B \in M_n(\mathbf{K}), {}^t B = B \right\}$$

of  $SL_n(\mathbf{K})$  with the symmetric matrices in  $M_n(\mathbf{K})$  has Property (T) for  $n \geq 3$ .

**Exercise 1.8.7** Let  $\mathbf{K}$  be a local field. Consider the action of  $SL_n(\mathbf{K})$  on  $M_{n,m}(\mathbf{K})$ , the space of the  $(n \times m)$  matrices, given by

$$(g, A) \mapsto gA, \quad g \in SL_n(\mathbf{K}), A \in M_{n,m}(\mathbf{K}).$$

Prove that the semidirect product  $SL_n(\mathbf{K}) \ltimes M_{n,m}(\mathbf{K})$  has Property (T) for  $n \geq 3$  and for every  $m \in \mathbf{N}$ .

**Exercise 1.8.8** Let  $\mathbf{K}$  be a local field, and consider a continuous representation of  $SL_n(\mathbf{K})$  on a finite dimensional vector space  $V$  over  $\mathbf{K}$ , without non-zero fixed vector. Show that the corresponding semidirect product  $SL_n(\mathbf{K}) \ltimes V$  has Property (T) for  $n \geq 3$ . For a more general result, see [Wang–82, Theorems 1.9 and 2.10].

**Exercise 1.8.9** Let  $\mathbf{K}$  be a local field and let  $n \geq 2$  be an integer. Show that the semidirect product  $G = Sp_{2n}(\mathbf{K}) \ltimes \mathbf{K}^{2n}$  has Property (T).

[Hint: Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  almost having invariant vectors. Since  $Sp_{2n}(\mathbf{K})$  has Property (T), there exists a non-zero  $Sp_{2n}(\mathbf{K})$ -invariant

vector  $\xi \in \mathcal{H}$ . Using Mautner's Lemma, show that  $\xi$  is fixed by every element  $e_i$  of the standard basis of  $\mathbf{K}^{2n}$ .]

**Exercise 1.8.10** Let  $\mathbf{K}$  be a local field and let  $n \geq 1$  be an integer. Consider the symplectic form  $\omega$  on  $\mathbf{K}^{2n}$  given by

$$\omega(x, y) = x^t J y, \quad x, y \in \mathbf{K}^{2n},$$

where  $J$  is the  $(2n \times 2n)$  matrix as in Section 1.5. The  $(2n + 1)$ -dimensional Heisenberg group over  $\mathbf{K}$  is the group  $H_{2n+1}(\mathbf{K})$  with underlying set  $\mathbf{K}^{2n} \times \mathbf{K}$  and product

$$(x, \lambda)(y, \mu) = (x + y, \lambda + \mu + \omega(x, y)), \quad x, y \in \mathbf{K}^{2n}, \lambda, \mu \in \mathbf{K}.$$

The symplectic group  $Sp_{2n}(\mathbf{K})$  acts by automorphisms of  $H_{2n+1}(\mathbf{K})$ :

$$g(x, \lambda) = (gx, \lambda), \quad g \in Sp_{2n}(\mathbf{K}), x \in \mathbf{K}^{2n}, \lambda \in \mathbf{K}.$$

Show that the corresponding semi-direct product  $G = Sp_{2n}(\mathbf{K}) \ltimes H_{2n+1}(\mathbf{K})$  has Property (T) for  $n \geq 2$ .

[Hint: Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  almost having invariant vectors. Since  $Sp_{2n}(\mathbf{K})$  has Property (T), there exists a non-zero  $Sp_{2n}(\mathbf{K})$ -invariant vector  $\xi \in \mathcal{H}$ . By Mautner's Lemma,  $\xi$  is fixed by the subgroups

$$A = \{((a, 0), 0) : a \in \mathbf{K}^n\} \quad \text{and} \quad B = \{((0, b), 0) : b \in \mathbf{K}^n\}$$

of  $H_{2n+1}(\mathbf{K})$ ; see the hint for Exercise 1.8.9. Now  $H_{2n+1}(\mathbf{K})$  is generated by  $A \cup B$ . Hence,  $\xi$  is fixed by  $H_{2n+1}(\mathbf{K})$ .]

**Exercise 1.8.11** Let  $G$  be a topological group. Assume that  $G$  is the union of an increasing sequence  $(H_n)_n$  of open subgroups. Show that if  $G$  has Property (T), then the sequence is stationary.

[Hint: Imitate the proof of Theorem 1.3.1.]

**Exercise 1.8.12** Let  $G$  be a locally compact group and let  $N$  be a closed normal subgroup. Show that the following properties are equivalent:

- (i)  $G$  has Property (T);
- (ii) the quotient  $G/N$  and the pair  $(G, N)$  have Property (T).

[Hint: To show that (ii) implies (i), look at the proof of Proposition 1.7.6.]

**Exercise 1.8.13** Let  $G_1, G_2$  be locally compact groups,  $\Gamma$  a lattice in  $G = G_1 \times G_2$ , and  $N$  a normal subgroup in  $\Gamma$ . Assume that  $\Gamma$  is irreducible, that is,  $p_i(\Gamma)$  is dense in  $G_i$  for  $i = 1, 2$ , where  $p_i : G \rightarrow G_i$  denotes the  $i$ -th projection.

(i) Show that  $\overline{p_i(N)}$  is normal in  $G_i$ .

Assume now that  $\Gamma/N$  has Property (T).

(ii) Show that  $G_i/\overline{p_i(N)}$  has Property (T).

(iii) Let  $\varphi : G \rightarrow \mathbf{R}$  be a continuous homomorphism with  $\varphi|_N = 0$ . Show that  $\varphi = 0$ .

[There is a converse, which is a deep result by Y. Shalom (Theorem 0.1 in [Shal-00a]): assume that  $G_1, G_2$  are compactly generated, and that  $\Gamma$  is a cocompact irreducible lattice in  $G$ . If (ii) and (iii) hold, then  $\Gamma/N$  has Property (T).]

**Exercise 1.8.14** Let  $\Gamma$  be either a free group, or a pure braid group, or a braid group, and let  $\Gamma_0$  be a subgroup of  $\Gamma$  not reduced to one element. Show that  $\Gamma_0$  does not have Property (T).

[Hint: If  $\Gamma$  is free, then  $\Gamma_0$  is free by the Nielsen–Schreier theorem. Let  $\Gamma = P_k$  be the pure braid group on  $k$  strings; if  $k \geq 2$ , the kernel of the natural homomorphism from  $P_k$  to  $P_{k-1}$  which “forgets the last string” is free (a theorem of E. Artin), and the claim of the exercise follows by induction on  $k$ . If  $\Gamma = B_k$  is Artin’s braid group on  $k$  strings, then  $\Gamma$  has  $P_k$  as a subgroup of finite index.]

**Exercise 1.8.15** Let  $G$  be a locally compact group and  $H$  a closed subgroup of  $G$ . Prove that  $(G, H)$  has Property (T) if and only if there exists a neighbourhood  $V$  of  $1_G$  in  $\widehat{G}$  such that  $\pi|_H$  contains  $1_H$  for every  $\pi \in V$ .

[Hint: Look at the proof of Lemma 1.2.4.]

**Exercise 1.8.16** (This exercise was suggested to us by D. Gaboriau.)

(i) Let  $G$  be a locally compact group and  $H, L$  two open subgroups of  $G$ . Denote by  $\pi$  the restriction to  $H$  of the quasi-regular representation of  $G$  on  $\ell^2(G/L)$ . Check that the following conditions are equivalent:

- the representation  $\pi$  contains  $1_H$ ;
- there exists a finite  $H$ -orbit in  $G/L$ .

(ii) Let  $G$  be a locally compact group and  $H$  be an open subgroup of  $G$ . Assume that the pair  $(G, H)$  has Property (T). Show that there exists a compactly generated subgroup of  $G$  which contains  $H$ .

[Hint: Imitate the proof of Theorem 1.3.1 and use the first part of the exercise.]

(iii) Consider an integer  $n \geq 2$ , the tautological action of  $SL_n(\mathbf{Q})$  on  $\mathbf{Q}^n$ , and the corresponding semidirect product  $SL_n(\mathbf{Q}) \ltimes \mathbf{Q}^n$ . Show that the pair  $(SL_n(\mathbf{Q}) \ltimes \mathbf{Q}^n, \mathbf{Q}^n)$  does not have Property (T).

**Exercise 1.8.17** Show that the group  $SL_3(\mathbf{Q})$  has the *local Property (T)*, by which we mean the following property: any finitely generated subgroup  $A$  of  $SL_3(\mathbf{Q})$  is contained in a subgroup  $B$  of  $SL_3(\mathbf{Q})$  which has Property (T).

[Hint: Let  $N$  be the greatest common divisor of all denominators of all matrices in some finite generating set of  $A$ , and let  $p_1, \dots, p_k$  be the list of the prime divisors of  $N$ . Set  $B = SL_3(\mathbf{Z}[1/N])$ . Then  $A$  is contained in  $B$ , and  $B$  embeds as a lattice in the direct product  $SL_3(\mathbf{R}) \times SL_3(\mathbf{Q}_{p_1}) \times \dots \times SL_3(\mathbf{Q}_{p_k})$ . In other words,  $A$  embeds in an  $S$ -arithmetic subgroup  $B$  of  $SL_3(\mathbf{Q})$ , and  $B$  has Property (T).]

**Exercise 1.8.18** Let  $M$  be an orientable compact 3-manifold and let

$$M_1, \dots, M_k$$

be the pieces of a canonical decomposition along embedded spheres, discs and tori. Assume that each  $M_j$  admits one of the eight geometric structures of 3-manifolds in the sense of Thurston. Show that the fundamental group of  $M$  has Property (T) if and only if this group is finite.

[Hint: See [Fujiw-99].]

**Exercise 1.8.19 (A Kazhdan group with an infinite outer automorphism group)** Let  $n$  be an integer with  $n \geq 3$  and let  $M_n(\mathbf{R})$  be the space of the  $(n \times n)$  matrices with real coefficients. Recall from Exercise 1.8.7 that the semidirect product  $G = SL_n(\mathbf{R}) \ltimes M_n(\mathbf{R})$  has Property (T) where  $SL_n(\mathbf{R})$  acts on  $M_n(\mathbf{R})$  by *left* multiplication.

The discrete group  $\Gamma = SL_n(\mathbf{Z}) \ltimes M_n(\mathbf{Z})$  is a lattice in  $G$  and has therefore Property (T).

On the other hand,  $SL_n(\mathbf{Z})$  acts also on  $M_n(\mathbf{Z})$  by *right* multiplication:

$$A \mapsto A\delta, \quad \delta \in SL_n(\mathbf{Z}), \quad A \in M_n(\mathbf{Z}).$$

(i) For  $\delta \in SL_n(\mathbf{Z})$ , let

$$s_\delta : \Gamma \rightarrow \Gamma, \quad (\alpha, A) \mapsto (\alpha, A\delta).$$

Show that  $s_\delta$  is an automorphism of  $\Gamma$  and that  $\delta \mapsto s_\delta$  is an injective group antihomomorphism from  $SL_n(\mathbf{Z})$  to the automorphism group  $\text{Aut}(\Gamma)$  of  $\Gamma$ .

(ii) For  $\delta \in SL_n(\mathbf{Z})$ , show that  $s_\delta$  is an inner automorphism of  $\Gamma$  if and only if  $\delta \in \{\pm I\}$ .

In particular, the outer automorphism group of  $\Gamma$  is infinite.

[The previous example appears in [Cornu-07]. For other examples, see [OllWi].]

**Exercise 1.8.20 (Property (T) for representations on Banach spaces)** Let  $B$  be a complex or real Banach space. Denote by  $\mathcal{O}(B)$  the group of all linear bijective isometries from  $B$  to  $B$ . A representation of a topological group  $G$  on  $B$  is a group homomorphism  $\pi : G \rightarrow \mathcal{O}(B)$  which is strongly continuous (that is, the mapping  $G \rightarrow B$ ,  $g \mapsto \pi(g)v$  is continuous for every  $v \in B$ ). Every representation  $\pi : G \rightarrow \mathcal{O}(B)$  induces in a natural way a representation  $\tilde{\pi} : G \rightarrow \mathcal{O}(B/B^G)$  on the quotient Banach space  $B/B^G$ , where  $B^G$  is the subspace of  $G$ -fixed vectors in  $B$ .

We say that  $G$  has *Property (T)<sub>B</sub>* if, for every representation  $\pi$  of  $G$  on  $B$ , the associated representation  $\tilde{\pi}$  on  $B/B^G$  does not almost have invariant vectors.

(i) Show that, when  $B$  is a Hilbert space, Property (T)<sub>B</sub> coincides with Property (T).

(ii) Let  $G$  be a compact group. Show that  $G$  has Property (T)<sub>B</sub> for every Banach space  $B$ .

[Hint: For every Banach space  $E$ , for every representation  $\pi : G \rightarrow \mathcal{O}(E)$ , and every  $v \in E$ , the  $E$ -valued integral  $\int_G \pi(g)vdg$  defines an invariant vector, where  $dg$  denotes a Haar measure on  $G$ .]

(iii) Let  $G$  be a second countable locally compact group. Let  $B = C_0(G)$  be the Banach space of the complex-valued functions on  $G$  which tend to zero at infinity, endowed with the supremum norm. Consider the representation  $\pi$  of  $G$  on  $C_0(G)$  given by left translations. Show that  $G$  does not have Property (T)<sub>B</sub> when  $G$  is not compact.

[Hint: Since  $G$  is not compact, there is no non-zero invariant function in  $C_0(G)$ . On the other hand,  $\pi$  almost has invariant vectors. To show this, let  $d$  be a left invariant distance on  $G$  (such a distance exists; see, e.g., [HewRo–63, Chapter II, (8.3)]). For every  $n \in \mathbb{N}$ , consider the function  $f_n \in C_0(G)$  defined by  $f_n(x) = \frac{1}{n+d(x,e)}$ .]

For more details on Property (T)<sub>B</sub>, see [BaFGM].

## 2

### Property (FH)

In this chapter, we introduce a new property: a topological group  $G$  is said to have Property (FH) if every continuous action of  $G$  by affine isometries on a real Hilbert space has a fixed point – the acronym (FH) stands for “Fixed point for affine isometric actions on Hilbert spaces”. As we will show in the last section, Property (FH) is equivalent to Property (T) for  $\sigma$ -compact locally compact groups (the Delorme–Guichardet Theorem 2.12.4).

Topological groups will play their part in three ways:

- in affine isometric actions on real Hilbert spaces (geometric data);
- in cohomology groups  $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$  with coefficients in orthogonal representations (algebraic data);
- as domains of functions conditionally of negative type on  $G$  (analytic data).

Here are the relevant links. Let  $\alpha$  be an affine isometric action of a topological group  $G$  on a real Hilbert space  $\mathcal{H}$ , with linear part  $\pi$  and translation part  $b$ . Then  $\pi$  is an orthogonal representation of  $G$  and  $g \mapsto b(g)$  is a 1-cocycle with coefficients in  $\pi$  defining a class in  $H^1(G, \pi)$ . Moreover,  $g \mapsto \psi(g) = \|b(g)\|^2$  is a function conditionally of negative type on  $G$ .

The first aim of this chapter is to establish the following dictionary:

Affine isometric actions	1-cocycles	Functions $\psi$ conditionally of negative type
Fixed point	1-coboundaries	$\psi = \varphi(e) - \varphi(g)$ , where $\varphi$ is a real function of positive type
Bounded orbits	Bounded 1-cocycles	$\psi$ bounded

It will be seen that an affine isometric action has a bounded orbit if and only if it has a fixed point (Proposition 2.2.9), so that the second and third lines of the

above dictionary are actually equivalent. It follows that topological groups  $G$  with Property (FH) are characterised by the fact that  $H^1(G, \pi) = 0$  for every orthogonal representation  $\pi$  of  $G$ ; equivalently, every function conditionally of negative type on  $G$  is bounded.

We will show that Property (FH) for a group  $G$  implies strong restrictions for its actions on trees, on real or complex hyperbolic spaces, and on the circle. This will provide us with several examples of groups *without* Property (FH). Moreover, we will characterise wreath products with Property (FH).

A remark is in order. Hilbert spaces are *complex* in our Chapter 1, which is essentially on unitary group representations and has an analytic flavour, whereas Hilbert spaces are *real* in most of our Chapter 2, which is geometric in nature. The correspondence between affine isometric actions and functions conditionally of negative type (Proposition 2.10.2) is simpler to formulate for real Hilbert spaces and orthogonal representations than in the complex case (compare [Guic–72b, Proposition 4.5]). In this context, recall also the important Mazur–Ulam Theorem [Banac–32, Chapitre XI, Théorème 2]: any isometry of a real Hilbert space is affine. Clearly, this does not hold for complex Hilbert spaces.

## 2.1 Affine isometric actions and Property (FH)

**Definition 2.1.1** An *affine real Hilbert space* is a set  $\mathcal{H}$  given together with a simply transitive action of the additive group of a real Hilbert space  $\mathcal{H}^0$ . The mapping  $T_\xi : \mathcal{H} \rightarrow \mathcal{H}$  induced by  $\xi \in \mathcal{H}^0$  is called a *translation*, and is denoted by  $x \mapsto x + \xi$ . For  $x, y \in \mathcal{H}$ , the unique vector  $\xi \in \mathcal{H}^0$  such that  $T_\xi x = y$  is denoted by  $y - x$ .

Observe that a real Hilbert space  $\mathcal{H}^0$  has a canonical structure as affine Hilbert space given by  $(\xi, x) \mapsto x + \xi$ .

Let  $\mathcal{H}$  be an affine real Hilbert space. The mapping  $(x, y) \mapsto \|x - y\|$  is a metric on  $\mathcal{H}$ . By the Mazur–Ulam Theorem quoted above, any isometry of  $\mathcal{H}$  is affine.

Let  $\mathcal{O}(\mathcal{H}^0)$  denote the *orthogonal group* of  $\mathcal{H}^0$ , that is, the group of invertible isometric linear operators on  $\mathcal{H}^0$ . There is a natural homomorphism  $p : \text{Isom}(\mathcal{H}) \rightarrow \mathcal{O}(\mathcal{H}^0)$  defined for  $g \in \text{Isom}(\mathcal{H})$  by

$$p(g)\xi = g(x + \xi) - g(x), \quad \text{for all } \xi \in \mathcal{H}^0,$$

where  $x$  is an arbitrary point in  $\mathcal{H}$ ; the kernel of  $p$  is the group  $\mathcal{H}^0$  of translations, and  $p$  is onto. The choice of an origin  $0 \in \mathcal{H}$  provides a section of  $p$  and a



semidirect product decomposition

$$\text{Isom}(\mathcal{H}) = \mathcal{O}(\mathcal{H}^0) \ltimes \mathcal{H}^0.$$

Let  $G$  be a topological group.

**Definition 2.1.2** An *affine isometric action* of  $G$  on  $\mathcal{H}$  is a group homomorphism  $\alpha : G \rightarrow \text{Isom}(\mathcal{H})$  such that the mapping

$$G \rightarrow \mathcal{H}, \quad g \mapsto \alpha(g)x$$

is continuous for every  $x \in \mathcal{H}$ .

Let  $\alpha$  be an affine isometric action of  $G$  on  $\mathcal{H}$ . Composing  $\alpha$  with the mapping  $\text{Isom}(\mathcal{H}) \rightarrow \mathcal{O}(\mathcal{H}^0)$ , we obtain a strongly continuous homomorphism  $\pi : G \rightarrow \mathcal{O}(\mathcal{H}^0)$ , that is, an *orthogonal representation* of  $G$  on  $\mathcal{H}^0$ . The strong continuity of  $\pi$  means that the mapping

$$G \rightarrow \mathcal{H}^0, \quad g \mapsto \pi(g)\xi$$

is continuous for every  $\xi \in \mathcal{H}^0$ . We call  $\pi$  the *linear part* of  $\alpha$ .

**Example 2.1.3** A continuous homomorphism  $b$  from  $G$  to the additive group of  $\mathcal{H}^0$  gives rise to an affine isometric action  $\alpha$  on  $\mathcal{H}$ , defined by

$$\alpha(g)x = x + b(g)$$

for all  $g \in G$  and  $x \in \mathcal{H}$ . The linear part  $\pi$  of  $\alpha$  is the trivial representation of  $G$  on  $\mathcal{H}^0$ , given by  $\pi(g) = I$  for all  $g \in G$ .

**Definition 2.1.4** A topological group  $G$  has *Property (FH)* if every affine isometric action of  $G$  on a real Hilbert space has a fixed point.

## 2.2 1-cohomology

In this section, we formulate Property (FH) in terms of 1-cohomology.

Let  $G$  be a topological group, and let  $\mathcal{H}$  be an affine real Hilbert space. In the previous section, we have seen that any affine isometric action  $\alpha$  of  $G$  on  $\mathcal{H}$  gives rise to an orthogonal representation of  $G$  on  $\mathcal{H}^0$ , called the linear part of  $\alpha$ . We address now the following question: Given an orthogonal representation  $\pi$  of  $G$  on  $\mathcal{H}^0$ , what are the affine isometric actions of  $G$  with linear part  $\pi$ ?

**Lemma 2.2.1** *Let  $\pi$  be an orthogonal representation of  $G$  on  $\mathcal{H}^0$ . For a mapping  $\alpha : G \rightarrow \text{Isom}(\mathcal{H})$  the following conditions are equivalent:*

- (i)  *$\alpha$  is an affine isometric action of  $G$  with linear part  $\pi$ ;*
- (ii) *there exists a continuous mapping  $b : G \rightarrow \mathcal{H}^0$  satisfying the 1-cocycle relation*

$$b(gh) = b(g) + \pi(g)b(h),$$

*and such that*

$$\alpha(g)x = \pi(g)x + b(g)$$

*for all  $g, h \in G$  and  $x \in \mathcal{H}$ .*

**Proof** Let  $\alpha$  be an affine isometric action of  $G$  with linear part  $\pi$ . For every  $g \in G$ , there exists  $b(g) \in \mathcal{H}^0$  such that  $\alpha(g)x = \pi(g)x + b(g)$  for all  $x \in \mathcal{H}$ . It is clear that  $g \mapsto b(g)$  is continuous. Moreover, since  $\alpha$  is a homomorphism,  $b$  satisfies the 1-cocycle relation.

Conversely, let  $b : G \rightarrow \mathcal{H}^0$  be a continuous mapping satisfying the 1-cocycle relation and such that  $\alpha(g)x = \pi(g)x + b(g)$  for all  $g \in G$  and  $x \in \mathcal{H}$ . It is straightforward to verify that  $\alpha$  is an affine isometric action of  $G$ . ■

**Example 2.2.2** Let  $\pi$  be an orthogonal representation of  $G$  on  $\mathcal{H}^0$ . For each  $\xi \in \mathcal{H}^0$ , the mapping

$$b : G \rightarrow \mathcal{H}^0, \quad g \mapsto \pi(g)\xi - \xi$$

is continuous and satisfies the 1-cocycle relation.

**Definition 2.2.3** Let  $\pi$  be an orthogonal representation of the topological group  $G$  on a real Hilbert space  $\mathcal{H}^0$ .

- (i) A continuous mapping  $b : G \rightarrow \mathcal{H}^0$  such that

$$b(gh) = b(g) + \pi(g)b(h), \quad \text{for all } g, h \in G$$

is called a *1-cocycle* with respect to  $\pi$ .

- (ii) A 1-cocycle  $b : G \rightarrow \mathcal{H}^0$  for which there exists  $\xi \in \mathcal{H}^0$  such that

$$b(g) = \pi(g)\xi - \xi, \quad \text{for all } g \in G,$$

is called a *1-coboundary* with respect to  $\pi$ .

- (iii) The space  $Z^1(G, \pi)$  of all 1-cocycles with respect to  $\pi$  is a real vector space under the pointwise operations, and the set  $B^1(G, \pi)$  of all 1-coboundaries is a subspace of  $Z^1(G, \pi)$ . The quotient vector space

$$H^1(G, \pi) = Z^1(G, \pi) / B^1(G, \pi)$$

is called the *first cohomology group* with coefficients in  $\pi$ .

- (iv) Let  $b \in Z^1(G, \pi)$ . The affine isometric action *associated to* a cocycle  $b \in Z^1(G, \pi)$  is the affine isometric action  $\alpha$  of  $G$  on  $\mathcal{H}$  defined by

$$\alpha(g)x = \pi(g)x + b(g), \quad g \in G, x \in \mathcal{H},$$

where  $\mathcal{H}$  is the canonical affine Hilbert space associated to  $\mathcal{H}^0$ .

**Example 2.2.4** Let  $\pi$  be the trivial representation of  $G$  on  $\mathcal{H}^0$ . Then  $B^1(G, \pi)$  is reduced to  $\{0\}$  and  $H^1(G, \pi) = \text{Hom}(G, \mathcal{H}^0)$  is the space of all continuous homomorphisms from  $G$  to the additive group of  $\mathcal{H}^0$ .

**Remark 2.2.5** Let  $\pi$  be an orthogonal representation  $\pi$  of  $G$ . For  $b \in Z^1(G, \pi)$ , we have

- (i)  $b(e) = 0$ ;
- (ii)  $b(g^{-1}) = -\pi(g^{-1})b(g)$ , for all  $g \in G$ .

The proof of these assertions is straightforward (Exercise 2.14.1).

We give a first characterisation of the 1-coboundaries in terms of their associated action.

**Lemma 2.2.6** *Let  $\pi$  be an orthogonal representation of the topological group  $G$  on a real Hilbert space  $\mathcal{H}^0$ . Let  $b \in Z^1(G, \pi)$ , with associated affine isometric action  $\alpha$ . The following properties are equivalent:*

- (i)  $b$  belongs to  $B^1(G, \pi)$ ;
- (ii)  $\alpha$  has a fixed point in  $\mathcal{H}$ ;
- (iii)  $\alpha$  is conjugate to  $\pi$  by some translation, that is, there exists  $\xi \in \mathcal{H}^0$  such that  $\alpha(g)x = \pi(g)(x + \xi) - \xi$  for all  $g \in G$  and  $x \in \mathcal{H}$ .

**Proof** Assume that  $b \in B^1(G, \pi)$ , that is, there exists  $\xi \in \mathcal{H}^0$  such that  $b(g) = \pi(g)\xi - \xi$  for all  $g \in G$ . Then

$$\alpha(g)x = \pi(g)(x + \xi) - \xi$$

for all  $g \in G$ ,  $x \in \mathcal{H}$ . In particular,  $-\xi$  is fixed by  $\alpha$ . This proves that (i) implies (ii).

Assume that  $\alpha$  has a fixed point  $-\xi$ . Then

$$b(g) = \alpha(g)(-\xi) - \pi(g)(-\xi) = \pi(g)\xi - \xi$$

and, hence,

$$\alpha(g)x = \pi(g)(x + \xi) - \xi$$

for all  $g \in G$ ,  $x \in \mathcal{H}$ , showing that (ii) implies (iii).

Finally, if there exists  $\xi \in \mathcal{H}^0$  such that  $\alpha$  is conjugate to  $\pi$  by the translation  $T_\xi$ , then  $b(g) = \pi(g)\xi - \xi$  for all  $g \in G$ . Hence,  $b \in B^1(G, \pi)$ . This shows that (iii) implies (i). ■

We want to improve the previous lemma by showing that the existence of a bounded orbit for the affine action  $\alpha$  already implies the existence of a fixed point. The following lemma is standard (compare with a lemma of Serre in [HarVa–89, Chapitre 3, Lemme 8] and a lemma of Bruhat–Tits [BruTi–72, 2.3.2]). It holds, more generally, for complete metric spaces satisfying suitable assumptions of non-positive curvature (see [HarVa–89, Chapitre 3, Exemples 10]). The proof given here is that of [Glasn–03].

**Lemma 2.2.7 (“Lemma of the centre”)** *Let  $\mathcal{H}$  be a real or complex Hilbert space, and let  $X$  be a non-empty bounded subset of  $\mathcal{H}$ . Among all closed balls in  $\mathcal{H}$  containing  $X$ , there exists a unique one with minimal radius.*

**Proof** Set

$$r = \inf_{y \in \mathcal{H}} \sup_{x \in X} \|x - y\|.$$

Observe that  $r < \infty$ , since  $X$  is bounded. We claim that there exists a closed ball of radius  $r$  containing  $X$ . Indeed, for  $t > r$ , the set

$$C_t = \{y \in \mathcal{H} : \sup_{x \in X} \|x - y\| \leq t\}$$

is non-empty, by definition of  $r$ . Moreover,  $C_t$  is a closed and bounded convex subset of  $\mathcal{H}$ . Hence,  $C_t$  is compact for the weak topology on  $\mathcal{H}$ . As  $C_t \subset C_s$  for  $t \leq s$ , it follows that the set

$$C = \bigcap_{t > r} C_t$$

is non-empty. It is clear that, for any  $y \in C$ , the closed ball with centre  $y$  and radius  $r$  contains  $X$  and that these are exactly the closed balls with minimal radius with this property.

We claim that  $C$  is reduced to a single point. Indeed, let  $y_1, y_2 \in C$ . Then  $\frac{y_1 + y_2}{2} \in C$ , since  $C$  is convex. On the other hand, we have, for every  $x \in X$ ,

$$\left\| x - \frac{y_1 + y_2}{2} \right\|^2 = \frac{1}{2} \|x - y_1\|^2 + \frac{1}{2} \|x - y_2\|^2 - \left\| \frac{y_1 - y_2}{2} \right\|^2,$$

by the parallelogram identity. Therefore,

$$\begin{aligned} \sup_{x \in X} \left\| x - \frac{y_1 + y_2}{2} \right\|^2 &\leq \sup_{x \in X} \frac{1}{2} \|x - y_1\|^2 + \sup_{x \in X} \frac{1}{2} \|x - y_2\|^2 - \left\| \frac{y_1 - y_2}{2} \right\|^2 \\ &= r^2 - \left\| \frac{y_1 - y_2}{2} \right\|^2. \end{aligned}$$

It follows that  $\left\| \frac{y_1 - y_2}{2} \right\|^2 = 0$ , that is,  $y_1 = y_2$ . This concludes the proof. ■

**Definition 2.2.8** The centre of the unique closed ball with minimal radius containing  $X$  in the previous lemma is called the *centre* of  $X$ .

The following theorem is a strengthening of Lemma 2.2.6. The fact that (ii) implies (i) was first proved by B. Johnson, in a more general context and with a different method (see [Johns–67, Theorem 3.4]).

**Proposition 2.2.9** *Let  $\pi$  be an orthogonal representation of the topological group  $G$  on a real Hilbert space  $\mathcal{H}$ . Let  $b \in Z^1(G, \pi)$ , with associated affine isometric action  $\alpha$ . The following properties are equivalent:*

- (i)  $b$  belongs to  $B^1(G, \pi)$ ;
- (ii)  $b$  is bounded;
- (iii) all the orbits of  $\alpha$  are bounded;
- (iv) some orbit of  $\alpha$  is bounded;
- (v)  $\alpha$  has a fixed point in  $\mathcal{H}$ .

**Proof** Properties (ii), (iii) and (iv) are equivalent since, for every  $g \in G$  and  $x \in \mathcal{H}$ ,

$$\alpha(g)x = \pi(g)x + b(g) \quad \text{and} \quad \|\pi(g)x\| = \|x\|.$$

Properties (i) and (v) are equivalent by Lemma 2.2.6. It is obvious that (v) implies (iv). It remains to show that (iv) implies (v). Assume that  $\alpha$  has a bounded orbit  $X$ . Let  $x_0$  be the centre of  $X$ , as in the previous lemma. For every  $g \in G$ , the centre of  $\alpha(g)X$  is  $\alpha(g)x_0$ . Since  $\alpha(g)X = X$ , it follows that  $\alpha(g)x_0 = x_0$ . Hence,  $x_0$  is a fixed point for  $\alpha$ . ■

The following equivalent reformulation of Property (FH) is an immediate consequence of Proposition 2.2.9; one further property will be added in Theorem 2.10.4.

**Proposition 2.2.10** *Let  $G$  be a topological group. The following properties are equivalent:*

- (i)  $G$  has Property (FH);
- (ii)  $H^1(G, \pi) = 0$  for every orthogonal representation  $\pi$  of  $G$ .

Compact groups have Property (FH). Indeed, every affine isometric action of such a group has bounded orbits and hence fixed points, by Proposition 2.2.9.

We introduced in Definition 1.4.3 Property (T) for a pair  $(G, H)$  consisting of a group  $G$  and a subgroup  $H$ . Similarly, we define Property (FH) for the pair  $(G, H)$ .

**Definition 2.2.11** Let  $H$  be a closed subgroup of the topological group  $G$ . The pair  $(G, H)$  has *Property (FH)* if every affine isometric action of  $G$  on a real Hilbert space has an  $H$ -fixed point.

As in the previous proposition, Property (FH) for the pair  $(G, H)$  can be reformulated in terms of 1-cohomology:  $(G, H)$  has Property (FH) if and only if, for every orthogonal representation  $\pi$  of  $G$ , the restriction mapping  $\text{Res}_G^H : H^1(G, \pi) \rightarrow H^1(H, \pi|_H)$  is the zero mapping (Exercise 2.14.7).

**Remark 2.2.12** The definition of  $H^1(G, \pi)$  carries over with the obvious modifications to the case of a unitary representation  $\pi$  of a topological group  $G$  on a complex Hilbert space.

## 2.3 Actions on trees

In this section, we discuss actions of groups with Property (FH) on trees, relate this property to Serre's Property (FA) and draw some consequences.

### Constructing affine isometric actions

We describe two general procedures to construct affine isometric actions of groups. The first procedure will be applied to actions on trees in this section as well as to actions on hyperbolic spaces in Section 2.6. The second procedure is described in Proposition 2.4.5.

Let  $G$  be a topological group. Assume we are given

- an action of  $G$  by homeomorphisms on a topological space  $X$ ,
- an orthogonal representation  $\pi$  of  $G$  on a real Hilbert space  $\mathcal{H}$ , and
- a continuous mapping  $c : X \times X \rightarrow \mathcal{H}$  satisfying the two conditions:

$$c(x, y) + c(y, z) = c(x, z) \text{ for all } x, y, z \in X \text{ (Chasles' relation)}$$

$$\pi(g)c(x, y) = c(gx, gy) \text{ for all } g \in G \text{ and } x, y \in X \text{ (} G\text{-equivariance).}$$

From this data, we construct a family of affine actions of  $G$  on  $\mathcal{H}$ .

**Proposition 2.3.1** *Let the notation be as above.*

(i) *For any  $x \in X$ , the mapping*

$$b_x : G \rightarrow \mathcal{H}, \quad g \mapsto c(gx, x)$$

*belongs to  $Z^1(G, \pi)$ .*

(ii) *For  $x_0$  and  $x_1$  in  $X$ , the affine isometric actions  $\alpha_{x_0}$  and  $\alpha_{x_1}$  of  $G$  with linear part  $\pi$  associated to  $b_{x_0}$  and  $b_{x_1}$  are conjugate under the translation  $T_{c(x_0, x_1)}$  given by  $c(x_0, x_1)$ . In particular, the cocycles  $b_{x_0}$  and  $b_{x_1}$  define the same class in  $H^1(G, \pi)$ .*

**Proof** (i) Since  $c$  is continuous,  $b_x$  is continuous. For  $g, h \in G$ , we have

$$\begin{aligned} b_x(gh) &= c(ghx, x) = c(ghx, gx) + c(gx, x) \\ &= \pi(g)c(hx, x) + c(gx, x) = \pi(g)b_x(h) + b_x(g), \end{aligned}$$

by Chasles' relation and the equivariance of  $c$ . Hence,  $b$  satisfies the 1-cocycle relation.

(ii) We have

$$\begin{aligned} T_{c(x_0, x_1)}\alpha_{x_0}(g)T_{c(x_0, x_1)}^{-1}\xi &= \pi(g)(\xi - c(x_0, x_1)) + c(gx_0, x_0) + c(x_0, x_1) \\ &= \pi(g)\xi - c(gx_0, gx_1) + c(gx_0, x_1) \\ &= \pi(g)\xi + c(gx_1, x_1) \\ &= \alpha_{x_1}(g)\xi \end{aligned}$$

for all  $g \in G$  and  $\xi \in \mathcal{H}$ . ■

### Actions on trees and Property (FA)

Let  $X = (V, \mathbb{E})$  be a graph. Our notation involves the set  $V$  of *vertices* and the set  $\mathbb{E}$  of *oriented edges* of  $X$ ; each  $e \in \mathbb{E}$  has a source  $s(e) \in V$  and a range  $r(e) \in V$ . There is a fixed-point free involution  $e \mapsto \bar{e}$  on  $\mathbb{E}$ , with  $s(\bar{e}) = r(e)$  and  $r(\bar{e}) = s(e)$  for all  $e \in \mathbb{E}$ . The set of all pairs  $\{e, \bar{e}\}$  is the set  $E$  of *geometric edges* of  $X$ ; if  $\mathbb{E}$  is finite, then  $\#\mathbb{E} = 2\#E$ .

Two vertices  $x, y \in V$  are *adjacent* if there exists  $e \in \mathbb{E}$  with  $x = s(e)$  and  $y = r(e)$ . For  $x, y \in V$ , we define the *distance*  $d(x, y)$  to be the smallest integer  $n$  (if it exists) such that there are pairs  $(x_0 = x, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n = y)$  of adjacent vertices. The graph  $X$  is *connected* if any two of its vertices are at a finite distance from each other; in this case  $d$  defines a distance on  $V$ . Graphs

which appear below have no loop (that is,  $s(e) \neq r(e)$  for all  $e \in \mathbb{E}$ ) and are *simple* (that is, for  $x, y \in V$ , there is at most one edge  $e \in \mathbb{E}$  with  $s(e) = x$  and  $r(e) = y$ ).

An *automorphism* of a connected graph  $X = (V, \mathbb{E})$  is an isometry of its vertex set, for the distance defined above. Endowed with the topology of pointwise convergence, the set  $\text{Aut}(X)$  of automorphisms of  $X$  is a topological group. An *action* of a topological group  $G$  on  $X$  is a continuous homomorphism  $G \rightarrow \text{Aut}(X)$ ; equivalently, it is an isometric action of  $G$  on  $V$  such that the mapping  $G \rightarrow V$ ,  $g \mapsto gx$  is continuous for all  $x \in V$ .

A *cycle* in a graph  $X = (V, \mathbb{E})$  is a sequence  $(e_1, e_2, \dots, e_n)$  of distinct edges such that  $r(e_i) = s(e_{i+1})$ ,  $e_{i+1} \neq \bar{e}_i$  for  $i \in \{1, n-1\}$  and  $r(e_n) = s(e_1)$ . A *tree* is a simple connected graph without cycle.

We will need the following standard lemma (see [Serre-77, No I.2.2, Corollaire to Proposition 10]).

**Lemma 2.3.2** *Let  $G$  be a group acting on a tree  $X$ . If  $G$  has a bounded orbit, then  $G$  fixes either a vertex or a geometric edge of  $X$ .*

**Proof** Let  $O$  be a bounded  $G$ -orbit in  $X$ . Consider the convex hull  $X_0 \subset X$  of  $O$ . Then  $X_0$  is a bounded subtree and is  $G$ -invariant. Let  $N \in \mathbf{N}$  be the diameter of  $X_0$ . We define inductively a sequence of subtrees  $X_j$  for  $j = 1, 2, \dots, M$ , where  $M$  is the integer part of  $\frac{N-1}{2}$  in the following way. If we assume that  $X_{j-1}$  is defined,  $X_j$  is obtained by removing from  $X_{j-1}$  every vertex which is adjacent to exactly one vertex and by removing the corresponding edge. It is clear that  $X_j$  is  $G$ -invariant and that the diameter of  $X_j$  is  $N - 2j$  for  $j = 1, 2, \dots, M$ . Hence,  $X_M$  has 1 or 2 vertices and this finishes the proof. ■

Let  $G$  be a topological group acting on a graph  $X = (V, \mathbb{E})$ . Let  $\mathcal{H}$  denote the Hilbert space of functions  $\xi : \mathbb{E} \rightarrow \mathbf{R}$  such that

$$\xi(\bar{e}) = -\xi(e), \quad \text{for all } e \in \mathbb{E}$$

and  $\sum_{e \in \mathbb{E}} |\xi(e)|^2 < \infty$ , with the inner product defined by

$$\langle \xi, \eta \rangle = \frac{1}{2} \sum_{e \in \mathbb{E}} \xi(e) \eta(e).$$

The action of  $G$  on  $X$  involves an action of  $G$  on  $\mathbb{E}$  and induces an orthogonal representation of  $G$  on  $\mathcal{H}$ , which will be denoted by  $\pi_X$ .

Let  $X = (V, \mathbb{E})$  be a tree. Given two vertices  $x$  and  $y$  in  $V$ , there exists a unique sequence of vertices  $x_0 = x, x_1, \dots, x_n = y$  such that  $d(x, x_j) = j$  for all  $j = 1, \dots, n$ . We denote by  $[x, y]$  the corresponding *interval* which is the set of



edges  $e \in \mathbb{E}$  with source and range in  $\{x_0, x_1, \dots, x_n\}$ . An edge  $e \in [x, y]$  *points from  $x$  to  $y$*  if  $d(s(e), x) = d(r(e), x) - 1$ , and points from  $y$  to  $x$  otherwise. For a vertex  $z$ , we write  $z \in [x, y]$  if  $z$  is the source of some  $e \in [x, y]$ .

Define a mapping  $c : V \times V \rightarrow \mathcal{H}$  by

$$c(x, y)(e) = \begin{cases} 0 & \text{if } e \text{ is not on } [x, y]; \\ 1 & \text{if } e \text{ is on } [x, y] \text{ and points from } x \text{ to } y; \\ -1 & \text{if } e \text{ is on } [x, y] \text{ and points from } y \text{ to } x. \end{cases}$$

It is clear that  $c$  satisfies the  $G$ -equivariance condition

$$\pi_X(g)c(x, y) = c(gx, gy), \quad \text{for all } g \in G, x, y \in V.$$

Moreover,  $c$  satisfies Chasles' relation  $c(x, y) + c(y, z) = c(x, z)$  for  $x, y, z \in V$ . Indeed, due to the shape of triangles in a tree, there is a unique vertex  $t$  lying simultaneously on  $[x, y]$ ,  $[y, z]$  and  $[x, z]$  (see Figure 2.1).

Then the contributions of the edges in  $[t, y]$  cancel out in  $c(x, y)$  and  $c(y, z)$ , that is,

$$c(x, y)(e) + c(y, z)(e) = 0$$

for every edge  $e$  in  $[t, y]$ . Hence,  $c(x, y) + c(y, z)$  and  $c(x, z)$  agree on edges in  $[t, y]$ . They also agree on edges which are not in  $[t, y]$ . This shows that  $c(x, y) + c(y, z) = c(x, z)$ .

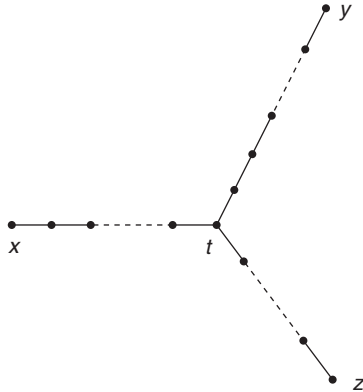


Figure 2.1 A triangle in a tree

Observe that, for all  $x, y \in V$ , we have

$$\|c(x, y)\|^2 = \frac{1}{2} \sum_{e \in \mathbb{E}} c(x, y)(e)^2 = d(x, y).$$

**Proposition 2.3.3** *Let  $G$  be a topological group acting on a tree  $X$ . Assume that  $G$  fixes no vertex and no geometric edge of  $X$ . Then, with the notation above,  $H^1(G, \pi_X) \neq 0$ . In particular,  $G$  does not have Property (FH).*

**Proof** Fix a base vertex  $x_0 \in V$ . By Proposition 2.3.1, the mapping

$$G \rightarrow \mathcal{H}, \quad g \mapsto c(gx_0, x_0)$$

is a 1-cocycle with respect to  $\pi_X$ .

Assume, by contradiction, that  $H^1(G, \pi_X) = 0$ . It follows from Proposition 2.2.9 that

$$g \mapsto \|c(gx_0, x_0)\|^2 = d(gx_0, x_0)$$

is a bounded function on  $G$ . In other words,  $G$  has a bounded orbit in  $V$ . By Lemma 2.3.2,  $G$  fixes either a vertex or a geometric edge of  $X$ . This is a contradiction to the hypothesis. ■

**Definition 2.3.4 (Serre)** A topological group  $G$  has *Property (FA)* if every action of  $G$  on a tree has either a fixed vertex or a fixed geometric edge.

The acronym (FA) stands for “points fixes sur les arbres”.

**Remark 2.3.5** It is proved in [Serre–77, No I.6.1] that a countable group  $G$  has Property (FA) if and only if the following three conditions are satisfied:

- (i)  $G$  is finitely generated;
- (ii)  $G$  has no quotient isomorphic to  $\mathbf{Z}$ ;
- (iii)  $G$  is not a non-trivial amalgamated product.

Observe that, if (i) is satisfied, then the abelianised group  $G/[G, G]$  is a finitely generated abelian group and, hence, a direct sum of finitely many cyclic groups. Therefore, we can replace condition (ii) above by the condition

(ii)' the abelianised group  $G/[G, G]$  is finite.

The following result is an immediate consequence of Proposition 2.3.3. It was proved by Y. Watatani [Watat–82]; it was previously shown by G. Margulis [Margu–81] that higher rank lattices have Property (FA).

**Theorem 2.3.6** *Any topological group with Property (FH) has Property (FA).*

**Remark 2.3.7** (i) There exist discrete groups with Property (FA) which do not have Property (FH). An example is given by *Schwarz' group*  $G$  with presentation

$$\langle x, y | x^a = y^b = (xy)^c = e \rangle,$$

where  $a, b, c \geq 2$  are integers. Such a group has Property (FA), by [Serre–77, No I.6.3.5]. If  $G$  is infinite or, equivalently, if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1,$$

then  $G$  contains a subgroup of finite index isomorphic to the fundamental group of an oriented surface of genus  $\geq 1$ . Such a subgroup has a quotient isomorphic to  $\mathbf{Z}$  (Example 1.3.7) and so does not have Property (FH) by Example 2.2.4. Since Property (FH) is inherited by finite index subgroups (Remark 2.5.8 below), it follows that  $G$  does not have Property (FH).

(ii) R. Alperin [Alper–82] proved the following strengthening of Theorem 2.3.6: let  $G$  be a separable locally compact group with Property (T). Then  $G$ , viewed as a *discrete* group, has Property (FA).

## 2.4 Consequences of Property (FH)

We draw some consequences for the structure of groups with Property (FH).

**Proposition 2.4.1** *Let  $G$  be a topological group which is the union of a strictly increasing sequence of open subgroups  $H_n$ ,  $n \in \mathbf{N}$ . Let  $\pi_n$  be the quasi-regular representation of  $G$  on  $\ell_{\mathbf{R}}^2(G/H_n)$ . Then*

$$H^1(G, \bigoplus_{n \in \mathbf{N}} \pi_n) \neq 0.$$

*In particular,  $G$  does not have Property (FH).*

**Proof** We define a graph  $X$  with set of vertices

$$V = \bigsqcup_{n \in \mathbf{N}} G/H_n,$$

the disjoint union of the  $G/H_n$ ,  $n \in \mathbf{N}$ , and with set of edges

$$\mathbb{E} = \{(gH_n, gH_{n+1}), (gH_{n+1}, gH_n) : n \in \mathbf{N}, g \in G\}.$$

The graph  $X$  is connected. Indeed, given  $gH_m, hH_n \in V$ , there exists  $k \geq m, n$  such that  $g^{-1}h \in H_k$ , and hence,  $gH_m$  and  $hH_m$  are connected to  $gH_k = hH_k$ .

For every  $gH_n \in V$ , the vertex  $gH_{n+1}$  is the unique vertex in  $G/H_{n+1}$  which is adjacent to  $gH_n$ . This implies that  $X$  is a tree.

The action of  $G$  by left translations on  $G/H_n$  induces an action by automorphisms of  $X$ . Since  $(H_n)_{n \in \mathbb{N}}$  is strictly increasing,  $G$  fixes no vertex and no edge of  $X$ . Hence, with the notation of Proposition 2.3.3,  $H^1(G, \pi_X) \neq 0$ .

Now,  $\pi_X$  and  $\bigoplus_{n \in \mathbb{N}} \pi_n$  are orthogonally equivalent. Indeed, the  $G$ -equivariant mapping

$$V \rightarrow \mathbb{E}, gH_n \mapsto (gH_n, gH_{n+1})$$

induces a  $G$ -equivariant orthogonal bijective linear mapping

$$\bigoplus_{n \in \mathbb{N}} \ell_{\mathbb{R}}^2(G/H_n) \rightarrow \mathcal{H},$$

where  $\mathcal{H}$  is the Hilbert space of  $\pi_X$ . ■

**Corollary 2.4.2** *Let  $G$  be a  $\sigma$ -compact locally compact group with Property (FH). Then  $G$  is compactly generated.*

**Proof** Since  $G$  is a  $\sigma$ -compact locally compact group, there exists a sequence  $(K_n)_n$  of compact neighbourhoods of  $e$  in  $G$  such that  $G = \bigcup_n K_n$ . The subgroup  $H_n$  generated by  $K_n$  is an open subgroup in  $G$ , and, of course,  $G = \bigcup_n H_n$ . By the previous proposition,  $G = H_n$  for some  $n$ . ■

**Remark 2.4.3** (i) In particular, a countable group which has Property (FH) is finitely generated. There are examples of uncountable groups (with the discrete topology) which have Property (FH), such as the group of all permutations of an infinite set (see [Cornu–06c]).

(ii) Corollary 2.4.2 can also be deduced from the Delorme–Guichardet Theorem 2.12.4 and the analogous result for groups with Property (T) in Theorem 1.3.1.

We record for later use the following consequence of Proposition 2.4.1. Recall that the *direct sum*

$$G = \bigoplus_{i \in I} G_i,$$

of a family of groups  $(G_i)_{i \in I}$  is the subgroup of the direct product  $\prod_{i \in I} G_i$  consisting of the families  $(g_i)_{i \in I}$  with  $g_i = e_i$  for all but finitely many indices  $i$ , where  $e_i$  denotes the neutral element of  $G_i$ .

**Corollary 2.4.4** *Let  $G = \bigoplus_{i \in I} G_i$  be a direct sum of an infinite family of groups  $(G_i)_{i \in I}$ , with no  $G_i$  reduced to one element. Then  $G$  does not have Property (FH).*

**Proof** Since  $I$  is infinite, there exists a mapping  $\varphi : I \rightarrow \mathbf{N}$  which is onto. For every  $n \in \mathbf{N}$ , let  $I_n = \varphi^{-1}\{1, \dots, n\}$  and  $H_n = \bigoplus_{i \in I_n} G_i$ . Then  $(H_n)_{n \in \mathbf{N}}$  is a strictly increasing sequence of subgroups and  $G = \bigcup_{n \in \mathbf{N}} H_n$ . Proposition 2.4.1 shows that  $G$  does not have Property (FH). ■

For another proof of the previous corollary, see Proposition 2.8.1.

Almost invariant vectors can be used to construct affine isometric actions. We are going to apply this procedure to show that Property (FH) implies Property (T) for  $\sigma$ -compact locally compact groups. For the converse, see Section 2.12.

Let  $\pi$  be an orthogonal representation of the topological group  $G$  on a real Hilbert space  $\mathcal{H}$ . As in the case of unitary representations (Definition 1.1.1), we say that  $\pi$  *almost has invariant vectors* if, for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists a unit vector  $\xi \in \mathcal{H}$  which is  $(Q, \varepsilon)$ -invariant, that is, such that

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| < \varepsilon.$$

Denote by  $\infty\pi$  an infinite countable multiple of  $\pi$ , that is, the representation  $\pi \oplus \pi \oplus \dots$  on  $\mathcal{H} \oplus \mathcal{H} \oplus \dots$ .

**Proposition 2.4.5** *Let  $G$  be a  $\sigma$ -compact locally compact group, and let  $\pi$  be an orthogonal representation of  $G$  on a real Hilbert space  $\mathcal{H}$  without non-zero invariant vectors. If  $\pi$  almost has invariant vectors, then  $H^1(G, \infty\pi) \neq 0$ . In particular,  $G$  does not have Property (FH).*

**Proof** Let  $(Q_n)_n$  be an increasing sequence of compact subsets of  $G$  such that  $G = \bigcup_n Q_n$ . Since  $\pi$  almost has invariant vectors, there exists, for every  $n$ , a unit vector  $\xi_n \in \mathcal{H}$  such that

$$\|\pi(g)\xi_n - \xi_n\|^2 < 1/2^n, \quad \text{for all } g \in Q_n.$$

For each  $g \in G$ , set

$$b(g) = \bigoplus_{n \in \mathbf{N}} n(\pi(g)\xi_n - \xi_n).$$

We claim that  $b(g)$  belongs to the Hilbert space  $\mathcal{H} \oplus \mathcal{H} \oplus \dots$ . Indeed, for fixed  $g \in G$ , there exists  $N$  such that  $g \in Q_N$  and, hence,  $\|\pi(g)\xi_n - \xi_n\|^2 < 1/2^n$  for all  $n \geq N$ . It follows that

$$\sum_{n=N}^{\infty} n^2 \|\pi(g)\xi_n - \xi_n\|^2 \leq \sum_{n=N}^{\infty} n^2 / 2^n < \infty,$$

and  $b(g)$  is a well defined element of  $\mathcal{H} \oplus \mathcal{H} \oplus \cdots$ . The same argument shows that the series  $\sum_{n=1}^{\infty} n^2 \|\pi(g)\xi_n - \xi_n\|^2$  is uniformly convergent on compact subsets of  $G$ , so that the mapping

$$b : G \rightarrow \mathcal{H} \oplus \mathcal{H} \oplus \cdots$$

is continuous. Moreover, it is clear that  $b$  satisfies the 1-cocycle relation.

We claim that  $b$  is unbounded. Indeed, since  $\pi$  has no non-zero invariant vectors, we can find, for every  $n \in \mathbf{N}$ , an element  $g_n \in G$  such that

$$\|\pi(g_n)\xi_n - \xi_n\| \geq 1$$

(see Proposition 1.1.5). Hence,

$$\|b(g_n)\| \geq n \|\pi(g_n)\xi_n - \xi_n\| \geq n.$$

This shows that  $b$  is not bounded, so that  $b \notin B^1(G, \infty\pi)$ . ■

**Remark 2.4.6** Let  $\pi$  be a unitary representation of the topological group  $G$  on the complex Hilbert space  $\mathcal{H}$ . Let  $\mathcal{H}_{\mathbf{R}}$  be the space  $\mathcal{H}$ , viewed as a real Hilbert space with the inner product

$$(\xi, \eta) \mapsto \operatorname{Re}\langle \xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

The representation  $\pi$ , considered as acting on  $\mathcal{H}_{\mathbf{R}}$ , becomes an orthogonal representation  $\pi_{\mathbf{R}}$ . It is obvious that  $\pi$  almost has invariant vectors if and only if  $\pi_{\mathbf{R}}$  almost has invariant vectors.

The following corollary is a consequence of Proposition 2.4.5 and the previous remark. It is due to Guichardet [Guic–72a, Théorème 1]; see also Theorem 2.12.4.ii below.

**Corollary 2.4.7** *Any  $\sigma$ -compact locally compact group with Property (FH) has Property (T).*

## 2.5 Hereditary properties

We investigate the behaviour of Property (FH) under quotients and extensions, as well as for appropriate subgroups and for abelian groups.

In analogy with Theorem 1.3.4, we have the following proposition.

**Proposition 2.5.1** *Let  $G_1$  and  $G_2$  be topological groups, and let  $\varphi : G_1 \rightarrow G_2$  be a continuous homomorphism with dense image. If  $G_1$  has Property (FH), then  $G_2$  has Property (FH).*

*In particular, Property (FH) is inherited by quotients: if  $G_1$  has Property (FH), then so does  $G_1/N$  for every closed normal subgroup  $N$  of  $G_1$ .*

**Proof** Let  $\alpha$  be an affine isometric action of  $G_2$  on a real Hilbert space  $\mathcal{H}$ . Then  $\alpha \circ \varphi$  is an affine isometric action of  $G_1$ . By assumption,  $\alpha \circ \varphi$  has a  $G_1$ -fixed point  $\xi \in \mathcal{H}$ . As the image of  $\varphi$  is dense,  $\xi$  is  $G_2$ -fixed. ■

Using Corollary 2.4.7 as well as Theorem 1.1.6, we obtain the following corollary.

**Corollary 2.5.2** *Let  $G_1$  be a topological group with Property (FH), and let  $G_2$  be an amenable locally compact group which is  $\sigma$ -compact. Every continuous homomorphism  $\varphi : G_1 \rightarrow G_2$  has a relatively compact image.*

*In particular, every continuous homomorphism  $\varphi : G_1 \rightarrow \mathbf{R}^n$  or  $\varphi : G_1 \rightarrow \mathbf{Z}^n$  is constant. If, moreover,  $G_1$  is locally compact, then  $G_1$  is unimodular.*

**Remark 2.5.3** (i) It has been shown by Alperin [Alper–82, Lemma] that, if  $G$  is a separable locally compact group, then every homomorphism  $G \rightarrow \mathbf{Z}^n$  is continuous.

(ii) The fact that  $\text{Hom}(G, \mathbf{R}^n) = 0$  for a topological group  $G$  with Property (FH) can be proved in a direct way: let  $\pi$  be the unit representation of  $G$  on the real Hilbert space  $\mathbf{R}^n$ . Then  $H^1(G, \pi) = \text{Hom}(G, \mathbf{R}^n)$ ; see Example 2.2.4.

The following proposition is to be compared with the analogous result for Property (T) in Proposition 1.7.6.

**Proposition 2.5.4** *Let  $G$  be a topological group, and let  $N$  be a closed normal subgroup of  $G$ . If  $N$  and  $G/N$  have Property (FH), then so does  $G$ .*

**Proof** Let  $\alpha$  be an affine isometric action of  $G$  on the real Hilbert space  $\mathcal{H}$ . Since  $N$  has Property (FH), the set  $\mathcal{H}^N$  of  $\alpha(N)$ -fixed points is non-empty. As  $N$  is normal in  $G$ , the subspace  $\mathcal{H}^N$  is  $\alpha(G)$ -invariant. It is clear that the action of  $G$  on  $\mathcal{H}^N$  factorises through  $G/N$ . Since  $G/N$  has Property (FH), there exists a point in  $\mathcal{H}^N$  which is fixed under  $G/N$  and, hence, under  $G$ . ■

We have seen in Theorem 1.7.1 that a closed subgroup  $H$  with finite covolume in a locally compact group  $G$  has Property (T) if and only if  $G$  has Property (T). In the next two propositions, we show that the “only if” part of the corresponding statement for Property (FH) remains true for an arbitrary topological group  $G$  and that the “if” part is true if  $H$  has finite index.

**Proposition 2.5.5** *Let  $G$  be a topological group, and let  $H$  be a closed subgroup. Assume that the homogeneous space  $G/H$  carries a  $G$ -invariant probability measure  $\mu$  defined on the Borel subsets of  $G/H$ . Then:*

- (i) *if an affine isometric action  $\alpha$  of  $G$  on a real Hilbert space  $\mathcal{H}$  has a  $H$ -fixed point, then it has a  $G$ -fixed point;*
- (ii) *if  $\pi$  is an orthogonal representation of  $G$ , then the restriction mapping  $\text{Res}_G^H : H^1(G, \pi) \rightarrow H^1(H, \pi|_H)$  is injective;*
- (iii) *if  $H$  has Property (FH), then so does  $G$ .*

**Proof** It suffices to prove (i).

Let  $\xi_0 \in \mathcal{H}$  be a fixed point under  $\alpha(H)$ . Consider the continuous mapping

$$\Phi : G \rightarrow \mathcal{H}, \quad g \mapsto \alpha(g)\xi_0.$$

Since  $\Phi(gh) = \Phi(g)$  for all  $g \in G$  and  $h \in H$ , this mapping factorises to a continuous mapping, still denoted by  $\Phi$ , from  $G/H$  to  $\mathcal{H}$ . Let  $\nu = \Phi_*(\mu)$  be the image of  $\mu$  under  $\Phi$ . This is an  $\alpha(G)$ -invariant probability measure on the Borel subsets of  $\mathcal{H}$ .

For  $R > 0$ , let  $B_R$  denote the closed ball in  $\mathcal{H}$  of radius  $R$  centred at  $\xi_0$ . Since  $\nu$  is a probability measure,

$$\lim_{R \rightarrow \infty} \nu(B_R) = 1.$$

Hence, there exists  $R_0$  such that  $\nu(B_{R_0}) > 1/2$ . For every  $g \in G$ , we have

$$\alpha(g)B_{R_0} \cap B_{R_0} \neq \emptyset.$$

Indeed, otherwise, we would have

$$\begin{aligned} 1 = \nu(\mathcal{H}) &\geq \nu(\alpha(g)B_{R_0} \cup B_{R_0}) \\ &= \nu(\alpha(g)B_{R_0}) + \nu(B_{R_0}) \\ &= 2\nu(B_{R_0}) > 1, \end{aligned}$$

a contradiction.

Now,  $\alpha(g)B_{R_0}$  is the closed ball of radius  $R_0$  centred at  $\alpha(g)\xi_0$ . We deduce that

$$\|\alpha(g)\xi_0 - \xi_0\| \leq 2R_0, \quad \text{for all } g \in G.$$

Therefore, the orbit of  $\xi_0$  under  $\alpha(G)$  is bounded. By Proposition 2.2.9, it follows that  $\alpha(G)$  has a fixed point. ■



Next, we show that Property (FH) is inherited by subgroups of finite index. For this, we introduce the notion of an induced affine action, which is analogous to the notion of an induced unitary representation (see Chapter E).

Let  $G$  be a topological group, and let  $H$  be a closed subgroup of  $G$ . We assume that  $H$  is of finite index in  $G$ . Let  $\alpha$  be an affine isometric action of  $H$  on a real Hilbert space  $\mathcal{K}$ . Let  $\mathcal{H}$  be the space of all mappings  $\xi : G \rightarrow \mathcal{K}$  such that

$$\xi(gh) = \alpha(h^{-1})\xi(g) \quad \text{for all } h \in H, g \in G.$$

This is an affine subspace of the space of all mappings  $G \rightarrow \mathcal{K}$ . Fix  $\eta_0 \in \mathcal{K}$ , and let  $\xi_0 : G \rightarrow \mathcal{K}$  be defined by

$$\xi_0(g) = \begin{cases} \alpha(g^{-1})\eta_0 & \text{if } g \in H \\ 0 & \text{if } g \notin H. \end{cases}$$

Observe that  $\xi_0$  belongs to  $\mathcal{H}$ . If  $\xi$  and  $\xi'$  are two elements in  $\mathcal{H}$ , then the function

$$x \mapsto \langle \xi(x) - \xi_0(x), \xi'(x) - \xi_0(x) \rangle$$

is invariant under right translations by elements of  $H$ , so that it factorises to a function on  $G/H$ . We define a Hilbert space structure on the linear space  $\mathcal{H}^0 = \{\xi - \xi_0 : \xi \in \mathcal{H}\}$  by

$$\langle \xi - \xi_0, \xi' - \xi_0 \rangle = \sum_{x \in G/H} \langle \xi(x) - \xi_0(x), \xi'(x) - \xi_0(x) \rangle, \quad \xi, \xi' \in \mathcal{H}.$$

In this way,  $\mathcal{H}$  becomes a real affine Hilbert space. The *induced* affine isometric action  $\text{Ind}_H^G \alpha$  of  $G$  on  $\mathcal{H}$  is defined by

$$(\text{Ind}_H^G \alpha(g)\xi)(x) = \xi(g^{-1}x), \quad g, x \in G, \quad \xi \in \mathcal{H}.$$

Observe that  $\text{Ind}_H^G \alpha$  is indeed isometric, since

$$\begin{aligned} \|\text{Ind}_H^G \alpha(g)\xi - \text{Ind}_H^G \alpha(g)\xi'\|^2 &= \sum_{x \in G/H} \|\xi(g^{-1}x) - \xi'(g^{-1}x)\|^2 \\ &= \sum_{x \in G/H} \|\xi(x) - \xi'(x)\|^2 = \|\xi - \xi'\|^2, \end{aligned}$$

for every  $g \in G$ . Observe also that  $\text{Ind}_H^G \alpha$  is continuous, since  $H$  is open in  $G$ .

**Remark 2.5.6** When  $G$  is a locally compact group, induced affine isometric actions can be defined if  $H$  is a cocompact lattice in  $G$  (see [Shal–00b, Section 3.III]).

**Proposition 2.5.7** *Let  $G$  be a topological group and let  $H$  be a closed subgroup of  $G$  with finite index. If  $G$  has Property (FH), then  $H$  has Property (FH).*

**Proof** Let  $\alpha$  be an affine isometric action of  $H$  on  $\mathcal{K}$ . Since  $G$  has Property (FH), the induced affine isometric action  $\text{Ind}_H^G \alpha$  has a fixed point  $\xi$  in the affine Hilbert space  $\mathcal{H}$  of  $\text{Ind}_H^G \alpha$ . Thus, we have

$$\xi(g^{-1}x) = \xi(x), \quad \text{for all } g, x \in G.$$

In particular,  $\xi(h^{-1}) = \xi(e)$  for all  $h \in H$ . Since  $\xi(h^{-1}) = \alpha(h)\xi(e)$ , it follows that  $\xi(e)$  is a fixed point for  $\alpha$ . ■

**Remark 2.5.8** Assume that  $G$  is a  $\sigma$ -compact locally compact group. By the Delorme–Guichardet Theorem 2.12.4 and Theorem 1.7.1, the previous proposition holds for every closed subgroup of finite covolume in  $G$ , that is, Property (FH) is inherited by any such subgroup.

It is a particular case of Corollary 2.5.2 that a  $\sigma$ -compact locally compact amenable group with Property (FH) is compact. Using some structure theory of abelian groups, we show that this result remains true for soluble groups without the  $\sigma$ -compactness assumption.

**Proposition 2.5.9** *Let  $G$  be a locally compact soluble group with Property (FH). Then  $G$  is compact.*

**Proof** The proof proceeds in two steps.

• *First step:* Let  $G$  be a discrete abelian group with Property (FH). We claim that  $G$  is finite.

Let  $H$  be a maximal free abelian subgroup of  $G$ . Then  $G_1 = G/H$  is an abelian torsion group and has Property (FH). For each  $n \in \mathbb{N}$ , set

$$T_n = \{x \in G_1 : x^{n!} = e\}.$$

Then  $(T_n)_{n \in \mathbb{N}}$  is a non-decreasing family of subgroups of  $G_1$  and  $G_1 = \bigcup_{n \in \mathbb{N}} T_n$ . By Proposition 2.4.1, it follows that  $G_1 = T_n$  for some  $n \in \mathbb{N}$ , that is,  $G_1$  is a periodic abelian group. By the structure theory of such groups,  $G_1$  is a direct sum of finite cyclic groups (see, e.g., [HewRo–63, Appendix, Theorem A.5]). Since  $G_1$  has Property (FH), it follows from Corollary 2.4.4

that  $G_1$  is finite, that is,  $H$  is of finite index in  $G$ . Hence,  $H$  has Property (FH) by Proposition 2.5.7. This implies that  $H = \{e\}$ . Indeed, a non-trivial free abelian group has  $\mathbf{Z}$  as quotient and therefore does not have Property (FH). Consequently,  $G = G_1$  is finite, as claimed.

• *Second step:* Let  $G$  be a locally compact soluble group with Property (FH). We claim that  $G$  is compact. By Corollary 2.5.2, it suffices to show that  $G$  is  $\sigma$ -compact.

Since  $G$  is soluble, there exists a sequence of *closed* normal subgroups

$$\{e\} = G_n \subset G_{n-1} \subset \cdots \subset G_1 \subset G_0 = G$$

with abelian quotients  $G_i/G_{i+1}$  for  $i = 0, \dots, n-1$ ; see [Bou–GLie, Chapitre III, §9, Corollaire 1].

Let  $Q$  be a compact neighbourhood of  $e$  in  $G$  and let  $H$  be the subgroup generated by  $Q$ ; notice that  $H$  is  $\sigma$ -compact. It is clear that  $HG_{i+1}$  is an open normal subgroup of  $HG_i$  with abelian quotient, for every  $i \in \{0, \dots, n-1\}$ . In particular,  $HG_1$  is an open normal subgroup of  $G$  and the quotient  $G/HG_1$  is a discrete abelian group with Property (FH). Hence  $HG_1$  has finite index in  $G$ , by the first step. Consequently,  $HG_1$  has Property (FH), so that the discrete abelian group  $HG_1/HG_2$  has Property (FH). As before, this implies that  $HG_1/HG_2$  is finite and that  $HG_2$  has Property (FH). Continuing this way, we see that  $HG_i/HG_{i+1}$  is finite for every  $i = 0, \dots, n-1$ . It follows that  $G/H$  is finite and  $G$  is  $\sigma$ -compact. ■

The following result is an immediate corollary of the previous proposition. In the case where  $G$  is  $\sigma$ -compact, it is also a consequence of Corollary 2.5.2. For the analogous result concerning groups with Property (T), see Corollary 1.3.6.

**Corollary 2.5.10** *Let  $G$  be a locally compact group with Property (FH). Then the Hausdorff abelianised group  $G/[\overline{G}, \overline{G}]$  is compact.*

## 2.6 Actions on real hyperbolic spaces

Let  $G = O(n, 1)$  be the subgroup of  $GL_{n+1}(\mathbf{R})$  preserving the quadratic form

$$\langle x, y \rangle = -x_{n+1}y_{n+1} + \sum_{i=1}^n x_i y_i$$

on  $\mathbf{R}^{n+1}$ . Let  $\mathbf{H}^n(\mathbf{R})$  be the *real  $n$ -dimensional hyperbolic space*, that is, the open subset of the projective space  $\mathbf{P}^n(\mathbf{R})$  which is the image of the set

$$\{x \in \mathbf{R}^{n+1} : \langle x, x \rangle < 0\}.$$

For  $x \in \mathbf{R}^{n+1}$ , we denote by  $[x]$  its image in  $\mathbf{H}^n(\mathbf{R})$ . The geodesic distance  $d([x], [y])$  between two points  $[x], [y] \in \mathbf{H}^n(\mathbf{R})$  is defined by

$$\cosh d([x], [y]) = \frac{|\langle x, y \rangle|}{|\langle x, x \rangle|^{1/2} |\langle y, y \rangle|^{1/2}}.$$

Observe that  $d([x], [y])$  is well defined since the right hand side is  $\geq 1$ . Indeed, we have

$$\begin{aligned} \langle x, x \rangle \langle y, y \rangle &= \left( -x_{n+1}^2 + \sum_{i=1}^n x_i^2 \right) \left( -y_{n+1}^2 + \sum_{i=1}^n y_i^2 \right) \\ &\leq \left( -x_{n+1}^2 + \sum_{i=1}^n x_i^2 \right) \left( -y_{n+1}^2 + \sum_{i=1}^n y_i^2 \right) \\ &\quad + \left( x_{n+1} \sqrt{\sum_{i=1}^n y_i^2} - y_{n+1} \sqrt{\sum_{i=1}^n x_i^2} \right)^2 \\ &= \left( x_{n+1} y_{n+1} - \sqrt{\sum_{i=1}^n x_i^2} \sqrt{\sum_{i=1}^n y_i^2} \right)^2 \\ &\leq \left| x_{n+1} y_{n+1} - \sum_{i=1}^n x_i y_i \right|^2 = |\langle x, y \rangle|^2, \end{aligned}$$

where we have used Cauchy–Schwarz inequality together with  $\langle x, x \rangle < 0$  and  $\langle y, y \rangle < 0$ . For a proof of the triangle inequality, and more generally for an introduction to real hyperbolic spaces, see, e.g. [BriHa–99, Chapter I.2].

The group  $G = O(n, 1)$  acts isometrically and transitively on  $\mathbf{H}^n(\mathbf{R})$ . The stabiliser of the point  $x_0 = [(0, \dots, 0, 1)]$  is the compact group

$$K = O(n) \times O(1),$$

so that  $\mathbf{H}^n(\mathbf{R})$  can be identified with the homogeneous space  $G/K$ .

A *hyperplane* in  $\mathbf{H}^n(\mathbf{R})$  is a non-empty intersection of  $\mathbf{H}^n(\mathbf{R})$  with the image of an  $n$ -dimensional subspace of  $\mathbf{R}^{n+1}$ . A hyperplane is isometric to  $\mathbf{H}^{n-1}(\mathbf{R})$ . The subset  $H_0$  of  $\mathbf{H}^n(\mathbf{R})$  consisting of the image of all points with first coordinate equal to zero is a hyperplane, and every hyperplane is a translate of  $H_0$  under some element of  $G$ . Hence, the space  $\mathcal{S}$  of all hyperplanes in  $\mathbf{H}^n(\mathbf{R})$  can be identified with the homogeneous space  $G/G_{H_0}$ , where  $G_{H_0} \cong O(1) \times O(n-1, 1)$  is the stabiliser of  $H_0$  in  $G$ .

**Lemma 2.6.1** *Let  $g, h \in G$ . The point  $x = gx_0 \in \mathbf{H}^n(\mathbf{R})$  lies on the hyperplane  $hH_0$  if and only if  $h \in gKG_{H_0}$ .*

**Proof** Indeed,  $gx_0 \in hH_0$  if and only if  $h^{-1}gx_0 \in H_0$ . Using the fact that  $G_{H_0}$  acts transitively on  $H_0$ , we have  $h^{-1}gx_0 \in H_0$  if and only if  $h^{-1}gx_0 = g_0x_0$  for some  $g_0 \in G_{H_0}$ . This is the case if and only if  $g^{-1}hg_0 \in K$  for some  $g_0 \in G_{H_0}$ , that is,  $h \in gKG_{H_0}$ . ■

Since  $G$  and  $G_{H_0}$  are both unimodular (Example A.3.5.x), the locally compact topological space  $\mathcal{S} \cong G/G_{H_0}$  carries a non-zero positive  $O(n, 1)$ -invariant regular Borel measure  $\nu$  (Corollary B.1.7).

For  $x, y \in \mathbf{H}^n(\mathbf{R})$ , let  $[x, y]$  denote the geodesic segment between  $x$  and  $y$ .

**Lemma 2.6.2** (i) *For all  $x, y \in \mathbf{H}^n(\mathbf{R})$ , the set*

$$\{H \in \mathcal{S} : H \cap [x, y] \neq \emptyset\}$$

*is compact and therefore has finite measure.*

(ii) *For all  $x \in \mathbf{H}^n(\mathbf{R})$ , the set  $\{H \in \mathcal{S} : x \in H\}$  has measure 0.*

**Proof** (i) Since  $[x, y]$  is a compact subset of  $\mathbf{H}^n(\mathbf{R})$ , there exists a compact subset  $Q$  of  $G$  such that  $[x, y] = Qx_0$  (Lemma B.1.1). Consider  $h \in G$  and  $H = hH_0 \in \mathcal{S}$ ; by the previous lemma, there exists  $q \in Q$  such that  $qx_0 \in H$  if and only if  $h \in qKG_{H_0}$ . This proves (i).

(ii) Let  $g \in G$  be such that  $x = gx_0$ . Consider again  $h \in G$  and  $H = hH_0 \in \mathcal{S}$ ; then  $x \in H$  if and only if  $h \in gKG_{H_0}$ . As  $\nu$  is  $G$ -invariant, we have

$$\nu(\{H \in \mathcal{S} : x \in H\}) = \nu(gKG_{H_0}/G_{H_0}) = \nu(KG_{H_0}/G_{H_0}).$$

Observe that  $KG_{H_0}/G_{H_0}$  is a submanifold of  $G/G_{H_0}$ , as it is the orbit of  $G_{H_0}$  under the compact group  $K$ . To show that  $\nu(KG_{H_0}/G_{H_0}) = 0$ , it suffices to show that  $\dim(KG_{H_0}/G_{H_0}) < \dim(G/G_{H_0}) = n$ ; but  $KG_{H_0}/G_{H_0} \cong K/K \cap G_{H_0}$  and

$$\begin{aligned} \dim(K/K \cap G_{H_0}) &= \dim K - \dim(K \cap G_{H_0}) \\ &= \dim O(n) - \dim O(n-1) \\ &= n-1 < n. \quad \blacksquare \end{aligned}$$

The complement of a hyperplane in  $\mathbf{H}^n(\mathbf{R})$  has two connected components. Such a connected component is called a *half-space*. We denote by  $\Omega$  the set of all half-spaces in  $\mathbf{H}^n(\mathbf{R})$ .

The subset  $\omega_0$  of  $\mathbf{H}^n(\mathbf{R})$  consisting of all points with first and last coordinates of the same sign is a half-space, and every other half-space is a translate of  $\omega_0$  under some element of  $G$ . Hence,  $\Omega$  can be identified with the homogeneous space  $G/G_{\omega_0}$ , where  $G_{\omega_0} \cong O(n-1, 1)$  is the closed subgroup of  $G$  consisting of all elements which leave  $\omega_0$  globally invariant.

Since  $G$  and  $G_{\omega_0}$  are both unimodular,  $\Omega$  carries a non-zero positive  $G$ -invariant regular Borel measure  $\mu$ . The canonical projection  $p : \Omega \rightarrow \mathcal{S}$ , which associates to a half-space its boundary, is a double covering. Since the measures  $\mu$  and  $\nu$  on  $\Omega$  and  $\mathcal{S}$  are unique up to a positive multiple (Corollary B.1.7), we can assume that  $\nu$  is the image of  $\mu$  under  $p$ .

For a point  $x$  in  $\mathbf{H}^n(\mathbf{R})$ , let  $\Sigma_x$  denote the set of half-spaces containing  $x$ . Observe that, for  $x, y \in \mathbf{H}^n(\mathbf{R})$ , the set of half-spaces separating  $x$  from  $y$  is the symmetric difference  $\Sigma_x \Delta \Sigma_y$ .

**Lemma 2.6.3** (i) *For all  $x, y \in \mathbf{H}^n(\mathbf{R})$ , we have*

$$\mu(\Sigma_x \Delta \Sigma_y) = \nu(\{H \in \mathcal{S} : H \cap [x, y] \neq \emptyset\}) < \infty.$$

(ii) *For all  $x \in \mathbf{H}^n(\mathbf{R})$ , we have  $\mu(\{\omega \in \Omega : x \in p(\omega)\}) = 0$ .*

(iii) *For all  $x, y \in \mathbf{H}^n(\mathbf{R})$  with  $x \neq y$ , we have  $\mu(\Sigma_x \Delta \Sigma_y) > 0$ .*

**Proof** Observe that

$$\Sigma_x \Delta \Sigma_y = p^{-1}(\{H \in \mathcal{S} : H \cap [x, y] \neq \emptyset \text{ and } [x, y] \not\subset H\}).$$

Since, by (ii) of the previous lemma,  $\nu(\{H \in \mathcal{S} : [x, y] \subset H\}) = 0$ , we have

$$\mu(\Sigma_x \Delta \Sigma_y) = \nu(\{H \in \mathcal{S} : H \cap [x, y] \neq \emptyset\}).$$

It is now clear that (i) and (ii) are consequences of the previous lemma.

To show (iii), we follow an argument shown to us by Y. de Cornulier. Let  $x, y \in \mathbf{H}^n(\mathbf{R})$  with  $x \neq y$ . Consider the subset  $\Sigma_{x,y}$  of  $\Sigma_x \Delta \Sigma_y$  consisting of the half-spaces  $\omega$  such that  $x$  belongs to  $\omega$  and such that  $y$  does not belong to the closure of  $\omega$  in  $\mathbf{H}^n(\mathbf{R})$ . We claim that  $\Sigma_{x,y}$  is a non-empty open subset of  $\Omega$ . Once proved, this will imply that  $\mu(\Sigma_{x,y}) > 0$  (see Proposition B.1.5) and therefore  $\mu(\Sigma_x \Delta \Sigma_y) > 0$ .

Every half-space is of the form  $g\omega_0$  for some  $g \in G$ , where  $\omega_0$  is the half-space introduced above. By continuity of the action of  $G$  on  $\mathbf{H}^n(\mathbf{R})$ , the subset

$$\{g \in G : x \in g\omega_0\} = \{g \in G : g^{-1}x \in \omega_0\}$$

is open in  $G$ , since  $\omega_0$  is open in  $\mathbf{H}^n(\mathbf{R})$ . This shows that  $\{\omega \in \Omega : x \in \omega\}$  is an open subset of  $\Omega$ . The same argument shows that  $\{\omega \in \Omega : y \notin \bar{\omega}\}$  is an open subset of  $\Omega$ . Therefore  $\Sigma_{x,y}$  is open in  $\Omega$ .

It remains to show that  $\Sigma_{x,y}$  is not empty. The space  $\mathbf{H}^n(\mathbf{R})$  is two-point homogeneous, that is,  $G$  acts transitively on pairs of equidistant points in  $\mathbf{H}^n(\mathbf{R})$ ; see [BriHa–99, Part I, Proposition 2.17]. It therefore suffices to prove that  $\Sigma_{x,y} \neq \emptyset$  for

$$x = [(r, 0, \dots, 0, 1)] \quad \text{and} \quad y = [(-r, 0, \dots, 0, 1)],$$

for some  $0 < r < 1$ . But this is clear since  $\omega_0 \in \Sigma_{x,y}$ . ■

The following formula for half-spaces is a variant of the Crofton formula for hyperplanes; see [Roben–98, Corollary 2.5] and [TaKuU–81, Proof of Theorem 1].

**Proposition 2.6.4 (Crofton formula)** *There is a constant  $k > 0$  such that*

$$\mu(\Sigma_x \Delta \Sigma_y) = kd(x, y),$$

for all  $x, y \in \mathbf{H}^n(\mathbf{R})$ .

**Proof** By the previous lemma,  $\mu(\Sigma_x \Delta \Sigma_y)$  is finite for all  $x, y \in \mathbf{H}^n(\mathbf{R})$ . We claim that the function

$$\mathbf{H}^n(\mathbf{R}) \times \mathbf{H}^n(\mathbf{R}) \rightarrow \mathbf{R}_+, \quad (x, y) \mapsto \mu(\Sigma_x \Delta \Sigma_y)$$

is continuous. Indeed, let  $x, x', y, y' \in \mathbf{H}^n(\mathbf{R})$ . Since

$$\Sigma_x \Delta \Sigma_y \subset (\Sigma_x \Delta \Sigma_z) \cup (\Sigma_z \Delta \Sigma_y),$$

the function

$$(x, y) \mapsto \mu(\Sigma_x \Delta \Sigma_y)$$

satisfies the triangle inequality. In particular, for  $x, y, x', y' \in \mathbf{H}^n(\mathbf{R})$ , we have

$$|\mu(\Sigma_x \Delta \Sigma_y) - \mu(\Sigma_{x'} \Delta \Sigma_{y'})| \leq \mu(\Sigma_x \Delta \Sigma_{x'}) + \mu(\Sigma_y \Delta \Sigma_{y'}).$$

Hence, it suffices to show that, for  $x \in \mathbf{H}^n(\mathbf{R})$ ,

$$\lim_{x' \rightarrow x} \mu(\Sigma_x \Delta \Sigma_{x'}) = 0.$$

This follows from (i) and (ii) of Lemma 2.6.2, since by regularity of the measure  $\nu$ , we have

$$\begin{aligned} \lim_{x' \rightarrow x} \mu(\Sigma_x \Delta \Sigma_{x'}) &= \lim_{x' \rightarrow x} \nu(\{H \in \mathcal{S} : H \cap [x, x'] \neq \emptyset\}) \\ &= \nu(\{H \in \mathcal{S} : x \in H\}). \end{aligned}$$

Moreover the function  $(x, y) \mapsto \mu(\Sigma_x \Delta \Sigma_y)$  is  $G$ -invariant, that is,

$$\mu(\Sigma_{gx} \Delta \Sigma_{gy}) = \mu(\Sigma_x \Delta \Sigma_y) \quad \text{for all } x, y \in \mathbf{H}^n(\mathbf{R}), g \in O(n, 1).$$

Since  $\mathbf{H}^n(\mathbf{R})$  is two-point homogeneous, it follows that there exists a continuous function  $\varphi : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  such that

$$\mu(\Sigma_x \Delta \Sigma_y) = \varphi(d(x, y))$$

for all  $x, y \in \mathbf{H}^n(\mathbf{R})$ .

Fix now  $r_1, r_2 > 0$ . Choose three collinear points  $x, y, z$  in  $\mathbf{H}^n(\mathbf{R})$  such that

$$d(x, y) = r_1, d(y, z) = r_2, \text{ and } d(x, z) = r_1 + r_2.$$

The set  $\Sigma_x \Delta \Sigma_z$  of half-spaces separating  $x$  and  $z$  is the disjoint union of three sets: the set  $\Sigma_x \Delta \Sigma_y$  of half-spaces separating  $x$  and  $y$ , the set  $\Sigma_y \Delta \Sigma_z$  of half-spaces separating  $y$  and  $z$ , and the set of all half-spaces containing  $y$  on their boundaries. Since, by the previous lemma, the latter set has measure 0, we have

$$\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2),$$

that is,  $\varphi$  is additive. Extend  $\varphi$  to a continuous group homomorphism  $\tilde{\varphi} : \mathbf{R} \rightarrow \mathbf{R}$  by setting  $\tilde{\varphi}(x) = -\varphi(-x)$  for  $x < 0$ . It is well known that such a homomorphism must be linear (Exercise C.6.12), that is, there exists  $k \geq 0$  such that  $\varphi(r) = kr$  for all  $r \in \mathbf{R}_+$ . By (iii) of the previous lemma,  $\varphi \neq 0$ . Hence,  $k > 0$  and this concludes the proof. ■

Let now  $L$  be a topological group, and let  $\sigma : L \rightarrow O(n, 1)$  be a continuous homomorphism. Then  $L$  acts by isometries on  $\mathbf{H}^n(\mathbf{R})$  and hence on  $\Omega$ , preserving the measure  $\mu$ . This gives rise to an orthogonal representation  $\pi_\sigma$  of  $L$  on



$L^2_{\mathbf{R}}(\Omega, \mu)$ , the real Hilbert space of real valued square integrable functions on  $\Omega$  (Proposition A.6.1).

**Proposition 2.6.5** *Let  $L$  be a topological group, and let  $\sigma : L \rightarrow O(n, 1)$  be a continuous homomorphism. Let  $\pi_\sigma$  be the associated orthogonal representation of  $L$  on  $L^2_{\mathbf{R}}(\Omega, \mu)$ . If  $\sigma(L)$  is not relatively compact in  $O(n, 1)$ , then  $H^1(L, \pi_\sigma) \neq 0$ . In particular,  $L$  does not have Property (FH).*

**Proof** For  $x \in \mathbf{H}^n(\mathbf{R})$ , let  $\chi_x$  denote the characteristic function of the set  $\Sigma_x$  of half-spaces containing  $x$ .

Let  $x, y \in \mathbf{H}^n(\mathbf{R})$ . By the Crofton formula,

$$\int_{\Omega} |\chi_x(\omega) - \chi_y(\omega)|^2 d\mu(\omega) = \mu(\Sigma_x \Delta \Sigma_y) = kd(x, y).$$

Hence, the function  $\chi_x - \chi_y$  belongs to  $L^2_{\mathbf{R}}(\Omega, \mu)$  and

$$\|\chi_x - \chi_y\|^2 = kd(x, y).$$

Define

$$c : \mathbf{H}^n(\mathbf{R}) \times \mathbf{H}^n(\mathbf{R}) \rightarrow L^2_{\mathbf{R}}(\Omega, \mu)$$

by  $c(x, y) = \chi_x - \chi_y$ . It is clear that  $c$  satisfies Chasles' relation and  $L$ -equivariance. Therefore, the mapping  $g \mapsto c(gx_0, x_0)$  is a 1-cocycle on  $L$  with respect to  $\pi_\sigma$ .

Assume, by contradiction, that  $H^1(L, \pi_\sigma) = 0$ . Then the above cocycle is bounded, that is,

$$g \mapsto \|c(gx_0, x_0)\|^2 = kd(gx_0, x_0)$$

is a bounded mapping on  $L$ . We deduce from this that the  $\sigma(L)$ -orbit of  $x_0$  in  $\mathbf{H}^n(\mathbf{R})$  is bounded and hence relatively compact. On the other hand, since the stabiliser  $K = O(n) \times O(1)$  of  $x_0$  in  $O(n, 1)$  is compact, the action of  $O(n, 1)$  on  $\mathbf{H}^n(\mathbf{R})$  is proper (Exercise 2.14.5). Hence,  $\sigma(L)$  is relatively compact in  $O(n, 1)$ , contradicting our assumption. ■

**Remark 2.6.6** As we will see next, Proposition 2.6.5 has an analogue for the group  $U(n, 1)$ , the adjoint group of which is the isometry group of the complex hyperbolic space  $\mathbf{H}^n(\mathbf{C})$ . However, the proof given for  $O(n, 1)$  does not carry over to  $U(n, 1)$ , since  $\mathbf{H}^{n-1}(\mathbf{C})$  does not separate  $\mathbf{H}^n(\mathbf{C})$  in two connected components. It is an open problem to find a proof for  $U(n, 1)$  using real hypersurfaces which disconnect  $\mathbf{H}^n(\mathbf{C})$ .

## 2.7 Actions on boundaries of rank 1 symmetric spaces

Recall that the rank of a Riemannian symmetric space  $X$  is the dimension of a flat, totally geodesic submanifold of  $X$ . By Cartan's classification (see [Helga-62, Chapter IX, §4]), there are four families of irreducible Riemannian symmetric spaces of the non-compact type and of rank one:

- $X = \mathbf{H}^n(\mathbf{R})$ , the real hyperbolic space, for  $n \geq 2$ ;
- $X = \mathbf{H}^n(\mathbf{C})$ , the complex hyperbolic space, for  $n \geq 2$ ;
- $X = \mathbf{H}^n(\mathbf{H})$ , the hyperbolic space over the quaternions  $\mathbf{H}$ , for  $n \geq 2$ ;
- $X = \mathbf{H}^2(\text{Cay})$ , the hyperbolic plane over the Cayley numbers Cay.

For the sake of simplicity, we will not treat the exceptional case, and give one model for each of the classical cases.

Let  $\mathbf{K}$  denote one of the three real division algebras  $\mathbf{R}, \mathbf{C}, \mathbf{H}$ . Equip the right vector space  $\mathbf{K}^{n+1}$  with the hermitian form

$$\langle z, w \rangle = -\overline{z_{n+1}}w_{n+1} + \sum_{i=1}^n \overline{z_i}w_i,$$

where  $x \mapsto \bar{x}$  is the standard involution on  $\mathbf{K}$ .

The *hyperbolic space*  $X$  of dimension  $n$  over  $\mathbf{K}$  is defined as the open subset of the projective space  $\mathbf{P}^n(\mathbf{K})$  consisting of all points  $[z] \in \mathbf{P}^n(\mathbf{K})$  such that  $\langle z, z \rangle < 0$ , where  $[z]$  is the image in  $\mathbf{P}^n(\mathbf{K})$  of  $z \in \mathbf{K}^{n+1}$ . The group of all  $\mathbf{K}$ -linear mappings from  $\mathbf{K}^{n+1}$  to  $\mathbf{K}^{n+1}$  preserving the given hermitian form is  $G = O(n, 1)$  in the case  $\mathbf{K} = \mathbf{R}$ ,  $G = U(n, 1)$  in the case  $\mathbf{K} = \mathbf{C}$ , and  $G = Sp(n, 1)$  in the case  $\mathbf{K} = \mathbf{H}$ . The manifold  $X$  is diffeomorphic to  $\mathbf{K}^n = \mathbf{R}^{nd}$ , where  $d = \dim_{\mathbf{R}} \mathbf{K}$ , and has a  $G$ -invariant metric for which the geodesic distance  $d(z, w)$  between two points  $[z], [w] \in X$  is defined by

$$\cosh d([z], [w]) = \frac{|\langle z, w \rangle|}{|\langle z, z \rangle|^{1/2} |\langle w, w \rangle|^{1/2}}.$$

(Observe that  $d([z], [w])$  is well-defined since the right hand side is  $\geq 1$ ; compare with the beginning of Section 2.6.) The *boundary* of  $X$ , denoted by  $\partial X$ , is the closed subset of  $\mathbf{P}^n(\mathbf{K})$  consisting of all points  $[z] \in \mathbf{P}^n(\mathbf{K})$  such that  $\langle z, z \rangle = 0$ . It is a sphere  $S^{dn-1}$  and  $G$  acts on  $\partial X$  by diffeomorphisms.

P. Julg has constructed a canonical affine action of  $G$  associated with its action on  $\partial X$  (see [CCJVV-01, Chapter 3]). This action is on an affine real space whose associated vector space is endowed with a quadratic form  $Q$  invariant under the linear part of the affine action. The positive-definiteness of  $Q$  will be an issue, and we will see that  $Q$  is positive-definite for  $\mathbf{K} = \mathbf{R}, \mathbf{C}$ , but *not* for  $\mathbf{K} = \mathbf{H}$ .

We now sketch Julg's construction and refer to his original work for more details. Let  $E$  be the real vector space of maximal degree smooth differential forms with integral 0 on  $\partial X$ . Denote by  $\pi$  the representation of  $G$  on  $E$  induced by the action of  $G$  on  $\partial X$ .

**Proposition 2.7.1** (i) *There exists a non-zero mapping  $c : X \times X \rightarrow E$  which is  $G$ -equivariant and satisfies Chasles' relation (as defined in the beginning of Section 2.3).*

(ii) *There exist a quadratic form  $Q : E \rightarrow \mathbf{R}$  which is  $G$ -invariant and a function  $\varphi : \mathbf{R}_+ \rightarrow \mathbf{R}$  with  $\lim_{r \rightarrow \infty} \varphi(r) = +\infty$  such that*

$$Q(c(x, y)) = \varphi(d(x, y)), \quad \text{for all } x, y \in X.$$

(iii) *For  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{C}$ , the quadratic form  $Q$  is positive definite.*

**Sketch of proof** The cocycle  $c$  is easy to describe. For  $x \in X$ , the unit sphere  $S_x$  in the tangent space  $T_x X$  can be identified with  $\partial X$  by means of the visual mapping (see [BriHa-99, Part II, 8.11]). We denote by  $\mu_x$  the push-forward, under the visual mapping, of the canonical volume form of volume 1 on  $S_x$ . Then  $\mu_x$  is a maximal degree differential form of volume 1 on  $\partial X$ , and we set

$$c(x, y) = \mu_x - \mu_y, \quad x, y \in X.$$

It is straightforward to check that the mapping  $c$  satisfies the conditions of (i).

Let  $\Delta$  be the diagonal in  $\partial X \times \partial X$ . For every  $x \in X$ , define a function  $f_x$  on  $(\partial X \times \partial X) \setminus \Delta$  by

$$f_x(p, q) = \log \left| \frac{\langle p, q \rangle \langle x, x \rangle}{\langle p, x \rangle \langle q, x \rangle} \right|.$$

Observe that  $f_x(p, q)$  diverges like  $\log | \langle p, q \rangle |$ , when  $p$  tends to  $q$ . For  $x, y$  in  $X$ , the *Busemann cocycle*  $\gamma_{xy}$  is the smooth function on  $\partial X$  defined by

$$\gamma_{xy}(p) = \log \left| \frac{\langle y, p \rangle}{\langle x, p \rangle} \left( \frac{\langle x, x \rangle}{\langle y, y \rangle} \right)^{1/2} \right|.$$

It clearly satisfies Chasles' relation  $\gamma_{xy} + \gamma_{yz} = \gamma_{xz}$ . Geometrically,  $\gamma_{xy}(p)$  is the limit of  $d(z, y) - d(z, x)$ , when  $z \in X$  tends to  $p \in \partial X$ . For  $x, y \in X$  and  $p, q \in \partial X$ ,  $p \neq q$ , we have

$$f_x(p, q) - f_y(p, q) = \gamma_{xy}(p) + \gamma_{xy}(q).$$

Fixing  $x \in X$ , we define then a quadratic form  $Q$  on  $E$  by

$$Q(\alpha) = - \int_{\partial X} \int_{\partial X} f_x(p, q) \alpha(p) \alpha(q), \quad \alpha \in E.$$

This integral makes sense because of the logarithmic divergence of  $f_x$  on the diagonal of  $\partial X \times \partial X$ . The form  $Q$  does not depend on the choice of  $x$ . Indeed, for  $\alpha \in E$  and  $x, y \in X$ , we have

$$\begin{aligned} \int_{\partial X} \int_{\partial X} f_x(p, q) \alpha(p) \alpha(q) - \int_{\partial X} \int_{\partial X} f_y(p, q) \alpha(p) \alpha(q) = \\ \int_{\partial X} \int_{\partial X} \gamma_{xy}(p) \alpha(p) \alpha(q) + \int_{\partial X} \int_{\partial X} \gamma_{xy}(q) \alpha(p) \alpha(q) = 0, \end{aligned}$$

since  $\int_{\partial X} \alpha = 0$ . This implies

$$\begin{aligned} Q(\pi(g)\alpha) &= \int_{\partial X} \int_{\partial X} f_{\pi(g)x}(p, q) \alpha(p) \alpha(q) \\ &= Q(\alpha) \end{aligned}$$

for all  $g \in G$ . Hence,  $Q$  is  $G$ -invariant.

Since  $X$  is two-point homogeneous, it follows that  $Q(c(x, y))$  only depends on the distance  $d(x, y)$ . Hence, there exists a function  $\varphi : \mathbf{R}_+ \rightarrow \mathbf{R}$  such that

$$Q(c(x, y)) = \varphi(d(x, y)), \quad \text{for all } x, y \in X.$$

An explicit computation given in [CCJJV-01, Proposition 3.2.4] shows that, denoting by  $S$  the unit sphere in  $\mathbf{K}^n$ , we have

$$\varphi(r) = 2 \int_S \log |\cosh r - u_1 \sinh r| dv(u),$$

where  $dv$  is the normalised volume on  $S$  and  $u_1$  is the first coordinate of  $u \in S$ . It can be checked that  $\varphi(r)$  behaves asymptotically as  $\log \cosh r$  for  $r \rightarrow \infty$ , in particular that  $\lim_{r \rightarrow \infty} \varphi(r) = +\infty$ , so that (ii) holds.

It remains to show that  $Q$  is positive-definite when  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ . In order to perform the computation, choose  $x \in X \subset \mathbf{P}^n(\mathbf{K})$  to be the class of  $(0, \dots, 0, 1) \in \mathbf{K}^{n+1}$ . Then

$$\left| \frac{\langle p, q \rangle \langle x, x \rangle}{\langle p, x \rangle \langle q, x \rangle} \right| = \left| \frac{\langle p, q \rangle}{p_{n+1} q_{n+1}} \right| = |1 - \langle z | w \rangle|,$$

where

$$z = (p_1/p_{n+1}, \dots, p_n/p_{n+1}), \quad w = (q_1/q_{n+1}, \dots, q_n/q_{n+1}) \in \mathbf{K}^n$$

and  $\langle z | w \rangle = \sum_{i=1}^n \bar{z}_i w_i$ . So, identifying  $\partial X$  and  $S$  by means of the diffeomorphism  $p \mapsto z$ , it is enough to show that, for any differential form

$\alpha$  with integral 0 on the unit sphere  $S$  of  $\mathbf{K}^n$ ,

$$-\int_S \int_S \log |1 - \langle z|w \rangle| \alpha(z) \alpha(w) \geq 0.$$

Since

$$-\log |1 - \langle z|w \rangle| = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Re}(\langle z|w \rangle^k),$$

it suffices to show that

$$\int_S \int_S \operatorname{Re}(\langle z|w \rangle^k) \alpha(z) \alpha(w) \geq 0.$$

Now, the kernel  $(z, w) \mapsto \langle z|w \rangle$  on  $\mathbf{K}^n = \mathbf{R}^n$  or  $\mathbf{K}^n = \mathbf{C}^n$  is of positive type, and hence the same is true for the kernels

$$(z, w) \mapsto \operatorname{Re}(\langle z|w \rangle^k)$$

(see Example C.1.3 and Proposition C.1.6). This shows that  $Q$  is a positive-definite quadratic form on  $E$ , and completes the sketch of proof. ■

Let  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ . Then, after completion, the pair  $(E, Q)$  as in the previous theorem becomes a real Hilbert space  $\mathcal{H}$  with an orthogonal representation  $\pi$  of  $G$ . Fix an origin  $x_0 \in X$ . By Proposition 2.3.1, the mapping

$$b : G \rightarrow E, \quad g \mapsto c(gx_0, x_0)$$

belongs to  $Z^1(G, \pi)$ . As

$$\|c(gx_0, x_0)\|^2 = Q(c(gx_0, x_0)) = \varphi(d(gx_0, x_0)),$$

and  $\lim_{r \rightarrow \infty} \varphi(r) = +\infty$ , the cocycle  $b$  is a *proper* cocycle on  $G$ , that is, for every bounded subset  $B$  of  $\mathcal{H}$ , the set  $b^{-1}(B)$  is relatively compact in  $G$ . As in the proof of Proposition 2.6.5, we obtain the following result.

**Theorem 2.7.2** *Let  $L$  be a topological group and  $\sigma : L \rightarrow G$  a continuous homomorphism, where  $G = O(n, 1)$  or  $G = U(n, 1)$ . Assume that  $\sigma(L)$  is not relatively compact in  $G$ . For the orthogonal representation  $\pi$  of  $G$  as above, we have  $H^1(L, \pi \circ \sigma) \neq 0$ . In particular,  $L$  does not have Property (FH).*

Here is a straightforward consequence of the previous theorem (compare Theorem 2.3.6).

**Corollary 2.7.3** *Any continuous isometric action of a topological group with Property (FH) on a real or complex hyperbolic space has a fixed point.*

**Remark 2.7.4** (i) The proof of the positive definiteness of  $Q$  in Proposition 2.7.1 fails for  $\mathbf{K} = \mathbf{H}$ . Indeed, due to the non-commutativity of  $\mathbf{H}$ , the kernels  $(z, w) \mapsto \operatorname{Re}(\langle z, w \rangle^k)$  are not of positive type.

(ii) For another proof of the last statement of Theorem 2.7.2, see Remark 2.11.4.

(iii) We will see later (Theorem 2.12.7) that the previous theorem implies that  $O(n, 1)$  and  $U(n, 1)$ , as well as their closed non-compact subgroups, do not have Property (T). On the other hand, we will also see that  $Sp(n, 1)$  does have Property (T) for  $n \geq 2$  (Section 3.3). This implies that, in the case of  $\mathbf{K} = \mathbf{H}$ , the quadratic form  $Q$  defined above cannot be positive-definite.

There is a class of groups generalising those appearing in Theorem 2.7.2.

**Definition 2.7.5** A topological group  $G$  has the *Haagerup Property*, or is  *$a$ -T-menable* in the sense of Gromov, if there exists an orthogonal representation  $\pi$  of  $G$  which has a *proper* 1-cocycle  $b \in Z^1(G, \pi)$ .

By what we have just seen, closed subgroups of  $O(n, 1)$  or  $U(n, 1)$  have the Haagerup Property. The class of groups with the Haagerup Property is a large class containing, moreover, amenable groups, free groups, Coxeter groups, and groups of automorphisms of locally finite trees [CCJIV–01]. Concerning Coxeter groups, see also Example 2.10.5 below.

## 2.8 Wreath products

In this section, we characterise wreath products which have Property (FH).

Let  $(H_i)_{i \in I}$  be a family of groups, and let

$$N = \bigoplus_{i \in I} H_i,$$

be the direct sum of the  $H_i$ 's. For  $i \in I$ , denote by  $p_i : N \rightarrow H_i$  the canonical projection, and set

$$\pi = \bigoplus_{i \in I} \lambda_{H_i} \circ p_i;$$

this is a unitary representation of  $N$  on the Hilbert space  $\mathcal{H} = \bigoplus_{i \in I} \ell^2(H_i)$ .

We define a 1-cocycle  $b \in Z^1(N, \pi)$  as follows. For  $n = (h_i)_{i \in I} \in N$ , set

$$b(n)_i = \delta_{h_i} - \delta_{e_i}.$$

Since  $h_i = e_i$  and therefore  $b(n)_i = 0$  for all but finitely many indices  $n$ , we have  $b(n) = (b(n)_i)_{i \in I} \in \mathcal{H}$ . To see that  $b$  is a 1-cocycle, consider the vector

$$x = (\delta_{e_i})_{i \in I} \in \prod_{i \in I} \ell^2(H_i).$$

Clearly  $x \notin \mathcal{H}$  if  $I$  is infinite but, formally, we have

$$b(n) = \pi(n)x - x,$$

showing that  $b \in Z^1(N, \pi)$ . As we now see, the 1-cocycle  $b$  is not a coboundary, unless the index set  $I$  is finite.

**Proposition 2.8.1** *The following conditions are equivalent:*

- (i)  $b \in B^1(N, \pi)$ ;
- (ii)  $H_i = \{e_i\}$  for all but finitely many indices  $i \in I$ .

**Proof** Assume that there exists a finite subset  $F$  of  $I$  such that  $H_i = \{e_i\}$  for every  $i \notin F$ . Let

$$\xi = \bigoplus_{i \in F} \delta_{e_i} \in \mathcal{H}.$$

Then  $b(n) = \pi(n)\xi - \xi$  for all  $n \in N$ , so that  $b \in B^1(N, \pi)$ .

Conversely, assume that  $H_i \neq \{e_i\}$  for infinitely many  $i \in I$ . For  $n = (h_i)_{i \in I} \in N$ , define the support of  $n$  as

$$\text{supp}(n) = \{i \in I : h_i \neq e_i\}.$$

Then clearly

$$\|b(n)\|^2 = 2 \# \text{supp}(n).$$

This shows that  $b$  is unbounded on  $N$ , so that  $b$  cannot be a coboundary. ■

Let  $H$  and  $\Gamma$  be two (discrete) groups. The *wreath product*  $\Gamma \wr H$  is the semidirect product

$$\Gamma \ltimes \left( \bigoplus_{\gamma \in \Gamma} H \right),$$

where  $\Gamma$  acts on the direct sum  $\bigoplus_{\gamma \in \Gamma} H$  by shifting on the left:

$$g(h_\gamma)_{\gamma \in \Gamma} = (h_{g^{-1}\gamma})_{\gamma \in \Gamma} \quad \text{for } g \in \Gamma \quad \text{and} \quad (h_\gamma)_{\gamma \in \Gamma} \in \bigoplus_{\gamma \in \Gamma} H.$$

The following result was obtained independently by F. Martin and M. Neuhauser, with different proofs.

**Proposition 2.8.2** *Let  $H$  be a group not reduced to one element. The following conditions are equivalent:*

- (i) *the wreath product  $\Gamma \wr H$  has Property (FH);*
- (ii)  *$H$  has Property (FH) and  $\Gamma$  is finite.*

The non-trivial part of the proof is based on the following lemma. For the notion of Property (FH) for pairs, see Definition 2.2.11.

**Lemma 2.8.3** *Let  $H$  be a group which is not reduced to one element, and let  $\Gamma$  be an infinite group. Let  $N = \bigoplus_{\gamma \in \Gamma} H$ . Then the pair  $(\Gamma \wr H, N)$  does not have Property (FH). In particular,  $\Gamma \wr H$  does not have Property (FH).*

**Proof** Let  $\pi$  be the unitary representation of  $N$  on  $\mathcal{H} = \bigoplus_{\gamma \in \Gamma} \ell^2(H)$  defined by

$$\pi((h_\gamma)_{\gamma \in \Gamma})((\xi_\gamma)_{\gamma \in \Gamma}) = (\lambda_H(h_\gamma)\xi_\gamma)_{\gamma \in \Gamma}$$

for  $n = (h_\gamma)_{\gamma \in \Gamma} \in N$  and  $\xi = (\xi_\gamma)_{\gamma \in \Gamma} \in \mathcal{H}$ . Let  $b \in Z^1(N, \pi)$  be the 1-cocycle described before Proposition 2.8.1:

$$b(n)_\gamma = \delta_{h_\gamma} - \delta_e \quad \text{for all } n = (h_\gamma)_{\gamma \in \Gamma} \in N.$$

Denote by  $\alpha$  the associated affine isometric action of  $N$  on  $\mathcal{H}$ .

Let  $\lambda$  be the unitary representation of  $\Gamma$  on  $\mathcal{H}$  defined by

$$\lambda(g)((\xi_\gamma)_{\gamma \in \Gamma}) = (\xi_{g^{-1}\gamma})_{\gamma \in \Gamma} \quad \text{for all } g \in \Gamma.$$

For  $n = (h_\gamma)_{\gamma \in \Gamma} \in N$ ,  $g \in \Gamma$ , and  $\xi = (\xi_\gamma)_{\gamma \in \Gamma} \in \mathcal{H}$ , we have

$$\begin{aligned} \alpha(gng^{-1})\lambda(g)\xi &= \pi(gng^{-1})\lambda(g)\xi + b(gng^{-1}) \\ &= \pi((h_{g^{-1}\gamma})_{\gamma \in \Gamma})((\xi_{g^{-1}\gamma})_{\gamma \in \Gamma}) + b((h_{g^{-1}\gamma})_{\gamma \in \Gamma}) \\ &= (\lambda_H(h_{g^{-1}\gamma})\xi_{g^{-1}\gamma} + \delta_{h_{g^{-1}\gamma}} - \delta_e)_{\gamma \in \Gamma} \\ &= \lambda(g)(\lambda_H(h_\gamma)\xi_\gamma + \delta_{h_\gamma} - \delta_e)_{\gamma \in \Gamma} \\ &= \lambda(g)\alpha(n)\xi \end{aligned}$$

that is,  $\alpha(gng^{-1}) = \lambda(g)\alpha(n)\lambda(g^{-1})$ . It follows that  $\alpha$  extends to an affine isometric action  $\tilde{\alpha}$  of  $\Gamma \wr H$  on  $\mathcal{H}$ . By Proposition 2.8.1, the orbits of  $\alpha$  are



unbounded. Hence,  $\tilde{\alpha}$  has no  $N$ -fixed point. This shows that  $(\Gamma \wr H, N)$  does not have Property (FH). ■

**Proof of Proposition 2.8.2** If  $H$  has Property (FH) and  $\Gamma$  is finite, then  $N = \bigoplus_{\gamma \in \Gamma} H$  has Property (FH), by Proposition 2.5.4. Since  $N$  has finite index in  $\Gamma \wr H$ , it follows that  $\Gamma \wr H$  has Property (FH).

To show the converse, assume that  $\Gamma \wr H$  has Property (FH). The previous lemma shows that  $\Gamma$  is finite. Hence, the finite index subgroup  $N = \bigoplus_{\gamma \in \Gamma} H$  has Property (FH) by Proposition 2.5.7, and  $H$  has Property (FH) since it is a quotient of  $N$ . ■

**Remark 2.8.4** It is a natural question to ask if, more generally, Property (FH) for a semidirect product  $\Gamma \ltimes N$  of two discrete groups  $\Gamma$  and  $N$  implies that  $N$  is finitely generated. The answer is negative. Indeed, the group  $SL_3(\mathbb{Z}[1/p]) \ltimes (\mathbb{Z}[1/p])^3$  has Property (FH) since it is a lattice in the Kazhdan group

$$(SL_3(\mathbb{R}) \ltimes \mathbb{R}^3) \times (SL_3(\mathbb{Q}_p) \ltimes \mathbb{Q}_p^3),$$

but the normal subgroup  $(\mathbb{Z}[1/p])^3$  is not finitely generated. (This example was drawn to our attention by Y. de Cornulier.)

## 2.9 Actions on the circle

Let  $\mathbf{S}^1$  be the unit circle in  $\mathbb{R}^2$ , parametrised by angle  $\theta \in [0, 2\pi)$ . We denote by  $\text{Homeo}_+(\mathbf{S}^1)$  the group of orientation preserving homeomorphisms of  $\mathbf{S}^1$ . Since  $\mathbb{R}$  is the universal covering space of  $\mathbf{S}^1$ , every  $f \in \text{Homeo}_+(\mathbf{S}^1)$  can be lifted to a strictly increasing mapping  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$  with the property  $\tilde{f}(x + 2\pi) = \tilde{f}(x) + 2\pi$  for all  $x \in \mathbb{R}$ . Observe that  $\tilde{f}$  is determined by  $f$  up to translations by integral multiples of  $2\pi$ . By abuse of notation, we will not distinguish between  $f$  and  $\tilde{f}$ .

A diffeomorphism  $f$  of  $\mathbf{S}^1$  is said to be of class  $1 + \alpha$  for a real number  $\alpha \geq 0$  if the first derivatives  $f'$  and  $(f^{-1})'$  are Hölder continuous with exponent  $\alpha$ , that is, if

$$\|f'\|_\alpha = \sup_{\theta \neq \varphi} \frac{|f'(\theta) - f'(\varphi)|}{|\theta - \varphi|^\alpha} < \infty$$

and if  $\|(f^{-1})'\|_\alpha < \infty$ , where  $|\theta - \varphi|$  denotes the arc length on  $\mathbf{S}^1$ . Let  $\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  be the subgroup of  $\text{Homeo}_+(\mathbf{S}^1)$  consisting of all orientation preserving diffeomorphisms of  $\mathbf{S}^1$  of class  $1 + \alpha$ . The aim of this section is to prove the following theorem due to A. Navas [Nava-02a]. This theorem generalises work by D. Witte [Witte-94], E. Ghys [Ghys-99, Théorème 1.1], and M. Burger

and N. Monod [BurMo–99, Corollary 1.5] on higher rank lattices as well as a result of A. Reznikov [Rezni–00, Chapter II, Theorem 1.7].

**Theorem 2.9.1** *Let  $\Gamma$  be a group with Property (FH), and let  $\alpha > 1/2$ . If  $\Phi : \Gamma \rightarrow \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  is any homomorphism, then  $\Phi(\Gamma)$  is a finite cyclic group.*

The proof will be given after some preliminary steps.

### 1-cocycles associated to actions on the circle

We consider the cartesian product  $\mathbf{S}^1 \times \mathbf{S}^1$  endowed with the Lebesgue measure  $d\theta d\varphi$ . Let  $\mathcal{H} = L_{\mathbf{R}}^2(\mathbf{S}^1 \times \mathbf{S}^1)$  be the real Hilbert space of square-integrable real valued kernels on  $\mathbf{S}^1$  (that is, square-integrable real valued functions on  $\mathbf{S}^1 \times \mathbf{S}^1$ ). Let  $\pi$  be the orthogonal representation of  $\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  on  $\mathcal{H}$  given by

$$(\pi(f)K)(\theta, \varphi) = \sqrt{(f^{-1})'(\theta)(f^{-1})'(\varphi)} K(f^{-1}(\theta), f^{-1}(\varphi)),$$

for  $f \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  and  $K \in \mathcal{H}$ . Consider the kernel  $F$  on  $\mathbf{S}^1$  (which is not square integrable) given by

$$F(\theta, \varphi) = \frac{1}{2 \tan((\theta - \varphi)/2)}, \quad (\theta, \varphi) \in \mathbf{S}^1 \times \mathbf{S}^1.$$

For  $f \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$ , we define a kernel  $b(f)$  on  $\mathbf{S}^1$  formally by

$$b(f) = \pi(f)F - F$$

or, more precisely, by

$$b(f)(\theta, \varphi) = \frac{\sqrt{(f^{-1})'(\theta)(f^{-1})'(\varphi)}}{2 \tan((f^{-1}(\theta) - f^{-1}(\varphi))/2)} - \frac{1}{2 \tan((\theta - \varphi)/2)}.$$

In the proof of the following lemma, the main point is to show that the kernels  $b(f)$  are square summable. The argument appears in Proposition 1.1 in [Rezni–00, Chapter 2] and is essentially that of Proposition 6.8.2 in [PreSe–86] (see also [Sega–81]).

**Lemma 2.9.2** *If  $\alpha > 1/2$ , then  $b(f) \in \mathcal{H}$  for every  $f \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$ , and  $b$  is a 1-cocycle with coefficients in  $\pi$ , that is,*

$$b \in Z^1(\text{Diff}_+^{1+\alpha}(\mathbf{S}^1), \pi).$$

**Proof** It is clear that  $b$  satisfies the 1-cocycle relation. We only have to show that  $b(f)$  is square-integrable, for  $f \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  if  $\alpha > 1/2$ .

The notation  $(\theta, \varphi) \mapsto \theta - \varphi$  does not make sense as a function on  $\mathbf{S}^1 \times \mathbf{S}^1$ , but it does on a small enough neighbourhood of the diagonal. We will write abusively  $\frac{1}{\theta - \varphi}$  for the value of a function  $\mathbf{S}^1 \times \mathbf{S}^1 \rightarrow \mathbf{R} \cup \{\infty\}$  which is  $\frac{1}{\theta - \varphi}$  near enough the diagonal, 0 far enough from the diagonal, and smooth outside the diagonal. Then

$$F(\theta, \varphi) = \frac{1}{\theta - \varphi} + K_0(\theta, \varphi),$$

where  $K_0$  is a continuous kernel on  $\mathbf{S}^1$ . Hence, we have to show that the function

$$(\theta, \varphi) \mapsto \frac{\sqrt{f'(\theta)f'(\varphi)}}{f(\theta) - f(\varphi)} - \frac{1}{\theta - \varphi}$$

is square-integrable on an appropriate neighbourhood of the diagonal.

For  $(\theta, \varphi)$  in this set,  $\theta \neq \varphi$ , we find by the mean value theorem some  $\psi$  belonging to the shortest arc between  $\theta$  and  $\varphi$  such that

$$f(\theta) - f(\varphi) = f'(\psi)(\theta - \varphi).$$

Then

$$\begin{aligned} & \left| \frac{\sqrt{f'(\theta)f'(\varphi)}}{f(\theta) - f(\varphi)} - \frac{1}{\theta - \varphi} \right| \\ &= \frac{1}{f'(\psi)|\theta - \varphi|} \left| \sqrt{f'(\theta)f'(\varphi)} - f'(\psi) \right| \\ &= \frac{1}{f'(\psi)|\theta - \varphi|} \frac{|f'(\theta)f'(\varphi) - f'(\psi)^2|}{\left( \sqrt{f'(\theta)f'(\varphi)} + f'(\psi) \right)} \\ &\leq \frac{1}{2 \inf(f')^2 |\theta - \varphi|} (|f'(\theta) - f'(\psi)|f'(\varphi) + f'(\psi)|f'(\varphi) - f'(\psi)|) \\ &\leq \frac{\sup(f') \|f'\|_\alpha}{2 \inf(f')^2 |\theta - \varphi|} (|\theta - \psi|^\alpha + |\varphi - \psi|^\alpha), \end{aligned}$$

since  $f'$  is Hölder continuous of exponent  $\alpha$ .

Since  $|\theta - \psi| + |\varphi - \psi| = |\varphi - \theta|$ , it follows that, for some constant  $C$  (depending on  $\alpha$  and  $f$ ), we have

$$\left| \frac{\sqrt{f'(\theta)f'(\varphi)}}{f(\theta) - f(\varphi)} - \frac{1}{\theta - \varphi} \right| \leq C |\varphi - \theta|^{\alpha-1}.$$

As  $\alpha > 1/2$ , the kernel

$$(\varphi, \theta) \mapsto |\varphi - \theta|^{\alpha-1}$$

is square-integrable on  $\mathbf{S}^1 \times \mathbf{S}^1$ . This concludes the proof. ■

### A cohomological criterion for the existence of invariant measures

Let  $G$  be a  $\sigma$ -compact locally compact group, acting in a measurable way on a measure space  $(\Omega, \mathcal{B}, \nu)$  such that  $\nu$  is quasi-invariant. Assume that  $\nu$  is  $\sigma$ -finite and that  $\mathcal{B}$  is generated by a countable family of subsets. The Hilbert space  $L_{\mathbf{R}}^2(\Omega, \nu)$  of square-integrable real-valued functions on  $\Omega$  is separable and an orthogonal representation  $\pi_\nu$  of  $G$  on  $L_{\mathbf{R}}^2(\Omega, \nu)$  is defined by

$$\pi_\nu(g)f(\omega) = c_\nu(g^{-1}, \omega)^{1/2}f(g^{-1}\omega), \quad \text{for all } f \in L_{\mathbf{R}}^2(\Omega, \nu), g \in G, \omega \in \Omega,$$

where  $c_\nu(g, \omega) = \frac{dg\nu}{d\nu}(\omega)$  and  $\frac{dg\nu}{d\nu}$  is the Radon–Nikodym derivative of the image  $g\nu$  of  $\nu$  under the action of  $g \in G$  (see Section A.6).

The following proposition gives a criterion for the existence of a  $G$ -invariant measure  $\mu$  on  $\Omega$  which is absolutely continuous with respect to  $\nu$  (that is,  $\mu(A) = 0$  for all  $A \in \mathcal{B}$  such that  $\nu(A) = 0$ ).

**Proposition 2.9.3** *With the notation as above, assume that there exists a measurable real-valued function  $F$  on  $\Omega$ , not in  $L_{\mathbf{R}}^2(\Omega, \nu)$ , such that for every  $g \in G$  the function*

$$\omega \mapsto c_\nu(g, \omega)^{1/2}F(g\omega) - F(\omega)$$

*belongs to  $L_{\mathbf{R}}^2(\Omega, \nu)$ . If  $H^1(G, \pi_\nu) = 0$ , there exists a positive measure  $\mu$  on  $(\Omega, \mathcal{B})$  which is  $G$ -invariant and absolutely continuous with respect to  $\nu$ .*

**Proof** For  $g \in G$ , let  $b(g) \in L_{\mathbf{R}}^2(\Omega, \nu)$  be defined by

$$b(g)(\omega) = c_\nu(g, \omega)^{1/2}F(g\omega) - F(\omega), \quad \text{for all } \omega \in \Omega.$$

Since we can formally write

$$b(g) = \pi_\nu(g)F - F, \quad \text{for all } g \in G,$$

we see that  $b$  is a 1-cocycle on  $G$  with respect to  $\pi_\nu$ . Moreover, for every  $\xi \in L_{\mathbf{R}}^2(\Omega, \nu)$ , the function

$$G \rightarrow \mathbf{R}, \quad g \mapsto \langle b(g), \xi \rangle = \int_{\Omega} \left( c_\nu(g, \omega)^{1/2}F(g\omega) - F(\omega) \right) \xi(\omega) d\nu(\omega)$$

is measurable. Hence, the 1-cocycle  $b : G \rightarrow L_{\mathbf{R}}^2(\Omega, \nu)$  is continuous (Exercise 2.14.3).

Since  $H^1(G, \pi_\nu) = 0$  by assumption, there exists  $\xi \in L_{\mathbf{R}}^2(\Omega, \nu)$  such that  $b(g) = \pi_\nu(g)\xi - \xi$  for all  $g \in G$ , that is,

$$c_\nu(g, \omega)^{1/2} F(g\omega) - F(\omega) = c_\nu(g, \omega)^{1/2} \xi(g\omega) - \xi(\omega), \quad \text{for all } g \in G, \omega \in \Omega.$$

Hence, we have

$$\frac{dg\nu}{d\nu}(\omega) (F - \xi)^2(g\omega) = (F - \xi)^2(\omega), \quad \text{for all } g \in G, \omega \in \Omega,$$

so that the measure

$$d\mu(\omega) = (F - \xi)^2(\omega) d\nu(\omega)$$

is  $G$ -invariant. Finally,  $\mu$  is non-zero since  $\xi \in L_{\mathbf{R}}^2(\Omega, \nu)$  and  $F \notin L_{\mathbf{R}}^2(\Omega, \nu)$ . ■

### Geodesic currents

Recall that the *Poincaré disc*, which is the unit disc in  $\mathbf{R}^2$  with the Riemannian metric

$$ds^2 = 2 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2},$$

is a model for the real hyperbolic plane  $\mathbf{H}^2(\mathbf{R})$ . The group  $SU(1, 1) \cong SL(2, \mathbf{R})$  acts isometrically on the Poincaré disc by Möbius transformations

$$z \mapsto gz = \frac{\alpha z + \beta}{\bar{\beta}z + \alpha}, \quad z \in \mathbf{S}^1, \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \alpha \end{pmatrix} \in SU(1, 1)$$

where  $\alpha, \beta \in \mathbf{C}$  are such that  $|\alpha|^2 - |\beta|^2 = 1$ . Let  $\Delta$  be the diagonal in  $\mathbf{S}^1 \times \mathbf{S}^1$ . For the action of the two-element group  $\mathbf{Z}/2\mathbf{Z}$  on  $(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta$ , given by

$$(\theta, \varphi) \mapsto (\varphi, \theta),$$

we observe that the quotient space

$$(\mathbf{S}^1 \times \mathbf{S}^1 \setminus \Delta) / (\mathbf{Z}/2\mathbf{Z})$$

(that is, the set of unordered pairs in  $(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta$ ) can be identified with the space of unoriented lines in the Poincaré disc. This motivates the following definition.

**Definition 2.9.4** A *geodesic current* is a positive regular Borel measure  $\mu$  on  $(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta$  such that

$$\mu([a, b] \times [c, d]) = \mu([c, d] \times [a, b])$$

whenever  $a, b, c, d$  are pairwise distinct and cyclically ordered points on  $\mathbf{S}^1$ .

**Example 2.9.5** Let

$$K : (\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta \rightarrow \mathbf{R}_+$$

be locally integrable and symmetric (that is,  $K(\theta, \varphi) = K(\varphi, \theta)$ ). Then

$$d\mu(\theta, \varphi) = K(\theta, \varphi) d\theta d\varphi$$

is a geodesic current. In particular, this is the case for the measure

$$d\mu_0(\theta, \varphi) = \frac{d\theta d\varphi}{4 \sin^2((\theta - \varphi)/2)}.$$

We claim that the geodesic current  $\mu_0$  is invariant under  $SU(1, 1)$  for the induced action on  $(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta$ . Indeed, since

$$\begin{aligned} \frac{\partial^2}{\partial \theta \partial \varphi} \left( \frac{1}{2} \log \left( \frac{1}{4} |e^{i\theta} - e^{i\varphi}|^2 \right) \right) &= \frac{\partial^2}{\partial \theta \partial \varphi} \left( \frac{1}{2} \log \sin^2((\theta - \varphi)/2) \right) \\ &= \frac{1}{4 \sin^2((\theta - \varphi)/2)}, \end{aligned}$$

we have, for  $0 \leq a < b < c < d \leq 2\pi$ ,

$$\begin{aligned} \int_c^d \int_a^b \frac{d\theta d\varphi}{4 \sin^2((\theta - \varphi)/2)} &= \frac{1}{2} \log \left( \frac{1}{4} |e^{i\theta} - e^{i\varphi}|^2 \right) \Big|_{\varphi=c}^{\varphi=d} \Big|_{\theta=a}^{\theta=b} \\ &= \log \frac{|e^{ib} - e^{id}| |e^{ia} - e^{ic}|}{|e^{ib} - e^{ic}| |e^{ia} - e^{id}|}. \end{aligned}$$

The argument  $|e^{ib} - e^{id}| |e^{ia} - e^{ic}| / |e^{ib} - e^{ic}| |e^{ia} - e^{id}|$  is the cross-ratio of the four points  $(e^{ia}, e^{ib}, e^{ic}, e^{id})$  and is invariant by Möbius transformations (see [Conwa–98, Chapter III, Proposition 3.8]). Since the rectangles  $[a, b] \times [c, d]$  generate the Borel subsets on  $(\mathbf{S}^1 \times \mathbf{S}^1 \setminus \Delta) / (\mathbf{Z}/2\mathbf{Z})$ , this proves the claim.

Now, let  $\Gamma$  be a group and let  $\Phi : \Gamma \rightarrow \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  be a homomorphism. Let  $\pi$  be the orthogonal representation of  $\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  defined before Lemma 2.9.2. Then

$$\pi_\Phi = \pi \circ \Phi : \gamma \mapsto \pi(\Phi(\gamma))$$

is an orthogonal representation of  $\Gamma$  on  $\mathcal{H}$ , and, for  $\alpha > 1/2$ , the mapping

$$b_\Phi = b \circ \Phi : \gamma \mapsto b(\Phi(\gamma))$$

is a 1-cocycle on  $\Gamma$  with coefficients in  $\pi_\Phi$ .

**Proposition 2.9.6** *Let  $\Gamma$  be a group, let  $\alpha > 1/2$ , and let  $\Phi : \Gamma \rightarrow \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$  be a homomorphism such that  $H^1(\Gamma, \pi_\Phi) = 0$ . Then there exists a geodesic current  $\mu$  which is invariant under  $\Phi(\Gamma)$  and which has the following properties:*

$$(*) \quad \mu([a, a] \times [b, c]) = 0$$

$$(**) \quad \mu([a, b] \times (b, c]) = +\infty,$$

whenever  $a, b, c$  are pairwise distinct and cyclically ordered points on  $\mathbf{S}^1$ .

**Proof** Let  $\mathcal{K}$  be the closed subspace of  $\mathcal{H} = L_{\mathbf{R}}^2(\mathbf{S}^1 \times \mathbf{S}^1)$  consisting of the antisymmetric kernels, that is, the kernels  $K$  on  $\mathbf{S}^1$  with the property

$$K(\theta, \varphi) = -K(\varphi, \theta), \quad \text{for almost all } (\theta, \varphi) \in \mathbf{S}^1 \times \mathbf{S}^1.$$

Observe that  $\mathcal{K}$  is invariant under  $\pi_\Phi(\Gamma)$ . Observe also that the cocycle  $b_\Phi$  takes its values in  $\mathcal{K}$  since the kernel  $F$  is antisymmetric.

By Proposition 2.9.3 and its proof, there exists  $K \in \mathcal{K}$  such that the positive measure  $\mu$  on  $(\mathbf{S}^1 \times \mathbf{S}^1) \setminus \Delta$  given by

$$d\mu(\theta, \varphi) = (F - K)^2(\theta, \varphi) d\theta d\varphi$$

is invariant under  $\Phi(\Gamma)$ . Since  $(F - K)^2$  is a symmetric kernel,  $\mu$  is a geodesic current.

As  $\mu$  is absolutely continuous with respect to the Lebesgue measure  $d\theta d\varphi$ , the relation  $(*)$  above is obvious.

To verify relation  $(**)$ , let  $a, b, c \in [0, 2\pi)$  with  $a < b < c$ . Choose  $x, y \in [0, 2\pi)$  with  $a < x < b$  and  $b < y < c$ . Then

$$\begin{aligned} \mu([a, x] \times [y, c])^{1/2} &= \left( \int_a^x \int_y^c \left( \frac{1}{2 \tan((\theta - \varphi)/2)} - K(\theta, \varphi) \right)^2 d\theta d\varphi \right)^{1/2} \\ &\geq \left( \int_a^x \int_y^c \frac{d\theta d\varphi}{4 \tan^2((\theta - \varphi)/2)} \right)^{1/2} \\ &\quad - \left( \int_a^x \int_y^c K(\theta, \varphi)^2 d\theta d\varphi \right)^{1/2}, \end{aligned}$$

by the Minkowski inequality in  $L^2([a, x] \times [y, c])$ . Hence

$$\mu([a, x] \times [y, c])^{1/2} \geq \left( \int_a^x \int_y^c \frac{d\theta d\varphi}{4 \tan^2((\theta - \varphi)/2)} \right)^{1/2} - \|K\|,$$

and it suffices to show that

$$\lim_{(x,y) \rightarrow (b,b)} \int_a^x \int_y^c \frac{d\theta d\varphi}{4 \tan^2((\theta - \varphi)/2)} = +\infty.$$

To prove this, write, as in the proof of Lemma 2.9.2,

$$\frac{1}{2 \tan((\theta - \varphi)/2)} = \frac{1}{\theta - \varphi} + K_0(\theta, \varphi),$$

where  $K_0$  is continuous on  $\mathbf{S}^1 \times \mathbf{S}^1$ . Since

$$\int_a^x \int_y^c \frac{d\theta d\varphi}{(\theta - \varphi)^2} = -\log(y - x) + \log(c - x) + \log(y - a) - \log(c - a),$$

the claim follows.  $\blacksquare$

The geodesic current  $\mu_0$  appearing in Example 2.9.5 is, as we saw, invariant under  $SL(2, \mathbf{R})$ . On the other hand, it is a standard fact that a Möbius transformation is determined by its action on three arbitrary points in the projective line  $\mathbf{R} \cup \{\infty\} \simeq \mathbf{S}^1$ . In particular, an element of  $SL(2, \mathbf{R})$  fixing three points of  $\mathbf{S}^1$  acts as the identity. The next lemma, due to A. Navas [Nava-02a], generalises this fact.

**Lemma 2.9.7** *Assume that  $h \in \text{Homeo}^+(\mathbf{S}^1)$  fixes three points of  $\mathbf{S}^1$  and leaves invariant a geodesic current with the properties (\*) and (\*\*) of the previous proposition. Then  $h$  is the identity on  $\mathbf{S}^1$ .*

**Proof** Assume, by contradiction, that  $h$  fixes three points  $a, b, c \in \mathbf{S}^1$  and that  $h$  is not the identity on  $\mathbf{S}^1$ . Let  $I$  be a connected component of the set

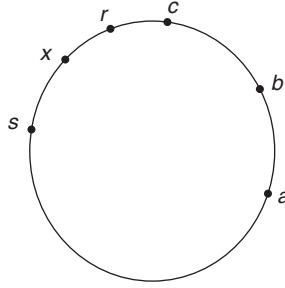
$$\{x \in \mathbf{S}^1 : h(x) \neq x\}.$$

Then  $I = (r, s)$  for some points  $r, s \in \mathbf{S}^1$ . We can assume that  $a, b, c, r, s$  are cyclically ordered.

Note that  $r$  and  $s$  are fixed under  $h$  and  $h[r, s] = [r, s]$ . Since  $h$  has no fixed points in  $[r, s]$ , except  $r$  and  $s$ , it follows that, for all  $x \in [r, s]$ ,

$$\text{either } \lim_{n \rightarrow \infty} h^n(x) = s \quad \text{or} \quad \lim_{n \rightarrow \infty} h^n(x) = r.$$



Figure 2.2 The cyclically ordered points  $a, b, c, r, s$ 

Fix  $x \in (r, s)$ . Upon replacing, if necessary,  $h$  by  $h^{-1}$ , we can assume that the first case occurs:

$$\lim_{n \rightarrow \infty} h^n(x) = s \quad \text{and} \quad \lim_{n \rightarrow \infty} h^{-n}(x) = r.$$

Let  $\mu$  be an  $h$ -invariant geodesic current with Properties (\*) and (\*\*). Since  $h(r) = r$  and  $h(s, b] = (s, b]$ , we have by  $h$ -invariance of  $\mu$ :

$$\mu([r, x) \times (s, b]) = \mu([r, h(x)) \times (s, b]).$$

Since  $[r, x) \subsetneq [r, h(x))$ , we have

$$\mu([x, h(x)) \times (s, b]) = 0.$$

Thus

$$\mu([h^n(x), h^{n+1}(x)) \times (s, b]) = 0$$

by  $h$ -invariance of  $\mu$  and, since  $(r, s) = \bigcup_{n \in \mathbb{Z}} [h^n(x), h^{n+1}(x))$ ,

$$\mu((r, s) \times (s, b]) = 0$$

by  $\sigma$ -additivity of  $\mu$ . Since  $r, s, b$  are cyclically ordered and pairwise distinct (it is here that we use the fact that  $h$  fixes three points), we have

$$\mu([r, r] \times (s, b]) = 0$$

by Condition (\*). It follows that

$$\mu([r, s) \times (s, b]) = 0.$$

This contradicts Condition (\*\*), and this ends the proof. ■

### Groups acting freely on $\mathbf{S}^1$

We intend to prove that, if a group  $G$  acts freely on  $\mathbf{S}^1$ , then  $G$  has to be abelian. This will follow from a result of Hölder about bi-orderable groups.

A group  $G$  is *bi-orderable* if there exists a total order relation  $\leq$  on  $G$  which is bi-invariant, that is, such that  $x \leq y$  implies  $axb \leq ayb$  for all  $x, y, a, b \in G$ . This order is said to be *archimedean* if, for any pair of elements  $a, b \in G$  with  $a > e$  and  $b > e$ , there exists an integer  $n \geq 1$  such that  $a^n > b$ .

**Example 2.9.8** Let  $\text{Homeo}_+(\mathbf{R})$  denote the group of orientation preserving homeomorphisms of the real line. Let  $G$  be a subgroup of  $\text{Homeo}_+(\mathbf{R})$  acting *freely* on  $\mathbf{R}$ . Then  $G$  is bi-orderable, with an archimedean order. Indeed, fix a base point  $x_0 \in \mathbf{R}$ . For  $f, g \in G$ , set  $f \leq g$  if  $f(x_0) \leq g(x_0)$ . Since  $G$  acts freely, this defines a total order  $\leq$  on  $G$ . It is clear that this order is left invariant. Moreover, this order is independent of the choice of  $x_0$ , again by freeness of the action of  $G$ . It follows that  $\leq$  is right invariant. We claim that  $\leq$  is archimedean. Indeed, let  $f, g \in G$  with  $f > e$ . Since  $f(x_0) > x_0$ , we have  $\lim_{n \rightarrow \infty} f^n(x_0) = +\infty$  and hence  $f^n > g$  for  $n$  large enough.

The following result is a weak version of a theorem due to Hölder. For a more general statement, see [BotRh–77, Theorem 1.3.4], or [HecHi–00, Theorem 3.1.6], or [Ghys–01, Theorem 6.10].

**Proposition 2.9.9** *Let  $(G, \leq)$  be a bi-ordered group. Assume that the order  $\leq$  is archimedean. Then  $G$  is abelian.*

**Proof** We follow a proof shown to us by A. Navas. We first claim that, for any  $a \in G$  with  $a > e$ , we have

$$(*) \quad a^{-2} < bcb^{-1}c^{-1} < a^2 \quad \text{for all } b, c \in G.$$

Indeed, since  $\leq$  is archimedean, there exist integers  $m, n$  such that

$$a^m \leq b < a^{m+1} \quad \text{and} \quad a^n \leq c < a^{n+1}.$$

Since  $\leq$  is bi-invariant, we have

$$a^{m+n} \leq bc < a^{m+n+2} \quad \text{and} \quad a^{m+n} \leq cb < a^{m+n+2},$$

and  $(*)$  follows.

• *First case:* there exists a minimal element  $a \in G$  with  $a > e$ . Then  $G$  is generated by  $a$  and therefore  $G \cong \mathbf{Z}$ . Indeed, for every  $b \in G$ , there exists  $m \in \mathbf{Z}$  with  $a^m \leq b < a^{m+1}$  and hence  $b = a^m$  by minimality of  $a$ .

• *Second case:* there exists no minimal element  $x \in G$  with  $x > e$ . Assume, by contradiction, that  $G$  is not abelian. Then there exist  $b, c \in G$  such that  $bcb^{-1}c^{-1} > e$ . Set  $\bar{a} = bcb^{-1}c^{-1}$ . By assumption, we can find  $a \in G$  with  $\bar{a} > a > e$ . Inequality (\*) above shows that  $\bar{a} < a^2$ . Hence,  $\bar{a}a^{-1} < a$  and therefore  $a^{-1}\bar{a}a^{-1} < e$ . It follows that

$$(\bar{a}a^{-1})^2 = \bar{a}(a^{-1}\bar{a}a^{-1}) < \bar{a} = bcb^{-1}c^{-1}.$$

This is a contradiction to (\*) applied to  $\bar{a}a^{-1}$  in place of  $a$ . ■

**Corollary 2.9.10** *If a group  $G$  acts freely on the circle  $\mathbf{S}^1$ , then  $G$  is abelian.*

**Proof** The group  $G$  can be seen as a subgroup of  $\text{Homeo}_+(\mathbf{S}^1)$ . Let  $\tilde{G}$  be the subgroup of  $\text{Homeo}_+(\mathbf{R})$  consisting of all orientation preserving homeomorphisms  $f$  of  $\mathbf{R}$  which are obtained by lifting some  $f \in G$ . There is a canonical surjective homomorphism  $\tilde{G} \rightarrow G$ . It is clear that  $\tilde{G}$  acts freely on  $\mathbf{R}$ . In view of Example 2.9.8, it follows from Proposition 2.9.9 that  $\tilde{G}$  is abelian. Hence,  $G$  is abelian. ■

### Proof of Theorem 2.9.1

• *First step:* We claim that  $\Phi(\Gamma)$  acts freely on  $\mathbf{S}^1$ , that is, if  $\Phi(\gamma)$  fixes some point of  $\mathbf{S}^1$ , then  $\Phi(\gamma)$  is the identity on  $\mathbf{S}^1$ .

To see this, we follow an argument due to D. Witte. Consider the triple covering mapping

$$p : \mathbf{S}^1 \rightarrow \mathbf{S}^1, \quad z \mapsto z^3.$$

Let  $\text{Aut}(p)$  be the group of automorphisms of the covering  $p$ , that is,

$$\text{Aut}(p) = \{\tilde{h} \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1) : p \circ \tilde{h} = h \circ p \text{ for some } h \in \text{Diff}_+^{1+\alpha}(\mathbf{S}^1)\}.$$

By standard covering theory, there is a central extension

$$0 \rightarrow \mathbf{Z}/3\mathbf{Z} \rightarrow \text{Aut}(p) \rightarrow \text{Diff}_+^{1+\alpha}(\mathbf{S}^1) \rightarrow 1$$

where the generator of  $\mathbf{Z}/3\mathbf{Z}$  acts by

$$z \mapsto \zeta z,$$

$\zeta$  being a non-trivial third root of unity. This gives rise to a pull-back diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbf{Z}/3\mathbf{Z} & \rightarrow & \text{Aut}(p) & \rightarrow & \text{Diff}_+^{1+\alpha}(\mathbf{S}^1) \rightarrow 1 \\ & & \uparrow \text{Id} & & \uparrow \tilde{\Phi} & & \uparrow \Phi \\ 0 & \rightarrow & \mathbf{Z}/3\mathbf{Z} & \rightarrow & \tilde{\Gamma} & \rightarrow & \Gamma \rightarrow 1 \end{array}$$

The group  $\tilde{\Gamma}$  has Property (FH) since it is an extension of  $\Gamma$  by a finite group (Proposition 2.5.4). Let  $b_{\tilde{\Phi}} \in Z^1(\tilde{\Gamma}, \pi_{\tilde{\Phi}})$  be the 1-cocycle associated to  $\tilde{\Phi}$  by Lemma 2.9.2. Since  $\tilde{\Gamma}$  has Property (FH), we have  $b_{\tilde{\Phi}} \in B^1(\tilde{\Gamma}, \pi_{\tilde{\Phi}})$ . By Proposition 2.9.6, it follows that  $\tilde{\Phi}(\tilde{\Gamma})$  preserves a geodesic current  $\tilde{\mu}$  such that

$$\tilde{\mu}([a, a] \times [b, c]) = 0 \quad \text{and} \quad \tilde{\mu}([a, b] \times (b, c]) = +\infty$$

whenever  $a, b, c$  are cyclically ordered on  $\mathbf{S}^1$ .

Assume that, for  $\gamma \in \Gamma$ , the diffeomorphism  $\Phi(\gamma)$  fixes the point  $z_0 \in \mathbf{S}^1$ . Then all the three inverse images of  $\gamma$  in  $\tilde{\Gamma}$  preserve the fibre  $p^{-1}(z_0)$ , and act on it by cyclically permuting its elements. Hence, one of these inverse images, call it  $\tilde{\gamma}$ , is such that  $\tilde{\Phi}(\tilde{\gamma})$  fixes all three points of  $p^{-1}(z_0)$ . By Lemma 2.9.7, the diffeomorphism  $\tilde{\Phi}(\tilde{\gamma})$  is the identity, so that  $\Phi(\gamma)$  is the identity.

• *Second step:* We claim that  $\Phi(\Gamma)$  is a finite cyclic group. Indeed, since  $\Phi(\Gamma)$  acts freely on the circle, it must be abelian by Corollary 2.9.10. On the other hand, as  $\Gamma$  has Property (FH), its abelianisation  $\Gamma/[\Gamma, \Gamma]$  is finite (Corollary 2.5.10). Hence,  $\Phi(\Gamma)$  is a finite abelian group.

It remains to show that  $\Phi(\Gamma)$  is cyclic. This is true for every finite group of diffeomorphisms of  $\mathbf{S}^1$  as shown in the next lemma. ■

**Lemma 2.9.11** *Let  $\Gamma$  be a finite group of orientation preserving diffeomorphisms of  $\mathbf{S}^1$ . Then  $\Gamma$  is conjugate in  $\text{Homeo}^+(\mathbf{S}^1)$  to a subgroup of  $SO(2)$  and is hence cyclic.*

**Proof** If  $d\theta$  is the normalised Lebesgue measure on  $\mathbf{S}^1$ , then the measure

$$\nu = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma_*(d\theta)$$

is a probability measure on  $\mathbf{S}^1$  which is preserved by  $\Gamma$  and equivalent to  $d\theta$ . Hence, by the Radon–Nikodym theorem, there is an orientation preserving homeomorphism  $f$  of  $\mathbf{S}^1$  which transforms  $\nu$  into  $d\theta$ . Then  $f \circ \gamma \circ f^{-1}$  belongs to the stabiliser of  $d\theta$  in  $\text{Homeo}^+(\mathbf{S}^1)$ , for every  $\gamma \in \Gamma$ . Now (Exercise 2.14.11) this stabiliser is exactly  $SO(2)$ . Thus,  $\Gamma$  is isomorphic to a finite subgroup of  $SO(2)$ , and so it is cyclic. ■

**Example 2.9.12** Let  $\Gamma$  be the group of *R.J. Thompson* (see [CaFIP–96], where this group is denoted by  $T$ ), namely the group of piecewise linear orientation preserving homeomorphisms  $f$  of  $\mathbf{R}/\mathbf{Z}$  such that:

- $f'$  has finitely many discontinuity points, which are all rational dyadics,
- the slopes of  $f$  are powers of 2,
- $f(0)$  is a rational dyadic.

It is known that  $\Gamma$  is a simple, infinite group with finite presentation. It has been shown by E. Ghys and V. Sergiescu [GhySe–87, Theorem A] that  $\Gamma$  is conjugate to a subgroup of  $\text{Diff}^\infty(\mathbf{S}^1)$ . The question was asked in [GhySe–87] (and repeated in [HarVa–89]) whether  $\Gamma$  has Property (T). Since  $\Gamma$  can be viewed as a subgroup of  $\text{Diff}_+^{1+\alpha}(\mathbf{S}^1)$ , we see that the answer is negative. It has recently been shown by D. Farley [Farle–03] that  $\Gamma$  has the Haagerup Property of Definition 2.7.5.

## 2.10 Functions conditionally of negative type

Recall that a continuous real valued kernel  $\Psi$  on a topological space  $X$  is conditionally of negative type if  $\Psi(x, x) = 0$ ,  $\Psi(x, y) = \Psi(y, x)$  for all  $x, y$  in  $X$ , and

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(x_i, x_j) \leq 0$$

for any elements  $x_1, \dots, x_n$  in  $X$ , and any real numbers  $c_1, \dots, c_n$  with  $\sum_{i=1}^n c_i = 0$ . Recall also that a continuous real valued function  $\psi$  on a topological group  $G$  is conditionally of negative type if the kernel on  $G$  defined by  $(g, h) \mapsto \psi(h^{-1}g)$  is conditionally of negative type. See Section C.2.

**Example 2.10.1** Let  $G$  be a topological group, and let  $\alpha$  be an affine isometric action of  $G$  on a real Hilbert space  $\mathcal{H}$ . It follows from Example C.2.2.ii that, for any  $\xi \in \mathcal{H}$ , the function

$$\psi : G \rightarrow \mathbf{R}, \quad g \mapsto \|\alpha(g)\xi - \xi\|^2$$

is conditionally of negative type. In particular, for any orthogonal representation  $\pi$  on  $\mathcal{H}$  and for any  $b \in Z^1(G, \pi)$ , the function  $g \mapsto \|b(g)\|^2$  is conditionally of negative type. Indeed, if  $\alpha$  is the affine isometric action of  $G$  associated to  $\pi$  and  $b$ , then  $b(g) = \alpha(g)\xi - \xi$  for  $\xi = 0$ .

It is a natural question to ask if, conversely, any function on  $G$  which is conditionally of negative type is of the form  $g \mapsto \|b(g)\|^2$ . Our next proposition provides the answer.

**Proposition 2.10.2** *Let  $\psi$  be a function conditionally of negative type on a topological group  $G$ . There exists a pair  $(\mathcal{H}_\psi, \alpha_\psi)$ , where  $\mathcal{H}_\psi$  is a real Hilbert space and  $\alpha_\psi$  is an affine isometric action of  $G$  on  $\mathcal{H}_\psi$ , with the following properties:*

- (i)  $\psi(g) = \|\alpha_\psi(g)(0)\|^2$  for every  $g \in G$ ;
- (ii) the linear span of  $\{\alpha_\psi(g)(0) : g \in G\}$  is dense in  $\mathcal{H}_\psi$ .

The pair  $(\mathcal{H}_\psi, \alpha_\psi)$  is unique in the following sense: if  $(\mathcal{H}, \alpha)$  is a pair with Properties (i) and (ii), then there exists a  $G$ -equivariant affine isometry from  $\mathcal{H}_\psi$  onto  $\mathcal{H}$ .

**Proof** By the existence part of Theorem C.2.3, there exists a real Hilbert space  $\mathcal{H}_\psi$  and a mapping  $f : G \rightarrow \mathcal{H}_\psi$  with the following properties:

- (a)  $\psi(y^{-1}x) = \|f(x) - f(y)\|^2$  for all  $x, y \in G$ ;
- (b) the linear span of  $\{f(x) - f(x_0) : x \in G\}$  is dense in  $\mathcal{H}_\psi$  for every  $x_0 \in G$ .

Replacing, if necessary,  $f$  by  $f - f(e)$ , we can assume that  $f(e) = 0$ . For every  $g \in G$ , the mapping

$$G \rightarrow \mathcal{H}_\psi, \quad x \mapsto f(gx)$$

has the Properties (a) and (b) above. By the uniqueness assertion of Theorem C.2.3, it follows that there is a unique affine isometry  $\alpha_\psi(g) : \mathcal{H}_\psi \rightarrow \mathcal{H}_\psi$  such that

$$f(gx) = \alpha_\psi(g)f(x) \quad \text{for all } x \in G.$$

We claim that  $\alpha_\psi$  is an affine isometric action of  $G$  on  $\mathcal{H}_\psi$ . To show this, observe first that  $g \mapsto \alpha_\psi(g)$  is a homomorphism from  $G$  to  $\text{Isom}(\mathcal{H}_\psi)$ , again by the uniqueness assertion. Moreover, the mapping  $G \rightarrow \mathcal{H}_\psi, g \mapsto \alpha_\psi(g)\xi$  is continuous for every  $\xi \in \mathcal{H}_\psi$ . Indeed, as the linear span of

$$\{f(x) : x \in G\} = \{f(x) - f(e) : x \in G\}$$

is dense in  $\mathcal{H}_\psi$ , it suffices to show this when  $\xi = f(x)$  for some  $x \in G$ . Since

$$\begin{aligned} \|\alpha_\psi(g)f(x) - \alpha_\psi(h)f(x)\|^2 &= \|f(gx) - f(hx)\|^2 \\ &= \psi(x^{-1}h^{-1}gx), \end{aligned}$$

for  $g, h \in G$ , the claim follows from the continuity of  $\psi$ .

The affine isometric action  $\alpha_\psi$  satisfies Properties (i) and (ii) of the present proposition since

$$\psi(g) = \|f(g) - f(e)\|^2 = \|f(g)\|^2 = \|\alpha_\psi(g)(0)\|^2$$

for every  $g \in G$  and since the linear span of

$$\{\alpha_\psi(g)(0) : g \in G\} = \{f(g) : g \in G\}$$

is dense in  $\mathcal{H}_\psi$ .

We leave the proof of the uniqueness statement as Exercise 2.14.6. ■

As a consequence, we obtain the following characterisation of bounded functions conditionally of negative type.

**Corollary 2.10.3** *Let  $\psi$  be a function conditionally of negative type on the topological group  $G$ , and let  $(\mathcal{H}_\psi, \alpha_\psi)$  be the pair associated to  $\psi$  by Proposition 2.10.2. The following properties are equivalent:*

- (i)  $\psi$  is bounded on  $G$ ;
- (ii)  $\alpha_\psi$  has a fixed point;
- (iii) there exists a real valued function of positive type  $\varphi$  on  $G$  and a constant  $c \geq 0$  such that  $\psi = c - \varphi$ .

**Proof** The equivalence of (i) and (ii) follows from Propositions 2.10.2 and 2.2.9. Since functions of positive type are bounded, it is obvious that (iii) implies (i).

Assume that  $\alpha_\psi$  has a fixed point  $x \in \mathcal{H}_\psi$ . Let  $\pi_\psi$  denote the linear part of  $\alpha_\psi$ . By (i) of Proposition 2.10.2 we have, for every  $g \in G$ ,

$$\begin{aligned} \psi(g) &= \|\alpha_\psi(g)0\|^2 \\ &= \|(\alpha_\psi(g)0 - x) + x\|^2 \\ &= \|(\alpha_\psi(g)0 - \alpha_\psi(g)x) + x\|^2 \\ &= \|\pi_\psi(g)x + x\|^2 \\ &= 2\|x\|^2 - 2\langle \pi_\psi(g)x, x \rangle. \end{aligned}$$

Since  $g \mapsto 2\langle \pi_\psi(g)x, x \rangle$  is a function of positive type, this shows that (iii) holds. ■

We give now a reformulation of Property (FH) in terms of functions conditionally of negative type:

**Theorem 2.10.4** *Let  $G$  be a topological group. The following statements are equivalent:*

- (i)  $G$  has Property (FH);
- (ii)  $H^1(G, \pi) = 0$  for every orthogonal representation  $\pi$  of  $G$ ;
- (iii) every function conditionally of negative type on  $G$  is bounded.

**Proof** The equivalence of (i) and (ii) is Proposition 2.2.10; the equivalence of (ii) and (iii) is a consequence of Proposition 2.10.2 and Example 2.10.1. ■

**Example 2.10.5** Let  $\Gamma$  be a group generated by a finite set  $S$ , and let  $d_S$  denote the word distance on  $\Gamma$  with respect to  $S$  (see Section G.5).

(i) If  $\Gamma$  is free on  $S$ , the kernel  $d_S$  is conditionally of negative type, as first shown in [Haage–78]. This can be reformulated in terms of the distance kernel on a regular tree. See Example C.2.2.iii.

(ii) Let  $(W, S)$  be a Coxeter system. It was proved by Bozejko, Januszkiewicz, and Spatzier that  $d_S$  is a kernel conditionally of negative type [BoJaS–88]. It follows that a Coxeter group has the Haagerup Property (see Definition 2.7.5). In particular, an infinite Coxeter group does not have Property (FH).

## 2.11 A consequence of Schoenberg’s Theorem

Let  $G$  be a topological group. Recall Schoenberg’s Theorem (Corollary C.4.19): a real valued continuous function  $\psi$  on  $G$  with  $\psi(e) = 0$  and  $\psi(g^{-1}) = \psi(g)$  for all  $g \in G$  is conditionally of negative type if and only if the function  $\exp(-t\psi)$  is of positive type for every  $t \geq 0$ . We single out an important consequence of Schoenberg’s Theorem.

**Proposition 2.11.1** *Let  $\pi$  be an orthogonal representation of  $G$  on a real Hilbert space  $\mathcal{H}$ . Let  $b \in Z^1(G, \pi)$ . Denote by  $\alpha$  the affine isometric action of  $G$  associated to  $\pi$  and  $b$ .*

*Fix  $t > 0$ . Then there exists a complex Hilbert space  $\mathcal{H}_t$ , a continuous mapping  $\Phi_t$  from  $\mathcal{H}$  to the unit sphere of  $\mathcal{H}_t$ , and a unitary representation  $\pi_t$  of  $G$  on  $\mathcal{H}_t$ , with the following properties:*

- (i)  $\langle \Phi_t(\xi), \Phi_t(\eta) \rangle = \exp(-t\|\xi - \eta\|^2)$  for all  $\xi, \eta \in \mathcal{H}$ ;
- (ii)  $\pi_t(g)\Phi_t(\xi) = \Phi_t(\alpha(g)\xi)$  for all  $g \in G$  and  $\xi \in \mathcal{H}$ ;
- (iii) the linear span of  $\Phi_t(\mathcal{H})$  is dense in  $\mathcal{H}_t$ .

*The pair  $(\mathcal{H}_t, \pi_t)$  is unique up to unique isomorphism. Moreover:*

- (iv) *if  $(\xi_n)_n$  is a sequence in  $\mathcal{H}$  tending to infinity (that is,  $\lim_n \|\xi_n\| = \infty$ ), then  $(\Phi_t(\xi_n))_n$  tends weakly to 0 in  $\mathcal{H}_t$ ;*
- (v) *the action  $\alpha$  has a fixed point in  $\mathcal{H}$  if and only if the representation  $\pi_t$  has a non-zero fixed vector in  $\mathcal{H}_t$ .*

**Proof** Let  $\mathbb{G}$  be the semidirect product  $G \ltimes \mathcal{H}$  of  $G$  with the additive group of  $\mathcal{H}$ , with the product

$$(g, \xi)(h, \eta) = (gh, \pi(g)\eta + \xi), \quad g, h \in G, \quad \xi, \eta \in \mathcal{H}.$$



Let  $\Pi$  be the orthogonal representation of  $\mathbb{G}$  on  $\mathcal{H}$  given by

$$\Pi(g, \xi) = \pi(g), \quad (g, \xi) \in \mathbb{G}.$$

Define a 1-cocycle  $B$  on  $\mathbb{G}$  by

$$B(g, \xi) = b(g) + \xi, \quad (g, \xi) \in \mathbb{G},$$

and let  $\Psi$  be the corresponding function conditionally of negative type on  $\mathbb{G}$

$$\Psi(g, \xi) = \|B(g, \xi)\|^2 = \|b(g) + \xi\|^2, \quad (g, \xi) \in \mathbb{G}.$$

By Schoenberg's Theorem, the function  $\exp(-t\Psi)$  is of positive type on  $\mathbb{G}$ . Let  $(\mathcal{H}_t, \Pi_t, \xi_t)$  be the GNS triple associated to  $\exp(-t\Psi)$ ; see Theorem C.4.10. Recall that  $\Pi_t$  is a unitary representation of  $\mathbb{G}$  in a Hilbert space  $\mathcal{H}_t$  and  $\xi_t$  is a cyclic unit vector in  $\mathcal{H}_t$  such that

$$\exp(-t\Psi)(g, \xi) = \langle \Pi_t(g, \xi)\xi_t, \xi_t \rangle, \quad (g, \xi) \in \mathbb{G}.$$

Let  $\pi_t$  be the unitary representation of  $G$  obtained by restriction of  $\Pi_t$  to  $G$ , that is,

$$\pi_t(g) = \Pi_t(g, 0), \quad g \in G.$$

Define  $\Phi_t : \mathcal{H} \rightarrow \mathcal{H}_t$  by

$$\Phi_t(\xi) = \Pi_t(e, \xi)(\xi_t), \quad \xi \in \mathcal{H}.$$

Then, for  $\xi, \eta \in \mathcal{H}$ , we have

$$\begin{aligned} \langle \Phi_t(\xi), \Phi_t(\eta) \rangle &= \langle \Pi_t(e, \xi)(\xi_t), \Pi_t(e, \eta)(\xi_t) \rangle \\ &= \langle \Pi_t(e, \eta)^{-1} \Pi_t(e, \xi)(\xi_t), \xi_t \rangle \\ &= \langle \Pi_t(e, \xi - \eta)(\xi_t), \xi_t \rangle \\ &= \exp(-t\Psi)(e, \xi - \eta) = \exp(-t\|\xi - \eta\|^2). \end{aligned}$$

This proves (i).

Observe first that, for  $g \in G$ , we have

$$\begin{aligned} \langle \Phi_t(b(g)), \pi_t(g)\xi_t \rangle &= \langle \Pi_t(g, 0)^{-1} \Pi_t(e, b(g))(\xi_t), \xi_t \rangle \\ &= \langle \Pi_t(g^{-1}, \pi(g^{-1})b(g))(\xi_t), \xi_t \rangle \\ &= \langle \Pi_t(g^{-1}, -b(g^{-1}))(\xi_t), \xi_t \rangle \\ &= \exp(-t\Psi)(g^{-1}, -b(g^{-1})) = 1, \end{aligned}$$

since  $\Psi(g^{-1}, -b(g^{-1})) = 0$ . Hence, as  $\Phi_t(b(g))$  and  $\pi_t(g)\xi_t$  are unit vectors, it follows from the equality case of the Cauchy–Schwarz inequality that

$$(*) \quad \Phi_t(b(g)) = \pi_t(g)\xi_t, \quad \text{for all } g \in G.$$

From this, we deduce that, for  $g \in G$  and  $\xi \in \mathcal{H}$ ,

$$\begin{aligned} \pi_t(g)\Phi_t(\xi) &= \Pi_t(g, 0)\Pi_t(e, \xi)(\xi_t) \\ &= \Pi_t(g, \pi(g)\xi)(\xi_t) \\ &= \Pi_t(e, \pi(g)\xi)\Pi_t(g, 0)(\xi_t) \\ &= \Pi_t(e, \pi(g)\xi)\pi_t(g)\xi_t \\ &= \Pi_t(e, \pi(g)\xi)\Phi_t(b(g)) \\ &= \Pi_t(e, \pi(g)\xi)\Pi_t(e, b(g))(\xi_t) \\ &= \Pi_t(e, \pi(g)\xi + b(g))\xi_t = \Phi_t(\alpha(g)\xi). \end{aligned}$$

This proves (ii).

From (\*), we compute also:

$$\begin{aligned} \Pi_t(g, \xi)\xi_t &= \Pi_t(e, \xi)\pi_t(g)\xi_t = \Pi_t(e, \xi)\Phi_t(b(g)) \\ &= \Pi_t(e, \xi)\Pi_t(e, b(g))\xi_t = \Pi_t(e, b(g) + \xi)\xi_t \\ &= \Phi_t(b(g) + \xi). \end{aligned}$$

Since  $\xi_t$  is a cyclic vector for  $\Pi_t$ , it follows that the linear span of  $\Phi_t(\mathcal{H})$  is dense in  $\mathcal{H}_t$  and this proves (iii). The uniqueness of the pair  $(\mathcal{H}_t, \pi_t)$  follows from the uniqueness of the GNS-construction (Theorem C.4.10).

Let  $(\xi_n)_n$  be a sequence in  $\mathcal{H}$  such that  $\lim_n \|\xi_n\| = \infty$ . Then, by (i), we have, for every  $\xi \in \mathcal{H}$ ,

$$\langle \Phi_t(\xi_n), \Phi_t(\xi) \rangle = \exp(-t\|\xi_n - \xi\|^2)$$

and hence  $\lim_n \langle \Phi_t(\xi_n), \Phi_t(\xi) \rangle = 0$ , because  $t > 0$ . Since the linear span of  $\{\Phi_t(\xi) : \xi \in \mathcal{H}\}$  is dense in  $\mathcal{H}_t$  by (iii) and since  $(\Phi_t(\xi_n))_n$  is bounded, it follows that  $\lim_n \langle \Phi_t(\xi_n), \zeta \rangle = 0$  for any  $\zeta \in \mathcal{H}_t$ . This proves (iv).

Assume that  $\alpha$  has a fixed point  $\xi \in \mathcal{H}$ . Then  $\Phi_t(\xi)$  is a fixed unit vector for  $\pi_t(G)$ , by (ii). Conversely, assume that  $\alpha$  has no fixed point in  $\mathcal{H}$ . Then, by Proposition 2.2.9, all orbits of  $\alpha$  are unbounded. Let  $\zeta \in \mathcal{H}_t$  be a fixed vector for  $\pi_t(G)$ . We want to show that  $\zeta = 0$ . Fix an arbitrary  $\xi \in \mathcal{H}$ , and let  $(g_n)_n$

be a sequence in  $G$  such that

$$\lim_n \|\alpha(g_n)\xi\| = \infty.$$

By (iv), the sequence  $(\Phi_t(\alpha(g_n)\xi))_n$  converges weakly to 0. Hence,

$$\begin{aligned} \langle \Phi_t(\xi), \zeta \rangle &= \lim_n \langle \Phi_t(\xi), \pi_t(g_n^{-1})\zeta \rangle \\ &= \lim_n \langle \pi_t(g_n)\Phi_t(\xi), \zeta \rangle \\ &= \lim_n \langle \Phi_t(\alpha(g_n)\xi), \zeta \rangle = 0, \end{aligned}$$

as  $\zeta$  is  $\pi_t(G)$ -fixed. Since the linear span of the  $\Phi_t(\xi)$ 's is dense in  $\mathcal{H}_t$ , it follows that  $\zeta = 0$ . ■

For later use (proofs of Theorem 2.12.4 and Theorem 2.12.9), we record the following fact.

**Proposition 2.11.2** *Let  $\alpha$  be an affine isometric action of the topological group  $G$ . For  $t > 0$ , let  $(\Phi_t, \mathcal{H}_t, \pi_t)$  be the triple associated to  $\alpha$  as in Proposition 2.11.1. If  $(t_n)_n$  is a sequence of positive real numbers with  $\lim_n t_n = 0$ , then  $\bigoplus_n \pi_{t_n}$  weakly contains  $1_G$ .*

**Proof** Set  $\pi = \bigoplus_n \pi_{t_n}$  and let  $\xi_n = \Phi_{t_n}(0) \in \mathcal{H}_{t_n}$ . Then, using (i) and (ii) of Proposition 2.11.1, we have  $\|\xi_n\| = 1$  and

$$\begin{aligned} \|\pi_{t_n}(g)\xi_n - \xi_n\|^2 &= \|\Phi_{t_n}(\alpha(g)(0)) - \Phi_{t_n}(0)\|^2 \\ &= 2 \left( 1 - \exp(-t_n \|\alpha(g)(0)\|^2) \right). \end{aligned}$$

It follows that

$$\lim_n \|\pi_{t_n}(g)\xi_n - \xi_n\| = 0$$

uniformly on compact subsets of  $G$ . Viewing each  $\xi_n$  as a vector in the Hilbert space of  $\pi$ , we see that, if  $Q$  is a compact subset of  $G$  and  $\varepsilon > 0$ , then  $\xi_n$  is  $(Q, \varepsilon)$ -invariant for  $n$  large enough, so that  $\pi$  almost has invariant vectors. ■

We give an application of Schoenberg's Theorem to real and complex hyperbolic spaces.

Let  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ . Recall that the  $n$ -dimensional hyperbolic space  $\mathbf{H}^n(\mathbf{K})$  over  $\mathbf{K}$  can be realized as the subset of the projective space  $\mathbf{P}^n(\mathbf{K})$  defined by the condition  $\langle z, z \rangle < 0$ , where

$$\langle z, w \rangle = -\overline{z_{n+1}}w_{n+1} + \sum_{i=1}^n \overline{z_i}w_i$$

(see Section 2.7). The distance between  $[z], [w] \in \mathbf{H}^n(\mathbf{K})$  is given by

$$\cosh d([z], [w]) = \frac{|\langle z, w \rangle|}{|\langle z, z \rangle|^{1/2} |\langle w, w \rangle|^{1/2}}.$$

We will assume that  $z, w \in \mathbf{K}^{n+1}$  are always choosed so that

$$\cosh d([z], [w]) = |\langle z, w \rangle|.$$

The following result is due to J. Faraut and K. Harzallah [FarHa-74, Proposition 7.3].

**Theorem 2.11.3** *Let  $\mathbf{K} = \mathbf{R}$  or  $\mathbf{K} = \mathbf{C}$ . The kernel on the hyperbolic space  $\mathbf{H}^n(\mathbf{K})$  defined by*

$$(x, y) \mapsto \log(\cosh d(x, y))$$

*is conditionally of negative type.*

**Proof** Let

$$X = \{z \in \mathbf{K}^{n+1} : \langle z, z \rangle < 0\}.$$

By Schoenberg's Theorem (Theorem C.3.2), we have to show that the kernel on  $X$

$$(z, w) \mapsto |\langle z, w \rangle|^{-t}$$

is of positive type for all  $t \geq 0$ . We have

$$|\langle z, w \rangle|^{-t} = (|z_{n+1}| |w_{n+1}|)^{-t} \left| 1 - (\overline{z_{n+1}} w_{n+1})^{-1} \sum_{i=1}^n \overline{z_i} w_i \right|^{-t}.$$

(Observe that  $|z_{n+1}| \neq 0$  for  $z \in X$ .) Since the kernel  $(z, w) \mapsto (|z_{n+1}| |w_{n+1}|)^{-t}$  is obviously of positive type, it suffices to show that the kernel  $\Phi$  defined by

$$\Phi(z, w) = \left| 1 - (\overline{z_{n+1}} w_{n+1})^{-1} \sum_{i=1}^n \overline{z_i} w_i \right|^{-t}$$

is of positive type for all  $t \geq 0$ . Set

$$\Phi_0(z, w) = (\overline{z_{n+1}} w_{n+1})^{-1} \sum_{i=1}^n \overline{z_i} w_i.$$

The kernel  $\Phi_0$  is of positive type, since  $\Phi_0(z, w) = \langle f(z), f(w) \rangle$  for the mapping

$$f : X \rightarrow \mathbf{K}^n, \quad z \mapsto (\overline{z_1/z_{n+1}}, \dots, \overline{z_n/z_{n+1}}).$$

(Here,  $\langle \cdot, \cdot \rangle$  denotes the usual inner product on  $\mathbf{K}^n$ .) Moreover, by the Cauchy–Schwarz inequality, we have

$$|\Phi_0(z, w)| \leq \frac{\sqrt{\sum_{i=1}^n |z_i|^2} \sqrt{\sum_{i=1}^n |w_i|^2}}{|z_{n+1}| |w_{n+1}|} < 1,$$

for all  $z, w \in X$ . By Lemma C.1.8, it follows that the kernels

$$\Phi_1 : (z, w) \mapsto (1 - \Phi_0(x, y))^{-t/2} \quad \text{and} \quad \Phi_2 : (z, w) \mapsto (1 - \overline{\Phi_0(x, y)})^{-t/2}$$

are of positive type. Therefore,  $\Phi = \Phi_1 \Phi_2$  is of positive type. ■

**Remark 2.11.4** Let  $G = O(n, 1)$  or  $G = U(n, 1)$ . The kernel  $\Psi = \log \cosh d$  on  $\mathbf{H}^n(\mathbf{K})$  is  $G$ -invariant. It therefore defines a function conditionally of negative type on  $G$ , denoted by  $\psi$  (see Remark C.4.18). Since  $\psi$  is proper, it is unbounded on every closed and non-compact subgroup  $L$  of  $G$ . It follows from Proposition 2.10.2 that such a subgroup has the Haagerup Property (see Definition 2.7.5), and in particular it does not have Property (FH); for another proof of this fact, see Theorem 2.7.2.

## 2.12 The Delorme–Guichardet Theorem

### 1-cohomology and weak containment

Recall that we defined in Section 2.4 the notion of “almost having invariant vectors” for orthogonal representations of a topological group.

**Remark 2.12.1** (i) Let  $\pi$  be an orthogonal representation of the topological group  $G$ . Let  $\pi_{\mathbf{C}}$  be the complexification of  $\pi$  (see Remark A.7.2). The orthogonal representation  $\pi$  almost has invariant vectors if and only if the unitary representation  $\pi_{\mathbf{C}}$  almost has invariant vectors.

Indeed, let  $\xi \in \mathcal{H}$  be a  $(Q, \varepsilon)$ -invariant unit vector for  $\pi$ . Then  $\xi \otimes 1 \in \mathcal{H}_{\mathbf{C}}$  is a  $(Q, \varepsilon)$ -invariant unit vector for  $\pi_{\mathbf{C}}$ . Conversely, let  $\eta \in \mathcal{H}_{\mathbf{C}}$  be a  $(Q, \varepsilon/2)$ -invariant unit vector for  $\pi_{\mathbf{C}}$ . We have  $\eta = \xi_1 \otimes 1 + \xi_2 \otimes i$  for vectors  $\xi_1, \xi_2 \in \mathcal{H}$ , where, say,  $\|\xi_1\| \geq 1/2$ . Then

$$\|\pi(g)(\xi_1/\|\xi_1\|) - (\xi_1/\|\xi_1\|)\| < \varepsilon/(2\|\xi_1\|) \leq \varepsilon, \quad \text{for all } g \in Q$$

that is,  $\xi_1/\|\xi_1\|$  be a  $(Q, \varepsilon)$ -invariant unit vector for  $\pi$ .

(ii) Let  $\alpha$  be an affine isometric action of the topological group  $G$  on a real Hilbert space. For  $t > 0$ , let  $(\Phi_t, \mathcal{H}_t, \pi_t)_t$  be the associated triple as in Proposition 2.11.1. The unitary representation  $\pi_t$  is the complexification of an orthogonal representation. Indeed,  $\pi_t$  is associated by GNS-construction to the *real-valued* function of positive type  $g \mapsto \exp(-t\|\alpha(g)(0)\|^2)$ .

Recall that, for an orthogonal representation of a topological group  $G$ , the real vector space  $Z^1(G, \pi)$  is endowed with the topology of uniform convergence on compact subsets of  $G$ , that is, with the locally convex topology given by the family of seminorms

$$p_Q(b) = \sup_{g \in Q} \|b(g)\|,$$

where  $Q$  runs over all compact subsets of  $G$ . Assume that  $G$  is  $\sigma$ -compact, that is,  $G$  is the union of an increasing sequence  $(Q_n)_n$  of compact subsets. Then the topology of  $G$  is given by the separating sequence of seminorms  $(p_{Q_n})_n$  and is therefore metrisable. Clearly,  $Z^1(G, \pi)$  is complete, so that  $Z^1(G, \pi)$  is a Fréchet space.

The following result establishes a link between 1-cohomology and Fell's topology and is due to A. Guichardet (see [Guic-72a, Théorème 1]).

**Proposition 2.12.2** *Let  $G$  be a locally compact group, and let  $\pi$  be an orthogonal representation of  $G$  on a real Hilbert space  $\mathcal{H}$ , without non-zero invariant vectors.*

- (i) *If  $\pi$  does not almost have invariant vectors, then  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ .*
- (ii) *If  $G$  is  $\sigma$ -compact, then the converse holds, namely: if  $\pi$  almost has invariant vectors, then  $B^1(G, \pi)$  is not closed in  $Z^1(G, \pi)$ . In particular,  $H^1(G, \pi) \neq 0$ .*

**Proof** Consider the mapping

$$\Phi : \mathcal{H} \rightarrow B^1(G, \pi), \quad \xi \mapsto \pi(\cdot)\xi - \xi.$$

It is clear that  $\Phi$  is linear, continuous and surjective. Since  $\pi$  has no non-zero invariant vectors,  $\Phi$  is also injective.

(i) Assume that  $\pi$  does not almost have invariant vectors. Then there exist a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$  such that

$$(*) \quad p_Q(\Phi(\xi)) = \sup_{g \in Q} \|\pi(g)\xi - \xi\| \geq \varepsilon \|\xi\|$$

for all  $\xi \in \mathcal{H}$ . Let  $(\xi_i)_i$  be a net of vectors in  $\mathcal{H}$  such that  $(\Phi(\xi_i))_i$  converges to some  $b \in Z^1(G, \pi)$ . Then there exists a subsequence  $(\xi_n)_n$  such that

$$\lim_n p_Q(\Phi(\xi_n) - b) = 0.$$

By (\*) above,  $(\xi_n)_n$  is a Cauchy sequence and, hence, converges to some vector  $\xi \in \mathcal{H}$ . It is clear that  $\Phi(\xi) = b$ , showing that  $B^1(G, \pi)$  is closed.

(ii) Assume that  $B^1(G, \pi)$  is closed in  $Z^1(G, \pi)$ . Since  $G$  is  $\sigma$ -compact,  $Z^1(G, \pi)$  and, hence,  $B^1(G, \pi)$  is a Fréchet space. The open mapping theorem (see [Rudin–73, Corollaries 2.12]) shows that  $\Phi$  is bicontinuous. Therefore, there exists a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$  such that

$$p_Q(\Phi(\xi)) = \sup_{g \in Q} \|\pi(g)\xi - \xi\| \geq \varepsilon \|\xi\|$$

for all  $\xi \in \mathcal{H}$ . This implies that  $\pi$  does not almost have invariant vectors. ■

**Example 2.12.3** (i) Let  $G$  be a non-amenable locally compact group, and let  $\lambda_G$  be the left regular representation of  $G$  on  $L^2_{\mathbf{R}}(G)$ . Then  $\lambda_G$  does not almost have invariant vectors (Theorem G.3.2). Therefore,  $B^1(G, \lambda_G)$  is closed in  $Z^1(G, \lambda_G)$ .

(ii) Let  $G$  be a  $\sigma$ -compact locally compact group. Assume that  $G$  is amenable and non-compact. Then  $\lambda_G$  almost has invariant vectors, but no non-zero invariant ones. Hence,  $B^1(G, \lambda_G)$  is not closed in  $Z^1(G, \lambda_G)$ .

### The Delorme–Guichardet Theorem

The following theorem shows that Property (T) and Property (FH) are closely related. Part (i) is due to P. Delorme [Delor–77, Théorème V.1]; see also [Wang–74, Theorem 1] for the finite dimensional case. Part (ii) was already shown in Corollary 2.4.7. We give here a different proof based on Proposition 2.12.2.

**Theorem 2.12.4 (Delorme–Guichardet)** *Let  $G$  be a topological group.*

- (i) *If  $G$  has Property (T), then  $G$  has Property (FH).*
- (ii) *If  $G$  is a  $\sigma$ -compact locally compact group and if  $G$  has Property (FH), then  $G$  has Property (T).*

**Proof** (i) Assume that  $G$  does not have Property (FH). Then there exists an affine isometric action  $\alpha$  of  $G$  without fixed points. For  $t > 0$ , let  $(\mathcal{H}_t, \Phi_t, \pi_t)$  be the triple associated to  $\alpha$  as in Proposition 2.11.1. Set

$$\pi = \bigoplus_{n=1}^{\infty} \pi_{1/n}.$$

By (v) of Proposition 2.11.1, the representation  $\pi_t$  has no non-zero invariant vectors, for every  $t > 0$ . Hence,  $\pi$  has no non-zero invariant vectors. On the other hand,  $\pi$  almost has invariant vectors, by Proposition 2.11.2. It follows that  $G$  does not have Property (T).

(ii) Assume that  $G$  is a  $\sigma$ -compact locally compact group and that  $G$  does not have Property (T). Then there exists a unitary representation  $\pi$  of  $G$  on a Hilbert space  $\mathcal{H}$  which almost has invariant vectors, but has no non-zero invariant vectors. Let  $\mathcal{H}_{\mathbf{R}}$  be the space  $\mathcal{H}$  viewed as real Hilbert space and  $\pi_{\mathbf{R}}$  the representation  $\pi$  considered as orthogonal representation on  $\mathcal{H}_{\mathbf{R}}$ . Then  $\pi_{\mathbf{R}}$  almost has invariant vectors (see Remark 2.4.6). Hence,  $H^1(G, \pi_{\mathbf{R}}) \neq 0$ , by Proposition 2.12.2.ii. ■

**Remark 2.12.5** (i) The assumption that  $G$  is  $\sigma$ -compact in Part (ii) of the Delorme–Guichardet Theorem is necessary. Indeed, as mentioned in Remark 2.4.3, there are examples of uncountable discrete groups which have Property (FH). Such groups do not have Property (T), since they are not finitely generated.

(ii) The Delorme–Guichardet Theorem extends to pairs  $(G, H)$  consisting of a group  $G$  and a subgroup  $H$ ; see Exercise 2.14.9.

Combining the previous theorem with Theorems 2.3.6 and 2.7.2, we obtain the following consequences.

**Theorem 2.12.6** *Let  $G$  be topological group with Property (T). Every action of  $G$  on a tree has either a fixed vertex or a fixed geometric edge.*

**Theorem 2.12.7** *Let  $G$  be a topological group with Property (T). Any continuous homomorphism  $G \rightarrow O(n, 1)$  or  $G \rightarrow U(n, 1)$  has relatively compact image. In particular, a closed non-compact subgroup of  $O(n, 1)$  or  $U(n, 1)$  does not have Property (T).*



**Remark 2.12.8** As  $SL_2(\mathbf{R}) \cong SU(1, 1)$ , we obtain another proof that  $SL_2(\mathbf{R})$  does not have Property (T); compare with Example 1.3.7, Example 1.7.4, and Proposition 2.6.5.

### Another characterisation of Property (T)

The following characterisation of  $\sigma$ -compact locally compact groups with Property (T) is Theorem 1 in [BekVa–93]. It is based on the Delorme–Guichardet Theorem and will be used in Chapter 6 (see Theorem 6.3.4).

**Theorem 2.12.9** *Let  $G$  be a  $\sigma$ -compact locally compact group. The following statements are equivalent:*

- (i)  $G$  has Property (T);
- (ii) any unitary representation  $(\pi, \mathcal{H})$  of  $G$  which weakly contains  $1_G$  contains a non-zero finite dimensional subrepresentation.

Given an affine action  $\alpha$  of  $G$  on a real Hilbert space  $\mathcal{H}$ , we first identify the family of unitary representations associated to the diagonal action  $\alpha \oplus \alpha$  of  $G$  on  $\mathcal{H} \oplus \mathcal{H}$ .

**Lemma 2.12.10** *Let  $\alpha$  be an affine isometric action of the topological group  $G$  on the real Hilbert space  $\mathcal{H}$ . Let  $A = \alpha \oplus \alpha$  be the diagonal affine isometric action of  $G$  on  $\mathcal{H} \oplus \mathcal{H}$ . Let  $(\pi_t)_{t>0}$  and  $(\Pi_t)_{t>0}$  be the families of unitary representations of  $G$  associated respectively to  $\alpha$  and  $A$  as in Proposition 2.11.1. Then  $\Pi_t$  is unitarily equivalent to  $\pi_t \otimes \pi_t$  for every  $t > 0$ .*

**Proof** Fix  $t > 0$ . Let  $\Phi_t : \mathcal{H} \rightarrow \mathcal{H}_t$  be as in Proposition 2.11.1. Define a mapping

$$\Psi_t : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}_t \otimes \mathcal{H}_t$$

by  $\Psi_t(\xi, \xi') = \Phi_t(\xi) \otimes \Phi_t(\xi')$ . For all  $g \in G$  and  $\xi, \xi', \eta, \eta' \in \mathcal{H}$ , we have

$$\begin{aligned} \langle \Psi_t(\xi, \xi'), \Psi_t(\eta, \eta') \rangle &= \exp(-t\|\xi - \eta\|^2) \exp(-t\|\xi' - \eta'\|^2) \\ &= \exp\left(-t\|(\xi, \xi') - (\eta, \eta')\|^2\right), \end{aligned}$$

and

$$\Psi_t(A(g)(\xi, \xi')) = (\pi_t \otimes \pi_t)(g)\Psi_t(\xi, \xi'), \quad \xi, \xi' \in \mathcal{H}.$$

Moreover, the image of  $\Psi_t$  is total in  $\mathcal{H}_t \otimes \mathcal{H}_t$ . By the uniqueness statement in Proposition 2.11.1, it follows that  $\Pi_t$  and  $\pi_t \otimes \pi_t$  are unitarily equivalent. ■

**Proof of Theorem 2.12.9** It is obvious that (i) implies (ii).

Assume that (ii) holds. By the Delorme–Guichardet Theorem, it suffices to show that  $G$  has Property (FH). Let  $\alpha$  be an affine isometric action of  $G$  on a real Hilbert space  $\mathcal{H}$ . For each  $t > 0$ , let  $(\Phi_t, \mathcal{H}_t, \pi_t)_t$  be the associated triple. Fix a sequence of positive real numbers with  $\lim_n t_n = 0$ , and write  $\pi_n$  for  $\pi_{t_n}$ . Let

$$\pi = \bigoplus_n \pi_n.$$

By Proposition 2.11.2, the representation  $\pi$  weakly contains  $1_G$ . Hence, by Assumption (ii),  $\pi$  contains a non-zero finite dimensional subrepresentation, that is,  $\pi \otimes \bar{\pi}$  contains  $1_G$ . Since

$$\pi \otimes \bar{\pi} = \bigoplus_{n_1, n_2} \pi_{n_1} \otimes \bar{\pi}_{n_2},$$

it follows that there is a pair  $(n_1, n_2)$  such that  $\pi_{n_1} \otimes \bar{\pi}_{n_2}$  contains  $1_G$ . Observe that, for every  $t > 0$ , the representations  $\pi_t$  and  $\bar{\pi}_t$  are unitarily equivalent since  $\pi_t$  is the complexification of an orthogonal representation (see Remark 2.12.1.ii). Hence,  $\pi_{n_1} \otimes \pi_{n_2}$  contains  $1_G$ . It follows that  $\pi_{n_1}$  contains a non-zero finite dimensional subrepresentation, and in turn, this implies that  $\pi_{n_1} \otimes \pi_{n_1}$  contains  $1_G$  (see Proposition A.1.12). By the previous lemma,  $\pi_{n_1} \otimes \pi_{n_1}$  is associated to the diagonal action  $\alpha \oplus \alpha$ . Proposition 2.11.1.v applied to  $\alpha \oplus \alpha$  shows that  $\alpha \oplus \alpha$  has a fixed point in  $\mathcal{H} \oplus \mathcal{H}$ . Hence,  $\alpha$  has a fixed point in  $\mathcal{H}$ . ■

**Remark 2.12.11** Let  $G$  be a  $\sigma$ -compact locally compact group without Property (T). The proof above shows that there exists a unitary representation  $\pi$  of  $G$  with the following properties:  $\pi$  is the complexification of an orthogonal representation,  $\pi$  contains weakly  $1_G$ , and  $\pi$  has no non-zero finite dimensional subrepresentation.

## 2.13 Concordance

There is some overlap between Chapter 1 on Property (T) and Chapter 2 on Property (FH), as a consequence of our choice to write large parts of them independently of each other. The following concordance table is designed to help the reader find corresponding facts. Notation:  $G$  is a topological group,  $H$  a closed subgroup of  $G$ , and  $\varphi$  a continuous homomorphism from  $G$  to some other group.

Item	Property (T)	Property (FH)
Definition for $G$	1.1.3	2.1.4
Equivalent definitions	1.2.5	2.2.10, 2.10.4
Definition for $(G, H)$	1.4.3	2.2.11
$G$ compact	1.1.5	2.2.9
$G$ amenable	1.1.6	2.5.2
Compact generation	1.3.1	2.4.2
$\varphi : G_1 \rightarrow G_2$ with dense image, quotients	1.3.4	2.5.1
$\varphi : G \rightarrow \mathbf{R}^n$ or $\mathbf{Z}^n$	1.3.5	2.5.2
$G/[G, G]$	1.3.6	2.5.10
Closed subgroups	1.7.1	2.5.7
$(G, N)$ and $N$	1.8.12	2.14.8
Overgroups	1.7.1	2.5.5
Short exact sequences	1.7.6	2.5.4
Property (FA)	2.12.6	2.3.6
Subgroups of $O(n, 1)$ and $SU(n, 1)$	2.12.7	2.6.5, 2.7.2
$G = \cup_n H_n$	1.8.11	2.4.1
Covering groups	1.7.11	3.5.3

## 2.14 Exercises

**Exercise 2.14.1** Let  $\pi$  be an orthogonal (or unitary) representation of a topological group  $G$  on a real (or complex) Hilbert space  $\mathcal{H}$ . For  $b \in Z^1(G, \pi)$ , prove that

- (i)  $b(e) = 0$ ;
- (ii)  $b(g^{-1}) = -\pi(g^{-1})b(g)$ , for all  $g \in G$ ;
- (iii)  $b(hgh^{-1}) = \pi(h)b(g)$  for all  $g \in \text{Ker } \pi$  and  $h \in G$ .

**Exercise 2.14.2** Let  $\pi$  be an orthogonal (or unitary) representation  $\pi$  of a topological group  $G$  on a real (or complex) Hilbert space  $\mathcal{H}$ . Let  $b : G \rightarrow \mathcal{H}$  be a 1-cocycle, that is, a mapping satisfying the cocycle relation  $b(gh) = b(g) + \pi(g)b(h)$  for all  $g, h \in G$ . Show that  $b$  is continuous on  $G$  if and only if  $b$  is continuous at the unit element  $e$  of  $G$ .

**Exercise 2.14.3** Let  $G$  be a locally compact group and let  $\pi$  be an orthogonal (or unitary) representation of  $G$  on a separable real (or complex) Hilbert space  $\mathcal{H}$ . Let  $b : G \rightarrow \mathcal{H}$  be a 1-cocycle. Assume that  $b$  is weakly measurable, that is, the function  $G \rightarrow \mathbf{R}$ ,  $g \mapsto \langle b(g), \xi \rangle$  is measurable for every  $\xi \in \mathcal{H}$ . We claim that  $b$  is continuous.

To show this, we imitate the proof of Lemma A.6.2. By Exercise 2.14.2, it suffices to show that  $b$  is continuous at the unit element  $e$ . Fix  $\varepsilon > 0$ , and let  $A = \{g \in G : \|b(g)\| < \varepsilon/2\}$ .

- (i) Show that  $A$  is measurable.
- (ii) Verify that  $A = A^{-1}$  and that  $A^2 \subset \{g \in G : \|b(g)\| < \varepsilon\}$ .
- (iii) Since  $\mathcal{H}$  is separable, there exists a sequence  $(g_n)_n$  in  $G$  such that  $(b(g_n))_n$  is dense in  $b(G)$ . Show that  $G = \bigcup_n g_n A$ .
- (iv) Show that  $b$  is continuous at  $e$  and is hence continuous on  $G$ .

**Exercise 2.14.4** Let  $F_2$  be the free group on two generators  $x$  and  $y$ , and let  $\pi$  be an orthogonal representation of  $F_2$  on a real Hilbert space  $\mathcal{H}$ .

- (i) Let  $f : Z^1(F_2, \pi) \rightarrow \mathcal{H} \oplus \mathcal{H}$  be the mapping defined by  $f(b) = (b(x), b(y))$ . Prove that  $f$  is a continuous linear bijection.
- (ii) Deduce from (i) that  $H^1(F_2, \pi) \neq 0$ . [This is a result from [Guic-72a].]

**Exercise 2.14.5** Let  $G$  be a locally compact group acting continuously and transitively on a locally compact space  $X$ . Assume that the stabiliser in  $G$  of some (and, hence, of any) point in  $X$  is compact. Show that the action of  $G$  (and, hence, of any closed subgroup of  $G$ ) on  $X$  is proper, that is, for any compact subsets  $Q$  and  $Q'$  of  $X$ , the set

$$\{g \in G : gQ \cap Q' \neq \emptyset\}$$

is compact. [This fact is used in the proof of Proposition 2.6.5.]

**Exercise 2.14.6** Prove the uniqueness assertion of Proposition 2.10.2.

**Exercise 2.14.7** Let  $G$  be a topological group and  $H$  a closed subgroup of  $G$ . Prove that the following properties are equivalent:

- (i)  $(G, H)$  has Property (FH);
- (ii) for every orthogonal representation  $\pi$  of  $G$ , the restriction mapping  $\text{Res}_G^H : H^1(G, \pi) \rightarrow H^1(H, \pi|_H)$  is the zero mapping.

**Exercise 2.14.8** Let  $G$  be a topological group and let  $N$  be a closed normal subgroup of  $G$ . Assume that the quotient group  $G/N$  has Property (FH) and that the pair  $(G, N)$  has Property (FH). Show that  $G$  has Property (FH).

**Exercise 2.14.9** Let  $G$  be a topological group and let  $H$  be a closed subgroup of  $G$ .

- (i) Assume that the pair  $(G, H)$  has Property (T). Show that  $(G, H)$  has Property (FH).

(ii) Assume that  $G$  is a  $\sigma$ -compact locally compact group and that  $(G, H)$  has Property (FH). Show that  $(G, H)$  has Property (T).

[Hint: To show (i), use a suitable version of Proposition 2.11.1. To show (ii), use the cocycle which appears in the proof of Proposition 2.4.5.]

**Exercise 2.14.10** (This exercise was suggested to us by Y. Shalom.)

Let  $\Gamma$  denote the matrix group  $SU(2)$  with the discrete topology. Show that  $\Gamma$  does not have Property (FH).

[Hint: Since the locally compact group  $SL_2(\mathbf{C})$  does not have Property (FH), there exists a continuous isometric action  $\alpha$  of  $SL_2(\mathbf{C})$  on a real Hilbert space  $\mathcal{H}$  without fixed point. Consider an action of  $\Gamma$  on  $\mathcal{H}$  of the form  $\alpha \circ \theta$ , where  $\theta$  is the automorphism of  $SL_2(\mathbf{C})$  induced by a wild automorphism of  $\mathbf{C}$ .]

**Exercise 2.14.11** Show that the stabiliser of the Lebesgue measure on the circle  $\mathbf{S}^1$  in  $\text{Homeo}^+(\mathbf{S}^1)$  consists exactly of the rotations. [This fact is used in the proof of Lemma 2.9.11.]

**Exercise 2.14.12 (Affine isometric actions on Banach spaces)** Let  $B$  be a real Banach space. Let  $G$  be a topological group. We say that  $G$  has *Property*  $(F_B)$  if every continuous action of  $G$  on  $B$  by affine isometries has a fixed point.

(i) Let  $G$  be a locally compact  $\sigma$ -compact group with Property  $(F_B)$ . Show that  $G$  has Property  $(T_B)$  from Exercise 1.8.20.

[Hint: Imitate the proof of 2.12.4.ii.]

(ii) Show that, in general, Property  $(T_B)$  does not imply Property  $(F_B)$ .

[Hint: Let  $B = \mathbf{R}$ . Show that every topological group has Property  $(T_B)$ . Show that  $\mathbf{Z}$  does not have Property  $(F_B)$ .]

Property  $(T_B)$  does imply Property  $(F_B)$  for some classes of Banach spaces. This is the case, for instance, when  $B = L^p(X, \mu)$  for a measure space  $(X, \mu)$  and  $1 \leq p \leq 2$  (see [BaFGM]).

### 3

## Reduced cohomology

Given an orthogonal representation  $\pi$  of a topological group  $G$ , the space  $Z^1(G, \pi)$  of 1-cocycles with coefficients in  $\pi$  has a natural topology (see Section 2.12). In this chapter, we study the reduced cohomogy group

$$\overline{H^1}(G, \pi) = Z^1(G, \pi) / \overline{B^1(G, \pi)},$$

where  $\overline{B^1(G, \pi)}$  is the closure of  $B^1(G, \pi)$  in  $Z^1(G, \pi)$ . We first characterise elements from  $\overline{B^1(G, \pi)}$  in terms of the associated affine isometric action and in terms of the corresponding function conditionally of negative type. This provides a new entry to the dictionary set up in the introduction to Chapter 2:

Affine isometric actions	1-cocycles	Functions $\psi$ conditionally of negative type
Almost having fixed points	Cocycles in $\overline{B^1(G, \pi)}$	Condition (ii) in Proposition 3.1.5

The central result in this chapter is Shalom's characterisation of Property (T) for compactly generated, locally compact groups by the vanishing of the *reduced* 1-cohomology of all their unitary representations.

As a first application of Shalom's Theorem, we give a new and short proof of Property (T) for  $Sp(n, 1)$ ,  $n \geq 2$ , and  $F_{4(-20)}$ , based on ideas due to Gromov. Combined with Chapter 1, this completes the classification of the semisimple real Lie groups with Property (T). As another application, we show that every group with Property (T) is a quotient of a finitely presented group with Property (T). The last section is devoted to the proof that Property (T) is not invariant under quasi-isometries.

### 3.1 Affine isometric actions almost having fixed points

Let  $G$  be a topological group, and let  $\pi$  be an orthogonal representation of  $G$  on a real Hilbert space  $\mathcal{H}$ . The space  $Z^1(G, \pi)$ , endowed with the topology of uniform convergence on compact subsets of  $G$ , is a Hausdorff topological real vector space (see Section 2.12). The closure  $\overline{B^1(G, \pi)}$  of  $B^1(G, \pi)$  in  $Z^1(G, \pi)$  is a closed linear subspace of  $Z^1(G, \pi)$ .

**Definition 3.1.1** The *reduced 1-cohomology* of  $G$  with coefficients in  $\pi$  is the quotient vector space

$$\overline{H^1(G, \pi)} = Z^1(G, \pi) / \overline{B^1(G, \pi)}.$$

We are going to characterise the elements in  $\overline{B^1(G, \pi)}$  by their associated affine actions, as we did for the elements in  $B^1(G, \pi)$  in Section 2.2.

**Definition 3.1.2** Let  $\alpha$  be an affine isometric action of  $G$  on  $\mathcal{H}$ . We say that  $\alpha$  *almost has fixed points* if, for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists  $x \in \mathcal{H}$  such that

$$\sup_{g \in Q} \|\alpha(g)x - x\| < \varepsilon.$$

**Proposition 3.1.3** Let  $\pi$  be an orthogonal representation of the topological group  $G$  on a real Hilbert space  $\mathcal{H}$ . Let  $b \in Z^1(G, \pi)$ , with associated affine isometric action  $\alpha$ . The following properties are equivalent:

- (i)  $b$  belongs to  $\overline{B^1(G, \pi)}$ ;
- (ii)  $\alpha$  almost has fixed points.

**Proof** Let  $Q$  be a compact subset of  $G$  and  $\varepsilon > 0$ . Observe that, for  $g \in G$  and  $\xi \in \mathcal{H}$ , we have

$$(*) \quad \|b(g) - (\pi(g)\xi - \xi)\| = \|\alpha(g)(-\xi) - (-\xi)\|.$$

Assume that  $b \in \overline{B^1(G, \pi)}$ . Then there exists  $\xi \in \mathcal{H}$  such that

$$\|b(g) - (\pi(g)\xi - \xi)\| < \varepsilon, \quad \text{for all } g \in Q,$$

and, hence,  $\alpha$  almost has fixed points. Conversely, if  $\alpha$  almost has fixed points, then there exists  $\xi \in \mathcal{H}$  such that

$$\|\alpha(g)(-\xi) - (-\xi)\| < \varepsilon, \quad \text{for all } g \in Q,$$

and  $(*)$  shows that  $b \in \overline{B^1(G, \pi)}$ . ■

To a function conditionally of negative type  $\psi$  on  $G$  is associated a pair  $(\mathcal{H}_\psi, \alpha_\psi)$  as in Proposition 2.10.2. Recall that  $\alpha_\psi$  is an affine isometric action of  $G$  on the real Hilbert space  $\mathcal{H}_\psi$  such that  $\psi(g) = \|\alpha_\psi(g)(0)\|^2$  for every  $g \in G$  and such that the linear span of  $\{\alpha_\psi(g)(0) : g \in G\}$  is dense in  $\mathcal{H}_\psi$ . We denote by  $\pi_\psi$  the linear part of  $\alpha_\psi$  and by  $b_\psi$  the 1-cocycle with coefficients in  $\pi_\psi$  associated to  $\alpha_\psi$ , that is,

$$b_\psi(g) = \alpha_\psi(g)(0), \quad g \in G.$$

We will call  $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$  the *triple associated to  $\psi$* .

We need a formula for the inner product on  $\mathcal{H}_\psi$ .

**Lemma 3.1.4** *Let  $\psi$  be a function conditionally of negative type on the topological group  $G$ , and let  $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$  be the associated triple. For  $g, h \in G$ , we have*

$$\langle b_\psi(g), b_\psi(h) \rangle = \frac{1}{2}(\psi(g) + \psi(h) - \psi(h^{-1}g)).$$

**Proof** We have

$$\begin{aligned} & \langle \alpha_\psi(g)(0), \alpha_\psi(h)(0) \rangle \\ &= \frac{1}{2} \left( \|\alpha_\psi(g)(0)\|^2 + \|\alpha_\psi(h)(0)\|^2 - \|\alpha_\psi(g)(0) - \alpha_\psi(h)(0)\|^2 \right) \\ &= \frac{1}{2} \left( \|\alpha_\psi(g)(0)\|^2 + \|\alpha_\psi(h)(0)\|^2 - \|\alpha_\psi(h^{-1}g)(0)\|^2 \right) \\ &= \frac{1}{2}(\psi(g) + \psi(h) - \psi(h^{-1}g)) \end{aligned}$$

for all  $g, h \in G$ . ■

We give a characterisation, due to Y. Shalom [Shal–00a, Corollary 6.6], of those functions conditionally of negative type  $\psi$  for which  $b_\psi$  belongs to  $\overline{B^1(G, \pi_\psi)}$ .

**Proposition 3.1.5** *Let  $\psi$  be a function conditionally of negative type on the topological group  $G$ , and let  $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$  be the associated triple. The following properties are equivalent:*

- (i)  $b_\psi \in \overline{B^1(G, \pi_\psi)}$ ;
- (ii) for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exist non-negative real numbers  $a_1, \dots, a_n$  with  $\sum_{i=1}^n a_i = 1$  and  $g_1, \dots, g_n \in G$  such that

$$(*) \quad \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left( \psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i) \right) < \varepsilon, \quad \text{for all } g \in Q.$$



Assume that  $G$  is a compactly generated, locally compact group, and let  $Q$  be a compact generating set of  $G$ . Then (i) and (ii) are equivalent to:

(ii') for every  $\varepsilon > 0$ , there exist non-negative real numbers  $a_1, \dots, a_n$  with  $\sum_{i=1}^n a_i = 1$  and  $g_1, \dots, g_n \in G$  such that

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \left( \psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i) \right) < \varepsilon, \quad \text{for all } g \in Q.$$

**Proof** By Proposition 3.1.3, we have to show that (ii) holds if and only if the action  $\alpha_\psi$  almost has fixed points.

Using the formula from the previous lemma, we have, for  $a_1, \dots, a_n \in \mathbf{R}$  with  $\sum_{i=1}^n a_i = 1$  and  $g, g_1, \dots, g_n \in G$ ,

$$\begin{aligned} (1) \quad \left\| \alpha_\psi(g) \left( \sum_{i=1}^n a_i b_\psi(g_i) \right) - \sum_{i=1}^n a_i b_\psi(g_i) \right\|^2 &= \left\| \sum_{i=1}^n a_i (b_\psi(g g_i) - b_\psi(g_i)) \right\|^2 \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \left( \psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i) \right). \end{aligned}$$

If Condition (\*) is satisfied, then  $\sum_{i=1}^n a_i b_\psi(g_i)$  is a point in  $\mathcal{H}_\psi$  which is moved under  $\alpha_\psi(g)$  by less than  $\varepsilon$  for all  $g \in Q$ . Hence,  $\alpha_\psi$  almost has fixed points, showing that (ii) implies (i).

Assume, conversely, that  $\alpha_\psi$  almost has fixed points. Let  $\mathcal{C}$  be the convex hull in  $\mathcal{H}_\psi$  of  $\{b_\psi(g) : g \in G\}$ . The set  $\mathcal{C}$  is invariant under  $\alpha_\psi(G)$ . The closure  $\overline{\mathcal{C}}$  of  $\mathcal{C}$  is a convex, closed subset of  $\mathcal{H}_\psi$ . As is well known (see, e.g., [Rudin-73, Theorem 12.3]), for every  $\xi \in \mathcal{H}_\psi$ , there exists a unique vector  $P_{\mathcal{C}}(\xi) \in \overline{\mathcal{C}}$  such that

$$\|P_{\mathcal{C}}(\xi) - \xi\| = \min\{\|\eta - \xi\| : \eta \in \mathcal{C}\}.$$

Moreover,  $P_{\mathcal{C}}$  is distance decreasing, that is,

$$\|P_{\mathcal{C}}(\xi) - P_{\mathcal{C}}(\eta)\| \leq \|\xi - \eta\|, \quad \text{for all } \xi, \eta \in \mathcal{H}_\psi.$$

Since  $\overline{\mathcal{C}}$  is  $\alpha_\psi(G)$ -invariant and  $\alpha_\psi$  is isometric, it is clear that  $P_{\mathcal{C}}$  is  $\alpha_\psi(G)$ -equivariant. Let  $Q$  be a compact subset of  $G$  and  $\varepsilon > 0$ . There exists  $\xi \in \mathcal{H}_\psi$  such that

$$\|\alpha_\psi(g)\xi - \xi\| < \sqrt{\varepsilon}$$

for all  $g \in Q$ . Then

$$\|\alpha_\psi(g)P_C(\xi) - P_C(\xi)\| < \sqrt{\varepsilon}, \quad \text{for all } g \in Q.$$

For  $\eta \in C$  close enough to  $P_C(\xi)$ , we have

$$\|\alpha_\psi(g)\eta - \eta\| < \sqrt{\varepsilon}, \quad \text{for all } g \in Q.$$

Now  $\eta$  can be written as  $\eta = \sum_{i=1}^n a_i b_\psi(g_i)$  for some  $g_1, \dots, g_n \in G$  and some  $a_1, \dots, a_n \geq 0$  with  $\sum_{i=1}^n a_i = 1$ . By Equation (1), it follows that

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j \left( \psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i) \right) < \varepsilon$$

for all  $g \in Q$ . Hence, (i) implies (ii).

It remains to show that (ii') implies (i); compare Proposition F.1.7. Assume that  $(*)$  holds for a compact generating subset  $Q$ . Let  $K$  be an arbitrary compact subset of  $G$  and  $\varepsilon > 0$ . Then  $K$  is contained in  $(Q \cup Q^{-1})^n$  for some  $n$  (see the proof of Proposition F.1.7). As shown above, we find a point  $v \in \mathcal{H}$  such that

$$\|\alpha_\psi(g)v - v\| < \varepsilon/n, \quad \text{for all } g \in Q.$$

Since  $\alpha_\psi(g^{-1})$  is an isometry, the same inequality holds for all  $g \in Q^{-1}$ . For  $g \in K$ , there exists  $g_1, g_2, \dots, g_n \in Q \cup Q^{-1}$  such that  $g = g_1 g_2 \cdots g_n$ . Using the triangle inequality and the fact that  $\alpha_\psi$  is an isometric action, we have

$$\begin{aligned} \|\alpha_\psi(g)v - v\| &\leq \|\alpha_\psi(g_1 \dots g_n)v - \alpha_\psi(g_1 \dots g_{n-1})v\| \\ &\quad + \|\alpha_\psi(g_1 \dots g_{n-1})v - \alpha_\psi(g_1 \dots g_{n-2})v\| \\ &\quad + \cdots + \|\alpha_\psi(g_1)v - v\| \\ &= \|\alpha_\psi(g_n)v - v\| + \cdots + \|\alpha_\psi(g_1)v - v\| \\ &< n \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

This shows that  $\alpha_\psi$  almost has fixed points. ■

### 3.2 A theorem by Y. Shalom

This section is devoted to a characterisation, due to Y. Shalom, of Property (T) for compactly generated locally compact groups in terms of their *reduced*

1-cohomology [Shal-00a, Theorem 0.2 and Theorem 6.1]; in the particular case of finitely presented groups, the result has been shown earlier by N. Mok, with a proof of another nature [Mok-95]. It is appropriate to formulate the theorem in terms of unitary representations; to apply the results of Chapter 2, we view any such representation as an orthogonal one on the underlying real Hilbert spaces (Remark 2.4.6).

Note that a compactly generated locally compact group is  $\sigma$ -compact, and so Property (T) and Property (FH) are equivalent for this class of groups, by the Delorme–Guichardet Theorem 2.12.4.

**Theorem 3.2.1** *Let  $G$  be a locally compact group which is second countable and compactly generated. The following conditions are equivalent:*

- (i)  $G$  has Property (T);
- (ii)  $H^1(G, \pi) = 0$  for every irreducible unitary representation  $\pi$  of  $G$ ;
- (iii)  $\overline{H^1}(G, \pi) = 0$  for every irreducible unitary representation  $\pi$  of  $G$ ;
- (iv)  $\overline{H^1}(G, \pi) = 0$  for every unitary representation  $\pi$  of  $G$ .

**Remark 3.2.2** (i) The fact that (ii) implies (i) answers positively a conjecture of Vershik and Karpushev [VerKa-82, page 514].

(ii) The equivalence of (iii) and (iv) in Theorem 3.2.1 follows from standard properties of reduced cohomology. See Lemma 3.2.4, which can be found on page 315 of [Guic-72a] and as Theorem 7.2 in [Blanc-79].

(iii) The hypothesis that  $G$  is compactly generated in the theorem is necessary, as the following example shows. Let  $G = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$  be the direct sum of countably many copies of  $\mathbb{Z}/2\mathbb{Z}$ . Then  $G$  is a countable group. As  $G$  is not finitely generated,  $G$  does not have Property (T), by Corollary 2.4.2 or Theorem 1.3.1. On the other hand, since  $G$  is abelian, every unitary irreducible representation of  $G$  is given by a unitary character of  $G$  (Corollary A.2.3). We claim that  $H^1(G, \chi) = 0$  for every unitary character  $\chi$  of  $G$ . If  $\chi = 1_G$ , then

$$H^1(G, \chi) = \text{Hom}(G, \mathbb{C}) = 0,$$

since  $G$  is a torsion group (that is, every element of  $G$  has finite order). Assume that  $\chi \neq 1_G$ , and choose  $g_0 \in G$  with  $\chi(g_0) \neq 1$ . For  $b \in Z^1(G, \chi)$ , we have, for every  $g \in G$ ,

$$\chi(g)b(g_0) + b(g) = b(gg_0) = b(g_0g) = \chi(g_0)b(g) + b(g_0),$$

and hence

$$b(g) = \chi(g) \left( \frac{b(g_0)}{\chi(g_0) - 1} \right) - \left( \frac{b(g_0)}{\chi(g_0) - 1} \right).$$

This shows that  $b \in B^1(G, \chi)$ .

(iv) The hypothesis of second countability of  $G$  in Theorem 3.2.1 has been removed in [LoStV-04].

For the proof of Shalom's Theorem, we need several preliminary lemmas. The first lemma, which is Theorem 2 in [Guic-72a], is a preparation for Lemma 3.2.4 below. It gives a characterisation of the elements from  $\overline{B^1(G, \pi)}$ .

**Lemma 3.2.3** *Let  $G$  be a locally compact group and let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Let  $L_c^2(G, \mathcal{H})$  denote the space of all square-integrable measurable mappings  $f : G \rightarrow \mathcal{H}$  with compact support. For a 1-cocycle  $b \in Z^1(G, \pi)$ , the following conditions are equivalent:*

- (i)  $b \in \overline{B^1(G, \pi)}$ ;
- (ii) for all  $f \in L_c^2(G, \mathcal{H})$  with  $\int_G (\pi(g^{-1}) - I)f(g)dg = 0$ , we have

$$\int_G \langle b(g), f(g) \rangle dg = 0.$$

**Proof** Let  $\delta : \mathcal{H} \rightarrow B^1(G, \pi)$  be the linear mapping defined by

$$\delta(\xi)(g) = \pi(g)\xi - \xi, \quad \xi \in \mathcal{H}, g \in G.$$

For  $f \in L_c^2(G, \mathcal{H})$  and  $\xi \in \mathcal{H}$ , we have

$$(*) \quad \int_G \langle \xi, (\pi(g^{-1}) - I)f(g) \rangle dg = \int_G \langle \delta(\xi)(g), f(g) \rangle dg.$$

This shows that Condition (i) implies Condition (ii).

To show that (ii) implies (i), fix a compact neighbourhood  $Q$  of  $e$ . Let  $C = Q^2$ . Then  $C$  is a compact subset containing  $Q$ . Let  $\delta_C : \mathcal{H} \rightarrow L^2(C, \mathcal{H})$  be defined by

$$\delta_C(\xi) = \delta(\xi)|_C, \quad \xi \in \mathcal{H}.$$

Equality (\*) shows that  $(\delta_C)^* = T$ , where  $T : L^2(C, \mathcal{H}) \rightarrow \mathcal{H}$  is the linear mapping given by

$$T(f) = \int_C (\pi(g^{-1}) - I)f(g)dg, \quad f \in L^2(C, \mathcal{H}).$$

It follows that the closure of  $\delta_C(\mathcal{H})$  coincides with the orthogonal complement of  $\text{Ker } T$  in  $L^2(C, \mathcal{H})$ .

Since  $b$  is continuous, its restriction  $b|_C$  belongs to the Hilbert space  $L^2(C, \mathcal{H})$ . Condition (ii) implies that  $b|_C \in (\text{Ker } T)^\perp$ . Hence,  $b|_C$  belongs to the

closure of  $\delta_C(\mathcal{H})$  in  $L^2(C, \mathcal{H})$ , that is, there exists a sequence of 1-coboundaries  $(b_n)_n$  with coefficients in  $\pi$  such that

$$\lim_n \int_C \|b(g) - b_n(g)\|^2 dg = 0.$$

We claim that  $(b_n)_n$  converges to  $b$  uniformly on  $Q$ . To show this, let  $\varphi$  be a non-negative, continuous function on  $G$  with support contained in  $Q$  and with  $\int_G \varphi(g) dg = 1$ . Set  $a_n = b - b_n$ . For every  $g \in G$ , we have

$$\begin{aligned} a_n(g) &= \int_G \varphi(x) a_n(g) dx \\ &= \int_G \varphi(x) a_n(gx) dx - \int_G \varphi(x) \pi(g) a_n(x) dx \\ &= \int_G \varphi(g^{-1}x) a_n(x) dx - \pi(g) \int_G \varphi(x) a_n(x) dx \\ &= \int_{gQ} \varphi(g^{-1}x) a_n(x) dx - \pi(g) \int_Q \varphi(x) a_n(x) dx. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain for every  $g \in Q$ ,

$$\begin{aligned} \|a_n(g)\| &\leq \left( \int_C \varphi(g^{-1}x)^2 dx \right)^{1/2} \left( \int_C \|a_n(x)\|^2 dx \right)^{1/2} \\ &\quad + \left( \int_C \varphi(x)^2 dx \right)^{1/2} \left( \int_C \|a_n(x)\|^2 dx \right)^{1/2}. \end{aligned}$$

Since  $\lim_n \int_C \|a_n(x)\|^2 dx = 0$ , the claim follows. ■

The next lemma shows that reduced 1-cohomology is better behaved than ordinary 1-cohomology with respect to direct integral decompositions (see Section F.5).

**Lemma 3.2.4** *Let  $G$  be a second countable locally compact group and let*

$$\pi = \int_Z^{\oplus} \pi(z) d\mu(z)$$

*be a direct integral of unitary representations  $\pi(z)$  of  $G$  over a standard Borel space  $Z$ , where  $\mu$  is a positive bounded measure on  $Z$ . If  $\overline{H^1}(G, \pi(z)) = 0$  for  $\mu$ -almost every  $z \in Z$ , then  $\overline{H^1}(G, \pi) = 0$ .*

**Proof** The representation  $\pi$  is defined in terms of a measurable field  $(\mathcal{H}(z))_{z \in Z}$  of Hilbert spaces over  $Z$  and of the direct integral  $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z) d\mu(z)$ . Let

$b \in Z^1(G, \pi)$ . There exists a family  $(b(z))_{z \in Z}$ , where  $b(z)$  is a mapping from  $G$  to  $\mathcal{H}(z)$  such that

$$\begin{aligned} & \text{the mapping } (g, z) \mapsto b(z)(g) \text{ is measurable on } G \times Z, \\ & b(z) \in Z^1(G, \pi(z)) \text{ for } \mu\text{-almost all } z \in Z, \\ & b(g) = \int_Z^{\oplus} b(z)(g) d\mu(z) \text{ for almost all } g \in G. \end{aligned}$$

Choose a compact subset  $Q$  of  $G$  and a function  $f \in L^2(Q, \mathcal{H})$  such that

$$\int_G (\pi(g^{-1}) - I)f(g)dg = 0.$$

It is enough to show that

$$\int_G \langle b(g), f(g) \rangle dg = 0,$$

by the previous lemma.

As  $L^2(Q, \mathcal{H}) = \int_Z^{\oplus} L^2(Q, H(z)) d\mu(z)$ , there exists a family  $(f(z))_{z \in Z}$  with  $f(z) \in L^2(Q, H(z))$ , such that

$$\begin{aligned} & \text{the mapping } (g, z) \mapsto f(z)(g) \text{ is measurable on } Q \times Z, \\ & f = \int_Z^{\oplus} f(z) d\mu(z). \end{aligned}$$

Since

$$\int_G (\pi(g^{-1}) - I)f(g)dg = \int_Z^{\oplus} \int_G (\pi(z)(g^{-1}) - I)f(z)(g)dg d\mu(z),$$

we also have

$$\int_G (\pi(z)(g^{-1}) - I)f(z)(g)dg = 0$$

for  $\mu$ -almost every  $z \in Z$ .

By assumption,  $b(z) \in \overline{B^1(G, \pi(z))}$  for  $\mu$ -almost every  $z \in Z$ , and it follows from the previous lemma that

$$\int_G \langle b(z)(g), f(z)(g) \rangle dg = 0$$

for  $\mu$ -almost every  $z \in Z$ . Hence,

$$\int_G \langle b(g), f(g) \rangle dg = \int_Z^{\oplus} \int_G \langle b(z)(g), f(z)(g) \rangle dg d\mu(z) = 0$$

and this ends the proof. ■

The following lemma is the crucial tool in the proof of Shalom's Theorem.

**Lemma 3.2.5** *Let  $G$  be a compactly generated locally compact group and let  $Q$  be a compact generating subset of  $G$ , with non-empty interior. Let  $\pi$  be an orthogonal representation on a Hilbert space  $\mathcal{H}$ , and define the function*

$$\delta : \mathcal{H} \rightarrow \mathbf{R}_+, \quad \xi \mapsto \max_{g \in Q} \|\pi(g)\xi - \xi\|.$$

*Assume that  $\pi$  almost has invariant vectors but no non-zero invariant vectors.*

*Then, for every  $M > 1$ , there exists a vector  $\xi_M \in \mathcal{H}$  with the following properties:*

- (i)  $\delta(\xi_M) = 1$ ;
- (ii)  $\delta(\eta) > 1/6$  for every  $\eta \in \mathcal{H}$  such that  $\|\eta - \xi_M\| < M$ .

*Moreover, the family of functions on  $G$*

$$\varphi_M : g \mapsto \|\pi(g)\xi_M - \xi_M\|, \quad M > 1,$$

*is equicontinuous.*

**Proof** • *First step:* We claim that, for every  $L > 1$ , there exist  $r > 0$  and  $\zeta_L \in \mathcal{H}$  such that  $\delta(\zeta_L) = r/L$  and  $\delta(\eta) > r/2L$  for all  $\eta \in \mathcal{H}$  with  $\|\eta - \zeta_L\| < 3r$ .

Indeed, since  $\pi$  almost has invariant vectors, there exists a unit vector  $\zeta \in \mathcal{H}$  with  $\delta(\zeta) < 1/6L$ . Then  $\zeta_1 = (2\delta(\zeta)L)^{-1}\zeta$  satisfies

$$\|\zeta_1\| > 3 \quad \text{and} \quad \delta(\zeta_1) = 1/2L.$$

If  $\delta(\eta) > 1/4L$  for every  $\eta \in \mathcal{H}$  with  $\|\eta - \zeta_1\| < 3/2$ , then we choose  $r = 1/2$  and  $\zeta_L = \zeta_1$ .

If not, there exists  $\eta \in \mathcal{H}$  with  $\|\eta - \zeta_1\| < 3/2$  and  $\delta(\eta) \leq 1/4L$ . Since  $\delta$  is continuous, we find  $\zeta_2 \in \mathcal{H}$  with

$$\|\zeta_2 - \zeta_1\| < 3/2 \quad \text{and} \quad \delta(\zeta_2) = 1/4L.$$

If  $\delta(\eta) > 1/8L$  for every  $\eta \in \mathcal{H}$  with  $\|\eta - \zeta_2\| < 3/4$ , then we choose  $r = 1/4$  and  $\zeta_L = \zeta_2$ .

If not, there exists  $\eta \in \mathcal{H}$  with  $\|\eta - \zeta_2\| < 3/4$  and  $\delta(\eta) \leq 1/8L$  and we find  $\zeta_3 \in \mathcal{H}$  with

$$\|\zeta_3 - \zeta_2\| < 3/4 \quad \text{and} \quad \delta(\zeta_3) = 1/8L.$$

Continuing this way, we claim that the process must stop after finitely many steps. Indeed, otherwise, we find a sequence  $(\zeta_i)_{i \geq 1}$  in  $\mathcal{H}$  with

$$\|\zeta_{i+1} - \zeta_i\| < 3/2^i \quad \text{and} \quad \delta(\zeta_{i+1}) = 1/2^{i+1}L$$

for all  $i \geq 1$ . Then  $(\zeta_i)_{i \geq 1}$  is a Cauchy sequence and, hence, converges to a vector  $\zeta \in H$ . We have  $\delta(\zeta) = 0$ , by continuity of  $\delta$ . Moreover  $\zeta \neq 0$ , since  $\|\zeta_1\| > 3$  and

$$\|\zeta_{i+1} - \zeta_1\| \leq \sum_{k=1}^i \|\zeta_{k+1} - \zeta_k\| < \sum_{k=1}^i 3/2^k < 3,$$

for all  $i \geq 1$ . So,  $\zeta$  is a non-zero vector which is invariant under  $\pi(g)$  for all  $g \in Q$ . Since  $Q$  generates  $G$ , it is invariant under  $\pi(g)$  for all  $g \in G$ . This contradicts our assumption.

• *Second step:* With  $\zeta_L$  as in the first step, set

$$\xi_L'' = \frac{L}{r} \zeta_L.$$

Then  $\delta(\xi_L'') = 1$  and

$$\delta(\eta) > 1/2, \quad \text{for all } \eta \in \mathcal{H} \quad \text{with} \quad \|\eta - \xi_L''\| < 3L.$$

• *Third step:* If the group is discrete, the equicontinuity condition is automatically satisfied, and the proof is finished by setting  $M = 3L$  and  $\xi_M = \xi_L''$ . In the general case, we have to regularise the vector  $\xi_L''$ . In order to do this, take a continuous non-negative real function  $f$  on  $G$  with support contained in the interior of  $Q$ , and such that  $\int_G f(g) dg = 1$ . Let

$$L = M + \frac{1}{3},$$

and let  $\xi_L' \in \mathcal{H}$  be defined by the  $\mathcal{H}$ -valued integral

$$\xi_L' = \int_G f(g) \pi(g) \xi_L'' dg.$$



Since  $\text{supp } f \subset Q$ , we have

$$\|\xi'_L - \xi''_L\| = \left\| \int_G f(g)(\pi(g)\xi''_L - \xi'_L)dg \right\| \leq \delta(\xi''_L) = 1,$$

and hence, by the triangle inequality,

$$\delta(\xi'_L) \leq 2\|\xi'_L - \xi''_L\| + \delta(\xi''_L) \leq 3.$$

Moreover, for every  $\eta \in \mathcal{H}$  with  $\|\eta - \xi'_L\| < 3L - 1$ , we have  $\|\eta - \xi''_L\| < 3L$  and hence  $\delta(\eta) > 1/2$ .

Set

$$\xi_M = \frac{\xi'_L}{\delta(\xi'_L)}.$$

Then  $\delta(\xi_M) = 1$  and, for every  $\eta \in \mathcal{H}$  with  $\|\eta - \xi_M\| < M$ , we have

$$\|\delta(\xi'_L)\eta - \xi'_L\| < \delta(\xi'_L)M \leq 3M = 3L - 1,$$

and hence

$$\delta(\eta) > \frac{1}{2\delta(\xi'_L)} \geq \frac{1}{6}.$$

• *Coda:* It remains to show that the family of functions

$$\varphi_M : g \mapsto \|\pi(g)\xi_M - \xi_M\|, \quad M > 1,$$

is equicontinuous. Let  $\varepsilon > 0$ . Choose a neighbourhood  $U$  of  $e$  such that, for every  $x \in U$ , the function  $y \mapsto f(x^{-1}y)$  has support in  $Q$ , and such that

$$\int_G |f(x^{-1}y) - f(y)|dy < \varepsilon/2, \quad \text{for all } x \in U.$$

Then, for all  $g \in G$  and  $x \in U$ , and with  $L = M + 1/3$ , we have

$$\begin{aligned} & \left| \|\pi(g)\pi(x)\xi'_L - \xi'_L\| - \|\pi(g)\xi'_L - \xi'_L\| \right| \\ & \leq \|\pi(g)\pi(x)\xi'_L - \pi(g)\xi'_L\| \\ & = \|\pi(x)\xi'_L - \xi'_L\| \\ & = \left\| \int_G f(y)\pi(xy)\xi''_L dy - \int_G f(y)\pi(y)\xi''_L dy \right\| \\ & = \left\| \int_G (f(x^{-1}y) - f(y))\pi(y)\xi''_L dy \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \int_G (f(x^{-1}y) - f(y))(\pi(y)\xi_L'' - \xi_L'') dy \right\| \\
&\leq \delta(\xi_L'') \int_G |f(x^{-1}y) - f(y)| dy \leq \varepsilon/2.
\end{aligned}$$

Since  $\|\xi_L' - \xi_L''\| \leq 1 < 3L - 1$ , we have  $\delta(\xi_L') > 1/2$  by the second step. Hence,

$$\begin{aligned}
|\varphi_M(gx) - \varphi_M(g)| &= \frac{1}{\delta(\xi_L')} \|\pi(g)\pi(x)\xi_L' - \xi_L'\| - \|\pi(g)\xi_L' - \xi_L'\| \\
&\leq \frac{\varepsilon}{2\delta(\xi_L')} \leq \varepsilon. \quad \blacksquare
\end{aligned}$$

**Proof of Theorem 3.2.1** That (i) implies (ii) follows from the Delorme–Guichardet Theorem 2.12.4. It is obvious that (ii) implies (iii).

Let us check that (iii) implies (iv). If the Hilbert space  $\mathcal{H}$  of  $\pi$  were separable, then  $\pi$  would have a decomposition as a direct integral of irreducible unitary representations (Theorem F.5.3) and the claim would follow directly from Lemma 3.2.4. In the general case, consider  $b \in Z^1(G, \pi)$ . The cocycle identity shows that the closure  $\mathcal{K}$  of the linear span of the set  $\{b(g) : g \in G\}$  is  $\pi(G)$ -invariant. If  $\pi'$  denotes the restriction of  $\pi$  to  $\mathcal{K}$ , it suffices to show that  $b \in \overline{B^1(G, \pi')}$ . Since  $G$  is second countable, the space  $\mathcal{K}$  is separable, so that Lemma 3.2.4 applies.

We proceed now to show that (iv) implies (i), which is the deep part of the theorem. Assume that  $G$  does not have Property (T). We will show the existence of a unitary representation  $\sigma$  of  $G$  with  $\overline{H^1(G, \sigma)} \neq 0$ .

Fix a compact generating subset  $Q$  of  $G$ , with non-empty interior. We can assume that  $Q$  is symmetric, that is,  $Q = Q^{-1}$ . Since  $G$  does not have Property (T), there exists a unitary representation  $(\pi, \mathcal{H})$  almost having invariant vectors, but without non-zero invariant vectors. It follows from the previous lemma that, for every integer  $n > 1$ , there exists a vector  $\xi_n \in \mathcal{H}$  such that  $\delta(\xi_n) = 1$  and

$$\delta(\eta) > 1/6 \quad \text{if} \quad \|\eta - \xi_n\| < n.$$

Set

$$\psi_n(g) = \|\pi(g)\xi_n - \xi_n\|^2, \quad g \in G.$$

The function  $\psi_n$  is a function conditionally of negative type on  $G$ .

We claim that, if  $K$  is a compact subset of  $G$ , the sequence  $(\psi_n)_n$  is uniformly bounded on  $K$ . Indeed, there exists  $m \in \mathbf{N}$  such that  $K \subset Q^m$ . Every  $g \in K$  can

therefore by written as a product  $g = g_1 \cdots g_m$  with  $g_i$  in  $Q$ ; using the triangle equality (compare with the proof of Proposition 3.1.5), we have

$$\begin{aligned}
 (*) \quad \|\pi(g)\xi_n - \xi_n\| &= \|\pi(g_1 \cdots g_m)\xi_n - \xi_n\| \\
 &\leq \sum_{i=1}^m \|\pi(g_i)\xi_n - \xi_n\| \\
 &\leq m\delta(\xi_n) = m,
 \end{aligned}$$

and this shows the claim.

By the previous lemma, the family  $(\psi_n)_n$  is equicontinuous. Hence, it follows from the classical Arzela–Ascoli Theorem (see, e.g., [Rudin–73, Appendix A 5]) that there exists a subsequence  $(\psi_{n_k})_k$  which converges, uniformly on compact subsets of  $G$ , to a continuous function  $\psi$  on  $G$ . It is clear that  $\psi$  is a function conditionally of negative type on  $G$  (Proposition C.2.4).

Let  $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$  be the triple associated to  $\psi$  by Proposition 2.10.2. We claim that  $b_\psi \notin \overline{B^1(G, \pi_\psi)}$  and, therefore,  $\overline{H^1(G, \pi_\psi)} \neq 0$ .

To prove this, assume, by contradiction, that  $b_\psi \in \overline{B^1(G, \pi_\psi)}$ . By Proposition 3.1.5, there exist  $a_1, \dots, a_N \geq 0$  with  $\sum_{i=1}^N a_i = 1$  and  $g_1, \dots, g_N \in G$  such that, for every  $g \in Q$ , we have

$$(**) \quad \sum_{i=1}^N \sum_{j=1}^N a_i a_j (\psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i)) < 1/6^2.$$

Choose  $m \in \mathbb{N}$  such that, for every  $g \in Q$ , all the elements  $g_i, g_j^{-1} g g_i$  and  $g_j^{-1} g_i$  are contained in  $Q^m$ .

On the one hand, Inequality (\*\*) holds with  $\psi$  replaced by  $\psi_{n_k}$  for  $n_k$  large enough, that is,

$$\sum_{i=1}^N \sum_{j=1}^N a_i a_j (\psi_{n_k}(g_j^{-1} g g_i) - \psi_{n_k}(g_j^{-1} g_i)) < \frac{1}{6^2}.$$

Set  $\eta_k = \sum_{i=1}^N a_i \pi(g_i) \xi_{n_k}$ . A computation similar to that in the proof of Proposition 3.1.5 shows that

$$\|\pi(g)\eta_k - \eta_k\|^2 = \sum_{i=1}^N \sum_{j=1}^N a_i a_j (\psi_{n_k}(g_j^{-1} g g_i) - \psi_{n_k}(g_j^{-1} g_i)).$$

Hence,  $\delta(\eta_k) < 1/6$ . On the other hand, since  $g_i \in Q^m$  for all  $i \in \{1, \dots, N\}$ , we have

$$\|\pi(g_i)\xi_{n_k} - \xi_{n_k}\| \leq m, \quad \text{for all } i = 1, \dots, N,$$

by Inequality (\*) above and, hence,

$$\|\xi_{n_k} - \eta_k\| = \|\xi_{n_k} - \sum_{i=1}^N a_i \pi(g_i) \xi_{n_k}\| \leq m.$$

If  $n_k > m$ , the previous inequality implies

$$\|\xi_{n_k} - \eta_k\| < n_k$$

and, hence,  $\delta(\eta_k) \geq 1/6$ , by construction of  $\xi_{n_k}$ . This is a contradiction.

To conclude the proof, we define the unitary representation  $\sigma$  to be the complexification of  $\pi_\psi$  (see Remark 2.12.1). Let

$$b : G \mapsto (\mathcal{H}_\psi)_\mathbb{C}, \quad g \mapsto b(g) = b_\psi(g) \otimes 1.$$

Then  $b \in Z^1(G, \sigma)$  and  $b \notin \overline{B^1(G, \sigma)}$ . ■

Recall that, for a topological group  $G$ , the unitary dual  $\widehat{G}$  of  $G$  is equipped with Fell's topology (see Section F.2). Modifying slightly the definition given in [VerKa–82, Definition 2], we say that  $\pi \in \widehat{G}$  is *infinitely small* if  $\pi$  is not Hausdorff separated from the unit representation  $1_G$ , that is, if  $U \cap V \neq \emptyset$ , for all neighbourhoods  $U$  of  $\pi$  and  $V$  of  $1_G$  in  $\widehat{G}$ . The *cortex* of  $G$ , denoted by  $\text{Cor}(G)$ , is the subset of  $\widehat{G}$  consisting of all infinitely small irreducible unitary representations of  $G$ .

Let  $G$  be a locally compact, compactly generated group and let  $\pi \in \widehat{G}$ . It has been shown in [VerKa–82, Theorem 2] that, if  $H^1(G, \pi) \neq 0$ , then  $\pi \in \text{Cor}(G)$ ; see also [Louve–01]. We obtain from the theorem above the following characterisation of locally compact second-countable groups with Property (T), given by Shalom [Shal–00a].

**Corollary 3.2.6** *Let  $G$  be a second-countable locally compact group. The following statements are equivalent:*

- (i)  $G$  has Property (T);
- (ii)  $G$  satisfies the following three conditions:
  - (ii.1)  $G$  is compactly generated;
  - (ii.2)  $\text{Hom}(G, \mathbf{R}) = 0$ ;
  - (ii.3)  $\text{Cor}(G) = \{1_G\}$ .

**Proof** If  $G$  has Property (T), then  $G$  satisfies the conditions of (ii) by Theorem 1.3.1, Corollary 1.3.5, and Theorem 1.2.3 (see also Corollaries 2.4.2 and 2.5.2).

Conversely, assume that  $G$  satisfies the conditions (ii.1), (ii.2), and (ii.3). Since  $G$  is compactly generated, to establish Property (T), it is enough to show that  $H^1(G, \pi) = 0$  for every  $\pi \in \widehat{G}$ , by Theorem 3.2.1. By the result of Vershik and Karpushev mentioned above, it follows from condition (ii.3) that  $H^1(G, \pi) = 0$  if  $\pi \neq 1_G$ . On the other hand,

$$H^1(G, 1_G) = \text{Hom}(G, \mathbf{C}) = 0,$$

by condition (ii.2). This proves that (ii) implies (i). ■

### 3.3 Property (T) for $Sp(n, 1)$

In this section, we present a proof of Kostant's result ([Kosta–69], [Kosta–75]) that the real rank 1 simple Lie groups  $Sp(n, 1)$ ,  $n \geq 2$ , and  $F_{4(-20)}$  have Property (T). Our proof rests on ideas of M. Gromov [Gromo–03] and Y. Shalom. It is based on the consideration of harmonic mappings and therefore has some similarities with geometric superrigidity (see [MoSiY–93]).

Our proof of Property (T) for  $Sp(n, 1)$  and  $F_{4(-20)}$  follows by combining two results:

**Theorem A** *Let  $G$  be a connected semisimple Lie group with finite centre, and let  $K$  be a maximal compact subgroup of  $G$ . If  $G$  does not have Property (T), there exists an affine isometric action of  $G$  on some Hilbert space  $\mathcal{H}$  and a mapping  $F : G/K \rightarrow \mathcal{H}$  which is  $G$ -equivariant, harmonic, and non-constant.*

We will see that this result holds under the more general assumptions that  $G$  is a connected Lie group,  $\text{Hom}(G, \mathbf{R}) = 0$ , and that  $K$  is a compact subgroup such that  $(G, K)$  is a Gelfand pair.

**Theorem B** *For  $G = Sp(n, 1)$  and  $K = Sp(n) \times Sp(1)$ ,  $n \geq 2$ , or  $G = F_{4(-20)}$  and  $K = \text{Spin}(9)$ , any harmonic mapping  $G/K \rightarrow \mathcal{H}$  which is  $G$ -equivariant with respect to some affine isometric action of  $G$  on a Hilbert space  $\mathcal{H}$  has to be constant.*

### Gelfand pairs

**Definition 3.3.1** Let  $G$  be a locally compact group, and let  $K$  be a compact subgroup of  $G$ . The pair  $(G, K)$  is a *Gelfand pair* if the convolution algebra

$C_c(G//K)$  of continuous  $K$ -bi-invariant functions on  $G$  with compact support is commutative.

**Example 3.3.2** Let  $G$  be a real semisimple Lie group with finite centre, and let  $K$  be a maximal compact subgroup of  $G$ . By a classical result of Gelfand, the pair  $(G, K)$  is a Gelfand pair (see, e.g., [Helga–84, Chapter IV, Theorem 3.1]).

Let  $(G, K)$  be a Gelfand pair, and let  $(\pi, \mathcal{H})$  be an irreducible unitary representation of  $G$ . The subspace  $\mathcal{H}^K$  of  $\pi(K)$ -invariant vectors in  $\mathcal{H}$  is invariant under the operators  $\pi(f)$  for  $f \in C_c(G//K)$ . (For the definition of  $\pi(f) \in \mathcal{L}(\mathcal{H})$ , see Section F.4.) If  $\mathcal{H}^K \neq 0$ , then

$$f \mapsto \pi(f)|_{\mathcal{H}^K}$$

is an irreducible  $*$ -representation of the commutative  $*$ -algebra  $C_c(G//K)$  (see [Helga–84, Chapter IV, Lemma 3.6]). By Schur's Lemma (Theorem A.2.2), it follows that  $\dim_{\mathbb{C}} \mathcal{H}^K = 1$ .

Summarizing, we see that, for  $\pi \in \widehat{G}$ , we have either  $\dim_{\mathbb{C}} \mathcal{H}^K = 0$  or  $\dim_{\mathbb{C}} \mathcal{H}^K = 1$ .

**Definition 3.3.3** Let  $(G, K)$  be a Gelfand pair. An irreducible unitary representation  $(\pi, \mathcal{H})$  of  $G$  is called a *spherical representation* if  $\dim_{\mathbb{C}} \mathcal{H}^K = 1$ .

**Lemma 3.3.4** Let  $G$  be a topological group, and let  $K$  be a compact subgroup of  $G$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  which does not contain the unit representation. Assume that  $\mathcal{H}^K$  is finite dimensional. Then  $\pi$  does not almost have invariant vectors.

*In particular, if  $(G, K)$  is a Gelfand pair and  $\pi$  is a non-trivial spherical representation of  $G$ , then  $\pi$  does not almost have invariant vectors.*

**Proof** Assume, by contradiction, that  $\pi$  almost has invariant vectors, that is, there exists a net  $(\xi)_i$  of unit vectors such that

$$\lim_i \|\pi(g)\xi_i - \xi_i\| = 0$$

uniformly on compact subsets of  $G$ . For  $i$  large enough, we have  $\int_K \pi(k)\xi_i dk \neq 0$  and we set

$$\eta_i = \frac{1}{\|\int_K \pi(k)\xi_i dk\|} \int_K \pi(k)\xi_i dk,$$

where  $dk$  is a Haar measure on  $K$ . Then  $\eta_i$  is a unit vector in  $\mathcal{H}^K$  such that

$$(*) \quad \lim_i \|\pi(g)\eta_i - \eta_i\| = 0$$

uniformly on compact subsets of  $G$ . Since  $\mathcal{H}^K$  is finite dimensional, its unit sphere is compact. Hence, a subnet of  $(\eta_i)_i$  converges to some unit vector  $\eta \in \mathcal{H}^K$ . It follows from  $(*)$  that  $\eta$  is invariant under  $\pi(G)$ . This is a contradiction, since  $\pi$  does not contain  $1_G$ . ■

The following result is due to P. Delorme [Delor–77, Proposition V.3]. The proof we give was found by Y. Shalom (unpublished).

**Proposition 3.3.5** *Let  $(G, K)$  be a Gelfand pair, and let  $(\pi, \mathcal{H})$  be a spherical representation of  $G$  distinct from the unit representation; assume that  $G$  is compactly generated. Then  $H^1(G, \pi) = 0$ .*

**Proof** Let  $\alpha$  be an affine isometric action of  $G$  on  $\mathcal{H}$ , with linear part  $\pi$ . We must show that  $\alpha$  has a globally fixed point.

Let  $b \in Z^1(G, \pi)$  be defined by

$$\alpha(g)\xi = \pi(g)\xi + b(g), \quad \xi \in \mathcal{H}.$$

For every compactly supported regular Borel measure  $\mu$  on  $G$ , we can define an affine transformation  $\alpha(\mu)$  on  $\mathcal{H}$  by the  $\mathcal{H}$ -valued integral

$$\alpha(\mu)\xi = \int_G \alpha(g)\xi d\mu(g) = \pi(\mu)\xi + \int_G b(g)d\mu(g), \quad \xi \in \mathcal{H}.$$

Let  $\mu_0$  be a symmetric,  $K$ -bi-invariant probability measure on  $G$  which is absolutely continuous with respect to a left Haar measure on  $G$  and has compact support. Assume, in addition, that the subgroup generated by the support of  $\mu_0$  is dense in  $G$ .

• *First claim:* the operator  $\alpha(\mu_0)$  has a unique fixed point  $\xi_0 \in \mathcal{H}$ . Indeed, by the previous lemma,  $\pi$  does not almost have invariant vectors. Hence,  $\|\pi(\mu_0)\| < 1$  by Proposition G.4.2. Now, for  $\xi, \eta \in \mathcal{H}$

$$\|\alpha(\mu_0)\xi - \alpha(\mu_0)\eta\| = \|\pi(\mu_0)\xi - \pi(\mu_0)\eta\| \leq \|\pi(\mu_0)\| \|\xi - \eta\|.$$

It follows that  $\alpha(\mu_0)$  is a strict contraction of the affine Hilbert space  $\mathcal{H}$ . Hence,  $\alpha(\mu_0)$  has a unique fixed point  $\xi_0$ .

• *Second claim:* the affine subspace of the  $\alpha(K)$ -fixed points has positive dimension. Indeed, observe first that  $\xi_0$  is fixed under  $\alpha(K)$  since

$$\alpha(k)\xi_0 = \alpha(k)\alpha(\mu_0)\xi_0 = \alpha(\mu_0)\xi_0 = \xi_0, \quad k \in K$$

by the left  $K$ -invariance of  $\mu_0$ . (Observe that we used the fact that  $\alpha(k)$  is an affine mapping.) Moreover, as  $\pi$  is spherical, there exists a non-zero  $\pi(K)$ -fixed vector  $\xi_1$  in  $\mathcal{H}$ . Then  $\xi_2 = \xi_0 + \xi_1$  is  $\alpha(K)$ -fixed and distinct from  $\xi_0$ .

• *Third claim:*  $\xi_0$  is fixed under  $\alpha(\mu)$  for every  $\mu$  in the space  $M(G//K)$  of  $K$ -bi-invariant, compactly supported probability measures on  $G$ . Indeed, since  $(G, K)$  is a Gelfand pair,  $C_c(G//K)$  is commutative. As every measure in  $M(G//K)$  can be approximated in the weak topology by absolutely continuous measures with a density from  $C_c(G//K)$ , it follows that  $M(G//K)$  is commutative. Hence, for every  $\mu \in M(G//K)$ ,

$$\alpha(\mu)\xi_0 = \alpha(\mu)\alpha(\mu_0)\xi_0 = \alpha(\mu_0)\alpha(\mu)\xi_0,$$

so that  $\alpha(\mu)\xi_0$  is fixed under  $\alpha(\mu_0)$ . By uniqueness of the fixed point of  $\alpha(\mu_0)$ , it follows that  $\alpha(\mu)\xi_0 = \xi_0$ .

• *Fourth claim:*  $\xi_0$  is fixed under  $\alpha(G)$ . Indeed, let  $\nu_K$  be the normalised Haar measure  $dk$  on  $K$  viewed as a measure on  $G$ . By the third claim,  $\xi_0$  is fixed by  $\alpha(\nu_K * \delta_g * \nu_K)$ , that is,

$$\int_K \alpha(k)\alpha(g)\xi_0 dk = \xi_0$$

for every  $g \in G$ . Hence, the closed affine subset

$$\mathcal{A} = \{\xi \in \mathcal{H} : \int_K \alpha(k)\xi dk = \xi_0\}.$$

of  $\mathcal{H}$  contains the  $\alpha(G)$ -orbit of  $\xi_0$ . By the irreducibility of  $\pi$ , it follows that  $\mathcal{A} = \mathcal{H}$  or  $\mathcal{A} = \{\xi_0\}$ . The first case cannot occur. Indeed, by the second claim,  $\alpha(K)$  has a fixed point  $\xi \neq \xi_0$  and

$$\int_K \alpha(k)\xi dk = \xi.$$

Hence,  $\mathcal{A} = \{\xi_0\}$  and  $\alpha(G)\xi_0 = \{\xi_0\}$ . This finishes the proof of the proposition. ■

**Corollary 3.3.6** *Let  $(G, K)$  be a Gelfand pair; assume that  $G$  is compactly generated. Let  $(\pi, \mathcal{H})$  be an irreducible unitary representation of  $G$ , distinct from the unit representation, such that  $H^1(G, \pi) \neq 0$ . Let  $b \in Z^1(G, \pi)$  be a 1-cocycle which is not a coboundary. Let  $\alpha$  be the affine isometric action of  $G$  on  $\mathcal{H}$  associated to  $(\pi, b)$ . Then there exists a unique fixed point under  $\alpha(K)$  in  $\mathcal{H}$ .*



**Proof** As  $K$  is compact,  $\alpha(K)$  has a fixed point  $\xi_0$  in  $\mathcal{H}$ . Assume that  $\alpha(K)$  has another fixed point  $\xi_1 \neq \xi_0$ . Then  $\xi_0 - \xi_1$  is a non-zero  $\pi(K)$ -fixed point. This implies that  $\pi$  is spherical. Since  $H^1(G, \pi) \neq 0$ , this is a contradiction to the previous proposition. ■

### A mean value property

Let  $G$  be a locally compact group which is second countable and compactly generated, and let  $K$  be a compact subgroup of  $G$ ; we make the following assumptions:

- (i)  $(G, K)$  is a Gelfand pair;
- (ii)  $\text{Hom}(G, \mathbf{R}) = 0$ ;
- (iii)  $G$  does not have Property (T).

By Shalom's Theorem 3.2.1, there exists an *irreducible* unitary representation  $(\pi, \mathcal{H})$  of  $G$  such that  $H^1(G, \pi) \neq 0$ . It follows from our assumption (ii) that  $\pi$  is distinct from the unit representation. Let  $\alpha$  be an affine isometric action of  $G$  on  $\mathcal{H}$ , with linear part  $\pi$  and without fixed point. By Corollary 3.3.6, there is a unique fixed point  $\xi_0$  under  $\alpha(K)$ . We consider the orbital mapping

$$F : G \rightarrow \mathcal{H}, \quad g \mapsto \alpha(g)\xi_0,$$

which is non-constant, and study its properties.

**Proposition 3.3.7** (i) For every left  $K$ -invariant probability measure  $\mu$  on  $G$ , we have

$$\int_G F(gh)d\mu(h) = F(g), \quad \text{for all } g \in G.$$

(ii) Fix  $g_0 \in G$ . The mapping

$$g \mapsto \int_K F(g_0kg)dk$$

is constant, with value  $F(g_0)$ .

**Proof** (i) Observe that, by uniqueness of  $\xi_0$ ,

$$\int_K \alpha(k)\xi dk = \xi_0, \quad \text{for all } \xi \in \mathcal{H},$$

since  $\int_K \alpha(k)\xi dk$  is fixed under  $\alpha(K)$ . Hence, using the left  $K$ -invariance of  $\mu$  and the fact that  $\alpha(g)$  is an affine mapping, we have

$$\begin{aligned} \int_G F(gh)d\mu(h) &= \int_G \int_K F(gkh)dkd\mu(h) \\ &= \alpha(g) \left( \int_G \left( \int_K \alpha(k)\alpha(h)\xi_0 dk \right) d\mu(h) \right) \\ &= \alpha(g)\xi_0 = F(g), \end{aligned}$$

for all  $g \in G$ .

(ii) Fix  $g \in G$ . Let  $\mu$  be the left  $K$ -invariant probability measure  $\nu_K * \delta_g$ , where  $\nu_K$  is the normalised Haar measure on  $K$  viewed as a measure on  $G$ . Then, by (i),

$$\int_K F(g_0kg)dk = \int_G F(g_0h)d\mu(h) = F(g_0),$$

as claimed. ■

Since  $\xi_0$  is fixed under  $\alpha(K)$ , the mapping  $F$  introduced above factorises to a  $G$ -equivariant mapping, also denoted by  $F$ , from  $G/K$  to  $\mathcal{H}$ . Let  $g_0 \in G$ ; set  $x_0 = g_0K$ . We have

$$\int_K F(g_0kg_0^{-1}x)dk = F(x_0), \quad \text{for all } x \in G/K$$

by Proposition 3.3.7.ii. This means that the value of  $F$  at  $x_0$  coincides with the mean value of  $F$  over the orbit of any point  $x \in G/K$  under the stabiliser  $g_0Kg_0^{-1}$  of  $x_0$  in  $G$ ; in other words,  $F(x_0)$  coincides with the mean value of  $F$  over any “sphere” around  $x_0$ . This is a well-known property of harmonic functions on the complex plane. Functions on symmetric spaces with this mean value property have been considered by R. Godement [Godem–52] and H. Furstenberg [Furst–63].

We are going to show that the mean value property implies that  $F$  is annihilated by  $G$ -invariant operators acting on appropriate spaces.

Let  $\mathcal{F}$  be a vector space of continuous mappings  $\Phi : G/K \rightarrow \mathcal{H}$  endowed with a locally convex topology such that  $\mathcal{F}$  is a Fréchet space and such that point evaluations are continuous on  $\mathcal{F}$ . Assume further that

- $\mathcal{F}$  contains  $F$ ;
- $\mathcal{F}$  contains the constant mappings;

- $\mathcal{F}$  is invariant under left translations by elements of  $G$ , that is,  ${}_g\Phi \in \mathcal{F}$  whenever  $\Phi \in \mathcal{F}$  and  $g \in G$ , where  ${}_g\Phi(x) = \Phi(gx)$  for  $x \in G/K$ ;
- for every  $\Phi \in \mathcal{F}$ , the mapping  $G \rightarrow \mathcal{F}$ ,  $g \mapsto {}_g\Phi$  is continuous.

**Proposition 3.3.8** *Let  $T : \mathcal{F} \rightarrow \mathcal{F}$  be a continuous linear operator. Assume that  $T$  commutes with left translations by elements of  $G$ , and that  $T$  annihilates the constants. Then  $T(F) = 0$ .*

**Proof** Fix  $g_0 \in G$ . Then  ${}_{g_0k}F$  belongs to  $\mathcal{F}$  for every  $k \in K$  and the mapping

$$K \rightarrow \mathcal{F}, \quad k \mapsto {}_{g_0k}F$$

is continuous. The integral  $\int_K {}_{g_0k}F dk$  is therefore a well defined element of  $\mathcal{F}$ , with the property that

$$\langle \varphi, \int_K {}_{g_0k}F dk \rangle = \int_K \langle \varphi, {}_{g_0k}F \rangle dk$$

for every continuous linear functional  $\varphi$  on  $\mathcal{F}$  (see [Rudin–73, Theorem 3.27]).

Since  $T$  is linear, continuous and since it commutes with left translations, we have

$$T\left(\int_K {}_{g_0k}F dk\right) = \int_K T({}_{g_0k}F) dk = \int_K {}_{g_0k}T(F) dk.$$

By Proposition 3.3.7.ii, the mapping

$$G \rightarrow \mathcal{H}, \quad x \mapsto \int_K {}_{g_0k}F(x) dk = \int_K F(g_0kx) dk$$

is constant, with value  $F(g_0)$ . On the other hand, point evaluations are continuous, so that

$$\left(\int_K {}_{g_0k}F dk\right)(x) = \int_K {}_{g_0k}F(x) dk$$

for every  $x \in G/K$ . This implies that  $\int_K {}_{g_0k}F dk$  is constant. Hence,  $T(\int_K {}_{g_0k}F dk) = 0$ , by assumption on  $T$ .

We conclude that, for every  $x \in G/K$ ,

$$0 = \int_K {}_{g_0k}T(F)(x) dk = \int_K T(F)(g_0kx) dk.$$

Let  $x_0 = K$  be the origin of  $G/K$ . For  $x = x_0$ , we obtain

$$0 = \int_K T(F)(g_0 k x_0) dk = T(F)(g_0 x_0),$$

since  $kx_0 = x_0$ . As this holds for every  $g_0 \in G$ , this means that  $T(F) = 0$ . ■

We apply the previous proposition in the following situation. Let  $G$  be a connected real Lie group. Let  $\mathcal{F}$  be the space of the mappings  $\Phi : G/K \rightarrow \mathcal{H}$  which are  $C^\infty$ . Equip  $\mathcal{F}$  with the Fréchet topology defined by the requirement:  $\Phi_n$  tends to 0 if  $\Phi_n$  as well as its derivatives of all orders tend to 0 uniformly on compact subsets of  $G/K$ .

**Lemma 3.3.9** *The mapping  $F$  is  $C^\infty$  on  $G/K$ , that is,  $F \in \mathcal{F}$ .*

**Proof** Recall that, by Proposition 3.3.7.i, for every left  $K$ -invariant probability measure  $\mu$  on  $G$ , we have

$$\int_G F(gh) d\mu(h) = F(g), \quad \text{for all } g \in G,$$

where we view  $F$  as function on  $G$ . Take for  $\mu$  a probability measure given by a smooth density on  $G$ . The integral formula above shows that  $F$  is  $C^\infty$  on  $G$ . This implies that  $F$  is  $C^\infty$  on  $G/K$ . ■

The space  $\mathcal{F}$  satisfies all the assumptions made before Proposition 3.3.8. Let  $D$  be a differential operator on  $G/K$ . Then  $D$  acts as a continuous linear operator on  $\mathcal{F}$ . We obtain therefore the following corollary.

**Corollary 3.3.10** *Let  $D$  be a differential operator on  $G/K$  commuting with left translations by elements of  $G$  and annihilating the constants. Then  $DF = 0$ .*

We will specialise to the case where  $D$  is the Laplacian on  $G/K$  and show that  $F$  is harmonic in the usual sense.

### Harmonicity

Let  $M$  be a Riemannian manifold of dimension  $n$ . For  $x \in X$ , let  $T_x M$  be the tangent space at  $x$  and

$$\exp_x : T_x M \rightarrow M$$

the Riemannian exponential mapping at  $x$ .

**Definition 3.3.11** Let  $\mathcal{H}$  be a Hilbert space, and let  $f : M \rightarrow \mathcal{H}$  be a  $C^2$ -mapping. The *Laplacian* of  $f$  is the mapping  $\Delta f : M \rightarrow \mathcal{H}$  given by

$$\Delta f(x) = - \sum_{i=1}^n \frac{d^2}{dt^2} f(\exp_x tX_i)|_{t=0},$$

where  $X_1, \dots, X_n$  is an orthonormal basis in  $T_x M$ .

**Remark 3.3.12** The definition of  $\Delta$  does not depend on the choice of the orthonormal basis  $X_1, \dots, X_n$ . In fact, if  $S_x$  denotes the unit sphere in  $T_x M$  and  $\nu_x$  the normalised rotation-invariant measure on  $S_x$ , then averaging over all orthonormal basis, we have (Exercise 3.7.1)

$$(*) \quad \Delta f(x) = -n \int_{S_x} \frac{d^2}{dt^2} f(\exp_x tX)|_{t=0} d\nu_x(X).$$

An important property of the Laplacian is that it commutes with isometries of  $M$ . Thus, if  $\varphi : M \rightarrow M$  is an isometry, then  $\Delta(f \circ \varphi) = (\Delta f) \circ \varphi$  for every  $C^2$ -mapping  $f : M \rightarrow \mathbf{C}$ .

**Definition 3.3.13** A  $C^2$ -mapping  $f : M \rightarrow \mathcal{H}$  is *harmonic* if  $\Delta f = 0$ . This is equivalent to saying that  $x \mapsto \langle f(x), \xi \rangle$  is a harmonic function on  $M$  for every  $\xi \in \mathcal{H}$ .

Given a mapping  $f : M \rightarrow \mathcal{H}$ , we will have to consider the Laplacian of the function  $\|f\|^2 : M \rightarrow \mathbf{R}$ .

**Proposition 3.3.14** Let  $f : M \rightarrow \mathcal{H}$  be a mapping of class  $C^2$ . Then, for every  $x \in M$ ,

$$\Delta(\|f\|^2)(x) = 2\operatorname{Re}\langle \Delta f(x), f(x) \rangle - 2\|df_x\|_{\text{HS}}^2,$$

where  $\|df_x\|_{\text{HS}} = (\operatorname{Tr}(df_x^* df_x))^{1/2}$  is the Hilbert–Schmidt norm of the differential of  $f$  at  $x$ .

**Proof** For  $X \in T_x M$ , we have

$$\begin{aligned} \frac{d}{dt} \langle f, f \rangle(\exp_x tX) &= \left\langle \frac{d}{dt} f(\exp_x tX), f(\exp_x tX) \right\rangle + \left\langle f(\exp_x tX), \frac{d}{dt} f(\exp_x tX) \right\rangle \\ &= 2\operatorname{Re} \left\langle \frac{d}{dt} f(\exp_x tX), f(\exp_x tX) \right\rangle \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \langle f, f \rangle (\exp_x tX) &= 2\operatorname{Re} \left\langle \frac{d^2}{dt^2} f(\exp_x tX), f(\exp_x tX) \right\rangle \\ &\quad + 2 \left\langle \frac{d}{dt} f(\exp_x tX), \frac{d}{dt} f(\exp_x tX) \right\rangle. \end{aligned}$$

Evaluating at  $t = 0$ , we obtain

$$\frac{d^2}{dt^2} \langle f, f \rangle (\exp_x tX)|_{t=0} = 2\operatorname{Re} \left\langle \frac{d^2}{dt^2} f(\exp_x tX)|_{t=0}, f(x) \right\rangle + 2\|df_x(X)\|^2.$$

Hence, if  $X_1, \dots, X_n$  is an orthonormal basis of  $T_x M$ , it follows that

$$\begin{aligned} \Delta(\|f\|^2)(x) &= - \sum_{i=1}^n \frac{d^2}{dt^2} \langle f, f \rangle (\exp_x tX_i)|_{t=0} \\ &= -2\operatorname{Re} \sum_{i=1}^n \left\langle \frac{d^2}{dt^2} f(\exp_x tX_i)|_{t=0}, f(x) \right\rangle - 2 \sum_{i=1}^n \|df_x(X_i)\|^2 \\ &= 2\operatorname{Re} \langle \Delta f(x), f(x) \rangle - 2\|df_x\|_{\text{HS}}^2. \quad \blacksquare \end{aligned}$$

We return to the mapping  $F : G/K \rightarrow \mathcal{H}$  introduced above. Let now  $x_0 = K$  be the origin of  $G/K$ .

We recall the construction of a  $G$ -invariant Riemannian structure on  $G/K$ . For  $g \in G$ , let  $\lambda(g)$  denote the diffeomorphism of  $G/K$  given by left translation by  $g$ :

$$\lambda(g)hK = ghK, \quad h \in G.$$

As  $K$  is compact, there exists a  $K$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on the tangent space  $T_{x_0}(G/K)$  of  $G/K$  at  $x_0$ . For every  $x = gx_0$ , we define an inner product  $\langle \cdot, \cdot \rangle_x$  on  $T_x(G/K)$  by

$$\langle d\lambda(g)_{x_0}(X), d\lambda(g)_{x_0}(Y) \rangle_x = \langle X, Y \rangle, \quad \text{for all } X, Y \in T_{x_0}(G/K).$$

This is well-defined, since  $\langle \cdot, \cdot \rangle$  is  $K$ -invariant. It is clear that  $x \mapsto \langle \cdot, \cdot \rangle_x$  is a  $G$ -invariant Riemannian structure on  $G/K$ .

We can now consider the Laplacian on  $G/K$  with respect to this structure.

**Proposition 3.3.15** *The mapping  $F : G/K \rightarrow \mathcal{H}$  defined above is harmonic.*

**Proof** See Corollary 3.3.10. ■

To summarise, in this subsection we have proved the following theorem.

**Theorem 3.3.16 (Shalom, unpublished)** *Let  $G$  be a connected Lie group, with  $\text{Hom}(G, \mathbf{R}) = 0$ . Let  $K$  be a compact subgroup of  $G$  such that  $(G, K)$  is a Gelfand pair. If  $G$  does not have Property (T), there exists an affine isometric action of  $G$  on some Hilbert space  $\mathcal{H}$ , and a  $G$ -equivariant mapping  $F : G/K \rightarrow \mathcal{H}$  which is harmonic and non-constant.*

Next, we specialise further to semisimple Lie groups.

### The case of a non-compact semisimple Lie group

Theorem 3.3.16 applies if  $G$  is a semisimple Lie group with finite centre, and  $K$  is a maximal compact subgroup of  $G$ . In this way, we obtain Theorem A stated at the beginning of this section. Actually, in this case, a stronger result is true, namely: the  $G$ -equivariant harmonic mapping  $F : G/K \mapsto \mathcal{H}$  can be taken to be locally isometric, that is, for every  $x \in G/K$ ,

$$\|dF_x(Y)\| = \|Y\|, \quad \text{for all } Y \in T_x(G/K).$$

To see this, we first recall some classical facts on semisimple Lie groups.

Let  $G$  be a semisimple Lie group with finite centre, and let  $K$  be a maximal compact subgroup of  $G$ . There exists a *Cartan involution*  $\theta$ , that is, an involutive automorphism of  $G$  such that  $K$  is the set of  $\theta$ -fixed elements in  $G$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , and let  $\mathfrak{k}$  be the subalgebra corresponding to  $K$ . Let

$$\mathfrak{p} = \{X \in \mathfrak{g} : d\theta_e(X) = -X\}.$$

The decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  is the *Cartan decomposition* of  $\mathfrak{g}$ . Fix a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ , and let  $A = \exp \mathfrak{a}$ . We have the Cartan decompositions

$$G = KAK \quad \text{and} \quad G = K \exp \mathfrak{p}$$

of  $G$  (see [Helga–62, Chapter V, §6, Chapter III, §7]). Every element  $X \in \mathfrak{p}$  defines a tangent vector  $D_X \in T_{x_0}(G/K)$  for  $x_0 = K$ , given by

$$D_X f(x_0) = \frac{d}{dt} f(\exp tXx_0)|_{t=0}, \quad f \in C^\infty(G/K).$$

The mapping  $X \rightarrow D_X$  is a linear bijection between  $\mathfrak{p}$  and  $T_{x_0}(G/K)$ . This allows us to identify  $T_{x_0}(G/K)$  with  $\mathfrak{p}$ .

Observe that, for  $k \in K$ , the linear automorphism  $d\lambda(k)_{x_0}$  of  $T_{x_0}(G/K)$  corresponds to the linear mapping  $\text{Ad}(k) : \mathfrak{p} \rightarrow \mathfrak{p}$ , since

$$\begin{aligned} \frac{d}{dt}f(k \exp tXx_0)|_{t=0} &= \frac{d}{dt}f(k \exp(tX)k^{-1}x_0)|_{t=0} \\ &= \frac{d}{dt}f(\exp(t\text{Ad}(k)X)x_0)|_{t=0} \end{aligned}$$

for all  $f \in C^\infty(G/K)$ .

**Proposition 3.3.17** *Let  $G/K$  be a Riemannian symmetric space of the non-compact type. Assume that the action of  $K$  on the tangent space  $T_{x_0}(G/K)$  at  $x_0 = K \in G/K$  is irreducible. Let  $\alpha$  be an affine isometric action of  $G$  on a Hilbert space  $\mathcal{H}$ , and let  $F : G/K \rightarrow \mathcal{H}$  be a  $G$ -equivariant  $C^1$ -mapping. If  $F$  is not constant, then there exists  $\lambda > 0$  such that  $\lambda F$  is a local isometry.*

**Proof** Let  $\pi$  be the linear part of  $\alpha$ . We identify the tangent space  $T_{x_0}(G/K)$  with  $\mathfrak{p}$  as above and we identify the tangent space at a vector  $\xi \in \mathcal{H}$  with  $\mathcal{H}$ . Using the  $G$ -equivariance of  $F$ , we have, for every  $k \in K$  and  $X \in \mathfrak{p}$ ,

$$\begin{aligned} dF_{x_0}(\text{Ad}(k)X) &= \frac{d}{dt}F(\exp(t\text{Ad}(k)X)x_0)|_{t=0} \\ &= \frac{d}{dt}F(k \exp tXx_0)|_{t=0} \\ &= \frac{d}{dt}(\alpha(k)0 + \pi(k)F(\exp tXx_0))|_{t=0} \\ &= \pi(k)\frac{d}{dt}F(\exp tXx_0)|_{t=0} \\ &= \pi(k)dF_{x_0}(X). \end{aligned}$$

Hence, the symmetric bilinear form  $Q$  on  $T_{x_0}(G/K) \cong \mathfrak{p}$  defined by

$$Q(X, Y) = \langle dF_{x_0}(X), dF_{x_0}(Y) \rangle$$

is  $\text{Ad}(K)$ -invariant. Let  $A$  be the non-negative symmetric linear operator defined on the Euclidean space  $T_{x_0}(G/K)$  by

$$Q(X, Y) = \langle AX, Y \rangle, \quad \text{for all } X, Y \in T_{x_0}(G/K).$$

Then  $A$  commutes with  $\text{Ad}(k)$ , for every  $k \in K$ . Hence, the eigenspaces of  $A$  are  $K$ -invariant. Since, by assumption,  $K$  acts irreducibly on  $T_{x_0}(G/K)$ , it follows that  $A$  has a unique eigenvalue, that is,  $A = cI$  for some  $c \geq 0$ . Therefore,

$$\|dF_{x_0}(X)\|^2 = c\|X\|^2, \quad \text{for all } X \in T_{x_0}(G/K).$$



By  $G$ -equivariance of  $F$ , we have, for  $g \in G$  and  $x = gx_0$ ,

$$F = \alpha(g) \circ F \circ \lambda(g^{-1})$$

and

$$dF_x = \pi(g)dF_{x_0}d\lambda(g^{-1})_x.$$

Since  $\pi(g)$  is an isometry, it follows that

$$\begin{aligned} \|dF_x(d\lambda(g)_{x_0}X)\| &= \|dF_{x_0}(X)\| \\ &= c\|X\| = c\|d\lambda(g)_{x_0}(X)\|_x \end{aligned}$$

for all  $X \in \mathfrak{p}$ . As  $F$  is non-constant, this shows that  $c \neq 0$  and that  $\frac{1}{\sqrt{c}}F$  is a local isometry. ■

**Remark 3.3.18** The assumption that the action of  $K$  on  $T_{x_0}(G/K)$  is irreducible is fulfilled when  $G$  is a simple Lie group (for more details, see [Helga–62, Chapter VIII]).

### Growth of harmonic mappings on rank 1 spaces

The presentation here follows ideas due to M. Gromov (see [Gromo–03, Section 3.7 D’]). We thank T. Delzant for his patient explanations.

Let  $X$  be an irreducible Riemannian symmetric space of the non-compact type and of rank one. As discussed in Section 2.7,  $X$  is isometric to  $\mathbf{H}^n(\mathbf{K})$  for a real division algebra  $\mathbf{K}$  and for an integer  $n$ . When  $\mathbf{K}$  is one of  $\mathbf{R}, \mathbf{C}, \mathbf{H}$ , we assume that  $n \geq 1$ ; the Euclidean line  $\mathbf{H}^1(\mathbf{R})$  is of course not a Riemannian symmetric space of the non-compact type, but we include it in the table below because we want to consider the standard inclusion mapping  $\mathbf{H}^1(\mathbf{R}) \rightarrow \mathbf{H}^2(\mathbf{R})$ . When  $\mathbf{K} = \text{Cay}$ , which is non-associative, the integer has to satisfy  $n \leq 2$ ; here again, we do consider the space  $\mathbf{H}^1(\text{Cay})$  and its standard inclusion in  $\mathbf{H}^2(\text{Cay})$ . There are isomorphisms for low values of  $n$ , since  $\mathbf{H}^1(\mathbf{C}), \mathbf{H}^1(\mathbf{H}), \mathbf{H}^1(\text{Cay})$  are isometric to  $\mathbf{H}^2(\mathbf{R}), \mathbf{H}^4(\mathbf{R}), \mathbf{H}^8(\mathbf{R})$  respectively. The space  $X$  can be identified with the homogeneous  $G/K$ , where  $G$  and  $K$  are described by the following list (see [Helga–62, Chapter IX, Section 4]):

$\mathbf{K}$	$X$	$G$	$K$
$\mathbf{R}$	$\mathbf{H}^n(\mathbf{R})$	$SO_0(n, 1)$	$SO(n)$
$\mathbf{C}$	$\mathbf{H}^n(\mathbf{C})$	$SU(n, 1)$	$S(U(n) \times U(1))$
$\mathbf{H}$	$\mathbf{H}^n(\mathbf{H})$	$Sp(n, 1)$	$Sp(n) \times Sp(1)$
Cay	$\mathbf{H}^1(\text{Cay})$	$SO_0(8, 1)$	$SO(8)$
Cay	$\mathbf{H}^2(\text{Cay})$	$F_{4(-20)}$	$\text{Spin}(9)$

It is crucial for our purpose to normalise metrics in such a way that, for  $m \leq n$ , the embedding  $\mathbf{H}^m(\mathbf{K}) \rightarrow \mathbf{H}^n(\mathbf{K})$  is an isometry. This can be achieved as follows.

Assume that  $\mathbf{K} \neq \text{Cay}$ , and view  $G$  as a subgroup of  $GL_{n+1}(\mathbf{K})$ , as in the previous table. The Cartan involution we choose on  $G$  is

$$\theta(g) = JgJ, \quad g \in G,$$

where

$$J = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

The Cartan decomposition of the Lie algebra  $\mathfrak{g}$  is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

with

$$\mathfrak{p} = \left\{ A_z = \begin{pmatrix} 0 & z \\ \bar{z}^t & 0 \end{pmatrix} : z = (z_1, \dots, z_n) \in \mathbf{K}^n \right\}.$$

The adjoint action of  $K$  on  $\mathfrak{p}$  is simply given by

$$\text{Ad}(k)A_z = A_{kz}, \quad k \in K, \quad z \in \mathbf{K}^n.$$

We endow  $\mathfrak{p}$  with the  $\text{Ad}(K)$ -invariant inner product

$$\langle A_z, A_{z'} \rangle = \text{Re} \sum_{i=1}^n \bar{z}_i z'_i, \quad z, z' \in \mathbf{K}^n.$$

Observe that  $K$  acts transitively on the unit sphere of  $\mathfrak{p}$ . We identify  $\mathfrak{p}$  with  $T_{x_0}(G/K)$  and define a  $G$ -invariant Riemannian structure on  $\mathbf{H}^n(\mathbf{K}) = G/K$ , as explained before Proposition 3.3.5 and Proposition 3.3.17.

A function  $f$  on  $G/K$  is said to be *radial* if it depends only on the geodesic distance to  $x_0$ , that is, if there exists a function  $\varphi$  on  $\mathbf{R}_+$  such that, for every  $x \in G/K$ ,

$$f(x) = \varphi(r), \quad r = d(x, x_0).$$

**Remark 3.3.19** Let  $X = G/K$  be an irreducible Riemannian symmetric space of rank 1, and let  $F : X \rightarrow \mathcal{H}$  be a  $G$ -equivariant mapping, with respect to an

affine isometric action  $\alpha$  of  $G$  on the Hilbert space  $\mathcal{H}$ . If  $F(x_0) = 0$ , then  $\|F\|^2$  is a radial function. Indeed, if  $d(x, x_0) = d(y, x_0)$ , then there exists  $k \in K$  such that  $y = kx$ , since  $K$  is transitive on the spheres centred at  $x_0$ . As  $F(x_0) = 0$ , we see that  $\alpha(k)$  is a linear isometry and

$$\|F(y)\|^2 = \|\alpha(k)F(x)\|^2 = \|F(x)\|^2.$$

The Laplacian of a radial function takes a very simple form on  $\mathbf{H}^n(\mathbf{K})$  :

$$(**) \quad \Delta f(x) = -\frac{d^2\varphi}{dr^2} - m(r)\frac{d\varphi}{dr},$$

where

$$m(r) = m_1 \coth r + 2m_2 \coth 2r$$

and

$$m_1 = d(n-1), \quad m_2 = d-1, \quad d = \dim_{\mathbf{R}} \mathbf{K}$$

(see [Farau–83, p.338] or [Helga–84, Chapter II, Section 3]).

We are going to apply this formula to  $\Delta(\|F\|^2)$ , where  $F$  is a mapping as in the previous remark. For this purpose, the following positivity result will be of crucial importance. We thank G. Skandalis for suggesting a simplification in our original proof.

**Lemma 3.3.20** *Let  $X = G/K$  be an irreducible Riemannian symmetric space of rank 1, and let  $\alpha$  be an affine isometric action of  $G$  on a Hilbert space  $\mathcal{H}$ . Let  $F : X \rightarrow \mathcal{H}$  be a  $G$ -equivariant mapping of class  $C^2$  with  $F(x_0) = 0$ . Then*

$$\operatorname{Re}\langle \Delta F(x), F(x) \rangle \geq 0,$$

for every  $x \in X$ .

**Proof** Let  $\pi$  be the linear part of  $\alpha$  and let  $b \in Z^1(G, \pi)$  be the corresponding 1-cocycle:

$$\alpha(g)\xi = \pi(g)\xi + b(g), \quad g \in G, \quad \xi \in \mathcal{H}.$$

Since  $F(x_0) = 0$ , we have, by  $G$ -equivariance of  $F$ ,

$$F(gx_0) = \alpha(g)(0) = b(g).$$

Set  $P = \exp(\mathfrak{p})$ , and let  $G = PK$  be the Cartan decomposition of  $G$ . For  $x \in X$ , there exists a unique  $g \in P$  such that  $g^{-1}x_0 = x$ . Fix  $Y$  in the unit sphere of  $\mathfrak{p} = T_{x_0}X$ .

For  $k \in K$ , we have

$$\begin{aligned}
 \frac{d}{dt} F(g^{-1} \exp(t \operatorname{Ad}(k) Y) x_0) &= \frac{d}{dt} F(g^{-1} k \exp(t Y) k^{-1} x_0) \\
 &= \frac{d}{dt} F(g^{-1} k \exp(t Y) x_0) \\
 &= \frac{d}{dt} b(g^{-1} k \exp t Y) \\
 &= \frac{d}{dt} \left( b(g^{-1} k) + \pi(g^{-1} k) b(\exp t Y) \right) \\
 &= \pi(g^{-1} k) \frac{d}{dt} b(\exp t Y).
 \end{aligned}$$

Set

$$\beta(Y) = \frac{d}{dt} b(\exp t Y)|_{t=0},$$

which is a vector in  $\mathcal{H}$ . (Observe that  $b$  is a  $C^2$ -mapping on  $G$  since  $F$  is  $C^2$  on  $G/K$ .) Then

$$\begin{aligned}
 \frac{d}{dt} b(\exp t Y) &= \lim_{s \rightarrow 0} \frac{b(\exp(t+s) Y) - b(\exp t Y)}{s} \\
 &= \pi(\exp t Y) \lim_{s \rightarrow 0} \frac{b(\exp s Y)}{s} = \pi(\exp t Y) \beta(Y).
 \end{aligned}$$

Set  $\pi(Y) = \frac{d}{dt} \pi(\exp t Y)|_{t=0}$ , an unbounded operator acting on the smooth vectors in  $\mathcal{H}$ . Then

$$\frac{d^2}{dt^2} F(g^{-1} \exp(t \operatorname{Ad}(k) Y) x_0)|_{t=0} = \pi(g^{-1} k) \pi(Y) \beta(Y).$$

On the other hand, by formula (\*) in Remark 3.3.12,

$$\Delta F(x) = -\dim X \int_{\mathfrak{p}_1} \frac{d^2}{dt^2} F(g^{-1} \exp t Z x_0)|_{t=0} d\nu(Z)$$

where  $\mathfrak{p}_1$  is the unit sphere in  $\mathfrak{p}$  and  $\nu$  is the normalised rotation-invariant measure on  $\mathfrak{p}_1$ . Since  $K$  acts transitively on  $\mathfrak{p}_1$ , it follows that

$$\begin{aligned}
 \Delta F(x) &= -\dim X \int_K \frac{d^2}{dt^2} F(g^{-1} \exp(t \operatorname{Ad}(k) Y) x_0)|_{t=0} dk \\
 &= -\dim X \int_K \pi(g^{-1} k) \pi(Y) \beta(Y) dk.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \langle \Delta F(x), F(x) \rangle &= -\dim X \int_K \langle \pi(g^{-1}k)\pi(Y)\beta(Y), b(g^{-1}) \rangle dk \\
 &= -\dim X \int_K \langle \pi(k)\pi(Y)\beta(Y), \pi(g)b(g^{-1}) \rangle dk \\
 &= \dim X \int_K \langle \pi(k)\pi(Y)\beta(Y), b(g) \rangle dk.
 \end{aligned}$$

To proceed, we decompose  $\pi$  as a direct integral

$$\pi = \int_{\Lambda}^{\oplus} \pi_{\lambda} d\mu(\lambda)$$

of irreducible unitary representations  $\pi_{\lambda}$  for a measure space  $(\Lambda, \mu)$ ; see Theorem F.5.3. The cocycle  $b$  decomposes accordingly

$$b = \int_{\Lambda}^{\oplus} b_{\lambda} d\mu(\lambda),$$

where  $b_{\lambda} \in Z^1(G, \pi_{\lambda})$  is  $C^2$  for  $\mu$ -almost all  $\lambda \in \Lambda$ . Set

$$\beta_{\lambda}(Y) = \frac{d}{dt} b_{\lambda}(\exp tY)|_{t=0}.$$

Then, using Fubini's Theorem,

$$\langle \Delta F(x), F(x) \rangle = \dim X \int_{\Lambda} \left\langle \int_K \pi_{\lambda}(k)\pi_{\lambda}(Y)\beta_{\lambda}(Y)dk, b_{\lambda}(g) \right\rangle d\mu(\lambda).$$

We now show that the integrand on the right-hand side has  $\mu$ -almost everywhere non-negative real part.

Observe that the operator

$$P_{\lambda} = \int_K \pi_{\lambda}(k)dk$$

is the orthogonal projection from the Hilbert space  $\mathcal{H}_{\lambda}$  of  $\pi_{\lambda}$  onto the subspace of  $\pi_{\lambda}(K)$ -invariant vectors. Two cases may occur:

- *First case:*  $\pi_{\lambda}$  is not spherical. Then  $P_{\lambda} = 0$  and the integrand vanishes.
- *Second case:*  $\pi_{\lambda}$  is spherical. Then, by Proposition 3.3.5, the cocycle  $b_{\lambda}$  is a coboundary, so we find  $\xi_{\lambda} \in \mathcal{H}_{\lambda}$  such that

$$b_{\lambda}(h) = \pi_{\lambda}(h)\xi_{\lambda} - \xi_{\lambda}, \quad h \in G.$$

Note that, since  $b(k) = 0$  for every  $k \in K$ , we have  $b_\lambda|_K = 0$  for  $\mu$ -almost every  $\lambda \in \Lambda$ . This means that  $\xi_\lambda$  is a  $\pi_\lambda(K)$ -invariant vector, for  $\mu$ -almost every  $\lambda \in \Lambda$ . Since, moreover,  $\beta_\lambda(Y) = \pi_\lambda(Y)\xi_\lambda$ , it follows that

$$\begin{aligned} \left\langle \int_K \pi_\lambda(k) \pi_\lambda(Y) \beta_\lambda(Y) dk, b_\lambda(g) \right\rangle &= \langle P_\lambda \pi_\lambda(Y) \beta_\lambda(Y), b_\lambda(g) \rangle \\ &= \langle \pi_\lambda(Y)^2 \xi_\lambda, P_\lambda (\pi_\lambda(g) \xi_\lambda - \xi_\lambda) \rangle \\ &= \langle \pi_\lambda(Y)^2 \xi_\lambda, P_\lambda \pi_\lambda(g) \xi_\lambda \rangle - \langle \pi_\lambda(Y)^2 \xi_\lambda, \xi_\lambda \rangle. \end{aligned}$$

Since the space of  $\pi_\lambda(K)$ -invariant vectors has dimension 1, we find  $c_\lambda \in \mathbf{C}$ , with  $|c_\lambda| \leq 1$ , such that

$$P_\lambda \pi_\lambda(g) \xi_\lambda = \overline{c_\lambda} \xi_\lambda.$$

Therefore,

$$\left\langle \int_K \pi_\lambda(k) \pi_\lambda(Y) \beta_\lambda(Y) dk, b_\lambda(g) \right\rangle = (c_\lambda - 1) \langle \pi_\lambda(Y)^2 \xi_\lambda, \xi_\lambda \rangle.$$

Since  $\pi_\lambda(Y)$  is anti-self-adjoint, we have

$$\langle \pi_\lambda(Y)^2 \xi_\lambda, \xi_\lambda \rangle \leq 0.$$

As  $\operatorname{Re}(c_\lambda - 1) \leq 0$ , it follows that

$$\operatorname{Re} \left( \left\langle \int_K \pi_\lambda(k) \pi_\lambda(Y) \beta_\lambda(Y) dk, b_\lambda(g) \right\rangle \right) \geq 0,$$

and this concludes the proof. ■

Here is a crucial observation of M. Gromov. It says that, among  $G$ -equivariant mappings, the harmonic ones achieve the fastest growth rate.

**Proposition 3.3.21** *Let  $X = G/K$  be an irreducible Riemannian symmetric space of rank 1, and let  $\alpha$  be an affine isometric action of  $G$  on a Hilbert space  $\mathcal{H}$ . Let  $F : X \rightarrow \mathcal{H}$  be a  $G$ -equivariant, locally isometric mapping of class  $C^2$  with  $F(x_0) = 0$ . Define a function  $\varphi$  on  $\mathbf{R}_+$  by*

$$\varphi(r) = \|F(x)\|^2, \quad r = d(x, x_0), \quad x \in X.$$

Then

$$\varphi(r) \leq \frac{2 \dim X}{m_1 + 2m_2} r + o(r), \quad \text{as } r \rightarrow +\infty$$

with equality if  $F$  is harmonic.

**Proof** By Proposition 3.3.14 and formula (\*\*) above, we have

$$\begin{aligned}\Delta(\|F\|^2)(x) &= 2\operatorname{Re}\langle \Delta F(x), F(x) \rangle - 2\|dF_x\|_{\text{HS}}^2 \\ &= -\varphi''(r) - m(r)\varphi'(r),\end{aligned}$$

so that

$$\varphi''(r) + m(r)\varphi'(r) = -2\operatorname{Re}\langle \Delta F, F \rangle(r) + 2\|dF_x\|_{\text{HS}}^2.$$

Set  $\psi = \varphi'$ . Since  $F$  is locally isometric, we have  $\|dF_x\|_{\text{HS}}^2 = \dim X$  and, therefore,

$$\psi'(r) + m(r)\psi(r) = -2\operatorname{Re}\langle \Delta F, F \rangle(r) + 2\dim X.$$

This is a first order ordinary differential equation for  $\psi$ , which we solve by the method of variations of constants. The general solution of the associated homogeneous equation is a constant multiple of the function

$$\psi_0(r) = (\sinh r)^{-m_1} (\sinh 2r)^{-m_2},$$

so that a particular solution of the inhomogeneous equation is

$$\psi(r) = \psi_0(r) \int_0^r (2\dim X - 2\operatorname{Re}\langle \Delta F, F \rangle)(s) \frac{1}{\psi_0(s)} ds.$$

By the previous lemma,  $\operatorname{Re}\langle \Delta F, F \rangle \geq 0$ . Hence, we have the estimate

$$\psi(r) \leq (2\dim X)\psi_0(r) \int_0^r \frac{1}{\psi_0(s)} ds,$$

with equality if  $F$  is harmonic. Setting

$$f(r) = \frac{1}{\psi_0(r)} = (\sinh r)^{m_1} (\sinh 2r)^{m_2},$$

we have

$$\lim_{r \rightarrow +\infty} \frac{1}{f(r)} \int_0^r f(s) ds = \lim_{r \rightarrow +\infty} \frac{f(r)}{f'(r)} = \frac{1}{m_1 + 2m_2}.$$

Therefore,

$$\psi(r) \leq \frac{2\dim X}{m_1 + 2m_2} + o(1) \quad \text{as } r \rightarrow +\infty$$

(again, with equality if  $F$  is harmonic). The result for  $\varphi$  follows by integrating. ■

At this point, we pause to compute the constant  $2 \dim X / (m_1 + 2m_2)$  for the various families of hyperbolic spaces.

$X$	$2 \dim X / (m_1 + 2m_2)$
$\mathbf{H}^n(\mathbf{R})$	$\frac{2n}{n-1} = 2 + \frac{2}{n-1}$
$\mathbf{H}^n(\mathbf{C})$	$\frac{4n}{2(n-1)+2} = 2$
$\mathbf{H}^n(\mathbf{H})$	$\frac{8n}{4(n-1)+6} = 2 - \frac{2}{2n+1}$
$\mathbf{H}^1(\text{Cay})$	$\frac{16}{14} = \frac{8}{7}$
$\mathbf{H}^2(\text{Cay})$	$\frac{32}{8+14} = \frac{16}{11}$

This table shows the main differences between the families  $\mathbf{H}^n(\mathbf{R})$ ,  $\mathbf{H}^n(\mathbf{C})$ , and  $\mathbf{H}^n(\mathbf{K})$  with  $\mathbf{K} = \mathbf{H}$  or Cay. More precisely, on  $\mathbf{H}^n(\mathbf{R})$  the growth rate of harmonic  $G$ -equivariant mappings decreases with dimension, on  $\mathbf{H}^n(\mathbf{C})$  it is independent of dimension, while on  $\mathbf{H}^n(\mathbf{K})$ , for  $\mathbf{K} = \mathbf{H}$  or Cay, it increases with dimension.

We are now in a position to prove that  $Sp(n, 1)$  for  $n \geq 2$  and  $F_{4(-20)}$  have Property (T).

**Proof of Theorem B** Let  $X = G/K$  be either  $\mathbf{H}^n(\mathbf{H})$  for  $n \geq 2$  or  $\mathbf{H}^2(\text{Cay})$ . Assume, by contradiction, that there exists an affine isometric action  $\alpha$  of  $G$  on a Hilbert  $\mathcal{H}$  and a non-constant  $G$ -equivariant harmonic mapping

$$F : X \mapsto \mathcal{H}$$

of class  $C^2$ . Upon replacing  $F$  by  $\lambda F$  for some  $\lambda > 0$ , we can assume by Proposition 3.3.17 that  $F$  is a local isometry. (To preserve the  $G$ -equivariance, we have to replace  $\alpha$  by the affine action which has the same linear part and is given by the 1-cocycle  $\lambda b$ , where  $b$  is the 1-cocycle associated to  $\alpha$ .)

Similarly, replacing  $F$  by  $F - F(x_0)$ , we can assume that  $F(x_0) = 0$ . (Here we have to replace  $\alpha$  by the affine action which has the same linear part and is given by the 1-cocycle  $g \mapsto b(g) + \pi(g)F(x_0) - F(x_0)$ .)



Suppose that  $X = \mathbf{H}^n(\mathbf{H})$ ,  $n \geq 2$ . Restrict  $F$  to a quaternionic hyperbolic line  $Y = \mathbf{H}^1(\mathbf{H})$ . Of course,  $F|_Y$  has no reason to remain harmonic, but it is still equivariant with respect to the isometry group  $Sp(1, 1)$  of  $Y$ . Since we have normalised metrics in such a way that the inclusion  $Y \rightarrow X$  is an isometric totally geodesic embedding,  $F|_Y$  must have the same growth rate as  $F$ . Hence, by Proposition 3.3.21, we have for  $x \in Y$  and  $r = d(x, x_0)$ ,

$$\|F|_Y\|^2(r) = \frac{4n}{2n+1}r + o(r), \quad \text{as } r \rightarrow +\infty.$$

On the other hand, by the same proposition,

$$\|F|_Y\|^2(r) \leq \frac{4}{3}r + o(r), \quad \text{as } r \rightarrow +\infty.$$

Since  $4/3 < 4n/(2n+1)$ , this is the desired contradiction.

The proof for  $X = \mathbf{H}^2(\text{Cay})$  and  $G = F_{4(-20)}$  is completely similar. ■

**Remark 3.3.22** (i) Theorem B is valid more generally for  $\Gamma$ -equivariant mappings, where  $\Gamma$  is a lattice in  $Sp(n, 1)$  or  $F_{4(-20)}$ ; see [JosLi-96].

(ii) It is proved in [Delor-77] that, for  $G = SU(n, 1)$ ,  $n \geq 1$ , there are exactly two inequivalent irreducible unitary representations  $\pi$  of  $G$  such that  $H^1(G, \pi) \neq 0$ ; these representations, called the *cohomological representations* of  $G$ , are contragredient of each other. For  $G = SO_0(n, 1)$ ,  $n \geq 3$ , there exists, up to unitary equivalence, a unique irreducible unitary representation  $\pi$  of  $G$  such that  $H^1(G, \pi) \neq 0$ .

### 3.4 The question of finite presentability

Kazhdan asked in his original paper [Kazhd-67, Hypothesis 1] whether a discrete group with Property (T) is necessarily *finitely presented*. The answer is negative, as it follows from each of the following examples. The first one appears in [Margu-91, Chapter III, (5.11)] and the second one has been pointed out to us by Y. de Cornulier. We show then a result of Shalom, according to which any discrete group with Property (T) is the quotient of a finitely presented group with Property (T).

**Example 3.4.1** Let  $p$  be a prime. Denote by  $\mathbf{F}_p((X))$  the field of Laurent series over the field with  $p$  elements; recall from Chapter D.4 that it is a local field. The subring  $\mathbf{F}_p[X^{-1}]$  of  $\mathbf{F}_p((X))$  of series of the form  $\sum_{k=0}^n a_k X^{-k}$  is naturally isomorphic to the ring of polynomials  $\mathbf{F}_p[X]$ . Denote by  $G$  the locally

compact group  $SL_3(\mathbf{F}_p((X)))$  and by  $\Gamma$  the subgroup  $SL_3(\mathbf{F}_p[X^{-1}])$ , which is a lattice. Then  $G$  has Property (T) by Theorem 1.4.15 and  $\Gamma$  has Property (T) by Theorem 1.7.1. Therefore, the group  $SL_3(\mathbf{F}_p[X])$ , which is isomorphic to  $\Gamma$ , has Property (T); on the other hand, it was shown in [Behr–79] that the group  $SL_3(\mathbf{F}_p[X])$  is *not* finitely presented. (Observe that, by [RehSo–76, Page 164], the group  $SL_n(\mathbf{F}_p[X])$  is finitely presented for  $n \geq 4$ . On the other hand,  $SL_2(\mathbf{F}_p[X])$  is not finitely generated; see [Serr–70b, II, §1.6, Exercice 2].)

Other examples of non-finitely presented groups with Property (T) are provided by infinite torsion quotients of uniform lattices in  $Sp(n, 1)$ ,  $n \geq 2$ ; see Corollary 5.5.E in [Gromo–87].

The following example is due to Yves de Cornulier.

**Theorem 3.4.2** *Let  $p$  be a prime. The countable group  $\Gamma_p = Sp_4(\mathbf{Z}[1/p]) \ltimes (\mathbf{Z}[1/p])^4$  has Property (T) and is not finitely presented.*

To prove this result, we need some notation. Let  $R$  be a commutative ring with unit. Consider a non-degenerate alternate bilinear form on  $R^4$ , for example the form  $\omega$  which is defined by

$$\omega(x, y) = x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2$$

and which corresponds to the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The symplectic group of  $(R^4, \omega)$  is

$$Sp_4(R) = \{g \in GL_4(R) : {}^t g J g = J\}$$

(compare Section 1.5). Denote by

$$G(R) = Sp_4(R) \ltimes R^4$$

the semidirect product corresponding to the standard action of  $Sp_4(R)$  on  $R^4$ ; in particular, we have  $\Gamma_p = G(\mathbf{Z}[1/p])$ .

Let  $H_5(R)$  denote the five-dimensional *Heisenberg group* over  $R$ , which is the set  $R^4 \times R$  equipped with the product

$$(x, \lambda)(y, \mu) = (x + y, \lambda + \mu + \omega(x, y)), \quad x, y \in R^4, \lambda, \mu \in R.$$

Assume that the mapping  $R \rightarrow R$ ,  $x \mapsto 2x$  is injective. Then the centre  $Z$  of  $H_5(R)$  is  $\{(0, \lambda) : \lambda \in R\} \cong R$  and  $H_5(R)/Z \cong R^4$ . We have therefore a central extension

$$0 \rightarrow R \rightarrow H_5(R) \rightarrow R^4 \rightarrow 1.$$

The symplectic group acts by automorphisms on  $H_5(R)$  :

$$g(x, \lambda) = (gx, \lambda), \quad \text{for all } g \in Sp_4(R), x \in R^4, \lambda \in R.$$

We denote the corresponding semidirect product by

$$\tilde{G}(R) = Sp_4(R) \ltimes H_5(R).$$

Since the action of  $Sp_4(R)$  on the centre of  $H_5(R)$  is trivial, we have another central extension

$$0 \rightarrow R \rightarrow \tilde{G}(R) \rightarrow G(R) \rightarrow 1.$$

Set  $\tilde{\Gamma}_p = \tilde{G}(\mathbf{Z}[1/p])$ .

**Lemma 3.4.3** *The group  $\tilde{\Gamma}_p$  has Property (T); in particular,  $\tilde{\Gamma}_p$  is finitely generated.*

**Proof** The group  $\tilde{G}(\mathbf{K})$  has Property (T), for any local field  $\mathbf{K}$  (Exercise 1.8.10). Hence, the claim will be proved if we show that  $\tilde{\Gamma}_p$  is isomorphic to a lattice in

$$G = \tilde{G}(\mathbf{Q}_p) \times \tilde{G}(\mathbf{R}).$$

The diagonal embedding  $\mathbf{Z}[1/p] \rightarrow \mathbf{Q}_p \times \mathbf{R}$  has a discrete and cocompact image; in particular, it induces an embedding of  $\tilde{\Gamma}_p$  into  $G$  with discrete image. Observe that  $G$  is isomorphic to the semidirect product

$$(Sp_4(\mathbf{Q}_p) \times Sp_4(\mathbf{R})) \ltimes (H_5(\mathbf{Q}_p) \times H_5(\mathbf{R})).$$

Hence, it suffices to show that  $Sp_4(\mathbf{Z}[1/p])$  and  $H_5(\mathbf{Z}[1/p])$  are lattices in  $Sp_4(\mathbf{Q}_p) \times Sp_4(\mathbf{R})$  and in  $H_5(\mathbf{Q}_p) \times H_5(\mathbf{R})$ , respectively (see Exercise B.3.5).

The group  $Sp_4$ , viewed as an algebraic group defined over  $\mathbf{Q}$ , is perfect and has therefore no non-trivial rational character. By the Borel–Harish–Chandra Theorem (Theorem 12.3 in [BorHa–62], see also Theorem 3.2.7 in Chapter I of [Margu–91]), it follows that  $Sp_4(\mathbf{Z}[1/p])$  is a lattice in  $Sp_4(\mathbf{Q}_p) \times Sp_4(\mathbf{R})$ .

The proof that  $H_5(\mathbf{Z}[1/p])$  is a lattice in  $H_5(\mathbf{Q}_p) \times H_5(\mathbf{R})$  is more elementary: if  $X \subset \mathbf{Q}_p \times \mathbf{R}$  is a compact fundamental domain for  $\mathbf{Z}[1/p]$ , then  $X^4 \times X \subset H_5(\mathbf{Q}_p) \times H_5(\mathbf{R})$  is a compact fundamental domain for  $H_5(\mathbf{Z}[1/p])$ . ■

We will need the following elementary fact, observed by P. Hall [Hall–54, page 421].

**Lemma 3.4.4** *Let  $G$  be a finitely generated group, and let  $N$  be a normal subgroup of  $G$ . Assume that  $G/N$  is finitely presented. Then  $N$  is finitely generated as a normal subgroup of  $G$ .*

**Proof** Since  $G$  is finitely generated, there exists a surjective homomorphism  $\beta : F_n \rightarrow G$ , where  $F_n$  denotes the free group on a finite set  $\{a_1, \dots, a_n\}$ . Set  $R = \beta^{-1}(N)$  and denote by  $p : G \rightarrow G/N$  the canonical projection. The kernel of the surjective homomorphism  $p \circ \beta : F_n \rightarrow G/N$  is  $R$ ; in other words,

$$\langle a_1, \dots, a_n \mid r \in R \rangle$$

is a presentation of  $G/N$ . Since  $G/N$  is finitely presented,  $R$  is generated as a normal subgroup of  $F_n$  by finitely many elements  $r_1, \dots, r_m$ . (Recall that, if a group has a finite presentation with respect to some finite set of generators, it has a finite presentation with respect to any other finite set of generators.) It follows that  $N$  is generated as a normal subgroup of  $G$  by the finite set  $\{\beta(r_1), \dots, \beta(r_m)\}$ . ■

**Proof of Theorem 3.4.2** By Lemma 3.4.3, the group  $\tilde{\Gamma}_p$  is finitely generated. Consider the central extension

$$0 \rightarrow \mathbf{Z}[1/p] \rightarrow \tilde{\Gamma}_p \rightarrow \Gamma_p \rightarrow 1.$$

The group  $\Gamma_p$  has Property (T), since it is a quotient of  $\tilde{\Gamma}_p$ . Observe that the kernel  $\mathbf{Z}[1/p]$  is not finitely generated and, since it is central, it is not finitely generated as a normal subgroup of  $\tilde{\Gamma}_p$ . Hence, by the previous lemma,  $\Gamma_p$  is not finitely presented. ■

We now turn to a consequence, due to Shalom [Shal–00a, Theorem 6.7], of the method of proof of Theorem 3.2.1.

**Theorem 3.4.5** *Let  $\Gamma$  be a discrete group with Property (T). Then there exists a finitely presented group  $\tilde{\Gamma}$  with Property (T) and a normal subgroup  $N$  of  $\tilde{\Gamma}$  such that  $\Gamma$  is isomorphic to  $\tilde{\Gamma}/N$ .*

**Proof** Since  $\Gamma$  has Property (T), it is finitely generated. Let

$$\beta : F_n \rightarrow \Gamma,$$

be a surjective homomorphism, where  $F_n$  is the free group on  $n$  generators  $a_1, \dots, a_n$ . Set  $N = \text{Ker } \beta$ , and let  $w_1, w_2, \dots$  be an enumeration of the elements of  $N$ . For each  $k \in \mathbf{N}$ , let  $N_k$  be the normal subgroup of  $F_n$  generated by  $w_1, \dots, w_k$ , and set

$$\Gamma_k = F_n / N_k.$$

The group  $\Gamma_k$  is finitely presented, and  $\beta$  factorises to a surjective homomorphism  $\Gamma_k \rightarrow \Gamma$ . It is enough to show that  $\Gamma_k$  has Property (T) for some  $k \in \mathbf{N}$ .

Assume, by contradiction, that, for every  $k \in \mathbf{N}$ , the group  $\Gamma_k$  does not have Property (T). Then there exists an orthogonal representation  $(\pi_k, \mathcal{H}_k)$  of  $\Gamma_k$  which almost has invariant vectors and no non-zero invariant vectors. We view  $\pi_k$  as representation of  $F_n$ . Define

$$\delta_k : \mathcal{H}_k \rightarrow \mathbf{R}_+, \quad \xi \mapsto \max_{1 \leq i \leq n} \|\pi_k(a_i)\xi - \xi\|.$$

By Lemma 3.2.5, there exists  $\xi_k \in \mathcal{H}_k$  with  $\delta_k(\xi_k) = 1$  and such that  $\delta_k(\eta) > 1/6$  for every  $\eta \in \mathcal{H}_k$  with  $\|\eta - \xi_k\| < k$ .

For every  $k \in \mathbf{N}$ , the function

$$\psi_k : g \mapsto \|\pi_k(g)\xi_k - \xi_k\|^2$$

is conditionally of negative type on  $F_n$ . As in the proof of Theorem 3.2.1, the sequence  $(\psi_k)_k$  is uniformly bounded on finite subsets of  $F_n$ . Hence, upon passing to a subsequence, we can assume that  $(\psi_k)_k$  converges pointwise to a function  $\psi$  conditionally of negative type on  $F_n$ .

Let  $(H_\psi, \pi_\psi, b_\psi)$  be the triple associated to  $\psi$ . As in the proof of Theorem 3.2.1, it can be checked that  $b_\psi$  does not belong to  $\overline{B^1(F_n, \pi_\psi)}$  and, in particular, not to  $B^1(F_n, \pi_\psi)$ .

Since  $\psi_k = 0$  on  $N_k$ , the function  $\psi$  vanishes on  $N$ . Hence, both the representation  $\pi_\psi$  and the cocycle  $b_\psi$  factorise through  $F_n/N$ . Since  $F_n/N$  is isomorphic to  $\Gamma$ , we obtain in this way an orthogonal representation of  $\Gamma$  with non-zero first cohomology, contradicting our assumption that  $\Gamma$  has Property (T). ■

### 3.5 Other consequences of Shalom's Theorem

In this section, using Theorem 3.2.1, we give another treatment of the behaviour of Property (T) under covering of groups (compare Section 1.7). We then give the classification of semisimple real Lie groups with Property (T).

**Lemma 3.5.1** *Let  $G$  be topological group, with centre  $Z(G)$ . Let  $(\pi, \mathcal{H})$  be an irreducible unitary representation of  $G$ . If  $\pi$  is non-trivial on  $Z(G)$ , then  $H^1(G, \pi) = 0$ .*

**Proof** Let  $b \in Z^1(G, \pi)$ . We have to show that  $b$  is a coboundary. By our assumption on  $\pi$ , there exists  $z_0 \in Z(G)$  such that  $\pi(z_0) \neq I_{\mathcal{H}}$ . By Schur's Lemma (Theorem A.2.2),  $\pi(z_0) = \lambda_0 I_{\mathcal{H}}$  for some complex number  $\lambda_0 \neq 1$ . We have

$$\pi(g)b(z_0) + b(g) = b(gz_0) = b(z_0g) = \pi(z_0)b(g) + b(z_0) = \lambda_0 b(g) + b(z_0),$$

and hence

$$b(g) = \pi(g) \left( \frac{b(z_0)}{\lambda_0 - 1} \right) - \left( \frac{b(z_0)}{\lambda_0 - 1} \right),$$

for all  $g \in G$ . ■

The proof of the following result was shown to us by Y. Shalom.

**Theorem 3.5.2** *Let  $G$  be a topological group. Assume that  $G/\overline{[G, G]}$  is compact, and let  $C$  be a closed subgroup contained in the centre of  $G$ .*

*If  $H^1(G/C, \sigma) = 0$  for every irreducible unitary representation  $\sigma$  of  $G/C$ , then  $H^1(G, \pi) = 0$  for every irreducible unitary representation  $\pi$  of  $G$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be an irreducible unitary representation of  $G$ . We consider the restriction  $\pi|_C$  of  $\pi$  to  $C$ . Two cases occur.

- $\pi|_C$  is non-trivial. Then  $H^1(G, \pi) = 0$  by the previous lemma.
- $\pi|_C$  is trivial, that is,  $\pi$  factorises through  $G/C$ . Let  $b \in Z^1(G, \pi)$ . We claim that  $b|_C \equiv 0$ . Indeed, assume, by contradiction that there exists  $z_0 \in C$  such that  $b(z_0) \neq 0$ . Then, expanding  $b(gz_0) = b(z_0g)$  in two ways as in the proof of the previous lemma, we obtain

$$\pi(g)b(z_0) + b(g) = \pi(z_0)b(g) + b(z_0)$$

and, hence,

$$\pi(g)b(z_0) = b(z_0)$$

for all  $g \in G$ , since  $\pi(z_0) = I_{\mathcal{H}}$ . Thus,  $\pi$  has a non-zero invariant vector. Since  $\pi$  is irreducible, it follows that  $\pi$  is the unit representation and  $b : G \rightarrow \mathbb{C}$  is a continuous homomorphism. Hence,  $b$  factorises through  $G/\overline{[G, G]}$ . As the latter group is compact, it follows that  $b \equiv 0$ . This contradicts the fact that  $b(z_0) \neq 0$ .

Therefore,  $b|_C \equiv 0$ . This means that, not only  $\pi$ , but also  $b$  factorises through  $G/C$ . Hence,  $b \in Z^1(G/C, \pi) = B^1(G/C, \pi)$  and therefore  $b \in B^1(G, \pi)$ . ■

From Theorem 3.2.1, we obtain immediately the following corollary, which we proved earlier by another method (Theorem 1.7.11).

**Corollary 3.5.3** *Let  $G$  be a second countable locally compact group. Assume that  $G$  is compactly generated, and that  $G/\overline{[G, G]}$  is compact. Let  $C$  be a closed subgroup contained in the centre of  $G$ . If  $G/C$  has Property (T), then  $G$  has Property (T).*

As a first consequence, we classify the semisimple real Lie groups with Property (T).

**Theorem 3.5.4** *Let  $G$  be a connected semisimple real Lie group, with Lie algebra  $\mathfrak{g}$  and universal covering group  $\tilde{G}$ . The following conditions are equivalent:*

- (i)  $G$  has Property (T);
- (ii)  $\tilde{G}$  has Property (T);
- (iii) no simple factor of  $\mathfrak{g}$  is isomorphic to  $\mathfrak{so}(n, 1)$  or  $\mathfrak{su}(n, 1)$ .

**Proof** To show that (i) and (ii) are equivalent, let  $Z = \pi_1(G)$  be the fundamental group of  $G$ . Then  $Z$  can be viewed as a discrete central subgroup of  $\tilde{G}$ , and  $\tilde{G}/Z$  is isomorphic to  $G$ . Since  $\tilde{G}$  is a connected locally compact group,  $\tilde{G}$  is compactly generated. Moreover, the Lie algebra of the analytic subgroup  $[\tilde{G}, \tilde{G}]$  of  $\tilde{G}$  is  $[\mathfrak{g}, \mathfrak{g}]$ . As  $\mathfrak{g}$  is semisimple, we have  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  and, hence,

$$\overline{[\tilde{G}, \tilde{G}]} = [\tilde{G}, \tilde{G}] = \tilde{G}.$$

Therefore, the previous corollary applies and shows the equivalence of (i) and (ii).

To show that (ii) and (iii) are equivalent, let  $\text{Ad } G = G/Z(G)$  be the adjoint group of  $G$ . The centre of  $\text{Ad } G$  is trivial and

$$\text{Ad } G = G_1 \times \cdots \times G_n$$

for simple Lie groups  $G_1, \dots, G_n$  with trivial centres. By the previous corollary,  $G$  has Property (T) if and only if  $\text{Ad } G$  has Property (T). On the other hand,  $\text{Ad } G$  has Property (T) if and only if every  $G_i$  has Property (T) by Corollary 1.7.8 or by Propositions 2.5.4 and 2.5.1. By Theorems 1.6.1, 2.12.7, and by Section 3.3, the simple Lie group  $G_i$  has Property (T) if and only if its Lie algebra is not isomorphic to  $\mathfrak{so}(n, 1)$  or  $\mathfrak{su}(n, 1)$ . ■

**Remark 3.5.5** (i) The previous theorem shows that, for connected semisimple real Lie groups, Property (T) is invariant under local isomorphisms. This is not true for more general Lie groups. Indeed, the circle group  $G = \mathbf{S}^1$  has Property (T) since it is compact, but its universal covering  $\tilde{G} = \mathbf{R}$  does not have Property (T).

(ii) Let  $G$  be a connected Lie group with *finite* fundamental group. Then  $G$  has Property (T) if and only if  $\tilde{G}$  has Property (T).

(iii) Let  $G$  be a simple Lie group with trivial centre. It is known (see [Helga-62, Chapter IX, Section 4]) that  $\pi_1(G)$  is infinite cyclic if and only if  $G$  is the connected component of the isometry group of some irreducible non-compact, hermitian symmetric space, and this is the case if and only if the Lie algebra of  $G$  is one of the following list:

$$\begin{array}{ll} \mathfrak{su}(p, q) & (p \geq q \geq 1) \\ \mathfrak{so}(2, r) & (r \geq 3) \\ \mathfrak{so}^*(2s) & (s \geq 4) \\ \mathfrak{sp}^*(2t, \mathbf{R}) & (t \geq 2) \\ \mathfrak{e}_{6(-14)} & \\ \mathfrak{e}_{7(-26)} & . \end{array}$$

With the exception of the adjoint group of  $\mathfrak{su}(p, 1)$ ,  $p \geq 1$ , such a group  $G$  has Property (T), as shown above. In this case,  $\tilde{G}$  is a Kazhdan group with infinite centre.

We give now a result about the non-vanishing of *second* cohomology.

**Corollary 3.5.6** *Let  $G$  be a simple Lie group with an infinite cyclic fundamental group, and with Property (T). If  $\Gamma$  is a lattice in  $G$ , then  $H^2(\Gamma, \mathbf{Z}) \neq 0$ .*

**Proof** Let  $\tilde{G}$  be the universal covering of  $G$  and  $p : \tilde{G} \rightarrow G$  the covering mapping. Set  $\tilde{\Gamma} = p^{-1}(\Gamma)$ . Then  $\tilde{\Gamma}$  is a lattice in  $\tilde{G}$  containing  $\mathbf{Z}(\tilde{G}) = \pi_1(G)$ . Since  $\pi_1(G)$  is infinite cyclic, we have a central extension

$$0 \rightarrow \mathbf{Z} \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

We claim that this central extension defines a non-zero element in  $H^2(\Gamma, \mathbf{Z})$ . For this, it is enough to prove that the group  $\tilde{\Gamma}$  is not isomorphic to the direct product  $\Gamma \times \mathbf{Z}$ .

By the previous corollary,  $\tilde{G}$  has Property (T). Hence,  $\tilde{\Gamma}$  has Property (T), since it is a lattice in  $\tilde{G}$  (Theorem 1.7.1). On the other hand,  $\Gamma \times \mathbf{Z}$  does not have Property (T) as it has  $\mathbf{Z}$  as quotient (Corollary 1.3.5 or Corollary 2.5.2). ■



**Remark 3.5.7** (i) The previous corollary is true and easy to prove when  $G$  is the connected component of the isometry group of an irreducible non-compact, hermitian symmetric space  $X$  and  $\Gamma$  is a *cocompact* torsion-free lattice in  $G$ . Indeed, the Kähler form on  $\Gamma \backslash X$  defines a non-zero element in  $H^2(\Gamma, \mathbf{Z}) = H^2(\Gamma \backslash X, \mathbf{Z})$ .

(ii) Let  $G$  be the connected component of the isometry group of an irreducible non-compact, hermitian symmetric space and let  $\Gamma$  be a non-uniform lattice. In view of the list of simple Lie groups given in Remark 3.5.5 (iii), we have  $H^2(\Gamma, \mathbf{Z}) \neq 0$  if  $G \neq PU(n, 1)$ , the adjoint group of  $SU(n, 1)$ .

Now  $SU(1, 1)$  contains the free group on two generators  $F_2$  as non-uniform lattice and  $H^2(F_2, \mathbf{Z}) = 0$ . Hence, the previous corollary fails for  $SU(1, 1)$ . On the other hand, B. Klingler and P. Pansu communicated to us a proof that the previous corollary remains true for non-uniform lattices in  $SU(n, 1)$ ,  $n \geq 2$ .

### 3.6 Property (T) is not geometric

A group-theoretic property (P) is said to be a *geometric* if, for a pair  $(\Gamma_1, \Gamma_2)$  of finitely generated groups which are quasi-isometric,  $\Gamma_1$  has (P) if and only if  $\Gamma_2$  has (P); for the notion of quasi-isometric groups, see the definition below. Examples of such properties are: being virtually free, being virtually infinite cyclic, being virtually nilpotent, being amenable (for this and other examples, see [Harpe–00, IV.50]).

A question due to E. Ghys [HarVa–89, page 133] is whether Property (T) is geometric. This remained open until early 2000, when suddenly news of counterexamples percolated through the geometric group theory community, without being attributed to a single person. However, the name of S. Gersten is frequently associated with the idea. Our aim is now to present a class of such counterexamples.

**Definition 3.6.1** Let  $(X_1, d_1)$  and  $(X_2, d_2)$  be metric spaces. We say that  $X_1$  and  $X_2$  are *quasi-isometric* if there exists mappings

$$f : X_1 \rightarrow X_2 \quad \text{and} \quad g : X_1 \rightarrow X_2,$$

and a constant  $C > 0$  such that, for all  $x_1, x'_1 \in X_1$  and  $x_2, x'_2 \in X_2$ , we have

- $\frac{1}{C}d_1(x_1, x'_1) - C \leq d_2(f(x_1), f(x'_1)) \leq Cd_1(x_1, x'_1) + C$
- $\frac{1}{C}d_2(x_2, x'_2) - C \leq d_1(g(x_2), g(x'_2)) \leq Cd_2(x_2, x'_2) + C$

- $d_1(x_1, g \circ f(x_1)) \leq C$
- $d_2(x_2, f \circ g(x_2)) \leq C$ .

**Example 3.6.2** (i) Assume that  $(X_1, d_1)$  and  $(X_2, d_2)$  are bi-Lipschitz equivalent metric spaces, that is, there exists a homeomorphism  $h : X_1 \rightarrow X_2$  and a constant  $C > 0$  such that

$$\frac{1}{C}d_1(x_1, x'_1) \leq d_2(h(x_1), h(x'_1)) \leq Cd_1(x_1, x'_1)$$

for all  $x_1, x'_1 \in X_1$ . Then  $X_1$  and  $X_2$  are quasi-isometric.

(ii) Let  $\Gamma$  be a finitely generated group, and let  $S_1$  and  $S_2$  be two finite generating subsets of  $\Gamma$ . Let  $d_1$  and  $d_2$  be the word metrics on  $\Gamma$  defined by  $S_1$  and  $S_2$  (see Section G.5). Then the metric spaces  $(\Gamma, d_1)$  and  $(\Gamma, d_2)$  are bi-Lipschitz equivalent (Exercise 3.7.3) and hence quasi-isometric.

(iii) Two finitely generated groups  $\Gamma$  and  $\Lambda$  are said to be *quasi-isometric* if the metric spaces  $(\Gamma, d)$  and  $(\Lambda, \delta)$  are quasi-isometric, where  $d$  and  $\delta$  are the word metrics defined by finite generating sets  $S$  and  $T$  of  $\Gamma$  and  $\Lambda$ . By (ii) above, this definition does not depend on the choices of  $S$  and  $T$ .

(iv) Let  $(X, d)$  be a metric space, and let  $\Gamma$  be a group acting properly by isometries on  $X$ . If the quotient space  $\Gamma \backslash X$  is compact, then  $\Gamma$  is finitely generated and quasi-isometric to  $X$  (for the proof, see [Harpe–00, IV.23]).

Let  $G$  be a connected semisimple Lie group with finite centre, and let  $K$  be a maximal compact subgroup of  $G$ . Recall that  $G$  has an *Iwasawa decomposition*  $G = ANK$ , where  $S = AN$  is a solvable simply connected closed subgroup of  $G$  (see, e.g., [Walla–73, Theorem 7.4.3]). Recall also that there exists a  $G$ -invariant Riemannian metric on  $G/K$ , and that, for any such metric,  $G/K$  is a Riemannian symmetric space (see [Helga–62, Chapter IV, Proposition 3.4]). The proof of the following lemma was explained to us by C. Pittet (see also Lemma 3.1 in [Pitte–02]).

**Lemma 3.6.3** *Let  $G$  be a connected semisimple Lie group with finite centre, and let  $K$  be a maximal compact subgroup of  $G$ . Let  $G$  and  $G/K$  be endowed with  $G$ -invariant Riemannian metrics, and let  $K$  be endowed with a Riemannian metric.*

*Then  $G$  and  $G/K \times K$  are bi-Lipschitz equivalent, where  $G/K \times K$  is endowed with the product Riemannian metric.*

**Proof** Let  $G = SK$  be an Iwasawa decomposition of  $G$  as above. Let

$$\psi : G \rightarrow S \times K$$

denote the inverse of the product mapping  $S \times K \rightarrow G$ ,  $(s, k) \mapsto sk$ ; observe that the diffeomorphism  $\psi$  is equivariant for the actions of  $S$  on  $G$  and on

$S \times K$  by left translation. We identify  $S$  with  $G/K$ , and denote by  $Q_0$  and  $Q_1$  the Riemannian metrics on  $G$  and  $S \times K$ , respectively. Fix  $x \in G$ . Since any two scalar products on  $\mathbf{R}^n$  are bi-Lipschitz equivalent, there exists a constant  $C > 0$  such that, for every  $X \in T_x(G)$ , we have

$$\frac{1}{C} \|X\|_{Q_0} \leq \|d\psi_x(X)\|_{Q_1} \leq C \|X\|_{Q_0}.$$

Set

$$C_x = \sup_{X \in T_x(G) \setminus \{0\}} \left( \frac{\|d\psi_x(X)\|_{Q_1}}{\|X\|_{Q_0}} + \frac{\|X\|_{Q_0}}{\|d\psi_x(X)\|_{Q_1}} \right) < \infty.$$

Since  $Q_0$  and  $Q_1$  are  $S$ -invariant, we have

$$C_{sx} = C_x, \quad \text{for all } s \in S.$$

As  $G = SK$ , this implies that

$$\sup_{x \in G} C_x = \sup_{x \in K} C_x.$$

Since  $K$  is compact,  $\sup_{x \in K} C_x < \infty$ . Hence,

$$\lambda = \sup_{x \in G} C_x < \infty$$

and we have, for all  $x \in G$  and all  $X \in T_x(G)$ ,

$$\frac{1}{\lambda} \|X\|_{Q_0} \leq \|d\psi_x(X)\|_{Q_1} \leq \lambda \|X\|_{Q_0}.$$

This implies that  $\psi$  is a bi-Lipschitz mapping. ■

Now, let  $G$  be a connected simple Lie group with infinite cyclic fundamental group (Remark 3.5.5), and let  $\tilde{G}$  be the universal covering group of  $G$ . Let  $\Gamma$  be a cocompact lattice in  $G$  (such lattices exist, see [Borel–63]). It is easy to see that  $\Gamma$  is finitely generated (Exercise 3.7.4). Let  $\tilde{\Gamma}$  be the inverse image of  $\Gamma$  in  $\tilde{G}$ , which is also a cocompact lattice in  $\tilde{G}$ . Since  $\tilde{\Gamma}$  contains  $\pi_1(G) \cong \mathbf{Z}$ , we have a central extension

$$0 \rightarrow \mathbf{Z} \rightarrow \tilde{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

**Lemma 3.6.4** *Let  $\Gamma$  and  $\tilde{\Gamma}$  be the finitely generated groups defined above. Then  $\tilde{\Gamma}$  and  $\Gamma \times \mathbf{Z}$  are quasi-isometric.*

**Proof** Let  $K$  be a maximal compact subgroup of  $G$ . As in the previous lemma, we endow  $G, K$  and  $G/K$  with appropriate Riemannian metrics. We lift the metrics on  $G$  and  $K$  to left-invariant Riemannian metrics on  $\tilde{G}$  and on  $\tilde{K}$ , the inverse image of  $K$  in  $\tilde{G}$ .

On the one hand,  $\tilde{\Gamma}$  acts freely by isometries (via left translations) on  $\tilde{G}$ , with compact quotient. Hence,  $\tilde{\Gamma}$  is quasi-isometric to the metric space  $\tilde{G}$  (see Example 3.6.2.iv).

On the other hand,  $\Gamma$  acts properly by isometries on  $G/K$ , with compact quotient. Moreover, since  $\pi_1(K) = \pi_1(G) \cong \mathbf{Z}$ , the group  $\mathbf{Z}$  acts freely by isometries on  $\tilde{K}$ , with compact quotient. Hence  $\Gamma \times \mathbf{Z}$  is quasi-isometric to  $G/K \times \tilde{K}$ .

By the previous lemma,  $G$  and  $G/K \times K$  are bi-Lipschitz equivalent. Hence  $\tilde{G}$  and  $G/K \times \tilde{K}$  are bi-Lipschitz equivalent and therefore quasi-isometric. It follows that  $\tilde{\Gamma}$  and  $\Gamma \times \mathbf{Z}$  are quasi-isometric. ■

Let  $G$  be a simple real Lie group with infinite cyclic fundamental group. Assume that  $G$  has Property (T); for examples of such groups, see Remark 3.5.5. Let  $\Gamma$  be a cocompact lattice in  $G$ . By Theorem 3.5.4,  $\tilde{G}$  and  $\tilde{\Gamma}$  have Property (T). On the other hand,  $\Gamma \times \mathbf{Z}$  does not have Property (T). By the previous lemma,  $\tilde{\Gamma}$  and  $\Gamma \times \mathbf{Z}$  are quasi-isometric.

We summarise the discussion as follows:

**Theorem 3.6.5** *Let  $G$  be a connected simple real Lie group with infinite cyclic fundamental group. Assume that  $G$  has Property (T). Let  $\Gamma$  be a cocompact lattice in  $G$ , and let  $\tilde{\Gamma}$  be the inverse image of  $\Gamma$  in the universal covering group  $\tilde{G}$ . Then  $\tilde{\Gamma}$  has Property (T) and is quasi-isometric to the group  $\Gamma \times \mathbf{Z}$  which does not have Property (T).*

### 3.7 Exercises

**Exercise 3.7.1** Let  $M$  be a Riemannian manifold of dimension  $n$ , and let  $\Delta$  be the Laplacian on  $M$ . Show that, for  $f \in C^\infty(M)$  and  $x \in M$ ,

$$\Delta f(x) = -n \int_{S_x} \frac{d^2}{dt^2} f(\exp_x tX)|_{t=0} d\nu_x(X),$$

where  $\nu_x$  is the normalised rotation-invariant measure on the unit sphere  $S_x$  in  $T_x M$ .

**Exercise 3.7.2 (A converse to Theorem A)** Let  $G$  be a connected semisimple Lie group with finite centre and let  $K$  be a maximal compact subgroup of  $G$ .

Assume that  $G$  has Property (T). Let  $\alpha$  be an affine isometric action of  $G$  on some Hilbert space  $\mathcal{H}$  and let  $F : G/K \rightarrow \mathcal{H}$  be a  $G$ -equivariant and harmonic mapping. We want to show that  $F$  is constant.

(i) Let  $x_0 = K$  be the base point in  $G/K$  and set  $\xi_0 = F(x_0)$ . Show that there exists  $\xi \in \mathcal{H}$  such that

$$F(gx_0) = \pi(g)(\xi_0 - \xi) + \xi \quad \text{for all } g \in G,$$

where  $\pi$  is the linear part of  $\alpha$ .

(ii) Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition of the Lie algebra  $\mathfrak{g}$  of  $G$ . Let  $Y_1, \dots, Y_m$  be an orthonormal basis of  $\mathfrak{p}$  with respect to the Killing form. Show that  $\sum_{i=1}^m \pi(Y_i)^2(\xi_0 - \xi) = 0$ . (For  $X \in \mathfrak{g}$ , recall that  $\pi(X)$  is the unbounded operator acting on the smooth vectors in  $\mathcal{H}$  by  $\pi(X) = \frac{d}{dt}\pi(\exp tX)|_{t=0}$ .)

(iii) Deduce from (ii) that

$$\sum_{i=1}^m \|\pi(Y_i)(\xi_0 - \xi)\|^2 = 0$$

and hence that  $\pi(Y_i)(\xi_0 - \xi) = 0$  for all  $i = 1, \dots, m$ .

[Hint: Use the fact that  $\pi(Y_i)$  is anti-self-adjoint.]

(iv) Show that  $\pi(X)(\xi_0 - \xi) = 0$  for all  $X \in \mathfrak{g}$ .

(v) Conclude that  $\xi_0 - \xi$  is  $G$ -invariant. Hence,  $F$  is constant.

**Exercise 3.7.3** Let  $\Gamma$  be a finitely generated group, and let  $S_1$  and  $S_2$  be two finite generating subsets of  $\Gamma$ . Let  $d_1$  and  $d_2$  be the word metrics on  $\Gamma$  defined by  $S_1$  and  $S_2$ . Show that the metric spaces  $(\Gamma, d_1)$  and  $(\Gamma, d_2)$  are bi-Lipschitz equivalent.

**Exercise 3.7.4** Let  $G$  be a locally compact, compactly generated group, and let  $\Gamma$  be a *cocompact* lattice in  $G$ . Prove that  $\Gamma$  is finitely generated.

[Hint: Take a compact generating subset  $Q$  of  $G$  with  $Q^{-1} = Q$  and such that  $G = Q\Gamma$ . Show that there exists a finite subset  $S$  of  $\Gamma$  containing  $Q \cap \Gamma$  and such that  $Q^2$  is contained in  $QS$ . Prove that  $S$  generates  $\Gamma$ .]

# 4

## Bounded generation

In this chapter, we explain a result of Y. Shalom [Shal–99a] which provides explicit Kazhdan constants for several groups, including  $SL_n(\mathbf{Z})$ ,  $n \geq 3$ . This gives a direct proof of Property (T) for  $SL_n(\mathbf{Z})$  which does *not* use the fact that  $SL_n(\mathbf{Z})$  is a lattice in  $SL_n(\mathbf{R})$ .

The main idea is to relate Property (T) to the following group-theoretic property. A group  $G$  has *bounded generation* (or is *boundedly generated*) if there exists a finite subset  $S$  of  $G$  and a positive integer  $\nu$  such that every element  $g$  in  $G$  can be written as a product

$$g = g_1^{k_1} g_2^{k_2} \cdots g_\nu^{k_\nu},$$

where the  $g_i$ 's are in  $S$  and the  $k_i$ 's are integers. Carter and Keller [CarKe–83] established this property for the groups  $SL_n(\mathcal{O})$  with respect to the set of elementary matrices as generating set, where  $\mathcal{O}$  is the ring of integers of a number field and  $n \geq 3$ . (See also [AdiMe–92] for the case  $\mathcal{O} = \mathbf{Z}$ .)

We discuss first  $SL_n(\mathbf{Z})$  and then  $SL_n(R)$  for a class of topological rings  $R$ . When  $R$  is the ring of continuous functions on the circle, this shows that the loop group of  $SL_n(\mathbf{C})$  has Property (T) for  $n \geq 3$ ; this is our first example of a non-locally compact group with Property (T).

In the first section, we give a complete proof of the bounded generation of  $SL_n(\mathbf{Z})$  for  $n \geq 3$ . In a first reading, this fact can be taken for granted and one can directly proceed to the next section for the proof of Property (T) of these groups.

### 4.1 Bounded generation of $SL_n(\mathbf{Z})$ for $n \geq 3$

Let  $R$  be a commutative ring with unit, and let  $SL_n(R)$  be the group of  $(n \times n)$  matrices with coefficients in  $R$  and with determinant 1. For integers  $i, j$  with

$1 \leq i, j \leq n, i \neq j$  and  $t \in R$ , recall from Section 1.4 that the *elementary matrix*  $E_{ij}(t) \in SL_n(R)$  is the matrix with 1 on the diagonal,  $t$  at the entry  $(i, j)$ , and 0 elsewhere.

**Definition 4.1.1** The group  $SL_n(R)$  has *bounded elementary generation* (or is *boundedly elementary generated*) if there exists a positive integer  $\nu$  such that every matrix in  $SL_n(R)$  can be written as a product of at most  $\nu$  elementary matrices. The minimal integer with this property will be denoted by  $\nu_n(R)$ .

**Remark 4.1.2** (i) If  $\mathbf{K}$  is a field,  $SL_n(\mathbf{K})$  is boundedly elementary generated for  $n \geq 1$ . Indeed, Gauss' elimination shows that  $\nu_n(\mathbf{K}) \leq n(n-1)$ ; see Exercise 4.4.2.

(ii) Let  $H$  be a subgroup of finite index of the group  $G$ . Then  $G$  is boundedly generated if and only if  $H$  is boundedly generated (Exercise 4.4.3). It is clear that a non-abelian free group cannot be boundedly generated. Since  $SL_2(\mathbf{Z})$  has a subgroup of finite index which is free on 2 generators (Example B.2.5), it follows that  $SL_2(\mathbf{Z})$  is not boundedly generated.

The following result is from [CarKe-84]

**Theorem 4.1.3 (Carter–Keller)** For  $n \geq 3$ , the group  $SL_n(\mathbf{Z})$  is boundedly elementary generated with

$$\nu_n(\mathbf{Z}) \leq \frac{1}{2}(3n^2 - n) + 36.$$

In particular, every matrix on  $SL_3(\mathbf{Z})$  can be written as a product of at most 48 elementary matrices.

The proof will be given in several steps. An *elementary operation* on a matrix  $A$  is the multiplication of  $A$  by an elementary matrix  $E_{ij}(t)$ , on the left or on the right. Multiplying  $A$  by  $E_{ij}(t)$  from the left amounts to adding  $t$  times the  $j$ -th row to the  $i$ -th row of  $A$ , and multiplying  $A$  by  $E_{ij}(t)$  from the right amounts to adding  $t$  times the  $i$ -th column to the  $j$ -th column of  $A$ .

Observe that, if  $k$  elementary operations transform  $A$  into a matrix  $B$ , then  $k$  elementary operations transform  $A^t$  into  $B^t$  and  $A^{-1}$  into  $B^{-1}$ .

**Lemma 4.1.4** Let  $n \geq 3$  and  $A \in SL_n(\mathbf{Z})$ . Then  $A$  can be transformed into a matrix of the form

$$\begin{pmatrix} a & b & 0 & \cdots & 0 \\ c & d & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{n-2} & \\ 0 & 0 & & & \end{pmatrix}$$

by at most  $\frac{1}{2}(3n^2 - n - 10)$  elementary operations.

**Proof** Let

$$A = \begin{pmatrix} * & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ * & * & \cdots & * \\ u_1 & u_2 & \cdots & u_n \end{pmatrix}.$$

The greatest common divisor  $\gcd(u_1, u_2, \dots, u_n)$  of  $u_1, u_2, \dots, u_n$  is 1, since  $A \in SL_n(\mathbf{Z})$ .

• *First step:* Using at most one elementary operation, we can transform  $A$  into a matrix

$$B = \begin{pmatrix} * & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ * & * & \cdots & * \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

with  $\gcd(v_1, v_2, \dots, v_{n-1}) = 1$ .

Indeed, if  $u_i = 0$  for all  $1 \leq i \leq n-1$ , then  $u_n = \pm 1$  and we can add the last column to the first one.

Assume now that  $(u_1, \dots, u_{n-1}) \neq (0, \dots, 0)$ . Using the Chinese Remainder Theorem, we find  $t \in \mathbf{Z}$  such that

$$\begin{cases} t \equiv 1 & \text{mod all primes which divide } \gcd(u_1, \dots, u_{n-1}) \\ t \equiv 0 & \text{mod all primes which divide } \gcd(u_2, \dots, u_{n-1}) \text{ but not } u_1. \end{cases}$$

Then  $\gcd(u_1 + tu_n, u_2, \dots, u_{n-1}) = 1$ . Indeed, let  $p$  be a prime divisor of  $\gcd(u_2, \dots, u_{n-1})$ . If  $p$  divides  $u_1$ , then  $t \equiv 1 \pmod p$  and  $u_1 + tu_n \equiv u_n \pmod p$ . Since  $\gcd(u_1, u_2, \dots, u_n) = 1$ , it follows that  $p$  does not divide  $u_1 + tu_n$ . If  $p$  does not divide  $u_1$ , then  $t \equiv 0 \pmod p$ , hence  $u_1 + tu_n \equiv u_1 \pmod p$ , and  $p$  does not divide  $u_1 + tu_n$ .



Now, adding  $t$  times the  $n$ -th column to the first one, we obtain a matrix  $B$  with last row  $(v_1, \dots, v_n)$  with  $v_1 = u_1 + tu_n, v_2 = u_2, \dots, v_n = u_n$ .

• *Second step:* As  $\gcd(v_1, v_2, \dots, v_{n-1}) = 1$ , we can find  $t_1, \dots, t_{n-1} \in \mathbf{Z}$  such that

$$t_1 v_1 + \dots + t_{n-1} v_{n-1} = 1 - v_n.$$

We then use  $n - 1$  elementary operations to transform  $B$  to a matrix  $C$  of the form

$$C = \begin{pmatrix} * & * & \cdots & * & * \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ * & * & \cdots & * & * \\ v_1 & v_2 & \cdots & v_{n-1} & 1 \end{pmatrix}.$$

Using at most  $n - 1$  further elementary operations, we can transform  $C$  into a matrix of the form

$$\begin{pmatrix} * & * & \cdots & * & * \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Thus, using at most  $1 + 3(n - 1)$  elementary operations, we can transform  $A \in SL_n(\mathbf{Z})$  into a matrix in  $SL_{n-1}(\mathbf{Z})$ . It follows that we can transform  $A$  into a matrix in  $SL_2(\mathbf{Z})$ , using at most

$$\sum_{k=1}^{n-2} (3(n-k) + 1) = \frac{1}{2}(3n^2 - n - 10)$$

elementary operations. ■

**Lemma 4.1.5** *Let  $s \in \mathbf{N}$ , and let*

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(\mathbf{Z}) \quad \text{and} \quad B = \begin{pmatrix} a^s & b & 0 \\ x & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(\mathbf{Z}).$$

*Then  $A^s$  can be transformed into  $B$  using at most 16 elementary operations.*

**Proof** Set

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}).$$

By the Cayley–Hamilton Theorem, there exist  $f, g \in \mathbf{Z}$  such that

$$L^s = fI + gL.$$

Since  $\det(L) = 1$ , we have  $\det(fI + gL) = 1$ . On the other hand,

$$\begin{aligned}\det(fI + gL) &\equiv \det(gL) \pmod{f} \\ &\equiv g^2 \det(L) \pmod{f} \\ &\equiv g^2 \pmod{f}.\end{aligned}$$

Hence,  $f$  divides  $g^2 - 1 = (g - 1)(g + 1)$ . Set  $f^+ = \gcd(f, g + 1)$ . There exist  $f^-, g_1, g_2 \in \mathbf{Z}$  such that

$$f = f^+ f^-, \quad g + 1 = f^+ g_1 \quad \text{and} \quad g - 1 = f^- g_2.$$

Set

$$G = E_{23}(f^-)E_{32}(g_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g & f^- \\ 0 & g_2 & 1 \end{pmatrix}$$

and

$$H = E_{31}(-1)E_{13}(1 - f^+)E_{21}(-f^-)E_{31}(g_1) = \begin{pmatrix} * & 0 & * \\ -f^- & 1 & 0 \\ g & 0 & f^+ \end{pmatrix}.$$

Then

$$\begin{aligned}J &= GH = \begin{pmatrix} h & 0 & * \\ 0 & g & f \\ 1 & * & * \end{pmatrix} \quad \text{for some } h, \\ J_1 &= E_{13}(1 - h)J = \begin{pmatrix} 1 & r & * \\ 0 & g & f \\ 1 & * & * \end{pmatrix} \quad \text{for some } r, \\ J_2 &= J_1 E_{12}(-r) = \begin{pmatrix} 1 & 0 & t \\ 0 & g & f \\ 1 & * & * \end{pmatrix} \quad \text{for some } t,\end{aligned}$$

$$J_3 = J_2 E_{13}(-t - b) = \begin{pmatrix} 1 & 0 & -b \\ 0 & g & f \\ 1 & * & * \end{pmatrix},$$

$$S = J_3 E_{23}(a) = \begin{pmatrix} 1 & 0 & -b \\ 0 & g & f + ga \\ 1 & v & w \end{pmatrix} \quad \text{for some } v \text{ and } w.$$

By counting, we see that  $S$  is the product of 10 elementary matrices.

Since  $L^s = fI + gL$ , we have

$$A^s = \begin{pmatrix} f + ga & bg & 0 \\ x_0 & y_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } x_0 \text{ and } y_0.$$

Since  $A$  is triangular when considered mod  $b$ , we also have

$$A^s \equiv \begin{pmatrix} a^s & 0 & 0 \\ * & d^s & 0 \\ 0 & 0 & 1 \end{pmatrix} \pmod{b}.$$

Hence,  $f + ag = a^s + bu$  for some integer  $u$ , and we have

$$B = \begin{pmatrix} f + ag - bu & b & 0 \\ x & y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, using the fact that  $\det B = a^s y - xb = 1$ , we have

$$B_1 = B E_{21}(u) S = \begin{pmatrix} f + ag & bg & 0 \\ x + uy & yg & 1 \\ 1 & v & w \end{pmatrix},$$

$$B_2 = E_{32}(1 - w) B_1 = \begin{pmatrix} f + ag & bg & 0 \\ x + uy & yg & 1 \\ u_1 & v_1 & 1 \end{pmatrix} \quad \text{for some } u_1 \text{ and } v_1,$$

$$B_3 = E_{23}(-1) B_2 = \begin{pmatrix} f + ag & bg & 0 \\ x + uy - u_1 & yg - v_1 & 0 \\ u_1 & v_1 & 1 \end{pmatrix},$$

$$B_4 = B_3 E_{31}(-u_1) E_{32}(-v_1) = \begin{pmatrix} f + ag & bg & 0 \\ x_1 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } x_1 \text{ and } y_1.$$

Since  $\det(A^s) = 1 = \det B_4$ , we have

$$(f + ag)y_0 - bgx_0 = (f + ag)y_1 - bgx_1$$

and

$$(f + ag)(y_0 - y_1) = bg(x_0 - x_1).$$

As  $\gcd(f + ag, bg) = 1$ , there exists an integer  $r$  such that

$$y_0 = y_1 + rbg \quad \text{and} \quad x_0 = x_1 + (f + ag)r.$$

Then

$$E_{21}(r)B_4 = \begin{pmatrix} f + ag & bg & 0 \\ x_0 & y_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A^s.$$

Therefore,  $B$  can be transformed into  $A^s$  using at most  $10 + 5 + 1 = 16$  elementary operations, as claimed. ■

For the proof of the next lemma, we will use Dirichlet's theorem on prime numbers in arithmetic progressions.

**Lemma 4.1.6** *Let*

$$A = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(\mathbf{Z})$$

*with  $b \equiv 3 \pmod{4}$ . Then  $A$  can be written as a product of at most 40 elementary matrices.*

**Proof** If  $c = 0$  or  $d = 0$ , then  $a \in \{\pm 1\}$  or  $b \in \{\pm 1\}$  and the claim follows from straightforward computations as

$$\begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence we can assume that  $c$  and  $d$  are non-zero.

Since  $\gcd(4d, b) = 1$ , there exists, by Dirichlet's prime number theorem, a positive prime  $p$  such that  $p \equiv b \pmod{4d}$ . Using one elementary operation,  $A$  can be transformed into

$$A' = \begin{pmatrix} u & p & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } u.$$

Observe that  $p \equiv 3 \pmod{4}$  and that  $u \neq 0$  (since, otherwise,  $\det(A') \neq 1$ ). By the Chinese Remainder Theorem, we can find an integer  $t$  such that

$$t \equiv c \pmod{u} \quad \text{and} \quad t \equiv -1 \pmod{r}$$

for all primes  $r$  which divide  $p - 1$  but not  $u$ .

Let  $s$  be a prime which divides both  $p - 1$  and  $u$ . Then

$$1 = \det \begin{pmatrix} u & p & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = ud - pc \equiv -c \pmod{s} \equiv -t \pmod{s}.$$

Hence,  $t \equiv -1 \pmod{r}$  for every prime  $r$  dividing  $p - 1$ . In particular,  $p - 1$  and  $t$  are relatively prime as well as  $(p - 1)u$  and  $t$ . Moreover,  $(p - 1)/2$  and  $t - 1$  are relatively prime, since  $(p - 1)/2$  is odd.

By Dirichlet's Theorem, there exists a positive prime  $q$  with

$$q \equiv t \pmod{(p - 1)u}.$$

As  $(p - 1)/2$  and  $t - 1$  are relatively prime,  $(p - 1)/2$  and  $q - 1$  are relatively prime.

Since  $t = mu + c$  and  $q = t + m'(p - 1)u$  for some integers  $m, m'$ , we have  $q = ((p - 1)m' + m)u + c$ . Hence, using one elementary operation, we can transform  $A'$  into

$$H = \begin{pmatrix} u & p & 0 \\ q & v & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{for some } v.$$

Since  $(p - 1)/2$  and  $q - 1$  are relatively prime, there exist integers  $k, l$  such that

$$k(p - 1)/2 - l(q - 1) = 1.$$

By Fermat's Theorem,  $v^{q-1} \equiv 1 \pmod{q}$  and we can find  $\alpha \in \mathbf{Z}$  such that

$$v^{(q-1)l} = 1 + \alpha q.$$

Set

$$B = E_{12}(-q)E_{21}(-\alpha) = \begin{pmatrix} v^{(q-1)l} & -q & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the previous lemma, using at most 16 elementary operations,  $B$  can be transformed into the matrix

$$\begin{pmatrix} v & -q & 0 \\ -p & u & 0 \\ 0 & 0 & 1 \end{pmatrix}^{(q-1)l}$$

which is the inverse transpose of the matrix  $H^{(q-1)l}$ . We conclude that  $H^{-(q-1)l}$  can be written as a product of at most 18 elementary matrices.

Again, by Fermat's Theorem,  $u^{p-1} \equiv 1 \pmod{p}$  and, hence,  $u^{k(p-1)/2} \equiv \pm 1 \pmod{p}$ .

• *First case:*  $u^{k(p-1)/2} \equiv 1 \pmod{p}$ , that is,  $u^{k(p-1)/2} = 1 + rp$  for some integer  $r$ . Set

$$V = E_{12}(p)E_{21}(r) = \begin{pmatrix} u^{k(p-1)/2} & p & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the previous lemma,  $V$  can be transformed into  $H^{k(p-1)/2}$ , using at most 16 elementary operations. Hence,  $H^{k(p-1)/2}$  is a product of at most 18 elementary matrices. Since

$$H = H^{k(p-1)/2-l(q-1)} = H^{k(p-1)/2}H^{-(q-1)l},$$

it follows that  $H$  is a product of at most  $18 + 18 = 36$  elementary matrices. We conclude that  $A$  can be written as a product of at most  $1 + 1 + 36 = 38$  elementary matrices.

• *Second case:*  $u^{k(p-1)/2} \equiv -1 \pmod{p}$ , that is,  $u^{k(p-1)/2} = -1 + rp$  for some integer  $r$ . Set

$$W = E_{12}(2)E_{21}(-1)E_{12}(2-p)E_{21}(r) = \begin{pmatrix} u^{k(p-1)/2} & p & 0 \\ pr - r - 1 & p - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the previous lemma,  $W$  can be transformed into  $H^{k(p-1)/2}$ , using at most 16 elementary operations. Hence,  $H^{k(p-1)/2}$  is a product of at most 20 elementary matrices. As before, we conclude that  $A$  can be written as a product of at most  $1 + 1 + 18 + 20 = 40$  elementary matrices. ■

**Proof of Theorem 4.1.3** For  $n \geq 3$ , let  $A \in SL_n(\mathbf{Z})$ . By Lemma 4.1.4,  $A$  can be transformed into a matrix of the form

$$B = \begin{pmatrix} a & b & 0 & \cdots & 0 \\ c & d & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & I_{n-2} & \\ 0 & 0 & & & \end{pmatrix}$$

using at most  $\frac{1}{2}(3n^2 - n - 10)$  elementary operations. Using one further elementary operation if necessary, we can assume that  $b$  is odd.

We have either  $b \equiv 3 \pmod{4}$  or  $-b \equiv 3 \pmod{4}$ . Hence, by the previous lemma, either  $B$  or  $B^{-1}$  can be written as a product of at most 40 elementary matrices. Therefore,  $B$  can be written as a product of at most 40 elementary matrices. We conclude that  $A$  can be written as a product of

$$\frac{1}{2}(3n^2 - n - 10) + 1 + 40 = \frac{1}{2}(3n^2 - n) + 36$$

elementary matrices, as claimed. ■

## 4.2 A Kazhdan constant for $SL_n(\mathbf{Z})$

In this section, we give a proof, due to Shalom, of Property (T) for  $SL_n(\mathbf{Z})$ ,  $n \geq 3$ . This proof does not use the embedding of  $SL_n(\mathbf{Z})$  as a lattice in  $SL_n(\mathbf{R})$  as in Example 1.7.4; it gives, moreover, an explicit Kazhdan constant. We start by establishing a quantitative version of Property (T) for the pair  $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ , due to M. Burger [Burge–91, §1, Examples]. We follow the simpler proof given in [Shal–99a, Theorem 2.1].

### Property (T) for $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$

Set

$$U^\pm = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L^\pm = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}.$$

The following lemma is due to M. Burger [Burge–91, Lemma 5]; compare with Proposition 1.4.12.

**Lemma 4.2.1** *Let  $\nu$  be a mean on the Borel sets of  $\mathbf{R}^2 \setminus \{0\}$ . There exists a Borel subset  $M$  of  $\mathbf{R}^2 \setminus \{0\}$  and an element  $\gamma \in \{U^\pm, L^\pm\}$  such that  $|\nu(\gamma M) - \nu(M)| \geq 1/4$  for the linear action of  $SL_2(\mathbf{Z})$  on  $\mathbf{R}^2$ .*

**Proof** Consider the eight domains in the plane  $\mathbf{R}^2$  limited by the four lines of equations  $x = 0, y = 0, x = y$  and  $x = -y$ . Pairing these domains, we define a partition of  $\mathbf{R}^2 \setminus \{0\}$  into four regions  $A, B, C, D$ , with

$$A = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{R}^2 \setminus \{0\} : 0 \leq y < x \text{ or } x < y \leq 0 \right\}$$

and  $B$  (respectively  $C, D$ ) the image of  $A$  by a counterclockwise rotation of angle  $\pi/4$  (respectively  $\pi/2, 3\pi/4$ ); see Figure 4.1. The equalities

$$U^+(A \cup B) = A \quad L^+(A \cup B) = B$$

$$U^-(C \cup D) = D \quad L^-(C \cup D) = C$$

are straightforward.

Assume, by contradiction, that  $|\nu(\gamma M) - \nu(M)| < 1/4$  for any Borel subset  $M$  of  $\mathbf{R}^2 \setminus \{0\}$  and for all  $\gamma \in \{U^\pm, L^\pm\}$ . This implies for  $M = A \cup B$

$$\nu(A) = \nu(A \cup B) - \nu(B) = \nu(A \cup B) - \nu(L^+(A \cup B)) < 1/4$$

and similarly  $\nu(B), \nu(C), \nu(D) < 1/4$ . This is impossible, since

$$1 = \nu(\mathbf{R}^2 \setminus \{0\}) = \nu(A) + \nu(B) + \nu(C) + \nu(D). \quad \blacksquare$$

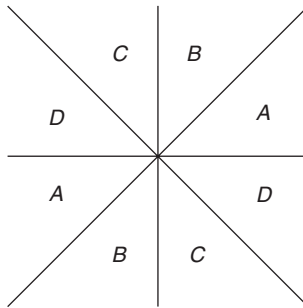


Figure 4.1 The four regions  $A, B, C, D$



The dual group of  $\mathbf{Z}^2$  will be identified with the 2-torus  $\mathbf{T}^2$  by associating to  $(e^{2\pi ix}, e^{2\pi iy}) \in \mathbf{T}^2$  the character  $(m, n) \mapsto e^{2\pi i(xm + yn)}$ . The dual action of  $SL_2(\mathbf{Z})$  on  $\widehat{\mathbf{Z}^2}$  corresponds to the transpose inverse of the natural action of  $SL_2(\mathbf{Z})$  on  $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ .

We consider the four vectors

$$e^\pm = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \quad \text{and} \quad f^\pm = \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$$

in  $\mathbf{Z}^2$  and the generating set

$$Q = \{U^\pm, L^\pm, e^\pm, f^\pm\}$$

of the semidirect product  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ .

Recall that Property (T) and Kazhdan constants for pairs of groups have been defined in Definition 1.4.3 and Remark 1.4.4.

**Theorem 4.2.2** *The pair  $(Q, 1/10)$  is a Kazhdan pair for  $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ . In particular,  $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$  has Property (T).*

**Proof** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$  which has a  $(Q, 1/10)$ -invariant vector. Assume, by contradiction, that  $\mathcal{H}$  has no non-zero vector which is invariant under  $\mathbf{Z}^2$ . Let

$$E : \mathcal{B}(\mathbf{T}^2) \rightarrow \mathcal{L}(\mathcal{H}), \quad B \mapsto E(B)$$

be the projection valued measure on  $\widehat{\mathbf{Z}^2} = \mathbf{T}^2$  associated to the unitary representation  $\pi|_{\mathbf{Z}^2}$  of  $\mathbf{Z}^2$  (see Theorem D.3.1). Thus,

$$\langle \pi(z)\xi, \eta \rangle = \int_{\mathbf{T}^2} \chi(z) d\langle E(\chi)\xi, \eta \rangle, \quad z \in \mathbf{Z}^2, \xi, \eta \in \mathcal{H}.$$

For any  $\gamma \in SL_2(\mathbf{Z})$  and any Borel subset  $B$  of  $\mathbf{T}^2$ , we have

$$(*) \quad E(\gamma B) = \pi(\gamma^{-1})E(B)\pi(\gamma).$$

Set  $\varepsilon = 1/10$ , and let  $\xi \in \mathcal{H}$  be a  $(Q, \varepsilon)$ -invariant unit vector. Let  $\mu_\xi$  be the probability measure on  $\mathbf{T}^2$  defined by

$$\mu_\xi(B) = \langle E(B)\xi, \xi \rangle$$

for any Borel subset  $B$  of  $\mathbf{T}^2$ . Observe that, since  $\pi|_{\mathbf{Z}^2}$  has no non-zero invariant vectors,  $E(\{0\}) = 0$  and hence  $\mu_\xi(\{0\}) = 0$ .

We identify now  $\widehat{\mathbf{Z}^2}$  with the square  $(-\frac{1}{2}, \frac{1}{2}]^2$  by assigning to  $(x, y) \in (-\frac{1}{2}, \frac{1}{2}]^2$  the character

$$\chi_{x,y} : (m, n) \mapsto \exp(2\pi i(xm + yn))$$

on  $\mathbf{Z}^2$ . Set  $X = (-\frac{1}{4}, \frac{1}{4}]^2$ .

• *First step:* We claim that  $\mu_\xi(X) \geq 1 - \varepsilon^2$ . Indeed, since  $\xi$  is  $(Q, \varepsilon)$ -invariant and since  $e^\pm, f^\pm \in Q$ , we have

$$\begin{aligned} \|\pi(e^\pm)\xi - \xi\|^2 &= \int_{(-\frac{1}{2}, \frac{1}{2}]^2} |e^{\pm 2\pi i x} - 1|^2 d\mu_\xi(x, y) \leq \varepsilon^2 \\ \|\pi(f^\pm)\xi - \xi\|^2 &= \int_{(-\frac{1}{2}, \frac{1}{2}]^2} |e^{\pm 2\pi i y} - 1|^2 d\mu_\xi(x, y) \leq \varepsilon^2. \end{aligned}$$

As

$$|e^{\pm 2\pi i t} - 1|^2 = 2 - 2\cos 2\pi t = 4\sin^2 \pi t \geq 2$$

for  $1/4 \leq |t| \leq 1/2$ , it follows that

$$\begin{aligned} \varepsilon^2 &\geq \int_{\substack{(x,y) \in (-\frac{1}{2}, \frac{1}{2}]^2 \\ |x| \geq 1/4}} 4\sin^2(\pi x) d\mu_\xi(x, y) \\ &\geq 2\mu_\xi\left(\left\{(x, y) \in \left(-\frac{1}{2}, \frac{1}{2}\right]^2 : |x| \geq 1/4\right\}\right) \end{aligned}$$

and consequently

$$\mu_\xi\left(\left\{(x, y) \in \left(-\frac{1}{2}, \frac{1}{2}\right]^2 : |x| \geq 1/4\right\}\right) \leq \frac{\varepsilon^2}{2}.$$

Similarly,

$$\mu_\xi\left(\left\{(x, y) \in \left(-\frac{1}{2}, \frac{1}{2}\right]^2 : |y| \geq 1/4\right\}\right) \leq \frac{\varepsilon^2}{2}$$

and the claim follows.

Let  $\nu$  be the probability measure on  $\mathbf{T}^2$  defined by

$$\nu(B) = \frac{\mu_\xi(B \cap X)}{\mu_\xi(X)}$$

for every Borel subset  $B$  of  $\mathbf{T}^2$ .

• *Second step:* We claim that

$$|\nu(\gamma B) - \nu(B)| < \frac{1}{4},$$

for any Borel subset  $B$  of  $\mathbf{T}^2$  and for any  $\gamma \in \{U^\pm, L^\pm\}$ . Indeed, using Equality (\*) above, we have

$$\begin{aligned} |\mu_\xi(\gamma B) - \mu_\xi(B)| &= |\langle \pi(\gamma^{-1})E(B)\pi(\gamma)\xi, \xi \rangle - \langle E(B)\xi, \xi \rangle| \\ &\leq |\langle \pi(\gamma^{-1})E(B)\pi(\gamma)\xi, \xi \rangle - \langle \pi(\gamma^{-1})E(B)\xi, \xi \rangle| \\ &\quad + |\langle \pi(\gamma^{-1})E(B)\xi, \xi \rangle - \langle E(B)\xi, \xi \rangle| \\ &= |\langle \pi(\gamma^{-1})E(B)(\pi(\gamma)\xi - \xi), \xi \rangle| + |\langle E(B)\xi, (\pi(\gamma)\xi - \xi) \rangle| \\ &\leq \|\pi(\gamma^{-1})E(B)\| \|\pi(\gamma)\xi - \xi\| + \|E(B)\| \|\pi(\gamma)\xi - \xi\| \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

By the first step,  $0 \leq \mu_\xi(B) - \mu_\xi(B \cap X) \leq \varepsilon^2$ . It follows that

$$\begin{aligned} \mu_\xi(\gamma B \cap X) - \mu_\xi(B \cap X) &= (\mu_\xi(\gamma B \cap X) - \mu_\xi(\gamma B)) \\ &\quad + (\mu_\xi(\gamma B) - \mu_\xi(B)) + (\mu_\xi(B) - \mu_\xi(B \cap X)) \\ &\leq 0 + 2\varepsilon + \varepsilon^2. \end{aligned}$$

Since this holds for both  $B$  and  $\gamma^{-1}B$ , we have

$$|\mu_\xi(\gamma B \cap X) - \mu_\xi(B \cap X)| \leq 2\varepsilon + \varepsilon^2.$$

Using the first step and recalling that  $\varepsilon = 1/10$ , we obtain

$$|\nu(\gamma B) - \nu(B)| \leq \frac{2\varepsilon + \varepsilon^2}{1 - \varepsilon^2} = \frac{21}{99} < \frac{1}{4}.$$

This proves the second step.

• *Third step:* As  $\mu_\xi(\{0\}) = 0$ , we can view  $\nu$  as a measure on  $\mathbf{R}^2 \setminus \{0\}$ . Observe that, for  $\gamma \in \{U^\pm, L^\pm\}$ , we have  $\gamma X \subset (-\frac{1}{2}, \frac{1}{2}]^2$  for the usual linear action of  $\gamma$

on  $\mathbf{R}^2$ . Since  $\nu(X) = 1$ , we have  $|\nu(\gamma B) - \nu(B)| < 1/4$  for every Borel subset  $B$  of  $\mathbf{R}^2 \setminus \{0\}$ , where  $\gamma$  acts in the usual way on  $\mathbf{R}^2$ . This is a contradiction to the previous lemma. ■

**Corollary 4.2.3** *Let  $\varepsilon > 0$ , and let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ . If  $\pi$  has a  $(Q, \varepsilon/20)$ -invariant unit vector  $\xi$ , then  $\xi$  is  $(\mathbf{Z}^2, \varepsilon)$ -invariant, that is,  $\|\pi(t)\xi - \xi\| < \varepsilon$  for every  $t \in \mathbf{Z}^2$ .*

**Proof** Denote by  $\mathcal{H}_0$  the subspace of  $\mathcal{H}$  consisting of the  $\pi(\mathbf{Z}^2)$ -invariant vectors, and let  $\mathcal{H}_1$  be its orthogonal complement. Since  $\mathbf{Z}^2$  is normal in  $\Gamma = SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ , both subspaces  $\mathcal{H}_0$  and  $\mathcal{H}_1$  are invariant under the whole group  $\Gamma$ . Let  $\xi = \xi_0 + \xi_1$  be the corresponding orthogonal decomposition. We have

$$\|\pi(\gamma)\xi - \xi\|^2 = \|\pi(\gamma)\xi_0 - \xi_0\|^2 + \|\pi(\gamma)\xi_1 - \xi_1\|^2$$

for every  $\gamma \in \Gamma$ , and therefore

$$\|\pi(\gamma)\xi_1 - \xi_1\|^2 \leq \|\pi(\gamma)\xi - \xi\|^2 < \left(\frac{\varepsilon}{20}\right)^2$$

for every  $\gamma \in Q$ . Since there exist no non-zero  $\pi(\mathbf{Z}^2)$ -invariant vectors in  $\mathcal{H}_1$ , it follows from the previous theorem that

$$\|\pi(\gamma)\xi_1 - \xi_1\|^2 \geq \left(\frac{\|\xi_1\|}{10}\right)^2$$

for some  $\gamma \in Q$ . Hence, combining the last two inequalities, we obtain

$$\left(\frac{\|\xi_1\|}{10}\right)^2 < \left(\frac{\varepsilon}{20}\right)^2,$$

that is,  $\|\xi_1\| < \varepsilon/2$ . Therefore, as  $\xi_0$  is invariant under  $\pi(\mathbf{Z}^2)$ ,

$$\|\pi(t)\xi - \xi\| = \|\pi(t)\xi_1 - \xi_1\| \leq 2\|\xi_1\| < \varepsilon$$

for all  $t \in \mathbf{Z}^2$ . ■

### Property (T) for $SL_n(\mathbf{Z})$ , $n \geq 3$

Fix an integer  $n \geq 3$ . The following lemma shows that every elementary matrix in  $SL_n(\mathbf{Z})$  is contained in an appropriate copy of  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ .

**Lemma 4.2.4** *Let  $R$  be a commutative ring with unit. Let  $i, j$  be integers with  $1 \leq i, j \leq n$  and  $i \neq j$ . There exists an injective homomorphism*

$\alpha : SL_2(R) \ltimes R^2 \rightarrow SL_n(R)$  such that  $E_{ij}(t) \in SL_n(R)$  is contained in  $\alpha(R^2)$  for every  $t \in R$  and such that

$$\alpha(SL_2(R)) = \begin{pmatrix} I_k & 0 & 0 \\ 0 & SL_2(R) & 0 \\ 0 & 0 & I_{n-k-2} \end{pmatrix}$$

for some  $k \in \{0, \dots, n-2\}$ .

**Proof** Assume first that  $n = 3$ . There are natural embeddings of  $SL_2(R) \ltimes R^2$  in  $SL_3(R)$ , respectively with images of the form

$$\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}.$$

The lemma for  $n = 3$  follows by inspection.

Let now  $n > 3$ . We proceed by induction, assuming that the lemma holds for  $n-1$ . If  $(i, j) \neq (1, n)$  and  $(i, j) \neq (n, 1)$ , consider one of the subgroups

$$\begin{pmatrix} SL_{n-1}(R) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & SL_{n-1}(R) \end{pmatrix}$$

of  $SL_n(R)$ . The conclusion follows by the induction hypothesis. If  $(i, j) = (1, n)$ , we can take the embedding of  $SL_2(R) \ltimes R^2$  with image the subgroup

$$\begin{pmatrix} SL_2(R) & 0 & R^2 \\ 0 & I_{n-4} & 0 \\ 0 & 0 & I_2 \end{pmatrix}$$

in  $SL_n(R)$  and if  $(i, j) = (n, 1)$  the embedding of  $SL_2(R) \ltimes R^2$  with image the transposed subgroup. ■

We are now ready to complete the proof of Property (T) for  $SL_n(\mathbf{Z})$ ,  $n \geq 3$ . Recall from Theorem 4.1.3 that  $SL_n(\mathbf{Z})$  is boundedly generated, with  $v_n \leq \frac{1}{2}(3n^2 - n) + 36$ , where  $v_n = v_n(\mathbf{Z})$  is the integer introduced in Definition 4.1.1.

**Theorem 4.2.5** *The group  $SL_n(\mathbf{Z})$  has Property (T) for  $n \geq 3$ .*

*More precisely, let  $Q_n$  be the subset of  $SL_n(\mathbf{Z})$  consisting of the  $n^2 - n$  elementary matrices  $E_{i,j}(1)$  for  $1 \leq i, j \leq n$  and  $i \neq j$ . Then  $(Q_n, 1/20v_n)$  is a Kazhdan pair for  $SL_n(\mathbf{Z})$ .*

**Proof** Let  $T_n = Q_n \cup Q_n^{-1}$ . It is enough to show that  $(T_n, 1/20v_n)$  is a Kazhdan pair (see Remark 1.1.2.iii).

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_n(\mathbf{Z})$  which has a  $(T_n, 1/20v_n)$ -invariant unit vector  $\xi$ . Let  $\gamma$  be any elementary matrix in  $SL_n(\mathbf{Z})$ . By the previous lemma, there exists an embedding  $\alpha$  of  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$  into  $SL_n(\mathbf{Z})$  such that  $\gamma \in \alpha(\mathbf{Z}^2)$  and such that  $\alpha(Q) = T_n \cap \text{Im}(\alpha)$ , where  $Q$  is the generating set of  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$  defined before Theorem 4.2.2. By Corollary 4.2.3, it follows that  $\xi$  is  $(\alpha(\mathbf{Z}^2), 1/v_n)$ -invariant. In particular,

$$\|\pi(\gamma)\xi - \xi\| < \frac{1}{v_n}$$

for any elementary matrix in  $SL_n(\mathbf{Z})$ .

Let now  $\gamma$  be an arbitrary matrix in  $SL_n(\mathbf{Z})$ . By bounded elementary generation of  $SL_n(\mathbf{Z})$ , there exist an integer  $N \leq v_n$  and elementary matrices  $\gamma_1, \dots, \gamma_N$  such that  $\gamma = \gamma_1 \gamma_2 \dots \gamma_N$ . We then have

$$\begin{aligned} \|\pi(\gamma)\xi - \xi\| &\leq \sum_{i=0}^{N-1} \|\pi(\gamma_1 \dots \gamma_{N-i})\xi - \pi(\gamma_1 \dots \gamma_{N-i-1})\xi\| \\ &= \sum_{j=1}^N \|\pi(\gamma_j)\xi - \xi\| \leq \frac{N}{v_n} \leq 1, \end{aligned}$$

so that  $\xi$  is  $(SL_n(\mathbf{Z}), 1)$ -invariant. It follows from Proposition 1.1.5 that  $\pi$  has non-zero invariant vectors. ■

**Remark 4.2.6** (i) The Kazhdan constant for  $SL_n(\mathbf{Z})$  from the previous theorem has been improved by Kassabov [Kassa] to the value  $1/(42\sqrt{n} + 860)$ .

(ii) As was observed by A. Zuk, the optimal Kazhdan constant for  $SL_n(\mathbf{Z})$ , with respect to the set  $Q_n$  introduced above, is bounded from above by  $\sqrt{2/n}$  (Exercise 4.4.4). This shows that the order  $1/\sqrt{n}$  in Kassabov's constant from (i) is optimal.

(iii) Bounded generation of  $SL_3(\mathbf{Z})$  has already been used by Colin de Verdière [ColVe–98, Théorème 3.9] to estimate a variant of Kazhdan constants related to the family of all finite dimensional unitary representations (instead of all unitary representations as in Theorem 4.2.5). Previously, Kazhdan constants for finite dimensional unitary representations have been estimated by M. Burger [Burge–91, Proposition 3].

### 4.3 Property (T) for $SL_n(R)$

In this section, we will show that  $SL_n(R)$ ,  $n \geq 3$ , has Property (T) for suitable topological rings  $R$ . We first deal with finitely generated discrete rings. The strategy of the proof is the same as in the case  $R = \mathbf{Z}$ , only the technical details are more involved.

#### Property (T) for $(SL_2(R[t]) \ltimes R[t]^2, R[t]^2)$

Let  $R$  be a commutative ring with unit. Let  $R[t]$  be the ring of polynomials over  $R$ . We view  $R$  and  $R[t]$  as discrete topological rings. The natural embedding of  $R$  into  $R[t]$  induces an embedding of  $SL_2(R) \ltimes R^2$  into  $SL_2(R[t]) \ltimes R[t]^2$ .

The following result is Theorem 3.1 in [Shal–99a]. It will provide us with an inductive procedure to treat finitely generated rings in the next section.

For a subset  $Q$  of  $SL_2(R)$  containing the four elementary matrices  $E_{12}(\pm 1)$  and  $E_{21}(\pm 1)$ , we define

$$Q_t = Q \bigcup \{E_{12}(\pm t), E_{21}(\pm t)\} \subset SL_2(R[t]).$$

**Proposition 4.3.1** *Let  $R$  be a discrete commutative ring with unit. Let  $Q$  be a subset of  $SL_2(R)$  containing the four elementary matrices  $E_{12}(\pm 1)$  and  $E_{21}(\pm 1)$ . Assume that  $(SL_2(R) \ltimes R^2, R^2)$  has Property (T) and that  $(Q, \varepsilon)$  is a Kazhdan pair for  $(SL_2(R) \ltimes R^2, R^2)$ . Let  $\delta \in (0, \varepsilon)$  be such that  $(\delta + 2\delta/\varepsilon)/(1 - \delta/\varepsilon) < 1/10$ .*

*Then the pair  $(SL_2(R[t]) \ltimes R[t]^2, R[t]^2)$  has Property (T), with  $(Q_t, \delta)$  as Kazhdan pair.*

For the proof, we need a suitable description of  $\widehat{R[t]}$ , the dual group of the discrete abelian group  $R[t]$ , in terms of  $\widehat{R}$ . It is convenient here to write *additively* the group law on  $\widehat{R}$  and  $\widehat{R[t]}$ . We write  $\langle \chi, r \rangle$  instead of  $\chi(r)$  for  $r \in R$  and  $\chi \in \widehat{R}$  (compare with  $\langle \Phi(\bar{\chi}), \bar{r} \rangle$  below). Observe that  $\widehat{R}$ , as the dual group of a discrete abelian group, is compact.

Let  $\widehat{R}[[t^{-1}]]$  denote the group of formal power series in  $t^{-1}$  with coefficients in  $\widehat{R}$ , with the topology of pointwise convergence, namely with the product topology of  $\widehat{R}[[t^{-1}]]$  identified with  $(\widehat{R})^\infty$ . It is a compact abelian group. We define a continuous homomorphism

$$(*) \quad \Phi : \widehat{R}[[t^{-1}]] \rightarrow \widehat{R[t]}, \quad \langle \Phi(\bar{\chi}), \bar{r} \rangle = \sum_n \langle \chi_n, r_n \rangle,$$

for

$$\bar{\chi} = \sum_n \chi_n t^{-n} \in \widehat{R}[[t^{-1}]] \quad \text{and} \quad \bar{r} = \sum_n r_n t^n \in R[t].$$

(There is no convergence problem since all but finitely many of the  $r_n$ 's are zero.)

**Lemma 4.3.2** *The homomorphism  $\Phi : \widehat{R}[[t^{-1}]] \rightarrow \widehat{R}[t]$  defined above is a topological isomorphism.*

**Proof** If  $\Phi(\bar{\chi}) = 0$  for  $\bar{\chi} = \sum_n \chi_n t^{-n} \in \widehat{R}[[t^{-1}]]$ , then

$$\langle \chi_n, r \rangle = \langle \Phi(\bar{\chi}), rt^n \rangle = 0,$$

for all  $n$  and all  $r \in R$ . Hence,  $\bar{\chi} = 0$ . This shows the injectivity of  $\Phi$ .

The image of  $\Phi$  is a compact and therefore closed subgroup of  $\widehat{R}[t]$ . To show that  $\Phi$  is onto, it is sufficient by Pontrjagin duality (Theorem D.1.3) to show that, if  $\bar{r} = \sum_n r_n t^n \in R[t]$  is such that  $\langle \Phi(\bar{\chi}), \bar{r} \rangle = 0$  for all  $\bar{\chi} \in \widehat{R}[[t^{-1}]]$ , then  $\bar{r} = 0$ . But this is straightforward because, for any  $n \geq 0$  and every  $\chi \in \widehat{R}$ ,

$$\langle \chi, r_n \rangle = \langle \Phi(\chi t^{-n}), \bar{r} \rangle = 0,$$

and therefore  $\bar{r} = 0$ . ■

Let us recall some general facts about dual groups of rings. Let  $A$  be an arbitrary commutative topological ring with unit. The abelian topological group  $\widehat{A}$  carries a module structure over  $A$  given by

$$(a, \chi) \mapsto a\chi \quad \text{with} \quad a\chi(b) = \chi(ab), \quad a, b \in A, \chi \in \widehat{A}.$$

The group  $SL_2(A)$  acts by automorphisms on  $A^2$  and, by duality, on  $\widehat{A}^2 = \widehat{A}^2 :$

$$(g, \chi) \mapsto g\chi \quad \text{with} \quad g\chi(a) = \chi(g^{-1}a), \quad g \in SL_2(A), a \in A^2, \chi \in \widehat{A}^2.$$

Moreover, the  $A$ -module structure of  $\widehat{A}$  induces a module structure of  $\widehat{A}^2$  over the ring  $M_2(A)$  of all  $(2 \times 2)$  matrices with coefficients in  $A$ . Let us write the group law of  $\widehat{A}$  additively. The action of  $g \in SL_2(A)$  on  $\widehat{A}^2$  defined above corresponds in the  $M_2(A)$ -structure to the multiplication by the transpose of the matrix  $g^{-1}$

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \mapsto \begin{pmatrix} d\chi_1 - c\chi_2 \\ -b\chi_1 + a\chi_2 \end{pmatrix} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$



Let us return to the dual group of  $R[t]$ . It will be convenient to embed  $\widehat{R}[[t^{-1}]]$  into the larger group  $\widehat{R}((t^{-1}))$  of all Laurent series in  $t^{-1}$  with coefficients in  $\widehat{R}$ , that is, the group of all formal series

$$\bar{\chi} = \sum_{n=k}^{\infty} \chi_n t^{-n}, \quad \chi_n \in \widehat{R}, \quad k \in \mathbf{Z}.$$

Endowed with the topology for which  $\widehat{R}[[t^{-1}]]$  is an open subgroup,  $\widehat{R}((t^{-1}))$  is a locally compact abelian group.

At the same time, we can also view  $\widehat{R}[[t^{-1}]]$  as a quotient of  $\widehat{R}((t^{-1}))$ . Indeed, the homomorphism  $\Phi$  extends to a continuous homomorphism, also denoted by  $\Phi$ , from  $\widehat{R}((t^{-1}))$  onto  $\widehat{R[t]}$ , defined by the same formula (\*). Observe that the kernel of  $\Phi$  is the subgroup of all formal series

$$\bar{\chi} = \sum_{n=-k}^{-1} \chi_n t^{-n}, \quad \chi_n \in \widehat{R}, \quad k \geq 1.$$

The point in introducing  $\widehat{R}((t^{-1}))$  is that the module structure of  $\widehat{R[t]}$  over the ring  $R[t]$  given by duality can now be easily described. Indeed, the group  $\widehat{R}((t^{-1}))$  carries a module structure over  $R[t]$  given by

$$\bar{r} \bar{\chi} = \sum_{n,m} r_n \chi_m t^{n-m},$$

for  $\bar{\chi} = \sum_{m=k}^{\infty} \chi_m t^{-m} \in \widehat{R}((t^{-1}))$  and  $\bar{r} = \sum_n r_n t^n \in R[t]$ . Observe that  $\widehat{R}[[t^{-1}]]$  is not a submodule of  $\widehat{R}((t^{-1}))$ .

**Lemma 4.3.3** *The mapping  $\Phi : \widehat{R}((t^{-1})) \rightarrow \widehat{R[t]}$  is a  $R[t]$ -module homomorphism.*

**Proof** By additivity and continuity of  $\Phi$ , it suffices to show that  $\Phi(\bar{r} \bar{\chi}) = \bar{r} \Phi(\bar{\chi})$  for  $\bar{\chi} = \chi_n t^{-n} \in \widehat{R}((t^{-1}))$  and  $\bar{r} = r_m t^m \in R[t]$ . This is indeed the case, since

$$\begin{aligned} \langle \bar{r} \Phi(\bar{\chi}), s_p t^p \rangle &= \langle \bar{\chi}_n, r_m s_p t^{m+p} \rangle \\ &= \begin{cases} \langle \chi_n, r_m s_p \rangle & \text{if } p = n - m \\ 1 & \text{if } p \neq n - m \end{cases} \\ &= \langle \Phi(\bar{r} \bar{\chi}), s_p t^p \rangle, \end{aligned}$$

for  $s_p t^p \in R[t]$ . ■

As a last ingredient for the proof of Proposition 4.3.1, we prove the following analogue of Lemma 4.2.1. The  $R[t]$ -module structure on  $\widehat{R}((t^{-1}))$  introduced above induces an action of the group  $SL_2(R[t])$  on  $\widehat{R}((t^{-1}))^2$ .

**Lemma 4.3.4** *Let  $\mu$  be a mean on the Borel sets of  $\widehat{R}((t^{-1}))^2 \setminus \{0\}$ . There exist a Borel subset  $M$  of  $\widehat{R}((t^{-1}))^2 \setminus \{0\}$  and an elementary matrix*

$$\gamma \in F\{E_{12}(\pm 1), E_{21}(\pm 1), E_{12}(\pm t), E_{21}(\pm t)\}$$

such that  $|\nu(\gamma M) - \nu(M)| \geq 1/5$ .

**Proof** We define an “absolute value” (compare with Section D.4) on  $\widehat{R}((t^{-1}))$  as follows. Set

$$|\overline{\chi}| = \begin{cases} 2^{-m} & \text{if } \overline{\chi} = \sum_{n=m}^{\infty} \chi_n t^{-n}, \chi_m \neq 0 \\ 0 & \text{if } \overline{\chi} = 0. \end{cases}$$

It is clear that  $|\cdot|$  is ultrametric in the sense that  $|\chi + \chi'| \leq \max\{|\chi|, |\chi'|\}$  for  $\chi, \chi' \in \widehat{R}((t^{-1}))$ . Observe that  $|\chi + \chi'| = \max\{|\chi|, |\chi'|\}$  if  $|\chi| \neq |\chi'|$  and that  $|t\chi| = 2|\chi|$ .

In  $\widehat{R}((t^{-1}))^2 \setminus \{0\}$ , we consider the following three pairwise disjoint Borel subsets:

$$\begin{aligned} A &= \{(\chi_1, \chi_2) : |\chi_1| < |\chi_2|\} \\ B &= \{(\chi_1, \chi_2) : |\chi_1| = |\chi_2|\} \\ C &= \{(\chi_1, \chi_2) : |\chi_1| > |\chi_2|\}. \end{aligned}$$

We have

$$(1) \quad E_{12}(t)(A \cup B) \subset C.$$

Indeed,

$$E_{12}(t) \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_1 + t\chi_2 \\ \chi_2 \end{pmatrix}$$

and, if  $(\chi_1, \chi_2) \neq (0, 0)$  and  $|\chi_1| \leq |\chi_2|$ , then  $|t\chi_2| = 2|\chi_2| > |\chi_1|$  and  $|\chi_1 + t\chi_2| = \max\{|\chi_1|, |t\chi_2|\} > |\chi_1|$ .

Similarly, we have

$$(2) \quad E_{21}(t)(C \cup B) \subset A$$

and

$$(3) \quad E_{12}(1)(A) \subset B.$$

Assume, by contradiction, that  $|\mu(gM) - \mu(M)| < 1/5$  for all Borel subsets  $M$  of  $\widehat{R}((t^{-1}))^2 \setminus \{0\}$  and all  $g \in F$ . Then, by (2),

$$\mu(C) + \mu(B) < \mu(A) + 1/5$$

and, by (3),

$$\mu(A) < \mu(B) + 1/5.$$

Hence,

$$\mu(C) + \mu(B) < \mu(B) + 2/5,$$

that is,  $\mu(C) < 2/5$ . It follows from (1) that

$$\mu(A) + \mu(B) < \mu(C) + 1/5 < 3/5.$$

Hence

$$1 = \mu(\widehat{R}((t^{-1}))^2 \setminus \{0\}) = \mu(A) + \mu(B) + \mu(C) < 3/5 + 2/5 = 1.$$

This is a contradiction. ■

**Proof of Proposition 4.3.1** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_2(R[t]) \ltimes R[t]^2$  which has a  $(Q_t, \delta)$ -invariant unit vector  $\xi \in \mathcal{H}$ . Assume, by contradiction, that  $\mathcal{H}$  has no non-zero vector which is invariant under  $R[t]^2$ .

Let  $\mathcal{H}_0$  be the subspace of all  $R^2$ -invariant vectors in  $\mathcal{H}$ .

• *First step:*  $\mathcal{H}_0$  contains a  $(Q_t, 1/10)$ -invariant unit vector. Indeed, let  $\mathcal{H}_1$  be the orthogonal complement of  $\mathcal{H}_0$  in  $\mathcal{H}$ , and let  $\xi = \xi_0 + \xi_1$  be the corresponding orthogonal decomposition of  $\xi$ . Since  $\mathcal{H}_0$  and  $\mathcal{H}_1$  are invariant under  $SL_2(R) \ltimes R^2$ , we have, for all  $g \in Q$ ,

$$\|\pi(g)\xi_1 - \xi_1\|^2 \leq \|\pi(g)\xi - \xi\|^2 \leq \delta^2 < \varepsilon^2.$$

As  $\mathcal{H}_1$  contains no non-zero  $R^2$ -invariant vector and  $(Q, \varepsilon)$  is a Kazhdan set for  $(SL_2(R) \ltimes R^2, R^2)$ , there exists  $g_0 \in Q$  such that

$$\varepsilon^2 \|\xi_1\|^2 \leq \|\pi(g_0)\xi_1 - \xi_1\|^2.$$

It follows that  $\|\xi_1\|^2 \leq \delta^2/\varepsilon^2$ . Hence,

$$\|\xi_0\|^2 = \|\xi\|^2 - \|\xi_1\|^2 \geq 1 - (\delta^2/\varepsilon^2).$$

Therefore  $\|\xi_0\| \geq 1 - (\delta/\varepsilon)$ , since  $\delta < \varepsilon$ .

Let  $\eta_0$  be the unit vector  $\xi_0/\|\xi_0\|$ . Then, for every  $g \in Q_t$ , we have

$$\begin{aligned} \|\pi(g)\eta_0 - \eta_0\| &\leq \frac{1}{\|\xi_0\|} (\|\pi(g)\xi - \xi\| + \|\pi(g)\xi_1 - \xi_1\|) \\ &\leq \frac{1}{1 - \delta/\varepsilon} (\delta + 2\delta/\varepsilon). \end{aligned}$$

Hence, by the choice of  $\delta$ ,

$$\|\pi(g)\eta_0 - \eta_0\| < 1/10,$$

that is,  $\eta_0$  is  $(Q_t, 1/10)$ -invariant. This proves the first step.

Let  $E$  be the projection valued measure on  $\widehat{R[t]^2}$  associated to the restriction of  $\pi$  to  $R[t]^2$ . Since  $\mathcal{H}$  has no non-zero  $R[t]^2$ -invariant vectors,  $E(\{0\}) = 0$ .

The orthogonal projection on the space  $\mathcal{H}_0$  of  $R^2$ -invariant vectors is  $E(X)$ , where  $X$  is the subset of  $\widehat{R[[t]]^2}$  of all  $\chi$  such that  $\chi|_{R^2} = 1$ . We identify  $\widehat{R[t]^2}$  with  $\widehat{R[[t^{-1}]]^2}$  as above and observe that  $X$  is the subset of  $\widehat{R[[t^{-1}]]^2}$  of all pairs  $(\chi_1, \chi_2)$  where  $\chi_1$  and  $\chi_2$  are of the form  $\sum_{n=1}^{\infty} \chi_n t^{-n}$ ,  $\chi_n \in \widehat{R}$ .

Let  $\mu$  be the probability measure on  $\widehat{R[[t^{-1}]]^2}$  defined by

$$\mu(B) = \langle E(B)\eta_0, \eta_0 \rangle$$

for any Borel set  $B$  in  $\widehat{R[[t^{-1}]]^2}$ . We have  $\mu(X) = 1$ . Indeed, this follows from the fact that  $\eta_0 \in \mathcal{H}_0$  and that  $E(X)$  is the orthogonal projection on  $\mathcal{H}_0$ . Observe also that  $\mu(\{0\}) = 0$  since  $E(\{0\}) = 0$ .

• *Second step:* For any Borel subset  $B$  of  $\widehat{R[[t^{-1}]]^2}$  and for any  $g$  in the subset

$$F = \{E_{12}(\pm 1), E_{21}(\pm 1), E_{12}(\pm t), E_{21}(\pm t)\}$$

of  $SL_2(R[t])$ , we have

$$|\mu(gB) - \mu(B)| < \frac{1}{5}.$$

Indeed, since  $\eta_0$  is  $(F, 1/10)$ -invariant, the argument is the same as the one given in the proof of Theorem 4.2.2:

$$\begin{aligned}
 |\mu(gB) - \mu(B)| &= |\langle \pi(g^{-1})E(B)\pi(g)\eta_0, \eta_0 \rangle - \langle E(B)\eta_0, \eta_0 \rangle| \\
 &\leq |\langle \pi(g^{-1})E(B)\pi(g)\eta_0, \eta_0 \rangle - \langle \pi(g^{-1})E(B)\eta_0, \eta_0 \rangle| \\
 &\quad + |\langle \pi(g^{-1})E(B)\eta_0, \eta_0 \rangle - \langle E(B)\eta_0, \eta_0 \rangle| \\
 &= |\langle \pi(g^{-1})E(B)(\pi(g)\eta_0 - \eta_0), \eta_0 \rangle| \\
 &\quad + |\langle E(B)\eta_0, (\pi(g)\eta_0 - \eta_0) \rangle| \\
 &\leq \|\pi(g^{-1})E(B)\| \|\pi(g)\eta_0 - \eta_0\| + \|E(B)\| \|\pi(g)\eta_0 - \eta_0\| \\
 &\leq 1/10 + 1/10 = 2/10.
 \end{aligned}$$

• *Third step:* We view  $\widehat{R}[[t^{-1}]]^2$  as a subset of  $\widehat{R}((t^{-1}))^2$ . Since  $\mu(\{0\}) = 0$ , we can view  $\mu$  as a probability measure on  $\widehat{R}((t^{-1}))^2 \setminus \{0\}$ . Observe that  $\mu$  is supported by  $X$ . For any  $g \in F$ , it is clear that

$$gX \subset \widehat{R}[[t^{-1}]]^2,$$

where now the action of  $g$  on  $\widehat{R}((t^{-1}))^2$  is given by the  $R[t]$ -module structure on  $\widehat{R}((t^{-1}))$  as previously defined. By the second step and by Lemma 4.3.3, we have

$$|\mu(gB) - \mu(B)| < \frac{1}{5},$$

for any Borel subset  $B$  of  $\widehat{R}((t^{-1}))^2$  and for any  $g$  in the subset  $F$  of  $SL_2(R[t])$ . This is a contradiction to Lemma 4.3.4 and finishes the proof. ■

### Property (T) for $SL_n(R)$

Let  $R$  be a topological commutative ring with unit. Then  $SL_n(R)$  is a topological group for the topology induced from the product topology on  $R^{n^2}$ .

The following result is [Shal–99a, Main Theorem].

**Theorem 4.3.5** *Fix an integer  $n \geq 3$ , and let  $R$  be a topological commutative ring with unit. Assume that  $SL_n(R)$  has bounded elementary generation. Assume, moreover, that there exist finitely many elements in  $R$  which generate a dense subring.*

*Then  $SL_n(R)$  has Property (T).*

**Remark 4.3.6** The proof of Theorem 4.3.5 will show that  $(Q, \varepsilon)$  is a Kazhdan pair for  $SL_n(R)$ , where  $Q$  and  $\varepsilon$  are defined as follows. Let  $r_1, \dots, r_m$  be elements in  $R$  generating a dense subring. Let  $Q_1$  be the set of the elementary matrices  $E_{ij}(1)$ ,  $1 \leq i \neq j \leq n$ , and let  $Q_2$  be the set of the elementary matrices  $E_{ij}(r_k)$  where  $j = i + 1$  or  $j = i - 1$  for  $1 \leq k \leq m$  and  $1 \leq i \leq n$ . Set  $Q = Q_1 \cup Q_2$  and  $\varepsilon = 1/\nu_n(R)22^{m+1}$ .

Before we give the proof, we establish some preliminary results. For  $m \geq 0$ , let  $R_m = \mathbf{Z}[X_1, \dots, X_m]$  be the ring of polynomials over  $\mathbf{Z}$  in the  $m$  indeterminates  $X_1, \dots, X_m$ .

Property (T) for the pair  $(SL_2(R_m) \ltimes R_m^2, R_m^2)$  is an immediate consequence of the results in the previous sections. More precisely, let  $Q$  be the subset of  $SL_2(R_m) \ltimes R_m^2$  consisting of the four elements  $(\pm 1, 0), (0, \pm 1) \in R_m^2$  and the  $4(m+1)$  elementary matrices

$$E_{12}(\pm 1), E_{21}(\pm 1), E_{12}(\pm X_1), E_{21}(\pm X_1), \dots, E_{12}(\pm X_m), E_{21}(\pm X_m)$$

in  $SL_2(R_m)$ .

**Proposition 4.3.7** *Let  $R_m = \mathbf{Z}[X_1, \dots, X_m]$ . The pair  $(SL_2(R_m) \ltimes R_m^2, R_m^2)$  has Property (T), with  $(Q, 2/22^{m+1})$  as Kazhdan pair.*

**Proof** We proceed by induction on  $m$ . The case  $m = 0$  follows from Theorem 4.2.2. Assume  $m \geq 1$  and the claim is true for  $m - 1$ . Set  $\varepsilon = 2/22^m$  and  $\delta = 2/22^{m+1}$ . Then

$$\frac{\delta + 2\delta/\varepsilon}{1 - \delta/\varepsilon} = \frac{2}{21} \left( \frac{1}{22^m} + 1 \right) \leq \frac{23}{231} < \frac{1}{10}.$$

Since  $R_m = R_{m-1}[X_m]$ , the claim for  $m$  follows from Proposition 4.3.1. ■

**Corollary 4.3.8** *Let  $R$  be a topological commutative ring with unit. Assume that there are finitely many elements  $r_0 = 1, r_1, \dots, r_m \in R$  generating a dense subring  $S$  of  $R$ . Let  $F$  be the subset of  $SL_2(R) \ltimes R^2$  consisting of  $(\pm 1, 0), (0, \pm 1) \in R^2$  and the elementary matrices  $E_{12}(\pm r_i), E_{21}(\pm r_i) \in SL_2(R)$  for  $0 \leq i \leq m$ .*

*Let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_2(R) \ltimes R^2$  which has a  $(F, \varepsilon/22^{m+1})$ -invariant unit vector  $\xi$  for some  $\varepsilon > 0$ . Then  $\|\pi(g)\xi - \xi\| < \varepsilon$  for every  $g \in R^2$ .*

**Proof** The mapping  $R_m \rightarrow S$  which sends  $1, X_1, \dots, X_m$  to  $1, r_1, \dots, r_m$  extends to a surjective ring homomorphism

$$\Phi : SL_2(R_m) \ltimes R_m^2 \rightarrow SL_2(S) \ltimes S^2.$$

Let  $\mathcal{H}_0$  be the subspace of the  $R^2$ -invariant vectors in  $\mathcal{H}$  and let  $\mathcal{H}_1$  be its orthogonal complement. Let  $\xi = \xi_0 + \xi_1$  be the corresponding decomposition of  $\xi$ . By density of  $S$ , there exists no non-zero  $S^2$ -invariant vector in  $\mathcal{H}_1$ . Hence, the unitary representation  $\pi \circ \Phi$  of  $SL_2(R_m) \ltimes R_m^2$  has no non-zero invariant vector in  $\mathcal{H}_1$ . It follows from the previous proposition that there exists  $g_0 \in F$  such that

$$\frac{2}{22^{m+1}} \|\xi_1\| \leq \|\pi(g_0)\xi_1 - \xi_1\|.$$

On the other hand, since  $\xi$  is  $(F, \varepsilon/22^{m+1})$ -invariant, we have

$$\|\pi(g_0)\xi_1 - \xi_1\|^2 \leq \|\pi(g_0)\xi - \xi\|^2 < (\varepsilon/22^{m+1})^2.$$

Hence,  $\|\xi_1\| < \varepsilon/2$ . This implies that, for every  $g \in R^2$ ,

$$\|\pi(g)\xi - \xi\| = \|\pi(g)\xi_1 - \xi_1\| \leq 2\|\xi_1\| < \varepsilon,$$

as claimed. ■

**Proof of Theorem 4.3.5** Let  $r_1, \dots, r_m$  be elements in  $R$  generating a dense subring. Let  $Q_1$  be the set of the elementary matrices  $E_{ij}(1)$ ,  $1 \leq i \neq j \leq n$ , and let  $Q_2$  be the set of the elementary matrices  $E_{ij}(r_k)$  where  $j = i + 1$  or  $j = i - 1$  for  $1 \leq k \leq m$  and  $1 \leq i \leq n$ . Set  $Q = Q_1 \cup Q_2$  and  $\varepsilon = 1/v_n(R)22^{m+1}$ .

We claim that  $(Q, \varepsilon)$  is a Kazhdan pair for  $SL_n(R)$ . The proof is similar to the proof of Theorem 4.2.5. Indeed, let  $(\pi, \mathcal{H})$  be a unitary representation of  $SL_n(R)$  which has a  $(Q, \varepsilon)$ -invariant unit vector  $\xi$ . Let  $g$  be any elementary matrix in  $SL_n(R)$ . By Lemma 4.2.4, there exists an embedding  $\alpha$  of  $SL_2(R) \ltimes R^2$  into  $SL_n(R)$  such that  $g \in \alpha(R^2)$  and such that  $\alpha(F) = (Q \cup Q^{-1}) \cap \text{Im}(\alpha)$ , where  $F$  is the subset of  $SL_2(R) \ltimes R^2$  defined in the previous corollary. Hence, by this corollary,  $\xi$  is  $(\alpha(R^2), 1/v_n(R))$ -invariant. In particular,

$$\|\pi(g)\xi - \xi\| < \frac{1}{v_n(R)}$$

for any elementary matrix  $g$  in  $SL_n(R)$ . As in the proof of Theorem 4.2.5, it follows from the bounded generation of  $SL_n(R)$  that  $\|\pi(g)\xi - \xi\| \leq 1$  for every  $g \in SL_n(R)$ , and this implies that  $\pi$  has non-zero invariant vectors. ■

**Remark 4.3.9** (i) The previous proof shows the following more general result. Let  $E(n, R)$  denote the subgroup of  $SL_n(R)$  generated by all elementary matrices. Assume that  $R$  contains a dense finitely generated subring and that  $E(n, R)$  is boundedly generated by the elementary matrices. Then  $E(n, R)$  has Property (T).

(ii) Let  $R$  be as in Theorem 4.3.5. Assume that there exists a fixed  $m \in \mathbf{N}$  such that, for every neighbourhood  $U$  of 0 in  $R$ , we can find  $m$  elements  $r_1, \dots, r_m \in U$  generating a dense subring. Then the set  $Q_1$  of the  $n^2 - n$  elementary matrices  $E_{ij}(1)$ , with  $1 \leq i \neq j \leq n$ , is already a Kazhdan set, with the same Kazhdan constant (see Exercise 4.4.5).

(iii) In a major breakthrough, Shalom has shown that  $SL_n(\mathbf{Z}[X_1, \dots, X_m])$  has Property (T) when  $n \geq m + 3$  [Shal–ICM]. The question, due to W. van der Kallen [Kalle–82], whether this group is boundedly generated is still open even for the case  $m = 1$ .

### Property (T) for the loop group of $SL_n(\mathbf{C})$

We apply now Theorem 4.3.5 to the loop group of  $SL_n(\mathbf{C})$ .

Let  $X$  be a topological space and  $G$  a topological group. The set  $G^X$  of continuous mappings from  $X$  to  $G$  is a group for the pointwise product, defined by  $(f_1 f_2)(x) = f_1(x) f_2(x)$  for  $f_1, f_2 \in G^X$  and  $x \in X$ . Endowed with the topology of uniform convergence,  $G^X$  is a topological group. In case  $X = \mathbf{S}^1$  is the unit circle, it is called the *loop group* of  $G$  and denoted by  $LG$ . The following simple proposition gives two necessary conditions for  $LG$  to have Property (T).

**Proposition 4.3.10** *Let  $G$  be a topological group and assume that  $LG$  has Property (T). Then*

- (i)  $G$  has Property (T);
- (ii) if  $G$  is a connected Lie group, the fundamental group  $\pi_1(G)$  of  $G$  is finite.

**Proof** (i) The mapping

$$LG \rightarrow G, f \mapsto f(e)$$

is a continuous surjective group homomorphism, and the claim follows from Theorem 1.3.4.

(ii) Let  $L_e G$  be the normal subgroup of  $LG$  consisting of all  $f \in LG$  with  $f(1) = e$ . Then  $LG$  is naturally isomorphic to the semidirect product  $G \ltimes L_e G$ . It is well known that  $\pi_1(G) = \pi_1(G, e)$  is the set of path-connected components of  $L_e G$ . Hence,  $\pi_1(G)$  is isomorphic to the quotient of  $L_e G$  by its (open) path-connected component  $(L_e G)_0$  of the identity. Since  $G$  is path-connected,  $LG/(LG)_0$  is isomorphic to  $L_e G/(L_e G)_0$  and hence to  $\pi_1(G)$ . It follows that  $\pi_1(G)$  has Property (T) as a discrete group. On the other hand,  $\pi_1(G)$  is abelian. Hence,  $\pi_1(G)$  is finite (Theorem 1.1.6). ■



**Remark 4.3.11** (i) Examples of Kazhdan Lie groups with infinite fundamental group are  $SO(n, 2)$  for  $n \geq 3$  and  $Sp_n(\mathbf{R})$  for  $n \geq 2$  (see Example 1.7.13). Thus, we see that the loop groups of such groups do not have Property (T).

(ii) Let  $G$  be a compact connected simple Lie group. Then its loop group  $LG$  does not have Property (T); see Exercises 4.4.6 and 4.4.7.

Let  $G = SL_n(\mathbf{C})$ . Then  $LG$  can be identified with  $SL_n(R)$ , where  $R = C(\mathbf{S}^1)$  is the ring of all continuous complex-valued functions on  $\mathbf{S}^1$ .

We first observe that  $R = C(\mathbf{S}^1)$  has a finitely generated dense subring. Indeed, fix an irrational real number  $\theta$  and let  $S$  be the subring generated by the functions

$$r_1 = \sin 2\pi x, \quad r_2 = \cos 2\pi x, \quad r_3 = e^{2\pi i\theta}.$$

By the Stone–Weierstraß Theorem, the subring generated by  $\mathbf{C} \cup \{r_1, r_2\}$  is dense in  $C(\mathbf{S}^1)$ . Moreover,  $r_3$  generates a dense subring of  $\mathbf{C}$ . Hence,  $S$  is dense in  $C(\mathbf{S}^1)$ .

**Theorem 4.3.12** *The loop group  $L(SL_n(\mathbf{C}))$  has Property (T) for  $n \geq 3$ .*

This will be a consequence of Theorem 4.3.5, once we have shown that  $SL_n(R)$  is boundedly elementary generated for  $R = C(\mathbf{S}^1)$ . The proof is based on the following elementary lemma.

**Lemma 4.3.13** *Let  $f, h \in C(\mathbf{S}^1)$  be two continuous functions on  $\mathbf{S}^1$  with no common zero. Then there exists a continuous function  $\varphi \in C(\mathbf{S}^1)$  such that  $h + \varphi f$  has no zero.*

**Proof** The following proof is due to M. Neuhauser. Consider the closed subset  $A = \{z \in \mathbf{S}^1 : |f(z)| = |h(z)|\}$  of  $\mathbf{S}^1$ . Since  $f$  and  $h$  have no common zero,

$$\varphi_1 : A \rightarrow \mathbf{S}^1, \quad z \mapsto \frac{h(z)}{f(z)}$$

is a well-defined continuous function on  $A$ . If  $A = \mathbf{S}^1$ , then  $h$  has no zero and we can take  $\varphi = 0$ .

Assume that  $A \neq \mathbf{S}^1$ . Then there exists a continuous function  $\psi_1 : A \rightarrow \mathbf{R}$  such that  $\varphi_1(z) = \exp(i\psi_1(z))$  for all  $z \in A$ . By Urysohn's Lemma,  $\psi_1$  extends to a continuous function  $\psi : \mathbf{S}^1 \rightarrow \mathbf{R}$ . Let  $\varphi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$  be defined by  $\varphi(z) = \exp(i\psi(z))$ . Then  $h(z) + \varphi(z)f(z) \neq 0$  for every  $z \in \mathbf{S}^1$ . Indeed, this is clear if  $z \in A$ , since  $h(z) + \varphi(z)f(z) = 2h(z) \neq 0$ . If  $z \notin A$ , then  $|f(z)| \neq |h(z)|$  and hence  $h(z) + \varphi(z)f(z) \neq 0$ . ■

**Proposition 4.3.14** *Let  $R = C(S^1)$ . For  $n \geq 2$ , the group  $LSL_n(\mathbb{C}) = SL_n(R)$  is elementary boundedly generated, with  $v_n(R) \leq \frac{3}{2}n^2$ .*

**Proof** Let

$$A = \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ * & * & \cdots & * \\ . & . & \cdots & . \\ * & * & \cdots & * \end{pmatrix} \in SL_n(R).$$

Since  $\det A = 1$ , the functions  $f_1, \dots, f_n \in R$  have no common zero. Set  $f = |f_1|^2 + \cdots + |f_{n-1}|^2$  and  $h = f_n$ . Then  $f$  and  $h$  have no common zero. By the previous lemma, there exists  $\varphi \in R$  such that the function  $\psi = h + \varphi f$  has no zero. Thus,  $\psi$  is invertible in  $R$ .

We perform the following  $n - 1$  elementary operations on  $A$ . First add to the last column  $\varphi \bar{f}_1$  times the first column. Then add to the last column  $\varphi \bar{f}_2$  times the second column. Continuing this way until the last but one column, we obtain at the end of these  $n - 1$  operations a matrix of the form

$$B = \begin{pmatrix} * & \cdots & * & \psi \\ * & \cdots & * & * \\ . & \cdots & . & . \\ * & \cdots & * & * \end{pmatrix}.$$

Since  $\psi$  is invertible, using at most one elementary operation, one can transform  $B$  into a matrix

$$C = \begin{pmatrix} * & \cdots & * & * \\ . & \cdots & . & . \\ * & \cdots & * & * \\ * & \cdots & * & 1 \end{pmatrix}.$$

Using  $(n - 1) + (n - 1) = 2n - 2$  elementary operations, one can transform  $C$  into a matrix of the form

$$D = \begin{pmatrix} * & \cdots & * & 0 \\ . & \cdots & . & . \\ * & \cdots & * & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Thus, using at most  $3n - 2$  elementary operations, we have transformed  $A \in SL_n(R)$  into a matrix  $D \in SL_{n-1}(R)$ . The assertion follows by induction on  $n$ . ■

**Remark 4.3.15** (i) Theorem 4.3.5 and the previous proposition show that  $\varepsilon = 2/(3n^2 \cdot 22^2) > 1/(10^6 n^2)$  is a Kazhdan constant for the set  $Q$  defined in Remark 4.3.6. Observe that every neighbourhood of 0 in  $C(\mathbf{S}^1)$  contains  $m = 3$  functions generating a dense subring: fix an irrational real number  $\theta$  and take

$$r_1 = \frac{1}{N} \sin(2\pi x), \quad r_2 = \frac{1}{N} \cos(2\pi x), \quad r_3 = \frac{e^{2\pi i\theta}}{N}$$

for suitable  $N \in \mathbf{N}$ . It follows from Remark 4.3.9.ii that the set  $Q_1$  of the  $n^2 - n$  constant functions  $z \mapsto E_{ij}(1)$ ,  $1 \leq i \neq j \leq n$ , is a Kazhdan set, with the same Kazhdan constant.

(ii) The loop group of  $SL_n(\mathbf{R})$  is not elementary boundedly generated. In fact, the elementary matrices do not even generate  $LSL_n(\mathbf{R})$ ; with the notation of Remark 4.3.9.i, this means that  $E(n, R) \neq LSL_n(\mathbf{R})$ , where  $R$  is the ring of all real-valued continuous functions on  $\mathbf{S}^1$ . Indeed, every mapping  $f \in E(n, R)$  is homotopic to a constant path (Exercise 4.4.8). On the other hand,  $\pi_1(SL_n(\mathbf{R}))$  is non-trivial. However, as observed in [Cornu-06e],  $LSL_n(\mathbf{R})$  has Property (T) for  $n \geq 3$ . In view of Remark 4.3.9.i, this follows from the fact that  $E(n, R)$  is boundedly generated by the elementary matrices, a result due to Vaserstein [Vaser-88], and that  $\pi_1(SL_n(\mathbf{R}))$  is finite.

(iii) Using similar methods, Neuhauser [Neuh-03b] showed that the loop group of  $Sp_{2n}(\mathbf{C})$  has Property (T) for  $n \geq 2$ .

## 4.4 Exercises

**Exercise 4.4.1** We know from Corollary 1.4.13 that the pair  $(SL_2(\mathbf{R}) \ltimes \mathbf{R}^2, \mathbf{R}^2)$  has Property (T). Prove that the pair  $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$  has Property (T), by using the fact that  $SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$  is a lattice in  $SL_2(\mathbf{R}) \ltimes \mathbf{R}^2$ .

[Hint: Imitate the proof of Theorem 1.7.1.]

**Exercise 4.4.2** Let  $K$  be a field and  $n \geq 2$ . Show that  $v_n(K) \leq n(n-1)$ , where  $v_n(K)$  is as in Definition 4.1.1.

[Hint: Use Gauss elimination.]

**Exercise 4.4.3** Let  $H$  be a subgroup of finite index of the group  $G$ . Then  $G$  is boundedly generated if and only if  $H$  is boundedly generated.

[Hint: The “if part” is straightforward. To show the “only if” part, let  $S$  be a finite set of generators of  $G$ . A finite set of generators for  $H$  is provided by the so-called Rademeister–Schreier method. More precisely, let  $T$  be a set of representatives for  $H \backslash G$  containing the group unit. For  $g \in G$ , denote by

$\tau(g) \in T$  the representative of the class  $Hg$ . Show that the set

$$U = \{ts(\tau(ts))^{-1} : s \in S, t \in T\}$$

is a generating set of  $H$ ; moreover, if  $G$  is boundedly generated with respect to  $S$ , then  $H$  is boundedly generated with respect to  $U$ . If necessary, look at the proof of Theorem 2.7 in [MagKS–66, §2.3].]

**Exercise 4.4.4** Let  $R$  be a commutative ring with unit. For  $n \in \mathbf{N}$ , let  $Q$  be the set of all elementary matrices  $E_{ij}(1)$  in  $\Gamma = SL_n(R)$ . Let  $\pi$  be the natural representation of  $\Gamma$  in  $\ell^2(R^n \setminus \{0\})$ . Let  $\xi \in \ell^2(R^n \setminus \{0\})$  be the characteristic function of the set  $\{e_1, \dots, e_n\}$ , where  $e_1, \dots, e_n$  is the standard basis of  $R^n$ .

(i) Show that for every  $\gamma \in Q$ ,

$$\|\pi(\gamma)\xi - \xi\| \leq \sqrt{\frac{2}{n}} \|\xi\|.$$

(ii) Show that, if  $SL_n(R)$  has Property (T), then the optimal Kazhdan constant for  $Q$  is bounded from above by  $\sqrt{2/n}$ .

**Exercise 4.4.5** Prove the claim from Remark 4.3.9.ii.

[Hint: Use continuity of the involved representations.]

**Exercise 4.4.6** Let  $\pi$  be the standard representation of  $G = SU(2)$  on  $\mathbf{C}^2$ . For every  $z \in \mathbf{S}^1$ , define an irreducible unitary representation  $\pi_z$  of the loop group  $LG$  on  $\mathbf{C}^2$  by

$$\pi_z(f) = \pi(f(z)), \quad f \in LG.$$

(i) Show that  $\pi_z$  and  $\pi_1$  are not unitarily equivalent for  $z \neq 1$ .

[Hint: Consider the characters of  $\pi_z$  and  $\pi_1$ .]

(ii) Show that the trivial representation  $1_{LG}$  is weakly contained in

$$\bigoplus_{z \in \mathbf{S}^1 \setminus \{1\}} \pi_z \otimes \overline{\pi_1}.$$

(iii) Show that  $LG$  does not have Property (T).

**Exercise 4.4.7** Let  $G$  be a compact connected simple Lie group. Show that the loop group  $LG$  of  $G$  does not have Property (T).

[Hint: Imitate the previous exercise, using a one-dimensional torus  $\mathbf{S}^1$  in  $G$ .]

**Exercise 4.4.8** Let  $X$  be a connected manifold and let  $f : X \rightarrow SL_n(\mathbf{C})$  be a continuous mapping. Assume that  $f = E_1 \cdots E_N$ , where  $E_1, \dots, E_N$  are

elementary matrices in  $SL_n(C(X))$ . Show that  $f$  is homotopic to a constant mapping  $X \rightarrow SL_n(\mathbf{C})$ .

**Exercise 4.4.9** Consider the ring  $R = C(S^3)$  of all continuous functions on the 3-sphere  $S^3$ , equipped with the topology of uniform convergence. Show that  $SL_2(R) = C(S^3, SL_2(\mathbf{C}))$  is not elementary boundedly generated.

[Hint: Observe that  $\pi_3(SL_2(\mathbf{C})) = \pi_3(SU(2)) = \pi_3(S^3) \neq 0$  and use the previous exercise.]

**Exercise 4.4.10** Let  $R = C(S^3)$ , where  $S^3$  is the 3-sphere. Show that  $SL_3(R) = C(S^3, SL_3(\mathbf{C}))$  does not have Property (T).

[Hint: Observe that  $\pi_3(SL_3(\mathbf{C})) = \pi_3(SU(3)) = \mathbf{Z}$  is infinite and imitate the proof of Proposition 4.3.10.ii.]

# 5

## A spectral criterion for Property (T)

The aim of this chapter is to give a spectral condition implying Property (T) and allowing the computation of Kazhdan constants for groups acting on a simplicial complex of dimension 2. This condition is due to A. Zuk [Zuk–96] and W. Ballmann and J. Swiatkowski [BalSw–97] (see also [Pansu–98] and [Wang–98]), after fundamental work by H. Garland [Garla–73] (see the Bourbaki report [Borel–73]). In [Zuk–03], an abstract form of this criterion is given involving only the spectrum of the Laplace operator of a finite graph associated to a generating set of a group.

The proof we give for the spectral criterion is based on the article of M. Gromov [Gromo–03] and on the exposition of this work given by E. Ghys [Ghys–04]. The main result Theorem 5.5.4 holds for arbitrary locally compact groups acting properly on a simplicial complex of dimension 2, while only discrete groups are treated in the papers quoted above. This generalisation was pointed out by G. Skandalis and allows us to give a new proof of Property (T) for  $SL_3(\mathbf{K})$  when  $\mathbf{K}$  is a non-archimedean local field.

Sections 5.1 and 5.2 describe our setting for a random walk  $\mu$  on a set  $X$ , including the corresponding Laplace operators  $\Delta_\mu$  acting on spaces  $\Omega_{\mathcal{H}}^0(X)$  of Hilbert-space-valued functions on  $X$ . In Section 5.3, the set  $X$  is finite. In Section 5.4, the set  $X$  can be infinite, and there is a group  $G$  acting on  $X$  with finitely many orbits. The interaction finite/infinite, or local/global, is analysed in Section 5.5, and applications follow in the two last sections.

More precisely, Proposition 5.3.1 establishes in the case of a finite set  $X$  the basic properties of the smallest positive eigenvalue  $\lambda_1$  of  $\Delta_\mu$  acting on  $\Omega_{\mathbb{C}}^0(X)$ . Proposition 5.4.5 relates Property (T) for  $G$  to minorations of the operators  $\Delta_\mu$ , and this is reformulated in Proposition 5.4.8 as Poincaré type majorations of the operators  $\Delta_{\mu^{*k}}$ . The situation for comparing “local” random walks on finite sets  $X_x$  and a “global” random walk on  $X$  is described in Section 5.5, together with the main example of a simplicial 2-complex  $X$  and the links  $X_x$

of its vertices; the comparison itself is the object of Theorems 5.5.2 (general situation) and 5.5.4 (a 2-complex and its links). The first application is Zuk's criterion (Theorem 5.6.1). In Section 5.7, there are applications to Euclidean buildings of type  $\tilde{A}_2$ .

## 5.1 Stationary measures for random walks

A *random walk* or *Markov kernel* on a non-empty set  $X$  is a kernel with non-negative values

$$\mu : X \times X \longrightarrow \mathbf{R}_+$$

such that  $\sum_{y \in X} \mu(x, y) = 1$  for all  $x \in X$ . Such a random walk is *irreducible* if, given any pair  $(x, y)$  of distinct points in  $X$ , there exist an integer  $n \geq 1$  and a sequence  $x = x_0, x_1, \dots, x_n = y$  of points in  $X$  such that  $\mu(x_{j-1}, x_j) > 0$  for any  $j \in \{1, \dots, n\}$ .

A *stationary measure* for a random walk  $\mu$  is a function

$$v : X \longrightarrow \mathbf{R}_+^*$$

such that

$$v(x)\mu(x, y) = v(y)\mu(y, x) \quad \text{for all } x, y \in X.$$

A random walk is *reversible* if it has at least one stationary measure.

Let  $\mu$  be a random walk on a set  $X$ . There are two obvious necessary conditions for  $\mu$  to have a stationary measure: the first is

$$(*) \quad \mu(y, x) \neq 0 \quad \text{if and only if} \quad \mu(x, y) \neq 0 \quad (x, y \in X)$$

and the second is

$$(**) \quad \mu(x_1, x_2) \cdots \mu(x_{n-1}, x_n)\mu(x_n, x_1) = \mu(x_n, x_{n-1}) \cdots \mu(x_2, x_1)\mu(x_1, x_n)$$

for any integer  $n \geq 3$  and any sequence  $x_1, \dots, x_n$  of points in  $X$ . We leave as Exercise 5.8.1 to check that Conditions (\*) and (\*\*) are also sufficient for the existence of a stationary measure, and for its uniqueness in case  $\mu$  is irreducible.

**Example 5.1.1** Let  $\mathcal{G} = (X, \mathbb{E})$  be a locally finite *graph*. It is convenient to adopt here a definition slightly different from that used in Section 2.3. Here, the

edge set  $\mathbb{E}$  is a subset of  $X \times X$  which contains  $\bar{e} = (y, x)$  whenever it contains  $e = (x, y)$ ; the *source* of  $e$  is  $x$  (also written  $e^-$ ) and the *range* of  $e$  is  $y$  (also written  $e^+$ ). Thus  $\mathcal{G}$  has no multiple edge (namely has at most one edge with given source and range in  $X$ ) and  $\mathcal{G}$  can have loops, namely edges of the form  $e = (x, x)$ , for which  $\bar{e} = e$ . The *degree* of a vertex  $x \in X$  is the integer

$$\deg(x) = \# \{y \in X : (x, y) \in \mathbb{E}\}.$$

For  $x, y \in X$ , set

$$\mu(x, y) = \begin{cases} 1/\deg(x) & \text{if } (x, y) \in \mathbb{E} \\ 0 & \text{otherwise.} \end{cases}$$

Then  $\mu$  is the so-called *simple random walk* on  $X$  and  $v : x \mapsto \deg(x)$  is a stationary measure for  $\mu$ . [It is important to allow loops in  $\mathcal{G}$ , since  $\mu(x, x) \neq 0$  should not be excluded.]

Conversely, to any random walk  $\mu$  on  $X$  for which Condition (\*) holds, we can associate a graph  $\mathcal{G}_\mu = (X, \mathbb{E}_\mu)$  with edge-set

$$\mathbb{E}_\mu = \{(x, y) \in X \times X : \mu(x, y) > 0\}.$$

The graph  $\mathcal{G}_\mu$  is connected if and only if the random walk  $\mu$  is irreducible; this graph is locally finite if and only if the random walk  $\mu$  has *finite range*, namely if and only if the set  $\{y \in X : \mu(x, y) \neq 0\}$  is finite for all  $x \in X$ . Here is a particular case of the criterion for the existence of a stationary measure: if  $\mu$  is a random walk for which Condition (\*) holds and for which the graph  $\mathcal{G}_\mu$  is a tree, then  $\mu$  has a stationary measure.

## 5.2 Laplace and Markov operators

Consider a set  $X$ , a random walk  $\mu$  on  $X$  which is reversible, the graph  $\mathcal{G}_\mu = (X, \mathbb{E}_\mu)$  defined in the previous section, a stationary measure  $v$  for  $\mu$ , and a complex Hilbert space  $\mathcal{H}$ . For an edge  $e \in \mathbb{E}_\mu$  with source and range  $e^-, e^+ \in X$ , set

$$c(e) = v(e^-)\mu(e^-, e^+).$$

Observe that  $c(\bar{e}) = c(e)$  for  $e \in \mathbb{E}_\mu$ . In the interpretation of  $(X, \mu)$  as an electrical network,  $c(e)$  is the *conductance* of  $e$ , the inverse of the *resistance*



of  $e$ , and

$$v(x) = \sum_{e \in \mathbb{E}_\mu : e^- = x} c(e) = \sum_{e \in \mathbb{E}_\mu : e^+ = x} c(e)$$

is the *total conductance at  $x$*  (see, e.g., [Soard–94] or [Woess–00]).

Consider the Hilbert spaces

$$\Omega_{\mathcal{H}}^0(X) = \left\{ f : X \rightarrow \mathcal{H} : \sum_{x \in X} \|f(x)\|_{\mathcal{H}}^2 v(x) < \infty \right\}$$

and

$$\Omega_{\mathcal{H}}^1(X) = \left\{ F : \mathbb{E}_\mu \rightarrow \mathcal{H} : \begin{array}{ll} F(\bar{e}) = -F(e) & \text{for all } e \in \mathbb{E}_\mu \\ \text{and} & \frac{1}{2} \sum_{e \in \mathbb{E}_\mu} \|F(e)\|_{\mathcal{H}}^2 \frac{1}{c(e)} < \infty \end{array} \right\},$$

with the natural inner products. We can view  $F$  as a mapping on  $X \times X$  by

$$F(x, y) = \begin{cases} F(e) & \text{if } e = (x, y) \in \mathbb{E}_\mu \\ 0 & \text{otherwise.} \end{cases}$$

Inner products in  $\Omega_{\mathcal{H}}^1(X)$  can be written as

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{(x, y) \in X^2} \langle F_1(x, y), F_2(x, y) \rangle \frac{1}{c(x, y)}.$$

Define a linear operator

$$d : \Omega_{\mathcal{H}}^0(X) \rightarrow \Omega_{\mathcal{H}}^1(X), \quad (df)(e) = c(e) (f(e^+) - f(e^-)).$$

**Remark 5.2.1** (i) Instead of our space  $\Omega_{\mathcal{H}}^1(X)$ , some authors deal with another space of functions from  $\mathbb{E}_\mu$  to  $\mathcal{H}$ , with a condition of the form  $\frac{1}{2} \sum_{e \in \mathbb{E}_\mu} \|F(e)\|_{\mathcal{H}}^2 c(e) < \infty$ , and therefore with another operator  $d$  defined by  $(df)(e) = f(e^+) - f(e^-)$ .

(ii) There is some analogy between two situations: on the one hand the set  $X$  (together with positive-valued functions  $v$  on  $X$  and  $c$  on  $\mathbb{E}_\mu$ ) and the Hilbert spaces  $\Omega_{\mathcal{H}}^0(X)$ ,  $\Omega_{\mathcal{H}}^1(X)$ ; and on the other hand a manifold, the Hilbert spaces of  $\mathcal{H}$ -valued  $L^2$ -functions and  $L^2$ -one-forms, respectively. The operator  $d$  is then the analogue in the present setting of the exterior differential.

**Proposition 5.2.2** *The notation being as above, the operator  $d$  is bounded, indeed*

$$(i) \quad \|d\| \leq \sqrt{2}.$$

*Its adjoint  $d^* : \Omega_{\mathcal{H}}^1(X) \rightarrow \Omega_{\mathcal{H}}^0(X)$  is given by*

$$(ii) \quad (d^*F)(y) = \frac{1}{v(y)} \sum_{x \in X} F(x, y).$$

*If  $\Delta_\mu$  and  $M_\mu$  are the operators defined on  $\Omega_{\mathcal{H}}^0(X)$  by*

$$\Delta_\mu = d^*d = I - M_\mu$$

*(with  $I$  the identity operator on  $\Omega_{\mathcal{H}}^0(X)$ ), we have*

$$(iii) \quad \Delta_\mu \geq 0 \quad \text{and} \quad \|M_\mu\| \leq 1.$$

*For  $f \in \Omega_{\mathcal{H}}^0(X)$ , we have*

$$(iv) \quad \langle \Delta_\mu f, f \rangle = \langle df, df \rangle = \frac{1}{2} \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 v(x) \mu(x, y)$$

$$(v) \quad (\Delta_\mu f)(x) = f(x) - (M_\mu f)(x) = f(x) - \sum_{y \sim x} f(y) \mu(x, y)$$

*where  $\sum_{y \sim x}$  indicates a summation over all neighbours of  $x$  in the graph  $\mathcal{G}_\mu$ .*

*We have  $\Delta_\mu f = 0$  if and only if  $f$  is locally constant. In particular, when the random walk  $\mu$  is irreducible,  $\Delta_\mu f = 0$  if and only if  $f$  is constant.*

**Proof** The proposition follows from straightforward computations. Here are some of them. For  $f \in \Omega_{\mathcal{H}}^0(X)$ , we have, by two applications of the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \sum_{(x,y) \in X^2} \langle f(y), f(x) \rangle c(x, y) \right| &\leq \left\{ \sum_{(u,y) \in X^2} \|f(y)\|^2 c(u, y) \sum_{(x,v) \in X^2} \|f(x)\|^2 c(x, v) \right\}^{\frac{1}{2}} \\ &= \left\{ \sum_{y \in X} \|f(y)\|^2 v(y) \sum_{x \in X} \|f(x)\|^2 v(x) \right\}^{\frac{1}{2}} = \|f\|^2. \end{aligned}$$

This implies

$$\begin{aligned}\langle df, df \rangle &= \frac{1}{2} \sum_{(x,y) \in X^2} \langle f(y), f(y) \rangle c(x, y) + \frac{1}{2} \sum_{(x,y) \in X^2} \langle f(x), f(x) \rangle c(x, y) \\ &\quad - \operatorname{Re} \left( \sum_{(x,y) \in X^2} \langle f(y), f(x) \rangle c(x, y) \right) \\ &\leq \frac{1}{2} \|f\|^2 + \frac{1}{2} \|f\|^2 + \|f\|^2 = 2 \|f\|^2,\end{aligned}$$

so that  $\|d\| \leq \sqrt{2}$ . The verification of (ii) is straightforward, and the positivity of  $\Delta_\mu$  follows from its definition as  $d^*d$ .

We leave the verification of formulae (iv) and (v) as an exercise. The last claim of the proposition follows from (iv). Now

$$\langle M_\mu f, f \rangle = \sum_{x \in X} \langle (M_\mu f)(x), f(x) \rangle \nu(x) = \sum_{x \in X} \sum_{y \sim x} \langle f(y), f(x) \rangle c(x, y)$$

and, by the inequality above, this implies that  $\|M_\mu\| \leq 1$ , so that the proof of (iii) is complete. ■

The operators  $\Delta_\mu$  and  $M_\mu$  on  $\Omega_{\mathcal{H}}^0(X)$  are the *Laplace operator* and the *Markov operator*, respectively. Observe that  $\Delta_\mu$  is positive by definition. A function  $f \in \Omega_{\mathcal{H}}^0(X)$  is *harmonic* if  $\Delta_\mu f = 0$ , namely if  $M_\mu f = f$ . When  $\mu$  is irreducible, a function  $f \in \Omega_{\mathcal{H}}^0(X)$  is harmonic if and only if it is constant, by the last claim of Proposition 5.2.2.

The formula defining  $df : \mathbb{E}_\mu \rightarrow \mathcal{H}$  makes sense for any function  $f : X \rightarrow \mathcal{H}$ . The function  $f$  is said to be *Dirichlet finite* if  $df \in \Omega_{\mathcal{H}}^1(X)$ , and the number

$$E_\mu(f) = \|df\|^2 = \frac{1}{2} \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \mu(x, y)$$

given by formula (iv) of Proposition 5.2.2 is the *Dirichlet energy* of  $f$ .

Assume that the random walk  $\mu$  has finite range. Formula (v) in Proposition 5.2.2 defines  $\Delta_\mu f$  and  $M_\mu f$  for any function  $f : X \rightarrow \mathcal{H}$ .

Let  $\mu, \mu'$  be two random walks on the same set  $X$ . The *convolution*  $\mu * \mu'$  is defined by

$$(\mu * \mu')(x, z) = \sum_{y \in X} \mu(x, y) \mu'(y, z) \quad \text{for all } x, z \in X$$

and is clearly again a random walk on  $X$ . For an integer  $k \geq 1$ , we write  $\mu^{*k}$  for the  $k$ -th convolution power of  $\mu$ ; moreover  $\mu^{*0}(x, y) = 1$  if  $x = y$  and  $\mu^{*0}(x, y) = 0$  if  $x \neq y$ . The following proposition is straightforward.

**Proposition 5.2.3** *If  $\nu$  is a stationary measure for two random walks  $\mu, \mu'$  on the same set  $X$ , then  $\nu$  is also a stationary measure for  $\frac{1}{2}(\mu * \mu' + \mu' * \mu)$ . The Markov operator for the convolution  $\mu * \mu'$  is the product of the Markov operators for the factors:*

$$M_{\mu * \mu'} = M_\mu M_{\mu'}.$$

*In particular,  $\nu$  is a stationary measure for  $\mu^{*k}$  and*

$$M_{\mu^{*k}} = (M_\mu)^k$$

*for any  $k \geq 0$ .*

### 5.3 Random walks on finite sets

Let  $X$  be a finite set of cardinality  $n$ , let  $\mu$  be a random walk on  $X$ , and let  $\nu$  be a stationary measure for  $\mu$ . We denote by

$$\Omega_{\mathcal{H}}^0(X)_0 = \left\{ f \in \Omega_{\mathcal{H}}^0(X) : \sum_{x \in X} f(x) \nu(x) = 0 \right\}$$

the orthogonal complement in  $\Omega_{\mathcal{H}}^0(X)$  of the space of constant functions.

In case  $\mathcal{H} = \mathbf{C}$ , the self-adjoint operator  $\Delta_\mu$  acting on  $\Omega_{\mathbf{C}}^0(X)$  has exactly  $n$  eigenvalues (with repetitions according to multiplicities) which are real and which can be enumerated in such a way that

$$0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{n-1}.$$

It follows from Proposition 5.2.2 that  $\lambda_1 > 0$  if and only if  $\mu$  is irreducible, namely if and only if the graph  $\mathcal{G}_\mu$  is connected.

For functions with values in a Hilbert space  $\mathcal{H}$ , the Laplace operator is the tensor product of the Laplace operator for  $\mathbf{C}$ -valued functions with the identity operator on  $\mathcal{H}$ . A number  $\lambda \in \mathbf{R}$  which is an eigenvalue of some multiplicity  $N$  in the scalar case is therefore an eigenvalue of multiplicity  $\dim(\mathcal{H})N$  in the  $\mathcal{H}$ -valued case.

**Proposition 5.3.1** *Let  $X$  be a finite set, let  $\mu$  be a reversible irreducible random walk on  $X$ , let  $\nu$  be a stationary measure for  $\mu$ , and let  $\mathcal{H}$  be a Hilbert space. Denote by  $\lambda_1$  the smallest positive eigenvalue of the Laplacian  $\Delta_\mu$  acting on  $\Omega_{\mathbb{C}}^0(X)$ . We have*

$$\lambda_1 = \inf \left\{ \frac{1}{2 \|f\|^2} \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \mu(x,y) : f \in \Omega_{\mathcal{H}}^0(X)_0, f \neq 0 \right\}$$

and

$$\begin{aligned} & \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \nu(y) \\ & \leq \frac{\sum_{z \in X} \nu(z)}{\lambda_1} \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \mu(x,y) \end{aligned}$$

for all  $f \in \Omega_{\mathcal{H}}^0(X)$ .

**Proof** Since  $\mu$  is irreducible, the space  $\Omega_{\mathcal{H}}^0(X)_0$  is the orthogonal complement in  $\Omega_{\mathcal{H}}^0(X)$  of the eigenspace of  $\Delta_\mu$  of eigenvalue  $\lambda_0 = 0$ . Hence, the first formula follows from Proposition 5.2.2.iv.

For the inequality, there is no loss of generality if we assume that  $f \in \Omega_{\mathcal{H}}^0(X)_0$ . On the one hand, we have just shown that

$$2\lambda_1 \|f\|^2 \leq \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \mu(x,y).$$

On the other hand, since  $\sum_{x \in X} f(x) \nu(x) = 0$ , we have

$$\begin{aligned} & \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \nu(x) \nu(y) \\ & = 2 \sum_{x \in X} \|f(x)\|^2 \nu(x) \sum_{y \in X} \nu(y) - 2 \operatorname{Re} \left( \left\langle \sum_{y \in X} f(y) \nu(y), \sum_{x \in X} f(x) \nu(x) \right\rangle \right) \\ & = 2 \|f\|^2 \sum_{x \in X} \nu(x). \end{aligned}$$

The inequality follows. ■

**Remark 5.3.2** For  $\mu$  reversible and irreducible, there is a unique stationary measure  $\nu$  for  $\mu$  normalised by the condition  $\sum_{z \in X} \nu(z) = 1$  (Exercise 5.8.1).

**Example 5.3.3** Let  $\mathcal{G} = (X, \mathbb{E})$  be a connected finite graph, let  $\mu$  be the corresponding simple random walk on  $X$  (Example 5.1.1), and let  $\nu$  be the stationary measure defined by  $\nu(x) = \deg(x)$  for all  $x \in X$ ; observe that  $c(e) = 1$  for all  $e \in \mathbb{E}$  and that  $\sum_{z \in X} \nu(z) = \#\mathbb{E}$ . The inequality in the previous proposition reads:

$$\sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \deg(x) \deg(y) \leq \frac{\#\mathbb{E}}{\lambda_1} \sum_{(x,y) \in \mathbb{E}} \|f(y) - f(x)\|^2.$$

Suppose moreover that  $\mathcal{G}$  is a regular graph; its degree is therefore  $\#\mathbb{E}/\#X$ . Then

$$(*) \quad \frac{1}{\#(X \times X)} \sum_{(x,y) \in X^2} \|f(y) - f(x)\|^2 \leq \frac{1}{\lambda_1} \frac{1}{\#\mathbb{E}} \sum_{(x,y) \in \mathbb{E}} \|f(y) - f(x)\|^2$$

or, in words: the average of the function  $(x, y) \mapsto \|f(y) - f(x)\|^2$  over all pairs of vertices in  $X$  is bounded by  $\lambda_1^{-1}$  times the average of the same function over all pairs of adjacent vertices.

## 5.4 $G$ -equivariant random walks on quasi-transitive free sets

Let  $X$  be a set and let  $G$  be a unimodular locally compact group acting on  $X$ . We assume that

- (Pi) the action is continuous, namely the subgroup  $G_x = \{g \in G : gx = x\}$  of  $G$  stabilizing  $x$  is open for all  $x \in X$ ;
- (Pii) the action is proper, namely the subset  $\{g \in G : gF \cap F' \neq \emptyset\}$  of  $G$  is compact for all pairs  $(F, F')$  of finite subsets of  $X$ ; in particular, the stabilisers  $G_x$  are compact subgroups of  $G$ ;
- (Piii) the action is cofinite, namely the set  $G \backslash X$  of orbits is finite.

Let moreover  $\mu$  be a reversible random walk on  $X$  and let  $\nu$  be a stationary measure for  $\mu$ . We assume that  $\mu$  and  $\nu$  are  $G$ -invariant

- (Piv)  $\mu(gx, gy) = \mu(x, y)$  and  $\nu(gx) = \nu(x)$  for all  $g \in G$  and  $x, y \in X$ .

**Example 5.4.1** (i) Let  $\Gamma$  be a finitely generated group. Let  $S$  be a finite generating set of  $\Gamma$  which is symmetric, namely such that  $s^{-1} \in S$  if and only if  $s \in S$ . We define a random walk  $\mu$  on  $\Gamma$  (or on the Cayley graph of  $\Gamma$  with

respect to  $S$ ) by

$$\mu(\gamma, \gamma') = \begin{cases} \frac{1}{\#S} & \text{if } \gamma^{-1}\gamma' \in S \\ 0 & \text{otherwise} \end{cases}$$

(this is a particular case of Example 5.1.1). Observe that  $\mu$  is irreducible, since  $S$  generates  $\Gamma$ . As  $\mu$  is symmetric and irreducible, the stationary measures for  $\mu$  are the constant functions on  $\Gamma$ . Properties (Pi) to (Piv) are straightforward to check.

(ii) Let  $A, B$  be two finite groups and let  $\Gamma$  be their free product. Let  $\mathcal{G} = (X, \mathbb{E})$  be the canonical tree on which  $\Gamma$  acts (see [Serre-77]). Recall that  $X$  is the disjoint union of the coset spaces  $\Gamma/A$  and  $\Gamma/B$ , and that each  $\gamma \in \Gamma$  corresponds to a pair  $(e, \bar{e})$  of edges,  $e$  having source  $\gamma A$  and range  $\gamma B$ . The simple random walk on  $\mathcal{G}$  has properties (Pi) to (Piv).

We return to the general case. We choose a Haar measure  $dg$  on  $G$ ; it is both left and right-invariant. For  $x \in X$ , we write  $|G_x|$  for the Haar measure of the compact open subgroup  $G_x$ ; observe that  $|G_{gx}| = |G_x|$  for each  $g \in G$  since  $G_{gx} = gG_xg^{-1}$ .

A *fundamental domain* for the action of  $G$  on  $X$  is a subset  $T$  of  $X$  such that  $X$  is equal to the disjoint union of the orbits  $(Gx)_{x \in T}$ ; such a  $T$  is finite by (Piii) and its cardinality is that of  $G \backslash X$ .

If  $\varphi : X \rightarrow \mathbf{C}$  is a  $G$ -invariant function, the value  $\sum_{x \in T} \varphi(x)$  does not depend on the choice of  $T$  in  $X$ ; we will write

$$\sum_{x \in G \backslash X} \varphi(x)$$

for such a sum (which is a finite sum).

**Lemma 5.4.2** *Let  $\Phi : X^2 \rightarrow \mathbf{C}$  be a function which is  $G$ -invariant for the diagonal action of  $G$  on  $X^2$ . Assume that the series  $\sum_{v \in X} \Phi(x, v)$  and  $\sum_{u \in X} \Phi(u, y)$  are absolutely convergent for all  $x, y \in X$ . Then*

$$(*) \quad \sum_{(x, y) \in G \backslash X^2} \frac{1}{|G_x \cap G_y|} \Phi(x, y) = \sum_{x \in T} \frac{1}{|G_x|} \sum_{y \in X} \Phi(x, y) = \sum_{y \in T} \frac{1}{|G_y|} \sum_{x \in X} \Phi(x, y).$$

**Proof** For each  $x \in T$ , choose a fundamental domain  $U_x \subset X$  for the action of  $G_x$  on  $X$ . Then

$$\{(x, u) : x \in T, u \in U_x\}$$

is a fundamental domain for the action of  $G$  on  $X^2$ . We have

$$\begin{aligned}
 \sum_{(x,y) \in G \backslash X^2} \frac{1}{|G_x \cap G_y|} \Phi(x,y) &= \sum_{x \in T} \sum_{u \in U_x} \frac{1}{|G_x \cap G_u|} \Phi(x,u) \\
 &= \sum_{x \in T} \sum_{u \in U_x} \frac{1}{|G_x \cap G_u|} \frac{1}{|G_x|} \int_{G_x} \Phi(g^{-1}x, u) dg \\
 &= \sum_{x \in T} \sum_{u \in U_x} \frac{1}{|G_x \cap G_u|} \frac{1}{|G_x|} \sum_{g \in G_x / (G_x \cap G_u)} \int_{G_x \cap G_u} \Phi(h^{-1}g^{-1}x, u) dh \\
 &= \sum_{x \in T} \sum_{u \in U_x} \frac{1}{|G_x \cap G_u|} \frac{1}{|G_x|} \sum_{g \in G_x / (G_x \cap G_u)} \int_{G_x \cap G_u} \Phi(g^{-1}x, hu) dh \\
 &= \sum_{x \in T} \sum_{u \in U_x} \frac{1}{|G_x \cap G_u|} \frac{1}{|G_x|} \sum_{g \in G_x / (G_x \cap G_u)} \int_{G_x \cap G_u} \Phi(g^{-1}x, u) dh \\
 &= \sum_{x \in T} \frac{1}{|G_x|} \sum_{u \in U_x} \sum_{g \in G_x / (G_x \cap G_u)} \Phi(g^{-1}x, u) \\
 &= \sum_{x \in T} \frac{1}{|G_x|} \sum_{u \in U_x} \sum_{g \in G_x / (G_x \cap G_u)} \Phi(x, gu) \\
 &= \sum_{x \in T} \frac{1}{|G_x|} \sum_{y \in X} \Phi(x, y).
 \end{aligned}$$

Using the fundamental domain  $\{(u, y) : y \in T, u \in U_y\}$  for the  $G$ -action on  $X^2$ , we also have by a similar computation

$$\sum_{(x,y) \in G \backslash X^2} \frac{1}{|G_x \cap G_y|} \Phi(x,y) = \sum_{y \in T} \frac{1}{|G_y|} \sum_{x \in X} \Phi(x,y). \quad \blacksquare$$

Let  $\pi$  be a unitary representation of  $G$  in a Hilbert space  $\mathcal{H}$ . Consider the vector spaces

$$\mathcal{E}_\pi^0(X) = \{f : X \rightarrow \mathcal{H} : f(gx) = \pi(g)(f(x)) \quad \text{for all } g \in G, x \in X\}$$

and

$$\mathcal{E}_\pi^1(X) = \left\{ F : \mathbb{E}_\mu \rightarrow \mathcal{H} : \begin{array}{ll} F(\bar{e}) = -F(e) & \text{and} \\ F(ge) = \pi(g)(F(e)) & \text{for all } e \in \mathbb{E}_\mu, g \in G \end{array} \right\}.$$



We view  $\mathcal{E}_\pi^0(X)$  and  $\mathcal{E}_\pi^1(X)$  as Hilbert spaces for the inner products defined by

$$\langle f_1, f_2 \rangle = \sum_{x \in T} \langle f_1(x), f_2(x) \rangle \frac{v(x)}{|G_x|} \quad \text{for } f_1, f_2 \in \mathcal{E}_\pi^0(X)$$

and

$$\langle F_1, F_2 \rangle = \frac{1}{2} \sum_{(x,y) \in G \setminus X^2} \langle F_1(x,y), F_2(x,y) \rangle \frac{1}{c(x,y)} \frac{1}{|G_x \cap G_y|}$$

for  $F_1, F_2 \in \mathcal{E}_\pi^1(X)$ .

Here as in Section 5.2, the conductance of an edge is defined by  $c(x, y) = v(x)\mu(x, y) = v(y)\mu(y, x)$ , and we have extended functions in  $\mathcal{E}_\pi^1(X)$  by zero on the complement of  $\mathbb{E}_\mu$  in  $X^2$ . The summation on  $G \setminus X^2$  is therefore a summation on  $G \setminus \mathbb{E}_\mu$ . Observe that  $\sum_{(x,y) \in G \setminus X^2} (\dots)$  is a finite sum if the random walk  $\mu$  has finite range.

Define a linear operator

$$d : \mathcal{E}_\pi^0(X) \rightarrow \mathcal{E}_\pi^1(X), \quad (df)(e) = c(e) (f(e^+) - f(e^-)).$$

The following proposition is the analogue in the present setting of Proposition 5.2.2.

**Proposition 5.4.3** *The notation being as above, the operator  $d$  is bounded, indeed*

$$(i) \quad \|d\| \leq \sqrt{2}.$$

*Its adjoint  $d^* : \mathcal{E}_\pi^1(X) \rightarrow \mathcal{E}_\pi^0(X)$  is given by*

$$(ii) \quad (d^*F)(y) = \frac{1}{v(y)} \sum_{x \in X} F(x, y).$$

*If  $\Delta_\mu$  and  $M_\mu$  are the operators defined on  $\mathcal{E}_\pi^0(X)$  by*

$$\Delta_\mu = d^*d = I - M_\mu$$

*(with  $I$  the identity operator on  $\mathcal{E}_\pi^0(X)$ ), we have*

$$(iii) \quad \Delta_\mu \geq 0 \quad \text{and} \quad \|M_\mu\| \leq 1.$$

For  $f \in \mathcal{E}_\pi^0(X)$ , we have

(iv)

$$\begin{aligned} \langle \Delta_\mu f, f \rangle &= \langle df, df \rangle = \frac{1}{2} \sum_{x \in X, y \in T} \|f(y) - f(x)\|^2 \frac{\nu(x)\mu(x, y)}{|G_y|} \\ &= \frac{1}{2} \sum_{x \in T, y \in T, g \in S_x} \|f(y) - \pi(g)f(x)\|^2 \frac{\nu(x)\mu(gx, y)}{|G_y|}, \end{aligned}$$

where  $S_x$  is a set of representatives for the coset space  $G/G_x$ .

(v)

$$(\Delta_\mu f)(x) = f(x) - (M_\mu f)(x) = f(x) - \sum_{y \sim x} f(y) \mu(x, y)$$

where  $\sum_{y \sim x}$  indicates a summation over all neighbours of  $x$  in the graph  $\mathcal{G}_\mu$ .

When the random walk  $\mu$  is irreducible, then  $\Delta_\mu f = 0$  if and only if the function  $f$  is constant with value in the subspace  $\mathcal{H}^G$  of  $\pi(G)$ -fixed points of  $\mathcal{H}$ .

**Proof** Let us check (i). For  $f \in \mathcal{E}_\pi^0(X)$ , we have, by Lemma 5.4.2,

$$\begin{aligned} \sum_{(u, y) \in G \setminus X^2} \|f(y)\|^2 \frac{c(u, y)}{|G_u \cap G_y|} &= \sum_{y \in T} \|f(y)\|^2 \frac{1}{|G_y|} \sum_{u \in X} c(y, u) \\ &= \sum_{y \in T} \|f(y)\|^2 \frac{\nu(y)}{|G_y|} = \|f\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} &\left| \sum_{(x, y) \in G \setminus X^2} \langle f(y), f(x) \rangle \frac{c(x, y)}{|G_x \cap G_y|} \right| \\ &\leq \left\{ \sum_{(u, y) \in G \setminus X^2} \|f(y)\|^2 \frac{c(u, y)}{|G_u \cap G_y|} \sum_{(x, v) \in G \setminus X^2} \|f(x)\|^2 \frac{c(x, v)}{|G_x \cap G_v|} \right\}^{\frac{1}{2}} = \|f\|^2 \end{aligned}$$

by the Cauchy–Schwarz inequality. Hence

$$\begin{aligned}
 \langle df, df \rangle &= \frac{1}{2} \sum_{(x,y) \in G \setminus X^2} \langle f(y), f(y) \rangle \frac{c(x,y)}{|G_x \cap G_y|} \\
 &\quad + \frac{1}{2} \sum_{(x,y) \in G \setminus X^2} \langle f(x), f(x) \rangle \frac{c(x,y)}{|G_x \cap G_y|} \\
 &\quad - \sum_{(x,y) \in G \setminus X^2} \langle f(y), f(x) \rangle \frac{c(x,y)}{|G_x \cap G_y|} \\
 &\leq \frac{1}{2} \|f\|^2 + \frac{1}{2} \|f\|^2 + \|f\|^2 = 2 \|f\|^2
 \end{aligned}$$

so that  $\|d\| \leq \sqrt{2}$ .

Let us now check (ii). For  $f \in \mathcal{E}_\pi^0(X)$  and  $F \in \mathcal{E}_\pi^1(X)$ , we have, using again Lemma 5.4.2,

$$\begin{aligned}
 \langle df, F \rangle &= \frac{1}{2} \sum_{(x,y) \in G \setminus X^2} c(x,y) \langle f(y) - f(x), F(x,y) \rangle \frac{1}{c(x,y)} \frac{1}{|G_x \cap G_y|} \\
 &= \frac{1}{2} \sum_{y \in T} \sum_{x \in X} \langle f(y), F(x,y) \rangle \frac{1}{|G_y|} + \frac{1}{2} \sum_{x \in T} \sum_{y \in X} \langle f(x), F(y,x) \rangle \frac{1}{|G_x|} \\
 &= \frac{1}{2} \sum_{y \in T} \sum_{x \in X} \langle f(y), F(x,y) \rangle \frac{1}{|G_y|} + \frac{1}{2} \sum_{y \in T} \sum_{x \in X} \langle f(y), F(x,y) \rangle \frac{1}{|G_y|} \\
 &= \sum_{y \in T} \sum_{x \in X} \left\langle f(y), \frac{1}{v(y)} F(x,y) \right\rangle \frac{v(y)}{|G_y|} \\
 &= \langle f, d^*F \rangle.
 \end{aligned}$$

We leave the proof of Claims (iii) to (v) as an exercise. Assume now that  $\mu$  is irreducible. Consider  $f \in \mathcal{E}_\pi^0(X)$  with  $\Delta_\mu(f) = 0$ . The equalities in (iv) implies that  $f(x) = f(y)$  for all  $x \in X, y \in T$  with  $\mu(x,y) > 0$ . By  $G$ -invariance of the function  $(x,y) \mapsto \|f(y) - f(x)\|^2 v(x)\mu(x,y)/|G_y|$ , it follows that  $f(x) = f(y)$  for all  $x, y \in X$  with  $\mu(x,y) > 0$ . By irreducibility of  $\mu$ , this implies that  $f$  is constant on  $X$ . Since  $f(gx) = \pi(g)f(x)$  for all  $g \in G, x \in X$ , the value of  $f$  belongs to  $\mathcal{H}^G$ . ■

The operators  $\Delta_\mu = d^*d$  and  $M_\mu = I - d^*d$  on  $\mathcal{E}_\pi^0(X)$  are the *Laplace operator* and the *Markov operator*, respectively. A function  $f \in \mathcal{E}_\pi^0(X)$  is again called *harmonic* if  $\Delta_\mu f = 0$ , namely if  $M_\mu f = f$ . When  $\mu$  is irreducible,

a function  $f \in \mathcal{E}_\pi^0(X)$  is harmonic if and only if it is constant with value in  $\mathcal{H}^G$ , by the last claim of the previous proposition.

**Example 5.4.4** Let  $\Gamma$  be a group generated by a finite symmetric  $S$  and let  $\mu$  be the symmetric irreducible random walk defined in Example 5.4.1.i. Moreover, let  $\mathcal{H}$  be a Hilbert space and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  be a unitary representation. Then

$$\mathcal{E}_\pi^0(\Gamma) = \{f : \Gamma \rightarrow \mathcal{H} : f(\gamma_1\gamma_2) = \pi(\gamma_1)(f(\gamma_2)) \text{ for all } \gamma_1, \gamma_2 \in \Gamma\}$$

and the linear mapping  $\mathcal{E}_\pi^0(\Gamma) \rightarrow \mathcal{H}, f \mapsto f(1)$  is a norm preserving isomorphism onto. We leave to the reader to check that  $\mathcal{E}_\pi^1(\Gamma) \cong \mathcal{H}^d$ , where  $d = \#S$  if  $1 \notin S$  and  $d = \#S - 1$  if  $1 \in S$ .

We will now assume that  $\mu$  has finite range, that is,

$$(Pv) \quad \#\{y \in X : \mu(x, y) > 0\} < \infty \text{ for all } x \in X.$$

(Recall that (Pi) to (Piv) have been defined at the beginning of the present Section 5.4.) This has the the following consequence:

(Pvi) The group  $G$  is compactly generated. More precisely, if  $T$  is a fundamental domain and if  $x_0 \in T$  is some base point, then  $G$  is generated by the union of the compact subgroup  $G_{x_0}$  and the compact subset

$$S = \{s \in G : \mu(x, sy) > 0 \text{ for some } x, y \in T\}.$$

Indeed, consider  $g \in G$ . Since  $\mathcal{G}_\mu$  is connected, there exists a sequence  $x_1, \dots, x_n = gx_0$  in  $X$  such that  $\mu(x_{i-1}, x_i) > 0$  for all  $i \in \{1, \dots, n\}$ . Let  $y_1, \dots, y_n$  be a sequence in  $T$  and  $g_1, \dots, g_n$  a sequence in  $G$  such that  $x_i = g_i y_i$  ( $1 \leq i \leq n$ ). If  $s_i = g_{i-1}^{-1} g_i$  (with  $g_0 = 1$ ), we have

$$\mu(y_{i-1}, s_i y_i) = \mu(g_{i-1} y_{i-1}, g_i y_i) > 0$$

and therefore  $s_i \in S$  for all  $i \in \{1, \dots, n\}$ . Thus  $(\prod_{1 \leq i \leq n} s_i) y_n = x_n = gx_0$ , so that  $y_n = x_0$  and  $g^{-1} \prod_{1 \leq i \leq n} s_i$  is in the isotropy group  $G_{x_0}$ .

**Proposition 5.4.5** *Let  $X$  be a set and  $G$  a unimodular locally compact group acting on  $X$ . Let  $\mu$  be a reversible random walk on  $X$  and let  $\nu$  be a stationary measure for  $\mu$ . Assume that Properties (Pi) to (Pvi) above hold, and moreover that  $\mu$  is irreducible. The following statements are equivalent.*

- (i)  $G$  has Property (T);
- (ii) for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors, there exists  $\varepsilon > 0$  such that

$$\langle \Delta_\mu f, f \rangle \geq \varepsilon \|f\|^2$$

for all  $f \in \mathcal{E}_\pi^0(X)$ ;

(iii) there exists  $\varepsilon > 0$  such that, for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors,

$$\langle \Delta_\mu f, f \rangle \geq \varepsilon \|f\|^2$$

for all  $f \in \mathcal{E}_\pi^0(X)$ .

**Remark 5.4.6** The only implication which is used below is (iii)  $\implies$  (i). Observe that the implication (iii)  $\implies$  (ii) is obvious.

**Proof** We choose as above a fundamental domain  $T$  for the action of  $G$  on  $X$  as well as, for each  $y \in T$ , a set  $S_y$  of representatives of  $G/G_y$ . Recall that  $S$  is the subset of  $G$  of those  $s \in G$  for which  $\mu(x, sy) > 0$  for some  $x, y \in T$ .

Several of the equalities written below are of the following kind: given  $x \in X$  and a measurable function  $\alpha : G \longrightarrow \mathbb{R}_+^*$  such that  $\alpha(gh) = \alpha(g)$  for all  $g \in G$  and  $h \in G_x$ , we have  $\frac{1}{|G_x|} \int_G \alpha(g) dg = \sum_{g \in S_x} \alpha(g)$ .

Let us show that (ii) implies (i). Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  without non-zero invariant vector. Let  $\varepsilon > 0$  be a positive number for which (ii) holds. Choose a unit vector  $\xi \in \mathcal{H}$  and define  $f \in \mathcal{E}_\pi^0(X)$  by

$$f(x) = \pi(g)\xi \quad \text{where } x = gy, y \in T, g \in S_y;$$

in particular,  $f(y) = \xi$  for all  $y \in T$ . Then  $\|f\|^2 = \sum_{y \in T} \frac{\nu(y)}{|G_y|}$ . Using Proposition 5.4.3.iv, we have

$$\begin{aligned} \varepsilon \|f\|^2 &\leq \langle \Delta_\mu f, f \rangle = \frac{1}{2} \sum_{x \in T, y \in T, g \in S_x} \|\xi - \pi(g)\xi\|^2 \frac{\nu(y)\mu(y, gx)}{|G_y|} \\ &= \frac{1}{2} \sum_{x \in T, y \in T} \frac{1}{|G_x|} \int_G \|\xi - \pi(g)\xi\|^2 \frac{\nu(y)\mu(y, gx)}{|G_y|} dg \\ &= \frac{1}{2} \int_S \|\xi - \pi(g)\xi\|^2 \left( \sum_{x \in T, y \in T} \frac{\nu(y)\mu(y, gx)}{|G_x| |G_y|} \right) dg \\ &\leq \frac{1}{2} \max_{g \in S} \|\xi - \pi(g)\xi\|^2 \int_S \left( \sum_{x \in T, y \in T} \frac{\nu(y)\mu(y, gx)}{|G_x| |G_y|} \right) dg. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \int_S \left( \sum_{x \in T, y \in T} \frac{\nu(y) \mu(y, gx)}{|G_x| |G_y|} \right) dg &= \sum_{y \in T} \frac{\nu(y)}{|G_y|} \sum_{x \in T} \frac{1}{|G_x|} \int_G \mu(y, gx) dg \\ &= \sum_{y \in T} \frac{\nu(y)}{|G_y|} \sum_{x \in X} \mu(y, x) = \|f\|^2. \end{aligned}$$

It follows that

$$\frac{1}{2} \max_{g \in S} \|\xi - \pi(g)\xi\|^2 \geq \varepsilon,$$

namely that  $(S, \sqrt{2\varepsilon})$  is a Kazhdan pair for  $G$ . Since  $S$  is compact, the group  $G$  has Property (T).

Let us now show that (i) implies (iii). Let  $x_0 \in T$ . Recall from (Pvi) that  $G_{x_0} \cup S$  is a compact generating set for  $G$ . Let  $\kappa > 0$  be such that  $(G_{x_0} \cup S, \kappa)$  is a Kazhdan pair for  $G$ . Let

$$\begin{aligned} \delta &= \min \left\{ \mu(y, z) \frac{\nu(y)}{|G_y|} : y \in T, z \in X, \mu(y, z) > 0 \right\} \\ &= \min \left\{ \mu(y, z) \frac{\nu(y)}{|G_y|} : y, z \in X, \mu(y, z) > 0 \right\}. \end{aligned}$$

Observe that  $\delta > 0$ , since  $\mu$  has finite range. Let

$$N = \max \{d(y, x_0) + d(x_0, sy) : y \in T, s \in G_{x_0} \cup S\},$$

where  $d$  is the distance on  $X$  associated to the graph  $\mathcal{G}_\mu$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  without non-zero invariant vectors. We claim that, for every  $f \in \mathcal{E}_\pi^0(X)$ , we have

$$\langle \Delta_\mu f, f \rangle \geq \frac{1}{2N^2} \delta^2 \kappa^2 \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right)^{-2} \|f\|^2.$$

Indeed, assume by contradiction that there exists  $f \in \mathcal{E}_\pi^0(X)$  with  $\|f\|^2 = 1$  such that

$$\langle \Delta_\mu f, f \rangle < \frac{1}{2N^2} \delta^2 \kappa^2 \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right)^{-2}.$$

Since

$$\sum_{y \in T} \|f(y)\|^2 \frac{\nu(y)}{|G_y|} = 1,$$

there exists  $y_0 \in T$  such that

$$\|f(y_0)\|^2 \geq \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right)^{-1}.$$

Fix  $s \in G_{x_0} \cup S$ . Choose paths of minimal length between  $y_0$  and  $x_0$  and between  $x_0$  and  $sy_0$ , that is, choose  $z_0, \dots, z_n \in X$  and  $w_0, \dots, w_m \in X$  such that

$$z_0 = y_0, z_n = x_0, n = d(y_0, x_0), \quad \text{and} \quad \mu(z_i, z_{i+1}) > 0$$

for  $0 \leq i \leq n-1$  and

$$w_0 = x_0, w_m = sy_0, m = d(x_0, sy_0), \quad \text{and} \quad \mu(w_i, w_{i+1}) > 0$$

for  $0 \leq i \leq m-1$ . By the choice of  $\delta$ , we have

$$\begin{aligned} \|f(y_0) - f(sy_0)\| \delta &\leq \|f(y_0) - f(x_0)\| \delta + \|f(x_0) - f(sy_0)\| \delta \\ &\leq \sum_{i=0}^{n-1} \|f(z_i) - f(z_{i+1})\| \mu(z_i, z_{i+1}) \frac{\nu(z_i)}{|G_{z_i}|} \\ &\quad + \sum_{i=0}^{m-1} \|f(w_i) - f(w_{i+1})\| \mu(w_i, w_{i+1}) \frac{\nu(w_i)}{|G_{w_i}|}. \end{aligned}$$

Each of the terms on the right-hand side is equal to a term of the form

$$\|f(y) - f(z)\| \mu(y, z) \frac{\nu(y)}{|G_y|}$$

for some  $y \in T, z \in X$ . Since  $n + m \leq N$ , it follows that

$$\|f(y_0) - f(sy_0)\| \delta \leq N \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\| \mu(y, z) \frac{\nu(y)}{|G_y|}.$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
 & \left( \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\| \mu(y, z) \frac{\nu(y)}{|G_y|} \right)^2 \\
 & \leq \left( \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu(y, z) \frac{\nu(y)}{|G_y|} \right) \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \sum_{z \in X} \mu(y, z) \right) \\
 & = \left( \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu(y, z) \frac{\nu(y)}{|G_y|} \right) \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right).
 \end{aligned}$$

Together with Proposition 5.4.3.iv, this implies

$$\|f(y_0) - f(sy_0)\|^2 \delta^2 \leq 2N^2 \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right) \langle \Delta_\mu f, f \rangle.$$

Since  $\langle \Delta_\mu f, f \rangle < \frac{1}{2N^2} \delta^2 \kappa^2 \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right)^{-2}$ , it follows that

$$\begin{aligned}
 \|f(y_0) - \pi(s)f(y_0)\|^2 &= \|f(y_0) - f(sy_0)\|^2 \\
 &< \kappa^2 \left( \sum_{y \in T} \frac{\nu(y)}{|G_y|} \right)^{-1} \leq \kappa^2 \|f(y_0)\|^2
 \end{aligned}$$

for all  $s \in G_{x_0} \cup S$ . This is a contradiction, since  $\kappa$  is a Kazhdan constant for  $G_{x_0} \cup S$ . ■

**Remark 5.4.7** Assume that there exists  $\varepsilon > 0$  such that

$$\langle \Delta_\mu f, f \rangle \geq \varepsilon \|f\|^2$$

for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors and for every  $f \in \mathcal{E}_\pi^0(X)$ . The proof above shows that  $(S, \sqrt{2\varepsilon})$  is a Kazhdan pair, where  $S$  is the set of all  $s \in G$  for which  $\mu(x, sy) > 0$  for some  $x, y \in T$ .

In the next proposition, (ii) and (iii) display variants of the *Poincaré inequality*.

**Proposition 5.4.8** *Let  $X$  be a set and  $G$  a unimodular locally compact group acting on  $X$ . Let  $\mu$  be a reversible random walk on  $X$  and let  $\nu$  be a stationary*



measure for  $\mu$ . Assume that Properties (Pi)–(Pvi) above hold, and moreover that  $\mu$  is irreducible. Let  $k \geq 2$  be an integer. The following statements are equivalent.

(i)  $G$  has Property (T);

(ii) for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors, there exists a constant  $C_k < k$  such that

$$\langle \Delta_{\mu^{*k}} f, f \rangle \leq C_k \langle \Delta_{\mu} f, f \rangle$$

for every  $f \in \mathcal{E}_{\pi}^0(X)$ ;

(iii) there exists a constant  $C_k < k$  such that

$$\langle \Delta_{\mu^{*k}} f, f \rangle \leq C_k \langle \Delta_{\mu} f, f \rangle$$

for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors and for every  $f \in \mathcal{E}_{\pi}^0(X)$ .

**Proof** Let  $\pi$  be a unitary representation of  $G$  without non-zero invariant vectors. The positive operator  $\Delta_{\mu}$  acting on  $\mathcal{E}_{\pi}^0(X)$  has a positive square root  $\Delta_{\mu}^{\frac{1}{2}}$ . Assume that  $\mu$  is irreducible so that 0 is not an eigenvalue of  $\Delta_{\mu}$  by Proposition 5.4.3. A fortiori, 0 is not an eigenvalue of  $\Delta_{\mu}^{\frac{1}{2}}$ . As  $(\text{Im}(\Delta_{\mu}^{\frac{1}{2}}))^{\perp} = \text{Ker}(\Delta_{\mu}^{\frac{1}{2}})$ , the range of  $\Delta_{\mu}^{\frac{1}{2}}$  is dense in  $\mathcal{E}_{\pi}^0(X)$ . Choose an integer  $k \geq 2$  and set

$$C_k = \sup \left\{ \frac{\langle \Delta_{\mu^{*k}} f, f \rangle}{\langle \Delta_{\mu} f, f \rangle} : f \in \mathcal{E}_{\pi}^0(X), f \neq 0 \right\}.$$

We claim that  $C_k < k$  if and only if there exist  $\varepsilon > 0$  such that

$$\langle \Delta_{\mu} f, f \rangle \geq \varepsilon \|f\|^2, \quad \text{for all } f \in \mathcal{E}_{\pi}^0(X).$$

Indeed, since

$$\Delta_{\mu^{*k}} = I - M_{\mu^{*k}} = (I - M_{\mu})(I + M_{\mu} + \cdots + M_{\mu^{*(k-1)}}),$$

we have

$$\Delta_{\mu^{*k}} = \Delta_{\mu}^{\frac{1}{2}}(I + M_{\mu} + \cdots + M_{\mu^{*(k-1)}})\Delta_{\mu}^{\frac{1}{2}}.$$

It follows from the density of  $\text{Im}(\Delta_{\mu}^{\frac{1}{2}})$  that

$$C_k = \sup\{\langle (I + M_{\mu} + \cdots + M_{\mu^{*(k-1)}})f, f \rangle : f \in \mathcal{E}_{\pi}^0(X), \|f\| = 1\}.$$

Since  $M_\mu$  is self-adjoint, this means that

$$C_k = \sup_{\lambda \in \sigma(M_\mu)} (1 + \lambda + \cdots + \lambda^{k-1}),$$

where  $\sigma(M_\mu)$  denotes the spectrum of  $M_\mu$ . Hence,  $C_k < k$  if and only if  $1 \notin \sigma(M_\mu)$ , namely if and only if  $0 \notin \sigma(\Delta_\mu)$ , or equivalently if and only if there exists  $\varepsilon > 0$  such that

$$\langle \Delta_\mu f, f \rangle \geq \varepsilon \|f\|^2 \quad \text{for all } f \in \mathcal{E}_\pi^0(X).$$

Proposition 5.4.5 shows now that (i), (ii), and (iii) are equivalent. ■

**Remark 5.4.9** Assume that there exists a constant  $C < 2$  such that

$$\langle \Delta_{\mu*2} f, f \rangle \leq C \langle \Delta_\mu f, f \rangle$$

for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without non-zero invariant vectors and for every  $f \in \mathcal{E}_\pi^0(X)$ . Then  $\sqrt{2(2-C)}$  is a Kazhdan constant for the generating set  $S$  of  $G$  defined above. Indeed, the proof above shows that  $\sup_{\lambda \in \sigma(M_\mu)} (1 + \lambda) \leq C$ . Hence

$$\inf_{\lambda \in \sigma(\Delta_\mu)} \lambda \geq 2 - C,$$

that is,

$$\langle \Delta_\mu f, f \rangle \geq (2 - C) \|f\|^2, \quad \text{for all } f \in \mathcal{E}_\pi^0(X),$$

and the claim follows by Remark 5.4.7.

## 5.5 A local spectral criterion

Let  $X$  be a set and  $G$  a unimodular locally compact group acting on  $X$ . Let  $\mu$  be a reversible random walk on  $X$  and let  $\nu$  be a stationary measure for  $\mu$ . Assume that Properties (Pi)–(Pvi) from the previous section hold, and moreover that  $\mu$  is irreducible.

We will give a condition in terms of a family of locally defined random walks on  $X$  implying the Poincaré inequality (iii) from Proposition 5.4.8 and hence Property (T) for  $G$ .

Assume that, for every  $x \in X$ , there exist a finite subset  $X_x$  of  $X$ , a reversible random walk  $\mu_x$  on  $X_x$ , a probability measure  $\nu_x$  on  $X_x$  which is stationary for

$\mu_x$ , and a  $G$ -invariant positive measure  $\tau$  on  $X$  (a “weight function”) with the following properties:

- (i)  $\mu_{gx}(gy, gz) = \mu_x(y, z)$  and  $\nu_{gx}(gy) = \nu_x(y)$ ,
- (ii)  $\sum_{x \in X} \tau(x) \nu_x(y) \mu_x(y, z) = \nu(y) \mu(y, z)$ ,
- (iii)  $\sum_{x \in X} \tau(x) \nu_x(y) \nu_x(z) = \nu(y) \mu^{*2}(y, z)$ ,

for all  $x \in X$ ,  $y, z \in X_x$ , and  $g \in G$ , where we view all  $\mu_x$ ’s as measures on  $X \times X$  and all  $\nu_x$ ’s as measures on  $X$ .

Here is a first example; a second one appears below in connection with Zuk’s criterion (Theorem 5.6.1).

**Example 5.5.1** Let  $\mathcal{X}$  be a simplicial complex such that any vertex belongs to some edge and any edge belongs to some triangle. Let  $X$  denote the set of vertices of  $\mathcal{X}$ . For vertices  $y, z \in X$ , denote by  $\tau(y, z)$  the number of triangles containing  $y$  and  $z$  when  $y \neq z$ , and the number 0 when  $y = z$ ; denote by  $\tau(y)$  the number of *oriented* triangles containing  $y$ , so that  $\tau(y) = \sum_{z \in X} \tau(y, z)$ . Define a random walk  $\mu$  on  $X$  by

$$\mu(y, z) = \frac{\tau(y, z)}{\tau(y)} \quad \text{for all } y, z \in X.$$

Then  $\mu$  is reversible, with stationary measure  $\nu$  given by

$$\nu(y) = \tau(y) \quad \text{for all } y \in X.$$

Moreover  $\mu$  is irreducible if and only if  $\mathcal{X}$  is connected.

Choose now a vertex  $x \in X$ . Denote by  $X_x$  the set of vertices  $y$  of  $\mathcal{X}$  distinct from  $x$  such that  $x$  and  $y$  are vertices of a triangle in  $\mathcal{X}$ . Let  $\mathbb{E}_x$  be the set of pairs  $(y, z) \in X_x \times X_x$  such that  $x, y, z$  are the vertices of a triangle in  $\mathcal{X}$ . Thus  $\mathcal{G}_x = (X_x, \mathbb{E}_x)$  is a graph which is called the *link* of  $x$  in  $\mathcal{X}$ . The simple random walk  $\mu_x$  on  $\mathcal{G}_x$  is given by

$$\mu_x(y, z) = \begin{cases} \frac{1}{\tau(x, y)} & \text{if } (x, y, z) \text{ is a triangle in } \mathcal{X} \\ 0 & \text{otherwise.} \end{cases}$$

It is reversible, with stationary measure  $\nu_x$  given by

$$\nu_x(y) = \frac{\tau(x, y)}{\tau(x)}$$

for all  $y \in X_x$ .

Let us verify that Conditions (ii) and (iii) above are satisfied. Let  $y, z \in X$ . When  $y \neq z$ , we have for  $x \in X$

$$v_x(y)\mu_x(y, z) = \begin{cases} \frac{1}{\tau(x)} & \text{if } x, y, z \text{ are the vertices of a triangle in } \mathcal{X} \\ 0 & \text{otherwise.} \end{cases}$$

Hence  $\sum_{x \in X} \tau(x)v_x(y)\mu_x(y, z)$  is the number of triangles containing  $y$  and  $z$  when  $y \neq z$ , and is the number 0 otherwise. It follows that

$$(ii) \quad \sum_{x \in X} \tau(x)v_x(y)\mu_x(y, z) = v(y)\mu(y, z) \quad \text{for all } y, z \in X.$$

We have also

$$\tau(x)v_x(y)v_x(z) = v(x)\mu(x, y)\mu(x, z) = v(y)\mu(y, x)\mu(x, z)$$

and therefore

$$(iii) \quad \sum_{x \in X} \tau(x)v_x(y)v_x(z) = v(y) \sum_{x \in X} \mu(y, x)\mu(x, z) = v(y)\mu^{*2}(y, z).$$

If a group  $G$  acts by automorphisms of the simplicial complex  $\mathcal{X}$ , then Condition (i) is satisfied.

**Theorem 5.5.2 (Local criterion for Property (T))** *Let  $X, G, \mu, v, (\mu_x)_{x \in X}$ , and  $(v_x)_{x \in X}$  be as in the beginning of this section; we assume that Properties (Pi)–(Pvi) of Section 5.4 hold. For each  $x \in X$ , denote by  $\mathcal{G}_x = (X_x, \mathbb{E}_x)$  the finite graph associated to  $\mu_x$  and by  $\Delta_x$  the corresponding Laplace operator.*

*If  $\mathcal{G}_x$  is connected and if the smallest non-zero eigenvalue of  $\Delta_x$  satisfies  $\lambda_1(x) > 1/2$  for all  $x \in X$ , then  $G$  has Property (T).*

**Proof** Let  $T$  be a fundamental domain for the action of  $G$  on  $X$ . Set

$$\lambda = \min_{x \in T} \lambda_1(x).$$

By assumption,  $\lambda > 1/2$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  without non-zero invariant vectors. Let  $\Delta_\mu$  be the Laplacian on  $\mathcal{E}_\pi^0(X)$  as defined in the previous section. We claim that

$$(*) \quad \langle \Delta_{\mu^{*2}} f, f \rangle \leq \lambda^{-1} \langle \Delta_\mu f, f \rangle$$

for all  $f \in \mathcal{E}_\pi^0(X)$ . As  $\lambda^{-1} < 2$ , it will follow from Proposition 5.4.8 that  $G$  has Property (T). Observe that, by Proposition 5.4.3.iv, inequality (\*) is equivalent to the following inequality:

$$\begin{aligned}
 (**) \quad & \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu^{*2}(y, z) \frac{\nu(y)}{|G_y|} \\
 & \leq \lambda^{-1} \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu(y, z) \frac{\nu(y)}{|G_y|}.
 \end{aligned}$$

Let  $f \in \mathcal{E}_\pi^0(X)$ . The local information provided by Proposition 5.3.1 is

$$\begin{aligned}
 & \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \nu_x(w) \nu_x(z) \\
 & \leq \lambda^{-1} \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \mu_x(w, z) \nu_x(w),
 \end{aligned}$$

for all  $x \in X$ , since the graph  $\mathcal{G}_x$  is connected.

Using Conditions (i)–(iii), let us rewrite the left-hand side of Inequality (\*\*) as a linear combination with positive coefficients of quantities which occur in the left-hand side of the local information. For  $y \in T$  and  $z \in X$ , we have

$$\begin{aligned}
 & \|f(y) - f(z)\|^2 \mu^{*2}(y, z) \frac{\nu(y)}{|G_y|} \\
 &= \frac{1}{|G_y|} \sum_{x \in X} \|f(y) - f(z)\|^2 \tau(x) \nu_x(y) \nu_x(z) \\
 &= \frac{1}{|G_y|} \sum_{x \in T} \sum_{g \in S_x} \|f(y) - f(z)\|^2 \tau(gx) \nu_{gx}(y) \nu_{gx}(z) \\
 &= \frac{1}{|G_y|} \sum_{x \in T} \frac{1}{|G_x|} \int_G \|f(y) - f(z)\|^2 \tau(gx) \nu_{gx}(y) \nu_{gx}(z) dg \\
 &= \frac{1}{|G_y|} \sum_{x \in T} \frac{1}{|G_x|} \int_G \|f(g^{-1}y) - f(g^{-1}z)\|^2 \tau(x) \nu_x(g^{-1}y) \nu_x(g^{-1}z) dg
 \end{aligned}$$

by our assumptions (i) and (iii) above and by the  $G$ -equivariance of  $f$ . (For  $x \in X$ , recall that  $S_x$  denotes a system of representatives for the cosets in  $G/G_x$ .) Summing up this identity over all  $y \in T$  and  $z \in X$ , we obtain

$$\begin{aligned}
& \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu^{*2}(y, z) \frac{\nu(y)}{|G_y|} \\
&= \sum_{y \in T} \sum_{z \in X} \frac{1}{|G_y|} \sum_{x \in T} \frac{1}{|G_x|} \int_G \|f(g^{-1}y) - f(g^{-1}z)\|^2 \tau(x) \nu_x(g^{-1}y) \nu_x(g^{-1}z) dg \\
&= \sum_{x \in T} \sum_{y \in T} \frac{1}{|G_y| |G_x|} \int_G \left( \sum_{z \in X} \|f(g^{-1}y) - f(g^{-1}z)\|^2 \tau(x) \nu_x(g^{-1}y) \nu_x(g^{-1}z) \right) dg \\
&= \sum_{x \in T} \sum_{y \in T} \frac{1}{|G_y| |G_x|} \int_G \left( \sum_{z \in X} \|f(g^{-1}y) - f(z)\|^2 \tau(x) \nu_x(g^{-1}y) \nu_x(z) \right) dg \\
&= \sum_{x \in T} \sum_{y \in T} \sum_{z \in X} \frac{1}{|G_y| |G_x|} \int_G \|f(gy) - f(z)\|^2 \tau(x) \nu_x(gy) \nu_x(z) dg \\
&= \sum_{x \in T} \sum_{y \in T} \frac{1}{|G_x|} \sum_{z \in X} \sum_{g \in S_y} \|f(gy) - f(z)\|^2 \tau(x) \nu_x(gy) \nu_x(z) \\
&= \sum_{x \in T} \frac{1}{|G_x|} \sum_{w \in X} \sum_{z \in X} \|f(w) - f(z)\|^2 \tau(x) \nu_x(w) \nu_x(z) \\
&= \sum_{x \in T} \frac{\tau(x)}{|G_x|} \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \nu_x(w) \nu_x(z).
\end{aligned}$$

Similarly for the right-hand side of (\*\*), we have

$$\begin{aligned}
& \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu(y, z) \frac{\nu(y)}{|G_y|} \\
&= \sum_{x \in T} \frac{\tau(x)}{|G_x|} \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \mu_x(w, z) \nu_x(w).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu^{*2}(y, z) \frac{\nu(y)}{|G_y|} \\
&= \sum_{x \in T} \frac{\tau(x)}{|G_x|} \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \nu_x(w) \nu_x(z)
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda^{-1} \sum_{x \in T} \frac{\tau(x)}{|G_x|} \sum_{w, z \in X_x} \|f(w) - f(z)\|^2 \mu_x(w, z) \nu_x(w) \\
&= \lambda^{-1} \sum_{y \in T} \sum_{z \in X} \|f(y) - f(z)\|^2 \mu(y, z) \frac{\nu(y)}{|G_y|}
\end{aligned}$$

as was to be shown. ■

**Remark 5.5.3** (i) Let  $T$  be a fundamental domain for the action of  $G$  on  $X$  and let  $S$  be as in Property (Pvi) of the last section. The proof above and Remark 5.4.9 show that  $\sqrt{2(2 - \lambda^{-1})}$  is a Kazhdan constant for  $S$ , where  $\lambda = \min_{x \in T} \lambda_1(x)$ .

(ii) The condition that  $\lambda_1(x) > 1/2$  for all  $x$  in the fundamental domain  $T$  is equivalent to the condition that  $\lambda_1(x) > 1/2$  for all  $x \in X$ . Indeed, every  $g \in G$  induces an isomorphism between the graphs  $\mathcal{G}_x$  and  $\mathcal{G}_{gx}$ , for any  $x \in X$ .

We apply now the local criterion 5.5.2 to a group acting on a simplicial complex as in Example 5.5.1.

**Theorem 5.5.4 (Local criterion for groups acting on a simplicial complex)**

*Let  $\mathcal{X}$  be a simplicial complex such that every vertex belongs to some edge and every edge belongs to some triangle. Assume that, for each  $x \in X$ , the link  $\mathcal{G}_x$  of  $x$  is connected and that the smallest non-zero eigenvalue of the Laplacian corresponding to the simple random walk on  $\mathcal{G}_x$  satisfies  $\lambda_1(x) > 1/2$ .*

*Let  $G$  be a unimodular locally compact group with a continuous, proper and cofinite action on  $X$ . Then  $G$  has Property (T).*

**Remark 5.5.5** (i) The condition  $\lambda_1(x) > 1/2$  cannot be improved. Indeed, let  $\mathcal{X}$  be the simplicial complex induced by the tiling of the Euclidean plane by equilateral triangles. The link  $\mathcal{G}_x$  at each point  $x \in X$  is a 6-cycle and has  $\lambda_1(x) = 1/2$  (Exercise 5.8.3). On the other hand, the group  $\mathbb{Z}^2$  acts freely and transitively on  $X$ , and does not have Property (T).

(ii) The assumption that  $G$  is unimodular is necessary; see Remark 5.7.9 below.

(iii) Theorem 5.5.4 generalises [BalSw-97, Corollary 1], where  $G$  is assumed to be discrete.

## 5.6 Zuk's criterion

Let  $\Gamma$  be a group generated by a finite set  $S$ . We assume that the group unit 1 does *not* belong to  $S$  and that  $S$  is symmetric, that is,  $S^{-1} = S$ . The graph

$\mathcal{G}(S)$  associated to  $S$  has vertex set  $S$  and edge set the pairs  $(s, t) \in S \times S$  such that  $s^{-1}t \in S$ , as in [Zuk-03]. Observe that  $(s, t, s^{-1}t) \in S^3$  if and only if  $(s^{-1}, s^{-1}t, t) \in S^3$ , so that  $(s, t)$  is an edge if and only if  $(s^{-1}, s^{-1}t)$  is an edge; in particular,  $\deg(s) = \deg(s^{-1})$ .

Let  $\mu_S$  be the simple random walk on  $\mathcal{G}(S)$  as in Example 5.1.1, that is, for  $s, t \in S$ ,

$$\mu_S(s, t) = \begin{cases} 1/\deg(s) & \text{if } t^{-1}s \in S \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$d = \sum_{s \in S} \deg(s).$$

We assume from now on that  $d \geq 1$ . The probability measure  $\nu_S$ , defined by

$$\nu_S(s) = \frac{\deg(s)}{d}, \quad s \in S,$$

is stationary for  $\mu_S$ .

Define now a left invariant random walk  $\mu$  on  $X = \Gamma$  by

$$\mu(\gamma, \gamma') = \begin{cases} \deg(\gamma^{-1}\gamma')/d & \text{if } \gamma^{-1}\gamma' \in S \\ 0 & \text{otherwise,} \end{cases}$$

where  $\deg(s)$  refers to the degree of  $s \in S$  in the graph  $\mathcal{G}(S)$ . Observe that  $\mu$  is symmetric, so that  $\nu = 1$  is a stationary measure for  $\mu$ . The graph  $\mathcal{G}_\mu$  associated to  $\mu$  is the Cayley graph  $\mathcal{G}(\Gamma, S)$  of  $G$  with respect to  $S$ . In particular,  $\mathcal{G}_\mu$  is connected.

For every  $\gamma \in \Gamma$ , define a random walk  $\mu_\gamma$  on  $X_\gamma = \gamma S$  by

$$\mu_\gamma(\gamma s, \gamma t) = \mu_S(s, t)$$

and a stationary measure  $\nu_\gamma$  by

$$\nu_\gamma(\gamma s) = \nu_S(s),$$

for all  $s, t \in S$ .

We verify that Conditions (i), (ii) and (iii) from the beginning of Section 5.5 are satisfied for the choice  $\tau = 1$ .



This is obvious for the invariance Condition (i). As to Condition (ii), observe first that

$$\sum_{x \in \Gamma} v_x(1) \mu_x(1, \gamma) = 0 \quad \text{and} \quad \mu(1, \gamma) = 0$$

if  $\gamma \notin S$ . For  $\gamma \in S$ , we have

$$\begin{aligned} \sum_{x \in \Gamma} v_x(1) \mu_x(1, \gamma) &= \sum_{x \in \Gamma} v_S(x^{-1}) \mu_S(x^{-1}, x^{-1} \gamma) \\ &= \sum_{s \in S} v_S(s^{-1}) \mu_S(s^{-1}, s^{-1} \gamma) \\ &= \sum_{s \in S: s^{-1} \gamma \in S} v_S(s^{-1}) \mu_S(s^{-1}, s^{-1} \gamma) \\ &= \sum_{s \in S: s^{-1} \gamma \in S} \frac{\deg(s^{-1})}{d} \frac{1}{\deg(s^{-1})} \\ &= \frac{1}{d} \# \{s \in S : s^{-1} \gamma \in S\} = \frac{1}{d} \deg(\gamma) \\ &= \mu(1, \gamma). \end{aligned}$$

The invariance Condition (i) shows that

$$\sum_{x \in \Gamma} v_x(\gamma) \mu_x(\gamma, \gamma') = \mu(\gamma, \gamma')$$

for all  $\gamma, \gamma' \in \Gamma$ .

Let us check Condition (iii). For every  $\gamma \in \Gamma$ , we have

$$\begin{aligned} \sum_{x \in \Gamma} v_x(1) v_x(\gamma) &= \sum_{s \in S} v_S(s^{-1}) v_S(s^{-1} \gamma) \\ &= \sum_{s \in S: s^{-1} \gamma \in S} v_S(s^{-1}) v_S(s^{-1} \gamma) \\ &= \sum_{s \in S: s^{-1} \gamma \in S} \frac{\deg(s^{-1})}{d} \frac{\deg(s^{-1} \gamma)}{d} \\ &= \sum_{s \in S: s^{-1} \gamma \in S} \mu(1, s) \mu(s, \gamma) \end{aligned}$$

$$\begin{aligned}
&= \sum_{x \in \Gamma} \mu(1, x) \mu(x, \gamma) \\
&= \mu^{*2}(1, \gamma).
\end{aligned}$$

As above, it follows that

$$\sum_{x \in \Gamma} v_x(\gamma) v_x(\gamma') = \mu^{*2}(\gamma, \gamma')$$

for all  $\gamma, \gamma' \in \Gamma$ , that is, Condition (iii) is satisfied.

The following result which is [Zuk–03, Theorem 1] is now a consequence of the local criterion Theorem 5.5.2; the claim about the Kazhdan constant follows from Remark 5.5.3.i.

**Theorem 5.6.1 (Zuk’s criterion)** *Let  $\Gamma$  be a group generated by a finite symmetric set  $S$  with  $1 \notin S$ . Let  $\mathcal{G}(S)$  be the graph associated to  $S$ . Assume that  $\mathcal{G}(S)$  is connected and that the smallest non-zero eigenvalue of the Laplace operator  $\Delta_{\mu_S}$  corresponding to the simple random walk on  $\mathcal{G}(S)$  satisfies*

$$\lambda_1(\mathcal{G}(S)) > \frac{1}{2}.$$

*Then  $\Gamma$  has Property (T) and*

$$\sqrt{2 \left( 2 - \frac{1}{\lambda_1(\mathcal{G}(S))} \right)}$$

*is a Kazhdan constant for  $S$ .*

**Remark 5.6.2** (i) The Kazhdan constant above is better than the constant

$$\frac{2}{\sqrt{3}} \left( 2 - \frac{1}{\lambda_1(\mathcal{G}(S))} \right),$$

obtained in [Zuk–03].

(ii) The condition  $\lambda_1(\mathcal{G}(S)) > 1/2$  cannot be improved. Indeed, let  $\Gamma = \mathbf{Z}$  with the set of generators  $S = \{\pm 1, \pm 2\}$ . Then  $\lambda_1(\mathcal{G}(S)) = 1/2$  (Exercise 5.8.4) and  $\mathbf{Z}$  does not have Property (T).

(iii) Assuming that  $\mathcal{G}(S)$  is connected is not a strong restriction. Indeed, if  $S$  is a finite generating set of  $\Gamma$ , then the graph  $\mathcal{G}(S')$  associated to the generating set  $S' = S \cup (S^2 \setminus \{1\})$  is connected (Exercise 5.8.5).

(iv) Theorem 5.6.1 can be deduced from Theorem 5.5.4 by considering the Cayley complex associated to a presentation of  $\Gamma$  (Exercise 5.8.8).

(v) The spectral criterion implies much more than Property (T): it ensures that every isometric action of  $\Gamma$  on a complete CAT(0)-space, with the property that every tangent cone is isometric to a closed convex cone in a Hilbert space, has a globally fixed point; see Theorem 1.2 in [IzeNa–05].

Note that a Riemannian symmetric space of the non-compact type is a CAT(0)-space with that property. If  $\Gamma$  is a lattice in a semisimple Lie group  $G$  (e.g.  $\Gamma = SL_3(\mathbf{Z})$ ), then  $\Gamma$  acts properly isometrically on the Riemannian symmetric space associated to  $G$ . This means that one *cannot* prove Property (T) for such a group  $\Gamma$  by means of the spectral criterion.

## 5.7 Groups acting on $\tilde{A}_2$ -buildings

We will apply Theorem 5.5.4 to groups acting on  $\tilde{A}_2$ -buildings. In particular, we give a new proof of the fact that  $SL_3(\mathbf{K})$  has Property (T) for a non-archimedean local field  $\mathbf{K}$  (compare with Theorem 1.4.15). Concerning basic facts on buildings, we refer to [Brown–89] and [Ronan–89].

**Definition 5.7.1** A *projective plane* consists of a set  $P$  of “points”, a set  $L$  of “lines”, and an *incidence relation* between points and lines, with the following properties:

- (i) given two distinct points, there is a unique line incident with them,
- (ii) given two distinct lines, there is a unique point incident with them,
- (iii) each point is incident with at least three lines, and each line is incident with at least three points.

The *incidence graph*  $\mathcal{G}_{(P,L)}$  of a projective plane  $(P, L)$  is defined as follows:

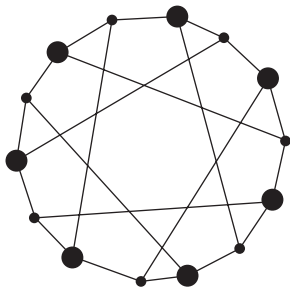
- the set of vertices of  $\mathcal{G}_{(P,L)}$  is the disjoint union  $P \coprod L$ ,
- the set of edges of  $\mathcal{G}_{(P,L)}$  is the set of pairs  $(p, \ell) \in P \times L$  and  $(\ell, p) \in L \times P$  such that  $p$  and  $\ell$  are incident.

Observe that the graph  $\mathcal{G}_{(P,L)}$  is connected, by Properties (i), (ii), (iii) above, and that it is bipartite.

**Remark 5.7.2** Let  $(P, L)$  be a finite projective plane. It is known that there exists an integer  $q \geq 2$ , called the *order* of  $(P, L)$ , such that each point is incident with exactly  $q + 1$  lines and each line is incident with exactly  $q + 1$  points. Therefore, the degree of each vertex of  $\mathcal{G}_{(P,L)}$  is  $q + 1$ . Moreover,

$$\#P = \#L = q^2 + q + 1.$$

For all this, see [HugPi–73].

Figure 5.1 The incidence graph of  $\mathbf{P}^2(\mathbf{F}_2)$ 

**Example 5.7.3** The most familiar example of a projective plane is the projective plane  $\mathbf{P}^2(\mathbf{F})$  over a field  $\mathbf{F}$ . In this case,  $P$  the set of one-dimensional subspaces of the vector space  $\mathbf{F}^3$  and  $L$  is the set of two-dimensional subspaces of  $\mathbf{F}^3$ . The incidence relation is defined by inclusion. In case  $\mathbf{F}$  is a finite field, the order of  $\mathbf{P}^2(\mathbf{F})$  is  $q = \#\mathbf{F}$ . Figure 5.1 shows the incidence graph of the projective plane  $\mathbf{P}^2(\mathbf{F}_2)$ .

**Definition 5.7.4** An  $\tilde{A}_2$ -building is a two-dimensional contractible connected simplicial complex  $X$  such that the link of any vertex of  $X$  has the structure of the incidence graph of a finite projective plane.

**Example 5.7.5** Let  $\mathbf{K}$  be a non-archimedean local field, with absolute value  $x \mapsto |x|$  (see Section D.4). We briefly review the construction of the  $\tilde{A}_2$ -building  $X_{\mathbf{K}}$  associated to  $\mathbf{K}$ . The closed unit ball

$$\mathcal{O} = \{x \in \mathbf{K} : |x| \leq 1\}$$

is an open and compact subring of  $\mathbf{K}$  with unique maximal ideal

$$\wp = \{x \in \mathbf{K} : |x| < 1\}.$$

The quotient ring  $\mathbf{F} = \mathcal{O}/\wp$  is a field, called the *residual field* of  $\mathbf{K}$ . Since  $\mathcal{O}$  is compact and  $\wp$  is open in  $\mathcal{O}$ , the residual field  $\mathbf{F}$  is finite. Let  $\pi \in \wp \setminus \wp^2$ . Then  $\wp = \pi\mathcal{O}$ .

A lattice in  $V = \mathbf{K}^3$  is an  $\mathcal{O}$ -submodule  $L$  of  $V$  of rank three, that is,

$$L = \mathcal{O}v_1 + \mathcal{O}v_2 + \mathcal{O}v_3,$$

where  $\{v_1, v_2, v_3\}$  is a basis of  $V$ . Two lattices  $L$  and  $L'$  are equivalent if  $L' = xL$  for some  $x \in \mathbf{K} \setminus \{0\}$ .

The set of vertices of the building  $X_{\mathbf{K}}$  is the set of equivalence classes  $[L]$  of lattices  $L$  in  $V$ . Three distinct vertices  $x_1, x_2, x_3$  in  $X_{\mathbf{K}}$  determine a triangle if there exists lattices  $L_1 \in x_1, L_2 \in x_2, L_3 \in x_3$  such that

$$\pi L_1 \subsetneq L_3 \subsetneq L_2 \subsetneq L_1.$$

This implies that the link  $X_x$  of  $x \in X_{\mathbf{K}}$  consists of the vertices  $y$  for which there exist lattices  $L_1 \in x$  and  $L_2 \in y$  such that  $\pi L_1 \subsetneq L_2 \subsetneq L_1$ . Since  $L_1/\pi L_1 \cong (\mathcal{O}/\pi\mathcal{O})^3$ , consideration of the quotient mapping  $L_1 \rightarrow L_1/\pi L_1$  shows that the vertices in  $X_x$  are in bijection with the one-dimensional subspaces and the two-dimensional subspaces of  $\mathbf{F}^3$  and that the graph structure of  $X_x$  corresponds to the incidence graph of the projective plane associated to the residual field  $\mathbf{F} = \mathcal{O}/\pi\mathcal{O}$  of  $\mathbf{K}$ . In particular,  $X_x$  has  $2(q^2 + q + 1)$  vertices and every vertex in  $X_x$  is adjacent to  $q + 1$  vertices in  $X_x$ , where  $q = \#\mathbf{F}$ .

Next, we determine the spectrum of the Laplace operator of the incidence graph of a finite projective plane. The result is due to Feit and Higman [FeiHi-64].

**Proposition 5.7.6** *Let  $(P, L)$  be a finite projective plane of order  $q \geq 2$ . Let  $\mathcal{G}_{(P,L)}$  be the associated incidence graph. The eigenvalues of the Laplace operator  $\Delta$  associated to the standard random walk on  $\mathcal{G}_{P,L}$  are*

$$0, 1 - \frac{\sqrt{q}}{q+1}, 1 + \frac{\sqrt{q}}{q+1}, 2,$$

with multiplicities  $1, q^2 + q, q^2 + q, 1$ .

*In particular, the smallest non-zero eigenvalue of  $\Delta$  is  $1 - \frac{\sqrt{q}}{q+1} > 1/2$ .*

**Proof** Recall that  $P$  as well as  $L$  have  $n = q^2 + q + 1$  elements, so that  $\mathcal{G}_{(P,L)}$  has  $2n$  vertices. Recall also that every point  $p \in P$  (respectively, every line  $\ell \in L$ ) is incident with  $q + 1$  lines (respectively,  $q + 1$  points). With respect to the basis

$$\{\delta_p : p \in P\} \cup \{\delta_\ell : \ell \in L\},$$

$\Delta$  has the matrix

$$I - \frac{1}{q+1} \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},$$

where  $A$  is the  $(n \times n)$  matrix  $(a_{p\ell})_{(p,\ell) \in P \times L}$  with

$$a_{p\ell} = \begin{cases} 1 & \text{if } p \text{ is incident with } \ell \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to determine the eigenvalues of the matrix

$$B = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},$$

which is the so-called adjacency matrix of the graph  $\mathcal{G}_{(P,L)}$ . We have

$$B^2 = \begin{pmatrix} AA^t & 0 \\ 0 & A^tA \end{pmatrix}.$$

For two points  $p, p' \in P$ , the entry  $(p, p')$  of  $AA^t$  is equal to the number of lines incident with both  $p$  and  $p'$ . Similarly, for two lines  $\ell, \ell' \in L$ , the entry  $(\ell, \ell')$  of  $A^tA$  is equal to the number of points incident with both  $\ell$  and  $\ell'$ . Hence

$$AA^t = A^tA = \begin{pmatrix} q+1 & 1 & \cdots & \cdots & 1 \\ 1 & q+1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & q+1 & 1 \\ 1 & \cdots & \cdots & 1 & q+1 \end{pmatrix}.$$

The eigenvalues of  $AA^t = A^tA$  are  $(q+1)^2$ , with multiplicity 1, and  $q$ , with multiplicity  $n-1 = q^2 + q$ . Indeed, the vector  ${}^t(1, 1, \dots, 1)$  is an eigenvector with eigenvalue  $(q+1)^2$ , and the  $n-1$  linearly independent vectors

$${}^t(1, -1, 0, 0, \dots, 0), {}^t(1, 0, -1, 0, \dots, 0), \dots, {}^t(1, 0, 0, \dots, 0, -1)$$

are eigenvectors with eigenvalue  $q$ . Hence, the eigenvalues of  $B^2$  are

$$\begin{aligned} &(q+1)^2 \quad \text{with multiplicity } 2 \\ &q \quad \text{with multiplicity } 2(q^2 + q). \end{aligned}$$

It follows that the eigenvalues of  $B$  are contained in  $\{\pm(q+1), \pm\sqrt{q}\}$ . On the other hand, due to the special structure of the matrix  $B$  (the graph  $\mathcal{G}_{(P,L)}$  being bipartite), the spectrum of  $B$  is symmetric about 0. Indeed,

if  ${}^t(x_1, \dots, x_n, y_1, \dots, y_n)$  is an eigenvector of  $B$  with eigenvalue  $\lambda$ , it is straightforward to see that  ${}^t(x_1, \dots, x_n, -y_1, \dots, -y_n)$  is an eigenvector of  $B$  with eigenvalue  $-\lambda$ . Therefore,  $\{\pm(q+1), \pm\sqrt{q}\}$  is the spectrum of  $B$  and

$$\left\{0, 1 - \frac{\sqrt{q}}{q+1}, 1 + \frac{\sqrt{q}}{q+1}, 2\right\}$$

is the spectrum of  $\Delta$ , with multiplicities  $1, q^2 + q, q^2 + q, 1$ . ■

**Theorem 5.7.7** *Let  $X$  be an  $\tilde{A}_2$ -building and let  $G$  be a unimodular locally compact group acting on  $X$ . Assume that the stabilisers of the vertices of  $X$  are compact and open subgroups of  $G$  and that the quotient  $G \backslash X$  is finite. Then  $G$  has Property (T). Moreover,*

$$\sqrt{2 \frac{(\sqrt{q}-1)^2}{(\sqrt{q}-1)^2 + \sqrt{q}}}$$

*is a Kazhdan constant for the compact generating set  $S$  as in Property (Pvi) of the last section.*

**Proof** The first statement is an immediate consequence of Proposition 5.7.6 and Theorem 5.5.4. The second statement follows from Proposition 5.7.6 and Remark 5.5.3. ■

**Example 5.7.8** Using the previous theorem, we obtain a new proof of Property (T) for  $SL_3(\mathbf{K})$ , when  $\mathbf{K}$  is a non-archimedean local field. Indeed, the natural action of  $GL_3(\mathbf{K})$  on the set of all lattices in  $\mathbf{K}^3$  is transitive. This gives rise to a transitive action of  $GL_3(\mathbf{K})$  on the  $\tilde{A}_2$ -building  $X_{\mathbf{K}}$  from Example 5.7.5. There are three  $SL_3(\mathbf{K})$ -orbits in  $X_{\mathbf{K}}$ : the orbit of the equivalence class of the standard lattice  $L_0 = \mathcal{O}^3$  of  $\mathbf{K}^3$ , the orbit of the equivalence class of the lattice  $L_1 = g_1 L_0$  and the orbit of the equivalence class of the lattice  $L_2 = g_2 L_0$ , where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix} \quad \text{and} \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}.$$

The stabilisers of  $[L_0], [L_1]$  and  $[L_2]$  in  $SL_3(\mathbf{K})$  are the compact subgroups  $SL_3(\mathcal{O})$ ,  $g_1 SL_3(\mathcal{O}) g_1^{-1}$  and  $g_2 SL_3(\mathcal{O}) g_2^{-1}$ . This shows that the assumptions of Theorem 5.7.7 are fulfilled and  $SL_3(\mathbf{K})$  has Property (T).

**Remark 5.7.9** The subgroup  $B$  of all triangular matrices in  $SL_3(\mathbf{K})$  has the same orbits on  $X_{\mathbf{K}}$  as  $SL_3(\mathbf{K})$  and the stabilisers in  $B$  of vertices are open and

compact. However,  $B$  does not have Property (T), since it is solvable and non-compact (see Theorem 1.1.6). This shows that the unimodularity assumption of  $G$  in Theorem 5.7.7 is necessary.

**Example 5.7.10** A (discrete) group is called an  $\tilde{A}_2$ -group if it acts freely and transitively on the vertices of an  $\tilde{A}_2$ -building, and if it induces a cyclic permutation of the type of the vertices. These groups were introduced and studied in [CaMSZ–93]. Some  $\tilde{A}_2$ -groups, but not all, can be embedded as cocompact lattices in  $PGL_3(\mathbf{K})$  for a non-archimedean local field  $\mathbf{K}$ .

It was shown in [CaMIS–93] through a direct computation that an  $\tilde{A}_2$ -group  $\Gamma$  has Property (T) in the case where the underlying projective plane is associated to a finite field. Theorem 5.7.7 is a generalisation of this result. Moreover, the Kazhdan constant from Theorem 5.7.7 for the given generating set  $S$  of  $\Gamma$  coincides with the one found in [CaMIS–93]; as shown in [CaMIS–93], this is the *optimal* Kazhdan constant for  $S$  in this case.

**Example 5.7.11** The following examples of groups described by their presentations and satisfying the spectral criterion in Theorem 5.5.4 are given in [BalSw–97]. Let  $G$  be a finite group, and let  $S$  be a set of generators of  $G$  with  $e \notin S$ . Assume that the Cayley graph  $\mathcal{G} = \mathcal{G}(G, S)$  of  $G$  has girth at least 6. (Recall that the girth of a finite graph  $\mathcal{G}$  is the length of a shortest closed circuit in  $\mathcal{G}$ .) Let  $\langle S \mid R \rangle$  be a presentation of  $G$ . Then the group  $\Gamma$  given by the presentation

$$\langle S \cup \{\tau\} \mid R \cup \{\tau^2\} \cup \{(s\tau)^3 : s \in S\} \rangle$$

acts transitively on the vertices of a CAT(0) two-dimensional simplicial complex, with finite stabilisers of the vertices, such that the link at every vertex of  $X$  is isomorphic to  $\mathcal{G}$ . Therefore, if  $\lambda_1(\mathcal{G}) > 1/2$ , then  $\Gamma$  has Property (T). Examples of finite groups  $G$  satisfying the conditions above are the groups  $PSL_2(\mathbf{F})$  over a finite field  $\mathbf{F}$  with a certain set of generators (see [Lubot–94] or [Sarna–90]).

## 5.8 Exercises

**Exercise 5.8.1** Let  $\mu$  be a random walk on a set  $X$ .

(i) Assume that  $\mu$  satisfies Conditions (\*) and (\*\*) of the beginning of Section 5.1. Show that  $\mu$  is reversible.

(ii) Assume moreover that  $\mu$  is irreducible. If  $v_1, v_2$  are two stationary measures for  $\mu$ , show that there exists a constant  $c > 0$  such that  $v_2(x) = cv_1(x)$  for all  $x \in X$ .



(iii) Let  $G$  be a group acting on  $X$ . Assume that  $\mu$  is reversible, irreducible, and  $G$ -invariant. If  $\nu$  is a stationary measure for  $\mu$ , show that  $\nu$  is  $G$ -invariant. [Statement (iii) shows that, in the hypothesis or Proposition 5.4.5, one of the hypothesis, namely the part of Property (Piv) concerning  $\nu$ , follows from the others.]

**Exercise 5.8.2** Show that the bound  $\|d\| \leq \sqrt{2}$  of Proposition 5.2.2 is optimal.

[Hint: Compute  $d$  for the simple random walk on the regular connected graph with one geometric edge, namely on the graph  $\mathcal{G} = (X, \mathbb{E})$  with  $X = \{x, y\}$  and with  $\mathbb{E} = \{(x, y), (y, x)\}$ .]

**Exercise 5.8.3** For  $n \in \mathbf{N}$ , let  $\mathcal{G}_n$  be the Cayley graph of the finite cyclic group  $\mathbf{Z}/n\mathbf{Z}$  for the generating set  $\{[1], [-1]\}$ . Thus  $\mathcal{G}_n$  can be represented as an  $n$ -cycle in the plane.

Show that the eigenvalues of the corresponding Laplace operator are

$$\lambda_k = 1 - \cos(2\pi k/n), \quad k = 0, 1, \dots, n-1.$$

[Hint: Note that  $(1, \omega, \dots, \omega^{n-1})$  is an eigenvector of the Laplace operator for any  $n$ -th root of unity  $\omega$ .]

**Exercise 5.8.4** Consider the generating set  $S = \{\pm 1, \pm 2\}$  of the group  $\mathbf{Z}$ , and let  $\mathcal{G}(S)$  be the finite graph associated to  $S$ .

(i) Compute the matrix of the Laplace operator  $\Delta$  with respect to the basis  $\{\delta_s : s \in S\}$ .

(ii) Show that eigenvalues of  $\Delta$  are 0,  $1/2$ ,  $3/2$ , 2. So, the smallest non-zero eigenvalue is  $\lambda_1(\mathcal{G}(S)) = 1/2$ .

(iii) Determine the random walk on  $\mathbf{Z}$  constructed before Zuk's criterion (Theorem 5.6.1) and observe that it is different from the simple random walk on the Cayley graph of  $\mathbf{Z}$  associated to  $S$ .

**Exercise 5.8.5** Let  $\Gamma$  be a group generated by a finite set  $S$ . Let  $S'$  be the finite generating set  $S \cup (S^2 \setminus \{e\})$ . Show that the graph  $\mathcal{G}(S')$  associated to  $S'$  is connected.

**Exercise 5.8.6** It is well-known that the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

generate the unimodular group  $SL_2(\mathbf{Z})$ . Let

$$S = \{-I, A, B, -A, -B, A^{-1}, B^{-1}, -A^{-1}, -B^{-1}\}.$$

- (i) Draw the graph  $\mathcal{G}(S)$  associated to  $S$ .
- (ii) Show that the smallest non-zero eigenvalue of  $\Delta$  is  $1/2$ .

**Exercise 5.8.7** Consider a simplicial complex  $\mathcal{X}$  of dimension 2 containing six vertices  $a, b, c, x, y, t$  and six oriented triangles

$$\begin{array}{lll} (a, x, z) & (b, y, x) & (c, z, y) \\ (a, z, x) & (b, x, y) & (c, y, z). \end{array}$$

Check that the conditions of Example 5.5.1 are satisfied. Draw the link of  $\mathcal{G}_x$  and compute the spectrum of the Laplace operator corresponding to the simple random walk on  $\mathcal{G}_x$ .

**Exercise 5.8.8** Deduce Theorem 5.6.1 from Theorem 5.5.4 by considering the Cayley complex associated to a presentation  $\langle S \mid R \rangle$  of  $\Gamma$  (see [BriHa–99, Chapter I, 8A.2]).

**Exercise 5.8.9** Consider a family of *expander graphs*, namely a sequence  $(\mathcal{G}_j = (X_j, \mathbb{E}_j))_{j \geq 1}$  of finite connected graphs such that  $\lim_{j \rightarrow \infty} \#X_j = \infty$ , all regular of the same degree, say  $k \geq 3$ , and such that the eigenvalues  $\lambda_1(\mathcal{G}_j)$  of the corresponding simple random walks are bounded below by a positive constant (see Section 6.1).

Consider moreover a dimension  $n \geq 1$  and a sequence  $f_j : X_j \rightarrow \mathbf{R}^n$  of mappings which are 1-Lipschitz, that is, such that  $\|f_j(u) - f_j(v)\| \leq 1$  for all  $(u, v) \in \mathbb{E}_j$  and  $j \geq 1$ .

Show that there exists a sequence  $(x_j, y_j)_{j \geq 1}$ , with  $x_j, y_j \in X_j$  for all  $j \geq 1$ , such that the combinatorial distance between  $x_j$  and  $y_j$  in  $\mathcal{G}_j$  is unbounded, and such that the  $\mathbf{R}^n$ -distance between  $f_j(x_j)$  and  $f_j(y_j)$  is bounded.

[Hint: See Example 5.3.3 and [Ghys–04, pages 916–22].]

## 6

### Some applications of Property (T)

In this chapter, we present some applications of Property (T). The first application is the by now classical construction of expander graphs, due to G. Margulis [Margu-73]. We then turn to the role of Property (T) in ergodic theory. It is first shown that Property (T) for a locally compact group  $G$  implies the existence of a spectral gap and provides an estimate for convolution operators associated to ergodic actions of  $G$ . We discuss the Schmidt–Connes–Weiss dynamical characterisation of Property (T) in terms of ergodic measure-preserving actions. Next, we study Property (T) in the context of orbit equivalence; in particular, we prove the Furman–Popa result showing that Property (T) is invariant under measure equivalence. In the last section, we show how Property (T) is used for a solution of the Banach–Ruziewicz problem for  $n \geq 5$ ; this problem asks whether the Lebesgue measure is the unique rotation-invariant finitely additive probability measure on the Lebesgue measurable subsets of the unit sphere in  $\mathbf{R}^n$ .

#### 6.1 Expander graphs

We introduce families of expander graphs defined by the existence of a uniform bound for their expanding constants. We describe two examples of such families.

##### Expander graphs

Let  $\mathcal{G} = (V, \mathbb{E})$  be a locally finite graph, where  $V$  is the set of vertices and  $\mathbb{E}$  the set of edges of  $\mathcal{G}$ . As in Example 5.1.1, we assume that the edge set  $\mathbb{E}$  is a subset of  $X \times X$  which contains  $\bar{e} = (y, x)$  whenever it contains  $e = (x, y)$ ; in particular,  $\mathcal{G}$  has no multiple edge but is allowed to have loops.

For a subset  $A$  of  $V$ , the *boundary*  $\partial A$  of  $A$  is the set of vertices in  $V \setminus A$  which are adjacent to a vertex in  $A$ , that is,

$$\partial A = \{y \in V \setminus A : y \sim x \text{ for some } x \in A\}.$$

The *expanding constant* or *isoperimetric constant* of  $\mathcal{G}$  is the positive real number

$$h(\mathcal{G}) = \min \left\{ \frac{\#\partial A}{\min\{\#A, \#(V \setminus A)\}} : A \subsetneq V, 0 < \#A < \infty \right\}.$$

**Remark 6.1.1** If we view the graph  $\mathcal{G}$  as modelling a communication network, then the expanding constant  $h(\mathcal{G})$  is a way to measure the spreading of information through the network.

**Example 6.1.2** The proof of the following claims is left as Exercise 6.5.1.

(i) Let  $\mathcal{G} = C_n$  be the cycle with  $n$  vertices. Then

$$h(C_n) = 2 / \left\lceil \frac{n}{2} \right\rceil \approx \frac{4}{n}.$$

(ii) Let  $\mathcal{G} = K_n$  be the complete graph with  $n$  vertices. Then

$$h(K_n) = 1.$$

(iii) Let  $T_k$  denote the regular tree of degree  $k \geq 2$ . Then

$$h(T_k) = k - 2.$$

**Definition 6.1.3** Let  $k \in \mathbf{N}$  be a *fixed* integer and let  $\varepsilon > 0$  be a *constant*. A family  $\mathcal{G}_n = (V_n, \mathbb{E}_n)$  of finite connected graphs is a family of  $(k, \varepsilon)$ -*expanders* if

- (i)  $\lim_{n \rightarrow \infty} \#V_n = \infty$ ;
- (ii)  $\deg(x) = \#\{y \in V_n : y \sim x\} \leq k$  for all  $n \in \mathbf{N}$  and all  $x \in V_n$ ;
- (iii)  $h(\mathcal{G}_n) \geq \varepsilon$  for all  $n \in \mathbf{N}$ .

The constant  $\varepsilon$  is called an *expanding constant* for  $(\mathcal{G}_n)_n$ .

The *existence* of expander graphs is settled by the following proposition, which can be proved by elementary counting arguments (see [Lubot–94, Proposition 1.2.1]).

**Proposition 6.1.4** *Let  $k$  be an integer with  $k \geq 5$  and  $\varepsilon = 1/2$ . There exists a family of  $(k, \varepsilon)$ -expanders.*

**Remark 6.1.5** (i) Let  $\mathcal{G} = (V, \mathbb{E})$  be a connected  $k$ -regular finite graph; consider the simple random walk associated to  $\mathcal{G}$  (see Example 5.1.1). Let  $\Delta$  be the corresponding Laplace operator acting on  $\ell^2(V)$ . We have the following lower and upper bounds for the expanding constant of  $\mathcal{G}$  in terms of the smallest non-zero eigenvalue  $\lambda_1$  of  $\Delta$ :

$$\frac{\lambda_1}{2} \leq h(\mathcal{G}) \leq \sqrt{2k\lambda_1}.$$

For a proof, see [DaSaV–03, Theorem 1.2.3]; for the first inequality, see also Exercise 6.5.3.

These bounds are the discrete analogues of isoperimetric inequalities in the context of Riemannian manifolds proved by Cheeger and Buser (see [Chave–93]).

(ii) For a fixed integer  $k$ , let  $(\mathcal{G}_n)_n$  be a family of connected  $k$ -regular finite graphs with  $\lim_{n \rightarrow \infty} \#V_n = \infty$ . For each  $n$ , let  $\lambda_1^{(n)}$  denote the smallest non-zero eigenvalue of the Laplace operator corresponding to the standard random walk on  $\mathcal{G}_n$ . In view of (i),  $(\mathcal{G}_n)_n$  is a family of expanders if and only if there exists a positive uniform lower bound for the  $\lambda_1^{(n)}$ 's.

We will give an explicit construction of a family of expander graphs using Property (T). As a preparation, we review the construction of the Schreier graph associated to a subgroup  $H$  of a group  $\Gamma$ .

Let  $\Gamma$  be finitely generated group, and let  $S$  be a finite generating set of  $\Gamma$  with  $S^{-1} = S$ . Let  $\Gamma$  act on a set  $V$ . Define the *Schreier graph*  $\mathcal{G}(V, S)$  as follows:

- the set of vertices is  $V$ ,
- $(x, y) \in V \times V$  is an edge if and only if  $y = sx$  for some  $s \in S$ .

Observe that, for every vertex  $x \in V$ , we have  $\deg(x) \leq k$ , where  $k = \#S$ . We assume further that the graph  $\mathcal{G}(V, S)$  is connected; this is the case if and only if  $\Gamma$  acts transitively on  $V$ .

**Remark 6.1.6** If  $H$  is a subgroup of  $\Gamma$ , we consider the natural action of  $\Gamma$  on  $\Gamma/H$ . The corresponding Schreier  $\mathcal{G}(\Gamma/H, S)$  is connected. Every connected Schreier graph  $\mathcal{G}(V, S)$  is of the form  $\mathcal{G}(\Gamma/H, S)$  for a subgroup  $H$  of  $G$ . Indeed, if we choose a base point  $v_0 \in V$ , the  $\Gamma$ -space  $V$  can be identified with  $\Gamma/H$ , where  $H$  is the stabiliser of  $v_0$  in  $\Gamma$ .

With the previous notation, let  $\pi_V$  be the quasi-regular representation of  $\Gamma$  on  $\ell^2(V)$ , thus

$$\pi_V(\gamma)\xi(x) = \xi(\gamma^{-1}x), \quad \xi \in \ell^2(V), x \in V, \gamma \in \Gamma.$$

From now on, we assume that  $V$  is *finite*. Then the constant functions on  $V$  belong to  $\ell^2(V)$  and the subspace

$$\ell_0^2(V) = \left\{ \xi \in \ell^2(V) : \sum_{x \in V} \xi(x) = 0 \right\} = \{1_V\}^\perp$$

is  $\Gamma$ -invariant. The corresponding representation  $\pi_V^0$  of  $\Gamma$  on  $\ell_0^2(V)$  has no non-zero invariant vectors.

The following crucial lemma establishes a link between the expanding constant of the graph  $\mathcal{G}(V, S)$  and the Kazhdan constant  $\kappa(\Gamma, S, \pi_V^0)$  associated to  $S$  and  $\pi_V^0$  (see Remark 1.1.4). Recall that

$$\kappa(\Gamma, S, \pi_V^0) = \inf \left\{ \max_{s \in S} \|\pi_V(s)\xi - \xi\| : \xi \in \ell_0^2(V), \|\xi\| = 1 \right\}.$$

**Lemma 6.1.7** *With the previous notation, we have*

$$h(\mathcal{G}(V, S)) \geq \frac{\kappa(\Gamma, S, \pi_V^0)^2}{4}.$$

**Proof** Let  $A$  be a proper non-empty subset of  $V$ . We have to show that there exists a unit vector  $\xi \in \ell_0^2(V)$  such that

$$\frac{\#\partial A}{\min\{\#A, \#(V \setminus A)\}} \geq \frac{1}{4} \max_{s \in S} \|\pi_V(s)\xi - \xi\|^2.$$

Set  $B = V \setminus A$ ,  $n = \#V$ ,  $a = \#A$ , and  $b = \#B = n - a$ . Let  $f : V \rightarrow \mathbf{C}$  be defined by

$$f(x) = \begin{cases} b & \text{if } x \in A \\ -a & \text{if } x \in B. \end{cases}$$

Then  $f \in \ell_0^2(V)$  and

$$\|f\|^2 = ab^2 + ba^2 = nab.$$

Fix  $s \in S$ . We have

$$f(s^{-1}x) - f(x) = \begin{cases} b + a & \text{if } s^{-1}x \in A \text{ and } x \in B \\ -a - b & \text{if } s^{-1}x \in B \text{ and } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\|\pi_V(s)f - f\|^2 = (a + b)^2 \#E_s = n^2 \#E_s,$$

where

$$E_s = (B \cap sA) \cup (A \cap sB) = (B \cap sA) \cup s(B \cap s^{-1}A).$$

It is clear that  $\#E_s \leq 2\#\partial A$ . Hence, with

$$\xi = \frac{1}{\sqrt{nab}}f,$$

we have  $\|\xi\| = 1$  and

$$\begin{aligned} \#\partial A &\geq \frac{\#E_s}{2} = \frac{1}{2n^2} \|\pi_V(s)f - f\|^2 \\ &= \frac{nab}{2n^2} \|\pi_V(s)\xi - \xi\|^2 = \frac{ab}{2n} \|\pi_V(s)\xi - \xi\|^2. \end{aligned}$$

As

$$\min\{\#A, \#(V \setminus A)\} = \min\{a, b\} \leq \frac{2ab}{a+b} = \frac{2ab}{n},$$

we have

$$\frac{\#\partial A}{\min\{\#A, \#(V \setminus A)\}} \geq \frac{1}{4} \max_{s \in S} \|\pi_V(s)\xi - \xi\|^2$$

and this ends the proof. ■

**Theorem 6.1.8** *Let  $\Gamma$  be a group generated by a finite set  $S$  with  $S^{-1} = S$ . Let  $(H_n)_n$  be a sequence of subgroups of  $\Gamma$  of finite index with*

$$\lim_n \#(\Gamma/H_n) = \infty.$$

*Assume that there exists  $\varepsilon > 0$  such that, for every  $n$ , there is no  $(S, \varepsilon)$ -invariant vector in  $\ell_0^2(\Gamma/H_n)$ .*

*Then the family of the Schreier graphs  $\mathcal{G}(\Gamma/H_n, S)$  is a family of  $(k, \varepsilon^2/4)$ -expanders, where  $k = \#S$ .*

**Proof** Let  $n \in \mathbf{N}$ . Write  $\pi_n$  for  $\pi_{\Gamma/H_n}$ . Since there is no  $(S, \varepsilon)$ -invariant vector in  $\ell_0^2(\Gamma/H_n)$ , we have

$$\max_{s \in S} \|\pi_n(s)\xi - \xi\| \geq \varepsilon$$

for all  $\xi \in \ell_0^2(\Gamma/H_n)$  with  $\|\xi\| = 1$ . Hence,  $h(\mathcal{G}_n) \geq \varepsilon^2/4$ , by the previous lemma. ■

Let now  $\Gamma$  be a group with Property (T), and let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ . Recall that there exists  $\varepsilon > 0$  such that  $(S, \varepsilon)$  is a Kazhdan pair (Proposition 1.3.2.i). Lemma 6.1.7 shows that, for any subgroup of finite index  $H$  of  $\Gamma$ , we have

$$h(\mathcal{G}(\Gamma/H, S)) \geq \varepsilon^2/4$$

for the corresponding Schreier graph.

**Corollary 6.1.9** *Let  $\Gamma$  be a residually finite, infinite group with Property (T). Let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ , and let  $\varepsilon > 0$  be a Kazhdan constant for  $S$ .*

*For every decreasing sequence  $(H_n)_n$  of finite index subgroups of  $\Gamma$  with  $\bigcap_n H_n = \{1\}$ , the family of Schreier graphs  $(\mathcal{G}(\Gamma/H_n, S))_n$ , is a family of  $(k, \varepsilon^2/4)$ -expanders, where  $k = \#S$ .*

**Remark 6.1.10** Let  $\Gamma$  be a group with Property (T); let  $(S, \varepsilon)$  be as in the previous corollary. Let  $H$  be a subgroup of finite index in  $\Gamma$ . In view of Remark 6.1.5, we obtain a uniform lower bound for the smallest non-zero eigenvalue  $\lambda_1$  of the Laplace operator  $\Delta$  of the Schreier graph  $\mathcal{G}(\Gamma/H, S)$  of  $H$ . In fact, a direct proof yields the better bound

$$\lambda_1 \geq \frac{\varepsilon^2}{2}.$$

Indeed, let  $f \in \ell_0^2(\Gamma/H)$  be an eigenfunction of  $\Delta$  with  $\|f\| = 1$ . Denoting by  $V$  and  $\mathbb{E}$  the set of vertices and the set of edges of  $\mathcal{G}(\Gamma/H, S)$ , we have by Proposition 5.2.2.iv,

$$\begin{aligned} \lambda_1 &= \frac{1}{2} \sum_{(x,y) \in \mathbb{E}} |f(y) - f(x)|^2 \\ &= \frac{1}{2} \sum_{s \in S} \sum_{x \in V} |f(s^{-1}x) - f(x)|^2 \\ &= \frac{1}{2} \sum_{s \in S} \|\pi_V(s)f - f\|^2 \\ &\geq \frac{1}{2} \max_{s \in S} \|\pi_V(s)f - f\|^2 \\ &\geq \frac{1}{2} \varepsilon^2. \end{aligned}$$

Observe that this inequality, combined with the first inequality in Remark 6.1.5, gives another proof of Theorem 6.1.8.



### Examples of expander graphs

We describe two families of expander graphs: the first one is given by quotients of  $SL_3(\mathbf{Z})$  and is based on Property (T) for this group; the second one is Margulis' original construction from [Margu-73] and uses Property (T) of the pair  $(SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ .

**Example 6.1.11** Let  $\Gamma = SL_3(\mathbf{Z})$ . Then

$$S = \left\{ E_{ij}^{\pm 1} : 1 \leq i, j \leq 3, i \neq j \right\}$$

is a generating set of  $\mathcal{G}$ , where  $E_{ij}^{\pm 1} = E_{ij}(\pm 1)$  is the elementary matrix as before Lemma 1.4.6. By Theorems 4.2.5 and 4.1.3,

$$\varepsilon = 1/960 \geq 1/(20v_3(\mathbf{Z}))$$

is a Kazhdan constant for the set  $S$ .

For every prime integer  $p$ , let

$$\Gamma(p) = \{A \in \Gamma : A \equiv I \pmod{p}\},$$

that is,  $\Gamma(p)$  is the kernel of the surjective homomorphism  $SL_3(\mathbf{Z}) \rightarrow SL_3(\mathbf{Z}/p\mathbf{Z})$  given by reduction modulo  $p$ . (The  $\Gamma(p)$ 's are the so-called principal congruence subgroups.) Since  $\Gamma/\Gamma(p) \cong SL_3(\mathbf{Z}/p\mathbf{Z})$ , the subgroup  $\Gamma(p)$  has finite index

$$p^3(p^3 - 1)(p^2 - 1) \approx p^8.$$

The family of Schreier graphs  $(\mathcal{G}(\Gamma/\Gamma(p), S))_p$  is a family of regular  $(k, \varepsilon')$ -expanders with

$$k = 12 \quad \text{and} \quad \varepsilon' \approx \frac{1}{4}10^{-6}.$$

**Example 6.1.12** Let  $\Gamma = SL_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ . The set  $S$  consisting of the four matrices

$$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \quad \text{in} \quad SL_2(\mathbf{Z})$$

and the four vectors

$$\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \quad \text{in} \quad \mathbf{Z}^2$$

is a generating set for  $\Gamma$ . By Theorem 4.2.2, the pair  $(\Gamma, \mathbf{Z}^2)$  has Property (T) and  $\varepsilon = 1/10$  is a Kazhdan constant for  $S$ .

The group  $\Gamma$  acts by affine transformations on  $\mathbf{Z}^2$ . This induces an action of  $\Gamma$  on  $V_n = (\mathbf{Z}/n\mathbf{Z})^2$  for every  $n \in \mathbf{N}$ . Observe that this action is transitive. In the associated Schreier graph  $\mathcal{G}(V_n, S)$ , the neighbours of a vertex  $(x, y) \in V_n$  are the eight vertices

$$(x \pm y, x), \quad (x, x \pm y), \quad (x \pm 1, y), \quad (x, y \pm 1).$$

The family  $(\mathcal{G}(V_n, S))_n$  is a family of  $(k, \varepsilon')$ -expanders with

$$k = 8 \quad \text{and} \quad \varepsilon' = \frac{1}{400}.$$

**Remark 6.1.13** (i) For a variation on Example 6.1.12 above, see [GabGa–81]; see also [BeCuH–02].

(ii) Families of expander graphs which are in some sense optimal are Ramanujan graphs; for an account on these graphs, see [DaSaV–03].

In contrast to the previous results, as we now show, finite quotients of an *amenable* group never give rise to a family of expander graphs. This is a result from [LubWe–93].

**Theorem 6.1.14 (Lubotzky–Weiss)** *Let  $\Gamma$  be an amenable discrete group with a finite generating set  $S$  with  $S^{-1} = S$ . Let  $(H_n)_{n \geq 1}$  be a family of finite index subgroups  $H_n$  of  $\Gamma$  with  $\lim_n \#(\Gamma/H_n) = \infty$ .*

*The family of Schreier graphs  $(\mathcal{G}(\Gamma/H_n, S))_{n \geq 1}$  is not a family of expanders.*

**Proof** Let  $\varepsilon > 0$ . Since  $\Gamma$  is amenable, there exists, by Følner's Property (Section G.5), a non-empty finite subset  $A$  of  $\Gamma$  such that

$$\#(sA \Delta A) \leq \varepsilon \#A, \quad \text{for all } s \in S.$$

Let  $n \geq 1$  be such that  $\#(\Gamma/H_n) > 2\#A$ . Define a function  $\varphi : \Gamma/H_n \rightarrow \mathbf{N}$  by

$$\varphi(X) = \#(X \cap A) = \sum_{h \in H_n} \chi_A(xh), \quad \text{for all } X = xH_n \in \Gamma/H_n,$$

where  $\chi_A$  is the characteristic function of  $A$ . Observe that

$$\|\varphi\|_1 = \sum_{X \in \Gamma/H_n} \varphi(X) = \sum_{X \in \Gamma/H_n} \#(X \cap A) = \#A.$$

For  $s \in S$ , let  ${}_s\varphi$  denote the function defined by  ${}_s\varphi(X) = \varphi(sX)$ . Let  $T \subset \Gamma$  be a set of representatives for the coset space  $\Gamma/H_n$ . We have

$$\begin{aligned} \|{}_s\varphi - \varphi\|_1 &= \sum_{X \in \Gamma/H_n} |\varphi(sX) - \varphi(X)| = \sum_{x \in T} \left| \sum_{h \in H_n} \chi_A(sxh) - \chi_A(xh) \right| \\ &\leq \sum_{x \in T} \sum_{h \in H_n} |\chi_A(sxh) - \chi_A(xh)| = \sum_{\gamma \in \Gamma} |\chi_A(s\gamma) - \chi_A(\gamma)| \\ &= \sum_{\gamma \in \Gamma} |\chi_{s^{-1}A}(\gamma) - \chi_A(\gamma)| = \#(s^{-1}A \Delta A) \end{aligned}$$

so that

$$(*) \quad \|{}_s\varphi - \varphi\|_1 \leq \varepsilon \|\varphi\|_1.$$

For  $j \in \mathbf{N}$ , define

$$B_j = \{X \in \Gamma/H_n : \varphi(X) \geq j\} \subset \Gamma/H_n.$$

We have (see Lemma G.5.2)

$$\|\varphi\|_1 = \sum_{j \geq 1} \#B_j$$

as well as

$$\|{}_s\varphi - \varphi\|_1 = \sum_{j \geq 1} \#(s^{-1}B_j \Delta B_j), \quad \text{for all } s \in S.$$

For  $s \in S$ , set

$$J_s = \left\{ j \geq 1 : \#(s^{-1}B_j \Delta B_j) \geq \sqrt{\varepsilon} \#B_j \right\}.$$

Using (\*), we have

$$\begin{aligned} \sum_{j \in J_s} \#B_j &\leq \frac{1}{\sqrt{\varepsilon}} \sum_{j \in J_s} \#(s^{-1}B_j \Delta B_j) \\ &\leq \frac{1}{\sqrt{\varepsilon}} \sum_{j \geq 1} \#(s^{-1}B_j \Delta B_j) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{\varepsilon}} \|_s \varphi - \varphi\|_1 \\
&\leq \sqrt{\varepsilon} \|\varphi\|_1.
\end{aligned}$$

Without loss of generality, we can assume that  $\sqrt{\varepsilon} \#S < 1$ . Then

$$\sum_{s \in S} \sum_{j \in J_s} \#B_j \leq \sqrt{\varepsilon} \#S \|\varphi\|_1 < \sum_{j \geq 1} \#B_j.$$

It follows that there exists  $j_0 \geq 1$  which does not belong to any  $J_s$  for  $s \in S$ . For such an integer  $j_0$ , we have, for every  $s \in S$ ,

$$\#(s^{-1}B_{j_0} \triangle B_{j_0}) < \sqrt{\varepsilon} \#B_{j_0}$$

and therefore

$$\#\partial B_{j_0} < \sqrt{\varepsilon} \#B_{j_0}.$$

Observe that  $0 < \#B_{j_0} \leq \#A < \frac{\#(\Gamma/H_n)}{2}$ . Hence  $(\mathcal{G}(\Gamma/H_m, S))_m$  is not a family of  $(\#S, \sqrt{\varepsilon})$ -expanders. ■

## 6.2 Norm of convolution operators

Let  $G$  be a locally compact group with left Haar measure  $dx$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  without non-zero invariant vectors and  $\mu(x)dx$  an absolutely continuous probability measure on  $G$ . We give estimates of the norm of the associated “convolution” operator  $\pi(\mu) \in \mathcal{L}(\mathcal{H})$  in terms of Kazhdan constants.

Recall from Section F.4 that  $\pi$  extends to a  $*$ -representation, again denoted by  $\pi$ , of the Banach  $*$ -algebra  $L^1(G)$  of all integrable functions on  $G$ : for  $\mu$  in  $L^1(G)$ , the operator  $\pi(\mu) \in \mathcal{L}(\mathcal{H})$  is defined by

$$\pi(\mu)\xi = \int_G \mu(x)\pi(x)\xi dx, \quad \text{for all } \xi \in \mathcal{H}.$$

Let  $C(G)_{1,+}$  be the set of all continuous non-negative integrable functions  $\mu$  on  $G$  with  $\int_G \mu(x)dx = 1$ . Observe that  $\|\pi(\mu)\| \leq 1$  for every  $\mu \in C(G)_{1,+}$ .

Assume that  $\pi$  does not weakly contain  $1_G$ . Proposition G.4.2 shows that  $\|\pi(\mu)\| < 1$  for  $\mu \in C(G)_{1,+}$  if  $\text{supp}(\mu^* * \mu)$  generates a dense subgroup in  $G$ . The following proposition is a quantitative version of this result. It is

a slight improvement upon [Margu–91, Chapter III, Lemma 1.1.(b)]; see also Lemma 3.3 in [FisMa–05].

**Proposition 6.2.1** *Let  $G$  be a locally compact group. Let  $Q$  be a compact subset of  $G$  with  $Q^{-1} = Q$  and  $1 \in Q$ . Let  $\mu \in C(G)_{1,+}$  be such that  $\mu(x) > 0$  for all  $x \in Q$ .*

*For every  $\varepsilon > 0$ , there exists a constant  $\delta = \delta(\varepsilon, \mu) < 1$  such that, for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without  $(Q, \varepsilon)$ -invariant vectors, we have  $\|\pi(\mu)\| \leq \delta$ .*

**Proof** Let  $f = \mu^* * \mu \in C(G)_{1,+}$ . Since  $\|\pi(f)\| = \|\pi(\mu)\|^2$ , it suffices to estimate the norm of  $\pi(f)$ . Since  $\mu > 0$  on  $Q$  and since  $Q$  is symmetric,  $f > 0$  on  $Q^2$ . Set

$$\alpha = \inf \{f(x) : x \in Q^2\} > 0.$$

For every  $x \in Q$ , we have

$$f \geq \frac{\alpha}{2}(\chi_{xQ} + \chi_Q),$$

since  $xQ$  and  $Q$  are contained in  $Q^2$ . Set

$$a = \frac{\alpha|Q|}{2} \quad \text{and} \quad \varphi = \frac{1}{|Q|}\chi_Q,$$

where  $|Q|$  is the Haar measure of  $Q$ . Then

$$(*) \quad f \geq a(x_{-1}\varphi + \varphi),$$

for all  $x \in Q$ , where  $x\varphi(y) = \varphi(xy)$  for all  $y \in G$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  without  $(Q, \varepsilon)$ -invariant vectors. Let  $\xi$  be a unit vector in  $\mathcal{H}$ . Then there exists  $x_0 \in Q$  such that

$$\|\pi(x_0)\pi(\varphi)\xi - \pi(\varphi)\xi\| \geq \varepsilon\|\pi(\varphi)\xi\|.$$

Using the parallelogram identity, we have

$$\begin{aligned} \|\pi(x_0^{-1}\varphi + \varphi)\xi\|^2 &= 4\|\pi(\varphi)\xi\|^2 - \|\pi(x_0)\pi(\varphi)\xi - \pi(\varphi)\xi\|^2 \\ &\leq (4 - \varepsilon^2)\|\pi(\varphi)\xi\|^2 \\ &\leq 4 - \varepsilon^2 \end{aligned}$$

since  $\|\pi(\varphi)\| \leq \|\varphi\|_1 = 1$ . On the other hand, since by (\*) the function  $f - a(x_{-1}\varphi + \varphi)$  is non-negative, we have

$$\|\pi(f - a(x_{-1}\varphi + \varphi))\xi\| \leq \int_G (f(x) - a(\varphi(x_{-1}x) + \varphi(x))) dx = 1 - 2a.$$

It follows that

$$\begin{aligned} \|\pi(f)\xi\| &\leq \|\pi(f - a(x_{-1}\varphi + \varphi))\xi\| + a\|\pi(x_{-1}\varphi + \varphi)\xi\| \\ &\leq (1 - 2a) + a\sqrt{4 - \varepsilon^2} = 1 - a(2 - \sqrt{4 - \varepsilon^2}). \end{aligned}$$

Setting  $\delta = \sqrt{1 - a(2 - \sqrt{4 - \varepsilon^2})}$ , we have  $\delta < 1$  and  $\|\pi(\mu)\| \leq \delta$ . ■

**Remark 6.2.2** The same proof yields the following variation of the previous result, in which the symmetry assumption on  $Q$  is removed. Let  $Q$  be a compact subset of  $G$  with  $1 \in Q$ . Let  $\mu \in C(G)_{1,+}$  be such that  $\mu(x) > 0$  for all  $x \in Q^2$ . Let  $\varepsilon > 0$ .

Then there exists a constant  $\delta = \delta(\varepsilon, \mu)$  with  $0 < \delta < 1$  such that, for every unitary representation  $(\pi, \mathcal{H})$  of  $G$  without  $(Q, \varepsilon)$ -invariant vectors,  $\|\pi(\mu)\| \leq \delta$ .

In case  $G$  has Property (T), the previous proposition yields, for fixed  $\mu$  as above, a *uniform* bound for the norms of the operators  $\pi(\mu)$  for every unitary representation  $\pi$  without non-zero invariant vectors.

**Corollary 6.2.3** *Let  $G$  be locally compact group with Property (T), and let  $(Q, \varepsilon)$  be a Kazhdan pair for  $G$ , where  $Q$  is a compact Kazhdan set with  $Q^{-1} = Q$  and  $1 \in Q$ . Let  $\mu \in C(G)_{1,+}$  with  $\mu > 0$  on  $Q$ .*

*Then there exists a constant  $\delta < 1$  such that, for every unitary representation  $\pi$  of  $G$  without non-zero invariant vectors, we have  $\|\pi(\mu)\| \leq \delta$ .*

### 6.3 Ergodic theory and Property (T)

Property (T) for a group  $G$  has strong implications for its measure preserving actions. Indeed, Property (T) can be characterised in terms of such actions.

Let  $G$  be a second countable locally compact group acting measurably on a measure space  $(\Omega, \nu)$ ; see Section A.6. Throughout this section, we assume that  $L^2(\Omega, \nu)$  is separable and that:

- $\nu$  is a probability measure on  $\Omega$ ;
- $\nu$  is  $G$ -invariant.

A unitary representation  $\pi_\nu$  of  $G$  is defined on  $L^2(\Omega, \nu)$  by

$$\pi_\nu(g)f(\omega) = f(g^{-1}\omega), \quad f \in L^2(\Omega, \nu), \quad g \in G, \quad \omega \in \Omega$$

(Proposition A.6.1). Observe that  $L^2(\Omega, \nu)$  contains the constant function  $1_\Omega$  on  $\Omega$ . The subspace orthogonal to  $1_\Omega$  is

$$L_0^2(\Omega, \nu) = \left\{ f \in L^2(\Omega, \nu) : \int_\Omega f(\omega) d\nu(\omega) = 0 \right\}$$

and is  $G$ -invariant. We denote by  $\pi_\nu^0$  the restriction of  $\pi_\nu$  to  $L_0^2(\Omega, \nu)$ .

There is a close link, first observed by Koopman [Koopm–31], between ergodic theory of group actions and unitary representations. In particular, the following properties are equivalent (Exercise 6.5.4):

- (i) the action of  $G$  on  $\Omega$  is *ergodic*, which means that any measurable subset  $A$  of  $\Omega$  which is  $G$ -invariant is trivial in the sense that  $\nu(A)=0$  or  $\nu(\Omega \setminus A)=0$ ;
- (ii) the unitary representation  $\pi_\nu^0$  has no non-zero invariant vectors.

The following result is a rephrasing of Corollary 6.2.3 in the context of ergodic group actions.

**Corollary 6.3.1** *Let  $G$  be a second countable locally compact group with Property (T). Let  $(Q, \varepsilon)$  be a Kazhdan pair for  $G$ , where  $Q$  is a compact Kazhdan set with  $Q^{-1} = Q$  and  $1 \in Q$ . Let  $\mu \in C(G)_{1,+}$  with  $\mu(x) > 0$  for all  $x \in Q$ .*

*Then there exists a constant  $\delta < 1$  such that, for every measure preserving ergodic action of  $G$  on a probability space  $(\Omega, \nu)$ , we have  $\|\pi_\nu^0(\mu)\| \leq \delta$ .*

The result above implies that there is a uniform exponential decay of the norms of the convolution powers of  $\mu$  : for every measure preserving ergodic action of  $G$  on a probability space  $(\Omega, \nu)$ , we have

$$\|\pi_\nu^0(\mu^{*n})\| \leq \delta^n \quad \text{for all } n \in \mathbf{N}.$$

Since  $f \mapsto (\int_\Omega f d\nu)1_\Omega$  is the orthogonal projection of  $L^2(\Omega, \nu)$  onto the constants, this is equivalent to

$$\left\| \int_G f(g^{-1}\omega) d\mu^{*n}(g) - \int_\Omega f d\nu \right\|_{L^2(\Omega, \nu)} \leq \delta^n \|f\|_{L^2(\Omega, \nu)}$$

for all  $f \in L^2(\Omega, \nu)$  and all  $n \in \mathbb{N}$ . Therefore, we have

$$\lim_{n \rightarrow \infty} \left\| \pi(\mu)^n f - \int_{\Omega} f d\nu \right\|_{L^2(\Omega, \nu)} = 0,$$

with exponential decay.

For later use, we recall that a measure preserving action of  $G$  on a probability space  $(\Omega, \nu)$  is said to be *weakly mixing* if the diagonal action of  $G$  on the product measure space  $(\Omega \times \Omega, \nu \otimes \nu)$  is ergodic. Since  $L^2(\Omega \times \Omega, \nu \otimes \nu) \cong L^2(\Omega, \nu) \otimes L^2(\Omega, \nu)$ , the following properties are equivalent (see Proposition A.1.12):

- (i) the action of  $G$  on  $\Omega$  is weakly mixing;
- (ii) the unitary representation  $(\pi_v^0, L_0^2(\Omega, \nu))$  has no non-zero finite dimensional subrepresentation.

In particular, this shows that every weakly mixing action is ergodic.

### The Schmidt–Connes–Weiss characterisation of Property (T)

Let  $G$  be a second countable locally compact group acting measurably on the probability space  $(\Omega, \nu)$  and preserving  $\nu$ . If this action is ergodic, there is no non-trivial invariant measurable subset of  $\Omega$ . Nevertheless, there might exist non-trivial asymptotically invariant subsets in the following sense.

A sequence of measurable subsets  $(A_n)_n$  of  $\Omega$  is said to be *asymptotically invariant* if, for every compact subset  $Q$  of  $G$ ,

$$\lim_n \sup_{g \in Q} \nu(gA_n \Delta A_n) = 0,$$

where  $\Delta$  denotes the symmetric difference. If either  $\nu(A_n) \rightarrow 0$  or  $\nu(A_n) \rightarrow 1$ , then  $(A_n)_n$  is clearly asymptotically invariant. An asymptotically invariant sequence  $(A_n)_n$  is said to be *non-trivial* if

$$\inf_n \nu(A_n)(1 - \nu(A_n)) > 0.$$

The action of  $G$  on  $\Omega$  is said to be *strongly ergodic* if there exists no non-trivial asymptotically invariant sequence in  $\Omega$ .

**Proposition 6.3.2** *Let  $G$  be a second countable locally compact group acting on the probability space  $(\Omega, \nu)$  and preserving  $\nu$ . If there exists a non-trivial asymptotically invariant sequence in  $\Omega$ , then the unit representation  $1_G$  is weakly contained in  $\pi_v^0$ .*



*Equivalently: if  $1_G$  is not weakly contained in  $\pi_\nu^0$ , then the action of  $G$  on  $(\Omega, \nu)$  is strongly ergodic.*

**Proof** If  $(A_n)_n$  is a non-trivial asymptotically invariant sequence in  $\Omega$ , set

$$f_n = \chi_{A_n} - \nu(A_n)1_\Omega.$$

Then  $f_n \in L_0^2(\Omega, \nu)$ ,

$$\|f_n\|^2 = \nu(A_n)(1 - \nu(A_n)) \quad \text{and} \quad \|\pi_\nu^0(g)f_n - f_n\|^2 = \nu(gA_n \Delta A_n).$$

It follows that  $\pi_\nu^0$  almost has invariant vectors. ■

**Remark 6.3.3** (i) The converse does not hold in the previous proposition: in [Schmi–81, (2.7) Example], an example is given of a strongly ergodic measure preserving action of the free group  $F_3$  on a probability space such that the associated representation  $\pi_\nu^0$  almost has invariant vectors.

(ii) Let  $G$  be a second countable locally compact group acting on the probability space  $(\Omega, \nu)$  and preserving  $\nu$ . Following [Schmi–81], we say that a sequence of measurable subsets  $(A_n)_n$  of  $\Omega$  is an *I-sequence*, if  $\nu(A_n) > 0$  for every  $n$ ,  $\lim_n \nu(A_n) = 0$ , and

$$\limsup_n \sup_{g \in Q} \frac{\nu(gA_n \Delta A_n)}{\nu(A_n)} = 0$$

for every compact subset  $Q$  of  $G$ . As in the previous proposition, it is easy to show that the existence of an I-sequence implies that the associated  $\pi_\nu^0$  almost has invariant vectors. In [Schmi–81], it is shown that the converse is true for discrete groups: if  $\pi_\nu^0$  almost has invariant vectors, then an I-sequence exists. This was generalized to arbitrary locally compact groups in [FurSh–99, Section 5]; see also [KleMa–99, Appendix].

Using Theorem 2.12.9, we give a slightly stronger version of the Connes–Weiss result from [ConWe–80], with a simplified proof.

**Theorem 6.3.4 (Connes–Weiss)** *Let  $G$  be a second countable locally compact group. The following properties are equivalent:*

- (i)  $G$  has Property (T);
- (ii) every measure preserving ergodic action of  $G$  is strongly ergodic;
- (iii) every measure preserving weakly mixing action of  $G$  is strongly ergodic.

**Proof** That (i) implies (ii) follows from Proposition 6.3.2. It is clear that (ii) implies (iii).

We assume that  $G$  does not have Property (T) and construct a weakly mixing action of  $G$  on a measure space  $(\Omega, \nu)$  with a non-trivial asymptotically invariant sequence.

There exists a unitary representation  $\pi$  of  $G$  on a separable Hilbert space  $\mathcal{H}$  which almost has invariant vectors but no non-zero invariant vectors. By Theorem 2.12.9 (see also Remark 2.12.11), we can assume that  $\pi$  has no non-zero finite dimensional subrepresentation and that  $\pi$  is the complexification of an orthogonal representation. Thus  $\mathcal{H} = \mathcal{H}' \otimes \mathbb{C}$  for a real Hilbert space  $\mathcal{H}'$  such that  $\pi(g)\mathcal{H}' = \mathcal{H}'$  for all  $g \in G$ .

Let  $\mathcal{K}$  be an infinite dimensional Gaussian Hilbert space in  $L^2_{\mathbf{R}}(\Omega, \nu)$  for some probability space  $(\Omega, \nu)$ ; this means that every  $X \in \mathcal{K}$  is a centred Gaussian random variable on  $\Omega$  (see Section A.7). We can assume that the  $\sigma$ -algebra generated by all  $X \in \mathcal{K}$  coincides with the  $\sigma$ -algebra of all measurable subsets of  $\Omega$ .

Let  $\Phi : \mathcal{H}' \rightarrow \mathcal{K}$  be an isometric isomorphism. By Theorem A.7.13 and Corollary A.7.15,  $\Phi$  extends to an isomorphism  $\tilde{\Phi}$  between the symmetric Fock space

$$S(\mathcal{H}') = \bigoplus_{n=0}^{\infty} S^n(\mathcal{H}')$$

of  $\mathcal{H}'$  and  $L^2_{\mathbf{R}}(\Omega, \nu)$ ; moreover, there exists a measure preserving action of  $G$  on  $\Omega$  such that  $\Phi$  intertwines the direct sum  $\bigoplus_{n=0}^{\infty} S^n(\pi)$  of the symmetric tensor powers of  $\pi$  and the representation  $\pi_{\nu}$  on  $L^2_{\mathbf{R}}(\Omega, \nu)$  associated to the action of  $G$  on  $\Omega$ .

Since  $(\pi, \mathcal{H}')$  has no non-zero finite dimensional subrepresentation, the space

$$\bigoplus_{n=1}^{\infty} S^n(\mathcal{H}) \cong \bigoplus_{n=1}^{\infty} (S^n(\mathcal{H}'))_{\mathbb{C}} \cong L^2_0(\Omega, \nu)$$

has no non-zero finite dimensional subrepresentation (apply Propositions A.1.8 and A.1.12). Hence, the action of  $G$  on  $(\Omega, \nu)$  is weakly mixing.

Let  $(\xi_n)_n$  be a sequence of unit vectors in  $\mathcal{H}'$  such that

$$\lim_n \|\pi(g)\xi_n - \xi_n\| = 0$$

uniformly on compact subsets of  $G$ . Then

$$\lim_n \langle \pi(g)\xi_n, \xi_n \rangle = 1$$

uniformly on compact subsets of  $G$ . For every  $n \in \mathbf{N}$  and every  $g \in G$ , set

$$X_n = \Phi(\xi_n) \quad \text{and} \quad X_n^g = \pi_v(g)X_n.$$

Then  $X_n \in \mathcal{K}$  and  $X_n^g \in \mathcal{K}$ . Hence,  $X_n$  and  $X_n^g$  are centred Gaussian random variables on  $\Omega$ ; they have variance 1, since  $\|\xi_n\| = 1$ . Define

$$A_n = \{\omega \in \Omega : X_n(\omega) \geq 0\}.$$

Since the distribution of  $X_n$  is symmetric around the origin in  $\mathbf{R}$ , we have  $\nu(A_n) = 1/2$ . Let  $\alpha_n(g) \in [0, \pi]$  be defined by

$$\langle \pi_n(g)\xi_n, \xi_n \rangle = \langle X_n^g, X_n \rangle_{L_{\mathbf{R}}^2(\Omega, \nu)} = \cos \alpha_n(g).$$

Observe that  $\lim_n \langle \pi_n(g)\xi_n, \xi_n \rangle = 1$  implies  $\lim_n \alpha_n(g) = 0$ . Upon disregarding small values of  $n$ , we can therefore assume that  $\alpha_n(g) \in [0, \pi/2]$  for all  $n$ .

We claim that

$$\nu(gA_n \Delta A_n) = \frac{\alpha_n(g)}{\pi}.$$

Indeed, observe first that

$$gA_n \Delta A_n = \{X_n^g \geq 0 \text{ and } X_n < 0\} \cup \{X_n^g < 0 \text{ and } X_n \geq 0\}.$$

Write the orthogonal decomposition of  $X_n^g$  with respect to the subspace  $\mathbf{R}X_n$ :

$$X_n^g = \langle X_n^g, X_n \rangle X_n + Z_n,$$

with  $Z_n \in (\mathbf{R}X_n)^\perp$ . This can also be written as

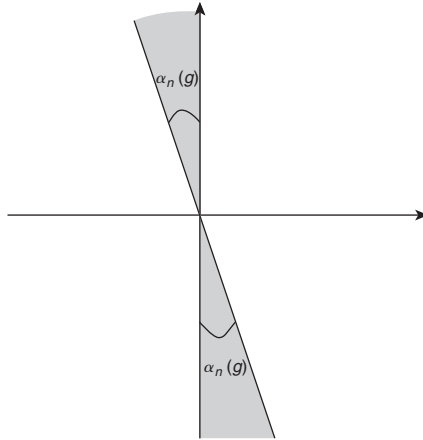
$$X_n^g = \cos(\alpha_n(g))X_n + \sin(\alpha_n(g))Y_n,$$

where

$$Y_n = \frac{1}{\sin(\alpha_n(g))}Z_n.$$

Then  $Y_n$  is a centred Gaussian variable, with variance 1. Moreover,  $X_n$  and  $Y_n$  are independent random variables, since  $Y_n$  is orthogonal to  $X_n$  and both are Gaussian variables. It follows that the joint distribution of  $X_n$  and  $Y_n$  is a probability measure  $m$  on  $\mathbf{R}^2$  which is rotation invariant (in fact,  $m$  is the two-dimensional standard Gaussian measure). The  $\nu$ -measure of  $gA_n \Delta A_n$  coincides with the  $m$ -measure of the following subset of  $\mathbf{R}^2$  (see Figure 6.1):

$$\begin{aligned} & \{(x, y) \in \mathbf{R}^2 : \cos(\alpha_n(g))x + \sin(\alpha_n(g))y \geq 0 \text{ and } x < 0\} \cup \\ & \{(x, y) \in \mathbf{R}^2 : \cos(\alpha_n(g))x + \sin(\alpha_n(g))y < 0 \text{ and } x \geq 0\}, \end{aligned}$$

Figure 6.1 The subset of  $\mathbf{R}^2$  associated to  $gA_n\Delta A_n$ 

that is, the subset

$$\left\{ (x, y) \in \mathbf{R}^2 : y \geq -\frac{1}{\tan(\alpha_n(g))}x \text{ and } x < 0 \right\} \cup \\ \left\{ (x, y) \in \mathbf{R}^2 : y < -\frac{1}{\tan(\alpha_n(g))}x \text{ and } x \geq 0 \right\}.$$

It follows that  $\nu(gA_n\Delta A_n) = \alpha_n(g)/\pi$ , as claimed.

The claim we have just proved implies that  $(A_n)_n$  is a non-trivial asymptotically invariant sequence in  $\Omega$ . ■

### Orbit equivalence and measure equivalence

Let  $(\Omega, \nu)$  be a Borel space with a non-atomic probability measure  $\nu$ . We will always assume that  $\Omega$  is an uncountable standard Borel space, so that there exists an isomorphism of Borel spaces  $f : [0, 1] \rightarrow \Omega$  (see Section F.5). We will also assume that  $\nu$  is the image under  $f$  of the Lebesgue measure on  $[0, 1]$ .

Given a measure preserving ergodic action of a countable group  $\Gamma$  on a standard Borel space  $(\Omega, \nu)$ , we consider the equivalence relation  $\mathcal{R}_\Gamma$  on  $\Omega$  associated to it and defined by

$$x\mathcal{R}_\Gamma y \quad \text{if} \quad \Gamma x = \Gamma y.$$

We can ask which properties of the group  $\Gamma$  and its action on  $(\Omega, \nu)$  are determined by the equivalence relation  $\mathcal{R}_\Gamma$ . This leads to the notion of orbit equivalence.

Two measure preserving ergodic actions of two countable groups  $\Gamma_1$  and  $\Gamma_2$  on standard Borel spaces  $(\Omega_1, \nu_1)$  and  $(\Omega_2, \nu_2)$  are *orbit equivalent* if there exist measurable subsets  $\Omega'_1$  and  $\Omega'_2$  with measure 1 in  $\Omega_1$  and  $\Omega_2$  and a Borel isomorphism  $f : \Omega'_1 \rightarrow \Omega'_2$  with  $f_*(\nu_1) = \nu_2$  such that, for  $\nu_1$ -almost every  $\omega \in \Omega'_1$ , we have

$$f(\Gamma_1 \omega) = \Gamma_2 f(\omega).$$

An action of a group  $\Gamma$  on a probability space is *essentially free* if, for  $\nu$ -almost every  $\omega \in \Omega$ , the stabiliser of  $\omega$  is reduced to  $\{e\}$ .

It has been shown by Dye [Dye–59] that any two essentially free, measure preserving ergodic actions of  $\mathbf{Z}$  on standard probability spaces are orbit equivalent. This has been generalized as follows by [OrnWe–80].

**Theorem 6.3.5** *Let  $\Gamma_1$  and  $\Gamma_2$  be infinite countable amenable groups. Any two essentially free, measure preserving ergodic actions of  $\Gamma_1$  and  $\Gamma_2$  on standard Borel spaces are orbit equivalent.*

For a further generalisation of this result, see [CoFeW–81]. In contrast to this, we have the following theorem of Hjorth [Hjort–05].

**Theorem 6.3.6** *Let  $\Gamma$  be an infinite discrete group with Property (T). There exist uncountably many essentially free, measure preserving ergodic actions of  $\Gamma$  which are pairwise not orbit equivalent.*

**Remark 6.3.7** (i) The previous result was obtained for some higher rank lattices in [GefGo–89] as a consequence of Zimmer’s superrigidity results for cocycles; see also [Zimm–84a, Example 5.2.13].

(ii) While the proof of Theorem 6.3.6 in [Hjort–05] is an existence result, [Popa–06c] gives an *explicit* construction of uncountably many non orbit equivalent ergodic actions, for any group  $\Gamma$  containing an infinite normal subgroup  $N$  such that  $(\Gamma, N)$  has Property (T).

As a consequence of Theorem 6.3.4, we obtain the following result from [Schmi–81, Corollary 3.6].

**Corollary 6.3.8** *Let  $\Gamma$  be a countable group which is non-amenable and does not have Property (T). Then  $\Gamma$  has two essentially free, measure preserving ergodic actions on standard Borel spaces which are not orbit equivalent.*

**Proof** Consider the *Bernoulli shift* of  $\Gamma$ . More precisely, let

$$\Omega_1 = \prod_{\gamma \in \Gamma} \mathbf{Z}/2\mathbf{Z}$$

be the direct product of copies of  $\mathbf{Z}/2\mathbf{Z}$  indexed by  $\Gamma$ , equipped with the product topology and the normalised Haar measure  $\nu_1$ . Consider the action of  $\Gamma$  on  $(\Omega_1, \nu_1)$  given by shifting the coordinates. The corresponding unitary representation on  $L_0^2(\Omega_1, \nu_1)$  is equivalent to a multiple of the regular representation of  $\Gamma$  (Exercise E.4.5). Since  $\Gamma$  is not amenable, there exists no non-trivial asymptotically invariant sequence in  $\Omega_1$  (Proposition 6.3.2), that is, the  $\Gamma$ -action on  $\Omega_1$  is strongly ergodic. Moreover, this action is essentially free.

On the other hand, since  $\Gamma$  does not have Property (T), it has a measure preserving weakly mixing action on a standard Borel space  $(\Omega_2, \nu_2)$  which is not strongly ergodic, by the Connes–Weiss Theorem 6.3.4. The diagonal action of  $\Gamma$  on the direct product  $(\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2)$  is ergodic. Indeed, we have the following decomposition of  $L^2(\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2) \cong L^2(\Omega_1, \nu_1) \otimes L^2(\Omega_2, \nu_2)$  into  $\Gamma$ -invariant subspaces:

$$\begin{aligned} L^2(\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2) \\ \cong \left( L_0^2(\Omega_1, \nu_1) \otimes L^2(\Omega_2, \nu_2) \right) \oplus \left( \mathbf{C}1_{\Omega_1} \otimes L_0^2(\Omega_2, \nu_2) \right) \oplus \mathbf{C}1_{\Omega_1 \times \Omega_2}. \end{aligned}$$

The unitary representation on  $L_0^2(\Omega_1, \nu_1) \otimes L^2(\Omega_2, \nu_2)$  is equivalent to a multiple of the regular representation of  $\Gamma$  (Corollary E.2.6). Hence, since  $\Gamma$  is infinite,  $L_0^2(\Omega_1, \nu_1) \otimes L^2(\Omega_2, \nu_2)$  contains no non-zero invariant function. The space  $\mathbf{C}1_{\Omega_1} \otimes L_0^2(\Omega_2, \nu_2) = L_0^2(\Omega_2, \nu_2)$  contains no non-zero invariant function, by ergodicity of the action of  $\Gamma$  on  $\Omega_2$ . It follows that the only  $\Gamma$ -invariant functions in  $L^2(\Omega_1 \times \Omega_2, \nu_1 \otimes \nu_2)$  are the constants.

Moreover, the action of  $\Gamma$  on  $\Omega_1 \times \Omega_2$  is obviously essentially free and not strongly ergodic. Since strong ergodicity is an invariant for orbit equivalence (Exercise 6.5.8), the claim follows. ■

From the previous corollary together with Theorems 6.3.5 and 6.3.6, we obtain the following characterisation of countable amenable groups.

**Corollary 6.3.9** *A countable group  $\Gamma$  is amenable if and only if any two essentially free, measure preserving ergodic actions of  $\Gamma$  on standard Borel spaces are orbit equivalent.*

We now turn to the notion of measure equivalent groups, introduced by Gromov in [Gromo–93, 0.5.E] and studied by Furman in [Furma–99a], [Furma–99b].

**Definition 6.3.10** Two countable groups  $\Gamma$  and  $\Lambda$  are *measure equivalent* if there exist commuting, essentially free, and measure preserving actions of  $\Gamma$  and  $\Lambda$  on a standard Borel space  $\Omega$ , equipped with a  $\sigma$ -finite measure  $\nu$ , such that the action of each of the groups  $\Gamma$  and  $\Lambda$  has a measurable fundamental domain with finite measure. The space  $(\Omega, \nu)$  is called a *coupling* of  $\Gamma$  and  $\Lambda$ .

**Example 6.3.11** Let  $\Gamma$  and  $\Lambda$  be lattices in a second countable locally compact group  $G$ . Then  $\Gamma$  and  $\Lambda$  are measure equivalent. Indeed, we can take as coupling  $(G, \nu)$ , where  $\nu$  is a left Haar measure of  $G$  and where  $\Gamma$  and  $\Lambda$  act on  $G$  by translations from the left and from the right, respectively. Observe that  $G$  is unimodular, since it contains a lattice (Proposition B.2.2). This shows that these actions are measure preserving.

**Remark 6.3.12** (i) Measure equivalence can be considered as a measure-theoretic analogue of quasi-isometry (see Example 3.6.2). Indeed, the following criterion for quasi-isometric groups holds: two finitely generated groups  $\Gamma$  and  $\Lambda$  are quasi-isometric if and only if there exist commuting, proper continuous actions of  $\Gamma$  and  $\Lambda$  on a locally compact space  $X$  such that the action of each of the groups  $\Gamma$  and  $\Lambda$  has a compact fundamental domain (see [Gromo–93, 0.2.C<sub>2</sub>'] and [Harpe–00, Chapter IV, Exercises 34 and 35]).

(ii) Measure equivalence is related to orbit equivalence as follows. The countable groups  $\Gamma$  and  $\Lambda$  are measure equivalent if and only if they admit essentially free, measure preserving actions on standard Borel spaces  $(\Omega_1, \nu_1)$  and  $(\Omega_2, \nu_2)$  which are *stably orbit equivalent*, that is, there exist measurable subsets of positive measure  $A$  of  $\Omega_1$  and  $B$  of  $\Omega_2$  and an isomorphism  $f : A \rightarrow B$  such that  $f_*(\nu_1/\nu_1(A)) = \nu_2/\nu_2(B)$  and  $f(\Gamma\omega \cap A) = \Lambda f(\omega) \cap B$  for almost every  $\omega \in A$  (see [Furman–99b, Lemma 3.2 and Theorem 3.3]).

Recall that Property (T) is not an invariant of quasi-isometry (see Theorem 3.6.5). In contrast to this, we have the following result, which is Corollary 1.4 in [Furman–99a] and which is also a consequence of Theorems 4.1.7 and 4.1.9 in [Popa].

**Theorem 6.3.13** *Let  $\Gamma$  and  $\Lambda$  be measure equivalent countable groups. If  $\Gamma$  has Property (T), then  $\Lambda$  has Property (T).*

Before we proceed with the proof, we show how to associate to a unitary representation of  $\Lambda$  a unitary representation of  $\Gamma$ .

Let  $(\Omega, \nu)$  be a coupling for  $\Gamma$  and  $\Lambda$ . We denote the actions of elements  $\gamma \in \Gamma$  and  $\lambda \in \Lambda$  on  $\Omega$  as

$$\omega \mapsto \gamma\omega \quad \omega \mapsto \omega\lambda^{-1}.$$

Let  $X$  and  $Y$  be measurable subsets of  $\Omega$  which are fundamental domains for the actions of  $\Lambda$  and  $\Gamma$ , respectively. We define a measurable cocycle  $\alpha : \Gamma \times X \rightarrow \Lambda$  as follows: for  $\gamma \in \Gamma$  and  $x \in X$ , let  $\alpha(\gamma, x)$  be the unique element  $\lambda \in \Lambda$  such that  $\gamma x \in X\lambda$ . The natural action of  $\Gamma$  on  $\Omega/\Lambda \cong X$  is described by

$$\gamma \cdot x = \gamma x \alpha(\gamma, x)^{-1} \quad x \in X, \quad \gamma \in \Gamma.$$

Let  $\mu = (v|_X)/v(X)$ , so that  $\mu$  is a  $\Gamma$ -invariant probability measure on  $X$ .

Similarly, we define a measurable cocycle  $\beta : Y \times \Lambda \rightarrow \Gamma$  and the natural action of  $\Lambda$  on  $\Gamma \backslash \Omega \cong Y$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $\Lambda$ . Let  $L^2(X, \mathcal{H})$  be the Hilbert space of all measurable mappings  $f : X \rightarrow \mathcal{H}$  such that

$$\int_X \|f(x)\|^2 d\mu(x) < \infty.$$

The formula

$$(\tilde{\pi}(\gamma)f)(x) = \pi(\alpha(\gamma^{-1}, x)^{-1})f(\gamma^{-1} \cdot x), \quad f \in L^2(X, \mathcal{H})$$

defines a unitary representation  $\tilde{\pi}$  of  $\Gamma$  on  $L^2(X, \mathcal{H})$ . We call  $\tilde{\pi}$  the representation of  $\Gamma$  induced by the representation  $\pi$  of  $\Lambda$ .

**Lemma 6.3.14** *Assume that  $\pi$  weakly contains  $1_\Lambda$ . Then  $\tilde{\pi}$  weakly contains  $1_\Gamma$ .*

**Proof** Let  $(\xi_n)_n$  be a sequence of unit vectors in  $\mathcal{H}$  with

$$\lim_n \|\pi(\lambda)\xi_n - \xi_n\| = 0$$

for all  $\lambda \in \Lambda$ . Let  $f_n \in L^2(X, \mathcal{H})$  be the constant mapping  $f_n \equiv \xi_n$ . Then  $\|f_n\| = 1$  and

$$\langle \tilde{\pi}(\gamma)f_n, f_n \rangle = \int_X \langle \pi((\alpha(\gamma^{-1}, x)^{-1})\xi_n, \xi_n) d\mu(x).$$

Fix  $\gamma \in \Gamma$ . Let  $\varepsilon > 0$ . There exists a finite subset  $F$  of  $\Lambda$  such that, for the measurable subset

$$X_\varepsilon = \left\{ x \in X : \alpha(\gamma^{-1}, x)^{-1} \in F \right\},$$

we have  $\mu(X_\varepsilon) \geq 1 - \varepsilon$  (compare with the hint in Exercise 6.5.8). There exists  $N_0$  such that, for all  $x \in X_\varepsilon$ ,

$$|\langle \pi((\alpha(\gamma^{-1}, x)^{-1})\xi_n, \xi_n) - 1| < \varepsilon \quad \text{for all } n \geq N_0.$$



Then, for all  $n \geq N_0$ , we have

$$|\langle \tilde{\pi}(\gamma)f_n, f_n \rangle - 1| \leq (1 - \varepsilon)\varepsilon + \varepsilon \leq 2\varepsilon.$$

This shows that  $\tilde{\pi}$  almost has  $\Gamma$ -invariant vectors. ■

**Proof of Theorem 6.3.13** Assume, by contradiction, that  $\Gamma$  has Property (T) and  $\Lambda$  does not have Property (T).

By Theorem 2.12.9, there exists a unitary representation  $(\pi, \mathcal{H})$  of  $\Lambda$  which weakly contains  $1_\Lambda$  and has no non-zero finite dimensional subrepresentation. By the previous lemma, the induced  $\Gamma$ -representation  $\tilde{\pi}$  weakly contains  $1_\Gamma$ . Since  $\Gamma$  has Property (T), it follows that there exists a non-zero mapping  $f \in L^2(X, \mathcal{H})$  which is  $\tilde{\pi}(\Gamma)$ -invariant, that is,

$$\pi(\alpha(\gamma^{-1}, x)^{-1})f(\gamma^{-1} \cdot x) = f(x) \quad \text{for all } x \in X, \gamma \in \Gamma.$$

We consider now the Hilbert space  $L^2(Y, \mathcal{H})$ , where  $Y$  is equipped with the probability measure  $\mu' = (\nu|_Y)/\nu(Y)$ . This space carries the unitary representation  $\rho$  of  $\Lambda$  defined by

$$(\rho(\lambda)\psi)(y) = \pi(\lambda)\psi(y \cdot \lambda), \quad \psi \in L^2(Y, \mathcal{H}).$$

Observe that  $\rho$  is unitarily equivalent to the tensor product  $\pi \otimes \pi_{\mu'}$  on  $\mathcal{H} \otimes L^2(Y, \mu')$ , where  $\pi_{\mu'}$  is the natural representation of  $\Lambda$  associated to the action of  $\Lambda$  on  $(Y, \mu')$ .

We claim that  $L^2(Y, \mathcal{H})$  contains a non-zero  $\rho(\Lambda)$ -invariant vector. This will imply that  $\pi \otimes \pi_{\mu'}$  contains  $1_\Lambda$  and will yield a contradiction, since  $\pi$  has no non-zero finite dimensional subrepresentation (see Proposition A.1.12).

Extend  $f$  to a measurable mapping  $F : \Omega \rightarrow \mathcal{H}$  by

$$F(x\lambda) = \pi(\lambda^{-1})f(x) \quad x \in X, \lambda \in \Lambda.$$

Then  $F(\omega\lambda) = \pi(\lambda^{-1})F(\omega)$  for all  $\omega \in \Omega$  and  $\lambda \in \Lambda$ . Moreover, we have, for all  $x \in X$  and all  $\gamma \in \Gamma$ ,

$$F(\gamma x) = F(\gamma \cdot x\alpha(\gamma, x)) = \pi(\alpha(\gamma, x)^{-1})f(\gamma \cdot x) = f(x) = F(x).$$

It follows that  $F(\gamma\omega) = F(\omega)$  for all  $\omega \in \Omega$  and  $\gamma \in \Gamma$ . Indeed, writing  $\omega = x\lambda$  for some  $x \in X$  and  $\lambda \in \Lambda$ , we have

$$\begin{aligned} F(\gamma\omega) &= F(\gamma(x\lambda)) = F((\gamma x)\lambda) = \pi(\lambda^{-1})F(\gamma x) \\ &= \pi(\lambda^{-1})F(x) = F(x\lambda) = F(\omega). \end{aligned}$$

Observe that the subset  $\{y \in Y : F(y) \neq 0\}$  has non-zero measure. Indeed, otherwise, we would have  $F = 0$  almost everywhere on  $\Omega$  and hence  $f = 0$  almost everywhere on  $X$ , by  $\Gamma$ -invariance and  $\Lambda$ -invariance of the mapping  $\omega \mapsto \|F(\omega)\|$ . As  $Y$  has finite measure, it follows that there exists constants  $C > 0$  and  $\varepsilon > 0$  such that the  $\Lambda$ -invariant measurable subset

$$Y_0 = \{y \in Y : \varepsilon < \|F(y)\| < C\}$$

has non-zero measure. Define  $\varphi : Y \rightarrow \mathcal{H}$  by

$$\varphi(y) = \chi_{Y_0}(y)F(y), \quad y \in Y$$

where  $\chi_{Y_0}$  is the characteristic function of  $Y_0$ . Then  $\varphi \neq 0$  and  $\varphi \in L^2(Y, \mathcal{H})$ , since  $\varphi$  is bounded on the probability space  $Y$ . For all  $y \in Y$  and  $\lambda \in \Lambda$ , we have

$$\begin{aligned} \varphi(y \cdot \lambda) &= \chi_{Y_0}(y \cdot \lambda)F(y \cdot \lambda) \\ &= \chi_{Y_0}(y)F(\beta(y, \lambda)^{-1}y\lambda) \\ &= \chi_{Y_0}(y)F(y\lambda) \\ &= \chi_{Y_0}(y)\pi(\lambda^{-1})F(y) \\ &= \pi(\lambda^{-1})\varphi(y). \end{aligned}$$

This shows that  $\varphi$  is a non-zero  $\rho(\Lambda)$ -invariant vector in  $L^2(Y, \mathcal{H})$ . As claimed, this finishes the proof. ■

**Remark 6.3.15** Zimmer introduced a notion of Property (T) for a measured equivalence relation and, in particular, for an essentially free ergodic action of a countable group  $\Gamma$  on a standard Borel space  $(\Omega, \nu)$  with a quasi-invariant probability measure  $\nu$  (see [Zimme–81] and [Moore–82]). He showed that, if  $\nu$  is invariant and the action is weakly mixing, then the action has Property (T) if and only if  $\Gamma$  has Property (T). For a generalization to the context of groupoids, see [Anant–05].

## 6.4 Uniqueness of invariant means

Let  $n$  be an integer,  $n \geq 1$ . The so-called *Banach–Ruziewicz problem* asks whether the normalised Lebesgue measure  $\lambda$ , defined on all Lebesgue measurable subsets of the unit sphere  $S^n$  in  $\mathbf{R}^{n+1}$ , is the unique normalised  $SO_{n+1}(\mathbf{R})$ -invariant *finitely additive* measure. In the case  $n = 1$ , it was shown by

Banach that the answer is negative. Moreover, Tarski proved, using the Banach–Tarski paradox, that every  $SO_{n+1}(\mathbf{R})$ -invariant finitely additive measure on the Lebesgue measurable subsets of  $S^n$  has to be absolutely continuous with respect to the Lebesgue measure  $\lambda$ . This shows that the Banach–Ruziewicz problem can be formulated as follows (see Section G.1): *Is integration against  $\lambda$  the unique  $SO_{n+1}(\mathbf{R})$ -invariant mean on  $L^\infty(S^n, \lambda)$ ?* The following result, due to Rosenblatt [Rosen–81], Schmidt [Schmi–81] and Losert-Rindler [LosRi–81], relates the Banach–Ruziewicz problem to Property (T).

**Proposition 6.4.1** *Let  $\Gamma$  be a countable group acting in a measure preserving way on a probability space  $(\Omega, \nu)$ . Assume that the associated unitary representation  $\pi_\nu^0$  of  $\Gamma$  on  $L_0^2(\Omega, \nu)$  does not weakly contains  $1_\Gamma$ . Then integration against  $\nu$  is the unique  $\Gamma$ -invariant mean on  $L^\infty(\Omega, \nu)$ .*

**Proof** Let  $m$  be a  $\Gamma$ -invariant mean on  $L^\infty(\Omega, \nu)$ . We show that  $m = \nu$ . The arguments are similar to arguments used in the proofs of Theorems G.3.1 and G.3.2.

Using the density of

$$L^1(\Omega, \nu)_{1,+} = \left\{ f \in L^1(\Omega, \nu) : f \geq 0 \text{ and } \int_\Omega f d\nu = 1 \right\}$$

in the set of means on  $L^\infty(\Omega, \nu)$ , we find a net  $(g_i)_i$  in  $L^1(\Omega, \nu)_{1,+}$ , converging to  $m$  in the weak\*-topology. By invariance of  $m$ , we have

$$(*) \quad \lim_i (\gamma g_i - g_i) = 0 \quad \text{for all } \gamma \in \Gamma,$$

in the weak topology of  $L^1(\Omega, \nu)$ , where  $\gamma g_i(\omega) = g_i(\gamma\omega)$ . Using Namioka's argument (see the proof of Theorem G.3.1), we can assume that the convergence in  $(*)$  holds in the  $L^1$ -norm.

Define  $f_i = \sqrt{g_i} \in L^2(\Omega, \nu)$ . Then  $\|f_i\|_2 = 1$  and, as in the proof of G.3.2,

$$\lim_i \|\pi_\nu(\gamma)f_i - f_i\|_2 = 0 \quad \text{for all } \gamma \in \Gamma.$$

Let  $f_i = \xi_i + c_i 1_\Omega$  be the orthogonal decomposition of  $f_i$ , where  $\xi_i \in L_0^2(\Omega, \nu)$  and  $c_i = \int_\Omega f_i d\nu$ . We have

$$\lim_i \|\pi_\nu^0(\gamma)\xi_i - \xi_i\|_2 = 0 \quad \text{for all } \gamma \in \Gamma.$$

Since  $1_\Gamma$  is not weakly contained in  $\pi_\nu^0$ , it follows that

$$\inf_i \|\xi_i\|_2 = 0.$$

Hence, upon passing to a subnet, we can assume that  $\lim_i \|\xi_i\|_2 = 0$ . It follows that  $\lim_i c_i = 1$  and hence  $\lim_i \|f_i - 1_\Omega\|_2 = 0$ . Since

$$\begin{aligned} \|g_i - 1_\Omega\|_1 &= \int_\Omega |f_i^2 - 1| d\nu = \int_\Omega |f_i - 1| |f_i + 1| d\nu \\ &\leq \|f_i - 1_\Omega\|_2 \|f_i + 1_\Omega\|_2 \leq 2\|f_i - 1_\Omega\|_2, \end{aligned}$$

we have  $\lim_i \|g_i - 1_\Omega\|_1 = 0$ . Hence  $(g_i)_i$  converges weakly to  $\nu$  and it follows that  $m = \nu$ . ■

**Remark 6.4.2** In the previous proposition, the converse is also true: if  $\nu$  is the unique invariant mean on  $L^\infty(\Omega, \nu)$ , then  $\pi_\nu^0$  does not weakly contains  $1_\Gamma$ . This implication holds for arbitrary second countable locally compact groups (see [FurSh–99, Theorem 1.6]).

**Corollary 6.4.3** *Let  $\Gamma$  be a discrete group with Property (T) acting in a measure preserving way on a probability space  $(\Omega, \nu)$ . Assume that this action is ergodic. Then  $\nu$  is the unique invariant mean on  $L^\infty(\Omega, \nu)$ .*

**Proof** Since the action of  $\Gamma$  is ergodic, the representation  $\pi_\nu^0$  has no non-zero fixed vectors, and the claim follows from the previous proposition. ■

The following theorem, due to Margulis [Margu–80] and Sullivan [Sulli–81], combined with the previous corollary, shows that the Banach–Ruziewicz problem for  $S^n$  has a positive answer when  $n \geq 4$ . Observe that a dense subgroup  $\Gamma$  of  $SO_{n+1}(\mathbf{R})$  acts ergodically on  $S^n$ . Indeed, if  $f \in L_0^2(S^n, \lambda)$  is a  $\Gamma$ -invariant function, then  $f$  is  $SO_{n+1}(\mathbf{R})$ -invariant, by density of  $\Gamma$ , and hence  $f = 0$ .

**Theorem 6.4.4** *For  $n \geq 5$ , the group  $SO_n(\mathbf{R})$  contains a dense subgroup  $\Gamma$  which has Property (T) as discrete group.*

**Proof** We follow the construction from [Margu–80]. Let  $p$  be a prime integer with  $p \equiv 1 \pmod{4}$ . Since the equation  $x^2 = -1$  has a solution in  $\mathbf{Q}_p$ , the group  $SO_n$  has  $\mathbf{Q}_p$ -rank equal to  $[n/2]$ . Hence,  $SO_n(\mathbf{Q}_p)$  has Property (T) for  $n \geq 5$  (see Theorem 1.6.1; observe that  $SO_4$  is not simple). The group  $\Lambda = SO_n(\mathbf{Z}[1/p])$  embeds diagonally into the Kazhdan group

$$G = SO_n(\mathbf{Q}_p) \times SO_n(\mathbf{R})$$

as a lattice in  $G$ , and therefore has Property (T). Denote by  $\Gamma$  the projection of  $\Lambda$  to  $SO_n(\mathbf{R})$ . Then  $\Gamma$  is a dense subgroup of  $SO_n(\mathbf{R})$  and has Property (T). ■

**Remark 6.4.5** (i) Zimmer [Zimm–84c] showed that  $SO_n(\mathbf{R})$  does not contain an infinite countable group with Property (T) for  $n = 3, 4$ . A stronger result was proved in [GuHiW–05]: For a field  $\mathbf{K}$ , every countable subgroup of  $GL_2(\mathbf{K})$  has the Haagerup property (see Definition 2.7.5).

(ii) The previous theorem has the following extension. Let  $G$  be a simple compact real Lie group which is not locally isomorphic to  $SO_3(\mathbf{R})$ . Then  $G$  contains a dense subgroup  $\Gamma$  which has Property (T) as discrete group (see [Margu–91, Chapter III, Proposition 5.7]).

(iii) Drinfeld [Drinf–84] proved, using deep estimates for Fourier coefficients of modular forms, that  $SO_n(\mathbf{R})$ , for  $n = 3, 4$ , contains a dense subgroup  $\Gamma$  such that the  $\Gamma$ -representation  $\pi_\lambda^0$  on  $L_0^2(S^{n-1}, \lambda)$  does not weakly contain  $1_\Gamma$ . So, Proposition 6.4.1 still applies and shows that the Banach–Ruziewicz problem has a positive answer for  $n = 2, 3$ .

In summary, we have the following result.

**Corollary 6.4.6** *For  $n \geq 2$ , the Lebesgue measure is the unique rotation-invariant, finitely additive normalised measure defined on all Lebesgue measurable subsets of  $S^n$ .*

**Remark 6.4.7** It is crucial that we considered above the *Lebesgue measurable* subsets of  $S^n$ . The corresponding problem for the Borel subsets of  $S^n$  is open.

## 6.5 Exercises

**Exercise 6.5.1** Prove the claims of Example 6.1.2.

[Hint: The proofs of (i) and (ii) are straightforward. To prove (iii), one can proceed as follows (compare [CeGrH–99, Item 47]). Let  $F$  be a finite subset of the  $k$  regular tree  $T_k$  for  $k \geq 2$ . We claim that

$$(*) \quad \#\partial F \geq (k-2)(\#F) + 2,$$

with equality when  $F$  is connected.

Let  $F_1, \dots, F_n$  be the connected components of  $F$ . Claim (\*) is proved by induction on  $n$ .

To start with, assume first that  $n = 1$ , that is,  $F$  is connected. Use induction on  $\#F$  to prove that  $\#\partial F = (k-2)(\#F) + 2$ .

Assume now that  $n \geq 2$ . Show that, changing the numbering of the  $F_i$ 's if necessary, we can assume that  $\partial F_1$  has at most one vertex in common with  $\partial(\bigcup_{2 \leq i \leq n} F_i)$ . Use the induction hypothesis to prove (\*).]

**Exercise 6.5.2** Construct a locally finite tree  $X = (V, \mathbb{E})$  such that

$$\sup \left\{ \frac{\#\partial A}{\min\{\#A, \#(V \setminus A)\}} : A \subsetneq V, 0 < \#A < \infty \right\} = +\infty.$$

**Exercise 6.5.3** Let  $\mathcal{G} = (V, \mathbb{E})$  be a finite  $k$ -regular connected graph. Prove that

$$\frac{\lambda_1}{2} \leq h(\mathcal{G}),$$

where  $\lambda_1$  is the smallest non-zero eigenvalue of the Laplace operator  $\Delta$  on  $V$  (see Remark 6.1.5).

[Hint: For a subset  $A$  of  $V$ , let  $f \in \ell_0^2(V)$  be defined as in the proof of Lemma 6.1.7. Using Proposition 5.2.2, show that

$$\frac{\lambda_1}{2} \leq \frac{\#\partial A}{\min\{\#A, \#(V \setminus A)\}}.]$$

**Exercise 6.5.4** Consider a measure preserving action of a second countable locally compact group  $G$  on a  $\sigma$ -finite measure space  $(\Omega, \nu)$  and the unitary representation  $\pi_\nu$  of  $G$  on  $L^2(\Omega, \nu)$ , as in Section A.6.

(i) Let  $f \in L^2(\Omega, \nu)$  be a  $\pi_\nu(G)$ -invariant function, that is, for every  $g \in G$ , there exists a measurable subset  $N_g$  of  $X$  such that  $\nu(N_g) = 0$  and  $\tilde{f}(gx) = f(x)$  for all  $x \in X \setminus N_g$ . Show that there exists a measurable function  $\tilde{f}$  on  $X$  such that  $\tilde{f} = f$  almost everywhere on  $X$  and such that  $\tilde{f}(gx) = \tilde{f}(x)$  for all  $g \in G$  and  $x \in X$ .

[Hint: Look at the proof of Theorem E.3.1.]

(ii) Show that the action of  $G$  on  $X$  is ergodic if and only if the restriction of  $\pi_\nu$  to  $L_0^2(\Omega, \nu)$  has no non-zero invariant vectors.

[Hint: Assume that the action of  $G$  is ergodic. Let  $f \in L^2(\Omega, \nu)$  be a  $\pi_\nu(G)$ -invariant real-valued function. We have to show that  $f$  is constant almost everywhere. By (i), we can assume that  $\tilde{f}(gx) = \tilde{f}(x)$  for all  $g \in G$  and  $x \in X$ . Now,  $\tilde{f}$  can be approximated in  $L^2(X)$  by linear combinations of characteristic functions  $\chi_A$  of subsets of the form  $A = f^{-1}([a, b))$  for intervals  $[a, b) \subset \mathbf{R}$ .]

**Exercise 6.5.5** Let  $\Gamma$  be a discrete group acting by continuous automorphisms on a compact abelian group  $A$ .

(i) Show that the normalised Haar measure  $\nu$  on  $A$  is preserved under  $\Gamma$ .

(ii) Show that the action of  $\Gamma$  is ergodic if and only if, except the unit character  $1_A$ , all the  $\Gamma$ -orbits for the dual action of  $\Gamma$  on  $\hat{A}$  are infinite.

[Hint: If  $B$  is a  $\Gamma$ -invariant measurable subset of  $A$ , observe that the Fourier transform of  $\chi_B$  is a  $\Gamma$ -invariant function in  $\ell^2(\widehat{A})$ .]

**Exercise 6.5.6** For  $n \geq 2$ , consider the natural action of  $SL_n(\mathbf{Z})$  on the  $n$ -torus  $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ .

(i) Show that this action is ergodic.

(ii) Show that this action is weakly mixing.

[Hint: Apply Exercise 6.5.5. Compare also with Exercise D.5.7]

**Exercise 6.5.7** Let  $\Gamma$  be an infinite group and let  $K$  be a compact abelian group. For each  $\gamma \in \Gamma$ , set  $K_\gamma = K$  and let  $\Omega = \prod_{\gamma \in \Gamma} K_\gamma$  be the direct product, equipped with the product topology and the Haar measure  $\nu$ . Then  $\Gamma$  acts on  $\Omega$  via shifting on the left:

$$\gamma'(k_\gamma)_{\gamma \in \Gamma} = (k_{\gamma'^{-1}\gamma})_{\gamma \in \Gamma}, \quad \gamma' \in \Gamma.$$

(i) Show that this action is ergodic.

[Hint: Apply Exercise 6.5.5.]

(ii) Show that this action is *strongly mixing*, that is, given measurable subsets  $A$  and  $B$  of  $\Omega$  and  $\varepsilon > 0$ , there exists a finite subset  $F$  of  $\Gamma$  such that

$$|\nu(\gamma A \cap B) - \nu(A)\nu(B)| < \varepsilon, \quad \text{for all } \gamma \in F.$$

[Hint: Apply Exercise E.4.5]

**Exercise 6.5.8** Let  $\Gamma_1$  and  $\Gamma_2$  be countable groups with essentially free, measure preserving actions on standard Borel spaces  $(\Omega_1, \nu_1)$  and  $(\Omega_2, \nu_2)$ . Assume that there exists an orbit equivalence  $\theta : \Omega_1 \rightarrow \Omega_2$ . Let  $(A_n)_n$  be an asymptotically invariant sequence in  $\Omega_2$ . Show that  $(\theta^{-1}(A_n))_n$  is an asymptotically invariant sequence in  $\Omega_1$ .

[Hint: For  $\gamma_1 \in \Gamma_1$  and  $\omega \in \Omega_1$ , let  $\alpha(\gamma_1, \omega)$  be the unique element in  $\Gamma_2$  such that  $\theta(\gamma_1 \omega) = \alpha(\gamma_1, \omega)\theta(\omega)$ . The mapping  $\alpha : \Gamma_1 \times \Omega_1 \rightarrow \Gamma_2$  is a measurable cocycle. Given  $\gamma_1 \in \Gamma_1$ , show that, for every  $\varepsilon > 0$ , there exists a finite subset  $F$  of  $\Gamma_2$  such that  $\nu_2(X_\varepsilon) \geq 1 - \varepsilon$ , where

$$X_\varepsilon = \{\omega \in \Omega_1 : \alpha(\gamma_1, \omega) \in F\}.$$

Write  $\theta^{-1}(A_n) = (\theta^{-1}(A_n) \cap X_\varepsilon) \cup (\theta^{-1}(A_n) \setminus X_\varepsilon)$ .]

# 7

## A short list of open questions

We have collected below a sample of open questions which are standard in the subject. There are several other ones in the various papers of our reference list.

### Open examples of groups

**(7.1)** It is not known whether the *automorphism group*  $\text{Aut}(F_k)$  of a non-abelian free group  $F_k$  on  $k$  generators has Property (T) for  $k$  large enough. The question is often asked for the related group  $\text{Out}(F_k)$  of *outer automorphisms*, which is the quotient of  $\text{Aut}(F_k)$  by the group of inner automorphisms (which is isomorphic to  $F_k$ ).

The relevance of the question is discussed in [LubPa–01]; see the end of our historical introduction. Observe that  $\text{Aut}(F_2)$  does not have Property (T) since it has a quotient isomorphic to  $GL_2(\mathbb{Z})$ , neither does  $\text{Aut}(F_3)$  by [McCoo–89]. See also Corollary 1.3 in [GruLu].

**(7.2)** Let  $\Sigma_g$  denote a closed oriented surface of genus  $g \geq 2$ . Denote by  $\text{Mod}_g$  the corresponding *mapping class group*, namely the group of homotopy classes of orientation preserving homeomorphisms of  $\Sigma_g$ . It is not known whether this group has Property (T) for  $g$  large enough.

There is in [Taher–00] a report of a GAP-assisted computation which shows that  $\text{Mod}_2$  does not have Property (T).

**(7.3)** Similarly, we do not know whether Property (T) holds for the *Burnside group*

$$B(k, n) = F_k / \langle x^n = 1 \mid x \in F_k \rangle,$$

or for some infinite quotient of it (it is natural to assume  $k \geq 2$  and  $n$  large enough for  $B(k, n)$  to be infinite). See the last section in [Shal–ICM] for a short discussion.



**(7.4)** Does there exist a subgroup of infinite index in  $SL_3(\mathbf{Z})$  which has Property (T)?

More generally, if  $G$  denotes one of the groups  $SL_3(\mathbf{R})$ ,  $Sp_4(\mathbf{R})$ , and  $Sp(2, 1)$ , does there exist a discrete subgroup of  $G$  which is not a lattice and which has Property (T)?

**(7.5)** What are the groups of homeomorphisms of the circle which have Property (FH)? See Section 2.9 for the result of Navas and Reznikov on the appropriate diffeomorphism group.

It is known that, for  $n \geq 1$ , the group of homeomorphisms of the sphere of dimension  $n$ , viewed as a discrete group, has Property (FH) [Cornu–06b].

## Open examples of pairs of groups

**(7.6)** We feel that we do not know enough examples of pairs  $(\Gamma, \Delta)$  which have Property (T). As a sample of specific pairs for which we do not know whether Property (T) holds or not, there are

- $(\text{Aut}(F_k), F_k)$ , where  $F_k$  is embedded via inner automorphisms;
- $(\text{Aut}(F_k) \rtimes F_k, F_k)$ , where  $F_k$  is embedded as the second factor;
- $(\text{Mod}_g \rtimes \pi_1(\Sigma_g), \pi_1(\Sigma_g))$ .

**(7.7)** Let  $G$  be a locally compact group; in case it helps for what follows, assume that  $G$  is compactly generated and second countable. Let  $H, K, L$  be closed subgroups of  $G$  such that  $L \subset K \subset H \subset G$ . As a first preliminary observation, let us point out that, if the pair  $(H, K)$  has Property (T), then so does  $(G, L)$ . From now on, assume that there exists a finite  $K$ -invariant measure on  $K/L$  and a finite  $G$ -invariant measure on  $G/H$ . Recall that, if the pair  $(G, L)$  has Property (T), then the pair  $(G, K)$  also has it.

If the pair  $(G, L)$  has Property (T), does it follow that  $(H, L)$  has it? The answer is known to be affirmative in the particular case where  $L$  is moreover a normal subgroup of  $G$ .

See Theorems 1.4 and 1.5 in [BatRo–99], as well as [Jolis–05].

## Properties of Kazhdan groups

**(7.8)** Is it true that a countable group which can be left ordered does not have Property (T)?

For comparison with Question (7.5), recall that a countable group is left orderable if and only if it is isomorphic to a subgroup of the group of orientation

preserving homeomorphisms of the real line (see for example Theorem 6.8 in [Ghys–01]).

A related result was proved in [ChMaV–04]: a left ordered locally compact group  $G$  with the property that segments  $\{x \in G : a \leq x \leq b\}$  are measurable with *finite* Haar measure cannot have Property (T).

**(7.9)** Does there exist an infinite group with Kazhdan Property (T) which is not of uniform exponential growth?

See the discussion in [Harpe–02].

**(7.10)** Does there exist a countable infinite Kazhdan group  $\Gamma$  such that the subset  $\{\pi \in \widehat{\Gamma} : \dim(\pi) < \infty\}$  is dense in the unitary dual  $\widehat{\Gamma}$  of  $\Gamma$ ?

See [Bekk–99] and [LubSh–04].

## Uniform Kazhdan property

**(7.11)** Say that a finitely generated group  $\Gamma$  has *Property (T) uniformly* if there exists a number  $\kappa > 0$  which is a Kazhdan constant for  $\Gamma$  and *every* finite generating set  $Q$  of  $\Gamma$  (Definition 1.1.3).

Does the group  $SL_3(\mathbf{Z})$  have Property (T) uniformly? (This appears for example as Problem 10.3.1 in [Lubot–94].)

A related question is to know whether there exists  $\kappa' > 0$  which is a Kazhdan constant for  $SL_3(\mathbf{Z})$  and every generating set of at most 100 (say) elements. For groups which have Property (T) but which do not have it uniformly, see [GelZu–02] and [Osin–02]. For an infinite group which has Property (T) uniformly, see [OsiSo].

## Kazhdan subsets of amenable groups

**(7.12)** The Kazhdan property involves compact Kazhdan sets, but arbitrary Kazhdan sets (Definition 1.1.3) could deserve more attention.

The question of knowing if a subset  $Q$  of  $\mathbf{Z}$  is a Kazhdan set is possibly related to the equidistribution in the circle of the sequence  $(e^{2i\pi n\theta})_{n \in Q}$  for  $\theta$  irrational, in the sense of Weyl [Weyl–16] (this was brought to our attention by Y. Shalom).

More generally, what are the Kazhdan subsets of  $\mathbf{Z}^k$ ,  $\mathbf{R}^k$ , the Heisenberg group, other infinite amenable groups? Observe that, for a countable group which has Property (T), a subset is a Kazhdan set if and only if it is generating.

## Fundamental groups of manifolds

**(7.13)** Let  $\Gamma$  be the fundamental group of a compact manifold  $M$ . If  $\Gamma$  has Property (T), what are the consequences for the topology and the geometry of  $M$ ?

For example, if  $M$  is a closed Riemannian manifold with pinched sectional curvature,  $-4 < K < -1$ , can  $\pi_1(M)$  be Kazhdan? (Problem 5.3 in [Spatz-95]).

**(7.14)** Let  $\Gamma$  be the fundamental group of a compact orientable 3-manifold. Is there an elementary proof which would show that  $\Gamma$  has Property (T) if and only if  $\Gamma$  is finite?

In [Fujiw-99] (see also Exercise 1.8.18), there is an argument reducing the claim to the geometrization conjecture of Thurston.

**(7.15)** If the manifold is hyperbolic, is it true that  $\Gamma$  does not have Property ( $\tau$ )?

This is the so-called Lubotzky–Sarnak conjecture. See Conjecture 4.2 in [Lubot-97], and [Lacke-06].



## PART II

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### Background on Unitary Representations



# Appendix A

## Unitary group representations

A linear representation, or simply a *representation*, of a group  $G$  in a vector space  $V$  is a group homomorphism from  $G$  to the group of linear automorphisms of  $V$ . Appendix A collects basic facts about unitary representations of topological groups in Hilbert spaces. *Hilbert spaces* are assumed to be complex, unless stated otherwise (as in Chapter 2, Section A.7, and Section C.2, where Hilbert spaces are most often real). Topological groups are *always assumed to be Hausdorff*.

### A.1 Unitary representations

The inner product of two vectors  $\xi, \eta$  in a Hilbert space  $\mathcal{H}$  is denoted by  $\langle \xi, \eta \rangle$ , and is linear in the first variable. We denote by  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  the vector space of all continuous linear operators from a Hilbert space  $\mathcal{H}_1$  to a Hilbert space  $\mathcal{H}_2$ ; an operator  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  has an *adjoint*  $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ . We write  $\mathcal{L}(\mathcal{H})$  for  $\mathcal{L}(\mathcal{H}, \mathcal{H})$ , and observe that it is naturally an involutive complex algebra, with unit the identity operator  $I$  of  $\mathcal{H}$ . An operator  $U : \mathcal{H} \rightarrow \mathcal{H}$  is *unitary* if

$$UU^* = U^*U = I$$

or, equivalently, if  $\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$  for all  $\xi, \eta \in \mathcal{H}$  and if  $U$  is onto. The *unitary group*  $\mathcal{U}(\mathcal{H})$  of  $\mathcal{H}$  is the group of all unitary operators in  $\mathcal{L}(\mathcal{H})$ .

**Definition A.1.1** A *unitary representation* of a topological group  $G$  in a Hilbert space  $\mathcal{H}$  is a group homomorphism  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$  which is *strongly continuous* in the sense that the mapping

$$G \rightarrow \mathcal{H}, \quad g \mapsto \pi(g)\xi$$

is continuous for every vector  $\xi$  in  $\mathcal{H}$ . We will often write  $(\pi, \mathcal{H})$  instead of  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$ . About continuity, see Exercise A.8.1.

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ , and let  $\mathcal{K}$  be a closed  $G$ -invariant subspace of  $\mathcal{H}$ . Denoting, for every  $g$  in  $G$ , by

$$\pi^{\mathcal{K}}(g) : \mathcal{K} \rightarrow \mathcal{K}$$

the restriction of the operator  $\pi(g)$  to  $\mathcal{K}$ , we obtain a unitary representation  $\pi^{\mathcal{K}}$  of  $G$  on  $\mathcal{K}$ . We say that  $\pi^{\mathcal{K}}$  is a *subrepresentation* of  $\pi$ .

Given a unitary representation  $(\pi, \mathcal{H})$  of  $G$  and vectors  $\xi, \eta$  in  $\mathcal{H}$ , the continuous function

$$G \rightarrow \mathbf{C}, \quad g \mapsto \langle \pi(g)\xi, \eta \rangle$$

is called a *matrix coefficient* of  $\pi$ .

An important feature of unitary representations is that they are completely reducible in the sense that every closed invariant subspace has a closed invariant complement. More precisely:

**Proposition A.1.2** *Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ , and let  $\mathcal{K}$  be a  $G$ -invariant subspace. Then  $\mathcal{K}^{\perp}$ , the orthogonal complement of  $\mathcal{K}$  in  $\mathcal{H}$ , is  $G$ -invariant.*

**Proof** We have

$$\langle \pi(g)\xi, \eta \rangle = \langle \xi, \pi(g)^* \eta \rangle = \langle \xi, \pi(g^{-1})\eta \rangle = 0,$$

for every  $g$  in  $G$ ,  $\xi$  in  $\mathcal{K}^{\perp}$ , and  $\eta$  in  $\mathcal{K}$ . ■

**Definition A.1.3** An *intertwining operator* between two unitary representations  $(\pi_1, \mathcal{H}_1)$  and  $(\pi_2, \mathcal{H}_2)$  of  $G$  is a continuous linear operator  $T$  from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  such that  $T\pi_1(g) = \pi_2(g)T$  for all  $g \in G$ . The representations  $\pi_1$  and  $\pi_2$  are *equivalent*, and we will write  $\pi_1 \simeq \pi_2$ , if there exists an intertwining operator  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  which is isometric and onto; the next proposition shows in particular that we would define the same notion by requiring only  $T$  to be invertible.

First, let us recall some terminology. An operator  $U$  from a Hilbert space  $\mathcal{H}_1$  to another Hilbert space  $\mathcal{H}_2$  is a *partial isometry* if there exists a closed subspace  $\mathcal{M}$  of  $\mathcal{H}_1$  such that the restriction of  $U$  to  $\mathcal{M}$  is an isometry (that is,  $\|U\xi\| = \|\xi\|$  for all  $\xi$  in  $\mathcal{M}$ ) and such that  $U = 0$  on  $\mathcal{M}^{\perp}$ . The subspace  $\mathcal{M} = (\text{Ker } U)^{\perp}$  is the *initial space* and  $U(\mathcal{M})$  the *final space* of the partial isometry  $U$ .

Given a continuous operator  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ , set  $|T| = (T^*T)^{1/2}$ . Since

$$\| |T| \xi \|^2 = \langle T^*T\xi, \xi \rangle = \|T\xi\|^2,$$



the mapping  $|T|\xi \mapsto T\xi$  extends to an isometry  $U$  from the closure of  $|T|(\mathcal{H}_1)$  onto the closure of  $T(\mathcal{H}_1)$ . Extend now  $U$  linearly to  $\mathcal{H}_1$  by setting  $U = 0$  on  $|T|(\mathcal{H}_1)^\perp = \text{Ker}T$ . Then  $U$  is a partial isometry with initial space  $(\text{Ker}T)^\perp$  and final space the closure of  $T(\mathcal{H}_1)$ , and we have  $T = U|T|$ . This is the so-called *polar decomposition* of  $T$  (for more details, see Problem 105 in [Halmo–67]).

**Proposition A.1.4** *Let  $(\pi_1, \mathcal{H}_1)$  and  $(\pi_2, \mathcal{H}_2)$  be two unitary representations of  $G$ . Let  $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  be an intertwining operator between  $\pi_1$  and  $\pi_2$ ; set  $\mathcal{M}_1 = (\text{Ker}T)^\perp$  and let  $\mathcal{M}_2$  denote the closure of the image of  $T$ .*

*Then  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are closed invariant subspaces of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively, and the subrepresentation of  $\pi_1$  defined by  $\mathcal{M}_1$  is equivalent to the subrepresentation of  $\pi_2$  defined by  $\mathcal{M}_2$ .*

**Proof** We first check that  $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$  intertwines  $\pi_2$  and  $\pi_1$ . Indeed, for all  $g \in G$ , we have

$$T^*\pi_2(g) = (\pi_2(g^{-1})T)^* = (T\pi_1(g^{-1}))^* = \pi_1(g)T^*.$$

It follows that  $T^*T \in \mathcal{L}(\mathcal{H}_1)$  intertwines  $\pi_1$  with itself. Since  $|T| = (T^*T)^{1/2}$  is a limit in the strong operator topology of polynomials in  $T^*T$  (see Problem 95 in [Halmo–67]),  $|T|$  also intertwines  $\pi_1$  with itself. Let  $T = U|T|$  be the polar decomposition of  $T$ ; then  $\text{Ker}U = \text{Ker}T$  and the restriction of  $U$  to  $\mathcal{M}_1 = (\text{Ker}T)^\perp$  is an isometry onto  $\mathcal{M}_2$ . It remains to check that  $U$  intertwines  $\pi_1$  with  $\pi_2$ .

Let  $g$  in  $G$ . On the one hand,

$$\pi_2(g)U|T|\xi = \pi_2(g)T\xi = T\pi_1(g)\xi = U|T|\pi_1(g)\xi = U\pi_1(g)|T|\xi,$$

for all  $\xi$  in  $\mathcal{H}_1$ . This shows that  $\pi_2(g)U$  and  $U\pi_1(g)$  coincide on the image of  $|T|$ , and therefore on its closure  $\mathcal{M}_1$ . On the other hand,  $U = 0$  on  $\text{Ker}T$  and  $\text{Ker}T$  is  $\pi_1(G)$ -invariant. Hence  $\pi_2(g)U = U\pi_1(g)$  on  $\mathcal{H}_1$ . ■

**Definition A.1.5** A unitary representation  $\rho$  of  $G$  is *strongly contained* or *contained* in a representation  $\pi$  of  $G$  if  $\rho$  is equivalent to a subrepresentation of  $\pi$ . This is denoted by  $\rho \subset \pi$ .

Let  $(\mathcal{H}_i, \langle \cdot, \cdot \rangle_i)_{i \in I}$  be a family of Hilbert spaces. The Hilbert direct sum of the  $\mathcal{H}_i$ 's, denoted by  $\bigoplus_{i \in I} \mathcal{H}_i$ , is the Hilbert space consisting of all families  $(\xi_i)_i$  with  $\xi_i \in \mathcal{H}_i$  such that  $\sum_i \langle \xi_i, \xi_i \rangle_i < \infty$  with inner product

$$\langle (\xi_i)_i, (\eta_i)_i \rangle = \sum_i \langle \xi_i, \eta_i \rangle_i.$$

We denote the elements of  $\bigoplus_{i \in I} \mathcal{H}_i$  by  $\oplus_i \xi_i$  instead of  $(\xi_i)_i$ .

Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ , and let  $\mathcal{K}$  be a closed  $G$ -invariant subspace. Then

$$\pi(g) = \pi^{\mathcal{K}}(g) \oplus \pi^{\mathcal{K}^\perp}(g),$$

for all  $g \in G$ . Thus,  $\pi$  is the direct sum of the two representations  $\pi^{\mathcal{K}}$  and  $\pi^{\mathcal{K}^\perp}$ , in the sense of the following definition.

**Definition A.1.6** Let  $(\pi_i, \mathcal{H}_i)_{i \in I}$  be a family of unitary representations of  $G$ . Let  $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$  be the Hilbert direct sum of the  $\mathcal{H}_i$ 's. The *direct sum of the representations*  $\pi_i$  is the unitary representation  $\pi$  of  $G$  on  $\mathcal{H}$ , defined by

$$\pi(g)(\oplus_i \xi_i) = \oplus_i \pi_i(g) \xi_i, \quad \text{for all } g \in G, \oplus_i \xi_i \in \mathcal{H}.$$

Viewing each  $\mathcal{H}_i$  as a subspace of  $\mathcal{H}$ , the representation  $(\pi_i, \mathcal{H}_i)$  can be identified with a subrepresentation of  $\pi$ . We write  $\pi = \bigoplus_{i \in I} \pi_i$ .

If all the representations  $\pi_i$  are equivalent to the same representation  $\sigma$ , we will sometimes write  $\pi = n\sigma$ , where  $n$  is the cardinality of  $I$ .

**Definition A.1.7** A unitary representation  $(\pi, \mathcal{H})$  of  $G$  is said to be *irreducible* if the only  $G$ -invariant closed subspaces of  $\mathcal{H}$  are the trivial ones, that is,  $\{0\}$  and  $\mathcal{H}$ .

The set of equivalence classes of irreducible representations of  $G$  is called the *unitary dual* of  $G$  and is denoted by  $\widehat{G}$ .

At this point, it is not clear that  $\widehat{G}$  is a *set*. It will be seen that this is indeed the case, in Corollary A.2.3 for abelian groups and in Remark C.4.13 in general; we will also equip  $\widehat{G}$  with a natural topology, in Definition A.2.4 for abelian groups and in Section F.2 in general.

We will use several times the following elementary fact.

**Proposition A.1.8** Let  $(\pi_i, \mathcal{H}_i)_{i \in I}$  be a family of unitary representations of  $G$  and let  $(\pi, \mathcal{K})$  be an irreducible unitary representation of  $G$ . Assume that  $\pi$  is strongly contained in  $\bigoplus_{i \in I} \pi_i$ . Then  $\pi$  is strongly contained in  $\pi_i$  for some  $i \in I$ .

**Proof** Set  $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$  and, for every  $i \in I$ , let  $p_i: \mathcal{H} \rightarrow \mathcal{H}_i$  denote the corresponding orthogonal projection.

Since  $\pi$  is strongly contained in  $\bigoplus_{i \in I} \pi_i$ , there exists a non-zero intertwining operator  $T: \mathcal{K} \rightarrow \bigoplus_{i \in I} \mathcal{H}_i$ . Hence, there exists  $i \in I$  such that  $T_i = p_i \circ T$  is non-zero. It is clear that  $T_i$  is an intertwining operator between  $\pi$  and

$\pi_i$ . As  $\pi$  is irreducible, we have  $\text{Ker } T_i = \{0\}$  and the claim follows from Proposition A.1.4. ■

One-dimensional unitary representations of  $G$  are obviously irreducible. They correspond to the unitary characters of  $G$ .

**Definition A.1.9** A *unitary character* of  $G$  is a continuous homomorphism  $\chi : G \rightarrow \mathbf{S}^1$ , where  $\mathbf{S}^1$  is the multiplicative group of all complex numbers of modulus 1. We will identify a one-dimensional representation  $\pi$  of  $G$  with its character  $g \mapsto \text{Trace}(\pi(g))$ .

The constant character

$$G \rightarrow \mathbf{S}^1, \quad g \mapsto 1$$

is called the *unit representation* or *unit character* of  $G$  and will be denoted by  $1_G$ .

Observe that  $1_G$  is contained in a unitary representation  $(\pi, \mathcal{H})$  if and only if the subspace

$$\mathcal{H}^G = \{\xi \in \mathcal{H} : \pi(g)\xi = \xi, \quad \text{for all } g \in G\}$$

of  $G$ -invariant vectors in  $\mathcal{H}$  is non-zero. Observe also that two one-dimensional unitary representations are equivalent if and only if their associated unitary characters coincide.

Let  $N$  be a closed normal subgroup of  $G$ . Let the quotient group  $G/N$  be given the quotient topology, and let  $p : G \rightarrow G/N$  denote the canonical projection. Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G/N$ . Then  $(\pi \circ p, \mathcal{H})$  is a unitary representation of  $G$  called the *lift* of  $\pi$  to  $G$ . We will often use the same notation for  $\pi$  and its lift to  $G$ , and in particular view  $\widehat{G/N}$  as a subset of  $\widehat{G}$ .

Let  $\mathcal{H}$  be a Hilbert space. The *conjugate Hilbert space*  $\bar{\mathcal{H}}$  is the Hilbert space with underlying additive group identical to that of  $\mathcal{H}$ , with scalar multiplication defined by

$$(\lambda, \xi) \mapsto \bar{\lambda}\xi, \quad \lambda \in \mathbf{C}, \xi \in \mathcal{H}$$

and with inner product  $[\cdot, \cdot]$  defined by

$$[\xi, \eta] = \langle \eta, \xi \rangle, \quad \xi, \eta \in \mathcal{H}.$$

By Riesz's Theorem, the mapping

$$\bar{\mathcal{H}} \rightarrow \mathcal{L}(\mathcal{H}, \mathbf{C}), \quad \eta \mapsto \varphi_\eta,$$

with  $\varphi_\eta(\xi) = \langle \xi, \eta \rangle$ , is a linear isomorphism of complex vector spaces.

Let  $G$  be a topological group and  $(\pi, \mathcal{H})$  a unitary representation of  $G$ . For  $g \in G$ , denote by  $\bar{\pi}(g)$  the operator on  $\bar{\mathcal{H}}$  which coincides with  $\pi(g)$  as a set-theoretical transformation. It is straightforward to check that  $\bar{\pi}$  is a unitary representation of  $G$  in  $\bar{\mathcal{H}}$ . Observe that  $\bar{\bar{\pi}} = \pi$ .

**Definition A.1.10** The representation  $\bar{\pi}$  is called the *contragredient* or *conjugate* representation of  $\pi$ .

We now discuss tensor products of unitary representations. Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces, and  $\mathcal{H} \otimes \mathcal{K}$  their algebraic tensor product. The completion of  $\mathcal{H} \otimes \mathcal{K}$ , with respect to the unique inner product for which

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \xi_2 \rangle \langle \eta_1, \eta_2 \rangle, \quad \xi_1, \xi_2 \in \mathcal{H}, \quad \eta_1, \eta_2 \in \mathcal{K},$$

is called the *Hilbert tensor product* of  $\mathcal{H}$  and  $\mathcal{K}$  and is again denoted by  $\mathcal{H} \otimes \mathcal{K}$ .

**Definition A.1.11** If  $\pi$  and  $\rho$  are unitary representations of a topological group  $G$  on  $\mathcal{H}$  and  $\mathcal{K}$ , their *tensor product* is the unitary representation  $\pi \otimes \rho$  of  $G$  defined on  $\mathcal{H} \otimes \mathcal{K}$  by

$$(\pi \otimes \rho)(g)(\xi \otimes \eta) = \pi(g)\xi \otimes \rho(g)\eta,$$

for all  $\xi \in \mathcal{H}$ ,  $\eta \in \mathcal{K}$ , and  $g \in G$ .

Observe that  $\pi \otimes \rho$  is the restriction to the diagonal subgroup  $\Delta \cong G$  of the *outer tensor product*  $\pi \times \rho$ , a unitary representation of  $G \times G$  defined on the same Hilbert space by

$$(\pi \times \rho)(g_1, g_2)(\xi \otimes \eta) = \pi(g_1)\xi \otimes \rho(g_2)\eta.$$

We discuss now another realisation of  $\pi \otimes \rho$ . Let  $\bar{\mathcal{K}}$  be the conjugate Hilbert space of  $\mathcal{K}$ , and denote by  $HS(\bar{\mathcal{K}}, \mathcal{H})$  the space of Hilbert–Schmidt operators from  $\bar{\mathcal{K}}$  to  $\mathcal{H}$ . Equipped with the inner product

$$(T, S) \mapsto \text{Trace}(S^*T), \quad T, S \in HS(\bar{\mathcal{K}}, \mathcal{H}),$$

$HS(\bar{\mathcal{K}}, \mathcal{H})$  is a Hilbert space. Let  $\Phi: \mathcal{H} \otimes \mathcal{K} \rightarrow HS(\bar{\mathcal{K}}, \mathcal{H})$  be the linear mapping defined, for all  $\xi \in \mathcal{H}$  and  $\eta \in \mathcal{K}$ , by

$$\Phi(\xi \otimes \eta)(\zeta) = \langle \eta, \zeta \rangle \xi, \quad \zeta \in \bar{\mathcal{K}},$$

where the inner product on the right is taken in  $\mathcal{K}$ . Then  $\Phi$  is easily seen to be an isomorphism of Hilbert spaces.

The tensor product  $\pi \otimes \rho$  corresponds to the unitary representation of  $G$  on  $HS(\overline{\mathcal{K}}, \mathcal{H})$  given by

$$T \mapsto \pi(g)T\overline{\rho}(g^{-1}), \quad T \in HS(\overline{\mathcal{K}}, \mathcal{H}), \quad g \in G.$$

This realisation of  $\pi \otimes \rho$  is often very useful. For instance, it shows immediately that if  $\pi$  is *finite dimensional* then  $\pi \otimes \overline{\pi}$  contains the unit representation: the identity operator  $I$  is obviously invariant. This is a particular case of the following proposition.

**Proposition A.1.12** *Let  $\pi$  and  $\rho$  be unitary representations of the topological group  $G$  on  $\mathcal{H}$  and  $\mathcal{K}$ . The following properties are equivalent:*

- (i)  $\pi \otimes \rho$  contains the unit representation  $1_G$  (that is,  $\pi \otimes \rho$  has a non-zero invariant vector);
- (ii) there exists a finite dimensional representation of  $G$  which is a subrepresentation of both  $\pi$  and  $\overline{\rho}$ .

**Proof** That (ii) implies (i) was already mentioned. Assume that  $\pi \otimes \rho$  contains  $1_G$ . Then there exists a non-zero  $T \in HS(\overline{\mathcal{K}}, \mathcal{H})$  with

$$\pi(g)T\overline{\rho}(g^{-1}) = T$$

for all  $g \in G$ . Thus,  $T$  intertwines  $\pi$  and  $\overline{\rho}$ . Since  $T^*T \in \mathcal{L}(\overline{\mathcal{K}})$  is a compact positive operator and is non-zero, it has an eigenvalue  $\lambda > 0$ . The corresponding eigenspace  $E_\lambda$  of  $T^*T$  is a finite dimensional closed subspace of  $\overline{\mathcal{K}}$ . As  $T^*T$  intertwines  $\pi$  with itself,  $E_\lambda$  is invariant. Since

$$\|T\xi\|^2 = \langle T^*T\xi, \xi \rangle = \lambda\|\xi\|^2, \quad \text{for all } \xi \in E_\lambda,$$

the restriction of  $\lambda^{-1/2}T$  to  $E_\lambda$  is a bijective isometry between the finite dimensional invariant subspaces  $E_\lambda$  and  $T(E_\lambda)$  of  $\overline{\mathcal{K}}$  and  $\mathcal{H}$ , intertwining  $\overline{\rho}$  and  $\pi$ . ■

**Corollary A.1.13** *Let  $\pi$  and  $\rho$  be unitary representations of the topological group  $G$ , and assume that  $\pi$  is irreducible. The following properties are equivalent:*

- (i)  $\pi \otimes \rho$  contains the unit representation  $1_G$ ;
- (ii)  $\pi$  is finite dimensional and  $\pi$  is contained in  $\overline{\rho}$ .

## A.2 Schur's Lemma

For a unitary representation  $(\pi, \mathcal{H})$  of  $G$ , the set of all operators  $T \in \mathcal{L}(\mathcal{H})$  which intertwine  $\pi$  with itself (that is, those such that  $T\pi(g) = \pi(g)T$  for all  $g \in G$ ) is called the *commutant* of  $\pi(G)$  and is denoted by  $\pi(G)'$ .

It is obvious that  $\pi(G)'$  is a subalgebra of  $\mathcal{L}(\mathcal{H})$  which is closed for the weak operator topology. Moreover,  $\pi(G)'$  is self-adjoint, that is, if  $T \in \pi(G)'$  then  $T^* \in \pi(G)'$ ; see the proof of Proposition A.1.4.

**Proposition A.2.1** *Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Let  $\mathcal{K}$  be a closed subspace of  $\mathcal{H}$  and let  $P \in \mathcal{L}(\mathcal{H})$  be the orthogonal projection onto  $\mathcal{K}$ . Then  $\mathcal{K}$  is  $G$ -invariant if and only if  $P \in \pi(G)'$ .*

**Proof** Assume that  $\mathcal{K}$  is  $G$ -invariant. Then  $\pi(g)P = P\pi(g)P$  for all  $g \in G$ . Hence

$$P\pi(g) = (\pi(g^{-1})P)^* = (P\pi(g^{-1})P)^* = P\pi(g)P$$

and therefore  $\pi(g)P = P\pi(g)$  for all  $g \in G$ .

Conversely, assume that  $\pi(g)P = P\pi(g)$  for all  $g$  in  $G$ . Then, clearly,  $\mathcal{K}$  is  $G$ -invariant. ■

For an operator  $T \in \mathcal{L}(\mathcal{H})$ , recall that the *spectrum*  $\sigma(T)$  of  $T$  is the set of all  $\lambda \in \mathbb{C}$  such that  $T - \lambda I$  is not invertible. It is a non-empty, compact subset of  $\mathbb{C}$ . We assume from now on that  $T$  is self-adjoint (that is,  $T = T^*$ ), so that  $\sigma(T)$  is contained in  $\mathbb{R}$ .

The *functional calculus* associates to any bounded Borel function  $f : \sigma(T) \rightarrow \mathbb{C}$ , an operator  $f(T) \in \mathcal{L}(\mathcal{H})$  in such a way that the following properties hold:

(i) if  $f(x) = \sum_{i=0}^n a_i x^i$  is a polynomial with complex coefficients, then  $f(T) = \sum_{i=0}^n a_i T^i$ ;

(ii) let  $\mathcal{B}(\sigma(T))$  denote the  $*$ -algebra of complex-valued bounded Borel functions on  $\sigma(T)$ , with pointwise operations and complex conjugation, and let  $W^*(T)$  denote the closure of the  $*$ -algebra

$$\left\{ \sum_{i=0}^n a_i T^i : n \geq 0, a_0, \dots, a_n \in \mathbb{C} \right\}$$

in  $\mathcal{L}(\mathcal{H})$  for the strong operator topology; then  $f \mapsto f(T)$  is a  $*$ -algebra homomorphism from  $\mathcal{B}(\sigma(T))$  onto  $W^*(T)$ ;

(iii) let  $C(\sigma(T))$  be the algebra of complex-valued continuous functions on  $\sigma(T)$  equipped with the supremum norm; the restriction of  $f \mapsto f(T)$  to  $C(\sigma(T))$  is isometric.

It follows from (ii) that, for  $f \in \mathcal{B}(\sigma(T))$ , the operator  $f(T)$  commutes with every bounded operator on  $\mathcal{H}$  which commutes with  $T$ . For all this, see [Rudin-73, 12.24].

**Theorem A.2.2 (Schur's Lemma)** *A unitary representation  $(\pi, \mathcal{H})$  of  $G$  is irreducible if and only if  $\pi(G)'$  consists of the scalar multiples of the identity operator  $I$ .*

**Proof** Assume that  $\pi(G)'$  consists of the multiples of  $I$ . Let  $\mathcal{K}$  be a closed  $G$ -invariant subspace of  $\mathcal{H}$ , with corresponding orthogonal projection  $P$ . By the previous proposition,  $P \in \pi(G)'$ . So,  $P = \lambda I$  for some complex number  $\lambda$ . As  $P^2 = P$ , we have  $\lambda = 0$  or  $\lambda = 1$ , that is,  $\mathcal{K} = \{0\}$  or  $\mathcal{K} = \mathcal{H}$ . Thus,  $\pi$  is irreducible.

Conversely, assume that  $\pi$  is irreducible and let  $T \in \pi(G)'$ . Set  $T_1 = (T + T^*)/2$  and  $T_2 = (T - T^*)/2i$ . Then  $T_1$  and  $T_2$  are self-adjoint,  $T = T_1 + iT_2$ , and  $T_1, T_2 \in \pi(G)'$ . If we can show that  $T_1$  and  $T_2$  are scalar multiples of  $I$ , then  $T$  is a scalar multiple of  $I$ . Hence, we can assume that  $T$  is self-adjoint.

We claim that the spectrum  $\sigma(T)$  of  $T$  consists of a single real number  $\lambda$ . Once proved, the functional calculus shows that  $T = \lambda I$ .

Assume, by contradiction, that there exist  $\lambda_1, \lambda_2 \in \sigma(T)$  with  $\lambda_1 \neq \lambda_2$ . Let  $U_1$  and  $U_2$  be disjoint neighbourhoods of  $\lambda_1$  and  $\lambda_2$  in  $\sigma(T)$ . We can find real-valued functions  $f_1, f_2 \in C(\sigma(T))$  with

$$0 \leq f_i \leq \chi_{U_i} \quad \text{and} \quad f_i \neq 0, \quad \text{for } i = 1, 2,$$

where  $\chi_{U_i}$  denotes the characteristic function of  $U_i$ . Let  $f_i(T), \chi_{U_i}(T) \in \mathcal{L}(\mathcal{H})$  be defined by functional calculus. Then, by (ii) and (iii) above, we have

$$0 \leq f_i(T) \leq \chi_{U_i}(T) \quad \text{and} \quad f_i(T) \neq 0, \quad \text{for } i = 1, 2.$$

Hence,  $\chi_{U_i}(T) \neq 0$  for  $i = 1, 2$ . Moreover,

$$\chi_{U_i}(T)^2 = \chi_{U_i}(T) = \chi_{U_i}(T)^*,$$

that is,  $\chi_{U_i}(T)$  is an orthogonal projection. Now  $\chi_{U_i}(T) \in \pi(G)'$ , since  $T$  commutes with every  $\pi(g)$ . Hence,  $\chi_{U_i}(T) = 0$  or  $\chi_{U_i}(T) = I$ , by irreducibility of  $\pi$ . As  $\chi_{U_i}(T) \neq 0$ , it follows that  $\chi_{U_i}(T) = I$  for  $i = 1, 2$ . This is a contradiction since  $\chi_{U_1}(T)\chi_{U_2}(T) = (\chi_{U_1}\chi_{U_2})(T) = 0$ . ■

**Corollary A.2.3** *Let  $G$  be an abelian topological group. Then any irreducible unitary representation of  $G$  is one-dimensional. Thus, the unitary dual  $\widehat{G}$  can be identified with the set of unitary characters of  $G$ .*

**Proof** Let  $(\pi, \mathcal{H})$  be an irreducible unitary representation of  $G$ . Since  $G$  is abelian,  $\pi(G)$  is contained in  $\pi(G)'$ . Thus, by Schur's Lemma, for every  $g \in G$ , the unitary operator  $\pi(g)$  is of the form  $\chi(g)I$ . It is clear that the dimension of  $\mathcal{H}$  is 1 and that  $g \mapsto \chi(g)$  is a unitary character of  $G$ . ■

For an arbitrary topological group  $G$ , the set of unitary characters of  $G$  is a group for pointwise multiplication

$$\chi_1 \chi_2(g) := \chi_1(g) \chi_2(g), \quad g \in G,$$

the group unit being the unit character, and the inverse of  $\chi$  being the conjugate character  $\bar{\chi}$  (which corresponds to the contragredient representation). Hence, if  $G$  is abelian, the unitary dual  $\widehat{G}$  is an abelian group.

**Definition A.2.4** Let  $G$  be an abelian topological group. The unitary dual  $\widehat{G}$  of  $G$  is called the *dual group* of  $G$ . Equipped with the topology of uniform convergence on compact subsets of  $G$ , the dual group  $\widehat{G}$  is a topological group.

Let  $G$  be again arbitrary; let  $\overline{[G, G]}$  denote the closure of the subgroup generated by all commutators  $[g, h] = ghg^{-1}h^{-1}$  for  $g, h \in G$ . Every unitary character of  $G$  is the lift of a character of  $G/\overline{[G, G]}$ . Thus, the set of unitary characters of  $G$  can be identified with the dual group of  $G/\overline{[G, G]}$ .

**Example A.2.5** (i) Every  $y \in \mathbf{R}$  defines a unitary character of  $\mathbf{R}$  by

$$\chi_y(x) = e^{2\pi ixy}, \quad x \in \mathbf{R}.$$

We claim that every unitary character of  $\mathbf{R}$  is of the form  $\chi_y$  for a unique  $y \in \mathbf{R}$ . This can be shown as follows (see [Folla-95, (4.5)] for a proof using differential equations).

Let  $\chi \in \widehat{\mathbf{R}}$ . By continuity of  $\chi$ , there exists a sequence  $(y_n)_n$  in  $\mathbf{R}$  with  $\lim_n y_n = 0$  such that  $\chi(2^{-n}) = e^{2\pi i y_n}$ , for all  $n$ . Since  $\chi(2^{-n}) = \chi(2^{-(n+1)})^2$ , we have  $e^{2\pi i(2y_{n+1} - y_n)} = 1$  and, hence,

$$2y_{n+1} - y_n \in \mathbf{Z}.$$

Thus, there exists some  $N$  such that  $2y_{n+1} = y_n$  for all  $n \geq N$ . Hence,  $2^n y_n = 2^N y_N$ , for all  $n \geq N$ . Set  $y = 2^N y_N$ . Then  $\chi(2^{-n}) = \chi_y(2^{-n})$  for all  $n \geq N$ . This implies that  $\chi(k2^{-n}) = \chi_y(k2^{-n})$  for all  $k \in \mathbf{Z}$  and all  $n \in \mathbf{N}$ . Hence, by density,  $\chi(x) = \chi_y(x)$  for all  $x \in \mathbf{R}$ . The uniqueness of  $y$  is clear, and the mapping  $y \mapsto \chi_y$  from  $\mathbf{R}$  to  $\widehat{\mathbf{R}}$  is a group isomorphism.



See Proposition D.4.5 for a generalisation of this to any local field.

(ii) Let  $G_1, \dots, G_n$  be abelian topological groups, and let  $G$  be their direct product. The mapping

$$(\chi_1, \dots, \chi_n) \mapsto \chi_1 \otimes \dots \otimes \chi_n,$$

where  $(\chi_1 \otimes \dots \otimes \chi_n)(x_1, \dots, x_n) = \chi_1(x_1) \dots \chi_n(x_n)$ , is an isomorphism between  $\widehat{G_1} \times \dots \times \widehat{G_n}$  and  $\widehat{G}$ .

Indeed,  $\chi_1 \otimes \dots \otimes \chi_n$  is a character of  $G$  for  $\chi_i \in \widehat{G_i}$ . Conversely, if  $\chi \in \widehat{G}$ , then  $\chi = \chi_1 \otimes \dots \otimes \chi_n$ , for  $\chi_i \in \widehat{G_i}$  defined by

$$\chi_i(x) = \chi(e, \dots, e, x, e, \dots, e), \quad x \in G_i.$$

(iii) Every  $y = (y_1, \dots, y_n) \in \mathbf{R}^n$  defines a unitary character of  $\mathbf{R}^n$  by

$$\chi_y(x) = \prod_{k=1}^n e^{2\pi i x_k y_k}, \quad x = (x_1, \dots, x_n) \in \mathbf{R}^n.$$

It follows from (i) and (ii) that  $y \mapsto \chi_y$  is an isomorphism between  $\mathbf{R}^n$  and  $\widehat{\mathbf{R}^n}$ . (Compare with Corollary D.4.6.)

(iv) A character  $\chi_y$  of  $\mathbf{R}^n$  factorises through the torus  $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$  if and only if  $y \in \mathbf{Z}^n$ . Hence,  $\widehat{\mathbf{T}^n}$  is isomorphic to  $\mathbf{Z}^n$ .

### A.3 The Haar measure of a locally compact group

A topological group  $G$  is locally compact if and only if there exists a compact neighbourhood of the group unit  $e$ . This class of groups includes all *discrete groups* (that is, groups with the discrete topology), all compact groups, and all (finite dimensional) Lie groups.

One of the features which distinguishes locally compact groups from other topological groups is the existence of a Haar measure. We first recall a few facts about measure theory on locally compact spaces. It should be mentioned that our measures are usually not assumed to be  $\sigma$ -finite. For what follows, some good references are [BeChR-84], [Bou-Int1], [Bou-Int2], [Halm-74], [HewRo-63], and [Rudin-3].

Let  $X$  be a locally compact space. A *Borel measure* on  $X$  is a (not necessarily finite) positive measure on the  $\sigma$ -algebra  $\mathcal{B}(X)$  of Borel subsets of  $X$  (this is the smallest  $\sigma$ -algebra containing all open subsets). A Borel measure  $\mu$  on  $X$  is said to be *regular* if

- (i)  $\mu(B) = \inf\{\mu(V) : V \text{ is open and } B \subset V\}$  for every Borel set  $B$ ,
- (ii)  $\mu(U) = \sup\{\mu(K) : K \text{ is compact and } K \subset U\}$  for every open subset  $U$  of  $X$ ,
- (iii)  $\mu(K) < \infty$  for every compact subset  $K$  of  $X$ .

If every open subset of  $X$  is  $\sigma$ -compact (that is, a countable union of compact subsets), then (iii) implies (i) and (ii); see [Rudin–3, Theorem 2.18].

Let  $C_c(X)$  denote the space of continuous functions on  $X$  with compact support. A linear functional  $\varphi : C_c(X) \rightarrow \mathbb{C}$  is said to be positive if  $\varphi(f) \geq 0$  for all non-negative real valued functions  $f$  in  $C_c(X)$ . Such a linear functional is called a *Radon measure* on  $X$ . A regular measure  $\mu$  defines a Radon measure on  $X$ , still denoted by  $\mu$ , by means of the formula

$$\mu(f) = \int_X f(x) d\mu(x), \quad \text{for all } f \in C_c(X).$$

Conversely, any Radon measure on  $X$  is obtained in this way from a unique regular Borel measure on  $X$ . This is the standard *Riesz Representation Theorem* (see [Rudin–3, Theorem 2.14]).

Let  $G$  be a locally compact group. There exists a non-zero regular Borel measure  $\mu$  on  $G$  which is left invariant, that is,

$$\mu(gB) = \mu(B), \quad \text{for all } B \in \mathcal{B}(G), g \in G,$$

or, equivalently,

$$\int_G f(g^{-1}x) d\mu(x) = \int_G f(x) d\mu(x), \quad \text{for all } f \in C_c(G), g \in G.$$

Moreover  $\mu$  is unique, up to a multiplicative constant, that is, if  $\mu'$  is another regular Borel measure which is left invariant, then  $\mu' = c\mu$  for some non-negative number  $c$ . For all this, see [HewRo–63, Chapter 15]. The measure  $\mu$  is called a left *Haar measure* on  $G$ .

**Remark A.3.1** A result of Weil says that the class of groups which admit a left invariant measure essentially coincides with the class of locally compact groups. See [Weil–65, pp. 140–146]; see also [Macke–57] and [Varad–68].

As the next proposition shows, the support of a Haar measure on  $G$  is  $G$ . Recall that the *support* of a regular Borel measure  $\mu$  on a locally compact space  $X$  is the smallest closed subset  $F$  of  $X$  such that  $\mu(X \setminus F) = 0$ .

**Proposition A.3.2** *Let  $\mu$  be a left Haar measure on  $G$ . Then,  $\mu(U) > 0$  for every non-empty open subset  $U$  of  $G$  and  $\int_G f(x) d\mu(x) > 0$  for every non-negative function  $f \in C_c(G)$  with  $f \neq 0$ .*

**Proof** Assume that  $\mu(U) = 0$  for some open non-empty subset  $U$  of  $G$ . Then  $\mu(gU) = 0$  for all  $g \in G$ , by left invariance of  $\mu$ . For any compact subset  $K$  of  $G$ , there exists  $g_1, \dots, g_n \in G$  such that  $K \subset \cup_{i=1}^n g_i U$ . Hence,  $\mu(K) = 0$  for every compact subset  $K$  of  $G$ . Since  $\mu$  is regular, this implies that  $\mu = 0$ , a contradiction.

Let  $f \in C_c(G)$  be non-negative and  $f \neq 0$ . Then, by continuity,  $f \geq \varepsilon$  on a non-empty open subset  $U$  of  $G$  and for some  $\varepsilon > 0$ . Hence,

$$\int_G f(x) d\mu(x) \geq \varepsilon \mu(U) > 0,$$

as claimed. ■

Sometimes, but not always, left Haar measures are also *right* invariant. Whether this happens or not depends on the modular function of the group, to be introduced below. For each fixed  $g \in G$ , the Borel measure  $\mu_g$  defined by

$$\mu_g(B) = \mu(Bg), \quad \text{for all } B \in \mathcal{B}(G),$$

is a non-zero regular positive Borel measure which is left invariant. Hence,  $\mu_g = \Delta_G(g)\mu$  for some positive number  $\Delta_G(g)$ . Observe that  $\Delta_G(g)$  does not depend on the choice of  $\mu$ . We have

$$\int_G f(xg^{-1}) d\mu(x) = \Delta_G(g) \int_G f(x) d\mu(x), \quad \text{for all } f \in C_c(G), g \in G.$$

The function  $\Delta_G$ , which is clearly a homomorphism from  $G$  to the multiplicative group  $\mathbf{R}_+^*$  of positive real numbers, is called the *modular function* of  $G$ . We now show that it is continuous.

A mapping  $f$  from a topological group  $G$  to some metric space  $(X, d)$  is said to be *left uniformly continuous* (respectively, *right uniformly continuous*) if for every  $\varepsilon > 0$ , there exists a neighbourhood  $U$  of  $e$  such that

$$\sup_{x \in G} d(f(ux), f(x)) < \varepsilon \quad (\text{respectively, } \sup_{x \in G} d(f(xu), f(x)) < \varepsilon)$$

for all  $u \in U$ . (These definitions are often reversed in the literature.)

A standard  $\varepsilon/2$ -argument shows that, if  $f$  is continuous and has compact support, then  $f$  is left and right uniformly continuous.

For a subset  $A$  of a group and an integer  $n \geq 1$ , denote by  $A^{-1}$  the set of all  $a^{-1}$  for  $a \in A$  and by  $A^n$  the set of all products  $a_1 a_2 \cdots a_n$  for  $a_1, \dots, a_n \in A$ . Set also  $A^{-n} = (A^{-1})^n$  and  $A^0 = \{e\}$ .

**Proposition A.3.3** *The modular function  $\Delta_G$  is continuous.*

**Proof** Let  $f \in C_c(G)$  be such that  $\int_G f(x) d\mu(x) \neq 0$ . Then

$$\Delta_G(g) = \frac{\int_G f(xg^{-1}) d\mu(x)}{\int_G f(x) d\mu(x)}.$$

We claim that  $\varphi : g \mapsto \int_G f(xg^{-1}) d\mu(x)$  is continuous. Indeed, let  $g \in G$  and  $\varepsilon > 0$ . Choose a compact neighbourhood  $U_0$  of  $e$ . As  $f$  is right uniformly continuous, there exists a neighbourhood  $U = U^{-1}$  of  $e$  contained in  $U_0$  such that

$$\sup_{x \in G} |f(xu^{-1}) - f(x)| < \varepsilon / \mu(KU_0g), \quad \text{for all } u \in U,$$

where  $K = \text{supp } f$ . Then

$$\begin{aligned} |\varphi(ug) - \varphi(g)| &\leq \int_G |f(xg^{-1}u^{-1}) - f(xg^{-1})| d\mu(x) \\ &\leq \mu(KU_0g) \sup_{x \in G} |f(xu^{-1}) - f(x)| < \varepsilon, \end{aligned}$$

for all  $u \in U$ . ■

**Lemma A.3.4** *We have*

$$\int_G f(x^{-1}) d\mu(x) = \int_G \Delta_G(x^{-1}) f(x) d\mu(x),$$

for all  $f \in C_c(G)$ , that is, symbolically,  $d\mu(x^{-1}) = \Delta_G(x^{-1}) d\mu(x)$ .

**Proof** The linear functional

$$f \mapsto \int_G f(x^{-1}) d\mu(x)$$

on  $C_c(G)$  defines a right invariant Haar measure on  $G$ . On the other hand, the same is true for the functional

$$f \mapsto \int_G \Delta_G(x^{-1}) f(x) d\mu(x),$$

as a consequence of the definition of  $\Delta_G$ . Hence, by uniqueness, there exists  $c > 0$  such that

$$(*) \quad \int_G f(x^{-1})d\mu(x) = c \int_G \Delta_G(x^{-1})f(x)d\mu(x), \quad \text{for all } f \in C_c(G).$$

It remains to show that  $c = 1$ . Replacing in this equality  $f$  by  $\check{f}$ , defined by  $\check{f}(x) = f(x^{-1})$ , we obtain

$$\int_G f(x)d\mu(x) = c \int_G \Delta_G(x^{-1})f(x^{-1})d\mu(x), \quad \text{for all } f \in C_c(G).$$

As

$$\int_G \Delta_G(x^{-1})f(x^{-1})d\mu(x) = c \int_G f(x)d\mu(x)$$

by (\*), it follows that  $c^2 = 1$ , that is,  $c = 1$ . ■

It is clear that a left Haar measure  $\mu$  on  $G$  is right invariant if and only if  $\Delta_G \equiv 1$ . In this case,  $G$  is said to be *unimodular*.

**Example A.3.5** (i) The Lebesgue measure is a Haar measure on  $\mathbf{R}^n$ . The same is true for the torus  $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ . Locally compact groups which are abelian are obviously unimodular.

(ii) If  $G$  is discrete, the counting measure  $\mu$  defined by  $\mu(B) = \#B$  for any subset  $B$  of  $G$  is a Haar measure. As  $\mu$  is also right invariant, discrete groups are unimodular.

(iii) Let

$$GL_n(\mathbf{R}) = \{X = (x_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbf{R}) : \det X \neq 0\}$$

be the group of all invertible real  $(n \times n)$  matrices, with its topology as open subset of  $M_n(\mathbf{R}) \cong \mathbf{R}^{n^2}$ . The measure  $d\mu(X) = |\det X|^{-n} \prod_{1 \leq i, j \leq n} dx_{ij}$ , defined by

$$\int_{GL_n(\mathbf{R})} f(X)d\mu(X) = \int_{GL_n(\mathbf{R})} f(x_{11}, \dots, x_{nn})|\det X|^{-n}dx_{11} \dots dx_{nn}$$

for  $f \in C_c(GL_n(\mathbf{R}))$ , is a left Haar measure which is also right invariant (Exercise A.8.9). So,  $GL_n(\mathbf{R})$  is unimodular.

(iv) Let  $G$  be the so-called  $(ax + b)$ -group (over  $\mathbf{R}$ ). Thus,

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{R}^*, b \in \mathbf{R} \right\}.$$

Then  $|a|^{-2}dad b$  is a left Haar measure, while  $|a|^{-1}dad b$  is a right Haar measure on  $G$  (Exercise A.8.5). This implies that the modular function of  $G$  is

$$\Delta_G \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = |a|^{-1}.$$

(v) Let  $G$  be the group of  $2 \times 2$  real upper triangular matrices with determinant 1. Then  $a^{-2}dad b$  is a left Haar measure on  $G$  and

$$\Delta_G \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^{-2}.$$

(vi) Let

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{R} \right\},$$

the so-called *Heisenberg group*. Then  $dx dy dz$  is a left and right Haar measure on  $G$  (Exercise A.8.6).

(vii) The existence of Haar measures on connected Lie groups is easy to establish. Indeed, let  $G$  be such a group, say of dimension  $n$ . Fix a non-zero alternating  $n$ -form  $\omega_e$  on the tangent space  $T_e(G)$  at the group unit  $e$ . For each  $g \in G$ , let  $\omega_g$  be the alternating  $n$ -form on  $T_g(G)$  which is the image of  $\omega_e$  under left translation by  $g^{-1}$ . In this way, we obtain a left invariant  $n$ -form  $\omega$  on  $G$ . This defines an orientation on  $G$ , and

$$f \mapsto \int_G f \omega, \quad f \in C_c(G),$$

is a left invariant Borel measure on  $G$ .

(viii) Any compact group  $G$  is unimodular. Indeed,  $\Delta_G(G)$  is a compact subgroup of  $\mathbf{R}_+^*$ . Since  $\{1\}$  is the only such subgroup of  $\mathbf{R}_+^*$ , it follows that  $\Delta_G \equiv 1$ .

(ix) More generally, if  $G/\overline{[G, G]}$  is compact, then  $G$  is unimodular. For instance,

$$G = SL_n(\mathbf{R}) = \{g \in M_n(\mathbf{R}) : \det g = 1\}$$

is unimodular, since  $[G, G] = G$ .

(x) By (ix), semisimple real Lie groups are unimodular.

The modular function of a connected Lie group  $G$  can be determined as follows. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , identified with the tangent space of  $G$  at  $e$ . For  $g$  in  $G$ , let  $\text{Ad}(g) : \mathfrak{g} \rightarrow \mathfrak{g}$  be the differential at  $e$  of the group automorphism

$G \rightarrow G, x \mapsto gxg^{-1}$ . Then  $\text{Ad}(g)$  is an automorphism of the Lie algebra  $\mathfrak{g}$ , and  $\text{Ad} : G \rightarrow GL(\mathfrak{g})$  is a representation of  $G$ , called the *adjoint representation* of  $G$ . The proof of the following proposition is left as Exercise A.8.7.

**Proposition A.3.6** *Let  $G$  be a connected Lie group. Then*

$$\Delta_G(g) = \det(\text{Ad}(g^{-1})), \quad g \in G.$$

**Example A.3.7** Let  $G$  be a connected nilpotent Lie group. Then  $\text{Ad}(g)$  is unipotent (that is  $\text{Ad}(g) - I$  is nilpotent) and, hence,  $\det(\text{Ad}(g)) = 1$  for every  $g$  in  $G$ . Thus, connected nilpotent Lie groups are unimodular. More generally, any locally compact nilpotent group is unimodular. Indeed, such a group  $G$  is of *polynomial growth*, that is, for every compact neighbourhood  $U$  of  $G$ , there exist a constant  $C > 0$  and an integer  $d$  such that  $\mu(U^n) \leq Cn^d$  for all  $n \in \mathbf{N}$ , where  $\mu$  is a left Haar measure on  $G$  [Guiva–73]. On the other hand, a locally compact group with polynomial growth is unimodular (Exercise A.8.10).

## A.4 The regular representation of a locally compact group

Throughout this section,  $G$  is a locally compact group, with fixed left Haar measure, usually denoted by  $dg$ .

For a function  $f : G \rightarrow \mathbf{C}$  on  $G$  and for  $a \in G$ , we define the left and right translates  ${}_af : G \rightarrow \mathbf{C}$  and  $f_a : G \rightarrow \mathbf{C}$  of  $f$  by

$${}_af(x) = f(ax) \quad \text{and} \quad f_a(x) = f(xa) \quad \text{for all } x \in G.$$

Let  $L^2(G) = L^2(G, dg)$  be the Hilbert space of (equivalence classes of) square integrable functions  $f : G \rightarrow \mathbf{C}$  with respect to the Haar measure. For  $g$  in  $G$ , the operator  $\lambda_G(g) : L^2(G) \rightarrow L^2(G)$  defined by

$$\lambda_G(g)\xi(x) = ({}_g\xi)(x) = \xi(g^{-1}x), \quad \xi \in L^2(G), x \in G,$$

is unitary, by left invariance of the Haar measure. Moreover,  $\lambda_G(gh) = \lambda_G(g)\lambda_G(h)$  for  $g, h$  in  $G$ , and the mapping

$$G \rightarrow L^2(G), \quad g \mapsto \lambda_G(g)\xi$$

is continuous for each  $\xi$  in  $L^2(G)$ ; see Exercise A.8.3. Thus,  $(\lambda_G, L^2(G))$  is a unitary representation, called the *left regular representation* of  $G$ .

The *right regular representation* of  $G$  is the unitary representation  $\rho_G$  on  $L^2(G)$  defined by

$$\rho_G(g)\xi(x) = \Delta_G(g)^{1/2}(\xi_g)(x) = \Delta_G(g)^{1/2}\xi(xg), \quad \xi \in L^2(G), \quad g, x \in G,$$

the constant factor  $\Delta_G(g)^{1/2}$  being introduced in order to insure the unitarity of  $\rho_G(g)$ .

**Proposition A.4.1** *The regular representations  $\lambda_G$  and  $\rho_G$  are equivalent.*

**Proof** Let  $T : L^2(G) \rightarrow L^2(G)$  be defined by

$$T\xi(x) = \Delta_G(x)^{-1/2}\xi(x^{-1}), \quad \xi \in L^2(G).$$

By Lemma A.3.4 above,  $T$  is a unitary operator. It is clear that  $T\lambda_G(g) = \rho_G(g)T$ , for all  $g$  in  $G$ . ■

Combining the representations  $\lambda_G$  and  $\rho_G$ , we obtain a unitary representation  $\pi$  of the product  $G \times G$  on  $L^2(G)$ , defined by

$$\begin{aligned} \pi(g_1, g_2)\xi(x) &= \lambda_G(g_1)\rho_G(g_2)\xi(x) \\ &= \Delta_G(g_2)^{1/2}\xi(g_1^{-1}xg_2), \quad \xi \in L^2(G), x \in G, \end{aligned}$$

and called the *left-right regular representation* of  $G \times G$ . The restriction of  $\pi$  to the diagonal subgroup  $\{(g, g) : g \in G\} \cong G$  is the *conjugation representation*  $\gamma_G$  in  $L^2(G)$ :

$$\gamma_G(g)\xi(x) = \Delta_G(g)^{1/2}\xi(g^{-1}xg), \quad \xi \in L^2(G), g, x \in G.$$

## A.5 Representations of compact groups

Let  $G$  be a compact group. Then its Haar measure  $dg$  is finite, that is,  $\int_G dg < \infty$  and the constant functions on  $G$  belong to  $L^2(G)$ . In particular, the unit representation  $1_G$  is contained in the regular representation  $\lambda_G$ . Each of these properties characterises compact groups:

**Proposition A.5.1** *For a locally compact group  $G$ , the following properties are equivalent:*

- (i)  $G$  is compact;
- (ii)  $1_G$  is contained in  $\lambda_G$ ;
- (iii) the Haar measure on  $G$  is finite.



**Proof** It is obvious that (i) implies (ii) and that (ii) implies (iii). To show that (iii) implies (i), denote by  $\mu$  a Haar measure on  $G$ . Let  $U$  be a compact neighbourhood of  $e$ . Assume that  $G$  is not compact. Then, we can find inductively an infinite sequence  $g_1, g_2, \dots$  in  $G$  with

$$g_{n+1} \notin \bigcup_{i=1}^n g_i U, \quad \text{for all } n \in \mathbf{N}.$$

Choose a neighbourhood  $V$  of  $e$  with  $V = V^{-1}$  and  $V^2 \subset U$ . Then

$$g_n V \cap g_m V = \emptyset, \quad \text{for all } n \neq m.$$

Hence,

$$\mu(G) \geq \mu\left(\bigcup_{n \in \mathbf{N}} g_n V\right) = \sum_{n \in \mathbf{N}} \mu(g_n V) = \sum_{n \in \mathbf{N}} \mu(V) = \infty,$$

since  $\mu$  is invariant and since  $\mu(V) > 0$ . ■

The central result in the “abstract” theory of unitary representations of compact groups is the Peter–Weyl Theorem (see [Robet–83, Chapter 5]). For the definition of the unitary dual  $\widehat{G}$  of  $G$ , see A.1.7.

**Theorem A.5.2 (Peter–Weyl)** *Let  $G$  be a compact group.*

- (i) *Every unitary representation of  $G$  is the direct sum of irreducible subrepresentations.*
- (ii) *Every irreducible unitary representation of  $G$  is finite dimensional.*
- (iii) *Every irreducible unitary representation of  $G$  is contained in the regular representation  $\lambda_G$  of  $G$ . More precisely,  $\lambda_G$  is the direct sum  $\bigoplus_{\pi \in \widehat{G}} \sigma_\pi$  of subrepresentations  $\sigma_\pi$ , where each  $\sigma_\pi$  is equivalent to  $(\dim \pi)\pi$ .*

## A.6 Unitary representations associated to group actions

The regular representation is an example of more general representations associated to group actions on measure spaces.

A *measure space*  $(\Omega, \mathcal{B}, \mu)$  is a set  $\Omega$  equipped with a  $\sigma$ -algebra  $\mathcal{B}$  and a positive measure  $\mu : \mathcal{B} \rightarrow \mathbf{R}$ .

Let  $G$  be a topological group. A group action of  $G$  on the measure space  $\Omega$  is *measurable* if the mapping

$$G \times \Omega \rightarrow \Omega, \quad (g, x) \mapsto gx$$

is measurable, when  $G$  is equipped with the  $\sigma$ -algebra of its Borel subsets. Given such an action, we define for each  $g \in G$  a measure  $g\mu$  on  $\Omega$  by  $g\mu(A) = \mu(g^{-1}A)$ , where  $A$  is a measurable subset of  $\Omega$ .

The measure  $\mu$  is invariant if  $g\mu = \mu$  for all  $g \in G$ . We say that  $\mu$  is *quasi-invariant* if  $\mu$  and  $g\mu$  are equivalent measures for all  $g \in G$ . Recall that two measures  $\mu_1$  and  $\mu_2$  on  $\Omega$  are *equivalent* if, for every measurable subset  $A$  of  $\Omega$ , we have  $\mu_1(A) = 0$  if and only if  $\mu_2(A) = 0$ .

Consider a measurable group action of  $G$  on  $\Omega$  such that  $\mu$  is quasi-invariant. Assume either that  $\mu$  is  $\sigma$ -finite (that is,  $\Omega$  is a countable union of measurable subsets of finite measure) or that  $\Omega$  is a locally compact space, that  $G$  acts continuously on  $\Omega$ , and that  $\mu$  is a regular Borel measure. Then there exists a non-negative measurable function  $\frac{dg\mu}{d\mu}$  on  $\Omega$ , called the *Radon–Nikodym derivative* of  $g\mu$  with respect to  $\mu$ , such that

$$\int_{\Omega} f(g\omega) \frac{dg\mu}{d\mu}(\omega) d\mu(\omega) = \int_{\Omega} f(\omega) d\mu(\omega), \quad \text{for all } f \in L^1(\Omega, \mu)$$

(see [Bou–Int1, Chapter 5, Section 5, No 5]). Moreover, the function

$$c_{\mu} : G \times \Omega \rightarrow \mathbf{R}_+^*, \quad c_{\mu}(g, \omega) = \frac{dg\mu}{d\mu}(\omega)$$

is measurable and satisfies the *cocycle relation*

$$(*) \quad c_{\mu}(g_1 g_2, \omega) = c_{\mu}(g_1, g_2 \omega) c_{\mu}(g_2, \omega)$$

for all  $g_1, g_2$  in  $G$  and for  $\mu$ -almost all  $\omega$  in  $\Omega$ .

In this situation, define for each  $g$  in  $G$  an operator  $\pi_{\mu}(g)$  on  $L^2(\Omega, \mu)$ , the Hilbert space of square-integrable complex-valued functions on  $\Omega$ , by

$$\pi_{\mu}(g)f(\omega) = c_{\mu}(g^{-1}, \omega)^{1/2} f(g^{-1}\omega)$$

for all  $f \in L^2(\Omega, \mu)$  and  $\omega \in \Omega$ . It follows from the defining property of the Radon–Nikodym derivative that the operator  $\pi_{\mu}(g)$  is unitary and from the cocycle relation  $(*)$  that

$$\pi_{\mu} : G \rightarrow \mathcal{U}(L^2(\Omega, \mu))$$

is a group homomorphism. The homomorphism  $\pi_{\mu}$  need not be a representation: see Remark A.6.3.ii below; but the following proposition shows an important case in which such pathologies do not occur.

**Proposition A.6.1** *Let  $G$  be a  $\sigma$ -compact locally compact group, let  $(\Omega, \mu)$  be a  $\sigma$ -finite measurable space such that  $L^2(\Omega, \mu)$  is separable, and let  $G \times \Omega \rightarrow \Omega$  be a measurable action such that  $\mu$  is quasi-invariant.*

*Then the homomorphism  $\pi_\mu$  defined above is a unitary representation of  $G$  on  $L^2(\Omega, \mu)$ .*

**Proof** Since  $G$  is  $\sigma$ -compact, for all  $f_1, f_2 \in L^2(\Omega, \mu)$ , the matrix coefficient

$$g \mapsto \langle \pi_\mu(g)f_1, f_2 \rangle = \int_{\Omega} c_\mu(g^{-1}, \omega)^{1/2} f_1(g^{-1}\omega) \overline{f_2(\omega)} d\mu(\omega)$$

is a measurable function on  $G$ , by Fubini's Theorem. The strong continuity of  $\pi_\mu$  follows from the next lemma. ■

**Lemma A.6.2** *Let  $G$  be a locally compact group, and let  $\mathcal{H}$  be a separable Hilbert space. Let  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$  be a homomorphism from  $G$  to the unitary group of  $\mathcal{H}$ . Assume that the function  $g \mapsto \langle \pi(g)\xi, \xi \rangle$  is measurable for all  $\xi \in \mathcal{H}$ . Then  $\pi$  is strongly continuous.*

**Proof** Let  $\xi$  be a vector in  $\mathcal{H}$ . It suffices to show that  $g \mapsto \pi(g)\xi$  is continuous at the unit element  $e$ .

Choose  $\varepsilon > 0$  and set  $A = \{g \in G : \|\pi(g)\xi - \xi\| < \varepsilon/2\}$ . Then  $A$  is measurable, since

$$A = \{g \in G : 2\operatorname{Re}\langle \pi(g)\xi, \xi \rangle > 2\|\xi\|^2 - \varepsilon^2/4\}.$$

Moreover,  $A = A^{-1}$  and

$$A^2 = AA^{-1} \subset \{g \in G : \|\pi(g)\xi - \xi\| < \varepsilon\},$$

since  $\|\pi(g_1g_2)\xi - \xi\| \leq \|\pi(g_1)\xi - \xi\| + \|\pi(g_2)\xi - \xi\|$  for all  $g_1, g_2 \in G$ .

The subset  $\pi(G)\xi = \{\pi(g)\xi : g \in G\}$  is separable, since  $\mathcal{H}$  is separable. Hence, there exists a sequence  $(g_n)_n$  in  $G$  such that  $(\pi(g_n)\xi)_n$  is dense in  $\pi(G)\xi$ . For any  $g \in G$ , we have

$$\|\pi(g_n^{-1}g)\xi - \xi\| = \|\pi(g)\xi - \pi(g_n)\xi\| < \varepsilon/2$$

for some  $n$ . Therefore,  $G = \bigcup_n g_n A$  and the Haar measure of  $A$  is non-zero. It follows that  $A^2 = AA^{-1}$  is a neighbourhood of  $e$  (Exercise C.6.11) and this proves the claim. ■

**Remark A.6.3** (i) The Radon–Nikodym Theorem is not valid for a general measure space  $\Omega$ : for instance, consider  $\Omega = \mathbf{R}$  with the  $\sigma$ -algebra of its Borel

subsets and the counting measure  $\mu$ . The Lebesgue measure  $\lambda$  on  $\mathbf{R}$  is obviously absolutely continuous with respect to  $\mu$  (in the sense that every  $\mu$ -null set is a  $\lambda$ -null set), but there exists no measurable function  $f$  such that  $\lambda = f\mu$ . For an extensive discussion of the Radon–Nikodym Theorem, see [Halmo–74, Section 31].

(ii) The separability condition on  $\mathcal{H}$  in the previous lemma is necessary: let  $\mathbf{R}_d$  denote the group  $\mathbf{R}$  equipped with the discrete topology, and let  $\pi$  be the regular representation of  $\mathbf{R}$  on  $\ell^2(\mathbf{R}_d)$ . Then  $x \mapsto \langle \pi(x)\xi, \xi \rangle$  is a measurable function on  $\mathbf{R}$  for all  $\xi \in \ell^2(\mathbf{R}_d)$ , but  $\pi$  is not strongly continuous; for more details, see Exercise A.8.4 and [HewRo–63, (22.20),(c)].

(iii) Let  $(\Omega, \mathcal{B}, \mu)$  be a measure space. The Hilbert space  $L^2(\Omega, \mu)$  is separable when  $\mathcal{B}$  is generated as  $\sigma$ -algebra by a countable family of subsets. More generally, say that two subsets  $A, B \in \mathcal{B}$  are equivalent if  $\mu(A \Delta B) = 0$ , where

$$A \Delta B = (A \setminus B) \cup (B \setminus A)$$

denotes the *symmetric difference*. The set  $S(\mu)$  of equivalence classes of elements in  $\mathcal{B}$  of finite measure is a metric space for the distance  $d(A, B) = \mu(A \Delta B)$ . Then  $L^2(\Omega, \mu)$  is separable if and only if  $S(\mu)$  is separable; see [Halmo–74, Section 42, Exercise 1].

**Example A.6.4** (i) Interesting geometrical examples arise when  $\Omega$  is an oriented manifold and  $G$  is a topological group acting by diffeomorphisms on  $\Omega$ . Let  $v$  be a volume form on  $\Omega$ . For  $g$  in  $G$ , let  $g^*v$  be the volume form which is the pull-back of  $v$  by the diffeomorphism  $g$ . Then we have

$$g^*v(\omega) = c(g^{-1}, \omega)v(\omega) \quad \text{for all } \omega \in \Omega,$$

for some continuous function  $c : G \times \Omega \rightarrow \mathbf{R}_+^*$ . Thus, the regular Borel measure  $\mu_v$  on  $\Omega$  associated to  $v$  is quasi-invariant under  $G$ .

(ii) As an example for the situation described in (i), the group  $G = SL_2(\mathbf{R})$  acts by fractional linear transformations on the real projective line  $\mathbf{R} \cup \{\infty\}$ . The Lebesgue measure  $\mu$  on  $\mathbf{R} \cup \{\infty\}$  is quasi-invariant (and *not* invariant) under the action of  $G$ . The associated unitary representation  $\pi_\mu$  is the representation of  $SL_2(\mathbf{R})$  on  $L^2(\mathbf{R} \cup \{\infty\})$  given by

$$\pi_\mu \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(\omega) = | -c\omega + a |^{-1} f \left( \frac{d\omega - b}{-c\omega + a} \right)$$

for all  $f \in L^2(\mathbf{R} \cup \{\infty\})$ . The representation  $\pi_\mu$  is a so-called principal series representation (see Example E.1.8).

Observe that  $\mathbf{R} \cup \{\infty\}$  is a homogeneous space since the action of  $SL_2(\mathbf{R})$  is transitive. Indeed,  $\mathbf{R} \cup \{\infty\}$  is diffeomorphic to  $SL_2(\mathbf{R})/P$ , where

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbf{R}, a \neq 0 \right\}$$

is the stabiliser of  $\{\infty\}$ .

We claim that there is *no* non-zero  $SL_2(\mathbf{R})$ -invariant Borel measure on  $\mathbf{R} \cup \{\infty\}$ . Indeed, let  $\nu$  be such a measure. Then  $\nu$  is invariant under the transformations  $x \mapsto x + t$  given by the matrices

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad t \in \mathbf{R}.$$

Because of the uniqueness of the Lebesgue measure  $\mu$  as normalised translation-invariant measure on  $\mathbf{R}$ , this implies that  $\nu = c_1\mu + c_2\delta_\infty$  for some constants  $c_1, c_2 \geq 0$ . On the other hand, neither  $\mu$  nor  $\delta_\infty$  is invariant under the transformation  $x \mapsto -1/x$  given by the matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Hence  $c_1 = c_2 = 0$  and  $\nu = 0$ . For another proof, see Example B.1.11.

As we will discuss in the next chapter, quasi-invariant measures on homogeneous spaces always exist and we will determine exactly when there are invariant ones.

## A.7 Group actions associated to orthogonal representations

Let  $G$  be a second countable locally compact group with a measure preserving action on a probability space  $(\Omega, \mu)$ . The space  $L^2_{\mathbf{R}}(\Omega, \mu)$  of all *real-valued* square-integrable functions on  $\Omega$ , modulo equality  $\mu$ -almost everywhere, is a real Hilbert space. As in Section A.6, we can define, for each  $g \in G$ , an orthogonal operator  $\pi_\mu(g)$  on  $L^2_{\mathbf{R}}(\Omega, \mu)$ . Assuming that  $L^2_{\mathbf{R}}(\Omega, \mu)$  is separable, the mapping  $g \mapsto \pi_\mu(g)$  is an orthogonal representation of  $G$  in the sense of the definition below. The aim of this section is to show that every orthogonal representation of  $G$  occurs as subrepresentation of a representation of the form  $(\pi_\mu, L^2_{\mathbf{R}}(\Omega, \mu))$  for some probability space  $(\Omega, \mu)$ .

**Definition A.7.1** An *orthogonal representation* of a topological group  $G$  in a real Hilbert space  $\mathcal{H}$  is a strongly continuous group homomorphism from  $G$  to the orthogonal group  $\mathcal{O}(\mathcal{H})$  of  $\mathcal{H}$ .

The notions of orthogonal equivalence, subrepresentation, direct sum, and tensor product of orthogonal representations are defined in the same way as for unitary representations.

**Remark A.7.2** To every orthogonal representation  $(\pi, \mathcal{H})$  of the topological group  $G$  is canonically associated a unitary representation  $\pi_{\mathbf{C}}$  of  $G$ , called the *complexification* of  $\pi$ ; it acts on the complexification  $\mathcal{H}_{\mathbf{C}} = \mathcal{H} \otimes_{\mathbf{R}} \mathbf{C}$  of  $\mathcal{H}$  by

$$\pi_{\mathbf{C}}(g)(\xi \otimes \lambda) = \pi(g)\xi \otimes \lambda, \quad g \in G, \xi \in \mathcal{H}, \lambda \in \mathbf{C}.$$

Let us recall some standard notions from probability theory (see, for instance, [Loeve-77]). Let  $(\Omega, \mathcal{B}, \mu)$  be a probability space.

**Definition A.7.3** A real-valued *random variable*  $X$  on  $\Omega$  is a measurable function from  $\Omega$  to the real line  $\mathbf{R}$ ; two random variables are identified if they are equal  $\mu$ -almost everywhere.

The *distribution* (or the *law*) of a random variable  $X$  on  $\Omega$  is the measure  $\mu_X$  on  $\mathbf{R}$  which is the image of  $\mu$  under  $X$ . If  $X$  is either integrable on  $\Omega$  or positive-valued, its *expectation* or *mean value*  $\mathbb{E}[X]$  is defined by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) d\mu(\omega) = \int_{\mathbf{R}} x d\mu_X(x).$$

A random variable  $X$  is *centred* if  $\mathbb{E}[X] = 0$ .

Assume that  $X$  is in  $L^2_{\mathbf{R}}(\Omega, \mu)$ ; then  $X \in L^1_{\mathbf{R}}(\Omega, \mu)$ , since  $\mu$  is a finite measure. The *variance*  $\sigma^2$  of  $X$  is  $\sigma^2 = \mathbb{E}[(X - m)^2]$ , where  $m = \mathbb{E}[X]$ . It is easy to check that  $\sigma^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ .

When the measure  $\mu_X$  on  $\mathbf{R}$  is absolutely continuous with respect to the Lebesgue measure,  $X$  has a *density function* which is the positive-valued function  $p: \mathbf{R} \rightarrow \mathbf{R}_+$  such that  $d\mu_X(x) = p(x)dx$  and  $\int_{-\infty}^{+\infty} p(x)dx = 1$ .

Let  $(X_i)_{i \in I}$  be a family of real-valued random variables on  $\Omega$ . The  $\sigma$ -algebra generated by  $(X_i)_{i \in I}$  is the smallest sub- $\sigma$ -algebra of  $\mathcal{B}$  which contains  $X_i^{-1}(B)$  for all  $i \in I$  and all Borel subsets  $B$  of  $\mathbf{R}$ . The family  $(X_i)_{i \in I}$  is *independent* if, for every finite subset  $F$  of  $I$  and all Borel subsets  $B_i$  of  $\mathbf{R}$ ,  $i \in F$ , we have

$$\mu \left( \bigcap_{i \in F} X_i^{-1}(B_i) \right) = \prod_{i \in F} \mu(X_i^{-1}(B_i)).$$

We will need a few facts about Gaussian Hilbert spaces (for more on this, see [Neveu-68] or [Janso-97]).

**Definition A.7.4** A real-valued random variable  $X$  on  $\Omega$  is *Gaussian* if either  $X$  is constant or the distribution of  $X$  is absolutely continuous with respect to Lebesgue measure, with a density  $p(x)$  of the form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2},$$

for some real numbers  $\sigma > 0$  and  $m$ . In this case,  $\mathbb{E}[X] = m$  and  $\mathbb{E}[(X - m)^2] = \sigma^2$ .

A *Gaussian Hilbert space* is a closed subspace  $\mathcal{K}$  of  $L^2_{\mathbf{R}}(\Omega, \mu)$  such that each  $X \in \mathcal{K}$  is a centred Gaussian random variable.

**Remark A.7.5** (i) Observe that a Gaussian random variable is in  $L^p_{\mathbf{R}}(\Omega, \mu)$  for all  $p \in [1, +\infty)$ , since it is constant or its density function is a function of rapid decay on the real line.

(ii) Let  $X$  be a random variable in  $L^2_{\mathbf{R}}(\Omega, \mu)$  which is the  $L^2$ -limit of a sequence of centred Gaussian random variables  $(X_i)_{i \geq 1}$ . Then  $X$  is a centred Gaussian random variable. Indeed,  $X$  is centred since  $\mathbb{E}[X] = \lim_i \mathbb{E}[X_i] = 0$ . Then, if  $\sigma_i^2 = \mathbb{E}[X_i^2] = \|X_i\|_2^2$  and  $\sigma^2 = \mathbb{E}[X^2] = \|X\|_2^2$ , we have  $\lim_i \sigma_i^2 = \sigma^2$ . If  $\sigma = 0$ , then  $X = 0$ ; otherwise, for every  $f \in C_c(\mathbf{R})$ , we have

$$\lim_i \frac{1}{\sigma_i\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2\sigma_i^2} f(x) dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbf{R}} e^{-x^2/2\sigma^2} f(x) dx,$$

which shows that the distribution of  $X$  is the measure  $\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} dx$ .

(iii) A product  $X_1 X_2 \cdots X_n$  of finitely many Gaussian random variables is also in  $L^p_{\mathbf{R}}(\Omega, \mu)$  for all  $p \in [1, \infty)$ . Indeed,  $\bigcap_{p \in [1, \infty)} L^p_{\mathbf{R}}(\Omega, \mu)$  is an algebra, as follows from the generalised Hölder's inequality: for  $p_1, \dots, p_n, r \in [1, \infty)$  with  $\sum_{i=1}^n 1/p_i = 1/r$  and  $X_1 \in L^{p_1}_{\mathbf{R}}(\Omega, \mu), \dots, X_n \in L^{p_n}_{\mathbf{R}}(\Omega, \mu)$ , we have

$$\|X_1 \cdots X_n\|_r \leq \|X_1\|_{p_1} \cdots \|X_n\|_{p_n}.$$

(iv) If  $X$  is a centred Gaussian variable with variance  $\sigma^2 = \mathbb{E}[X^2] > 0$ , then  $\exp X \in L^2_{\mathbf{R}}(\Omega, \mu)$  and

$$\mathbb{E}[\exp X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^x e^{-x^2/2\sigma^2} dx = e^{\sigma^2/2} = \exp\left(\frac{1}{2}\mathbb{E}[X^2]\right).$$

**Example A.7.6** Let  $I$  be an arbitrary index set. Let

$$dv(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

be the *standard Gaussian measure on  $\mathbf{R}$* , defined on the  $\sigma$ -algebra  $\mathcal{F}$  of the Borel subsets of  $\mathbf{R}$ . Let  $\Omega = \prod_{i \in I} \mathbf{R}$  be the product of copies of  $\mathbf{R}$ , equipped with the product  $\sigma$ -algebra  $\mathcal{B} = \otimes_{i \in I} \mathcal{F}$  and the product measure  $\mu = \otimes_{i \in I} \nu$ .

For each  $i \in I$ , let  $X_i : \Omega \rightarrow \mathbf{R}$  be the projection on the  $i$ -th factor:

$$X_i((\omega_j)_{j \in I}) = \omega_i.$$

The  $X_i$ 's are independent centred Gaussian random variables with variance 1. It follows that every linear combination of the  $X_i$ 's is a centred Gaussian random variable. Let  $\mathcal{K}$  be the closed linear subspace of  $L^2_{\mathbf{R}}(\Omega, \mu)$  generated by  $\{X_i : i \in I\}$ . By (ii) of the previous remark,  $\mathcal{K}$  is a Gaussian Hilbert space; its dimension is the cardinality of  $I$ .

**Remark A.7.7** Every real Hilbert space is isometrically isomorphic to a Gaussian Hilbert space. Indeed, as the previous example shows, Gaussian Hilbert spaces of arbitrary dimensions exist.

**Lemma A.7.8** *Let  $\mathcal{K}$  be a Gaussian Hilbert space in  $L^2_{\mathbf{R}}(\Omega, \mathcal{B}, \mu)$ . Assume that the  $\sigma$ -algebra generated by all the random variables  $X \in \mathcal{K}$  coincides with  $\mathcal{B}$ . Then the set  $\{\exp(X) : X \in \mathcal{K}\}$  is total in  $L^2_{\mathbf{R}}(\Omega, \mathcal{B}, \mu)$ .*

**Proof** Let  $Y \in L^2_{\mathbf{R}}(\Omega, \mu)$  be orthogonal to  $\{\exp(X) : X \in \mathcal{K}\}$ . Consider the (non-necessarily positive) bounded measure  $Y(\omega)d\mu(\omega)$  on  $\Omega$ .

Let  $(X_i)_{i \in I}$  be an orthonormal basis of  $\mathcal{K}$  and let  $\{i_1, \dots, i_n\}$  be a finite subset of  $I$ . Denote by  $\nu$  the measure on  $\mathbf{R}^n$  which is the image of  $Y(\omega)d\mu(\omega)$  under the mapping

$$\Omega \rightarrow \mathbf{R}^n, \quad \omega \mapsto (X_{i_1}(\omega), \dots, X_{i_n}(\omega)).$$

Then

$$\int_{\mathbf{R}^n} \exp\left(\sum_{k=1}^n u_k x_k\right) d\nu(x) = \int_{\Omega} \exp\left(\sum_{k=1}^n u_k X_{i_k}(\omega)\right) Y(\omega) d\mu(\omega) = 0,$$

for all  $u_1, \dots, u_n \in \mathbf{R}$ . By analytic continuation, this holds for all  $u_1, \dots, u_n \in \mathbf{C}$ . In particular, the Fourier transform of  $\nu$  is identically 0. Hence,  $\nu = 0$ . This implies that the measure  $Y(\omega)d\mu(\omega)$  vanishes on the  $\sigma$ -algebra  $\sigma\{X_{i_1}, \dots, X_{i_n}\}$  generated by  $\{X_{i_1}, \dots, X_{i_n}\}$ .

The union  $\mathcal{R}$  of all  $\sigma\{X_{i_1}, \dots, X_{i_n}\}$ , where  $\{i_1, \dots, i_n\}$  runs over all the finite subsets of  $I$ , is a ring of subsets of  $\Omega$ . (For the definition of a ring of sets, see the beginning of G.1.) We have just shown that the non-negative measures  $Y^+(\omega)d\mu(\omega)$  and  $Y^-(\omega)d\mu(\omega)$  agree on  $\mathcal{R}$ , where  $Y^+$  and  $Y^-$  denote the positive and negative parts of the real-valued function  $Y$ . It follows that



$Y^+(\omega)d\mu(\omega)$  and  $Y^-(\omega)d\mu(\omega)$  agree on the  $\sigma$ -algebra generated by  $\mathcal{R}$  (see, for instance, [Halmo–74, §13, Theorem A]). Since, by assumption, this  $\sigma$ -algebra coincides with  $\mathcal{B}$ , the measure  $Y(\omega)d\mu(\omega)$  vanishes on  $\mathcal{B}$ . Therefore  $Y = 0$  in  $L^2_{\mathbf{R}}(\Omega, \mu)$ . ■

Let  $\mathcal{H}$  be a real or complex Hilbert space. We review the construction of the symmetric Fock space of  $\mathcal{H}$ .

For every integer  $n \geq 1$ , let  $\mathcal{H}^{\otimes n}$  be the Hilbert  $n$ -th tensor power of  $\mathcal{H}$ . The symmetric group  $\Sigma_n$  acts by orthogonal transformations  $U_\sigma$  on  $\mathcal{H}^{\otimes n}$  defined by

$$U_\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}, \quad \sigma \in \Sigma_n, \quad \xi_1, \dots, \xi_n \in \mathcal{H}.$$

The closed subspace

$$S^n(\mathcal{H}) = \{\xi \in \mathcal{H}^{\otimes n} : U_\sigma \xi = \xi \text{ for all } \sigma \in \Sigma_n\}$$

is the  $n$ -th symmetric tensor power of  $\mathcal{H}$ . The orthogonal projection  $P_n : \mathcal{H}^{\otimes n} \rightarrow S^n(\mathcal{H})$  is given by

$$P_n = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} U_\sigma.$$

For  $\xi_1, \dots, \xi_n \in \mathcal{H}$ , we write  $\xi_1 \odot \cdots \odot \xi_n$  for  $P_n(\xi_1 \otimes \cdots \otimes \xi_n)$ . Set  $S^0(\mathcal{H}) = \mathbf{R}$  or  $\mathbf{C}$ , depending on whether  $\mathcal{H}$  is a real or a complex Hilbert space. The symmetric Fock space  $S(\mathcal{H})$  of  $\mathcal{H}$  is

$$S(\mathcal{H}) = \bigoplus_{n \geq 0} S^n(\mathcal{H}),$$

the Hilbert space direct sum of the  $S^n(\mathcal{H})$ 's.

Let  $\text{EXP} : \mathcal{H} \rightarrow S(\mathcal{H})$  be the mapping defined by

$$\text{EXP}(\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^{\otimes n}, \quad \xi \in \mathcal{H}.$$

Observe that

$$(1) \quad \langle \text{EXP}(\xi), \text{EXP}(\eta) \rangle = \exp \langle \xi, \eta \rangle \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

We will need the following elementary lemma.

**Lemma A.7.9** *The set  $\{\text{EXP}(\xi) : \xi \in \mathcal{H}\}$  is total in  $S(\mathcal{H})$ .*

**Proof** Let  $\mathcal{L}$  be the closed linear subspace generated by  $\{\text{EXP}(\xi) : \xi \in \mathcal{H}\}$ . For  $\xi \in \mathcal{H}$ , consider the function  $f : \mathbf{R} \rightarrow S(\mathcal{H})$  defined by  $f(t) = \text{EXP}(t\xi)$ ; for every  $n \geq 0$ , the  $n$ -th derivative of  $f$  at 0 is

$$f^{(n)}(0) = \sqrt{n!} \xi^{\odot n}.$$

Since  $f(t) \in \mathcal{L}$  for every  $t \in \mathbf{R}$ , it follows that  $\xi^{\odot n} \in \mathcal{L}$  for every  $\xi \in \mathcal{H}$  and  $n \geq 0$ . Hence,  $\mathcal{L} = S(\mathcal{H})$ . ■

**Remark A.7.10** The set  $\{\text{EXP}(\xi) : \xi \in \mathcal{H}\}$  is linearly independent (see Proposition 2.2 in [Guic–72b]); we will not need this fact.

Given an isomorphism  $\Phi$  between a real Hilbert space  $\mathcal{H}$  and an appropriate Hilbert space of Gaussian variables on a probability space  $(\Omega, \mathcal{B}, \mu)$ , we will extend  $\Phi$  to an isomorphism between  $S(\mathcal{H})$  and  $L^2_{\mathbf{R}}(\Omega, \mu)$ , with the following property: every orthogonal operator on  $\mathcal{H}$  induces a measure preserving transformation of  $(\Omega, \mu)$ .

By a *measure preserving transformation* of  $(\Omega, \mu)$ , we mean a measurable bijective mapping  $\theta : \Omega \rightarrow \Omega$  such that  $\theta_*(\mu) = \mu$  and such that  $\theta^{-1}$  is measurable. Observe that any such transformation gives rise to an orthogonal operator

$$\theta^* : L^2_{\mathbf{R}}(\Omega, \mu) \rightarrow L^2_{\mathbf{R}}(\Omega, \mu), \quad X \mapsto X \circ \theta^{-1},$$

with the property  $\theta^*(YZ) = \theta^*(Y)\theta^*(Z)$  for every pair  $(Y, Z)$  of characteristic functions of measurable subsets. The next lemma shows that the converse is true if  $\Omega$  is a standard Borel space.

Let  $\tilde{\mathcal{B}}$  be the quotient space of  $\mathcal{B}$  by the equivalence relation  $B_1 \sim B_2$  if  $\mu(B_1 \triangle B_2) = 0$ . Then  $\tilde{\mathcal{B}}$  is in a natural way a Boolean  $\sigma$ -algebra under the operations of complementation, countable unions and intersections, and the measure  $\mu$  induces a measure  $\tilde{\mu}$  on  $\tilde{\mathcal{B}}$  defined by  $\tilde{\mu}(\tilde{B}) = \mu(B)$  if  $\tilde{B}$  is the equivalence class of  $B \in \mathcal{B}$ . The pair  $(\tilde{\mathcal{B}}, \tilde{\mu})$  is called the *measure algebra* of  $(\Omega, \mathcal{B}, \mu)$ . An automorphism of  $(\tilde{\mathcal{B}}, \tilde{\mu})$  is a bijection  $\tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}$  which preserves complements, countable unions, and the measure  $\tilde{\mu}$ . A measure preserving transformation of  $\Omega$  induces in a natural way an automorphism of  $(\tilde{\mathcal{B}}, \tilde{\mu})$ .

For the notion of a standard Borel space, see Definition F.5.2.

**Lemma A.7.11** *Let  $(\Omega, \mathcal{B})$  be a standard Borel space equipped with a non-atomic probability measure  $\mu$ . Let  $V : L^2(\Omega) \rightarrow L^2(\Omega)$  be a bijective orthogonal linear mapping. Assume that  $V(YZ) = V(Y)V(Z)$  and  $V^{-1}(YZ) = V^{-1}(Y)V^{-1}(Z)$  for every pair  $(Y, Z)$  of characteristic functions of measurable subsets.*

Then there exists a measure preserving transformation  $\theta$  of  $\Omega$  such that  $V = \theta^*$ .

**Proof** Let  $B \in \mathcal{B}$ . Then, by assumption, we have  $V(\chi_B)^2 = V(\chi_B^2) = V(\chi_B)$ . Hence, there exists  $B' \in \mathcal{B}$  such that  $V(\chi_B) = \chi_{B'}$ . This defines a mapping

$$\lambda: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}, \quad \tilde{B} \mapsto \tilde{B}',$$

where  $\tilde{B}$  is the class of  $B$  in the measure algebra  $(\tilde{\mathcal{B}}, \tilde{\mu})$ . Using the same procedure for  $V^{-1}$  and the fact that  $V$  is orthogonal, it is straightforward to check (see the proof of Theorem 2.4 in [Walte–82]) that  $\lambda$  is an automorphism of  $(\tilde{\mathcal{B}}, \tilde{\mu})$ .

Since  $\Omega$  is a standard Borel space and  $\mu$  is non-atomic, we can assume without loss of generality that  $\Omega = [0, 1]$  and  $\mu$  is the Lebesgue measure (see [Sri–98, Theorem 3.4.2.3]). It is then known that  $\lambda$  is induced by a measure preserving transformation  $\theta$  of  $\Omega$  (see [vNeum–32, Satz 1]). We clearly have  $V = \theta^*$ . ■

**Example A.7.12** We illustrate the construction given in the proof of Theorem A.7.13 below in the case  $\mathcal{H} = \mathbf{R}$ . Let  $\mu$  be the standard Gaussian measure on  $\mathbf{R}$ . Let  $X \in L^2_{\mathbf{R}}(\mathbf{R}, \mu)$  denote the identity mapping  $x \mapsto x$  on  $\mathbf{R}$ . Let  $\Phi: \mathbf{R} \rightarrow \mathbf{R}X$  be the isomorphism defined by  $\Phi(1) = X$ . Recall that

$$\left\{ \frac{1}{\sqrt{n!}} H_n : n \geq 0 \right\}$$

is an orthonormal basis of  $L^2_{\mathbf{R}}(\mathbf{R}, \mu)$ , where the  $H_n$ 's are the Hermite polynomials. Since  $\{1^{\odot n} : n \geq 0\}$  is an orthonormal basis of  $S(\mathbf{R})$ , we can extend  $\Phi$  to an isometric isomorphism  $\tilde{\Phi}: S(\mathbf{R}) \rightarrow L^2_{\mathbf{R}}(\mathbf{R}, \mu)$  by defining  $\tilde{\Phi}(1^{\odot n}) = \frac{1}{\sqrt{n!}} H_n(X)$  for all  $n \geq 0$ .

We give another description of  $\tilde{\Phi}$ . It is known that

$$e^{ux - u^2/2} = \sum_{n=0}^{\infty} \frac{u^n}{n!} H_n(x) \quad \text{for all } u, x \in \mathbf{R}.$$

Hence,  $\tilde{\Phi}(\text{EXP}(u)) = \exp(ux - u^2/2)$  for every  $u \in \mathbf{R}$ . By Lemma A.7.9, this formula entirely determines  $\tilde{\Phi}$ .

Let  $\mathcal{H}$  be a real Hilbert space; every  $U \in \mathcal{O}(\mathcal{H})$  extends in a canonical way to an orthogonal operator  $S(U)$  on  $S(\mathcal{H})$  defined on  $S^n(\mathcal{H})$  by

$$S(U)(\xi_1 \odot \cdots \odot \xi_n) = U\xi_1 \odot \cdots \odot U\xi_n, \quad \xi_1, \dots, \xi_n \in \mathcal{H}.$$

Observe that  $S(U^{-1}) = S(U)^{-1}$  and that  $S(U)(\text{EXP}(\xi)) = \text{EXP}(U\xi)$  for all  $\xi \in \mathcal{H}$ .

**Theorem A.7.13** *Let  $\mathcal{H}$  be a separable real Hilbert space. Let  $\Phi: \mathcal{H} \rightarrow \mathcal{K}$  be an isometric isomorphism between  $\mathcal{H}$  and a Gaussian Hilbert space  $\mathcal{K} \subset L^2_{\mathbf{R}}(\Omega, \mu)$  for a probability space  $(\Omega, \mathcal{B}, \mu)$ . Assume that  $\Omega$  is a standard Borel space, that  $\mu$  is non-atomic, and that the  $\sigma$ -algebra generated by all the random variables  $X \in \mathcal{K}$  coincides with  $\mathcal{B}$ .*

*Then  $\Phi$  extends to an isometric isomorphism  $\tilde{\Phi}: S(\mathcal{H}) \rightarrow L^2_{\mathbf{R}}(\Omega, \mu)$  with the following property: for every orthogonal operator  $U: \mathcal{H} \rightarrow \mathcal{H}$ , there exists a unique (up to a subset of measure 0) measure preserving transformation  $\theta_U$  of the probability space  $(\Omega, \mu)$  such that  $\tilde{\Phi} \circ S(U) = \theta_U^* \circ \tilde{\Phi}$ .*

**Proof** • *First step:* For  $\xi, \eta \in \mathcal{H}$ , set  $X = \Phi(\xi), Y = \Phi(\eta) \in \mathcal{K}$ . We claim that

$$(2) \quad \left\langle \exp\left(X - \frac{1}{2}\mathbb{E}[X^2]\right), \exp\left(Y - \frac{1}{2}\mathbb{E}[Y^2]\right) \right\rangle = \langle \text{EXP}(\xi), \text{EXP}(\eta) \rangle.$$

Indeed, we have

$$\begin{aligned} & \left\langle \exp\left(X - \frac{1}{2}\mathbb{E}[X^2]\right), \exp\left(Y - \frac{1}{2}\mathbb{E}[Y^2]\right) \right\rangle \\ &= \mathbb{E} \left[ \exp\left(X - \frac{1}{2}\mathbb{E}[X^2]\right) \exp\left(Y - \frac{1}{2}\mathbb{E}[Y^2]\right) \right] \\ &= \mathbb{E} \left[ \exp\left(X + Y - \frac{1}{2}\mathbb{E}[X^2] - \frac{1}{2}\mathbb{E}[Y^2]\right) \right] \\ &= \exp\left(-\frac{1}{2}(\mathbb{E}[X^2] + \mathbb{E}[Y^2])\right) \mathbb{E}[\exp(X + Y)] \\ &= \exp\left(-\frac{1}{2}(\mathbb{E}[X^2] + \mathbb{E}[Y^2])\right) \exp\left(\mathbb{E}\left[\frac{1}{2}(X + Y)^2\right]\right) \\ &= \exp \mathbb{E}[XY] = \exp \langle X, Y \rangle \\ &= \exp \langle \Phi(\xi), \Phi(\eta) \rangle = \exp \langle \xi, \eta \rangle \\ &= \langle \text{EXP}(\xi), \text{EXP}(\eta) \rangle, \end{aligned}$$

where we used Remark A.7.5.iv, the fact that  $\Phi$  is an isometry, and formula (1) above.

• *Second step:* We extend  $\Phi$  to the symmetric Fock space  $S(\mathcal{H})$  of  $\mathcal{H}$ . Define a mapping  $\tilde{\Phi}$  from the set  $\{\text{EXP}(\xi) : \xi \in \mathcal{H}\}$  to  $L^2_{\mathbf{R}}(\Omega, \mu)$  by

$$\tilde{\Phi}(\text{EXP}(\xi)) = \exp\left(\Phi(\xi) - \frac{1}{2}\mathbb{E}[\Phi(\xi)^2]\right).$$

Lemma A.7.9 and formula (2) above show that  $\tilde{\Phi}$  extends to an isometric linear mapping  $\tilde{\Phi} : S(\mathcal{H}) \rightarrow L^2_{\mathbf{R}}(\Omega, \mu)$ . By Lemma A.7.8,  $\tilde{\Phi}$  is onto.

Let now  $U : \mathcal{H} \rightarrow \mathcal{H}$  be an orthogonal operator and  $S(U)$  its canonical extension to  $S(\mathcal{H})$ . Then  $V = \tilde{\Phi} \circ S(U) \circ \tilde{\Phi}^{-1}$  is an orthogonal operator on  $L^2_{\mathbf{R}}(\Omega, \mu)$ .

• *Third step:* Let  $Y = \tilde{\Phi}(\text{EXP}(\xi))$  and  $Z = \tilde{\Phi}(\text{EXP}(\eta))$  for  $\xi, \eta \in \mathcal{H}$ . We claim that  $YZ \in L^2_{\mathbf{R}}(\Omega, \mu)$  and that

$$(3) \quad V(YZ) = V(Y)V(Z) \quad \text{and} \quad V^{-1}(YZ) = V^{-1}(Y)V^{-1}(Z).$$

Indeed, by definition of  $V$ , we have

$$\begin{aligned} V(Y) &= V(\tilde{\Phi}(\text{EXP}(\xi))) = \tilde{\Phi}(S(U)(\text{EXP}(\xi))) \\ &= \tilde{\Phi}(\text{EXP}(U\xi)) \\ &= \exp\left(\Phi(U\xi) - \frac{1}{2}\mathbb{E}[\Phi(U\xi)^2]\right). \end{aligned}$$

Similarly, we have

$$V(Z) = \exp\left(\Phi(U\eta) - \frac{1}{2}\mathbb{E}[\Phi(U\eta)^2]\right),$$

so that

$$V(Y)V(Z) = \exp\left(\Phi(U(\xi + \eta)) - \frac{1}{2}\mathbb{E}[\Phi(U\xi)^2 + \Phi(U\eta)^2]\right).$$

On the other hand, we have

$$\begin{aligned} YZ &= \exp\left(\Phi(\xi) - \frac{1}{2}\mathbb{E}[\Phi(\xi)^2]\right) \exp\left(\Phi(\eta) - \frac{1}{2}\mathbb{E}[\Phi(\eta)^2]\right) \\ &= \exp\left(\Phi(\xi + \eta) - \frac{1}{2}\mathbb{E}[(\Phi(\xi + \eta))^2]\right) \exp(\mathbb{E}[\Phi(\xi)\Phi(\eta)]) \\ &= \exp(\mathbb{E}[\Phi(\xi)\Phi(\eta)]) \tilde{\Phi}(\text{EXP}(\xi + \eta)). \end{aligned}$$

This shows that  $YZ \in L_{\mathbf{R}}^2(\Omega, \mu)$  and that

$$\begin{aligned} V(YZ) &= \exp(\mathbb{E}[\Phi(\xi)\Phi(\eta)]) \\ &\quad \times \exp\left(\Phi(U(\xi + \eta)) - \frac{1}{2}\mathbb{E}[\Phi(U(\xi + \eta))^2]\right). \end{aligned}$$

Since  $\Phi$  and  $U$  are isometric, we have

$$\begin{aligned} \mathbb{E}[\Phi(\xi)\Phi(\eta)] &= \langle \xi, \eta \rangle = \langle U\xi, U\eta \rangle \\ &= \mathbb{E}[\Phi(U\xi)\Phi(U\eta)]. \end{aligned}$$

It follows that

$$\begin{aligned} V(YZ) &= \exp(\mathbb{E}[\Phi(U\xi)\Phi(U\eta)]) \\ &\quad \times \exp\left(\Phi(U(\xi + \eta)) - \frac{1}{2}\mathbb{E}[\Phi(U(\xi + \eta))^2]\right) \\ &= V(Y)V(Z). \end{aligned}$$

Since  $V^{-1} = \tilde{\Phi} \circ S(U^{-1}) \circ \tilde{\Phi}^{-1}$ , the same computation applied to  $U^{-1}$  in place of  $U$  shows that  $V^{-1}(YZ) = V^{-1}(Y)V^{-1}(Z)$ .

• *Fourth step:* There exists a measure preserving transformation  $\theta$  of the probability space  $(\Omega, \mu)$  such that  $\theta^* = V$ . The set  $\{\tilde{\Phi}(\text{EXP}(\xi)) : \xi \in \mathcal{H}\}$  is total in  $L_{\mathbf{R}}^2(\Omega, \mu)$  as well as in  $L_{\mathbf{R}}^4(\Omega, \mu)$ ; indeed, the proof of Lemma A.7.8 shows that this set is total in  $L_{\mathbf{R}}^p(\Omega, \mu)$  for every  $p \in [1, \infty)$ . It follows from the third step that Equation (3) holds for any pair  $(Y, Z)$  of characteristic functions of measurable subsets of  $\Omega$ . We conclude with Lemma A.7.11. ■

**Remark A.7.14** If  $(\pi, \mathcal{H})$  is an orthogonal representation of  $G$ , then, for every  $n \geq 0$ , the subspace  $S^n(\mathcal{H})$  is a closed  $G$ -invariant subspace of  $\mathcal{H}^{\otimes n}$  and defines a subrepresentation of  $\pi^{\otimes n}$ , which is called the  $n$ -th *symmetric tensor power* of  $\pi$  and which we denote by  $S^n(\pi)$ .

**Corollary A.7.15** *Let  $G$  be a second countable locally compact group, and let  $\pi$  be an orthogonal representation of  $G$  in a separable real Hilbert space  $\mathcal{H}$ . There exist a probability space  $(\Omega, \mu)$  and a measure preserving action of  $G$  on  $\Omega$  such that the associated orthogonal representation of  $G$  on  $L_{\mathbf{R}}^2(\Omega, \mu)$  is equivalent to the direct sum  $\bigoplus_{n=0}^{\infty} S^n(\pi)$  of all symmetric tensor powers of  $\pi$ .*

**Proof** Let  $\Phi$  be an isometric isomorphism between  $\mathcal{H}$  and a Gaussian Hilbert space  $\mathcal{K} \subset L_{\mathbf{R}}^2(\Omega, \mu)$ . As Example A.7.6 shows, we can assume that  $\Omega$  is a

standard Borel space and that  $\mu$  is non-atomic; we can also assume that the  $\sigma$ -algebra generated by all  $X \in \mathcal{K}$  coincides with the  $\sigma$ -algebra of all measurable subsets of  $\Omega$ .

Let  $\tilde{\Phi} : S(\mathcal{H}) \rightarrow L^2_{\mathbf{R}}(\Omega, \mu)$  be the extension of  $\Phi$  as in the previous theorem. For  $g \in G$ , the orthogonal operator  $\pi(g)$  on  $\mathcal{H}$  induces a measure preserving mapping  $\theta_g : \Omega \rightarrow \Omega$ . Since  $\pi$  is a representation of  $G$ , we have for every  $g, h \in G$ ,

$$\begin{aligned}\theta_g(\theta_h(\omega)) &= \theta_{gh}(\omega) \quad \text{for almost all } \omega \in \Omega, \\ \theta_{g^{-1}}(\theta_g(\omega)) &= \omega \quad \text{for almost all } \omega \in \Omega.\end{aligned}$$

It follows that there exists a measurable subset  $\Omega_0$  of  $\Omega$  with measure 1, which is  $\theta_g$ -invariant for every  $g \in G$  and such that the mapping

$$G \times \Omega_0 \rightarrow \Omega_0, \quad (g, \omega) \mapsto \theta_g(\omega)$$

defines a measurable action of  $G$  on  $\Omega_0$ . Indeed, this is easily proved in case  $G$  is countable; for the general case, see [Zimm–84a, Appendix B.10].

The isomorphism  $\tilde{\Phi}$  intertwines  $\bigoplus_{n=0}^{\infty} S^n(\pi)$  and the representation  $\pi_{\mu}$  of  $G$  on  $L^2_{\mathbf{R}}(\Omega_0, \mu) = L^2_{\mathbf{R}}(\Omega, \mu)$  associated to this action. ■

## A.8 Exercises

**Exercise A.8.1** Let  $G$  be a locally compact group,  $\mathcal{H}$  a Hilbert space,  $\mathcal{U}(\mathcal{H})$  its unitary group, and  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$  a group homomorphism. Show that the following properties are equivalent:

- (i) the mapping  $G \times \mathcal{H} \rightarrow \mathcal{H}, (g, \xi) \mapsto \pi(g)\xi$  is continuous;
- (ii) for each  $\xi \in \mathcal{H}$ , the mapping  $G \rightarrow \mathcal{H}, g \mapsto \pi(g)\xi$  is continuous, that is,  $\pi$  is a unitary representation;
- (iii) for each  $\xi \in \mathcal{H}$ , the mapping  $G \rightarrow \mathcal{C}, g \mapsto \langle \pi(g)\xi, \xi \rangle$  is continuous.

[If necessary, see Chapter 13 of [Robet–83].]

**Exercise A.8.2** Let  $G$  be a locally compact group, and let  $\pi : G \rightarrow \mathcal{U}(\mathcal{H})$  be a homomorphism into the unitary group of a Hilbert space  $\mathcal{H}$ . Let  $\mathcal{V}$  be a total set in  $\mathcal{H}$ , and assume that

$$G \rightarrow \mathcal{H}, \quad g \mapsto \pi(g)\xi$$

is continuous for all  $\xi$  in  $\mathcal{V}$ . Prove that  $\pi$  is strongly continuous (and hence a unitary representation of  $G$ ).

**Exercise A.8.3** Let  $G$  be a locally compact group. For  $p \in [1, +\infty)$ , let  $f \in L^p(G)$ . Prove that  $a \mapsto {}_a f$  is continuous mapping from  $G$  to  $L^p(G)$ . Show that this is no longer true for  $p = +\infty$ .

[Hint: Show this first for  $f \in C_c(G)$ .]

**Exercise A.8.4** Let  $G$  be a non-discrete locally compact group. Let  $G_d$  be the group  $G$  endowed with the discrete topology. Let  $\pi$  be the left regular representation of  $G_d$  on  $\ell^2(G_d)$ .

(i) Show that  $g \mapsto \langle \pi(g)\xi, \xi \rangle$  is a measurable function on  $G$  for all  $\xi \in \ell^2(G_d)$ .

[Hint: The function  $g \mapsto \langle \pi(g)\xi, \xi \rangle$  is zero on the complement of a countable set.]

(ii) Show that  $g \mapsto \langle \pi(g)\xi, \xi \rangle$  is not continuous on  $G$  if  $\xi \neq 0$ .

**Exercise A.8.5** Let  $G$  be the real  $(ax + b)$ -group, as in Example A.3.5.iv. Prove that  $dadb/|a|^2$  is a left Haar measure and that  $dadb/|a|$  is a right Haar measure on  $G$ .

**Exercise A.8.6** Let  $G$  be the real Heisenberg group, as in Example A.3.5.vi. Prove that  $dx dy dz$  is a left and right Haar measure on  $G$ .

**Exercise A.8.7** Let  $G$  be a connected Lie group. For  $g \in G$ , let  $L_g : x \mapsto gx$  and  $R_g : x \mapsto xg$  be the left and right translations by  $g$ . Recall that  $(d(R_g^{-1}L_g))_e = \text{Ad}(g)$ . Set  $n = \dim G$ .

(i) Let  $\omega$  be an  $n$ -form on  $G$ . Show that

$$(d(R_g^{-1}L_g)\omega)_e = \omega_e \circ \text{Ad}(g) = \det(\text{Ad}(g))\omega_e.$$

(ii) Let  $\omega$  be a non-zero left invariant  $n$ -form on  $G$ . Show that  $dR_g\omega$  is left invariant. Hence, there exists  $\lambda(g) > 0$  such that  $dR_g\omega = \lambda(g)\omega$ .

(iii) Show that, for the modular function  $\Delta_G$  on  $G$ , we have

$$\Delta_G(g) = \lambda(g^{-1}) = \det(\text{Ad}(g^{-1})).$$

**Exercise A.8.8** Let  $\mathbf{K}$  be a local field, that is,  $\mathbf{K}$  is a topological field for a non-discrete locally compact topology (compare Chapter D.4). Let  $\mu$  be a Haar measure on  $\mathbf{K}$ .

(i) Show that, for every  $a \in \mathbf{K} \setminus \{0\}$ , there exists a positive real number  $c(a)$  such that

$$\int_{\mathbf{K}} f(a^{-1}x) d\mu(x) = c(a) \int_{\mathbf{K}} f(x) d\mu(x)$$

for all  $f \in C_c(\mathbf{K})$ . Determine  $c(a)$  for  $\mathbf{K} = \mathbf{R}$  and  $\mathbf{K} = \mathbf{C}$ .



(ii) Setting  $c(0) = 0$ , show that  $c$  is a continuous function on  $\mathbf{K}$  and that  $c(ab) = c(a)c(b)$  for all  $a, b \in \mathbf{K}$ .

(iii) Show that  $\mathbf{K}$  is not compact.

**Exercise A.8.9** With the notation as in the previous exercise, let  $dx$  denote the measure  $\mu \otimes \cdots \otimes \mu$  on  $\mathbf{K}^n$ .

(i) Let  $A \in GL_n(\mathbf{K})$ . Show that

$$\int_{\mathbf{K}^n} f(Ax) dx = c(\det A)^{-1} \int_{\mathbf{K}^n} f(x) dx,$$

for all  $f \in C_c(\mathbf{K}^n)$ .

[Hint: Consider first the case where  $A$  is a diagonal matrix and then the case where  $A$  is an elementary matrix, that is, a matrix with 1 on the diagonal,  $a \in \mathbf{K}$  at the entry  $(i, j)$ , and 0 otherwise.]

(ii) Deduce from (i) that a left and right Haar measure on  $G = GL_n(\mathbf{K})$  is given by

$$f \mapsto \int_G f(X) c(\det X)^{-n} dX,$$

where  $G$  is viewed as an open subset of  $\mathbf{K}^{n^2}$  and  $dX$  denotes the measure  $\mu \otimes \cdots \otimes \mu$  on  $\mathbf{K}^{n^2}$ .

**Exercise A.8.10** A locally compact group  $G$  with left Haar measure  $\mu$  is of *subexponential growth* if  $\lim_n (\mu(U^n))^{1/n} = 1$  for every compact neighbourhood  $U$  of  $G$ . This is a generalisation of the notion of a group of polynomial growth (see Example A.3.7). For examples of groups of subexponential growth and not of polynomial growth, see [Harpe-00, Chapter VIII].

Show that a locally compact group of subexponential growth is unimodular.

[Hint: Assume, by contradiction, that  $\Delta_G(x) > 1$  for some  $x \in G$ . Let  $U$  be a compact neighbourhood of  $e$  containing  $x$ . Observe that  $Ux^n$  is contained in  $U^{n+1}$ .]

# Appendix B

## Measures on homogeneous spaces

Let  $G$  be a topological group, and let  $H$  be a closed subgroup of  $G$ . In the present chapter, we recall how, in case  $G$  is locally compact,  $G/H$  has also a measure which is quasi-invariant by  $G$ , and canonically defined up to appropriate equivalence. It follows that any closed subgroup provides the corresponding *quasi-regular representation* of  $G$ , which is a basic example of a unitary representation. More generally, quasi-invariant measures enter the definition of induced representations: see Appendix E.

We discuss the existence of *invariant* measures on  $G/H$  in Section B.1. The important case where  $H$  is a lattice is treated in Section B.2.

### B.1 Invariant measures

For the quotient topology on the homogeneous space  $G/H$ , the canonical projection  $p : G \rightarrow G/H$  is continuous and open.

**Lemma B.1.1** *Let  $Q$  be a compact subset of  $G/H$ . Then there exists a compact subset  $K$  of  $G$  with  $p(K) = Q$ .*

**Proof** Let  $U$  be a compact neighbourhood of  $e$  in  $G$ . As  $Q$  is compact and  $p(xU)$  is a neighbourhood of  $xH$  for each  $x$  in  $G$ , there exists  $x_1, \dots, x_n$  in  $G$  such that  $Q \subset \bigcup_{i=1}^n p(x_i U)$ . The set

$$K = p^{-1}(Q) \cap \bigcup_{i=1}^n x_i U$$

is compact and  $p(K) = Q$ . ■

**Lemma B.1.2** *Let  $dh$  be a left Haar measure on  $H$ . The linear mapping*

$$T_H : C_c(G) \rightarrow C_c(G/H), \quad (T_H f)(xH) = \int_H f(xh)dh,$$

*is surjective.*

**Proof** Let  $\varphi \in C_c(G/H)$  and  $Q = \text{supp } \varphi$ . By the previous lemma,  $Q = p(K)$  for some compact subset  $K$  of  $G$ . Choose  $\psi \in C_c(G)$ ,  $\psi \geq 0$  with  $\psi = 1$  on  $K$ . Define a function  $f$  on  $G$  by

$$f(x) = \frac{\varphi(p(x))}{(T_H \psi)(p(x))} \psi(x)$$

if  $(T_H \psi)(p(x)) \neq 0$  and  $f(x) = 0$  otherwise. Then  $f$  is continuous since  $T_H \psi > 0$  on a neighbourhood of  $Q = \text{supp } \varphi$ , its support is compact, and  $T_H f = \varphi$ . ■

The next elementary lemma is crucial for all what follows.

**Lemma B.1.3** *Let  $\rho : G \rightarrow \mathbf{R}_+^*$  be a continuous function on  $G$ . The following properties are equivalent:*

(i) *the formula*

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(x)$$

*holds for all  $x \in G$  and all  $h \in H$ ;*

(ii) *the formula*

$$\int_G f(xh^{-1})\rho(x)dx = \Delta_H(h) \int_G f(x)\rho(x)dx,$$

*holds for all  $f \in C_c(G)$  and all  $h \in H$ ;*

(iii) *the mapping*

$$T_H f \mapsto \int_G f(x)\rho(x)dx, \quad f \in C_c(G),$$

*is a well-defined positive linear functional on  $C_c(G/H)$ .*

*If these properties hold, the associated regular Borel measure  $\mu$  on  $G/H$  is quasi-invariant with Radon–Nikodym derivative*

$$\frac{d\mu}{d\mu}(xH) = \frac{\rho(yx)}{\rho(x)}, \quad y, x \in G.$$

**Proof** Since, for any  $f \in C_c(G)$  and  $h \in H$ ,

$$\int_G f(xh^{-1})\rho(x)dx = \Delta_G(h) \int_G f(x)\rho(xh)dx,$$

it is clear that (i) and (ii) are equivalent.

We want to show that (i) and (iii) are equivalent. Assume that (i) holds. Let  $f, g \in C_c(G)$ . We have

$$\begin{aligned} \int_G f(x)(T_H g)(p(x))\rho(x)dx &= \int_G f(x) \int_H g(xh)dh \rho(x)dx \\ &= \int_H \int_G f(x)g(xh)\rho(x)dx dh \\ &= \int_H \Delta_G(h^{-1}) \int_G f(xh^{-1})g(x)\rho(xh^{-1})dx dh \\ &= \int_H \Delta_H(h^{-1}) \int_G f(xh^{-1})g(x)\rho(x)dx dh \\ &= \int_G \int_H \Delta_H(h^{-1})f(xh^{-1})dh g(x)\rho(x)dx \\ &= \int_G \int_H f(xh)dh g(x)\rho(x)dx \\ &= \int_G T_H f(p(x))g(x)\rho(x)dx, \end{aligned}$$

where we have used successively the definition of  $T_H g$ , Fubini's Theorem, the definition of  $\Delta_G$ , Property (i), Fubini's Theorem again, Lemma A.3.4, and the definition of  $T_H f$ . Observe that, even though the Haar measures on  $G$  or  $H$  are not necessarily  $\sigma$ -finite, we can apply Fubini's Theorem since the function

$$(x, h) \mapsto f(x)g(xh)\rho(x)$$

vanishes outside a  $\sigma$ -finite subset of  $G \times H$  (see [HewRo–63, (13.10) Theorem]).

Suppose that  $T_H f = 0$ . It follows from the formula above that

$$\int_G f(x)(T_H g)(p(x))\rho(x)dx = 0, \quad \text{for all } g \in C_c(G).$$

By the previous lemma, there exists  $g \in C_c(G)$  with  $T_H g = 1$  on  $\text{supp } f$ . Hence,  $\int_G f(x)\rho(x)dx = 0$ . This and the previous lemma show that

$$T_H f \mapsto \int_G f(x)\rho(x)dx$$

is a well-defined positive linear functional on  $C_c(G/H)$ . Hence, (i) implies (iii).

Assume that (iii) holds. The Borel measure  $\mu$  on  $G/H$  associated to the given linear functional satisfies the identity

$$\int_G f(x) \rho(x) dx = \int_{G/H} T_H f(xH) d\mu(xH) = \int_{G/H} \int_H f(xk) dk d\mu(xH)$$

for all  $f \in C_c(G)$ . Hence, for  $h \in H$ , we have

$$\begin{aligned} \int_G f(xh^{-1}) \rho(x) dx &= \int_{G/H} \int_H f(xkh^{-1}) dk d\mu(xH) \\ &= \Delta_H(h) \int_{G/H} \int_H f(xk) dk d\mu(xH) \\ &= \Delta_H(h) \int_G f(x) \rho(x) dx. \end{aligned}$$

This shows that (ii) is satisfied.

For the last claim, consider  $\varphi \in C_c(G/H)$ . Choose  $f \in C_c(G)$  with  $T_H f = \varphi$ . Let  $y \in G$ . Observe that the function  $x \mapsto \frac{\rho(yx)}{\rho(x)}$  can be viewed as a function on  $G/H$ , as a consequence of (i). Observe also that, if  $\tilde{f}$  denotes the function  $x \mapsto f(yx) \frac{\rho(yx)}{\rho(x)}$ , we have

$$\begin{aligned} (T_H \tilde{f})(xH) &= \int_H f(yxh) \frac{\rho(yxh)}{\rho(xh)} dh = \int_H f(yxh) \frac{\rho(yx)}{\rho(x)} dh \\ &= \varphi(yxH) \frac{\rho(yx)}{\rho(x)}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{G/H} \varphi(yxH) \frac{\rho(yx)}{\rho(x)} d\mu(xH) &= \int_G f(yx) \frac{\rho(yx)}{\rho(x)} \rho(x) dx \\ &= \int_G f(yx) \rho(yx) dx = \int_G f(x) \rho(x) dx \\ &= \int_{G/H} \varphi(xH) d\mu(xH). \end{aligned}$$

This shows the formula of the last claim for the Radon–Nikodym derivative. ■

A continuous function  $\rho : G \rightarrow \mathbf{R}_+^*$  satisfying the identity (i) in Lemma B.1.3

$$(*) \quad \rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(x), \quad \text{for all } x \in G, h \in H$$

exists for any locally compact group  $G$  and any closed subgroup  $H$ . For the proof, see [Reiter–68, Chapter 8, Section 1] or [Folla–95, (2.54)]. Such a function is called a *rho-function* for the pair  $(G, H)$ .

Taking this for granted, Part (i) of the following theorem follows from Lemma B.1.3. Part (iii) is straightforward. For the proof of Part (ii), see [Bou–Int2] or [Folla–95, (2.59)].

**Theorem B.1.4** (i) *Quasi-invariant regular Borel measures always exist on  $G/H$ . More precisely, given a rho-function  $\rho$  for  $(G, H)$ , there exists a quasi-invariant regular Borel measure  $\mu$  on  $G/H$  such that*

$$\int_G f(x) \rho(x) dx = \int_{G/H} \int_H f(xh) dh d\mu(xH), \quad f \in C_c(G)$$

and with Radon–Nikodym derivative

$$\frac{d\mu}{d\mu}(xH) = \frac{\rho(gx)}{\rho(x)}, \quad g, x \in G.$$

(ii) *Any quasi-invariant regular Borel measure on  $G/H$  is associated as above to a rho-function for  $(G, H)$ .*

(iii) *If  $\mu_1$  and  $\mu_2$  are quasi-invariant regular Borel measures on  $G/H$ , with corresponding rho-functions  $\rho_1$  and  $\rho_2$ , then  $\mu_1$  and  $\mu_2$  are equivalent, with Radon–Nikodym derivative  $d\mu_1/d\mu_2 = \rho_1/\rho_2$ .*

**Proposition B.1.5** *Let  $\mu$  be a quasi-invariant regular Borel measure on  $G/H$  as above. Then the support of  $\mu$  is  $G/H$ .*

**Proof** The proof is similar to the proof of Proposition A.3.2. Indeed, assume that  $\mu(U) = 0$  for some open non-empty subset  $U$  of  $G/H$ . Then  $\mu(gU) = 0$  for all  $g \in G$ , by quasi-invariance of  $\mu$ . For any compact subset  $K$  of  $G/H$ , there exists  $g_1, \dots, g_n \in G$  such that  $K \subset \bigcup_{i=1}^n g_i U$ . Hence,  $\mu(K) = 0$  for every compact subset  $K$  of  $G/H$ . Since  $\mu$  is regular, this implies that  $\mu = 0$ , a contradiction. ■

Under suitable conditions, there exist relatively invariant measures on homogeneous spaces. A measure  $\mu$  on  $G/H$  is said to be *relatively invariant* if, for every  $g \in G$ , there exists a constant  $\chi(g) > 0$  such that  $g^{-1}\mu = \chi(g)\mu$ , that

is, the Radon–Nikodym derivative  $\frac{dg\mu}{d\mu}$  is a constant, possibly depending on  $g$ . It is clear that  $\chi$  is then a continuous homomorphism from  $G$  to  $\mathbf{R}_+^*$ , called the *character* of  $\mu$ . Relatively invariant measures are also called semi-invariant measures (see [Raghu–72, page 18]).

**Proposition B.1.6** (i) Assume that there exists a relatively invariant regular Borel measure  $\mu$  on  $G/H$ . Then the character of  $\mu$  is a continuous extension to  $G$  of the homomorphism  $h \mapsto \Delta_H(h)/\Delta_G(h)$  of  $H$ .

(ii) Assume that the homomorphism  $H \rightarrow \mathbf{R}_+^*$ ,  $h \mapsto \Delta_H(h)/\Delta_G(h)$  extends to a continuous homomorphism  $\chi : G \rightarrow \mathbf{R}_+^*$ . Then there exists a relatively invariant regular Borel measure  $\mu$  with character  $\chi$ . Moreover, the measure  $\mu$  is unique up to a constant: if  $\mu'$  is another relatively invariant regular Borel measure on  $G/H$  with character  $\chi$ , then  $\mu' = c\mu$  for a constant  $c > 0$ .

**Proof** (i) Let  $\chi$  be the character of the relatively invariant measure  $\mu$ . The linear functional on  $C_c(G)$  given by

$$f \mapsto \int_{G/H} T_H \left( \frac{f}{\chi} \right) (xH) d\mu(xH)$$

defines a left invariant measure on  $G$ . Indeed, for every  $a \in G$ , we have

$$\begin{aligned} \int_{G/H} T_H \left( \frac{af}{\chi} \right) (xH) d\mu(xH) &= \chi(a) \int_{G/H} T_H \left( \frac{af}{a\chi} \right) (xH) d\mu(xH) \\ &= \chi(a) \int_{G/H} T_H \left( \frac{f}{\chi} \right) (axH) d\mu(xH) \\ &= \int_{G/H} T_H \left( \frac{f}{\chi} \right) (xH) d\mu(xH). \end{aligned}$$

Hence, by uniqueness of Haar measure, there exists  $c > 0$  such that

$$(*) \quad \int_{G/H} T_H f(xH) d\mu(xH) = c \int_G f(x) \chi(x) dx, \quad \text{for all } f \in C_c(G).$$

Replacing  $\mu$  by  $c^{-1}\mu$ , we can assume that  $c = 1$ . Then, for  $f \in C_c(G)$  and  $k \in H$ , we have

$$\begin{aligned} \Delta_G(k) \int_G f(x) \chi(x) dx &= \int_G f(xk^{-1}) \chi(xk^{-1}) dx \\ &= \chi(k^{-1}) \int_{G/H} \int_H f(xhk^{-1}) dh d\mu(xH) \end{aligned}$$

$$\begin{aligned}
&= \Delta_H(k) \chi(k^{-1}) \int_{G/H} \int_H f(xh) dh d\mu(xH) \\
&= \Delta_H(k) \chi(k^{-1}) \int_G f(x) \chi(x) dx.
\end{aligned}$$

Hence,  $\chi|_H = (\Delta_H)/(\Delta_G|_H)$ , as claimed.

(ii) Let  $\chi : G \rightarrow \mathbf{R}_+^*$  be a continuous homomorphism extending  $(\Delta_H)/(\Delta_G|_H)$ . Then  $\rho = \chi$  satisfies Condition (i) in Lemma B.1.3. Hence, there exists a relatively invariant Borel measure on  $G/H$  with character  $\chi$ . Moreover, Lemma B.1.2 shows that Formula (\*) determines  $\mu$ . This shows the uniqueness of  $\mu$  up to a constant. ■

Taking  $\chi = 1$  in the previous proposition, we obtain the following result.

**Corollary B.1.7** *An invariant Borel measure exists on  $G/H$  if and only if  $\Delta_G|_H = \Delta_H$ . Moreover, such a measure  $\mu$  is unique up to a constant factor and, for a suitable choice of this factor, we have*

$$\int_G f(x) dx = \int_{G/H} T_H f(xH) d\mu(xH) = \int_{G/H} \int_H f(xh) dh d\mu(xH)$$

for all  $f \in C_c(G)$ .

In practice, the formula

$$\int_G f(x) dx = \int_{G/H} \int_H f(xh) dh d\mu(xH)$$

is often used to determine a Haar measure of  $G$ , once an invariant measure on  $G/H$  and a Haar measure on  $H$  are known (Example B.1.11.iii).

**Corollary B.1.8** *Let  $H$  be a unimodular subgroup of the locally compact group  $G$ .*

- (i) *Then  $G/H$  has a relatively invariant regular measure  $\mu$ , with character  $(\Delta_G)^{-1}$ .*
- (ii) *If  $\mu$  is finite (this happens, for instance, if  $G/H$  is compact), then  $\mu$  is invariant, and  $G$  is unimodular.*

**Proof** (i) This is a particular case of Proposition B.1.6.

(ii) Assume that  $\mu(G/H) < \infty$ . For  $g \in G$ , we have

$$\mu(G/H) = \mu(gG/H) = (g^{-1})\mu(G/H) = (\Delta_G(g^{-1}))\mu(G/H)$$

and therefore  $\Delta_G(g) = 1$ . ■



**Definition B.1.9** Let  $H$  be a closed subgroup of the locally compact group  $G$ . Let  $\rho$  be a rho-function for  $(G, H)$  and let  $\mu$  be the corresponding quasi-invariant measure on  $G/H$ . The unitary representation  $\lambda_{G/H}$  of  $G$  defined on  $L^2(G/H) = L^2(G/H, \mu)$  by

$$\lambda_{G/H}(g)\xi(x) = \left( \frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x), \quad g, x \in G, \quad \xi \in L^2(G/H)$$

is called the *quasi-regular representation* of  $G$  associated to  $H$ .

**Remark B.1.10** (i) The quasi-regular representation  $\lambda_{G/H}$  coincides with the representation  $\pi_\mu$  defined in Section A.6.

(ii) If  $H$  is normal in  $G$ , then  $\lambda_{G/H}$  coincides with the lift to  $G$  of the left regular representation of the quotient group.

**Example B.1.11** (i) Let  $G = SL_2(\mathbf{R})$  and  $P$  the subgroup of upper triangular matrices. Then  $G$  is unimodular, but  $P$  is not (Example A.3.5). Thus, we recover the fact, mentioned at the end of Section A.6, that  $G/P$  has no invariant measure.

Moreover,  $G$  has no character different from  $1_G$ , since  $[G, G] = G$ . Hence, there is no relatively invariant measure on  $G/P$ .

(ii) Let  $K$  be a compact subgroup of the locally compact group  $G$ . Then  $G/K$  has an invariant Borel measure. Indeed,  $\Delta_G(K)$ , being a compact subgroup of  $\mathbf{R}_+^*$ , is trivial. For the same reason,  $K$  is unimodular.

The fact that  $G/K$  has an invariant measure can also be seen by the following argument. The projection  $p : G \rightarrow G/K$  is proper (that is,  $p^{-1}(Q)$  is compact for every compact subset  $Q$  of  $G/K$ ). For any regular Borel measure  $\mu$  on  $G$ , the image  $p_*(\mu)$ , defined by  $p_*(\mu)(A) = \mu(p^{-1}(A))$  for Borel subsets  $A$  in  $G/K$ , is a regular Borel measure on  $G/K$ . It is clear that  $p_*(\mu)$  is invariant if  $\mu$  is a left Haar measure on  $G$ .

(iii) Let  $G = SL_2(\mathbf{R})$  and  $K = SO(2)$ . The invariant measure on  $G/K$  can be determined as follows. Let

$$\mathbf{P} = \{z \in \mathbf{C} : \text{Im} z > 0\}$$

be the *Poincaré half-plane*. The group  $G$  acts by fractional linear transformations on  $\mathbf{P}$ :

$$(g, z) \mapsto gz = \frac{az + b}{cz + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad z \in \mathbf{P}.$$

The stabiliser of  $i$  being  $K$ , the mapping

$$G/K \rightarrow \mathbf{P}, \quad gK \mapsto gi$$

identifies  $G/K$  with  $\mathbf{P}$ . The measure  $d\mu(x, y) = y^{-2}dxdy$  on  $\mathbf{P}$  is  $G$ -invariant. Indeed, let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $G$ ; denote by  $M_g$  the associated transformation of  $\mathbf{P}$ . On the one hand,  $\text{Im}(M_g z) = |cz + d|^{-2}\text{Im}(z)$ , by a straightforward computation. On the other hand, when viewed as a linear transformation of  $\mathbf{R}^2$ , the Jacobian of  $M_g$  at  $z \in \mathbf{P}$  is equal to  $|cz + d|^{-4}$ . Thus, by change of variable, we find

$$\begin{aligned} \int_{\mathbf{P}} \varphi(g^{-1}z) \frac{dxdy}{y^2} &= \int_{\mathbf{P}} \varphi(z) \frac{1}{|cz + d|^4} \frac{|cz + d|^4}{y^2} dxdy \\ &= \int_{\mathbf{P}} \varphi(z) \frac{dxdy}{y^2}. \end{aligned}$$

A left Haar measure  $dg$  on  $G = SL_2(\mathbf{R})$  is given by

$$\int_G f(g) dg = \int_{\mathbf{P}} \frac{dxdy}{y^2} \int_{SO(2)} f(gk) dk,$$

where  $gK$  is identified with  $gi = x + iy$  and  $dk$  is the normalised Lebesgue measure on  $SO(2) = \mathbf{S}^1$ .

## B.2 Lattices in locally compact groups

Let  $G$  be a locally compact group.

**Definition B.2.1** A *lattice* in  $G$  is a discrete subgroup  $\Gamma$  of  $G$  such that  $G/\Gamma$  carries a *finite invariant* regular Borel measure. Observe that such a measure is then unique up to a constant factor, by Corollary B.1.7.

As discrete groups are unimodular, the following proposition is an immediate consequence of Corollary B.1.8.

**Proposition B.2.2** Let  $\Gamma$  be a discrete subgroup of  $G$ .

- (i) If  $\Gamma$  is cocompact (that is, if  $G/\Gamma$  is compact), then  $\Gamma$  is a lattice.
- (ii) If  $\Gamma$  is a lattice, then  $G$  is unimodular.

A more direct argument for (ii) is as follows. Since  $\Gamma$  is unimodular, it is contained in the normal subgroup  $\text{Ker } \Delta_G$  of  $G$ . Hence, the Haar measure of the locally compact group  $G/\text{Ker } \Delta_G$  is finite. This implies that  $G/\text{Ker } \Delta_G$  is compact (see Proposition A.5.1). Hence,  $G/\text{Ker } \Delta_G$  is topologically isomorphic to a compact subgroup of  $\mathbf{R}_+^*$ . It follows that  $G = \text{Ker } \Delta_G$ .

**Definition B.2.3** Let  $G$  be a topological group and let  $H$  be a subgroup of  $G$ . A *Borel fundamental domain* for  $H$  is a Borel subset  $\Omega$  of  $G$  such that  $G = \bigcup_{h \in H} \Omega h$  and  $\Omega h_1 \cap \Omega h_2 = \emptyset$  for all  $h_1, h_2 \in G$  with  $h_1 \neq h_2$ .

The following proposition is often used to show that a discrete subgroup is a lattice.

**Proposition B.2.4** *Let  $G$  be a  $\sigma$ -compact locally compact group and let  $\Gamma$  be a discrete subgroup of  $G$ .*

- (i) *There exists a Borel fundamental domain for  $\Gamma$ .*
- (ii) *If  $\Gamma$  is a lattice in  $G$ , then every Borel fundamental domain for  $\Gamma$  has finite Haar measure.*
- (iii) *Assume that  $G$  is unimodular and that there exists a Borel subset  $\Omega$  of  $G$  with finite Haar measure such that  $G = \Omega\Gamma$ . Then  $\Gamma$  is a lattice in  $G$ .*

**Proof** Since  $\Gamma$  is discrete, there exists an open neighbourhood  $U$  of  $e$  in  $G$  such that  $U \cap \Gamma = \{e\}$ . Let  $V$  be an open neighbourhood of  $e$  such that  $V^{-1}V \subset U$ . As  $G$  is  $\sigma$ -compact, there exists a sequence  $(g_n)_{n \in \mathbb{N}}$  in  $G$  such that  $G = \bigcup_{n \in \mathbb{N}} g_n V$ . We define inductively a sequence  $(\Omega_n)_{n \in \mathbb{N}}$  of pairwise disjoint Borel subsets of  $G$  as follows. Set  $\Omega_0 = g_0 V$ . For every  $n \geq 1$ , consider the Borel set

$$\Omega_n = g_n V \setminus \left( \bigcup_{0 \leq m < n} g_m V \Gamma \right).$$

We claim that

$$\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$$

is a Borel fundamental domain for  $\Gamma$ . Indeed,  $\Omega$  is a Borel subset of  $G$ . Moreover, since  $G = \bigcup_{n \in \mathbb{N}} g_n V \Gamma$ , we have  $G = \bigcup_{n \in \mathbb{N}} \Omega_n \Gamma$ , that is,  $G = \Omega \Gamma$ . Let  $\gamma_1, \gamma_2 \in \Gamma$  be such that  $\Omega \gamma_1 \cap \Omega \gamma_2 \neq \emptyset$ . Then there exist  $m, n \in \mathbb{N}$  such that  $\Omega_n \gamma_1 \cap \Omega_m \gamma_2 \neq \emptyset$ . Since  $\Omega_i \cap \Omega_j \Gamma = \emptyset$  for all  $i, j$  with  $i > j$ , it follows that  $n = m$  and that there exist  $v, w \in V$  such that  $g_n v \gamma_1 = g_n w \gamma_2$ . This implies that

$$w^{-1}v = \gamma_2 \gamma_1^{-1} \in V^{-1}V \cap \Gamma \subset U \cap \Gamma.$$

It follows that  $\gamma_2^{-1} \gamma_1 = e$ , that is,  $\gamma_1 = \gamma_2$ . This proves (i).

To show (ii), assume that  $\Gamma$  is a lattice. Then  $G$  is unimodular (Corollary B.1.8). Hence, by Corollary B.1.7, there exists a finite  $G$ -invariant regular Borel measure  $\mu$  on  $G/\Gamma$  such that

$$(*) \quad \int_G f(x) d\nu(x) = \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x\gamma) d\mu(x\Gamma), \quad \text{for all } f \in C_c(G),$$

where  $\nu$  is a Haar measure on  $G$ . Let  $\chi_\Omega$  denote the characteristic function of a Borel fundamental domain  $\Omega$  for  $\Gamma$ . We have

$$\nu(\Omega) = \sup \left\{ \int_G f(x) d\nu(x) : f \in C_c(G), 0 \leq f \leq \chi_\Omega \right\}$$

and, since  $\Omega$  is a fundamental domain for  $\Gamma$ ,

$$\sum_{\gamma \in \Gamma} \chi_\Omega(x\gamma) = 1, \quad \text{for all } x \in G.$$

Let  $f \in C_c(G)$  with  $0 \leq f \leq \chi_\Omega$ . Then

$$\begin{aligned} \int_G f(x) d\nu(x) &= \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x\gamma) d\mu(x\Gamma) \\ &\leq \int_{G/\Gamma} \sum_{\gamma \in \Gamma} \chi_\Omega(x\gamma) d\mu(x\Gamma) \\ &= \int_{G/\Gamma} d\mu(x\Gamma) = \mu(G/\Gamma) \end{aligned}$$

and therefore  $\nu(\Omega) \leq \mu(G/\Gamma) < \infty$ .

To show (iii), let  $\Omega$  be a Borel subset of  $G$  such that  $G = \Omega\Gamma$  and  $\nu(\Omega) < \infty$ , where  $\nu$  is a Haar measure on  $G$ . As above, there exists a  $G$ -invariant regular Borel measure  $\mu$  on  $G/H$  such that Formula (\*) holds. We claim that  $\mu$  is finite, that is,

$$\sup \left\{ \int_{G/\Gamma} \varphi(x\Gamma) d\mu(x\Gamma) : \varphi \in C_c(G/\Gamma) \text{ with } 0 \leq \varphi \leq 1 \right\} < \infty.$$

Let  $\varphi \in C_c(G/\Gamma)$  with  $0 \leq \varphi \leq 1$ . As the proof of Lemma B.1.2 shows, there exists  $f \in C_c(G)$  with  $f \geq 0$  and such that

$$\varphi(x\Gamma) = \sum_{\gamma \in \Gamma} f(x\gamma), \quad \text{for all } x \in G.$$

Since  $G = \bigcup_{\gamma \in \Gamma} \Omega\gamma$ , and  $\Gamma$  is countable (because  $G$  is  $\sigma$ -compact), we have

$$\begin{aligned}
 \int_{G/\Gamma} \varphi(x\Gamma) d\mu(x\Gamma) &= \int_{G/\Gamma} \sum_{\gamma \in \Gamma} f(x\gamma) d\mu(x\Gamma) \\
 &= \int_G f(x) dv(x) \\
 &\leq \sum_{\gamma \in \Gamma} \int_{\Omega\gamma} f(x) dv(x) \\
 &= \sum_{\gamma \in \Gamma} \int_G \chi_{\Omega\gamma}(x) f(x) dv(x) \\
 &= \sum_{\gamma \in \Gamma} \int_G \chi_{\Omega\gamma}(x\gamma) f(x\gamma) dv(x) \\
 &= \sum_{\gamma \in \Gamma} \int_{\Omega} f(x\gamma) dv(x) \\
 &= \int_{\Omega} \sum_{\gamma \in \Gamma} f(x\gamma) dv(x) \\
 &\leq \int_{\Omega} dv(x) = v(\Omega),
 \end{aligned}$$

where we used the fact that  $v$  is also right invariant (recall that  $G$  is unimodular). Hence,  $\mu$  is finite and  $\Gamma$  is a lattice in  $G$ . ■

**Example B.2.5** (i) The group  $\mathbf{Z}^n$  is a cocompact lattice in  $G = \mathbf{R}^n$ .

(ii) The discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z} \right\}$$

of the Heisenberg group  $G$  (see Example A.3.5) is a cocompact lattice. Indeed,

$$\Omega = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in [0, 1) \right\}$$

is a Borel fundamental domain for  $\Gamma$ . Moreover  $G/\Gamma$  is compact, since  $\Omega$  is relatively compact in  $G$  and  $G/\Gamma = p(\Omega) = p(\overline{\Omega})$ , where  $p : G \rightarrow G/\Gamma$  is the canonical projection.

(iii) The *modular group*  $\Gamma = SL_2(\mathbf{Z})$  is a discrete subgroup of  $G = SL_2(\mathbf{R})$ . Consider the domain

$$F = \{z \in \mathbf{P} : |z| \geq 1, |\operatorname{Re} z| \leq 1/2\}$$

in the Poincaré half-plane  $\mathbf{P}$  (see Example B.1.11). Then  $F$  intersects every orbit of  $\Gamma$  in  $\mathbf{P}$  (see [BeMa-00b, Chapter II, 2.7] or [Serr-70a, Chapter VII, 1.2]). It follows that  $G = \Gamma\varphi^{-1}(F)$ , where  $\varphi$  is the mapping

$$G \rightarrow \mathbf{P}, g \mapsto gi.$$

Moreover  $F$  has finite measure for the  $G$ -invariant measure  $\mu$  on  $\mathbf{P}$ . Indeed, we compute

$$\mu(F) = \int_F \frac{dx dy}{y^2} = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{dx dy}{y^2} = \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \frac{\pi}{3}.$$

If we identify  $\mathbf{P}$  with  $G/K$  for  $K = SO(2)$  and  $\varphi$  with the canonical mapping  $G \rightarrow G/K$ , the Haar measure of  $\varphi^{-1}(F)$  is

$$\begin{aligned} \int_G \chi_{\varphi^{-1}(F)}(g) dg &= \int_{G/K} \int_K \chi_{\varphi^{-1}(F)}(gk) dk d\mu(gK) \\ &= \int_{G/K} \chi_F(gK) \int_K dk d\mu(gK) \\ &= \mu(F) < \infty, \end{aligned}$$

where  $\chi_{\varphi^{-1}(F)}$  is the characteristic function of  $\varphi^{-1}(F)$ . Hence  $\Gamma$  is a lattice in  $G$ , by Proposition B.2.4.iii. Observe that  $\Gamma$  is not cocompact in  $G$ . Indeed, the interior of  $F$  intersects any  $\Gamma$ -orbit at most once and is not relatively compact.

(iv) Let  $H$  be the subgroup of  $\Gamma = SL_2(\mathbf{Z})$  generated by the two matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

Then  $H$  has index 12 in  $SL_2(\mathbf{Z})$ . More precisely, let  $\Gamma(2)$  be the kernel of the surjective homomorphism  $SL_2(\mathbf{Z}) \rightarrow SL_2(\mathbf{Z}/2\mathbf{Z})$  defined by reduction modulo 2. Since  $SL_2(\mathbf{Z}/2\mathbf{Z})$  is isomorphic to the group of permutations of three elements,  $\Gamma(2)$  has index 6 in  $SL_2(\mathbf{Z})$ . On the other hand,  $H$  is a subgroup of index 2 in  $\Gamma(2)$ ; see [Lehne-64, Chapter VII, 6C]. It follows from (iii) that  $H$  is a lattice in  $SL_2(\mathbf{R})$  which is not cocompact.

Moreover,  $H$  is isomorphic to  $F_2$ , the non-abelian free group on two generators (Exercise G.6.8). This shows that  $F_2$  embeds as a lattice in  $SL_2(\mathbf{R})$ .

(v) Let  $\Sigma$  be a closed Riemann surface of genus  $g \geq 2$ . Then, by uniformization theory,  $\mathbf{P}$  is a universal covering for  $\Sigma$ . Hence, the fundamental group  $\pi_1(\Sigma)$  of  $\Sigma$  can be identified with a cocompact lattice in  $PSL_2(\mathbf{R})$ ; see [FarKr-92, Chapter IV].

(vi) The subgroup  $\Gamma = SL_n(\mathbf{Z})$  is a lattice in  $G = SL_n(\mathbf{R})$ . This classical fact (see, e.g., [Bore-69b, 1.11 Lemme]) is due to H. Minkowski. The homogeneous space  $G/\Gamma$  – which can be identified in a natural way with the set of unimodular lattices in  $\mathbf{R}^n$  – is not compact.

### B.3 Exercises

**Exercise B.3.1** Let  $D = \{z \in \mathbf{C} : |z| < 1\}$  be the unit disc in  $\mathbf{C}$ . The group

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbf{C}, |\alpha|^2 - |\beta|^2 = 1 \right\}$$

acts by Möbius transformations on  $D$ .

(i) Show that the mapping

$$\Phi : \mathbf{P} \rightarrow D, z \mapsto \frac{z - i}{z + i}$$

is a biholomorphic isomorphism between the Poincaré half-plane  $\mathbf{P}$  and  $D$ .

(ii) Show that  $SU(1, 1)$  is conjugate to  $SL_2(\mathbf{R})$  inside  $SL_2(\mathbf{C})$ .

(iii) Show that  $dx dy / (1 - x^2 - y^2)^2$  is an  $SU(1, 1)$ -invariant measure on  $D$ .

**Exercise B.3.2** Recall that the algebra  $\mathbf{H}$  of the *Hamilton quaternions* over  $\mathbf{R}$  can be defined as the subalgebra

$$\left\{ \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} : x_1, \dots, x_4 \in \mathbf{R} \right\}$$

of  $M_2(\mathbf{C})$ . Let  $S^3$  be the unit sphere in  $\mathbf{R}^4$ .

(i) Verify that the mapping  $\Phi : \mathbf{R}^4 \rightarrow \mathbf{H}$  defined by

$$\Phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + ix_2 & -x_3 + ix_4 \\ x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

is an  $\mathbf{R}$ -linear isomorphism which, by restriction, induces a homeomorphism between  $S^3$  and the compact group  $SU(2)$ .

(ii) Verify that  $\|x\|^2 = \det \Phi(x)$  for all  $x \in \mathbf{R}^4$ .

(iii) For  $g \in SU(2)$ , let  $\pi(g)$  be the linear mapping on  $\mathbf{R}^4$  defined by

$$\pi(g)x = \Phi^{-1}(g\Phi(x)), \quad x \in \mathbf{R}^4.$$

Show that  $\pi(g) \in SO(4)$ .

(iv) Using the existence of a rotation invariant measure on  $S^3$  given in polar coordinates

$$x_1 = \cos \theta$$

$$x_2 = \sin \theta \cos \varphi$$

$$x_3 = \sin \theta \sin \varphi \cos \psi$$

$$x_4 = \sin \theta \sin \varphi \sin \psi$$

$$(0 \leq \theta \leq \pi, 0 \leq \varphi \leq \pi, 0 \leq \psi \leq 2\pi)$$

by  $\sin^2 \theta \sin \varphi d\theta d\varphi d\psi$ , show that this is the normalised Haar measure on  $SU(2)$ .

**Exercise B.3.3** Let  $\Phi$  be the linear isomorphism

$$\Phi : (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix}$$

between  $\mathbf{R}^3$  and the space of all real symmetric  $(2 \times 2)$  matrices. For  $g \in SL_2(\mathbf{R})$ , let  $\pi(g)$  be the linear mapping on  $\mathbf{R}^3$  defined by

$$\pi(g)x = \Phi^{-1}(g\Phi(x)g^t), \quad x \in \mathbf{R}^3.$$

(i) Show that  $\pi(g) \in O(1, 2)$ , the group of all linear transformations on  $\mathbf{R}^3$  preserving the quadratic form  $x_1^2 - x_2^2 - x_3^2$ .

(ii) Show that  $\text{Ker} \pi = \{\pm I\}$  and  $\pi(SL_2(\mathbf{R})) = SO_0(1, 2)$ , the connected component of  $SO(1, 2)$ .

(iii) Show that the homogeneous space  $SL_2(\mathbf{R})/SO(2)$  can be identified with the hyperboloid sheet

$$X = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 - x_2^2 - x_3^2 = 1, x_1 > 0\}.$$



(iv) Show that the mapping

$$(x_1, x_2, x_3) \mapsto \frac{x_2}{x_1} + i \frac{x_3}{x_1}$$

is a homeomorphism between  $X$  and the unit disc  $D$  in  $\mathbb{C}$ .

(v) Consider the coordinates

$$\begin{aligned} x_1 &= \cosh \theta, & x_2 &= \sinh \theta \cos \varphi, & x_3 &= \sinh \theta \sin \varphi, \\ (0 \leq \theta &\leq +\infty, & 0 \leq \varphi &\leq \pi) \end{aligned}$$

on  $X$ . Show that  $\sinh^2 \theta d\theta d\varphi$  is an  $SL_2(\mathbf{R})$ -invariant measure on  $X$ . Determine from this a Haar measure on  $SL_2(\mathbf{R})$ .

**Exercise B.3.4** Let  $G$  be a locally compact group. Suppose that  $G = H_1 H_2$  for two closed subgroups  $H_1$  and  $H_2$  such that the mapping  $(x, y) \mapsto xy$  is a homeomorphism between  $H_1 \times H_2$  and  $G$ . Show that

$$\frac{\Delta_G(y)}{\Delta_{H_2}(y)} dx dy$$

is a left Haar measure on  $G$ , where  $dx$  and  $dy$  are left Haar measures on  $H_1$  and  $H_2$ . In particular, if  $G$  is unimodular, then  $dx d_r y$  is a left Haar measure on  $G$ , where  $d_r y$  is a right Haar measure on  $H_2$ .

**Exercise B.3.5** Let  $G = H \ltimes N$  be a semidirect product of the locally compact groups  $H$  and  $N$ . Let  $\Gamma$  and  $\Delta$  be lattices in  $H$  and  $N$ , respectively. Assume that  $\Delta$  is normalised by  $\Gamma$ . Show that the semidirect product  $\Gamma \ltimes \Delta$  is a lattice in  $G$ .

# Appendix C

## Functions of positive type and GNS construction

Two of the most important notions for unitary group representations are functions of positive type and induced representations, defined respectively in the present appendix and in Appendix E.

We first discuss two kinds of kernels on a topological space  $X$ : those of positive type and those conditionally of negative type; the crucial difference is the presence or not of “conditionally”, whereas the difference between “positive” and “negative” is only a matter of sign convention. For each of these two types of kernels, there is a so-called GNS construction (for Gelfand, Naimark, and Segal) which shows how kernels are simply related to appropriate mappings of  $X$  to Hilbert spaces. Moreover, a theorem of Schoenberg establishes a relation between the two types of kernels.

A function  $\varphi$  on a topological group  $G$  is of positive type if the kernel  $(g, h) \mapsto \varphi(g^{-1}h)$  is of positive type. Functions of positive type provide an efficient tool to prove some basic general results, such as the Gelfand–Raikov Theorem according to which a locally compact group has sufficiently many irreducible unitary representations to separate its points.

### C.1 Kernels of positive type

Let  $X$  be a topological space.

**Definition C.1.1** A kernel of *positive type* on  $X$  is a continuous function  $\Phi: X \times X \rightarrow \mathbf{C}$  such that, for any  $n$  in  $\mathbf{N}$ , any elements  $x_1, \dots, x_n$  in  $X$ , and any complex numbers  $c_1, \dots, c_n$ , the following inequality holds:

$$\sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \Phi(x_i, x_j) \geq 0.$$

**Proposition C.1.2** *Let  $\Phi$  be a kernel of positive type on  $X$ . Then, for all  $x, y$  in  $X$ :*

- (i)  $\Phi(y, x) = \overline{\Phi(x, y)}$ ;
- (ii)  $|\Phi(x, y)|^2 \leq \Phi(x, x)\Phi(y, y)$ .

**Proof** For  $n = 2$ ,  $x_1 = x$ ,  $x_2 = y$ , the matrix

$$\begin{pmatrix} \Phi(x, x) & \Phi(x, y) \\ \Phi(y, x) & \Phi(y, y) \end{pmatrix}$$

is positive, by definition. Hence, it is hermitian with positive diagonal values, and

$$\Phi(x, x)\Phi(y, y) - \Phi(y, x)\Phi(x, y) \geq 0.$$

This proves the claim. ■

**Example C.1.3** Let  $\mathcal{H}$  be a Hilbert space and  $f: X \rightarrow \mathcal{H}$  a continuous mapping. Then  $\Phi$ , defined by

$$\Phi(x, y) = \langle f(x), f(y) \rangle,$$

is a kernel of positive type on  $X$ . Indeed,

$$\sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \Phi(x_i, x_j) = \left\langle \sum_{i=1}^n c_i f(x_i), \sum_{i=1}^n c_i f(x_i) \right\rangle \geq 0$$

for all  $x_1, \dots, x_n$  in  $X$  and  $c_1, \dots, c_n$  in  $\mathbb{C}$ .

The so-called *GNS construction* (named after Gelfand, Naimark and Segal) shows that the above example is universal.

**Theorem C.1.4 (GNS construction)** *Let  $\Phi$  be a kernel of positive type on the topological space  $X$ . Then there exists a Hilbert space  $\mathcal{H}$  and a continuous mapping  $f: X \rightarrow \mathcal{H}$  with the following properties:*

- (i)  $\Phi(x, y) = \langle f(x), f(y) \rangle$  for all  $x, y$  in  $X$ ;
- (ii) the linear span of  $\{f(x): x \in X\}$  is dense in  $\mathcal{H}$ .

Moreover, the pair  $(\mathcal{H}, f)$  is unique, up to canonical isomorphism, that is, if  $(\mathcal{K}, g)$  is another pair satisfying (i) and (ii), then there exists a unique Hilbert space isomorphism  $T: \mathcal{H} \rightarrow \mathcal{K}$  such that  $g = T \circ f$ .

**Proof** For every  $x \in X$ , denote by  $\Phi_x$  the continuous function on  $X$  defined by  $\Phi_x(y) = \Phi(x, y)$ . Let  $V$  be the linear span of the subset

$$\{\Phi_x : x \in X\}$$

of  $C(X)$ , the space of continuous functions on  $X$ . For  $\varphi = \sum_{i=1}^m a_i \Phi_{x_i}$  and  $\psi = \sum_{j=1}^n b_j \Phi_{x_j}$ , define

$$\langle \varphi, \psi \rangle_\Phi := \sum_{i=1}^m \sum_{j=1}^n a_i \bar{b}_j \Phi(x_i, x_j).$$

Observe that

$$\sum_{i=1}^m \sum_{j=1}^n a_i \bar{b}_j \Phi(x_i, x_j) = \sum_{j=1}^n \bar{b}_j \varphi(x_j) = \sum_{i=1}^m a_i \overline{\psi(x_i)}.$$

It follows that the common value of these sums does not depend on the representation of  $\varphi$  (respectively  $\psi$ ) in  $V$  by the sum  $\sum_{i=1}^m a_i \Phi_{x_i}$  (respectively  $\sum_{j=1}^n b_j \Phi_{x_j}$ ).

The mapping

$$(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_\Phi$$

is a positive hermitian form on  $V$ . We claim that it is definite. Indeed, for all  $\varphi$  in  $V$ , we have  $\langle \varphi, \Phi_x \rangle_\Phi = \varphi(x)$  and, by the Cauchy–Schwarz inequality,

$$(*) \quad |\varphi(x)|^2 \leq \Phi(x, x) \langle \varphi, \varphi \rangle_\Phi, \quad \text{for all } x \in X.$$

Thus,  $(V, \langle \cdot, \cdot \rangle_\Phi)$  is a prehilbert space. Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle_\Phi)$  be the Hilbert space completion of  $V$ . Observe that  $\mathcal{H}$  can be realised as a space of functions on  $X$ . Indeed, if  $(\varphi_n)_n$  is a Cauchy sequence in  $V$  for the norm  $\| \cdot \|_\Phi$  induced by the inner product, then  $(*)$  implies that  $(\varphi_n(x))_n$  converges for every  $x \in X$ . The Hilbert space completion of  $V$  is a subspace of the space of all functions  $\varphi$  on  $X$  which are pointwise limits of Cauchy sequences in  $(V, \| \cdot \|_\Phi)$ .

Let  $f : X \rightarrow V$ ,  $x \mapsto \Phi_x$ . Then

$$\langle f(x), f(y) \rangle_\Phi = \Phi(x, y).$$

Moreover,  $f$  is continuous since

$$\|f(x) - f(y)\|_\Phi^2 = \Phi(x, x) - 2\operatorname{Re}\Phi(x, y) + \Phi(y, y)$$

and since  $\Phi$  is continuous.

This settles the existence of the pair  $(\mathcal{H}, f)$  with Properties (i) and (ii). Let  $(\mathcal{K}, g)$  be another pair consisting of a Hilbert space  $\mathcal{K}$  and a continuous function  $g : X \rightarrow \mathcal{K}$  satisfying (i) and (ii). As

$$\begin{aligned} \left\| \sum_{i=1}^n a_i \Phi_{x_i} \right\|_{\Phi}^2 &= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \Phi(x_i, x_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n a_i \overline{a_j} \langle g(x_i), g(x_j) \rangle \\ &= \left\| \sum_{i=1}^n a_i g(x_i) \right\|^2, \end{aligned}$$

the linear mapping

$$V \rightarrow \mathcal{K}, \quad \left( \sum_{i=1}^n a_i \Phi_{x_i} \right) \mapsto \sum_{i=1}^n a_i g(x_i)$$

is well defined and extends to an isometry  $T : \mathcal{H} \rightarrow \mathcal{K}$ . Moreover,  $T$  is surjective, since  $T(V)$  is the linear span of  $\{g(x) : x \in X\}$  which is dense in  $\mathcal{K}$ . Clearly,  $T \circ f = g$  and this relation uniquely determines  $T$ . ■

**Remark C.1.5** (i) With the above notation, it follows from the continuity of  $\Phi$  that  $\Phi$  is locally bounded, that is, for every  $x \in X$ , there exists a neighbourhood  $U_x$  of  $x$  and a constant  $c > 0$  such that  $\Phi(y, y) \leq c$  for all  $y$  in  $U_x$ . This shows that the associated Hilbert space  $\mathcal{H}$  can be realised as a space of *continuous* functions on  $X$ . Indeed, the above inequality (\*) shows that, if  $(\varphi_n)_n$  is a Cauchy sequence in  $V$ , then  $(\varphi_n)_n$  converges locally uniformly on  $X$ .

(ii) An alternative construction of the Hilbert space  $\mathcal{H}$  associated to  $\Phi$  is as follows (see the proof of Theorem C.2.3). Let  $V$  be the vector space of all functions on  $X$  with finite support, endowed with the semidefinite, positive hermitian form

$$\left\langle \sum_{i=1}^m a_i \delta_{x_i}, \sum_{j=1}^n b_j \delta_{x_j} \right\rangle = \sum_{i=1}^m \sum_{j=1}^n a_i \overline{b_j} \Phi(x_i, x_j),$$

where  $\delta_x$  denotes the Dirac function at  $x$ . Then  $\mathcal{H}$  is the completion of the quotient space of  $V$  by the radical of this form.

We state for later use some permanence properties of kernels of positive type.

**Proposition C.1.6** *Let  $\Phi$  and  $\Psi$  be kernels of positive type on  $X$ . Then the following are kernels of positive type:*

- (i)  $\overline{\Phi} : (x, y) \mapsto \overline{\Phi(x, y)}$ ,
- (ii)  $t\Phi : (x, y) \mapsto t\Phi(x, y)$ , for all  $t \geq 0$ ,
- (iii)  $\Phi + \Psi : (x, y) \mapsto \Phi(x, y) + \Psi(x, y)$ ,
- (iv)  $\Phi\Psi : (x, y) \mapsto \Phi(x, y)\Psi(x, y)$ .
- (v) *Let  $(\Phi_t)_t$  be a family of kernels of positive type on  $X$  converging pointwise on  $X \times X$  to a continuous kernel  $\Phi : X \times X \rightarrow \mathbf{C}$ . Then  $\Phi$  is a kernel of positive type.*

**Proof** Assertions (i), (ii), (iii), and (v) are obvious from the definition. As to (iv), let  $f : X \rightarrow \mathcal{H}$  and  $g : X \rightarrow \mathcal{K}$  be the mappings given by Theorem C.1.4. Let  $\mathcal{H} \otimes \mathcal{K}$  be the Hilbert space tensor product, and let

$$h : X \rightarrow \mathcal{H} \otimes \mathcal{K}, \quad x \mapsto f(x) \otimes g(x).$$

Then

$$(\Phi\Psi)(x, y) = \langle h(x), h(y) \rangle,$$

for all  $x, y \in X$ . ■

In other words, the set of kernels of positive type is a convex cone in the vector space of all continuous kernels on  $X$  which is closed under complex conjugation, under pointwise product, and under the topology of pointwise convergence.

**Remark C.1.7** Property (iv) of Proposition C.1.6 is essentially a result due to I. Schur. Recall that the *Schur product* of two matrices  $A = (a_{i,j})_{1 \leq i, j \leq n}$  and  $B = (b_{i,j})_{1 \leq i, j \leq n}$  is the matrix  $S = (s_{i,j})_{1 \leq i, j \leq n}$  defined by  $s_{i,j} = a_{i,j}b_{i,j}$ . A complex matrix  $A \in M_n(\mathbf{C})$  is *positive* if  $\sum_{i,j=1}^n c_i \overline{c_j} a_{i,j} \geq 0$  for all  $c_1, \dots, c_n \in \mathbf{C}$ . Here is Schur's result: if  $A$  and  $B$  in  $M_n(\mathbf{C})$  are positive, then so is their Schur product  $S$ . And here is a proof.

There exists a matrix  $C = (c_{i,j})_{1 \leq i, j \leq n} \in M_n(\mathbf{C})$  such that  $B = CC^*$ . For  $c_1, \dots, c_n \in \mathbf{C}$ , we have

$$\begin{aligned} \sum_{i,k=1}^n c_i \overline{c_k} s_{i,k} &= \sum_{i,k=1}^n c_i \overline{c_k} a_{i,k} \sum_{j=1}^n c_j \overline{c_{k,j}} \\ &= \sum_{j=1}^n \sum_{i,k=1}^n c_i c_{i,j} \overline{c_k c_{k,j}} a_{i,k} \geq 0 \end{aligned}$$

and  $S$  is positive, as claimed.

The following lemma is used in the proof of Theorem 2.11.3.

**Lemma C.1.8** *Let  $\Phi$  be a kernel of positive type on a set  $X$  such that  $|\Phi(x, y)| < 1$  for all  $x, y \in X$ . Then the kernel*

$$(x, y) \mapsto (1 - \Phi(x, y))^{-t}$$

*is of positive type for every  $t \geq 0$ .*

**Proof** Since  $|\Phi(x, y)| < 1$ , the series

$$1 + t\Phi(x, y) + \frac{t(t+1)}{2!}\Phi(x, y)^2 + \frac{t(t+1)(t+2)}{3!}\Phi(x, y)^3 + \cdots$$

converges to  $(1 - \Phi(x, y))^{-t}$  for every  $x, y \in X$ . The claim follows from the previous proposition. ■

## C.2 Kernels conditionally of negative type

Let  $X$  be a topological space.

**Definition C.2.1** A kernel *conditionally of negative type* on  $X$  is a continuous function  $\Psi: X \times X \rightarrow \mathbf{R}$  with the following property:

- (i)  $\Psi(x, x) = 0$  for all  $x$  in  $X$ ;
- (ii)  $\Psi(y, x) = \Psi(x, y)$  for all  $x, y$  in  $X$ ;
- (iii) for any  $n$  in  $\mathbf{N}$ , any elements  $x_1, \dots, x_n$  in  $X$ , and any real numbers  $c_1, \dots, c_n$  with  $\sum_{i=1}^n c_i = 0$ , the following inequality holds:

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(x_i, x_j) \leq 0.$$

**Example C.2.2** (i) Let  $\mathcal{H}$  be a *real* Hilbert space. The kernel

$$\Psi: \mathcal{H} \times \mathcal{H} \rightarrow \mathbf{R}, \quad (\xi, \eta) \mapsto \|\xi - \eta\|^2$$

is conditionally of negative type. Indeed, for  $\xi_1, \dots, \xi_n \in \mathcal{H}$  and  $c_1, \dots, c_n \in \mathbf{R}$  with  $\sum_{i=1}^n c_i = 0$ , we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(\xi_i, \xi_j) = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \|\xi_i - \xi_j\|^2 = -2 \left\| \sum_{i=1}^n c_i \xi_i \right\|^2.$$

(ii) Let  $X$  be a topological space,  $\mathcal{H}$  a real Hilbert space, and  $f : X \rightarrow \mathcal{H}$  a continuous mapping. It follows from (i) that the kernel

$$\psi : X \times X \rightarrow \mathbf{R}, \quad (x, y) \mapsto \|f(x) - f(y)\|^2$$

is conditionally of negative type.

(iii) On a tree  $X$  (see Section 2.3), the distance  $d : X \times X \rightarrow \mathbf{R}_+$  is a kernel which is conditionally of negative type. Indeed, denoting by  $\mathbb{E}$  the set of edges of  $X$ , let  $\mathcal{H}$  be the Hilbert space of all functions  $\xi : \mathbb{E} \rightarrow \mathbf{R}$  such that

$$\xi(\bar{e}) = -\xi(e), \quad \text{for all } e \in \mathbb{E},$$

with the inner product defined by

$$\langle \xi, \eta \rangle = \frac{1}{2} \sum_{e \in \mathbb{E}} \xi(e) \eta(e).$$

Fix a base vertex  $x_0 \in X$  and define a mapping  $f : X \times X \rightarrow \mathcal{H}$  by

$$f(x)(e) = \begin{cases} 0 & \text{if } e \text{ is not on the segment } [x_0, x]; \\ 1 & \text{if } e \text{ is on } [x_0, x] \text{ and points from } x_0 \text{ to } x; \\ -1 & \text{if } e \text{ is on } [x_0, x] \text{ and points from } x \text{ to } x_0. \end{cases}$$

One checks that

$$d(x, y) = \|f(x) - f(y)\|^2, \quad \text{for all } x, y \in X$$

(compare with the proof of Proposition 2.3.3).

(iv) Let  $X$  be a real or complex hyperbolic space and denote by  $d$  the geodesic distance on  $X$  (see Section 2.6). By Theorem 2.11.3, the kernel  $(x, y) \mapsto \log(\cosh d(x, y))$  is conditionally of negative type on  $X$ . It is shown in [FarHa-74, Proposition 7.4] that  $d$  is conditionally of negative type.

Example C.2.2.ii is universal (compare with Theorem C.1.4).

**Theorem C.2.3 (GNS construction)** *Let  $\Psi$  be a kernel conditionally of negative type on the topological space  $X$  and let  $x_0 \in X$ . Then there exists a*



real Hilbert space  $\mathcal{H}$  and a continuous mapping  $f : X \rightarrow \mathcal{H}$  with the following properties:

- (i)  $\Psi(x, y) = \|f(x) - f(y)\|^2$  for all  $x, y$  in  $X$ ;
- (ii) the linear span of  $\{f(x) - f(x_0) : x \in X\}$  is dense in  $\mathcal{H}$ .

Moreover, the pair  $(\mathcal{H}, f)$  is unique, up to canonical isomorphism. That is, if  $(\mathcal{K}, g)$  is another pair satisfying (i) and (ii), then there exists a unique affine isometry  $T : \mathcal{H} \rightarrow \mathcal{K}$  such that  $g(x) = T(f(x))$  for all  $x \in X$ .

**Proof** For every  $x \in X$ , let  $\delta_x : X \rightarrow \mathbf{R}$  be the Dirac function at  $x$ . Let  $V$  be the real vector space consisting of linear combinations  $\sum_{i=1}^m c_i \delta_{x_i}$  with  $\sum_{i=1}^m c_i = 0$ , for  $x_i \in X$  and  $c_i \in \mathbf{R}$ .

For  $\varphi = \sum_{i=1}^m a_i \delta_{x_i}$  and  $\psi = \sum_{j=1}^n b_j \delta_{x_j}$  in  $V$ , define

$$\langle \varphi, \psi \rangle_\Psi := -\frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n a_i b_j \Psi(x_i, x_j).$$

The mapping

$$(\varphi, \psi) \mapsto \langle \varphi, \psi \rangle_\Psi$$

is a positive symmetric form on  $V$ . Set

$$N_\Psi = \{\varphi \in V : \langle \varphi, \varphi \rangle_\Psi = 0\}.$$

The Cauchy–Schwarz inequality shows that  $N_\Psi$  is a linear subspace of  $V$ . On the quotient space  $V/N_\Psi$ , define

$$\langle [\varphi], [\psi] \rangle_\Psi := \langle \varphi, \psi \rangle_\Psi,$$

where  $[\varphi], [\psi]$  denote the images in  $V/N_\Psi$  of  $\varphi, \psi \in V$ . This is a well-defined scalar product on  $V/N_\Psi$ . Let  $(\mathcal{H}, \langle \cdot, \cdot \rangle_\Psi)$  be the Hilbert space completion of  $V/N_\Psi$ .

Set

$$f : X \rightarrow \mathcal{H}, \quad x \mapsto [\delta_x - \delta_{x_0}].$$

Then

$$\|f(x) - f(y)\|_\Psi^2 = \Psi(x, y), \quad \text{for all } x, y \in X,$$

since  $\Psi(x, x) = \Psi(y, y) = 0$  and  $\Psi(y, x) = \Psi(x, y)$ . Moreover,  $f$  is continuous, since  $\Psi$  is continuous. This settles the existence of the pair  $(\mathcal{H}, f)$  with Properties (i) and (ii).

Let  $(\mathcal{K}, g)$  be another pair consisting of a Hilbert space  $\mathcal{K}$  and a continuous function  $g : X \rightarrow \mathcal{K}$  satisfying (i) and (ii). We claim that the affine mapping

$$V/N_\Psi \rightarrow \mathcal{K}, \quad \left[ \sum_{i=1}^n a_i \delta_{x_i} \right] \mapsto \sum_{i=1}^n a_i g(x_i)$$

is well defined and extends to an isometry  $T : \mathcal{H} \rightarrow \mathcal{K}$ . Indeed, since  $\sum_{i=1}^n a_i = 0$ , we have

$$\begin{aligned} \left\| \left[ \sum_{i=1}^n a_i \delta_{x_i} \right] \right\|_\Psi^2 &= \left\| \left[ \sum_{i=1}^n a_i (\delta_{x_i} - \delta_{x_0}) \right] \right\|_\Psi^2 \\ &= -\frac{1}{2} \sum_{i,j=1}^n a_i a_j (\Psi(x_i, x_j) - \Psi(x_i, x_0) - \Psi(x_j, x_0)) \\ &= -\frac{1}{2} \sum_{i,j=1}^n a_i a_j \left( \|g(x_i) - g(x_j)\|^2 - \|g(x_i) - g(x_0)\|^2 - \|g(x_j) - g(x_0)\|^2 \right) \\ &= \sum_{i,j=1}^n a_i a_j \langle g(x_i) - g(x_0), g(x_j) - g(x_0) \rangle \\ &= \left\| \sum_{i=1}^n a_i (g(x_i) - g(x_0)) \right\|^2 \\ &= \left\| \sum_{i=1}^n a_i g(x_i) \right\|^2. \end{aligned}$$

Moreover,  $T$  is surjective, since the linear span of  $\{g(x) - g(x_0) : x \in X\}$  is dense in  $\mathcal{K}$  by Condition (ii). Clearly,  $T \circ f = g$  and this relation uniquely determines  $T$ . ■

We state some elementary facts about kernels conditionally of negative type; we leave the proof as Exercise C.6.9.

**Proposition C.2.4** *Let  $X$  be a topological space.*

- (i) *The set of all kernels conditionally of negative type on  $X$  is a convex cone, that is, if  $\Psi_1$  and  $\Psi_2$  are kernels conditionally of negative type, then so is  $s\Psi_1 + t\Psi_2$  for all real numbers  $s, t \geq 0$ .*

- (ii) Let  $(\Psi_t)_t$  be a family of kernels conditionally of negative type on  $X$  converging pointwise on  $X \times X$  to a continuous kernel  $\Psi: X \times X \rightarrow \mathbf{R}$ . Then  $\Psi$  is a kernel conditionally of negative type.
- (iii) Let  $\Phi$  be a real valued kernel of positive type on  $X$ . Then

$$(x, y) \mapsto \Phi(x, x) - \Phi(x, y)$$

is a kernel conditionally of negative type.

### C.3 Schoenberg's Theorem

Let  $X$  be a topological space. We have the following connection between kernels conditionally of negative type and kernels of positive type on  $X$ .

**Lemma C.3.1** *Let  $X$  be a topological space, and let  $\Psi: X \times X \rightarrow \mathbf{R}$  be a continuous kernel with  $\Psi(x, x) = 0$  and  $\Psi(y, x) = \Psi(x, y)$  for all  $x, y$  in  $X$ . Fix  $x_0 \in X$ , and define  $\Phi: X \times X \rightarrow \mathbf{R}$  by*

$$\Phi(x, y) = \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y).$$

*Then  $\Psi$  is a kernel conditionally of negative type if and only if  $\Phi$  is a kernel of positive type.*

**Proof** Assume that  $\Phi$  is of positive type. For  $x_1, \dots, x_n \in X$  and  $c_1, \dots, c_n \in \mathbf{R}$  with  $\sum_{i=1}^n c_i = 0$ , we have

$$0 \leq \sum_{i=1}^n \sum_{j=1}^n c_i c_j \Phi(x_i, x_j) = - \sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(x_i, x_j),$$

showing that  $\Psi$  is conditionally of negative type.

Conversely, assume that  $\Psi$  is conditionally of negative type. By Theorem C.2.3, there exists a real Hilbert space  $\mathcal{H}$  and a continuous mapping  $f: X \rightarrow \mathcal{H}$  such that

$$\Psi(x, y) = \|f(x) - f(y)\|^2, \quad \text{for all } x, y \in X.$$

It follows that

$$\begin{aligned} \Phi(x, y) &= \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y) \\ &= \|f(x) - f(x_0)\|^2 + \|f(y) - f(x_0)\|^2 - \|f(x) - f(y)\|^2 \\ &= 2 \langle f(x) - f(x_0), f(y) - f(x_0) \rangle, \end{aligned}$$

for all  $x, y \in X$ . Hence, for  $x_1, \dots, x_n \in X$  and  $c_1, \dots, c_n \in \mathbf{R}$ ,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Phi(x_i, x_j) = 2 \left\| \sum_{i=1}^n c_i (f(x_i) - f(x_0)) \right\|^2 \geq 0.$$

Since  $\Psi$  is symmetric, we deduce that

$$\sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \Phi(x_i, x_j) \geq 0$$

for  $x_1, \dots, x_n \in X$  and complex numbers  $c_1, \dots, c_n$ . Hence,  $\Phi$  is of positive type. ■

A more important connection between kernels conditionally of negative type and kernels of positive type is given by Schoenberg's Theorem [Schoe–38].

**Theorem C.3.2 (Schoenberg)** *Let  $X$  be a topological space, and let  $\Psi : X \times X \rightarrow \mathbf{R}$  be a continuous kernel on  $X$  such that  $\Psi(x, x) = 0$  and  $\Psi(y, x) = \Psi(x, y)$  for all  $x, y \in X$ . The following properties are equivalent:*

- (i)  $\Psi$  is conditionally of negative type;
- (ii) the kernel  $\exp(-t\Psi)$  is of positive type, for every  $t \geq 0$ .

**Proof** Assume that  $\exp(-t\Psi)$  is of positive type for every  $t \geq 0$ . Since the kernel

$$1 - \exp(-t\Psi)$$

is conditionally of negative type by Proposition C.2.4.iii, so is the pointwise limit

$$\Psi = \lim_{t \rightarrow 0} \frac{1 - \exp(-t\Psi)}{t}$$

by Proposition C.2.4.ii.

Assume, conversely, that  $\Psi$  is conditionally of negative type. It suffices to show that  $\exp(-\Psi)$  is of positive type. Fix some point  $x_0 \in X$ . By the previous lemma, the kernel on  $X$  defined by

$$\Phi(x, y) = \Psi(x, x_0) + \Psi(y, x_0) - \Psi(x, y)$$

is of positive type. Now

$$\exp(-\Psi(x, y)) = \exp(\Phi(x, y)) \exp(-\Psi(x, x_0)) \exp(-\Psi(y, x_0)).$$

Since  $\Phi$  is of positive type, the same is true for  $\Phi(x, y)^n$  for any integer  $n \geq 0$  and, hence,  $\exp(\Phi)$  is of positive type (Proposition C.1.6). The kernel

$$(x, y) \mapsto \exp(-\Psi(x, x_0)) \exp(-\Psi(y, x_0))$$

is of positive type, since

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \exp(-\Psi(x_i, x_0)) \exp(-\Psi(x_j, x_0)) = \left| \sum_{i=1}^n c_i \exp(-\Psi(x_i, x_0)) \right|^2 \geq 0$$

for  $x_1, \dots, x_n \in X$  and complex numbers  $c_1, \dots, c_n$ . It follows that  $\exp(-\Psi)$  is of positive type, since it is the product of two kernels of positive type (Proposition C.1.6). ■

## C.4 Functions on groups

Let  $G$  be a topological group.

### Functions of positive type

**Definition C.4.1** A function of *positive type* on  $G$  is a continuous function  $\varphi : G \mapsto \mathbf{C}$  such that the kernel on  $G$  defined by

$$(g_1, g_2) \mapsto \varphi(g_2^{-1} g_1)$$

is of positive type, that is,

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \varphi(g_j^{-1} g_i) \geq 0$$

for all  $g_1, \dots, g_n$  in  $G$  and  $c_1, \dots, c_n$  in  $\mathbf{C}$ . Functions of positive type are often called *positive definite* functions.

**Proposition C.4.2** Let  $\varphi$  be a function of positive type on  $G$ . Then, for  $g, h \in G$ , we have

- (i)  $\overline{\varphi(g)} = \varphi(g^{-1})$ ,
- (ii)  $|\varphi(g)| \leq \varphi(e)$ ,
- (iii)  $|\varphi(g) - \varphi(h)|^2 \leq 2\varphi(e) (\varphi(e) - \operatorname{Re}\varphi(g^{-1}h))$ .

**Proof** Properties (i) and (ii) are immediate consequences of Proposition C.1.2. Let us show (iii) in case  $\varphi(e) = 1$ ; by (ii), we can also assume that  $\varphi(g) \neq \varphi(h)$ .

For  $x, y, z \in \mathbf{C}$ , we have

$$|x|^2 + |y|^2 + |z|^2 + 2\operatorname{Re} \left( \varphi(g^{-1})\bar{y}x + \varphi(h^{-1})\bar{z}x + \varphi(g^{-1}h)\bar{y}z \right) \geq 0$$

because  $\varphi$  is of positive type. In the particular case with

$$x = |\varphi(g) - \varphi(h)|, \quad y = -\frac{|\varphi(g) - \varphi(h)|}{\varphi(g) - \varphi(h)}, \quad z = \frac{|\varphi(g) - \varphi(h)|}{\varphi(g) - \varphi(h)},$$

this reads

$$|\varphi(g) - \varphi(h)|^2 + 2 + 2\operatorname{Re} \left( \left( -\overline{\varphi(g)} + \overline{\varphi(h)} \right) \frac{|\varphi(g) - \varphi(h)|^2}{\varphi(g) - \varphi(h)} - \varphi(g^{-1}h) \right) \geq 0$$

which is but a rewriting of (iii). ■

Inequality (iii) above shows that continuous functions of positive type are (left and right) uniformly continuous. For another proof of (iii), see Remark C.4.14.

Examples of functions of positive type arise from diagonal matrix coefficients of unitary representations.

**Proposition C.4.3** *Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ , and let  $\xi$  be a vector in  $\mathcal{H}$ . Then the diagonal matrix coefficient*

$$g \mapsto \langle \pi(g)\xi, \xi \rangle$$

*is of positive type.*

**Proof** This is a special case of Example C.1.3, with  $f : G \rightarrow \mathcal{H}$  defined by  $f(g) = \pi(g)\xi$ . ■

The functions of the type  $\langle \pi(\cdot)\xi, \xi \rangle$  are said to be the functions of positive type *associated to*  $\pi$ .

**Definition C.4.4** Let  $G$  be a locally compact group, with Haar measure  $dx$ . Given two functions  $f$  and  $g$  on  $G$ , their *convolution*  $f * g$  is the function on  $G$  defined by

$$f * g(x) = \int_G f(xy)g(y^{-1})dy = \int_G f(y)g(y^{-1}x)dy, \quad x \in G,$$

whenever these integrals make sense.

For instance, if  $f, g \in L^1(G)$  then  $f * g$  is defined almost everywhere and  $f * g \in L^1(G)$ . More generally, if  $f \in L^1(G)$  and  $g \in L^p(G)$  for  $p \in [1, \infty]$ , then  $f * g$  is defined almost everywhere and  $f * g \in L^p(G)$ .

**Example C.4.5** Let  $G$  be a locally compact group. Let  $f \in L^2(G)$ . Then  $f * \tilde{f}$  is a function of positive type associated to the regular representation  $\lambda_G$ , where  $\tilde{f}(x) = \overline{f(x^{-1})}$ . Indeed, more generally, for  $f, g \in L^2(G)$ , we have

$$\begin{aligned} \langle \lambda_G(x)f, g \rangle &= \int_G f(x^{-1}y) \overline{g(y)} dy = \int_G f(y) \overline{g(xy)} dy \\ &= \overline{g} * \tilde{f}(x). \end{aligned}$$

So,

$$\{f * \tilde{g} : f, g \in L^2(G)\} = \{f * \check{g} : f, g \in L^2(G)\}$$

is the set of all matrix coefficients of the regular representation of  $G$ .

Recall that a continuous function  $f$  on a topological space  $X$  is said to *vanish at infinity* if, for all  $\varepsilon > 0$ , the set  $\{x \in X : |f(x)| \geq \varepsilon\}$  is compact. The space of all such functions is denoted by  $C_0(X)$ . Observe that  $C_0(X)$  is closed in  $C(X)$  in the topology of uniform convergence on  $X$ .

**Proposition C.4.6** *Every matrix coefficient of the regular representation of a locally compact group  $G$  vanishes at infinity.*

**Proof** Let  $f * \check{g}$  be a coefficient of the regular representation of  $G$ , where  $f, g \in L^2(G)$ . Since  $C_c(G)$  is dense in  $L^2(G)$ , there exists  $f_n, g_n \in C_c(G)$  with  $\lim_n f_n = f$  and  $\lim_n g_n = g$  in  $L^2(G)$ . Then  $\lim_n f_n * \check{g}_n = f * \check{g}$ , uniformly on  $G$ . Hence, to show that  $f * \check{g}$  belongs to  $C_0(G)$ , we can, by the observation above, assume that  $f, g \in C_c(G)$ . If  $K = \text{supp } f$  and  $L = \text{supp } g$ , the support of  $f * \check{g}$  is contained in  $KL^{-1}$ . Thus,  $f * \check{g} \in C_c(G)$ . ■

The following corollary generalises the implication “(ii)  $\implies$  (i)” in Proposition A.5.1.

**Corollary C.4.7** *Let  $G$  be a non-compact locally compact group. Then  $\lambda_G$  has no finite dimensional subrepresentation.*

**Proof** Let  $(\pi, \mathcal{H})$  be a finite dimensional unitary representation of  $G$ . Fix an orthonormal basis  $(v_i)_i$  of  $(\pi, \mathcal{H})$ . By the previous proposition, it suffices to show that some matrix coefficient  $\langle \pi(\cdot)v_i, v_j \rangle$  of  $\pi$  does not vanish at infinity.

For this, observe that  $(\langle \pi(g)v_i, v_j \rangle)_{i,j}$ , which is the matrix of  $\pi(g)$  with respect to the basis  $(v_i)_i$ , is a unitary matrix. Hence, for any  $i$ , we have

$$\sum_j |\langle \pi(g)v_i, v_j \rangle|^2 = 1, \quad \text{for all } g \in G,$$

from which the claim follows. ■

The above corollary shows in particular that the regular representation of a non-compact locally compact abelian group has *no* irreducible subrepresentation. It follows that the regular representation of such a group cannot be decomposed as a direct sum of irreducible subrepresentations. However, it can be decomposed as a direct *integral* of irreducible representations, that is, of unitary characters (see F.5.4).

Using the GNS-construction of Proposition C.1.4, we will show that every function of positive type on  $G$  is the diagonal matrix coefficient of an essentially unique cyclic unitary representation of  $G$ .

**Definition C.4.8** A unitary representation  $(\pi, \mathcal{H})$  of  $G$  is *cyclic* if there exists a vector  $\xi$  in  $\mathcal{H}$  such that the linear span of  $\pi(G)\xi = \{\pi(g)\xi : g \in G\}$  is dense in  $\mathcal{H}$ . In this case,  $\xi$  is said to be a *cyclic vector* for  $\pi$ .

Observe that a unitary representation  $(\pi, \mathcal{H})$  of  $G$  is irreducible if and only if *every* non-zero vector  $\xi$  in  $\mathcal{H}$  is cyclic for  $\pi$ .

As the next proposition shows, an arbitrary unitary representation can always be decomposed as direct sum of cyclic representations.

**Proposition C.4.9** Let  $(\pi, \mathcal{H})$  be a unitary representation of the topological group  $G$ . Then  $\mathcal{H}$  can be decomposed as a direct sum  $\mathcal{H} = \bigoplus_i \mathcal{H}_i$  of mutually orthogonal, closed, and invariant subspaces  $\mathcal{H}_i$ , such that the restriction of  $\pi$  to  $\mathcal{H}_i$  is cyclic for every  $i$ .

**Proof** Let  $X$  be the set of all families  $(\mathcal{H}_i)_i$  of mutually orthogonal, closed, and invariant subspaces  $\mathcal{H}_i$  of  $\mathcal{H}$ , such that the restriction of  $\pi$  to  $\mathcal{H}_i$  is cyclic. We equip  $X$  with the ordering given by set inclusion. By Zorn's lemma,  $X$  contains a maximal family  $(\mathcal{H}_i)_i$ . We claim that  $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ . Indeed, otherwise, there exists a non-zero vector  $\xi$  which is orthogonal to all  $\mathcal{H}_i$ 's. Let  $\mathcal{K}$  be the closure of the linear span of  $\pi(G)\xi$ . Then  $\mathcal{K}$  is orthogonal to all  $\mathcal{H}_i$ 's. Hence, the family  $\{(\mathcal{H}_i)_i, \mathcal{K}\}$  is in  $X$  and strictly contains  $(\mathcal{H}_i)_i$ . This is a contradiction. ■

**Theorem C.4.10 (GNS construction)** Let  $\varphi$  be a function of positive type on the topological group  $G$ . Then there exists a triple  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  consisting of a



cyclic unitary representation  $\pi_\varphi$  in a Hilbert space  $\mathcal{H}_\varphi$  and a cyclic vector  $\xi_\varphi$  in  $\mathcal{H}_\varphi$  such that

$$\varphi(g) = \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle, \quad g \in G.$$

Moreover, this triple is unique in the following sense: if  $(\pi, \mathcal{H}, \xi)$  is another such triple, then there exists a Hilbert space isomorphism  $T: \mathcal{H}_\varphi \rightarrow \mathcal{H}$  intertwining  $\pi_\varphi$  and  $\pi$ , and such that  $T\xi_\varphi = \xi$ .

**Proof** Let  $\Phi$  be the kernel of positive type on  $G$  associated to  $\varphi$ . By Theorem C.1.4, there exists a Hilbert space  $\mathcal{H}_\varphi$  and a continuous mapping  $f: G \rightarrow \mathcal{H}_\varphi$  such that

$$\langle f(x), f(y) \rangle = \Phi(x, y) = \varphi(y^{-1}x), \quad \text{for all } x, y \in G.$$

For every  $g \in G$ , we have

$$\langle f(gx), f(gy) \rangle = \Phi(gx, gy) = \Phi(x, y).$$

Hence, by the uniqueness result in Theorem C.1.4, there exists a unitary operator  $\pi_\varphi(g)$  on  $\mathcal{H}_\varphi$  such that  $\pi_\varphi(g)f(x) = f(gx)$  for all  $x$  in  $G$ . By density of the linear span of  $\{f(x): x \in G\}$ , we have

$$\pi_\varphi(g_1 g_2) = \pi_\varphi(g_1) \pi_\varphi(g_2)$$

for all  $g_1, g_2$  in  $G$ . Moreover, since  $f$  is continuous,  $g \mapsto \pi_\varphi(g)\xi$  is continuous for every  $\xi$  in  $\mathcal{H}_\varphi$ . Thus,  $\pi_\varphi$  is a unitary representation of  $G$ . Set  $\xi_\varphi = f(e)$ . Then  $\xi_\varphi$  is a cyclic vector for  $\mathcal{H}_\varphi$  and

$$\varphi(g) = \Phi(g, e) = \langle f(g), f(e) \rangle = \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle$$

for all  $g \in G$ .

If  $(\pi, \mathcal{H}, \xi)$  is a triple with the same properties as  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ , then

$$\Phi(g_1, g_2) = \langle \pi(g_1)\xi, \pi(g_2)\xi \rangle, \quad \text{for all } g_1, g_2 \in G.$$

By the uniqueness result in Theorem C.1.4 again, there exists an isomorphism  $T: \mathcal{H}_\varphi \rightarrow \mathcal{H}$  such that  $T\pi_\varphi(g)\xi_\varphi = \pi(g)\xi$  for all  $g \in G$ . In particular,  $T\xi_\varphi = \xi$  and  $T$  intertwines the representations  $\pi_\varphi$  and  $\pi$ , since  $\xi_\varphi$  is cyclic. ■

**Definition C.4.11** The triple  $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$  as above is called the *GNS triple* associated to the function of positive type  $\varphi$  on  $G$ .

**Example C.4.12** Let  $G$  be a discrete group, and let  $\varphi = \delta_e$  be the Dirac function at  $e$ . Then  $\varphi$  is a function of positive type on  $G$  (Exercise C.6.7) and the GNS-representation associated to  $\varphi$  is equivalent to the regular representation of  $G$  (Exercise C.6.8).

**Remark C.4.13** The family of equivalence classes of all *cyclic* unitary representations of  $G$  is a set. Indeed, let  $(\pi, \mathcal{H})$  be such a representation, with cyclic vector  $\xi$ . Then, by Theorem C.4.10,  $\pi$  is unitary equivalent to  $\pi_\varphi$ , where  $\varphi = \langle \pi(\cdot)\xi, \xi \rangle$ . Thus, the family of equivalence classes of cyclic representations of  $G$  can be parametrised by a subset of the set of all functions of positive type on  $G$ .

In particular, the unitary dual  $\widehat{G}$  of  $G$  is indeed a set.

**Remark C.4.14** Here is another proof of inequality (iii) in Proposition C.4.2. Let  $\varphi$  be a function of positive type on a topological group  $G$ , with  $\varphi(e) = 1$ . By Theorem C.4.10, there exists a unitary representation  $(\pi, \mathcal{H})$  of  $G$  and a unit vector  $\xi \in \mathcal{H}$  such that  $\varphi(x) = \langle \pi(x)\xi, \xi \rangle$  for all  $x \in G$ . Then

$$\begin{aligned} |\varphi(x) - \varphi(y)|^2 &= |\langle \pi(x)\xi - \pi(y)\xi, \xi \rangle|^2 \\ &\leq \|\pi(x)\xi - \pi(y)\xi\|^2 = 2(1 - \operatorname{Re}\varphi(y^{-1}x)) \end{aligned}$$

for all  $x, y \in G$ .

**Corollary C.4.15** Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  and  $\xi \in \mathcal{H}$ . Let  $\varphi$  be the function of positive type defined by  $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ . Then  $\pi_\varphi$  is contained in  $\pi$ .

**Proof** The uniqueness assertion of Theorem C.4.10 implies that  $\pi_\varphi$  is equivalent to the subrepresentation of  $\pi$  defined on the closure of the linear span of  $\pi(G)\xi$ . ■

**Corollary C.4.16** Let  $\varphi$  and  $\varphi'$  be continuous functions of positive type on  $G$ , and let  $\psi = \varphi\varphi'$ . Then  $\pi_\psi$  is contained in  $\pi_\varphi \otimes \pi_{\varphi'}$ .

**Proof** Since

$$\psi(g) = \langle (\pi_1 \otimes \pi_2)(g)(\xi_\varphi \otimes \xi_{\varphi'}), (\xi_\varphi \otimes \xi_{\varphi'}) \rangle,$$

the claim follows from the previous corollary. ■

Observe that, in general,  $\pi_\psi$  and  $\pi_\varphi \otimes \pi_{\varphi'}$  are not unitarily equivalent (Exercise C.6.8).

### Functions conditionally of negative type

Let  $G$  be a topological group.

**Definition C.4.17** A continuous function  $\psi : G \rightarrow \mathbf{R}$  is *conditionally of negative type* if the kernel  $\Psi$  on  $G$ , defined by  $\Psi(g, h) = \psi(h^{-1}g)$  for  $g, h$  in  $G$ , is conditionally of negative type.

**Remark C.4.18** Let  $G$  be a topological group acting continuously on the topological space  $X$ . Let  $\Psi$  be a kernel conditionally of negative type on  $X$  which is  $G$ -invariant, that is,  $\Psi(gx, gy) = \Psi(x, y)$  for all  $g \in G$  and  $x, y \in X$ . For fixed  $x_0 \in X$ , the function  $\psi : G \rightarrow \mathbf{R}$  defined by  $\psi(g) = \Psi(gx_0, x_0)$  is a function conditionally of negative type on  $G$ . For examples, see Sections 2.10 and 2.11.

Specialising to functions conditionally of negative type on a group, Theorem C.3.2 takes the following form.

**Corollary C.4.19 (Schoenberg)** *Let  $G$  be a topological group, and let  $\psi : G \rightarrow \mathbf{R}$  be a continuous function with  $\psi(e) = 0$  and  $\psi(g^{-1}) = \psi(g)$  for all  $g \in G$ . The following properties are equivalent:*

- (i)  $\psi$  is conditionally of negative type;
- (ii) the function  $\exp(-t\psi)$  is of positive type, for every  $t \geq 0$ .

## C.5 The cone of functions of positive type

Let  $\mathcal{P}(G)$  denote the set of all functions of positive type on the topological group  $G$ . It follows from Proposition C.1.6 that  $\mathcal{P}(G)$  is a convex cone, closed under complex conjugation and pointwise product.

We will also consider the convex set  $\mathcal{P}_1(G)$  of all  $\varphi$  in  $\mathcal{P}(G)$  normalised by the Condition  $\varphi(e) = 1$ . We continue to use the notation  $\pi_\varphi, \mathcal{H}_\varphi$  of Theorem C.4.10.

**Proposition C.5.1** *Let  $\varphi_1$  and  $\varphi_2$  be in  $\mathcal{P}(G)$ , and let  $\varphi = \varphi_1 + \varphi_2$ . Then  $\pi_{\varphi_1}$  is contained in  $\pi_\varphi$ . Moreover, if  $\pi_\varphi$  is irreducible, then  $\varphi_1 = t\varphi$  for some  $t \geq 0$ .*

**Proof** Since  $\varphi - \varphi_1$  is of positive type, we have, for  $g_1, \dots, g_n$  in  $G$  and complex numbers  $c_1, \dots, c_n$ ,

$$\begin{aligned} \left\| \sum_{i=1}^n c_i \pi_{\varphi_1}(g_i) \xi_{\varphi_1} \right\|^2 &= \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \varphi_1(g_j^{-1} g_i) \\ &\leq \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \varphi(g_j^{-1} g_i) \\ &= \left\| \sum_{i=1}^n c_i \pi_{\varphi}(g_i) \xi_{\varphi} \right\|^2. \end{aligned}$$

Hence, the mapping

$$\sum_{i=1}^n c_i \pi_{\varphi}(g_i) \xi_{\varphi} \mapsto \sum_{i=1}^n c_i \pi_{\varphi_1}(g_i) \xi_{\varphi_1}$$

is well defined and extends to a continuous operator  $T: \mathcal{H}_{\varphi} \rightarrow \mathcal{H}_{\varphi_1}$ . Clearly,  $T$  intertwines  $\pi_{\varphi}$  and  $\pi_{\varphi_1}$ , and  $T(\mathcal{H}_{\varphi})$  is dense in  $\mathcal{H}_{\varphi_1}$ . By Proposition A.1.4, the subrepresentation of  $\pi_{\varphi}$  defined on  $(\text{Ker } T)^{\perp}$  is equivalent to  $\pi_{\varphi_1}$ .

Assume now that  $\pi_{\varphi}$  is irreducible. As  $T^*T$  intertwines  $\pi_{\varphi}$  with itself,  $T^*T = tI$  for some  $t \geq 0$ , by Schur's Lemma A.2.2. Therefore, we have

$$\begin{aligned} \varphi_1(g) &= \langle T\pi_{\varphi}(g)\xi_{\varphi}, T\xi_{\varphi} \rangle = \langle T^*T\pi_{\varphi}(g)\xi_{\varphi}, \xi_{\varphi} \rangle \\ &= t\varphi(g), \end{aligned}$$

for all  $g$  in  $G$ . ■

Let  $\text{ext}(\mathcal{P}_1(G))$  be the set of extreme points of the convex set  $\mathcal{P}_1(G)$ . Thus,  $\varphi \in \mathcal{P}_1(G)$  lies in  $\text{ext}(\mathcal{P}_1(G))$  if and only if, whenever  $\varphi = t\varphi_1 + (1-t)\varphi_2$  for some  $\varphi_1, \varphi_2 \in \mathcal{P}_1(G)$  and  $0 \leq t \leq 1$ , then  $\varphi = \varphi_1$  or  $\varphi = \varphi_2$ . The functions in  $\text{ext}(\mathcal{P}_1(G))$  are called *pure functions of positive type*.

**Theorem C.5.2** *Let  $\varphi \in \mathcal{P}_1(G)$ . Then  $\varphi \in \text{ext}(\mathcal{P}_1(G))$  if and only if the unitary representation  $\pi_{\varphi}$  is irreducible.*

**Proof** Suppose that  $\pi_{\varphi}$  is not irreducible. Then  $\mathcal{H}_{\varphi} = \mathcal{K} \oplus \mathcal{K}^{\perp}$  for some non-trivial closed invariant subspace  $\mathcal{K}$ . Let  $\xi_{\varphi} = \xi_1 + \xi_2$  be the corresponding orthogonal decomposition of  $\xi_{\varphi}$ . We have  $\xi_{\varphi} \notin \mathcal{K}$  and  $\xi_{\varphi} \notin \mathcal{K}^{\perp}$ , since  $\xi_{\varphi}$  is a cyclic vector for  $\mathcal{H}_{\varphi}$ . Therefore,  $\xi_1 \neq 0$  and  $\xi_2 \neq 0$ . Set  $s = \|\xi_1\|^2$  and  $t = \|\xi_2\|^2$ . Then  $s + t = \|\xi_{\varphi}\|^2 = \varphi(e) = 1$ , the functions  $\varphi_1$  and  $\varphi_2$  defined by

$$\varphi_1(g) = \langle \pi_{\varphi}(g)s^{-1/2}\xi_1, s^{-1/2}\xi_1 \rangle \quad \text{and} \quad \varphi_2(g) = \langle \pi_{\varphi}(g)t^{-1/2}\xi_2, t^{-1/2}\xi_2 \rangle$$

belong to  $\mathcal{P}_1(G)$ , and

$$\begin{aligned}\varphi(g) &= \langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle \\ &= \langle \pi_\varphi(g)\xi_1, \xi_1 \rangle + \langle \pi_\varphi(g)\xi_2, \xi_2 \rangle \\ &= s\varphi_1(g) + t\varphi_2(g).\end{aligned}$$

We claim that  $\varphi \neq \varphi_1$  and  $\varphi \neq \varphi_2$ . Indeed, assume that, say,  $\varphi = \varphi_1$ , that is,  $\langle \pi_\varphi(g)\xi_\varphi, \xi_\varphi \rangle = s^{-1}\langle \pi_\varphi(g)\xi_1, \xi_1 \rangle$  for all  $g \in G$ . Then

$$\begin{aligned}\left\langle \sum_{i=1}^n c_i \pi_\varphi(g_i) \xi_\varphi, \xi_\varphi \right\rangle &= s^{-1} \left\langle \sum_{i=1}^n c_i \pi_\varphi(g_i) \xi_1, \xi_1 \right\rangle \\ &= s^{-1} \left\langle \sum_{i=1}^n c_i \pi_\varphi(g_i) \xi_\varphi, \xi_1 \right\rangle,\end{aligned}$$

for all  $g_1, \dots, g_n \in G$  and  $c_1, \dots, c_n \in \mathbf{C}$ . Since  $\xi_\varphi$  is a cyclic vector, it follows that

$$\langle \eta, \xi_\varphi \rangle = s^{-1} \langle \eta, \xi_1 \rangle, \quad \text{for all } \eta \in \mathcal{H}_\varphi,$$

that is  $\xi_1 = s\xi_\varphi$ . This is a contradiction, since  $\xi_2 \neq 0$ . Hence,  $\varphi$  is not pure.

Conversely, assume that  $\pi_\varphi$  is irreducible and that  $\varphi = \varphi_1 + \varphi_2$  for some  $\varphi_1, \varphi_2$  in  $\mathcal{P}(G)$ . Then, by Proposition C.5.1,  $\varphi_1 = t\varphi$  for some  $t \geq 0$  and  $\varphi$  is pure. ■

Recall that, by Proposition C.4.2, every  $\varphi$  in  $\mathcal{P}(G)$  is bounded and that  $\|\varphi\|_\infty = \varphi(e)$ .

Assume from now on that  $G$  is *locally compact*, so that we can consider  $\mathcal{P}(G)$  as a subset of  $L^\infty(G) = L^\infty(G, \mu)$ , for a Haar measure  $\mu$  on  $G$ . Due to the fact that  $G$  is not necessarily  $\sigma$ -finite, the definition of  $L^\infty(G)$  has to be phrased as follows:

A subset  $A$  of  $G$  is *locally Borel* if  $A \cap B$  is a Borel set for every Borel set  $B$  with  $\mu(B) < \infty$ . Such a set  $A$  is *locally null* if  $\mu(A \cap B) = 0$  for every Borel set  $B$  with  $\mu(B) < \infty$ . A function  $f : G \mapsto \mathbf{C}$  is *locally measurable* if  $f^{-1}(E)$  is locally Borel for all Borel subsets  $E$  of  $\mathbf{C}$ . A property is said to hold *locally almost everywhere* on  $G$ , if it is satisfied outside a locally null set. Then  $L^\infty(G)$  is defined to be the space of all locally measurable functions which are bounded locally almost everywhere, two such functions being identified if they agree locally almost everywhere.

With this definition,  $L^\infty(G)$  can be identified with the (topological) dual space of  $L^1(G)$  by means of the formula

$$\langle \varphi, f \rangle = \int_G f(x) \varphi(x) dx, \quad \varphi \in L^\infty(G), f \in L^1(G)$$

(see [Bou–Int1, Chapter 5, Section 5, No 8]). Equipped with convolution and with the involution

$$f^*(x) = \Delta_G(x^{-1}) \overline{f(x^{-1})}, \quad f \in L^1(G),$$

$L^1(G)$  is a Banach  $*$ -algebra.

The following lemma is obvious when  $G$  is discrete, and is proved by an approximation procedure in the general case. The proof can be found in [Folla–95, (3.35)] or [Dixmi–69, 13.4.4 Proposition]; see also Exercise C.6.1.

**Lemma C.5.3** *For a function  $\varphi \in L^\infty(G)$ , the following properties are equivalent:*

- (i)  $\varphi$  agrees locally almost everywhere on  $G$  with a continuous function of positive type;
- (ii)  $\langle \varphi, f^* * f \rangle \geq 0$  for all  $f \in L^1(G)$ .

By the Banach–Alaoglu Theorem, the unit ball in  $L^\infty(G)$  is compact in the weak\* topology. Let  $\mathcal{P}_{\leq 1}(G)$  be the convex set consisting of all  $\varphi$  in  $\mathcal{P}(G)$  with  $\|\varphi\|_\infty = \varphi(e) \leq 1$ .

**Lemma C.5.4** *The set  $\mathcal{P}_{\leq 1}(G)$  is compact in the weak\* topology on  $L^\infty(G)$ .*

**Proof** As  $\mathcal{P}_{\leq 1}(G)$  is contained in the unit ball of  $L^\infty(G)$ , it suffices to show that  $\mathcal{P}_{\leq 1}(G)$  is weak\* closed. This is the case, by Lemma C.5.3. ■

It follows from the previous lemma and from the Krein–Milman Theorem [Rudin–73, 3.21] that the convex hull of  $\text{ext}(\mathcal{P}_{\leq 1}(G))$  is weak\* dense in  $\mathcal{P}_{\leq 1}(G)$ . If  $G$  is discrete,  $\mathcal{P}_1(G)$  is compact in the weak\* topology (indeed, the equality  $\varphi(e) = \langle \varphi, \delta_e \rangle$  shows that  $\mathcal{P}_1(G)$  is weak\* closed in the unit ball of  $\ell^\infty(G)$ ). This is not true in general for non-discrete groups (see Exercise C.6.10). Nevertheless, the following result holds for  $\mathcal{P}_1(G)$ .

**Theorem C.5.5** *Let  $G$  be a locally compact group. With the notation as above, we have:*

- (i)  $\text{ext}(\mathcal{P}_{\leq 1}(G)) = \text{ext}(\mathcal{P}_1(G)) \cup \{0\}$ ;
- (ii) the convex hull of  $\text{ext}(\mathcal{P}_1(G))$  is weak\* dense in  $\mathcal{P}_1(G)$ .

**Proof** (i) • *First step:* We claim that  $0 \in \text{ext}(\mathcal{P}_{\leq 1}(G))$ . Indeed, let  $\varphi_1, \varphi_2 \in \mathcal{P}_{\leq 1}(G)$  and  $0 \leq t \leq 1$  be such that  $t\varphi_1 + (1-t)\varphi_2 = 0$ . Then  $\varphi_1(e) = 0$  or  $\varphi_2(e) = 0$  and hence  $\varphi_1 = 0$  or  $\varphi_2 = 0$ .

• *Second step:* We show that  $\text{ext}(\mathcal{P}_1(G)) \subset \text{ext}(\mathcal{P}_{\leq 1}(G))$ . For this, let  $\varphi \in \text{ext}(\mathcal{P}_1(G))$ ,  $\varphi_1, \varphi_2 \in \mathcal{P}_{\leq 1}(G)$ , and  $0 < t < 1$  be such that  $t\varphi_1 + (1-t)\varphi_2 = \varphi$ . Then  $t\varphi_1(e) + (1-t)\varphi_2(e) = 1$ . Hence  $\varphi_1(e) = \varphi_2(e) = 1$ , that is,  $\varphi_1, \varphi_2 \in \mathcal{P}_1(G)$ . This implies that  $\varphi \in \text{ext}(\mathcal{P}_{\leq 1}(G))$ .

• *Third step:* We claim that  $\text{ext}(\mathcal{P}_{\leq 1}(G)) \subset \text{ext}(\mathcal{P}_1(G)) \cup \{0\}$ . Indeed, let  $\varphi \in \mathcal{P}_{\leq 1}(G) \setminus (\mathcal{P}_1(G) \cup \{0\})$ , and set  $t = \varphi(e)$ . Then  $0 < t < 1$  and  $\frac{1}{t}\varphi \in \mathcal{P}_1(G)$ . Since

$$\varphi = t \left( \frac{1}{t}\varphi \right) + (1-t)0,$$

it follows that  $\varphi \notin \text{ext}(\mathcal{P}_{\leq 1}(G))$ . This shows that  $\text{ext}(\mathcal{P}_{\leq 1}(G))$  is contained in  $\text{ext}(\mathcal{P}_1(G)) \cup \{0\}$  and finishes the proof of (i).

(ii) Let  $\varphi \in \mathcal{P}_1(G)$ . Since the convex hull of

$$\text{ext}(\mathcal{P}_{\leq 1}(G)) = \text{ext}(\mathcal{P}_1(G)) \cup \{0\}$$

is weak\* dense in  $\mathcal{P}_{\leq 1}(G)$ , there exists a net  $(\psi_i)_i$  in  $\mathcal{P}_{\leq 1}(G)$  converging to  $\varphi$  in the weak\* topology, where the  $\psi_i$ 's are functions of the form

$$\psi_i = t_1\varphi_1 + \cdots + t_n\varphi_n + t_{n+1}0,$$

with  $\varphi_1, \dots, \varphi_n \in \text{ext}(\mathcal{P}_1(G))$ ,  $t_1, \dots, t_{n+1} \geq 0$ , and  $t_1 + \cdots + t_{n+1} = 1$ . Since balls in  $L^\infty(G)$  are weak\* closed, and since  $\|\varphi\|_\infty = 1$  and  $\|\psi_i\|_\infty \leq 1$ , we have

$$\lim_i \|\psi_i\|_\infty = 1.$$

Set  $\psi'_i = \psi_i / \psi_i(e)$ . Since  $\|\psi_i\|_\infty = \psi_i(e)$ , we have  $\lim_i \psi'_i = \varphi$ . Moreover, each  $\psi'_i$  belongs to the convex hull of  $\text{ext}(\mathcal{P}_1(G))$ , since  $t_1 + \cdots + t_n = \psi_i(e)$  and

$$\psi'_i = \frac{t_1}{\psi_i(e)}\varphi_1 + \cdots + \frac{t_n}{\psi_i(e)}\varphi_n.$$

This shows that  $\varphi$  is in the weak\*-closure of  $\text{ext}(\mathcal{P}_1(G))$ . ■

Another useful topology on  $\mathcal{P}_1(G)$  and on  $\mathcal{P}_{\leq 1}(G)$  is the topology of uniform convergence on compact subsets of  $G$ . It is a remarkable fact, due to Raikov, that this topology coincides with the weak\* topology on  $\mathcal{P}_1(G)$ .

**Theorem C.5.6 (Raikov)** *The weak\* topology and the topology of uniform convergence on compact subsets coincide on  $\mathcal{P}_1(G)$ .*

**Proof** We first show the easy part of the theorem, namely that the topology of uniform convergence on compact subsets is finer than the weak\* topology on  $\mathcal{P}_1(G)$ . Let  $(\varphi_i)_i$  be a net in  $\mathcal{P}_1(G)$  converging uniformly on compact subsets to  $\varphi \in \mathcal{P}_1(G)$ . Let  $f \in L^1(G)$  and  $\varepsilon > 0$ . Choose a compact subset  $Q$  of  $G$  such that  $\int_{G \setminus Q} |f(x)| dx \leq \varepsilon$ . For  $i$  large enough, we have  $\sup_{x \in Q} |\varphi_i(x) - \varphi(x)| \leq \varepsilon$  and, since  $\|\varphi_i\|_\infty = \|\varphi\|_\infty = 1$ ,

$$\begin{aligned} |\langle \varphi_i, f \rangle - \langle \varphi, f \rangle| &= \left| \int_G (\varphi_i(x) - \varphi(x)) f(x) dx \right| \\ &\leq \varepsilon \|f\|_1 + 2 \int_{G \setminus Q} |f(x)| dx \\ &\leq \varepsilon \|f\|_1 + 2\varepsilon. \end{aligned}$$

This proves the claim.

Conversely, let  $(\varphi_i)_i$  be a net in  $\mathcal{P}_1(G)$  converging to  $\varphi \in \mathcal{P}_1(G)$  in the weak\* topology. Observe that, since  $\|\varphi_i\|_\infty = 1$ , this implies that  $\lim_i \langle \varphi_i, f \rangle = \langle \varphi, f \rangle$  uniformly when  $f$  runs over a fixed compact subset of  $L^1(G)$ .

Let  $Q$  be a compact subset of  $G$  and  $\varepsilon > 0$ . Since  $\varphi$  is continuous at  $e$ , there exists a compact neighbourhood  $V$  of  $e$  such that

$$(*) \quad \sup_{x \in V} |\varphi(x) - \varphi(e)| = \sup_{x \in V} |\varphi(x) - 1| \leq \varepsilon.$$

Let  $\chi_V$  be the characteristic function of  $V$  and let  $|V| > 0$  be the Haar measure of  $V$ . Set  $f = |V|^{-1} \chi_V$ . Observe that  $f \in L^1(G)$ ,  $f \geq 0$ , and  $\int_G f(x) dx = 1$ .

We claim that the net of bounded continuous functions  $(f * \varphi_i)_i$  converges uniformly on  $Q$  to  $f * \varphi$ . Indeed, we have, for every  $x \in Q$ ,

$$f * \varphi_i(x) = \int_G f(xy) \varphi_i(y^{-1}) dy = \langle \check{\varphi}_i, {}_x f \rangle,$$

and similarly  $f * \varphi(x) = \langle \check{\varphi}, {}_x f \rangle$ . Since the mapping

$$G \rightarrow L^1(G), \quad x \mapsto {}_x f$$

is continuous (Exercise A.8.3), the set  $\{{}_x f : x \in Q\}$  is a compact subset of  $L^1(G)$ . As  $(\check{\varphi}_i)_i$  converges to  $\check{\varphi}$  in the weak\* topology, it follows that  $\lim_i \langle \check{\varphi}_i, {}_x f \rangle = \langle \check{\varphi}, {}_x f \rangle$  uniformly for all  $x \in Q$ , that is,

$$\lim_i f * \varphi_i(x) = f * \varphi(x)$$



uniformly for  $x \in Q$ . Hence, we can assume that

$$(**) \quad \sup_{x \in Q} |f * \varphi_i(x) - f * \varphi(x)| \leq \varepsilon,$$

for every  $i$ .

Consider the subset

$$X = \{\psi \in \mathcal{P}_1(G) : |\langle \psi - \varphi, f \rangle| \leq \varepsilon\}.$$

Since  $X$  is a neighbourhood of  $\varphi$  in the weak\* topology, we can also assume that  $\varphi_i \in X$  for every  $i$ .

Let  $\psi \in X$ . Then

$$\begin{aligned} |\langle (1 - \psi), f \rangle| &\leq |\langle (1 - \varphi), f \rangle| + |\langle (\psi - \varphi), f \rangle| \\ &= \left| |V|^{-1} \int_V (1 - \varphi(x)) dx \right| + |\langle (\psi - \varphi), f \rangle| \\ &\leq |V|^{-1} \int_V |1 - \varphi(x)| dx + |\langle (\psi - \varphi), f \rangle|, \end{aligned}$$

and hence, by inequality (\*),

$$(***) \quad |\langle (1 - \psi), f \rangle| \leq 2\varepsilon.$$

On the other hand, we have for every  $x \in G$ ,

$$\begin{aligned} |f * \psi(x) - \psi(x)| &= \left| |V|^{-1} \int_G \chi_V(y) \psi(y^{-1}x) dy - |V|^{-1} \int_V \psi(x) dy \right| \\ &= \left| |V|^{-1} \int_V (\psi(y^{-1}x) - \psi(x)) dy \right| \\ &\leq |V|^{-1} \int_V |\psi(y^{-1}x) - \psi(x)| dy. \end{aligned}$$

Now, by Proposition C.4.2.iii,

$$|\psi(y^{-1}x) - \psi(x)| \leq \sqrt{2(1 - \operatorname{Re} \psi(y))}.$$

for all  $x, y \in G$ . It follows that, for every  $x \in Q$ ,

$$\begin{aligned} |f * \psi(x) - \psi(x)| &\leq |V|^{-1} \int_V \sqrt{2(1 - \operatorname{Re} \psi(y))} dy \\ &\leq |V|^{-1} \sqrt{2} \left( \int_V (1 - \operatorname{Re} \psi(y)) dy \right)^{1/2} \left( \int_V dy \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
&\leq |V|^{-1/2} \sqrt{2} \left| \int_V (1 - \psi(y)) dy \right|^{1/2} \\
&= \sqrt{2} |\langle 1 - \psi, f \rangle|^{1/2},
\end{aligned}$$

where we used the Cauchy–Schwarz inequality. Hence, by (\*\*\*), we have

$$|f * \psi(x) - \psi(x)| \leq 2\sqrt{\varepsilon}.$$

Combining this with inequality (\*\*), we have therefore, for every  $x \in Q$  and every  $i$ ,

$$\begin{aligned}
|\varphi_i(x) - \varphi(x)| &\leq |\varphi_i(x) - f * \varphi_i(x)| + |f * \varphi_i(x) - f * \varphi(x)| \\
&\quad + |f * \varphi(x) - \varphi(x)| \\
&\leq \varepsilon + 4\sqrt{\varepsilon},
\end{aligned}$$

since  $\varphi_i, \varphi \in X$ . This finishes the proof of the theorem. ■

**Remark C.5.7** Raikov’s Theorem is not true when  $\mathcal{P}_1(G)$  is replaced by  $\mathcal{P}_{\leq 1}(G)$ . Indeed, for the circle  $G = \mathbf{S}^1$ , the sequence of characters  $\chi_n : t \mapsto e^{int}$  belongs to  $\mathcal{P}_1(G)$ , tends to 0 in the weak\* topology (by the Riemann–Lebesgue Lemma, see Section D.1), but  $\chi_n(1) = 1$  for all  $n$ .

The following proposition is a consequence of C.5.6 and C.5.5.

**Proposition C.5.8** *Let  $G$  be a locally compact group. The convex hull of  $\text{ext}(\mathcal{P}_1(G))$  is dense in  $\mathcal{P}_1(G)$  for the topology of uniform convergence on compact subsets of  $G$ .*

**Corollary C.5.9 (Gelfand–Raikov Theorem)** *Let  $G$  be a locally compact group. Then, for  $x$  and  $y$  in  $G$  with  $x \neq y$ , there exists an irreducible unitary representation  $\pi$  of  $G$  with  $\pi(x) \neq \pi(y)$ .*

**Proof** Assume, by contradiction, that  $\pi(x) = \pi(y)$  for all irreducible unitary representations  $\pi$  of  $G$ . Then, by Theorem C.5.2, we have  $\varphi(x) = \varphi(y)$  for all  $\varphi \in \text{ext}(\mathcal{P}_1(G))$ . Therefore,  $\varphi(x) = \varphi(y)$  for all  $\varphi \in \mathcal{P}(G)$ , by the previous proposition. It follows that  $f_1 * f_2(x) = f_1 * f_2(y)$  for all  $f_1, f_2 \in C_c(G)$ , since  $f_1 * f_2$  is a matrix coefficient of the regular representation of  $G$  (see Example C.4.5).

On the other hand, observe that the set  $\{f_1 * f_2 : f_1, f_2 \in C_c(G)\}$  is dense in  $C_c(G)$  for the topology of uniform convergence. Indeed, take  $f$  in  $C_c(G)$ . For each compact neighbourhood  $U$  of  $e$ , set  $f_U = \frac{1}{|U|} \chi_U$ , where  $\chi_U$  is the

characteristic function of  $U$  and  $|U|$  the Haar measure of  $U$ . Then

$$\lim_{U \rightarrow e} f_U * f = f$$

uniformly on  $G$ , by the uniform continuity of  $f$ . In particular, there exist  $f_1, f_2 \in C_c(G)$  such that  $f_1 * f_2(x) \neq f_1 * f_2(y)$ . This is a contradiction. ■

A topological group need not have any unitary representation besides multiples of the identity representation [Banás–83]. A topological group which has a faithful unitary representation need not have any irreducible unitary representation distinct from the identity representation. The following example was drawn to our attention by V. Pestov.

**Example C.5.10** Consider the Hilbert space  $L^2([0, 1])$ . The abelian von Neumann algebra  $\mathcal{A} = L^\infty([0, 1])$  is an algebra of operators on  $L^2([0, 1])$ , a function  $\varphi \in \mathcal{A}$  being viewed as a multiplication operator  $\xi \mapsto \varphi\xi$  on  $L^2([0, 1])$ . Let  $\mathcal{U}$  denote the unitary group of  $\mathcal{A}$ , together with the strong operator topology which makes it a Polish abelian group; elements of  $\mathcal{U}$  are measurable functions from  $[0, 1]$  to the unit circle  $\mathbf{S}^1$ , up to equality almost everywhere. The tautological representation of  $\mathcal{U}$  on  $L^2([0, 1])$  is faithful.

Let  $\pi$  be an irreducible unitary representation of  $\mathcal{U}$ . As  $\mathcal{U}$  is abelian,  $\pi$  is one-dimensional, and  $(u, z) \mapsto \pi(u)z$  defines a continuous action of  $\mathcal{U}$  on the circle. Now the group  $\mathcal{U}$  is *extremely amenable* [Glasn–98], which means that any continuous action of  $\mathcal{U}$  on a compact space has a fixed point. In particular, there exists  $z \in \mathbf{S}^1$  such that  $\pi(u)z = z$  for all  $u \in \mathcal{U}$ . It follows that  $\pi$  is the unit representation of  $\mathcal{U}$ .

## C.6 Exercises

**Exercise C.6.1** Prove Lemma C.5.3.

[Hint: In order to show that (i) implies (ii), use the fact that the measure induced by the Haar measure on a compact subset of  $G$  can be approximated in the weak\* topology by positive measures of finite support which are norm-bounded.]

**Exercise C.6.2** Let  $\varphi$  be a continuous function of positive type on a topological group  $G$ . Show that, for  $g \in G$ , each of the left translate  ${}_g\varphi$  and the right translate  $\varphi_g$  of  $\varphi$  can be written as a linear combination of four continuous functions of positive type on  $G$ .

**Exercise C.6.3** Let  $\varphi$  be a continuous function of positive type on a topological group  $G$ , and let  $\pi$  be the unitary representation associated to  $\varphi$  by the GNS

construction. Show that every function of positive type associated to  $\pi$  is the uniform limit on  $G$  of functions of the form

$$x \mapsto \sum_{i=1}^n \sum_{j=1}^n c_i \bar{c}_j \varphi(x_j^{-1} x x_i),$$

for  $x_1, \dots, x_n \in G$  and  $c_1, \dots, c_n \in \mathbf{C}$ .

**Exercise C.6.4** Let  $\varphi$  be a continuous function of positive type on a topological group  $G$ . Show that

$$H = \{x \in G : \varphi(x) = \varphi(e)\}$$

is a closed subgroup of  $G$  and that  $\varphi$  is constant on every double coset  $HxH$ ,  $x \in G$ .

**Exercise C.6.5** Show that the kernel  $\Phi: \mathbf{R}_+ \times \mathbf{R}_+ \rightarrow \mathbf{R}$  defined by  $\Phi(s, t) = \min\{s, t\}$  is of positive type.

[Hint: Consider the mapping

$$f: \mathbf{R}_+ \rightarrow L^2(\mathbf{R}), \quad s \mapsto \chi_{[0,s]},$$

where  $\chi_{[0,s]}$  is the characteristic function of  $[0, s]$ .]

**Exercise C.6.6** Let  $X$  be a non-empty set and let  $R$  be a subset of  $X \times X$ . Let  $\Phi$  be the characteristic function of  $R$ .

(i) Show that  $\Phi$  is a kernel of positive type if  $R$  is an equivalence relation.

[Hint: Consider the mapping

$$f: X \rightarrow \ell^2(X/R), \quad x \mapsto \delta_{[x]},$$

where  $[x]$  denotes the equivalence class of  $x \in X$ .]

(ii) Assume that  $\Phi$  is a kernel of positive type and that  $R$  contains the diagonal  $\Delta$ . Show that  $R$  is an equivalence relation.

[Hint: Let  $\mathcal{H}$  be a Hilbert space and let  $f: X \rightarrow \mathcal{H}$  be such that  $\Phi(x, y) = \langle f(x), f(y) \rangle$ . Show that  $f(x) = f(y)$  if and only if  $(x, y) \in R$ .]

(iii) Give an example showing that the assumption  $\Delta \subset R$  in (ii) is necessary.

[Hint: Let  $X$  be a set with more than one element and  $R = \{(x_0, x_0)\}$  for  $x_0 \in X$ .]

**Exercise C.6.7** Let  $H$  be an open subgroup of the topological group  $G$ , and let  $\varphi$  be a function of positive type on  $H$ . Show that the trivial extension of  $\varphi$  to  $G$ , that is, the function on  $G$  defined by

$$x \mapsto \begin{cases} \varphi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H \end{cases}$$

is of positive type. In particular, the characteristic function  $\chi_H$  of  $H$  is a function of positive type on  $G$ .

**Exercise C.6.8** Let  $G$  be a discrete group, and let  $\varphi = \delta_e$  be the Dirac function at the group unit.

(i) Show that the unitary representation  $\pi_\varphi$  obtained by GNS-construction is equivalent to the left regular representation of  $G$ .

(ii) Observe that  $\varphi^2 = \varphi$ . Show that  $\pi_{\varphi^2} = \pi_\varphi$  is not equivalent to  $\pi_\varphi \otimes \pi_\varphi$ , when  $G$  has more than one element (compare with Corollary C.4.16).

**Exercise C.6.9** Prove the assertions of Proposition C.2.4.

**Exercise C.6.10** Let  $G = \mathbf{S}^1$  be the circle group. By considering the unitary characters of  $G$ , show that  $\mathcal{P}_1(G)$  is not weak\* closed in  $L^\infty(G)$ .

**Exercise C.6.11** Let  $G$  be a locally compact group with Haar measure  $\mu$ , and let  $A$  be a measurable subset of  $G$  with  $\mu(A) > 0$ . Show that  $AA^{-1}$  is a neighbourhood of  $e$ .

[Hint: One can assume that  $\mu(A) < \infty$ . The function  $\chi_A * \widetilde{\chi}_A$  is a function of positive type associated to the regular representation.]

**Exercise C.6.12** (i) Show that there exist discontinuous real valued functions  $f$  on  $\mathbf{R}$  which are additive, that is, such that  $f(x + y) = f(x) + f(y)$  for all  $x, y \in \mathbf{R}$ . As a consequence,  $\mathbf{R}$  has discontinuous unitary characters.

[Hint: View  $\mathbf{R}$  as a vector space over  $\mathbf{Q}$ .]

Let  $f$  be an additive complex-valued, measurable function on  $\mathbf{R}$ .

(ii) Show that  $f$  is continuous

[Hint: Use the previous exercise and the proof of A.6.2.]

(iii) Show that  $f$  has a continuous derivative.

[Hint: Let  $\varphi \in C_c(\mathbf{R})$  be non-negative, with support contained in a neighbourhood of 0 and with  $\int_{\mathbf{R}} \varphi(x) dx = 1$ . Consider the function  $g = \varphi * f$ .]

(iv) Show that  $f$  is linear.

**Exercise C.6.13** Let  $X$  be a topological space. A complex-valued kernel  $\Psi: X \times X \rightarrow \mathbf{C}$  is conditionally of negative type if

- (i)  $\Psi(x, y) = \overline{\Psi(y, x)}$  for all  $x, y$  in  $X$  and  
 (iii)  $\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \Psi(x_i, x_j) \leq 0$  for any  $n$  in  $\mathbf{N}$ , any elements  $x_1, \dots, x_n$  in  $X$ , and any complex numbers  $c_1, \dots, c_n$  with  $\sum_{i=1}^n c_i = 0$ .

Formulate and prove a version of Schoenberg's Theorem for such kernels.

[Hint: See, e.g., Theorem 3.2.2 and Proposition 3.3.2 in [BeChR-84].]

**Exercise C.6.14** For  $r, h > 0$ , consider the helix

$$f: \mathbf{R} \rightarrow \mathbf{R}^3, \quad x \mapsto (r \cos x, r \sin x, hx).$$

Compute the kernel conditionally of negative type  $\Psi$  on  $\mathbf{R}$  defined by  $f$  and observe that  $\Psi$  is translation-invariant.

**Exercise C.6.15** Let  $\Psi: X \times X \rightarrow \mathbf{R}_+$  be a kernel conditionally of negative type on the topological space  $X$ . Let  $0 < \alpha < 1$ . Show that  $\Psi^\alpha$  is a kernel conditionally of negative type on  $X$ .

[Hint: Use the formula

$$z^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-tz}) t^{-(\alpha+1)} dt,$$

for  $z \in \mathbf{C}$  with  $\operatorname{Re} z \geq 0$ .]

# Appendix D

## Unitary representations of locally compact abelian groups

Classical harmonic analysis rests on a few general and powerful theorems concerning the Fourier transform. These are Plancherel's Theorem on  $L^2$ -functions and Pontrjagin's Duality, recalled without proofs in Section D.1 (overall references are [Bou–Spec] and [Folla–95]), and Bochner's Theorem on Fourier transforms of positive measures, stated and proved here in Section D.2.

It was Hermann Weyl who realised around 1925 that classical harmonic analysis is naturally expressed in terms of unitary group representations (classically, the groups are  $\mathbf{R}^n$ ,  $\mathbf{Z}^n$ , and  $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ ). Then Stone described in 1930 all unitary representations of the additive group  $\mathbf{R}$ ; the generalisation to arbitrary locally compact abelian groups is due independently to Naimark, Ambrose, and Godement, and is the so-called SNAG Theorem of Section D.3.

The SNAG Theorem, concerning abelian groups, is necessary in Chapter 1 for our analysis of the unitary representations of groups like  $SL_n(\mathbf{K})$ , where  $\mathbf{K}$  is a local field. We recall in Section D.4 a few facts about local fields.

In Sections D.1–D.3, we denote by  $G$  a locally compact *abelian* group, with fixed Haar measure  $dx$ . The dual group  $\widehat{G}$  of  $G$  (see Section A.2) is contained in the set  $\mathcal{P}_1(G)$  of normalised functions of positive type (see Section C.5). More precisely, with appropriate identifications (see Corollary A.2.3, Remark C.4.13, and Theorem C.5.2), we have  $\widehat{G} = \text{ext}(\mathcal{P}_1(G))$ . The topology of uniform convergence on compact subsets of  $G$  makes  $\widehat{G}$  a topological group, and it is shown in Section D.1 that  $\widehat{G}$  is locally compact.

### D.1 The Fourier transform

Let us first recall part of Gelfand's theory of commutative Banach algebras. Let  $A$  be a (not necessarily unital) Banach algebra. The *spectrum*  $\sigma(x)$  of an element  $x$  in  $A$  is the set of all  $\lambda \in \mathbf{C}$  such that  $x - \lambda e$  is not invertible in  $\widetilde{A}$ , the

Banach algebra obtained from  $A$  by adjunction of a unit  $e$ . The *spectral radius*  $r(x)$  of  $x$  is  $\sup\{|\lambda| : \lambda \in \sigma(x)\}$ . By the *spectral radius formula*, we have

$$r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

From now on,  $A$  will denote a *commutative* Banach algebra. A *character* of  $A$  is a *non-zero* algebra homomorphism from  $A$  to  $\mathbf{C}$ . The set  $\Delta(A)$  of all characters of  $A$  is a locally compact subspace in the unit ball of  $A^*$ , the topological dual space of  $A$  (where  $A^*$  is equipped with the weak\* topology). Every element  $x \in A$  defines through

$$\widehat{x} : \Delta(A) \rightarrow \mathbf{C}, \quad \Phi \mapsto \Phi(x)$$

a continuous function  $\widehat{x}$  on  $\Delta(A)$ , which vanishes at infinity.

The mapping  $x \mapsto \widehat{x}$ , called the *Gelfand transform* on  $A$ , is an algebra homomorphism from  $A$  to  $C_0(\Delta(A))$ . For every  $x \in A$ , the uniform norm of  $\widehat{x}$  coincides with the spectral radius of  $x$ , that is,

$$\|\widehat{x}\|_\infty = \lim_{n \rightarrow \infty} \|x^n\|^{1/n}.$$

Let  $G$  be a locally compact abelian group. Consider the Banach algebra  $L^1(G)$ . Every  $\chi \in \widehat{G}$  defines a character  $\Phi_\chi$  of  $L^1(G)$  by

$$\Phi_\chi(f) = \int_G f(x) \chi(x) dx.$$

The mapping  $\chi \mapsto \Phi_\chi$  is known to be a homeomorphism from  $\widehat{G}$  onto  $\Delta(L^1(G))$ . In particular, the dual group  $\widehat{G}$  is a locally compact abelian group.

Let  $M(G)$  denote the Banach  $*$ -algebra (under convolution and total variation) of all finite complex regular Borel measures on  $G$ , endowed with the involution  $d\mu^*(x) = d\mu(x^{-1})$ . By the Riesz Representation Theorem,  $M(G)$  can be identified with the Banach space dual of  $C_0(G)$ . Observe that  $L^1(G)$  can be viewed as subset of  $M(G)$ .

**Definition D.1.1** The *Fourier transform* (also called the *Fourier–Stieltjes transform*) of  $\mu \in M(G)$  is the function  $\widehat{\mu}$  on  $\widehat{G}$  defined by

$$\widehat{\mu}(\chi) = \int_G \overline{\chi(x)} d\mu(x), \quad \chi \in \widehat{G}.$$

The Fourier transform  $\mathcal{F} : \mu \rightarrow \widehat{\mu}$  is a norm decreasing injective  $*$ -algebra homomorphism from  $M(G)$  to  $C^b(\widehat{G})$ , the Banach  $*$ -algebra of all continuous



bounded functions on  $\widehat{G}$  with the supremum norm and complex conjugation. The Fourier transform of  $L^1(G)$  is obtained by restriction:

$$(\mathcal{F}f)(\chi) = \widehat{f}(\chi) = \int_G \overline{\chi(x)} f(x) dx, \quad f \in L^1(G).$$

Observe that, when  $\widehat{G}$  is identified with the set of characters  $\Delta(L^1(G))$  as above, the Fourier transform coincides with the Gelfand transform on  $L^1(G)$ . (In fact, there is an unessential difference:  $\mathcal{F}f(\chi)$  coincides rather with the Gelfand transform of  $f$  at  $\overline{\chi}$ .)

In particular,  $\widehat{f}$  belongs to  $C_0(\widehat{G})$  for all  $f$  in  $L^1(G)$ . This is a generalisation of the *Riemann–Lebesgue Lemma* of classical Fourier analysis on  $\mathbf{R}^n$ .

**Theorem D.1.2** *For a suitable normalisation of the Haar measure  $d\widehat{x}$  on  $\widehat{G}$ , we have:*

**(Plancherel's Theorem)** *The Fourier transform  $f \mapsto \widehat{f}$  from  $L^1(G) \cap L^2(G)$  to  $L^2(\widehat{G})$  extends to an isometry from  $L^2(G)$  onto  $L^2(\widehat{G})$ .*

**(Fourier Inversion Theorem)** *If  $f \in L^1(G)$  and  $\widehat{f} \in L^1(\widehat{G})$ , then for almost every  $x \in G$ ,*

$$f(x) = \int_{\widehat{G}} \widehat{x}(x) \widehat{f}(\widehat{x}) d\widehat{x}.$$

Every  $x \in G$  defines a unitary character  $\eta(x)$  on  $\widehat{G}$  by the formula

$$\eta(x)(\chi) = \chi(x), \quad \chi \in \widehat{G}.$$

The continuous group homomorphism  $\eta : G \rightarrow \widehat{\widehat{G}}$  is injective by the Gelfand–Raikov Theorem C.5.9. Pontrjagin's Duality Theorem shows that  $\eta$  is surjective. More precisely, we have the following result.

**Theorem D.1.3 (Pontrjagin Duality)** *The canonical group homomorphism  $\eta : G \rightarrow \widehat{\widehat{G}}$  is an isomorphism of topological groups.*

So, we can and will always identify  $\widehat{\widehat{G}}$  with  $G$ .

**Remark D.1.4** We have

$$\mathcal{F}(\lambda_G(x)f)(\widehat{x}) = \overline{\widehat{x}(x)} \mathcal{F}f(\widehat{x}), \quad f \in L^2(G), x \in G, \widehat{x} \in \widehat{G}.$$

Hence, by Plancherel's Theorem, the Fourier transform  $\mathcal{F} : L^2(G) \rightarrow L^2(\widehat{G})$  is a unitary equivalence between the regular representation  $\lambda_G$  on  $L^2(G)$  and the unitary representation  $\pi$  of  $G$  on  $L^2(\widehat{G})$  defined by

$$(\pi(x)\xi)(\widehat{x}) = \overline{\widehat{x}(x)} \xi(\widehat{x}), \quad \xi \in L^2(\widehat{G}), \widehat{x} \in \widehat{G}, x \in G.$$

We can use this equivalence in order to describe the closed invariant subspaces of  $L^2(G)$  in case  $G$  is second countable. Indeed, for every Borel subset  $B$  of  $\widehat{G}$ , let  $T_B$  be the operator on  $L^2(\widehat{G})$  given by multiplication with the characteristic function of  $B$ . Then  $T_B$  is a projection and commutes with all  $\pi(x)$ 's. On the other hand, every projection in  $\mathcal{L}(L^2(\widehat{G}))$  which commutes with all  $\pi(x)$ 's is of the form  $T_B$  (Exercise D.5.1). Hence, the closed invariant subspaces of  $L^2(G)$  are of the form  $\mathcal{F}^{-1}(V_B)$ , where

$$V_B = T_B(L^2(\widehat{G})) = \{\varphi \in L^2(\widehat{G}) : \varphi(x) = 0, \text{ for all } x \notin B\}.$$

## D.2 Bochner's Theorem

Let  $\mu$  be a finite positive Borel measure on  $G$ . Its Fourier transform  $\widehat{\mu}$  is a continuous function of positive type on  $\widehat{G}$ , that is,  $\widehat{\mu} \in \mathcal{P}(\widehat{G})$ , with the notation as in Section C.5. Indeed, for  $\chi_1, \dots, \chi_n$  in  $\widehat{G}$  and  $c_1, \dots, c_n$  in  $\mathbb{C}$ ,

$$\sum_{i,j=1}^n c_i \overline{c_j} \widehat{\mu}(\overline{\chi_j} \chi_i) = \int_G \left| \sum_{i=1}^n c_i \overline{\chi_i(x)} \right|^2 d\mu(x) \geq 0.$$

**Example D.2.1** Let  $G = \mathbf{R}^n$ . Recall that  $\widehat{G} = \{x \mapsto e^{2\pi ixy} : y \in \mathbf{R}^n\}$  where  $xy = \sum_{i=1}^n x_i y_i$  for  $x, y \in \mathbf{R}^n$  (Example A.2.5). For fixed  $t > 0$ , set

$$f(x) = (4\pi t)^{-n/2} \exp(-\|x\|^2/4t).$$

A computation shows that

$$\widehat{f}(y) = \int_{\mathbf{R}^n} e^{-2\pi ixy} f(x) dx = \exp(-4\pi^2 t \|y\|^2).$$

Hence, the function  $y \mapsto \exp(-t\|y\|^2)$  is of positive type on  $\mathbf{R}^n$ . This also follows from Schoenberg's Theorem (Corollary C.4.19).

Bochner's Theorem says that every function of positive type on  $\widehat{G}$  arises as a Fourier transform of a finite positive measure on  $G$ . We give a proof of this theorem, using C.5.5 and Pontrjagin's Duality Theorem D.1.3.

**Theorem D.2.2 (Bochner)** *Let  $\varphi$  be a continuous function of positive type on  $\widehat{G}$ . There exists a finite positive Borel measure  $\mu$  on  $G$  such that  $\widehat{\mu} = \varphi$ .*

**Proof** Let  $M_{\leq 1}(G)$  denote the set of all positive Borel measures  $\mu$  on  $G$  with  $\|\mu\| = \mu(G) \leq 1$ . This is a weak\* closed and hence compact subset of the space  $M(G) \cong C_0(G)^*$  of all finite complex regular Borel measures on  $G$ .

On the other hand,  $\mathcal{P}_{\leq 1}(\widehat{G})$  is a weak\* closed and hence compact subset of  $L^\infty(\widehat{G})$ .

We claim that the Fourier transform  $\mathcal{F} : M_{\leq 1}(G) \mapsto \mathcal{P}_{\leq 1}(\widehat{G})$  is continuous when both  $M_{\leq 1}(G)$  and  $\mathcal{P}_{\leq 1}(\widehat{G})$  are equipped with their weak\* topology. Indeed, this follows from the formula

$$\begin{aligned} \int_{\widehat{G}} f(\widehat{x}) \widehat{\mu}(\widehat{x}) d\widehat{x} &= \int_{\widehat{G}} \int_G f(\widehat{x}) \overline{\widehat{x}(x)} d\mu(x) d\widehat{x} \\ &= \int_G \widehat{f}(x) d\mu(x), \end{aligned}$$

for all  $f \in L^1(\widehat{G})$ .

The image  $\mathcal{F}(M_{\leq 1}(G))$  is therefore a weak\* compact subset of  $\mathcal{P}_{\leq 1}(\widehat{G})$ . Observe that  $\mathcal{F}(M_{\leq 1}(G))$  contains  $\widehat{\widehat{G}} = G$ , since  $\widehat{\delta_x} = x^{-1}$ , where  $x^{-1} \in G$  is viewed as a character on  $\widehat{G}$ . Moreover, we have  $\text{ext}(\mathcal{P}_1(\widehat{G})) = G$  and  $\mathcal{F}(M_{\leq 1}(G))$  is convex. It follows from Theorem C.5.5.i and from the Krein–Milman Theorem that  $\mathcal{F}(M_{\leq 1}(G))$  contains  $\mathcal{P}_{\leq 1}(\widehat{G})$ . ■

**Remark D.2.3** Let  $(\pi, \mathcal{H})$  be a unitary representation of the topological group  $G$ . Let  $\varphi$  be a matrix coefficient of  $\pi$ , that is,  $\varphi = \langle \pi(\cdot)\xi, \eta \rangle$  for some  $\xi$  and  $\eta$  in  $\mathcal{H}$ . Then  $\varphi$  is a linear combination of diagonal matrix coefficients of  $\pi$ . Indeed, by polarization,

$$\begin{aligned} 4\langle \xi, \eta \rangle &= \langle \xi + \eta, \xi + \eta \rangle - \langle \xi - \eta, \xi - \eta \rangle \\ &\quad + i\langle \xi + i\eta, \xi + i\eta \rangle - i\langle \xi - i\eta, \xi - i\eta \rangle. \end{aligned}$$

As diagonal matrix coefficients of unitary representations are functions of positive type, we obtain the following corollary of Bochner's Theorem.

**Corollary D.2.4** *Let  $(\pi, \mathcal{H})$  be a unitary representation of a locally compact abelian group  $G$ , and let  $\varphi$  be a matrix coefficient of  $\pi$ . Then there exists a finite complex regular Borel measure  $\widehat{\mu}$  on  $\widehat{G}$  such that  $\widehat{\mu} = \varphi$ .*

### D.3 Unitary representations of locally compact abelian groups

Let  $G$  be a locally compact abelian group. We will show that every unitary representation of  $G$  can be described by a projection valued measure on  $\widehat{G}$ .

Let  $X$  be a locally compact space, equipped with the  $\sigma$ -algebra  $\mathcal{B}(X)$  of its Borel subsets. Let  $\mathcal{H}$  be a Hilbert space. Denote by  $\text{Proj}(\mathcal{H})$  the set of orthogonal

projections in  $\mathcal{L}(\mathcal{H})$ . A *projection valued measure* on  $X$  is a mapping

$$E : \mathcal{B}(X) \rightarrow \text{Proj}(\mathcal{H})$$

with the following properties:

- (i)  $E(\emptyset) = 0$  and  $E(X) = I$ ;
- (ii)  $E(B \cap B') = E(B)E(B')$  for all  $B, B'$  in  $\mathcal{B}(X)$ ;
- (iii) if  $(B_n)_n$  is a sequence of pairwise disjoint sets from  $\mathcal{B}(X)$ , then

$$E\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \sum_{n \in \mathbb{N}} E(B_n),$$

where the sum is convergent in the strong operator topology on  $\mathcal{L}(\mathcal{H})$ .

Let  $E$  be a projection valued measure on  $X$ . For  $\xi, \eta$  in  $\mathcal{H}$ , the mapping

$$B \mapsto \langle E(B)\xi, \eta \rangle$$

is a complex Borel measure on  $X$ , denoted by  $dE_{\xi, \eta}$ . Clearly, the measure  $dE_{\xi, \xi}$  is positive and  $dE_{\xi, \xi}(X) = \|\xi\|^2$ . Since, by polarization,

$$4dE_{\xi, \eta} = dE_{\xi+\eta, \xi+\eta} - dE_{\xi-\eta, \xi-\eta} + idE_{\xi+i\eta, \xi+i\eta} - idE_{\xi-i\eta, \xi-i\eta},$$

it follows that every measure  $dE_{\xi, \eta}$  is finite.

For each bounded Borel function  $f$  on  $X$ , the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad (\xi, \eta) \mapsto \int_X f(x) dE_{\xi, \eta}(x)$$

is bounded. Hence, there exists an operator in  $\mathcal{L}(\mathcal{H})$ , denoted by  $\int_X f(x) dE(x)$ , such that

$$\left\langle \left( \int_X f(x) dE(x) \right) \xi, \eta \right\rangle = \int_X f(x) dE_{\xi, \eta}(x), \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

The mapping

$$f \mapsto \int_G f(x) dE(x)$$

is a  $*$ -algebra homomorphism from the  $*$ -algebra of bounded Borel functions on  $X$  to  $\mathcal{L}(\mathcal{H})$ . In particular, we have

$$(*) \quad \left\| \left( \int_X f(x) dE(x) \right) \xi \right\|^2 = \int_X |f(x)|^2 dE_{\xi, \xi}(x),$$

for every bounded Borel function  $f$  on  $X$  and every  $\xi \in \mathcal{H}$ .

We say that the projection valued measure  $E$  is *regular* if the measure  $\mu_{\xi,\xi}$  is regular for every  $\xi$  in  $\mathcal{H}$ . For all this, see [Rudin–73, 12.17].

The following result was proved by Stone for the case  $G = \mathbf{R}$  and independently generalised by Naimark, Ambrose and Godement.

**Theorem D.3.1 (SNAG Theorem)** (i) Let  $(\pi, \mathcal{H})$  be a unitary representation of the locally compact abelian group  $G$ . There exists a unique regular projection valued measure  $E_\pi : \mathcal{B}(\widehat{G}) \rightarrow \text{Proj}(\mathcal{H})$  on  $\widehat{G}$  such that

$$\pi(x) = \int_{\widehat{G}} \overline{\widehat{x}(x)} dE_\pi(\widehat{x}), \quad \text{for all } x \in G.$$

Moreover, an operator  $T$  in  $\mathcal{L}(\mathcal{H})$  commutes with  $\pi(x)$  for all  $x \in G$  if and only if  $T$  commutes with  $E(B)$  for all  $B \in \mathcal{B}(\widehat{G})$ .

(ii) Conversely, if  $E$  is a regular projection valued measure

$$E : \mathcal{B}(\widehat{G}) \rightarrow \text{Proj}(\mathcal{H})$$

on  $\widehat{G}$ , then

$$\pi(x) = \int_{\widehat{G}} \overline{\widehat{x}(x)} dE(\widehat{x}), \quad \text{for all } x \in G$$

defines a unitary representation  $\pi$  of  $G$  on  $\mathcal{H}$ .

**Proof** (i) Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . For  $\xi, \eta \in \mathcal{H}$ , denote by  $\varphi_{\xi,\eta}$  the corresponding matrix coefficient of  $\pi$ :

$$\varphi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle.$$

By Bochner's Theorem D.2.2 (see also Corollary D.2.4), there exists a finite complex regular Borel measure  $\mu_{\xi,\eta}$  on  $\widehat{G}$  such that

$$\varphi_{\xi,\eta}(x) = \widehat{\mu}_{\xi,\eta}(x) = \int_{\widehat{G}} \overline{\widehat{x}(x)} d\mu_{\xi,\eta}(\widehat{x}).$$

For every Borel subset  $B$  of  $\widehat{G}$ , the sesquilinear form

$$\mathcal{H} \times \mathcal{H} \rightarrow \mathbf{C}, \quad (\xi, \eta) \mapsto \mu_{\xi,\eta}(B)$$

is bounded. Therefore, there exists an operator  $E_\pi(B)$  in  $\mathcal{L}(\mathcal{H})$  such that

$$\langle E_\pi(B)\xi, \eta \rangle = \mu_{\xi,\eta}(B).$$

It is readily verified that  $B \mapsto E_\pi(B)$  is a regular projection valued measure on  $\widehat{G}$ . Moreover, for  $x$  in  $G$  and  $\xi, \eta$  in  $\mathcal{H}$ ,

$$\begin{aligned} \left\langle \int_{\widehat{G}} \overline{\widehat{x}(x)} dE_\pi(\widehat{x}) \xi, \eta \right\rangle &= \int_{\widehat{G}} \overline{\widehat{x}(x)} d\mu_{\xi, \eta}(\widehat{x}) = \widehat{\mu}_{\xi, \eta}(x) \\ &= \varphi_{\xi, \eta}(x) = \langle \pi(x) \xi, \eta \rangle, \end{aligned}$$

that is,

$$\pi(x) = \int_{\widehat{G}} \overline{\widehat{x}(x)} dE_\pi(\widehat{x}).$$

Let  $T \in \mathcal{L}(\mathcal{H})$ . If  $T\pi(x) = \pi(x)T$  for all  $x$  in  $G$ , then  $\mu_{T\xi, \eta} = \mu_{\xi, T^*\eta}$ , and therefore  $TE_\pi(B) = E_\pi(B)T$  for all Borel sets  $B$ , by definition of  $E_\pi$ . Conversely, if  $TE_\pi(B) = E_\pi(B)T$  for all Borel sets  $B$ , then  $\mu_{T\xi, \eta} = \mu_{\xi, T^*\eta}$  for all  $\xi, \eta$  in  $\mathcal{H}$  and, hence,  $T\pi(x) = \pi(x)T$  for all  $x$  in  $G$ .

(ii) Formula (\*) above shows that  $\pi(x)$  is a unitary operator for every  $x \in G$ . Since  $f \mapsto \int_{\widehat{G}} f(\widehat{x}) dE(\widehat{x})$  is a  $*$ -algebra homomorphism from the algebra of bounded Borel functions on  $\widehat{G}$  to  $\mathcal{L}(\mathcal{H})$ , the mapping  $x \mapsto \pi(x)$  is a group homomorphism from  $G$  to  $\mathcal{U}(\mathcal{H})$ . It remains to show that  $\pi$  is strongly continuous. It suffices to check this at the group unit  $e$ .

Let  $\xi \in \mathcal{H}$ . By Formula (\*) again, we have

$$\|\pi(x)\xi - \xi\|^2 = \int_{\widehat{G}} |\overline{\widehat{x}(x)} - 1|^2 dE_{\xi, \xi}(\widehat{x}).$$

Let  $\varepsilon > 0$ . Since  $dE_{\xi, \xi}$  is a regular measure, we can find a compact subset  $Q$  of  $\widehat{G}$  such that  $dE_{\xi, \xi}(\widehat{G} \setminus Q) < \varepsilon$ . The set

$$V = \{x \in G : \sup_{\widehat{x} \in Q} |\overline{\widehat{x}(x)} - 1| < \varepsilon\}$$

is a neighbourhood of  $e$  in  $G$ , since the mapping  $G \rightarrow \widehat{\widehat{G}}$  is continuous. For every  $x \in V$ , we have

$$\begin{aligned} \|\pi(x)\xi - \xi\|^2 &= \int_Q |\overline{\widehat{x}(x)} - 1|^2 dE_{\xi, \xi}(\widehat{x}) + \int_{\widehat{G} \setminus Q} |\overline{\widehat{x}(x)} - 1|^2 dE_{\xi, \xi}(\widehat{x}) \\ &\leq \varepsilon^2 dE_{\xi, \xi}(Q) + 4dE_{\xi, \xi}(\widehat{G} \setminus Q) \\ &\leq \varepsilon \|\xi\|^2 + 4\varepsilon. \end{aligned}$$

This shows the continuity of  $x \mapsto \pi(x)\xi$  at  $e$ . ■

**Example D.3.2** Recall that, by Plancherel's Theorem, the Fourier transform  $\mathcal{F} : L^2(G) \rightarrow L^2(\widehat{G})$  is a unitary equivalence between the regular representation  $\lambda_G$  and the unitary representation  $\pi$  of  $G$  on  $L^2(\widehat{G})$  defined by

$$(\pi(x)\xi)(\widehat{x}) = \overline{\widehat{x}(x)}\xi(\widehat{x}), \quad \xi \in L^2(\widehat{G})$$

(see Remark D.1.4). For a Borel set  $B$  in  $\widehat{G}$ , let  $T_B$  denote the operator on  $L^2(\widehat{G})$  given by multiplication with the characteristic function of  $B$ . The projection valued measure associated to  $\pi$  is  $B \mapsto T_B$ . Hence, the projection valued measure associated to  $\lambda_G$  is  $B \mapsto \mathcal{F}^{-1}T_B\mathcal{F}$ .

## D.4 Local fields

Let  $\mathbf{K}$  be a field. An *absolute value* on  $\mathbf{K}$  is a real-valued function  $x \mapsto |x|$  such that, for all  $x$  and  $y$  in  $\mathbf{K}$  :

- (i)  $|x| \geq 0$ , and  $|x| = 0$  if and only if  $x = 0$ ;
- (ii)  $|xy| = |x||y|$ ;
- (iii)  $|x + y| \leq |x| + |y|$ .

An absolute value defines a topology on  $\mathbf{K}$  given by the metric  $d(x, y) = |x - y|$ .

**Definition D.4.1** A field  $\mathbf{K}$  is a *local field* if  $\mathbf{K}$  can be equipped with an absolute value for which  $\mathbf{K}$  is locally compact and not discrete.

For a given absolute value,  $\mathbf{K}$  is locally compact if and only if the ball  $U = \{x \in \mathbf{K} : |x| \leq 1\}$  is compact, and  $\mathbf{K}$  is not discrete if and only if the absolute value is non trivial (that is,  $|x| \neq 1$  for some  $x \in \mathbf{K}$ ,  $x \neq 0$ ).

**Example D.4.2** (i)  $\mathbf{K} = \mathbf{R}$  and  $\mathbf{K} = \mathbf{C}$  with the usual absolute value are local fields.

(ii) Fix a prime  $p \in \mathbf{N}$ . For  $x \in \mathbf{Q} \setminus \{0\}$ , write  $x = p^m a/b$  with  $a, b \in \mathbf{Z} \setminus \{0\}$ ,  $m \in \mathbf{Z}$ , and  $a, b$  prime to  $p$ . Define  $|x|_p = p^{-m}$ ; set  $|0|_p = 0$ . Then

$$x \mapsto |x|_p$$

is a non-trivial absolute value on  $\mathbf{Q}$ , called the *p-adic absolute value*. The completion of  $\mathbf{Q}$  for the corresponding distance

$$d_p(x, y) = |x - y|_p, \quad x, y \in \mathbf{Q}$$

is the field  $\mathbf{Q}_p$  of *p-adic numbers*.

The  $p$ -adic absolute value on  $\mathbf{Q}$  extends to an absolute value on  $\mathbf{Q}_p$ . The subset

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p : |x|_p \leq 1\} = \{x \in \mathbf{Q}_p : |x|_p < p\}$$

is an open and compact subring of  $\mathbf{Q}_p$ , called the ring of  $p$ -adic integers. This shows that  $\mathbf{Q}_p$  is locally compact, and is a local field.

Observe that the  $p$ -adic absolute value is *non-archimedean*: it satisfies the following ultrametric strengthening of the triangle inequality:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \quad x, y \in \mathbf{Q}_p.$$

**Remark D.4.3** Every non-discrete locally compact topological field is a local field. Indeed, let  $\mu$  be a Haar measure on the additive group  $(\mathbf{K}, +)$ . It follows from the uniqueness of  $\mu$  (up to a multiplicative constant) that, for every  $x \in \mathbf{K}$ ,  $x \neq 0$ , there exists a real number  $c(x) > 0$  such that  $\mu(xB) = c(x)\mu(B)$  for every Borel subset  $B$  of  $\mathbf{K}$  (see Exercise A.8.8); we set  $c(0) = 0$ . The function  $x \mapsto c(x)$  (respectively,  $x \mapsto c(x)^{1/2}$ ) is an absolute value on  $\mathbf{K}$  if  $\mathbf{K}$  is not isomorphic to  $\mathbf{C}$  (respectively, if  $\mathbf{K}$  is isomorphic to  $\mathbf{C}$ ). The topology induced by this function coincides with the original topology of  $\mathbf{K}$ .

Moreover, a local field is isomorphic to one of the following fields:

- (i)  $\mathbf{R}$  or  $\mathbf{C}$  with the usual absolute value,
- (ii) a finite extension of the field of  $p$ -adic numbers  $\mathbf{Q}_p$  with an extension of the  $p$ -adic absolute value,
- (iii) the field  $\mathbf{K} = k((X))$  of Laurent series over a finite field  $k$  with absolute value  $|\sum_{i=m}^{\infty} a_i X^i| = e^{-m}$  with  $a_m \in k \setminus \{0\}$ .

For all this, see [Weil–73] or [Bou–AlCo].

If  $\mathbf{K}$  is a local field, let us show that the dual group of the additive group  $\mathbf{K}$  can be identified with  $\mathbf{K}$ , in a non-canonical way. For this, we need the following general elementary fact.

**Lemma D.4.4** *Let  $G$  be a topological group, and let  $H$  be a subgroup. Assume that  $H$  is locally compact, for the induced topology from  $G$ . Then  $H$  is closed in  $G$ .*

**Proof** Since  $H$  is locally compact, there exists a neighbourhood  $U_0$  of  $e$  in  $G$  such that  $U_0 \cap H$  is compact. As  $G$  is Hausdorff,  $U_0 \cap H$  is closed in  $G$ . Fix a neighbourhood  $V_0$  of  $e$  in  $G$  with  $V_0^{-1} = V_0$  and  $V_0^2 \subset U_0$ .

Let  $x$  be in the closure  $\overline{H}$  of  $H$  in  $G$ . Then  $x \in V_0 H$ ; so we can choose  $v_0 \in V_0$  such that  $v_0 x \in H$ . We claim that  $v_0 \in H$  and hence that  $x \in H$ .



Indeed, for any neighbourhood  $W$  of  $e$  in  $G$  with  $W^{-1} = W$  and  $W \subset V_0$ , we have  $x^{-1} \in HW$ , since  $x^{-1} \in \overline{H}$ . This implies that

$$v_0 = (v_0x)x^{-1} \in H(HW) = HW.$$

So there exists  $w \in W$  such that  $v_0w \in H$ . Since  $V_0W \subset V_0^2 \subset U_0$ , we have  $v_0w \in U_0 \cap H$ . Hence  $v_0 \in (U_0 \cap H)W$ . As  $W$  is arbitrary,  $v_0$  is in the closure of  $U_0 \cap H$  in  $G$ . Since  $U_0 \cap H$  is closed,  $v_0 \in H$ . ■

**Proposition D.4.5** *Let  $\mathbf{K}$  be a local field. Let  $\chi$  be a fixed unitary character on the additive group of  $\mathbf{K}$ , distinct from the unit character. For each  $y \in \mathbf{K}$ , let  $\chi_y \in \widehat{\mathbf{K}}$  be defined by  $\chi_y(x) = \chi(yx)$  for all  $x$  in  $\mathbf{K}$ . The mapping*

$$\mathbf{K} \rightarrow \widehat{\mathbf{K}}, \quad y \mapsto \chi_y$$

*is a topological group isomorphism.*

**Proof** The mapping  $\Phi : y \mapsto \chi_y$  is clearly a continuous injective group homomorphism. Let  $H$  be the image of  $\Phi$ . Then  $H$  is a dense subgroup of  $\widehat{\mathbf{K}}$ . Indeed, otherwise, the quotient group  $\widehat{\mathbf{K}}/\overline{H}$  would be a non-trivial locally compact abelian group. Hence, by Pontrjagin's Duality Theorem D.1.3, there would exist  $x \in \mathbf{K}$ ,  $x \neq 0$ , with  $\chi_y(x) = 1$  for all  $y$  in  $\mathbf{K}$ . This is impossible, since  $\chi \neq 1_{\mathbf{K}}$ .

We claim that the isomorphism  $\Phi^{-1} : H \rightarrow \mathbf{K}$  is continuous. It suffices to show that  $\Phi^{-1}$  is continuous at the unit character  $1_{\mathbf{K}}$ . Let  $\varepsilon > 0$ . Since  $\chi$  is non-trivial,  $\chi(x_0) \neq 1$  for some  $x_0 \in \mathbf{K}$ . Set  $\delta = |\chi(x_0) - 1| > 0$  and

$$Q = \{x \in \mathbf{K} : |x| \leq |x_0|/\varepsilon\},$$

where  $|\cdot|$  is an absolute value on  $\mathbf{K}$ . Then  $Q$  is a compact subset of  $\mathbf{K}$ , and

$$U_{Q,\delta} = \{\chi_y : |\chi(yx) - 1| < \delta, \text{ for all } x \in Q\}$$

is a neighbourhood of  $1_{\mathbf{K}}$  in  $H$ . Let  $y \in \mathbf{K}$  with  $\chi_y \in U_{Q,\delta}$ . Then, since

$$|\chi(y(y^{-1}x_0)) - 1| = |\chi(x_0) - 1| = \delta,$$

we have  $y^{-1}x_0 \notin Q$ , that is,  $|y| < \varepsilon$ . Thus,  $\Phi^{-1}$  is continuous and is therefore a topological isomorphism between  $H$  and  $\mathbf{K}$ . Hence,  $H$  is locally compact and, by the previous lemma,  $H$  is closed in  $\widehat{\mathbf{K}}$ . Since  $H$  is dense,  $H = \widehat{\mathbf{K}}$ . ■

Recall that, for abelian topological groups  $G_1, \dots, G_n$ , the dual group of  $G_1 \times \dots \times G_n$  can be naturally identified with  $\widehat{G_1} \times \dots \times \widehat{G_n}$  (Example A.2.5.ii). The following corollary is an immediate consequence of the previous proposition.

**Corollary D.4.6** *Let  $\mathbf{K}$  be a local field, and  $n \geq 1$ . Let  $\chi$  be a fixed unitary character of  $\mathbf{K}$ , distinct from the unit character. The mapping*

$$\mathbf{K}^n \rightarrow \widehat{\mathbf{K}^n}, y \mapsto \chi_y$$

*is a topological group isomorphism, where  $\chi_y$  is the character of  $\mathbf{K}^n$  defined by  $\chi_y(x) = \chi(\sum_{i=1}^n x_i y_i)$ .*

**Remark D.4.7** The existence of a non-trivial unitary character on a locally compact abelian group follows from the Gelfand-Raikov Theorem (Corollary C.5.9). But, for local fields, it is possible to give explicit examples of such characters. For instance, let  $\mathbf{K} = \mathbf{Q}_p$  be the field of  $p$ -adic numbers. Every  $x \in \mathbf{Q}_p$ ,  $x \neq 0$ , can be represented uniquely as a convergent series in  $\mathbf{Q}_p$

$$x = \sum_{j=m}^{\infty} a_j p^j$$

for integers  $m \in \mathbf{Z}$ ,  $a_j \in \{0, \dots, p-1\}$ , with  $a_m \neq 0$ ; we have  $|x|_p = p^{-m}$ . In particular,  $x \in \mathbf{Z}_p$  if and only if  $m \geq 0$ . Define

$$\chi : \mathbf{Q}_p \rightarrow \mathbf{S}^1, \quad \sum_{j=m}^{\infty} a_j p^j \mapsto \exp \left( 2\pi i \sum_{j=m}^{-1} a_j p^j \right).$$

Then  $\chi$  is a homomorphism and  $\chi$  is not the unit character of  $\mathbf{Q}_p$ . To prove the continuity of  $\chi$ , observe that  $\chi$  takes the value 1 on  $\mathbf{Z}_p$  and that  $\mathbf{Z}_p$  is a neighbourhood of 0.

## D.5 Exercises

**Exercise D.5.1** Let  $G$  be a second countable locally compact abelian group. For  $\varphi \in L^\infty(G)$ , let  $T_\varphi \in \mathcal{L}(L^2(G))$  denote the operator given by multiplication with  $\varphi$ . Let  $T$  be an operator in  $\mathcal{L}(L^2(G))$  commuting with  $T_\chi$  for all  $\chi \in \widehat{G}$ .

(i) Show that  $T$  commutes with  $T_\varphi$  for all  $\varphi \in L^\infty(G)$ .

(ii) Show that  $T = T_\psi$  for some  $\psi \in L^\infty(G)$ .

[Hint: Choose a continuous function  $f_0$  on  $G$  with  $f_0(x) > 0$  for all  $x \in G$  and with  $f_0 \in L^2(G)$ . Set  $\psi = \frac{Tf_0}{f_0}$ .]

Observe that (ii) shows that  $\{T_\varphi : \varphi \in L^\infty(G)\}$  is a maximal abelian subalgebra of  $\mathcal{L}(L^2(G))$ . This result holds for more general measure spaces; see for instance Theorem 6.6 in Chapter IX of [Conwa-87].

**Exercise D.5.2** Let  $G$  be a locally compact abelian group, and let  $H$  be a subgroup of  $G$ . Show that the following properties are equivalent:

- (i)  $H$  is dense in  $G$ ;
- (ii) If  $\chi|_H = 1_H$  for  $\chi \in \widehat{G}$ , then  $\chi = 1_G$ .

**Exercise D.5.3** Let  $G$  be a locally compact abelian group, and let  $H$  be a closed subgroup of  $G$ . Set

$$H^\perp = \{\chi \in \widehat{G} : \chi|_H = 1_H\},$$

a closed subgroup of  $\widehat{G}$ .

- (i) Show that  $\widehat{G/H}$  is topologically isomorphic to  $H^\perp$ .
- (ii) Show that  $\widehat{H}$  is topologically isomorphic to  $\widehat{G}/H^\perp$ .

**Exercise D.5.4** Let  $G$  be a compact abelian group. Show that  $\{\chi : \chi \in \widehat{G}\}$  is an orthonormal basis of the Hilbert space  $L^2(G, dx)$ , where  $dx$  is the normalised Haar measure on  $G$ .

[This is a special case of the Peter–Weyl Theorem A.5.2.]

**Exercise D.5.5** Let  $G$  be a locally compact abelian group.

- (i) Assume that  $G$  is discrete. Show that the dual group  $\widehat{G}$  is compact.
- (ii) Assume that  $G$  is compact. Show that  $\widehat{G}$  is discrete.

[Hint: Use Exercise D.5.4.]

**Exercise D.5.6** Let  $(G_i)_{i \in I}$  be a family of compact abelian groups, and let  $G = \prod_{i \in I} G_i$  be the direct product, equipped with the product topology. Let  $X = \bigoplus_{i \in I} \widehat{G}_i$  be the direct sum of the dual groups  $\widehat{G}_i$ , that is,  $X$  is the subgroup of  $\prod_{i \in I} \widehat{G}_i$  consisting of the families  $(\chi_i)_{i \in I}$  with  $\chi_i = 1_{G_i}$  for all but finitely many indices  $i$ . Let  $X$  be equipped with the discrete topology. Define  $\Phi : X \rightarrow \widehat{G}$  by

$$\Phi((\chi_i)_{i \in I})((g_j)_{j \in I}) = \prod_{i \in I} \chi_i(g_i).$$

Show that  $\Phi$  is a topological group isomorphism.

**Exercise D.5.7** Let  $\Gamma$  be a discrete group acting by continuous automorphisms on a compact group  $G$ . Let  $G$  be equipped with the normalised Haar measure. For  $\gamma \in \Gamma$ , define  $\pi(\gamma) \in \mathcal{L}(L^2(G))$  by

$$\pi(\gamma)\xi(x) = \xi(\gamma^{-1}(x)), \quad \text{for all } \xi \in L^2(G), \gamma \in \Gamma, x \in G.$$

(i) Show that  $\pi$  is a unitary representation of  $G$ .

(ii) Assume, moreover, that  $G$  is abelian. Show that  $\pi$  is equivalent to the unitary representation  $\sigma$  of  $\Gamma$  on  $\ell^2(\widehat{G})$  defined by duality:

$$\sigma(\gamma)\xi(\widehat{x}) = \xi(\gamma^{-1}(\widehat{x})), \quad \text{for all } \xi \in \ell^2(\widehat{G}), \gamma \in \Gamma, \widehat{x} \in \widehat{G},$$

where  $\gamma^{-1}(\widehat{x})(x) = \widehat{x}(\gamma(x))$ .

# Appendix E

## Induced representations

Let  $G$  be a locally compact group. To any closed subgroup  $H$  and any unitary representation  $(\sigma, \mathcal{K})$  of  $H$  is associated the *induced representation*  $\text{Ind}_H^G \sigma$  of  $G$ . This is one of the most important notions of the theory.

In case  $G$  is a finite group, the construction goes back to Frobenius; the space  $\mathcal{H}$  of  $\text{Ind}_H^G \sigma$  consists of mappings  $\xi : G \rightarrow \mathcal{K}$  such that  $\xi(xh) = \sigma(h^{-1})\xi(x)$  for all  $x$  in  $G$  and  $h$  in  $H$ , and  $\text{Ind}_H^G \sigma(g)$  is given by left translation by  $g^{-1}$  on  $\mathcal{H}$  for every  $g$  in  $G$  (see [Curti–99, Chapter II, § 4]). In case  $G$  is locally compact and  $\sigma$  unitary, Section E.1 shows how to modify this construction in order to obtain a unitary representation of  $G$ .

The second section of this chapter shows two useful facts concerning induced representations. The first one is the important result of induction by stages: if  $\sigma$  is itself of the form  $\text{Ind}_K^H \tau$  for a unitary representation  $\tau$  of a closed subgroup  $K$  of  $H$ , then  $\text{Ind}_H^G (\text{Ind}_K^H \tau)$  and  $\text{Ind}_K^G \tau$  are equivalent. The second one deals with tensor products.

In the third section, we give a simple necessary and sufficient condition for  $\text{Ind}_H^G \sigma$  to have non-zero invariant vectors.

For more details on this chapter, see [Folla–95] or [Gaal–73].

### E.1 Definition of induced representations

Let  $G$  be a locally compact group,  $H$  a closed subgroup of  $G$ , and  $(\sigma, \mathcal{K})$  a unitary representation of  $H$ .

We first define a (usually non-complete) space of functions from  $G$  to  $\mathcal{K}$  on which  $G$  acts in a natural way. Let  $p : G \rightarrow G/H$  be the canonical projection, and let  $\mathcal{A}$  be the vector space of all mappings  $\xi : G \rightarrow \mathcal{K}$  with the following properties:

- (i)  $\xi$  is continuous,

- (ii)  $p(\text{supp } \xi)$  is compact,
- (iii)  $\xi(xh) = \sigma(h^{-1})\xi(x)$  for all  $x \in G$  and all  $h \in H$ .

The induced representation of  $\sigma$  has two ingredients: the natural left action of  $G$  on  $\mathcal{A}$  (and on its completion, to be defined) and the Radon–Nikodym factor which appears in formula (\*\*) below.

The following proposition shows how to construct “many” elements from  $\mathcal{A}$ . For this, we need to consider *vector valued integrals*. Let  $X$  be a locally compact space with a regular Borel measure  $\mu$ , and let  $\xi : X \rightarrow \mathcal{K}$  be a continuous function with compact support from  $X$  to some Hilbert space  $\mathcal{K}$ . Then  $\int_X \xi(x) d\mu(x)$  is defined as the unique element in  $\mathcal{K}$  such that

$$\left\langle \int_X \xi(x) d\mu(x), v \right\rangle = \int_X \langle \xi(x), v \rangle d\mu(x)$$

for all  $v \in \mathcal{K}$ .

Fix left Haar measures  $dx$  and  $dh$  on  $G$  and  $H$ , respectively. For  $f \in C_c(G)$  and  $v \in \mathcal{K}$ , define a mapping  $\xi_{f,v} : G \rightarrow \mathcal{K}$  by

$$\xi_{f,v}(x) = \int_H f(xh) \sigma(h) v dh, \quad x \in G.$$

(Observe that this is well defined, since the mapping  $h \mapsto f(xh)\sigma(h)v$  is continuous with compact support.)

**Proposition E.1.1** *The mapping  $\xi_{f,v}$  belongs to  $\mathcal{A}$  and is left uniformly continuous for all  $f \in C_c(G)$  and  $v \in \mathcal{K}$ .*

**Proof** For  $x \in G$  and  $k \in H$ , we have

$$\begin{aligned} \xi_{f,v}(xk) &= \int_H f(xkh) \sigma(h) v dh = \int_H f(xh) \sigma(k^{-1}h) v dh \\ &= \sigma(k^{-1}) \int_H f(xh) \sigma(h) v dh = \sigma(k^{-1}) \xi_{f,v}(x). \end{aligned}$$

Let  $Q$  denote the support of  $f$ . It is clear that  $\xi_{f,v}$  vanishes outside  $QH$ . It remains to show that  $\xi_{f,v}$  is left uniformly continuous. Since  $\xi_{f,v}(xh) = \sigma(h^{-1})\xi_{f,v}(x)$  and since  $\xi_{f,v}$  vanishes outside  $QH$ , it suffices to show that  $\xi_{f,v}$  is uniformly continuous on  $Q$ . Without loss of generality, we can assume that  $v \neq 0$ .

Let  $\varepsilon > 0$ , and fix a compact neighbourhood  $U_0$  of  $e$  in  $G$ . Let  $|Q^{-1}U_0Q \cap H|$  denote the measure of the compact subset  $Q^{-1}U_0Q \cap H$  of  $H$ . Since  $f$  is left

uniformly continuous, there exists a neighbourhood  $U$  of  $e$  contained in  $U_0$ , with  $U^{-1} = U$ , and such that

$$\sup_{x \in G} |f(ux) - f(x)| < \frac{\varepsilon}{|Q^{-1}U_0Q \cap H| \|v\|}, \quad \text{for all } u \in U.$$

Then

$$\begin{aligned} \|\xi_{f,v}(ux) - \xi_{f,v}(x)\| &\leq \int_H \|f(uxh)\sigma(h)v - f(xh)\sigma(h)v\| dh \\ &= \|v\| \int_H |f(uxh) - f(xh)| dh \\ &\leq \|v\| |Q^{-1}U_0Q \cap H| \sup_{x \in G} |f(ux) - f(x)| < \varepsilon \end{aligned}$$

for all  $x \in Q$  and  $u \in U$ . ■

Let  $d\mu$  be a quasi-invariant regular Borel measure on  $G/H$ . We equip  $\mathcal{A}$  with an inner product as follows. Let  $\xi$  and  $\eta$  in  $\mathcal{A}$ . Observe that

$$\langle \xi(xh), \eta(xh) \rangle = \langle \sigma(h^{-1})\xi(x), \sigma(h^{-1})\eta(x) \rangle = \langle \xi(x), \eta(x) \rangle$$

for all  $x \in G$  and all  $h \in H$ . This shows that  $x \mapsto \langle \xi(x), \eta(x) \rangle$  is constant on the cosets of  $G$  modulo  $H$ , and hence can be viewed as a function on  $G/H$ . For  $\xi$  and  $\eta$  in  $\mathcal{A}$ , we define

$$\langle \xi, \eta \rangle = \int_{G/H} \langle \xi(x), \eta(x) \rangle d\mu(xH).$$

This integral is finite, since  $xH \mapsto \langle \xi(x), \eta(x) \rangle$  is continuous and has compact support in  $G/H$ . It is clear that  $(\xi, \eta) \mapsto \langle \xi, \eta \rangle$  is a positive hermitian form on  $\mathcal{A}$ . Moreover, it is definite, since the support of  $\mu$  is  $G/H$  (Proposition B.1.5). Let  $\mathcal{H}_\mu$  be the Hilbert space completion of  $\mathcal{A}$ .

**Remark E.1.2** As in the case of the usual  $L^2$ -spaces,  $\mathcal{H}_\mu$  can be identified with the space of all locally measurable mappings  $\xi : G \rightarrow \mathcal{K}$  such that:

- (i)  $\xi(xh) = \sigma(h^{-1})\xi(x)$  for all  $h \in H$  and for locally almost every  $x \in G$ ;
- (ii)  $\|\xi\|^2 = \int_{G/H} \|\xi(x)\|^2 d\mu(xH) < \infty$ ;

see [Gaal–73, Chapter VI]. Observe that, when  $G$  is  $\sigma$ -compact, “locally measurable” and “locally almost all  $x \in G$ ” can be replaced by “measurable” and “almost all  $x \in G$ ”.

For most arguments, it is however sufficient to consider the following total subspace of  $\mathcal{H}_\mu$ .

**Lemma E.1.3** *Let  $\mathcal{V}$  be a total set in  $\mathcal{K}$ . The set*

$$\{\xi_{f,v} : f \in C_c(G), v \in \mathcal{V}\}$$

*is total in  $\mathcal{H}_\mu$ .*

**Proof** It suffices to show that the linear span of

$$\{\xi_{f,v} : f \in C_c(G), v \in \mathcal{K}\}$$

is dense in  $\mathcal{A}$ .

Let  $\xi \in \mathcal{A}$ . By Lemma B.1.2, there exists  $\varphi \in C_c(G)$  such that  $T_H \varphi = 1$  on the compact set  $p(\text{supp } \xi)$ . Set  $\eta = \varphi \xi$ . Then  $\eta : G \rightarrow \mathcal{K}$  is continuous with compact support in  $G$ , and

$$(*) \quad \int_H \sigma(h) \eta(xh) dh = \int_H \varphi(xh) \sigma(h) \xi(xh) dh = \int_H \varphi(xh) \xi(x) dh = \xi(x),$$

for all  $x \in G$ .

Set  $Q = \text{supp } \eta$  and fix a compact neighbourhood  $K$  of  $Q$ . Let  $\varepsilon > 0$ . Since  $\eta$  is uniformly continuous, there exists a neighbourhood  $U$  of  $e$  in  $G$  such that, for all  $u$  in  $U$ ,

$$\sup_{x \in G} \|\eta(ux) - \eta(x)\| \leq \varepsilon.$$

Let  $x_1, \dots, x_n$  in  $Q$  be such that  $Q \subset \bigcup_{i=1}^n Ux_i$ . We can assume that  $Ux_i \subset K$  for every  $i$ . Choose a function  $f \in C_c(G)$  with  $0 \leq f \leq 1$ , with  $\text{supp } f \subset \bigcup_{i=1}^n Ux_i$ , and such that  $f = 1$  on  $Q$ . Choose  $f_i \in C_c(Ux_i)$  with  $0 \leq f_i \leq 1$  and  $\sum_{i=1}^n f_i = f$ , and set  $v_i = \eta(x_i) \in \mathcal{K}$ . Since  $\|\eta(x) - v_i\| \leq \varepsilon$  when  $f_i(x) > 0$ , we have

$$\begin{aligned} \left\| \eta(x) - \sum_{i=1}^n f_i(x) v_i \right\| &= \left\| \sum_{i=1}^n f_i(x) (\eta(x) - v_i) \right\| \\ &\leq \sum_{i=1}^n f_i(x) \|\eta(x) - v_i\| \\ &\leq \varepsilon \sum_{i=1}^n f_i(x) = \varepsilon f(x) \leq \varepsilon, \end{aligned}$$



for all  $x \in G$ . Since  $\eta$  and  $f$  have their supports in  $K$ , it follows from (\*) that

$$\begin{aligned} \left\| \xi - \sum_{i=1}^n \xi_{f_i, v_i} \right\|^2 &\leq \int_{p(K)} \left( \int_H \|\eta(xh) - \sum_{i=1}^n f_i(xh) v_i\| dh \right)^2 d\mu(xH) \\ &\leq \mu(p(K)) |K^{-1}K \cap H|^2 \varepsilon^2, \end{aligned}$$

and this completes the proof. ■

For every  $g$  in  $G$ , we define an operator  $\pi_\mu(g)$  on  $\mathcal{A}$  by

$$(**) \quad \pi_\mu(g)\xi(x) = \left( \frac{dg^{-1}\mu(xH)}{d\mu(xH)} \right)^{1/2} \xi(g^{-1}x), \quad \xi \in \mathcal{A}, x \in G.$$

Then  $\pi_\mu(g)$  preserves the inner product on  $\mathcal{A}$ , since

$$\begin{aligned} \langle \pi_\mu(g)\xi_1, \pi_\mu(g)\xi_2 \rangle &= \int_{G/H} \langle \xi_1(g^{-1}x), \xi_2(g^{-1}x) \rangle \frac{dg^{-1}\mu(xH)}{d\mu(xH)} d\mu(xH) \\ &= \int_{G/H} \langle \xi_1(x), \xi_2(x) \rangle d\mu(xH) = \langle \xi_1, \xi_2 \rangle. \end{aligned}$$

Hence,  $\pi_\mu(g)$  extends to a unitary operator on  $\mathcal{H}_\mu$ .

**Proposition E.1.4** *The assignment  $g \mapsto \pi_\mu(g)$  defines a unitary representation of  $G$  on  $\mathcal{H}_\mu$ .*

**Proof** The group law  $\pi_\mu(g_1g_2) = \pi_\mu(g_1)\pi_\mu(g_2)$  is a consequence of the cocycle identity for the Radon–Nikodym derivative; compare with Formula (\*) in Section A.6. It remains to show that the mapping  $g \mapsto \pi_\mu(g)\xi$  is continuous for all  $\xi \in \mathcal{H}_\mu$ , and indeed by Lemma E.1.3 for vectors of the form  $\xi_{f,v}$  (see Exercise A.8.2). But the continuity of  $g \mapsto \pi_\mu(g)\xi_{f,v}$  follows from the uniform continuity of  $\xi_{f,v}$  (Proposition E.1.1). ■

Next, we check that the equivalence class of  $\pi_\mu$  is independent of the choice of the quasi-invariant measure  $\mu$  on  $G/H$ .

**Proposition E.1.5** *Let  $\mu_1$  and  $\mu_2$  be quasi-invariant regular Borel measures on  $G/H$ . Then  $\pi_{\mu_1}$  and  $\pi_{\mu_2}$  are equivalent.*

**Proof** If  $\rho_1$  and  $\rho_2$  are the rho-functions associated to  $\mu_1$  and  $\mu_2$ , then  $d\mu_1/d\mu_2 = \rho_1/\rho_2$  (Theorem B.1.4). Define

$$\Phi : \mathcal{A} \rightarrow \mathcal{A}, \quad \xi \mapsto \sqrt{\frac{\rho_1}{\rho_2}} \xi.$$

Then  $\Phi$  is a linear bijection. Since

$$\int_{G/H} \frac{\rho_1}{\rho_2}(x) \|\xi(x)\|^2 d\mu_2(xH) = 3 \int_{G/H} \|\xi(x)\|^2 d\mu_1(xH),$$

$\Phi$  extends to a unitary operator from  $\mathcal{H}_{\mu_1}$  to  $\mathcal{H}_{\mu_2}$ . Moreover,

$$\begin{aligned} (\pi_{\mu_2}(g)\Phi\xi)(x) &= \left( \frac{dg^{-1}\mu_2(xH)}{d\mu_2(xH)} \frac{\rho_1(g^{-1}x)}{\rho_2(g^{-1}x)} \right)^{1/2} \xi(g^{-1}x) \\ &= \left( \frac{\rho_2(g^{-1}x)}{\rho_2(x)} \frac{\rho_1(g^{-1}x)}{\rho_2(g^{-1}x)} \right)^{1/2} \xi(g^{-1}x) \\ &= (\Phi\pi_{\mu_1}(g)\xi)(x). \end{aligned}$$

This shows that  $\Phi$  is a unitary equivalence between  $\pi_{\mu_1}$  and  $\pi_{\mu_2}$ . ■

**Definition E.1.6** The unitary representation  $\pi_\mu$  on  $\mathcal{H}_\mu$  defined above is called the representation of  $G$  *induced* by the representation  $\sigma$  of  $H$ , and is denoted by  $\text{Ind}_H^G \sigma$ .

**Remark E.1.7** In the case where  $G/H$  has an invariant measure  $\mu$ , the induced representation  $\text{Ind}_H^G \sigma$  is given on  $\mathcal{H}_\mu$  simply by left translations:

$$\text{Ind}_H^G \sigma(g)\xi(x) = \xi(g^{-1}x), \quad g, x \in G.$$

**Example E.1.8** (i) If  $H = \{e\}$  and  $\sigma = 1_H$ , then  $\text{Ind}_H^G \sigma$  is the left regular representation  $\lambda_G$  of  $G$ .

(ii) More generally, for any closed subgroup  $H$ , the representation  $\text{Ind}_H^G 1_H$  is the quasi-regular representation  $\lambda_{G/H}$  introduced in Definition B.1.9.

(iii) The group  $G = SL_2(\mathbf{R})$  acts on the real projective line  $\Omega = \mathbf{R} \cup \{\infty\}$ , with the Lebesgue measure as quasi-invariant measure (see Example A.6.4).

Fix  $t \in \mathbf{R}$ . The *principal series representation*  $\pi_{it}^\pm$  of  $G$  is the unitary representation on  $L^2(\Omega)$ , given by

$$\begin{aligned} \pi_{it}^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(\omega) &= |-c\omega + a|^{-1-it} f\left(\frac{d\omega - b}{-c\omega + a}\right) \\ \pi_{it}^- \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(\omega) &= \text{sgn}(-c\omega + a) |-c\omega + a|^{-1-it} f\left(\frac{d\omega - b}{-c\omega + a}\right). \end{aligned}$$

It is equivalent to  $\text{Ind}_P^G \chi_t^\pm$ , where  $P$  is the subgroup of all upper triangular matrices and  $\chi_t^\pm$  is the unitary character of  $P$  defined by

$$\chi_t^\pm \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \varepsilon^\pm(a) |a|^it,$$

with  $\varepsilon^+(a) = 1$  and  $\varepsilon^-(a) = \text{sgn}(a)$ ; see Exercise E.4.1.

The representations  $\pi_{it}^+$  and  $\pi_{it}^-$  are called the *spherical principal series representations* and the *non-spherical principal series representations*, respectively.

## E.2 Some properties of induced representations

The following proposition shows that inducing preserves unitary equivalence of representations.

**Proposition E.2.1** *Let  $(\sigma_1, \mathcal{K}_1)$  and  $(\sigma_2, \mathcal{K}_2)$  be equivalent representations of  $H$ . Then  $\text{Ind}_H^G \sigma_1$  and  $\text{Ind}_H^G \sigma_2$  are equivalent.*

**Proof** Let  $U : \mathcal{K}_1 \rightarrow \mathcal{K}_2$  be a unitary equivalence between  $\sigma_1$  and  $\sigma_2$ . Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be the Hilbert spaces of  $\text{Ind}_H^G \sigma_1$  and  $\text{Ind}_H^G \sigma_2$ . Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be the dense subspaces of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  introduced above. Define a linear bijection  $\tilde{U} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$  by

$$\tilde{U}\xi(x) = U(\xi(x)), \quad \xi \in \mathcal{A}_1, x \in G.$$

It can be checked that  $\tilde{U}$  is an isometry intertwining  $\text{Ind}_H^G \sigma_1$  and  $\text{Ind}_H^G \sigma_2$ , so that  $\tilde{U}$  extends to a unitary equivalence between  $\text{Ind}_H^G \sigma_1$  and  $\text{Ind}_H^G \sigma_2$ . ■

**Proposition E.2.2** *Let  $(\sigma_i, \mathcal{K}_i)_i$  be a family of unitary representations of  $H$ . Then  $\text{Ind}_H^G (\bigoplus_i \sigma_i)$  is equivalent to  $\bigoplus_i \text{Ind}_H^G \sigma_i$ .*

**Proof** Let  $\mathcal{H}$  be the space of  $\text{Ind}_H^G (\bigoplus_i \sigma_i)$  and  $\mathcal{H}_i$  the space of  $\text{Ind}_H^G \sigma_i$ . It is easily seen that there is a unitary bijective intertwining operator between the dense subspace  $\mathcal{A}$  of  $\mathcal{H}$  (defined as above) and the direct sum  $\bigoplus_i \mathcal{A}_i$  of the dense subspaces  $\mathcal{A}_i$  of the  $\mathcal{H}_i$ 's. As  $\bigoplus_i \mathcal{A}_i$  is dense in  $\bigoplus_i \mathcal{H}_i$  the claim follows. ■

**Corollary E.2.3** *Let  $(\sigma, \mathcal{K})$  be a unitary representation of  $H$ . If  $\text{Ind}_H^G \sigma$  is irreducible, then  $\sigma$  is irreducible.*

The following fundamental result has a proof which rests on simple principles, but which is technically involved, and we will only sketch it; details are left to the reader and can be found in [Folla–95] or [Gaal–73].

**Theorem E.2.4 (Induction by stages)** *Let  $H$  and  $K$  be closed subgroups of  $G$  with  $K \subset H$ , and let  $\tau$  be a unitary representation of  $K$ . Then  $\text{Ind}_H^G(\text{Ind}_K^H \tau)$  is equivalent to  $\text{Ind}_K^G \tau$ .*

**Proof** Let  $\mathcal{L}_\tau$  be the space of  $\tau$ . Denote by  $\mathcal{H}$  and  $\mathcal{H}'$  the spaces of  $\text{Ind}_K^G \tau$  and  $\text{Ind}_H^G(\text{Ind}_K^H \tau)$ , and by  $\mathcal{K}$  the space of  $\text{Ind}_K^H \tau$ .

Let  $\rho_1$  and  $\rho_2$  be two rho-functions on  $G$  defining the quasi-invariant measures  $\mu_1$  and  $\mu_2$  on  $G/K$  and  $G/H$ . Then  $\rho = (\rho_1/\rho_2)|_H$  is a rho-function for the pair  $(H, K)$ . Let  $\mu$  be the quasi-invariant measure on  $H/K$  defined by  $\rho$ .

Let  $\mathcal{A}$  be the dense subspace of  $\mathcal{H}$  introduced at the beginning of Section E.1. For  $\xi \in \mathcal{A}$  and  $x \in G$ , define a mapping  $\varphi(\xi, x) : H \rightarrow \mathcal{L}_\tau$  by

$$\varphi(\xi, x)(h) = \left( \frac{dx\mu(h)}{d\mu(h)} \right)^{1/2} \xi(xh), \quad h \in H.$$

It is easily verified that  $\varphi(\xi, x)$  belongs to  $\mathcal{K}$ , the space of  $\text{Ind}_K^H \tau$ .

Moreover, for every  $\xi$  in  $\mathcal{A}$ , the mapping

$$\Phi_\xi : G \rightarrow \mathcal{K}, \quad x \mapsto \varphi(\xi, x)$$

belongs to  $\mathcal{H}'$ , the Hilbert space of  $\text{Ind}_H^G(\text{Ind}_K^H \sigma)$ . Finally, the mapping

$$\mathcal{A} \rightarrow \mathcal{H}', \quad \xi \mapsto \Phi_\xi$$

extends to an isometry between  $\mathcal{H}$  and  $\mathcal{H}'$  which intertwines  $\text{Ind}_K^G \tau$  and  $\text{Ind}_H^G(\text{Ind}_K^H \sigma)$ . ■

The following formula shows that the tensor product of an arbitrary representation and an induced representation is again an induced representation.

**Proposition E.2.5** *Let  $(\pi, \mathcal{H}_\pi)$  be a unitary representation of  $G$  and  $(\sigma, \mathcal{K}_\sigma)$  a unitary representation of  $H$ . The representation  $\pi \otimes \text{Ind}_H^G \sigma$  is equivalent to  $\text{Ind}_H^G((\pi|_H) \otimes \sigma)$ .*

**Proof** Let  $\mathcal{H}$  and  $\mathcal{L}$  be the Hilbert spaces of  $\text{Ind}_H^G \sigma$  and  $\text{Ind}_H^G((\pi|_H) \otimes \sigma)$ , respectively. We define a linear mapping  $U : \mathcal{H}_\pi \otimes \mathcal{H} \rightarrow \mathcal{L}$  by

$$U(\theta \otimes \xi)(x) = \pi(x^{-1})\theta \otimes \xi(x), \quad x \in G,$$

and check that  $U$  is a bijective isometry from  $\mathcal{H}_\pi \otimes \mathcal{H}$  onto  $\mathcal{L}$ . Moreover, for all  $g$  in  $G$ ,

$$\begin{aligned} & \left( \left( \text{Ind}_H^G((\pi|_H) \otimes \sigma)(g) \right) (U(\theta \otimes \xi)) \right) (x) \\ &= \left( \frac{dg^{-1}\mu(xH)}{d\mu(xH)} \right)^{1/2} \pi(x^{-1}g)\theta \otimes \xi(g^{-1}x) = U \left( \pi(g)\theta \otimes (\text{Ind}_H^G \sigma)(g)\xi \right) (x), \end{aligned}$$

so that  $U$  intertwines  $\text{Ind}_H^G((\pi|_H) \otimes \sigma)$  and  $\pi \otimes \text{Ind}_H^G \sigma$ . ■

**Corollary E.2.6** (i) Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . The representation  $\text{Ind}_H^G(\pi|_H)$  is equivalent to  $\pi \otimes \lambda_{G/H}$ , where  $\lambda_{G/H}$  is the quasi-regular representation of  $G$  on  $L^2(G/H)$ .

(ii) In particular,  $\pi \otimes \lambda_G$  is equivalent to  $(\dim \pi)\lambda_G$ .

### E.3 Induced representations with invariant vectors

The following important theorem is a characterisation of induced representations which have non-zero invariant vectors. The result is used in the proof of Theorem 1.7.1.

**Theorem E.3.1** Let  $G$  be a  $\sigma$ -compact locally compact group and  $H$  a closed subgroup of  $G$ . Let  $(\sigma, \mathcal{K})$  be a unitary representation of  $H$ . The following properties are equivalent:

- (i) the unit representation  $1_G$  of  $G$  is contained in  $\text{Ind}_H^G \sigma$ , that is, the Hilbert space  $\mathcal{H}$  of  $\text{Ind}_H^G \sigma$  contains a non-zero invariant vector;
- (ii) the space  $G/H$  has a finite invariant regular Borel measure and  $1_H$  is contained in  $\sigma$ .

**Proof** We first show that (ii) implies (i), which is the easier implication. By assumption,  $1_H$  is contained in  $\sigma$ . Hence, by Proposition E.2.2, the quasi-regular representation  $\text{Ind}_H^G 1_H$  is contained in  $\text{Ind}_H^G \sigma$ . Since  $G/H$  has finite measure, the constant function 1 on  $G/H$  belongs to  $L^2(G/H)$ , the Hilbert space of  $\text{Ind}_H^G 1_H$ , and is invariant.

To show that (i) implies (ii), we realise the Hilbert space  $\mathcal{H}$  of  $\pi = \text{Ind}_H^G \sigma$  as a space of measurable mappings from  $G$  to  $\mathcal{K}$  (see Remark E.1.2).

Let  $\xi$  be a non-zero invariant vector in  $\mathcal{H}$ . Thus,  $\xi : G \rightarrow \mathcal{K}$  is a measurable non-zero mapping such that, for every  $h \in H$ ,

$$\xi(xh) = \sigma(h^{-1})\xi(x)$$

for almost every  $x \in G$ , and such that

$$\int_{G/H} \|\xi(x)\|^2 d\mu(xH) < \infty.$$

Since  $\xi$  is invariant, we have, for every  $g \in G$  and for almost every  $x \in G$ ,

$$(*) \quad \left( \frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x) = \xi(x),$$

where  $\rho$  is a rho-function defining the quasi-invariant measure  $\mu$  on  $G/H$ , which we assume to be normalised by  $\rho(e) = 1$ .

We claim that we can modify  $\xi$  on a subset of measure zero such that  $(*)$  holds for every  $g \in G$  and every  $x \in G$ . Indeed, let

$$Q = \left\{ (g, x) \in G \times G : \left( \frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x) \neq \xi(x) \right\}.$$

For every  $n \in \mathbf{N}$ , the set

$$Q_n = \left\{ (g, x) \in G \times G : \left\| \left( \frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x) - \xi(x) \right\| \geq 1/n \right\},$$

is measurable. Hence,  $Q = \bigcup_{n \in \mathbf{N}} Q_n$  is measurable. Since  $G$  is  $\sigma$ -compact, Fubini's Theorem applies and shows that  $Q$  has  $(\lambda \times \lambda)$ -measure zero, where  $\lambda$  denotes a Haar measure on  $G$ . Let  $X$  be the set of all  $x \in G$  such that

$$\lambda \left( \left\{ g \in G : \left( \frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x) \neq \xi(x) \right\} \right) = 0.$$

By Fubini's Theorem again,  $X$  is measurable and  $\lambda(G \setminus X) = 0$ . Let  $x, y \in X$ . By definition of  $X$ , there exist subsets  $N_x$  and  $N_y$  of measure zero such that

$$\rho(z)^{1/2} \xi(z) = \rho(x)^{1/2} \xi(x), \quad \text{for all } z \in G \setminus N_x$$

and

$$\rho(z)^{1/2} \xi(z) = \rho(y)^{1/2} \xi(y), \quad \text{for all } z \in G \setminus N_y.$$

It follows that  $\rho(x)^{1/2} \xi(x) = \rho(y)^{1/2} \xi(y)$ . This shows that the mapping  $x \mapsto \rho(x)^{1/2} \xi(x)$  is constant on  $X$  and proves the claim.

We can therefore assume that Equation (\*) holds for every  $g \in G$  and every  $x \in G$ . In particular,  $\xi(x) \neq 0$  for every  $x \in G$ , since  $\xi \neq 0$ . Choosing  $h \in H$  and  $x = e$ , we now obtain

$$\rho(h^{-1})^{1/2} \sigma(h) \xi(e) = \xi(e)$$

and, hence,

$$\rho(h^{-1}) \|\xi(e)\|^2 = \|\xi(e)\|^2 \quad \text{for all } h \in H.$$

It follows that  $\rho = 1$  on  $H$ . On the other hand, we have

$$\rho(h) = \frac{\Delta_H(h)}{\Delta_G(h)} \quad \text{for all } h \in H,$$

by the functional equation of  $\rho$ . Hence,  $\Delta_G|_H = \Delta_H$ . Corollary B.1.7 shows that  $\mu$  is an invariant measure, and we can assume that  $\rho = 1$  everywhere. Then, by Equation (\*) again,  $\xi(x) = \xi(e) \neq 0$  for all  $x \in G$ . Hence,

$$\|\xi(e)\|^2 \int_{G/H} d\mu(xH) = \int_{G/H} \|\xi(x)\|^2 d\mu(xH) < \infty$$

and

$$\sigma(h) \xi(e) = \xi(h^{-1}) = \xi(e),$$

that is,  $\mu$  is finite and  $\xi(e)$  is an invariant non-zero vector in  $\mathcal{K}$ . ■

It is worth stating the above result for the case where  $H$  is a lattice in  $G$ .

**Corollary E.3.2** *Let  $\Gamma$  be a lattice in the  $\sigma$ -compact locally compact group  $G$ .*

- (i) *The quasi-regular representation  $\text{Ind}_\Gamma^G 1_\Gamma$  contains the unit representation  $1_G$ .*
- (ii) *Let  $\sigma$  be a unitary representation of  $\Gamma$ . If  $\text{Ind}_\Gamma^G \sigma$  contains  $1_G$ , then  $\sigma$  contains  $1_\Gamma$ .*

## E.4 Exercises

**Exercise E.4.1** Let  $\pi_t^\pm$  be the principal series representation of  $G = SL_2(\mathbf{R})$  defined in Example E.1.8. Prove, with the notation there, that  $\pi_t^\pm$  is equivalent to  $\text{Ind}_P^G \chi_t^\pm$ .

[Hint: If necessary, see [Knapp-86, Chapter VII, p.167–168].]

**Exercise E.4.2** Let  $H$  be a closed subgroup of a locally compact group  $G$ . Let  $(\sigma, \mathcal{K})$  be a unitary representation of  $H$ . Prove that  $\text{Ind}_H^G \bar{\sigma}$  is equivalent to  $\overline{\text{Ind}_H^G \sigma}$ .

**Exercise E.4.3** For  $i = 1, 2$ , let  $H_i$  be a closed subgroup of a locally compact group  $G_i$  and let  $\sigma_i$  be a unitary representation of  $H_i$ . Let  $\sigma_1 \times \sigma_2$  be their outer tensor product. Prove that  $\text{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\sigma_1 \times \sigma_2)$  is equivalent to  $\text{Ind}_{H_1}^{G_1} \sigma_1 \times \text{Ind}_{H_2}^{G_2} \sigma_2$ .

**Exercise E.4.4** Let  $G$  be a locally compact group, and let  $K$  be a compact subgroup of  $G$ . Let  $\sigma$  be an irreducible unitary representation of  $K$ . Show that  $\text{Ind}_K^G \sigma$  is strongly contained in the regular representation  $\lambda_G$ .

[Hint: Observe that  $\lambda_G$  is equivalent to  $\text{Ind}_K^G(\lambda_K)$ .]

**Exercise E.4.5** Let  $\Gamma$  be a discrete group and let  $K$  be a compact abelian group. For each  $\gamma \in \Gamma$ , set  $K_\gamma = K$  and let  $G = \prod_{\gamma \in \Gamma} K_\gamma$  be the direct product, equipped with the product topology. Then  $\Gamma$  acts by continuous automorphisms of  $G$  via shifting on the left:

$$\gamma'(k_\gamma)_{\gamma \in \Gamma} = (k_{\gamma'^{-1}\gamma})_{\gamma \in \Gamma}, \quad \gamma' \in \Gamma.$$

Let  $\pi$  be the unitary representation of  $\Gamma$  on  $L^2(G)$  defined as in Exercise D.5.7.

(i) Show that  $\pi$  is unitarily equivalent to a direct sum  $1_\Gamma \oplus \bigoplus_{i \in I} \pi_i$ , where each  $\pi_i$  is the quasi-regular representation  $\lambda_{\Gamma/\Gamma_i}$  corresponding to a finite subgroup  $\Gamma_i$  of  $\Gamma$ .

[Hint: By Exercise D.5.6,  $\widehat{G}$  can be identified with the direct sum  $\bigoplus_{\gamma \in \Gamma} \widehat{K}_\gamma$ , where  $\widehat{K}_\gamma = \widehat{K}$  for all  $\gamma \in \Gamma$  and, by Exercise D.5.7,  $\pi$  is equivalent to the representation of  $\Gamma$  on  $\ell^2(\widehat{G})$  defined by duality. Show that the stabilizer of  $\chi = \bigoplus_{\gamma \in \Gamma} \chi_\gamma \in \widehat{G}$  with  $\chi \neq 1$  has finite index in  $\Gamma$ .]

(ii) Let  $\pi_0$  be the unitary representation of  $\Gamma$  defined by restriction of  $\pi$  to the invariant subspace  $L_0^2(G) = \{f \in L^2(G) : \int_G f(x) dx = 0\}$  (the orthogonal complement to the constants). Deduce from (i) that  $\pi_0$  is equivalent to a subrepresentation of a multiple of the regular representation  $\lambda_\Gamma$ . (In particular,  $\pi_0$  is weakly contained in  $\lambda_\Gamma$ , in the sense of Definition F.1.1.)

[Hint: Exercise E.4.4.]



# Appendix F

## Weak containment and Fell's topology

The notion of subrepresentation, which is convenient for comparing representations of compact groups, is far too rigid for other groups. The appropriate notion for locally compact groups is that of *weak containment*, introduced by Godement [Godem–48] and then much developed by Fell, both for locally compact groups and for  $C^*$ -algebras (see [Fell–60], [Fell–62], and [Dixmi–69]).

Weak containment is defined in Section F.1. In particular, consider two unitary representations  $\pi, \rho$  of a locally compact group  $G$ , with  $\pi$  irreducible; if  $\pi$  is a subrepresentation of  $\rho$ , then  $\pi$  is weakly contained in  $\rho$ ; the converse holds when  $G$  is compact, but not in general.

In Section F.2, we analyse the notion of weak containment in the context of *Fell's topology*. In particular, the usual operations (direct sums, tensor products, restrictions, inductions) are shown to be continuous for this topology in Section F.3. Section F.4 expresses weak containment in terms of group  $C^*$ -algebras. In Section F.5, we discuss briefly direct integral decompositions of unitary representations.

### F.1 Weak containment of unitary representations

If  $(\pi, \mathcal{H})$  is a representation of a topological group, recall that the diagonal matrix coefficients  $\langle \pi(\cdot)\xi, \xi \rangle$  for  $\xi$  in  $\mathcal{H}$  are called the functions of positive type associated to  $\pi$ ; see Proposition C.4.3.

**Definition F.1.1** Let  $(\pi, \mathcal{H})$  and  $(\rho, \mathcal{K})$  be unitary representations of the topological group  $G$ . We say that  $\pi$  is *weakly contained* in  $\rho$  if every function of positive type associated to  $\pi$  can be approximated, uniformly on compact subsets of  $G$ , by finite sums of functions of positive type associated to  $\rho$ . This means: for every  $\xi$  in  $\mathcal{H}$ , every compact subset  $Q$  of  $G$ , and every  $\varepsilon > 0$ , there

exist  $\eta_1, \dots, \eta_n$  in  $\mathcal{K}$  such that, for all  $x \in Q$ ,

$$\left| \langle \pi(x)\xi, \xi \rangle - \sum_{i=1}^n \langle \rho(x)\eta_i, \eta_i \rangle \right| < \varepsilon.$$

We write for this  $\pi < \rho$ .

If  $\pi < \rho$  and  $\rho < \pi$ , we say that  $\pi$  and  $\rho$  are *weakly equivalent* and denote this by  $\pi \sim \rho$ .

**Remark F.1.2** (i) For  $\pi < \rho$ , it clearly suffices that the condition stated in the above definition holds for any *normalised* function of positive type  $\varphi = \langle \pi(\cdot)\xi, \xi \rangle$  associated to  $\pi$  (that is, with  $\varphi(e) = \|\xi\|^2 = 1$ ).

On the other hand, if this condition holds for a normalised function of positive type  $\varphi = \langle \pi(\cdot)\xi, \xi \rangle$ , then  $\varphi$  can be approximated, uniformly on compact subsets of  $G$ , by *convex combinations* of normalised functions of positive type associated to  $\rho$ . Indeed, let  $\sum_{i=1}^n \psi_i$  be close to  $\varphi$ , uniformly on a compact set  $Q$ . We can assume that  $e \in Q$ . Hence,  $\sum_{i=1}^n \psi_i(e)$  is close to  $\varphi(e) = 1$ . This implies that the convex sum

$$\sum_{i=1}^n \frac{\psi_i(e)}{\psi_1(e) + \dots + \psi_n(e)} \left( \frac{\psi_i}{\psi_i(e)} \right)$$

is close to  $\varphi$ , uniformly on  $Q$ .

(ii) The relation  $\pi < \rho$  depends only on the equivalence classes of the representations  $\pi$  and  $\rho$ .

(iii) If  $\pi$  is contained in  $\rho$ , then  $\pi < \rho$ . Indeed, in this case, every function of positive type associated to  $\pi$  is a function of positive type associated to  $\rho$ .

(iv) For unitary representations  $\pi$ ,  $\rho$ , and  $\sigma$  of  $G$ , the weak containments  $\pi < \rho$  and  $\rho < \sigma$  imply  $\pi < \sigma$ .

(v) For unitary representations  $\pi$  and  $\rho$  and cardinal numbers  $m, n > 0$ , we have  $\pi < \rho$  if and only if  $m\pi < n\rho$ . Thus, the weak containment relation does not take multiplicities into account.

(vi) If  $\pi$  is finite dimensional and if  $\rho$  is irreducible, then  $\rho < \pi$  implies that  $\rho$  is contained in  $\pi$  (Corollary F.2.9 below). For this reason, the notion of weak containment is not relevant for finite dimensional representations.

(vii) For unitary representations  $(\pi)_{i \in I}$  and  $\rho$  of  $G$ , we have  $\bigoplus_{i \in I} \pi_i < \rho$  if and only if  $\pi_i < \rho$  for all  $i \in I$ . See also Proposition F.2.7 below.

(viii) The notion of weak containment is best understood in terms of representations of the  $C^*$ -algebra of  $G$ , for which it appears in a natural way. For this, see Section F.4.

(ix) Definition 7.3.5 of [Zimm–84a] offers a *different* definition of weak containment. Let  $(\pi, \mathcal{H})$  and  $(\rho, \mathcal{K})$  be unitary representations of the topological group  $G$ . To a finite orthonormal set  $\xi_1, \dots, \xi_n$  of  $\mathcal{H}$  is associated a continuous function

$$\Phi_{(\xi_1, \dots, \xi_n)} : G \rightarrow M_n(\mathbb{C}), \quad g \mapsto (\langle \pi(g)\xi_i, \xi_j \rangle)_{1 \leq i, j \leq n}$$

which is called a *n-by-n submatrix of  $\pi$* . Define then  $\pi$  to be *weakly contained in  $\rho$  in the sense of Zimmer*, and write  $\pi \prec_Z \rho$ , if any submatrix of  $\pi$  is a limit of submatrices of  $\rho$ , uniformly on compact subsets of  $G$ .

If  $\pi$  and  $\rho$  are finite dimensional, then  $\pi \prec_Z \rho$  if and only if  $\pi$  is contained in  $\rho$ ; in particular,  $\rho \oplus \rho$  is not weakly contained in  $\rho$  in the sense of Zimmer, whereas  $\rho \oplus \rho \prec \rho$ . In general, it can be shown that  $\pi \prec \rho$  if and only if  $\pi \prec_Z \infty \rho$ ; in case  $\pi$  is irreducible, then  $\pi \prec \rho$  if and only if  $\pi \prec_Z \rho$ .

The following lemma, due to Fell [Fell–63], will be used several times (proofs of Propositions F.1.10, F.3.2 and of Theorem F.3.5). Its proof is surprisingly less elementary than one might first expect.

**Lemma F.1.3** *Let  $(\pi, \mathcal{H})$  and  $(\rho, \mathcal{K})$  be unitary representations of the topological group  $G$ . Let  $\mathcal{V}$  be a subset of  $\mathcal{H}$  such that  $\{\pi(x)\xi : x \in G, \xi \in \mathcal{V}\}$  is total in  $\mathcal{H}$ . The following are equivalent:*

(i)  $\pi \prec \rho$ ;

(ii) *every function of positive type of the form  $\langle \pi(\cdot)\xi, \xi \rangle$  with  $\xi \in \mathcal{V}$  can be approximated, uniformly on compact subsets of  $G$ , by finite sums of functions of positive type associated to  $\rho$ .*

**Proof** We only have to show that (ii) implies (i).

Let  $\mathcal{X}$  be the set of all vectors  $\xi$  in  $\mathcal{H}$  such that the associated function of positive type  $\langle \pi(\cdot)\xi, \xi \rangle$  can be approximated, uniformly on compact subsets of  $G$ , by finite sums of functions of positive type associated to  $\rho$ . We have to show that  $\mathcal{X} = \mathcal{H}$ . This will be done in several steps.

- $\mathcal{X}$  is  $G$ -invariant and closed under scalar multiplication. This is obvious.

- For  $\xi \in \mathcal{X}$  and  $g_1, g_2 \in G$ , we have  $\pi(g_1)\xi + \pi(g_2)\xi \in \mathcal{X}$ . Indeed, set  $\varphi(x) = \langle \pi(x)\xi, \xi \rangle$  and  $\psi(x) = \langle \pi(x)(\pi(g_1)\xi + \pi(g_2)\xi), (\pi(g_1)\xi + \pi(g_2)\xi) \rangle$ . Then

$$\psi = g_1^{-1}\varphi_{g_1} + g_2^{-1}\varphi_{g_2} + g_1^{-1}\varphi_{g_2} + g_2^{-1}\varphi_{g_1}.$$

The claim now follows.

- $\mathcal{X}$  is closed in  $\mathcal{H}$ . Indeed, let  $\xi$  be in the closure of  $\mathcal{X}$ . We can assume that  $\|\xi\| = 1$ . Let  $Q$  be a compact subset of  $G$  and  $\varepsilon > 0$ . Choose  $\xi' \in \mathcal{X}$  with

$\|\xi - \xi'\| < \varepsilon$ . Then, for all  $x$  in  $G$ ,

$$\begin{aligned} |\langle \pi(x)\xi, \xi \rangle - \langle \pi(x)\xi', \xi' \rangle| &\leq |\langle \pi(x)\xi, \xi - \xi' \rangle| + |\langle \pi(x)(\xi - \xi'), \xi' \rangle| \\ &\leq \|\xi\| \|\xi - \xi'\| + \|\xi'\| \|\xi - \xi'\| < (2 + \varepsilon)\varepsilon. \end{aligned}$$

As  $\xi' \in \mathcal{X}$ , there exists  $\eta_1, \dots, \eta_n$  in  $\mathcal{K}$  such that

$$\sup_{x \in Q} \left| \langle \pi(x)\xi', \xi' \rangle - \sum_{i=1}^n \langle \rho(x)\eta_i, \eta_i \rangle \right| < \varepsilon.$$

Then, for every  $x \in Q$ , we have

$$\begin{aligned} \left| \langle \pi(x)\xi, \xi \rangle - \sum_{i=1}^n \langle \rho(x)\eta_i, \eta_i \rangle \right| &\leq |\langle \pi(x)\xi, \xi \rangle - \langle \pi(x)\xi', \xi' \rangle| \\ &\quad + \left| \langle \pi(x)\xi', \xi' \rangle - \sum_{i=1}^n \langle \rho(x)\eta_i, \eta_i \rangle \right| \\ &< (3 + \varepsilon)\varepsilon. \end{aligned}$$

•  $\mathcal{X}$  is closed under addition. Indeed, let  $\xi_1$  and  $\xi_2$  be elements in  $\mathcal{X}$ . Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be the closed invariant subspaces of  $\mathcal{H}$  generated by  $\xi_1$  and  $\xi_2$ . By the previous steps,  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are contained in  $\mathcal{X}$ . Let  $\mathcal{L}$  denote the closure of  $\mathcal{H}_1 + \mathcal{H}_2$  in  $\mathcal{H}$ . Observe that  $\mathcal{L}$  is also an invariant subspace of  $\mathcal{H}$ . Let

$$P : \mathcal{H}_2 \rightarrow \mathcal{H}_1^\perp$$

be the orthogonal projection from  $\mathcal{H}_2$  to the orthogonal complement  $\mathcal{H}_1^\perp$  of  $\mathcal{H}_1$  in  $\mathcal{L}$ . It is clear that  $P(\mathcal{H}_2)$  is dense in  $\mathcal{H}_1^\perp$ . As  $\mathcal{H}_1^\perp$  is invariant,  $P$  intertwines  $\pi$  with itself. Hence, by Proposition A.1.4, the restriction of  $\pi$  to  $\mathcal{H}_1^\perp$  is equivalent to the restriction of  $\pi$  to the orthogonal complement of  $\text{Ker } P$  in  $\mathcal{H}_2$ .

As  $\mathcal{H}_2$  is contained in  $\mathcal{X}$ , it follows that  $\mathcal{H}_1^\perp$  is also contained in  $\mathcal{X}$ . Now, let

$$\xi'_1 = P(\xi_1 + \xi_2) \in \mathcal{H}_1^\perp \quad \text{and} \quad \xi'_2 = (I - P)(\xi_1 + \xi_2) \in \mathcal{H}_1.$$

Then  $\xi'_1$  and  $\xi'_2$  are in  $\mathcal{X}$ . On the other hand, for every  $x \in G$ , we have

$$\langle \pi(x)(\xi_1 + \xi_2), (\xi_1 + \xi_2) \rangle = \langle \pi(x)\xi'_1, \xi'_1 \rangle + \langle \pi(x)\xi'_2, \xi'_2 \rangle.$$

This shows that  $\xi_1 + \xi_2$  is in  $\mathcal{X}$ .

• By the previous steps,  $\mathcal{X}$  is a closed subspace of  $\mathcal{H}$ . Since  $\mathcal{X}$  contains the total set  $\{\pi(x)\xi : x \in G, \xi \in \mathcal{V}\}$ , it follows that  $\mathcal{X} = \mathcal{H}$ . ■

In the important case where an irreducible representation  $\pi$  is weakly contained in  $\rho$ , we can approximate the functions of positive type associated to  $\pi$  by functions of positive type associated to  $\rho$ , and not just by sums of such functions.

**Proposition F.1.4** *Let  $(\pi, \mathcal{H})$  and  $(\rho, \mathcal{K})$  be unitary representations of the locally compact group  $G$  such that  $\pi \prec \rho$ . Assume that  $\pi$  is irreducible. Let  $\xi$  be a unit vector in  $\mathcal{H}$ . Then  $\langle \pi(\cdot)\xi, \xi \rangle$  can be approximated, uniformly on compact subsets of  $G$ , by normalised functions of positive type associated to  $\rho$ .*

**Proof** Let  $\mathcal{F}$  be the set of normalised functions of positive type on  $G$  which are associated to  $\rho$ . Let  $\mathcal{C}$  be the closure of the convex hull of  $\mathcal{F}$  in the weak\* topology of  $L^\infty(G)$ . Then  $\mathcal{C}$  is a compact convex set and, by assumption,  $\varphi = \langle \pi(\cdot)\xi, \xi \rangle$  is in  $\mathcal{C}$ .

On the other hand, since  $\pi$  is irreducible,  $\varphi \in \text{ext}(\mathcal{P}_{\leq 1}(G))$  by Theorem C.5.2 and, hence,  $\varphi \in \text{ext}(\mathcal{C})$ . It is then a standard fact that  $\varphi$  is in the weak\* closure of  $\mathcal{F}$  in  $\mathcal{C}$  (see, e.g., [Conwa–87, Theorem V.7.8]). By Raikov’s Theorem C.5.6, it follows that  $\varphi$  is in the closure of  $\mathcal{F}$  with respect to the topology of uniform convergence on compact subsets of  $G$ . ■

The case of the unit representation  $1_G$  deserves special attention.

**Corollary F.1.5** *Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Then  $1_G \prec \pi$  if and only if, for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists a unit vector  $\xi$  in  $\mathcal{H}$  such that*

$$\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon.$$

**Proof** For  $x \in G$  and a unit vector  $\xi \in \mathcal{H}$ , we have

$$\begin{aligned} (1) \quad & \|\pi(x)\xi - \xi\|^2 = 2(1 - \text{Re}\langle \pi(x)\xi, \xi \rangle) \leq 2|1 - \langle \pi(x)\xi, \xi \rangle| \\ (2) \quad & |1 - \langle \pi(x)\xi, \xi \rangle|^2 \leq 2(1 - \text{Re}\langle \pi(x)\xi, \xi \rangle) = \|\pi(x)\xi - \xi\|^2. \end{aligned}$$

(The inequality in (2) holds for any complex number of modulus at most 1 instead of  $\langle \pi(x)\xi, \xi \rangle$ .)

Assume that  $1_G \prec \pi$ . Let  $Q$  and  $\varepsilon$  be as above. By Proposition F.1.4, there exists a unit vector  $\xi \in \mathcal{H}$  such that  $\sup_{x \in Q} |1 - \langle \pi(x)\xi, \xi \rangle| < \varepsilon^2/2$ . Hence,  $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon$  by (1).

Conversely, if  $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon$ , then  $1_G \prec \pi$  by (2). ■

**Remark F.1.6** Using the terminology introduced in Definition 1.1.1, we see that  $1_G \prec \pi$  if and only if  $\pi$  almost has invariant vectors.

Recall that a locally compact group  $G$  is *compactly generated* if  $G$  is generated by a compact subset, that is, if there exists a compact subset  $Q$  of  $G$  such that  $G = \bigcup_{n \in \mathbf{N}} Q^n$ .

**Proposition F.1.7** *Let  $G$  be a compactly generated locally compact group, with compact generating subset  $Q$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . Then  $1_G \prec \pi$  if and only if, for every  $\varepsilon > 0$ , there exists a unit vector  $\xi$  in  $\mathcal{H}$  such that,*

$$(*) \quad \sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon.$$

**Proof** The “only if” part being clear, assume that, for every  $\varepsilon > 0$ , there exists a unit vector  $\xi_\varepsilon$  in  $\mathcal{H}$  such that  $(*)$  holds for all  $x \in Q$ . Then for every  $x$  in  $Q$ ,

$$\|\pi(x^{-1})\xi_\varepsilon - \xi_\varepsilon\| = \|\xi_\varepsilon - \pi(x)\xi_\varepsilon\| < \varepsilon.$$

Let  $n \in \mathbf{N}$  and  $x_1, \dots, x_n \in Q \cup Q^{-1}$ . Then

$$\begin{aligned} & \|\pi(x_1 \cdots x_n)\xi_{\varepsilon/n} - \xi_{\varepsilon/n}\| \\ & \leq \|\pi(x_1 \cdots x_n)\xi_{\varepsilon/n} - \pi(x_1 \cdots x_{n-1})\xi_{\varepsilon/n}\| \\ & \quad + \|\pi(x_1 \cdots x_{n-1})\xi_{\varepsilon/n} - \pi(x_1 \cdots x_{n-2})\xi_{\varepsilon/n}\| + \cdots + \|\pi(x_1)\xi_{\varepsilon/n} - \xi_{\varepsilon/n}\| \\ & = \|\pi(x_n)\xi_{\varepsilon/n} - \xi_{\varepsilon/n}\| + \|\pi(x_{n-1})\xi_{\varepsilon/n} - \xi_{\varepsilon/n}\| + \cdots + \|\pi(x_1)\xi_{\varepsilon/n} - \xi_{\varepsilon/n}\| \\ & < n \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

Therefore, for every  $n \in \mathbf{N}$ , there exists a unit vector  $\xi$  such that  $(*)$  holds for all  $x \in \tilde{Q}^n$ , where  $\tilde{Q} = Q \cup Q^{-1} \cup \{e\}$ . Next, observe that, since  $G = \bigcup_{n \in \mathbf{N}} \tilde{Q}^n$  is locally compact,  $\tilde{Q}^n$  contains a neighbourhood of  $e$  for some  $n$ , by Baire's Category Theorem. This implies that every compact subset is contained in  $\tilde{Q}^m$  for some  $m \in \mathbf{N}$ . Therefore  $1_G \prec \pi$ . ■

As the following proposition shows, the relation of weak containment essentially coincides with the relation of containment in the case of compact groups.

**Proposition F.1.8** *Let  $G$  be a compact group, and let  $\pi$  and  $\rho$  be unitary representations of  $G$ . Then  $\pi \prec \rho$  if and only if every irreducible subrepresentation of  $\pi$  is contained in  $\rho$ .*

**Proof** Assume that  $\pi \prec \rho$ . Since  $\pi$  can be decomposed as a direct sum of irreducible representations, we can assume that  $\pi$  is irreducible. Let  $\varphi$  be a normalised function of positive type associated to  $\pi$ . Then, as  $\pi \prec \rho$ , there

exists a sequence  $(\psi_n)_n$  of functions of positive type associated to  $\rho$  which converges to  $\varphi$  uniformly on  $G$ . For  $n$  large enough, we have

$$\int_G \varphi(x) \overline{\psi_n(x)} dx \neq 0.$$

Decomposing  $\rho$  as a direct sum  $\rho = \bigoplus_i \rho_i$  of irreducible subrepresentations  $\rho_i$ , we see that each  $\psi_n$  is a sum  $\psi_n = \sum_i \psi_n^{(i)}$  for functions  $\psi_n^{(i)}$  of positive type associated to  $\rho_i$ . Hence for some  $i$  and  $n$ ,

$$\int_G \varphi(x) \overline{\psi_n^{(i)}(x)} dx \neq 0.$$

By Schur's orthogonality relations for compact groups (see, e.g. [Robet–83, (5.6) Theorem]), this implies that  $\pi$  is equivalent to  $\rho_i$ . ■

**Remark F.1.9** Let  $\Gamma$  be a discrete group and  $H$  a subgroup of  $\Gamma$ . The restriction  $\lambda_\Gamma|_H$  of the regular representation of  $\Gamma$  to  $H$  is a multiple of the regular representation  $\lambda_H$  of  $H$ . Indeed, let  $T$  be a set of representatives for the right coset space  $H \backslash \Gamma$ . Then  $\ell^2(\Gamma)$  has a direct sum decomposition  $\ell^2(\Gamma) = \bigoplus_{t \in T} \ell^2(Ht)$  into  $\lambda_\Gamma(H)$ -invariant subspaces. Since the restriction of  $\lambda_\Gamma|_H$  to each subspace  $\ell^2(Ht)$  is equivalent to  $\lambda_H$ , this proves the claim.

The next proposition gives a partial generalisation of this fact to an arbitrary locally compact group (see also Remark F.1.11 below).

**Proposition F.1.10** *Let  $G$  be a locally compact group, and let  $H$  be a closed subgroup of  $G$ . Then  $\lambda_G|_H \prec \lambda_H$ .*

**Proof** Let  $f \in C_c(G)$ . Since the left regular representation of  $H$  is equivalent to the right regular representation  $\rho_H$  of  $H$  (Proposition A.4.1), it suffices, by Lemma F.1.3, to show that  $x \mapsto \langle \lambda_G(x)f, f \rangle$  can be approximated, uniformly on compact subsets of  $H$ , by sums of functions of positive type associated to  $\rho_H$ .

Let  $dx$  and  $dh$  be left Haar measures on  $G$  and  $H$ . Let  $\mu$  be a quasi-invariant Borel measure on  $G/H$ , with corresponding rho-function  $\rho$ . For  $k \in H$ , we have

$$\begin{aligned} \langle \lambda_G(k)f, f \rangle &= \int_G f(k^{-1}x) \overline{f(x)} dx = \int_G \check{f}(x^{-1}k) \overline{\check{f}(x^{-1})} dx \\ &= \int_G \Delta_G(x^{-1}) \check{f}(xk) \overline{\check{f}(x)} dx \\ &= \int_{G/H} \int_H \Delta_G(h^{-1}x^{-1}) \check{f}(xhk) \overline{\check{f}(xh)} \rho(xh)^{-1} dh d\mu(xH), \end{aligned}$$

using Lemma A.3.4 and Theorem B.1.4. For  $x \in G$ , define  $f_x \in C_c(H)$  by

$$f_x(h) = \sqrt{\Delta_G(h^{-1}x^{-1})} \check{f}(xh) \rho(xh)^{-1/2}, \quad h \in H.$$

Then

$$\begin{aligned} \Delta_G(h^{-1}x^{-1}) \check{f}(xhk) \overline{\check{f}(xh)} \rho(xh)^{-1} \\ &= \frac{1}{\sqrt{\Delta_G(k^{-1})}} f_x(hk) \overline{f_x(h)} \frac{\rho(xhk)^{1/2}}{\rho(xh)^{1/2}} \\ &= \sqrt{\Delta_H(k)} f_x(hk) \overline{f_x(h)} \\ &= (\rho_H(k) f_x)(h) \overline{f_x(h)}, \end{aligned}$$

using Lemma B.1.3. Thus

$$\langle \lambda_G(k)f, f \rangle = \int_{G/H} \langle \rho_H(k) f_x, f_x \rangle d\mu(xH).$$

Since the mapping  $(x, k) \mapsto \langle \rho_H(k) f_x, f_x \rangle$  is continuous, it follows that  $\langle \lambda_G(\cdot)f, f \rangle$  is a uniform limit on compact subsets of  $H$  of linear combinations with positive coefficients of functions of the form  $\langle \rho_H(\cdot) f_{x_i}, f_{x_i} \rangle$  with  $x_i \in G$ . This shows that  $\lambda_G|_H \prec \rho_H$ . ■

**Remark F.1.11** Conversely,  $\lambda_H$  is weakly contained in  $\lambda_G|_H$  (Exercise F.6.1) so that  $\lambda_G|_H$  and  $\lambda_H$  are actually weakly equivalent.

**Example F.1.12** As shown in Remark 1.1.2.vi, the unit representation  $1_{\mathbf{R}}$  of  $\mathbf{R}$  is weakly contained in the regular representation  $\lambda_{\mathbf{R}}$ . Let us see that every character  $\chi \in \widehat{\mathbf{R}}$  is weakly contained in  $\lambda_{\mathbf{R}}$ . Indeed, given a compact subset  $Q$  of  $\mathbf{R}$  and  $\varepsilon > 0$ , there exists a function  $f$  in  $L^2(\mathbf{R})$  with  $\|f\|_2 = 1$  such that

$$\|\lambda_{\mathbf{R}}(x)f - f\| < \varepsilon, \quad \text{for all } x \in Q.$$

Let  $g = \overline{\chi}f$ . Then, since  $\langle \lambda_{\mathbf{R}}(x)g, g \rangle = \chi(x) \langle \lambda_{\mathbf{R}}(x)f, f \rangle$ , we obtain

$$|\chi(x) - \langle \lambda_{\mathbf{R}}(x)g, g \rangle| = |1 - \langle \lambda_{\mathbf{R}}(x)f, f \rangle| = |\langle f, \lambda_{\mathbf{R}}(x)f - f \rangle| < \varepsilon,$$

for all  $x \in Q$ . For a more general fact, see Theorem G.3.2.

## F.2 Fell topology on sets of unitary representations

Let  $G$  be a topological group. One would like to define a topology on the family of equivalence classes of unitary representations of  $G$ . There is a problem



since this family is not a set. For this reason, we have to restrict ourselves to sets of such classes. One standard way is to consider only equivalence classes of unitary representations in Hilbert spaces with dimension bounded by some cardinal number; another is to consider equivalence classes of irreducible representations, namely to consider the unitary dual  $\widehat{G}$ ; a third way is to consider cyclic representations (see Remark C.4.13).

Let  $\mathcal{R}$  be a fixed set of equivalence classes of unitary representations of  $G$ .

**Definition F.2.1** For a unitary representation  $(\pi, \mathcal{H})$  in  $\mathcal{R}$ , functions of positive type  $\varphi_1, \dots, \varphi_n$  associated to  $\pi$ , a compact subset  $Q$  of  $G$ , and  $\varepsilon > 0$ , let  $W(\pi, \varphi_1, \dots, \varphi_n, Q, \varepsilon)$  be the set of all unitary representations  $\rho$  in  $\mathcal{R}$  with the following property:

for each  $\varphi_i$ , there exists a function  $\psi$  which is a sum of functions of positive type associated to  $\rho$  and such that

$$|\varphi_i(x) - \psi(x)| < \varepsilon, \quad \text{for all } x \in Q.$$

The sets  $W(\pi, \varphi_1, \dots, \varphi_n, Q, \varepsilon)$  form a basis for a topology on  $\mathcal{R}$ , called *Fell's topology*.

Fell's topology can be described in terms of convergence of nets as follows.

**Proposition F.2.2** Let  $\pi \in \mathcal{R}$ . A net  $(\pi_i)_i$  in  $\mathcal{R}$  converges to  $\pi$  if and only if  $\pi < \bigoplus_j \pi_j$  for every subnet  $(\pi_j)_j$  of  $(\pi_i)_i$ .

**Proof** This is clear since the sets  $W(\pi, \varphi_1, \dots, \varphi_n, Q, \varepsilon)$  form a basis for the family of neighbourhoods of  $\pi$ . ■

**Remark F.2.3** If the net  $(\pi_i)_i$  converges to  $\pi$ , then  $(\pi_i)_i$  converges also to every unitary representation  $\rho$  which is weakly contained in  $\pi$ .

Proposition F.1.4 implies that, in the important case  $\mathcal{R} = \widehat{G}$ , the  $\psi$ 's occurring in Definition F.2.1 can be taken as functions of positive type associated to  $\rho$ , and not just sums of such functions. More generally, the following proposition is an immediate consequence of Proposition F.1.4.

**Proposition F.2.4** Let  $\mathcal{R}$  be a set of unitary representations of  $G$ , and let  $\pi \in \mathcal{R}$  be an irreducible representation. A basis for the family of neighbourhoods of  $\pi$  in the Fell topology on  $\mathcal{R}$  is given by the sets  $\widetilde{W}(\pi, \varphi_1, \dots, \varphi_n, Q, \varepsilon)$  consisting of all  $\rho \in \mathcal{R}$  such that, for each  $\varphi_i$ , there exists a function of positive type  $\psi$  associated to  $\rho$  for which

$$|\varphi_i(x) - \psi(x)| < \varepsilon, \quad \text{for all } x \in Q.$$

**Example F.2.5** (i) Let  $G$  be an abelian topological group. The topology of its dual group  $\widehat{G}$  (see Definition A.2.4) coincides with the Fell topology on  $\widehat{G}$ . Moreover,  $\widehat{G}$  is locally compact when  $G$  is locally compact (see Section D.1).

(ii) Let  $G$  be a compact group. Then Fell's topology on  $\widehat{G}$  is the discrete topology. Indeed, let  $\pi \in \widehat{G}$ . By Propositions F.1.8 and F.2.2, a net of irreducible unitary representations  $\pi_i$  converges to  $\pi$  if and only if eventually  $\pi_i = \pi$ .

(iii) Let  $G$  be the group of affine transformations of the line  $x \mapsto ax + b$ , with  $a \in \mathbf{R}_+^*$  and  $b \in \mathbf{R}$  compare with (Example A.3.5.iv). Its unitary dual consists of the equivalence classes of the following unitary representations:

- the unitary characters  $\chi_t$  for  $t \in \mathbf{R}$  defined by

$$\chi_t \left( \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right) = a^{it};$$

- the infinite dimensional representation  $\pi = \text{Ind}_N^G \delta$ , where  $N$  is the normal subgroup

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbf{R} \right\},$$

and  $\delta$  is the character of  $N$  defined by

$$\delta \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = e^{ib}.$$

Thus, as a set,  $\widehat{G}$  can be identified with the disjoint union of  $\mathbf{R}$  and a point  $\pi$ . For a subset  $W$  of  $\widehat{G}$ , the closure of  $W$  for the Fell topology is  $\widehat{G}$  itself if  $\pi \in W$ , and the closure in the usual sense if  $W \subset \mathbf{R}$ . In particular,  $\widehat{G}$  is not a Hausdorff space.

For this and other examples, see [Fell–62, Section 5]; see also Exercise F.6.2.

As already mentioned (see the remark after Corollary C.4.7), a given unitary representation of a non-compact group does not admit in general a direct sum decomposition into irreducible subrepresentations. One can consider direct integral decompositions as defined in Section F.5. However, in many situations, a sufficient substitute is provided by the technically less involved notion of the support of a unitary representation.

**Definition F.2.6** Let  $\pi$  be a unitary representation of the topological group  $G$ . The *support* of  $\pi$ , denoted by  $\text{supp } \pi$ , is the set of all  $\sigma$  in  $\widehat{G}$  with  $\sigma \prec \pi$ .

**Proposition F.2.7** Let  $G$  be a locally compact group and  $(\pi, \mathcal{H})$  a unitary representation of  $G$ . Then  $\pi$  is weakly equivalent to  $\bigoplus \{\sigma : \sigma \in \text{supp } \pi\}$ .

**Proof** It is clear that

$$\bigoplus \{ \sigma : \sigma \in \text{supp } \pi \} \prec \pi.$$

To show the converse, we can assume that  $\pi$  is cyclic, since it is a direct sum of cyclic representations (see Proposition C.4.9). Let  $\xi$  be a unit cyclic vector, and  $\varphi_0 = \langle \pi(\cdot)\xi, \xi \rangle$ .

Let  $\mathcal{C}$  be the smallest weak\* closed convex subset of  $L^\infty(G)$  containing all normalised functions of positive type associated to  $\pi$ . Then  $\mathcal{C}$  is compact and, by the Krein–Milman Theorem,  $\mathcal{C}$  is the weak\* closure of the convex hull of  $\text{ext}(\mathcal{C})$ . We claim that  $\text{ext}(\mathcal{C})$  is contained in  $\text{ext}(\mathcal{P}_{\leq 1}(G))$ . Indeed, let  $\varphi \in \text{ext}(\mathcal{C})$ ; assume that  $\varphi = t\varphi_1 + (1-t)\varphi_2$  for  $0 < t < 1$  and  $\varphi_1, \varphi_2 \in \mathcal{P}_{\leq 1}(G)$ . Then, by Proposition C.5.1,  $\varphi_1 = \langle \pi(\cdot)\xi_1, \xi_1 \rangle$  for some unit vector  $\xi_1$  in  $\mathcal{H}$ . Let  $\varepsilon > 0$ . Since  $\xi$  is cyclic, there exists  $x_1, \dots, x_n$  in  $G$  and  $\lambda_1, \dots, \lambda_n$  in  $\mathbb{C}$  such that

$$\left\| \xi_1 - \sum_{i=1}^n \lambda_i \pi(x_i) \xi \right\| < \varepsilon.$$

Then, as in the proof of Lemma F.1.3, it follows that

$$\left| \langle \pi(x)\xi_1, \xi_1 \rangle - \left\langle \pi(x) \left( \sum_{i=1}^n \lambda_i \pi(x_i) \xi \right), \left( \sum_{i=1}^n \lambda_i \pi(x_i) \xi \right) \right\rangle \right| < (2 + \varepsilon)\varepsilon,$$

for all  $x$  in  $G$ . This shows that  $\varphi_1$  is in  $\mathcal{C}$ . Similarly,  $\varphi_2$  is in  $\mathcal{C}$ . Since  $\varphi$  is an extreme point in  $\mathcal{C}$ , it follows that  $\varphi = \varphi_1 = \varphi_2$ , showing that  $\text{ext}(\mathcal{C})$  is contained in  $\text{ext}(\mathcal{P}_{\leq 1}(G))$ .

Now,  $\text{ext}(\mathcal{C}) \setminus \{0\}$  is contained in  $\text{ext}(\mathcal{P}_1(G))$ , by (i) of Theorem C.5.5. Hence, for every  $\psi$  in  $\text{ext}(\mathcal{C}) \setminus \{0\}$ , the associated unitary representation  $\pi_\psi$  is irreducible and, by Raikov's Theorem C.5.6, is weakly contained in  $\pi = \pi_{\varphi_0}$ . On the other hand, as  $\varphi$  is contained in the weak\* closure of the convex hull of  $\text{ext}(\mathcal{C}) \setminus \{0\}$ , again by Raikov's Theorem,  $\pi$  is weakly contained in the direct sum of all such  $\pi_\psi$ 's. ■

The following lemma describes in particular the support of a finite dimensional unitary representation; it is used in the proof of Lemma 1.2.4.

**Lemma F.2.8** *Let  $G$  be a topological group, and let  $\pi$  be a finite dimensional unitary representation of  $G$ . Let  $\varphi$  be a normalised function of positive type on  $G$ . Assume that  $\varphi$  can be approximated, uniformly on finite subsets of  $G$ , by finite sums of functions of positive type associated to  $\pi$ .*

*Then  $\varphi$  is a finite sum of functions of positive type associated to  $\pi$ .*

**Proof** By assumption, there exists a net  $(\varphi_i)_{i \in I}$  such that  $\lim_i \varphi_i = \varphi$  uniformly on finite subsets of  $G$ , where each  $\varphi_i$  is a finite sum of functions of positive type associated to  $\pi$ . We can clearly assume that  $\|\varphi_i\|_\infty = \varphi_i(e) = 1$  for every  $i \in I$ .

The Hilbert space  $\mathcal{H}$  of  $\pi$  has a decomposition

$$\mathcal{H} = \mathcal{H}^{(1)} \oplus \cdots \oplus \mathcal{H}^{(m)}$$

into irreducible  $G$ -invariant subspaces  $\mathcal{H}^{(1)}, \dots, \mathcal{H}^{(m)}$ . For  $j \in \{1, \dots, m\}$ , denote by  $\pi^{(j)}$  the restriction of  $\pi$  to  $\mathcal{H}^{(j)}$ . For each  $i \in I$ , we can write

$$\varphi_i = \sum_{j=1}^m \varphi_i^{(j)},$$

where  $\varphi_i^{(j)}$  is a finite sum of functions of positive type associated to  $\pi^{(j)}$ .

Since

$$\|\varphi_i^{(j)}\|_\infty = \varphi_i^{(j)}(e) \leq \varphi_i(e) = 1,$$

upon passing to a subnet, we can assume that, for every  $j \in \{1, \dots, m\}$ ,

$$\lim_i \varphi_i^{(j)} = \varphi^{(j)} \quad \text{uniformly on finite subsets of } G$$

for some bounded function  $\varphi^{(j)}$  on  $G$ . It is clear that each  $\varphi^{(j)}$  is a function of positive type and that

$$\varphi = \sum_{j=1}^m \varphi^{(j)}.$$

We claim that every function  $\varphi^{(j)}$  is a finite sum of functions of positive type associated to  $\pi^{(j)}$ . This will finish the proof.

Let  $j \in \{1, \dots, m\}$ . The Hilbert space  $\mathcal{H}^{(j)}$  is isomorphic to  $\mathbf{C}^n$  for some  $n \geq 1$ . Since  $\pi^{(j)}$  is irreducible, the linear span of  $\{\pi^{(j)}(g) : g \in G\}$  coincides with  $\mathcal{L}(\mathcal{H}^{(j)}) \cong M_n(\mathbf{C})$ , by Wedderburn's Theorem.

The formula

$$\tilde{\varphi}^{(j)} \left( \sum_{k=1}^l c_k \pi^{(j)}(g_k) \right) = \sum_{k=1}^l c_k \varphi^{(j)}(g_k), \quad \text{for } c_1, \dots, c_l \in \mathbf{C}, g_1, \dots, g_l \in G$$

defines a positive linear functional  $\tilde{\varphi}^{(j)}$  on  $\mathcal{L}(\mathcal{H}^{(j)})$ . Indeed, let  $c_1, \dots, c_l \in \mathbf{C}$  and  $g_1, \dots, g_l \in G$  be such that  $\sum_{k=1}^l c_k \pi^{(j)}(g_k) = 0$ . Then

$$\sum_{k=1}^l c_k \psi^{(j)}(g_k) = 0$$

for every function of positive type  $\psi^{(j)}$  associated to  $\pi^{(j)}$ . Since  $\varphi^{(j)}$  is a pointwise limit of sums of such functions, we have

$$\sum_{k=1}^l c_k \varphi^{(j)}(g_k) = 0.$$

This shows that  $\tilde{\varphi}^{(j)}$  is a well-defined linear functional on  $\mathcal{L}(\mathcal{H}^{(j)})$ .

Since  $\varphi^{(j)}$  is of positive type, it is clear that  $\tilde{\varphi}^{(j)}$  is positive, that is,  $\tilde{\varphi}^{(j)}(T^*T) \geq 0$  for all  $T \in \mathcal{L}(\mathcal{H}^{(j)})$ . On the other hand, it is well-known and easy to prove that every positive linear functional on  $M_n(\mathbf{C})$  is of the form  $T \mapsto \text{Trace}(TS)$  for some positive matrix  $S$ . Let  $\eta_1, \dots, \eta_n$  be an orthonormal basis of  $\mathcal{H}$  consisting of eigenvectors of  $S$  for eigenvalues  $\alpha_1, \dots, \alpha_n$ . With  $\xi_i = \sqrt{\alpha_i} \eta_i$ , we have

$$\tilde{\varphi}^{(j)}(T) = \sum_{k=1}^n \langle T \xi_k, \xi_k \rangle \quad \text{for all } T \in \mathcal{L}(\mathcal{H}^{(j)}),$$

that is  $\varphi^{(j)} = \sum_{k=1}^n \langle \pi^{(j)}(\cdot) \xi_k, \xi_k \rangle$ . ■

**Corollary F.2.9** *Let  $G$  be a topological group, and let  $\pi$  and  $\rho$  be unitary representations of  $G$ . Assume that  $\pi$  is finite dimensional and that  $\rho$  is irreducible. If  $\rho \prec \pi$ , then  $\rho$  is contained in  $\pi$ .*

*In particular, every finite dimensional irreducible unitary representation of  $G$  is a closed point in  $\hat{G}$ .*

**Proof** Let  $\varphi$  be a normalised function of positive type associated to  $\rho$ . Since  $\rho \prec \pi$ , it follows from the previous lemma that  $\varphi$  is a finite sum of functions of positive type associated to  $\pi$ . As  $\rho$  is irreducible, this implies that  $\rho$  is contained in  $\pi$  (Proposition C.5.1). ■

## F.3 Continuity of operations

We are going to show that the usual operations on unitary representations are continuous with respect to Fell's topology. It suffices to show that any one of these operations preserves the weak containment relation.

The continuity of the operation of taking direct sums of representation is obvious from the definitions.

**Proposition F.3.1** *If  $(\pi_i)_i$  and  $(\rho_i)_i$  are families of unitary representations of the topological group  $G$  such that  $\pi_i \prec \rho_i$  for every  $i$ , then  $\bigoplus_i \pi_i \prec \bigoplus_i \rho_i$ .*

The continuity of the tensor product operation is less obvious, but it follows from Lemma F.1.3.

**Proposition F.3.2** *Let  $\pi_1, \pi_2, \rho_1, \rho_2$  be unitary representations of the topological group  $G$  such that  $\pi_1 \prec \rho_1$  and  $\pi_2 \prec \rho_2$ . Then  $\pi_1 \otimes \pi_2 \prec \rho_1 \otimes \rho_2$ .*

**Proof** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be the Hilbert spaces of  $\pi_1$  and  $\pi_2$ . It is clear that every function of positive type of the form

$$x \mapsto \langle ((\pi_1 \otimes \pi_2)(g))(\xi_1 \otimes \xi_2), \xi_1 \otimes \xi_2 \rangle = \langle \pi_1(g)\xi_1, \xi_1 \rangle \langle \pi_2(g)\xi_2, \xi_2 \rangle$$

can be approximated by sums of functions of positive type associated to  $\rho_1 \otimes \rho_2$ . Since the set  $\{\xi_1 \otimes \xi_2 : \xi_1 \in \mathcal{H}_1, \xi_2 \in \mathcal{H}_2\}$  is total in  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , the claim follows from Lemma F.1.3. ■

**Corollary F.3.3** *For a locally compact group  $G$ , the following properties are equivalent:*

- (i)  $1_G \prec \lambda_G$ ;
- (ii)  $\pi \prec \lambda_G$ , for every unitary representation  $\pi$  of  $G$ .

**Proof** If  $1_G \prec \lambda_G$ , then  $\pi = 1_G \otimes \pi \prec \lambda_G \otimes \pi$ , by the above proposition. As  $\lambda_G \otimes \pi$  is a multiple of  $\lambda_G$  (see Corollary E.2.6), the claim follows. ■

As we will see in Theorem G.3.2, each of the Properties (i) and (ii) in the previous corollary characterises amenable locally compact groups.

The continuity of the operation of restriction to closed subgroups is clear from the definition of weak containment.

**Proposition F.3.4** *Let  $H$  be a closed subgroup of the topological group  $G$ , and let  $\pi$  and  $\rho$  be unitary representations of  $G$  such that  $\pi \prec \rho$ . Then  $\pi|_H \prec \rho|_H$ .*

We turn to the continuity of induction. This fact plays a crucial role in the proof of Theorem 1.7.1.

**Theorem F.3.5 (Continuity of induction)** *Let  $H$  be a closed subgroup of the locally compact group  $G$ . Let  $\sigma$  and  $\tau$  be unitary representations of  $H$  such that  $\sigma \prec \tau$ . Then  $\text{Ind}_H^G \sigma \prec \text{Ind}_H^G \tau$ .*

**Proof** Let  $\mathcal{K}_\sigma$  and  $\mathcal{K}_\tau$  be the Hilbert spaces of  $\sigma$  and  $\tau$ . Let  $\pi = \text{Ind}_H^G \sigma$  and  $\rho = \text{Ind}_H^G \tau$ , and denote by  $\mathcal{H}_\pi$  and  $\mathcal{H}_\rho$  their Hilbert spaces. For  $f \in C_c(G)$  and  $v \in \mathcal{K}_\sigma$ , recall that  $\xi_{f,v}$  denotes the element from  $\mathcal{H}_\pi$  defined by

$$\xi_{f,v}(x) = \int_H f(xh) \sigma(h) v dh, \quad x \in G.$$

By Lemma E.1.3, the set

$$\{\xi_{f,v} : f \in C_c(G), v \in \mathcal{K}_\sigma\}$$

is total in  $\mathcal{H}_\pi$ . Hence, by Lemma F.1.3, it suffices to show that the functions of positive type of the form  $\langle \pi(\cdot) \xi_{f,v}, \xi_{f,v} \rangle$  can be approximated by functions of positive type associated to  $\rho$ .

Let  $f \in C_c(G)$  and  $v \in \mathcal{K}_\sigma$ . Choose a quasi-invariant measure  $\mu$  on  $G/H$  and set  $c_\mu(g, xH) = \frac{dg\mu}{d\mu}(xH)$  as in Section A.6. We have

$$\begin{aligned} \langle \pi(g) \xi_{f,v}, \xi_{f,v} \rangle &= \int_{G/H} c_\mu(g^{-1}, xH)^{1/2} \langle \xi_{f,v}(g^{-1}x), \xi_{f,v}(x) \rangle d\mu(xH) \\ &= \int_{G/H} c_\mu(g^{-1}, xH)^{1/2} \int_H \int_H f(g^{-1}xh) \\ &\quad \times \overline{f(xk)} \langle \sigma(k^{-1}h)v, v \rangle dh dk d\mu(xH), \end{aligned}$$

for every  $g \in G$ .

Fix a compact subset  $Q$  of  $G$  and a positive number  $\varepsilon > 0$ . Denote by  $K$  the support of  $f$ . Set  $L = (K^{-1}QK) \cap H$ , which is a compact subset of  $H$ . Since  $\sigma \prec \tau$ , there exists  $w_1, \dots, w_n \in \mathcal{K}_\tau$  such that

$$\sup_{h \in L} \left| \langle \sigma(h)v, v \rangle - \sum_{i=1}^n \langle \tau(h)w_i, w_i \rangle \right| < \varepsilon.$$

For  $g \in Q$ , we have

$$\begin{aligned} \langle \pi(g) \xi_{f,v}, \xi_{f,v} \rangle &- \sum_{i=1}^n \langle \rho(g) \xi_{f,w_i}, \xi_{f,w_i} \rangle \\ &= \int_{G/H} c_\mu(g^{-1}, xH)^{1/2} \int_H \int_H f(g^{-1}xh) \overline{f(xk)} D(k^{-1}h) dh dk d\mu(xH), \end{aligned}$$

where

$$D(k^{-1}h) = \langle \sigma(k^{-1}h)v, v \rangle - \sum_{i=1}^n \langle \tau(k^{-1}h)w_i, w_i \rangle.$$

Observe that  $f(g^{-1}xh)\overline{f(xk)} = 0$  unless  $g^{-1}xh \in K$  and  $xk \in K$ ; these inclusions imply  $xh \in QK$  and  $k^{-1}x^{-1} \in K^{-1}$ , and therefore also  $k^{-1}h \in (K^{-1}QK) \cap H = L$ . Hence,

$$\begin{aligned} & \left| \langle \pi(g)\xi_{f,v}, \xi_{f,v} \rangle - \sum_{i=1}^n \langle \rho(g)\xi_{f,w_i}, \xi_{f,w_i} \rangle \right| \\ & \leq \varepsilon \int_{G/H} c_\mu(g^{-1}, xH)^{1/2} \int_H |f(g^{-1}xh)| dh \int_H |f(xk)| dk d\mu(xH) \\ & = \varepsilon \int_{G/H} c_\mu(g^{-1}, xH)^{1/2} (T_H|f|)(g^{-1}xH) (T_H|f|)(xH) d\mu(xH). \end{aligned}$$

By the Cauchy–Schwarz inequality, we have finally

$$\begin{aligned} & \sup_{g \in Q} \left| \langle \pi(g)\xi_{f,v}, \xi_{f,v} \rangle - \sum_{i=1}^n \langle \rho(g)\xi_{f,w_i}, \xi_{f,w_i} \rangle \right| \\ & \leq \varepsilon \left( \int_{G/H} c_\mu(g^{-1}, xH) ((T_H|f|)(g^{-1}xH))^2 d\mu(xH) \right)^{1/2} \\ & \quad \times \left( \int_{G/H} ((T_H|f|)(xH))^2 d\mu(xH) \right)^{1/2} \\ & = \varepsilon \|T_H|f|\|_2^2, \end{aligned}$$

so that  $\pi$  is weakly contained in  $\rho$ . ■

**Example F.3.6** Let  $\pi_t^-$  be the non-spherical principal series representation of  $G = SL_2(\mathbf{R})$ , as in Example E.1.8. Since  $\pi_t^- = \text{Ind}_P^G \chi_t^-$  and  $\lim_{t \rightarrow 0} \chi_t^- = \chi_0^-$ , where

$$\chi_t^- \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \text{sgn}(a) |a|^{it},$$

we have  $\lim_{t \rightarrow 0} \pi_t^- = \pi_0^- = \text{Ind}_P^G \chi_0^-$ .

It is known that all  $\pi_t^-$  are irreducible for  $t \neq 0$  and that  $\pi_0^- = \sigma^+ \oplus \sigma^-$  for two irreducible representations  $\sigma^+$  and  $\sigma^-$ , the so-called *mock discrete series representations* (see [Knapp–86, Chapters II and VII]). It follows that  $\lim_{t \rightarrow 0} \pi_t^- = \sigma^+$  and  $\lim_{t \rightarrow 0} \pi_t^- = \sigma^+$ . In particular,  $\widehat{G}$  is not a Hausdorff space.



## F.4 The $C^*$ -algebras of a locally compact group

The study of the unitary representations of a locally compact group can be cast into the general framework of  $C^*$ -algebras. An overall reference for what follows is [Dixmi–69].

A Banach  $*$ -algebra  $A$  is called a  $C^*$ -algebra if the norm on  $A$  satisfies

$$(*) \quad \|x^*x\| = \|x\|^2, \quad \text{for all } x \in A.$$

**Example F.4.1** (i) For any locally compact space  $X$ , the algebra  $C_0(X)$ , with the obvious operations and the uniform norm, is a  $C^*$ -algebra.

(ii) If  $A$  is a commutative  $C^*$ -algebra, then the Gelfand transform (see D.1) is an isometric  $*$ -isomorphism between  $A$  and  $C_0(\Delta(A))$ . So, any commutative  $C^*$ -algebra occurs as in (i).

(iii) Let  $\mathcal{H}$  be a Hilbert space. It is easy to verify that  $\|T^*T\| = \|T\|^2$  for all  $T \in \mathcal{L}(\mathcal{H})$ . Hence, every norm closed  $*$ -subalgebra of  $\mathcal{L}(\mathcal{H})$  is a  $C^*$ -algebra.

(iv) By a theorem of Gelfand and Naimark, any  $C^*$ -algebra occurs as in (iii).

Let  $G$  be a *locally compact* group, fixed throughout this section. To every unitary representation  $(\pi, \mathcal{H})$  of  $G$  is associated a  $*$ -representation of the Banach  $*$ -algebra  $L^1(G)$  in  $\mathcal{H}$ , that is, a continuous  $*$ -algebra homomorphism  $L^1(G) \rightarrow \mathcal{L}(\mathcal{H})$ , again denoted by  $\pi$  and defined by

$$\pi(f) = \int_G f(x)\pi(x)dx \in \mathcal{L}(\mathcal{H}),$$

namely by

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx \in \mathbf{C}, \quad \xi, \eta \in \mathcal{H},$$

for  $f \in L^1(G)$ . This  $*$ -representation of  $L^1(G)$  is *non-degenerate*, which means that, for every  $\xi \in \mathcal{H} \setminus \{0\}$ , there exists  $f \in L^1(G)$  such that  $\pi(f)\xi \neq 0$ .

Conversely, any non-degenerate  $*$ -representation of  $L^1(G)$  is of this form. This is straightforward if the algebra  $L^1(G)$  has a unit, namely if the group  $G$  is discrete. In the general case, this follows from a standard argument using “approximate units” in  $L^1(G)$ ; see Proposition 13.4.2 in [Dixmi–69].

To any unitary representation  $(\pi, \mathcal{H})$  of  $G$ , we can associate the sub- $C^*$ -algebra of  $\mathcal{L}(\mathcal{H})$  generated by  $\pi(L^1(G))$ . Definitions F.4.3 and F.4.6 below refer to the two most important cases.

**Example F.4.2** For the left regular representation  $\lambda_G$  of  $G$ , the  $*$ -representation  $\lambda_G : L^1(G) \rightarrow \mathcal{L}(L^2(G))$  is given by convolution:

$$\lambda_G(f)\xi = f * \xi, \quad \text{for } f \in L^1(G) \quad \text{and} \quad \xi \in L^2(G).$$

Define the *universal representation*  $\pi_{\text{univ}}$  of  $G$  to be the direct sum of all cyclic unitary representations of  $G$ . We have a  $*$ -representation  $\pi_{\text{univ}} : L^1(G) \rightarrow \mathcal{L}(\mathcal{H}_{\text{univ}})$  and we define the *maximal norm* of  $f \in L^1(G)$  by

$$\|f\|_{\max} = \|\pi_{\text{univ}}(f)\|.$$

Observe that  $\|\lambda_G(f)\| \leq \|f\|_{\max} \leq \|f\|_1$ .

**Definition F.4.3** The completion of  $L^1(G)$  with respect to the norm  $f \mapsto \|f\|_{\max}$  is a  $C^*$ -algebra called the *maximal  $C^*$ -algebra* of  $G$ , and is denoted by  $C^*(G)$ .

Let  $\pi$  be a unitary representation of  $G$ . As  $\pi$  is a direct sum of cyclic representations (Proposition C.4.9),  $\|\pi(f)\| \leq \|f\|_{\max}$  for all  $f$  in  $L^1(G)$ . Hence,  $f \mapsto \pi(f)$  extends to a  $*$ -representation of  $C^*(G)$ , also denoted by  $\pi$ . In this way, we obtain a one-to-one correspondence between unitary representations of  $G$  and non-degenerate  $*$ -representations of the  $C^*$  algebra  $C^*(G)$ .

The notion of weak containment introduced in Section F.1 has the following neat interpretation in terms of  $C^*(G)$ . For the proof, see [Dixmi–69, Section 18].

**Theorem F.4.4** Let  $\pi$  and  $\rho$  be unitary representations of  $G$ . Denote by  $C^*\text{Ker}\pi$  and  $C^*\text{Ker}\rho$  the kernels of the corresponding representations of  $C^*(G)$ . The following properties are equivalent:

- (i)  $\pi \prec \rho$ ;
- (ii)  $C^*\text{Ker}\rho \subset C^*\text{Ker}\pi$ ;
- (iii)  $\|\pi(f)\| \leq \|\rho(f)\|$  for all  $f$  in  $L^1(G)$ .

**Remark F.4.5** Let  $\text{Prim}(C^*(G))$  be the *primitive ideal space* of  $C^*(G)$ , that is, the set of the kernels  $C^*\text{Ker}\pi$  of all irreducible representations  $\pi$  of  $G$ . The *Jacobson topology* on  $\text{Prim}(C^*(G))$  is a natural topology defined as follows: the closure of a subset  $\mathcal{S}$  of  $\text{Prim}(C^*(G))$  is the set of all  $C^*\text{Ker}\pi$  in  $\text{Prim}(C^*(G))$  such that

$$\bigcap_{\rho \in \mathcal{S}} C^*\text{Ker}\rho \subset C^*\text{Ker}\pi.$$

Define the Jacobson topology on  $\widehat{G}$  to be the inverse image of the mapping

$$\Phi : \widehat{G} \rightarrow \text{Prim}(C^*(G)), \quad \pi \mapsto C^*\text{Ker}\pi$$

(that is, the closed subsets in  $\widehat{G}$  are the sets  $\Phi^{-1}(\mathcal{S})$  where  $\mathcal{S}$  is closed in  $\text{Prim}(C^*(G))$ ). It can be shown that Fell's topology on  $\widehat{G}$  as defined in Section F.2 coincides with Jacobson's topology (see [Dixmi-69, Section 18]).

There is another  $C^*$ -algebra one can associate to  $G$ .

**Definition F.4.6** The norm closure of  $\{\lambda_G(f) : f \in L^1(G)\}$  in  $\mathcal{L}(\mathcal{H})$  is a  $C^*$ -algebra called the *reduced  $C^*$ -algebra* of  $G$ , and is denoted by  $C_{\text{red}}^*(G)$ .

The  $C^*$ -algebra  $C_{\text{red}}^*(G)$  can also be described as the completion of  $L^1(G)$  with respect to the norm  $f \mapsto \|\lambda_G(f)\|$ .

**Example F.4.7** Let  $G$  be a locally compact abelian group. The Fourier transform  $\mathcal{F} : L^2(G) \rightarrow L^2(\widehat{G})$  is a unitary equivalence between  $\lambda_G$  and the unitary representation  $\pi$  of  $G$  on  $L^2(\widehat{G})$  defined by

$$(\pi(x)\xi)(\widehat{x}) = \overline{\widehat{x}(x)}\xi(\widehat{x}), \quad \xi \in L^2(\widehat{G}), \widehat{x} \in \widehat{G}, x \in G$$

(see Remark D.1.4). The representation of  $L^1(G)$  in  $L^2(\widehat{G})$  associated to  $\pi$  is given, for  $f$  in  $L^1(G)$ , by  $\pi(f) = T_{\mathcal{F}f}$ , where  $T_{\mathcal{F}f}$  is the multiplication operator by the bounded function  $\mathcal{F}f$ . By the Stone–Weierstraß Theorem,  $\{\mathcal{F}f : f \in L^1(G)\}$  is a dense subalgebra of  $C_0(\widehat{G})$ . Hence,  $\lambda_G(f) \mapsto T_{\mathcal{F}f}$  extends to an isomorphism between  $C_{\text{red}}^*(G)$  and  $C_0(\widehat{G})$ .

The regular representation defines a surjective  $*$ -homomorphism

$$\lambda_G : C^*(G) \rightarrow C_{\text{red}}^*(G),$$

so that  $C_{\text{red}}^*(G)$  is a quotient of  $C^*(G)$ . Observe that, by the previous theorem,  $\lambda_G$  is an isomorphism if and only if  $\|f\|_{\max} = \|\lambda_G(f)\|$  for all  $f$  in  $L^1(G)$ , that is, if and only if every unitary representation of  $G$  is weakly contained in  $\lambda_G$ . We will characterise the groups for which this holds in Appendix G.

## F.5 Direct integrals of unitary representations

As previously mentioned (see remark after Corollary C.4.7), direct sums are not sufficient in order to decompose a given unitary representation into irreducible ones. One needs the notion of a direct integral of unitary representations.

Let  $(Z, \mu)$  be a measure space, where  $\mu$  is a  $\sigma$ -finite positive measure on  $Z$ . A *field of Hilbert spaces over  $Z$*  is a family  $(\mathcal{H}(z))_{z \in Z}$ , where  $\mathcal{H}(z)$  is a Hilbert

space for each  $z \in Z$ . Elements of the vector space  $\prod_{z \in Z} \mathcal{H}(z)$  are called *vector fields over  $Z$* .

To define a measurable field of Hilbert spaces over  $Z$ , we have to specify the measurable vector fields. This depends on the choice of a *fundamental family of measurable vector fields*; by definition, this is a sequence  $(x_n)_{n \in \mathbf{N}}$  of vector fields over  $Z$  with the following properties:

- (i) for any  $m, n \in \mathbf{N}$ , the function  $z \mapsto \langle x_m(z), x_n(z) \rangle$  is measurable;
- (ii) for every  $z \in Z$ , the linear span of  $\{x_n(z) : n \in \mathbf{N}\}$  is dense in  $\mathcal{H}(z)$ .

Fix a fundamental family of measurable vector fields. A vector field  $x \in \prod_{z \in Z} \mathcal{H}(z)$  is said to be a *measurable vector field* if all the functions

$$z \mapsto \langle x(z), x_n(z) \rangle, \quad n \in \mathbf{N}$$

are measurable. As is easily shown (Exercise F.6.6), the set  $M$  of measurable vector fields is a linear subspace of  $\prod_{z \in Z} \mathcal{H}(z)$ . Moreover, if  $x, y \in M$ , then the function

$$z \mapsto \langle x(z), y(z) \rangle$$

is measurable. The pair  $((\mathcal{H}(z))_{z \in Z}, M)$ , simply denoted by  $z \mapsto \mathcal{H}(z)$ , is called a *measurable field of Hilbert spaces over  $Z$* .

In the sequel, we identify two measurable vector fields which are equal  $\mu$ -almost everywhere. A measurable vector field  $x$  is a *square-integrable vector field* if

$$\int_Z \|x(z)\|^2 d\mu(z) < \infty.$$

The set  $\mathcal{H}$  of all square-integrable vector fields is a Hilbert space for the inner product

$$\int_Z \langle x(z), y(z) \rangle d\mu(z), \quad x, y \in \mathcal{H}.$$

We write

$$\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z) d\mu(z)$$

and call  $\mathcal{H}$  the *direct integral* of the field  $(\mathcal{H}(z))_{z \in Z}$  of Hilbert spaces over  $Z$ .

**Example F.5.1** (i) Let  $Z$  be a countable set and let  $\mu$  be a measure on  $Z$  such that  $\mu(z) \neq 0$  for all  $z \in Z$ . Then every vector field is measurable and

$$\int_Z^{\oplus} \mathcal{H}(z) d\mu(z) = \bigoplus_{z \in Z} \mathcal{H}(z)$$

is the direct sum of the Hilbert spaces  $\mathcal{H}(z)$ ,  $z \in Z$ .

(ii) Let  $(Z, \mu)$  be a  $\sigma$ -finite measure space and let  $\mathcal{H}(z) = \mathbb{C}$  for all  $z \in Z$ . We can choose a fundamental family of measurable vector fields such that the measurable vector fields are the measurable complex-valued functions on  $Z$ . Then

$$\int_Z^{\oplus} \mathcal{H}(z) d\mu(z) = L^2(Z, \mu).$$

(iii) Let  $(Z, \mu)$  be a  $\sigma$ -finite measure space and let  $\mathcal{K}$  be a fixed separable Hilbert space. Set  $\mathcal{H}(z) = \mathcal{K}$  for all  $z \in Z$ . We can choose a fundamental family of measurable vector fields such that the measurable vector fields are the measurable mappings  $Z \rightarrow \mathcal{K}$ , with respect to the Borel structure on  $\mathcal{K}$  given by the weak topology. Then

$$\int_Z^{\oplus} \mathcal{H}(z) d\mu(z) = L^2(Z, \mathcal{K}),$$

the Hilbert space of all square-integrable measurable mappings  $Z \rightarrow \mathcal{K}$ .

Let  $z \mapsto \mathcal{H}(z)$  be a measurable field of Hilbert spaces over  $Z$ . Let  $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z) d\mu(z)$ . For every  $z \in Z$ , let  $T(z)$  be a bounded operator on  $\mathcal{H}(z)$ . We say that  $(T(z))_{z \in Z}$  is a *measurable field of operators* on  $Z$  if all the functions

$$z \mapsto \langle T(z)x(z), y(z) \rangle, \quad x, y \in \mathcal{H},$$

are measurable. If, moreover,  $z \mapsto \|T(z)\|$  is  $\mu$ -essentially bounded, then we can define a bounded operator  $T : \mathcal{H} \rightarrow \mathcal{H}$  by

$$(Tx)(z) = T(z)x(z), \quad x \in \mathcal{H}, z \in Z.$$

In this case, we write  $T = \int_Z^{\oplus} T(z) d\mu(z)$ .

Let now  $G$  be a second countable locally compact group. Let  $(Z, \mu)$  be a  $\sigma$ -finite measure space, let  $z \mapsto \mathcal{H}(z)$  be a measurable field of Hilbert spaces over  $Z$ , and let  $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}(z) d\mu(z)$ . A *measurable field of unitary representations* of  $G$  on the  $\mathcal{H}(z)$ 's is a family  $z \mapsto \pi(z)$ , where  $\pi(z)$  is a unitary representation of  $G$  on  $\mathcal{H}(z)$ , such that  $(\pi(z)(g))_{z \in Z}$  is a measurable field of operators on  $\mathcal{H}$  for every  $g \in G$ .

Let  $g \in G$ . Since  $\pi(z)(g)$  is a unitary operator on  $\mathcal{H}(z)$ , we can define a unitary operator  $\pi(g)$  on  $\mathcal{H}$  by

$$\pi(g) = \int_Z^{\oplus} \pi(z)(g) d\mu(z).$$

It is clear that  $g \mapsto \pi(g)$  is a homomorphism from  $G$  to the unitary group of  $\mathcal{H}$ . It can be shown [Dixmi-69, Proposition 18.7.4] that the mapping  $g \mapsto \pi(g)$  is

strongly continuous, so that  $\pi$  is a unitary representation of  $G$  on  $\mathcal{H}$ . We write

$$\pi = \int_Z^{\oplus} \pi(z) d\mu(z)$$

and  $\pi$  is called the *direct integral* of the unitary representations  $\pi(z)$ ,  $z \in Z$ .

**Definition F.5.2** A Borel space is a space together with a  $\sigma$ -algebra of subsets. Such a space  $Z$  is a *standard Borel space*, if  $Z$  is isomorphic, as a Borel space, to a Borel subspace of a complete separable metric space.

A standard Borel space is either finite, or Borel isomorphic to  $\mathbf{Z}$ , or Borel isomorphic to the interval  $[0, 1]$  (see [Sriva–98, Corollary 3.3.16]).

The following theorem shows that every unitary representation of  $G$  can be decomposed as a direct integral of irreducible unitary representations over a standard Borel space; for the proof, see Theorem 8.5.2 and Section 18.7 in [Dixmi–69].

**Theorem F.5.3** *Let  $G$  be a second countable locally compact group and let  $\pi$  be a unitary representation of  $G$  on a separable Hilbert space  $\mathcal{H}$ . Then there exist a standard Borel space  $Z$ , a bounded positive measure  $\mu$  on  $Z$ , a measurable field of Hilbert spaces  $z \mapsto \mathcal{H}(z)$  over  $Z$ , and a measurable field of irreducible unitary representations  $z \mapsto \pi(z)$  of  $G$  on the  $\mathcal{H}(z)$ 's such that  $\pi$  is unitarily equivalent to the direct integral  $\int_Z^{\oplus} \pi(z) d\mu(z)$ .*

**Example F.5.4** Let  $G$  be a second countable locally compact abelian group. Choose the normalisation of the Haar measures on  $G$  and on the dual group  $\widehat{G}$  so that Plancherel's Theorem D.1.2 holds. The Fourier transform  $\mathcal{F} : L^2(G) \rightarrow L^2(\widehat{G})$  is then an isometric isomorphism which intertwines the regular representation  $\lambda_G$  on  $L^2(G)$  and the unitary representation  $\pi$  of  $G$  on  $L^2(\widehat{G})$  defined by

$$(\pi(x)\xi)(\chi) = \overline{\chi}(x)\xi(\chi), \quad \xi \in L^2(\widehat{G}), \chi \in \widehat{G}, x \in G.$$

For every  $\chi \in \widehat{G}$ , let  $\mathcal{H}(\chi) = \mathbf{C}$  and  $\pi(\chi) = \overline{\chi}$ . Then, denoting by  $\mu$  the Haar measure on  $\widehat{G}$ , we have

$$L^2(\widehat{G}) = \int_{\widehat{G}}^{\oplus} \mathcal{H}(\chi) d\mu(\chi) \quad \text{and} \quad \pi = \int_{\widehat{G}}^{\oplus} \pi(\chi) d\mu(\chi),$$

so that  $\lambda_G$  is unitarily equivalent to  $\int_{\widehat{G}}^{\oplus} \chi d\mu(\chi)$ .

## F.6 Exercises

**Exercise F.6.1** Let  $G$  be a locally compact group, and let  $H$  be a subgroup of  $G$ .

(i) Show that, if  $H$  is open, then the restriction  $\lambda_G|_H$  to  $H$  of the regular representation  $\lambda_G$  of  $G$  is equivalent to a multiple of  $\lambda_H$ .

[Hint: Look at Remark F.1.9.]

(ii) Show that, if  $H$  is closed, then  $\lambda_H$  is weakly contained in  $\lambda_G|_H$ .

**Exercise F.6.2** Let  $\mathbf{K}$  be a local field, and let  $G$  be the  $(ax+b)$ -group over  $\mathbf{K}$ :

$$G = \left\{ g_{a,b} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbf{K}^*, b \in \mathbf{K} \right\}.$$

Let  $N$  be the normal subgroup  $\{g_{1,b} \in G : b \in \mathbf{K}\}$ .

(i) For a unitary character  $\chi$  of  $N$  with  $\chi \neq 1_N$ , show that the induced representation  $\text{Ind}_N^G \chi$  is equivalent to the unitary representation  $\pi_\chi$  of  $G$  defined on  $L^2(\mathbf{K}^*, d^*x)$  by

$$\pi_\chi(g_{a,b})\xi(x) = \chi(bx)\xi(ax), \quad \xi \in L^2(\mathbf{K}^*, d^*x),$$

where  $d^*x$  is a Haar measure for the multiplicative group  $\mathbf{K}^*$  of  $\mathbf{K}$ .

(ii) Show that  $\pi_\chi$  is irreducible for  $\chi \neq 1_N$ .

[Hint: Use Schur's Lemma and Exercise D.5.1.]

(iii) Show that  $\pi_\chi$  and  $\pi_{\chi'}$  are equivalent for  $\chi, \chi' \in \widehat{N} \setminus \{1_N\}$ .

(iv) For  $\sigma \in \widehat{\mathbf{K}^*}$ , let  $\chi_\sigma$  be the unitary character of  $G$  defined by

$$\chi_\sigma(g_{a,b}) = \sigma(a), \quad g_{a,b} \in G.$$

It follows from Mackey's theory (see [Macke-76, §3.8]) that the unitary dual of  $G$  is

$$\widehat{G} = \{\pi_\chi\} \cup \{\chi_\sigma : \sigma \in \widehat{\mathbf{K}^*}\},$$

for any  $\chi \in \widehat{N} \setminus \{1_N\}$ .

Show that  $\pi_\chi$  is a dense open point in  $\widehat{G}$ , for any  $\chi \in \widehat{N} \setminus \{1_N\}$ .

**Exercise F.6.3** Let

$$\Gamma = \left\{ g_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbf{Z} \right\}$$

be the Heisenberg group over  $\mathbf{Z}$ . Let

$$a = g_{1,0,0}, \quad b = g_{0,1,0}, \quad c = g_{0,0,1}.$$

(i) Show that the centre of  $\Gamma$  is  $Z = \{c^n : n \in \mathbf{Z}\}$  and that  $\Gamma$  has the presentation

$$\langle a, b, c \mid aba^{-1}b^{-1} = c, ac = ca, bc = cb \rangle.$$

(ii) Let  $\pi$  be a unitary representation of  $\Gamma$ , with central character given by  $\pi(c) = \theta I$  for  $\theta \in \mathbf{S}^1$ . Let  $U = \pi(a)$  and  $V = \pi(b)$ . Show that

$$UV = \theta VU.$$

(iii) Conversely, let  $U, V$  be two unitary operators acting irreducibly on a Hilbert space  $\mathcal{H}$ , that is, the only closed subspaces of  $\mathcal{H}$  which are invariant under  $U$  and  $V$  are  $\{0\}$  and  $\mathcal{H}$ . Assume that  $UV = \theta VU$  for some  $\theta \in \mathbf{S}^1$ . Show that

$$\pi(a) = U, \quad \pi(b) = V, \quad \pi(c) = \theta I,$$

define an irreducible unitary representation  $\pi$  of  $\Gamma$  with central character  $c^n \mapsto \theta^n$ .

Let  $N$  be the subgroup of  $\Gamma$  generated by  $a$  and  $c$ . For  $\theta_1, \theta_2 \in \mathbf{S}^1$ , let  $\chi_{\theta_1, \theta_2}$  be the unitary character of  $N$  defined by

$$\chi_{\theta_1, \theta_2}(a) = \theta_1 \quad \text{and} \quad \chi_{\theta_1, \theta_2}(c) = \theta_2.$$

Assume that  $\theta_2$  is not a root of unity.

(iv) Show that  $\text{Ind}_N^\Gamma \chi_{\theta_1, \theta_2}$  is equivalent to the representation  $\pi_{\theta_1, \theta_2}$  of  $\Gamma$  on  $\ell^2(\mathbf{Z})$  defined by

$$(\pi_{\theta_1, \theta_2}(a)\xi)(n) = \theta_1^n \xi(n),$$

$$(\pi_{\theta_1, \theta_2}(b)\xi)(n) = \xi(n-1),$$

$$(\pi_{\theta_1, \theta_2}(c)\xi)(n) = \theta_2 \xi(n)$$

for all  $\xi \in \ell^2(\mathbf{Z})$  and  $n \in \mathbf{Z}$ .

(v) Show that  $\pi_{\theta_1, \theta_2}$  is irreducible.

(vi) Show that  $\pi_{\theta_1, \theta_2}$  and  $\pi_{\theta'_1, \theta'_2}$  are equivalent if and only if  $(\theta_1, \theta_2) = (\theta'_1, \theta'_2)$ .

**Exercise F.6.4** Let  $G$  be a topological group,  $\widehat{G}$  its unitary dual, and  $S, T$  two subsets of  $\widehat{G}$ . Show that the representation  $\bigoplus_{\sigma \in S} \sigma$  is weakly contained in  $\bigoplus_{\tau \in T} \tau$  if and only if  $S$  is contained in the closure of  $T$  for the Fell topology. In particular,  $\bigoplus_{\sigma \in S} \sigma$  and  $\bigoplus_{\tau \in T} \tau$  are weakly equivalent if and only if  $\overline{S} = \overline{T}$ .

**Exercise F.6.5** Consider a separable locally compact group  $G$ , a standard Borel space  $\Omega$ , a bounded positive measure  $\mu$  on  $\Omega$ , a measurable field  $(\pi_\omega)_{\omega \in \Omega}$



of unitary representations of  $G$  in a measurable field  $(\mathcal{H}_\omega)_{\omega \in \Omega}$  of Hilbert spaces, and the direct integral representation  $\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$  in  $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$ .

Show that the representation  $\pi_\omega$  is weakly contained in  $\pi$  for  $\mu$ -almost all  $\omega$  in  $\Omega$ .

[Hint: See [BekHa, Lemma 15].]

**Exercise F.6.6** Let  $(Z, \mu)$  be a measure space where  $\mu$  is  $\sigma$ -finite. Let  $z \mapsto \mathcal{H}(z)$  be a field of Hilbert spaces over  $Z$ . Let  $(x_n)_n$  be a fundamental family of measurable vector fields over  $Z$  and let  $M$  be the set of all measurable vector fields.

(i) Show that  $M$  is a linear subspace of  $\prod_{z \in Z} \mathcal{H}(z)$ .

(ii) Let  $x \in M$ . Show that  $z \mapsto \|x(z)\|$  is measurable.

[Hint: The  $\mathbf{Q}$ -linear span of  $\{x_n(z) : n \in \mathbf{N}\}$  is dense in  $\mathcal{H}(z)$ , for every  $z \in Z$ .]

(iii) Let  $x, y \in M$ . Show that  $z \mapsto \langle x(z), y(z) \rangle$  is measurable.

# Appendix G

## Amenability

Lebesgue proposed his view on integration in a short note (1901) and a famous book (1904). It was then a natural question to know whether Lebesgue's measure could be extended as a finitely additive measure defined on *all* subsets of  $\mathbf{R}^n$  which is invariant under isometric transformations. Hausdorff answered negatively for  $n \geq 3$  in 1914 and Banach positively for  $n \leq 2$  in 1923. Von Neumann showed in 1929 that the deep reason for this difference lies in the group of isometries of  $\mathbf{R}^n$  (viewed as a discrete group) which is amenable for  $n \leq 2$  and which is not so for  $n \geq 3$ . In 1950, Dixmier extended the notion of amenability to topological groups [Dixmi–50].

In Section G.1, amenability is defined for a topological group in two equivalent ways, by the existence of an invariant mean on an appropriate space and as a fixed point property. Examples are given in Section G.2. For a locally compact group  $G$ , it is shown in Section G.3 that  $G$  is amenable if and only if  $1_G < \lambda_G$ ; a consequence is that amenability is inherited by closed subgroups of locally compact groups.

It is remarkable that amenability can be given a very large number of equivalent definitions, at least for locally compact groups. In Section G.4, we give Kesten's characterisation in terms of appropriate operators, which is crucial for the study of random walks on groups. In Section G.5, we give Følner's characterisation in terms of asymptotically invariant sets from [Foeln–55]. For other equivalences, see the standard expositions which include [Green–69], [Eymar–75], Chapter 4 of [Zimm–84a], and [Pater–88]; see also [CeGrH–99] and [AnDRe–00].

The notion of amenability has spread over many domains of mathematics: Banach algebras, operator algebras (nuclearity, exactness, injectivity), metric spaces, group actions, foliations, groupoids, etc. In the limited context of this book, its importance is due to its relation with Property (T); in particular to the fact (Theorem 1.1.6) that a locally compact group is

both an amenable group and a group with Property (T) if and only if it is compact.

## G.1 Invariant means

Let  $X$  be a set. A *ring*  $\mathcal{R}$  of subsets of  $X$  is a non-empty class of subsets of  $X$  which is closed under the formation of union and differences of sets: if  $A, B \in \mathcal{R}$ , then  $A \cup B \in \mathcal{R}$  and  $A \setminus B \in \mathcal{R}$ .

**Definition G.1.1** A *mean*  $m$  on a ring  $\mathcal{R}$  of subsets of  $X$  containing  $X$  is a finitely additive probability measure on  $\mathcal{R}$ , that is,  $m$  is a function from  $\mathcal{R}$  to  $\mathbf{R}$  with the following properties:

- (i)  $m(A) \geq 0$  for all  $A \in \mathcal{R}$ ;
- (ii)  $m(X) = 1$ ;
- (iii)  $m(A_1 \cup \dots \cup A_n) = m(A_1) + \dots + m(A_n)$  if  $A_1, \dots, A_n \in \mathcal{R}$  are pairwise disjoint.

If a group  $G$  acts on  $X$  leaving  $\mathcal{R}$  invariant, then  $m$  is said to be a  *$G$ -invariant mean* if

- (iv)  $m(gA) = m(A)$  for all  $g \in G$  and  $A \in \mathcal{R}$ .

For a ring  $\mathcal{R}$  of subsets of  $X$ , denote by  $E$  the vector space of complex-valued functions on  $X$  generated by the characteristic functions  $\chi_A$  of subsets  $A$  in  $\mathcal{R}$ . There is a natural bijective correspondence between means  $m$  on  $\mathcal{R}$  and linear functionals  $M$  on  $E$  such that

$$m(A) = M(\chi_A) \quad \text{for all } A \in \mathcal{R}.$$

In case  $\mathcal{R}$  is  $G$ -invariant for some group  $G$  acting on  $X$ , the mean  $m$  is  $G$  invariant if and only if  $M({}_g\varphi) = M(\varphi)$  for all  $g \in G$  and  $\varphi \in E$ , where  ${}_g\varphi$  is the function on  $X$  defined by  ${}_g\varphi(x) = \varphi(gx)$  for  $x \in X$ .

Assume now that there is given a  $\sigma$ -algebra  $\mathcal{B}$  of subsets of  $X$  and a measure  $\mu$  on  $(X, \mathcal{B})$ . There is again a bijective correspondence between appropriate linear functionals on  $L^\infty(X, \mu)$  and appropriate means on  $\mathcal{B}$ , as we now explain.

**Definition G.1.2** Let  $(X, \mathcal{B}, \mu)$  be a measure space and let  $E$  be a closed subspace of  $L^\infty(X, \mathcal{B}, \mu)$  which contains the constant functions and is closed under complex conjugation. A *mean* on  $E$  is a linear functional  $M : E \rightarrow \mathbf{C}$  with the following properties:

- (i)  $M(1_X) = 1$ ;
- (ii)  $M(\varphi) \geq 0$  for all  $\varphi \in E$  with  $\varphi \geq 0$ .

Let  $G$  be a group acting on  $E$ . We say that  $M$  is  $G$ -invariant if, moreover,

(iii)  $M(g\varphi) = M(\varphi)$  for all  $g \in G$  and  $\varphi \in E$ .

**Remark G.1.3** (i) A mean  $M$  on  $E$  is automatically continuous. Indeed,

$$-\|\varphi\|_{\infty} 1_X \leq \varphi \leq \|\varphi\|_{\infty} 1_X.$$

Hence,  $|M(\varphi)| \leq \|\varphi\|_{\infty}$  by (i) and (ii).

(ii) A mean  $M$  on  $L^{\infty}(X, \mathcal{B}, \mu)$  defines a mean  $m$  on the  $\sigma$ -algebra  $\mathcal{B}$  by  $m(A) = M(\chi_A)$  for all  $A \in \mathcal{B}$ . Observe that  $m$  is absolutely continuous with respect to  $\mu$ , in the sense that  $m(A) = 0$  for all  $A \in \mathcal{B}$  with  $\mu(A) = 0$ .

Conversely, if  $m$  is a mean on  $\mathcal{B}$  which is absolutely continuous with respect to  $\mu$ , then there exists a unique mean  $M$  on  $L^{\infty}(X, \mathcal{B}, \mu)$  such that  $m(A) = M(\chi_A)$  for all  $A \in \mathcal{B}$ . Indeed, define

$$M(\varphi) = \sum_{i=1}^m \alpha_i m(A_i)$$

if  $\varphi = \sum_{i=1}^m \alpha_i \chi_{A_i}$  is a measurable simple function on  $X$ . By finite additivity of  $m$ , this definition does not depend on the given representation of  $\varphi$  as linear combination of characteristic functions of measurable subsets. Let  $\varphi$  be a measurable bounded function on  $X$ . There exists a sequence  $(\varphi_n)_n$  of measurable simple functions on  $X$  converging uniformly on  $X$  to  $\varphi$ . It is easily verified that  $(M(\varphi_n))_n$  is a Cauchy sequence in  $\mathbb{C}$  and that its limit does not depend on the particular choice of  $(\varphi_n)_n$ . Define then

$$M(\varphi) = \lim_n M(\varphi_n).$$

Since  $m$  is absolutely continuous with respect to  $\mu$ , the number  $M(\varphi)$  depends only on the equivalence class  $[\varphi]$  of  $\varphi$  in  $L^{\infty}(X, \mu)$ , and we can define  $M([\varphi]) = M(\varphi)$ . One checks that  $M$  is a mean on  $L^{\infty}(X, \mu)$ . For more details, see [HewSt-69, (20.35) Theorem].

From now on, we will identify a mean  $M$  on a space  $E$  in the sense of Definition G.1.2 and the corresponding mean  $m$  on  $\mathcal{B}$  (or on  $\mathcal{R}$ ), and we will use the same notation  $m$  in both cases.

Let  $G$  be a topological group. Let  $\ell^{\infty}(G)$  be the Banach space of bounded functions on  $G$ . The group  $G$  acts on  $\ell^{\infty}(G)$  by left translations:

$$\varphi \mapsto {}_g\varphi, \quad \varphi \in \ell^{\infty}(G), g \in G.$$

Recall that  ${}_g\varphi$  is defined by  ${}_g\varphi(x) = \varphi(gx)$ .

Let  $UCB(G)$  be the closed subspace of  $\ell^\infty(G)$ , consisting of all left uniformly continuous functions on  $G$ . Observe that a function  $\varphi$  in  $\ell^\infty(G)$  is in  $UCB(G)$  if and only if the mapping  $x \mapsto {}_x\varphi$  from  $G$  to  $\ell^\infty(G)$  is continuous. Observe also that  $UCB(G)$  is invariant under left translation by elements from  $G$ .

**Definition G.1.4** We consider the group  $G$  acting on itself by left translations. The topological group  $G$  is said to be *amenable* if there exists an invariant mean on  $UCB(G)$ .

**Example G.1.5** Let  $G$  be a compact group. Then  $UCB(G) = C(G)$ , and an invariant mean  $m$  on  $C(G)$  is an invariant regular Borel measure on  $G$  with  $m(1_G) = 1$ . Hence, the normalised Haar measure is the unique invariant mean on  $C(G)$ . In particular, compact groups are amenable.

For a non-compact group  $G$ , invariant means on  $UCB(G)$  – when they exist – are in general not unique.

Let  $X$  be convex subset of a locally convex topological vector space. A continuous action  $G \times X \rightarrow X$  of  $G$  on  $X$  is said to be an *affine action* if

$$X \rightarrow X, \quad x \mapsto gx$$

is an affine mapping for all  $g$  in  $G$ . Recall that a mapping  $\alpha : X \rightarrow X$  is affine if

$$\alpha(tx + (1-t)y) = t\alpha(x) + (1-t)\alpha(y), \quad \text{for all } x, y \in X, 0 \leq t \leq 1.$$

**Remark G.1.6** The topological dual space  $UCB(G)^*$  of  $UCB(G)$ , endowed with the weak\* topology, is a locally convex vector space. The set  $\mathcal{M}$  of all means on  $UCB(G)$  is a weak\* closed and, hence, compact convex subset of the unit ball of  $UCB(G)^*$ . Observe that  $\mathcal{M}$  is non-empty: for instance, the point evaluation  $f \mapsto f(e)$  is a mean on  $UCB(G)$ . There is a continuous affine action  $m \mapsto gm$  of  $G$  on  $\mathcal{M}$  given by  $gm(\varphi) = m({}_g{}^{-1}\varphi)$  for all  $g \in G$  and  $\varphi \in UCB(G)$ . The group  $G$  is amenable if and only if this action has a fixed point in  $\mathcal{M}$ .

The remarkable fact is that, if this is the case, *any* affine action of  $G$  on a compact convex has a fixed point.

**Theorem G.1.7** For a topological group  $G$ , the following properties are equivalent:

- (i)  $G$  is amenable;
- (ii) **(Fixed point property)** any continuous affine action of  $G$  on a non-empty compact convex subset  $X$  of a locally convex topological vector space has a fixed point.

**Proof** By the previous remark, (ii) implies (i).

Assume that  $G$  is amenable, and that a continuous affine action of  $G$  on a non-empty compact convex set  $X$  in a locally convex vector space  $V$  is given. Fix an element  $x_0$  in  $X$ . Let  $t : G \rightarrow X$ ,  $g \mapsto gx_0$  be the corresponding orbital mapping. For every  $f \in C(X)$ , the function  $f \circ t$  is in  $UCB(G)$ . Indeed, as  $X$  is compact,  $f \circ t$  is bounded. Since, moreover, the action of  $G$  is continuous, for every  $\varepsilon > 0$ , there exists a neighbourhood  $U$  of  $e$  such that, for all  $u \in U$ ,

$$\sup_{x \in X} |f(ux) - f(x)| \leq \varepsilon.$$

Let  $m$  be a mean on  $UCB(G)$ . Then a probability measure  $\mu_m$  on  $X$  is defined by

$$\mu_m(f) = m(f \circ t), \quad \text{for all } f \in C(X).$$

Let  $b_m \in X$  be the *barycentre* of  $\mu_m$ . Recall that  $b_m$  is defined as the  $X$ -valued integral  $b_m = \int_X x d\mu_m(x)$  and that  $b_m$  is the unique element in  $X$  with the property that  $\varphi(b_m) = \mu_m(\varphi)$  for every  $\varphi \in V^*$  (see [Rudin-73, Theorem 3.27]). In particular, the mapping  $\Phi : \mathcal{M} \rightarrow X$ ,  $m \mapsto b_m$  is continuous, when the set  $\mathcal{M}$  of all means on  $UCB(G)$  is endowed with the weak\* topology. Observe that  $b_m = gx_0$  if  $m = \delta_g$  is the evaluation at  $g \in G$ .

We claim that  $b_{gm} = gb_m$  for every  $g \in G$ . Indeed, this is clearly true if  $m$  is a convex combination of point evaluations. By the Hahn–Banach Theorem, the set of all convex combinations of point evaluations is dense in  $\mathcal{M}$  (Exercise G.6.1). Since  $G$  acts by continuous affine mappings on  $X$  and since  $\Phi$  is continuous, the claim follows.

If now  $m$  is an invariant mean on  $UCB(G)$ , then  $b_m \in X$  is a fixed point for the action of  $G$ . ■

## G.2 Examples of amenable groups

Besides compact groups, the first examples of amenable groups are the abelian groups.

**Theorem G.2.1 (Markov–Kakutani)** *Every abelian topological group  $G$  is amenable.*

**Proof** Given a continuous affine action of  $G$  on a non-empty compact convex subset  $X$  in a locally convex vector space  $V$ , define for every integer  $n \geq 0$  and

every  $g$  in  $G$ , a continuous affine transformation  $A_n(g) : X \rightarrow X$  by

$$A_n(g)x = \frac{1}{n+1} \sum_{i=0}^n g^i x, \quad x \in X.$$

Let  $\mathcal{G}$  be the semigroup of continuous affine transformations of  $X$  generated by the set  $\{A_n(g) : n \geq 0, g \in G\}$ . Since  $X$  is compact,  $\gamma(X)$  is a closed subset of  $X$  for every  $\gamma$  in  $\mathcal{G}$ .

We claim that  $\bigcap_{\gamma \in \mathcal{G}} \gamma(X)$  is non-empty. Indeed, since  $X$  is compact, it suffices to show that  $\gamma_1(X) \cap \dots \cap \gamma_n(X)$  is non-empty for all  $\gamma_1, \dots, \gamma_n$  in  $\mathcal{G}$ . Let  $\gamma = \gamma_1 \gamma_2 \dots \gamma_n \in \mathcal{G}$ . Then, since  $\mathcal{G}$  is abelian,  $\gamma(X)$  is contained in  $\gamma_i(X)$  for all  $i = 1, \dots, n$ . Thus,  $\gamma_1(X) \cap \dots \cap \gamma_n(X)$  contains  $\gamma(X)$ , and this proves the claim.

Let  $x_0 \in \bigcap_{\gamma \in \mathcal{G}} \gamma(X)$ . We claim that  $x_0$  is a fixed point for  $G$ . Indeed, for every  $g$  in  $G$  and every  $n \geq 0$ , there exists some  $x$  in  $X$  such that  $x_0 = A_n(g)x$ . Hence, for every  $\varphi$  in  $V^*$ ,

$$|\varphi(x_0 - gx_0)| = \frac{1}{n+1} |\varphi(x) - \varphi(g^{n+1}x)| \leq \frac{2C}{n+1},$$

where  $C = \sup_{y \in X} |\varphi(y)|$  (which is finite, since  $X$  is compact). As this holds for all  $n$ , it follows that  $\varphi(x_0) = \varphi(gx_0)$  for all  $\varphi$  in  $V^*$ . Hence,  $x_0 = gx_0$  for all  $g$  in  $G$ . ■

We discuss now the behaviour of amenability under exact sequences and directed unions.

**Proposition G.2.2** *Let  $G$  be a topological group.*

(i) *Let  $N$  be a closed normal subgroup of  $G$ . If  $G$  is amenable, then  $G/N$  is amenable.*

(ii) *Let  $N$  be a closed normal subgroup of  $G$ . If  $N$  and  $G/N$  are amenable, then  $G$  is amenable.*

(iii) *Let  $(G_i)_{i \in I}$  be a directed family of closed subgroups of  $G$  such that  $\bigcup_{i \in I} G_i$  is dense in  $G$ . If  $G_i$  is amenable for every  $i \in I$ , then  $G$  is amenable.*

**Proof** (i) follows from the fact that  $UCB(G/N)$  can be viewed as subspace of  $UCB(G)$ .

(ii) Assume that  $N$  and  $G/N$  are amenable, and that a continuous affine action of  $G$  on a non-empty compact convex set  $X$  is given. Let  $X^N$  be the closed subspace of all fixed points of  $N$  in  $X$ . It is clear that  $X^N$  is a compact convex subset of  $X$ , that  $X^N$  is invariant under  $G$ , and that the action of  $G$  on  $X^N$  factorises to an action of  $G/N$ . Since  $N$  is amenable,  $X^N$  is non-empty, by Theorem G.1.7.

Hence,  $G/N$  has a fixed point  $x_0$  in  $X^N$ , by the amenability of  $G/N$ . Clearly,  $x_0$  is a fixed point for  $G$ . The claim follows now from Theorem G.1.7.

(iii) Let  $G'$  denote the union  $\bigcup_{i \in I} G_i$ , with the inductive limit topology. The inclusion  $G' \rightarrow G$  is uniformly continuous with dense image, so that the restriction mapping  $UCB(G) \rightarrow UCB(G')$  is an isomorphism. We can therefore assume without loss of generality that  $G = \bigcup_{i \in I} G_i$ .

For every  $i \in I$ , let  $\mathcal{M}_i$  be the set of all  $G_i$ -invariant means on  $UCB(G)$ . We claim that  $\bigcap_{i \in I} \mathcal{M}_i$  is non-empty. This will prove (iii), since any  $m \in \bigcap_{i \in I} \mathcal{M}_i$  is a  $G$ -invariant mean on  $UCB(G)$ .

Observe that every  $\mathcal{M}_i$  is a closed subset of the compact set  $\mathcal{M}$  of all means on  $UCB(G)$  with respect to the weak\* topology. Hence, it suffices to show that the family  $(\mathcal{M}_i)_{i \in I}$  has the finite intersection property.

Let  $F$  be a finite subset of  $I$ . Let  $i \in I$  be such that  $G_j \subset G_i$  for all  $j \in F$ . Choose a  $G_i$ -invariant mean  $m_i$  on  $UCB(G_i)$ . Define a mean  $\tilde{m}_i$  on  $UCB(G)$  by

$$\tilde{m}_i(f) = m_i(f|_{G_i}), \quad \text{for all } f \in UCB(G).$$

Then  $\tilde{m}_i$  is  $G_i$ -invariant and therefore  $\tilde{m}_i \in \mathcal{M}_j$  for all  $j \in F$ . Thus,  $\bigcap_{j \in F} \mathcal{M}_j \neq \emptyset$ , as claimed. ■

**Corollary G.2.3** *Every compact extension of a soluble topological group is amenable. Every locally finite group is amenable.*

We now give examples of *non*-amenable groups.

**Example G.2.4** (i) The group  $G = SL_2(\mathbf{R})$  is non-amenable. Indeed, there is no invariant measure on the projective real line  $\mathbf{R} \cup \{\infty\} \cong G/P$  (see Example A.6.4). In particular, the action of  $G$  on the compact convex set of all probability measures on  $\mathbf{R} \cup \{\infty\}$  has no fixed points.

(ii) Let  $F_2$  be the non-abelian free group on two generators  $a$  and  $b$ , with the discrete topology. Then  $F_2$  is non-amenable.

Indeed, assume, by contradiction, that there exists an invariant mean  $m$  on  $UCB(F_2) = \ell^\infty(F_2)$ . For each subset  $B$  of  $F_2$ , write  $m(B)$  for the value of  $m$  on the characteristic function of  $B$ . Every element in  $F_2$  can be written as a reduced word in  $a, a^{-1}, b, b^{-1}$ . Let  $A$  be the subset of all words in  $F_2$  beginning with a non-zero power of  $a$ . Then  $F_2 = aA \cup A$ . As  $m(aA) = m(A)$  and  $m(F_2) = 1$ , it follows that  $m(A) \geq 1/2$ . On the other hand,  $A$ ,  $bA$  and  $b^2A$  are mutually disjoint subsets of  $F_2$ . Hence

$$1 = m(F_2) \geq m(A) + m(bA) + m(b^2A) = 3m(A) \geq \frac{3}{2}.$$

This is a contradiction.



(iii) The fundamental group  $\Gamma_g$  of a closed orientable Riemann surface of genus  $g \geq 2$  has a presentation

$$\Gamma_g = \left\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \left| \prod_{i=1}^g [a_i, b_i] \right. \right\rangle.$$

A surjective homomorphism  $\Gamma_g \rightarrow F_2$  is defined by  $a_1 \mapsto a$ ,  $a_2 \mapsto b$ ,  $a_i \mapsto e$  and  $b_j \mapsto e$  for all  $i = 3, \dots, g$  and  $j = 1, \dots, g$ . It follows from (ii) and from Proposition G.2.2 that  $\Gamma_g$  is not amenable.

### G.3 Weak containment and amenability

Let  $G$  be a *locally compact* group. We will always assume that  $G$  is equipped with a fixed *left* Haar measure; the spaces  $L^p(G)$  are taken with respect to this measure.

For such groups, it is more convenient to characterise amenability of  $G$  by the existence of invariant means on  $L^\infty(G)$ .

For  $f$  in  $L^1(G)$  and  $\varphi$  in  $L^\infty(G)$ , the convolution product  $f * \varphi$  belongs to  $UCB(G)$ . Let  $L^1(G)_{1,+}$  denote the convex set of all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ . Observe that  $L^1(G)_{1,+}$  is closed under convolution.

The set  $\mathcal{M}$  of all means on  $L^\infty(G)$  is a weak\* closed (and hence compact) convex subset of the unit ball of  $L^\infty(G)^*$ . We can view  $L^1(G)_{1,+}$  as a subset of  $\mathcal{M}$ , since every  $f$  in  $L^1(G)_{1,+}$  defines an element in  $\mathcal{M}$ , via integration against  $f$ . The Hahn–Banach Theorem shows that  $L^1(G)_{1,+}$  is weak\* dense in  $\mathcal{M}$  (Exercise G.6.2).

The following theorem is due to Reiter [Reiter–65] and Hulanicki [Hulan–66]. The proof we give for the equivalence of (iii) and (iv) is shorter than the proofs that we have found in the literature.

**Theorem G.3.1** *Let  $G$  be a locally compact group. The following properties are equivalent:*

- (i)  $G$  is amenable;
- (ii) there exists a topological invariant mean on  $L^\infty(G)$ , that is, a mean  $m$  on  $L^\infty(G)$  such that  $m(f * \varphi) = m(\varphi)$  for all  $f$  in  $L^1(G)_{1,+}$  and  $\varphi$  in  $L^\infty(G)$ ;
- (iii) (**Reiter’s Property** ( $P_1$ ))) for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists  $f$  in  $L^1(G)_{1,+}$  such that

$$\sup_{x \in Q} \|x^{-1}f - f\|_1 \leq \varepsilon;$$

(iv) (**Reiter's Property** ( $P_1^*$ )) for every finite subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists  $f$  in  $L^1(G)_{1,+}$  such that

$$\sup_{x \in Q} \|x^{-1}f - f\|_1 \leq \varepsilon;$$

(v) there exists an invariant mean on  $L^\infty(G)$ .

**Proof** Let us first show that (i) implies (ii). Let  $m$  be an invariant mean on  $UCB(G)$ . Since the mapping  $G \rightarrow UCB(G)$ ,  $y \mapsto y^{-1}\varphi$  is norm continuous for  $\varphi \in UCB(G)$ , the integral

$$f * \varphi = \int_G f(y) y^{-1} \varphi dy$$

is norm convergent in  $UCB(G)$ , for all  $f \in L^1(G)$ . It follows that

$$m(f * \varphi) = m(\varphi), \quad \text{for all } f \in L^1(G)_{1,+}, \varphi \in UCB(G).$$

Let  $(f_i)_i$  be a net in  $L^1(G)_{1,+}$  with  $\text{supp } (f_i) \rightarrow \{e\}$ . Then, for each  $\varphi \in L^\infty(G)$  and  $f \in L^1(G)_{1,+}$ ,

$$f * \varphi \in UCB(G) \quad \text{and} \quad \lim_i \|f * f_i * \varphi - f * \varphi\| = 0$$

and, hence,  $m(f * \varphi) = \lim_i m(f * f_i * \varphi) = \lim_i m(f_i * \varphi)$ . This shows that  $m(f * \varphi) = m(f' * \varphi)$  for all  $f, f' \in L^1(G)_{1,+}$  and all  $\varphi \in L^\infty(G)$ .

Fix any  $f_0 \in L^1(G)_{1,+}$ , and define a mean  $\tilde{m}$  on  $L^\infty(G)$  by

$$\tilde{m}(\varphi) = m(f_0 * \varphi), \quad \varphi \in L^\infty(G).$$

Then  $\tilde{m}$  is a topological invariant mean. Indeed,

$$\tilde{m}(f * \varphi) = m(f_0 * f * \varphi) = m(f_0 * \varphi) = \tilde{m}(\varphi),$$

for all  $\varphi \in L^\infty(G)$  and  $f \in L^1(G)_{1,+}$ .

Let us now show that (ii) implies (iii). Let  $m$  be a topological invariant mean on  $L^\infty(G)$ . Since  $L^1(G)_{1,+}$  is weak\* dense in the set of all means on  $L^\infty(G)$ , there exists a net  $(f_i)_i$  in  $L^1(G)_{1,+}$  converging to  $m$  in the weak\* topology. The topological invariance of  $m$  implies that, for every  $f \in L^1(G)_{1,+}$ ,

$$(*) \quad \lim_i (f * f_i - f_i) = 0$$

in the weak topology on  $L^1(G)$ .

For each  $f \in L^1(G)_{1,+}$ , take a copy of  $L^1(G)$  and consider the product space

$$E = \prod_{f \in L^1(G)_{1,+}} L^1(G),$$

with the product of the norm topologies. Then  $E$  is a locally convex space, and the weak topology on  $E$  is the product of the weak topologies. The set

$$\Sigma = \{(f * g - g)_{f \in L^1(G)_{1,+}} : g \in L^1(G)_{1,+}\} \subset E$$

is convex and, by  $(*)$ , its closure in the weak topology contains 0. Since  $E$  is locally convex, the closure of  $\Sigma$  in the weak topology coincides with the closure of  $\Sigma$  in the original topology of  $E$  (see [Rudin-73, Theorem 3.12]). Hence, there exists a net  $(g_j)_j$  in  $L^1(G)_{1,+}$  such that, for every  $f$  in  $L^1(G)_{1,+}$ ,

$$(**) \quad \lim_j \|f * g_j - g_j\|_1 = 0.$$

Since the  $g_j$ 's have bounded  $L^1$ -norm,  $(**)$  holds *uniformly* for all  $f$  in any norm compact subset  $K$  of  $L^1(G)$ . Indeed, let  $\varepsilon > 0$ . There exists  $\xi_1, \dots, \xi_n$  such that, for every  $f \in K$ , we have  $\|f - \xi_k\|_1 \leq \varepsilon$  for some  $k \in \{1, \dots, n\}$ . Choose  $j_0$  such that

$$\|\xi_l * g_j - g_j\|_1 \leq \varepsilon \quad \text{for all } l \in \{1, \dots, n\} \quad \text{and } j \geq j_0.$$

We have, for every  $j \geq j_0$  and for  $f \in K$ ,

$$\begin{aligned} \|f * g_j - g_j\|_1 &\leq \|f * g_j - \xi_k * g_j\|_1 + \|\xi_k * g_j - g_j\|_1 \\ &\leq \|f - \xi_k\|_1 + \|\xi_k * g_j - g_j\|_1 \leq 2\varepsilon, \end{aligned}$$

where  $k$  is such that  $\|f - \xi_k\|_1 \leq \varepsilon$ .

Let now  $Q$  be a compact subset of  $G$  containing  $e$ , and let  $\varepsilon > 0$ . Fix any  $f$  in  $L^1(G)_{1,+}$ . Since the mapping

$$G \rightarrow L^1(G), \quad x \mapsto {}_{x^{-1}}f$$

is continuous (Exercise A.8.3),  $\{{}_{x^{-1}}f : x \in Q\}$  is a compact subset of  $L^1(G)$ . Hence, there exists  $j$  such that

$$\|{}_{x^{-1}}f * g_j - g_j\|_1 \leq \varepsilon$$

for all  $x \in Q$ . Set  $g = f * g_j$ . Then  $g \in L^1(G)_{1,+}$  and, for all  $x \in Q$ , we have

$$\|x^{-1}g - g\|_1 \leq \|x^{-1}f * g_j - g_j\|_1 + \|f * g_j - g_j\|_1 \leq 2\varepsilon.$$

The fact that (iii) implies (iv) is obvious. Assume that (iv) holds. Then there is a net  $(f_i)_i$  in  $L^1(G)_{1,+}$  such that

$$(***) \quad \lim_i \|x^{-1}f_i - f_i\|_1 = 0.$$

for all  $x \in G$ . Let  $m$  be a weak\* limit point of  $(f_i)_i$  in the set of all means on  $L^\infty(G)$ . It follows from (\*\*\*) that  $m$  is invariant. This shows that (iv) implies (v).

That (v) implies (i) is obvious, since  $UCB(G)$  can be viewed as subspace of  $L^\infty(G)$ . ■

The following result gives a characterisation of amenability of a locally compact group  $G$  in terms of a weak containment property of  $\lambda_G$ . Its proof is based on the equivalence between amenability and Reiter's Property  $(P_1)$  from the previous lemma.

**Theorem G.3.2 (Hulanicki–Reiter)** *Let  $G$  be a locally compact group. The following properties are equivalent:*

- (i)  $G$  is amenable;
- (ii)  $1_G \prec \lambda_G$ ;
- (iii)  $\pi \prec \lambda_G$  for every unitary representation  $\pi$  of  $G$ .

**Proof** The equivalence of (ii) and (iii) was already proved in Proposition F.3.3. In view of the previous theorem, it suffices to show that (ii) is equivalent to Reiter's Property  $(P_1)$ .

Assume that (ii) holds. Then, given a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$ , there exists  $f_{Q,\varepsilon} \in L^2(G)$  with  $\|f_{Q,\varepsilon}\|_2 = 1$  such that

$$\sup_{x \in Q} \|\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}\|_2 < \varepsilon$$

(see Corollary F.1.5). Set  $g_{Q,\varepsilon} = |f_{Q,\varepsilon}|^2$ . Then  $g_{Q,\varepsilon} \in L^1(G)_{1,+}$  and, by the Cauchy–Schwarz inequality,

$$\begin{aligned} \|x^{-1}g_{Q,\varepsilon} - g_{Q,\varepsilon}\|_1 &\leq \|\lambda_G(x)\overline{f_{Q,\varepsilon}} + \overline{f_{Q,\varepsilon}}\|_2 \|\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}\|_2 \\ &\leq 2\|\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}\|_2 < 2\varepsilon \end{aligned}$$

for all  $x \in Q$ . Hence,  $G$  has Reiter's Property  $(P_1)$ .

Conversely, assume that  $G$  has Reiter's Property  $(P_1)$ . For a compact subset  $Q$  of  $G$  and  $\varepsilon > 0$ , let  $f \in L^1(G)_{1,+}$  be such that

$$\sup_{x \in Q} \|x^{-1}f - f\|_1 < \varepsilon.$$

Let  $g = \sqrt{f}$ . Then  $g \in L^2(G)$  and  $\|g\|_2 = 1$ . Moreover, using the inequality  $|a - b|^2 \leq |a^2 - b^2|$  for all non-negative real numbers  $a$  and  $b$ , we have

$$\begin{aligned} \|\lambda_G(x)g - g\|_2^2 &\leq \int_G |g(x^{-1}y)^2 - g(y)^2| dy = \|x^{-1}f - f\|_1 \\ &< \varepsilon, \end{aligned}$$

for all  $x \in Q$ . This shows that  $1_G \prec \lambda_G$ . ■

**Remark G.3.3** The amenability of  $G$  is equivalent to the fact that  $1_G$  is weakly contained in  $\lambda_G$ , when both representations are viewed as unitary representations of  $G_d$ , the group  $G$  viewed as *discrete* group. Indeed, the proof of the previous theorem shows that Reiter's Property  $(P_1^*)$  from Theorem G.3.1 is equivalent to the fact that  $1_G$  is weakly contained in  $\lambda_G$  when both are viewed as representations of  $G_d$ .

**Corollary G.3.4** *Closed subgroups of amenable locally compact groups are amenable.*

**Proof** Since the restriction of  $\lambda_G$  to  $H$  is weakly contained in  $\lambda_H$  (Proposition F.1.10), the claim follows from the previous theorem. ■

Since the free group on two generators  $F_2$  is not amenable (see Example G.2.4), this corollary has the following consequence.

**Corollary G.3.5** *Let  $G$  be a locally compact group containing  $F_2$  as a closed subgroup. Then  $G$  is not amenable.*

**Example G.3.6** (i) The previous corollary can be applied to give another proof of the non-amenability of  $SL_2(\mathbf{R})$ ; see Example G.2.4. Indeed,  $SL_2(\mathbf{R})$  contains  $F_2$  as a lattice (Example B.2.5)

(ii) More generally, a non-compact semisimple real Lie group is non-amenable. Indeed, such a group has a quotient which contains a closed subgroup isomorphic to  $PSL_2(\mathbf{R})$ .

(iii) Let  $\mathbf{K}$  be a local field (see D.4). Then  $SL_2(\mathbf{K})$  is not amenable. Indeed,  $SL_2(\mathbf{K})$  contains a discrete group which is isomorphic to  $F_2$  (see Exercises G.6.8 and G.6.9).

**Remark G.3.7** The previous corollary is, in general, not true for non-locally compact groups.

Indeed, let  $G = \mathcal{U}(\mathcal{H})$  be the unitary group of an infinite dimensional separable Hilbert space. Equipped with the weak operator topology,  $\mathcal{U}(\mathcal{H})$  is a topological group (Exercise G.6.3). It is shown in [Harpe–73] that  $\mathcal{U}(\mathcal{H})$  is amenable.

On the other hand, realising  $\mathcal{U}(\mathcal{H})$  as  $\mathcal{U}(\ell^2(F_2))$ , we see that  $\mathcal{U}(\mathcal{H})$  contains the subgroup

$$\Gamma = \{\lambda_{F_2}(\gamma) : \gamma \in F_2\}$$

which is isomorphic to  $F_2$ . The claim follows, as  $\Gamma$  is closed in  $\mathcal{U}(\ell^2(F_2))$ ; see Exercise G.6.4.

As we now show, the amenability of a closed subgroup  $H$  of a locally compact group  $G$  implies the amenability of  $G$  if  $G/H$  is an *amenable homogeneous space* in Eymard’s sense, that is, if the quasi-regular representation  $\lambda_{G/H}$  weakly contains  $1_G$ . This is the case, for example, if  $H$  is a lattice in  $G$ , or, more generally, if  $G/H$  has a finite invariant regular Borel measure. See [Eymard–72], where this notion is extensively studied.

**Corollary G.3.8** *Let  $H$  be a closed subgroup of a locally compact group  $G$  such that  $G/H$  is amenable. The following are equivalent:*

- (i)  $G$  is amenable;
- (ii)  $H$  is amenable.

**Proof** In view of Corollary G.3.4, we only have to show that if  $H$  is amenable, then so is  $G$ . By the Hulanicki–Reiter Theorem G.3.2, we have  $1_H \prec \lambda_H$ . Hence,

$$\lambda_{G/H} = \text{Ind}_H^G 1_H \prec \text{Ind}_H^G \lambda_H = \lambda_G,$$

by continuity of induction (Theorem F.3.5). Since  $G/H$  is amenable, we have  $1_G \prec \lambda_{G/H}$  and therefore  $1_G \prec \lambda_G$ . ■

The previous corollary, in combination with Remark G.3.6.ii, shows that a lattice in a non-compact semisimple real Lie group is never amenable.

The following characterisation of amenability in terms of  $C^*$ -algebras follows immediately from Theorem F.4.4 and the Hulanicki–Reiter Theorem.

**Corollary G.3.9** *For a locally compact group  $G$ , the following properties are equivalent:*

- (i)  $G$  is amenable;
- (ii)  $\lambda_G$  extends to an isomorphism between the maximal  $C^*$ -algebra  $C^*(G)$  and the reduced  $C^*$ -algebra  $C_r^*(G)$  of  $G$ .

## G.4 Kesten's characterisation of amenability

Let  $G$  be a locally compact group, and let  $\mu$  be a complex-valued finite regular Borel measure on  $G$ . For a unitary representation  $(\pi, \mathcal{H})$  of  $G$ , let  $\pi(\mu)$  be the operator in  $\mathcal{L}(\mathcal{H})$  defined by

$$\langle \pi(\mu)\xi, \eta \rangle = \int_G \langle \pi(x)\xi, \eta \rangle d\mu(x), \quad \xi, \eta \in \mathcal{H}.$$

Then  $\|\pi(\mu)\| \leq \|\mu\|$ . Observe that  $\pi(\mu^* * \mu) = \pi(\mu)^* \pi(\mu)$  is a positive self-adjoint operator and that  $\|\pi(\mu^* * \mu)\| = \|\pi(\mu)\|^2$ . (For a measure  $\mu$  on  $G$ , the measure  $\mu^*$  is defined by  $d\mu^*(x) = \overline{d\mu(x^{-1})}$ .) In case  $\mu$  is absolutely continuous with respect to a Haar measure  $dx$  on  $G$ , that is,  $d\mu(x) = f(x)dx$  for some  $f \in L^1(G)$ , we write  $\pi(f)$  instead of  $\pi(\mu)$ .

Assume now that  $\mu$  is a probability measure on  $G$ , (that is,  $\mu$  is positive with total mass  $\mu(G) = 1$ ). Then  $\pi(\mu)$  is a linear contraction on  $\mathcal{H}$ . By a *linear contraction* on a normed vector space  $V$ , we mean a bounded operator  $T : V \rightarrow V$  with  $\|T\| \leq 1$ . We show how weak containment of  $1_G$  in  $\pi$  is related to the spectrum of the operator  $\pi(\mu)$ ; compare [HaRoV-93, Proposition 2]. We first need the following elementary lemma.

**Lemma G.4.1** *Let  $T$  be a linear contraction on a Hilbert space  $\mathcal{H}$ . Then 1 belongs to the spectrum  $\sigma(T)$  if and only if 1 is an approximate eigenvalue of  $T$  (that is, there exists a sequence of unit vectors  $\xi_n$  in  $\mathcal{H}$  such that  $\lim_n \|T\xi_n - \xi_n\| = 0$ ).*

**Proof** It is clear that if 1 is an approximate eigenvalue of  $T$  then  $1 \in \sigma(T)$ . Conversely, assume that  $1 \in \sigma(T)$ . Two cases can occur. If the image of  $T - I$  is dense in  $\mathcal{H}$ , then 1 is an approximate eigenvalue of  $T$ . If the image of  $T - I$  is not dense in  $\mathcal{H}$ , then  $\text{Ker}(T^* - I) \neq 0$ , that is,  $T^*$  has a non-zero fixed vector  $\xi$ . As  $T$  is a contraction,  $\xi$  is also a fixed vector of  $T$ . Indeed,

$$\langle T\xi, \xi \rangle = \langle \xi, T^*\xi \rangle = \|\xi\|^2,$$

and it follows from the equality case of the Cauchy–Schwarz inequality that  $T\xi = \xi$ . Thus, 1 is an eigenvalue of  $T$ . ■

**Proposition G.4.2** *Let  $G$  be a locally compact group and let  $\mu$  be a probability measure on  $G$  which is absolutely continuous with respect to the Haar measure on  $G$ . For a unitary representation  $(\pi, \mathcal{H})$  of  $G$ , consider the*

following properties:

- (i)  $1_G \prec \pi$ ;
- (ii) 1 belongs to the spectrum  $\sigma(\pi(\mu))$  of the operator  $\pi(\mu)$ ;
- (iii)  $\|\pi(\mu)\| = 1$ .

Then (i) implies (ii) and (ii) implies (iii). If the subgroup generated by the support of  $\mu$  is dense in  $G$ , then (ii) implies (i). If the subgroup generated by the support of  $\mu^* * \mu$  is dense in  $G$ , then (iii) implies (i).

**Proof** To show that (i) implies (ii), assume that  $1_G \prec \pi$ . For every  $n \in \mathbf{N}$ , let  $Q_n$  be a compact subset of  $G$  such that  $\mu(G \setminus Q_n) \leq 1/n$ . There exists a unit vector  $\xi_n$  such that  $\|\pi(x)\xi_n - \xi_n\| \leq 1/n$  for all  $x \in Q_n$ . Then

$$\|\pi(\mu)\xi_n - \xi_n\| \leq \int_{Q_n} \|\pi(x)\xi_n - \xi_n\| d\mu(x) + \frac{2}{n} \leq \frac{3}{n}.$$

Hence, 1 is an approximate eigenvalue of  $\pi(\mu)$ .

To show that (ii) implies (i), when the support of  $\mu$  generates a dense subgroup of  $G$ , assume that  $1 \in \sigma(\pi(\mu))$ . By the previous lemma, 1 is an approximate eigenvalue of  $\pi(\mu)$ . Thus, for every  $n \in \mathbf{N}$ , there exists a unit vector  $\xi_n$  in  $\mathcal{H}$  such that

$$\|\pi(\mu)\xi_n - \xi_n\| \leq \frac{1}{n}.$$

It follows that  $\|\pi(\mu)\xi_n\| \geq 1/2$  for all  $n \geq 2$ . Set  $\eta_n = \|\pi(\mu)\xi_n\|^{-1}\pi(\mu)\xi_n$ . Then  $\eta_n$  is a unit vector and

$$\|\pi(\mu)\eta_n - \eta_n\| \leq \frac{1}{\|\pi(\mu)\xi_n\|} \|\pi(\mu)\xi_n - \xi_n\| \leq \frac{2}{n}.$$

Since

$$|1 - \operatorname{Re}\langle \pi(\mu)\eta_n, \eta_n \rangle| \leq |\langle \pi(\mu)\eta_n - \eta_n, \eta_n \rangle| \leq \|\pi(\mu)\eta_n - \eta_n\|,$$

it follows that

$$\lim_n \int_G (1 - \operatorname{Re}\langle \pi(x)\eta_n, \eta_n \rangle) d\mu(x) = \lim_n (1 - \langle \pi(\mu)\eta_n, \eta_n \rangle) = 0.$$

Hence, as  $1 - \operatorname{Re}\langle \pi(x)\eta_n, \eta_n \rangle \geq 0$  for all  $x \in G$ , there exists a subsequence, still denoted by  $(\eta_n)_n$ , such that

$$\lim_n \operatorname{Re}\langle \pi(x)\eta_n, \eta_n \rangle = 1$$



for  $\mu$ -almost every  $x$  in  $G$ . Since  $\|\pi(x)\eta_n - \eta_n\|^2 = 2(1 - \operatorname{Re}\langle \pi(x)\eta_n, \eta_n \rangle)$ , it follows that

$$(*) \quad \lim_n \|\pi(x)\eta_n - \eta_n\| = 0$$

for  $\mu$ -almost every  $x$  in  $G$ . The set of all  $x$  for which  $(*)$  holds is clearly a measurable subgroup  $H$  of  $G$ . Since  $\mu(H) = 1$ , the support of  $\mu$  is contained in the closure  $\overline{H}$  of  $H$ . Hence,  $\overline{H} = G$ , by our assumption on  $\mu$ .

Let  $Q$  be a compact subset of  $G$  and  $\varepsilon > 0$ . Since  $\mu$  is absolutely continuous with respect to the Lebesgue measure on  $G$ , we have  $d\mu(x) = f(x)dx$  for some  $f \in L^1(G)_{1,+}$ . The mapping

$$G \rightarrow L^1(G), x \mapsto x^{-1}f$$

is continuous (Exercise A.8.3). We can therefore find a neighbourhood  $U$  of  $e$  such that

$$\|u^{-1}f - f\|_1 \leq \varepsilon, \quad \text{for all } u \in U.$$

Since  $H$  is dense in  $G$  and  $Q$  is compact, there exists  $x_1, \dots, x_r$  in  $H$  such that  $Q \subset \bigcup_{i=1}^r x_i U$ . By  $(*)$ , there exists  $n$  such that

$$\|\pi(x_i)\eta_n - \eta_n\| \leq \varepsilon, \quad \text{for all } i = 1, \dots, r.$$

Let  $x \in Q$ . Then  $x = x_i u$  for some  $1 \leq i \leq r$  and some  $u \in U$ ; we then have

$$\begin{aligned} \|\pi(x)\eta_n - \eta_n\| &\leq \|\pi(x_i)(\pi(u)\eta_n - \eta_n)\| + \|\pi(x_i)\eta_n - \eta_n\| \\ &= \|\pi(u)\eta_n - \eta_n\| + \|\pi(x_i)\eta_n - \eta_n\| \\ &= \|\pi(\mu)\xi_n\|^{-1} \|\pi(u)\pi(\mu)\xi_n - \pi(\mu)\xi_n\| + \|\pi(x_i)\eta_n - \eta_n\| \\ &= \|\pi(\mu)\xi_n\|^{-1} \|\pi(u^{-1}f - f)\xi_n\| + \|\pi(x_i)\eta_n - \eta_n\| \\ &\leq 2\|u^{-1}f - f\|_1 + \|\pi(x_i)\eta_n - \eta_n\| \leq 3\varepsilon. \end{aligned}$$

This shows that  $1_G < \pi$ .

On (ii) and (iii): since

$$r(\sigma(\pi(\mu))) \leq \|\pi(\mu)\| \leq 1,$$

where  $r(\sigma(\pi(\mu)))$  is the spectral radius of the operator  $\sigma(\pi(\mu))$ , it is obvious that (ii) implies (iii). Assume that  $\|\pi(\mu)\| = 1$ . Then

$$\|\pi(\mu^* * \mu)\| = \|\pi(\mu)\|^2 = 1.$$

Since  $\pi(\mu^* * \mu)$  is self-adjoint and positive, it follows that  $1 \in \sigma(\pi(\mu^* * \mu))$ . If the subgroup generated by the support of  $\mu^* * \mu$  is dense, we can apply the argument used for (ii)  $\implies$  (i) with  $\mu^* * \mu$  in place of  $\mu$  and we obtain that  $1_G \prec \pi$ . ■

**Remark G.4.3** (i) In the previous proposition, the assumption that  $\mu$  is absolutely continuous with respect to the Haar measure on  $G$  is necessary; see Exercise G.6.6.

(ii) Let  $\mu$  be as in the previous proposition. Denote by  $X$  the support of the probability measure  $\mu$  and by  $Y$  that of  $\mu^* * \mu$ ; denote by  $\langle X \rangle$  and  $\langle Y \rangle$  the subgroups of  $G$  generated by  $X$  and  $Y$ .

Then  $\langle Y \rangle$  is a subgroup of  $\langle X \rangle$ . If  $e \in X$ , then  $Y$  contains  $X$  and  $\langle Y \rangle = \langle X \rangle$ . Otherwise, the inclusion  $\langle Y \rangle \subset \langle X \rangle$  can be strict: this is the case when  $G = \mathbf{Z}$  and  $\mu = \delta_1$ , since  $\mu^* * \mu = \delta_0$ .

(iii) With the notation of (ii),  $\langle Y \rangle$  is not dense in  $G$  if and only if  $X$  is contained in the left coset of a proper closed subgroup of  $G$ .

(iv) As the proof shows, the fact that (i) implies (ii) in the previous proposition is true if  $\mu$  is *any* probability measure on  $G$ .

The following characterisation of the amenability of a locally compact group was given by Kesten (see [Kest–59a]) in the case of a countable group  $G$  and a symmetric probability measure on  $G$ . The general case is due to [DerGu–73], where the result is proved without the absolute continuity assumption on the probability measure  $\mu$ ; see also [BeCh–74a].

**Theorem G.4.4 (Kesten)** *Let  $G$  be a locally compact group, and let  $\mu$  be a probability measure on  $G$ . Assume that  $\mu$  is absolutely continuous with respect to the Haar measure on  $G$  and that its support generates a dense subgroup in  $G$ . The following properties are equivalent:*

- (i)  $G$  is amenable;
- (ii) 1 belongs to the spectrum of the operator  $\lambda_G(\mu)$ ;
- (iii) the spectral radius  $r(\lambda_G(\mu))$  of  $\lambda_G(\mu)$  is 1.

**Proof** The equivalence of (i) and (ii) follows from the previous proposition together with the Hulanicki–Reiter Theorem G.3.2.

Since  $r(\lambda_G(\mu)) \leq \|\lambda_G(\mu)\| \leq 1$ , it is obvious that (ii) implies (iii).

Assume that  $r(\lambda_G(\mu)) = 1$ . Then  $\lambda_G(\mu)$  has a spectral value  $c$  with  $|c| = 1$ . The operator  $T = \bar{c}\lambda_G(\mu)$  is a contraction and has 1 as spectral value. Hence, by Lemma G.4.1, there exists a sequence of unit vectors  $f_n$  in  $L^2(G)$  such that

$\lim_n \|Tf_n - f_n\| = 0$ . Then

$$\lim_n \|\lambda_G(\mu)f_n - cf_n\| = 0,$$

or, equivalently,

$$\lim_n \int_G \langle \lambda_G(x)f_n, f_n \rangle d\mu(x) = \lim_n \langle \lambda_G(\mu)f_n, f_n \rangle = c.$$

In particular, we have

$$\lim_n \left| \int_G \langle \lambda_G(x)f_n, f_n \rangle d\mu(x) \right| = 1.$$

Since

$$\begin{aligned} 1 &= \int_G \|\lambda_G(x)f_n\| \|f_n\| d\mu(x) \geq \int_G \langle \lambda_G(x)|f_n|, |f_n| \rangle d\mu(x) \\ &\geq \int_G |\langle \lambda_G(x)f_n, f_n \rangle| d\mu(x) \\ &\geq \left| \int_G \langle \lambda_G(x)f_n, f_n \rangle d\mu(x) \right| \end{aligned}$$

for all  $n \in \mathbb{N}$ , it follows that

$$\lim_n \langle \lambda_G(\mu)|f_n|, |f_n| \rangle = \lim_n \int_G \langle \lambda_G(x)|f_n|, |f_n| \rangle d\mu(x) = 1,$$

that is,  $\lim_n \|\lambda_G(\mu)|f_n| - |f_n|\| = 0$ . Hence, 1 is a spectral value of  $\lambda_G(\mu)$ , showing that (iii) implies (ii). ■

**Remark G.4.5** Condition (ii) and Condition (iii) in the previous theorem cannot be replaced by the condition  $\|\lambda_G(\mu)\| = 1$ ; see Exercise G.6.7.

**Corollary G.4.6** Let  $\Gamma$  be a finitely generated group, with a finite generating set  $S$ . Let  $\mu_S = \frac{1}{|S|} \sum_{s \in S} \delta_s$ . The following properties are equivalent:

- (i)  $\Gamma$  is amenable;
- (ii) 1 belongs to the spectrum of  $\lambda_\Gamma(\mu_S)$ ;
- (iii)  $r(\lambda_\Gamma(\mu_S)) = 1$ .

**Remark G.4.7** The previous corollary admits the following probabilistic interpretation.

Let  $\Gamma$  be a finitely generated group. Let  $S$  be a finite generating set of  $G$  with  $S^{-1} = S$ . The *Cayley graph*  $\mathcal{G}(\Gamma, S)$  of  $\Gamma$  with respect to  $S$  is the graph where the vertices are the elements of  $\Gamma$ , and where  $x$  and  $y$  in  $\Gamma$  are connected by an edge if  $x^{-1}y \in S$ . Consider the random walk on  $\mathcal{G}(\Gamma, S)$  in which every step consists of left multiplication by  $s \in S$  with probability  $1/|S|$ . This random walk defines a Markov chain, with associated Markov operator

$$M = \frac{1}{|S|} \sum_{s \in S} \lambda_{\Gamma}(\delta_s)$$

acting on  $\ell^2(\Gamma)$ . Observe that  $M = \lambda_{\Gamma}(\mu_S)$ , with  $\mu_S$  as in the previous corollary.

For  $n \in \mathbb{N}$ , let  $\mu_S^{*n}$  be the  $n$ -fold convolution product of  $\mu_S$  with itself. Then,

$$\langle M^n \delta_e, \delta_e \rangle = \mu_S^{*n}(e)$$

is the probability  $p_n$  of the random walk having started at  $e$  to return to  $e$  at the  $n$ -th step. As Lemma G.4.8 below shows,

$$\limsup_n \mu_S^{*n}(e) = r(\lambda_{\Gamma}(\mu_S)) = \|\lambda_{\Gamma}(\mu_S)\|.$$

In the case where  $\Gamma$  is amenable, we have therefore  $\limsup_n p_n^{1/n} = 1$ ; in particular, the sequence  $(p_n/\alpha^n)_n$  is unbounded for any  $0 < \alpha < 1$ .

**Lemma G.4.8** *For  $\mu \in \ell^1(\Gamma)$ , we have*

$$\|\lambda_{\Gamma}(\mu)\| = \limsup_n ((\mu^* * \mu)^{*n}(e))^{1/2n}.$$

**Proof** We follow a proof shown to us by C. Anantharaman. Set

$$T = \lambda_{\Gamma}(\mu^* * \mu) = \lambda_{\Gamma}(\mu)^* \lambda_{\Gamma}(\mu).$$

Since  $T$  is a self-adjoint positive operator, its spectrum  $\sigma(T)$  is contained in  $[0, \|T\|]$ . Let  $\nu$  be the probability measure on  $\sigma(T)$  defined by

$$\int_{\sigma(T)} f(t) d\nu(t) = \langle f(T) \delta_e, \delta_e \rangle \quad \text{for all } f \in C(\sigma(T)),$$

where  $f(T) \in \mathcal{L}(\ell^2(\Gamma))$  is given by the functional calculus (see the comments before Theorem A.2.2).

We claim that the support of  $\nu$  is  $\sigma(T)$ . Indeed, otherwise there exists  $f \in C(\sigma(T))$  with  $f \geq 0$  and  $f \neq 0$  such that  $\int_{\sigma(T)} f(t) d\nu(t) = 0$ . Then

$$\|f^{1/2}(T) \delta_e\|^2 = \langle f(T) \delta_e, \delta_e \rangle = 0,$$

so that  $f^{1/2}(T)\delta_e = 0$ . Since  $T$  and hence  $f^{1/2}(T)$  commute with right translations by elements from  $\Gamma$ , it follows that

$$f^{1/2}(T)\delta_\gamma = f^{1/2}(T)\rho_\Gamma(\gamma)\delta_e = \rho_\Gamma(\gamma)f^{1/2}(T)\delta_e = 0, \quad \text{for all } \gamma \in \Gamma,$$

where  $\rho_\Gamma$  is the right regular representation of  $\Gamma$ . This implies that  $f^{1/2}(T) = 0$  and hence  $f = 0$ , which is a contradiction.

Let  $\varepsilon > 0$ . Set  $r = v(\|T\| - \varepsilon, \|T\|)$ . Then  $r > 0$ , since  $\text{supp}(v) = \sigma(T)$ . For every  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \|T\| &\geq \langle T^n \delta_e, \delta_e \rangle^{1/n} = \left( \int_{\sigma(T)} t^n d\nu(t) \right)^{1/n} \\ &\geq \left( \int_{\|T\|-\varepsilon}^{\|T\|} t^n d\nu(t) \right)^{1/n} \geq (r(\|T\| - \varepsilon)^n)^{1/n} \\ &= r^{1/n}(\|T\| - \varepsilon). \end{aligned}$$

Since  $\lim_n r^{1/n} = 1$ , it follows that

$$\|T\| \geq \limsup_n \langle T^n \delta_e, \delta_e \rangle^{1/n} \geq \|T\| - \varepsilon$$

for every  $\varepsilon > 0$ . Hence,

$$\|T\| = \limsup_n \langle T^n \delta_e, \delta_e \rangle^{1/n} = \limsup_n ((\mu^* * \mu)^{*n}(e))^{1/n}.$$

Since  $\|\lambda_\Gamma(\mu)\|^2 = \|T\|$ , this proves the claim. ■

For a version of the previous lemma, valid for arbitrary locally compact groups, see [BeCh-74b].

**Example G.4.9** (i) Let  $\Gamma = \mathbb{Z}$ , with generating set  $S = \{1, -1\}$ . Then  $\mu_S$  is the probability measure defined by  $\mu_S(1) = \mu_S(-1) = 1/2$ . The probability  $\mu_S^{*n}(0)$  of returning to 0 after  $n$  steps is obviously 0 if  $n$  is odd. If  $n = 2k$ , it is the probability of moving  $k$  times to the right and  $k$  times to the left, that is,

$$\mu_S^n(0) = \frac{1}{4^k} \binom{2k}{k} \approx \frac{1}{(\pi k)^{1/2}}.$$

So,  $\lim_n \sup \mu_S^{*n}(0)^{1/n} = 1$ , and we recover the fact that  $\|\lambda_\Gamma(\mu_S)\| = 1$ .

(ii) For  $k > 1$ , let  $\Gamma = F_k$  be the free group on  $k$  generators  $a_1, \dots, a_k$ , and  $S = \{a_1, \dots, a_k, a_1^{-1}, \dots, a_k^{-1}\}$ . For the probability measure  $\mu_S$ , we have

$$\|\lambda_\Gamma(\mu_S)\| = \limsup_n \mu_S^{*n}(e) = \frac{\sqrt{2k-1}}{k}.$$

This result has the following converse. Let  $\Gamma$  be a group generated by  $k \geq 2$  elements  $a_1, \dots, a_k$ . Assume that all  $a_i$ 's are of order greater than two and let  $\mu$  be the probability measure defined by  $\mu(a_i) = \mu(a_i^{-1}) = 1/2k$  for all  $i = 1, \dots, k$ . If  $\|\lambda_\Gamma(\mu)\| = \frac{\sqrt{2k-1}}{k}$ , then  $\Gamma$  is the free group on  $a_1, \dots, a_k$ . For all this, see [Kest–59b, Theorem 3].

## G.5 Følner's property

A useful refinement of Reiter's Property  $(P_1)$  of Theorem G.3.1 is Følner's Property. Recall that, for a measurable subset  $A$  of a locally compact group,  $|A|$  denotes the measure of  $A$  with respect to a *left* invariant measure.

**Theorem G.5.1** *Let  $G$  be a locally compact group. The following properties are equivalent:*

- (i)  $G$  is amenable;
- (ii) (**Følner's Property**) for every compact subset  $Q$  of  $G$  and every  $\varepsilon > 0$ , there exists a Borel subset  $U$  of  $G$  with  $0 < |U| < \infty$  such that

$$\frac{|xU \Delta U|}{|U|} \leq \varepsilon, \quad \text{for all } x \in Q,$$

where  $\Delta$  denotes the symmetric difference.

We will show that Følner's Property is equivalent Reiter's Property  $(P_1)$ . If  $f_U = |U|^{-1}\chi_U \in L^1(G)_{1,+}$  is the normalised characteristic function of a measurable subset  $U$  of  $G$  with  $|U| \neq 0$ , observe that

$$x^{-1}f_U(y) = |U|^{-1}\chi_U(x^{-1}y) = |U|^{-1}\chi_{xU}(y)$$

for all  $x, y \in G$ , so that

$$\|x^{-1}f_U - f_U\|_1 = \frac{|xU \Delta U|}{|U|}.$$

This shows that Følner's Property implies Reiter's Property  $(P_1)$ . The point is proving the converse. We will need the following elementary lemma.

**Lemma G.5.2** *Let  $(X, \mu)$  be a measure space. Let  $f, f'$  be non-negative functions in  $L^1(X)$ . For every  $t \geq 0$ , let  $E_t = \{x \in X : f(x) \geq t\}$  and  $E'_t = \{x \in X : f'(x) \geq t\}$ . Then*

$$\|f - f'\|_1 = \int_0^\infty \mu(E_t \Delta E'_t) dt.$$

*In particular,  $\|f\|_1 = \int_0^\infty \mu(E_t) dt$ .*

**Proof** Denote by  $\chi_t : [0, \infty) \rightarrow \mathbf{R}$  the characteristic function of  $[t, \infty)$ . We have

$$\int_0^\infty |\chi_t(s) - \chi_t(s')| dt = \int_0^\infty |\chi_s(t) - \chi_{s'}(t)| dt = |s - s'|$$

for  $s, s' \in [0, \infty)$ , and therefore

$$\int_0^\infty |\chi_t \circ f(x) - \chi_t \circ f'(x)| dt = |f(x) - f'(x)|$$

for  $x \in X$ . By Fubini's Theorem, it follows that

$$\begin{aligned} \int_0^\infty \|\chi_t \circ f - \chi_t \circ f'\|_1 dt &= \int_0^\infty \int_X |\chi_t \circ f(x) - \chi_t \circ f'(x)| d\mu(x) dt \\ &= \int_X |f(x) - f'(x)| d\mu(x) = \|f - f'\|_1. \end{aligned}$$

On the other hand,  $\chi_t \circ f(x) = \chi_t(f(x))$  is 1 if  $f(x) \geq t$  and 0 if  $f(x) < t$ , so that  $|\chi_t \circ f - \chi_t \circ f'|$  is the characteristic function of  $E_t \Delta E'_t$ . It follows that

$$\|\chi_t \circ f - \chi_t \circ f'\|_1 = \mu(E_t \Delta E'_t),$$

and this proves the claim. ■

**Proof of Theorem G.5.1** We have to show that Reiter's Property  $(P_1)$  implies Følner's Property. Let  $Q$  be a compact neighbourhood of  $e$ , and let  $\varepsilon > 0$ . Then  $K = Q^2$  is a compact subset of  $G$ , and there exists  $f \in L^1(G)_{1,+}$  such that

$$\sup_{x \in K} \|x^{-1}f - f\|_1 \leq \frac{\varepsilon|Q|}{2|K|}.$$

For  $t \geq 0$ , let as above  $E_t = \{y \in G : f(y) \geq t\}$ . Then

$$xE_t = \{y \in G : x^{-1}f(y) \geq t\}.$$

By the previous lemma,

$$\|_{x^{-1}f} - f\|_1 = \int_0^\infty |xE_t \Delta E_t| dt.$$

for every  $x \in G$ . Hence, for every  $x \in K$ , we have

$$\begin{aligned} \int_0^\infty |E_t| \left( \int_K \frac{|xE_t \Delta E_t|}{|E_t|} dx \right) dt &= \int_K \|_{x^{-1}f} - f\|_1 dx \\ &\leq \frac{\varepsilon|Q|}{2}. \end{aligned}$$

Since  $\int_0^\infty |E_t| dt = \|f\|_1 = 1$ , it follows that there exists  $t$  such that  $0 < |E_t| < \infty$  and

$$\int_K \frac{|xE_t \Delta E_t|}{|E_t|} dx \leq \frac{\varepsilon|Q|}{2}.$$

For the set

$$A = \left\{ x \in K : \frac{|xE_t \Delta E_t|}{|E_t|} \leq \varepsilon, \right\}$$

we have  $|K \setminus A| < |Q|/2$ .

We claim that  $Q \subset AA^{-1}$ . Indeed, let  $x \in Q$ . Then  $|xK \cap K| \geq |xQ| = |Q|$  and, hence,

$$\begin{aligned} |Q| &\leq |xK \cap K| \leq |xA \cap A| + |K \setminus A| + |x(K \setminus A)| \\ &= |xA \cap A| + 2|K \setminus A| \\ &< |xA \cap A| + |Q|. \end{aligned}$$

Therefore,  $|xA \cap A| > 0$ , and this implies that  $x \in AA^{-1}$ .

Now, for  $x_1, x_2 \in A$ , we have

$$x_1 x_2^{-1} E_t \Delta E_t \subset (x_1 x_2^{-1} E_t \Delta x_1 E_t) \cup (x_1 E_t \Delta E_t),$$

and hence

$$\begin{aligned} \left| x_1 x_2^{-1} E_t \Delta E_t \right| &\leq \left| x_2^{-1} E_t \Delta E_t \right| + |x_1 E_t \Delta E_t| \\ &= |x_2 E_t \Delta E_t| + |x_1 E_t \Delta E_t| \leq 2\varepsilon |E_t|. \end{aligned}$$

This finishes the proof. ■

**Remark G.5.3** Assume that  $G$  is compactly generated. It follows, by the arguments given in the proof of Proposition F.1.7, that the amenability of  $G$  is



equivalent to Følner's Property for *some* compact generating set  $Q$  of  $G$ , that is, to the existence of a sequence of Borel subsets  $U_n$  with  $0 < |U_n| < \infty$  such that

$$\limsup_n \sup_{x \in Q} \frac{|xU_n \Delta U_n|}{|U_n|} = 0.$$

Such a sequence  $(U_n)_n$  is called a *Følner sequence*.

**Example G.5.4** Let  $\Gamma = \mathbf{Z}$  with the set of generators  $S = \{\pm 1\}$ . The sequence of intervals  $\{-n, -(n-1), \dots, 0, \dots, n-1, n\}$  is easily seen to be a Følner sequence for  $\mathbf{Z}$ .

We give an application of Følner's Property to groups of subexponential growth. Let  $\Gamma$  be a finitely generated group, and let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ . The *word metric* on  $\Gamma$  with respect to  $S$  is the metric  $d_S$  on  $\Gamma$  associated to the Cayley graph  $\mathcal{G}(\Gamma, S)$ ; see Remark G.4.7. For each  $r > 0$ , let  $B_r$  be the ball of radius  $r$  centred at  $e$ , that is,  $B_r$  is the set of  $\gamma \in \Gamma$  which can be expressed as words of length  $\leq r$  in elements from  $S$ . The limit

$$\gamma_S = \lim_{r \rightarrow \infty} (\#B_r)^{1/r},$$

which is known to exist, is the *growth* of  $\Gamma$  (with respect to  $S$ ). It is also known that *independently* of  $S$ , we have

$$\text{either } \gamma_S > 1 \quad \text{or} \quad \gamma_S = 1.$$

In the first case,  $\Gamma$  is said to be of *exponential growth* and it is said to be of *subexponential growth* in the second one. For instance, nilpotent groups are of *polynomial growth* (that is, there exist a constant  $C$  and an integer  $n$  such that  $\#B_r \leq Cr^n$ , for all  $r > 0$ ) and hence of subexponential growth. For more details, see [Harpe-00, Chapter VI].

The following result appeared for the first time in [AdVSr-57].

**Corollary G.5.5** *If  $\Gamma$  is a finitely generated group of subexponential growth, then  $\Gamma$  is amenable. In particular, finitely generated groups of polynomial growth are amenable.*

**Proof** Let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ . We claim that a subsequence of the sequence of balls  $(B_n)_{n \geq 1}$  is a Følner sequence. Indeed, assume, by contradiction, that this is not the case. Then there exists  $\varepsilon > 0$  and  $s \in S$  such that  $\#(sB_n \Delta B_n) > \varepsilon \#B_n$  for infinitely many  $n$ . We have

$$\#(sB_n \setminus B_n) > \frac{\varepsilon}{2} \#B_n \quad \text{or} \quad \#(B_n \setminus sB_n) > \frac{\varepsilon}{2} \#B_n.$$

Since  $sB_n \setminus B_n$  and  $s^{-1}(B_n \setminus sB_n)$  are contained in  $B_{n+1} \setminus B_n$ , it follows that  $\#B_{n+1} \geq \left(1 + \frac{\varepsilon}{2}\right) \#B_n$ . Hence, there exists a constant  $C > 0$  such that

$$\#B_n \geq C \left(1 + \frac{\varepsilon}{2}\right)^n$$

for infinitely many  $n$ 's. This contradicts the fact that  $\Gamma$  is of subexponential growth. ■

Let  $\Gamma$  be a finitely generated group, and let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ . The existence of a Følner sequence  $F_n$  in  $\Gamma$  has the following interpretation in terms of the Cayley graph  $\mathcal{G}(\Gamma, S)$ .

Let  $\mathcal{G}$  be a graph and a set  $F$  of vertices of  $\mathcal{G}$ . The *boundary*  $\partial F$  of  $F$  is the set of all vertices of  $\mathcal{G}$  which are connected to some vertex in  $F$  but do not belong to  $F$ .

**Corollary G.5.6** *Let  $\Gamma$  be a finitely generated group, and let  $S$  be a finite generating subset of  $\Gamma$  with  $S^{-1} = S$ . The following properties are equivalent:*

- (i)  $\Gamma$  is amenable;
- (ii) there exists a sequence of non-empty finite subsets  $F_n$  of  $\Gamma$  such that, for every  $s \in S$ ,

$$\lim_n \frac{\#(sF_n \triangle F_n)}{\#F_n} = 0;$$

- (iii) there exists a sequence of non-empty finite subsets  $F_n$  of  $\Gamma$  such that

$$\lim_n \frac{\#\partial F_n}{\#F_n} = 0,$$

where  $\partial F_n$  is the boundary of  $F_n$  in the Cayley graph  $\mathcal{G}(\Gamma, S)$ .

**Proof** The equivalence of (i) and (ii) is a particular case of Theorem G.5.1 and Remark G.5.3.

For a finite subset  $F$  of  $\Gamma$ , we have

$$\partial F = \bigcup_{s \in S} (sF \setminus F),$$

by the definition of  $\mathcal{G}(\Gamma, S)$ . For every  $s \in S$ , we have

$$\#(sF \triangle F) = \#(sF \setminus F) + \#(F \setminus sF) = \#(sF \setminus F) + \#(s^{-1}F \setminus F).$$

Since  $S = S^{-1}$ , it follows that

$$\sum_{s \in S} \#(sF \triangle F) = 2 \sum_{s \in S} \#(sF \setminus F) \leq 2\#S \max_{s \in S} \#(sF \setminus F).$$

Therefore

$$\frac{1}{2\#S} \sum_{s \in S} \#(sF \triangle F) \leq \#\partial F \leq \frac{1}{2} \sum_{s \in S} \#(sF \triangle F),$$

so that (ii) and (iii) are equivalent. ■

## G.6 Exercises

**Exercise G.6.1** Let  $G$  be a topological group, and let  $\mathcal{M}$  be the set of all means on  $UCB(G)$ .

(i) Prove that  $\mathcal{M}$  is a weak\* closed (and hence compact) convex subset of the unit ball of  $UCB(G)^*$ .

(ii) Let  $\mathcal{M}_0$  be the convex hull of all point evaluations. Prove that  $\mathcal{M}_0$  is weak\* dense in  $\mathcal{M}$ .

**Exercise G.6.2** Let  $G$  be a locally compact group, and let  $\mathcal{M}$  be the set of all means on  $L^\infty(G)$ . Let  $L^1(G)_{1,+}$  denote the convex set of all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ .

Prove that  $L^1(G)_{1,+}$  is weak\* dense in  $\mathcal{M}$ , when  $L^1(G)$  is viewed as subspace of  $L^\infty(G)^*$  in the canonical way.

**Exercise G.6.3** (Compare Remark G.3.7.) Let  $\mathcal{U}(\mathcal{H})$  be the unitary group of a Hilbert space  $\mathcal{H}$ , with the *weak operator topology*, that is, a net  $(T_i)_i$  in  $\mathcal{U}(\mathcal{H})$  converges to  $T \in \mathcal{U}(\mathcal{H})$  if

$$\lim_i \langle T_i \xi, \eta \rangle = \langle T \xi, \eta \rangle, \quad \text{for all } \xi, \eta \in \mathcal{U}(\mathcal{H}).$$

(i) Show that  $\mathcal{U}(\mathcal{H})$  is a topological group.

(ii) Show that  $\mathcal{U}(\mathcal{H})$  is not locally compact if  $\mathcal{H}$  is infinite dimensional.

**Exercise G.6.4** (Compare Remark G.3.7.) Let  $G$  be a locally compact group, and let  $\mathcal{U}(L^2(G))$  be the unitary group of the Hilbert space  $L^2(G)$ , with the weak operator topology. Show that

$$\tilde{G} = \{\lambda_G(x) : x \in G\}$$

is a *closed* subgroup of  $\mathcal{U}(L^2(G))$ .

[Hint: Use Proposition C.4.6.]

**Exercise G.6.5** Let  $G$  be a locally compact group, and let  $H$  be a closed subgroup of  $G$ . Assume that  $H$  is amenable. Show that, for every unitary representation  $\sigma$  of  $H$ , the induced representation  $\text{Ind}_H^G \sigma$  is weakly contained in the regular representation  $\lambda_G$ .

**Exercise G.6.6** (Compare Remark G.4.3.) Fix an irrational real number  $\theta$ . The subgroup of  $\mathbf{S}^1$  generated by  $e^{2\pi i\theta}$  is dense. Let  $\mu$  be the probability measure on  $\mathbf{S}^1$  defined by

$$\mu = \sum_{n \geq 0} 2^{-n} \delta_{e^{2\pi i n \theta}}.$$

Let

$$\pi = \bigoplus_{k \neq 0} \chi_k$$

be the direct sum of all unitary characters  $\chi_k \neq 1_{\mathbf{S}^1}$  of  $\mathbf{S}^1$ .

(i) Show that 1 belongs to the spectrum of the operator  $\pi(\mu)$ .

(ii) Show that  $1_{\mathbf{S}^1}$  is not weakly contained in  $\pi$ .

[Hint: Use Proposition F.1.8.]

**Exercise G.6.7** (Compare Remark G.4.5.) Let  $\Gamma$  be the free group on two generators  $a, b$  and consider the probability measure  $\mu = (\delta_a + \delta_b)/2$  on  $\Gamma$ . Show that  $\|\lambda_\Gamma(\mu)\| = 1$ , although  $1_\Gamma$  is not weakly contained in  $\lambda_\Gamma$ .

[Hint: Let  $\nu = \mu * \delta_{a^{-1}}$ . Then  $\|\lambda_\Gamma(\nu)\| = \|\lambda_\Gamma(\mu)\|$ . Determine the subgroup generated by the support of  $\nu$  and deduce that  $\|\lambda_\Gamma(\nu)\| = 1$ .]

**Exercise G.6.8** Let  $A$  and  $B$  be the following matrices of  $GL_2(\mathbf{R})$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

and let  $\Gamma$  be the subgroup generated by  $A$  and  $B$ .

(i) Let  $\Omega_1 = \{(x, y) \in \mathbf{R}^2 \setminus \{(0, 0)\} : |x| \leq |y|\}$  and  $\Omega_2 = \{(x, y) \in \mathbf{R}^2 \setminus \{(0, 0)\} : |y| < |x|\}$ . Check that  $A^{\pm 1}\Omega_1 \subset \Omega_2$  and  $B^{\pm 1}\Omega_2 \subset \Omega_1$ .

(ii) Fix  $\omega \in \Omega_1$ . Show that  $A^{i_1}B^{i_2}\dots B^{i_{n-1}}A^{i_n}\omega \neq \omega$  for all integers  $i_1, \dots, i_n \in \mathbf{Z} \setminus \{0\}$ .

(iii) Show that  $\Gamma$  is the free group on  $A$  and  $B$ .

**Exercise G.6.9** Let  $\mathbf{K}$  be a local field. Choose  $\lambda \in \mathbf{K}$  with  $|\lambda| > 2$ , and let  $A$  and  $B$  be the following matrices of  $SL_2(\mathbf{K})$

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad B = CAC^{-1},$$

where  $C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . Show that the subgroup generated by  $A$  and  $B$  is a discrete subgroup of  $SL_2(\mathbf{K})$ , and is isomorphic to  $F_2$ .

[Hint: Use the method of the previous exercise.]

**Exercise G.6.10** For an integer  $n \geq 1$ , define a function  $f_n \in \ell^1(\mathbf{Z})_{1,+}$  by  $f_n(k) = (2n+1)^{-1}$  if  $|k| \leq n$  and  $f_n(k) = 0$  if  $|k| > n$ . Let  $m \in \mathcal{M}$  denote a mean on  $\ell^\infty(\mathbf{Z})$  which is a weak\* limit point of  $(f_n)_n$ , and which is therefore invariant, as in the proof of Theorem G.3.1. Show that there does not exist any subsequence of  $(f_n)_n$  which converges to  $m$  in the weak\* topology of  $\mathcal{M}$ .

[Hint: Suppose (ab absurdo) that there exists a subsequence  $(f_{l(n)})_n$  of  $(f_n)_n$  which converges to  $m$ . Upon replacing it by a subsequence, we can assume that  $l(n) \geq 3l(n-1)$  for all  $n \geq 2$ . Define a bounded function  $g$  on  $\mathbf{Z}$  by  $g(k) = 1$  if  $|k| \leq l(1)$  and

$$g(k) = (-1)^n \quad \text{if } l(n) < |k| \leq l(n+1) \quad \text{with } n \geq 1.$$

Check that the numerical sequence of  $n$ -th terms

$$f_{l(n)}(g) = \frac{1}{2l(n)+1} \sum_{k=-l(n)}^{l(n)} g(k)$$

is not convergent.]

**Exercise G.6.11** (This exercise was suggested to us by N. Monod.) Let  $G, H$  be two Hausdorff topological groups. A mapping  $\sigma : G \rightarrow H$  is *subadditive* if

$$\sigma(Q^2) \subset \sigma(Q)^2 \quad \text{for any compact subset } Q \subset G.$$

From now on, we assume that  $G$  has the *Baire property*, namely that any countable union of closed subsets of  $G$  without interior points is itself without interior points. Recall that locally compact spaces and complete metric spaces have the Baire Property.

(i) Let  $\sigma_n : G \rightarrow H, n \geq 1$ , be a sequence of continuous subadditive mappings which converges towards  $\varepsilon$  pointwise; here  $\varepsilon$  denotes the constant function on  $G$  with value the identity element  $e$  of  $H$ . Show that  $(\sigma_n)_n$  converges toward  $\varepsilon$  uniformly on compact subsets of  $G$ .

[Hint: Let  $Q$  be a compact subset of  $G$  and let  $U$  be a neighbourhood of  $e$  in  $H$ . Choose a closed neighbourhood  $A$  of  $e$  such that  $A^2 \subset U$ . Define  $F_n = \bigcap_{k \geq n} \sigma_k^{-1}(A)$  for all  $n \geq 1$  and observe that  $\bigcup_{n \geq 1} F_n = G$ . Deduce from the Baire property that there exists  $M \geq 1$  such that  $F_M \cap Q$  has interior points, from the compactness of  $Q$  that there exist  $g_1, \dots, g_j \in G$  such that  $g_1 F_M \cup \dots \cup g_j F_M \supset Q$ , and consequently that there exists  $N \geq 1$  such that  $F_N F_M \supset Q$ . Hence,  $(F_n)^2 \supset Q$  and therefore  $\sigma_n(Q) \subset U$ , for  $n \geq \max\{N, M\}$ .]

(ii) Let  $\psi_n : G \rightarrow \mathbf{R}, n \geq 1$ , be a sequence of functions conditionally of negative type such that  $\lim_{n \rightarrow \infty} \psi_n(g) = 0$  for all  $g \in G$ . Show that the convergence holds uniformly on compact subsets of  $G$ .

[Hint: Use Proposition 2.10.2 and check that  $g \mapsto \sqrt{\psi_n(g)}$  is subadditive.]

(iii) Let  $G$  be a compact group (for example,  $G = \mathbf{R}/\mathbf{Z}$ ) and let  $G_d$  be the group  $G$  endowed with the discrete topology. Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G_d$  which weakly contains  $1_{G_d}$  but not strongly. Show that there does not exist any sequence  $(\xi_n)_n$  of unit vectors in  $\mathcal{H}$  such that  $\lim_{n \rightarrow \infty} \|\pi(g)\xi_n - \xi_n\| = 0$  for all  $g \in G$ .

(iv) Let  $A_n : G \rightarrow G, n \geq 1$ , be a sequence of continuous endomorphisms such that  $\lim_{n \rightarrow \infty} A_n(g) = e$  for all  $g \in G$ . Show that the convergence holds uniformly on compact subsets of  $G$ .

# Bibliography

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(...) Je veux seulement remarquer combien il est difficile aujourd'hui de faire une bibliographie ayant quelque valeur historique. Il serait peut-être exact de dire que la moitié des attributions sont fausses, et que bien souvent on ne cite pas le premier inventeur. L'histoire des sciences deviendra de plus en plus difficile à écrire ; je ne compte guère, pour remonter le courant, sur les Encyclopédies où l'historien risque de se noyer dans un flot de citations au milieu desquelles disparaît celui qui a eu la première idée. Vous rendez, cher ami, un grand service dans les *Rendiconti* en faisant réviser et compléter souvent les citations des auteurs, en partie peut-être responsables de cet état de choses par le peu de soin qu'ils apportent aux indications bibliographiques ; soyez d'ailleurs assuré que je me dis en ce moment que celui qui est sans péché lui jette la première pierre !

E. Picard [Picar–13].

- [AdiMe–92] S.I. Adian and J. Mennicke. On bounded generation of  $SL_n(\mathbf{Z})$ . *Int. J. Alg. Comp.*, **2**:357–365, 1992.
- [AdVSr–57] G.M. Adel'son-Vel'skii and Yu.A. Sreider. The Banach mean on groups. *Uspehi Mat. Nauk. (N.S)*, **12**:131–136, 1957.
- [AkeWa–81] C.A. Akemann and M.E. Walter. Unbounded negative definite functions. *Can. J. Math.*, **33**:862–871, 1981.
- [AlLuW–01] N. Alon, A. Lubotzky, and A. Wigderson. Semi-direct product in groups and zig-zag product in graphs: connections and applications (extended abstract). 42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001), IEEE Computer Soc., 630–637, Los Alamitos, CA 2001.
- [Alper–82] R. Alperin. Locally compact groups acting on trees and Property T. *Monatsh. Math.*, **93**:261–265, 1982.
- [AnDRe–00] C. Anantharaman-Delaroche and J. Renault. *Amenable Groupoids*, Enseignement Mathématique, Monographie 36, 2000.
- [Anant–87] C. Anantharaman-Delaroche. On Connes' property T for von Neumann algebras. *Math. Japon.*, **32**:337–355, 1987.
- [Anant–05] C. Anantharaman-Delaroche. Cohomology of property (T) groupoids and applications. *Ergod. Th. & Dynam. Sys.*, **25**:977–1013, 2005.

- [BacHa–94] R. Bacher and P. de la Harpe. Exact values of Kazhdan constants for some finite groups. *J. Algebra*, **163**:495–515, 1994.
- [BaFGM] U. Bader, A. Furman, T. Gelander, and N. Monod. Property (T) and rigidity for actions on Banach spaces. Preprint, 2005.
- [BadSh–06] U. Bader and Y. Shalom. Factor and normal subgroup theorems for lattices in products of groups. *Inventiones Math.*, **163**:415–454, 2006.
- [Bagno–04] E. Bagno. Kazhdan constants of some colored permutation groups. *J. Algebra*, **282**:205–231, 2004.
- [BaGrS–85] W. Ballmann, M. Gromov, and V. Schroeder. *Manifolds of Nonpositive Curvature*, Basel, Birkhäuser, 1985.
- [BalSw–97] W. Ballmann and J. Świątkowski. On  $L^2$ -cohomology and property (T) for automorphism groups of polyhedral cell complexes. *Geom. Funct. Anal. (GAFA)*, **7**:615–645, 1997.
- [Banac–32] S. Banach. *Théorie des opérations linéaires*, Hafner Publishing Company, 1932.
- [Banas–83] W. Banaszczyk. On the existence of exotic Banach–Lie groups. *Math. Ann.*, **264**:485–493, 1983.
- [Bargm–47] V. Bargmann. Irreducible unitary representations of the Lorentz group. *Ann. Math.*, **48**:568–640, 1947.
- [BatRo–99] T. Bates and A.G. Robertson. Positive definite functions and relative property (T) for subgroups of discrete groups. *Bull. Austral. Math. Soc.*, **52**:31–39, 1999.
- [BeCh–74a] C. Berg and J.P.R. Christensen. On the relation between amenability of locally compact groups and the norms of convolution operators. *Math. Ann.*, **208**:149–153, 1974.
- [BeCh–74b] C. Berg and J.P.R. Christensen. Sur la norme des opérateurs de convolution. *Inventiones Math.*, **23**:173–178, 1974.
- [BeChJ–98] B. Bekka, P.-A. Cherix, and P. Jolissaint. Kazhdan constants associated with Laplacian on connected Lie groups. *J. Lie Theory*, **8**:95–110, 1998.
- [BeChR–84] C. Berg, J.P.R. Christensen, and P. Ressel. *Harmonic Analysis on Semigroups*, Berlin, Springer, 1984.
- [Behr–79] H. Behr.  $SL_3(\mathbb{F}_q[T])$  is not finitely presentable. In: Homological group theory, Proceedings Durham 1977, *London Math. Soc. Lecture Notes*, **36**:213–224, Cambridge University Press, 1979.
- [BekHa] B. Bekka and P. de la Harpe. Irreducibly represented groups. Preprint 2006.
- [Bekk–06] B. Bekka. A Property (T) for  $C^*$ -algebras. *Bull. London Math. Soc.*, **38**:857–867, 2006.
- [Bekk–99] B. Bekka. On the full  $C^*$ -algebras of arithmetic groups and the congruence subgroup problem. *Forum Math.*, **11**:705–715, 1999.
- [Bekka–03] B. Bekka. Kazhdan’s property (T) for the unitary group of a separable Hilbert space. *Geom. Funct. Anal. (GAFA)*, **13**:509–520, 2003.
- [BeCuH–02] B. Bekka, R. Curtis, and P. de la Harpe. Familles de graphes expenseurs et paires de Hecke. *C.R. Acad. Sci. Paris*, **335**:463–468, 2002.
- [BekLo–97] B. Bekka and N. Louvet. On a variant of Kazhdan’s Property (T) for subgroups of semisimple groups. *Ann. Inst. Fourier*, **47**:1065–1078, 1997.



- [BekNe–02] B. Bekka and M. Neuhauser. On Kazhdan’s property (T) for  $Sp(n, \mathbf{K})$ . *J. Lie Theory*, **12**:31–39, 2002.
- [BekVa–93] B. Bekka and A. Valette. Kazhdan’s property (T) and amenable representations. *Math. Z.*, **212**:293–299, 1993.
- [BeMa–00a] B. Bekka and M. Mayer. On Kazhdan’s property (T) and Kazhdan constants associated to a Laplacian for  $SL(3, \mathbf{R})$ . *J. Lie Theory*, **10**:93–105, 2000.
- [BeMa–00b] B. Bekka and M. Mayer. *Ergodic and Topological Dynamics of Group Actions on Homogeneous Spaces*, Cambridge University Press, 2000.
- [BezGo–81] S.I. Bezuglyui and V.Ya. Golodets. Hyperfinite and  $II_1$  actions for nonamenable groups. *J. Funct. Anal.*, **40**:30–44, 1981.
- [Blanc–79] P. Blanc. Sur la cohomologie continue des groupes localement compacts. *Ann. Scient. Éc. Norm. Sup.*, **12**:137–168, 1979.
- [BoJaS–88] M. Bozejko, T. Januszkiewicz, and R. Spatzier. Infinite Coxeter groups do not have Property (T). *J. Op. Theory*, **19**:63–67, 1988.
- [Borel–63] A. Borel. Compact Clifford–Klein forms of symmetric spaces. *Topology*, **2**:111–122, 1963.
- [Bore–69a] A. Borel. Sous-groupes discrets de groupes semi-simples (d’après D.A. Kazhdan et G.A. Margulis). *Séminaire Bourbaki*, exposé 358, 1969.
- [Bore–69b] A. Borel. *Introduction aux groupes arithmétiques*, Paris, Hermann, 1969.
- [Borel–73] A. Borel. Cohomologie de certains groupes discrets et laplacien  $p$ -adique (d’après H. Garland). *Séminaire Bourbaki*, exposé 437, 1973.
- [BorHa–62] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. Math.*, **75**:485–535, 1962.
- [BorWa–80] A. Borel and N. Wallach. *Continuous Cohomology, Discrete Groups, and Representations of Reductive Groups*, Princeton University Press, 1980.
- [BotRh–77] R. Botto Mura and A. Rhemtulla. *Orderable Groups*, Basel, Marcel Dekker, 1977.
- [Bou–Int1] N. Bourbaki. *Intégration*, Chapitres 4–5, Paris, Hermann, 1963.
- [Bou–Int2] N. Bourbaki. *Intégration*, Chapitres 7–8, Paris, Hermann, 1963.
- [Bou–Spec] N. Bourbaki. *Théories spectrales*, Paris, Hermann, 1967.
- [Bou–Top1] N. Bourbaki. *Topologie générale*, Chapters 1–2, Paris, Hermann, 1965.
- [Bou–Top2] N. Bourbaki. *Topologie générale*, Chapter 9, Paris, Hermann, 1958.
- [Bou–AlCo] N. Bourbaki. *Algèbre commutative*, Chapters 5–6, Paris, Hermann, 1964.
- [Bou–GLie] N. Bourbaki. *Groupes et algèbres de Lie*, Chapters 2–3, Paris, Hermann, 1972.
- [BouGa–06] J. Bourgain and A. Gamburd. New results on expanders. *C. R. Acad. Sci. Paris*, **342**:717–721, 2006.
- [BriHa–99] M. Bridson and A. Haefliger. *Metric Spaces of Non-positive Curvature*, Berlin, Springer, 1999.
- [Brook–86] R. Brooks. The spectral geometry of a tower of coverings. *J. Diff. Geometry*, **23**:97–107, 1986.
- [Brown–89] K.S. Brown. *Buildings*, Berlin, Springer, 1989.
- [BruTi–72] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. I. Données radicielles valuées. *Publ. Math. I.H.E.S.*, **41**:5–251, 1972.

- [Burge–91] M. Burger. Kazhdan constants for  $SL_3(\mathbf{Z})$ . *J. reine angew. Math.*, **413**:36–67, 1991.
- [BurMo–99] M. Burger and N. Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc.*, **1**:199–235 and 338, 1999.
- [BurMo–00] M. Burger and S. Mozes. Lattices in products of trees. *Publ. Math. I.H.É.S.*, **92**:151–194, 2000.
- [CalVe–60] E. Calabi and E. Vesentini. On compact, locally symmetric Kähler manifolds. *Ann. Math.*, **71**:472–507, 1960.
- [CalFr–06] D. Calegari and M.H. Freedman. Distortion in transformation groups. *Geom. Topol.*, **10**:267–293, 2006.
- [CaFlP–96] J.W. Cannon, W.J. Floyd, and W.R. Pary. Introductory notes on Richard Thompson’s groups. *Enseignement Math.*, **42**:215–256, 1996.
- [CarGa–99] L. Carbone and H. Garland. Lattices in Kac–Moody groups. *Math. Res. Lett.*, **6**:439–447, 1999.
- [CarKe–83] D. Carter and G. Keller. Bounded elementary generation of  $SL_n(\mathcal{O})$ . *Amer. J. Math.*, **105**:673–687, 1983.
- [CarKe–84] D. Carter and G. Keller. Elementary expressions for unimodular matrices. *Comm. Alg.*, **12**:379–389, 1984.
- [CaMSZ–93] D. Cartwright, A.M. Mantero, T. Steger, and A. Zappa. Groups acting simply transitively on the vertices of a building of type  $\tilde{A}_2$ . *Geom. Dedicata*, **47**:143–166, 1993.
- [CaMIS–93] D. Cartwright, W. Mlotkowski, and T. Steger. Property (T) and  $\tilde{A}_2$  groups. *Ann. Inst. Fourier*, **44**:213–248, 1993.
- [Chaba–50] C. Chabauty. Limite d’ensembles et géométrie des nombres. *Bull. Soc. Math. France*, **78**:143–151, 1950.
- [CCJJV–01] P.-A. Chérix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property: Gromov’s  $a$ -T-menability*, Basel, Birkhäuser, 2001.
- [CeGrH–99] T. Ceccherini-Silberstein, R. Grigorchuk and P. de la Harpe. Amenability and paradoxes for pseudogroups and for discrete metric spaces. *Proc. Sketlov Inst. Math.*, **224**:57–95, 1999.
- [Champ–91] C. Champetier. Propriétés génériques des groupes de présentation finie. Thèse de doctorat, Université de Lyon I, 1991.
- [Champ–95] C. Champetier. Propriétés statistiques des groupes de présentation finie. *Adv. Math.*, **116**:197–262, 1995.
- [Champ–00] C. Champetier. L’espace des groupes de type fini. *Topology*, **39**:657–680, 2000.
- [Chave–93] I. Chavel. *Riemannian Geometry: A Modern Introduction*, Cambridge University Press, 1993.
- [Cheri–95] P.-A. Chérix. Property (T) and expanding constants for semidirect products. *Linear and Multilinear Algebra*, **39**:153–160, 1995.
- [ChLaR–85] C. Chou, A.T. Lau, and J. Rosenblatt. Approximation of compact operators by sums of translations. *Illinois J. Math.*, **29**:340–350, 1995.
- [ChMaV–04] P.-A. Chérix, F. Martin, and A. Valette. Spaces with measurable walls, Haagerup property and property (T). *Ergod. Th. & Dynam. Sys.*, **24**:1895–1908, 2004.

- [Cloze–03] L. Clozel. Démonstration de la conjecture  $\tau$ . *Inventiones Math.*, **151**:297–328, 2003.
- [CoFeW–81] A. Connes, J. Feldman, and B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergod. Th. & Dynam. Sys.*, **1**:431–450, 1981.
- [ColVe–98] Y. Colin de Verdière. *Spectres de graphes*, Cours spécialisés, Soc. Math. France, 1998.
- [ConJo–85] A. Connes and V. Jones. Property  $T$  for von Neumann algebras. *Bull. London Math. Soc.*, **17**:57–62, 1985.
- [Conne–74] A. Connes. Almost periodic states and factors of type  $III_1$ . *J. Funct. Anal.*, **16**:415–445, 1974.
- [Conne–76] A. Connes. Classification of injective factors. Cases  $II_1, II_\infty, III_\lambda, \lambda \neq 1$ . *Ann. Math.*, **104**:73–115, 1976.
- [Conne–80] A. Connes. A factor of type  $II_1$  with countable fundamental group. *J. Operator Theory*, **4**:151–153, 1980.
- [Conne–82] A. Connes. Classification des facteurs. In: “Operator algebras and applications”, Proc. Symp. Pure Math. 38, Part 2, 43–109, Amer. Math. Soc., 1982.
- [Conne–04] A. Connes. Nombres de Betti  $L^2$  et facteurs de type  $II_1$ . *Séminaire Bourbaki* (2002/2003), Astérisque **294**, 323–333, Soc. Math. France, 2004.
- [Conwa–87] J.B. Conway. *A Course in Functional Analysis*, New York, Springer, 1987.
- [Conwa–98] J.B. Conway. *Functions of One Complex Variable*, Second edition, New York, Springer, 1998.
- [ConWe–80] A. Connes and B. Weiss. Property (T) and asymptotically invariant sequences. *Israel J. Math.*, **37**:209–210, 1980.
- [Corle–92] K. Corlette. Archimedean superrigidity and hyperbolic geometry. *Ann. Math.*, **135**:165–182, 1992.
- [Cornu–06a] Y. de Cornulier. Kazhdan and Haagerup properties in algebraic groups over local fields. *J. Lie Theory*, **16**:67–82, 2006.
- [Cornu–06b] Y. de Cornulier. Strong boundedness of  $\text{Homeo}(S^n)$ . Appendix to [CalFr–06].
- [Cornu–06c] Y. de Cornulier. Strongly bounded groups and infinite powers of finite groups. *Comm. Algebra*, **34**:2337–2345, 2006.
- [Cornu–06d] Y. de Cornulier. Relative Kazhdan Property. *Ann. Sci. École Norm. Sup.*, **39**:301–333, 2006.
- [Cornu–06e] Y. de Cornulier. Kazhdan property for spaces of continuous functions. *Bull. Belg. Math. Soc. – Simon Stevin*, **13**:899–902, 2006.
- [Cornu–07] Y. de Cornulier. Finitely presentable, non-Hopfian groups with Kazhdan’s Property (T) and infinite outer automorphism groups. *Proc. Amer. Math. Soc.*, **135**:951–959, 2007.
- [CowHa–89] M. Cowling and U. Haagerup. Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. *Inventiones Math.*, **96**:507–542, 1989.
- [Cowli–90] M. Cowling. Review of [HarVa–89]. Math. Reviews: 90m22001.

- [Curti–99] C.W. Curtis. *Pioneers of Representation theory: Frobenius, Burnside, Schur, and Brauer*, Providence, American Mathematical Society, 1999.
- [DaSaV–03] G. Davidoff, P. Sarnak, and A. Valette. *Elementary Number Theory, Group Theory, and Ramanujan Graphs*, Cambridge University Press, 2003.
- [DeJRo–79] A. del Junco and J. Rosenblatt. Counterexamples in ergodic theory and number theory. *Math. Ann.*, **245**:185–197, 1979.
- [DelKi–68] C. Delaroche and A. Kirillov. Sur les relations entre l’espace dual d’un groupe et la structure de ses sous-groupes fermés. *Séminaire Bourbaki*, exposé 43, 1968.
- [Delor–77] P. Delorme. 1-cohomologie des représentations unitaires des groupes de Lie semi-simples et résolubles. Produits tensoriels continus et représentations. *Bull. Soc. Math. France*, **105**:281–336, 1977.
- [DerGu–73] Y. Derriennic and Y. Guivarc’h. Théorème de renouvellement pour les groupes non moyennables. *C. R. Acad. Sci. Paris*, **277**:613–615, 1973.
- [DeuVa–95] A. Deutsch and A. Valette. On diameters of orbits of compact groups in unitary representations. *J. Austral. Math. Soc.*, **59**:308–312, 1995.
- [Dixmi–50] J. Dixmier. Les moyennes invariantes dans les semi-groupes et leurs applications. *Acta Scient. Math. (Szeged)*, **12**:213–227, 1950.
- [Dixmi–69] J. Dixmier. *Les  $C^*$ -algèbres et leurs représentations*, Paris, Gauthier-Villars, 1969.
- [Drinf–84] V.G. Drinfeld. Finitely additive measures on  $S^2$  and  $S^3$ , invariant with respect to rotations. *Funct. Anal. Appl.*, **18**:245–246, 1984.
- [Dye–59] H. Dye. On groups of measure preserving transformations I. *Amer. J. Math.*, **81**:119–159, 1959.
- [Dye–63] H. Dye. On groups of measure preserving transformations II. *Amer. J. Math.*, **85**:551–576, 1963.
- [DymJa–00] J. Dymara and T. Januszkiewicz. New Kazhdan groups. *Geom. Dedicata*, **80**:311–317, 2000.
- [DymJa–02] J. Dymara and T. Januszkiewicz. Equivariant cohomology of buildings and of their automorphism groups. *Inventiones Math.*, **150**:579–627, 2002.
- [Effro–75] E.G. Effros. Property  $\Gamma$  and inner amenability. *Proc. Amer. Math. Soc.*, **47**:483–486, 1975.
- [Engel–89] R. Engelking. *General Topology*, revised and completed edition. Heldermann, 1989.
- [Ershov] M. Ershov. Golod–Shafarevich groups with Property (T) and Kac–Moody groups. Preprint, 2006.
- [Eymar–72] P. Eymard. *Moyennes Invariantes et Représentations Unitaires*, Lecture Notes in Math. **300**, Berlin, Springer, 1972.
- [Eymar–75] P. Eymard. Initiation à la théorie des groupes moyennables. In: *Analyse harmonique sur les groupes de Lie*, Lecture Notes in Math. **497**:88–107, Berlin, Springer, 1975.
- [Farau–83] J. Faraut. Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques. In: *Analyse harmonique*, Ed.: J.L. Clerc, P. Eymard, J. Faraut, M. Rais, R. Takahashi, Les cours du C.I.M.P.A, Nice 1983.
- [FarHa–74] J. Faraut and K. Harzallah. Distances hilbertiennes invariantes sur un espace homogène. *Ann. Inst. Fourier*, **24**:171–217, 1974.

- [FarKr–92] H.M. Farkas and I. Kra. *Riemann Surfaces*, Berlin, Springer, 1992.
- [Farle–03] D. Farley. A proper isometric action of Thompson group  $V$  on Hilbert space. *Int. Math. Res. Not.*, **45**:2409–2414, 2003.
- [FeiHi–64] W. Feit and G. Higman. The nonexistence of certain generalized polygons. *J. Algebra*, **1**:114–131, 1964.
- [Fell–60] J.M.G. Fell. The dual space of  $C^*$ -algebras. *Trans. Amer. Math. Soc.*, **94**:365–403, 1960.
- [Fell–62] J.M.G. Fell. Weak containment and induced representations. *Canadian J. Math.*, **14**:437–268, 1962.
- [Fell–63] J.M.G. Fell. Weak containment and Kronecker tensor products of group representations. *Pacific J. Math.*, **13**:503–510, 1963.
- [Ferno–06] T. Fernos. Kazhdan’s relative property (T): Some new examples. *Ann. Inst. Fourier*, **56**:1767–1804, 2006.
- [FisMa–05] D. Fisher and G.A. Margulis. Almost isometric actions, Property (T), and local rigidity. *Inventiones Math.* **162**:19–80. 2005.
- [Folla–95] G.B. Folland. *A Course in Abstract Harmonic Analysis*, Boca Rator, CRC Press, 1995.
- [Foeln–55] E. Følner. On groups with full Banach mean value. *Math. Scand.*, **3**:243–254, 1955.
- [Fried–91] J. Friedman. On the second eigenvalue and random walks in random  $d$ -regular graphs. *Combinatorica*, **11**:331–362, 1991.
- [Fujiw–99] K. Fujiwara. 3-manifolds groups and property (T) of Kazhdan. *Proc. Japan Acad. Ser. A Math. Sci.*, **75**:103–104, 1999.
- [Furma–99a] A. Furman. Gromov’s measure equivalence and rigidity of higher rank lattices. *Ann. Math.*, **150**:1059–1081, 1999.
- [Furma–99b] A. Furman. Orbit equivalence rigidity. *Ann. Math.*, **150**:1083–1108, 1999.
- [FurSh–99] A. Furman and Y. Shalom. Sharp ergodic theorems for group actions and strong ergodicity. *Ergod. Th. & Dynam. Sys.*, **19**:1037–1061, 1999.
- [Furst–63] H. Furstenberg. A Poisson formula for semisimple Lie groups. *Ann. Math.*, **77**:335–386, 1963.
- [Gaal–73] S.A. Gaal. *Linear Analysis and Representation Theory*, Berlin, Springer, 1973.
- [GabPo–05] D. Gaboriau and S. Popa. An uncountable family of non orbit equivalent actions of  $F_n$ . *J. Amer. Math. Soc.* **18**:547–559, 2005.
- [GaJaS–99] A. Gamburd, D. Jakobson, and P. Sarnak. Spectra of elements in the group ring of  $SU(2)$ . *J. Eur. Math. Soc.*, **1**:51–85, 1999.
- [GabGa–81] O. Gabber and Z. Galil. Explicit construction of linear-sized superconcentrators. *J. Comp. and Syst. Sci.*, **22**:407–420, 1981.
- [Garla–73] H. Garland.  $p$ -adic curvature and the cohomology of discrete subgroups of  $p$ -adic groups. *Ann. Math.*, **97**:375–423, 1973.
- [GarRa–70] H. Garland and M.S. Raghunathan. Fundamental domains for lattices in  $(\mathbf{R})$ -rank 1 semisimple Lie groups. *Ann. Math.*, **92**:279–326, 1970.
- [GefGo–89] S.L. Gefter and V.Ya. Golodets. Fundamental groups for ergodic actions and actions with unit fundamental groups. *Publ. Res. Inst. Math. Sci.*, **24**:821–847, 1989.

- [Gelan–04] T. Gelander. Homotopy type and volume of locally symmetric manifolds. *Duke Math. J.*, **124**:459–515, 2004.
- [GelJa–78] S. Gelbart and H. Jacquet. A relation between automorphic representations of  $GL(2)$  and  $GL(3)$ . *Ann. Sci. École Normale Sup.*, **11**:471–542, 1978.
- [GelNa–47] I.M. Gel'fand and M.A. Naimark. Unitary representations of the Lorentz group. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, **11**:411–504, 1947. [= I.M. Gelfand, *Collected papers*, Volume II, 41–123.]
- [GelZu–02] T. Gelander and A. Zuk. Dependence of Kazhdan constants on generating subsets. *Israel J. Math.*, **129**:93–98, 2002.
- [Ghys–99] E. Ghys. Actions de réseaux sur le cercle. *Inventiones Math.*, **137**:199–231, 1999.
- [Ghys–01] E. Ghys. Groups acting on the circle. *Enseignement. Math.*, **47**:329–407, 2001.
- [Ghys–04] E. Ghys. Groupes aléatoires. *Séminaire Bourbaki*, exposé 916 (2002–2003), Astérisque **294**, 173–204, Soc. Math. France, 2004.
- [GhySe–87] E. Ghys and V. Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, **62**:185–239, 1987.
- [Glasn–98] E. Glasner. On minimal actions of Polish groups. *Topology Appl.*, **85**:119–125, 1998.
- [Glasn–03] E. Glasner. *Ergodic Theory via Joinings*, Providence, American Mathematical Society, 2003.
- [GlaWe–97] E. Glasner and B. Weiss. Kazhdan's property (T) and the geometry of the collection of invariant measures. *Geom. Funct. Anal. (GAFA)*, **7**:917–935, 1997.
- [Godem–48] R. Godement. Les fonctions de type de positif et la théorie des groupes. *Trans. Amer. Math. Soc.*, **63**:1–84, 1948.
- [Godem–52] R. Godement. Une généralisation du théorème de la moyenne pour les fonctions harmoniques. *C.R. Acad. Sci. Paris*, **234**:136–139, 1952.
- [Green–69] F.P. Greenleaf. *Invariant Means on Topological Groups and their Applications*, New York, Van Nostrand, 1969.
- [Grigo–84] R.I. Grigorchuk. Degrees of growth of finitely generated groups, and the theory of invariant means. *Math. USSR Izv.*, **25**:259–300, 1985. [Russian original: *Izv. Akad. Nauk. SSSR Ser. Mat.*, **48**:939–985, 1984.]
- [Gromo–81] M. Gromov. Groups of polynomial growth and expanding maps. *Publ. Math. I.H.É.S.*, **53**:53–73, 1981.
- [Gromo–87] M. Gromov. Hyperbolic groups. In: *Essays in Group Theory*, p.75–263, Ed: S.M. Gersten, Berlin, Springer, 1987.
- [Gromo–93] M. Gromov. Asymptotic invariants of infinite groups. In: *Geometric Group Theory, Vol. 2 (Sussex 1991)*, p. 1–295, Cambridge University Press, 1993.
- [Gromo–00] M. Gromov. Spaces and questions. *Geom. Funct. Anal. (GAFA)*, Special Volume, 118–161, 2000.
- [Gromo–01] M. Gromov.  $CAT(\kappa)$ -spaces: construction and concentration. *J. Math. Sci. (N. Y.)*, **119**:178–200, 2004. [Russian original: 2001.]
- [Gromo–03] M. Gromov. Random walks in random groups. *Geom. Funct. Anal. (GAFA)*, **13**:73–146, 2003.

- [GroSc–92] M. Gromov and R. Schoen. Harmonic maps into singular spaces and  $p$ -adic superrigidity for lattices in groups of rank one. *Publ. Math. I.H.É.S.*, **76**:165–246, 1992.
- [GruLu] F. Grunewald and A. Lubotzky. Linear representations of the automorphism group of a free group. Preprint, 2006. arXiv:math.GR/0606182v1.
- [GuHiW–05] E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. *Publ. Math. I.H.É.S.*, **101**:243–268, 2005.
- [Guic–72a] A. Guichardet. Sur la cohomologie des groupes topologiques II. *Bull. Soc. Math.*, **96**:305–332, 1972.
- [Guic–72b] A. Guichardet. *Symmetric Hilbert Spaces and Related Topics*, Lecture Notes in Math. **266**, Berlin, Springer, 1972.
- [Guic–77a] A. Guichardet. Étude de la 1-cohomologie et de la topologie du dual pour les groupes de Lie à radical abélien. *Math. Ann.* **228**:215–232, 1977.
- [Guic–77b] A. Guichardet. Représentations de  $G^X$  selon Gelfand et Delorme. *Séminaire Bourbaki* (1975–1976), exposé 486, Lecture Notes in Math. **567**, 238–255, Berlin, Springer 1977.
- [Guiva–73] Y. Guivarc’h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math.*, **101**:333–379, 1973.
- [Haage–78] U. Haagerup. An example of a nonnuclear  $C^*$ -algebra, which has the metric approximation property. *Inventiones Math.*, **50**:279–293, 1978.
- [Hall–54] P. Hall. Finiteness conditions for soluble groups. *Proc. London Math. Soc.*, **4**:419–436, 1954.
- [Halmo–74] P.R. Halmos. *Measure Theory*, New York, Springer, 1974.
- [Halmo–67] P.R. Halmos. *A Hilbert Space Problem Book*, New York, Van Nostrand, 1967.
- [Haris–47] Harish-Chandra. Infinite irreducible representations of the Lorentz group. *Proc. Royal Soc. A*, **189**:372–401, 1947. [= *Collected Papers*, Volume I, 162–191.]
- [Harpe–73] P. de la Harpe. Moyennabilité de quelques groupes topologiques de dimension infinie. *C.R. Acad. Sci. Paris*, **277**:A1037–1040, 1973.
- [Harpe–00] P. de la Harpe. *Topics in Geometric Group Theory*, University of Chicago Press, 2000. Second printing, with corrections and updates, 2003.
- [Harpe–02] P. de la Harpe. Uniform growth in groups of exponential growth. *Geom. Dedicata*, **95**:1–17, 2002.
- [HarVa–89] P. de la Harpe and A. Valette. *La propriété (T) pour les groupes localement compacts*, Astérisque **175**, Soc. Math. France, 1989.
- [HaRoV–93] P. de la Harpe, G. Robertson, and A. Valette. On the spectrum of the sum of generators of a finitely generated group. *Israel J. Math.*, **81**:65–96, 1993.
- [HechHi–00] G. Hector and U. Hirsch. *The Geometry of Foliations, Part B*, Vieweg, 1983.
- [Helga–62] S. Helgason. *Differential Geometry and Symmetric Spaces*, New York, Academic Press, 1962.
- [Helga–84] S. Helgason. *Groups and Geometric Analysis*, New York, Academic Press, 1984.
- [HewRo–63] E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis I*, New York, Springer, 1963.

- [HewSt–69] E. Hewitt and K. Stromberg. *Real and Abstract Analysis*, New York, Springer, 1969.
- [HiLaS–02] N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum–Connes conjecture. *Geom. Funct. Anal. (GAFA)*, **12**:320–354, 2002.
- [Hjort–05] G. Hjorth. A converse to Dye’s theorem. *Trans. Amer. Math. Soc.*, **357**:3083–3103, 2005.
- [HowMo–79] R. Howe and C.C. Moore. Asymptotic properties of unitary representations. *J. Funct. Anal.*, **32**:72–99, 1979.
- [HowTa–92] R. Howe and E.C. Tan. *Non-abelian Harmonic Analysis*, New York, Springer, 1992.
- [HubPo–79] J. Hubbard and I. Poureza. The space of closed subgroups of  $\mathbf{R}^2$ . *Topology*, **18**:143–146, 1979.
- [HugPi–73] D.R. Hughes and F.C. Piper. *Projective Planes*, New York, Springer, 1973.
- [Hulan–66] A. Hulanicki. Means and Følner condition on locally compact groups. *Studia Math.*, **27**:87–104, 1966.
- [IoPeP] A. Ioana, J. Peterson, and S. Popa. Amalgamated free products of  $w$ -rigid factors and calculation of their symmetry groups. Preprint, 2005.
- [IzeNa–05] H. Izei and S. Nayatani. Combinatorial harmonic maps and discrete-group actions on Hadamard spaces. *Geom. Dedicata*, **114**:147–188, 2005.
- [IvaOl–98] S.V. Ivanov and A.Yu. Ol’sanskii. Hyperbolic groups and their quotients of bounded exponents. *Trans. Amer. Math. Soc.*, **348**:2091–2138, 1998.
- [Jacob–85] N. Jacobson. *Basic Algebra I*, San Francisco, Freeman, 1985.
- [Janso–97] S. Janson. *Gaussian Hilbert Spaces*, Cambridge University Press, 1997.
- [Johns–67] B.E. Johnson. Cohomology in Banach algebras. *Memoirs Amer. Math. Soc.*, **127**, 1972.
- [Jolis–93] P. Jolissaint. Property  $T$  for discrete groups in terms of their regular representation. *Math. Ann.*, **297**:539–551, 1993.
- [Jolis–05] P. Jolissaint. On Property (T) for pairs of topological groups. *Enseignement Math.*, **51**:31–45, 2005.
- [JosLi–96] J. Jost and J. Li. Finite energy and totally geodesic maps from locally symmetric spaces of finite volume. *Calc. Var. Partial Diff. Eq.*, **4**:409–420, 1996.
- [Kaima–97] V.A. Kaimanovich. Amenability, hyperfiniteness, and isoperimetric inequalities. *C. R. Acad. Sci. Paris*, **325**:999–1004, 1997.
- [Kalle–82] W. van der Kallen.  $SL_3(\mathbf{C}[x])$  does not have bounded word length. *Lecture Notes in Math.*, **966**:357–361, Berlin, Springer 1982.
- [Kassa] M. Kassabov. Kazhdan constants for  $SL_n(\mathbf{Z})$ . Preprint, 2003.
- [KasNi–06] M. Kassabov and N. Nikolov. Universal lattices and property  $(\tau)$ . *Inventiones Math.*, **165**:209–224, 2006.
- [Kazhd–67] D. Kazhdan. Connection of the dual space of a group with the structure of its closed subgroups. *Funct. Anal. Appl.*, **1**:63–65, 1967.
- [KazMa–68] D. Kazhdan and G.A. Margulis. A proof of Selberg’s conjecture. *Math. USSR-Sbornik*, **4**:147–152, 1968.
- [Kechr–95] A.S. Kechris. *Classical Descriptive Set Theory*, Berlin, Springer, 1995.



- [KecMi-04] A.S. Kechris and D. Miller. *Topics in Orbit Equivalence*, Lecture Notes in Math. **1852**, Berlin, Springer, 2004.
- [Kest-59a] H. Kesten. Full Banach mean value on countable groups. *Math. Scand.*, **7**:146–156, 1959.
- [Kest-59b] H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, **92**:336–354, 1959.
- [Kirch-94] E. Kirchberg. Discrete groups with Kazhdan property T and factorization property are residually finite. *Math. Ann.*, **299**:551–563, 1994.
- [KorSc-93] J.N. Korevaar and R.M. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, **1**:561–659, 1993.
- [KleMa-99] D.Y. Kleinbock and G.A. Margulis. Logarithm laws for flows on homogeneous spaces. *Inventiones Math.*, **138**:451–494, 1999.
- [Knapp-86] A.W. Knap. *Representation Theory of Semisimple Groups*, Princeton University Press, 1986.
- [Koopm-31] B.O. Koopman. Hamiltonian systems and transformations in Hilbert spaces. *Proc. Nat. Acad. Sci.(U.S.A)*, **18**:315–318, 1931.
- [Kosta-69] B. Kostant. On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.*, **75**:627–642, 1969.
- [Kosta-75] B. Kostant. On the existence and irreducibility of certain series of representations. In: *Lie Groups and their Representations*, p. 231–329. Ed: I.M. Gelfand, Kaido, 1975.
- [Krawc-90] J. Krawczyk. A remark on continuity of translation invariant functionals on  $L_p(G)$  for compact Lie groups  $G$ . *Monatsh. Math.*, **109**:63–67, 1990.
- [Lacke-06] M. Lackenby. Heegard splittings, the virtually Haken conjecture and Property  $(\tau)$ . *Inventiones Math.*, **164**:317–359, 2006.
- [Lehne-64] J. Lehner. *Discontinuous Groups and Automorphic Functions*, Providence, American Mathematical Society, 1964.
- [Leuzi-03] E. Leuzinger. Kazhdan's property (T),  $L^2$ -spectrum and isoperimetric inequalities for locally symmetric spaces. *Comment. Math. Helv.*, **78**:116–133, 2003.
- [Loeve-77] M. Loève. *Probability Theory*, Vol. I, Berlin, Springer, 1977.
- [LosRi-81] V. Losert and H. Rindler. Almost invariant sets. *Bull. London Math. Soc.*, **13**:145–148, 1981.
- [Louve-01] N. Louvet. A propos d'un théorème de Vershik et Karpushev. *Enseignement Math.*, **47**:287–314, 2001.
- [LoStV-04] N. Louvet, Y. Stalder, and A. Valette. Fonctions conditionnellement de type négatif, représentations irréductibles et propriété (T). *Enseignement Math.*, **50**:239–266, 2004.
- [Lubot-89] A. Lubotzky. Trees and discrete subgroups of Lie groups over local fields. *Bull. Amer. Math. Soc.*, **20**:27–30, 1989.
- [Lubot-91] A. Lubotzky. Lattices in rank one Lie groups over local fields. *Geom. Funct. Anal. (GAFA)*, **1**:405–431, 1991.
- [Lubot-94] A. Lubotzky. *Discrete Groups, Expanding Graphs and Invariant Measures*, Basel, Birkhäuser, 1994.
- [Lubot-97] A. Lubotzky. Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. *Ann. Math.*, **145**:441–452, 1997.

- [LubPa-01] A. Lubotzky and I. Pak. The product replacement algorithm and Kazhdan's property (T). *J. Amer. Math. Soc.*, **14**:347–363, 2001.
- [LubSh-04] A. Lubotzky and Y. Shalom. Finite representations in the unitary dual and Ramanujan groups. In: *Discrete Geometric Analysis*. Proceedings of the 1st JAMS symposium, Sendai, Japan, December 12–20, 2002, p.173–189. Contemporary Mathematics **347**, Amer. Math. Soc. 2004.
- [LubWe-93] A. Lubotzky and B. Weiss. Groups and expanders. In: *Expanding Graphs*, p. 95–109. DIMACS Series in Discrete Mathematics, Volume 10, Amer. Math. Soc. 1993.
- [LubZi-89] A. Lubotzky and R.J. Zimmer. Variants of Kazhdan's Property for subgroups of semisimple groups. *Israel J. Math.*, **66**:289–299, 1989.
- [LubZu] A. Lubotzky and A. Zuk. *On Property*, Book in progress.
- [Macke-57] G.W. Mackey. Borel structures in groups and their duals. *Trans. Amer. Math. Soc.*, **85**:134–165, 1957.
- [Macke-76] G. W. Mackey. *The Theory of Unitary Group Representations*, University of Chicago Press, 1976.
- [MacSw-60] A. Macbeath and S. Swierczkowski. Limits of lattices in a compactly generated group. *Can. J. Math.*, **12**:427–437, 1960.
- [MagKS-66] W. Magnus, A. Karrass, and D. Solitar. *Combinatorial Group Theory*, New York, John Wiley, 1966.
- [Margu-69] G. Margulis. On the arithmeticity of discrete groups. *Soviet Math. Dokl.* **10**:900–902, 1969.
- [Margu-73] G.A. Margulis. Explicit constructions of concentrators. *Problems of Inform. Transm.*, **10**:325–332, 1975. [Russian original: *Problemy Peredači Informacii*, **9**:71–80, 1973.]
- [Margu-78] G.A. Margulis. Quotient groups of discrete subgroups and measure theory. *Funct. Anal. Appl.*, **12**:295–305, 1978.
- [Margu-79] G.A. Margulis. Finiteness of quotient groups of discrete subgroups. *Funct. Anal. Appl.*, **13**:178–187, 1979.
- [Margu-80] G.A. Margulis. Some remarks on invariant means. *Monatsh. Math.*, **90**:233–235, 1980.
- [Margu-81] G.A. Margulis. On the decomposition of discrete subgroups into amalgams. *Selecta Math. Sov.*, **1**:197–213, 1981.
- [Margu-82] G.A. Margulis. Finitely-additive invariant measures on Euclidean spaces. *Ergod. Th. & Dynam. Sys.*, **1**:383–396, 1982.
- [Margu-84] G. Margulis. Arithmeticity of the irreducible lattices in the semi-simple groups of rank greater than 1. *Inventiones Math.*, **76**:93–120, 1984.
- [Margu-91] G.A. Margulis. *Discrete Subgroups of Semisimple Lie Groups*, Berlin, Springer-Verlag, 1991.
- [Masse-67] W.S. Massey. *Algebraic Topology: An Introduction*, New York, Springer-Verlag, 1967.
- [Matsu-62] Y. Matsushima. On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces. *Ann. Math.*, **75**:312–330, 1972.
- [Mautn-57] F.I. Mautner. Geodesic flows on symmetric Riemannian spaces. *Ann. Math.*, **65**:416–430, 1957.
- [McCoo-89] J. McCool. A faithful polynomial representation of  $Out F_3$ . *Math. Proc. Cambridge Phil. Soc.*, **106**:207–213, 1989.

- [Mok–95] N. Mok. Harmonic forms with values in locally constant Hilbert bundles. In: Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), *J. Fourier Anal. Appl.*, Special Issue, 433–453, 1995.
- [Monod–01] N. Monod. *Continuous Bounded Cohomology of Locally Compact Groups*, Lecture Notes in Math. **1758**, Berlin, Springer, 2001.
- [MonSh] N. Monod and Y. Shalom. Orbit equivalence rigidity and bounded cohomology. *Ann. Math.*, **164**:825–878, 2006.
- [Moore–82] C.C. Moore. Ergodic theory and von Neumann algebras. In: *Operator Algebras and Applications*, Proc. Symposia Pure Math. 38, Part 2, 179–226, Amer. Math. Soc. 1982.
- [MoSiY–93] N. Mok, Y.T. Siu, and S.K. Yeung. Geometric superrigidity. *Inventiones Math.*, **113**:57–83, 1993.
- [Nava–02a] A. Navas. Actions de groupes de Kazhdan sur le cercle. *Ann. Scient. Éc. Norm. Sup.*, **35**:749–758, 2002.
- [Nava–02b] A. Navas. Groupes de Neretin et propriété (T) de Kazhdan. *C. R. Acad. Sci. Paris*, **335**:789–792, 2002.
- [Neuha–01] M. Neuhauser. Neue Methoden zur Bestimmung expliziter Schranken von Kazhdan-Konstanten. PhD Dissertation, Technische Universität München, 2001.
- [Neuh–03a] M. Neuhauser. Kazhdan constants for conjugacy classes of compact groups. *J. Algebra*, **270**:564–582, 2003.
- [Neuh–03b] M. Neuhauser. Kazhdan’s Property T for the symplectic group over a ring. *Bull. Belg. Math. Soc. Simon Stevin*, **10**:537–550, 2003.
- [Neuh–03c] M. Neuhauser. Kazhdan constants and matrix coefficients of  $Sp(n, \mathbf{R})$ . *J. Lie Theory*, **13**:133–154, 2003.
- [Neveu–68] J. Neveu. *Processus aléatoires Gaussiens*, Presses de l’Université de Montreal, 1968.
- [NiPoS–07] R. Nicoara, S. Popa, and R. Sasyk. On  $II_1$ -factors arising from 2-cocycles of w-rigid groups. *J. Funct. Anal.*, **242**:230–246, 2007.
- [Oh–02] Hee Oh. Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants. *Duke Math. J.*, **113**:133–192, 2002.
- [Olliv–04] Y. Ollivier. Sharp phase transition theorems for hyperbolicity of random groups. *Geom. Funct. Anal. (GAFA)*, **14**:595–679, 2004.
- [Olliv–05] Y. Ollivier. A January 2005 invitation to random groups. *Ensaio Matemáticos [Mathematical Surveys]*, 10. Sociedade Brasileira de Matemática, Rio de Janeiro, 2005.
- [OllWi] Y. Ollivier and D. Wise. Kazhdan groups with infinite outer automorphism group. *Trans. Amer. Math. Soc.*, **359**:1959–1976, 2007.
- [Ol’sh–92] A.Yu. Ol’shanskii. Periodic factor groups of hyperbolic groups. *Math. USSR Sbornik*, **72**:519–541, 1992.
- [O’Mea–78] O.T. O’Meara. *Symplectic Groups*, Providence, American Mathematical Society, 1978.
- [OrnWei–80] D. Ornstein and B. Weiss. Ergodic theory of amenable group actions I. The Rohlin lemma. *Bull. Amer. Math. Soc.*, **2**:107–139, 1980.
- [Osin–02] D.V. Osin. Kazhdan constants of hyperbolic groups. *Funct. Anal. Appl.*, **36**:290–297, 2002.

- [OsiSo] D.V. Osin and D. Sonkin. Uniform Kazhdan groups. Preprint 2006. arXiv: math.GR/0606012.
- [Ozawa–04] N. Ozawa. There is no separable universal  $II_1$ -factor. *Proc. Amer. Math. Soc.*, **132**:487–490, 2004.
- [PakZu–02] I. Pak and A. Zuk. On Kazhdan constants and mixing of random walks. *Int. Math. Res. Not.*, **36**:1891–1905, 2002.
- [Pansu–95] P. Pansu. Sous-groupes discrets des groupes de Lie : rigidité, arithmétique. *Séminaire Bourbaki*, exposé 778, Astérisque **227**, 69–105, Soc. Math. France, 1995.
- [Pansu–98] P. Pansu. Formules de Matsushima, de Garland, et propriété (T) pour des groupes agissant sur des espaces symétriques ou des immeubles. *Bull. Soc. Math. France*, **126**:107–139, 1998.
- [Pater–88] A.T. Paterson. *Amenability*, Providence, American Mathematical Society, 1988.
- [Pauli–91] F. Paulin. Outer automorphisms of hyperbolic groups and small actions on  $\mathbf{R}$ -trees. In: *Arboreal Group Theory* (Berkeley, CA, 1988), Math. Sci. Res. Inst. Publ. 19, 331–343, New York, Springer 1991.
- [Pesto] V. Pestov. Private communication. June 2004.
- [PetPo–05] J. Peterson and S. Popa. On the notion of relative Property (T) for inclusions of von Neumann algebras. *J. Funct. Anal.*, **219**:469–483, 2005.
- [Picar–13] E. Picard. Extrait d’une lettre au Directeur des Rendiconti. *Rend. Circ. Mat. Palermo*, **36**:277–278, 1913.
- [Picho–03] M. Pichot. Conditions simpliciales de rigidité pour les relations de type  $II_1$ . *C. R. Acad. Sci. Paris*, **337**:7–12, 2003.
- [PimPo–86] M. Pimsner and S. Popa. Sur les sous-facteurs d’indice fini d’un facteur de type  $II_1$  ayant la propriété T. *C. R. Acad. Sci. Paris*, **303**:359–361, 1986.
- [Pitte–02] C. Pittet. The isoperimetric profile of homogeneous Riemannian manifolds. *J. Diff. Geom.*, **54**:255–302, 2000.
- [Popa] S. Popa. Correspondences. INCREST preprint, 1986.
- [Popa–04] S. Popa. On the fundamental group of type  $II_1$  factors. *Proc. Nat. Acad. Sci.*, **101**:723–726, 2004.
- [Popa–06a] S. Popa. On a class of type  $II_1$  factors with Betti numbers invariants. *Ann. Math.*, **163**:767–807, 2006.
- [Popa–06b] S. Popa. Strong rigidity of  $II_1$  factors arising from malleable actions of w-rigid groups, I and II. *Inventiones Math.*, **165**:369–408 and **165**:409–452, 2006.
- [Popa–06c] S. Popa. Some computations of 1-cohomology groups and construction of non orbit equivalent actions. *J. Institut Math. Jussieu*, **5**:309–332, 2006.
- [PreSe–86] A. Pressley and G. Segal. *Loop Groups*, Oxford University Press, 1986.
- [Radul–92] F. Radulescu. The fundamental group of the von Neumann algebra of a free group with infinitely many generators is  $\mathbf{R}^+ \setminus 0$ . *J. Amer. Math. Soc.*, **5**:517–532, 1992.
- [Raghu–72] M.S. Raghunathan. *Discrete Subgroups of Lie Groups*, Berlin, Springer-Verlag, 1979.

- [Raghu–89] M.S. Raghunathan. Discrete Subgroups of algebraic groups over local fields of positive characteristic. *Proc. Indian Acad. Sci.*, **99**:127–146, 1989.
- [RehSo–76] U. Rehmann and C. Soulé. Finitely presented groups of matrices. In: *Algebraic K-theory* Proceedings, Evanston 1976. *Lecture Notes in Math.*, **551**:164–169, Berlin, Springer 1976.
- [Reiter–65] H. Reiter. On some properties of locally compact groups. *Nederl. Akad. Wetensch. Indag. Math.*, **27**:697–701, 1965.
- [Reiter–68] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*, Oxford University Press, 1968.
- [Rémy–99] B. Rémy. Construction de réseaux en théorie de Kac–Moody. *C. R. Acad. Sci. Paris*, **329**:475–478, 1999.
- [Rémy–02] B. Rémy. Classical and non-linearity properties of Kac–Moody lattices. In: *Rigidity in Dynamics and Geometry*, Ed: M. Burger and A. Iozzi, 391–406, Berlin, Springer 2002.
- [ReVaW–02] O. Reingold, S. Vadhan, and A. Wigderson. Entropy waves, the zig-zag product, and new constant degree expanders. *Ann. Math.*, **155**:157–187, 2002.
- [Rezni–00] A. Reznikov. Analytic topology of groups, actions, strings and varieties. Preprint, 2000. arXiv: math. DG/0001135.
- [Rezni–01] A. Reznikov. Analytic topology. In: *European Congress of Mathematics (Barcelona 2000)*, Vol. 1, 519–532, Basel, Birkhäuser 2001.
- [ROIV] F.J. Murray and J. von Neumann. On rings of operators, IV. *Ann. Math.*, **44**:716–808, 1943.
- [Robet–83] A. Robert. *Introduction to the Representation Theory of Compact and Locally Compact Groups*, Cambridge University Press, 1983.
- [Roben–98] G. Robertson. Crofton formulae and geodesic distance in hyperbolic spaces. *J. Lie Theory*, **8**:163–172, 1998.
- [RobSt–98] G. Robertson and T. Steger. Negative definite kernels and a dynamical characterization of property (T) for countable groups. *Ergod. Th. & Dynam. Sys.*, **18**:247–253, 1998.
- [RoeDi–81] W. Roelcke and S. Dierolf. *Uniform Structures on Topological Groups and their Quotients*, New York, McGraw-Hill, 1981.
- [Ronan–89] M. Ronan. *Lectures on Buildings*, New York, Academic Press, 1989.
- [Rosen–81] J. Rosenblatt. Uniqueness of invariant means for measure preserving transformations. *Trans. Amer. Math. Soc.*, **265**:623–636, 1981.
- [Rosen–85] J. Rosenblatt. Translation-invariant linear forms on  $L_p(G)$ . *Proc. Amer. Math. Soc.*, **94**:339–348, 1985.
- [Rudin–3] W. Rudin. *Real and Complex Analysis*, New York, McGraw-Hill, 1966.
- [Rudin–73] W. Rudin. *Functional analysis*, New York, McGraw-Hill, 1973.
- [Sarna–83] P. Sarnak. The arithmetic and geometry of some hyperbolic three manifolds. *Acta Math.*, **151**:253–295, 1983.
- [Sarna–90] P. Sarnak. *Some Applications of Modular Forms*, Cambridge University Press, 1990.
- [Schmi–80] K. Schmidt. Asymptotically invariant sequences and an action of  $SL(2, \mathbb{Z})$  on the 2-sphere. *Israel J. Math.*, **37**:193–208, 1980.

- [Schmi–81] K. Schmidt. Amenability, Kazhdan's property (T), strong ergodicity and invariant means for group actions. *Ergod. Th. & Dynam. Sys.*, **1**:223–236, 1981.
- [Schmi–85] K. Schmidt. Cohomology and the absence of strong ergodicity for ergodic group actions. In: *Operator Algebras and their Connections with Topology and Ergodic Theory* (Busteni 1983), Lecture Notes in Math. **1132**, 486–496, Berlin, Springer 1985.
- [Schoe–38] I.J. Schoenberg. Metric spaces and positive definite functions. *Trans. Amer. Math. Soc.*, **44**:522–536, 1938.
- [Sega–81] G.B. Segal. Unitary representations of some infinite dimensional groups. *Comm. Math. Phys.*, **80**:301–342, 1981.
- [Serr–70a] J-P. Serre. *Cours d'arithmétique*, Presses Universitaires de France, 1970.
- [Serr–70b] J-P. Serre. Le problème des groupes de congruence pour  $SL_2$ . *Ann. Math.*, **92**:489–527, 1970.
- [Serre–71] J-P. Serre. Cohomologie des groupes discrets. In: *Prospects in mathematics*, Annals of Math. Studies **70**, 77–169, Princeton University Press, 1971.
- [Serre–74] J-P. Serre. Amalgames et points fixes. In: *Proc. Conf. Canberra 1973*, Lecture Notes in Mathematics **372**, 633–640, Berlin, Springer, 1974.
- [Serre–77] J-P. Serre. *Arbres, amalgames,  $SL_2$* , Astérisque **46**, Soc. Math. France, 1977.
- [Shal–99a] Y. Shalom. Bounded generation and Kazhdan's Property (T). *Publ. Math. I.H.E.S.*, **90**:145–168, 1999.
- [Shal–99b] Y. Shalom. Invariant measures for algebraic actions, Zariski dense subgroups and Kazhdan's Property (T). *Trans. Amer. Math. Soc.*, **351**:3387–3412, 1999.
- [Shal–00a] Y. Shalom. Rigidity of commensurators and irreducible lattices. *Inventiones Math.*, **141**:1–54, 2000.
- [Shal–00b] Y. Shalom. Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation groups. *Ann. Math.*, **152**:113–182, 2000.
- [Shal–00c] Y. Shalom. Explicit Kazhdan constants for representations of semisimple and arithmetic groups. *Ann. Inst. Fourier*, **50**:833–863, 2000.
- [Shal–ICM] Y. Shalom. The algebraization of Kazhdan's Property (T). In: *Proceedings of the ICM, Madrid 2006*. Volume II, p. 1283–1310. European Mathematical Society (EMS). 2006.
- [Siege–43] K.L. Siegel. Discontinuous groups. *Ann. Math.*, **44**:674–689, 1943.
- [Skand–88] G. Skandalis. Une notion de nucléarité en  $K$ -théorie (d'après J. Cuntz). *K-Theory*, **1**:549–573, 1988.
- [Soard–94] P.M. Sordi. *Potential Theory on Infinite Networks*, Lecture Notes in Math. **1590**, Berlin, Springer, 1994.
- [Spatz–95] R. Spatzier. Harmonic analysis in rigidity theory. In: *Ergodic Theory and its Connections with Harmonic Analysis* (Alexandria 1993), London Math. Soc. Lecture Note Ser. **205**, 153–205, Cambridge University Press, 1995.
- [Sriva–98] S.M. Srivastava. *A Course on Borel Sets*, Berlin, Springer, 1998.

- [Sulli–81] D. Sullivan. For  $n > 3$ , there is only one finitely-additive measure on the  $n$ -sphere defined on all Lebesgue measurable sets. *Bull. Amer. Math. Soc.*, **1**:121–123, 1981.
- [Taher–00] F. Taherkhani. The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of its cofinite subgroups. *Experimental Mathematics*, **9**:261–274, 2000.
- [Takah–63] R. Takahashi. Sur les représentations unitaires des groupes de Lorentz généralisés. *Bull. Soc. Math. France*, **91**:289–433, 1963.
- [TaKuU–81] S. Takenada, I. Kubo, and H. Urakawa. Brownian motion parametrized with metric space of constant curvature. *Nagoya. Math. J.*, **82**:131–140, 1981.
- [Thoma–01] S. Thomas. On the complexity of the classification problem for torsion-free abelian groups of finite rank. *Bull. Symbolic Logic*, **7**:329–344, 2001.
- [Thoma–02] S. Thomas. On the complexity of the classification problem for torsion-free abelian groups of rank two. *Acta Math.*, **189**:287–305, 2002.
- [Thoma–03] S. Thomas. The classification problem for torsion-free abelian groups of finite rank. *J. Amer. Math. Soc.*, **16**:233–258, 2003.
- [Thurs–74] W.P. Thurston. A generalization of the Reeb stability theorem. *Topology*, **13**:347–352, 1974.
- [Valet–94] A. Valette. Old and new about Kazhdan’s Property (T). In: *Representations of Lie Groups and Quantum Groups* (Trento, 1993), Res. Notes Math. Ser. **311**, 271–333, London, Pitman, 1994.
- [Valet–97] A. Valette. Graphes de Ramanujan et applications. *Séminaire Bourbaki*, exposé 829, Astérisque **245**, 247–276, Soc. Math. France, 1997.
- [Valet–04] A. Valette. Nouvelles approches de la propriété (T) de Kazhdan. *Séminaire Bourbaki* (2002 / 2003) , Astérisque **294**, 97–124, Soc. Math. France, 2004.
- [Valet–05] A. Valette. Group pairs with Property (T), from arithmetic lattices. *Geom. Dedicata*, **112**:183–196, 2005.
- [Varad–68] V.S. Varadarajan. *Geometry of Quantum Theory I*, New York, Van Nostrand, 1968.
- [Vaser–68] L.N. Vaserstein. Groups having the property (T). *Funct. Anal. Appl.*, **2**:174, 1968.
- [Vaser–88] L.N. Vaserstein. Reduction of a matrix depending on parameters to a diagonal form by addition operators. *Proc. Amer. Math. Soc.*, **103**:741–746, 1988.
- [VeGeG–73] A.M. Vershik, I.M. Gel’fand, and M.I. Graev. Representations of the group  $SL(2, R)$  where  $R$  is a ring of functions. *Russian Math. Surveys*, **28**:87–132, 1973. [=I.M. Gelfand, *Collected Papers*, Vol. II, 793–838.]
- [VeGeG–77] A.M. Vershik, I.M. Gel’fand, and M.I. Graev. Representations of the group of smooth mappings of a manifold  $X$  into a compact Lie group. *Compos. Math.*, **35**:299–334, 1977. [= I.M. Gelfand, *Collected Papers*, Vol. II, 839–874.]
- [VeGeG–74] A.M. Vershik, I.M. Gel’fand, and M.I. Graev. Irreducible representations of the group  $G^X$  and cohomologies. *Funct. Anal. and its Appl.*, **8**:151–153, 1974. [= I.M. Gelfand, *Collected Papers*, Vol. II, 875–877.]

- [Venka-93] T.N. Venkataramana. On the arithmeticity of certain rigid subgroups. *C. R. Acad. Sci. Paris*, **316**:321–326, 1993.
- [VerKa-82] A.M. Vershik and S.I. Karpushev. Cohomology of groups in unitary representations, the neighbourhood of the identity, and conditionally positive definite functions. *Math. USSR Sbornik*, **47**:513–526, 1984. [Russian original: *Mat. Sb. (N.S.)* 119(161) (1982), no. 4, 521–533.]
- [Vilen-68] N.J. Vilenkin. *Special Functions and the Theory of Group Representations*, Translations of Math. Monographs **22**, Providence, American Mathematical Society, 1968.
- [Voic-90a] D. Voiculescu. Circular and semicircular systems and free product factors. In: *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory*, 45–60, Basel, Birkhäuser, 1990.
- [Voic-90b] D. Voiculescu. Property T and approximation of operators. *Bull. London Math. Soc.*, **22**:25–30, 1990.
- [vNeum-32] J. von Neumann. Einige Sätze über messbare Abbildungen. *Ann. Math.*, **33**:574–586, 1932.
- [Walla-73] N. Wallach. *Harmonic Analysis on Homogeneous Spaces*, New York, Marcel Dekker, 1973.
- [Walte-82] P. Walters. *An Introduction to Ergodic Theory*, New York, Springer, 1982.
- [Wang-69] S.P. Wang. The dual space of semi-simple Lie groups. *Amer. J. Math.*, **23**:921–937, 1969.
- [Wang-74] S.P. Wang. On the first cohomology of discrete groups with property (T). *Proc. Amer. Math. Soc.*, **42**:621–624, 1974.
- [Wang-75] S.P. Wang. On isolated points in the dual spaces of locally compact groups. *Math. Ann.*, **219**:19–34, 1975.
- [Wang-82] S.P. Wang. On the Mautner phenomenon and groups with Property (T). *Amer. J. Math.*, **104**:1191–1210, 1982.
- [Wang-98] M.-T. Wang. On a fixed point theorem of discrete group actions on Riemannian manifolds. *J. Diff. Geom.*, **50**:249–267, 1998.
- [Watat-82] Y. Watatani. Property (T) of Kazhdan implies property (FA) of Serre. *Math. Japonica*, **27**:97–103, 1982.
- [Weil-65] A. Weil. *L'intégration dans les groupes topologiques et ses applications*, seconde édition, Paris, Hermann, 1965.
- [Weil-73] A. Weil. *Basic Number Theory*, Berlin, Springer-Verlag, 1973.
- [Weyl-16] H. Weyl. Über die Gleichverteilung von Zahlen mod. 1. *Math. Ann.*, **77**:313–352, 1916.
- [Willi-88] G.A. Willis. Continuity of translation invariant linear functionals on  $C_0(G)$  for certain locally compact groups  $G$ . *Monatsh. Math.*, **105**: 61–164, 1988.
- [Witte-94] D. Witte. Arithmetic groups of higher  $\mathbf{Q}$ -rank cannot act on 1-manifolds. *Proc. Amer. Math. Soc.*, **122**:333–340, 1994.
- [Woess-00] W. Woess. *Random Walks on Infinite Graphs and Groups*, Cambridge University Press, 2000.
- [Zimme-81] R.J. Zimmer. On the cohomology of ergodic actions of semisimple Lie groups and discrete subgroups. *Amer. J. Math.*, **103**:937–950, 1981.



- [Zimm–84a] R.J. Zimmer. *Ergodic Theory and Semisimple Groups*, Basel, Birkhäuser, 1984.
- [Zimm–84b] R.J. Zimmer. Volume preserving actions of lattices in semisimple groups on compact manifolds. *Publ. Math. I.H.É.S.*, **59**:5–33, 1984.
- [Zimm–84c] R.J. Zimmer. Kazhdan groups acting on compact manifolds. *Inventiones Math.*, **75**:425–436, 1984.
- [Zimme–86] R.J. Zimmer. On the automorphism group of a compact Lorentz manifold and other geometric manifolds. *Inventiones Math.*, **83**:411–424, 1986.
- [Zuk–96] A. Zuk. La propriété (T) de Kazhdan pour les groupes agissant sur les polyèdres. *C. R. Acad. Sci. Paris*, **323**:453–458, 1996.
- [Zuk–03] A. Zuk. Property (T) and Kazhdan constants for discrete groups. *Geom. Funct. Anal (GAFA)*, **13**:643–670, 2003.

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