ON THE THEORY OF THE EISENSTEIN INTEGRAL

by

Harish-Chandra

The Institute for Advanced Study
Princeton, N. J.

§1. Introduction

Let F be a field and $\underset{\sim}{G}$ a connected, reductive algebraic group defined over F. Let G_F denote the subgroup of all F-rational points of G. If F is a local field (i.e. $F = \underset{\sim}{R}$, $\underset{\sim}{C}$ or a $\mathfrak{p}$-adic field), G_F has a natural locally compact topology. (We write $G_F = G_{\mathfrak{p}}$ when F is a $\mathfrak{p}$-adic field.) On the other hand if F is a finite field, G_F is a finite group. Finally if F is a global field (i.e. a number-field or a function-field in one variable over a finite field), we have the adèle group $G_{\underset{\sim}{A}}$ associated to $\underset{\sim}{G}$. From the point of view of harmonic analysis and the theory of automorphic forms, the study of the following five cases is important.

1) $G_{\underset{\sim}{R}}/\Gamma$ where Γ is an arithmetic subgroup of G,
2) $G_{\underset{\sim}{R}}$,
3) $G_{\mathfrak{p}}$,
4) G_F where F is a finite field,
5) $G_{\underset{\sim}{A}}/G_F$ where F is a global field.

Actually 5) is the most difficult case and the other four may, in fact, be regarded merely as its several facets. Nevertheless it is useful to pursue their study individually since a knowledge of one case enables us to guess, often quite accurately, analogous results in another case. This similarity from the standpoint of harmonic analysis, is most striking between $G_{\underset{\sim}{R}}$ and $G_{\mathfrak{p}}$ (see [4(e)]). But in this lecture I propose to bring out the resemblance between $G_{\underset{\sim}{R}}/\Gamma$ and $G_{\underset{\sim}{R}}$.

The theory of Eisenstein Series over $G_{\underset{\sim}{R}}/\Gamma$, which is largely due to Selberg, Gelfand-Piatetsky-Shapiro and Langlands (see [4(a)]), has an exact counterpart in the theory of Eisenstein Integrals over $G_{\underset{\sim}{R}}$. These integrals have functional equations (see §6) which are governed by the coefficients

$c_{P_2|P_1}(s:\nu)$ appearing in their asymptotic expansion. Following Gelfand [2] (see also [4(b), §5]), one is tempted to call these coefficients the local zeta-functions at infinity. Since a similar theory holds for $G_{\mathfrak{p}}$, one gets in this way local zeta-functions at each prime. It seems likely that the global zeta-functions (i.e. the corresponding coefficients appearing in the Eisenstein Series for $G_{\underset{\sim}{A}}/G_F$) are actually products built out of the local factors. This leads one to expect that the local factors are "elementary" or "Eulerian." Therefore, in particular, $c_{P_2|P_1}(s:\nu)$ should be expressible in terms of gamma factors. Although this conjecture remains unproved, we do obtain a rather simple formula for the "absolute value" of this operator and therefore for the Plancherel measure $\mu_\omega(\nu)d\nu$ (see §13). Moreover there is some evidence (see §14) that the zeros of the function μ_ω are related to the occurrence of the exceptional series (see also [6(a), (b)] and [7]).

In the first few sections of this paper, we restate in a precise form those results of [4(c)] and [4(d)], which are necessary for the understanding of our main theorems.

§2. The assumptions on G

For any Lie group G, we denote by G^o the connected component of 1 in G and by $X(G)$ the group of all continuous homomorphisms of G into $\underset{\sim}{R}^\times$. Put

$$^oG = \bigcap_{\chi \in X(G)} \ker|\chi| .$$

Then $G/^oG$ is an abelian Lie group. By the parabolic rank of G we mean $\dim G/^oG$ and denote it by prk G. A split component of G is a closed subgroup A of G such that

$$G = {}^oG.A , \quad {}^oG \cap A = \{1\} .$$

Note that A, if it exists, is abelian. In fact $A \simeq G/^oG$ and therefore

$\dim A = \operatorname{prk} G$. By a vector subgroup V of G, we mean a closed subgroup which is topologically isomorphic to the additive group of $\mathbb{R}^n$ for some $n \geq 0$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let G_c denote the connected complex adjoint group[1)] of $\mathfrak{g}_c$. We make the following assumptions on G.

1) $\mathfrak{g}$ is reductive and $\operatorname{Ad}(G) \subset G_c$.

2) Let G_1 denote the analytic subgroup of G corresponding to $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. Then the center of G_1 is finite.

3) $[G : G^o] < \infty$.

Fix a maximal compact subgroup K of G. (Such a subgroup exists and is unique up to conjugacy by G^o.) Let C be the center of G^o. Fix a maximal vector subgroup C_2 of C. Then $C = C_1 . C_2$ where $C_1 = C \cap K$. Let $\mathfrak{k}$, $\mathfrak{c}_1$, $\mathfrak{c}_2$ be the Lie algebras of K, C_1, C_2 respectively and θ the Cartan involution of $\mathfrak{g}_1$ with respect to $\mathfrak{k}_1 = \mathfrak{k} \cap \mathfrak{g}_1$. Extend θ to an involution of $\mathfrak{g}$ by setting

$$\theta(X_1 + X_2) = X_1 - X_2 \qquad (X_i \in \mathfrak{c}_i,\ i = 1,\ 2) \ .$$

Let $\mathfrak{p}$ be the set of all points $X \in \mathfrak{g}$ such that $\theta(X) = -X$.

Lemma 1. <u>The mapping</u> $(k, X) \longmapsto k \exp X$ $(k \in K,\ X \in \mathfrak{p})$ <u>is an analytic diffeomorphism of</u> $K \times \mathfrak{p}$ <u>onto</u> G <u>and</u> θ <u>extends to an automorphism of</u> G <u>such that</u>

$$\theta(k \exp X) = k \exp(-X) \qquad (k \in K,\ X \in \mathfrak{p}) \ .$$

We denote by $\log$ the inverse of the mapping $\exp : \mathfrak{p} \longrightarrow \exp \mathfrak{p}$.

Lemma 2. <u>There exists a real symmetric bilinear form</u> B <u>on</u> $\mathfrak{g}$ <u>such that:</u>

1) $B(\theta X, \theta Y) = B(X, Y)$ <u>and</u>

$$B([X, Y], Z) + B(Y, [X, Z]) = 0 \qquad (X, Y, Z \in \mathfrak{g}) \ .$$

2) <u>The quadratic form</u>

$$\|X\|^2 = -B(X, \theta X) \qquad (X \in \mathfrak{g})$$

<u>is positive-definite on</u> $\mathfrak{g}$.

We fix K, θ and B as above, once for all and define $\sigma(x) = \|X\|$ for $x = k \exp X$ $(k \in K,\ X \in \mathfrak{p})$.

§3. Parabolic subgroups[2)]

A subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called parabolic, if $\mathfrak{q}_c$ contains a Borel subalgebra (i.e. a maximal solvable subalgebra) of $\mathfrak{g}_c$. A subgroup Q of G is called parabolic, if it is the normalizer in G of some parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. Then Q is closed and its Lie algebra is $\mathfrak{q}$.

Lemma 3. <u>Let</u> Q <u>be a parabolic subgroup</u> (psgp) <u>of</u> G. <u>Then</u> $G = KQ$.

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$. By the radical of $\mathfrak{q}$, we mean the maximal ideal $\mathfrak{n}$ of $\mathfrak{q}_1 = \mathfrak{q} \cap [\mathfrak{g}, \mathfrak{g}]$ such that ad X is nilpotent for every $X \in \mathfrak{n}$. If Q is the psgp corresponding to $\mathfrak{q}$ and N the analytic subgroup corresponding to $\mathfrak{n}$, then N is called the radical of Q. Put $M_1 = Q \cap \theta(Q)$ and $M = {}^0M_1$ (see §2). Let A be the maximal θ-stable vector subgroup lying in the center of M_1. Then A is a split component both for Q and M_1. We call A <u>the</u> split component of Q. Then $M_1 = MA$ is the centralizer of A in G and

$$Q = MAN\ ,$$

the mapping $(m, a, n) \longmapsto man$ being an analytic diffeomorphism of $M \times A \times N$ onto Q. We call this the Langlands decomposition of Q.

Let $\mathfrak{m}$, $\mathfrak{a}$ denote the Lie algebra of M, A respectively and put $\mathfrak{m}_1 = \mathfrak{m} + \mathfrak{a}$, $K_M = K \cap M$. Let θ_M and B_M be the restrictions of θ and B respectively on $\mathfrak{m}$. Then if we replace (G, K, θ, B) by

(M, K_M, θ_M, B_M) all the conditions of §2 are fulfilled. The same holds for $(M_1, K_M, \theta_1, B_1)$ where θ_1 and B_1 are the restrictions of θ and B respectively on $\mathfrak{m}_1$.

By a p-pair (Q, A), we mean a psgp Q and its split component A. Then $Q = MAN$. By a root of Q (or (Q, A)), we mean an element α in $\mathfrak{a}^*$ with the following property. Let $\mathfrak{n}_\alpha$ denote the set of all $X \in \mathfrak{n}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}$. Then $\mathfrak{n}_\alpha \neq \{0\}$. It is clear that $\operatorname{prk} Q = \dim A \geq \operatorname{prk} G$.

Let $\ell = \operatorname{prk} Q - \operatorname{prk} G$ and let $\Sigma(Q)$ denote the set of all roots of Q. A root $\alpha \in \Sigma(Q)$ is called simple if it cannot be written in the form $\alpha = \beta + \gamma$ with $\beta, \gamma \in \Sigma(Q)$. Let $\Sigma^o(Q)$ be the set of all simple roots. Then[5] $\ell = [\Sigma^o(Q)]$. Let $\alpha_1, \ldots, \alpha_\ell$ be all the simple roots. Then they are linearly independent over $\underset{\sim}{R}$ and every $\alpha \in \Sigma(Q)$ can be written in the form

$$\alpha = m_1\alpha_1 + \ldots + m_\ell\alpha_\ell$$

where $m_i \in \underset{\sim}{Z}$ and $m_i \geq 0$ $(1 \leq i \leq \ell)$.

Fix a subset F of $\Sigma^o(Q)$ and let $\mathfrak{a}_F$ denote the set of all $H \in \mathfrak{a}$ such that $\alpha(H) = 0$ for all $\alpha \in F$. Let Σ_F be the set of all $\alpha \in \Sigma = \Sigma(Q)$ which vanish identically on $\mathfrak{a}_F$. Put

$$\mathfrak{n}_F = \sum_{\alpha \in \Sigma'_F} \mathfrak{n}_\alpha$$

where Σ'_F is the complement of Σ_F in Σ. Then $\mathfrak{n}_F$ is an ideal in $\mathfrak{n}$. Put $N_F = \exp \mathfrak{n}_F$, $A_F = \exp \mathfrak{a}_F$ and let Q_F denote the normalizer of N_F in G. Then (Q_F, A_F) is a p-pair in G and

$$Q_F = M_F A_F N_F$$

where $M_F = {}^oZ(A_F)$ and $Z(A_F) = M_F A_F$ is the centralizer of A_F in G. We write $(Q, A)_F = (Q_F, A_F)$.

Let (Q', A') be any p-pair in G. We write $(Q', A') \succ (Q, A)$ if

$Q' \supset Q$. This implies that $A' \subset A$. Every p-pair $(Q', A') \succ (Q, A)$ is of the form $(Q', A') = (Q, A)_F$ for a unique $F \subset \Sigma^o(Q)$.

Lemma 4. *There is a one-one correspondence between psgps* P *of* G *which are contained in* Q *and psgps* ${}^*P$ *of* M. *This correspondence is given by the relation* ${}^*P = P \cap M$. *If*

$$P = M'A'N' \ , \qquad {}^*P = {}^*M\,{}^*A\,{}^*N$$

are the corresponding Langlands decompositions, then

$$M' = {}^*M \ , \quad A' = {}^*A . A \ , \quad N' = {}^*N . N \ ,$$

$${}^*A = M \cap A' \ , \quad {}^*N = M \cap N' \ .$$

Lemma 5. *Any two minimal psgps of* G *are conjugate under* K^o. *Let* P_1, P_2 *and* P *be three psgps of* G. *Suppose* $P_1 \cap P_2 \supset P$ *and* P_2 *is conjugate to* P_1 *under* G. *Then* $P_1 = P_2$.

Let (P_i, A_i) $(i = 1, 2)$ be two p-pairs. We denote by $\mathfrak{w}(A_2 | A_1) = \mathfrak{w}(\mathfrak{a}_2 | \mathfrak{a}_1)$ the set of all linear mappings

$$s : \mathfrak{a}_1 \longrightarrow \mathfrak{a}_2$$

satisfying the following condition. There should exist an element $y \in G$ such that $sH = \mathrm{Ad}(y)H$ for all $H \in \mathfrak{a}_1$. y is then called a representative of s in G. (One can always choose a representative $y \in K$.) We also write $a^s = yay^{-1}$ $(a \in A_1)$.

P_1, P_2 are said to be associated if $\mathfrak{w}(A_2 | A_1)$ and $\mathfrak{w}(A_1 | A_2)$ are both nonempty. This is equivalent to saying that A_1 and A_2 are conjugate under G.

Let (P, A) be a p-pair. We write $\mathfrak{w}(A) = \mathfrak{w}(A|A)$. Then $\mathfrak{w}(A)$ is a finite group and in fact

$$\mathfrak{w}(A) = (\text{Normalizer of } A \text{ in } K)/(\text{Centralizer of } A \text{ in } K) \ .$$

We call $\mathfrak{W}(A)$ the Weyl group of A in G and sometimes denote it by $\mathfrak{W}(G/A)$.

Let $P = MAN$ be the Langlands decomposition of P. Define $\rho_P \in \mathfrak{a}^*$ by

$$\rho_P(H) = \tfrac{1}{2}\mathrm{tr}(\mathrm{ad}\, H)_{\mathfrak{n}} \qquad (H \in \mathfrak{a}) \ ,$$

where $(\mathrm{ad}\, H)_{\mathfrak{n}}$ denotes the restriction of $\mathrm{ad}\, H$ on $\mathfrak{n}$. Since $G = KP$, every $x \in G$ can be written in the form $x = kman$ ($k \in K$, $m \in M$, $a \in A$, $n \in N$). Here a is uniquely determined and we put $H_P(x) = \log a$. Then $H_P : G \longrightarrow \mathfrak{a}$ is an analytic mapping.

Let $d_\ell p$ and $d_r p$ denote the left- and right-invariant Haar measures on P so that $d_r p = d_\ell p^{-1}$. Then

$$d_r p = \delta_P(p) d_\ell p$$

where δ_P is a continuous homomorphism of P into $\mathbb{R}_+^\times$. In fact

$$\delta_P(man) = e^{2\rho(\log a)} \qquad (m \in M, \ a \in A, \ n \in N)$$

where $\rho = \rho_P$. Note that $\delta_P = 1$ on $K \cap P = K \cap M$.

Now suppose P is a minimal psgp of G. We extend δ_P to a function on G by setting $\delta_P(kp) = \delta_P(p)$ ($k \in K$, $p \in P$). Now put

$$\Xi_G(x) = \Xi(x) = \int_K \delta_P(xk)^{-1/2} dk$$

where dk is the normalized Haar measure on the compact group K. It follows from Lemma 5 that Ξ is actually independent of the choice of P.

§4. Cusp forms and the space $\mathcal{A}(G, \tau)$

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_c$. We regard elements of $\mathfrak{G}$ as left-invariant differential operators on G. There is an obvious anti-isomorphism $g \longmapsto g'$ of $\mathfrak{G}$ onto the algebra $\mathfrak{G}'$ of right-invariant differential operators on G. If $g_1, g_2 \in \mathfrak{G}$, we write

$$f(g_1 ; x; g_2) = (g_1{}' g_2 f)(x) \qquad (x \in G)$$

for any $f \in C^\infty(G)$. Put

$$\nu_{g_1, g_2, r}(f) = \sup_G |f(g_1; x; g_2)| \Xi(x)^{-1}(1+\sigma(x))^r$$

for $r \geq 0$. Then the Schwartz space $\mathcal{C}(G)$ consists of all functions $f \in C^\infty(G)$ such that

$$\nu_{g_1, g_2, r}(f) < \infty$$

for all $g_1, g_2 \in \mathfrak{G}$ and $r \geq 0$. The set of all seminorms $\nu_{g_1, g_2, r}$ defines the structure of a locally convex Hausdorff space on $\mathcal{C}(G)$ which is complete. Moreover $\mathcal{C}(G)$ is contained in $L_2(G)$.

If $P = MAN$ is a psgp of G and $f \in \mathcal{C}(G)$, we put

$$f^P(x) = \int_N f(xn)dn \qquad (x \in G) \quad ,$$

where dn is the Haar measure on N. (This integral is always convergent.) f is said to be a cusp form if $f^P = 0$ whenever prk $P > 0$. Let ${}^\circ\mathcal{C}(G)$ denote the space of all cusp forms. Then ${}^\circ\mathcal{C}(G)$ is a closed subspace of $\mathcal{C}(G)$.

Let τ be a unitary double representation of K on a finite-dimensional Hilbert space V. Let $C^\infty(G, \tau)$ denote the subspace of all $f \in C^\infty(G) \otimes V$ such that

$$f(k_1 x k_2) = \tau(k_1) f(x) \tau(k_2) \qquad (k_1, k_2 \in K, x \in G) \quad .$$

Put ${}^\circ\mathcal{C}(G, \tau) = C^\infty(G, \tau) \cap ({}^\circ\mathcal{C}(G) \otimes V)$.

Theorem 1. $\dim {}^\circ\mathcal{C}(G, \tau) < \infty$.

Let $\mathfrak{Z}$ be the algebra of all differential operators on G which commute with both left and right translations of G. Then $\mathfrak{Z}$ is the center of $\mathfrak{G}$ and therefore abelian.

Corollary. Every element in ${}^{\circ}\mathcal{C}(G, \tau)$ is $\mathfrak{Z}$-finite.

A continuous function f on G is said to satisfy the weak inequality, if there exist numbers c, $r \geq 0$ such that

$$|f(x)| \leq c\,\Xi(x)\,(1+\sigma(x))^r$$

for all $x \in G$.

Let $\mathcal{A}(G, \tau)$ denote the space of all $f \in C^\infty(G, \tau)$ such that:

1) f is $\mathfrak{Z}$-finite;

2) $|f|$ satisfies the weak inequality.

Let $P = MAN$ be a psgp of G and put

$$\gamma(a) = \inf_{\alpha} \alpha(\log a) \qquad (a \in A) \quad ,$$

where α runs over all roots of (P, A). a being a variable element of A, we write $a \underset{P}{\longrightarrow} \infty$ if 1) $\sigma(a) \longrightarrow \infty$ and 2) we can choose $\varepsilon > 0$ such that $\gamma(a) \geq \varepsilon\sigma(a)$. Let τ_M denote the restriction of τ on $K_M = K \cap M$.

Theorem 2. Given $f \in \mathcal{A}(G, \tau)$, there exists a unique element $f_P \in \mathcal{A}(MA, \tau_M)$ such that

$$\lim_{a \underset{P}{\to} \infty} \{\delta_P(ma)^{1/2} f(ma) - f_P(ma)\} = 0$$

for $m \in MA$ and $a \in A$.

We call f_P the constant term of f along P.

Let ${}^*P = {}^*M\,{}^*A\,{}^*N$ be a psgp of M and $P' = M'A'N'$ the psgp of G contained in P, which corresponds to ${}^*P$ according to Lemma 4.

Lemma 6. Fix $f \in \mathcal{A}(G, \tau)$, $a \in A$ and put

$$g(m) = f_P(ma) \qquad (m \in M) \quad .$$

Then $g \in \mathcal{A}(M, \tau_M)$ _and_

$$g_{{}^*P}({}^*m\,{}^*a) = f_{P'}({}^*m.\,{}^*a.a)$$

for ${}^*m \in {}^*M = M'$ _and_ ${}^*a \in {}^*A$.

We write $S^x = xSx^{-1}$ $(x \in G)$ for any subset S of G. If $k \in K$, it is clear that $P^k = M^kA^kN^k$ is the Langlands decomposition of the psgp P^k.

Lemma 7. _Fix_ $f \in \mathcal{A}(G, \tau)$ _and_ $k \in K$. _Then_

$$f_{P^k}(m^k) = \tau(k)f_P(m)\tau(k^{-1})$$

for $m \in MA$.

Let $f \in \mathcal{A}(G, \tau)$. We write $f_P \sim 0$ if

$$\int_M (\phi(m), f_P(ma))_V dm = 0$$

for all $\phi \in {}^\circ\mathcal{C}(M, \tau_M)$ and $a \in A$. Here the scalar product is in V, dm is the Haar measure on M and the integral is always convergent.

Theorem 3. _Suppose_ $f \in \mathcal{A}(G, \tau)$ _and_ $f_P \sim 0$ _for all psgps_ P _of_ G _(including_ $P = G$). _Then_ $f = 0$.

Theorem 4. _Fix_ $f \neq 0$ _in_ $\mathcal{A}(G, \tau)$ _and choose a psgp_ $P = MAN$ _of_ G _such that:_

1) $f_P \not\sim 0$,

2) P _is minimal with respect to condition_ 1).

Then for any $a \in A$, _the function_ $m \mapsto f_P(ma)$ _lies in_ ${}^\circ\mathcal{C}(M, \tau_M)$. _Let_ Q _be a psgp of_ G _such that_ $Q \subset P$ _and_ $Q \neq P$. _Then_ $f_Q = 0$.

§5. The Eisenstein Integral and its constant term

Let $P = MAN$ be a psgp of G. Fix $\psi \in {}^{\circ}\mathcal{C}(M, \tau_M)$ and extend it to a function on $G = KP$ as follows:

$$\psi(kman) = \tau(k)\psi(m) \qquad (k \in K,\ m \in M,\ a \in A,\ n \in N) \ .$$

Put

$$E(P : \psi : \nu : x) = \int_K \psi(xk)\tau(k^{-1})\exp\{((-1)^{1/2}\nu - \rho_P)(H_P(xk))\}dk$$

for $\nu \in \mathfrak{a}_c^*$ and $x \in G$. Then E is an analytic function on $\mathfrak{a}_c^* \times G$ which, for a fixed ν, is $\mathfrak{Z}$-finite and for a fixed x, an entire function of ν.

Lemma 8. Fix $\psi \in {}^{\circ}\mathcal{C}(M, \tau_M)$ and $\nu \in \mathfrak{a}^*$. Then $E(P : \psi : \nu) \in \mathcal{A}(G, \tau)$. Let $P' = M'A'N'$ be another psgp of G. Then

$$E_{P'}(P : \psi : \nu) \sim 0$$

unless P and P' are associated.

Let $\mathcal{P}(A)$ be the set of all psgps P' of G such that A is the split component of P'. Then $\mathcal{P}(A)$ is a finite set and if N' is the radical of P', it is clear that $P' = MAN'$ is the Langlands decomposition of P'. Put

$$\varpi_P = \prod_{\alpha>0} H_\alpha$$

where α runs over all roots of (P, A) and H_α is the element of $\mathfrak{a}$ given by

$$B(H, H_\alpha) = \alpha(H) \qquad (H \in \mathfrak{a}) \ .$$

Then ϖ_P may be regarded as a polynomial function on $\mathfrak{a}_c^*$. Put $\mathfrak{W} = \mathfrak{W}(A)$ and $L = {}^{\circ}\mathcal{C}(M, \tau_M)$. Then by Theorem 1, $\dim L < \infty$.

Theorem 5. We can choose an open connected neighborhood U of zero in $\mathfrak{a}^*$ and an integer $r \geq 0$ with the following properties. Fix $P_1, P_2 \in \mathcal{P}(A)$

and a point $\nu \in \mathfrak{a}^*$ *such that* $\varpi_P(\nu) \neq 0$. *Then there exist unique elements* $c_{P_2|P_1}(s:\nu) \in \text{End } L$ $(s \in \mathfrak{W})$ *such that*

$$E_{P_2}(P_1 : \psi : \nu : ma) = \sum_{s \in \mathfrak{W}} (c_{P_2|P_1}(s:\nu)\psi)(m) e^{(-1)^{1/2} s\nu(\log a)}$$

for $\psi \in L$, $m \in M$ *and* $a \in A$. *Moreover for any* $s \in \mathfrak{W}$, *the function*

$$\nu \longmapsto \varpi_P(\nu)^r c_{P_2|P_1}(s:\nu)$$

extends to a holomorphic function of ν *on* $\mathfrak{a}^* + (-1)^{1/2}U$.

As usual $s\nu(H) = \nu(s^{-1}H)$ for $\nu \in \mathfrak{a}_c^*$ and $H \in \mathfrak{a}_c$.

For $\nu \in \mathfrak{a}_c^*$, define ν_R and ν_I in $\mathfrak{a}^*$ by $\nu = \nu_R + (-1)^{1/2}\nu_I$. Let $\mathfrak{F}_c(P)$ denote the set of all $\nu \in \mathfrak{a}_c^*$ such that $\nu_I(H_\alpha) > 0$ for every root α of (P, A). (The definition of $\mathfrak{F}_c(P)$ given in [4(d), p. 546] is incorrect.) Put $\overline{P} = \theta(P)$, $\overline{N} = \theta(N)$, $\rho = \rho_P$ and $H(x) = H_P(x)$ $(x \in G)$. Any $x \in G$ can be written uniquely in the form $x = kman$ where $k \in K$, $m \in M \cap \exp \mathfrak{p}$, $a \in A$, $n \in N$ (in the notation of Lemma 1). Put $\kappa(x) = k$, $\mu(x) = m$.

Lemma 9. $c_{\overline{P}|P}(1:\nu)$ *and* $c_{P|P}(1:-\nu)$ *extend to holomorphic functions of* ν *on* $\mathfrak{F}_c(P)$ *and they are given there by the following convergent integrals.*

$$(c_{\overline{P}|P}(1:\nu)\psi)(m) = \int_{\overline{N}} \tau(\kappa(\bar{n}))\psi(\mu(\bar{n})m) e^{((-1)^{1/2}\nu - \rho)(H(\bar{n}))} d\bar{n} \quad ,$$

$$(c_{P|P}(1:-\nu)\psi)(m) = \int_{\overline{N}} \psi(m\mu(\bar{n})^{-1})\tau(\kappa(\bar{n}))^{-1} e^{((-1)^{1/2}\nu - \rho)(H(\bar{n}))} d\bar{n} \quad .$$

Here $\psi \in {}^\circ\mathcal{C}(M, \tau_M)$, $\nu \in \mathfrak{F}_c(P)$, $m \in M$ *and the Haar measure* $d\bar{n}$ *on* $\overline{N}$ *is so normalized that*

$$\int_{\overline{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1 \quad .$$

The following consequence of the above integral representation was pointed out to me by R. P. Langlands.

Corollary. det $c_{P|P}(1:-\nu)$ is not identically zero in ν.

§6. The functional equations

Put

$$\|\psi\|_M^2 = \int_M |\psi(m)|^2 dm$$

for $\psi \in L = {}^{\circ}\mathcal{C}(M, \tau_M)$. This defines the structure of a Hilbert space on L.

Theorem 6. Fix $P_1, P_2 \in \mathcal{P}(A)$. Then for any $s \in \mathfrak{W}$, $c_{P_2|P_1}(s:\nu)$ extends to a meromorphic function of ν on $\mathfrak{a}_c^*$. Put

$$c^{\circ}_{P_2|P_1}(s:\nu) = c_{P_2|P_1}(s:\nu)c_{P_1|P_1}(1:\nu)^{-1}$$

and

$$E^{\circ}(P:\psi:\nu) = E(P:c_{P|P}(1:\nu)^{-1}\psi:\nu) \qquad (\psi \in L,\ P \in \mathcal{P}(A)) .$$

Then

1) $c^{\circ}_{P_2|P_1}(s:\nu)$ is holomorphic and unitary on $\mathfrak{a}^*$,

2) $E^{\circ}(P:\psi:\nu)$ is holomorphic for $\nu \in \mathfrak{a}^*$,

3) $c_{P|P}(1:\nu) = (c_{\overline{P}|\overline{P}}(1:\overline{\nu}))^*$, $c_{\overline{P}|P}(1:\nu) = (c_{P|\overline{P}}(1:\overline{\nu}))^*$

where $\overline{\nu} = \nu_R - (-1)^{1/2}\nu_I$ and the star denotes the adjoint of a linear transformation. Moreover we have the following functional equations.

a) $c^{\circ}_{P_3|P_1}(ts:\nu) = c^{\circ}_{P_3|P_2}(t:s\nu)c^{\circ}_{P_2|P_1}(s:\nu)$,

b) $E^{\circ}(P_1:\psi:\nu) = E^{\circ}(P_2:c^{\circ}_{P_2|P_1}(s:\nu)\psi:s\nu)$

for $P_1, P_2, P_3 \in \mathcal{P}(A)$ and $s, t \in \mathfrak{W}$.

Theorem 7. Fix $P_1, P_2 \in \mathcal{P}(A)$ and suppose $P' = M'A'N'$ is a psgp of G such that $P' \supset P_1 \cup P_2$ and prk $P' \geq 1$. Put ${}^*P_i = M' \cap P_i$ (i = 1, 2), ${}^*A = M' \cap A$ and let ${}^*\mathfrak{W}$ denote the subgroup of all $s \in \mathfrak{W}$ which leave A' pointwise fixed. Then $({}^*P_i, {}^*A)$ $(i = 1, 2)$ are parabolic pairs in M' and ${}^*\mathfrak{W}$ may be identified with the Weyl group of ${}^*A$ in M'. For any $\nu \in \mathfrak{a}^*$, let ${}^*\nu$ denote the restriction of ν on ${}^*\mathfrak{a}$. Fix $\psi \in L$ and $\nu \in \mathfrak{a}^*$ and put $f(\nu) = E^o(P_1 : \psi : \nu)$. Then

$$f_{P'}(\nu : m'a') = \sum_{s \in {}^*\mathfrak{W}\backslash\mathfrak{W}} E^o({}^*P_2 : c^o_{P_2|P_1}(s : \nu)\psi : {}^*(s\nu) : m')e^{(-1)^{1/2}s\nu(\log a')}$$

for $m' \in M'$, $a' \in A'$. Moreover

$$c^o_{P_2|P_1}(t : \nu) = c^o_{{}^*P_2|{}^*P_1}(t : {}^*\nu)$$

for $t \in {}^*\mathfrak{W}$.

Here s runs over a complete system of representatives in $\mathfrak{W}$ for ${}^*\mathfrak{W}\backslash\mathfrak{W}$. Theorems 6 and 7 reduce the determination of $c^o_{P_2|P_1}(s : \nu)$ to that of $c^o_{\bar{P}|P}(1 : \nu)$ in case $\operatorname{prk} P = 1$.

§7. The Maass-Selberg relations and their consequences

Let $\mathcal{A}_P(G, \tau)$ denote the space of all $f \in \mathcal{A}(G, \tau)$ with the following property. If Q is a psgp of G, then $f_Q \sim 0$ unless Q is associated to P. The following theorem plays a decisive role in the proof of Theorems 6 and 7. The case when $\operatorname{prk} P = 1$ is especially important.

Theorem 8. Fix $\nu \in \mathfrak{a}^*$ such that $\varpi_P(\nu) \neq 0$ and suppose $f \in \mathcal{A}_P(G, \tau)$ and $\phi_{Q,s} \in L$ $(Q \in \mathcal{P}(A),\ s \in \mathfrak{W})$ are given functions satisfying the relation

$$f_Q(ma) = \sum_{s \in \mathfrak{W}} \phi_{Q,s}(m)e^{(-1)^{1/2}s\nu(\log a)} \quad (m \in M,\ a \in A)$$

for every $Q \in \mathcal{P}(A)$. Then

$$\|\phi_{P_1, s_1}\|_M = \|\phi_{P_2, s_2}\|_M$$

for $P_1, P_2 \in \mathcal{P}(A)$ *and* $s_1, s_2 \in \mathfrak{W}$.

Corollary. *Suppose* $\phi_{Q,s} = 0$ *for some pair* (Q, s). *Then* $f = 0$.

This theorem is a consequence of, what I call, the Maass-Selberg relations when prk $P = 1$. (These relations are similar to those discussed in [4(a), Chap. IV, §2].) The rest follows by induction on prk P. In fact Theorems 6, 7 and 8 are proved together in this induction (cf. [4(a), Chap. V]).

§8. The evaluation of $(\phi_\alpha)_\nu^{(P)}$

Put $\mathfrak{F} = \mathfrak{a}^*$ and let $\mathfrak{F}'$ be the set of all $\nu \in \mathfrak{F}$ such that $\varpi_P(\nu) \neq 0$. Let $\mathcal{C}(\mathfrak{F})$ denote the Schwartz space on the finite-dimensional vector space $\mathfrak{F}$.

Lemma 10. *For any* $\alpha \in \mathcal{C}(\mathfrak{F}) \otimes L$, *put*

$$\phi_\alpha(x) = \int E(P : \alpha(\nu) : \nu : x)d\nu \qquad (x \in G) ,$$

where $d\nu$ *denotes the Euclidean measure on* $\mathfrak{F}$. *Suppose* α *satisfies the following condition. For any* $s \in \mathfrak{W}$ *and* $P_1, P_2 \in \mathcal{P}(A)$, *the function*

$$\nu \longmapsto \| c_{P_2|P_1}(s : \nu)\alpha(\nu) \|_M \qquad (\nu \in \mathfrak{F}')$$

remains locally bounded on $\mathfrak{F}$. *Then* $\phi_\alpha \in \mathcal{C}(G, \tau)$.

In particular the condition of the lemma is fulfilled if $\alpha \in C_c^\infty(\mathfrak{F}') \otimes L$. For any $f \in \mathcal{C}(G, \tau)$, define a function $f^{(P)}$ on MA by

$$f^{(P)}(m) = \delta_P(m)^{1/2} \int_N f(mn)dn \qquad (m \in MA) .$$

Then $f^{(P)} \in \mathcal{C}(MA, \tau_M)$ and $f \longmapsto f^{(P)}$ is a continuous mapping of $\mathcal{C}(G, \tau)$ into $\mathcal{C}(MA, \tau_M)$.

Theorem 9. *Fix* $P_1, P_2 \in \mathcal{P}(A)$ *and suppose* $\alpha \in \mathcal{C}(\mathfrak{F}) \otimes L$ *satisfies the condition of Lemma* 10. *Put*

$$\phi_\alpha(x) = \int_{\mathfrak{F}} E(P_1 : \alpha(\nu) : \nu : x)d\nu \qquad (x \in G)$$

and

$$\phi_{P_2,\alpha}(m) = \int_{\mathfrak{F}} E_{P_2}(P_1 : \alpha(\nu) : \nu : m)d\nu \qquad (m \in MA) \enspace ,$$

Extend $\phi_{P_2,\alpha}$ *to a function on* G *by the rule*

$$\phi_{P_2,\alpha}(kmn) = \tau(k)\phi_{P_2,\alpha}(m) \qquad (k \in K,\ m \in MA,\ n \in N_2) \enspace .$$

Then

$$\phi_\alpha^{(\overline{P}_2)}(m) = \int_{\overline{N}_2} e^{-\rho_2(H_2(\overline{n}))} \phi_{P_2,\alpha}(\overline{n}m)d\overline{n} \qquad (m \in MA) \enspace .$$

Here $P_i = MAN_i$ $(i = 1, 2)$, $\rho_2 = \rho_{P_2}$, $H_2(x) = H_{P_2}(x)$ $(x \in G)$, and all the integrals are convergent.

Put

$$f_\nu^{(P)}(m) = \int_A f^{(P)}(ma)e^{-(-1)^{1/2}\nu(\log a)}da \qquad (m \in M)$$

for $f \in \mathscr{C}(G, \tau)$ and $\nu \in \mathfrak{F}$.

Theorem 10. *Fix* $\alpha \in C_c^\infty(\mathfrak{F}')$, $\psi \in L$ *and put*

$$\phi_\alpha(x) = \int \alpha(\nu)E(P : \psi : \nu : x)d\nu \qquad (x \in G) \enspace .$$

Then $\phi_\alpha \in \mathscr{C}(G, \tau)$ *and*

$$(\phi_\alpha)_\nu^{(P)} = \gamma(P) \sum_{s \in \mathfrak{W}} \alpha(s^{-1}\nu)(c_{P|\overline{P}}(1 : \nu)c_{\overline{P}|P}(s : s^{-1}\nu)\psi)$$

for $\nu \in \mathfrak{F}$. *Here*

$$\gamma(P) = \int_N e^{-2\rho(H(\theta(n)))}dn$$

and the measures da *and* $d\nu$ *are assumed to be dual to each other.*

§9. Normalization of the Haar measures

$\mathfrak{q}$ being any linear subspace of $\mathfrak{g}$, we denote by $d\mathfrak{q}$ the Euclidean measure on $\mathfrak{q}$ corresponding to the Euclidean norm on $\mathfrak{g}$ defined in Lemma 2. Let $P_o = M_oA_oN_o$ be a minimal psgp of G. Then $G = KA_oN_o$. Put $\rho_o = \rho_{P_o}$ and normalize the Haar measure dx on G in such a way that

$$dx = e^{2\rho_o(\log a_o)} dk\, da_o\, dn_o \ .$$

Here $x = ka_on_o$ ($k \in K$, $a_o \in A_o$, $n_o \in N_o$) and da_o and dn_o are the Haar measures on A_o and N_o respectively which correspond to the Euclidean measures on their Lie algebras under the exponential mapping. dk is the normalized Haar measure on K (so that the total measure of K is 1). This normalization of dx is independent of the choice of P_o. We call dx the standard Haar measure on G.

Now let $P = MAN$ as in §8. We can apply the above procedure to M instead of G and thus obtain the standard Haar measure dm on M.

Lemma 11. Let $P = MAN$ be any psgp of G and dx, dm the standard Haar measures on G and M respectively. Let da and dn denote the Haar measures on A and N which correspond to $d\mathfrak{a}$ and $d\mathfrak{n}$ respectively under the exponential mapping. Then

$$\int_G f(x)dx = \int_{K\times M\times A\times N} f(kman)e^{2\rho(\log a)} dk\, dm\, da\, dn$$

for $f \in C_c(G)$. Here dk is the normalized Haar measure on K and $\rho = \rho_P$.

From now on we shall always assume that the various Haar measures are normalized as in the above lemma.

§10. The space $\mathscr{C}_\omega(G, \tau)$

Let $\mathcal{E}(G)$ be the set of all equivalence classes of irreducible unitary representations of G and $\mathcal{E}_2(G)$ the subset of those classes ω which are square-integrable. For any $\omega \in \mathcal{E}_2(G)$, let $d(\omega)$ denote the formal degree of ω.

Let $G_1 = ZG^o$ where Z is the centralizer of G^o in G. Then G/G_1 is a finite group. Fix $\omega_1 \in \mathcal{E}_2(G_1)$ and let $\mathrm{Ind}\,\omega_1$ denote the class of the representation of G obtained from ω_1 by inducing from G_1 to G. Then $\mathrm{Ind}\,\omega_1$ is irreducible and $\omega_1 \longrightarrow \mathrm{Ind}\,\omega_1$ is a surjective mapping of $\mathcal{E}_2(G_1)$ on $\mathcal{E}_2(G)$. Taking into account the results of [4(f), §41], one can give an explicit formula for $d(\omega)$ $(\omega \in \mathcal{E}_2(G))$.

Now fix $\omega \in \mathcal{E}_2(G)$ and let $\mathcal{H}_\omega$ denote the smallest closed subspace of $L_2(G)$ containing all K-finite matrix coefficients of the class ω. Put $\mathcal{C}_\omega(G) = \mathcal{C}(G) \cap \mathcal{H}_\omega$ and

$$\mathcal{C}_\omega(G, \tau) = \mathcal{C}(G, \tau) \cap (\mathcal{C}_\omega(G) \otimes V) .$$

Then $\mathcal{C}_\omega(G, \tau) \subset {}^\circ\mathcal{C}(G, \tau)$.

§11. The characters $\Theta_{\omega,\nu}$

Let $P = MAN$ be a psgp of G. Fix $\omega \in \mathcal{E}_2(M)$, $\nu \in \mathfrak{a}^*$ and define a tempered distribution $\Theta_{\omega,\nu}$ on G as follows:

$$\Theta_{\omega,\nu}(f) = \theta_\omega(g_{f,\nu}) \qquad (f \in \mathcal{C}(G)) .$$

Here θ_ω is the character of ω,

$$g_{f,\nu}(m) = \int_{A \times N} \bar{f}(man)\exp\{((-1)^{1/2}\nu + \rho)(\log a)\}da\,dn \qquad (m \in M)$$

and

$$\bar{f}(x) = \int_K f(kxk^{-1})dk .$$

Then $\Theta_{\omega,\nu}$ is the character of a unitary representation of G. Let $\Omega(P, \nu)$ denote the class of this representation. Then

1) $\Omega(P, \nu)$ is irreducible if $\varpi_P(\nu) \neq 0$,

2) $\Omega(P_1, \nu) = \Omega(P_2, \nu)$ for $P_1, P_2 \in \mathcal{P}(A)$.

The group $\mathfrak{W} = \mathfrak{W}(A)$ operates on $\mathcal{E}_2(M)$ as follows. Fix $s \in \mathfrak{W}$ and $\omega \in \mathcal{E}_2(M)$. Choose a representative y in G for s and a representation β of M in the class ω. Put

$$\beta^y(m) = \beta(y^{-1}my) \qquad (m \in M) \ .$$

Since y normalizes M, β^y is a representation of M. We define ω^s to be the class of β^y. It is easy to see that $\omega^s \in \mathcal{E}_2(M)$ and it is independent of the choice of y and β.

Lemma 12. <u>Fix</u> $\omega_1, \omega_2 \in \mathcal{E}_2(M)$ <u>and</u> $\nu_1, \nu_2 \in \mathfrak{a}^*$. <u>Then</u>

$$\Theta_{\omega_1, \nu_1} = \Theta_{\omega_2, \nu_2}$$

<u>if and only if there exists an element</u> $s \in \mathfrak{W}$ <u>such that</u>

$$\omega_2 = \omega_1^s, \ \nu_2 = s\nu_1 \ .$$

The distributions θ_ω and $\Theta_{\omega, \nu}$ are actually functions. We write[3)]

$$(\Theta_{\omega, \nu}, f) = \int_G \operatorname{conj} \Theta_{\omega, \nu} . f dx \qquad (f \in \mathcal{C}(G)) \ ,$$

the integral being convergent. A similar notation is used also for $f \in \mathcal{C}(G, \tau)$.

Lemma 13. <u>Let</u> F <u>denote the projection in</u> V <u>given by</u>

$$Fv = \int_K \tau(k)v\tau(k^{-1})dk \qquad (v \in V) \ .$$

<u>Then</u>

$$(\Theta_{\omega, \nu}, f) = F(\theta_\omega, f_\nu^{(P)}) \qquad (f \in \mathcal{C}(G, \tau))$$

<u>for</u> $\omega \in \mathcal{E}_2(M)$ <u>and</u> $\nu \in \mathfrak{a}^*$.

Here

$$(\theta_\omega, g) = \int_M \operatorname{conj} \theta_\omega . g dm$$

for g in $\mathcal{C}(M)$ or $\mathcal{C}(M, \tau_M)$.

Fix $\nu_o \in \mathfrak{F}'$. Then $s\nu_o \neq \nu_o$ for $s \neq 1$ in W. Hence we can choose an open neighborhood U of ν_o in $\mathfrak{F}'$ such that $U \cap sU = \emptyset$ for $s \neq 1$ in W.

Lemma 14. *Put* $L_\omega = \mathcal{C}_\omega(M, \tau_M)$ *and suppose* $\alpha \in C_c^\infty(U)$ *and* $\psi \in L_\omega$. *Fix* $\omega' \in \mathcal{E}_2(M)$ *and* $\nu \in \mathfrak{F}$. *Then*

$$(\Theta_{\omega', \nu}, \phi_\alpha) = 0$$

unless $\nu \in \bigcup_{s \in W} sU$. *Now suppose* $\nu \in U$. *Then*

$$(\Theta_{\omega', \nu}, \phi_\alpha) = \begin{cases} \gamma(P)\alpha(\nu)F(\theta_\omega, c_{P|\bar{P}}(1:\nu)c_{\bar{P}|P}(1:\nu)\psi) & \text{if } \omega' = \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Here ϕ_α and $\gamma(P)$ have the same meaning as in Theorem 10.

§12. The Plancherel measure μ_ω

Let $\mathcal{C}_A(G)$ denote the set of all $f \in \mathcal{C}(G)$ with the following property. If $P' = M'A'N'$ is any psgp of G, then $f^{P'} \sim 0$ (see [4(d), p. 538]) unless P' is associated to P. For $f \in \mathcal{C}(G)$, put

$$\hat{f}(\omega : \nu) = (\Theta_{\omega, \nu}, f) \qquad (\omega \in \mathcal{E}_2(M), \nu \in \mathfrak{a}^*) .$$

We give $\mathcal{E}_2(M)$ the discrete topology.

Theorem 11.[4] *There exists a unique continuous function* μ *on* $\mathcal{E}_2(M) \times \mathfrak{a}^*$ *with the following properties:*

1) $\mu(\omega^s : s\nu) = \mu(\omega : \nu)$ *for* $s \in W$.

2) *For any* $f \in \mathcal{C}(G)$, *the series*

$$\sum_{\omega \in \mathcal{E}_2(M)} d(\omega) \int_{\mathfrak{a}^*} |\hat{f}(\omega : \nu)\mu(\omega : \nu)| d\nu$$

is convergent.

3) $f(1) = \sum_{\omega \in \mathcal{E}_2(M)} d(\omega) \int_{\mathfrak{a}^*} \hat{f}(\omega : \nu)\mu(\omega : \nu) d\nu$

for all $f \in \mathcal{C}_A(G)$.

Fix $\omega \in \mathcal{E}_2(M)$ and put $\mu_\omega(\nu) = \mu(\omega : \nu)$.

Lemma 15. μ_ω extends to a meromorphic function on $\mathfrak{a}_c^*$ which is holomorphic on $\mathfrak{a}^*$. Moreover $\mu_\omega(\nu) > 0$ on $\mathfrak{F}'$ and[5)]

$$[\mathfrak{W}]\mu_\omega(\nu).\gamma(P)c_{P|\bar{P}}(1:\nu)c_{\bar{P}|P}(1:\nu)\psi = \psi$$

for $\nu \in \mathfrak{F}'$ and $\psi \in L_\omega$.

§13. Explicit determination of μ_ω

We now make $\mathfrak{W}$ operate on L as follows. Fix $s \in \mathfrak{W}$ and $\psi \in L$ and let y be a representative of s in K. Then $s\psi$ is the function $m \longmapsto \tau(y)\psi(y^{-1}my)\tau(y^{-1})$ on M. We also put $P^s = P^y$.

Lemma 16. Let $P_1, P_2 \in \mathcal{P}(A)$ and $s, t \in \mathfrak{W}$. Then

$$sc_{P_2|P_1}(t : \nu) = c_{P_2{}^s|P_1}(st : \nu), \quad c_{P_2|P_1}(t : \nu)s^{-1} = c_{P_2|P_1{}^s}(ts^{-1} : s\nu)$$

and

$$sc^o_{P_2|P_1}(t : \nu) = c^o_{P_2{}^s|P_1}(st : \nu), \quad c^o_{P_2|P_1}(t : \nu)s^{-1} = c^o_{P_2|P_1{}^s}(ts^{-1} : s\nu)$$

for $\nu \in \mathfrak{a}_c^*$.

Fix $P = MAN$ in $\mathcal{P}(A)$ and for any $P_1 \in \mathcal{P}(A)$ $(P_1 = MAN_1)$ and $\nu \in \mathfrak{F}_c(P)$, define a linear transformation $J_{P_1|P}(\nu)$ on L as follows.

$$(J_{P_1|P}(\nu)\psi)(m) = \int_{\overline{N}\cap N_1} \psi(\overline{n}m)e^{((-1)^{1/2}\nu-\rho)(H(\overline{n}))}d\overline{n} \qquad (\psi \in L,\ m \in M) .$$

Here $d\overline{n}$ is the Haar measure on $\overline{N} \cap N_1$ which corresponds, under the exponential mapping, to the Euclidean measure on its Lie algebra. The function ψ is extended on $G = KP$ as before by the rule

$$\psi(kman) = \tau(k)\psi(m) \qquad (k \in K,\ m \in M,\ a \in A,\ n \in N) .$$

The above integral is convergent when $\nu \in \mathfrak{F}_c(P)$. We note that from Lemma 9,

$$J_{P|P}(\nu) = 1 , \quad J_{\overline{P}|P}(\nu) = \gamma(P)c_{\overline{P}|P}(1 : \nu) ,$$

where $\gamma(P)$ has the same meaning as in Theorem 10.

Let Σ be the set of all roots of (P, A). A root $\alpha \in \Sigma$ is called reduced if $t\alpha \notin \Sigma$ for $0 < t < 1$ $(t \in \underset{\sim}{R})$. Let Φ be the set of all reduced roots. For any $\alpha \in \Phi$, let $\Sigma(\alpha)$ denote the set of all roots in Σ of the form $t\alpha$ $(t \geq 1)$. Put

$$\mathfrak{n}_\alpha = \sum_{\beta\in\Sigma(\alpha)} \mathfrak{n}(\beta) ,$$

where $\mathfrak{n}(\beta)$ is the set of all $X \in \mathfrak{n}$ such that $[H, X] = \beta(H)X$ for all $H \in \mathfrak{a}$. Put $N_\alpha = \exp \mathfrak{n}_\alpha$.

Let σ_α denote the hyperplane $\alpha = 0$ on $\mathfrak{a}$ and Z_α the centralizer of σ_α in G. Put $M_\alpha = {}^o(Z_\alpha)$, $A_\alpha = M_\alpha \cap A$ and $\overline{N}_\alpha = \theta(N_\alpha)$. Then

$${}^*P_\alpha = MA_\alpha N_\alpha , \quad {}^*\overline{P}_\alpha = MA_\alpha \overline{N}_\alpha$$

are maximal psgps of M_α. Put $\rho_\alpha = \rho_{*P_\alpha}$, $H_\alpha(y) = H_{*P_\alpha}(y)$ $(y \in M_\alpha)$ and define

$$\gamma({}^*P_\alpha) = \int_{\overline{N}_\alpha} e^{-2\rho_\alpha(H_\alpha(\overline{n}))} d_\alpha\overline{n} ,$$

where $d_\alpha\overline{n}$ is the Haar measure on $\overline{N}_\alpha$ which corresponds to the Euclidean measure on its Lie algebra.

A point $H \in \mathfrak{a}$ is called regular if $\alpha(H) \neq 0$ for every $\alpha \in \Sigma$ and it is called semiregular if there is exactly one root $\alpha \in \Phi$ such that $\alpha(H) = 0$. Let C be the set of all points $H \in \mathfrak{a}$ where $\alpha(H) > 0$ for all $\alpha \in \Sigma$. Then we can choose two points H_o, H_1 in $\mathfrak{a}$ such that the following conditions hold.

1) $H_o \in C$ and $-H_1 \in C$.

2) Put $H(t) = (1-t)H_o + tH_1$ $(0 \leq t \leq 1)$. Then $H(t)$ is either regular or semiregular.

Let $0 < t_1 < t_2 < \dots < t_r < 1$ be all the values of t such that $H(t)$ is semiregular. Let α_i be the root in Φ which vanishes at $H(t_i)$ $(1 \leq i \leq r)$. It is clear that $r = [\Phi]$. Put

$$c_i(\nu) = \gamma({}^*P_{\alpha_i})c_{{}^*\overline{P}_{\alpha_i}|{}^*P_{\alpha_i}}(1 : \nu_{\alpha_i}) \qquad (\nu \in \mathfrak{F}_c(P))$$

where ν_α is the restriction of ν on $\mathfrak{a}_\alpha = \mathbb{R}H_\alpha$ $(\alpha \in \Phi)$.

Lemma 17. $\gamma(P)c_{\overline{P}|P}(1 : \nu) = c_r(\nu)c_{r-1}(\nu) \dots c_1(\nu)$ $(\nu \in \mathfrak{F}_c(P))$.

Since both sides are meromorphic on $\mathfrak{F}_c$, this relation must hold for all ν. Lemma 17 may be regarded as a generalization of a result of Gindikin and Karpelevič [3].

Let $\mathfrak{W}_\alpha$ denote the Weyl group of A_α in M_α and $\mu_{\omega,\alpha}$ the function on $\mathfrak{a}_\alpha^*$ which corresponds to μ_ω when we replace (G, P) by $(M_\alpha, {}^*P_\alpha)$. The following theorem is an immediate consequence of Lemmas 15 and 17.

Theorem 12. $\mu_\omega(\nu) = c \prod_{\alpha\in\Phi} \mu_{\omega,\alpha}(\nu_\alpha)$ $\qquad (\omega \in \mathcal{E}_2(M),\ \nu \in \mathfrak{a}^*)$

<u>where</u>

$$c = \gamma(P)[\mathfrak{W}]^{-1} \prod_{\alpha\in\Phi} [\mathfrak{W}_\alpha]\gamma({}^*P_\alpha)^{-1} .$$

Observe that $\operatorname{prk} M_\alpha = 0$ and $\operatorname{prk} {}^*P_\alpha = 1$. Hence, in order to compute μ_ω, it is enough to consider the case when $\operatorname{prk} G = 0$ and $\operatorname{prk} P = 1$.

Then there are two possibilities.

1) $\mathcal{E}_2(G) = \emptyset$.

2) $\mathcal{E}_2(G) \neq \emptyset$.

The first case is easier and there one can show that μ_ω is a polynomial function on $\mathfrak{a}^*$. On the other hand the second case can be dealt with by the method of [4(g), §24]. In this way one obtains an explicit formula for μ_ω.

§14. Relation with the exceptional series

Now we assume that prk $G = 0$ and prk $P = 1$. Fix $\omega \in \mathcal{E}_2(M)$ and put $\mathfrak{W} = \mathfrak{W}(A)$.

Lemma 18.[6] $\Theta_{\omega, 0}$ *is an irreducible character unless the following two conditions hold.*

1) $\mu_\omega(0) > 0$, $[\mathfrak{W}] = 2$ *and* $\omega^s = \omega$ $(s \in \mathfrak{W})$.

2) $\mathcal{E}_2(G) \neq \emptyset$.

Actually 2) is a consequence of 1). Moreover it seems likely that $\Theta_{\omega, 0}$ is the sum of two distinct irreducible characters when these conditions are fulfilled (cf. [4(b), Theorem 1] and [6(b)]).

Lemma 19. *Fix* $\omega \in \mathcal{E}_2(M)$ *such that* $\mu_\omega(0) = 0$. *Then* $[\mathfrak{W}] = 2$ *and* $\omega^s = \omega$ *where* s *is the element of* $\mathfrak{W}$ *other than* 1. *Put*

$$C(\nu)\psi = -c^0_{P|P}(s : (-1)^{1/2}\nu)\psi \qquad (\nu \in \mathfrak{a}_c^*, \ \psi \in L_\omega) \ .$$

Then $C(\nu)$ *is self-adjoint for* $\nu \in \mathfrak{a}^*$ *and*

$$C(\nu)C(-\nu) = 1 \ .$$

Finally $C(0) = 1$.

Let α denote the unique simple root of (P, A). We identify $\mathfrak{a}_c^*$ with $\mathbb{C}$ by means of the mapping $\nu \mapsto \langle \nu, \alpha \rangle$. Let $\delta_0 > 0$ be the distance, from the origin, of the nearest pole of C on the real axis. Then $C(\nu)$ is a positive-definite operator for $|\nu| < \delta_o$ $(\nu \in \mathbb{R})$. This enables us to prove the following theorem (cf. [6(a), Theorem 3.3]).

Theorem 13. *Fix* $\omega \in \mathcal{E}_2(M)$ *such that* $\mu_\omega(0) = 0$. *Then we can choose* $\delta > 0$ *with the following property. Suppose* $\nu \in (-1)^{1/2}\mathfrak{a}^*$, $|\langle \nu, \alpha \rangle| < \delta$ *and* $\nu \neq 0$. *Then* $\Theta_{\omega,\nu}$ *is the character of an irreducible unitary representation of* G *belonging to the exceptional series.*

It would be interesting to extend the above results to the case $\operatorname{prk} P > 1$.

References

1. A. Borel and J. Tits, Groupes rèductifs, Inst. Hautes Études Sci. Publ. Math. No. 27 (1965), pp. 55-150.

2. I. M. Gelfand, Automorphic functions and the theory of representations, Proc. Internat. Congress Math. (1962), pp. 74-85.

3. S. G. Gindikin and F. I. Karpelevič, Plancherel measure of Riemannian symmetric spaces of nonpositive curvature, Sov. Math. vol. 3 (1962), pp. 962-965.

4. Harish-Chandra, (a) Automorphic forms on semisimple Lie groups, Lecture notes in Math., no. 62, Springer-Verlag (1968).
   (b) Eisenstein Series over finite fields, Functional Analysis and Related Fields, pp. 76-88, Springer-Verlag (1970).
   (c) Some applications of the Schwartz space of a semisimple Lie group, Lecture notes in Math., no. 140 (1970), pp. 1-7, Springer-Verlag.
   (d) Harmonic Analysis on semisimple Lie groups, Bull. Amer. Math. Soc. vol. 78 (1970), pp. 529-551.
   (e) Harmonic Analysis on reductive p-adic groups, Lecture notes in Math., no. 162 (1970), Springer-Verlag.
   (f) Discrete series for semisimple Lie groups II, Acta Math. vol. 116 (1966), pp. 1-111.
   (g) Two theorems on semisimple Lie groups, Ann. of Math. vol. 83 (1966), pp. 74-128.

5. S. Helgason, A duality theory for symmetric spaces with applications to group representations, Advances in Math. vol. 5 (1970), pp. 1-154.

6. A. W. Knapp and E. M. Stein, (a) Existence of complementary series, Problems in Analysis, pp. 249-259, Princeton University Press, 1970.
   (b) Singular Integrals and the Principal Series II, Proc. Nat. Acad. Sci. U.S.A. vol. 66 (1970), pp. 13-17.

7. B. Kostant, On the existence and irreducibility of certain series of representations, Bull. Amer. Math. Soc. vol 75 (1969), pp. 627-642.

Footnotes

[1]For any finite-dimensional vector space V over $\mathbb{R}$, we denote by V^* its dual and by V_c its complexification. Moreover $V_c^* = (V^*)_c$.

[2]See Borel and Tits [1].

[3]conjc stands for the complex conjugate of $c \in \mathbb{C}$.

[4]Cf. [4(d), §12].

[5][F] denotes the number of elements in a finite set F.

[6]The fact that there is a rather direct connection between questions of reducibility and the theory of the Eisenstein Integral was first pointed out to me by J. G. Arthur (cf. his thesis "Harmonic analysis of tempered distributions on semisimple Lie groups of real rank one," Yale, 1970). See also [5].