DISCRETE SERIES

4(a) Discrete series

We again assume (G,X) is a Shimura datum, with G^{der} anisotropic, so the associated Shimura varieties are compact. Thus all the automorphic representations π of G are unitary, modulo the action of the center (which I will attempt to ignore). Recall the isomorphisms

$$H^\bullet(S(G),\tilde{V}) \xrightarrow{\sim} \bigoplus_\pi H^\bullet(\mathfrak{g},K;\pi_\infty\otimes V)\otimes\pi_f,$$

$$H^q(S(G,X),[W]) \xrightarrow{\sim} \bigoplus_\pi m(\pi)H^q(Lie(P_h),K_h;\pi_\infty\otimes W)\otimes\pi_f,$$

and the isomorphism from Hodge theory

$$H^\bullet(\mathfrak{g},K;\pi_\infty\otimes V) \xrightarrow{\sim} \mathcal{H}^\bullet(\pi_\infty\otimes V).$$

The next step is to explain how harmonic forms are computed in the most interesting cases, but first I add a fourth isomorphism expressing Hodge theory locally for coherent cohomology:

$$H^\bullet(Lie(P_h),K_h;\pi_\infty\otimes W) \xrightarrow{\sim} \mathcal{H}^\bullet(\pi_\infty\otimes W).$$

This is if anything even easier than the previous case, because every representation of K_h is a representation of P_h and carries a P_h-invariant hermitian metric. The Laplacian for $\overline{\partial}$ is denoted $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ rather than Δ but it also satisfies

$$(\Box\eta,\eta) = (\overline{\partial}\eta,\overline{\partial}\eta) + (\overline{\partial}^*\eta,\overline{\partial}^*\eta).$$

There is also a relation between $H^\bullet(\mathfrak{g},K_h;*)$ and $H^\bullet(Lie(P_h),K_h;*)$. Indeed, we have

$$\begin{aligned} C^i(\mathfrak{g},K_h;\pi_\infty\otimes V) &= Hom_K(\bigwedge^i\mathfrak{p},\pi_\infty\otimes V) = Hom_K(\bigwedge^i(\mathfrak{p}^+\oplus\mathfrak{p}^-),\pi_\infty\otimes V) \\ &= \oplus_{p+q=i} Hom_K(\bigwedge^p\mathfrak{p}^+\otimes\bigwedge^q\mathfrak{p}^-,\pi_\infty\otimes V) = \oplus_{p+q=i} C^{p,q}(\mathfrak{g},K_h;\pi_\infty\otimes V) \end{aligned}$$

where

$$C^{p,q}(\mathfrak{g},K_h;\pi_\infty\otimes V) = Hom_K(\bigwedge^q\mathfrak{p}^-, Hom(\bigwedge^p\mathfrak{p}^+,\pi_\infty\otimes V)).$$

We write $d = \partial + \overline{\partial}$ as in Kähler geometry, with ∂ of degree $(1,0)$ and $\overline{\partial}$ of degree $(0,1)$. Each of them has an adjoint with respect to the hermitian form, and we construct

$$\Box^+ = \partial\partial^* + \partial^*\partial; \Box^- = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$$

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Proposition. $\Delta = \Box^+ + \Box^-$.

This is equivalent to the equations

$$\partial\overline{\partial}^* + \overline{\partial}\partial^* = \overline{\partial}^*\partial + \partial^*\overline{\partial} = 0$$

which is a computation we omit. (This is in the 1963 Annals paper of Matsushima-Murakami. The point is that ∂ is given by derivatives with respect to $\mathfrak{p}^+$ whereas $\overline{\partial}^*$ is given by the adjoints of derivatives with respect to $\mathfrak{p}^-$, but these can be expressed as derivatives with respect to $\mathfrak{p}^+$ again, because of the nature of the hermitian form, so the derivatives in ∂ and $\overline{\partial}^*$ commute, as do those of $\overline{\partial}$ and ∂^*, and these give the relation when the signs are taken into account.

Corollary. *There is an isomorphism* $H^i(\mathfrak{g}, K_h; \pi_\infty \otimes V) \xrightarrow{\sim} \oplus_{p+q=i}\mathcal{H}^{p,q}(\pi_\infty \otimes V)$ *where* $\mathcal{H}^{p,q}$ *is given by the forms harmonic with respect to both* $\Box^+$ *and* $\Box^-$.

This is a formal consequence, as in Kähler geometry. The next statement is less formal.

Theorem. *There are canonical isomorphisms*

$$\mathcal{H}^{p,q}(\pi_\infty \otimes V) \xrightarrow{\sim} \mathcal{H}^q(\pi_\infty \otimes H^p(\mathfrak{p}^+, V)) = H^q(Lie(P_h), K_h; \pi_\infty \otimes H^p(\mathfrak{p}^+, V)).$$

The proof is omitted because it is long and here is only intended to motivate the study of coherent cohomology as a way to understand the Hodge theory of cohomology of local systems.

Kuga's formula.

We continue to assume π unitary. Kuga's formula calculates the action of Δ in terms of a specific element, the Casimir element $C \in Z(\mathfrak{g})$. The existence of C in the image of $\mathfrak{g} \otimes \mathfrak{g}$ in $U(\mathfrak{g})$ is equivalent to the existence of the Killing form. Via B we can identify $\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}$ as $Ad(\mathfrak{g})$-modules, and thus we have an isomorphism

$$B : \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}.$$

Let $X_1, \ldots, X_m$ be a basis for $\mathfrak{g}$, Y_i the dual basis of $\mathfrak{g}^*$. Then the element $Tr = \sum_i X_i \otimes Y_i \in \mathfrak{g} \otimes \mathfrak{g}^*$ is invariant under the representation of $\mathfrak{g}$ (or G) and corresponds to the trace in the given basis; it does not depend on the choice of basis. Then

$$B(Tr) = \sum X_i \otimes X_i^*$$

where $B(X_i, X_j^*) = \delta_{ij}$. Let C denote the image of $B(Tr)$ under the natural map $\mathfrak{g} \otimes \mathfrak{g} \to U(\mathfrak{g})$. Then C is invariant under the adjoint action of $\mathfrak{g}$, which means that $C \in Z(\mathfrak{g})$. This can also be checked by hand using the definition of B.

Now π and ρ have been assumed irreducible; thus by Schur's lemma they have infinitesimal characters (this needs proof).

Kuga's Formula. *Assume* $\pi(C) = s$ *and* $\rho(C) = r$. *Then* Δ *acts as* $r - s$ *on* $C^q(\pi \otimes V)$ *for any* q.

The proof is a long computation (see pp. 49-51) based on the Cartan decomposition; one writes C in terms of a basis of $\mathfrak{k}$ plus a basis of $\mathfrak{p}$.

Corollary. *If $r \neq s$ then $\mathcal{H}^q(\pi \otimes V) = 0$. If $r = s$ then every element of $C^q(\pi \otimes V) = Hom_K(\bigwedge^q \mathfrak{p}, \pi \otimes V)$ is harmonic, so that*

$$H^q(\mathfrak{g}, K; \pi \otimes V) = Hom_K(\bigwedge^q \mathfrak{p}, \pi \otimes V).$$

The first statement follows because $C \in Z(\mathfrak{g})$ and we have already seen that if the infinitesimal characters of π and V^* do not coincide then there is no cohomology, hence no harmonic forms. But the Laplacian is self-adjoint, so the eigenvalues on V and V^* coincide. (Alternatively: if $r \neq s$ then C just multiplies by $r - s \neq 0$, so there are no harmonic forms.) The second statement is then obvious.

A similar calculation shows that, if π is unitary, then $H^q(Lie(P_h), K_h; \pi \otimes W) = C^q(Lie(P_h), K_h; \pi \otimes W) = Hom_K(\bigwedge^q \mathfrak{p}^-, \pi \otimes W)$ (all cochains are harmonic). Using the bigrading, the previous corollary already gives

$$H^{p,q}(\mathfrak{g}, K; \pi \otimes V) = Hom_K(\bigwedge^q \mathfrak{p}^-, Hom(\bigwedge^q \mathfrak{p}^+, \pi \otimes V)) = Hom_K(\bigwedge^q \mathfrak{p}^-, \pi \otimes Hom(\bigwedge^p \mathfrak{p}^+, V))$$

and the latter can be identified by a similar calculation with

$$Hom_K(\bigwedge^q \mathfrak{p}^-, \pi \otimes H^p(\mathfrak{p}^+, V))) = C^q(Lie(P_h), K_h; \pi \otimes H^p(\mathfrak{p}^+, V)$$

as promised above.

To motivate the next section, I state Kostant's theorem.

Kostant's theorem. *Let G be any reductive Lie group, V an irreducible finite-dimensional representation of G, $P = L \cdot U$ a parabolic subgroup. Let W and W_L be the Weyl groups of G and L, respectively, relative to a maximal torus contained in L. Choose a positive root system Δ^+ so that μ is the highest weight of V and let ρ^+ be the half sum of positive roots. There is a unique set of (shortest) coset representatives W^P for W/W_L so that $w \star \mu := w(\mu + \rho^+) - \rho^+$ is a highest weight of a representation $V_{w \star \mu}$ for L relative to this positive root system. Then for any p,*

$$H^p(Lie(U), V) \xrightarrow{\sim} \oplus_{\ell(w)=p, w \in W^P} V_{w \star \mu}$$

as representation of L.

Kostant's original proof was by Hodge theory for the finite-dimensional $Lie(U)$-cohomology complex, and a calculation. Apply this to G as above and $P = P_h^+$ with Lie algebra $\mathfrak{k}_h \oplus \mathfrak{p}_h^+$. It follows that when π is unitary and V has highest weight μ,

$$H^i(\mathfrak{g}, K_h; \pi \otimes V) = \oplus_{p+q=i} \oplus_{\ell(w)=p} Hom_K(\bigwedge^q \mathfrak{p}^-, \pi \otimes V_{w \star \mu}) = \oplus_{p+q=i} \oplus_{\ell(w)=p} H^q(P_h, K_h; \pi \otimes V_{w \star \mu}).$$

In the next section, I describe a collection of $|W_G/W_{K_h}|$ unitary representations π_w of $G(\mathbb{R})$, with the property that $\dim H^q(P_h, K_h; \pi_w \otimes V_{w \star \mu}) = 1$ when $q + \ell(w) = \dim \mathfrak{p}$ and is 0 otherwise.

Discrete series.

Suppose G is a semisimple Lie group with maximal compact subgroup K. We consider $L_2(G)$ with respect to a Haar measure dg. Since dg is both right and left invariant under the action of G, there is an action of $G \times G$ on $L_2(G)$. If G is compact, $L_2(G)$ is the Hilbert space completion of $\mathbb{C}[G]$ (functions on G, or the group algebra of G) and then the Peter-Weyl theorem (generalizing the theory for finite groups) states that

$$L_2(G) = \hat{\oplus}_{\tau \in \hat{G}} \tau^* \otimes \tau.$$

In general, $L_2(G)$ includes a Hilbert sum of subrepresentations – this is the discrete spectrum – and a continuous spectrum. Note that if G is non-compact, the constant functions are not in L_2. If the discrete spectrum is non-trivial it is called the *discrete series*. Harish-Chandra proved that G has a discrete spectrum if and only if a maximal torus T of K is a maximal torus of G. We have seen that this is always true if G/K has a hermitian structure (because then the center of K contains a non-trivial torus, and it is contained in a maximal torus which is necessarily contained in its centralizer).

Harish-Chandra also classified the discrete series. Fix a maximal torus $T \subset K = K_h$ with Lie algebra $\mathfrak{t}$ and sets of positive roots $\Delta_K^+ \subset \Delta^+$ for T in K and G, respectively; let

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha; \;\; \rho_k = \frac{1}{2} \sum_{\alpha \in \Delta_K^+} \alpha.$$

Write $W = W(G,T)$, $W_K = W(K,T)$. The most natural parametrization of the discrete series starts with a finite-dimensional representation (σ, V) of G with highest weight Λ. The Harish-Chandra isomorphism identifies

$$Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W$$

where $w \in W$ acts on characters of $\mathfrak{t}$ by the twisted action $w \star (\chi) = w(\chi + \rho) - \rho$. Thus the character of $Z(\mathfrak{g})$ on V corresponds to the W-orbit of $\lambda := \Lambda + \rho \in \mathfrak{t}^*$. Since Λ is dominant integral, λ is *regular*: $< \lambda, \alpha > \neq 0$ for all roots α. Indeed $< \lambda, \alpha >> 0$ for all $\alpha \in \Delta^+$. We want to consider π such that $H^*(\mathfrak{g}, K; \pi \otimes V^*) \neq 0$ where V^* is the contragredient of V, and such that π is in the discrete series. (Every member of the discrete series arises in this way.) Letting χ_π and χ_V be the infinitesimal characters of π and V respectively, we know that $\chi_\pi = \chi_V = \chi_\lambda$. The set Π_V of discrete series π with infinitesimal character χ_λ is then indexed by $w \in W^{P_h^+} \equiv W/W_K$; in Harish-Chandra's notation these can be denoted $\pi_{w^{-1}\lambda}$; then for λ', λ'' in the W-orbit of λ, $\pi_{\lambda'} \simeq \pi_{\lambda''}$ if and only if $\lambda' = w(\lambda'')$ for some $w \in W_K$.

By *K-type* we mean an irreducible (finite-dimensional) representation of K. The following theorem is valid for reductive as well as semi-simple groups; a discrete series for a reductive group G is a finite sum of square-integrable representations of G^{der}.

Theorem. *Let $d = \frac{1}{2} dim_{\mathbb{R}} G/K$, so that $d = \dim X$ if G/K is of hermitian type. Let $\lambda' \in W(\lambda)$ and let $\Delta_{\lambda'}^+$ be the set of roots α such that $< \lambda', \alpha >> 0$ (it is a W-translate of the original $\Delta^+ = \Delta_\lambda^+$, and let*

$$\rho_{\lambda'} = \frac{1}{2} \sum_{\alpha \in \Delta_{\lambda'}^+} \alpha.$$

Then

(i) (Weak Blattner formula) The K-type with highest weight $\tau_{\lambda'} := \lambda' + \rho_{\lambda'} - 2\rho_k$ occurs in $\pi_{\lambda'}$ with multiplicity one, and any K-type that occurs in $\pi_{\lambda'}$ has highest weight of the form $\lambda' + \rho_{\lambda'} - 2\rho_k + Q'$ where Q' is a sum of elements of $\Delta^+_{\lambda'}$.

(ii) If $V' \neq V$ is a finite-dimensional irreducible representation of G, then $Ext^*_{\mathfrak{g},K}(V', \pi) = H^*(\mathfrak{g}, K; \pi \otimes (V')^*) = 0$, whereas $Ext^i_{\mathfrak{g},K}(V, \pi) = 0$ unless $i = d$; $\dim Ext^d_{\mathfrak{g},K}(V, \pi) = 1$.

Proof. Part (i) is due to Schmid (after work of many others). I derive (ii) from (i). The first statement follows from the fact that two distinct finite-dimensional irreducible representations of G have distinct infinitesimal characters. Let $\Delta^+_n = \Delta^+_{\lambda'} \setminus \Delta_K$, $\Delta_n = \Delta^+_n \coprod -\Delta^+_n$, the set of roots of T acting on $\mathfrak{p}$. The weights of T on $\bigwedge^i \mathfrak{p}$ are thus of the form $\alpha_1 + \cdots + \alpha_i$ for $\alpha_\bullet \in \Delta_n$ distinct. Write $\rho_n = \rho_{\lambda'} - \rho_k$. The weights on $\bigwedge^i \mathfrak{p}$ are then of the form

$$\alpha_1 + \cdots + \alpha_j - (\alpha_{j+1} + \cdots + \alpha_i)$$

where all the $\alpha_\bullet$ are now in Δ^+_n. We can write

$$\alpha_1 + \cdots + \alpha_j = 2\rho_n - (\gamma_1 + \cdots + \gamma_t)$$

where $\Delta^+_n = \{\alpha_1, \ldots, \alpha_j; \gamma_1, \ldots, \gamma_t\}$. It follows that every weight of T on $\bigwedge^i \mathfrak{p}$ is of the form $2\rho_n - Q$ where W is a sum of elements of Δ^+_n. Moreover, if $2\rho_n$ is itself a weight of $\bigwedge^i \mathfrak{p}$ then $i = d = |\Delta^+_n|$ and $2\rho_n$ occurs in $\bigwedge^d \mathfrak{p}$ with multiplicity 1.

On the other hand, the weights of (σ, V) are all of the form $\lambda' - \rho_{\lambda'} - Q$ with Q a sum of positive roots and $\Lambda' = \lambda' - \rho_{\lambda'}$ has mutliplicity 1 by the theorem of the highest weight (for the positive root system $\Delta^+_{\lambda'}$). Thus

(i) The weights of T on $\bigwedge^i \mathfrak{p} \otimes V$ are of the form $2\rho_n + \lambda' - \rho_{\lambda'} - Q$ where Q is a sum of elements of $\Delta^+_{\lambda'}$. If $2\rho_n + \lambda' - \rho_{\lambda'} + Q'$ is a weight on $\bigwedge^i \mathfrak{p} \otimes V$ (with Q' a sum of positive roots) then $Q' = 0$, $i = d$, and the multiplicity equals 1.

(ii) Now suppose τ is a dominant integral weight for Δ^+_K (still relative to λ') such that $Hom_K(W_\tau, \bigwedge^i \mathfrak{p} \otimes V) \neq 0$. Then $\tau = 2\rho_n + \lambda' - \rho_{\lambda'} - Q$ as before, and if $\tau = 2\rho_n + \lambda' - \rho_{\lambda'} + Q'$ then $Q' = 0$, $i = d$, and $\dim Hom_K(W_\tau, \bigwedge^q \mathfrak{p} \otimes V) = 1$. But now we are done by the weak Blattner formula.

Consequences for the cohomology and coherent cohomology of Shimura varieties.

Note that we have fixed K above in order to view π as a $(\mathfrak{g}, K)$-module. In what follows, $K = K_h$ for some point $h \in X$.

We continue to assume that π is in the discrete series and that V is the representation in the theorem above. The bigrading shows that

$$\begin{aligned} H^d(\mathfrak{g}, K; \pi \otimes V^*) &= \oplus_{p+q=d} H^{p,q}(\mathfrak{g}, K; \pi \otimes V^*) \\ &= \oplus_{p+q=d} H^q(Lie(P_h); K; \pi \otimes H^p(\mathfrak{p}^+, V^*)). \\ &= \oplus_{p+q=d} \oplus_{\ell(w)=p} H^q(Lie(P_h); K; \pi \otimes (V^*)_w)). \end{aligned}$$

Here we are using Kostant's theorem: if μ is the highest weight of V^* (relative to any positive root system compatible with P^+_h) then V^*_w is the representation of K with highest weight $w(\mu + \rho) - \rho$. But since the left-hand side is of dimension 1, only one term is non-trivial. Thus for each π in the discrete series, there is a unique representation σ of K_h such that

$$H^d(\mathfrak{g}, K; \pi \otimes V^*) \xrightarrow{\sim} H^q(Lie(P_h); K; \pi \otimes \sigma))$$

where σ has highest weight $w(\mu + \rho) - \rho$ as above for some (unique) w, and then $q = d - \ell(w)$. Moreover, this is an isomorphism of 1-dimensional spaces.

Returning to the calculation of cohomology of automorphic vector bundles, we find

$$\begin{aligned} H^q(S(G,X),[W]) &\xrightarrow{\sim} \bigoplus_{\pi} m(\pi) H^q(Lie(P_h), K_h; \pi_\infty \otimes W) \otimes \pi_f. \\ &\xrightarrow{\sim} \bigoplus_{\pi_\infty \notin \hat{G}_d} m(\pi) H^q(Lie(P_h), K_h; \pi_\infty \otimes W) \otimes \pi_f \oplus \bigoplus_{\pi_\infty \in \hat{G}_d} m(\pi)\pi_f. \end{aligned}$$

Here we write $\hat{G}_d$ for the collection of discrete series representations of $G(\mathbb{R})$. The second isomorphism is not canonical, because it depends on a choice of basis of the 1-dimensional space $H^q(Lie(P_h), K_h; \pi_\infty \otimes W)$. However, it is not hard to see that this space is defined over a number field (and I've even written a paper about refining this definition).

Now we can use the existence of canonical models to draw conclusions about rationality. (Here mention that the Ω^i and their tensor products are all automorphic vector bundles.)