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Intertwining Operators for Semisimple Groups, II

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The purpose of the present paper is to expand the use of intertwining operators for semisimple Lie groups. In an earlier form (see [20]), these operators were meromorphic continuations of integral operators and exhibited equivalences among nonunitary principal series representations, those induced from a minimal parabolic subgroup. We demonstrated that there was an intimate connection between these operators and the irreducibility of the principal series, on the one hand, and the unitarity of the analytically continued representations (the complementary series), on the other hand.

In the intervening years we have generalized the setting for intertwining operators significantly, and we have learned of new applications. For the most part, the expansion in the setting is that minimal parabolic subgroups have now been replaced by arbitrary parabolic subgroups MAN, and the representations that are studied are induced from a representation of MAN that is irreducible unitary on M, is one-dimensional on A, and is trivial on N. The expanded theory allows us to determine the degree of reducibility of all series of representations appearing in the Plancherel formula of the group, and to study complementary series attached to them. Such intertwining operators have also now found major applications in classifying representations (see Langlands [29] and Knapp-Zuckerman [25]) and appear to have significance for some problems in number theory.

Our objective in this paper is threefold: to develop the analytic properties of intertwining operators in what seems to be the appropriate degree of generality, to give a dimension formula for the commuting algebras of the unitary representations induced when the representation of M is in the discrete series and the character of A is unitary (and to give some further insights into these representations), and to illustrate a technique for dealing with complementary series. To make it possible to be more specific, we introduce some notation.

Let G be a reductive group whose identity component has compact center; the precise axioms for G are given in §1. Fix a Cartan involution θ for G, and let P be a parabolic subgroup of G. Then $P \cap \theta P$ decomposes into the product of commuting subgroups M and A, where A is a vector group and M satisfies the

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same axioms as G. There are finitely many distinct nilpotent groups N such that MAN is a parabolic subgroup of G. To each of these and to data consisting of an irreducible unitary representation ξ of M and a one-dimensional representation $\exp \Lambda$ of A, we can associate the induced representation of G given by

$$U_{MAN}(\xi, \Lambda, \cdot) = \operatorname*{ind}_{MAN \uparrow G} (\xi \otimes \exp \Lambda \otimes 1).$$

We adopt the convention that G operates on the left and that the parameters are arranged so that U is unitary when $\exp \Lambda$ is unitary.

To each pair of choices N_1 and N_2 for N, there is a formal expression for an operator intertwining the representations induced from MAN_1 and MAN_2 in the presence of the same data (ξ, Λ) for MA, namely

$$A(MAN_2 : MAN_1 : \xi : \Lambda) f(x) = \int_{N_2 \cap \theta N_1} f(xv)\, dv. \tag{0.1}$$

There is not always an element w in the normalizer of A with $N_2 = w^{-1} N_1 w$, but if there is, then right translation $R(w)$ by w carries representations induced from MAN_2 to representations induced from MAN_1 and then the composition

$$A_{MAN_1}(w, \xi, \Lambda) = R(w)\, A(MAN_2 : MAN_1 : \xi : \Lambda) \tag{0.2}$$

is the kind of operator that was studied in [20] under the additional hypothesis that MAN_1 is minimal parabolic (and hence M is compact).

Part I of this paper deals with the development and analytic properties of these operators (0.1) and (0.2) in the generality noted above. In the cases of interest, the integral (0.1) is usually divergent, and the operator is defined by analytic continuation from values of Λ for which there is convergence. A direct attack on the question of analytic continuation seems now to be quite difficult, because the singularity in the integral is complicated and the analytic behavior of the integral depends on the full asymptotic expansion at infinity of the matrix coefficients for the representation ξ of M. Our approach instead will be to capture this asymptotic expansion in the form of an imbedding on the algebraic level of ξ in a nonunitary principal series representation of M; such an imbedding exists by a theorem of Casselman quoted in §5. By means of this imbedding, we are able to reduce the analysis of the operators (0.1) and (0.2) substantially to the case of the operators considered in [20]; the details of the analysis are carried out in §§6–7. Only a small part of [20] needs to be used after the reduction – and not exactly in the form in [20]; for this reason, we have included some material in §§3–4 similar to that in [20].

After the development of the operators, the next step is their normalization. For this purpose, we make the following

Basic Assumption.[1] The infinitesimal character of ξ is a real linear combination of the roots of M.

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[1] Vogan has pointed out to us that for many purposes the Basic Assumption is no restriction since every irreducible unitary representation of G is a full induced representation $U_{MAN}(\xi, \Lambda, \cdot)$ for some parabolic subgroup MAN and some ξ satisfying the Basic Assumption.

Under this assumption, any system of normalizing factors satisfying certain properties will serve to produce nice relations among the intertwining operators, as is shown in § 8, and we readily prove the existence of such a system. However, different systems of normalizing factors are useful for different purposes. Our own treatment of complementary series here and in [20] requires a system free from unnecessary zeros and singularities. Kunze and Stein [28], Stein [33] and Gelbart [7] had earlier used a different normalization in some special cases that lends itself to Euclidean Fourier analysis. Arthur [3] has exhibited one useful in number theory, verifying a conjecture of Langlands.

Part II combines the techniques of Part I with results of Harish-Chandra to analyze the commuting algebras of the induced representations corresponding to ξ in the discrete series and $\exp \Lambda$ unitary. Briefly, Harish-Chandra's theory of c-functions led him to recognize a spanning set for the commuting algebra. This spanning set is identified in §9 as a set of intertwining operators of the kind in Part I. Starting in §11 from further results of Harish-Chandra, we are able to identify in §13 a subset of these operators that forms a basis for the commuting algebra. This identification can be given either in terms of the nonvanishing of certain "Plancherel factors" defined in §10 or in terms of a certain finite group $R_{\xi,\Lambda}$ described in §13. The core of the proof is the proof of linear independence, and this step is carried out in §12. In §15 we identify $R_{\xi,\Lambda}$ as the direct sum of two-element groups for the case that G is a linear connected semisimple group split over $\mathbb{R}$ and the parabolic subgroup is minimal; this result had been announced in [21].

Part III establishes a basic existence theorem for complementary series, to illustrate a general technique. Our result is far from best possible, and the state of knowledge in this area is nowhere near complete. Further progress in this area will doubtless have to precede a classification theorem for irreducible unitary representations.

We list here the main results of our paper:

(i) the meromorphic continuation of the intertwining operators (Theorem 6.6),

(ii) the basic cocycle relations of the normalized intertwining operators (Theorem 8.4),

(iii) the determination of the "reducibility group" (the R group) and the formula for the dimension of the commuting algebra in terms of the nonvanishing of Plancherel measures for the case of representations induced from discrete series (Theorem 13.4),

(iv) the construction of a wide class of complementary series (Theorem 16.2).

Substantially all of the present work was announced in [22] and [23]. The idea behind Part II and the theorem of §15 date from [21]. All this material was organized into its current form in 1975, and it was the subject of a lecture series by the first author at the Institute for Advanced Study in the fall of 1975. The press of other matters has delayed publication until now.

At a number of points the present work touches on the work of others. In some instances, results of ours (or at least special cases of them) have been obtained independently by other people. In addition, some authors have found it useful to quote some of our announced results and, in some cases, to supply proofs. This circumstance has created problems of exposition for us. Our choice has been to maintain the continuity of our presentation, even though an occasional lemma may be found elsewhere in the literature.

Several aspects of our work can be done in alternate ways, at least in special cases, and it is useful to understand the roles of the different methods:

(a) The connection between c-functions and intertwining operators is described in detail in §9. In the context of discrete series representations ξ of M (which for us is not the general case), the two concepts are equivalent. However, their thrust is quite different, as are the methods used to develop them, and the theorems they lead to are complementary. Harish-Chandra [15] naturally obtains a spanning set of self-intertwining operators, whereas we are naturally led to linear independence.

(b) Arthur [2] independently discovered parts of the connection between c-functions and intertwining operators and used it to develop intertwining operators. While his development has several advantages, it does impose two limitations: it restricts ξ to the discrete series and it restricts the domain of the intertwining operators to "K-finite" functions. The first restriction limits applicability of the theory in classification problems, such as in [25]. The second rules out dealing with questions of linear independence like that in §12.

(c) Wallach [35] independently developed some of the material of §6 by making use of Harish-Chandra's subquotient theorem in place of Casselman's subrepresentation theorem.

(d) Vogan [34] has recently developed an algebraic approach to the group $R_{\xi,A}$ of §13. His approach seems very different from ours, and the exact connection between the two approaches seems to be an interesting question.

We are indebted to several people for making known to us their own unpublished theorems at an early date - Borel and Tits for Theorem 12.2, Casselman for Theorem 5.1, and Harish-Chandra for Theorem 9.7. We are also happy to acknowledge valuable help given us by Langlands, Wallach, and Zuckerman; we were influenced both in our research and in this exposition by several of their suggestions.

Contents

I. Construction and Normalization of Intertwining Operators

§1. Notation, Formal Intertwining Operators

The Lie groups G that we deal with will all satisfy the following axioms:
(i) the Lie algebra $\mathfrak{g}$ of G is reductive,
(ii) the identity component G_0 of G has compact center,
(iii) G has finitely many components
(iv) $\mathrm{Ad}(G)$ is contained in the connected adjoint group of the complexification $\mathfrak{g}^{\mathbb{C}}$. (1.1)

Except for requirement (ii), these are Harish-Chandra's axioms ([12] and [13]), and the groups we consider are the same as Harish-Chandra's groups that have trivial split component.

We collect some notation and properties for such groups. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the corresponding Cartan decomposition, and let θ be the Cartan involution. By (ii) the center of $\mathfrak{g}$ is contained in $\mathfrak{k}$. We introduce an inner product B_θ on $\mathfrak{g}$ in the standard way, with the properties that $B_\theta(X,Y)=-B(X,\theta Y)$ and B is an $\mathrm{Ad}(G)$-invariant θ-invariant symmetric bilinear form on $\mathfrak{g}$. Let $\mathfrak{a}_p$ be a maximal abelian subspace of $\mathfrak{p}$. If we fix a notion of positivity for $\mathfrak{a}_p$-roots, we can let $\mathfrak{n}_p$ be the nilpotent subalgebra given as the sum of the root spaces for the positive roots and we can let $\mathfrak{v}_p=\theta\mathfrak{n}_p$. The Iwasawa decomposition is $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}_p\oplus\mathfrak{n}_p$. Let $\mathfrak{m}_p=Z_{\mathfrak{k}}(\mathfrak{a}_p)$ be the centralizer of $\mathfrak{a}_p$ in $\mathfrak{k}$.

On the group level, let $K=N_G(\mathfrak{k})$ be the normalizer of $\mathfrak{k}$ in G, let $M'_p=N_K(\mathfrak{a}_p)$, let $M_p=Z_K(\mathfrak{a}_p)$, and let A_p, N_p, and V_p be the analytic subgroups corresponding to $\mathfrak{a}_p$, $\mathfrak{n}_p$, and $\mathfrak{v}_p$, respectively. These groups have the following properties:
(i) K has Lie algebra $\mathfrak{k}$ and is a maximal compact subgroup. The identity component K_0 equals $N_{G_0}(\mathfrak{k})$.
(ii) $G=G_0M_p$
(iii) The map $(k,X)\in K\times\mathfrak{p}\to k\exp X\in G$ is a diffeomorphism onto.
(iv) The map $(k,a,n)\in K\times A_p\times N_p\to kan\in G$ is a diffeomorphism onto. (1.2)

Any conjugate of $\mathfrak{m}_p\oplus\mathfrak{a}_p\oplus\mathfrak{n}_p$ is called a *minimal parabolic subalgebra*, and any Lie subalgebra $\mathfrak{s}$ that contains a minimal parabolic subalgebra is called *parabolic*. Then $\mathfrak{s}$ has a Langlands decomposition (relative to θ) $\mathfrak{s}=\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$; see, e.g., [17]. Here $\mathfrak{m}\oplus\mathfrak{a}=Z_{\mathfrak{g}}(\mathfrak{a})$, and we can impose an ordering on the $\mathfrak{a}$-roots so that $\mathfrak{n}$ is built from the positive $\mathfrak{a}$-roots. Let $\mathfrak{v}=\theta\mathfrak{n}$. If $\mathfrak{a}_M$ is a maximal abelian subspace of $\mathfrak{m}\cap\mathfrak{p}$, then $\mathfrak{a}\oplus\mathfrak{a}_M$ is a maximal abelian subspace of $\mathfrak{p}$ and can be taken as $\mathfrak{a}_p$ in our theory. When we introduce an ordering on the $\mathfrak{a}_p$-roots so that $\mathfrak{a}$ comes before $\mathfrak{a}_M$, then the positive $\mathfrak{a}$-roots are the nonzero restrictions to $\mathfrak{a}$ of the positive $\mathfrak{a}_p$-roots. The sum of the root spaces for the positive $\mathfrak{a}_p$-roots that vanish on $\mathfrak{a}$ is denoted $\mathfrak{n}_M$.

Let M_0, A, A_M, N, V, N_M be analytic subgroups corresponding to $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{a}_M$, $\mathfrak{n}$, $\mathfrak{v}$, $\mathfrak{n}_M$, and define

$$M = M_0 M_p. \tag{1.3}$$

The group $P = MAN$ is a parabolic subgroup. The subgroups under discussion have the following properties. See [13].

(i) $MA = Z_G(\mathfrak{a})$, $MAN = N_G(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$, MAN is closed, and $(m, a, n) \in M \times A \times N \to man \in MAN$ is a diffeomorphism onto,
(ii) M satisfies the axioms (1.1), $\theta|_{\mathfrak{m}}$ is a Cartan involution of $\mathfrak{m}$, and $K_M = K \cap M$ is the corresponding maximal compact subgroup of M,
(iii) $M = K_M A_M N_M$ is an Iwasawa decomposition of M,
(iv) $A_p = A_M A$ and $N_p = N_M N$ diffeomorphically,
(v) $G = KMAN$ with the KM, A, and N components unique,
(vi) $K \cap MA = K \cap M$,
(vii) $V \cap MAN = \{1\}$, and
(viii) the M_p group for M equals the M_p group for G. (1.4)

Let us prove (viii). We have

$$\begin{aligned} Z_{K \cap M}(\mathfrak{a}_M) &= Z_K(\mathfrak{a}_M) \cap (K \cap M) = Z_K(\mathfrak{a}_M) \cap (K \cap MA) \\ &= Z_K(\mathfrak{a}_M) \cap Z_G(\mathfrak{a}) = Z_K(\mathfrak{a}_p), \end{aligned}$$

with the second and third equalities following from (vi) and (i), respectively. Then (viii) is proved.

Two parabolic subgroups with the same MA are *associated*. The choices for N are in obvious one-to-one correspondence with the Weyl chambers of $\mathfrak{a}$ formed by the $\mathfrak{a}$-root hyperplanes. Let $M' = N_K(\mathfrak{a})M$. The "Weyl group" for this situation is $W(\mathfrak{a}) = M'/M$. The group $W(\mathfrak{a})$ permutes the Weyl chambers, and a nontrivial element of $W(\mathfrak{a})$ does not leave any Weyl chamber stable. However, $W(\mathfrak{a})$ does not necessarily act transitively on the set of Weyl chambers. In terms of the groups N, if w is in M', then $w^{-1}Nw$ is another choice of the N group, but not every choice of N group arises by conjugating a particular one.

M' acts on characters of A and representations of M by

$$\begin{aligned} w\lambda(a) &= \lambda(w^{-1}aw) \\ w\xi(m) &= \xi(w^{-1}mw). \end{aligned}$$

Then $W(\mathfrak{a})$ acts on characters of A and classes of representations of M. Later we shall denote the class of the representation ξ of M by $[\xi]$.

It is not known in general whether $W(\mathfrak{a})$ is isomorphic with the Weyl group of a full root system. However, it is known in a special case that will be sufficient for the purposes of Part II of this paper. See [17].

There is another kind of construction that leads to subgroups of G satisfying the axioms (1.1). Fix MA arising from a parabolic subgroup. Let β be an $\mathfrak{a}$-root in the dual $\mathfrak{a}'$, H_β the corresponding member of $\mathfrak{a}$ under the identification set up by B_θ, and $(H_\beta)^\perp$ the orthogonal complement of $\mathbb{R}H_\beta$ in $\mathfrak{a}$. Then

$$Z_{\mathfrak{g}}((H_\beta)^\perp)=\mathfrak{a}\oplus\mathfrak{m}\oplus\sum_{c\neq 0}\mathfrak{g}_{c\beta},$$

where $\mathfrak{g}_{c\beta}$ is a root space. Define

$$\mathfrak{g}^{(\beta)}=\mathbb{R}H_\beta\oplus\mathfrak{m}\oplus\sum_{c\neq 0}\mathfrak{g}_{c\beta},$$

and let $G_0^{(\beta)}$ be the corresponding analytic subgroup. We shall be interested in

$$G^{(\beta)}=G_0^{(\beta)}M. \tag{1.5}$$

Since M normalizes $\mathfrak{g}^{(\beta)}$, $G^{(\beta)}$ is a group. Essentially by Lemma 2.3 of [15], $G^{(\beta)}$ satisfies the axioms (1.1). Also $\theta|_{\mathfrak{g}^{(\beta)}}$ is a Cartan involution, and the corresponding maximal compact subgroup of $G^{(\beta)}$ is $K^{(\beta)}=K\cap G^{(\beta)}$.

Let $\mathfrak{n}^{(\beta)}=\sum_{c>0}\mathfrak{g}_{c\beta}$, $\mathfrak{v}^{(\beta)}=\theta\mathfrak{n}^{(\beta)}=\sum_{c<0}\mathfrak{g}_{c\beta}$, and $\mathfrak{a}^{(\beta)}=\mathbb{R}H_\beta$; let $A^{(\beta)}$, $N^{(\beta)}$, and $V^{(\beta)}$ be the analytic subgroups corresponding to $\mathfrak{a}^{(\beta)}$, $\mathfrak{n}^{(\beta)}$, and $\mathfrak{v}^{(\beta)}$. Then $MA^{(\beta)}N^{(\beta)}$ and $MA^{(\beta)}V^{(\beta)}$ are maximal parabolic subgroups of $G^{(\beta)}$.

We now introduce classes of representations. Let $P=MAN$ be a parabolic subgroup, and let $\rho=\rho_P$ be $\frac{1}{2}\sum_{\beta>0}(\dim\mathfrak{g}_\beta)\beta$. To each continuous representation ξ of M on a Hilbert space, say H^ξ, and each complex-valued real-linear functional Λ on $\mathfrak{a}$ we associate a representation $U_P(\xi,\Lambda,\cdot)$ of G as follows. A dense subspace of the representation space is

$$\{f\colon G\to H^\xi\mid f \text{ is } C^\infty \text{ and } f(xman)=e^{-(\rho+\Lambda)\log a}\xi(m)^{-1}f(x)\},$$

the action is

$$U_P(\xi,\Lambda,g)f(x)=f(g^{-1}x),$$

and the norm is the L^2 norm of the restriction to K. This representation is unitary if Λ is imaginary and ξ is unitary.

What we have just described is the "induced picture" for $U_P(\xi,\Lambda,\cdot)$. The "compact picture" is the restriction of the induced picture to K. Here the dense subspace is

$$\{f\colon K\to H^\xi\mid f \text{ is } C^\infty \text{ and } f(km)=\xi(m)^{-1}f(k)\}$$

and is independent of Λ. For the action, we introduce notation for the $G=KMAN$ decomposition of (1.4), writing

$$g=\kappa(g)\,\mu(g)(\exp H(g))\,n. \tag{1.6}$$

Then

$$U_P(\xi,\Lambda,g)f(k)=e^{-(\rho+\Lambda)H(g^{-1}k)}\,\xi(\mu(g^{-1}k))^{-1}f(\kappa(g^{-1}k)).$$

Since the space is independent of Λ, we can speak of holomorphic functions with values in the space, and it is clear that $U_P(\xi,\Lambda,g)f$ depends holomorphically on Λ.

Next we introduce formal expressions, often divergent, for operators that implement equivalences among some of these representations. For now, we work in the induced picture. Let $P=MAN$ and $P'=MAN'$, and set

$$A(P':P:\xi:\Lambda)f(x)=\int_{V\cap N'} f(xv)\,dv, \tag{1.7}$$

with the normalization of Haar measure to be specified in §2.

Proposition 1.1. *When the indicated integrals are convergent,*

$$U_{P'}(\xi,\Lambda,g)\,A(P':P:\xi:\Lambda)=A(P':P:\xi:\Lambda)\,U_P(\xi,\Lambda,g) \tag{1.8}$$

for all g in G.

Sketch. The point is that $A(P':P:\xi:\Lambda)$ carries members of the representation space for U_P into members of the representation space for $U_{P'}$. If f transforms appropriately under MAN on the right, the image function is to transform appropriately under MAN'. One verifies this separately for M, A, $N'\cap N$, and $N'\cap V$ as in [28], page 395. The action of G is on the left and does not interfere with the transformation laws on the right, and the proposition follows.

For w in $K\cap M'$, let $R(w)f(x)=f(xw)$. Then it follows from Proposition 1.1 that

$$A_P(w,\xi,\Lambda)=R(w)\,A(w^{-1}Pw:P:\xi:\Lambda) \tag{1.9}$$

satisfies

$$U_P(w\xi,w\Lambda,\cdot)\,A_P(w,\xi,\Lambda)=A_P(w,\xi,\Lambda)\,U_P(\xi,\Lambda,\cdot) \tag{1.10}$$

whenever the indicated integrals are convergent.

We shall want to relate induced representations and intertwining operators defined relative to different subgroups of G. For this purpose it will be necessary to know how the various quantities ρ (half the sum of the positive roots) are related. First, in the case of a parabolic subgroup MAN containing a minimal parabolic $M_pA_pN_p$, so that $\mathfrak{a}_p=\mathfrak{a}\oplus\mathfrak{a}_M$, we have

$$\rho_p=\rho+\rho_M. \tag{1.11}$$

The ordering on $\mathfrak{a}_p$-roots here is such that an $\mathfrak{a}_p$-root α is positive if $\alpha|_{\mathfrak{a}}$ is positive or if $\alpha|_{\mathfrak{a}}=0$ and α is a positive $\mathfrak{a}_M$-root. To see (1.11), we note first that $W(\mathfrak{a}_M)$ fixes no nonzero members of $\mathfrak{a}_M$ and consequently the sum of the elements in any $W(\mathfrak{a}_M)$-orbit of $\mathfrak{a}_M$ is 0. From this we see that if β is an $\mathfrak{a}$-root, then the sum of all $\mathfrak{a}_p$-roots α with $\alpha|_{\mathfrak{a}}=\beta$ has 0 component in $\mathfrak{a}_M$. Formula (1.11) then follows.

The situation with $G^{(\beta)}$ and its associated $\rho^{(\beta)}$ is more complicated and is given in the next proposition. In the case of minimal parabolics, this result appears on p. 399 of [28]. In that case, root reflections are available, but in the general case they are not.

Proposition 1.2. *Let $P=MAN$ and $P'=MAN'$ be associated parabolic subgroups such that $V\cap N'=V^{(\beta)}$ for an $\mathfrak{a}$-root β. Let $\rho=\rho_P$, and let H_β be the member of* $\mathfrak{a}$

corresponding to β. Then

$$\rho(H_\beta)=\rho^{(\beta)}(H_\beta). \tag{1.12}$$

If β is reduced (i.e., *$c\beta$ is not an $\mathfrak{a}$-root for $0<c<1$*), *then β is simple for P and is the restriction to $\mathfrak{a}$ of an $\mathfrak{a}_p$-root simple for P_p.*

A proof is given in Appendix A.

§2. Normalization of Haar Measures on Nilpotent Groups

In this section we shall describe normalizations simultaneously for the Haar measures of all relevant nilpotent subgroups of all groups G under consideration. Our construction is motivated by the one given by Schiffmann [32].[2]

To define the normalization, we proceed as follows. To each $\mathfrak{a}_p$-root α, we normalize Haar measure on $V^{(\alpha)}$ by the requirement

$$\int_{V^{(\alpha)}} \exp\{-2\rho^{(\alpha)} H^{(\alpha)}(v)\}\, dv=1.$$

For any simply-connected nilpotent group, the exponential map carries Lebesgue measure to a Haar measure, and we use this fact to transfer normalized Haar measure on $V^{(\alpha)}$ to a multiple of Lebesgue measure on $\mathfrak{v}^{(\alpha)}$. Each of the nilpotent groups that we work with will have as its Lie algebra the direct sum of some $\mathfrak{v}^{(\alpha)}$'s with the α's lying in an open half space. Accordingly we form the product measure on the Lie algebra and then carry it to the group by the exponential map. Our normalization is then well defined and completely determined.

Lemma 2.1. *Let $\mathfrak{u}$ be a nilpotent Lie algebra contained in the vector space $\mathfrak{n}_p \oplus \mathfrak{v}_p$, and suppose that $\mathfrak{u}$ is of the form $\mathfrak{u}=\sum_{\alpha\in S}\mathfrak{v}^{(\alpha)}$ for a set S of $\mathfrak{a}_p$-roots α lying in an open half space. Put $U=\exp\mathfrak{u}$, and let w be in the normalizer of $\mathfrak{a}_p$ in K. Then the normalized Haar measures on U and $w^{-1}Uw$ satisfy*

$$\int_U F(u)\,du=\int_{w^{-1}Uw} F(wu'w^{-1})\,du'.$$

Proof. In view of our construction, we pass to the Lie algebras of U and $w^{-1}Uw$ and compare the Lebesgue measure there. Each is given as a product, and consequently the lemma will follow if we show that

$$\int_{V^{(\alpha)}} f(v)\,dv=\int_{w^{-1}V^{(\alpha)}w} f(wv'w^{-1})\,dv'. \tag{2.1}$$

Here $w^{-1}V^{(\alpha)}w=V^{(w^{-1}\alpha)}$, and it is enough to prove (2.1) for the single function

$$f(v)=\exp\{-2\rho^{(\alpha)}\mathrm{H}^{(\alpha)}(v)\}.$$

[2] Two slips need correction on p. 35 of [32]. Formula (2.2.8) should have an additional factor of $c_w(\rho)^{-1}$ on the right side, and the right side of formula (2.2.9) should be $c_w(\rho)$. The corrected (2.2.8) is by induction from [8] and (2.2.5).

The left side of (2.1) is then 1 by definition of dv. For g in $G^{(w^{-1}\alpha)}$, we have

$$\rho^{(\alpha)} H^{(\alpha)}(w g w^{-1}) = \rho^{(w^{-1}\alpha)} H^{(w^{-1}\alpha)}(g),$$

and hence the right side of (2.1) is 1 by definition of dv'. The lemma follows.

Lemma 2.2. *Let MAN_0 and MAN_1 be associated parabolic subgroups, and let w be a representative of a member of $W(\mathfrak{a}_p)$ such that $\operatorname{Ad}(w)(\mathfrak{n}_1+\mathfrak{n}_M)=\mathfrak{n}_0+\mathfrak{n}_M$. Then*

$$(V_p)_0 \cap w^{-1}(N_p)_0 w = V_0 \cap N_1.$$

Remark. This lemma allows one to relate the integral formulas in [32] and in the present paper.

Proof. The Lie algebra of the intersection on the left is

$$(\mathfrak{v}_0+\mathfrak{v}_M)\cap(\mathfrak{n}_1+\mathfrak{n}_M),$$

which is just $\mathfrak{v}_0 \cap \mathfrak{n}_1$.

Proposition 2.3. *Suppose that $P_i = MAN_i$, $0 \leqq i \leqq 2$, are three associated parabolic subgroups such that $\mathfrak{n}_2 \cap \mathfrak{n}_0 \subseteq \mathfrak{n}_1 \cap \mathfrak{n}_0$. Then*

$$\int_{V_0 \cap N_2} f(v)\,dv = \int_{(V_1\cap N_2)\times(V_0\cap N_1)} f(u_1 u_2)\,du_1\,du_2.$$

Proof. We have

$$\begin{aligned}\mathfrak{v}_1 \cap \mathfrak{n}_2 &= \mathfrak{v}_1 \cap \mathfrak{n}_2 \cap \mathfrak{v}_0 \oplus \mathfrak{v}_1 \cap \mathfrak{n}_2 \cap \mathfrak{n}_0 \\ &\subseteq \mathfrak{v}_1 \cap \mathfrak{n}_2 \cap \mathfrak{v}_0 + \mathfrak{v}_1 \cap \mathfrak{n}_1 \cap \mathfrak{n}_0 = \mathfrak{v}_1 \cap \mathfrak{n}_2 \cap \mathfrak{v}_0\end{aligned}$$

and

$$\begin{aligned}\mathfrak{v}_0 + \mathfrak{n}_1 &= \mathfrak{v}_0 \cap \mathfrak{n}_1 \cap \mathfrak{n}_2 \oplus \mathfrak{v}_0 \cap \mathfrak{n}_1 \cap \mathfrak{v}_2 \\ &\subseteq \mathfrak{v}_0 \cap \mathfrak{n}_1 \cap \mathfrak{n}_2 + \mathfrak{v}_0 \cap \mathfrak{n}_1 \cap \mathfrak{v}_1 = \mathfrak{v}_0 \cap \mathfrak{n}_1 \cap \mathfrak{n}_2.\end{aligned}$$

Thus

$$\mathfrak{v}_1 \cap \mathfrak{n}_2 = \mathfrak{v}_1 \cap (\mathfrak{v}_0 \cap \mathfrak{n}_2) \quad \text{and} \quad \mathfrak{v}_0 \cap \mathfrak{n}_1 = \mathfrak{n}_1 \cap (\mathfrak{v}_0 \cap \mathfrak{n}_2),$$

and

$$\mathfrak{v}_0 \cap \mathfrak{n}_2 = \mathfrak{v}_1 \cap (\mathfrak{v}_0 \cap \mathfrak{n}_2) \oplus \mathfrak{n}_1 \cap (\mathfrak{v}_0 \cap \mathfrak{n}_2) = (\mathfrak{v}_1 \cap \mathfrak{n}_2) \oplus (\mathfrak{v}_0 \cap \mathfrak{n}_1).$$

In view of our normalization of Haar measures, we therefore have

$$\begin{aligned}\int_{V_0\cap N_2} f(v)\,dv &= \int_{\mathfrak{v}_0\cap\mathfrak{n}_2} f(\exp x)\,dx \\ &= \int_{(\mathfrak{v}_1\cap\mathfrak{n}_2)\oplus(\mathfrak{v}_0\cap\mathfrak{n}_1)} f(\exp(x_1+x_2))\,dx_1\,dx_2 \\ &= \int_{(\mathfrak{v}_1\cap\mathfrak{n}_2)\oplus(\mathfrak{v}_0\cap\mathfrak{n}_1)} f(\exp y_1 \exp y_2)\,dy_1\,dy_2 \\ &= \int_{(V_1\cap N_2)\times(V_0\cap N_1)} f(u_1 u_2)\,du_1\,du_2,\end{aligned}$$

with the third equality holding since $(y_1, y_2) \to \exp^{-1}(\exp y_1 \exp y_2)$ is a diffeomorphism whose Jacobian determinant is identically one. (See [36], pages 95 and 235.)

Lemma 2.4. *Let MAN and MAN' be associated parabolic subgroups. Then the normalized Haar measures on N, $N\cap V'$, and $N\cap N'$ satisfy*

$$dn_N = dn_{N\cap V'}\, dn_{N\cap N'}.$$

Proof. We apply Proposition 2.3 to the three parabolic subgroups MAV, MAN', and MAN. The result applies since

$$0 = \mathfrak{n}\cap\mathfrak{v} \subseteq \mathfrak{n}'\cap\mathfrak{v}.$$

Lemma 2.5. *Let MAN be a parabolic subgroup. Then the normalized Haar measures on NN_M, N, and N_M satisfy*

$$dn_{NN_M} = dn_N\, dn_{N_M}.$$

Proof. This follows from Proposition 2.3 applied to the three parabolic subgroups $M_p A_p V V_M$, $M_p A_p V N_M$, and $M_p A_p N N_M$.

Lemma 2.6. *Let MAN and MAN' be associated parabolic subgroups and let s be in $N_K(\mathfrak{a})$. Then the normalized Haar measures satisfy*

$$\int_{V\cap N'} f(v)\,dv = \int_{s(V\cap N')s^{-1}} f(s^{-1}us)\,du.$$

Proof. Under the assumption that s is also in $N_K(\mathfrak{a}_p)$, this result follows from Lemma 2.1. In the general case, by Lemma 8 of [17], we can write $s = mt$ with t in $N_K(\mathfrak{a}_p)$ and m in M. The special case applies to t, and we have

$$\begin{aligned}\int_{V\cap N'} f(v)\,dv &= \int_{t(V\cap N')t^{-1}} f(t^{-1}ut)\,du\\ &= \int_{s(V\cap N')s^{-1}} f(t^{-1}m^{-1}umt)\,du\\ &= \int_{s(V\cap N')s^{-1}} f(s^{-1}us)\,du,\end{aligned}$$

the middle equality holding since m is in M. This proves the lemma.

Corollary 2.7. *Let MAN and MAN' be associated parabolic subgroups and let s be in $N_K(\mathfrak{a})$ and satisfy $sNs^{-1} = N'$. Then the normalized Haar measures on N and N' satisfy*

$$\int_N f(n)\,dn = \int_{N'} f(s^{-1}n's)\,dn'.$$

Proof. This follows from Lemma 2.6 applied to the parabolics MAV and MAN and to the same element s.

Lemma 2.8. *Let MAN and MAN' be associated parabolic subgroups, and let dn and dn' be the normalized Haar measures of N and N'. Then the integrals*

$$\int_{KMNA} f(kmna)\,dk\,dm\,dn\,da \quad \text{and} \quad \int_{KMN'A} f(kmn'a)\,dk\,dm\,dn'\,da$$

define the same normalization of Haar measure on G.

Proof. We can write $dm = dk_M dn_M da_M$ for a suitable normalization of da_M. Conjugation of N by a_M does not affect dn, since a_M is in M, and thus we can regroup the first integral as

$$\int_K \int_{K_M} \int_{A_M A} \int_{N_M N} f(k k_M n_M n a_M a)\, da_M\, da\, dn_M\, dn\, dk_M\, dk.$$

The dk_M goes away after a change of variables, $da_M da$ can be grouped as da_p, and $dn_M dn$ can be grouped as dn_p according to Lemma 2.5. Thus Lemma 2.8 reduces to the identity

$$\int_{KN_pA_p} f(kn_p a_p)\, dk\, dn_p\, da_p = \int_{KN'_pA_p} f(kn'_p a_p)\, dk\, dn'_p\, da_p,$$

which follows from Corollary 2.7 and a change of variables.

§ 3. Existence of Operators, Real-rank One Minimal Case

In this section we shall assume that G has $\dim \mathfrak{a}_p = 1$, and we shall drop the subscripts p. The theory of intertwining operators for this real-rank one minimal case was developed in [20] when G is a linear connected semisimple group. Much of that development is unnecessary for our current purposes, and we shall now isolate the essential results, referring to [20] for most of the proofs.

Our basic minimal parabolic subgroup is $P = MAN$, and M is contained in K since P is minimal. The assumption that $\dim A = 1$ enters in the following way: The group $M' = N_K(\mathfrak{a})$ has $W(\mathfrak{a}) = M'/M$ of order 2. Let w be in M' but not M. By the Bruhat decomposition, we have

$$G = MAN \cup MAN\, w\, MAN,$$

and it follows that every $v \neq 1$ in V has the property that $w^{-1}v$ is in $VMAN$. [To see this, we note that $V \cap MAN = 1$. Thus $v \neq 1$ implies that v is in $MAN\, w\, MAN = N\, w\, MAN$ and hence that $w^{-1}v$ is in $w^{-1} N\, w\, MAN = VMAN$.] This property of V means that the basic convolution operator given by (3.6) and (3.7) below has a one-point singularity. The analysis of the operator is therefore relatively simple.

Convergence and analytic continuation follow after a computation giving a number of equivalent forms for the intertwining operator $A_P(w, \xi, \Lambda)$. Before giving the computation, we assemble some identities. Corresponding to the decomposition $G = KAN$, we write $g = \kappa(g) e^{H(g)} n$ as in § 1. Corresponding to the decomposition of most of G as $VMAN$, we write $g = v m(g) a(g) n$. Then we have

$$e^{\Lambda H(v)} = e^{-\Lambda \log a(\kappa(v))} \tag{3.1}$$

$$m(\kappa(v)) = 1. \tag{3.2}$$

The unique positive reduced $\mathfrak{a}$-root is denoted α, and we let $p = \dim \mathfrak{g}_\alpha$ and $q = \dim \mathfrak{g}_{2\alpha}$. The linear functional $\rho = \rho_P$ is given as $\rho = \frac{1}{2}(p + 2q)\alpha$. With the normalization of Haar measure as in §2, we have

$$\int_V e^{-2\rho H(v)}\,dv = 1$$

since $V = V^{(\alpha)}$. Consequently we see from [10, p. 287] that

$$\int_K f(k)\,dk = \int_{V\times M} f(\kappa(v)m)\,e^{-2\rho H(v)}\,dm\,dv. \tag{3.3}$$

We can now do our computation for a function f on G satisfying

$$f(x\,m\,a\,n) = e^{-(\rho+\Lambda)\log a}\,\xi(m)^{-1} f(x),$$

provided one of the integrals in question is absolutely convergent:

$$\int_V f(x\,w\,v)\,dv = \int_V e^{-(\rho+\Lambda)H(v)} f(x\,w\,\kappa(v))\,dv \tag{3.4}$$

$$= \int_V e^{-(\rho-\Lambda)\log a(\kappa(v))}\,e^{-2\rho H(v)}\,\xi(m(\kappa(v)))\,f(x\,w\,\kappa(v))\,dv$$

$$= \int_{V\times M} e^{-(\rho-\Lambda)\log a(\kappa(v)m_0)}\,e^{-2\rho H(v)}\,\xi(m(\kappa(v)m_0))\,f(x\,w\,\kappa(v)m_0)\,dm_0\,dv$$

$$= \int_K e^{-(\rho-\Lambda)\log a(k)}\,\xi(m(k))\,f(x\,w\,k)\,dk$$

$$= \int_K e^{-(\rho-\Lambda)\log a(w^{-1}k)}\,\xi(m(w^{-1}k))\,f(x\,k)\,dk \tag{3.5}$$

$$= \int_{V\times M} e^{-(\rho-\Lambda)\log a(w^{-1}\kappa(v)m_0)}\,\xi(m(w^{-1}\kappa(v)m_0))\,f(x\,\kappa(v)m_0)\,e^{-2\rho H(v)}\,dm_0\,dv$$

$$= \int_V e^{-(\rho-\Lambda)\log a(w^{-1}\kappa(v))}\,\xi(m(w^{-1}\kappa(v)))\,f(x\,\kappa(v))\,e^{-2\rho H(v)}\,dv$$

$$= \int_V e^{-(\rho-\Lambda)\log a(w^{-1}v)}\,e^{-(\rho+\Lambda)H(v)}\,\xi(m(w^{-1}v))\,f(x\,\kappa(v))\,dv \tag{3.6}$$

$$= \int_V e^{-(\rho-\Lambda)\log a(w^{-1}v)}\,\xi(m(w^{-1}v))\,f(x\,v)\,dv. \tag{3.7}$$

The left side of (3.4) is, of course, the formal expression for $A_p(w,\xi,\Lambda)f(x)$. Formulas (3.5) and (3.7) show how to write this expression as a convolution on K and V, respectively, and (3.6) is the useful form of the expression in the proof of the theorem below.

Theorem 3.1. *For the rank-one minimal case, let $\Lambda = z\rho$ and $\rho = \frac{1}{2}(p+2q)\alpha$. Suppose f is a C^∞ function in the space of the representation $U_p(\xi,\Lambda,\cdot)$, realized in the "compact picture", i.e., f is a C^∞ function on K satisfying $f(km) = \xi(m)^{-1}f(k)$ for k in K, m in M. Then*

(i) *for* $\operatorname{Re} z > 0$, $A_p(w,\xi,z\rho)f$ *is convergent,*

(ii) *for general* z, $A_P(w, \xi, z\rho)f$ *extends to a meromorphic function of* z *with at most simple poles at nonnegative integral multiples of* $-(p+2q)^{-1}$; *except on this set, the map* $(z, f) \to A_P(w, \xi, z\rho)f$ *is continuous from* $\mathbb{C} \times C^\infty$ *into* C^∞.

Conclusion (i) was observed in [28]. It follows from (3.4), the finiteness of $\int_V e^{-(1+\varepsilon)\rho H(v)}\,dv$ (see [10], p. 290), and the boundedness of $f(x w \kappa(v))$. Conclusion (ii) is Theorem 3 of [20] if G is linear connected semisimple, and the proof is the same for general G.

As soon as we have the convergence in conclusion (i), we obtain the intertwining property (1.10). By conclusion (ii), (1.10) extends to be valid for all Λ, as an identity of meromorphic functions (when applied to a C^∞ function f).

We now introduce the function $\eta_\xi(z)$, which was called $c_\xi(z)$ in [20]. The operator $A_P(w^{-1}, w\xi, w\Lambda) A_P(w, \xi, \Lambda)$ commutes with $U_P(\xi, \Lambda, \cdot)$, which by Bruhat's theorem ([5], p. 193) has no nonscalar self-intertwining operators continuous in the C^∞ topologies if Λ is imaginary. Letting $\Lambda = z\rho$ and noting that $w\Lambda = -\Lambda$, we have

$$A_P(w^{-1}, w\xi, -z\rho) A_P(w, \xi, z\rho) = \eta_\xi(z) I \tag{3.8}$$

for z imaginary. From Theorem 3.1 (ii), this identity extends to be valid for all z, with $\eta_\xi(z)$ meromorphic in $\mathbb{C}$.

The operator $A_P(w, \xi, z\rho)$ commutes with left translation by K, and each K-space of functions is finite-dimensional. In other words, there are disjoint finite-dimensional spaces left stable by $A_P(w, \xi, z\rho)$ whose sum is a dense subspace. In the sense of K-space by K-space, straightforward computation from (3.5) yields the adjoint formula

$$A_P(w, \xi, -\bar{\Lambda})^* = A_P(w^{-1}, w\xi, w\Lambda). \tag{3.9}$$

Proposition 3.2. *The function* $\eta_\xi(z)$ *has the following properties:*

(a) η_ξ *is independent of the choice of* w *and depends only on the class of* ξ,

(b) $\eta_\xi(z) = \overline{\eta_\xi(-\bar{z})}$,

(c) $\eta_\xi(z) \geqq 0$ *on the imaginary axis, and* $\eta_\xi(z)$ *is not identically* 0,

(d) $\eta_{w\xi}(-z) = \eta_\xi(z)$,

(e) $\eta_{\xi^\varphi}(z) = \eta_\xi(z)$ *if* φ *is an automorphism of* G *leaving* K *stable and the positive chamber of* A *fixed and if* $\xi^\varphi(m) = \xi(\varphi^{-1}(m))$,

(f) $\eta_\xi(z) = \eta_\xi(-z)$.

Proof. Conclusions (a), (b), (c), and (d) are contained in Proposition 27 of [20]. For (e), we can check directly that

$$(A_P(w, \xi, \Lambda) f)(\varphi^{-1}(k)) = A_P(\varphi(w), \xi^\varphi, \Lambda)(f \circ \varphi)(k),$$

and then (e) follows readily. For (f), we combine (d) and (e), using $\varphi(g) = w^{-1}(\theta g) w$.

Finally we define $A(\bar{P} : P : \xi : \Lambda)$ by means of (1.9), where $\bar{P} = MAV$. Namely

$$A(\bar{P}:P:\xi:\Lambda)=R(w)^{-1}A_P(w,\xi,\Lambda). \tag{3.10}$$

Proposition 3.3. *For the rank-one minimal case*

(i) $A(P:\bar{P}:\xi:\Lambda)=R(w^{-1})A(\bar{P}:P:w\xi:w\Lambda)R(w)$,

(ii) $A(P:\bar{P}:\xi:\Lambda)A(\bar{P}:P:\xi:\Lambda)=\eta_\xi(z)I$ if $\Lambda=z\rho_P$.

Proof. In (i), apply each side to f and evaluate at k, under the assumption that $\Lambda = -z\rho_P$ with $\operatorname{Re} z>0$. The left side is

$$\int_N f(kn)\,dn,$$

and the right side is

$$\int_V f(kw^{-1}vw)\,dv,$$

and these are equal by Lemma 2.6. Then (ii) follows from (i), (3.10), and (3.8).

§ 4. Existence of Operators, Higher-rank Minimal Case

We drop the assumption that G has $\dim \mathfrak{a}_p=1$. But since we shall work in this section only with minimal parabolic subgroups, we continue to omit the subscripts p.

The operators $A_P(w,\xi,\Lambda)$ and their normalizations were dealt with in [32] and [20], and the operator $A(P_2:P_1:\xi:\Lambda)$ may be defined in terms of them by (1.9). However, following a suggestion due to N. Wallach, we shall rederive the theory by dealing with $A(P_2:P_1:\xi:\Lambda)$ first; this approach is one that can be adapted easily to the case of nonminimal parabolics and will make it possible to omit a number of proofs in §6 and §7.

The inner product B_θ on $\mathfrak{g}$ induces an inner product on the dual $\mathfrak{a}'$ of $\mathfrak{a}$, which we denote by $\langle\cdot,\cdot\rangle$.

Recall that the formal expression for an intertwining operator is

$$A(P_2:P_1:\xi:\Lambda)f(x)=\int_{V_1\cap N_2} f(xv)\,dv$$

if $P_1=MAN_1$ and $P_2=MAN_2$. For a reduced $\mathfrak{a}$-root α, we recall the definition of $G^{(\alpha)}$ in (1.5) and of the subgroups $N^{(\alpha)}$ and $V^{(\alpha)}$.

Proposition 4.1. *Let $P_1=MAN_1$ and $P_2=MAN_2$ be minimal parabolic subgroups such that $V_1\cap N_2=V^{(\alpha)}$ for a P_1-positive reduced root α. Let f be in the C^∞ space for $U_{P_1}(\xi,\Lambda,\cdot)$, and let f_k be the restriction to $G^{(\alpha)}$ of the left translate of f by k in K. Then f_k is in the C^∞ space for $U_{P^{(\alpha)}}(\xi,\Lambda|_{\mathfrak{a}^{(\alpha)}},\cdot)$, and*

$$A(P_2:P_1:\xi:\Lambda)f(k)=A^{(\alpha)}(\theta P^{(\alpha)}:P^{(\alpha)}:\xi:\Lambda|_{\mathfrak{a}^{(\alpha)}})f_k(1). \tag{4.1}$$

Consequently $A(P_2:P_1:\xi:\Lambda)f$ is given by an absolutely convergent integral if $\langle \operatorname{Re}\Lambda,\alpha\rangle>0$, and it continues to a global meromorphic function in Λ. Moreover,

$$A(P_1:P_2:\xi:\Lambda)A(P_2:P_1:\xi:\Lambda)=\eta(P_2:P_1:\xi:\Lambda)I, \tag{4.2}$$

where η is the meromorphic function of Λ given by

$$\eta(P_2:P_1:\xi:\Lambda)=\eta_\xi^{(\alpha)}\left(\frac{\langle\Lambda,\rho^{(\alpha)}\rangle}{\langle\rho^{(\alpha)},\rho^{(\alpha)}\rangle}\right).$$

Proof. It is clear that f_k satisfies the appropriate transformation law, apart from the relationship between ρ_{P_1} and $\rho^{(\alpha)}$. Here $\rho_{P_1}H_{P_1}(v)=\rho^{(\alpha)}H^{(\alpha)}(v)$ by p. 399 of [28] or by Proposition 1.2. The two sides of (4.1) are, respectively,

$$\int_{V_1\cap N_2} f(kv)\,dv \quad\text{and}\quad \int_{V^{(\alpha)}} f(ku)\,du.$$

Since $V_1\cap N_2=V^{(\alpha)}$, formula (4.1) is a question of whether the normalizations of Haar measure, the one for $V_1\cap N_2\subseteq G$ and the one for $V^{(\alpha)}\subseteq G^{(\alpha)}$, are the same. But both normalizations are defined the same way, in terms of the integral of $\exp\{-2\rho^{(\alpha)}H^{(\alpha)}(v)\}$, and hence the two normalizations are the same.

In view of (4.1), the absolute convergence of the integral when $\langle\operatorname{Re}\Lambda,\alpha\rangle>0$ is a consequence of Theorem 3.1 (i), and the meromorphic continuation follows from Theorem 3.1 (ii). Formula (4.2) then follows from Proposition 3.3 (ii).

Let $P=MAN$ and $P'=MAN'$ be minimal parabolic subgroups. A sequence $P_i=MAN_i$, $0\leqq i\leqq r$, is called a *string* from P to P' if there are P-positive reduced $\mathfrak{a}$-roots α_i, $1\leqq i\leqq r$, such that

$$\begin{gathered} V_{i-1}\cap N_i=V^{(\alpha_i)} \quad\text{or}\quad N^{(\alpha_i)}, \qquad 1\leqq i\leqq r,\\ P_0=P \quad\text{and}\quad P_r=P'. \end{gathered}\tag{4.3}$$

The string P_i is called a *minimal string* from P to P' if

$$\begin{gathered} V_{i-1}\cap N_i=V^{(\alpha_i)}, \qquad 1\leqq i\leqq r,\\ P_0=P \quad\text{and}\quad P_r=P'. \end{gathered}\tag{4.4}$$

The parabolics P and P' can always be connected by a minimal string ([12], p. 145). Namely, we choose H and H' in the respective open positive Weyl chambers of $\mathfrak{a}$ for P and P' so that

$$H(t)=(1-t)H+tH', \qquad 0\leqq t\leqq 1,$$

is never annihilated by more than one P-positive reduced $\mathfrak{a}$-root for a given t. Let $0<t_1<\ldots<t_r<1$ be the values of t such that $H(t_i)$ is annihilated by some P-positive reduced $\mathfrak{a}$-root α_i. Let $P_i=MAN_i$ be associated to the Weyl chamber in which (t_i,t_{i+1}) lies, for $0\leqq i\leqq r$, with $t_0=0$ and $t_{r+1}=1$. Then P_i, $0\leqq i\leqq r$, is a minimal string from P to P'.

Theorem 4.2. *Suppose that $P=MAN$ and $P'=MAN'$ are minimal parabolic subgroups and $P_i=MAN_i$, $0\leqq i\leqq r$, is a minimal string from P to P', with associated reduced P-positive $\mathfrak{a}$-roots $\{\alpha_i\}$. Then*

(i) *the set $\{\alpha_i\}$ is characterized as the set of reduced $\mathfrak{a}$-roots α that are positive for P and negative for P'.*

(ii) *r is characterized as the number of roots described in* (i).

(iii) *the unnormalized intertwining operators satisfy*

$$A(P':P:\xi:\Lambda)=A(P_r:P_{r-1}:\xi:\Lambda)\cdot\ldots\cdot A(P_1:P_0:\xi:\Lambda),$$

they converge when $\langle \operatorname{Re}\Lambda, \alpha_i\rangle > 0$ *for each* i, *and they have global meromorphic continuations in* Λ *that satisfy the intertwining identity* (1.8). *The meromorphic continuation of* $A(P':P:\xi:\Lambda)$ *is holomorphic at* Λ_0 *unless*

$$\frac{2\langle \Lambda_0, \alpha\rangle}{|\alpha|^2}\in\mathbb{Z} \tag{4.5}$$

for some reduced $\mathfrak{a}_p$*-root* α *that is positive for* P *and negative for* P'.

(iv) P_{r-i}, $0\leqq i\leqq r$, *is a minimal string from* P' *to* P, *with associated reduced* P'*-positive* $\mathfrak{a}$*-roots* $\{-\alpha_i\}$.

(v) $A(P:P':\xi:\Lambda)A(P':P:\xi:\Lambda)=\eta(P':P:\xi:\Lambda)I$, *where* η *is the scalar-valued function meromorphic in* Λ *given by*

$$\eta(P':P:\xi:\Lambda)=\prod_{\substack{\alpha\ \text{reduced}\\ \alpha>0\ \text{for}\ P\\ \alpha<0\ \text{for}\ P'}}\eta_\xi^{(\alpha)}\left(\frac{\langle \Lambda, \rho^{(\alpha)}\rangle}{\langle \rho^{(\alpha)}, \rho^{(\alpha)}\rangle}\right).$$

The function η *is holomorphic at* Λ_0 *unless* (4.5) *holds for some* $\mathfrak{a}$*-root* α *that is positive for* P *and negative for* P', *and it satisfies*

$$\eta(P:P':\xi:\Lambda)=\eta(P':P:\xi:\Lambda). \tag{4.6}$$

Proof. First we show by downward induction on i that

$$\begin{aligned}&\{\delta=\text{reduced }\mathfrak{a}\text{-root}\mid \mathfrak{g}_\delta\subseteq\mathfrak{n}_i\cap\mathfrak{n}\}\\ &\quad=\{\delta=\text{reduced }\mathfrak{a}\text{-root}\mid \mathfrak{g}_\delta\subseteq\mathfrak{n}'\cap\mathfrak{n}\}\cup\{\alpha_r\}\cup\ldots\cup\{\alpha_{i+1}\}\end{aligned} \tag{4.7}$$

disjointly. Formula (4.7) is clear for $i=r$. Assume (4.7) holds inductively for $i=j+1$; we prove (4.7) holds for $i=j$. Then we have

$$\mathfrak{n}_j\cap\mathfrak{n}=(\mathfrak{n}_j\cap\mathfrak{v}_{j+1}\cap\mathfrak{n})+(\mathfrak{n}_j\cap\mathfrak{n}_{j+1}\cap\mathfrak{n})\subseteq\mathfrak{n}^{(\alpha_{j+1})}+(\mathfrak{n}_{j+1}\cap\mathfrak{n}),$$

which says that the left side of (4.7) is contained in the right side for $i=j$. Also

$$\mathfrak{n}_{j+1}\cap\mathfrak{n}=(\mathfrak{n}_{j+1}\cap\mathfrak{v}_j\cap\mathfrak{n})+(\mathfrak{n}_{j+1}\cap\mathfrak{n}_j\cap\mathfrak{n})\subseteq 0+(\mathfrak{n}_j\cap\mathfrak{n})$$

and

$$\mathfrak{n}^{(\alpha_{j+1})}=\mathfrak{n}_j\cap\mathfrak{v}_{j+1}\subseteq\mathfrak{n}_j,$$

which says that the right side of (4.7) is contained in the left side for $i=j$. If the union stopped being disjoint for $i=j$, we would have

$$\mathfrak{n}^{(\alpha_{j+1})}\subseteq\mathfrak{n}_{j+1}\cap\mathfrak{n},$$

but this inclusion contradicts the inclusion $\mathfrak{n}^{(\alpha_{j+1})}\subseteq\mathfrak{v}_{j+1}$. Thus the induction goes through, and (4.7) holds.

Taking $i=0$ in (4.7) and identifying the left side as the set of reduced $\mathfrak{a}$-roots δ that are positive for P and the right side as the set of reduced $\mathfrak{a}$-roots δ that are positive for both P and P', we obtain conclusion (i) of the theorem. Then (ii) is immediate.

For (iii), if we disregard convergence questions, we can use (4.7) to see that $\mathfrak{n}_i \cap \mathfrak{n} \subseteq \mathfrak{n}_{i-1} \cap \mathfrak{n}$. Applying Proposition 2.3 inductively on i downward, with

$$MAN, \quad MAN_{i-1}, \quad \text{and} \quad MAN_i$$

as the three parabolics in the proposition, we obtain

$$\begin{aligned}\int_{V \cap N'} f(xv)\,dv &= \int_{(V_{r-1} \cap N_r) \times (V \cap N_{r-1})} f(xv_r v)\,dv_r\,dv \\ &= \int_{V^{(\alpha_r)} \times (V \cap N_{r-1})} f(xv_r v)\,dv_r\,dv \\ &= \ldots = \int_{V^{(\alpha_r)} \times \ldots \times V^{(\alpha_1)}} f(xv_r \ldots v_1)\,dv_r \ldots dv_1,\end{aligned}$$

which is formally the identity in conclusion (iii). To deal with the convergence, we note that if f is in the space for $U_P(\xi, \Lambda, \cdot)$, then $|f|$ is in the space for $U_P(1, \operatorname{Re}\Lambda, \cdot)$. We apply our formal computation to $|f|$, and the i^{th} integration on the right produces a finite result since $\langle \operatorname{Re}\Lambda, \alpha_i \rangle > 0$. (Here the relevant part of the proof of Proposition 4.1 is valid, even though the function acted on by the intertwining operator need not be C^∞.) Thus the integral on the left is absolutely convergent, and our formal computation is justified. As soon as we have convergence, the intertwining identity (1.8) is valid, and the meromorphic continuation follows from the continuation of each factor of (iii), known from Proposition 4.1. The singularities of $A(P':P:\xi:\Lambda)$ are limited in location to (4.5) by the product decomposition of the operator and by Theorem 3.1. This proves (iii).

For (iv), let $Q_i = P_{r-i}$. Then

$$\begin{aligned}V_{Q_{i-1}} \cap N_{Q_i} &= V_{r-i+1} \cap N_{r-i} = \theta(N_{r-i+1} \cap V_{r-i}) \\ &= \theta V^{(\alpha_{r-i+1})} = V^{(-\alpha_{r+1-i})}, \qquad 1 \leqq i \leqq r.\end{aligned}$$

This proves (iv). For (v), we expand the two factors on the left by means of (iii) and (iv) and then collapse pairs of factors (starting from the center) by means of (4.2). This proves the product formula. In (4.6), at any Λ where either side of (4.6) is nonzero the two operators $A(P':P:\xi:\Lambda)$ and $A(P:P':\xi:\Lambda)$ must commute with each other on each K-space, hence globally. Then (4.6) follows. This proves (v).

Lemma 4.3. *Suppose that f is a C^∞ function on K such that*

$$f(km) = \xi(m)^{-1} f(k) \quad \text{for } k \text{ in } K,\ m \text{ in } M,$$

and such that $\Lambda \to A(P':P:\xi:\Lambda)f$ fails to be holomorphic at $\Lambda = \Lambda_0$. Let $f = \sum f_\tau$ be the Fourier expansion of f on K, with f_τ the projection of f to the space of the K-type τ. Then $\Lambda \to A(P':P:\xi:\Lambda)f_\tau$ fails to be holomorphic at $\Lambda = \Lambda_0$ for some τ.

Proof. Expand $A(P':P:\xi:\Lambda)$ as in Theorem 4.2(iii) and let $A(P_i:P_{i-1}:\xi:\Lambda)$ be the term corresponding to the P-positive reduced root α. By Theorem 3 of [20] and Proposition 4.1, the operator

$$\langle\Lambda-\Lambda_0,\alpha\rangle A(P_i:P_{i-1}:\xi:\Lambda)$$

is holomorphic at Λ_0 on all of C^∞ and is continuous on C^∞. Consequently

$$\Big\{\prod_{\substack{\alpha \text{ reduced}\\ \alpha>0 \text{ for } P\\ \alpha<0 \text{ for } P'}}\langle\Lambda-\Lambda_0,\alpha\rangle\Big\}A(P':P:\xi:\Lambda) \tag{4.7}$$

is holomorphic at Λ_0 on all of C^∞ and is continuous on C^∞. Applying (4.7) to both sides of the identity $f=\sum f_\tau$, we obtain an equality of holomorphic functions near Λ_0:

$$\begin{aligned}&\{\prod\langle\Lambda-\Lambda_0,\alpha\rangle\}A(P':P:\xi:\Lambda)f\\&\quad=\sum\{\prod\langle\Lambda-\Lambda_0,\alpha\rangle\}A(P':P:\xi:\Lambda)f_\tau.\end{aligned} \tag{4.8}$$

Assuming by way of contradiction that our operator is holomorphic at Λ_0 on K-finite functions, we see that the right side of (4.8) is 0 term by term for all Λ near Λ_0 such that $\prod\langle\Lambda-\Lambda_0,\alpha\rangle=0$. The factors in $\prod\langle\Lambda-\Lambda_0,\alpha\rangle$ are of the first order, and their zero varieties are distinct. From the theory of one complex variable, we may divide through the right side of (4.8) by one factor at a time, retaining the holomorphic behavior. Thus the left side of (4.8) is holomorphic at Λ_0 if the factor $\prod\langle\Lambda-\Lambda_0,\alpha\rangle$ is dropped, and we have arrived at a contradiction.

At this time we could derive further properties of the operators $A(P':P:\xi:\Lambda)$, and we could introduce their normalizations, as well as the operators $A_P(w,\xi,\Lambda)$ and their normalizations. But these further properties will not be needed to develop the operators for nonminimal parabolics, and consequently we shall obtain the further properties as special cases of the theory for nonminimal parabolics.

§ 5. Subrepresentation Theorem

Casselman [6] has proved the following subrepresentation theorem. A detailed exposition has been given by Miličić [31].

Theorem 5.1. (Casselman). *Let G be a connected semisimple group with finite center, and let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. Let ξ be an irreducible admissible representation of G on a Banach space H^ξ, and let H^ξ_K be the subspace of K-finite vectors. Then there exists a nonunitary principal series representation $U(\sigma,\Lambda_0,\cdot)$ such that H^ξ_K imbeds in $\mathscr{U}(\mathfrak{g})$-equivariant fashion in the space of K-finite vectors for $U(\sigma,\Lambda_0,\cdot)$.*

In unpublished work, Casselman and Miličić have extended this result to all groups satisfying the basic axioms of §1. We shall give such an extension here, providing a proof that starts from Casselman's theorem and keeps track of the parameters in the imbedding. We need the result only for irreducible unitary representations and shall limit ourselves to that context. The result we seek is Theorem 5.4. Before stating the result, we put matters into perspective with Lemma 5.3 below, giving a proof based on Lemma 5.2, which is well known.

Lemma 5.2. *Let ξ be an irreducible unitary representation on a Hilbert space X of a group G satisfying the axioms of §1. For each z in the center $\mathscr{Z}$ of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$, $\xi(z)$ acts as a scalar on the C^∞ space of X.*

Proof. Since $\mathrm{Ad}(G)$ is contained in the connected adjoint group of the complexification $\mathfrak{g}^{\mathbb{C}}$, $\mathrm{Ad}(G)$ fixes every element z of $\mathscr{Z}$. It follows then from standard arguments that $\xi(z)$ is scalar on the C^∞ space of X.

Lemma 5.3.[3] *Let ξ be an irreducible unitary representation on a Hilbert space X of a group G satisfying the axioms of §1. Then under $\xi|_{G_0}$, X admits a splitting into the finite orthogonal sum of irreducible closed subspaces, with at most $[G:G_0]$ terms in the orthogonal sum.*

Proof. By Lemma 5.2, $\xi|_{G_0}$ is quasisimple. We shall apply Corollary 2, p. 229, of [9], which is proved in [9] for connected semisimple groups and is easily seen to be valid for G_0. By this corollary, there exist closed G_0-invariant subspaces $V \subseteq U$ such that $\xi|_{G_0}$ is irreducible on U/V. Since ξ is unitary, $F = U \cap V^\perp$ is a closed subspace of X invariant and irreducible under G_0.

Let g_i, $1 \leqq i \leqq n$, be coset representatives of G/G_0. Since G_0 is normal, $E_i = \xi(g_i)F$ is another closed irreducible subspace under G_0. Since ξ is irreducible, it follows that $\sum_{i=1}^{n} E_i$ is dense in X. Here the K_0-finite vectors of E_i are irreducible under $\mathfrak{g}$ and are dense in E_i. If P_1 denotes the orthogonal projection on $E_1^\perp$, then P_1 is a bounded G_0-intertwining operator from E_i into $E_1^\perp$. So

$$\sum E_i = E_1 + P_1 E_2 + \dots + P_1 E_n,$$

and the K_0-finite vectors in $P_1 E_i$, $2 \leqq i \leqq n$, are dense in $P_1 E_i$ and irreducible (or 0) under $\mathfrak{g}$. Let P_2 be the orthogonal projection on $(P_1 E_2)^\perp$. Proceeding as above, we obtain

$$\sum E_i = E_1 + P_1 E_2 + P_2 P_1 E_3 + \dots + P_2 P_1 E_n,$$

with each space on the right invariant under G_0 and such that its K_0-finite vectors are dense in the space and are irreducible (or 0) under $\mathfrak{g}$. In similar fashion, we obtain

$$\sum E_i = E_1 + P_1 E_2 + P_2 P_1 E_3 + P_3 P_2 P_1 E_4 + \dots + P_{n-1} \cdot \dots \cdot P_1 E_n.$$

The closure of each space on the right is irreducible under G_0. Since the sum is orthogonal, the sum of the closures is the closure of the sum. Then X is

[3] This is a relatively easy special case of Lemma 1.16 of Miličić [38]: If G is a locally compact group and H is an open subgroup of finite index and ξ is an irreducible unitary representation on X, then, under $\xi|_H$, X admits a splitting into the finite orthogonal sum of irreducible closed subspaces, with at most $[G:H]$ terms.

exhibited as the orthogonal sum of at most n closed G_0-irreducible subspaces, as required.

It follows from Lemma 5.3 that, in an irreducible unitary representation of G, the K-finite vectors for each K-type form a finite-dimensional space and are necessarily analytic vectors. We can therefore form an infinitesimal representation in the subspace of K-finite vectors; this consists of the algebra action by $\mathscr{U}(\mathfrak{g})$ and the group action by K, and the two are consistent in the obvious fashion. The space of K-finite vectors is irreducible in the sense that there are no proper subspaces invariant under $\mathscr{U}(\mathfrak{g})$ and K.

Theorem 5.4 (Casselman-Miličić). *Let G be a group satisfying the axioms of* §1, *let ξ be an irreducible unitary representation of G on a Hilbert space H^ξ, and let H_K^ξ be the subspace of K-finite vectors. Then there exists a nonunitary principal series representation $U(\sigma, \Lambda_0, \cdot)$ such that H_K^ξ imbeds in the space of K-finite vectors for $U(\sigma, \Lambda_0, \cdot)$, equivariantly with respect to $\mathfrak{g}$ and K.*

Proof. Let MAN be a minimal parabolic subgroup of G. Then $(M \cap G_0)AN$ is a minimal parabolic subgroup of G_0. We have $G_0 = Z_{G_0} \cdot G_s$, where Z_{G_0} is the compact center of G_0 and G_s is the (semisimple) commutator subgroup of G_0. The group $(M \cap G_s)AN$ is a minimal parabolic subgroup of G_s, and $(M \cap G_0) = Z_{G_0} \cdot (M \cap G_s)$.

By Lemma 5.3, we can choose a constituent ξ_0 of ξ, irreducible under G_0. Then Z_{G_0} acts as scalars under ξ_0 (say $\xi_0(z) = \chi(z)I$), and $\xi_0|_{G_s}$ is irreducible. By Theorem 5.1, we form an equivariant imbedding of the $(K \cap G_s)$-finite vectors of $\xi_0|_{G_s}$ in a nonunitary principal series $U(\sigma_s, \Lambda_0, \cdot)$ of G_s. The latter representation extends to a representation of G_0 given as $U(\sigma_s, \Lambda_0, \cdot)$ on G_s and the scalar $\chi(z)$ on z in Z_{G_0}, and this extended representation is easily seen to be canonically identified with $U(\sigma_0, \Lambda_0, \cdot)$ of G_0, where

$$\sigma_0 = \begin{cases} \sigma_s & \text{on } (M \cap G_s) \\ \chi & \text{on } Z_{G_0}. \end{cases}$$

Thus we have the required result for G_0: ξ_0 imbeds infinitesimally in $U(\sigma_0, \Lambda_0, \cdot)$.

Now we pass to G. By Theorem 4′ of [30], Frobenius reciprocity holds for G and G_0, provided we count only discrete occurrences of representations. The formula gives

$$[\xi|_{G_0} : \xi_0] = [\operatorname*{ind}_{G_0 \uparrow G} \xi_0 : \xi].$$

Since the left side is nonzero, ξ occurs in $\operatorname*{ind}_{G_0 \uparrow G} \xi_0$, which in turn imbeds in equivariant fashion in $\operatorname*{ind}_{G_0 \uparrow G} U(\sigma_0, \Lambda_0, \cdot)$. This last representation is canonically identified with $U(\operatorname*{ind}_{(M \cap G_0) \uparrow M} \sigma_0, \Lambda_0, \cdot)$, which in turn is identified with a direct sum

$$\sum_{i=1}^{n} U(\sigma_i, \Lambda_0, \cdot),$$

where

$$\operatorname*{ind}_{(M\cap G_0)\uparrow M} \sigma_0 = \sum_{i=1}^{n} \sigma_i,$$

σ_i irreducible. Since ξ imbeds in the direct sum and is irreducible, it must imbed into one of the factors, by projection. This proves the theorem.

In Lemma 5.2 we saw that the center $\mathscr{Z}$ of $\mathscr{U}(\mathfrak{g})$ acts by scalars, for an irreducible unitary representation, and thereby defines a character of $\mathscr{Z}$. It is well known that such characters are classified by complex-linear functionals on a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, modulo the action of the complex Weyl group.

Lemma 5.5. *Let the irreducible unitary representation ξ of G imbed in the nonunitary principal series representation of G with parameters (σ, Λ_0). If the infinitesimal character of ξ is a real linear combination of roots, then Λ_0 is real.*

Proof. The infinitesimal character of ξ must match that of the nonunitary principal series representation, which is $\chi_{\Lambda^- + \rho^- + \Lambda_0}$, where Λ^- is the highest weight of σ and ρ^- is half the sum of the positive roots of M_p. Since the infinitesimal character of ξ has been assumed real, it follows that Λ_0 is real.

§6. Existence of Operators, General Case

We turn to the general case with $P=MAN$ a parabolic subgroup, and we deal with MAN and its associated parabolics. We fix an Iwasawa decomposition $M = K_M A_M N_M$ of M such that K_M is $K\cap M$ and $\mathfrak{a}_M$ is in $\mathfrak{p}$. Then $G=KA_pN_p$ is an Iwasawa decomposition of G if we put $\mathfrak{a}_p = \mathfrak{a} \oplus \mathfrak{a}_M$ and $\mathfrak{n}_p = \mathfrak{n} \oplus \mathfrak{n}_M$, as in §1.

We shall construct intertwining operators that go with representations induced from P and associated parabolics. An approach that works when the representation ξ of M is in the discrete series is to connect intertwining operators with Harish-Chandra's c-functions (cf. §9). However, we shall follow a different approach, for two reasons:

(1) there would be no evident way of handling a representation ξ that is not in the discrete series

(2) there would be no clear way of dealing with functions in the induced space that are not K-finite, and such functions are critical to the proof of the linear independence in Theorem 12.1.

The different approach is to use an imbedding of ξ in the nonunitary principal series of M, by means of Theorem 5.4 applied to M. On an algebraic level, we shall see that

$$U_P(\xi, \Lambda, \cdot) = \operatorname*{ind}_{MAN\uparrow G} (\xi \otimes e^{\Lambda} \otimes 1)$$

then imbeds in the nonunitary principal series of G and its intertwining operators are identified with restrictions of the operators constructed in §4 and §5. The results in this section were announced in [22].

Let ξ be an irreducible unitary representation of M, and let $P_1 = MAN_1$ and $P_2 = MAN_2$ be given. The formal integral for the intertwining operator is

$$A(P_2:P_1:\xi:\Lambda)F(x) = \int_{V_1 \cap N_2} F(xv)\,dv. \tag{6.1}$$

By means of Theorem 5.4 applied to M, let the space $H^{\xi}_{K_M}$ of K_M-finite vectors in ξ imbed by a mapping ι in the space $H^{\omega}_{K_M}$ of K_M-finite vectors of

$$\omega = \operatorname*{ind}_{M_M A_M N_M \uparrow M} (\sigma \otimes e^{\Lambda_M} \otimes 1),$$

with the K_M and $\mathscr{U}(\mathrm{m})$ actions equivariant. Here ω acts on the space[4]

$$H^{\omega} = \{f: M \to H^{\sigma} \mid f(x m_M a_M n_M) = e^{-(\Lambda_M + \rho_M)\log a_M}\,\sigma(m_M)^{-1} f(x)\},$$

whose norm is the L^2 norm on K_M. Form

$$U_P(\omega, \Lambda, \cdot) = \operatorname*{ind}_{MAN \uparrow G} (\omega \otimes e^{\Lambda} \otimes 1)$$

acting on

$$H^{U_P} = \{F: G \to H^{\omega} \mid F(xman) = e^{-(\Lambda + \rho_P)\log a}\,\omega(m)^{-1} F(x)\} \tag{6.2}$$

with norm squared

$$\int_K |F(k)|^2_{H^{\omega}}\,dk.$$

Evaluation $F(\cdot) \to F(\cdot)(1)$ exhibits $U_P(\omega, \Lambda, \cdot)$ as equivalent via a unitary mapping with the nonunitary principal series representation

$$\operatorname*{ind}_{M_p A_p N_p \uparrow G} (\sigma \otimes e^{(\Lambda_M \oplus \Lambda)} \otimes 1).$$

(This is routine to check and uses the equalities $M_M = M_p$ and $\rho_p = \rho + \rho_M$.)

Now we pass to intertwining operators. In terms of N_1 and N_2, we let

$$(N_1)_p = N_1 N_M \quad \text{and} \quad (N_2)_p = N_2 N_M,$$

so that

$$(V_1)_p = V_1 V_M \quad \text{and} \quad (V_2)_p = V_2 V_M.$$

Then the set of integration for (6.1) is

$$V_1 \cap N_2 = (V_1)_p \cap (N_2)_p.$$

Therefore, using our evaluation map, we see that we have an equality of two formal expressions:

$$\begin{aligned} &A(P_2:P_1:\omega:\Lambda)F(x)(m) \\ &\quad = (A((P_2)_p:(P_1)_p:\sigma:\Lambda_M \oplus \Lambda)F(\cdot)(1))(xm). \end{aligned} \tag{6.3}$$

[4] To avoid measure-theoretic complications, we may stick to smooth functions.

The expression on the right has a definition and analytic continuation whenever $F(\cdot)(1)$ is smooth, and we can therefore use it to define the left side of (6.3) for *smooth* F and give an analytic continuation for it. (The left side of (6.3) is not identically singular in Λ; we shall identify in Theorem 6.6 a dense set of Λ in which it is holomorphic.)

Next we bring in ξ. For now, we shall be content with the effect of $A(P_2:P_1:\xi:\Lambda)$ on K-finite functions.[5] We have

$$\imath(H^{\xi}_{K_M}) \subseteq H^{\omega}_{K_M}.$$

Let

$$H^{\omega}_0 = \text{closure of } \imath(H^{\xi}_{K_M}) \text{ in norm topology of } H^{\omega}.$$

The space H^{ω}_0 is ω-stable under M_0 and K_M, by [9], hence is stable under $\omega(M)$. (But beware: *A priori*, the C^∞ vectors of H^{ω}_0 need not be related to C^∞ vectors of H^{ξ}, since no continuity of $\imath$ has been established.)

Let

$$\begin{aligned} C^\infty(H^U_0) &= \{\text{smooth } F \text{ in } H^U \mid F(x) \text{ is in } H^{\omega}_0 \text{ for all } x \text{ in } G\} \\ &= \{\text{smooth } F \text{ in } H^U \mid F(k) \text{ is in } H^{\omega}_0 \text{ for all } k \text{ in } K\}. \end{aligned}$$

Lemma 6.1. *The closure of $C^\infty(H^U_0)$ in the norm topology of H^U yields a representation of G in a Hilbert space, and the C^∞ subspace is $C^\infty(H^U_0)$. The topology that $C^\infty(H^U_0)$ gets as the C^∞ subspace is the same as the topology it inherits as a subset of the C^∞ subspace $C^\infty(H^U)$ of H^U.*

Proof. $C^\infty(H^U_0)$ is G-invariant, and hence so is its closure $\operatorname{cl} C^\infty(H^U_0)$. If $g \to U(g) F_0$ is C^∞ and F_0 is in the closure, then F_0 is in $C^\infty(H^U) \cap \operatorname{cl} C^\infty(H^U_0)$. Let l be any continuous linear functional on H^{ω} that vanishes on H^{ω}_0 and let h be in $L^2(K, \mathbb{C})$. Then

$$F \to \int_K h(k) F(k)\, dk \to l\Big(\int_K h F\, dk\Big)$$

is a composition of continuous functions in the norm topology and so vanishes on F_0. Peaking h, we obtain $l(\text{image } F_0) = 0$. Allowing l to vary, we obtain image $F_0 \subseteq H^{\omega}_0$. Hence F_0 is in $C^\infty(H^U_0)$.

In the reverse direction, if F_0 is in $C^\infty(H^U_0)$, then F_0 is in $C^\infty(H^U)$ and $g \to U(g) F_0$ is C^∞. Hence F_0 is a C^∞ vector.

To see that the topologies coincide,[6] we note that the representations in question are quasisimple. A Laplacian on G can therefore be given in terms of the scalar Casimir operator and a Laplacian Δ on K. This means that both topologies in question are given in terms of seminorms $\|\Delta^n F\|$, and they coincide.

[5] It will be crucial in §12 and in Appendix B to consider a wider class of functions than the K-finite functions. The wider class that we use will be smooth functions on K that transform appropriately under K_M and have their values in a finite-dimensional K_M-finite subspace of H^{ξ}.

[6] This style of argument for dealing with C^∞ topologies was pointed out to us by Roe Goodman.

Lemma 6.2. *The imbedding ι induces an imbedding $\iota^{\#}$ of the K-finite space for* $U_P(\xi, \Lambda, \cdot)$ onto *the K-finite space for $U_P(\omega, \Lambda, \cdot)|_{\mathrm{cl}C^\infty(H_0^\omega)}$. The map $\iota^{\#}$ is equivariant with respect to K and* $\mathfrak{g}$.

Proof. The K-finite space for $U_P(\xi, \Lambda, \cdot)$ consists of

$$\{F\colon G \to H^\xi \mid F(xman) = e^{-(\rho+\Lambda)\log a}\xi(m)^{-1}F(x) \text{ and } \dim\{\operatorname*{span}_{k\in K} F(k\,\cdot)\} < \infty\}.$$

These functions are determined by their restrictions to K, which satisfy

$$F(km) = \xi(m)^{-1}F(k) \quad \text{for } k\in K,\ m\in K\cap M$$

and

$$\dim\{\operatorname*{span}_{k\in K} F(k\,\cdot)\} < \infty.$$

For such an F, $F(k)$ is in $H^\xi_{K_M}$ because

$$\begin{aligned}\operatorname*{span}_{m\in K\cap M} \xi(m)^{-1}(F(k)) &= \operatorname*{span}_{m\in K\cap M} F(km)\\ &= \mathrm{eval}_1\{\operatorname*{span}_{m\in K\cap M} F(km\,\cdot)\} \subseteq \mathrm{eval}_1\{\operatorname*{span}_{k\in K} F(k\,\cdot)\}.\end{aligned}$$

Hence these F are characterized as

$$\left\{F: K \to H^\xi_{K_M} \,\middle|\, \begin{array}{l} F(km) = \xi(m)^{-1}F(k) \text{ for } k\in K,\ m\in K_M \\ \dim\{\operatorname*{span}_{k\in K} F(k\,\cdot)\} < \infty \end{array}\right\}. \tag{6.4}$$

Similarly the K-finite space for $U_P(\omega, \Lambda, \cdot)|_{\mathrm{cl}C^\infty(H_0^\omega)}$ is

$$\left\{F: K \to (H_0^\omega)_{K_M} \,\middle|\, \begin{array}{l} F(km) = \omega(m)^{-1}F(k) \text{ for } k\in K,\ m\in K_M \\ \dim\{\operatorname*{span}_{k\in K} F(k\,\cdot)\} < \infty \end{array}\right\}. \tag{6.5}$$

If F is in the space (6.4), then $\iota^{\#}F = \iota\circ F$ is in the space (6.5), since ι is K_M-equivariant. It is obvious that $\iota^{\#}$ is K-equivariant. For the $\mathfrak{g}$-equivariance, let X be in $\mathfrak{g}$. Then we have

$$\begin{aligned}U_P(\xi, \Lambda, X)F(k) &= \frac{d}{dt}F((\exp tX)^{-1}k)|_{t=0}\\ &= \frac{d}{dt}F(k\exp(-t\operatorname{Ad}(k^{-1})X))|_{t=0}.\end{aligned}$$

Thus if we decompose $-\operatorname{Ad}(k^{-1})X$ according to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we see that that above expression is given in terms of differentiation on K, the transformation law of $\xi(\mathfrak{m})$, and the transformation law for $\mathfrak{a}$. These are compatible with the corresponding formula for $U_P(\omega, \Lambda, X)$, and thus $\iota^{\#}$ is $\mathfrak{g}$-equivariant. Finally $\iota^{\#}$ is one-one onto since $(\iota^{\#})^{-1}F = \iota^{-1}\circ F$ provides a two-sided inverse.

For F in the K-finite space of $U_{P_1}(\xi, \Lambda, \cdot)$, we define

$$A(P_2 : P_1 : \xi : \Lambda)F = (\iota^{\#})^{-1}A(P_2 : P_1 : \omega : \Lambda)\,\iota^{\#}F. \tag{6.6}$$

The right side is meaningful, in view of the following lemma.

Lemma 6.3. *$A(P_2:P_1:\omega:\Lambda)$ carries the space $C^\infty(H_0^U)$, with transformation laws according to P_1, ω, and Λ, continuously into $C^\infty(H_0^U)$, with transformation laws according to P_2, ω, and Λ, provided Λ is a regular value.*

Proof. By (6.3) and Theorem 4.2(iii), $A(P_2:P_1:\omega:\Lambda)F(k)$ is given by a convergent integral of values $F(kv)$, provided Λ is in a suitable region. Each $F(kv)$ is in H_0^ω and hence so is the integral. Then $A(P_2:P_1:\omega:\Lambda)F(k)$ must be in H_0^ω for all Λ, by analytic continuation. The continuity of the intertwining operator follows easily from Lemma 6.1 and the technique at the end of its proof.

Formula (6.6) gives us a definition and analytic continuation for the left side of (6.6) for *K-finite* F. Before deriving the basic properties of the operators, we shall show that this definition is independent of the choice of nonunitary principal series ω into which ξ is imbedded. We do so by giving an intrinsic definition for the left side of (6.6) by means of a convergent integral for Λ in a suitable region; the definition extends to be intrinsic for all Λ because of the analyticity that has already been proved.

Lemma 6.4. *The trivial representation of M is imbedded as a subspace of the nonunitary principal series when $\sigma=1$ and $\Lambda_M=-\rho_M$.*

Proof. The space for this induced representation of M is

$$\begin{aligned}&\{f: M\to\mathbb{C}\mid f(mm_M a_M n_M)=e^{-(\rho_M+\Lambda_M)\log a_M} f(m)\}\\ &=\{f: M\to\mathbb{C}\mid f \text{ is right invariant under } M_M A_M N_M\},\end{aligned}$$

and the space of constant functions is an M-invariant subspace.

Lemma 6.5. *For each $\mathfrak{a}$-root β, let*

$$c_\beta=\max\{\rho_M(H_\alpha)\}, \tag{6.7}$$

where the maximum is taken over all $\mathfrak{a}_p$-roots α such that $\alpha|_{\mathfrak{a}}=\beta$. If MAN and MAN' are associated parabolics, then $\exp(-(\Lambda+\rho_P)H_P(v))$ is integrable on $V\cap N'$, provided $\langle \operatorname{Re}\Lambda,\beta\rangle>c_\beta$ for all β with $\mathfrak{g}_\beta\subseteq\mathfrak{n}\cap\mathfrak{v}'$.

Remark. For a minimal parabolic, each c_β should be interpreted as 0.

Proof. Taking note of Lemma 6.4, we apply our theory to the representation $U_P(1,\Lambda,\cdot)$ of G. The function $F: G\to\mathbb{C}$ given by

$$F(x)=\exp(-(\Lambda+\rho_P)H_P(x))$$

is in the space of this representation, and $\iota^\# F(x)$ is the constant function on M given by

$$\iota^\# F(x)(m)=F(x).$$

By (6.3), we have, for k in K and m in K_M,

$$\begin{aligned}(A(P':P:\omega:\Lambda)\,\iota^{\#}F)(k)(m)\\ =(A(P'_p:P_p:1:-\rho_M\oplus\Lambda)\,\iota^{\#}F(\cdot)(1))(km),\end{aligned}$$

with the expression on the right given by the absolutely convergent integral

$$\begin{aligned}&=\int_{V_p\cap N'_p}\iota^{\#}F(kmv)(1)\,dv=\int_{V\cap N'}F(kmv)\,dv\\ &=\int_{V\cap N'}\exp(-(\Lambda+\rho_P)H_P(v))\,dv,\end{aligned}\tag{6.8}$$

provided the condition of Theorem 4.2(iii) is satisfied,

$$\langle\operatorname{Re}(-\rho_M+\Lambda),\alpha\rangle>0,$$

for every $\mathfrak{a}_p$-root α such that $\mathfrak{g}_\alpha\subseteq\mathfrak{n}_p\cap\mathfrak{v}'_p$. The α's with $\mathfrak{g}_\alpha\subseteq\mathfrak{n}_p\cap\mathfrak{v}'_p=\mathfrak{n}\cap\mathfrak{v}'$ are the α's with $\alpha|_{\mathfrak{a}}=\beta$ and $\beta\subseteq\mathfrak{n}\cap\mathfrak{v}'$. For such an α, our hypothesis gives

$$\begin{aligned}\langle\operatorname{Re}(-\rho_M+\Lambda),\alpha\rangle&=-\rho_M(H_\alpha)+\langle\operatorname{Re}\Lambda,\beta\rangle\\ &\geqq -c_\beta+\langle\operatorname{Re}\Lambda,\beta\rangle>0.\end{aligned}$$

Hence the integral on the right of (6.8) is indeed convergent.

Theorem 6.6. *Let $P_1=MAN_1$ and $P_2=MAN_2$ be associated parabolics, and suppose that $\langle\operatorname{Re}\Lambda,\beta\rangle>c_\beta$ for every $\mathfrak{a}$-root β such that $\mathfrak{g}_\beta\subseteq\mathfrak{n}_1\cap\mathfrak{v}_2$, where c_β is given by (6.7). If F is a smooth function in the space for $U_{P_1}(\xi,\Lambda,\cdot)$, then the integral*

$$\int_{V_1\cap N_2}F(xv)\,dv$$

is a convergent H^ξ-valued integral. If, in addition, F is K-finite, then

$$A(P_2:P_1:\xi:\Lambda)F(x)=\int_{V_1\cap N_2}F(xv)\,dv.\tag{6.9}$$

In this case, if the restriction of F to K is regarded as a member of a space independent of Λ and if x is in K, then the integral has an analytic continuation to a global meromorphic function in Λ. Moreover, there exist a rational number c and finitely many complex numbers $d_i(\xi)$ such that the meromorphic continuation of $A(P_2:P_1:\xi:\Lambda)$ is holomorphic at Λ_0 unless

$$\frac{2\langle\Lambda_0,\beta\rangle}{|\beta|^2}\in c\mathbb{Z}+d_i(\xi)\tag{6.10}$$

for some $\mathfrak{a}$-root β that is positive for P_1 and negative for P_2. If $\operatorname{rank}M=\operatorname{rank}K_M$, *the number c can be taken to be $1/3$. If the infinitesimal character of ξ is a real linear combination of roots of M, then the numbers $d_i(\xi)$ are all real.*

Proof. For convergence of the integral, we have

$$F(xv)=\exp\{-(\Lambda+\rho_{P_1})H_{P_1}(xv)\}\,\xi(\mu(xv))^{-1}F(\kappa(xv))$$

and

$$|F(xv)|_{H^\xi}\leqq\exp\{-(\operatorname{Re}\Lambda+\rho_{P_1})H_{P_1}(xv)\}\sup_{k\in K}|F(k)|_{H^\xi}.$$

Write $x=kman_1v_1$, with n_1 in $N_1\cap N_2$ and v_1 in $V_1\cap N_2$. For a suitable constant C depending on a, we have

$$\begin{aligned}\int_{V_1\cap N_2}|F(xv)|_{H^\xi}dv&\leqq C\int_{V_1\cap N_2}\exp\{-(\operatorname{Re}\Lambda+\rho_{P_1})H_{P_1}(n_1v_1v)\}\,dv\\&=C\int_{V_1\cap N_2}\exp\{-(\operatorname{Re}\Lambda+\rho_{P_1})H_{P_1}(n_1v)\}\,dv.\end{aligned}$$

Since $N_2=(N_1\cap N_2)(V_1\cap N_2)=(V_1\cap N_2)(N_1\cap N_2)$, we can write $n_1v=\tilde{v}\tilde{n}_1$, and it is known that $v\to\tilde{v}$ is unimodular (cf. [28] or [8]). Thus the above integral equals

$$=C\int_{V_1\cap N_2}\exp\{-(\operatorname{Re}\Lambda+\rho_{P_1})H_{P_1}(v)\}\,dv,$$

which is finite by Lemma 6.5. This proves convergence of the integral.

To prove (6.9), it is enough to handle $x=k$ in K, since both sides transform the same way under MAN_2 on the right. Set

$$AF(k)=\int_{V_1\cap N_2}F(kv)\,dv.$$

Here F is K-finite, and it follows that AF and $A(P_2:P_1:\xi:\Lambda)$ are K-finite. Taking the spaces of left K-translates of F and AF and evaluating at 1, we see that

$$\operatorname{span}\{\text{all}\,F(k),\ \text{all}\,AF(k),\ \text{all}\,A(P_2:P_1:\xi:\Lambda)F(k)\}\tag{6.11}$$

is finite-dimensional in $H^\xi_{K_M}$. Then we can find an orthogonal projection E on H^ξ that is 1 on (6.11), has finite-dimensional image, and is 1 on every vector of a given K_M-type whenever it is 1 on one nonzero vector of that K_M-type. Let $E'=\iota E\iota^{-1}$ be the corresponding projection for H_0^ω.

We shall show that

$$E'\omega(m)E'=\iota E\xi(m)E\iota^{-1}\tag{6.12}$$

for all m in M. The K_M-equivariance of ι, E, and E' implies (6.12) for m in K_M, and it is enough to prove (6.12) for m in M_0. For X in $\mathscr{U}(\mathfrak{m})$, the $\mathfrak{m}$-equivariance of ι implies

$$\omega(X)E'=\omega(X)\iota E\iota^{-1}=\iota\xi(X)E\iota^{-1}.$$

Left multiplication of both sides by E' gives (6.12), but with m replaced by X. This means that the two sides of (6.12) are analytic functions on M_0 (since image E' is finite-dimensional) with respective derivatives of all orders equal at $m=1$. This proves (6.12) for M_0, hence for M.

Thus we obtain

$$\begin{aligned}
AF(k) &= EAF(k) = E \int_{V_1 \cap N_2} \exp\{-(\Lambda+\rho_{P_1}) H_{P_1}(v)\} E\xi(\mu(v))^{-1} F(k\kappa(v))\, dv \\
&\quad \text{since } E \text{ is bounded} \\
&= E \int_{V_1 \cap N_2} \exp\{-(\Lambda+\rho_{P_1}) H_{P_1}(v)\} E\xi(\mu(v))^{-1} EF(k\kappa(v))\, dv \\
&= E \int_{V_1 \cap N_2} \exp\{-(\Lambda+\rho_{P_1}) H_{P_1}(v)\} \iota^{-1} E'\omega(\mu(v))^{-1} E'\iota F(k\kappa(v))\, dv \\
&= \int_{V_1 \cap N_2} \exp\{-(\Lambda+\rho_{P_1}) H_{P_1}(v)\} E\iota^{-1} E'\omega(\mu(v))^{-1} \iota EF(k\kappa(v))\, dv \\
&= E\iota^{-1} E' \int_{V_1 \cap N_2} \exp\{-(\Lambda+\rho_{P_1}) H_{P_1}\} \omega(\mu(v))^{-1} \iota F(k\kappa(v))\, dv \\
&\quad \text{since } E\iota^{-1}E' \text{ is bounded} \\
&= E\iota^{-1} E' A(P_2 : P_1 : \omega : \Lambda)(\iota^{\#} F)(k) \\
&= E\iota^{-1} E'(\iota^{\#}(A(P_2 : P_1 : \xi : \Lambda) F)(k)) \\
&= E\iota^{-1}(\iota^{\#}(A(P_2 : P_1 : \xi : \Lambda) F)(k)) \\
&= A(P_2 : P_1 : \xi : \Lambda) F(k).
\end{aligned}$$

The qualitative statements about analytic continuation were proved earlier and center about Theorem 4.2(iii) and formulas (6.3) and (6.6). We still have to prove the quantitative estimate (6.10). Imbed ξ in a nonunitary principal series representation of M with parameters (σ, Λ_M). Theorem 4.2(iii) and formulas (6.3) and (6.6) show that $A(P_2 : P_1 : \xi : \Lambda)$ is holomorphic at Λ_0 unless

$$\frac{2\langle \Lambda_0 + \Lambda_M, \alpha \rangle}{|\alpha|^2} \in \mathbb{Z}$$

for some $\mathfrak{a}_p$-root α that is positive for $(P_1)_p$ and negative for $(P_2)_p$. For such an α, write $\alpha = \beta + \gamma$, where $\beta = \alpha|_{\mathfrak{a}}$ and $\gamma = \alpha|_{\mathfrak{a}_M}$. The condition is that the operator is holomorphic at Λ_0 unless

$$\frac{2\langle \Lambda_0, \beta \rangle}{|\beta+\gamma|^2} \in \mathbb{Z} - \frac{2\langle \Lambda_M, \gamma \rangle}{|\beta+\gamma|^2}$$

for some $\beta + \gamma$. To obtain (6.10), we are to show that $|\beta+\gamma|^2 = cn|\beta|^2$ for some integer n and fixed rational c. Since $W(\mathfrak{a}_M)$ fixes no nonzero member of $\mathfrak{a}_M$, we know that

$$\sum_{w \in W(\mathfrak{a}_M)} w\gamma = 0.$$

Then

$$\sum_{w \in W(\mathfrak{a}_M)} \frac{2\langle\beta+\gamma, w(\beta+\gamma)\rangle}{|\beta+\gamma|^2} = \sum_w \frac{2\langle\beta+\gamma, \beta+w\gamma\rangle}{|\beta+\gamma|^2} = \frac{2|\beta|^2 |W(\mathfrak{a}_M)|}{|\beta+\gamma|^2}.$$

The left side of this expression is an integer $\leqq 2|W(\mathfrak{a}_M)|$, and (6.10) follows. If $\operatorname{rank} M = \operatorname{rank} K_M$, we can compute c directly from [17]. In fact, $|\beta+\gamma|^2 = r|\beta|^2$ with $r=1,2,4$, or $4/3$; hence $c=1/3$. Finally if ξ has a real infinitesimal character, Λ_M is real by Lemma 5.5, and then $2\langle\Lambda_M, \gamma\rangle/|\beta+\gamma|^2$ is real.

§ 7. Properties of Unnormalized Operators

We retain the notation of § 6, working with general parabolic subgroups. The results in this section were announced in [22].

Proposition 7.1. *The analytic continuations of the operators $A(P_2:P_1:\xi:\Lambda)$, defined on K-finite functions for $U_{P_1}(\xi, \Lambda, \cdot)$, have the following properties:*

(i) *Let F be a finite set of K-types, and let E be the orthogonal projection onto the span of all functions of one of the K-types in F. Then*

$$E_F U_{P_2}(\xi, \Lambda, x) E_F A(P_2:P_1:\xi:\Lambda) = A(P_2:P_1:\xi:\Lambda) E_F U_{P_1}(\xi, \Lambda, x) E_F$$

for all x in G.

(ii) *$A(P_2:P_1:E\xi E^{-1}:\Lambda) = E A(P_2:P_1:\xi:\Lambda) E^{-1}$ if E is a unitary operator on H^ξ (carrying $H^\xi_{K_M}$ onto $H^{E\xi E^{-1}}_{K_M}$).*

(iii) *If w is in $N_K(\mathfrak{a})$, then*

$$A(P_2:P_1:\xi:\Lambda) = R(w)^{-1} A(wP_2w^{-1}:wP_1w^{-1}:w\xi:w\Lambda) R(w). \tag{7.1}$$

(iv) *$A(P_2:P_1:\xi:\Lambda)^* = A(P_1:P_2:\xi:-\bar{\Lambda})$, with the adjoint defined K-space by K-space.*

Proof. For (i), let Λ be in the region of convergence of Theorem 6.6, and define $A(P_2:P_1:\xi:\Lambda)F$ by (6.9) for F in C^∞. By Proposition 1.1,

$$U_{P_2}(\xi, \Lambda, x) A(P_2:P_1:\xi:\Lambda) = A(P_2:P_1:\xi:\Lambda) U_{P_1}(\xi, \Lambda, x)$$

for all x in G. Multiplying by E_F on the right and left and commuting E_F past $A(P_2:P_1:\xi:\Lambda)$, we obtain (i) for Λ in the region of convergence. On the range of E_F, both sides of (i) are operators in a finite-dimensional space varying meromorphically in Λ. Hence (i) extends to all Λ.

For (ii), let F be in the K-finite space for $U_{P_1}(E\xi E^{-1}, \Lambda, \cdot)$. Then $E^{-1}(F(\cdot))$ satisfies

$$\begin{aligned} E^{-1}F(xman) &= e^{-(\rho_{P_1}+\Lambda)\log a} E^{-1}(E\xi(m)^{-1}E^{-1})F(x) \\ &= e^{-(\rho_{P_1}+\Lambda)\log a} \xi(m)^{-1}(E^{-1}F(x)) \end{aligned}$$

and

$$\dim\{\operatorname{span} E^{-1}F(k\cdot)\} = \dim\{\operatorname{span} F(k\cdot)\},$$

so that $E^{-1}(F(\cdot))$ is in the K-finite space for $U_{P_1}(\xi, \Lambda, \cdot)$. For Λ in the region of convergence,

$$\begin{aligned}
&EA(P_2:P_1:\xi:\Lambda)E^{-1}F(x)\\
&=E\int_{V_1\cap N_2} E^{-1}F(xv)\,dv\\
&=\int_{V_1\cap N_2} EE^{-1}F(xv)\,dv \quad \text{since } E \text{ is bounded}\\
&=\int_{V_1\cap N_2} F(xv)\,dv\\
&=A(P_2:P_1:E\xi E^{-1}:\Lambda)F(x),
\end{aligned}$$

and (ii) follows by analytic continuation.

For (iii), suppose Λ is in the region of convergence of Theorem 6.6 for the operator on the left side of (7.1). This means that

$$\langle \operatorname{Re}\Lambda, \beta\rangle > c_\beta \quad \text{whenever } \mathfrak{g}_\beta \subseteq \mathfrak{n}_1 \cap \mathfrak{v}_2.$$

Then

$$\langle \operatorname{Re} w\Lambda, w\beta\rangle > c_\beta \quad \text{whenever } \mathfrak{g}_{w\beta} \subseteq \operatorname{Ad}(w)(\eta_1 \cap \mathfrak{v}_2).$$

Setting $\beta' = w\beta$, we obtain

$$\langle \operatorname{Re} w\Lambda, \beta'\rangle > c_{w^{-1}\beta'} \quad \text{whenever } \mathfrak{g}_{\beta'} \subseteq \operatorname{Ad}(w)(\mathfrak{n}_1 \cap \mathfrak{v}_2).$$

It is easy to see from Lemma 8 of [17] that $c_{w^{-1}\beta'} = c_{\beta'}$, and it follows that $w\Lambda$ is in the region of convergence of Theorem 6.6 for the operator on the right side of (7.1). For such Λ, the left and right sides of (7.1), applied to F at 1, respectively are

$$\int_{V_1\cap N_2} F(v)\,dv \quad \text{and} \quad \int_{w(V_1\cap N_2)w^{-1}} F(w^{-1}uw)\,du,$$

and these are equal by Lemma 2.6. Translating the functions by members of K on the left, we obtain (7.1) for Λ in the region of convergence. The conclusion for general Λ then follows by analytic continuation.

For the proof of (iv), we need the lemma below, which is also needed for the detailed proof of Proposition 1.1.

Lemma 7.2. $\displaystyle\int_{V_1\cap N_2} f(ava^{-1})\,dv = e^{(\rho_1-\rho_2)\log a}\int_{V_1\cap N_2} f(v)\,dv$ *for a in A.*

Proof. We have

$$c(a)\int_{V_1\cap N_2} f(ava^{-1})\,dv = \int_{V_1\cap N_2} f(v)\,dv,$$

where

$$c(a) = \det \operatorname{Ad}(a)|_{\mathfrak{v}_1\cap\mathfrak{n}_2} = \exp\Big\{\sum_{\mathfrak{g}_\beta\subseteq \mathfrak{v}_1\cap\mathfrak{n}_2} (\dim \mathfrak{g}_\beta)\,\beta(\log a)\Big\}.$$

But

$$2\sum_{\mathfrak{g}_\beta\subseteq\mathfrak{v}_1\cap\mathfrak{n}_2}(\dim\mathfrak{g}_\beta)\beta=\sum_{\mathfrak{g}_\beta\subseteq\mathfrak{v}_1}-\sum_{\mathfrak{g}_\beta\subseteq\mathfrak{v}_1\cap\mathfrak{v}_2}+\sum_{\mathfrak{g}_\beta\subseteq\mathfrak{n}_2}-\sum_{\mathfrak{g}_\beta\subseteq\mathfrak{n}_1\cap\mathfrak{n}_2}.$$

The second and fourth terms on the right cancel, and the lemma follows.

Proof of Proposition 7.1(iv).[7] Let Λ be in the region of convergence for $A(P_2:P_1:\xi:\Lambda)$ given in Theorem 6.6. Then $-\bar{\Lambda}$ is in the region of convergence for $A(P_1:P_2:\xi:-\bar{\Lambda})$. Suppose f and g transform according to

$$g(xman_1)=e^{-(\rho_1+\Lambda)\log a}\,\xi(m)^{-1}f(x) \tag{7.2}$$

$$f(xman_2)=e^{-(\rho_2-\bar{\Lambda})\log a}\,\xi(m)^{-1}f(x). \tag{7.3}$$

Choose $\varphi\geqq 0$ on G with

$$\int_{MAN_2}\varphi(xs)\,d_l s=1$$

for all x in G, where $d_l s$ denotes left Haar measure. Then

$$\begin{aligned}
&(g, A(P_2:P_1:\xi:\Lambda)^*f)\\
&=(A(P_2:P_1:\xi:\Lambda)g,f)\\
&=\int_K\Big(\int_{V_1\cap N_2}g(kv)\,dv,f(k)\Big)_{H^\xi}dk\\
&=\int_{KMAN_2}\Big(\int_{V_1\cap N_2}g(kv)\,dv,f(k)\Big)_{H^\xi}\varphi(kman_2)\,d_l(man_2)\,dk\\
&=\int_{KMAN_2}\Big(\int_{V_1\cap N_2}e^{-(\rho_1+\Lambda)\log a}\,\xi(m)^{-1}g(kv)\,dv,\ e^{-(\rho_2-\bar{\Lambda})\log a}\,\xi(m)^{-1}f(k)\Big)_{H^\xi}\\
&\qquad\cdot e^{(\rho_1+\rho_2)\log a}\,\varphi(kman_2)\,d_l(man_2)\,dk\\
&=\int_{KMAN_2}\Big(\int_{V_1\cap N_2}g(kmav^{a^{-1}})\,dv,f(kma)\Big)_{H^\xi}\\
&\qquad\cdot e^{(\rho_1+\rho_2)\log a}\,\varphi(kman_2)\,d_l(man_2)\,dk\\
&=\int_{KMAN_2}\Big(\int_{V_1\cap N_2}g(kmav)\,dv,f(kma)\Big)_{H^\xi}\\
&\qquad\cdot e^{2\rho_2\log a}\,\varphi(kman_2)\,d_l(man_2)\,dk\qquad\text{by Lemma 7.2}\\
&=\int_{KMAN_2}\Big(\int_{V_1\cap N_2}g(kma(n_2)_{V_1\cap N_2}(n_2)_{N_1\cap N_2}v)\,dv,f(kman_2)\Big)_{H^\xi}\\
&\qquad\cdot e^{2\rho_2\log a}\,\varphi(kman_2)\,d_l(man_2)\,dk
\end{aligned}$$

with n_2 introduced into f by (7.3), $(n_2)_{V_1\cap N_2}$ introduced into g by translation, and $(n_2)_{N_1\cap N_2}$ introduced into g by (7.2) and the same change of variables as in the proof of Theorem 6.6. Then the above expression is

[7] This proof evolved from an argument by Schiffmann [32].

$$= \int_{KMAN_2} \Big(\int_{V_1 \cap N_2} g(kman_2 v)\, dv, f(kman_2)\Big)_{H^\xi} \varphi(kman_2)\, e^{2\rho_2 \log a} d_l(man_2)\, dk$$

$$= \int_G \int_{V_1 \cap N_2} (g(xv), f(x))_{H^\xi} \varphi(x)\, dv\, dx$$

since $e^{2\rho_2 \log a} d_l(man_2)\, dk = dx$

$$= \int_G \int_{V_1 \cap N_2} (g(x), f(x))_{H^\xi} \varphi(xv)\, dv\, dx$$

under $xv \to x$ and $v \to v^{-1}$

$$= \int_{KMAN_1} \int_{V_1 \cap N_2} (g(kman_1), f(kman_1))_{H^\xi} \varphi(kman_1 v)$$

$$\cdot e^{2\rho_1 \log a} dm\, da\, dn_1\, dv\, dk \quad \text{by Lemma 2.8}$$

$$= \int_{KMA(N_1 \cap V_2)(N_1 \cap N_2)} \int_{V_1 \cap N_2} (g(kma), f(kman_{N_1 \cap V_2}))_{H^\xi} \varphi(kman_{N_1 \cap V_2} n_{N_1 \cap N_2} v)$$

$$\cdot e^{2\rho_1 \log a} dm\, da\, dn_{N_1 \cap V_2}\, dn_{N_1 \cap N_2}\, dv\, dk$$

by Lemma 2.4 with $n_1 = n_{N_1 \cap V_2} n_{N_1 \cap N_2}$ and by (7.2) and (7.3), and this is

$$= \int_K \int_{MAN_2} \int_{N_1 \cap V_2} (e^{-(\rho_1 + \Lambda)\log a} g(k), e^{-(\rho_2 - \bar{\Lambda})\log a} f(k n^{ma}_{N_1 \cap V_2}))_{H^\xi}$$

$$\cdot \varphi(kman_{N_1 \cap V_2} n_2)\, e^{2\rho_1 \log a} dn_{N_1 \cap V_2}\, dm\, da\, dn_2\, dk$$

by Lemma 2.4 and (7.2) and (7.3)

$$= \int_K \int_{N_1 \cap V_2} \int_{MAN_2} (g(k), f(kn_{N_1 \cap V_2}))_{H^\xi} \varphi(kn_{N_1 \cap V_2} man_2)\, d_l s\, dn_{N_1 \cap V_2}\, dk$$

by Lemma 7.2

$$= \int_K \int_{N_1 \cap V_2} (g(k), f(kn_{N_1 \cap V_2}))_{H^\xi} dn_{N_1 \cap V_2}\, dk$$

by the defining property of φ

$$= (g, A(P_1 : P_2 : \xi : -\bar{\Lambda}) f).$$

The result for general Λ follows by analytic continuation. This completes the proof of Proposition 7.1.

Proposition 7.3. *Let $P_1 = MAN_1$ and $P_2 = MAN_2$ be associated parabolic subgroups. Then there exists a scalar-valued function $\eta(P_2 : P_1 : \xi : \Lambda)$ meromorphic in Λ such that*

$$A(P_1 : P_2 : \xi : \Lambda) A(P_2 : P_1 : \xi : \Lambda) = \eta(P_2 : P_1 : \xi : \Lambda) I. \tag{7.4}$$

The function η satisfies

$$\eta(P_2 : P_1 : \xi : \Lambda) = \eta(P_1 : P_2 : \xi : \Lambda). \tag{7.5}$$

Moreover, there exist a rational number c and finitely many complex numbers $d_i(\xi)$ *such that the meromorphic continuation of* $\eta(P_2:P_1:\xi:\Lambda)$ *is holomorphic at* Λ_0 *unless*

$$\frac{2\langle\Lambda_0,\beta\rangle}{|\beta|^2}\in c\mathbb{Z}+d_i(\xi) \tag{7.6}$$

for some $\mathfrak{a}$*-root* β *that is positive for* P_1 *and negative for* P_2. *If* $\operatorname{rank} M=\operatorname{rank} K_M$, *the number c can be taken to be* $1/3$. *If the infinitesimal character of* ξ *is a real linear combination of roots of* M, *then the numbers* $d_i(\xi)$ *are all real. If* ξ *imbeds in the nonunitary principal series representation of* M *with parameters* (σ,Λ_M), *then*

$$\eta(P_2:P_1:\xi:\Lambda)=\eta((P_2)_p:(P_1)_p:\sigma:(\Lambda\oplus\Lambda_M)). \tag{7.7}$$

Proof. By means of (6.6), it is enough to prove (7.4) and (7.7) with ξ replaced by the nonunitary principal series representation ω of M. We compute the left side of (7.4) from (6.3), obtaining

$$\begin{aligned}
&(A(P_1:P_2:\omega:\Lambda)A(P_2:P_1:\omega:\Lambda)F)(x)(m)\\
&\quad=A((P_1)_p:(P_2)_p:\sigma:\Lambda\oplus\Lambda_M)((A(P_2:P_1:\omega:\Lambda)F)(\cdot)(1))(xm)\\
&\quad=A((P_1)_p:(P_2)_p:\sigma:\Lambda\oplus\Lambda_M)\{[A((P_2)_p:(P_1)_p:\sigma:\Lambda\oplus\Lambda_M)F(\cdot\,\cdot)(1)](\cdot)\}(xm)\\
&\quad=[A((P_1)_p:(P_2)_p:\sigma:\Lambda\oplus\Lambda_M)\,A((P_2)_p:(P_1)_p:\sigma:\Lambda\oplus\Lambda_M)F(\cdot\,\cdot)(1)](xm)\\
&\quad=\eta((P_2)_p:(P_1)_p:\sigma:\Lambda\oplus\Lambda_M)F(xm)(1)\\
&\qquad\text{by Theorem 4.2(v)}\\
&\quad=\eta((P_2)_p:(P_1)_p:\sigma:\Lambda\oplus\Lambda_M)F(x)(m),
\end{aligned}$$

and (7.4) and (7.7) follow. Applying (4.6), we obtain (7.5). Then (7.6) follows from Theorem 6.6, provided we enlarge the set $d_i(\xi)$ so as to be closed under multiplication by -1.

Proposition 7.4. *Let* $\dim A=1$ *and let* $P=MAN$ *and* $\bar{P}=MAV$ *be associated parabolic subgroups. Define*

$$\eta_\xi(z)=\eta(\bar{P}:P:\xi:z\rho_P).$$

Then $\eta_\xi(z)$ *is a scalar-valued meromorphic function of one complex variable with the following properties:*

(a) η_ξ *depends only on the class of* ξ,

(b) $\eta_\xi(z)=\overline{\eta_\xi(-\bar{z})}$,

(c) $\eta_\xi(z)\geqq 0$ *on the imaginary axis, and* $\eta_\xi(z)$ *is not identically* 0,[8]

(d) $\eta_{w\xi}(-z)=\eta_\xi(z)$ *if* w *is an element of* $N_K(\mathfrak{a})$ *satisfying* $wPw^{-1}=\bar{P}$,

[8] Actually η_ξ vanishes nowhere on the imaginary axis. The proof of this fact is deferred to Appendix B.

(e) $\eta_{\xi^\varphi}(z)=\eta_\xi(z)$ *if* φ *is an automorphism of* G *leaving* K *stable and the positive chamber of* A *fixed and if* $\xi^\varphi(m)=\xi(\varphi^{-1}(m))$.

(f) $\eta_\xi(z)=\eta_\xi(-z)$ *if the infinitesimal character of* ξ *is a real linear combination of the roots of* M. *In this case all the poles of* $\eta_\xi(z)$ *are real.*

Proof. Part (a) is immediate from (7.4) and Proposition 7.1 (ii). For (b), we take the adjoint of (7.4) on a single K-space and then apply Proposition 7.1 (iv). For (c), we apply Proposition 7.1 (iv) to the first factor in (7.4) to exhibit $\eta_\xi(z)I$ as an operator times its adjoint for z imaginary. Then $\eta_\xi(z)\geqq 0$ for z imaginary. The proof that $\eta_\xi(z)$ is not identically 0 is harder.

If $\eta_\xi(z)$ is identically 0, then the vanishing on the imaginary axis of $\eta_\xi(z)$ implies that $A(\bar{P}:P:\xi:z\rho_P)=0$ for all imaginary z. By analytic continuation, $A(\bar{P}:P:\xi:z\rho_P)=0$ for all z, and then Theorem 6.6 gives

$$0=\int_V F(v)\,dv=\int_V e^{-(1+z)\rho_P H_P(v)}\,\xi(\mu(v))^{-1}F(\kappa(v))\,dv \tag{7.8}$$

whenever F is in the K-finite space of the induced representation and $\operatorname{Re} z$ is sufficiently large. Passing to the limit by dominated convergence and the integrability in Lemma 6.5, we obtain (7.8) also for all smooth F on K such that $F(km)=\xi(m)^{-1}F(k)$ for k in K and m in K_M: there are many such F, by the techniques of [5]. If F is one such function and φ is the lift to K of a function on K/K_M, then φF is another such function. Now Lemma B.1 assures us that we can make $\varphi(\kappa(v))$ have compact support in V and then peak $\varphi(\kappa(v))$ for $v=1$, and it follows from (7.8) that $F(1)=0$. Since this equality holds for all the left K-translates of F, F is 0. Thus (7.8) for all F implies $F=0$ is the only function in the space, contradiction. This finishes the proof of (c).

For (d), we apply Proposition 7.1 (iii) to obtain

$$A(\bar{P}:P:\xi:\Lambda)=R(w)^{-1}A(P:\bar{P}:w\xi:w\Lambda)R(w)$$

and

$$A(P:\bar{P}:\xi:\Lambda)=R(w)^{-1}A(\bar{P}:P:w\xi:w\Lambda)R(w).$$

Here if $\Lambda=z\rho_P$, we have $w\Lambda=-z\rho_P$. Then (d) follows by making use of (7.5).

For (e), let $\Lambda=z\rho_P$ be in the region of convergence. If F is in the space for $U_P(\xi,\Lambda,\cdot)$, then $F\circ\varphi$ is in the space for $U_P(\xi^\varphi,\Lambda,\cdot)$. In the region of convergence for Λ,

$$\begin{aligned}(A(\bar{P}:P:\xi:\Lambda)F)\circ\varphi(k)&=\int_V F(\varphi(k)v)\,dv\\&=\int_V F(\varphi(k\varphi^{-1}(v)))\,dv\\&=\int_V F(\varphi(kv))\,dv\\&=A(\bar{P}:P:\xi^\varphi:\Lambda)(F\circ\varphi)(k),\end{aligned}\tag{7.9}$$

since φ acts in unimodular fashion on V (because a finite power of φ must be $\operatorname{Ad}(k_0)$ on $[\mathfrak{g},\mathfrak{g}]$ for some k_0 in K). By analytic continuation, (7.9) extends to all Λ, and then (e) follows.

For (f), let ξ imbed in the nonunitary principal series of M with parameters (σ, Λ_M). By Lemma 5.5, Λ_M is real. Thus we apply (a) to our parabolic, then (7.7), then (a) to a minimal parabolic, then (f) to a minimal parabolic, and finally (7.7) again to find

$$\begin{aligned}\eta_\xi(z) &= \overline{\eta_\xi(-\bar{z})} = \bar{\eta}(\bar{P}:P:\xi:-\bar{\Lambda})\\ &= \bar{\eta}(M_p A_p V N_M : M_p A_p N N_M : \sigma : -\bar{\Lambda}+\Lambda_M)\\ &= \eta(M_p A_p V N_M : M_p A_p N N_M : \sigma : \Lambda-\Lambda_M) \quad \text{since } \Lambda_M \text{ is real}\\ &= \eta(M_p A_p V N_M : M_p A_p N N_M : \sigma : -\Lambda+\Lambda_M)\\ &= \eta(\bar{P}:P:\xi:-\Lambda) = \eta_\xi(-z).\end{aligned}$$

In this case, the poles of $\eta_\xi(z)$ occur only for z real by Proposition 7.3.

Proposition 7.5. *Let $P_1 = MAN_1$ and $P_2 = MAN_2$ be associated parabolic subgroups such that $V_1 \cap N_2 = V^{(\beta)}$ for a P_1-positive reduced $\mathfrak{a}$-root β. Let f be in the K-finite space for $U_{P_1}(\xi, \Lambda, \cdot)$, and let f_k be the restriction to $G^{(\beta)}$ of the left translate of f by k in K. Then f_k is in the $K^{(\beta)}$-finite space for $U_{P^{(\beta)}}(\xi, \Lambda|_{\mathfrak{a}^{(\beta)}}, \cdot)$, and*

$$A(P_2:P_1:\xi:\Lambda)\, f(k) = A^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}^{(\beta)}})\, f_k(1). \tag{7.10}$$

Moreover,

$$\eta(P_2:P_1:\xi:\Lambda) = \eta^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}^{(\beta)}}).$$

Proof. The transformation law for f_k under $MA^{(\beta)}N^{(\beta)}$ is immediate if we take into account Proposition 1.2. Also

$$\operatorname*{span}_{k_\beta \in K^{(\beta)}} \{f_k(k_\beta\, \cdot)\} \subseteq \operatorname*{span}_{k \in K} \{f(k\, \cdot)\}$$

shows f_k is $K^{(\beta)}$-finite. As in Proposition 4.1, the remainder of the proof comes down to the question of whether the normalized Haar measure for $V_1 \cap N_2$ within G is the same as the normalized Haar measure for $V^{(\beta)}$ within $G^{(\beta)}$. If α is an $\mathfrak{a}_p$-root whose restriction to $\mathfrak{a}$ is a nonzero multiple of β, then the proof of Proposition 4.1 notes that $V^{(\alpha)}$ gets the same Haar measure whether considered as in G or in $G^{(\alpha)}$. Applying this result to $G^{(\beta)}$ in place of G, we see that $V^{(\alpha)}$ gets the same Haar measure whether considered as in G or in $G^{(\beta)}$. We therefore run through the proof of the first three parts of Theorem 4.2, applied to minimal parabolics, to see that the Haar measure in $V_1 \cap N_2 = V^{(\beta)}$ can be written as the product of the Haar measures for the various $V^{(\alpha)}$. Since the normalizations of the measures for $V^{(\alpha)}$ are the same in G as in $G^{(\beta)}$, the normalizations of the measures for $V^{(\beta)}$ are the same in G as in $G^{(\beta)}$.

Let $P = MAN$ and $P' = MAN'$ be associated parabolic subgroups. Proceeding as in §4 after Proposition 4.1, we can define *strings* $\{MAN_i\}$ and *minimal strings* from P to P'. The parabolics P and P' can always be connected by a minimal string, just as in §4 (see [12], p. 145).

Theorem 7.6. *Suppose that $P = MAN$ and $P' = MAN'$ are associated parabolic subgroups and $P_i = MAN_i$, $0 \leqq i \leqq r$, is a minimal string from P to P', with associated reduced P-positive $\mathfrak{a}$-roots $\{\beta_i\}$. Then*

(i) *the set* $\{\beta_i\}$ *is characterized as the set of reduced* $\mathfrak{a}$*-roots* β *that are positive for* P *and negative for* P'.

(ii) r *is characterized as the number of roots described in* (i).

(iii) *the unnormalized intertwining operators satisfy*

$$A(P':P:\xi:\Lambda)=A(P_r:P_{r-1}:\xi:\Lambda)\cdot\ldots\cdot A(P_1:P_0:\xi:\Lambda).$$

(iv) P_{r-i}, $0\leqq i\leqq r$, *is a minimal string from* P' *to* P, *with associated reduced* P'*-positive* $\mathfrak{a}$*-roots* $\{-\beta_i\}$.

(v) *the* η*-functions satisfy*

$$\eta(P':P:\xi:\Lambda)=\prod_{\substack{\beta\ \text{reduced}\\ \beta>0\ \text{for}\ P\\ \beta<0\ \text{for}\ P'}}\eta^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}^{(\beta)}}).$$

Proof. Conclusions (i) and (ii) are proved just as in Theorem 4.2. For (iii), let Λ be in the region of convergence of Theorem 6.6 for the operator on the left. Then Λ is in the region of convergence for each of the operators on the right. The proof of the formula for (iii) is then the same as in Theorem 4.2. Conclusion (iv) is proved as in Theorem 4.2. For (v), we obtain, by the method of Theorem 4.2,

$$\eta(P':P:\xi:\Lambda)=\prod_{i=1}^{r}\eta(P_i:P_{i-1}:\xi:\Lambda),$$

and (v) then follows from Proposition 4.1.

Corollary 7.7. *Suppose that* $P=MAN$, $P'=MAN'$, *and* $P''=MAN''$ *are associated parabolic subgroups such that* $\mathfrak{n}''\cap\mathfrak{n}\subseteq\mathfrak{n}'\cap\mathfrak{n}$. *Then the unnormalized intertwining operators satisfy*

$$A(P'':P:\xi:\Lambda)=A(P'':P':\xi:\Lambda)A(P':P:\xi:\Lambda).$$

Proof. Let $P_0,\ldots,P_k$ be a minimal string from P to P', and let $P_k,\ldots,P_r$ be a minimal string from P' to P''. Applying Theorem 7.6(iii) to the three operators in the statement of the corollary, we see that it is enough to prove that $P_0,\ldots,P_r$ is a minimal string from P to P''. Referring to the definition of minimal string, we see that we are to show that the P'-positive $\mathfrak{a}$-roots β_i associated to the string $P_k,\ldots,P_r$ are P-positive. From Theorem 7.6(i), we see that these β_i are characterized as the reduced $\mathfrak{a}$-roots β that are positive for P' and negative for P''. If such a β_i were P-negative, then we would have $\mathfrak{g}_{-\beta_i}\subseteq\mathfrak{n}''\cap\mathfrak{n}$ and $\mathfrak{g}_{-\beta_i}\nsubseteq\mathfrak{n}'\cap\mathfrak{n}$, in contradiction to hypothesis. Thus β_i is P-positive, as required.

Let $P=MAN$ be a parabolic subgroup. Recall from §1 the definition

$$A_P(w,\xi,\Lambda)=R(w)A(w^{-1}Pw:P:\xi:\Lambda),$$

where w is assumed to be in $N_K(\mathfrak{a})$ and where $R(w)$ is defined by $R(w)f(x)=f(xw)$.

Proposition 7.8. *The analytically continued operators* $A_P(w,\xi,\Lambda)$, *defined on* K*-finite functions for* $U_P(\xi,\Lambda,\cdot)$, *have the following properties:*

(i) *Let F be a finite set of K-types, and let E_F be the orthogonal projection onto the span of all functions of some K-type in F. Then*

$$E_F U_P(w\xi, wA, x)\, E_F A_P(w, \xi, A) = A_P(w, \xi, A)\, E_F\, U_P(\xi, A, x)\, E_F$$

for all x in G.

(ii) $A_P(w, E\xi E^{-1}, A) = E A_P(w, \xi, A) E^{-1}$ *if E is a unitary operator on H^ξ.*

(iii) $A_P(w, \xi, A)^* = A_P(w^{-1}, w\xi, -w\bar{A})$, *with the adjoint defined K-space by K-space.*

(iv) $A_P(w_1 w_2, \xi, A) = A_P(w_1, w_2\xi, w_2 A) A_P(w_2, \xi, A)$ *if every P-positive $\mathfrak{a}$-root β such that $w_1 w_2 \beta$ is P-positive has $w_2\beta$ P-positive.*

Proof. For (i), start with the formula of Proposition 7.1 (i) with $P_1 = P$ and $P_2 = w^{-1} P w$, and multiply on the left by $R(w)$. Since $R(w)$ commutes with E_F, the identity in question will follow from knowing

$$R(w)\, U_{w^{-1}Pw}(\xi, A, x) = U_P(w\xi, wA, x)\, R(w), \tag{7.11}$$

which is readily verified. (The identity $\rho_{w^{-1}Pw} = w^{-1}\rho_P$ is relevant here.)

Conclusion (ii) is immediate from Proposition 7.1 (ii) and the fact that $R(w)$ commutes with E. For (iii), we note that $R(w)$ is unitary on $L^2(K)$, since it is given by translation, and hence

$$\begin{aligned} A_P(w, \xi, A)^* &= A(w^{-1}Pw : P : \xi : A)^* R(w)^{-1} \\ &= A(P : w^{-1}Pw : \xi : -\bar{A}) R(w)^{-1} \\ &\quad \text{by Proposition 7.1 (iv)} \\ &= R(w)^{-1} A(wPw^{-1} : P : w\xi : -w\bar{A}) \\ &\quad \text{by Proposition 7.1 (iii)} \\ &= A_P(w^{-1}, w\xi, -w\bar{A}). \end{aligned}$$

For (iv), we apply Corollary 7.7 with $P' = w_2^{-1} P w_2$ and $P'' = w_2^{-1} w_1^{-1} P w_1 w_2$. Suppose $\mathfrak{g}_\beta$ is in $\mathfrak{n}'' \cap \mathfrak{n}$. Then β is a P-positive root such that $w_1 w_2 \beta$ is P-positive, and by assumption $w_2\beta$ is P-positive. Therefore $\mathfrak{g}_\beta$ is in $\mathfrak{n}' \cap \mathfrak{n}$. Thus

$$\begin{aligned} &A_P(w_1 w_2, \xi, A) \\ &\quad = R(w_1 w_2) A(w_2^{-1} w_1^{-1} P w_1 w_2 : P : \xi : A) \\ &\quad = R(w_1) R(w_2) A(w_2^{-1} w_1^{-1} P w_1 w_2 : w_2^{-1} P w_2 : \xi : A) R(w_2)^{-1} \\ &\qquad \cdot R(w_2) A(w_2^{-1} P w_2 : P : \xi : A) \qquad \text{by the corollary} \\ &\quad = R(w_1) A(w_1^{-1} P w_1 : P : w_2\xi : w_2 A) R(w_2) A(w_2^{-1} P w_2 : P : \xi : A) \\ &\qquad \text{by Proposition 7.1 (iii)} \\ &\quad = A_P(w_1, w_2\xi, w_2 A) A_P(w_2, \xi, A). \end{aligned}$$

In dealing with questions of reducibility and complementary series, we shall use also a slight variant of $A_P(w, \xi, A)$ addressed in Proposition 7.10 below. We first need the construction in Lemma 7.9.

Lemma 7.9.[9] *Let $H \subseteq H'$ be locally compact groups with H closed and normal and with H'/H cyclic of order n, let x be an element of H' whose powers meet all cosets of H'/H, and let L be an irreducible unitary representation of H on a Hilbert space V such that L and xL are equivalent. Then it is possible to define $L(x)$ as an operator on V in exactly n ways, differing only by an n^{th} root of unity as a factor, such that L extends to a unitary representation of H' on V.*

Proof. For existence, let E be an operator, which we may take to be unitary, such that $L(x^{-1}hx)=E^{-1}L(h)E$ for h in H, and put $L(x)=e^{i\theta}E$, with θ to be specified shortly. According to Lemma 59 of [20], it suffices to check that

$$L(x)\,L(h)\,L(x)^{-1}=L(xhx^{-1}) \tag{7.12}$$

and

$$L(x)^n=L(x^n). \tag{7.13}$$

Now (7.12) follows from

$$L(x)\,L(h)\,L(x)^{-1}=e^{i\theta}E\,L(h)\,e^{-i\theta}E^{-1}=E\,L(h)\,E^{-1}=L(xhx^{-1}).$$

For (7.13), we have

$$L(x)^n\,L(x^{-n})=e^{in\theta}E^n\,L(x^{-n}), \tag{7.14}$$

and the conjugate of $L(h)$ by $E^n L(x^{-n})$ is

$$E^n\,L(x^{-n})\,L(h)\,L(x^n)\,E^{-n}=E^n\,L(x^{-n}hx^n)\,E^{-n}.$$

By successive applications of the definition of E, we see that the right side simplifies to $L(h)$. By Schur's Lemma, $E^n L(x^{-n})=cI$ for a constant c (necessarily of modulus 1). Define $e^{i\theta}$ by $e^{in\theta}=c^{-1}$. Then the right side of (7.14) is the identity, and so (7.13) holds. This proves existence.

For the uniqueness result, we observe that any choice of $L(x)$ exhibits xL and L as equivalent and hence by Schur's Lemma is a multiple of E. The existence proof found all possible multiples of E. The proof of the lemma is complete.

Returning to the situation with $P=MAN$ a parabolic subgroup and ξ an irreducible unitary representation of M, let us suppose that w is a member of $N_K(\mathfrak{a})$ such that $w\xi$ is equivalent with ξ. Taking H in the lemma to be M and H' to be the smallest group containing M and w, we see that we can define $\xi(w)$ so as to extend ξ to a larger group acting on the same space H^ξ.

Proposition 7.10. *If w is in $N_K(\mathfrak{a})$ and if $w\xi$ is equivalent with ξ, then the operators $\xi(w)A_P(w,\xi,\Lambda)$, defined on K-finite functions for $U_P(\xi,\Lambda,\cdot)$, have the following properties:*

(i) *Let F be a finite set of K-types, and let E_F be the orthogonal projection onto the span of all functions of some K-type in F. Then*

[9] S. Lichtenbaum has pointed out to us that when $\dim V=1$, the lemma is an immediate consequence of the long exact sequence for cohomology of groups and the easy fact that H^2 of a finite cyclic group with coefficients in the circle is 0.

$$E_F U_P(\xi, wA, x) E_F \xi(w) A_P(w, \xi, A) = \xi(w) A_P(w, \xi, A) E_F U_P(\xi, A, x) E_F$$

for all x in G.

(ii) $[\xi(w) A_P(w, \xi, A)]^* = \xi(w)^{-1} A_P(w^{-1}, \xi, -w\bar{A})$, *with the adjoint defined K-space by K-space.*

Proof. For conclusion (i), we note that if f is in the space for $U_P(w\xi, wA, \cdot)$, then $\xi(w)f$ is in the space for $U_P(\xi, A, \cdot)$. Then (i) follows from Proposition 7.8(i). For (ii) we use Proposition 7.8, parts (ii) and (iii), to find

$$\begin{aligned}
[\xi(w) A_P(w, \xi, A)]^* &= A_P(w, \xi, A)^* \, \xi(w)^{-1} \\
&= A_P(w^{-1}, w\xi, -w\bar{A}) \, \xi(w)^{-1} \\
&= \xi(w)^{-1} A_P(w^{-1}, \xi(w)(w\xi)\xi(w)^{-1}, -w\bar{A}) \\
&= \xi(w)^{-1} A_P(w^{-1}, \xi, -w\bar{A}).
\end{aligned}$$

A final remark is in order. The operator $\xi(w) A_P(w, \xi, A)$ is unchanged if w is replaced by wm with m in K_M. In other words, the operator depends only on the element of $W(\mathfrak{a})$ that w represents. We shall occasionally use notation that incorporates this fact.

§ 8. Normalization of Operators, General Case

We retain the notation of § 6, working with general parabolic subgroups. The results in this section were announced in [23]. Taking into account the functions $\eta_\xi(z)$ and their properties as given in Proposition 7.4, we recall the following lemma.

Lemma 8.1 (Lemma 36 of [20]). *If $\eta(z)$ is a meromorphic function in the plane that is not identically 0 and is such that*

(i) $\eta(z) = \overline{\eta(-\bar{z})}$ *for all z and*

(ii) $\eta(z) \geq 0$ *on the imaginary axis,*

then there exists a meromorphic function $\gamma(z)$ in the plane such that

$$\eta(z) = \gamma(z)\overline{\gamma(-\bar{z})}. \tag{8.1}$$

The function $\gamma(z)$ can be chosen to be regular and nonvanishing whenever $\eta(z)$ is and to have all its zeros in the closed right half plane and all its poles in the closed left half plane. If also $\eta(z)$ satisfies

(iii) $\eta(z) = \eta(-z)$ *for all z,*

then $\gamma(z)$ can be chosen to be real for real z.

We shall apply this lemma to our functions $\eta_\xi(z)$ to obtain normalizing factors $\gamma_\xi(z)$. A construction of such factors was made in [20], but we shall redo the construction even there because, by error, the functions $\gamma_\xi(z)$ in [20] may not have had the important invariance property suggested by Proposition 3.2(e).[10]

Thus assume now that $P=MAN$ and $P'=MAN'$ are any parabolic subgroups. We first treat the case that $\dim A=1$. Guided by Proposition 7.4(f), we introduce the following

Basic Assumption. The infinitesimal character of the irreducible unitary representation ξ of M is a real linear combination of the roots of M.

What we do is select simultaneously for each scalar-valued meromorphic function $\eta(z)$ of one complex variable satisfying (i), (ii), and (iii) in Lemma 8.1 a meromorphic $\gamma(z)$ that satisfies (8.1) and is real for real z. (When it is convenient to do so, we shall assume that $\gamma(z)$ is regular and nonvanishing whenever $\eta(z)$ is and that $\gamma(z)$ has all its zeros in the closed right half plane and all its poles in the closed left half plane.)

Proposition 7.4 says that $\eta_\xi(z)$ is such a function $\eta(z)$, and we let $\gamma_\xi(z)$ be the corresponding function $\gamma(z)$. In this way, we obtain $\gamma_\xi(z)=\gamma_{\xi'}(z)$ whenever $\eta_\xi(z)=\eta_{\xi'}(z)$, in particular when $\xi'=\xi^\varphi$ (see Proposition 7.4(e)).

Restoring the notation that carries all the variables, let us write

$$\gamma(\bar{P}:P:\xi:\Lambda) \qquad \text{for } \gamma_\xi(z)$$

whenever $\Lambda=z\rho_P$ and $\eta_\xi(z)$ is given by $\eta(\bar{P}:P:\xi:\Lambda)$.

Lemma 8.2. *If* $\dim A=1$, *then*

(i) $\gamma(P:\bar{P}:\xi:\Lambda)=\gamma(\bar{P}:P:\xi:-\Lambda)$,

(ii) $\gamma(P:\bar{P}:\xi:\Lambda)\,\gamma(\bar{P}:P:\xi:\Lambda)=\eta(\bar{P}:P:\xi:\Lambda)$.

Proof. Let $\Lambda=z\rho_P=-z\rho_{\bar{P}}$. By Proposition 7.3

$$\eta(P:\bar{P}:\xi:\Lambda)=\eta(\bar{P}:P:\xi:\Lambda),$$

and hence

$$\eta(P:\bar{P}:\xi:-z\rho_{\bar{P}})=\eta(\bar{P}:P:\xi:z\rho_P).$$

Because of the Basic Assumption and Proposition 7.4(f),

$$\eta(P:\bar{P}:\xi:-z\rho_{\bar{P}})=\eta(P:\bar{P}:\xi:z\rho_{\bar{P}}).$$

Therefore

$$\eta(P:\bar{P}:\xi:z\rho_{\bar{P}})=\eta(\bar{P}:P:\xi:z\rho_P).$$

That is, the function $\eta_\xi(z)$ for $(P:\bar{P})$ is the same as for $(\bar{P}:P)$, and the same must be true of $\gamma_\xi(z)$. Consequently

$$\gamma(P:\bar{P}:\xi:\Lambda)=\gamma(P:\bar{P}:\xi:-z\rho_{\bar{P}})=\gamma(\bar{P}:P:\xi:-z\rho_P)=\gamma(\bar{P}:P:\xi:-\Lambda)$$

[10] This error becomes troublesome when we pass to G of higher rank. In a construction in that case, we recognize a certain subgroup as being of real-rank one, and we do not want our normalizing factors to depend on what isomorphism we choose between this subgroup and a standard real-rank one group.

and (i) follows. The right term here equals the complex conjugate of

$$\gamma(\bar{P}:P:\xi:-\bar{z}\rho_P),$$

since $\gamma_\xi(z)$ is real for real z. Substituting in (8.1), we obtain (ii).

For the general case with $\dim A$ arbitrary, we define

$$\gamma(P':P:\xi:\Lambda)=\prod_{\substack{\beta \text{ reduced} \\ \beta>0 \text{ for } P \\ \beta<0 \text{ for } P'}} \gamma^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)}),$$

where $\gamma^{(\beta)}$ is a factor attached to data for the group $G^{(\beta)}$, β an $\mathfrak{a}$-root. The normalized intertwining operators are defined, *under our Basic Assumption*, by

$$\mathscr{A}(P':P:\xi:\Lambda)=\gamma(P':P:\xi:\Lambda)^{-1}A(P':P:\xi:\Lambda)$$
$$\mathscr{A}_P(w,\xi,\Lambda)=\gamma(w^{-1}Pw:P:\xi:\Lambda)^{-1}A_P(w,\xi,\Lambda).$$

Once the initial choice of functions $\gamma(z)$ has been made, these definitions are unambiguous. Notice that $\gamma(P':P:\xi:\Lambda)$ is not identically 0 since it is a product of functions that satisfy (8.1), and $\eta_\xi(z)$ is not identically 0 by Proposition 7.4(c).

Lemma 8.3. *Let $P=MAN$ and $P'=MAN'$ be associated parabolic subgroups. Then*

(i) $\mathscr{A}(P:P':\xi:\Lambda)\mathscr{A}(P':P:\xi:\Lambda)=I$,

(ii) *for any minimal string $P_i=MAN_i$, $0\leqq i\leqq r$, from P to P',*

$$\mathscr{A}(P':P:\xi:\Lambda)=\mathscr{A}(P_r:P_{r-1}:\xi:\Lambda)\cdot\ldots\cdot\mathscr{A}(P_1:P_0:\xi:\Lambda).$$

Proof. Conclusion (ii) is trivial from Theorem 7.6, parts (iii) and (i), and from the definition of the normalization. For (i), we use (ii) and Theorem 7.6(iv) to see that it is enough to conclude that

$$\mathscr{A}(P_{i-1}:P_i:\xi:\Lambda)\mathscr{A}(P_i:P_{i-1}:\xi:\Lambda)=I$$

for each i. Let β be the associated P_{i-1}-positive reduced $\mathfrak{a}$-root here. The left side, in view of Proposition 7.3 and the definition of the normalizing factors, is

$$\gamma^{(\beta)}(P^{(\beta)}:\theta P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)})^{-1}\gamma^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)})^{-1}\eta(P_i:P_{i-1}:\xi:\Lambda)I, \tag{8.2}$$

and the η factor here equals

$$\eta^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)})$$

by Proposition 7.5. Thus (8.2) collapses to I by Lemma 8.2(ii), and the result follows.

Theorem 8.4. *For any three associated parabolic subgroups $P_i=MAN_i$ with $1\leqq i\leqq 3$,*

$$\mathscr{A}(P_3:P_1:\xi:\Lambda)=\mathscr{A}(P_3:P_2:\xi:\Lambda)\mathscr{A}(P_2:P_1:\xi:\Lambda).$$

Proof. We shall suppress ξ and Λ in the notation. By Lemma 8.3(i), we are to show that

$$\mathscr{A}(P_1:P_3)\mathscr{A}(P_3:P_2)\mathscr{A}(P_2:P_1)=I. \tag{8.3}$$

Choose minimal strings from P_3 to P_1, from P_2 to P_3, and from P_1 to P_2, and replace each of the factors on the left of (8.3) by the product of operators given in Lemma 8.3(ii). Then we can change notation and reformulate the result in question as follows: If $P_0, P_1, \ldots, P_n$ is a string with $P_0=P_n$, then

$$\mathscr{A}(P_n:P_{n-1})\cdot \ldots \cdot \mathscr{A}(P_2:P_1)\mathscr{A}(P_1:P_0)=I. \tag{8.4}$$

We shall prove this result by induction on n. We may assume that no two consecutive parabolics in the string are the same, since $\mathscr{A}(P:P)=I$.

First we show that n is even. Arguing as in the first part of the proof of Theorem 4.2, we see that

$$\mathfrak{n}_i\cap\mathfrak{n}_0=\mathfrak{n}^{(\beta_{i+1})}\oplus(\mathfrak{n}_{i+1}\cap\mathfrak{n}_0) \quad \text{if } V_i\cap N_{i+1}=V^{(\beta_{i+1})}$$

and

$$\mathfrak{n}_{i+1}\cap\mathfrak{n}_0=\mathfrak{n}^{(\beta_{i+1})}\oplus(\mathfrak{n}_i\cap\mathfrak{n}_0) \quad \text{if } V_i\cap N_{i+1}=N^{(\beta_{i+1})}.$$

That is, the number of reduced $\mathfrak{a}$-roots β such that $\mathfrak{g}_\beta\subseteq\mathfrak{n}_i\cap\mathfrak{n}_0$ either increases or decreases by one in passing from i to $i+1$. Since $P_0=P_n$, the number of such roots is the same for $i=0$ as it is for $i=n$, and it follows that n must be even.

The same argument shows that if P and P' are parabolics that are connected by a nonminimal string $P=P^{(0)}, P^{(1)}, \ldots, P^{(r)}=P'$, than any minimal string $P=Q^{(0)}, Q^{(1)}, \ldots, Q^{(s)}=P'$ between P and P' must have $s<r$. In fact, the counting argument of the previous paragraph shows that the number of reduced $\mathfrak{a}$-roots β such that $\mathfrak{g}_\beta\subseteq\mathfrak{n}$ exceeds the number such that $\mathfrak{g}_\beta\subseteq\mathfrak{n}'\cap\mathfrak{n}$ by less than r (and also by exactly s). Thus $s<r$.

We return to (8.4). For $n=2$, the result follows from Lemma 8.3. Fix n and the expression on the left of (8.4), and assume that such an identity holds for all shorter "circular" strings. We examine the strings

$$P_0, P_1, \ldots, P_{\frac{n}{2}} \tag{8.5a}$$

$$P_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \ldots, P_n \tag{8.5b}$$

and distinguish three cases.

If both (8.5a) and (8.5b) are minimal, then Lemma 8.3(ii) allows us to collapse the left side of (8.4) to

$$\mathscr{A}(P_n:P_{n/2})\mathscr{A}(P_{n/2}:P_0),$$

and this collapses to I by Lemma 8.3(i) since $P_0=P_n$.

If (8.5a) is not minimal, let $Q_0, \ldots, Q_r$ be a minimal string with $P_0=Q_0$ and $P_{n/2}=Q_r$. We have seen that $r<n/2$. Then

$$P_0=Q_0, Q_1, \ldots, Q_{r-1}, Q_r=P_{\frac{n}{2}}, P_{\frac{n}{2}-1}, \ldots, P_0$$

is a string with $r+\frac{n}{2}<n$ members, beginning and ending with P_0. By inductive hypothesis

$$\mathscr{A}(Q_0:Q_1)\mathscr{A}(Q_2:Q_3)\ldots\mathscr{A}(Q_{r-1}:Q_r)\mathscr{A}(P_{\frac{n}{2}}:P_{\frac{n}{2}-1})\ldots\mathscr{A}(P_1:P_0)=I$$

and hence, by Lemma 8.3(i),

$$\mathscr{A}(P_{\frac{n}{2}}:P_{\frac{n}{2}-1})\ldots\mathscr{A}(P_1:P_0)=\mathscr{A}(Q_r:Q_{r-1})\ldots\mathscr{A}(Q_1:Q_0).$$

We make this substitution on the left side of (8.4). Then

$$P_0=P_n,\, P_{n-1},\, \ldots,\, P_{\frac{n}{2}}=Q_r,\, \ldots,\, Q_0=P_0$$

is another string of $r+\frac{n}{2}$ members to which we can apply the inductive hypothesis. Thus the left side of (8.4) collapses to I.

Finally if (8.5b) is not minimal, nor is the string in the reverse order. Then no string of that length can be a minimal string from P_0 to $P_{n/2}$, and (8.5a) cannot be minimal. Thus we are reduced to the previous case. This proves the theorem.

Proposition 8.5. *The operators $\mathscr{A}(P_2:P_1:\xi:\Lambda)$, defined on K-finite functions for $U_{P_1}(\xi,\Lambda,\cdot)$, have the following properties:*

(i) *Let F be a finite set of K-types, and let E_F be the orthogonal projection onto the span of all functions of one of the K-types in F. Then*

$$E_F\,U_{P_2}(\xi,\Lambda,x)\,E_F\,\mathscr{A}(P_2:P_1:\xi:\Lambda)=\mathscr{A}(P_2:P_1:\xi:\Lambda)\,E_F\,U_{P_1}(\xi,\Lambda,\cdot)\,E_F$$

for all x in G.

(ii) $\mathscr{A}(P_2:P_1:E\xi E^{-1}:\Lambda)=E\mathscr{A}(P_2:P_1:\xi:\Lambda)E^{-1}$ *if E is a unitary operator on* H^ξ.

(iii) *If w is in $N_K(\mathfrak{a})$, then*

$$\mathscr{A}(P_2:P_1:\xi:\Lambda)=R(w)^{-1}\mathscr{A}(w P_2 w^{-1}:w P_1 w^{-1}:w\xi:w\Lambda)\,R(w).$$

(iv) $\mathscr{A}(P_2:P_1:\xi:\Lambda)^*=\mathscr{A}(P_1:P_2:\xi:-\bar{\Lambda})$, *with the adjoint defined K-space by K-space.*

(v) $\mathscr{A}(P_2:P_1:\xi:\Lambda)$ *extends to a holomorphic function of Λ for Λ imaginary, is unitary for every imaginary value of Λ, and, for such Λ, exhibits $U_{P_1}(\xi,\Lambda,\cdot)$ and $U_{P_2}(\xi,\Lambda,\cdot)$ as unitarily equivalent.*

Proof. Property (i) is immediate from Proposition 7.1, as is (ii) if we take into account Proposition 7.4(a). For (iii), Proposition 7.1 shows it is enough to prove

$$\gamma(P_2:P_1:\xi:\Lambda)=\gamma(w P_2 w^{-1}:w P_1 w^{-1}:w\xi:w\Lambda). \tag{8.6}$$

The left side here is

$$\prod_{\substack{\beta \text{ reduced}\\ \beta>0 \text{ for } P_1\\ \beta<0 \text{ for } P_2}} \gamma^{(\beta)}(\theta P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)}),$$

and the right side is

$$\prod_{\substack{\beta' \text{ reduced} \\ \beta'>0 \text{ for } wP_1w^{-1} \\ \beta'<0 \text{ for } wP_2w^{-1}}} \gamma^{(\beta')}(\theta P^{(\beta')} : P^{(\beta')} : w\xi : wA|_{\mathfrak{a}^{(\beta')}}).$$

For $\beta' = w\beta$, let us notice that $G^{(\beta)}$ and $G^{(\beta')}$ are isomorphic since $G^{(\beta')} = wG^{(\beta)}w^{-1}$. This isomorphism matches the parameters in $\gamma^{(\beta')}$ and $\gamma^{(\beta)}$ and shows the two products are equal, factor by factor. This proves (8.6) and (iii).

For (iv), Proposition 7.1 (iv) shows we want

$$\gamma(P_2 : P_1 : \xi : A) = \overline{\gamma(P_1 : P_2 : \xi : -\bar{A})}.$$

Expanding each side as a product of factors $\gamma^{(\beta)}$, we see we are to prove that

$$\gamma^{(\beta)}(\theta P^{(\beta)} : P^{(\beta)} : \xi : A|_{\mathfrak{a}^{(\beta)}}) = \overline{\gamma^{(\beta)}(P^{(\beta)} : \theta P^{(\beta)} : \xi : -\bar{A}|_{\mathfrak{a}^{(\beta)}})}. \tag{8.7}$$

These factors are real for A a real multiple of $\rho^{(\beta)}$, and the right side of (8.7) is therefore

$$= \gamma^{(\beta)}(P^{(\beta)} : \theta P^{(\beta)} : \xi : -A|_{\mathfrak{a}^{(\beta)}}),$$

which equals the left side of (8.7) by Lemma 8.2 (i). This proves (iv).

For (v), Theorem 6.6 and Proposition 7.3 show that $A(P_2 : P_1 : \xi : A)$ and $\eta(P_2 : P_1 : \xi : A)$ are both holomorphic on a dense open set of imaginary A. Moreover, Proposition 7.4(c) and Theorem 7.6(v) show that $\eta(P_2 : P_1 : \xi : A)$ is nonzero on a dense open set of imaginary A, and therefore the same thing is true for $\gamma(P_2 : P_1 : \xi : A)$. Consequently $\mathscr{A}(P_2 : P_1 : \xi : A)$ is holomorphic on a dense open set of imaginary A. Applying (iv) and Lemma 8.3 (i), we obtain (v) for such A, with the statement about equivalence following from (i).

The full conclusion of (v) will follow by a passage to the limit, provided we show that $\mathscr{A}(P_2 : P_1 : \xi : A)$ extends to a holomorphic function for A imaginary. If $\dim A = 1$, we can go over the above argument to see that the normalized operator can fail to be holomorphic only on a discrete set. At each point of this set, the possible singularity is at worst a pole but then is removable since $\mathscr{A}(P_2 : P_1 : \xi : A)$ is unitary for those imaginary A where it is defined. Applying Proposition 7.5 and the definition of normalizing factors, we see that $\mathscr{A}(P' : P : \xi : A)$ extends to be holomorphic for A imaginary, provided $V \cap N' = V^{(\beta)}$ for a reduced $\mathfrak{a}$-root β. By Lemma 8.3 (ii), $\mathscr{A}(P_2 : P_1 : \xi : A)$ itself extends to be holomorphic for A imaginary in all cases.

Proposition 8.6. *The operators $\mathscr{A}_P(w, \xi, A)$, defined on K-finite functions for $U_P(\xi, A, \cdot)$, have the following properties:*

(i) *Let F be a finite set of K-types, and let E_F be the orthogonal projection onto the span of all functions of some K-type in F. Then*

$$E_F U_P(w\xi, wA, x) E_F \mathscr{A}_P(w, \xi, A) = \mathscr{A}_P(w, \xi, A) E_F U_P(\xi, A, x) E_F$$

for all x in G.

(ii) $\mathscr{A}_P(w, E\xi E^{-1}, A) = E\mathscr{A}_P(w, \xi, A)E^{-1}$ *if E is a unitary operator on H^ξ.*

(iii) $\mathscr{A}_P(w, \xi, \Lambda)^* = \mathscr{A}_P(w^{-1}, w\xi, -w\bar{\Lambda})$, *with the adjoint defined K-space by K-space.*

(iv) $\mathscr{A}_P(w, \xi, \Lambda)$ *is defined and unitary for all* Λ *imaginary.*

(v) $\mathscr{A}_P(w_1 w_2, \xi, \Lambda) = \mathscr{A}_P(w_1, w_2\xi, w_2\Lambda)\, \mathscr{A}_P(w_2, \xi, \Lambda)$.

Proof. Conclusion (i) is immediate from Proposition 7.8, and (ii) and (iii) are obtained by repeating the proofs of Proposition 7.8 but relying on Proposition 8.5 instead of Proposition 7.1. Conclusion (iv) follows from Proposition 8.5(v) and the fact the $R(w)$ is unitary. Conclusion (v) is obtained by repeating the calculation of Proposition 7.8 but relying on Theorem 8.4 instead of Corollary 7.7 and on Proposition 8.5(iii) instead of Proposition 7.1(iii).

Corollary 8.7. *If* w *is in* $N_K(\mathfrak{a})$ *and if* $w\xi$ *is equivalent with* ξ, *then the operators* $\xi(w)\mathscr{A}_P(w, \xi, \Lambda)$, *defined on K-finite functions for* $U_P(\xi, \Lambda, \cdot)$, *have the following properties:*

(i) *Let* F *be a finite set of K-types, and let* E_F *be the orthogonal projection onto the span of all functions of some K-type in* F. *Then*

$$E_F U_P(\xi, w\Lambda, x) E_F \xi(w) \mathscr{A}_P(w, \xi, \Lambda) = \xi(w) \mathscr{A}_P(w, \xi, \Lambda) E_F U_P(\xi, \Lambda, x) E_F$$

for all x *in* G.

(ii) $[\xi(w)\mathscr{A}_P(w, \xi, \Lambda)]^* = \xi(w)^{-1} \mathscr{A}_P(w^{-1}, \xi, -w\bar{\Lambda})$, *with the adjoint defined K-space by K-space.*

Proof. This follows by comparing Propositions 7.10 and 8.6.

II. Applications to Reducibility Questions

§ 9. Connection with Eisenstein Integrals and c-functions

We mentioned in the introduction that the theory of intertwining operators and the proof of the Plancherel theorem should be regarded as complementary theories. One's first inclination might be to expect the theories to be identical. Indeed, there is a parallel structure to them. Matrix coefficients of induced representations lead, via asymptotics, to intertwining operators, as was pointed out in a special case in [19], and composition of intertwining operators leads to the η-functions. Meanwhile, within Harish-Chandra's theory, Eisenstein integrals lead, via asymptotics, to c-functions, and composition of c-functions leads to the Plancherel measure. It was already observed in [18] that intertwining operators and c-functions can be expressed in terms of each other in some special cases, and [19] gave a connection between η-functions and Plancherel measures. In the year 1971–72, Nolan Wallach (in unpublished work) extended these matters, explicitly relating matrix coefficients to Eisenstein integrals and relating intertwining operators to c-functions. In particular, he obtained formula (9.4) below.

But it is apparent that the parallel between the two theories leads not to similar results, but to complementary ones. For example, Harish-Chandra's results lead to an upper bound on the dimension of the space of self-intertwining operators, while the theory of intertwining operators itself leads to a lower bound.

The purpose of this section is to bring Harish-Chandra's results to bear on our own theory, after showing in detail the correspondence between the two theories. This material was announced in [22].

The notation for this section will be as follows. We shall work with a *cuspidal* parabolic subgroup $P=MAN$ (i.e., one in which rank $M=\operatorname{rank} K_M$) and its associated parabolics with the same MA. The representation ξ will always be in the discrete series of M. Then the Basic Assumption of §8 is satisfied: the infinitesimal character of ξ is a real linear combination of the roots of M. For the most part, our notation will be similar to Harish-Chandra's ([12] and [15]), except that our $\mathfrak{a}$-parameter and his will be off by a factor of i.

Let F be a finite set of irreducible representations of K, and let α_F be the sum of the degrees times characters of the members of F. Let V_F be the space of complex-valued functions f on $K\times K$ such that

$$\alpha_F * f(\cdot, k_2)=f(\cdot, k_2) \quad \text{and} \quad \bar{\alpha}_F * f(k_1,\cdot)=f(k_1,\cdot).$$

Define a double representation τ of K on V_F by

$$\tau(k_1) f \tau(k_2)(k, k')=f(k_1^{-1} k, k_2 k').$$

Let ${}^0\mathscr{C}_\xi(M, \tau_M)$ be the space of all functions ψ from M to V_F such that

$$\psi(k_1 m k_2)=\tau(k_1)\psi(m)\tau(k_2) \qquad \text{for } m\in M,\ k_1\in K_M,\ k_2\in K_M$$

and such that the entries of ψ are linear combinations of matrix entries of ξ. Finally let H_F be the subspace of functions f in the representation space of $U(\xi, \Lambda, \cdot)$, regarded as H^ξ-valued functions on K, such that $\alpha_F * f=f$. It is easy to check that the linear map $T\to\psi_T$ given by

$$\psi_T(m)(k_1, k_2)=d_\xi \operatorname{Tr}(e^* \xi(m)\, e\, L(k_2)\, T L(k_1^{-1})),$$

where

$e=$ evaluation at 1
$d_\xi=$ formal degree of ξ
$L=$ left regular representation of K,

carries $\operatorname{End} H_F$ into ${}^0\mathscr{C}_\xi(M, \tau_M)$. The lemma below is proved in [15], p. 133 and §9. See also [35].

Lemma 9.1. *The linear map $T\to\psi_T$ of* $\operatorname{End}(H_F)$ *into ${}^0\mathscr{C}_\xi(M,\tau_M)$ is an isomorphism onto. Apart from a scalar factor, this linear isomorphism is an isometry under the definitions*

$$(T_1, T_2)=\operatorname{Tr}(T_1 T_2^*) \qquad \textit{for } T_1, T_2\in \operatorname{End} H_F$$

$$(\psi_1, \psi_2)=\int_{K\times K}\int_M \psi_1(m)(k_1, k_2)\,\overline{\psi_2(m)(k_1, k_2)}\, dm\, dk_1\, dk_2$$

$$\text{for } \psi_1, \psi_2\in {}^0\mathscr{C}_\xi(M, \tau_M).$$

Moreover, $\psi_T * \psi_S = \psi_{ST}$ if a product on V_F is defined by

$$f \cdot g(k_1, k_2) = \int_K f(k_1, k) g(k, k_2) dk.$$

Using notation that is off by a factor of i from Harish-Chandra's, we define Eisenstein integrals by

$$E(P:\psi:\Lambda:x) = \int_K \psi(xk)\tau(k)^{-1} e^{(\Lambda-\rho_P)H_P(xk)} dk$$

for ψ in ${}^0\mathscr{C}_\xi(M, \tau_M)$, under the convention $\psi(kman) = \tau(k)\psi(m)$. The lemma below is proved in §9 of [15]. See also [35].

Lemma 9.2. *Let E_F denote the orthogonal projection on H_F. Then*

$$E(P:\psi_T:\Lambda:x)(k_1, k_2) = d_\xi \operatorname{Tr}(E_F U_P(\xi, \Lambda, k_1^{-1} x k_2) T E_F)$$

for ψ_T in ${}^0\mathscr{C}_\xi(M, \tau_M)$.

Before proceeding, let us underline the connection between Eisenstein integrals and matrix coefficients. In one direction, Lemma 9.1 says that every ψ is of the form ψ_T, and therefore Lemma 9.2 expresses all Eisenstein integrals in terms of matrix coefficients. In the reverse direction, take $k_1 = k_2 = 1$, let f and g be members of H_F, and define $Th = (h, g) f$. Then we see from Lemma 9.2 that

$$E(P:\psi_T:\Lambda:x)(1, 1) = d_\xi (U_P(\xi, \Lambda, x) f, g)_{H^\xi}.$$

Hence all matrix coefficients can be obtained from Eisenstein integrals.

Harish-Chandra (p. 131 of [12]) obtained asymptotics for Eisenstein integrals, showing, for Λ imaginary and for $f = E(P:\psi:\Lambda:\cdot)$, that there exists a unique $f_{P'}$ on MA transforming appropriately and satisfying

$$\lim_{\substack{a \to \infty \\ P'}} \{e^{\rho_{P'} \log a} f(ma) - f_{P'}(ma)\} = 0.$$

For the proof, see Theorem 21.1 and Lemma 19.1 of [13] and Lemma 17.1 of [14]. Harish-Chandra gave an expansion for $f_{P'}$ for Λ regular (p. 134 of [12]):

$$E_{P'}(P:\psi:\Lambda:ma) = \sum_{s \in W(\mathfrak{a})} (c_{P'|P}(s:\Lambda)\psi)(m) e^{s\Lambda \log a},$$

where $c_{P'|P}(s:\Lambda)$ is in $\operatorname{End}({}^0\mathscr{C}_\xi(M, \tau_M))$ and the dependence on Λ is meromorphic. See Theorem 18.1 of [14]. For $c_{P|P}(1:\Lambda)$, an integral formula is given as Theorem 19.1 of [14] for $\operatorname{Re}\Lambda$ sufficiently far out in the P-positive Weyl chamber:

$$\begin{aligned} &c_{P|P}(1:\Lambda)\psi(m)(k_1, k_2) \\ &\quad = C_P^{-1} \int_V \psi(m\mu(v)^{-1}) \tau(\kappa(v))^{-1}(k_1, k_2) e^{-(\rho_P+\Lambda)H_P(v)} dv, \end{aligned} \tag{9.1}$$

$$C_P = \int_V e^{-2\rho_P H_P(v)} dv. \tag{9.2}$$

In applying these results, we shall work with ξ and its transforms $w\xi$ by $N_K(\mathfrak{a})$, and we shall be led outside the space ${}^0\mathscr{C}_\xi(M, \tau_M)$. To emphasis the space to which ψ belongs, we shall write ψ^ξ or $\psi^{w\xi}$, as needed, and the c functions should be understood to be those appropriate to the space of ψ's being acted upon.

Proposition 9.3. *Let T be in* $\operatorname{End} H_F$, *let s be in $W(\mathfrak{a})$ with w in $N_K(\mathfrak{a})$ as a representative, and denote θP by $\bar{P}$. Then*

(i) $E(P_2 : \psi^\xi_{A(P_2:P_1:\xi:\Lambda)T} : \Lambda : x) = E(P_1 : \psi^\xi_{TA(P_2:P_1:\xi:\Lambda)} : \Lambda : x)$

and

$$c_{P_2|P_2}(s:\Lambda)\,\psi^\xi_{A(P_2:P_1:\xi:\Lambda)T} = c_{P_2|P_1}(s:\Lambda)\,\psi^\xi_{TA(P_2:P_1:\xi:\Lambda)}.$$

(ii) $E(P : \psi^\xi_T : \Lambda : x) = E(w^{-1}Pw : \psi^{w\xi}_{R(w)TR(w)^{-1}} : s\Lambda : x)$

and

$$c_{P|P}(s:\Lambda)\,\psi^\xi_T = c_{P|wPw^{-1}}(1:s\Lambda)\,\psi^{w\xi}_{R(w)TR(w)^{-1}}.$$

(iii) $c_{P|P}(1:\Lambda)\,\psi^\xi_T = C_P^{-1}\,\psi^\xi_{A(P:P:\xi:\Lambda)T}$, *where C_P is defined as in* (9.2).

(iv) $\psi^{w\xi}_S = \psi^\xi_T$ *if $w\xi$ is equivalent with ξ and $S = \xi(w)^{-1}T\xi(w)$, where $\xi(w)$ is defined as in Lemma* 7.9.

Remarks. From (i) we can express $c_{P|P'}(s:\Lambda)$ in terms of $c_{P|P}(s:\Lambda)$, which by (ii) can be expressed in terms of $c_{P|wPw^{-1}}(1:s\Lambda)$, which by (i) can be expressed in terms of $c_{P|P}(1:s\Lambda)$, which can be evaluated by (iii) in terms of intertwining operators. Thus all c-functions for G can be computed in terms of intertwining operators for G. When $w\xi$ is equivalent with ξ, (iv) shows that all the computations can be made in terms of a single mapping $T \to \psi^\xi_T$.

Proof. Starting from Lemma 9.2, we see that

$$\begin{aligned}
&E(P_2 : \psi^\xi_{A(P_2:P_1:\xi:\Lambda)T} : \Lambda : x)(k_1, k_2)\\
&= d_\xi \operatorname{Tr}(E_F U_{P_2}(\xi, \Lambda, k_1^{-1} x k_2) A(P_2 : P_1 : \xi : \Lambda) T E_F)\\
&= d_\xi \operatorname{Tr}(A(P_2 : P_1 : \xi : \Lambda) E_F U_{P_1}(\xi, \Lambda, k_1^{-1} x k_2) T E_F)\\
&\qquad \text{by Proposition 7.1(i)}\\
&= d_\xi \operatorname{Tr}(E_F U_{P_1}(\xi, \Lambda, k_1^{-1} x k_2) T E_F A(P_2 : P_1 : \xi : \Lambda))\\
&\qquad \text{since } \operatorname{Tr}(AB) = \operatorname{Tr}(BA)\\
&= d_\xi \operatorname{Tr}(E_F U_{P_1}(\xi, \Lambda_1^{-1} x k_2) T A(P_2 : P_1 : \xi : \Lambda) E_F)\\
&= E(P_1 : \psi^\xi_{TA(P_2:P_1:\xi:\Lambda)} : \Lambda : x)(k_1, k_2),
\end{aligned}$$

as a meromorphic identity in Λ. Suppressing (k_1, k_2), we form the P_2-limit of both Eisenstein integrals when Λ is imaginary and regular, obtaining

$$\begin{aligned}
&\sum_{s\in W} (c_{P_2|P_2}(s:\Lambda)\,\psi^\xi_{A(P_2:P_1:\xi:\Lambda)T})(m)\, e^{s\Lambda \log a}\\
&\quad = \sum_{s\in W} (c_{P_2|P_1}(s:\Lambda)\,\psi_{TA(P_2:P_1:\xi:\Lambda)})(m)\, e^{s\Lambda \log a}.
\end{aligned}$$

Since Λ is regular, we may equate coefficients to obtain (i) for Λ imaginary. Analytic continuation gives (i) for all Λ.

For (ii), the operator $T' = R(w) T R(w)^{-1}$ is in $\mathrm{End}(H'_F)$ for the associated space H'_F of functions transforming according to $w\xi$, and

$$U_P(\xi, \Lambda, x) T = R(w)^{-1} U_{wPw^{-1}}(w\xi, s\Lambda, x) T' R$$

by (7.11). Hence

$$\begin{aligned} E(P : \psi^\xi_T : \Lambda : x) &= d_\xi \operatorname{Tr}(E_F R(w)^{-1} U_{wPw^{-1}}(w\xi, s\Lambda, k_1^{-1} x k_2) T' R(w) E_F) \\ &= d_\xi \operatorname{Tr}(R(w)^{-1} E'_F U_{wPw^{-1}}(w\xi, s\Lambda, k_1^{-1} x k_2) T' E'_F R(w)) \\ &= d_\xi \operatorname{Tr}(E'_F U_{wPw^{-1}}(w\xi, s\Lambda, k_1^{-1} x k_2) T' E'_F) \\ &= E(wPw^{-1} : \psi^{w\xi}_{T'} : s\Lambda : x). \end{aligned}$$

For regular imaginary Λ, the P-limit of both sides gives

$$\sum_{t \in W} c_{P|P}(t : \Lambda) \psi^\xi_T e^{t\Lambda \log a} = \sum_{r \in W} c_{P|wPw^{-1}}(r : s\Lambda) \psi^{w\xi}_{T'} e^{rs\Lambda \log a}.$$

Equating the terms in which $t = s$ and $r = 1$ and using analytic continuation, we obtain (ii).

For (iii) we start from (9.1) for $\operatorname{Re}\Lambda$ sufficiently far out in the P-positive Weyl chamber and find

$$\begin{aligned} & C_P c_{P|P}(1 : \Lambda) \psi^\xi_T(m)(k_1, k_2) \\ &= \int_V \psi^\xi_T(m \mu(v)^{-1}) \tau(\kappa(v))^{-1}(k_1, k_2) e^{-(\rho_P + \Lambda) H_P(v)} \, dv \\ &= \int_V \psi^\xi_T(m \mu(v)^{-1})(k_1, \kappa(v)^{-1} k_2) e^{-(\rho_P + \Lambda) H_P(v)} \, dv \\ &= d_\xi \int_V Tr(e^* \xi(m) \xi(\mu(v))^{-1} e L(\kappa(v)^{-1} k_2) T L(k_1^{-1})) e^{-(\rho_P + \Lambda) H_P(v)} \, dv \\ &= d_\xi \sum_i \int_V (\xi(m) \xi(\mu(v))^{-1} T h_i(k_2^{-1} \kappa(v)), h_i(k_1^{-1}))_{H^\xi} e^{-(\rho_P + \Lambda) H_P(v)} \, dv \end{aligned}$$

where h_i runs through a (finite) orthonormal basis of H_F, and this is

$$\begin{aligned} &= d_\xi \sum_i (\xi(m) A(\bar{P} : P : \xi : \Lambda) T h_i(k_2^{-1}), h_i(k_1^{-1}))_{H^\xi} \\ &= \psi^\xi_{A(\bar{P} : P : \xi : \Lambda) T}(m)(k_1, k_2). \end{aligned}$$

Analytic continuation gives the result for general Λ.

Finally for (iv), with h_i running through an orthonormal basis of the space H_F for ξ, we have

$$\begin{aligned}\psi_S^{w\xi}(m)(k_1,k_2) &= d_\xi \operatorname{Tr}(e^* w\xi(m) e L(k_2) S L(k_1^{-1})) \\ &= d_\xi \sum_i (w\xi(m) S h_i(k_2^{-1}), h_i(k_1^{-1}))_{H^\xi} \\ &= d_\xi \sum_i (\xi(m)\, \xi(w) S h_i(k_2^{-1}), \xi(w) h_i(k_1^{-1}))_{H^\xi} \\ &= d_\xi \sum_i (\xi(m) T \xi(w) h_i(k_2^{-1}), \xi(w) h_i(k_1^{-1}))_{H^\xi} \\ &= d_\xi \operatorname{Tr}(e^* \xi(m) e L(k_2) T L(k_1)^{-1}) \\ &= \psi_T^\xi(m)(k_1,k_2),\end{aligned}$$

the next to last equality holding since $\xi(w) h_i$ runs through an orthonormal basis of the space for $w\xi$.

Corollary 9.4. *Let T be in $\operatorname{End} H_F$, let s be in $W(\mathfrak{a})$ with w in $N_K(\mathfrak{a})$ as a representative, and denote θP by $\bar{P}$. Then*

$$c_{P|P}(s:\Lambda)\,\psi_T^\xi = C_{\bar{P}}^{-1}\,\psi_{T'}^{w\xi},$$

where

$$T' = R(w) A(w^{-1}\bar{P}w : P : \xi : \Lambda) T A(P : w^{-1}Pw : \xi : \Lambda) R(w)^{-1}.$$

Remarks. Qualitatively the corollary says that each $c_{P|P}$ is given by a pair of "complementary" unnormalized intertwining operators, one operating on the left and one on the right. This result was obtained independently by Arthur [2].

Proof. We have

$$\begin{aligned} & c_{P|P}(s:\Lambda)\,\psi_T^\xi \\ &\quad = c_{P|wPw^{-1}}(1:s\Lambda)\,\psi^{w\xi}_{R(w)TR(w)^{-1}} \\ &\qquad \text{by Proposition 9.3(ii)} \\ &\quad = c_{P|P}(1:s\Lambda)\,\psi^{w\xi}_{A(P:wPw^{-1}:w\xi:s\Lambda)R(w)TR(w)^{-1}A(P:wPw^{-1}:w\xi:s\Lambda)^{-1}} \\ &\qquad \text{by Proposition 9.3(i)} \\ &\quad = c_{P|P}(1:s\Lambda)\,\psi^{w\xi}_{R(w)A(w^{-1}Pw:P:\xi:\Lambda)TA(w^{-1}Pw:P:\xi:\Lambda)^{-1}R(w)^{-1}} \\ &\qquad \text{by Proposition 7.1(iii)} \\ &\quad = C_{\bar{P}}^{-1}\,\psi^{w\xi}_{A(\bar{P}:P:w\xi:s\Lambda)R(w)A(w^{-1}Pw:P:\xi:\Lambda)TA(w^{-1}Pw:P:\xi:\Lambda)^{-1}R(w)^{-1}} \\ &\qquad \text{by Proposition 9.3(iii)} \\ &\quad = C_{\bar{P}}^{-1}\,\psi^{w\xi}_{R(w)A(w^{-1}\bar{P}w:w^{-1}Pw:\xi:\Lambda)A(w^{-1}Pw:P:\xi:\Lambda)TA(w^{-1}Pw:P:\xi:\Lambda)^{-1}R(w)^{-1}} \end{aligned} \tag{9.3}$$

by Proposition 7.1(iii). Now

$$\begin{aligned} & A(w^{-1}\bar{P}w : w^{-1}Pw : \xi : \Lambda) A(w^{-1}Pw : P : \xi : \Lambda) \\ &\quad = A(w^{-1}\bar{P}w : P : \xi : \Lambda) A(P : w^{-1}Pw : \xi : \Lambda) A(w^{-1}Pw : P : \xi : \Lambda) \\ &\qquad \text{by Corollary 7.7} \\ &\quad = \eta(w^{-1}Pw : P : \xi : \Lambda) A(w^{-1}\bar{P}w : P : \xi : \Lambda) \\ &\qquad \text{by Proposition 7.3} \end{aligned}$$

and

$$\begin{aligned} &A(w^{-1}Pw:P:\xi:\Lambda)^{-1}\\ &\quad=\eta(w^{-1}Pw:P:\xi:\Lambda)^{-1}A(P:w^{-1}Pw:\xi:\Lambda)\\ &\quad\quad\text{by Proposition 7.3.}\end{aligned}$$

Upon substitution, the η's cancel and the result follows.

Corollary 9.5. *Let T be in* $\operatorname{End} H_F$, *let s be in $W(\mathfrak{a})$ with w in $N_K(\mathfrak{a})$ as a representative, and denote θP by $\bar{P}$. If $w\xi$ and ξ are equivalent, then*

$$c_{P|P}(s:\Lambda)\psi_T^\xi = C_{\bar{P}}^{-1}\psi_{T'}^\xi,$$

where

$$T' = A(\bar{P}:P:\xi:s\Lambda)(\xi(w)\mathscr{A}_P(w,\xi,\Lambda))\,T(\xi(w)\mathscr{A}_P(w,\xi,\Lambda))^{-1}.$$

Proof. Starting from (9.3), we have

$$\begin{aligned} c_{P|P}(s:\Lambda)\psi_T^\xi &= c_{P|P}(1:s\Lambda)\psi^{w\xi}_{A_P(w,\xi,\Lambda)TA_P(w,\xi,\Lambda)^{-1}}\\ &= c_{P|P}(1:s\Lambda)\psi^{w\xi}_{\mathscr{A}_P(w,\xi,\Lambda)T\mathscr{A}_P(w,\xi,\Lambda)^{-1}}\\ &= c_{P|P}(1:s\Lambda)\psi^{\xi}_{\xi(w)\mathscr{A}_P(w,\xi,\Lambda)T(\xi(w)\mathscr{A}_P(w,\xi,\Lambda))^{-1}} \end{aligned} \tag{9.4}$$

by Proposition 9.3(iv). The corollary now follows from Proposition 9.3(iii).

Proposition 9.6. *If f and g are in H_F, then*

$$(A(\bar{P}:P:\xi:\Lambda)f,g)_{L^2(K)} = d_\xi^{-1}\,C_{\bar{P}}\int_K (c_{P|P}(1:\Lambda)\psi_T^\xi(1))(k,k)\,dk,$$

where $Th=(h,g)f$.

Remarks. From this result we can express $A(\bar{P}:P:\xi:\Lambda)$ for G in terms of c-functions for G. By Proposition 7.5 and Theorem 7.6, we can therefore express all unnormalized intertwining operators for G in terms of c-functions for G and its subgroups $G^{(\alpha)}$.

Proof. Take T as above, and let $\{h_i\}$ be an orthonormal basis of H_F with $h_1 = |g|^{-1}g$.

$$\begin{aligned} &\psi^\xi_{A(\bar{P}:P:\xi:\Lambda)T}(m)(k_1,k_2)\\ &\quad= d_\xi \operatorname{Tr}(e^*\xi(m)eL(k_2)A(\bar{P}:P:\xi:\Lambda)TL(k_1^{-1}))\\ &\quad= d_\xi\sum_i(\xi(m)A(\bar{P}:P:\xi:\Lambda)Th_i(k_2^{-1}),h_i(k_1^{-1}))_{H^\xi}\\ &\quad= d_\xi(\xi(m)A(\bar{P}:P:\xi:\Lambda)f(k_2^{-1}),g(k_1^{-1}))_{H^\xi}. \end{aligned}$$

By Proposition 9.3(iii), the left side is $C_{\bar{P}}c_{P|P}(1:\Lambda)\psi_T^\xi(m)(k_1,k_2)$. Hence the result follows.

Following Harish-Chandra, put

$${}^0c_{P_2|P_1}(s:\Lambda) = c_{P_2|P_2}(1:s\Lambda)^{-1}c_{P_2|P_1}(s:\Lambda).$$

Harish-Chandra's completeness theorem is the following. For a proof see Lemma 38.2 and Theorem 38.1 of [15].

Theorem 9.7 (Harish-Chandra). *Let Λ be imaginary and let*

$$W_{\xi,\Lambda}=\{s\in W(\mathfrak{a})\,|\,s[\xi]=[\xi]\text{ and } s\Lambda=\Lambda\}.$$

To each s in $W_{\xi,\Lambda}$ corresponds an operator γ_s in $GL(H_F)$ such that

$${}^0c_{P|P}(s,\Lambda)\psi_T^{\xi}=\psi_{\gamma_s T\gamma_s^{-1}}^{\xi}$$

for all T in $\operatorname{End} H_F$. *Moreover, the set of such γ_s, for s in $W_{\xi,\Lambda}$, spans the commuting algebra of $E_F U_P(\xi,\Lambda,\cdot)E_F$.*

Corollary 9.8. *Let Λ be imaginary and let $W_{\xi,\Lambda}$ be as in Theorem 9.7. Then the commuting algebra of $U_P(\xi,\Lambda,\cdot)$ is the linear span of the unitary operators $\xi(s)\mathscr{A}_P(s,\xi,\Lambda)$ for s in $W_{\xi,\Lambda}$.*[11]

Proof. It is enough to identify the commuting algebra of the operators projected to H_F. Corollary 9.5 and Proposition 9.3(iii) combine to give

$${}^0c_{P|P}(s:\Lambda)\psi_T^{\xi}=\psi_{T'}^{\xi},$$

where

$$T'=(\xi(w)\mathscr{A}_P(w,\xi,\Lambda))\,T(\xi(w)\mathscr{A}_P(w,\xi,\Lambda))^{-1}$$

if w is a representative of s. Hence we can take $\gamma_s=\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$ in Theorem 9.7. The operators $\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$ are unitary by Proposition 8.6(iv).

§10. Identities with Plancherel Factors

We continue to work with cuspidal parabolic subgroups built from the same MA, and we continue to assume that ξ is a discrete series representation of M. In this section we shall use Harish-Chandra's formula relating c-functions to the Plancherel measure in order to give formulas for the η-functions. As a consequence of these results and the known form of the Plancherel measure when G has $\dim\mathfrak{a}_p=1$, we obtain explicit formulas for the η-functions. These explicit formulas will be used in a later paper by the first author and G. Zuckerman.

According to Harish-Chandra [12], Theorem 11 and Lemma 15, the contribution to the Plancherel measure of G from the series $U_P(\xi,\Lambda,\cdot)$, with ξ in the discrete series of M and Λ imaginary, is

$$c\,d_\xi\,\mu_\xi(\Lambda), \tag{10.1}$$

where c is a constant, d_ξ is the formal degree of ξ, and $\mu_\xi(\Lambda)$ is a meromorphic function that is holomorphic for Λ imaginary, is positive for Λ imaginary and regular, satisfies $\mu_{w\xi}(w\Lambda)=\mu_\xi(\Lambda)$ for w representing a member of $W(\mathfrak{a})$, and is

[11] As noted at the end of §7, $\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$ does not depend upon the choice of the representative w of s, and we may therefore write the operator as $\xi(s)\mathscr{A}_P(s,\xi,\Lambda)$.

given by

$$|W(\mathfrak{a})|\, C_P \mu_\xi(\Lambda) c_{P|\bar P}(1:\Lambda) c_{\bar P|P}(1:\Lambda)\psi=\psi \tag{10.2}$$

for ψ in ${}^0\mathscr{C}_\xi(M,\tau_M)$, in the notation of §9.

Proposition 10.1. *The contribution $\mu_\xi(\Lambda)$ to the Plancherel measure of G is related to the η-function of* §7 *by*

$$\mu_\xi(\Lambda)^{-1}=|W(\mathfrak{a})|\, C_P^{-1}\eta(\bar P:P:\xi:\Lambda).$$

Proof. We apply Proposition 9.3. To simplify notation, let us suppress $(1:\Lambda)$ in the c-functions and $(\xi:\Lambda)$ in the intertwining operators and η-functions. Then we have

$$\begin{aligned}
c_{P|\bar P}\, c_{\bar P|P}\, \psi_T &= c_{P|\bar P}\, c_{\bar P|P}\, \psi_{A(\bar P:P)TA(\bar P:P)^{-1}}\\
&= C_{\bar P}^{-1}\, c_{P|\bar P}\, \psi_{A(P:\bar P)A(\bar P:P)TA(\bar P:P)^{-1}}\\
&= C_{\bar P}^{-1}\, \eta(\bar P:P)\, c_{P|\bar P}\, \psi_{TA(\bar P:P)^{-1}}\\
&= C_{\bar P}^{-1}\, \eta(\bar P:P)\, c_{P|\bar P}\, \psi_{A(P:\bar P)TA(\bar P:P)^{-1}A(P:\bar P)^{-1}}\\
&= C_{\bar P}^{-1}\, C_P^{-1}\, \eta(\bar P:P)\, \psi_{A(\bar P:P)A(P:\bar P)TA(\bar P:P)^{-1}A(P:\bar P)^{-1}}\\
&= C_{\bar P}^{-1}\, C_P^{-1}\, \eta(\bar P:P)\eta(P:\bar P)\eta(\bar P:P)^{-1}\psi_T.
\end{aligned}$$

Now $C_{\bar P}=C_P$, as follows from Lemma 2.8 by an easy argument, and $\eta(\bar P:P)=\eta(P:\bar P)$ by Proposition 7.3. The result follows.

We shall not use the full generality of Proposition 10.1 but shall use it only for the groups $G^{(\beta)}$. Let $\mu_{\xi,\beta}$ be the function of Proposition 10.1 in this case. Then we have

$$\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})^{-1}=|W(\mathfrak{a}^{(\beta)})|\, C_{P(\beta)}^{-1}\eta^{(\beta)}(\bar P^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)}). \tag{10.3}$$

We shall work with $\mu_{\xi,\beta}$ instead of $\eta^{(\beta)}$ in order to convert poles into zeros. For each pair of parabolics built from MA we define the *Plancherel factor* $\mu_{P'|P}$ by

$$\mu_{P'|P}(\xi:\Lambda)=\prod_{\substack{\beta \text{ reduced } \mathfrak{a}\text{-root}\\ \beta>0 \text{ for } P\\ \beta<0 \text{ for } P'}}\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)}). \tag{10.4}$$

Proposition 10.2. *The Plancherel factors have the following properties.*

(a) $\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})$ *is holomorphic for* $\Lambda|_{\mathfrak{a}(\beta)}$ *imaginary and is nonvanishing for* $\operatorname{Re}\Lambda|_{\mathfrak{a}(\beta)}\neq 0$.

(b) $\mu_{P'|P}(\xi:\Lambda)$ *is holomorphic for Λ imaginary.*

(c) $\mu_{P'|P}(\xi:\Lambda)=c\,\eta(P':P:\xi:\Lambda)^{-1}$, *with c a positive constant depending on P and P'.*

(d) $\mu_{P'|P}(\xi:\Lambda)=c'\mu_{P'_p|P_p}(\sigma:\Lambda\oplus\Lambda_M)$, *with c' a positive constant depending on P and P', if ξ imbeds in the nonunitary principal series representation of M with parameters (σ,Λ_M).*

Proof. In (a), $\mu_{\xi,\beta}$ is holomorphic by a result of Harish-Chandra quoted above or by Proposition B.2. Where it vanishes, $\eta^{(\beta)}$ has a pole, and Proposition 7.4(f) says this can happen only at real points. Then (b) follows from (a) and (10.4), (c) follows from Theorem 7.6(v) and (10.4), and (d) follows from Proposition 7.3 and (10.4).

§11. Intertwining Operators Corresponding to Reflections

Throughout this section ξ will denote a discrete series representation of M. The Plancherel factors μ will be those of §10. The point of this section will be to single out certain intertwining operators corresponding to reflections in $W(\mathfrak{a})$ that have to be scalar.

Our results use two lemmas announced by Harish-Chandra [12] and given as Lemmas 11.1 and 11.2 below. (The proofs of these lemmas appeared later in [15] in Lemmas 3.1 and 39.1.) The first lemma for $\Lambda=0$ has a long history. It was first proved for minimal parabolics in [19]. Arthur in his thesis [1] gave a completely different proof, which Harish-Chandra was able to extend to the case at hand.

Lemma 11.1. *Suppose $P=MAN$ has $\dim A=1$, and suppose Λ is imaginary. Then $U_P(\xi,\Lambda,\cdot)$ is irreducible unless $\Lambda=0$, $\mu_\xi(0)>0$, $|W(\mathfrak{a})|=2$, and $s[\xi]=[\xi]$, where s is the nontrivial element of $W(\mathfrak{a})$.*[12]

Lemma 11.2. *Suppose $P=MAN$ has $\dim A=1$. If $\mu_\xi(0)=0$, then $|W(\mathfrak{a})|=2$ and $s[\xi]=[\xi]$, where s is the nontrivial element in $W(\mathfrak{a})$.*[12]

Lemma 11.3. *Let $P=MAN$ and $P'=MAN'$ be parabolic subgroups such that $V\cap N'=V^{(\beta)}$ for a reduced $\mathfrak{a}$-root β. If the root reflection p_β exists in $W(\mathfrak{a})$, then $P'=p_\beta^{-1}Pp_\beta$.*

Proof. We are in the situation of Proposition 1.2. The problem is to show that the only P-positive roots γ such that $p_\beta\gamma$ is P-negative are the multiples of β. By Proposition 1.2 there exists a P_p-simple $\mathfrak{a}_p$-root α such that $\alpha|_{\mathfrak{a}}=\beta$. Let all the P_p-simple roots be

$$\alpha_1,\dots,\alpha_k,\alpha,\mu_1,\dots,\mu_l,$$

where $\mu_1,\dots,\mu_l$ are simple for $\mathfrak{a}_M$. Choose by Lemma 8 of [17] an element w of $W(\mathfrak{a}_p)$ such that $w|_{\mathfrak{a}}=p_\beta$. Then w fixes $\ker\beta$ in $\mathfrak{a}$ and must be the product of reflections in $W(\mathfrak{a}_p)$ fixing $\ker\beta$, by Chevalley's Lemma. Each such reflection must then carry each α_j into the sum of α_j and a linear combination of the roots $\alpha,\mu_1,\dots,\mu_l$, by the formula for a reflection, and then w must have the same property. Thus

$$w\left(\sum c_i\alpha_i+c\alpha+\sum c_j'\mu_j\right)=\sum c_i\alpha_i+d\alpha+\sum d_j'\mu_j.$$

If one of the c_i's is >0, the $\mathfrak{a}_p$-root on the right is >0. Thus the only positive $\mathfrak{a}$-roots that can go into negative $\mathfrak{a}$-roots under w are the multiples of β.

Lemma 11.4. *Let $P=MAN$ and $P'=MAN'$ be parabolic subgroups such that $V\cap N'=V^{(\beta)}$ for a reduced $\mathfrak{a}$-root β. If $\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})=0$ with Λ imaginary, then the*

[12] $[\xi]$ denotes the equivalence class of ξ.

root reflection p_β exists in $W(\mathfrak{a})$ and is such that $p_\beta[\xi]=[\xi]$, $p_\beta\Lambda=\Lambda$, and $\xi(p_\beta)\mathscr{A}_P(p_\beta,\xi,\Lambda)$ is scalar.

Proof. Let $\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})=0$. By Proposition 10.2(a), $\Lambda|_{\mathfrak{a}(\beta)}$ must be real. On the other hand, Λ is by assumption imaginary, and we conclude $\Lambda|_{\mathfrak{a}(\beta)}=0$. Hence $\mu_{\xi,\beta}(0)=0$. By Lemma 11.2, p_β exists in $W(\mathfrak{a}^{(\beta)})$, hence in $W(\mathfrak{a})$, and it fixes $[\xi]$. Since $\Lambda|_{\mathfrak{a}(\beta)}=0$, $p_\beta\Lambda=\Lambda$. By Lemma 11.1, $U^{(\beta)}_{P(\beta)}(\xi,0,\cdot)$ is irreducible. Then Corollary 8.7 shows that

$$\xi(p_\beta)\mathscr{A}^{(\beta)}_{P(\beta)}(p_\beta,\xi,0)=cI \quad \text{with} \quad |c|=1.$$

By abuse of notation, let us write p_β for a representative of the Weyl group element p_β within $K^{(\beta)}$. For k in K, we have

$$\begin{aligned}
cR(p_\beta)^{-1}f(k)&=cf(kp_\beta^{-1})=cf_k(p_\beta^{-1})\\
&=\xi(p_\beta)\mathscr{A}^{(\beta)}_{p(\beta)}(p_\beta,\xi,0)f_k(p_\beta^{-1})\\
&=\xi(p_\beta)R(p_\beta)\mathscr{A}(\bar{P}^{(\beta)}:P^{(\beta)}:\xi:0)f_k(p_\beta^{-1})\\
&=\xi(p_\beta)\gamma^{(\beta)}(\bar{P}^{(\beta)}:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)})^{-1}A(p_\beta^{-1}P^{(\beta)}p_\beta:P^{(\beta)}:\xi:\Lambda|_{\mathfrak{a}(\beta)})f_k(1)\\
&=\xi(p_\beta)\gamma(P':P:\xi:\Lambda)^{-1}A(P':P:\xi:\Lambda)f(k)\\
&\quad\text{by Proposition 7.5}\\
&=\xi(p_\beta)\mathscr{A}(p_\beta^{-1}Pp_\beta:P:\xi:\Lambda)f(k) \qquad \text{by Lemma 11.3.}
\end{aligned}$$

Multiplying through on the left by $R(p_\beta)$ and commuting $R(p_\beta)$ past $\xi(p_\beta)$, we obtain

$$\xi(p_\beta)\mathscr{A}_P(p_\beta:\xi:\Lambda)f(k)=cf(k),$$

as required.

Lemma 11.5. *Suppose ε is a $\mathfrak{a}$-root such that $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)})=0$, where Λ is imaginary. Then the root reflection p_ε exists in $W(\mathfrak{a})$ and is such that $p_\varepsilon[\xi]=[\xi]$, $p_\varepsilon\Lambda=\Lambda$, and $\xi(p_\varepsilon)\mathscr{A}_P(p_\varepsilon,\xi,\Lambda)$ is scalar for every choice of P.*

Proof. First suppose that p_ε exists in $W(\mathfrak{a})$ and fixes $[\xi]$ and Λ. Fix P, let $P'=p_\varepsilon^{-1}Pp_\varepsilon$, and choose a minimal string from P to P', say $P=P_0, P_1, \ldots, P_r=P'$. Write, by Lemma 8.3(ii),

$$\mathscr{A}(P':P:\xi:\Lambda)=\mathscr{A}(P_r:P_{r-1}:\xi:\Lambda)\cdot\ldots\cdot\mathscr{A}(P_1:P_0:\xi:\Lambda). \tag{11.1}$$

Multiplying ε by a scalar if necessary, we may assume that ε is P-positive and reduced. According to Theorem 7.6(i) the reduced $\mathfrak{a}$-roots γ_i such that $V_{i-1}\cap N_i=V^{(\gamma_i)}$ are those such that $\mathfrak{g}_{\gamma_i}\subseteq\mathfrak{v}'\cap\mathfrak{n}$, and ε satisfies this condition. Hence one factor on the right of (11.1), say

$$\mathscr{A}(P_i:P_{i-1}:\xi:\Lambda),$$

has $V_{i-1}\cap N_i=V^{(\varepsilon)}$. By Lemma 11.3, $P_i=p_\varepsilon^{-1}P_{i-1}p_\varepsilon$. Since $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)})=0$, Lemma 11.4 yields (with w denoting a representative of p_ε in $N_K(\mathfrak{a})$)

$$\xi(w)R(w)\mathscr{A}(w^{-1}P_{i-1}w:P_{i-1}:\xi:\Lambda)=\xi(w)\mathscr{A}_{P_{i-1}}(w,\xi,\Lambda)=cI \tag{11.2}$$

for a scalar c. With our given parabolic P, we then have

$$\begin{aligned}
&\xi(w)\mathscr{A}_P(w,\xi,\Lambda)\\
&\quad=\xi(w)R(w)\mathscr{A}(w^{-1}Pw:P:\xi:\Lambda)\\
&\quad=\xi(w)R(w)\mathscr{A}(w^{-1}Pw:w^{-1}P_{i-1}w:\xi:\Lambda)R(w)^{-1}\xi(w)^{-1}\\
&\qquad\cdot\xi(w)R(w)\mathscr{A}(w^{-1}P_{i-1}w:P_{i-1}:\xi:\Lambda)\cdot\mathscr{A}(P_{i-1}:P:\xi:\Lambda)\\
&\qquad\text{by Theorem 8.4}\\
&\quad=c\,\xi(w)\mathscr{A}(P:P_{i-1}:w\xi:w\Lambda)\xi(w)^{-1}\cdot\mathscr{A}(P_{i-1}:P:\xi:\Lambda)\\
&\qquad\text{by (11.2) and Proposition 8.5(iii)}\\
&\quad=c\,\mathscr{A}(P:P_{i-1}:\xi,w\Lambda)\cdot\mathscr{A}(P_{i-1}:P:\xi:\Lambda)\\
&\qquad\text{by Proposition 8.5(ii)}\\
&\quad=cI\qquad\text{by Theorem 8.4 since } w\Lambda=\Lambda.
\end{aligned}$$

We still have to prove that p_ε exists in $W(\mathfrak{a})$ and fixes $[\xi]$ and Λ. In view of Lemma 11.4, we will be done if we can produce $P_1=MAN_1$ and $P_2=MAN_2$ with $V_1\cap N_2=V^{(\varepsilon)}$. Choose a basis $\mu_1,\dots,\mu_k$ for the dual of $\mathfrak{a}_M$, adjoin ε, and extend to a basis for the dual of $\mathfrak{a}_p$ by adjoining $\alpha_1,\dots,\alpha_j$. Write this basis in the order

$$\alpha_1,\dots,\alpha_j,\varepsilon,\mu_1,\dots,\mu_k,$$

and use it to define positive $\mathfrak{a}$-roots; the N for this ordering will be denoted N_1. It is easy to see that ε is a simple $\mathfrak{a}$-root in this ordering. If instead we use the basis

$$\alpha_1,\dots,\alpha_j,-\varepsilon,\mu_1,\dots,\mu_k$$

to define another ordering (with N group N_2), then N_1 and N_2 have the required properties.

Lemma 11.6. *If s is a member of $W(\mathfrak{a})$ that fixes $[\xi]$ and Λ and satisfies*

$$\mu_{s^{-1}Ps|P}(\xi:\Lambda)=0$$

for an imaginary value of Λ, then there exists a P-positive $\mathfrak{a}$-root ε such that the reflection p_ε exists in $W(\mathfrak{a})$ and fixes $[\xi]$ and Λ, such that $\xi(p_\varepsilon)\mathscr{A}_P(p_\varepsilon,\xi,\Lambda)$ is scalar, and such that $s\varepsilon$ is P-negative.

Proof. Choose a minimal string from P to $s^{-1}Ps$, say $P=P_0,P_1,\dots,P_r=s^{-1}Ps$. We have

$$\mu_{s^{-1}Ps|P}(\xi:\Lambda)=\prod_{\substack{\beta\text{ reduced}\\ \beta>0\text{ for }P\\ \beta<0\text{ for }s^{-1}Ps}}\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})=\prod_{j=1}^{r}\mu_{P_i|P_{i-1}}(\xi:\Lambda)$$

by (10.4) and Theorem 7.6(i). Since each μ function in question is holomorphic at Λ by Proposition 10.2(b), the hypothesis implies that there is a value of i with

$$\mu_{P_i|P_{i-1}}(\xi:\Lambda)=0.$$

Letting ε be the reduced P-positive root associated with this factor, we then have $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)})=0$. Since ε is negative for $s^{-1}Ps$, we see that $s\varepsilon$ is negative for P. By Lemma 11.5, p_ε exists in $W(\mathfrak{a})$, fixes $[\xi]$ and Λ, and is such that $\xi(p_\varepsilon)\mathscr{A}_P(p_\varepsilon, \xi, \Lambda)$ is scalar.

§ 12. Linear Independence Theorem

For this section we fix a cuspidal parabolic subgroup $P=MAN$, and we let ξ be a discrete series representation of M. The Basic Assumption of § 8 is then satisfied. For Λ imaginary, let

$$W_{\xi,\Lambda}=\{w\in W(\mathfrak{a})\,|\,w[\xi]=[\xi] \text{ and } w\Lambda=\Lambda\}.$$

The goal of this section is to prove Theorem 12.1 below, which was announced in [23].

Theorem 12.1. *For Λ_0 imaginary, let*

$$R'=\{r\in W_{\xi,\Lambda_0}\,|\,\mu_{r^{-1}Pr|P}(\xi:\Lambda_0)\neq 0\}.$$

Then the unitary operators $\xi(r)\mathscr{A}_P(r, \xi, \Lambda_0)$ for r in R' are linearly independent.

In this form, the theorem was announced in [23]. For minimal P the result had already been obtained in connection with [21], and the idea behind the proof is implicit in Lemma 64 of [20].

The idea is that the operators in question are given in terms of distributions on K whose supports are filtered according to the complexity of the elements of R'. A linear relation among the operators then implies that the operator with the largest support makes no contribution and is absent. The result readily follows.

The technical aspects of the proof involve two ingredients. One is a description, due to Borel and Tits, of the closure of a Bruhat $M_pA_pN_p$ double coset in G. The other is an analysis of our intertwining operators on a larger domain of functions than the K-finite ones – large enough so that we can shape them to suit our needs and small enough so that we do not have to cope with questions of continuity of $\iota^\#$ or its inverse.[13]

We begin with the result of Borel and Tits [4]. For w in $W(\mathfrak{a}_p)$, let

$$B(w)=M_pA_pN_pwM_pA_pN_p. \qquad (12.1)^{14}$$

The Bruhat decomposition theorem says that the $B(w)$, $w\in W(\mathfrak{a}_p)$, disjointly exhaust G.

Theorem 12.2 (Borel and Tits). *For w in $W(\mathfrak{a}_p)$, the closure of $B(w)$ is the union of $B(w')$ for all w' in $W(\mathfrak{a}_p)$ described as follows: Fix a decomposition of w into the*

[13] In [23] it was asserted that $(\iota^*)^{-1}$ is continuous. We have since realized that there is a gap in our argument. However, the proof of Theorem 12.1 that we give here will not require that we know this continuity.

[14] Of course, w should be replaced on the right by any of its representatives in $N_K(\mathfrak{a}_p)$.

minimal product of simple reflections; then w' can be any element obtained from w by deleting some subset of the factors of w.

Before applying this result, we introduce some notation. For t in $W(\mathfrak{a}_p)$, let $l(t)$ be the length of t relative to $N_p = N N_M$. This is the number of reduced P_p-positive $\mathfrak{a}_p$-roots α such that $t\alpha$ is P_p-negative. For w in $W(\mathfrak{a})$, define

$$|w| = \max_{s \in W(\mathfrak{a}_M)} l(s\tilde{w}),$$

where $\tilde{w}$ is any member of $W(\mathfrak{a}_p)$ with $w = \tilde{w}|_{\mathfrak{a}}$. Such an element $\tilde{w}$ exists by Lemma 8 of [17], and $s\tilde{w}$ runs through all such choices as s runs through $W(\mathfrak{a}_M)$. For w in $W(\mathfrak{a})$, let

$$C(w) = MANwMAN.$$

Lemma 12.3. *For w in $W(\mathfrak{a})$, $C(w) = \bigcup_{s \in W(\mathfrak{a}_M)} B(s\tilde{w})$.*

Proof. We may choose $\tilde{w}$ so that $\tilde{w}^{-1}\delta > 0$ for all $\mathfrak{a}_M$-roots $\delta > 0$. Then

$$\begin{aligned} C(w) &= MAN\tilde{w}N \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} M_p A_M N_M s N_M A N \tilde{w} N \\ &\quad \text{by the Bruhat decomposition for } M \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} M_p A_p N_M s N_M N \tilde{w} N \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} M_p A_p N_M s N N_M \tilde{w} N \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} M_p A_p N_M N s\tilde{w} N_M N \\ &\quad \text{since} \quad sNs^{-1} = N \quad \text{and} \quad \tilde{w}^{-1} N_M \tilde{w} = N_M \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} B(s\tilde{w}). \end{aligned}$$

Lemma 12.4. *If p and w are members of $W(\mathfrak{a})$ with $|p| \geqq |w|$ and p in $\overline{C(w)}$, then $p = w$.*

Proof. Choose $\tilde{p}$ in $W(\mathfrak{a}_p)$ with $\tilde{p}|_{\mathfrak{a}} = p$ and $|p| = l(\tilde{p})$. Then

$$\begin{aligned} \overline{C(w)} &= \bigcup_{s \in W(\mathfrak{a}_M)} \overline{B(s\tilde{w})} \quad \text{by Lemma 12.3} \\ &= \bigcup_{s \in W(\mathfrak{a}_M)} (B(s\tilde{w}) \cup \text{certain } B(t) \text{ with } l(t) < l(s\tilde{w})) \\ &\quad \text{by Theorem 12.2.} \end{aligned}$$

If $\tilde{p}$ is in $\overline{C(w)}$, then $\tilde{p} = s\tilde{w}$ for some s or $l(\tilde{p}) < l(s\tilde{w})$ for some s. In the first case, we obtain $p = w$ by restriction to $\mathfrak{a}$. In the second case, we have

$$|p| = l(\tilde{p}) < l(s\tilde{w}) \leqq |w|,$$

contradiction. So $p = w$.

Lemma 12.5. *For Λ_0 imaginary if r is in R', then the normalizing factor for $\mathscr{A}_P(r, \xi, \Lambda)$ is holomorphic and nonvanishing at $\Lambda=\Lambda_0$.*

Proof. The factor in question is $\gamma(r^{-1}Pr:P:\xi:\Lambda)$. By (10.4) and Proposition 10.2(a), each $\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})$ is holomorphic and nonvanishing at Λ_0 when $\beta>0$ and $r\beta<0$. Hence each reciprocal $\eta^{(\beta)}$ has the same property, and so must each $\gamma^{(\beta)}$ and the product $\gamma(r^{-1}Pr:P:\xi:\Lambda)$.

By Lemma 12.5 the linear independence theorem really concerns the unnormalized operators $\xi(r)A_P(r, \xi, \Lambda_0)$. Instead of limiting ourselves to K-finite functions on K, we shall work with C^∞ functions on K whose values lie in a finite-dimensional space of K_M-finite vectors in H^ξ. Let F' and F'' be finite sets of K_M-types, and let $E_{F'}$ and $E_{F''}$ be the orthogonal projections onto the span of all functions of one of the K_M-types in F' and F'', respectively. Imbed the K_M-finite vectors for ξ in the K_M-finite vectors of a nonunitary principal series representation ω of M, via a mapping ι, and let $E'_{F'}$ and $E'_{F''}$ be the corresponding projections for the space H_0^ω defined in §6. Let X and Y be the images of H^ξ under $E_{F'}$ and $E_{F''}$, and let $C_\xi^\infty(K,X)$ and $C_\xi^\infty(K,Y)$ be the spaces of smooth functions transforming under K_M according to ξ and having values in X and Y, respectively. Let $C_0^\infty(K,X')$ and $C_\omega^\infty(K,Y')$ be the corresponding spaces for ω.

Lemma 12.6. *For r in R' there is an analytic family of operators $B(r:\xi:\Lambda)$ from $C_\xi^\infty(K,X)$ into $C_\xi^\infty(K,Y)$ such that*

(a) *the family is holomorphic in an open connected set containing Λ_0 and all values of Λ with* $\operatorname{Re}\Lambda$ *sufficiently far out in the positive Weyl chamber*

(b) *for* $\operatorname{Re}\Lambda$ *sufficiently far out in the positive Weyl chamber,*

$$B(r:\xi:\Lambda)f(k)=\int_{V\cap r^{-1}Nr} e^{-(\Lambda+\rho)H(v)}E_{F''}\xi(\mu(v))^{-1}E_{F'}f(k\kappa(v))\,dv$$

(c) *for* $\Lambda=\Lambda_0$

$$B(r:\xi:\Lambda_0)f(k)=E_{F''}A(r^{-1}Pr:P:\xi:\Lambda_0)E_{F'}f(k).$$

Proof. Let

$$B(r:\xi:\Lambda)f(k)=\iota^{-1}E'_{F''}(A(r^{-1}Pr:P:\omega:\Lambda)(\iota\circ E_{F'}\circ f)(k)(\cdot)) \tag{12.2a}$$
$$=\iota^{-1}E'_{F''}(A((r^{-1}Pr)_p:P_p:\sigma:\Lambda+\Lambda_M)(\iota\circ E_{F'}\circ f)(\cdot)(1))(k\,\cdot), \tag{12.2b}$$

where ω has parameters (σ, Λ_M). Here the variable $\cdot$ in (12.2a) is in K_M, and the variables $\cdot$ in (12.2b) are in K and K_M, respectively.

First we check on the continuity and analyticity. The map $(f,\Lambda)\to B(r:\xi:\Lambda)f$ is to be continuous from $C_\xi^\infty(K,X)\times\{\Lambda\}$ into $C_\xi^\infty(K,Y)$ and is to be holomorphic in Λ for fixed f. In (12.2b), we have maps

$$f\to\iota\circ E_{F''}\circ f=h \tag{12.3a}$$

and

$$h\to h(\cdot)(1)\to A((r^{-1}Pr)_p:P_p:\sigma:\Lambda+\Lambda_M)h(\cdot)(1)=H \tag{12.3b}$$

and

$$H \to H(k\,\cdot) = L(k) \tag{12.3c}$$

and

$$L(k) \to E'_{F'} L(k) \to \iota^{-1} E'_{F'} L(k). \tag{12.3d}$$

Line (12.3a) represents a continuous map of $C_\xi^\infty(K, X)$ into $C_\omega^\infty(K, X')$ since X and X' are finite-dimensional and $\iota: X \to X'$ is therefore continuous. In (12.3b), the maps are evaluation at 1 followed by the intertwining operator for the minimal parabolic, and (12.3b) is continuous from $C_\omega^\infty(K, X')$ into $C_\sigma^\infty(K, H^\sigma)$ by Theorems 3.1 and 4.2, provided Λ is a regular value of the intertwining operator. In (12.3c), the variable $\cdot$ is in K_M, and the map is continuous from $C_\sigma^\infty(K, H^\sigma)$ into $C_\omega^\infty(K, H^\omega) \subseteq C^\infty(K, C^\infty(K_M, H^\sigma))$. Finally in (12.3d), the maps are the projection from $C_\omega^\infty(K, H^\omega)$ to $C_\omega^\infty(K, Y')$ and the effect of ι^{-1}, which is to carry $C_\omega^\infty(K, Y')$ to $C_\xi^\infty(K, Y)$. These are continuous since Y and Y' are finite-dimensional and $\iota^{-1}: Y' \to Y$ has to be continuous.

The above argument proves $f \to B(r:\xi:\Lambda) f$ is continuous. To bring in Λ, we simply cross the map in (12.3a) and the first map in (12.3b) with the identity operator in Λ, and we use the joint properties of the second map of (12.3b) in the function and Λ given in Theorems 3.1 and 4.2. Then the joint continuity in (f, Λ) and the holomorphicity in Λ follow, but only for regular values Λ of the intertwining operator in (12.3b).

The argument that $B(r:\xi:\Lambda)$ is given by the integral formula for $\operatorname{Re}\Lambda$ sufficiently far out in the positive Weyl chamber is the argument of Theorem 6.6, except that the single E there is now replaced by two projections $E_{F'}$ and $E_{F''}$.

Now let us consider a value of Λ near Λ_0 for which the intertwining operator in (12.3b) is holomorphic. For K-finite f, the right side of (12.2a) is equal to

$$E_{F''} A(r^{-1} P r : P : \xi : \Lambda) E_{F'} f(k)$$

by formula (6.6). Lemma 12.5 shows that this expression is regular at $\Lambda = \Lambda_0$. Therefore the expression in (12.3b) is regular for $\Lambda = \Lambda_0$ on K-finite functions. By Lemma 4.3, the expression in (12.3b) remains regular for $\Lambda = \Lambda_0$ when we pass to C^∞ functions. This means that $\Lambda = \Lambda_0$ is in the domain in which $B(r:\xi:\Lambda)$ is holomorphic and provides a continuous operator. We have

$$B(r:\xi:\Lambda_0) = E_{F''} A(r^{-1} P r : P : \xi : \Lambda_0) E_{F'}$$

on K-finite functions with both sides continuous on $C_\xi^\infty(K, X)$. (The right side is a nonzero scalar times a unitary operator that commutes with translation by K and so is continuous on C^∞.) Hence the two are equal on $C_\xi^\infty(K, X)$.

Proof of Theorem 12.1. In view of Lemma 12.5, we are to show that

$$\sum_{r \in R'} c_r \xi(r) A_P(r, \xi, \Lambda_0) = 0$$

implies $c_r = 0$ for all r. Pick representatives in $N_K(\mathfrak{a})$ for each r, calling them r also. Among all members r of R' with $c_r \neq 0$, let p be one with $|p|$ as large as possible. We shall derive a contradiction by producing a smooth function f so that

$$\xi(w) A_P(w, \xi, \Lambda_0) f(1) = 0 \quad \text{when } c_w \neq 0 \text{ and } w \neq p$$
$$\xi(p) A_P(p, \xi, \Lambda_0) f(1) \neq 0.$$

First, we claim that p is not in the closure of

$$\bigcup_{\substack{w \neq p \\ c_w \neq 0}} w\kappa(V \cap w^{-1} N w) K_M \tag{12.4}$$

within G. Here $\kappa(\cdot)$ is the K-component relative to $G = KMAN$. In fact,

$$w\kappa(V \cap w^{-1} N w) K_M \subseteq w\kappa(w^{-1} N w) K_M = \kappa(N w) K_M \subseteq N w MANM = C(w).$$

Consequently if p were in the closure of (12.4), then p would be in $\overline{C(w)}$ for some $w \neq p$, and Lemma 12.4 and the maximality of $|p|$ would give a contradiction.

Because of the result of the previous paragraph, we can choose a complex-valued C^∞ function on K/K_M that vanishes on (12.4) but does not vanish at p; let φ_0 be its lift to K.

Next, let $X \subseteq H^\xi$ be a finite-dimensional subspace of the kind discussed before Lemma 12.6. Construct F_0 in $C_\xi^\infty(K, X)$ such that $F_0(p) \neq 0$. (To do so, choose a C^∞ function F_1 from K into X that is sufficiently peaked at p, and set

$$F_0(k) = \int_{K_M} \xi(k_M) F_1(k k_M) \, dk_M.$$

Then this F_0 has the required properties.) Put $F = \varphi_0 F_0$, so that F is in $C_\xi^\infty(K, X)$. Our function f will be $f = \varphi F$, where φ is a complex-valued right K_M-invariant C^∞ function on K to be specified.

Suppose $w \neq p$. In the notation of Lemma 12.6 we have

$$E_{F''} A_P(w, \xi, \Lambda_0) f(1) = B(w : \xi : \Lambda_0) f(w). \tag{12.5}$$

However, when $k = w$, the integral in (b) of Lemma 12.6 vanishes for $\operatorname{Re}\Lambda$ sufficiently far out in the positive Weyl chamber. Thus the analyticity asserted in the lemma implies that $B(w, \xi, \Lambda_0) f(w) = 0$, and (12.5) gives

$$E_{F''} A_P(w, \xi, \Lambda_0) f(1) = 0.$$

This equality holds for all F'', and thus

$$\xi(w) A_P(w, \xi, \Lambda_0) f(1) = 0.$$

Now we consider p. For $\operatorname{Re}\Lambda$ sufficiently far out in the positive Weyl chamber, we have

$$B(p : \xi : \Lambda) f(p) = \int_{V \cap p^{-1} N p} e^{-(\Lambda + \rho) H(v)} E_{F''} \xi(\mu(v))^{-1} E_{F'} \varphi(p\kappa(v)) F(p\kappa(v)) \, dv.$$

By Lemma B.1 choose φ at least to have the property that $\varphi(p\kappa(v))$ has compact support. Then this integral provides its own analytic continuation, and Lemma

12.6 implies

$$E_{F''} A_P(p, \xi, \Lambda_0) f(1) = \int_{V \cap p^{-1} N p} e^{-(\Lambda_0+\rho) H(v)} E_{F''} \xi(\mu(v))^{-1} \varphi(p\kappa(v)) F(p\kappa(v))\, dv.$$

Choose $F'' = F'$, and then

$$E_{F''} \xi(\mu(1))^{-1} F(p\kappa(1)) = F(p) \neq 0.$$

Then if we choose φ to be sufficiently peaked about p so that all of the mass of $\varphi(p\kappa(v))$ is concentrated near $v=1$, we conclude that

$$E_{F''} A_P(p, \xi, \Lambda_0) f(1) \neq 0.$$

This inequality persists if we remove $E_{F''}$, and thus we have

$$\xi(p) A_P(p, \xi, \Lambda_0) f(1) \neq 0.$$

This completes the proof of Theorem 12.1.

Corollary 12.7. *Suppose that ε is an $\mathfrak{a}$-root such that p_ε exists in $W(\mathfrak{a})$ and is in $W_{\xi,\Lambda}$, where Λ is imaginary. Then $\xi(p_\varepsilon)\mathscr{A}_P(p_\varepsilon, \xi, \Lambda)$ is scalar if and only if $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)}) = 0$.*

Proof. When $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)}) = 0$, the result is contained in Lemma 11.5. When $\mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)}) \neq 0$, we shall apply Theorem 12.1. We cannot do so immediately because $\mu_{p_\varepsilon^{-1} P p_\varepsilon | P}$ may have several factors, i.e., ε may not be simple. We proceed as in the proof of Lemma 11.5. Forming a minimal string $P = P_0, P_1, \ldots, P_r = p_\varepsilon^{-1} P p_\varepsilon$ and decomposing $\mathscr{A}(p_\varepsilon^{-1} P p_\varepsilon : P : \xi : \Lambda)$ accordingly, we find a factor

$$\mathscr{A}(P_i : P_{i-1} : \xi : \Lambda)$$

in which $V_{i-1} \cap N_i = V^{(\varepsilon)}$. By Lemma 11.3, $P_i = p_\varepsilon^{-1} P_{i-1} p_\varepsilon$. Here

$$\mu_{p_\varepsilon^{-1} P_{i-1} p_\varepsilon | P_{i-1}}(\xi : \Lambda) = \mu_{\xi,\varepsilon}(\Lambda|_{\mathfrak{a}(\varepsilon)}) \neq 0,$$

and p_ε is in the set R' defined relative to P_{i-1}. Thus the unitary operator

$$\xi(p_\varepsilon) \mathscr{A}_{P_{i-1}}(p_\varepsilon, \xi, \Lambda) \tag{12.6}$$

is not scalar. The computation in the middle of the proof of Lemma 11.5 shows that

$$\begin{aligned} \xi(p_\varepsilon)\mathscr{A}_P(p_\varepsilon, \xi, \Lambda) = \mathscr{A}(P : P_{i-1} : \xi : \Lambda) \cdot \xi(p_\varepsilon) \mathscr{A}_{P_{i-1}}(p_\varepsilon, \xi, \Lambda) \\ \cdot \mathscr{A}(P_{i-1} : P : \xi : \Lambda). \end{aligned}$$

That is, the operator of interest is the conjugate of the nonscalar operator (12.6) by the unitary operator $\mathscr{A}(P : P_{i-1} : \xi : \Lambda)$ and hence is nonscalar.

Corollary 12.8. *Suppose $P = MAN$ has dimension $A = 1$, and suppose Λ is imaginary. Then $U_P(\xi, \Lambda, \cdot)$ is reducible if $\Lambda = 0$, $\mu_\xi(0) > 0$, $|W(\mathfrak{a})| = 2$, and $s[\xi] = [\xi]$, where s is the nontrivial element of $W(\mathfrak{a})$.*

Remark. This is a converse to Lemma 11.1.

Proof. By Corollary 12.7, the intertwining operator corresponding to the element s (which is a reflection in this case) is not scalar and therefore exhibits the reducibility.

§ 13. *R* Group and the Commuting Algebra

Throughout this section ξ will denote a discrete series representation of M, and Λ will be imaginary-valued on $\mathfrak{a}$. The parabolic subgroup $P=MAN$ will be fixed. Writing $[\xi]$ for the class of ξ, we recall the definition

$$W_{\xi,\Lambda}=\{s\in W(\mathfrak{a})\,|\,s[\xi]=[\xi] \text{ and } s\Lambda=\Lambda\}.$$

We shall describe below a decomposition of $W_{\xi,\Lambda}$ as a semidirect product $W_{\xi,\Lambda}=W'_{\xi,\Lambda}R_{\xi,\Lambda}$, where $W'_{\xi,\Lambda}$ is normal and is a Weyl group. The group $R_{\xi,\Lambda}$ will turn out to coincide with the set R' of Theorem 12.1, and the operators $\xi(r)\mathscr{A}_P(r,\xi,\Lambda)$, for r in $R_{\xi,\Lambda}$, will turn out to be a linear basis for the commuting algebra $\mathscr{C}_P(\xi,\Lambda)$ of $U_P(\xi,\Lambda;\cdot)$. This result is given here as Theorem 13.4 and was announced in [23].

Lemma 13.1. *Let w_1 and w_2 be representatives in $N_K(\mathfrak{a})$ of members of $W_{\xi,\Lambda}$.*

(a) *If w_1 and w_2 are in a cyclic extension of K_M and if $\xi(w_1)$ and $\xi(w_2)$ are compatibly defined, then*

$$\xi(w_1)\mathscr{A}_P(w_1,\xi,\Lambda)\,\xi(w_2)\mathscr{A}_P(w_2,\xi,\Lambda)=\xi(w_1w_2)\mathscr{A}_P(w_1w_2,\xi,\Lambda).$$

(b) *Whether or not w_1 and w_2 are in a cyclic extension of K_M,*

$$\xi(w_1)\mathscr{A}_P(w_1,\xi,\Lambda)\,\xi(w_2)\mathscr{A}_P(w_2,\xi,\Lambda)=c\,\xi(w_1w_2)\mathscr{A}_P(w_1w_2,\xi,\Lambda)$$

with c a constant satisfying $|c|=1$ and given by

$$\xi(w_1w_2)^{-1}\,\xi(w_1)\,\xi(w_2)=cI.$$

Remarks. Without further proof, it might not be possible to choose the constant c in (b) to be 1, because $\xi(w_1w_2)$ is determined up to an n^{th} root of unity if w_1w_2 has order n modulo K_M. We shall take up this matter further in § 14.

Proof. We shall drop Λ from the notation since it is unaffected throughout. We have

$$\begin{aligned}&\mathscr{A}_P(w_1w_2,\xi)^{-1}\,\xi(w_1w_2)^{-1}\,\xi(w_1)\mathscr{A}_P(w_1,\xi)\,\xi(w_2)\mathscr{A}_P(w_2,\xi)\\&\quad=\mathscr{A}_P(w_1w_2,\xi)^{-1}\,\xi(w_1w_2)^{-1}\,\xi(w_1)\,\xi(w_2)\mathscr{A}_P(w_1,w_2\xi)\,\mathscr{A}_P(w_2,\xi)\\&\quad=\mathscr{A}_P(w_1w_2,\xi)^{-1}\,\xi(w_1w_2)^{-1}\,\xi(w_1)\,\xi(w_2)\mathscr{A}_P(w_1w_2,\xi).\end{aligned}$$

Then (a) is immediate. For (b), we need only prove

$$\xi(w_1w_2)^{-1}\,\xi(w_1)\,\xi(w_2)=cI,$$

and we do this by Schur's Lemma, showing commutativity with $\xi(m)$ for m in M. Thus we compute step by step that

$$\xi(w_1 w_2)^{-1}\,\xi(w_1)\,\xi(w_2)\,\xi(m)\,\xi(w_2)^{-1}\,\xi(w_1)^{-1}\,\xi(w_1 w_2)=\xi(m),$$

and the lemma follows.

In order to proceed, we need the main theorem from [17]. A root α of $\mathfrak{a}_p$ can be written as $\alpha=\beta+\gamma$, where β is the projection on $\mathfrak{a}'$ and γ is the projection on $\mathfrak{a}'_M$. By Lemma 9a of [17], $\bar{\alpha}=\beta-\gamma$ is again an $\mathfrak{a}_p$-root. We say that α is *useful* if $2\langle\alpha,\bar{\alpha}\rangle/|\alpha|^2 \neq +1$, and a root of $\mathfrak{a}$ is *useful* if it is the nonzero restriction to $\mathfrak{a}$ of a useful root of $\mathfrak{a}_p$.

Theorem 13.2. *The useful roots of* $\mathfrak{a}$ *form a possibly nonreduced root system* Δ_0 *in a subspace of* $\mathfrak{a}'$. *A reflection* p_β *in a root of* $\mathfrak{a}$ *is in* $W(\mathfrak{a})$ *if and only if* $t\beta$ *is useful for some* $t>0$, *if and only if* β *itself is useful in case* $\mathfrak{g}$ *has no split* G_2 *factors. Moreover,* $W(\mathfrak{a})$ *coincides with the Weyl group of* Δ_0.

We come to the fundamental definitions. We would like to define

$$\Delta'=\{\beta=\mathfrak{a}\text{-root}\,|\,\mu_{\xi,\beta}(\Lambda|_{\mathfrak{a}(\beta)})=0\}. \tag{13.1}$$

This formula will suffice as definition unless $\mathfrak{g}$ has a split G_2 factor. In that case, we include in the definition the additional assumption that β is useful in the sense described above; Lemma 11.5 and Theorem 13.2 then assure us that all members of Δ' are useful in every case. Lemma 11.5 tells us also that if β is in Δ', then the reflection p_β is in $W_{\xi,\Lambda}$. Bringing in Corollary 12.7, we obtain an alternate characterization of Δ' as

$$\Delta'=\{\beta=\text{useful } \mathfrak{a}\text{-root}\,|\,p_\beta\in W_{\xi,\Lambda} \text{ and } \xi(p_\beta)\mathscr{A}_P(p_\beta,\xi,\Lambda) \text{ is scalar}\}. \tag{13.2}$$

We shall see in Lemma 13.3 that Δ' is a (possibly nonreduced) root system that is mapped into itself by every element of $W_{\xi,\Lambda}$. Thus we can define groups $W'_{\xi,\Lambda}$ and $R_{\xi,\Lambda}$ by

$$W'_{\xi,\Lambda}=\text{Weyl group of } \Delta'\subseteq W(\mathfrak{a})$$

and

$$R_{\xi,\Lambda}=\{p\in W_{\xi,\Lambda}|p\beta>0 \text{ for every } \beta>0 \text{ in } \Delta'\}.$$

Lemma 13.3. *If* Δ' *is nonempty,* Δ' *is a (possibly nonreduced) root system in a subspace of* $\mathfrak{a}'$, *and* $W_{\xi,\Lambda}$ *carries* Δ' *into itself. Consequently* $W'_{\xi,\Lambda}$ *is a normal subgroup of* $W_{\xi,\Lambda}$, *and* $R_{\xi,\Lambda}$ *is a group.*

Proof. By (13.2), Δ' is contained in the root system Δ_0 of Theorem 13.2. Consequently Δ' is a root system if it is closed under its own reflections. Let β and ε be in Δ'. In view of (13.2), we are to show that the operator for $p_{p_\varepsilon\beta}$ is scalar if the operators for p_β and p_ε are. But $p_{p_\varepsilon\beta}=p_\varepsilon p_\beta p_\varepsilon$, and so this result follows from Lemma 13.1.

More generally if s is in $W_{\xi,\Lambda}$ and β is in Δ', then we can apply (13.2) and Lemma 13.1 to $p_{s\beta}=sp_\beta s^{-1}$ to see that $s\beta$ is in Δ'. The rest follows immediately.

Theorem 13.4. *$W_{\xi,\Lambda}$ is the semidirect product $W_{\xi,\Lambda}=W'_{\xi,\Lambda}R_{\xi,\Lambda}$ with $W'_{\xi,\Lambda}$ normal. This decomposition has the following properties:*

(i) *$W'_{\xi,\Lambda}$ is the set of s in $W_{\xi,\Lambda}$ for which $\xi(s)\mathscr{A}_P(s,\xi,\Lambda)$ is scalar.*

(ii) *$R_{\xi,\Lambda}$ is the set of r in $W_{\xi,\Lambda}$ for which $\mu_{r^{-1}Pr|P}(\xi:\Lambda)\neq 0$.*

(iii) *The unitary operators $\xi(r)\mathscr{A}_P(r,\xi,\Lambda)$ for r in $R_{\xi,\Lambda}$ are linearly independent and span the commuting algebra $\mathscr{C}_P(\xi,\Lambda)$ of $U_P(\xi,\Lambda,\cdot)$.*

(iv) *The dimension of the commuting algebra of $U_P(\xi,\Lambda,\cdot)$ is given by*

$$\dim\mathscr{C}_P(\xi,\Lambda)=|R_{\xi,\Lambda}|=|\{r\in W_{\xi,\Lambda}|\mu_{r^{-1}Pr|P}(\xi:\Lambda)\neq 0\}|.$$

Proof. The first step is to decompose every element of $W_{\xi,\Lambda}$ into the product of a member of $R_{\xi,\Lambda}$ by a member of $W'_{\xi,\Lambda}$. Let C^+ be the positive Weyl chamber of Δ' in the subspace of $\mathfrak{a}$ spanned by vectors H_β, $\beta\in\Delta'$. Let w be given in $W_{\xi,\Lambda}$. Since $w\Delta'\subseteq\Delta'$, w carries C^+ into another chamber wC^+. By the transitivity of a genuine Weyl group on its chambers, we can find w' in the Weyl group $W'_{\xi,\Lambda}$ of Δ' such that $w'wC^+=C^+$. Then $w'w$ is in $R_{\xi,\Lambda}$ and $w=(w')^{-1}(w'w)$ exhibits w as the required product.

Now $W'_{\xi,\Lambda}$ is normal by Lemma 13.3, and it leads to scalar operators by (13.2) and Lemma 13.1. Lemma 11.6 implies that no element r of $R_{\xi,\Lambda}$ can have $\mu_{r^{-1}Pr|P}(\xi:\Lambda)=0$. That is, $R_{\xi,\Lambda}$ is contained in R', in the notation of Theorem 12.1. By that theorem, the operators corresponding to $R_{\xi,\Lambda}$ are linearly independent.

If w is in R', we can write $w=w'r$ with w' in $W'_{\xi,\Lambda}$ and r in $R_{\xi,\Lambda}$. Applying Lemma 13.1, we see that the operator for w is a scalar times the operator for r. Since Theorem 12.1 says that the operators for R' are linearly independent, we conclude $R_{\xi,\Lambda}=R'$. This proves (ii) and the second equality in (iv).

The same argument applied to a general element of $W_{\xi,\Lambda}$ shows that scalar operators come only from $W'_{\xi,\Lambda}$. (This proves (i).) It shows also that the span of the operators for $R_{\xi,\Lambda}$ is the same as that for $W_{\xi,\Lambda}$, which proves part of (iii). Obviously $W'_{\xi,\Lambda}\cap R_{\xi,\Lambda}=\{1\}$, and so we have a semidirect product. By Corollary 9.8 to Harish-Chandra's completeness theorem, the span of the operators for $W_{\xi,\Lambda}$ is all of $\mathscr{C}_P(\xi,\Lambda)$. This proves the remaining part of (iii), and the first equality in (iv) follows from it.

§ 14. Reduction Lemmas

Fix a cuspidal parabolic subgroup $P=MAN$, let ξ be a discrete series representation of M, and let Λ be an imaginary-valued linear functional on $\mathfrak{a}$. Under some additional assumptions on G, it will be shown in a later paper [37] that the group $R_{\xi,\Lambda}$ of § 13 is a finite direct sum of copies of the two-element group $\mathbb{Z}_2$. Briefly we write $R_{\xi,\Lambda}=\sum\mathbb{Z}_2$ for this conclusion. (In fact, the number of factors $\mathbb{Z}_2$ will be shown not to exceed $\dim A$.) In Lemma 14.1 we shall show that it suffices to prove this result for $\Lambda=0$.

For $\Lambda=0$, the proof is long and ultimately reduces to the case that G is split over $\mathbb{R}$ and the parabolic subgroup is minimal. This case will be treated in §15, but the reduction is deferred to [37].

Even after it is shown that $R_{\xi,\Lambda}=\sum \mathbb{Z}_2$, it does not follow directly that the intertwining operators $\xi(r)\mathscr{A}_P(r,\xi,\Lambda)$, $r\in R$, commute and hence that the commuting algebra $\mathscr{C}_P(\xi,\Lambda)$ is commutative. A hint of the difficulty is given in Lemma 13.1(b). We shall isolate the problem in Lemma 14.2 but shall defer the resolution of the problem to [37].

Lemma 14.1. *If* $R_{\xi,0}=\sum \mathbb{Z}_2$ *and* Λ_0 *is imaginary, then* $R_{\xi,\Lambda_0}=\sum \mathbb{Z}_2$ *and* $|R_{\xi,\Lambda_0}|\leqq |R_{\xi,0}|$.

Proof. Let w be in $N_K(\mathfrak{a})$, let $[w]$ be its class in $W(\mathfrak{a})$, and suppose that $[w]$ is in W_{ξ,Λ_0} and $[w]$ has order k. The set of Λ for which $w\Lambda=\Lambda$ is connected since such Λ form a vector subspace. Form $\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$ for these Λ. By Lemma 13.1(a),

$$[\xi(w)\mathscr{A}_P(w,\xi,\Lambda)]^l=\xi(w^l)\mathscr{A}_P(w^l,\xi,\Lambda) \tag{14.1}$$

for each l. Consider the restriction of $\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$ to a large finite-dimensional sum of K-spaces. Equation (14.1) with $l=k$ says $[\xi(w)\mathscr{A}_P(w,\xi,\Lambda)]^k=I$. Hence there are only finitely many possibilities for the characteristic polynomial of the unitary operator $\xi(w)\mathscr{A}_P(w,\xi,\Lambda)$, and it follows that the characteristic polynomial of this operator must be independent of Λ. In particular, the order of the operator is independent of Λ. Taking $\Lambda=0$ and applying Theorem 12.4 and the assumption that $R_{\xi,0}=\sum \mathbb{Z}_2$, we see from Lemma 13.1(a) that

$$[\xi(w)\mathscr{A}_P(w,\xi,0)]^2$$

is scalar. Hence $[\xi(w)\mathscr{A}_P(w,\xi,\Lambda_0)]^2$ is scalar. If $[w]$ is in R_{ξ,Λ_0}, this contradicts Theorem 12.4(iii) unless $[w]^2=1$. Thus every element in R_{ξ,Λ_0} has order at most 2, and it follows that $R_{\xi,\Lambda_0}=\sum \mathbb{Z}_2$.

The same argument with characteristic polynomials shows that $\xi(w)\mathscr{A}_P(w,\xi,0)$ is nonscalar for each $w\neq 1$ in R_{ξ,Λ_0}. By Lemma 13.1(b), the $R_{\xi,0}$ component of such an element w must be nontrivial in the decomposition $W_{\xi,0}=W'_{\xi,0}R_{\xi,0}$. However, the map of R_{ξ,Λ_0} into $R_{\xi,0}$ defined by inclusion into $W_{\xi,0}$ followed by projection into $R_{\xi,0}$ is a group homomorphism. Thus this map is one-one, and $|R_{\xi,\Lambda_0}|\leqq |R_{\xi,0}|$.

Lemma 14.2. *Under the assumption that Λ is imaginary and $R_{\xi,\Lambda}$ is abelian, the following conditions are equivalent:*

(a) *The commuting algebra $\mathscr{C}_P(\xi,\Lambda)$ of $U_P(\xi,\Lambda,\cdot)$ is commutative.*

(b) *For every pair $[r]$ and $[s]$ in a set of generators of $R_{\xi,\Lambda}$,*

$$\xi(r)\,\xi(s)\,\xi(r)^{-1}\,\xi(s)^{-1}=\xi(rsr^{-1}s^{-1}).$$

(c) *The representation ξ extends to a representation, still on H^ξ, of the group generated by M and a representative in $N_K(\mathfrak{a})$ for each member of $R_{\xi,\Lambda}$.*

Proof. First we show (a)$\Leftrightarrow$(b). Let $[r]$ and $[s]$ be in $R_{\xi,\Lambda}$. By Lemma 13.1 we have

$$\xi(r)\mathscr{A}_P(r,\xi,\Lambda)\,\xi(s)\mathscr{A}_P(s,\xi,\Lambda)=c\,\xi(rs)\mathscr{A}_P(rs,\xi,\Lambda),$$

where $cI = \xi(rs)^{-1}\,\xi(r)\,\xi(s)$. An analogous formula holds for sr, with a constant d on the right. Now rs and sr are representatives of the same member of $R_{\xi,\Lambda}$ since $R_{\xi,\Lambda}$ is abelian, and so the corresponding operators are the same. Thus the operators for r and s commute if and only if $c=d$, i.e., if and only if

$$\xi(rs)^{-1}\,\xi(r)\,\xi(s) = \xi(sr)^{-1}\,\xi(s)\,\xi(r).$$

Multiplying through, we obtain the condition

$$\xi(r)\,\xi(s)\,\xi(r)^{-1}\,\xi(s)^{-1} = \xi(rs)\,\xi(sr)^{-1}.$$

On the right side, rs and sr are in the same cyclic extension of M, and so the operator on the right equals $\xi(rsr^{-1}s^{-1})$. Hence (a)$\Leftrightarrow$(b).

Obviously (c) implies (b). To see that (b) implies (c), we invoke Lemma 59 of [20]. We know from Lemma 7.9 that we can handle a cyclic extension of M. To handle an extension by a direct sum of cyclic groups, Lemma 59 says that it is enough to verify the relations in M' that correspond to the commutativity of $R_{\xi,\Lambda}$. These are the relations of (b). Thus (b) and (c) are equivalent.

§ 15. Structure of *R*-group, Split Minimal Case

The theorem in this section was announced in [21]. As will be seen in [37], the structure theorem for $R_{\xi,\Lambda}$ in the general case is reduced to the special case considered here.

Theorem 15.1. *Let G be a connected split semisimple Lie group of matrices, and let MAN be a minimal parabolic subgroup. In this case $R_{\xi,\Lambda}$ is always $\sum \mathbb{Z}_2$.*

Proof. By Lemma 14.1, we may take $\Lambda=0$. For this G and for the minimal parabolic, $\mathfrak{a}$ is a Cartan subalgebra. Also M is a finite abelian group and is generated by the elements

$$\gamma_\alpha = \exp 2\pi i |\alpha|^{-2} H_\alpha$$

for all $\mathfrak{a}$-roots α. Each γ_α has order at most 2. Direct computation gives

$$p_\beta \gamma_\alpha p_\beta^{-1} = \gamma_{p_\beta \alpha} = \gamma_\alpha \gamma_\beta^{2\langle\alpha,\beta\rangle/|\alpha|^2}.$$

Let Δ be the set of roots. For a minimal parabolic all roots are useful. For α in Δ, $\mathfrak{g}^{(\alpha)}$ is isomorphic with $sl(2,\mathbb{R})$ because $\mathfrak{g}$ is split, and the element γ_α is the image of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ under the corresponding mapping of $SL(2,\mathbb{R})$ into G. Within $SL(2,\mathbb{R})$ a representation (=character) σ of M has

$$\sigma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = +1 \text{ if and only if } \mu_\sigma(0)=0$$

$$\sigma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1 \text{ if and only if } \mu_\sigma(0)\neq 0,$$

as is well known (see, e.g., p. 544 of [20]). By (13.1),

$$\Delta' = \{\alpha \mid \xi(\gamma_\alpha) = +1\}.$$

We shall show that

$$W_{\xi,0} = \{w \in W(\mathfrak{a}) \mid w\Delta' = \Delta'\}. \tag{15.1}$$

In fact, each member of $W_{\xi,0}$ preserves Δ' by Lemma 13.3. Conversely, if w preserves Δ', then w preserves also its complement, and we have

$$\alpha \in \Delta' \Rightarrow w\xi(\gamma_\alpha) = \xi(w^{-1}\gamma_\alpha w) = \xi(\gamma_{w^{-1}\alpha}) = +1 = \xi(\gamma_\alpha)$$

$$\alpha \notin \Delta' \Rightarrow w\xi(\gamma_\alpha) = \xi(w^{-1}\gamma_\alpha w) = \xi(\gamma_{w^{-1}\alpha}) = -1 = \xi(\gamma_\alpha).$$

Thus $w\xi = \xi$. A corollary of (15.1) is that

$$R_{\xi,0} = \{w \in W(\mathfrak{a}) \mid w\Delta'^+ = \Delta'^+\}, \tag{15.2}$$

where Δ'^+ denotes the set of positive elements in Δ'.

We shall introduce a dual situation. We associate to the data

($\mathfrak{g}$ split, $\mathfrak{a}$ most noncompact Cartan subalgebra, Δ^+, ξ on M)

a set of data

($\mathfrak{g}^\vee$ with $\operatorname{rank}\mathfrak{g}^\vee = \operatorname{rank} k^\vee$, compact Cartan subalgebra $\mathfrak{b}$, $(\Delta^\vee)^+$, Cartan involution),

as follows: $(\mathfrak{g}^\vee)^{\mathbb{C}}$ is a complex semisimple Lie algebra whose Cartan matrix is the transpose of the Cartan matrix for $\mathfrak{g}$.[15] We single out a Cartan subalgebra $\mathfrak{b}^{\mathbb{C}}$ of $(\mathfrak{g}^\vee)^{\mathbb{C}}$ and a positive system $(\Delta^\vee)^+$ of roots, and we form in the standard way[16] a compact form $\mathfrak{u}$ of $(\mathfrak{g}^\vee)^{\mathbb{C}}$. Let $\mathfrak{b} = \mathfrak{u} \cap \mathfrak{b}^{\mathbb{C}}$. The simple roots in $\mathfrak{g}$ and $(\mathfrak{g}^\vee)^{\mathbb{C}}$ correspond, and the Weyl groups correspond. The correspondence of simple roots extends to a correspondence $\alpha \to \alpha^\vee$ of roots via the Weyl group action, not via addition.

We shall use ξ to isolate a real form $\mathfrak{g}^\vee$ of $(\mathfrak{g}^\vee)^{\mathbb{C}}$. To do so, we designate a simple root $\alpha^\vee$ as *noncompact* if $\xi(\gamma_\alpha) = -1$, otherwise *compact*. Choose H_0 in $\mathfrak{b}^{\mathbb{C}}$ such that

$$\alpha^\vee(H_0) = \begin{cases} 1 & \text{if } \alpha^\vee \text{ is a simple root designated noncompact,} \\ 0 & \text{if } \alpha^\vee \text{ is a simple root designated compact,} \end{cases}$$

and define $\theta = \operatorname{Ad}(\exp \pi i H_0)$. Then θ is an involution of $(\mathfrak{g}^\vee)^{\mathbb{C}}$ leaving $\mathfrak{u}$ stable. Consequently if we define

$$\mathfrak{k}^\vee = \mathfrak{u} \cap (+1 \text{ eigenspace of } \theta \text{ in } (\mathfrak{g}^\vee)^{\mathbb{C}})$$

$$\mathfrak{p}^\vee = i\mathfrak{u} \cap (-1 \text{ eigenspace of } \theta \text{ in } (\mathfrak{g}^\vee)^{\mathbb{C}})$$

$$\mathfrak{g}^\vee = \mathfrak{k}^\vee + \mathfrak{p}^\vee,$$

---

[15] For example, B_n leads to C_n, and vice-versa.

[16] See [36], page 155.

then $\mathfrak{g}^\vee$ is a real form of $(\mathfrak{g}^\vee)^{\mathbb{C}}$ with Cartan decomposition $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{p}^\vee$. Clearly $\mathfrak{h}$ is in $\mathfrak{k}$, and thus $\mathfrak{h}$ is a compact Cartan subalgebra of $\mathfrak{g}^\vee$.

Since $\mathfrak{h}$ is a compact Cartan subalgebra, we have a standard notion of compact and noncompact roots, and we can check readily for simple roots that this notion coincides with the one above used to define θ.

We shall show that a general root $\alpha^\vee$ is compact if and only if $\xi(\gamma_\alpha) = +1$. To do so, we write $\alpha = p_{\beta_1} \dots p_{\beta_n} \varepsilon$ with $\beta_1, \dots, \beta_n$, ε simple and with n as small as possible, and we proceed by induction on n. The case $n=0$ was noted in the previous paragraph. Thus let $\alpha = p_\beta \delta$ and suppose it is known that

$$\delta^\vee \text{ compact} \Leftrightarrow \xi(\gamma_\delta) = +1 \quad \text{and} \quad \beta^\vee \text{ compact} \Leftrightarrow \xi(\gamma_\beta) = +1.$$

We compare

$$\alpha^\vee = \delta^\vee - \frac{2\langle \delta^\vee, \beta^\vee \rangle}{|\beta^\vee|^2} \beta^\vee = \delta^\vee - \frac{2\langle \delta, \beta \rangle}{|\delta|^2} \beta^\vee \tag{15.3}$$

with

$$\xi(\gamma_\alpha) = \xi(\gamma_{p_\beta \delta}) = \xi(\gamma_\delta)\, \xi(\gamma_\beta)^{2\langle \delta, \beta \rangle / |\delta|^2}. \tag{15.4}$$

If $\beta^\vee$ is compact, (15.3) says $\alpha^\vee$ and $\delta^\vee$ are the same type, compact or noncompact, and (15.4) says $\xi(\gamma_\alpha) = \xi(\gamma_\delta)$. If $\beta^\vee$ is noncompact, (15.3) says $\alpha^\vee$ and $\delta^\vee$ are of the same type if and only if $2\langle \delta, \beta \rangle / |\delta|^2$ is even, and (15.4) says $\xi(\gamma_\alpha) = \xi(\gamma_\delta)$ if and only if $2\langle \delta, \beta \rangle / |\delta|^2$ is even. The assertion follows.

Thus we have

$$(\Delta')^\vee = \{\text{compact roots}\}. \tag{15.5}$$

Let s be in $W_{\xi,0}$, regard $W(\mathfrak{a})$ also as the complex Weyl group of $(\mathfrak{g}^\vee)^{\mathbb{C}}$, and let $\tilde{s}$ be a representative of s in the complex adjoint group $(G^\vee)^{\mathbb{C}}$; we may assume that $\tilde{s}$ is in the compact form U. Then (15.1) and the fact that the correspondence $\alpha \to \alpha^\vee$ is implemented by the action of the Weyl group together mean that

$$\operatorname{Ad}(\tilde{s})(\mathfrak{k}^\vee)^{\mathbb{C}} \subseteq (\mathfrak{k}^\vee)^{\mathbb{C}}.$$

Since $\tilde{s}$ is in U, $\tilde{s} U \tilde{s}^{-1} = U$. Thus

$$\operatorname{Ad}(\tilde{s})\mathfrak{k}^\vee = \operatorname{Ad}(\tilde{s})(\mathfrak{u} \cap (\mathfrak{k}^\vee)^{\mathbb{C}}) \subseteq \mathfrak{u} \cap (\mathfrak{k}^\vee)^{\mathbb{C}} = \mathfrak{k}^\vee$$

and $\operatorname{Ad}(\tilde{s})$ normalizes $\mathfrak{k}^\vee$. Similarly $\operatorname{Ad}(\tilde{s})$ normalizes $\mathfrak{p}^\vee$ and so it normalizes $\mathfrak{g}^\vee$. By Proposition 2 of Kostant-Rallis [27], $\operatorname{Ad}(\tilde{s}^2)$ is in $\operatorname{Ad}(K^\vee)$. In other words, s^2 is in the Weyl group generated by reflections in the compact roots. By (15.5), this means s^2 is in $W'_{\xi,0}$.

Applying this conclusion to s in $R_{\xi,0}$, we have

$$s^2 \in R_{\xi,0} \cap W'_{\xi,0} = \{1\}.$$

Hence every element of $R_{\xi,0}$ has order at most two, and consequently $R_{\xi,0} = \sum \mathbb{Z}_2$.

III. Complementary Series

§16. Existence

Under the hypotheses given in part I for G, P and ξ, we say that $U_P(\xi, \Lambda, \cdot)$ is in the complementary series if its K-finite vectors possess a nontrivial semidefinite Hermitian inner product with respect to which K acts by unitary operators and $\mathfrak{g}$ acts by skew-Hermitian operators. If s is an element of order 2 in $W(\mathfrak{a})$ that fixes ξ, then Corollary 8.7 says that

$$[\xi(s)\mathscr{A}_P(s, \xi, \Lambda)]^* = \xi(s)\mathscr{A}_P(s, \xi, -s\bar{\Lambda}).$$

Thus when $s\Lambda = -\bar{\Lambda}$, the form

$$\langle f, g\rangle = (\xi(s)\mathscr{A}_P(s, \xi, \Lambda)f, g)_{L^2(K)}$$

is Hermitian, and it is shown in Lemma 62 of [20] that it possesses the correct invariance properties. The question is whether the form is semidefinite.

We shall not attempt a comprehensive answer to this question now but shall be content with a general result that illustrates a method.

Lemma 16.1. *Suppose $P = MAN$ is a cuspidal parabolic subgroup and ξ is a discrete series representation of M. If ξ imbeds in the nonunitary principal series representation of M with parameters (σ, Λ_M) and if G_0 has a faithful matrix representation, then*

$$\frac{2\langle \Lambda_M, \alpha\rangle}{|\alpha|^2} \in \frac{1}{2}\mathbb{Z}$$

for every $\mathfrak{a}_p$-root α.

Proof. Since P is cuspidal, Lemma 4 of [17] shows that $\mathfrak{a}'_M$ has an orthogonal basis of roots δ_i. These roots may be taken to be "inessential" in the nomenclature of that paper; this means that if $\mathfrak{a}_M$ is extended to a Cartan subalgebra of $\mathfrak{m}$ and if each δ_i is extended to be 0 on the orthogonal complement of $\mathfrak{a}_M$, then the extended δ_i is a root relative to the Cartan subalgebra.

Since G_0 (and therefore M_0) has a faithful matrix representation, the parameter of the infinitesimal character of ξ is algebraically integral, by [11]. This parameter is $\Lambda^- + \rho^- + \Lambda_M$, where $\Lambda^- + \rho^-$ is a part corresponding to σ that is in the orthogonal complement of $\mathfrak{a}'_M$. Therefore

$$\frac{2\langle \Lambda_M, \delta_i\rangle}{|\delta_i|^2} \in \mathbb{Z} \quad \text{for all } i.$$

By Parseval's theorem,

$$\Lambda_M = \sum \frac{\langle \Lambda_M, \delta_i\rangle}{|\delta_i|^2}\delta_i,$$

and therefore

$$\frac{2\langle \Lambda_M, \alpha\rangle}{|\alpha|^2} = \sum \frac{\langle \Lambda_M, \delta_i\rangle}{|\delta_i|^2}\,\frac{2\langle \delta_i, \alpha\rangle}{|\alpha|^2} \in \frac{1}{2}\mathbb{Z}.$$

Theorem 16.2. *Under the assumptions on G in §1, let $P=MAN$ be a cuspidal parabolic subgroup, let ξ be a discrete series representation of M, and suppose $W'_{\xi,0}$ is not the one-element group. If s is any element of order 2 in $W'_{\xi,0}$, then every complex Λ that is not purely imaginary and satisfies*

(i) $s\Lambda=-\bar{\Lambda}$

(ii) $\left|\operatorname{Re}\frac{\langle\Lambda,\beta\rangle}{|\beta|^2}\right|<c_\xi$ *for every $\mathfrak{a}$-root β is such that $U_P(\xi,\Lambda,\cdot)$ is in the complementary series. Here c_ξ is a positive constant depending on ξ. If G_0 has a faithful matrix representation, the number c_ξ can be taken to be 1/4, independently of ξ.*

Remarks. (a) $W'_{\xi,0}$ is a Weyl group; if it is nontrivial, it must contain elements of order 2.

(b) For a minimal parabolic when G_0 is a matrix group, c_ξ can be taken to be 1/2, as will be apparent from the proof; 1/2 is the best possible universal constant, as is shown by $SL(2,\mathbb{R})$.

(c) This theorem was announced in [23]. For earlier results in the direction of this theorem, see [26], Theorems 8 and 9 of [20], and Theorem 13 of [12].

Proof. By the techniques of [20], it is enough to show that the unnormalized intertwining operator

$$A(s^{-1}Ps:P:\xi:\Lambda) \tag{16.1}$$

is holomorphic in the region

$$0<\left|\operatorname{Re}\frac{\langle\Lambda,\beta\rangle}{|\beta|^2}\right|<c_\xi \tag{16.2}$$

and that $\eta(s^{-1}Ps:P:\xi:\Lambda)$ has no singularities or zeros in this region.

To see that (16.1) is holomorphic in the region (16.2), we use Theorem 6.6 and also examine the proof of that theorem. Singularities of the operator can occur only when there is some β such that

$$\frac{2\langle\Lambda,\beta\rangle}{|\alpha|^2}\in\mathbb{Z}+d_i(\xi),$$

where $d_i(\xi)=-2\langle\Lambda_M,\alpha\rangle/|\alpha|^2$ for an $\mathfrak{a}_p$-root α whose restriction to $\mathfrak{a}$ is β. For a suitable choice of c_ξ, there are no singularities in (16.2). If G_0 is a matrix group, Lemma 16.1 says $d_i(\xi)$ is in $\frac{1}{2}\mathbb{Z}$. Thus the condition is that

$$\frac{2\langle\Lambda,\beta\rangle}{|\alpha|^2}\in\frac{1}{2}\mathbb{Z}.$$

Here $|\alpha|^2/|\beta|^2$ is one of the numbers 1, 2, 4, or 4/3, and so singularities can occur only when

$$\frac{2\langle\Lambda,\beta\rangle}{|\beta|^2}\in\frac{1}{2}\mathbb{Z}\quad\text{or}\quad\mathbb{Z}\quad\text{or}\quad2\mathbb{Z}\quad\text{or}\quad\frac{2}{3}\mathbb{Z}.$$

Therefore c_ξ can be taken to be 1/4.

Next, we deal with η. The singularities of η can arise only from singularities of one or another unnormalized intertwining operator, and there are none in the region (16.2).

As for the zeros of η, the question is one of singularities of Plancherel factors. By Proposition 10.2(c) and formula (10.4), we are to examine the singularities of

$$\mu_{\sigma,\alpha}((\Lambda+\Lambda_M)|_{\mathfrak{a}(\alpha)})=\mu_{\sigma,\alpha}\left(\frac{\langle\Lambda+\Lambda_M,\alpha\rangle}{|\alpha|^2}\alpha\right)$$

for every reduced $\mathfrak{a}_p$-root α that does not vanish identically on $\mathfrak{a}$. A Plancherel factor for a real-rank one group gets its singularities (poles) only from a factor of tangent or cotangent. Letting p and q be the dimensions of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{2\alpha}$, respectively, we are to look at singularities of

$$\tan\tfrac{1}{2}\pi piz \text{ or } \cot\tfrac{1}{2}\pi piz \qquad \text{if } q=0$$

or

$$\tan\tfrac{1}{4}\pi(p+2q)iz \text{ or } \cot\tfrac{1}{4}\pi(p+2q)iz \qquad \text{if } q\neq 0.$$

Here z is a parameter such that $z=1$ corresponds to $\rho^{(\alpha)}=\frac{1}{2}(p+2q)\alpha$. (See §12 of [20].) The singularities are all contained in the set of z's that are integral multiples of $(p+2q)^{-1}$, which corresponds to the set of multiples of $\alpha/2$. Thus the singularities of $\mu_{\sigma,\alpha}$ are limited to those Λ for which

$$\frac{\langle\Lambda+\Lambda_M,\alpha\rangle}{|\alpha|^2}\in\frac{1}{2}\mathbb{Z},$$

and this is the same set as before for the intertwining operator. Thus the η function has no zeros in the region (16.2).

Appendix A. Proof of Proposition 1.2

The Lie algebras $\mathfrak{n}+\mathfrak{n}_M$ and $\mathfrak{n}'+\mathfrak{n}_M$ are the sums of the root spaces for positive $\mathfrak{a}_p$-roots in two different orderings, and there exists a member w of the Weyl group $W(\mathfrak{a}_p)$ such that $w(\mathfrak{n}'+\mathfrak{n}_M)=\mathfrak{n}+\mathfrak{n}_M$. Let positivity of $\mathfrak{a}$-roots be defined relative to $\mathfrak{n}+\mathfrak{n}_M$. Without loss of generality, we may assume β is reduced. Then $V\cap N'=V^{(\beta)}$ implies

$$\mathfrak{n}=\sum_{\substack{\gamma>0\\ \gamma\neq c\beta}}\mathfrak{g}_\gamma+\sum_{c\geq 1}\mathfrak{g}_{c\beta} \tag{A.1}$$

and

$$\mathfrak{n}'=\sum_{\substack{\gamma>0\\ \gamma\neq c\beta}}\mathfrak{g}_\gamma+\sum_{c\geq 1}\mathfrak{g}_{-c\beta}. \tag{A.2}$$

We claim that β cannot be decomposed as the sum of positive $\mathfrak{a}$-roots $\beta=\beta_1+\beta_2$, i.e., β is simple. Assuming the contrary let α_1 and α_2 be $\mathfrak{a}_p$-roots with $\alpha_1|_\mathfrak{a}=\beta_1$ and $\alpha_2|_\mathfrak{a}=\beta_2$, and write $\alpha_1=\beta_1+\mu_1$, $\alpha_2=\beta_2+\mu_2$. Since $\beta_1\neq c\beta$ with $c\geq 1$, the $\mathfrak{a}_p$-root space $\mathfrak{g}_{\alpha_1}$ must be included in the first term of (A.1), and similarly for

$\mathfrak{g}_{\alpha_2}$. Thus $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{\alpha_2}$ are in $\mathfrak{n}'$. Since w carries $\mathfrak{n}'$ into $\mathfrak{n}+\mathfrak{n}_M$, it follows that

$$w(\beta_1+\mu_1)>0 \quad \text{and} \quad w(\beta_2+\mu_2)>0.$$

If s is in $W(\mathfrak{a}_M)$, then $s\alpha_1=\beta_1+s\mu_1$ is also in $\mathfrak{n}'$, and similarly for $s\alpha_2$. Hence

$$w(\beta_1+s\mu_1)>0 \quad \text{and} \quad w(\beta_2+s\mu_2)>0, \qquad s\in W(\mathfrak{a}_M).$$

Summing over s, we obtain $w\beta_1>0$ and $w\beta_2>0$ and therefore $w\beta>0$. Let $\alpha=\beta+\mu$ be an $\mathfrak{a}_p$-root with $\alpha|_{\mathfrak{a}}=\beta$. Applying s in $W(\mathfrak{a}_M)$ and summing over s in $W(\mathfrak{a}_M)$, we see from $w\beta>0$ that $w(\beta+s\mu)>0$ for some s in $W(\mathfrak{a}_M)$. But $\mathfrak{g}_{-(\beta+s\mu)}$ is contained in the second term of $\mathfrak{n}'$ in (A.2), and w carries $\mathfrak{n}'$ into $\mathfrak{n}+\mathfrak{n}_M$, so that $-w(\beta+s\mu)>0$. This is a contradiction, and we conclude that β is simple.

Let α be the least $\mathfrak{a}_p$-root with $\alpha|_{\mathfrak{a}}=\beta$. We show α is simple. Assuming the contrary, write $\alpha=\alpha_1+\alpha_2$ with α_1 and α_2 positive. Then α_1 and α_2 cannot both have nonzero restrictions to $\mathfrak{a}$, by what was shown in the previous paragraph. Thus assume $\alpha_2|_{\mathfrak{a}}=0$. Then α_1 is smaller than α and has $\alpha_1|_{\mathfrak{a}}=\beta$, contradiction. We conclude that α is a simple $\mathfrak{a}_p$-root.

The simple $\mathfrak{a}_M$-roots $\mu_1,\ldots,\mu_k$ are simple for $\mathfrak{a}_p$, and thus $\alpha,\mu_1,\ldots,\mu_k$ are all simple $\mathfrak{a}_p$-roots. From this set of simple $\mathfrak{a}_p$-roots, we can form a new parabolic subgroup $M^*A^*N^*$ containing MAN. Applying (1.11) twice, first to the parabolic $M^*A^*N^*$ of G and then to the parabolic $MA^{(\beta)}N^{(\beta)}$ of M^*, we obtain

$$\rho_p=\rho^*+\rho_{M^*}=\rho^*+\rho^{(\beta)}+\rho_M.$$

Evaluation of both sides on H_β completes the proof of Proposition 1.2.

Appendix B. Further Property of the η Function

We give here a further property of $\eta_\xi(z)$ that could have been included in Proposition 7.4 but for the length of its proof. We require a lemma.

Lemma B.1. *Let MAN and MAN' be associated parabolic subgroups. If φ is a C^∞ function on K/K_M that is supported in a sufficiently small neighborhood of the identity coset, then $\varphi(\kappa(v))$ is a well-defined function of compact support in $V\cap N'$.*

Proof. The ambiguity with $\kappa(v)\mu(v)$ is with elements of K_M; since φ is right K_M-invariant, $\varphi(\kappa(v))$ is well-defined. Since N' is closed, it is enough to find that $\varphi(\kappa(v))$ has compact support in V. Now $v\in V\to\kappa(v)K_M$ is a smooth map such that

$$\int_{K/K_M} f(k)\,dk=\int_V f(\kappa(v))\,e^{-2\rho_P H_P(v)}\,dv,$$

which says that $\exp\{-2\rho_P H_P(v)\}$ is the Jacobian determinant. Moreover, the map is one-one since $V\cap MAN=\{1\}$. Hence the image, all $\kappa(v)K_M$, is open and the inverse map is smooth onto V. The result follows.

Proposition B.2. *Let $\dim A=1$ and let $P=MAN$. Then the function $\eta_\xi(z)$ defined in Proposition 7.4 is nowhere vanishing for z imaginary.*

Proof. Suppose $\eta_\xi(z_0)=0$ with z_0 imaginary. As in the proof of Proposition 7.4(c), this means that $A(\bar{P}:P:\xi:z_0\rho_P)$ is the 0 operator on K-finite functions. To show that this is impossible, we shall give a simplified variant of the argument in §12. We introduce again the notation that follows Lemma 12.5: ξ is imbedded infinitesimally in ω, and there are projections $E_{F'}$ and $E_{F''}$ in H^ξ onto finite-dimensional subspaces X and Y.

We restate and reprove Lemma 12.6, replacing $r^{-1}Pr$ by $\bar{P}$ and $r^{-1}Nr$ by V throughout. Call the analytic family $B(\xi:\Lambda)f(k)$. To apply this lemma, we proceed as in the proof of Theorem 12.1. Construct F in $C_\xi^\infty(K,X)$ with $F(1)\neq 0$. Consider the function $f=\varphi F$, where φ is a complex-valued right K_M-invariant C^∞ function on K to be specified.

For $\Lambda=z\rho_P$ and $\operatorname{Re} z$ sufficiently large, we have

$$B(\xi:\Lambda)f(1)=\int_V e^{-(1+z)H(v)}E_{F''}\,\xi(\mu(v))^{-1}E_{F'}\,\varphi(\kappa(v))F(\kappa(v))\,dv.$$

By Lemma B.1 choose φ at least to have the property that $\varphi(\kappa(v))$ has compact support. Then this integral provides its own analytic continuation, and our variant of Lemma 12.6 implies

$$E_{F''}A(\bar{P}:P:\xi:z_0\rho_P)f(1)=\int_V e^{-(1+z_0)\rho_P H(v)}E_{F''}\,\xi(\mu(v))^{-1}\,\varphi(\kappa(v))F(\kappa(v))\,dv.$$

Choose $F''=F'$. Since $F(1)\neq 0$, if we choose φ to be sufficiently peaked about 1, we conclude that

$$E_{F''}A(\bar{P}:P:\xi:z_0\rho_P)f(1)\neq 0.$$

Thus there is a function in $C_\xi^\infty(K,X)$ on which the continuous operator $A(\bar{P}:P:\xi:z_0\rho_P)$ is non-vanishing. Composing with projections to K-finite spaces, we obtain a contradiction to our assumption that $A(\bar{P}:P:\xi:z_0\rho_P)$ vanishes on K-finite functions.

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