Appendix
Final remarks

A.1 Separable algebras

The notion of separable algebra is everywhere present in this book, even if it does not appear explicitly. For example, in the classical context of Galois theory, a K-algebra A on a field K is separable when it is split by some field extension $K \subseteq L$.

In order to throw more light on the historical development of the Galois theories presented in this book, we will take the notion of a separable algebra, and look at its evolution through various lines, from the classical theory to the most abstract level of general categories.

Most textbooks on algebra define this notion only in the special case of field extensions. They explain that for a finite dimensional (usually called just "finite") field extension the following conditions are equivalent.

(i) Every element of the extension is separable over the ground field; recall that an element is said to be separable if it is a simple root of its minimal polynonial, or equivalently, if that polynomial has a non-zero discriminant.

(ii) The extension is generated by its separable elements.

(iii) The separable degree of the extension coincides with its degree; recall that the separable degree is the number of algebra homomorphisms from the extension to the algebraic closure of the ground field, and the degree is the dimension of the extension as a vector space over that field.

And then they define the finite separable extensions as those which satisfy the equivalent conditions (i)–(iii). For the infinite (dimensional) extensions (iii) does not make sense, but (i) and (ii) are still equivalent and can be used as definitions (and in fact (i) is used in chapter 1 of this book). However sometimes it is also convenient to think of the transcendental extensions as being separable, and speak of algebraic separable and (general) separable extension; since non-algebraic separable extensions have no use in this book, we will not do that.

The first step of generalization is to replace the field extensions by the commutative algebras over fields. For a finite dimensional (commutative) algebra A over a field K, the following conditions are equivalent.

(iv) A is a finite product of separable field extensions of K.

(v) $L \otimes_K A$ has no nilpotent elements for any field extension L of K.

(vi) The L-algebra $L \otimes_K A$ is of the form $L \times \cdots \times L$ for some field extension L of K.

(vii) A is projective as an $(A \otimes_K A)$-module.

The equivalence (iv)$\Leftrightarrow$(v) is well known, and motivates (v), which is used for example by N. Bourbaki (see [9]), also in the case of infinite dimensional algebras, as the definition of a separable algebra; it is also used in the case of field extensions, in the definition of what we called above general (not necessarily algebraic) separable field extension. Each of the conditions (vi) and (vii) actually implies that A is finite dimensional, and N. Bourbaki uses (vi) as the definition of étale algebra, which agrees with A. Grothendieck's notion of étale covering of a scheme.

The second step of generalization is to replace the commutative algebras over fields by the commutative algebras over connected commutative rings. Recall that "connected" refers to the connectedness of the Zariski spectrum, which is equivalent to the absense of non-trivial idempotents. In other words a commutative ring is said to be connected if it is indecomposable into a non-trivial product of two rings (which also agrees with the categorical notion of connectedness). When K is such a ring, or even an arbitrary commutative ring, the condition (vii) above is used as the definition (also for non-commutative A, replacing then $A \otimes_K A$ by $A \otimes_K A^{\mathrm{op}}$, where A^{op} is A with the opposite multiplication) of separable algebra. Note that such separable algebras have a section in S. Mac Lane's *Homology* (see [64]), and F. DeMeyer and E. Ingraham's book [24] is about them. However in order to develop their Galois theory, one has to require an additional condition, namely that A is projective also as K-module. At this point we should also mention that the notion of Galois extension of commutative rings was first defined (independently of A. Grothendieck) by M. Auslander and

O. Goldman (see [3]), and the Galois theory of those extensions was developed by S.U. Chase, D.K. Harrison, and A. Rosenberg (see [21]) and G.J. Janusz (see [54]), and many others (see the references in [24]). As suggested by the Galois theory of commutative rings, "the right notion" is what A.R. Magid (see [67]) later called a strongly separable algebra. It is a commutative algebra A over a connected commutative ring K satisfying the following equivalent conditions.

(viii) A is projective as a K-module and as an $(A \otimes_K A)$-module.

(ix) The L-algebra $L \otimes_K A$ is of the form $L \times \cdots \times L$ for some commutative K-algebra L which is (non-zero) finitely generated and projective as a K-module.

(x) The same condition with faithfully flat instead of finitely generated and projective.

(xi) The same condition with $K \longrightarrow L$ being a pure monomorphism of K-modules instead of L being finitely generated and projective.

Those algebras correspond to the finite dimensional separable algebras over fields, and their filtered colimits (="directed unions") which A.R. Magid calls locally strongly separable algebras correspond to the infinite dimensional ones. Note that the equivalences (viii)–(xi) actually involve A. Grothendieck's descent theory, and in particular (x)$\Leftrightarrow$(xi) even a more recent unpublished result of A. Joyal and M. Tierney ("effective descent=pure").

Before discussing the third step of generalization – to arbitrary commutative rings – let us come back to the old work on covering spaces, already commented on in the introduction.

If we begin again with the standard textbooks, in this case on algebraic topology, the notion we are looking for is the notion of a covering map of topological spaces as defined in chapter 6: a continuous map $f\colon A \longrightarrow B$ satisfying the condition:

(xii) Every point in B has an open neighbourhood U whose inverse image is a disjoint union of open subsets each of which is mapped homeomorphically onto U by f.

A covering space is the same thing as a covering map: $A = (A, f)$ is a covering space over B (or of B) when $f\colon A \longrightarrow B$ is a covering map.

One now understands better the analogy between the covering spaces and the separable algebras. If B is connected and locally connected, and has a universal (i.e. the "largest" connected) covering (E, p), then all connected coverings of B are quotients of (E, p) and there is a bijection between (the isomorphic classes of) them and the subgroups of the automorphism group $\mathsf{Aut}(E, p)$. That bijection is constructed precisely like the standard Galois correspondence for separable field extensions, but in the dual category $(\mathsf{Top}/B)^{\mathsf{op}}$ of bundles over B.

We recall that this result appears in most books on algebraic topology only in the special case when the Chevalley fundamental group $\mathsf{Aut}(E, p)$ is isomorphic to the usual Poincaré fundamental group of B. The general case was first studied by C. Chevalley (see [23], where however the Galois correspondence is not explicitly mentioned) who actually called $\mathsf{Aut}(E, p)$ the Poincaré group.

The work of A. Grothendieck (see [35] and other "SGA") in abstract algebraic geometry is most amazing by the number of important notions he discovered. One of them is the notion of étale covering of a scheme which is a "combination" of a separable algebra and a covering space. And what is called Grothendieck's Galois theory is the result of his idea that one should describe not just the above-mentioned Galois correspondence, but the whole category of coverings of a given space/scheme/field. Omitting the precise definition of étale covering, let us just observe that it generalizes to the context of Grothendieck toposes, and then to elementary toposes. The generalization makes the definition very simple: a morphism $f\colon A \longrightarrow B$ (in a topos with coproducts) with connected B is a covering morphism if

(xiii) (A, f) is locally a coproduct of terminal objects, i.e. there exists an epimorphism $p\colon E \twoheadrightarrow B$ such that the pullback functor p^* sends (A, f) to a coproduct of objects of the form $(E, 1_E)$ (although there is a better definition equivalent to this one for "good" toposes).

Speaking of a theory of covering morphisms in a topos, one would refer first of all refer to the locally connected (="molecular") and "locally simply connected" cases described by M. Barr and R. Diaconescu in [7], and then in order to get a reasonably complete list of references for more complicated cases (also for ordinary topological spaces), to the more recent paper [60] of J. Kennison. M. Barr's work (see [5] and [6]) is also to be mentioned here.

Another thing is that since the Grothendieck toposes first of all generalize the categories of sheaves over topological spaces, we have to mention that the following conditions for a sheaf F over a "good" topological space B are equivalent.

(xiv) $F \longrightarrow 1$ is a covering in the topos of sheaves over B.

(xv) The étale space over B corresponding to F is a covering of B.

(xvi) F is a locally constant sheaf.

An important area of investigation in homotopy theory is to study the category of simplicial sets and other categories admitting the Quillen homotopy structure (see [70]) and/or its various modifications. Accordingly the notion of a covering has been introduced for some of them (see [12] and [33]). Most importantly there are coverings of simplicial sets and of groupoids; the definitions are recalled in section A.3 (it is far in the book, but almost no previous material is required to understand those definitions).

As already mentioned in the introduction, the central extensions of groups are not usually considered as a part of Galois theory. However, not to speak of their deep relationship with the covering spaces, we recall that they turn out to be precisely the covering morphisms in a certain "non-Grothendieck" special case of categorical Galois theory, as explained in section 5.2. Moreover, the same is true for the central extensions of Ω-groups as defined by A.S.-T. Lue in [63] (see also A. Fröhlich [30] for the earlier definition for algebras over rings, and [31] for a complete version of the theory).

Now we return to commutative rings. A.R. Magid in [67] develops Grothendieck's Galois theory of commutative rings in full generality, and according to his work the "right notion" now becomes what he calls the componentially locally strongly separable (algebras), i.e. the K-algebras A such that

(xvii) The Pierce representation of A (see section 4.2) is a sheaf of locally strongly separable algebras.

Note that A.R. Magid uses what he calls Galois groupoids instead of Galois groups, and those groupoids have a profinite topology which makes them not equivalent (as topological groupoids) to any kind of a topological family of groups – unlike the ordinary groupoids (although R. Brown has good reasons to say that even ordinary groupoids should never be replaced by families of groups!). Note also that the groupoids appear in Galois theory of commutative rings first in the papers of O.E. Villamayor and D. Zelinsky (see [73] and [74].

Finally, the categorical Galois theory defines a covering morphism in an abstract category with respect to a pair of adjoint functors between that category and another one. Although all the notions considered above are special cases of it, the most important observation which actually was the starting point of the categorical approach (see [36]) is

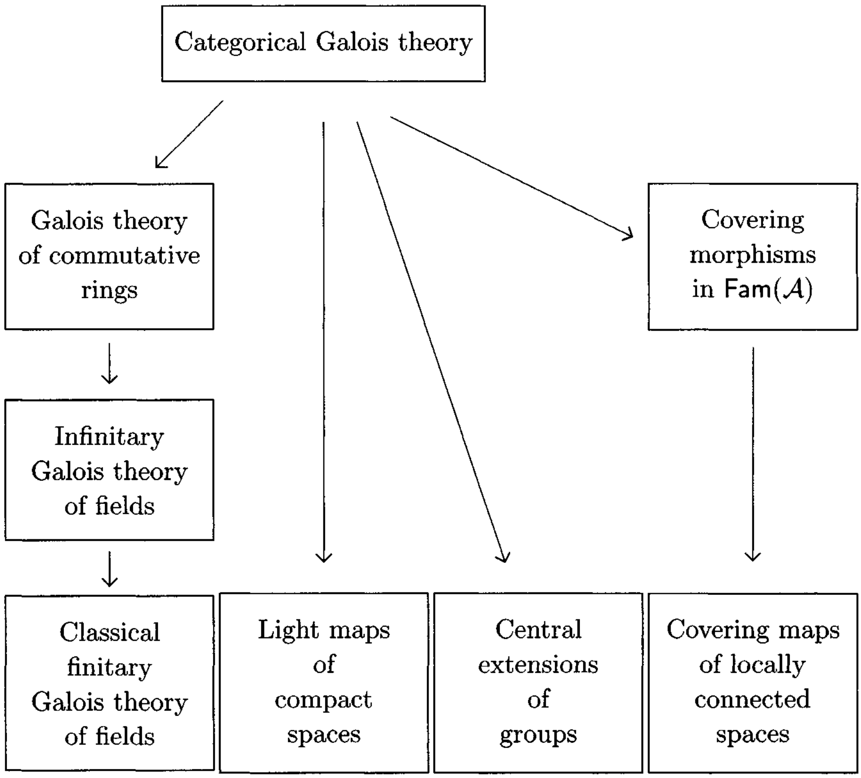
> *the covering morphisms (= "objects split over extensions" – see chapters 5 and 6 for the details) with respect to the Pierce spectrum functor*
>
> $$\mathsf{Sp}\colon \mathsf{Ring}^{\mathsf{op}} \longrightarrow \mathsf{Prof}$$
>
> *and its right adjoint are precisely the componentially locally strongly separable algebras in the sense of A.R. Magid.*

We would like to conclude that the last generalization (from commutative rings to general categories) simply indicates that the Galois theory has moved from classical algebra to category theory – just as a long time ago it moved from numbers to abstract fields. We know that those who believe that the fundamental theorem of Galois theory is a triviality and the "real" Galois theory begins with the class field theory will not agree with us. And yet, in this book we have tried to support the conclusion above by showing that the fundamental theorem of Galois theory has a purely categorical formulation and a purely categorical proof (as originally shown in various versions in [36], [37], [39], [41]).

An additional remark for category-theorists: three special cases of "Galois theory in variable categories" are mentioned in [48]:

(i) What we called above the categorical Galois theory.
(ii) R.H. Street's general theory of torsors (see [72]). Unfortunately neither the present book nor [48] or [72] describes the long list of investigations on various special types of torsors and their connections with Galois theories (another book is to be written about that!). Let us just say that various interesting articles of M. Bunge, J. Funk, J. Kennison, A. Kock, I. Moerdijk and others are devoted to the torsor approach to the fundamental groups in toposes, and that S.U. Chase and M.E. Sweedler's book (see [22]) devoted to so-called Galois objects (a modified notion of a torsor) had opened a new area of research in Galois theory of rings – on so-called Hopf Galois extensions.
(iii) The A. Joyal–M. Tierney theory in [57] (see section 7.10), which plays an important role in topos theory.

These three cases by no means cover each other, and only the first one really gives the fundamental theorem of Galois theory as an immediate corollary. Note also that [49], whose level of generality is strictly between



Scheme 2: Levels of generality in Galois theory

[48] and the categorical Galois theory, has nothing to do with [62] (which is related to [22]), but provides an approach to the Tannaka duality different from, for example, P. Deligne [26]. On the other hand [26] itself is in some sense similar to [57].

A.2 Back to the classical Galois theory

Excluding the "non-galoisian" situations studied in chapter 7, scheme 2 on this page presents roughly the levels of generality in Galois theory considered in this book so far.

In this section we will explain that the classical Galois theory is a special case of the theory of covering morphisms in $\mathsf{FinFam}(\mathcal{A})$, which is the obvious finite version of $\mathsf{Fam}(\mathcal{A})$. The important conclusion is that the classical theories of separable algebras and covering maps are much more similar to each other than one can conclude from the scheme on page 310 – and, not going into details, let us just mention that we are in fact speaking of a well-known similarity which Grothendieck's theory of étale coverings of schemes is based on.

Let K be a field and $\mathcal{C}_K$ the opposite category of finite dimensional commutative K-algebras; that is, a morphism $\alpha\colon A \longrightarrow B$ in $\mathcal{C}_K$ is a K-algebra homomorphism $\alpha\colon B \longrightarrow A$, and we assume $\alpha(1) = 1$ as in the previous chapters.

If e is an idempotent in some $A \in \mathcal{C}_K$, then

$$Ae = \{ae \mid a \in A\} \subseteq A$$

on the one hand is the ideal in A generated by e, and on the other hand can be considered as an object in $\mathcal{C}_K$ with $1 = e$. Note that the map $A \longrightarrow Ae$ defined by $a \mapsto ae$ is a K-algebra homomorphism, although the inclusion $Ae \longrightarrow A$ is not. Also note that

$$A \cong Ae \times A(1-e)$$

as K-algebras, and more generally, given idempotents $e_1, \ldots, e_n$ in A with $e_1 + \cdots + e_n = 1$ and $e_i e_j = 0$ for $i \neq j$, we have

$$A \cong \prod_{i=1}^{n} A_{e_i},$$

which becomes

$$A \cong \coprod_{i=1}^{n} Ae_i$$

in $\mathcal{C}_K$. Conversely, any decomposition into a coproduct in $\mathcal{C}_K$ is up to isomorphism of this form: indeed, given $A \cong \coprod_{i\in I}^{n} A_i$ we take $e_1, \ldots, e_n$ to be the idempotents in A corresponding to $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ in $\coprod_{i\in I}^{n} A_i$ which in fact is the cartesian product of the K-algebras $A_1, \ldots, A_n$. Of course, all possible non-trivial coproducts in $\mathcal{C}_K$ (= products of K-algebras) must be finite since we are dealing here with the finite dimensional algebras.

Since for the decompositions above we have $n \leq \dim\,[A : K]$, and every non-trivial idempotent in any Ae_i would give a further decomposition, we obtain

Lemma A.2.1 *Every $A \in \mathcal{C}_K$ has a finite system $e_1, \ldots, e_n$ of idempotents such that*

(i) *each Ae_i has no non-trivial idempotents,*

(ii) $e_1 + \cdots + e_n = 1$,

(iii) $i \neq j \Rightarrow e_i e_j = 0$. □

Readers familiar with chapter 4 will immediately conclude (using lemma 4.2.5, or directly), that $\{e_1, \ldots, e_n\}$ in lemma A.2.1 is precisely the set of all minimal (non-zero) idempotents in A, and that therefore the boolean algebra of all idempotents in A is the unique (up to isomorphism) boolean algebra with 2^n elements – and that $\{e_1, \ldots, e_n\}$ can be identified with its Stone space. However, our intention in this section is not to examine the finite version of the results of "profinite" arguments of chapter 4, but to show a simple direct way of applying the finite version of the results of section 6.6 to obtain the fundamental theorem of classical Galois theory.

Lemma A.2.2 *If $C \in \mathcal{C}_K$ is a non-zero ring with no non-trivial idempotents, then the functor $\mathsf{Hom}(C, -)\colon \mathcal{C}_K \longrightarrow \mathsf{Set}$ preserves finite coproducts.*

Proof Since there is no K-algebra homomorphism from the zero K-algebra to C, we know that $\mathsf{Hom}(C, -)$ preserves the empty coproduct. In order to prove that it preserves the binary coproducts, we have to show that every K-algebra homomorphism $A_1 \times A_2 \longrightarrow C$ (where X is the cartesian product for K-algebras) factors through one of the projections $A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$. Since C has no non-trivial idempotents and

- $(1, 0)$ and $(0, 1)$ are idempotents in $A_1 \times A_2$,
- $(1, 0)(0, 1) = (0, 0)$ which is the 0 in $A_1 \times A_2$,
- $(1, 0) + (0, 1) = (1, 1)$ which is the 1 in $A_1 \times A_2$

– we conclude that under the homomorphism above we have either $(1, 0) \mapsto 1$ and $(0, 1) \mapsto 0$, or $(1, 0) \mapsto 0$ and $(0, 1) \mapsto 1$. That is, that homomorphism factors through one of the projections. □

From lemmas A.2.1 and A.2.2, and the finite version of proposition 6.1.5, we obtain

Theorem A.2.3 *The category $\mathcal{C}_K$ is equivalent to $\mathsf{FinFam}(\mathcal{A}_K)$, where $\mathcal{A}_K$ is the full subcategory in $\mathcal{C}_K$ with objects all $A \in \mathcal{C}_K$ satisfying the following equivalent conditions:*

(i) *A is connected in $\mathcal{C}_K$;*

(ii) *A has no non-trivial idempotents;*

(iii) *A is indecomposable into a non-trivial product as a K-algebra (or, equivalently, as a ring).* □

This theorem of course suggests identifying $\mathcal{C}_K$ with $\mathsf{FinFam}(\mathcal{A}_K)$ and examining the Galois theory in $\mathcal{C}_K$ corresponding to the adjunction

$$\mathcal{C}_K = \mathsf{FinFam}(\mathcal{A}_K) \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathsf{FinSet}$$

which is constructed similarly to $\mathsf{Fam}(\mathcal{A}) \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathsf{Set}$ used in sections 6.2–6.7, but with $\mathcal{A}_K$ instead of the abstract $\mathcal{A}$, and the finite families and finite sets instead of all families and all sets.

Bearing in mind that we intend to apply the Galois theorem only to the field extensions, the first step of the examination is

Observations A.2.4

(i) A morphism $\alpha\colon A \longrightarrow B$ in $\mathcal{C}_K$ with connected B is a trivial covering morphism if and only if (A, α) is a coproduct of a finite number of copies of $(B, 1_B)$ in $\mathcal{C}_K$, and therefore if and only if

$$A \cong B \times \cdots \times B = B^n \quad \text{(for some finite } n\text{)}$$

as B-algebras.

(ii) Let $p\colon E \longrightarrow B$ and $\alpha\colon A \longrightarrow B$ be morphisms in $\mathcal{C}_K$ with connected E. Then by (i), and since the pullbacks in $\mathcal{C}_K$ are tensor products of algebras, (A, α) is split by $p\colon E \longrightarrow B$ if and only if

$$E \otimes_B A \cong E \times \cdots \times E$$

as E-algebras.

(iii) Let $p\colon E \longrightarrow B$ be a morphism in $\mathcal{C}_K$, in which E and B are fields. Then by (ii), and since

- p is an effective descent morphism in $\mathcal{C}_K$, which easily follows from corollary 4.4.5 (or can be proved directly using the same arguments),
- therefore p is a morphism of Galois descent if and only if $(E, p) \in \mathsf{Split}_B(p)$ – see the remarks between the proof of proposition 6.6.6 and theorem 6.6.7 –

we conclude that p is a morphism of Galois descent if and only if

$$E \otimes_B E \cong E \times \cdots \times E$$

as E-algebras.

(iv) Let $\alpha\colon A \longrightarrow B$ be a morphism in $\mathcal{C}_K$, in which B is a field. Then, as easily follows from 6.6.3(i), the following conditions are equivalent:

(a) $\alpha\colon A \longrightarrow B$ is a covering morphism;
(b) $\alpha\colon A \longrightarrow B$ is split by a field extension $B \subseteq E$ (considered as a morphism $E \longrightarrow B$ in $\mathcal{C}_K$);
(c) $\alpha\colon A \longrightarrow B$ is split by a Galois field extension. $\square$

The second step is to translate these algebraic conditions further into the classical language, which only needs picking up the appropriate simple results from chapter 2:

Observations A.2.5

(i) Theorem 2.3.3 together with A.2.4(ii) tells us that if $p\colon E \longrightarrow B$ and $\alpha\colon A \longrightarrow B$ are morphisms in $\mathcal{C}_K$ in which E and B are fields, then (A, α) is split by $p\colon E \longrightarrow B$ if and only if the field extension $B \subseteq E$ splits the B-algebra A in the sense of definition 2.3.1 (here $B \subseteq E$ up to isomorphism of course).

(ii) From (i) and A.2.4(iii) we conclude that if $p\colon E \longrightarrow B$ is a morphism in $\mathcal{C}_K$ correspondidng to a field extension $B \subseteq E$, then the following conditions are equivalent:

p is a morphism of Galois descent;
$(E, p) \in \mathsf{Split}_B(p)$;
$B \subseteq E$ is a (finite dimensional) Galois extension. $\square$

The third step is to give the following description of algebras split by a Galois extension (which could have been done already in chapter 2):

Proposition A.2.6 *Let $B \subseteq E$ be a finite dimensional Galois field extension, and A a finite dimensional B-algebra. Then the following conditions are equivalent:*

(i) *$B \subseteq E$ splits A;*
(ii) *there exist intermediate field extensions*

$$B \subseteq E_i \subseteq E, \quad i = 1, \dots, m$$

such that

$$A \cong E_1 \times \cdots \times E_m$$

as B-algebras (in particular $B \in \mathcal{C}_K$ implies $A \in \mathcal{C}_K$).

Proof It is easy to see that if $A \cong A_1 \times A_2$ then $B \subseteq E$ splits A if and only if it splits A_1 and A_2. This makes (ii) $\Rightarrow$ (i) obvious (just use the definition of Galois extension) and reduces (i) $\Rightarrow$ (ii) to the case of connected A. Since (i) implies that A is a B-subalgebra of $E \otimes_B A \cong E \times \cdots \times E$ (by A.2.5(i) and A.2.4(ii)), the connectedness easily implies $B \subseteq A \subseteq E$ as desired. □

Now we are ready to make the final conclusion:

Conclusion A.2.7 The abstract Galois theorem 5.1.24 applied to the adjunction $\mathsf{FinFam}(\mathcal{A}_K) \rightleftarrows \mathsf{FinSet}$ gives the Galois theorem 2.4.3. Note that the isomorphism between the Galois groups used in 5.1.24 (the groupoid from 5.1.24 is of course a group in this case) and 2.4.3 can be deduced from theorem 5.1.24 just as it is done in theorem 6.7.4 for a universal covering. Also note that the explicit construction of the equivalence given in 2.4.3 is in fact "automatic": any category equivalence of the form $\mathcal{D}^{\mathsf{op}} \approx \mathsf{FinSet}^G$, where G is a finite group, can be described as $D \mapsto \mathsf{Hom}(D, L)$, where L is any fixed object in $\mathcal{D}^{\mathsf{op}}$ corresponding to G (acting on itself by the multiplication) under the equivalence. □

Remark A.2.8

(i) In addition to the conclusion above, A.2.4(iv) tells us that the covering morphisms $\alpha\colon A \longrightarrow B$ in $\mathcal{C}_K$ with B a field are precisely the separable B-algebras, i.e. the B-algebras of the form $A_1 \times \cdots \times A_m$, where each A_i is a finite dimensional separable field extension of B. In fact this can also be extended for an arbitrary B, but replacing "separable" by "strongly separable" defined as follows: a B-algebra A is strongly separable if it is projective as B-module and as $(A \otimes_B A)$-module (recall again that all algebras are commutative and with unit).

(ii) The similarity with the covering maps of locally connected topological spaces suggests asking if there are universal covering morphisms in $\mathcal{C}_K$. The answer, obvious for readers familiar with field theory, is that this is not the case unless K is obtained from a separably closed field $\overline{K}$ by taking all elements in $\overline{K}$ fixed under a finite group of automorphisms of $\overline{K}$ (for example as for $K = \mathbb{R}$ and $\overline{K} = \mathbb{C}$). Since the "obvious good candidate" for the universal covering of an arbitrary field K is the separable closure $\overline{K}$ of K, which is always the "union" of all finite dimensional (and also of all) Galois extensions of K, it is reasonable to "add" it to $\mathcal{C}_K$. However, the tensor product $\overline{K} \otimes_K \overline{K}$ will (in general) have an infinite number of idempotents, which will force us to replace FinSet by the category of profinite topological spaces, and then there is no good reason not to work with the category of all commutative rings instead of $\mathcal{C}_K$. This conclusion is also supported by the existence of a good notion of strongly separable algebra (see above) over an arbitrary commutative ring. We will thus arrive at the Galois theory with respect to the adjunction between commutative rings and profinite spaces, i.e. at the situation considered in section 4.5. As shown in [37] (first briefly in [36], see also [19]) this brings us to A.R. Magid's Galois theory [67] with the so-called componentially locally strongly separable algebras as the covering morphisms. In that setting, every object (i.e. every commutative ring) has a "minimal" universal covering (connected when the object is connected), which is the separable closure in the sense of [67].

A.3 Exhibiting some links

No proofs are given in this section; we briefly describe various important links between concrete Galois/covering theories provided by a well-known abstract categorical theorem (see theorem A.3.1 below), and make some remarks on the corresponding viewpoints on the fundamental group.

Let us introduce some notation essentially already used in chapter 7: Given a functor $T\colon \mathcal{S} \longrightarrow \mathcal{A}$ from a small category S to a category $\mathcal{A}$ with (small Hom-sets and small) colimits, we construct the diagram

$$\begin{tikzcd}
{[\mathcal{S}^{\mathsf{op}}, \mathsf{Set}]} \arrow[r, shift left, "L"] & \mathcal{A} \arrow[l, shift left, "R"] \\
\mathcal{S} \arrow[u, "Y"] \arrow[ur, "T"'] &
\end{tikzcd}$$

which we will call the fundamental diagram of the functor T, as follows:

- $[S^{\mathsf{op}}, \mathsf{Set}]$ denotes the category of all functors $S^{\mathsf{op}} \longrightarrow \mathsf{Set}$, and the functor $Y\colon S \longrightarrow [S^{\mathsf{op}}, \mathsf{Set}]$ is the Yoneda embedding, i.e. $Y(s) = \mathsf{Hom}(-, s)$ for each object s in S;

- L is the functor defined by

$$L(F) = \mathsf{colim}\big(\mathsf{Elts}(F) \xrightarrow{\phi_H} \mathcal{S} \xrightarrow{T} \mathcal{A}\big),$$

where $\mathsf{Elts}(F)$ is the category of elements of F and ϕ_F the forgetful functor (see section 7.8);
- R is the functor defined by

$$R(A) = \mathsf{Hom}\big(T(-), A\big).$$

Theorem A.3.1 *The fundamental diagram above has the following properties:*

(i) *$L \dashv R$, i.e. L is left adjoint of R;*
(ii) *$LY \cong T$ and L is the unique (up to isomorphism) colimit preserving functor with this property.* □

Many special cases of this theorem were already used in this book; let us list (most of) them:

Example A.3.2

(i) For an arbitrary functor $f\colon \mathcal{S} \longrightarrow \mathcal{S}'$ between small categories take T to be the composite of f and the Yoneda embedding $\mathcal{S}' \longrightarrow [\mathcal{S}'^{\mathsf{op}}, \mathsf{Set}]$. Then for an object A in $\mathcal{A} = [\mathcal{S}'^{\mathsf{op}}, \mathsf{Set}]$ we have

$$R(A)(s) = \mathsf{Nat}\Big(\mathsf{Hom}\big(-, f(s)\big), A\Big) \cong Af(s),$$

and so the adjunction $L \dashv R$ can be identified with

$$[\mathcal{S}^{\mathsf{op}}, \mathsf{Set}] \underset{\text{Composition with } f}{\overset{\text{Left Kan extension along } f}{\rightleftarrows}} [\mathcal{S}'^{\mathsf{op}}, \mathsf{Set}]$$

where we identify f with the dual functor f^{op}, as we already did many times above.

(ii) In the special case $\mathcal{S}' = \mathcal{S}$, $f = 1_{\mathcal{S}}$, independently of (i) the property A.3.1(ii) tells us that L must be isomorphic to the identity functor, and then the formula defining L shows once again that every functor $\mathcal{S}^{\mathsf{op}} \longrightarrow \mathsf{Set}$ is a colimit of representable functors. Moreover, A.3.1(i) then tells us that R must also be isomorphic to the identity functor, which is nothing but the Yoneda lemma.

(iii) The adjunction $L \dashv R$ from (i) was in fact used in the proof of lemma 7.8.2 in the case where f was a geometric morphism of locales.

(iv) If we take $S' = \mathbf{1}$ in (i), then $L \dashv R$ becomes $\mathsf{colim} \dashv \Delta$ from the proof of lemma 7.8.4.

(v) In (iv) we had $T(s) = 1 \in \mathsf{Set}$ for every object s in S. If we take another constant functor $S \longrightarrow \mathsf{Set}$, namely with $T(s) = \pi_0(S)$ (= the set of connected components of S) instead of 1, then $L \dashv R$ becomes $I \dashv H$ from section 6.2 in the case $\mathsf{Fam}(\mathcal{A}) = [S^{\mathsf{op}}, \mathsf{Set}]$.

(vi) If we return to an arbitrary $\mathcal{A}$, but take $S = \mathbf{1}$, then the formulæ for L and R become

$$L(X) = X \cdot T(*), \quad R(A) = \mathsf{Hom}\big(T(*), A\big),$$

where $T(*)$ is the image in $\mathcal{A}$ of the unique object of $\mathbf{1}$ under T, and $X \cdot T(*)$ is the coproduct in $\mathcal{A}$ of the X-indexed family of $T(*)$ (where X is an arbitrary set). When $T(*) = 1$, this gives the adjunction considered in 6.2.2(iv), and therefore generalizes $H \dashv \Gamma$ of section 6.2 (in this case we do not need all colimits in $\mathcal{A}$ of course – just copowers).

(vii) For a moment let us write L_T and R_T instead of L and R respectively; since any $T\colon S \longrightarrow \mathcal{A}$ determines the dual functor $T^{\mathsf{op}}\colon S^{\mathsf{op}} \longrightarrow \mathcal{A}^{\mathsf{op}}$, the whole story above can be dualized, and the fundamental diagram of T^{op} displays as

$$\begin{tikzcd}
{[S, \mathsf{Set}]} \arrow[rr, shift left, "L_{T^{\mathsf{op}}}"] & & \mathcal{A}^{\mathsf{op}} \arrow[ll, shift left, "R_{T^{\mathsf{op}}}"] \\
& & \\
S^{\mathsf{op}} \arrow[uu, "S \mapsto \mathsf{Hom}(s,-)"] \arrow[uurr, "T^{\mathsf{op}}"'] & &
\end{tikzcd}$$

with

$$\begin{aligned}
L_{T^{\mathsf{op}}}(F) &= \mathsf{lim}\big(\mathsf{Elts}(F) \longrightarrow S \xrightarrow{T} \mathcal{A}\big), \\
R_{T^{\mathsf{op}}}(A) &= \mathsf{Hom}\big(A, T(-)\big)
\end{aligned}$$

(with the appropriate definition of $\mathsf{Elts}(F)$ and the forgetful functor from it to S). In particular the finite version of this adjunction (in a very special situation) turns out to be a category equivalence in the Galois theorem 2.4.3 – also see the last remark in A.2.7. $\square$

Example A.3.3 Let B be a topological space, S the locale $\mathcal{O}(B)$ of open subsets in B, considered as a category, and

$$T\colon S \longrightarrow \mathsf{Top}/B$$

the functor defined by $T(U) = (U, i_U)$, where $i_U : U \longrightarrow B$ is the inclusion map.

(i) $[S^{\mathsf{op}}, \mathsf{Set}]$ is the (ordinary) category of presheaves over the space B.

(ii) For $F \in [S^{\mathsf{op}}, \mathsf{Set}]$ and $L(F) = (A, \alpha)$, the space A can be identified with the space of germs $[U, b, x]$ of F with $\alpha\colon [U, b, x] \mapsto b$. Recall that such a germ is the equivalence class of a triple (U, b, x) in which U is an open subset of B, $b \in U$, and $x \in F(U)$, where (U, b, x) and (U', b', x') are said to be equivalent when $b = b'$ and there exists an open $V \subseteq U \cap U'$ with $b \in V$ and $x|_V = x'|_V$ – denoting by $x|_V$ the image of x under the map $F(U) \longrightarrow F(V)$. Briefly

$$A = \operatorname*{colim}_{(U,b,x)} U,$$

the (filtered!) colimit of U over all (U, b, x), and this also determines the topology in A as the colimit topology; readers of course noticed that the triples (U, b, x) are nothing but the objects of $\mathsf{Elts}(F)$ and that $U = \phi_F(U, b, x)$.

(iii) For $(A, \alpha) \in \mathsf{Top}/B$, $R(A, \alpha)$ is what is called the sheaf of local sections (or cross-sections) of (A, α) since

$$\begin{aligned} R(A,\alpha)(U) &= \mathsf{Hom}\big(T(U), (A,\alpha)\big) \\ &= \mathsf{Hom}\big((U, i_U), (A,\alpha)\big) \\ &= \big\{u\colon U \longrightarrow A \text{ in } \mathsf{Top} \;\big|\; \alpha(b) = b \text{ for each } b \in U\big\}. \end{aligned}$$

(iv) The adjunction $L \dashv R$ induces precisely the equivalence between the category $\mathsf{Sh}(B)$ of sheaves over B and the category $\mathsf{Et}(B)$, which is the full subcategory in Top/B with objects all (A, α) with étale α. This can be displayed as

$$\begin{tikzcd} {[\mathcal{O}(B)^{\mathsf{op}}, \mathsf{Set}]} \arrow[r, shift left, "L"] \arrow[dr, dotted] & \mathsf{Top}/B \arrow[l, shift left, "R"] \arrow[dl, dotted] \\ \mathsf{Sh}(B) \arrow[u, "\subseteq"] \arrow[r, leftrightarrow, "\approx"] & \mathsf{Et}(B) \arrow[u, "\subseteq"'] \end{tikzcd}$$

and means that a presheaf F is a sheaf if and only if the canonical morphism $F \longrightarrow RL(F)$ is an isomorphism, and a continuous map $\alpha\colon A \longrightarrow B$ is étale if and only if the canonical map $LR(A,\alpha) \longrightarrow (A,\alpha)$ is an isomorphism.

(v) By (iv), the two inclusion functors $\mathsf{Sh}(B) \longrightarrow [\mathcal{O}(B)^{\mathsf{op}}, \mathsf{Set}]$ and $\mathsf{Et}(B) \longrightarrow \mathsf{Top}/B$ have a left and a right adjoint respectively. The left adjoint of $\mathsf{Sh}(B) \longrightarrow [\mathcal{O}(B)^{\mathsf{op}}, \mathsf{Set}]$ is the associated sheaf functor, and it is of course isomorphic to the one constructed in the proof of theorem 7.7.3 when the locale involved is $\mathcal{O}(B)$. □

Here is another "famous" example playing a central role in homotopy theory, just as the previous one does in sheaf theory:

Example A.3.4 First we observe that since a finite dimensional vector space $\mathbb{R}^n$ can be considered as the coproduct of n copies of $\mathbb{R}$, there is a well-defined functor $\mathbb{R}^{(-)}$ from the category of finite sets to the category of real vector spaces. The set

$$\mathbf{\Delta} = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \big| x_i \geq 0 \text{ for all } i, \text{ and } x_1 + \cdots + x_{n+1} = 1\big\}$$

is called the standard n dimensional simplex; it is an n dimensional simplex which is "chosen nicely": in particular for every map

$$f\colon \{1, \ldots, n+1\} \longrightarrow \{1, \ldots, m+1\}$$

the map

$$\mathbb{R}^{n+1} = \mathbb{R}^{\{1,\ldots,n+1\}} \xrightarrow{\mathbb{R}^f} \mathbb{R}^{\{1,\ldots,m+1\}} = \mathbb{R}^{m+1}$$

induces a map $\mathbf{\Delta}_f\colon \mathbf{\Delta}_n \longrightarrow \mathbf{\Delta}_m$. Moreover, since $\mathbb{R}$ is a topological field, its topology makes each $\mathbf{\Delta}_n$ a topological space and each $\mathbf{\Delta}_f$ a continuous map.

Now we take S to be the simplicial category $\mathbf{\Delta}$ whose objects are all (non empty) sets of the form $[n] = \{1, \ldots, n+1\}$ and morphisms all maps f between them satisfying

$$i \leq j \Rightarrow f(i) \leq f(j),$$

and $T\colon \mathbf{\Delta} \longrightarrow \mathsf{Top}$ to be the functor defined by $T\big([n]\big) = \mathbf{\Delta}_n$, $T(f) = \mathbf{\Delta}_f$ in the notation above. The corresponding adjunction $L \dashv R$ according to the standard terminology has

- $[\mathbf{\Delta}^{\mathsf{op}}, \mathsf{Set}] = \mathsf{SimplSet}$, the category of simplicial sets,
- $L = |-|\colon \mathsf{SimplSet} \longrightarrow \mathsf{Top}$, the geometric realization functor,
- $R\colon \mathsf{Top} \longrightarrow \mathsf{SimplSet}$ the "singular complex" or "nerve" functor.

Note that the adjunction $L \dashv R$ here does not induce an equivalence between the "images" of L and R as it did in example A.3.3. However

- it induces an equivalence between certain categories built up from Top and $\mathsf{SimplSet}$, and the resulting category is the main object of investigation in homotopy theory,
- the image of L is still important: it produces the notion of CW-complex, which is a "homotopy-theoretic candidate" for the notion of a "good" space. □

Replacing spaces by small categories we obtain

Example A.3.5 Regarding the objects of $\mathbf{\Delta}$ as small categories (each $[n]$ being an ordered set is a category!) and the morphisms as functors, we take $T\colon \mathbf{\Delta} \longrightarrow \mathsf{Cat}$ to be the inclusion of $\mathbf{\Delta}$ into the category Cat of all small categories. The corresponding adjunction $L \dashv R$ has R full and faithful, and so induces an equivalence between Cat and a full subcategory in $\mathsf{SimplSet}$ (which can be described as the "image" of R). The functor R is again called the nerve. Note that for a (small) category A we have

- $R(A)\big([0]\big)$ is the set of objects in A,
- $R(A)\big([1]\big)$ is the set of morphisms in A,
- $R(A)\big([2]\big)$ is the set of composable pairs, and, more generally, $R(A)\big([n]\big)$, $n \geq 2$ is the set of composable sequences

$$\bullet \xrightarrow{\ a_1\ } \bullet \xrightarrow{\ a_2\ } \bullet \cdots \bullet \xrightarrow{\ a_{n-1}\ } \bullet \xrightarrow{\ a_n\ } \bullet$$

of morphisms in A. □

Omitting many similar examples, let us consider only groupoids:

Example A.3.6 Consider the commutative (up to isomorphism) diagram A.1: in which

- Grpd is the category of small groupoids,
- the fraction groupoid functor is the left adjoint of the inclusion functor $\mathsf{Grpd} \longrightarrow \mathsf{Cat}$ – it is the universal construction making all morphisms of a given category invertible –
- the functor "codiscrete" sends a set X to the codiscrete groupoid on X, i.e. to the groupoid whose objects are all the elements of X, with $\mathsf{Hom}(x, y)$ having exactly one element for every x and every y in X.

$$\begin{tikzcd}
\mathsf{Cat} \arrow[r, "\text{fraction groupoid}"] & \mathsf{Grpd} \\
\mathbf{\Delta} \arrow[u, "\text{inclusion as in A.3.5}"] \arrow[ur, dotted] \arrow[r, "\text{forgetful}"'] & \mathsf{Set} \arrow[u, "\text{codiscrete}"']
\end{tikzcd}$$

Diagram A.1

We take $T\colon \mathbf{\Delta} \longrightarrow \mathsf{Grpd}$ to be the resulting dotted arrow, i.e. any of the two isomorphic composites above. The corresponding adjunction $L \dashv R$ again has R full and faithful, and it is again called the nerve functor; for L the standard notation is

$$L(X) = \pi_1(X), \text{ or } \Pi_1(X),$$

and that groupoid is called the fundamental groupoid of the simplicial set X. □

Among the links between Galois/covering theories provided by the examples above are

A.3.7 Let us first apply the results of sections 6.6 and 6.7 to the case $\mathcal{C} = \mathsf{Set}^G$, the category of G-sets, where G is a group; and let us consider only the case $B = 1$. It is easy to prove the following.

(i) A morphism $A \longrightarrow 1$ is a trivial covering morphism if and only if the action of G on A is trivial, i.e. $ga = a$ for all $g \in G$ and $a \in A$.

(ii) Every morphism $A \longrightarrow 1$ is a covering split by $G \longrightarrow 1$ (where G is considered as an object in Set^G with the G-action via the multiplication in G – as already used in the proof of theorem 6.7.4). In particular $G \longrightarrow 1$ is the unique (up to isomorphism) connected universal covering of 1.

(iii) Applying theorems 6.6.7 and 6.7.4 to $G \longrightarrow 1$ we obtain the trivial equivalence $\mathsf{Set}^G \approx \mathsf{Set}^G$.

Now let us compare this with the same theorems for an abstract $\mathcal{C}$ via the adjunction

$$\mathsf{Set}^G = [G, \mathsf{Set}] \underset{R}{\overset{L}{\rightleftarrows}} \big((\mathcal{C}/B)^{\mathsf{op}}\big)^{\mathsf{op}} = \mathcal{C}/B$$

with $G = \mathsf{Aut}(p)^{\mathsf{op}}$ $\big(\approx \mathsf{Aut}(p)\big)$ obtained from the inclusion functor $\mathsf{Aut}(p)^{\mathsf{op}} \longrightarrow (\mathcal{C}/B)^{\mathsf{op}}$ as in A.3.2(vii); note that L and R here play the roles of $L_{\mathcal{T}^{\mathsf{op}}}$ and $R_{\mathcal{T}^{\mathsf{op}}}$ of A.3.2(vii). We observe the following.

- It is easy to deduce from the results of sections 6.6 and 6.7 that R induces an equivalence between the category of coverings of B in $\mathcal{C}$ and the category of coverings of 1 in Set^G itself (the finite version of this again agrees with the Galois theorem 2.4.3!).
- An independent investigation of the adjunction $L \dashv R$ could be called the "external Galois theory" in contrast with the "internal" one developed in chapter 5. Readers can judge now to what extent these two approaches agree. □

A.3.8 The equivalence $\mathsf{Sh}(B) \approx \mathsf{Et}(B)$ described in example A.3.3 tells us that $\mathsf{Et}(B)$ is a topos for every topological space B, and in particular so is LoCo/B for a locally connected B – which we could use to replace various topological arguments by the topos-theoretic ones; for example the fact that every epimorphism is an effective descent morphism holds in any topos. We could also use sheaf theory itself, expressing all constructions in the language of sheaves instead of étale maps; in particular the trivial covering maps correspond to the constant sheaves, and the covering maps to the locally constant sheaves. A good example where the competition between the two languages really occurs is the construction of the (connected) universal covering of a "good" space, say a CW-complex. Such a universal covering $p\colon E \longrightarrow B$ has E the set of equivalence classes of all paths f in B with $f(0) = b_0$ for a fixed $b_0 \in B$, and p is defined by $p\big([f]\big) = f(1)$ – but for the topology on E there is a choice between the following two definitions.

(i) The set of all paths f in B with $f(0) = b_0$ being a subset of all paths in B has the compact-open topology, and then we take the quotient topology on E.

(ii) Let $\mathcal{U}$ be the set of all open subsets U in B in which every path f with $f(0) = f(1)$ is equivalent in B to a constant path (if B were a CW-complex, or more generally locally simply connected and locally path-connected and connected, then we could simply take the set of all open simply connected subsets; however, the algebraic topologists prefer a still more general situation of semi-locally simply connected instead of locally simply connected). For $U \in \mathcal{U}$ we take $F(U)$ to be the set of all equivalence classes $[f]$ of paths f in B with $f(1) \in U$. It can be shown that the collection of all $F(U)$ produces a sheaf on B whose étale space is the E above.

Textbooks in algebraic topology usually give sufficient information to make the proof of equivalence of the two definitions an easy exercise; they also manage to avoid the words "sheaf" and "étale" in the second one. □

A.3.9 As shown in Appendix One of the book [33] of P. Gabriel and M. Zisman (which we certainly recommend our readers to look at!) the functors L and R from example A.3.4 as well as those from example A.3.6 send covering morphisms to covering morphisms. Moreover, it is shown there that for every simplicial set B the induced adjunctions,

$$\mathsf{SimplSet}/B \rightleftarrows \mathsf{Grpd}/\Pi_1(B),$$

$$\mathsf{SimplSet}/B \rightleftarrows \mathsf{Top}/|B|$$

themselves induce equivalences between the categories of coverings: of B and of its fundamental groupoid $\Pi_1(B)$ in the first case, and of B and its geometric realization $|B|$ in the second case. The resulting equivalence between the coverings of $|B|$ in Top and the coverings of $\Pi_1(B)$ in Grpd easily gives an independent proof of the classification theorem of coverings of $|B|$ in terms of the Poincaré fundamental group (in the connected case, to which the general case easily reduces since $|B|$ is localy connected – but also in the general case with the Poincaré groupoid equivalent to $\Pi_1(B)$). In fact a more general classification theorem is proved in [33]. For those who prefer more geometrical language we would recommend R. Brown's book [12].

Let us also observe:

(i) A slightly different construction of what we call T in example A.3.4 is used in [33] (and also in [65]).

(ii) According to [33] a covering morphism $\alpha\colon A \longrightarrow B$ in Grpd can be defined as a functor α from the groupoid A to the groupoid B satisfying the following condition:

for every morphism $f: b \longrightarrow b'$ in B and every object a in A with $\alpha(a) = b$ there exists a unique morphism $g: a' \longrightarrow a$ in A with $\alpha(g) = f$.

This definition on the one hand clearly imitates the unique path-lifting property for topological spaces, and on the other hand tells us that α determines a functor

$$F_\alpha: B^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

with $F_\alpha(b) = \alpha^{-1}(b)$. In fact $(-)^{\mathsf{op}}$ is irrelevant here since $B^{\mathsf{op}} \approx B$ and $\alpha: A \longrightarrow B$ satisfies the condition above if and only if $\alpha^{\mathsf{op}}: A^{\mathsf{op}} \longrightarrow B^{\mathsf{op}}$ does so (in the modern language, a functor between groupoids is a discrete fibration if and only if it is a discrete opfibration). This passage $\alpha \mapsto F_\alpha$ suggests that the fundamental groupoid (= the "largest" Galois groupoid classifying all coverings) of B must be B itself, or equivalent to it. This can be shown independently, but we also need to prove that the definition above of a covering morphism of groupoids does agree with the general one. The proof is similar to the same proof for simplicial sets sketched in (iii) below, but uses in addition the description of effective descent morphisms in Grpd: they turn out to be the functors surjective on composable triples of morphisms. For the simplicial sets, on the one hand that description is automatic since they form a topos, and on the other hand it helps to obtain the result in Grpd.

(iii) Again imitating the geometrical unique lifting property, P. Gabriel and M. Zisman define the covering morphisms of simplicial sets via the liftings for the representable simplicial sets, i.e. those of the form

$$\mathsf{Hom}(-, [n]): \mathbf{\Delta} \longrightarrow \mathsf{Set},$$

but how does this definition agree with definition 6.5.9? The answer is beautiful:

(a) Let X be the set of all morphisms from all $\mathsf{Hom}(-, [n])$ to B; by the Yoneda lemma it can be identified with the set of all elements of B (i.e. of all $B([n])$) and there is a canonical epimorphism

$$p: X \cdot \mathsf{Hom}(-, [n]) \longrightarrow\!\!\!\!\!\rightarrow B,$$

which is an effective descent morphism since $\mathsf{SimplSet}$ is a topos.

(b) It is easy to see (and it is "almost" mentioned in [33] – see the first proposition in [33], Appendix 1, 2.2) that a morphism $\alpha\colon A \longrightarrow B$ satisfies the "lifting definition" in [33] if and only if it is split by the morphism p from (a).

(c) The splitting by p above is equivalent to being a covering, as follows from 6.6.3(i) and the fact that $X \cdot \mathsf{Hom}(-, [n])$ is a projective object in $\mathsf{SimplSet}$. □

Remark A.3.10 The reformuation of the definition of a covering in terms of "unique lifting" is also useful: it says that $\alpha\colon A \longrightarrow B$ is a covering morphism (in the sense of definition 6.5.9, when A and B are connected) if and only if there exists an effective descent morphism $p\colon E \longrightarrow B$ such that for every commutative diagram

$$\begin{tikzcd} X \arrow[r, "v"] \arrow[d, "u"'] & A \arrow[d, "\alpha"] \\ E' \arrow[r, "p'"'] \arrow[ur, dotted] & B \end{tikzcd}$$

of solid arrows, where $p'\colon E' \longrightarrow B$ is a connected component of the morphism $p\colon E \longrightarrow B$, there exists a unique dotted arrow as above making the whole diagram commute.

The reader can once again return to finite dimensional separable field extensions and see that this lifting property (in the FinFam context) gives one of the classical definitions.

The geometrical unique lifting property used in section 6.8 is "less categorical": it also can be reformulated as a condition on a certain diagram being a pullback, but just in Set – not in Top. □

It is time now to ask J. Kennison's question (see [60]): *What is the fundamental group?*

The most general answer in the context of section 5.1 could be that the "fundamental groupoid" of an object B in $\mathcal{C}$ is the internal "pro-groupoid" in $\mathcal{X}$ formed by the Galois descent morphisms. However, this definition would only make good sense if there were enough Galois descent morphisms to split "all coverings", or otherwise we would be forced to replace groupoids by certain precategories as in chapter 7. Yet, under certain additional conditions on $\mathcal{X}$ (which hold when $\mathcal{X} = \mathsf{Set}$), the precategories can be themselves replaced by their fundamental groupoids defined now as for the simplicial sets in example A.3.6. And having any notion of a fundamental groupoid, one can define the fundamental group $\pi_1(B, b)$ for b being an object in the fundamental groupoid of B (i.e. a morphism from 1 to the object of objects in the internal context), as the internal automorphism group of b.

Omitting various intermediate levels of generality with many interesting examples, some of which are mentioned in [60], let us consider the case where $\mathcal{C} = \mathsf{Fam}(\mathcal{A})$ and B is connected and has a universal covering as in section 6.7. In this case we have the following.

- The fundamental groupoid of B is to be defined as the Galois groupoid of any universal covering morphism $p\colon E \longrightarrow B$, and since $\mathsf{Set}^{\mathsf{Gal}\,[p]} \approx \mathsf{Cov}(B)$ for each such p, the fundamental groupoid is then uniquely determined up to equivalence.
- The fundamental group of B is to be defined either as the automorphism group of any object of the fundamental groupoid, or as $\mathsf{Gal}\,[p]$ for a connected universal covering (E, p) of B. It is then determined uniquely up to isomorphism.
- As shown in section 6.7, the group $\mathsf{Gal}\,[p]$, for a connected universal covering (E, p) of B (which itself is determined uniquely up to isomorphism), is isomorphic to $\mathsf{Aut}(p)$ which is what we call the Chevalley fundamental group. However, depending on the way $p\colon E \longrightarrow B$ was constructed, it might be even easier to calculate $\mathsf{Gal}\,[p]$ using just its definition, i.e. as

$$\begin{tikzcd} I(E \times_B E \times_B E) \arrow[r] & I(E \times_B E) \arrow[loop below] \arrow[r, shift left=2] \arrow[r, shift right=2] & I(E), \arrow[l] \end{tikzcd}$$

  which since $\mathsf{Gal}\,[p]$ is a group now is convenient to write simply as $I(E \times_B E)$ (topologists would write $\pi_0(E \times_B E)$, or even $\pi_0(\tilde{B} \times_B \tilde{B})$ since $\tilde{B} \longrightarrow B$ seems to be a standard way to denote (the) universal covering of B). For instance, using the construction of $p\colon E \longrightarrow B$ via A.3.8(ii) (when $\mathcal{C} = \mathsf{LoCo}$ and B is a 'good' space) one arrives at the Poincaré fundamental group as $I(E \times_B E)$, and the same can be done via A.3.9.
- Another approach, which actually also works for 'less good' spaces, is to use proposition 6.4.2, which tells us that the surjective étale maps $\coprod_{\lambda \in \Lambda} U_\lambda \longrightarrow B$, obtained from the families $(U_\lambda)_{\lambda \in \Lambda}$ of open $U_\lambda \subseteq B$ whose union is B, are enough to split all coverings of B in LoCo. Using as above mentioned the corresponding precategories

and their fundamental groupoids, it can be shown that when each U_λ has no non-trivial coverings, the (Chevalley) fundamental group of B is the fundamental group(oid) (in the sense of A.3.6) of the simplicial set S defined by

$$S([n]) = I\left(\underbrace{\left(\coprod_{\lambda\in\Lambda} U\lambda\right) \times_B \cdots \times_B \left(\coprod_{\lambda\in\Lambda} U\lambda\right)}_{n+1 \text{ times}}\right).$$

There are also similar constructions without the existence of universal coverings involving progroups which occur in M. Artin and B. Masur's book (see [2]) and later in a topos-theoretic context (see again [60] and references there) – not to mention A. Grothendieck's original ideas on descent and coverings.

A.4 A short summary of further results and developments

We list now some works closely related to the present book. Most of them could have constituted an additional chapter. We invite the reader to consult them.

A.4.1 As shown in A.2, (the Grothendieck form of) the fundamental theorem of classical Galois theory not only can be considered as a special case of the categorical one, but can actually be deduced from it. For Magid's Galois theory of commutative rings (see [67]) the same is done in [37]; also see [16] and [17] for the further categorical simplifications in two directions. Moreover, as shown in [19], the categorical approach helps to improve the description of Galois correspondence for commutative ring extensions with profinite Galois groupoids.

A.4.2 Reference [38] answers one of the questions kindly suggested by S. Mac Lane. And it turns out that not the Picard–Vessiot extensions themselves, but their subalgebras generated by a fundamental system of solutions of a given linear differential equation and the inverse of the corresponding Wronski determinant, occur in the Galois theory of the adjunction

$$\begin{pmatrix} \text{Differential} \\ \text{commutative rings} \end{pmatrix}^{\mathrm{op}} \underset{\supseteq}{\overset{\text{constants}}{\rightleftarrows}} \begin{pmatrix} \text{Commutative} \\ \text{rings} \end{pmatrix}^{\mathrm{op}}.$$

A.4.3 As the title of [40] shows, it answers a question of R. Brown. The double central extensions are defined as the covering morphisms with respect to a Galois structure (= relatively admissible adjunction in the sense of section 5.1) involving

$$\begin{pmatrix} \text{Group} \\ \text{extensions} \end{pmatrix} \underset{\supseteq}{\overset{\text{“centralization”}}{\rightleftarrows}} \begin{pmatrix} \text{Central group} \\ \text{extensions} \end{pmatrix}.$$

This further extends to higher dimensions, and using the Brown–Ellis–Hopf formula from [13] presents the homology groups of groups as certain fundamental groups (see [43]).

A.4.4 The level of generality in the Barr–Diaconescu covering theory (see [7]) is stricly between the level considered in chapter 6 above and the level of topological spaces; this follows from the results of [42].

A.4.5 The theory of central extensions of universal algebras (and more generally, of objects in a Barr exact category) which on the one hand is a special case of the Galois theory, and on the other hand contains the case of Ω-groups studied by A. Fröhlich's school (the original definition is due to A.S.-T.Lue in [63]; also see J. Furtado-Coelho in [31] and references there), and in particular the ordinary central extensions of groups, is developed in [44]. A further comparison with the universal-algebraic notion of a centre is carried out in [46].

A.4.6 As already mentioned earlier, chapter 5 above should provide a good help for readers not familiar with category theory to understand [18], where the study of the relationship between the Galois theory and factorization systems (on the general categorical level) begins. Further results for the cases of "less well-behaved" coverings are obtained in [45], [51], [52], [53]. For example, as follows from the results of [45], the (purely inseparable, separable)-factorization for the finite dimensional field extensions extends to the category of commutative rings (although the class of homomorphisms imitating the purely inseparable extensions is yet to be investigated).

A.4.7 A categorical version of a so-called tautological proof of the van Kampen theorem is described in [14].

A.4.8 A theory of coverings with all Galois groupoids being (internal) equivalence relations, containing the case of light maps of compact spaces, the case of ring homomorphisms with semisimple kernels, and some others, is developed in [47].

A.4.9 The Galois theory of the adjunction

$$\begin{pmatrix} \text{Simplicial sets,} \\ \text{Kan fibrations} \end{pmatrix} \underset{\text{nerve}}{\overset{\pi_1}{\rightleftarrows}} \begin{pmatrix} \text{Groupoids,} \\ \text{fibrations} \end{pmatrix},$$

which produces a new notion of second order covering map of simplicial sets, is studied in [15].

A.4.10 The approach to Galois theory in various "boolean" cases developed by Y. Diers (see [25]) can be deduced from the categorical one (described in chapter 5 above), as follows from results of [17].

A.4.11 The Galois theory in symmetric monoidal categories (see [49]) has a level of generality strictly between [48] and [41]; its main prupose is to provide a simplified categorical framework for the so-called Tannaka duality.

A.4.12 We would like to mention two Ph.D. theses: of B. Mesablishvili on Galois theory of commutative rings in toposes (see [68]), where in particular the relationship of the results of S.U. Chase and M.E. Sweedler (see [22]), of M. Barr (see [5], [6]) and of Th. Ligon (see [62]) was investigated; and of M. Gran (see [34]) who described the central extensions with respect to various adjunctions involving internal categories in so-called Mal'tsev categories.