The Novikov conjecture and geometry of Banach spaces

Gennadi Kasparov and Guoliang Yu*

Abstract: In this paper, we prove the strong Novikov conjecture for groups coarsely embeddable into Banach spaces satisfying a geometric condition called Property (H).

1 Introduction

An important problem in higher dimensional topology is the Novikov conjecture on the homotopy invariance of higher signature. The Novikov conjecture is a consequence of the Strong Novikov Conjecture on K-theory of group C^*-algebras. The main purpose of this paper is to prove the Strong Novikov conjecture (with coefficients) for any group coarsely embeddable into a Banach space satisfying a geometric condition called Property (H).

Definition 1.1. *A real Banach space X is said to have Property (H) if there exists an increasing sequence of finite dimensional subspaces $\{V_n\}$ of X and an increasing sequence of finite dimensional subspaces $\{W_n\}$ of a real Hilbert space such that*

(1) $V = \cup_n V_n$ is dense in X;

(2) if W denotes $\cup_n W_n$, and $S(V)$, $S(W)$ denote respectively the unit spheres of V, W, then there exists a uniformly continuous map $\psi : S(V) \to S(W)$ such that the restriction of ψ to $S(V_n)$ is a homeomorphism (or more generally a degree one map) onto $S(W_n)$ for each n.

*Both authors are partially supported by NSF.

As an example, let X be the Banach space $l^p(\mathbb{N})$ for some $p \geq 1$. Let V_n and W_n be respectively the subspaces of $l^p(\mathbb{N})$ and $l^2(\mathbb{N})$ consisting of all sequences whose coordinates are zero after the n-th terms. We define a map ψ from $S(V)$ to $S(W)$ by:

$$\psi(c_1, ..., c_k, ...) = (c_1|c_1|^{p/2-1}, ..., c_k|c_k|^{p/2-1}, \cdots).$$

ψ is called the Mazur map [4]. It is not difficult to verify that ψ satisfies the conditions in the definition of Property (H). For each $p \geq 1$, we can similarly prove that the Banach space of all Schatten p-class operators on a Hilbert space has Property (H). We can also check that uniformly convex Banach spaces with certain unconditional bases have Property (H) (with the help of results on the uniform homeomorphism classification of Banach space spheres in Chapter 9 of [4]).

We also recall that a metric space Γ is said to be coarsely embeddable into a Banach space X [7] if there exists a map $h : \Gamma \to X$ for which there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}$ such that

(1) $\rho_1(d(x, y)) \leq \|h(x) - h(y)\| \leq \rho_2(d(x, y))$ for all $x, y \in \Gamma$;

(2) $\lim_{r \to +\infty} \rho_i(r) = +\infty$ for $i = 1, 2$.

In the case of a countable group Γ, we endow Γ with a proper (left invariant) length metric. If Γ is finitely generated, the word length metric is an example of a proper length metric. The issue of coarse embeddability of a countable group into a Banach space X is independent of the choice of the proper length metric.

The following theorem is the main result of this paper.

Theorem 1.2. *Let Γ be a countable discrete group and A any Γ-C^*-algebra. If Γ admits a coarse embedding into a Banach space with Property (H), then the Strong Novikov conjecture with coefficients in A holds for Γ, i.e. if $\mathcal{E}\Gamma$ is the universal space for proper Γ-actions and $A \rtimes_r \Gamma$ is the reduced crossed product C^*-algebra, then the Baum-Connes assembly map*

$$\mu : KK_*^{\Gamma}(\mathcal{E}\Gamma, A) \to K_*(A \rtimes_r \Gamma)$$

is injective.

The special case when the Banach space is the Hilbert space is proved in [22] and [20].

If we replace the degree one condition by a nonzero degree condition in the definition of Property (H), we say that X has rational Property (H).

Theorem 1.3. *Let Γ be a countable discrete group and A any Γ-C^*-algebra. If Γ admits a coarse embedding into a Banach space with rational Property (H), then the rational Strong Novikov conjecture with coefficients in A holds for Γ, i.e. if $\mathcal{E}\Gamma$ is the universal space for proper Γ-actions and $A \rtimes_r \Gamma$ is the reduced crossed product C^*-algebra, then the Baum-Connes assembly map*

$$\mu : KK_*^{\Gamma}(\mathcal{E}\Gamma, A) \otimes \mathbb{Q} \to K_*(A \rtimes_r \Gamma) \otimes \mathbb{Q}$$

is injective.

We remark that the rational Strong Novikov conjecture implies the Novikov conjecture on homotopy invariance of higher signatures and the Gromov-Lawson-Rosenberg conjecture regarding nonexistence of positive scalar curvature metrics on closed aspherical manifolds.

It is conjectured that any countable subgroup of the diffeomorphism group of a compact smooth manifold is coarsely embeddable into the Banach space of all Schatten p-class operators for some $p \geq 1$. If $p > 2$, then $l^p(\mathbb{N})$ is not coarsely embeddable into a Hilbert space [13]. More generally, $l^p(\mathbb{N})$ does not coarsely embed into $l^q(\mathbb{N})$ if $p > q \geq 2$ [17]. Let C_0 be the Banach space consisting of all sequences of real numbers that are convergent to 0. It is an open question if C_0 has (rational) Property (H). By the above theorems, a positive answer to this question would imply the Novikov conjecture since every countable group admits a coarse embedding into C_0 [5].

This paper is organized as follows. In section 2, we construct a C^*-algebra associated to a Banach space with (rational) Property (H) and study its K-groups. In section 3, we reformulate the Baum-Connes map and discuss its connection with the localization algebra. In section 4, we introduce the Bott map for K-groups. In section 5, we give a proof of the main theorem.

In this paper, K-groups of a graded C^*-algebra are defined to be the K-groups of the underlying ungraded C^*-algebra obtained by forgetting the grading structure. The same comment applies to KK-groups.

The authors wish to thank Misha Gromov for inspiring discussions and the referee for very helpful comments.

2 A C^*-algebra associated to a Banach space with Property (H)

In this section, we construct a C^*-algebra associated to a Banach space with (rational) Property (H) and study its K-groups.

Let ψ be as in the definition of (rational) Property (H). We extend ψ to a map $\phi : V \to W$ by:

$$\phi(v) = ||v||\psi(\frac{v}{||v||})$$

for any $v \in V$, where $\phi(0)$ should be interpreted as 0.

Let $Clifford(W_n)$ be the complex Clifford algebras of W_n, satisfying the relation $w^2 = ||w||^2$ for all $w \in W_n$. We define the complex Clifford algebra $Clifford(W)$ to be the C^*-algebra inductive limit of $Clifford(W_n)$. Let $C_0(V, Clifford(W))$ be the graded C^*-algebra of all bounded and uniformly continuous functions on V with values in the Clifford algebra $Clifford(W)$ which vanish at infinity, where the grading is given by the natural grading structure on the Clifford algebra. Let $\mathcal{S} = C_0(\mathbb{R})$ be the graded C^*-algebra of all complex-valued continuous functions on $\mathbb{R}$ vanishing at infinity (graded by even and odd functions). We define $\mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$ to be the graded C^*-algebra tensor product of $\mathcal{S}$ and $C_0(V, Clifford(W))$.

For any $f \in C_0(\mathbb{R})$, we can define an element

$$f((s, \phi(v))) \in \mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$$

by:

$$f((s, \phi(v))) = f(s\hat{\otimes}1 + 1\hat{\otimes}\phi(v)),$$

where s and $\phi(v)$ should be respectively viewed as unbounded degree one multipliers of $\mathcal{S}$ and $C_0(V, Clifford(W))$, $(s, \phi(v))$ is defined to be $s\hat{\otimes}1 + 1\hat{\otimes}\phi(v)$ as an unbounded degree one multiplier of $\mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$, and $f(s\hat{\otimes}1 + 1\hat{\otimes}\phi(v))$ is defined using functional calculus.

More concretely, $f((s, \phi(v)))$ can be defined as follows:

(1) if $f(t) = g(t^2)$ for some $g \in C_0(\mathbb{R})$, we define a scalar valued function $f((s, \phi(v)))$ of the variable $(s, v) \in \mathbb{R} \times V$ by:

$$(s, v) \to g(s^2 + ||\phi(v)||^2) = g(s^2 + ||v||^2)$$

for every $(s, v) \in \mathbb{R} \times V$;

(2) if $f(t) = tg(t^2) \in C_0(\mathbb{R})$ for some $g \in C_0(\mathbb{R})$, we define an element $f((s, \phi(v))) \in \mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$ by:

$$\begin{aligned} f((s, \phi(v))) &= g(s^2 + ||\phi(v)||^2)(s\hat{\otimes}1 + 1\hat{\otimes}\phi(v)) \\ &= g(s^2 + ||v||^2)(s\hat{\otimes}1 + 1\hat{\otimes}\phi(v)) \end{aligned}$$

for every $s \in \mathbb{R}, v \in V$, here $s\hat{\otimes}1 + 1\hat{\otimes}\phi(v)$ should viewed as an unbounded degree one multiplier of $\mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$;

(3) the general definition of $f((s, \phi(v)))$ follows using approximation of f by linear combinations of special functions of the above two types in $C_0(\mathbb{R})$.

Now we are ready to define a C^*-algebra asscociated to a Banach space with (rational) Property (H).

Definition 2.1. *Let X be a Banach space with (rational) Property (H). Let ϕ be as above. We define $\mathcal{A}(X)$ to be the graded C^*-subalgebra of $\mathcal{S}\hat{\otimes}C_0(V, Clifford(W))$ generated by all $f((s, \phi(v-v_0)))$ for $s \in \mathbb{R}$, all $v_0 \in V$ and $f \in C_0(\mathbb{R})$.*

It is not difficult to compute $K_*(\mathcal{A}(X))$ when X is an l^p-space for some $1 \leq p < \infty$ (cf. [12] for p=2). In general, it is an open question how to compute $K_*(\mathcal{A}(X))$. The following result provides a partial solution.

Proposition 2.2. *Let X be a Banach space with Property (H) and let $\mathcal{A}(X)$ be the C^*-algebra associated to X. If B is a (graded) C^*-algebra, then the homomorphism from $\mathcal{S}$ to $\mathcal{A}(X)$: $f(s) \to f((s, \phi(v))$, induces an injection:*

$$K_*(\mathcal{S}\hat{\otimes}B) \to K_*(\mathcal{A}(X)\hat{\otimes}B).$$

Proof. Let $\mathcal{A}(X, V_n)$ be the C^*-subalgebra of $\mathcal{A}(X)$ generated by all elements $f((s, \phi(v - v_0)))$ for $f \in C_0(\mathbb{R})$ and $s \in \mathbb{R}, v_0 \in V_n$. Note that $\mathcal{A}(X)\hat{\otimes}B$ is the inductive limit of $\mathcal{A}(X, V_n)\hat{\otimes}B$. It suffices to prove that the homomorphism β_n:

$$K_*(\mathcal{S}\hat{\otimes}B) \to K_*(\mathcal{A}(X, V_n)\hat{\otimes}B)$$

is injective for each n, where β_n is induced by the homomorphism from $\mathcal{S}$ to $\mathcal{A}(X, V_n)$: $f(t) \to f((s, \phi(v))$.

Let $\mathcal{C}(V_n)$ be the graded C^*-algebra of all continuous functions on V_n with values in the Clifford algebra $Clifford(W_n)$ which vanish at infinity. Define $\mathcal{A}(V_n)$ to be the graded C^*-algebra tensor product $\mathcal{S}\hat{\otimes}\mathcal{C}(V_n)$. Let r_n be the restriction homomorphism from $\mathcal{A}(X, V_n)\hat{\otimes}B$ to $\mathcal{A}(V_n)\hat{\otimes}B$. By the definition of Property (H), such restriction homomorphism is well defined. By Bott periodicity (cf. [12]) and the degree one condition in the definition of Property (H), we observe that the composition $(r_n)_* \circ \beta_n$ is an isomorphism. Proposition 2.2 follows from this observation. □

The following result follows from Proposition 2.2 and the Green-Julg theorem.

Corollary 2.3. *Let X be a Banach space with Property (H) and let $\mathcal{A}(X)$ be the C^*-algebra associated to X. If H is a compact topological group and B is a (graded) H-C^*-algebra, then we have a natural injective homomorphism:*

$$K_*^H(\mathcal{S}\hat{\otimes}B) \to K_*^H(\mathcal{A}(X)\hat{\otimes}B),$$

where H acts on $\mathcal{A}(X)$ trivially.

We can prove the following results using essentially the same argument.

Proposition 2.4. *Let X be a Banach space with rational Property (H) and let $\mathcal{A}(X)$ be the C^*-algebra associated to X. If B is a (graded) C^*-algebra, then we have a natural injective homomorphism*

$$K_*(\mathcal{S}\hat{\otimes}B) \otimes \mathbb{Q} \to K_*(\mathcal{A}(X)\hat{\otimes}B) \otimes \mathbb{Q}.$$

Corollary 2.5. *Let X be a Banach space with rational Property (H) and let $\mathcal{A}(X)$ be the C^*-algebra associated to X. If H is a compact topological group and B is a (graded) H-C^*-algebra, then we have a natural injection:*

$$K_*^H(\mathcal{S}\hat{\otimes}B)\otimes\mathbb{Q}\to K_*^H(\mathcal{A}(X)\hat{\otimes}B)\otimes\mathbb{Q}.$$

Let B be a graded C^*-algebra, let $\mathcal{K}$ be the graded C^*-algebra of all compact operators on a graded separable and infinite dimensional Hilbert space. Let $C_c(\mathbb{R})\hat{\otimes}_{alg}(B\hat{\otimes}\mathcal{K})$ be the algebraic graded tensor product of $C_c(\mathbb{R})$ with $B\hat{\otimes}\mathcal{K}$, where $C_c(\mathbb{R})$ is considered as an algebra graded by even and odd functions.

We define $\mathcal{S}_b^\infty(B)$ to be the graded C^*-algebra of all bounded sequences of uniformly equi-continuous functions $\{f_k\}$ in $\mathcal{S}\hat{\otimes}B\hat{\otimes}\mathcal{K}$ such that for each $\epsilon>0$, there exists $R>0$ for which there exists a bounded sequence of uniformly equi-continuous functions $\{g_k\}$ in $C_c(\mathbb{R})\hat{\otimes}_{alg}(B\hat{\otimes}\mathcal{K})$ satisfying $diameter(support(g_k))<R$ and $||f_k-g_k||<\epsilon$ for all k, where the support of g_k is a subset of $\mathbb{R}$. Let $\mathcal{S}_0^\infty(B)$ be the graded C^*-subalgebra of $\mathcal{S}_b^\infty(B)$ consisting of all sequence (f_k) in $\mathcal{S}_b^\infty(B)$ which are convergent to 0 in the sup norm. Define $\mathcal{S}^\infty(B)$ to be the quotient graded C^*-algebra $\mathcal{S}_b^\infty(B)/\mathcal{S}_0^\infty(B)$.

The proof of the following result is straightforward and is therefore omitted.

Proposition 2.6. *If H is a finite group and B is a graded H-C^*-algebra, then $K_*^H(\mathcal{S}^\infty(B))$ is naturally isomorphic to $(\prod K_{*+1}^H(B))/(\oplus K_{*+1}^H(B))$.*

We remark that by the Green-Julg theorem the equivariant case of the above proposition follows from the non-equivariant case.

Let $\mathcal{C}(V_k)$ be the graded C^*-algebra of all continuous functions on V_k with values in the Clifford algebra $Clifford(W_k)$ which vanish at infinity. Define $\mathcal{A}(V_n)$ to be the graded C^*-algebra tensor product $\mathcal{S}\hat{\otimes}\mathcal{C}(V_n)$. Endow $\mathbb{R}\times V_k$ with the product metric of the standard metric on $\mathbb{R}$ and the Banach norm metric on V_k.

We define $\mathcal{A}_b^\infty(X,B)$ to be the graded C^*-algebra of all bounded sequences (a_k) such that

(1) $a_k \in \mathcal{A}(V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$ and (a_k) is uniformly equi-continuous in $\{(s, v_k)\}$ in the sense that for any $\epsilon > 0$ there exists a $\delta > 0$ (independent of k) such that

$$||a_k(s, v_k) - a_k(s', v'_k)|| < \epsilon$$

if $d((s, v_k), (s', v'_k)) < \delta$;
(2) for each $\epsilon > 0$, there exists $R > 0$ for which there exists a sequence $\{g_k\}$ in $\mathcal{A}(V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$ satisfying

$$diameter(support(g_k)) < R,$$

$$||a_k(s, v_k) - g_k(s, v_k)|| < \epsilon$$

for all k and $(s, v_k) \in \mathbb{R} \times V_k$, where the support of g_k is a subset of $\mathbb{R} \times V_k$.

Let $\mathcal{A}_0^\infty(X, B)$ be the graded C^∞-subalgebra of $\mathcal{A}_b^\infty(X, B)$ consisting of all sequences which are convergent to 0 in norm. Define $\mathcal{A}^\infty(X, B)$ to be the quotient graded C^*-algebra $\mathcal{A}_b^\infty(X, B)/\mathcal{A}_0^\infty(X, B)$.

Let β_k be the homomorphism from $\mathcal{S}$ to $\mathcal{A}(V_k)$ defined by:

$$(\beta_k(f))(s, v_k)) = f((s, \phi(v_k)))$$

for each $f \in \mathcal{S}$, $s \in \mathbb{R}$ and $v_k \in V_k$.

β_k induces a homomorphism (still denoted by β_k) from $\mathcal{S}\hat{\otimes}B\hat{\otimes}\mathcal{K}$ to $\mathcal{A}(V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$. We define a homomorphism

$$\beta : \mathcal{S}^\infty(B) \to \mathcal{A}^\infty(X, B)$$

by:

$$\beta([(f_1, \cdots, f_k, \cdots)]) = [(\beta_1(f_1), \cdots, \beta_k(f_k), \cdots)]$$

for all $[(f_1, \cdots, f_k, \cdots)] \in \mathcal{S}^\infty(B)$.

Theorem 2.7. *If X is a Banach space with Property (H), then the homomorphism β induces an isomorphism:*

$$\beta_* : K_*(\mathcal{S}^\infty(B)) \to K_*(\mathcal{A}^\infty(X, B)).$$

Proof. For each k, endow V_k with a Euclidean metric and let D_k be the corresponding Dirac operator on V_k. Let α_k be the asymptotic morphism from $\mathcal{A}(V_k)$ to $\mathcal{S}\hat{\otimes}\mathcal{K}$ associated to D_k:

$$\alpha_k(t) : f\hat{\otimes}h \to f(t^{-1}(s\hat{\otimes}1 + 1\hat{\otimes}D_k))\pi(h)$$

for all $f \in \mathcal{S}$ and $h \in \mathcal{C}(V_k)$, where $\mathcal{C}(V_k)$ is as in the defintion of $\mathcal{A}(V_k)$, W_k (in the definition of $\mathcal{C}(V_k)$) is identified with V_k as Euclidean vector spaces, and $\pi(h)$ is the multiplication operator associated to h. α_k induces an asymptotic morphism (still called α_k) from $\mathcal{A}(V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$ to $\mathcal{S}\hat{\otimes}B\hat{\otimes}\mathcal{K}$. This asymptotic morphism was first introduced by Higson-Kasparov-Trout in [12].

For each $t \in [1, \infty)$, we define a map:

$$\alpha(t) : \mathcal{A}^{\infty}(X, B) \to \mathcal{S}^{\infty}(B)$$

by:

$$(\alpha(t))([(a_1, \cdots, a_k, \cdots)]) = [((\alpha_1(t))(a_1), \cdots, (\alpha_k(t))(a_k), \cdots)].$$

After rescaling $\alpha_k(t)$ for each k, α is an asymptotic morphism from $\mathcal{A}^{\infty}(X, B)$ to $\mathcal{S}^{\infty}(B)$, where the rescaling constant for each k can be chosen as the Lipschitz constant of the Lipschitz equivalence between the Euclidean norm on V_k and the Banach norm on V_k. α is called the Dirac morphism. More precisely, let C_k be the Lipschitz constant of the Lipschitz equivalence between the Euclidean norm $||\cdot||_E$ on V_k and the Banach norm on $||\cdot||$ on V_k, i.e. C_k is a positive constant satisfying $C_k^{-1}||v||_E \le ||v|| \le C_k||v||_E$ for all $v \in V_k$. For each k, the scaled asymptotic morphism is defined to be $\alpha_k(C_k t)$ (still denoted by α_k).

Let $\mathcal{A}^{\infty}(X \times X, B)$ be defined as $\mathcal{A}^{\infty}(X, B)$ using the sequence of finite dimensional subspaces $V_n \times V_n$ and $W_n \times W_n$.

We define a homomorphism β'_k from $\mathcal{A}(V_k)$ to $\mathcal{A}(V_k \times V_k)$ by:

$$(\beta'_k(f\hat{\otimes}g))(s, v_k, v'_k) = f((s, \phi(v'_k)))\hat{\otimes}g(v_k),$$

where $f \in \mathcal{S}$, $g \in C_0(V_k, Clifford(W_k))$, $s \in \mathbb{R}, (v_k, v'_k) \in V_k \times V_k$, and $f((s, \phi(v'_k)))$ is defined to be $f(s\hat{\otimes}1 + 1\hat{\otimes}\phi(v'_k))$.

β'_k induces a homomorphism (still denoted by β'_k) from $\mathcal{A}(V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$ to $\mathcal{A}(V_k \times V_k)\hat{\otimes}B\hat{\otimes}\mathcal{K}$. Using $\{\beta'_k\}$ we can define a homomorphism β' from $\mathcal{A}^\infty(X, B)$ to $\mathcal{A}^\infty(X \times X, B)$. Similarly we define the Dirac morphism α' from $\mathcal{A}^\infty(X \times X, B)$ to $\mathcal{A}^\infty(X, B)$ using the Dirac operator on the first copy of V_k in $V_k \times V_k$ for each k.

We now apply Atiyah's rotation trick [1]. For any $\theta \in [0, \frac{\pi}{2}]$, we define the rotation R_θ by:

$$R_\theta(v, w) = (cos\theta v - sin\theta w, sin\theta v + cos\theta w).$$

R_θ induces an automorphism (still denoted by R_θ) on the algebra $\mathcal{A}^\infty(X \times X, B)$. In particular, for any sequence of vectors $\{(c_k, c'_k)\}$ such that $(c_k, c'_k) \in V_k \times V_k$, we have

$$R_\theta : [(f_k((s, (\phi(v_k - c_k), \phi(v'_k - c'_k)))] \to$$

$$[(f_k((s, (cos\theta\phi(cos\theta v_k + sin\theta v'_k - c_k) - sin\theta\phi(-sin\theta v_k + cos\theta v'_k - c'_k),$$

$$sin\theta\phi(cos\theta v_k + sin\theta v'_k - c_k) + cos\theta\phi(-sin\theta v_k + cos\theta v'_k - c'_k)))))],$$

where $f_k \in \mathcal{S}$, $\{f_k\}$ is equi-continuous, and $s \in \mathbb{R}, (v_k, v'_k) \in V_k \times V_k$.

By homotopy invariance, we have

$$(Id)_* = (R_0)_* = (R_{\frac{\pi}{2}})_*,$$

where $(R_\theta)_*$ is the automorphism on $K_*(\mathcal{A}^\infty(X \times X, B))$ induced by R_θ.

For any $x \in K_*(\mathcal{S}^\infty(B))$, we have:

$$\alpha_*(\beta_*(x)) = x.$$

For any $y \in K_*(\mathcal{A}^\infty(X, B))$, we have

$$\beta_*(\alpha_*(y)) = \alpha'_*(\beta'_*(y)) = \alpha'_*(R_{\frac{\pi}{2}})_*(\beta'_*(y)) = \tilde{y},$$

where the map $y \to \tilde{y}$ is induced by the map: $(v_k) \to (-v_k)$.

The above two identities imply that β and α are isomorphisms, inverse to each other, and $y \to \tilde{y}$ is the identity map. □

The following result follows from Proposition 2.7 and the Green-Julg theorem.

Corollary 2.8. *Let H be a finite group and let B be a graded H-C^*-algebra. If X is a Banach space with Property (H), then β induces an isomorphism:*

$$\beta_* : K_*^H(\mathcal{S}^\infty(B)) \to K_*^H(\mathcal{A}^\infty(X, B)).$$

Proposition 2.9. *If X is a Banach space with rational Property (H), then the homomorphism β induces an isomorphism:*

$$\beta_* : K_*(\mathcal{S}^\infty(B)) \otimes \mathbb{Q} \to K_*(\mathcal{A}^\infty(X, B)) \otimes \mathbb{Q}.$$

Corollary 2.10. *Let H be a finite group and let B be a graded H-C^*-algebra. If X is a Banach space with rational Property (H), then β induces an isomorphism:*

$$\beta_* : K_*^H(\mathcal{S}^\infty(B)) \otimes \mathbb{Q} \to K_*^H(\mathcal{A}^\infty(X, B)) \otimes \mathbb{Q}.$$

3 The Baum-Connes map and localization

In this section, we briefly recall the Baum-Connes map and its relation to the localization algebra. Our reformulation of the Baum-Connes map follows the work of Roe [19] and uses a localization technique introduced in [21]. This reformulation will be useful in the next section.

Let Γ be a countable discrete group. Let Δ be a locally compact metric space with a proper and cocompact isometric action of Γ. Let $C_0(\Delta)$ be the algebra of all complex valued continuous functions on Δ which vanish at infinity. Let B be a Γ-C^*-algebra.

The following definition is due to John Roe [19].

Definition 3.1. *Let H be a Hilbert module over B and let φ be a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$, the C^*-algebra of all bounded (adjointable) operators on H. Let T be an operator in $B(H)$.*

(1) The support of T is defined to be the complement (in $\Delta \times \Delta$) of the set of all points $(x, y) \in \Delta \times \Delta$ for which there exists $f \in C_0(\Delta)$ and $g \in C_0(\Delta)$ satisfying $\varphi(f)T\varphi(g) = 0$ and $f(x) \neq 0$ and $g(y) \neq 0$;

(2) *The propagation of T is defined to be: $sup\{d(x,y) : (x,y) \in Supp(T)\}$;*

(3) *T is said to be locally compact if $\varphi(f)T$ and $T\varphi(f)$ are in $K(H)$ for all $f \in C_0(\Delta)$, where $K(H)$ is defined to be the operator norm closure of all finite rank operators on the Hilbert module H.*

Let H be a (countably generated) Γ-Hilbert module over B and let φ be a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$ which is covariant in the sense that $\varphi(\gamma f)h = (\gamma(\varphi(f))\gamma^{-1})h$ for all $\gamma \in \Gamma$, $f \in C_0(\Delta)$ and $h \in H$. Such a triple $(C_0(\Delta), \Gamma, \varphi)$ is called a covariant system.

Definition 3.2. *We define the covariant system $(C_0(\Delta), \Gamma, \varphi)$ to be admissible if*

(1) *the Γ-action on Δ is proper and cocompact;*

(2) *there exist a Γ-Hilbert space H_Δ and a separable and infinite dimensional Γ-Hilbert space E such that*

(a) H is isomorphic to $H_\Delta \otimes E \otimes B$ as Γ-Hilbert modules over B;

(b) $\varphi = \varphi_0 \otimes I$ for some Γ-equivariant $*$-homomorphism φ_0 from $C_0(\Delta)$ to $B(H_\Delta)$ such that $\varphi_0(f)$ is not in $K(H_\Delta)$ for any nonzero function $f \in C_0(\Delta)$ and φ_0 is nondegenerate in the sense that $\{\varphi_0(f)H_\Delta : f \in C_0(\Delta)\}$ is dense in H_Δ,

(c) for each $x \in \Delta$, E is isomorphic to $l^2(\Gamma_x) \otimes H_x$ as Γ_x-Hilbert spaces for some Hilbert space H_x with a trival Γ_x action, where Γ_x is the finite isotropy subgroup of Γ at x.

In the above definition, the Γ_x-action on $l^2(\Gamma_x)$ is regular, i.e. $(\gamma\xi)(z) = \xi(\gamma^{-1}z)$ for every $\gamma \in \Gamma_x$, $\xi \in l^2(\Gamma_x)$, and $z \in \Gamma_x$, B is the Γ-Hilbert module over B with the inner product $< a, b >= a^*b$, and I is the identity operator on $E \otimes B$.

We remark that such an admissible covariant system always exists (for example we can choose E to be $l^2(\Gamma)$ with a regular action Γ). We point out that condition (2) implies that E contains all unitary representations of the finite isotropy groups and this point is important for Proposition 3.4 of this section.

Definition 3.3. *Let $(C_0(\Delta), \Gamma, \varphi)$ be an admissible covariant system. We define $\mathbb{C}(\Gamma, \Delta, B)$ to be the algebra of Γ-invariant locally compact operators in $B(H)$ with finite propagation. The C^*-algebra $C_r^*(\Gamma, \Delta, B)$ is the operator norm closure of $\mathbb{C}(\Gamma, \Delta, B)$.*

The following result is essentially due to John Roe.

Proposition 3.4. *If $(C_0(\Delta), \Gamma, \varphi)$ is an admissible covariant system, then $C_r^*(\Gamma, \Delta, B)$ is $*$-isomorphic to $(B \rtimes_r \Gamma) \otimes \mathcal{K}$, where $B \rtimes_r \Gamma$ is the reduced crossed product C^*-algebra and $\mathcal{K}$ is the algebra of all compact operators on a separable and infinite dimensional Hilbert space.*

Proof. Proposition 3.4 follows from the definitions of the admissible covariant system, $C_r^*(\Gamma, \Delta, B)$, and the reduced crossed product C^*-algebra. $\square$

Next we will describe the Baum-Connes map.

Let H be a Γ-Hilbert module over B, let F be an operator in $B(H)$, let φ be a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$ such that F is Γ-equivariant, i.e. $\gamma F \gamma^{-1} = F$ for all $\gamma \in \Gamma$, $\varphi(f)F - F\varphi(f)$, $\varphi(f)(FF^* - I)$ and $\varphi(f)(F^*F - I)$ are in $K(H)$ for all $f \in C_0(\Delta)$.

We denote the group $KK_0^\Gamma(C_0(\Delta), B)$ by $KK_0^\Gamma(\Delta, B)$. (H, φ, F) gives a KK-cycle representing a class in $KK_0^\Gamma(\Delta, B)$. It is not difficult to prove that every class in $KK_0^\Gamma(\Delta, B)$ is equivalent to (H, φ, F) such that $(C_0(\Delta), \Gamma, \varphi)$ is an admissible covariant system. This can be seen as follows. We define a new KK-group $\tilde{KK}_*^\Gamma(\Delta, B)$ using KK-cycles (H, φ, F) such that $(C_0(\Delta), \Gamma, \varphi)$ is an admissible covariant system. By the proof of Proposition 5.5 in [15], we can show that there exists a Γ-Hilbert space H_Δ satisfying the conditions of Definition 3.2 such that $H \oplus (H_\Delta \otimes l^2(\Gamma) \otimes B)$ is isomorphic to $H_\Delta \otimes l^2(\Gamma) \otimes B$ as Γ-Hilbert modules over B. Using this stabilization result, we can prove that the natural homomorphism from $\tilde{KK}_*^\Gamma(\Delta, B)$ to $KK_0^\Gamma(\Delta, B)$ is an isomorphism.

For any $\epsilon > 0$, let $\{U_i\}_{i \in I}$ be a locally finite and Γ-equivariant open cover of Δ satisfying $diameter(U_i) < \epsilon$ for all i. Let $\{\psi_i\}$ be a Γ-equivariant partition of unity subordinate to $\{U_i\}_{i\in I}$. We define

$$F_\epsilon = \sum_{i\in I} \varphi(\sqrt{\psi_i}) F \varphi(\sqrt{\psi_i}),$$

where the convergence is in the strict topology.

Note that F_ϵ has propagation ϵ and (H, φ, F_ϵ) is equivalent to (H, φ, F) in $KK_0^\Gamma(\Delta, B)$.

F_ϵ is a multiplier of $C_r^*(\Gamma, \Delta, B)$. F_ϵ is invertible modulo $C_r^*(\Gamma, \Delta, B)$.

Let ∂ be the boundary map in K-theory:

$$K_1(M(C_r^*(\Gamma, \Delta, B))/C_r^*(\Gamma, \Delta, B)) \to K_0(C_r^*(\Gamma, \Delta, B)),$$

where $M(C_r^*(\Gamma, \Delta, H))$ is the multiplier algebra of $C_r^*(\Gamma, \Delta, H)$. We can define the Baum-Connes map

$$\mu : KK_0^\Gamma(\Delta, B) \to K_0(C_r^*(\Gamma, \Delta, B)) \cong K_0(B \rtimes_r \Gamma)$$

by:

$$\mu([(H, \varphi, F)]) = \partial([F_\epsilon]).$$

More precisely the Baum-Connes map can be implemented as follows.

Let p_ϵ be the idempotent:

$$\begin{pmatrix} F_\epsilon F_\epsilon^* + (I - F_\epsilon F_\epsilon^*) F_\epsilon F_\epsilon^* & F_\epsilon (I - F_\epsilon^* F_\epsilon) + (I - F_\epsilon F_\epsilon^*) F_\epsilon (I - F_\epsilon^* F_\epsilon) \\ (I - F_\epsilon^* F_\epsilon) F_\epsilon^* & (I - F_\epsilon^* F_\epsilon)^2 \end{pmatrix}.$$

Observe that the propagation of p_ϵ is at most 5ϵ.

Let

$$p_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

We have

$$\mu([(H, \varphi, F)]) = [p_\epsilon] - [p_0].$$

Similarly we can define the Baum-Connes map:

$$\mu : KK_1^\Gamma(\Delta, B) \to K_1(C_r^*(\Gamma, \Delta, B)) \cong K_1(B \rtimes_r \Gamma).$$

This induces the Baum-Connes map:

$$\mu : KK_*^\Gamma(\mathcal{E}\Gamma, B) \to K_*(B \rtimes_r \Gamma),$$

where $KK_*^\Gamma(\mathcal{E}\Gamma, B)$ is defined to be the inductive limit of $KK_*^\Gamma(\Delta, B)$ over all Γ-invariant and cocompact subspaces Δ of $\mathcal{E}\Gamma$ which are finite dimensional simplicial polyhedra. Here we choose a model of $\mathcal{E}\Gamma$ so that $\mathcal{E}\Gamma$ is equal to the union of Γ-invariant and cocompact subspaces Δ of $\mathcal{E}\Gamma$ which are finite dimensional simplicial polyhedra (the existence of such model follows from the construction of $\mathcal{E}\Gamma$ in the proof of Proposition 1.7 in [3] which is based on Milnor's join construction).

Let Δ be a locally compact and finite dimensional simplicial polyhedron. We endow Δ with the simplicial metric. Let $(C_0(\Delta), \Gamma, \varphi)$ be an admissible covariant system as before, where φ is a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$ for some Hilbert module H over B.

Definition 3.5. *(1) The algebraic localization algebra $\mathbb{C}_L(\Gamma, \Delta, B)$ is defined to be the algebra of all bounded and uniformly continuous functions $f : [0, \infty) \to \mathbb{C}(\Gamma, \Delta, B)$ such that the propagation of $f(t)$ goes to 0 as $t \to \infty$, where $\mathbb{C}(\Gamma, \Delta, B)$ is as in Definition 3.3.*

(2) The localization algebra $C_L^*(\Gamma, \Delta, B)$ is the norm closure of $\mathbb{C}_L(\Gamma, \Delta, B)$ with respect to the following norm:

$$||f|| = sup_{t\in[0,\infty)}||f(t)||.$$

It is not difficult to prove that, up to a $*$-isomorphism, $\mathbb{C}_L(\Gamma, \Delta, B)$ and $C_L^*(\Gamma, \Delta, B)$ are independent of the choice of the admissible covariant system $(C_0(\Delta), \Gamma, \varphi)$. The localization algebra is an equivariant analogue of the algebra introduced in [21].

Any class in $KK_0^\Gamma(\Delta, B)$ can be represented by (H, φ, F) such that the covariant system $(C_0(\Delta), \Gamma, \varphi)$ is admissible, where H is a Γ-Hilbert module over B, F is an operator in $B(H)$, φ is a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$ such that F is Γ-equivariant, and $\varphi(f)F - F\varphi(f)$, $\varphi(f)(FF^* - I)$ and $\varphi(f)(F^*F - I)$ are in $K(H)$ for all $f \in C_0(\Delta)$.

For each natural number n, we let $F_{\frac{1}{n}}$ be defined as above. We define an operator valued function $F(t)$ on $[0, \infty)$ by:

$$F(t) = (t-n+1)F_{\frac{1}{n}} + (t-n)F_{\frac{1}{n+1}}$$

for all $t \in [n, n+1]$.

$F(t)$ is a multiplier of $C_L^*(\Gamma, \Delta, B)$ and is invertible modulo $C_L^*(\Gamma, \Delta, B)$. We define the local Baum-Connes map:

$$\mu_L : KK_0^\Gamma(\Delta, B) \to K_0(C_L^*(\Gamma, \Delta, B)),$$

by

$$\mu_L[H, \varphi, F)] = \partial[F(t)],$$

where

$$\partial : K_1(M(C_L^*(\Gamma, \Delta, B))/C_L^*(\Gamma, \Delta, B)) \to K_0^*(C_L^*(\Gamma, \Delta, B)),$$

is the boundary map in K-theory and $M(C_L^*(\Gamma, \Delta, B))$ is the multiplier algebra of $C_L^*(\Gamma, \Delta, B)$.

Similarly we can define the local Baum-Connes map:

$$\mu_L : KK_1^\Gamma(\Delta, B) \to K_1(C_L^*(\Gamma, \Delta, B)).$$

We remark that the local Baum-Connes map is very much in the spirit of the local index theory of elliptic differential operators.

Theorem 3.6. *Let B be a Γ-C^*-algebra. The local Baum-Connes map μ_L is an isomorphism from $KK_*^\Gamma(\Delta, B)$ to $K_*(C_L^*(\Gamma, \Delta, B))$ if Δ is a finite dimensional simplicial polyhedron with the Γ-invariant simplicial metric.*

Proof. This theorem is a consequence of the Mayer-Vietoris and five lemma argument (essentially similar to the proof of the non-equivariant analogue in [21]). □

Let $C_L^*(\Gamma, \mathcal{E}\Gamma, B)$ be the C^*-algebra inductive limit of $C_L^*(\Gamma, \Delta, B)$, where the limit is taken over all Γ-invariant and cocompact subspaces Δ of $\mathcal{E}\Gamma$ which are finite dimensional simplicial polyhedra.

The above local Baum-Connes map induces a map (still called the local Baum-Connes map):

$$\mu_L : KK_*^{\Gamma}(\mathcal{E}\Gamma, B) \to K_*(C_L^*(\Gamma, \mathcal{E}\Gamma, B)).$$

Corollary 3.7. *The local Baum-Connes map μ_L is an isomorphism from $KK_*^{\Gamma}(\mathcal{E}\Gamma, B)$ to $K_*(C_L^*(\Gamma, \mathcal{E}\Gamma, B))$.*

Next we shall discuss the relation between the Baum-Connes map and an evaluation map. This connection will be useful in the proof of the main theorem in Section 5.

Let e be the evaluation map:

$$C_L^*(\Gamma, \Delta, B) \to C_r^*(\Gamma, \Delta, H) \cong (B \rtimes_r \Gamma) \otimes \mathcal{K}$$

defined by:

$$e(f) = f(0)$$

for all $f \in C_L^*(\Gamma, \Delta, B)$.

The above evaluation maps induce an evaluation homomorphism (still denoted by e):

$$C_L^*(\Gamma, \mathcal{E}\Gamma, B) \to (B \rtimes_r \Gamma) \otimes \mathcal{K}.$$

We have

$$\mu = e_* \circ \mu_L.$$

4 The Bott map

In this section, we introduce a Bott map for K-groups. The Bott map will play an essential role in the proof of the main result of this paper.

Let X be a Banach space with (rational) Property (H). Let Γ be a countable group with a left invariant proper length metric. Let $h : \Gamma \to X$ be a coarse embedding. Without loss of generality we can assume that the image of h is contained in V, where V is as in the definition of the (rational) Property (H).

For each $v \in V, w \in W$, and $\gamma \in \Gamma$, we define bounded functions $\xi_{v,\gamma}$ and $\eta_{v,w,\gamma}$ on Γ by:

$$\xi_{v,\gamma}(x) = ||\phi(v + h(x) - h(x\gamma))|| = ||v + h(x) - h(x\gamma)||,$$

$$\eta_{v,w,\gamma}(x) =< \phi(v + h(x) - h(x\gamma)), w >$$

for all $x \in \Gamma$, where ϕ is defined as in Section 2. The boundedness of $\xi_{v,\gamma}$ and $\eta_{v,w,\gamma}$ follows from the fact that h is a coarse embedding.

Let $c_0(\Gamma)$ be the algebra of all functions on Γ vanishing at infinity. We define the Γ action on $l^\infty(\Gamma)$ as follows: $(\gamma(\eta))(x) = \eta(x\gamma)$ for each $\eta \in l^\infty(\Gamma), \gamma \in \Gamma, x \in \Gamma$. Let Y be the spectrum of the unital commutative Γ-invariant C^*-subalgebra of $l^\infty(\Gamma)$ generated by $c_0(\Gamma)$, all constant functions on Γ, all $\xi_{v,\gamma}$ and $\eta_{v,w,\gamma}$, and their translations by group elements of Γ. Notice that Y is a separable compact space and is a quotient space of $\beta\Gamma$, the Stone Cech compactification of Γ.

Let $\mathcal{A}(X)$ be the C^*-algebra associated to X (as defined in Section 2). If A is a Γ-C^*-algebra, then Γ acts on the C^*-algebra

$$C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A \subseteq l^\infty(\Gamma)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$$

as follows:

$$\begin{gathered}\gamma(\eta(x)\hat{\otimes}f((s, \phi(v - v_0)))\hat{\otimes}a) \\ = \eta(x\gamma)\hat{\otimes}f((s, \phi(v - v_0 + h(x) - h(x\gamma)))\hat{\otimes}\gamma(a)\end{gathered}$$

for each $\gamma \in \Gamma, \eta \in C(Y) \subseteq l^\infty(\Gamma), x \in \Gamma, s \in \mathbb{R}, f \in C_0(\mathbb{R}), v \in V, v_0 \in V$ and $a \in A$, where $f((s, \phi(v - v_0)))$ is as in the definition of $\mathcal{A}(X)$.

The above Γ action on $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ has its origin in [22] and [20]. We can see that the Γ action on $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ is well defined as follows. Recall that, for any net $\{w_i\}_{i\in I}$ in a Hilbert space, if w_i converges to a vector w of the Hilbert space in weak topology and $||w_i||$ converges to $||w||$, then w_i converges to w in norm. Using this fact and the definition of Y, we know that $\phi(v - v_0 + h(x) - h(x\gamma))$ can be extended to a norm continuous function on Y with values in H. It follows that $f((s, \phi(v - v_0 + h(x) - h(x\gamma)))$ can be identified with a norm continuous function on Y with values in $\mathcal{A}(X)\hat{\otimes}A$. This, together with the compactness of Y, implies that $\gamma(\eta(x)\hat{\otimes}f((s,\phi(v-v_0)))\hat{\otimes}a)$ is an element in $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$. Let $\mathcal{D}$ be the subalgebra of $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ consisting of all linear combinations of products of elements of the type $\eta(x)\hat{\otimes}f((s,\phi(v-v_0)))\hat{\otimes}a$. Note that, by the definition of $\mathcal{A}(X)$, $\mathcal{D}$ is dense in the C^*-algebra $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$. By extending linearly and multiplicatively, for each $d\in\mathcal{D}$, we can define $\gamma(d)$ as an element of $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$. By the definition of Y, Γ is a dense subset of Y. This, together with the definition of $\gamma(d)$, implies that $||\gamma(d)||=||d||$ for each $d\in\mathcal{D}$. Finally the Γ action extends continuously to $C(Y)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$.

Let Z be the space of all probability measures on Y with the weak topology. Note that Z is a convex and compact topological space with the weak* topology. The idea of using the space of probability measures is due to Nigel Higson [10]. The action of Γ on $C(Y)\hat{\otimes}\mathcal{A}(X)\otimes A$ induces an action of Γ on $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ by:

$$\begin{aligned}&\gamma(u(\mu)\hat{\otimes}f((s,\phi(v-v_0)))\hat{\otimes}a)\\ &=u(\gamma(\mu))\hat{\otimes}f((s,\phi(v-v_0+\int_Y(h(y)-h(y\gamma))d\mu)))\hat{\otimes}\gamma(a)\end{aligned}$$

for each $\gamma\in\Gamma, u\in C(Z), \mu\in Z, f\in C_0(\mathbb{R}), s\in\mathbb{R}, v\in V, v_0\in V, a\in A$, where the Γ action on Z is induced by the Γ action on Y. The assumption that h is a coarse embedding implies that $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ is a Γ-proper C^*-algebra. This can be seen as follows: let $\mathcal{C}(X)$ be the abelian C^*-subalgebra of $C(Z)\hat{\otimes}\mathcal{A}(X)$ generated by all elements $b\hat{\otimes}g(s^2+||v-v_0||^2)$, where $b\in C(Z)$, $g\in C_0(\mathbb{R})$ and $v_0\in V$. $\mathcal{C}(X)$ is isomorphic to a commutative C^*-algebra $C_0(T)$ for some locally compact space T. Notice that T is a proper Γ-space (as a consequence of the fact that h is a coarse embedding), $\mathcal{C}(X)$ is in the center of the multiplier algebra of $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$, and $\mathcal{C}(X)(C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A)$ is dense in $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$.

For each $f\in\mathcal{S}$, let $f_t\in\mathcal{A}(X)$ be defined by

$$f_t((s,v))=f(t^{-1}(s,\phi(v)))$$

for all $s\in\mathbb{R}, v\in V$.

We define the Bott map

$$\beta : K_*((C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \rtimes_r \Gamma) \to K_*((C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A) \rtimes_r \Gamma)$$

to be the homomorphism induced by the following asymptotic morphism from $(C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \rtimes_r \Gamma$ to $(C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A) \rtimes_r \Gamma$:

$$\beta_t((u\hat{\otimes}f\hat{\otimes}a)\gamma) = (u\hat{\otimes}f_t\hat{\otimes}a)\gamma$$

for $t \in [1, \infty), u \in C(Z), f \in \mathcal{S}, a \in A, \gamma \in \Gamma$.

The fact that β_t is an asymptotic morphism follows from the identity $\phi(cv) = |c|\phi(v)$ for any scalar c and the assumption that the restriction of ϕ to the sphere of V is uniformly continuous.

Let Δ be a locally compact metric space with a proper and cocompact isometric action of Γ. We define an asymptotic morphism

$$\beta_t : C_L^*(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \to C_L^*(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A)$$

induced by the homomorphism from $\mathcal{S}$ to $\mathcal{A}(X)$: $f \to f_t$, where

$$f_t((s, v)) = f(t^{-1}(s, \phi(v)))$$

for all $f \in \mathcal{S}$ and $s \in \mathbb{R}, v \in V$.

More precisely β_t is defined as follows. Let $(C_0(\Delta), \Gamma, \varphi)$ be an admissible covariant system, where φ is a $*$-homomorphism from $C_0(\Delta)$ to $B(H)$ for some Hilbert module H over $C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A$. By the definition of admissible covariant system, we have

$$K(H) \cong C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A\hat{\otimes}\mathcal{K},$$

where $K(H)$ is the operator norm closure of all finite rank operators on the Hilbert module H and $\mathcal{K}$ is the graded C^*-algebra of all compact operators on a graded separable and infinite dimensional Hilbert space. Let β'_t be the asymptotic morphism from $C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A\hat{\otimes}\mathcal{K}$ to $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A\hat{\otimes}\mathcal{K}$ induced by the homomorphism from $\mathcal{S}$ to $\mathcal{A}(X)$: $f \to f_t$, where f_t is defined as in the previous paragraph.

Let c be a Γ cut-off function on Δ, i.e. c is a compactly supported non-negative continuous function on Δ satisfying

$$\sum_{\gamma\in\Gamma} c(\gamma^{-1}x) = 1$$

for all $x \in \Delta$. The existence of such cut-off function follows from properness and cocompactness of the Γ action on Δ.

For each $T \in \mathbb{C}_L(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A)$, we define $\beta_t(T)$ in $\mathbb{C}_L(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A)$ by:

$$\beta_t(T) = \sum_{\gamma\in\Gamma} \beta'_t(\varphi(\gamma(c))\gamma T\gamma^{-1}),$$

where β'_t is the asymptotic morphism defined above, $(\gamma(c))(x) = c(\gamma^{-1}x)$ for all $x \in \Delta$, and the sum converges in the strong operator topology because T has finite propagation.

Note that $\beta_t(T)$ is Γ-invariant and is an element of $\mathbb{C}_L(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A)$. It is not difficult to see that β_t can be extended to an asymptotic morphism:

$$\beta_t : C^*_L(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \to C^*_L(\Gamma, \Delta, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A).$$

By Corollary 3.7, the above asymptotic morphism induces a homomorphism (called the Bott map):

$$\beta : KK^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \to KK^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A).$$

Proposition 4.1. *The Bott map*

$$\beta : KK^\Gamma_*(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \to KK^\Gamma_*(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A)$$

is injective, where $\mathcal{E}\Gamma$ is the universal space for proper Γ-actions.

Proof. Let $\mathcal{A}^\infty(X, A)$ be as defined in Section 2. For any fixed $v_0 \in V_k \subseteq V$, we note that the following translation by v_0 on $\mathcal{A}^\infty(X, A)$ is well defined:

$$[(f_1(s, v_1), \cdots, f_{k-1}(s, v_{k-1}), f_k(s, v_k), f_{k+1}(s, v_{k+1}), \cdots)]$$

$$\to [(0, \cdots, 0, f_k(s, v_k + v_0), f_{k+1}(s, v_{k+1} + v_0), \cdots)]$$

for all $[(f_1, \cdots, f_{k-1}, f_k, f_{k+1} \cdots)] \in \mathcal{A}^\infty(X, A)$.

By the uniform equi-continuity condition in the definition of $\mathcal{A}^\infty(X, A)$, this translation operator is norm-continuous in v_0. Hence a Γ

action on $C(Z)\hat{\otimes}\mathcal{A}^\infty(X,A)$ can be defined exactly in the same way as the Γ action on $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$.

Now we can define the Bott map:

$$\beta^\infty : KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}^\infty(A)) \to KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}^\infty(X,A)),$$

in a way similar to the definition of the Bott map:

$$\beta : KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A) \to KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A),$$

where $\mathcal{S}^\infty(A)$ is defined in Section 2.

The Bott map β^∞ is induced by the asymptotic homomorphism from $C(Z)\hat{\otimes}\mathcal{S}^\infty(A)$ to $C(Z)\hat{\otimes}\mathcal{A}^\infty(X,A)$:

$$[u\hat{\otimes}(f_1\hat{\otimes}b_1,\cdots,f_k\hat{\otimes}b_k,\cdots)] \to [u\hat{\otimes}((f_1)_t\hat{\otimes}b_1,\cdots,(f_k)_t\hat{\otimes}b_k,\cdots)],$$

where $u\in C(Z)$, $f_k\in\mathcal{S}$, $b_k\in A$, and $(f_k)_t(s,v_k) = f_k(t^{-1}(s,\phi(v_k)))$ for all $(s,v_k)\in\mathbb{R}\times V_k$.

We claim that the Bott map:

$$\beta^\infty : KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}^\infty(A)) \to KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}^\infty(X,A))$$

is an isomorphism.

This claim follows from the standard Mayer-Vietoris and five lemma argument, Corollary 2.8, and the fact that, for any finite subgroup H of Γ, Z is H-equivariantly homotopy equivalent to a point $\mu_0\in Z$ fixed by H (using the linear homotopy). The point μ_0 can be obtained by averaging the H-orbit of a point in Z using the assumption that H is a finite group and the fact that Z is a convex and compact topological space. Then

$$g[u(\mu)\hat{\otimes}(f_k((s,\phi(v_k-v_0^k)))\hat{\otimes}a_k)]$$

evaluated at $\mu=\mu_0$ is equal to

$$[u(\mu)\hat{\otimes}(f_k((s,\phi(v_k-v_0^k)))\hat{\otimes}g(a_k))]$$

evaluated at $\mu=\mu_0$ for all $g\in H, u\in C(Z), f_k\in\mathcal{S}, v_0^k\in V_k, s\in\mathbb{R}, v_k\in V_k, a_k\in A$, and $[(f_k\hat{\otimes}a_k)]\in\mathcal{S}^\infty(A)$. The last equation follows from the definition of the Γ action on $C(Z)\hat{\otimes}\mathcal{A}^\infty(X,A)$.

We now consider the following commutative diagram:

$$\begin{array}{ccc} KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}\hat{\otimes}A) & \xrightarrow{\beta} & KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A) \\ \downarrow \sigma & & \downarrow \sigma' \\ KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}^\infty(A)) & \xrightarrow{\beta^\infty} & KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}^\infty(X, A)), \end{array}$$

where σ is induced by the homomorphism $\mathcal{S}\hat{\otimes}A \to C(Z)\hat{\otimes}\mathcal{S}^\infty(A)$ mapping each element f to $1\hat{\otimes}[(f\hat{\otimes}p_0)]$ (here 1 is the constant 1 function on Z, $[(f\hat{\otimes}p_0)]$ is the element represented by the constant sequence consisting of $f\hat{\otimes}p_0$, and p_0 is a rank one projection of grading degree 0 in $\mathcal{K}$) and σ' is induced by the homomorphism from $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ to $C(Z)\hat{\otimes}\mathcal{A}^\infty(X, A)$ which maps each element $u\hat{\otimes}f$ to the element represented by the sequence $u\hat{\otimes}(f_k\hat{\otimes}p_0)$ (here $u \in C(Z)$, $f \in \mathcal{A}(X)\hat{\otimes}A$, and f_k is the restriction of f to V_k).

We observe that σ is injective. This can be seen as follows. Let $\tilde{\sigma}$ be the homomorphism:

$$KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}\hat{\otimes}A) \to KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}^\infty(C(Z)\hat{\otimes}A))$$

obtained by composing σ with the homomorphism from $KK_*^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}^\infty(A))$ to $KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}^\infty(C(Z)\hat{\otimes}A))$ induced by the inclusion homomorphism from $C(Z)\hat{\otimes}\mathcal{S}^\infty(A)$ to $\mathcal{S}^\infty(C(Z)\hat{\otimes}A)$. It suffices to prove that $\tilde{\sigma}$ is injective. There exists a natural homomorphism:

$$\prod KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A) \to KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}^\infty(C(Z)\hat{\otimes}A)),$$

where $\prod KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A)$ is defined to be the inductive limit of $\prod KK_{*+1}^\Gamma(\Delta, C(Z)\hat{\otimes}A)$ over all Γ-cocompact subsets Δ of $\mathcal{E}\Gamma$. This homomophism induces a homomorphism (denoted by τ):

$$(\prod KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))/(\oplus KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))$$

$$\to KK_{*+1}^\Gamma(\mathcal{E}\Gamma, \mathcal{S}^\infty(C(Z)\hat{\otimes}A)).$$

Proposition 2.6, together with a standard Mayer-Vietoris and five lemma argument, implies that τ is an isomorphism. There is also a natural isomorphism (denoted by θ) from $KK_*^\Gamma(\mathcal{E}\Gamma, \mathcal{S}\hat{\otimes}A)$ to $KK_{*+1}^\Gamma(\mathcal{E}\Gamma, A)$. Let ς be the homomorphism:

$$KK_{*+1}^\Gamma(\mathcal{E}\Gamma, A) \to (\prod KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))/(\oplus KK_{*+1}^\Gamma(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))$$

obtained by composing the homomorphism from $KK^{\Gamma}_{*+1}(\mathcal{E}\Gamma, A)$ to $KK^{\Gamma}_{*+1}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A)$ induced by the inclusion map from A to $C(Z)\hat{\otimes}A$ with the group homomorphism mapping each element $z \in KK^{\Gamma}_{*+1}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A)$ to the element in $(\prod KK^{\Gamma}_{*+1}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))/(\oplus KK^{\Gamma}_{*+1}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}A))$ represented by the constant sequence consisting of z. By contractibility of Z, we know that ς is injective. We have

$$\tilde{\sigma} = \tau \circ \varsigma \circ \theta.$$

Now the injectivity of $\tilde{\sigma}$ follows from the injectivity of ς and the fact that τ and θ are isomorphisms.

Finally Proposition 4.1 follows from the injectivity of σ, the claim that β^{∞} is an isomorphism, and the commutative diagram in this proof. □

5 Proof of the main result

In this section, we give the proof of Theorem 1.2, the main result of this paper. The proof of Theorem 1.3 is essentially similar.

Proof. Let A be any Γ-C^*-algebra. We consider the following commutative diagram:

$$\begin{array}{ccc} KK^{\Gamma}_{*}(\mathcal{E}\Gamma, \mathcal{S}\hat{\otimes}A) & \xrightarrow{\mu_1} & K_*((\mathcal{S}\hat{\otimes}A) \rtimes_r \Gamma) \\ \downarrow \beta & & \downarrow \beta' \\ KK^{\Gamma}_{*}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A) & \xrightarrow{\mu_2} & K_*((C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A) \rtimes_r \Gamma), \end{array}$$

where μ_1 and μ_2 are the Baum-Connes assembly maps, β and β' are respectively the homomorphisms induced by the inclusion: $\mathbb{C} \to C(Z)$ composed with the Bott maps defined in Section 4. The commutativity of the above diagram follows from the definitions of the Bott maps and the relation between the Baum-Connes map and the evaluation map (cf. the discussion after Corollary 3.7 at the end of Section 3).

Notice that the inclusion map $\mathbb{C} \to C(Z)$ induces an isomorphism

$$KK^{\Gamma}_{*}(\mathcal{E}\Gamma, \mathcal{S}\hat{\otimes}A) \to KK^{\Gamma}_{*}(\mathcal{E}\Gamma, C(Z)\hat{\otimes}\mathcal{S}\hat{\otimes}A).$$

This, together with Proposition 4.1, implies that β is injective. The fact that $C(Z)\hat{\otimes}\mathcal{A}(X)\hat{\otimes}A$ is a Γ-proper C^*-algebra implies that μ_2 is an isomorphism (it can be seen by the Mayer-Vietoris and five lemma argument [9]). The above facts, together with commutativity of the above diagram, imply that μ_1 is injective. It follows that the Baum-Connes assembly map μ in Theorem 1.2 is injective. $\square$

References

[1] M. Atiyah, Bott periodicity and the index of elliptic operators. Quart. J. Math. Oxford Ser. (2) 19, 1968, 113–140.

[2] P. Baum and A. Connes, K-theory for discrete groups, Operator Algebras and Applications, (D. Evans and M. Takesaki, editors), Cambridge University Press (1989), 1–20.

[3] P. Baum, A. Connes and N. Higson, Classifying space for proper actions and K-theory of group C^*-algebras. C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[4] Y. Benyamini and J. Lindenstrauss, Geometric nonlinear functional analysis, American Mathematical Society Colloquium Publications, Volume 48, 2000.

[5] N. Brown and E. Guentner, Uniform embeddings of bounded geometry spaces into reflexive Banach space. Proc. Amer. Math. Soc. 133 (2005), no. 7, 2045–2050.

[6] A. Connes, M. Gromov and H. Moscovici, Group cohomology with Lipschitz control and higher signatures, Geometric and Functional Analysis, 3 (1993), 1–78.

[7] M. Gromov, Asymptotic invariants for infinite groups, Geometric Group Theory, (G. A. Niblo and M. A. Roller, editors), Cambridge University Press, (1993), 1–295.

[8] M. Gromov, Spaces and questions. GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part I, 118–161.

[9] E. Guentner, N. Higson and J. Trout, Equivariant E-theory for C^*-algebras. Mem. Amer. Math. Soc. 148 (2000), no. 703.

[10] N. Higson, Bivariant K-theory and the Novikov conjecture. Geom. Funct. Anal. 10 (2000), no. 3, 563–581.

[11] N. Higson and G. G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space. Invent. Math. 144 (2001), no. 1, 23–74.

[12] N. Higson, G. G. Kasparov and J. Trout, A Bott periodicity theorem for infinite-dimensional Euclidean space. Adv. Math. 135 (1998), no. 1, 1–40.

[13] W. B. Johnson and N. L. Randrianarivony, $l_p (p > 2)$ does not coarsely embed into a Hilbert space. Proc. Amer. Math. Soc. 134 (2006), no. 4, 1045–1050.

[14] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1988), 147–201.

[15] G. G. Kasparov and G. Skandalis, Groups acting properly on "bolic" spaces and the Novikov conjecture. Ann. of Math. (2) 158 (2003), no. 1, 165-206.

[16] G. G. Kasparov and G. Yu, The coarse geometric Novikov conjecture and uniform convexity, Adv. in Math. 206 (2006), no. 1, 1–56.

[17] M. Mendel and A. Naor, Metric Cotype, Ann. of Math. (2) 168 (2008), no. 1, 247–298.

[18] N. Ozawa, A note on non-amenability of $B(l_p)$ for $p = 1, 2$. Internat. J. Math. 15 (2004), no. 6, 557–565.

[19] J. Roe, Coarse cohomology and index theory on complete Riemannian manifolds. Mem. Amer. Math. Soc. 104 (1993), no. 497.

[20] G. Skandalis, J.-L. Tu and G. Yu, The coarse Baum-Connes conjecture and groupoids, Topology, 41(2002), 807-834.

[21] G. Yu, Localization algebras and the coarse Baum–Connes conjecture, K-Theory 11 (1997), 307–318.

[22] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Invent. Math. 139 (2000), no. 1, 201–240.

[23] G. Yu, Hyperbolic groups admit proper affine isometric actions on l^p-spaces, Geom. Funct. Anal. 15 (2005), no. 5, 1144-1151.

[24] G. Yu, A characterization of the image of the Baum-Connes map, Quanta of Maths, Clay Mathematics Proccedings, American Mathematical Society, Vol. 11 (2010), 649-657.

Department of Mathematics, 1326 Stevenson Center,
Vanderbilt University, Nashville, TN 37240, USA
e-mail: guoliang.yu@vanderbilt.edu and gennadi.kasparov@vanderbilt.edu