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Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **1125**

Unique Paper Code : **235201** **G**

Name of the Paper : **Differential Equation & Modeling-I**

Name of the Course : **B.Sc. (Hons.) Maths.**

Semester : **II**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Use of Scientific calculators is allowed.

Section I

1. Attempt any *three* of the following : 5+5+5

(*a*) Find particular solution of the differential equation :

$$\frac{dy}{dx} = 2xy + 3x^2 \exp\left(x^2\right), \quad y(0) = 5.$$

(*b*) Find a general solution of the differential equation :

$$x\frac{dy}{dx} - 4x^2 y + 2y \ln y = 5.$$

(*c*) Find the general solution of the differential equation :

$$\frac{d^2y}{dx^2} = \left(x + \frac{dy}{dx}\right)^2.$$

(*d*) State and prove the criterion for exactness.

2. Attempt any *two* of the following : 5+5

(*a*) Suppose that sodium pentobarbitol is used to anesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbitol per kilogram of the dog's body weight. Suppose also that sodium pentobarbital is eliminated exponentially from the dog's bloodstream, with a half-life of 5 hours. What single dose should be administered in order to anesthetize a 50 kg dog for 1 hour ?

(*b*) Suppose that a cylindrical tank initially containing V_0 gallons of water drains (through a bottom hole) in T minutes. Use Torricelli's law to show that the volume of water in the tank after $t \leq T$ minutes is $V = V_0[1 - (t/T)]^2$.

(c) Consider a body that moves horizontally through a medium whose resistance is proportional to the square of the velocity, so that $dv/dt = -kv^2$. Show that :

$$v(t) = \frac{v_0}{1 + v_0 kt}$$

and that

$$x(t) = x_0 + \frac{1}{k}\ln(1 + v_0 kt).$$

Section II

3. Attempt any *two* of the following : 8+8

(a) Consider the problem of population in the Lake Burley Griffin. Assume the lake has a constant volume :

(i) Write down a differential equation describing the concentration of pollution, using V for the volume of the lake, F for Flow, $c(t)$ of concentration at time t and c_{in} for concentration of population entering the lake.

(*ii*) Further, $V = 28 \times 10^6$ m^3, $F = 4 \times 10^6$ m^3/month, find how long would it take for the lake with pollution concentration of 10^7 parts/m^3 to drop below the safety threshold (4×10^6 parts/m^3) if :

Only fresh water enters in the lake.

Water enters the lake has a pollution concentration of 3×10^4 parts/m^3.

(*b*) A public bar opens at 6 pm and is rapidly filled with clients of whom the majority are smokers. The bar is equipped with ventilators which exchange the smoke-air mixture with fresh air. Cigarette smoke contains 4% carbon monoxide and a prolonged exposure to a concentration of more than 0.012% can be fatal. The bar has a floor area of 20 m by 15 m, and a height of 4 m. It is estimated that smoke enters the room at a constant rate of 0.0006 m^3/min, and that the ventilators remove the mixture of smoke and air at 10 times the rate at which smoke is produced. The problem is to establish a wise time to leave the bar. That is, sometimes before the concentration of carbon monoxide reaches the lethal limit.

(c) The following model describes the levels of drugs in a patient taking a course of cold pills :

$$\frac{dx}{dt} = 1 - k_1 x, \quad x(0) = 0,$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0,$$

Where k_1 and $k_2 (k_1 > 0, k_2 > 0$ and $k_1 \neq k_2)$ describe rates at which the drug moves between the two sequential compartments (the GI-tract and the bloodstream) and I denotes the amount of drug released into the GI-tract in each time step. The levels of the drug in the GI-tract and bloodstream are x and y respectively :

(i) Show by solving the equations sequentially that the solutions are :

$$x(t) = \frac{1}{k_1}\left(1 - e^{-k_1 t}\right), \; y(t) = \frac{1}{k_2}\left[1 - \frac{1}{k_2 - k_1}\left(k_2 e^{-k_1 t} - k_1 e^{-k_2 t}\right)\right].$$

(ii) Find the levels of the drug in the GI-tract and the bloodstream at $t \to \infty$.

Section III

4. Attempt any *three* of the following : 6+6+6

(*a*) Solve the initial value problem :

$$3y^{(3)} + 2y^{(2)} = 0; \ \ y(0) = -1, \ \ y^{(1)}(0) = 0, \ \ y^{(2)}(0) = 1.$$

(*b*) Use the method of undetermined coefficients to find the particular solution of the differential equation :

$$y^{(3)} + 6y^{(1)} + 13y = e^{-3x} \cos x.$$

(*c*) Use method of variation of parameter to find a particular solution of the differential equation :

$$y^{(2)} + ay = \sin 3x.$$

(*d*) A mass of 3 kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$ m/s. Find the amplitude, period and frequency of the resulting motion.

Section IV

5. Attempt any *two* of the following : 8+8

(*a*) (*i*) Develop a model with three differential equations describing a predator-prey interaction, where there are two different non-competing species of prey and one species of predator.

Draw the compartmental diagram, write the word equations, define the variables appropriately and hence derive the differential equations.

(*ii*) The following is the model describing one prey and two predators interaction :

$$\frac{dX}{dt} = a_1X - b_1XY - c_1XZ,$$

$$\frac{dY}{dt} = a_2XY - b_2Y,$$

$$\frac{dZ}{dt} = a_3XZ - b_3Z$$

Where a_i, b_i, c_i for $i = 1, 2, 3$, are all positive constants.

Find all the possible equilibrium points.

It is possible for all three populations to coexit in equilibrium ?

(*b*) A model for the spread of disease, where one susceptible infected, confers life-long immunity, is given by the coupled differential equation :

$$\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} = \beta SI - \alpha I$$

where α and β are positive constants, $S(t)$ denotes the number of susceptibles and $I(t)$ denotes the number of infectives at time t :

(*i*) Use the chain rule to find a relation between S and I, given the initial number of susceptible and infectives are s_0 and i_0 respectively.

(*ii*) Find and sketch directions of trajectories in the phase plane.

(*c*) A simple model for a battle between two armies red and blue, where both are the armies used aimed fire, is given by the coupled differential equations :

$$\frac{dR}{dt} = -a_1 B, \quad \frac{dB}{dt} = -a_2 R$$

where R and B are the numbers of soldiers in the red and blue armies respectively, a_1 and a_2 are positive constants.

If both the armies have equal attrition coefficients *i.e.* $a_1 = a_2$ and there are 10000 soldiers in the red army and 8000 in the blue army, determine who wins, if :

(*i*) There is one battle between the two armies.

(*ii*) There are two battles, first battle with half the red army against the entire blue army and second with the other half of the red army against the blue army survivors of the first battle.