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22/5/17

[This question paper contains 8 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **1824** **GC-4**

Unique Paper Code : 32351202

Name of the Course : **B.Sc.(Hons.) Mathematics–I**

Name of the Paper : Differential Equations

Semester : II

Time : 3 Hours ***Maximum Marks : 75***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Use of non-programmable scientific calculator is allowed.

SECTION – A

1. Attempt any **three** parts, each part is of **5** marks.

(a) Solve the initial value problem :

$$x\frac{dy}{dx}+y = xy^{3/2},\ y(1)=4.$$

(b) Determine the most general function M (x, y) such that the equation

M (x, y) dx + $(x^2y^3 + x^4y)$ dy =0, is exact and hence solve it.

(c) Solve the differential equation :

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0.$$

(d) Check the exactness of the differential equation :

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0.$$

Hence solve it by finding the integrating factor of the form $x^p y^q$.

2. Attempt any **two** parts; each part is of **5** marks.

(a) A certain moon rock was found to contain equal numbers of potassium and argon atoms. Assume that all the argon is the result of radioactive decay of potassium (its half-life is about 1.28×10^9 years) that one of every nine potassium atom disintegrations yields an argon atom. What is the age of the rock, measured from the time it contained only potassium ?

(b) A hemispherical bowl has top radius 4 ft and at time t = 0 is full of water. At that moment a circular hole with diameter 1 inch is opened in the bottom of the tank. How long will it take for all the water to drain from the tank ?

(c) A motor boat starts from rest (initial velocity $v(0) = v_0 = 0$). Its motor provides a constant acceleration of 4 ft/s^2, but water resistance causes a deceleration of $\frac{v^2}{400}$ ft/s^2. Find v when $t = 10$ s, and also find the limiting velocity as $t \to +\infty$ (that is, the maximum possible speed of the boat).

SECTION – B

3. Attempt any **two** parts; each part is of **7.5** marks.

(a) Consider the American system of two lakes: Lake Erie feeding into Lake Ontario. Assuming that volume in each lake to remain constant and that Lake Erie is the only source of pollution for Lake Ontario.

(i) Write down a differential equation describing the concentration of pollution in each of two lakes, using the variables V for volume, F for flow, c(t) for concentration at time t and subscripts 1 for Lake Erie and 2 for Lake Ontario.

(ii) Suppose that only unpolluted water flows into Lake Erie. How does this change the model proposed ?

(iii) Solve the system of equations to get expression for the pollution concentration $c_1(t)$ and $c_2(t)$.

(b) The following model describes the levels of a drug in a patient taking a course of cold pills :

$$\frac{dx}{dt} = I - k_1 x, \quad x(0) = 0$$

$$\frac{dy}{dt} = k_1 x - k_2 y, \quad y(0) = 0$$

Where k_1 and k_2 ($k_1 > 0$, $k_1 > 0$ and $k_1 \neq k_2$) describes rate at which the drug moves between the two sequential compartments (the GI-tract and the bloodstream) and I denotes the amount of drug released into the GI-tract in each step. At time t, x and y are the levels of the drug in the GI-tract and bloodstream respectively.

(i) Find solution expressions for x and y which satisfies this pair of differential equations.

(ii) Find the levels of the drug in the GI-tract and the bloodstream as $t \to \infty$.

(c) In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotas which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0 X.$$

(i) Find the non-zero equilibrium population.

(ii) At what critical harvesting rate can extinction occur ?

SECTION – C

4. Attempt any **four** parts; each part is of **5** marks.

(a) Use the method of variation of parameters to find a particular solution of the differential equation

$$y'' - 4y' + 4y = 2e^x.$$

(b) Use the method of undetermined coefficients to solve the differential equation

$$y'' + y = \sin x.$$

(c) A body with mass $m = \frac{1}{2}$ kg is attached to the end of the spring that is stretched 2 m (meters) by a force of 100 N (Newtons). It is set in motion with initial position $x_0 = 1$ m and initial velocity $v_0 = -5$m/s. Find the position function of the body as well as the amplitude, frequency and period of the oscillation.

(d) Show that the two solutions $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$ of the differential equation $y'' - 2y' + 2y = 0$ are linearly independent on the open interval I. Then find a particular solution of the above differential equation with initial condition

$$y(0) = 1 \text{ and } y'(0) = 5.$$

(e) Find the general solution of the Euler equation $x^2 y'' + 7xy' + 25y = 0$.

SECTION – D

5. Attempt any **two** parts; each part is of **7.5** marks.

(a) Consider a disease where all those who are infected remain contagious for life. Assume that there are no births and deaths :

(i) Write down suitable word equations for the rate of change of numbers of susceptibles and infectives. Hence develop a pair of differential equations.

(ii) Draw a sketch of typical phase-plane trajectories for this model. Determine the direction of travel along the trajectories.

(b) A simple model for a battle between two army red and blue, where both the army used aimed fire, is given by the coupled differential equations –

$$\frac{dR}{dt} = -a_1B, \quad \frac{dB}{dt} = -a_2R$$

Where R and B are the number of soldiers in the red and blue army respectively and a_1 and a_2 are the positive constants.

(i) Use the chain rule to find a relationship between R and B, given the initial numbers of soldiers for the two armies are r_0 and b_0 respectively.

(ii) Draw a rough sketch of phase-plane trajectories.

(iii) If both the army have equal attrition coefficients i.e. $a_1 = a_2$ and there are 10,000 soldiers in the red army and 8000 in blue army. Determine who wins if there is one battle between the two army.

(c) Consider the Lotka – Volterra model describing the simple predator prey model :

$$\frac{dx}{dt} = b_1X - c_1XY \quad \text{and} \quad \frac{dY}{dt} = c_2XY - a_2Y$$

where b_1, c_1, c_2, a_2 are positive constants and X and Y denotes the prey and predator populations respectively at time t.

(i) Find the equilibrium solutions of the above model.

(ii) Find the directions of trajectories in the phase plane.