Simulating ZF, MMSE, MLD Receivers in a Rayleigh Fading, MIMO Environment
Project #2

Intelligent Communication Systems (ICS) Lab.
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Winter Intern Seminar (2023-1)

Contents

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1 Implementation

다음의 조건을 만족하는 환경에 해당한다.

$$N_t \leq N_r \tag{1}$$

```
placeholder = 4

% Environment Variables
M = placeholder
Nt = placeholder
Nr =placeholder
NormalizationFactor = sqrt(2/3*(M-1)*Nt);

% Signal Generation
SignalSequence = randi([0 M-1], Nt, 1);
SignalBinary = de2bi(SignalSequence, log2(M), 'left-msb');
SymbolSequence = qammod(SignalSequence, M) / NormalizationFactor;

% Noise (n) Generation
NoiseSequence = (randn(Nt, 1) + 1j * randn(Nt, 1)) / sqrt(2);
```

```
H = (randn(Nr, Nt) + 1j * randn(Nr, Nt)) ./ sqrt(2);
```

*Normalization Factor*를 $\sqrt{\frac{2}{3}(M-1)N_t}$로 설정한 이유는 하나의 trasmitter가 $\frac{1}{N_t}$W의 전력을 갖도록 하기 위해서이다.

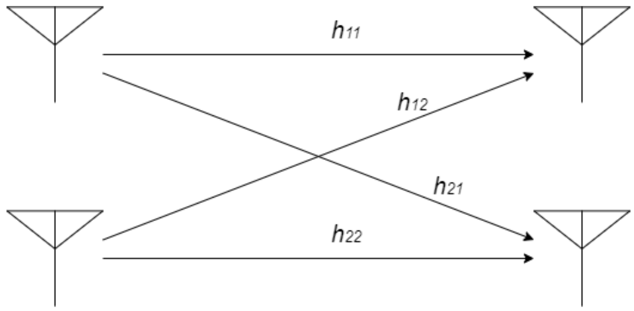


Figure 1

$$H = \begin{bmatrix} h_{11} & \dots & h_{1N_T} \\ \vdots & \ddots & \vdots \\ h_{N_R1} & \dots & h_{N_RN_T} \end{bmatrix} \tag{2}$$

1.1 ZF(Zero-forcing)

Moore-Penrose Pseudo Inverse

$\boldsymbol{W_{ZF}H} = \boldsymbol{I}$를 만족하는 W_{ZF}를 찾으려고 한다. $\boldsymbol{W_{ZF}} = (\boldsymbol{H}^H\boldsymbol{H})^{-1}\boldsymbol{H}^H$라고 하자.

$$\begin{aligned} \boldsymbol{W_{ZF}H} &= (\boldsymbol{H^H H})^{-1}\boldsymbol{H^H H} \\ &= (\boldsymbol{H}^H\boldsymbol{H})^{-1}(\boldsymbol{H}^H\boldsymbol{H}) \quad (\because \textit{associative property}) \\ &= \boldsymbol{I} \end{aligned} \tag{3}$$

이것이 성립하기 위해서 $(H^HH)^{-1}$ 값이 존재해야한다. 이 값이 존재하기 위해서는 $(H^HH)^{-1}$ 값이 존재해야한다.

H의 특성을 살펴보자.

1. H는 linearly independent column을 갖고 있다. (full column rank)
2. $N_r \geq N_t$이므로 행의 개수가 열의 개수보다 많다.

H의 크기는 $H_{N_R} \times H_{N_T}$이므로 H^HH의 크기는 $H_{N_T} \times H_{N_T}$로 square matrix이다.

$\vec{v} \in N(H^H H)$를 만족하는 $\vec{v}$가 있다고 하자.

$$\begin{aligned} H^H H\vec{v} &= \vec{0} \\ \vec{v}^H(H^H H\vec{v}) &= \vec{v}^H\vec{0} \\ (H\vec{v})^H(H\vec{v}) &= 0 \\ ||H\vec{v}||^2 &= 0 \\ H\vec{v} &= \vec{0} \end{aligned} \tag{4}$$

$$\therefore \vec{v} \in N(H) \tag{5}$$

즉, $\vec{v} \in N(H^H H)$를 만족하는 모든 $\vec{v}$는 $\vec{v} \in N(H)$를 만족한다. H가 *full column rank*이므로 $N(H) = \vec{0}$이며 $\vec{v}$로 가능한 값은 $\vec{0}$밖에 없다.

$$N(H^H H) = \vec{0} \tag{6}$$

$N(H^H H) = \vec{0}$라는 것은 $H^H H$가 *linearly independent column(full column rank)*을 갖고 있다는 것을 의미한다.

지금까지의 도출한 $H^H H$의 특성은 다음과 같다.

1. Linearly independent column을 갖고 있다. (full column rank)
2. $N_t \geq N_t$의 크기를 가지며, square matrix이다.

위 특성들로 하여금 $H^H H$의 역행렬이 존재한다는 것을 알 수 있다.

$$\therefore \exists W_{ZF}(= (H^H H)^{-1} H^H)$$

W_{ZF}를 H의 *left pseudo-inverse*라고 할 수 있다.

$ReceivedSymbolSequence = H * SymbolSequence + NoiseSequence/\sqrt{EsN0}$ 의 경우:

```
NormalizationFactor = sqrt(2/3*(M-1) * Nt);

w_zf = pinv(H); % pinv(H) = inv(H' * H) * H'
DetectedSymbolSequence_ZF = w_zf * ReceivedSymbolSequence;

DetectedSignalSequence_ZF = qamdemod(DetectedSymbolSequence_ZF*
    NormalizationFactor, M);
DetectedBinary_ZF = de2bi(DetectedSignalSequence_ZF, log2(M), 'left-msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_ZF, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_ZF, 'all');
```

$ReceivedSymbolSequence = \sqrt{EsN0} * H * SymbolSequence + NoiseSequence$ 의 경우:

```
NormalizationFactor = sqrt(2/3*(M-1)*Nt);

w_zf = NormalizationFactor /sqrt(EsN0) * pinv(H); % pinv(H) = inv(H' * H)
    * H'
DetectedSymbolSequence_ZF = w_zf * ReceivedSymbolSequence;

DetectedSignalSequence_ZF = qamdemod(DetectedSymbolSequence_ZF, M);
DetectedBinary_ZF = de2bi(DetectedSignalSequence_ZF, log2(M), 'left-msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_ZF, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_ZF, 'all');
```

1.2 MMSE(Minimum Mean Square Error)

$$\boldsymbol{W_{MMSE}} = \sqrt{\frac{N_t}{E_s}}(\boldsymbol{H}^H + \frac{N_t}{\rho})^{-1}\boldsymbol{H}^H \tag{7}$$

$ReceivedSymbolSequence = H * SymbolSequence + NoiseSequence/\sqrt{EsN0}$ 의 경우:

```
NormalizationFactor = sqrt(2/3*(M-1) * Nt); % size(H,1) = Nt

w_mmse = NormalizationFactor * inv(H' * H + Nt / EsN0 * eye(Nt)) * H';
DetectedSymbolSequence_MMSE = w_mmse * ReceivedSymbolSequence;

DetectedSignalSequence_MMSE = qamdemod(DetectedSymbolSequence_MMSE, M);
DetectedBinary_MMSE = de2bi(DetectedSignalSequence_MMSE, log2(M), 'left-
    msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_MMSE, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_MMSE, 'all')
    ;
```

$ReceivedSymbolSequence = \sqrt{EsN0} * H * SymbolSequence + NoiseSequence$ 의 경우:

```
Nt = size(H,2);
NormalizationFactor = sqrt(2/3*(M-1) * Nt); % size(H,1) = Nt

w_mmse = NormalizationFactor / sqrt(EsN0) * inv(H' * H + Nt / EsN0 * eye(
    Nt)) * H';
DetectedSymbolSequence_MMSE = w_mmse * ReceivedSymbolSequence; % Detection
     (Zero-Forcing: y / h)

DetectedSignalSequence_MMSE = qamdemod(DetectedSymbolSequence_MMSE, M); %
    Detection
DetectedBinary_MMSE = de2bi(DetectedSignalSequence_MMSE, log2(M), 'left-
    msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_MMSE, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_MMSE, 'all')
    ;
```

1.3 MLD(Maximum Likelihood Detection)

1.3.1 Creating All Possible Signal Combinations

```
% Creating Matrix for all possible combinations of signals (M^Nt possible combinations)
AllNumbers = de2bi([0:M^Nt-1], Nt*log2(M), 'left-msb');
Candidates = zeros(M^Nt, Nt);
for ii = 1 : M^Nt
    for jj = 1 : Nt
        Candidates(ii,jj) = bi2de(AllNumbers(ii,log2(M)*(jj-1)+1:log2(M)*jj), 'left-msb');
    end
end
Candidates = qammod(Candidates',M) / NormalizationFactor;
```

*AllNumbers*는 각 열마다 0부터 $M^{N_t}-1$를 이진수로 나태낸 matrix이다.

```
AllNumbers =

     0     0     0     0
     0     0     0     1
     0     0     1     0
     0     0     1     1
     0     1     0     0
     0     1     0     1
     0     1     1     0
     0     1     1     1
     1     0     0     0
     1     0     0     1
     1     0     1     0
     1     0     1     1
     1     1     0     0
     1     1     0     1
     1     1     1     0
     1     1     1     1
```

Figure 2: $M = 4$, $N_t = 2$

Figure 2은 $M = 4$, $N_t = 2$일 때의 *AllNumbers*의 예시이다. *AllNumbers*의 각 열을 $\log_2 M$ 숫자의 묶음이 N_t개로 있는 것으로 생각한다. 그렇다면, 각 열은 가능한 하나의 signal combination으로 볼 수 있다. *Candidates*의 dimension은 $M^{N_t} \times N_t$이다. 이는 N_t개의 signal로 만들 수 있는 M^{N_t}개의 signal combination을 나타낸다.

1.3.2 Solving For Minimum Euclidean Distance

$$\hat{\boldsymbol{s}} = \underset{s}{\operatorname{argmin}} \|\boldsymbol{y} - \sqrt{\frac{E_s}{N_T}} \boldsymbol{H}\boldsymbol{s}\|^2 \tag{8}$$

$ReceivedSymbolSequence = H * SymbolSequence + NoiseSequence/\sqrt{EsN0}$ 의 경우:

```
% 'EuclideanDistance' results in Nt x M^Nt, each column representing each
    candidate symbol combination
EuclideanDistance = abs(ReceivedSymbolSequence * ones(1,M^Nt) - H*
    Candidates).^2;
[val, idx] = min(sum(EuclideanDistance, 1));

DetectedBinary_MLD = reshape(de2bi(idx-1, log2(M)*Nt, 'left-msb'),log2(M)
    ,[])';
DetectedSequence_MLD = bi2de(DetectedBinary_MLD, 'left-msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_MLD, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSequence_MLD, 'all');
```

$ReceivedSymbolSequence = \sqrt{EsN0} * H * SymbolSequence + NoiseSequence$ 의 경우:

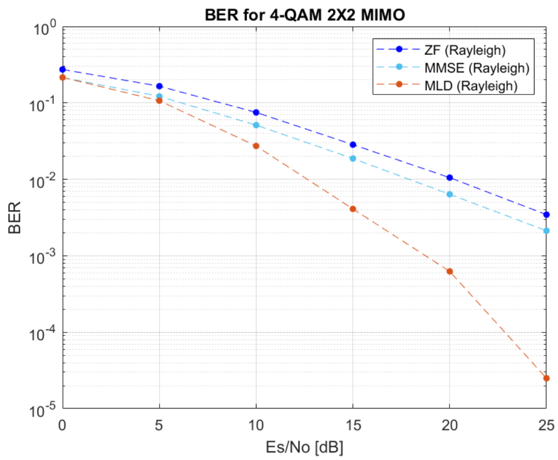
```
% 'EuclideanDistance' results in Nt x M^Nt, each column representing each
    candidate symbol combination
EuclideanDistance = abs(ReceivedSymbolSequence/sqrt(EsN0) * ones(1,M^Nt) -
     H*Candidates).^2;
[val, idx] = min(sum(EuclideanDistance, 1));

DetectedBinary_MLD = reshape(de2bi(idx-1, log2(M)*Nt, 'left-msb'),log2(M)
    ,[])';
DetectedSequence_MLD = bi2de(DetectedBinary_MLD, 'left-msb');

BitErrorCount = sum(SignalBinary~=DetectedBinary_MLD, 'all');
SignalErrorCount = sum(SignalSequence~=DetectedSequence_MLD, 'all');
```

$$ReceivedSymbolSequence = [y_1 \ \ \dots \ \ y_{N_R}]^T \tag{9}$$

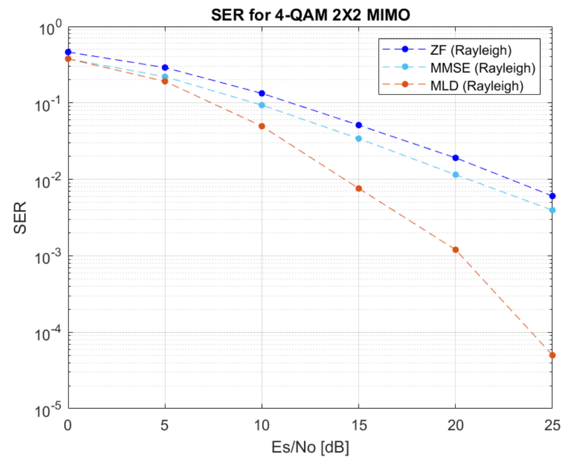
*sum(EuclideanDistance, 1)*는 $\|y - Hs\|_F^2$을 의미한다.

2 결과 및 분석

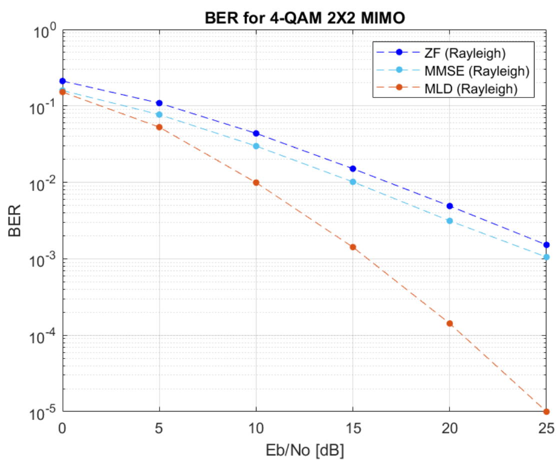
2.1 Simulation Result



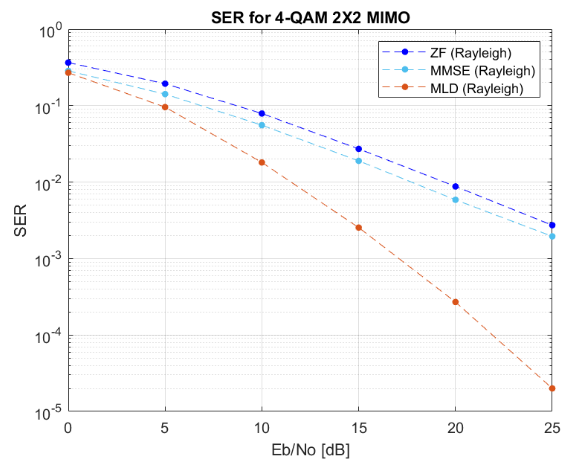
(a) BER



(b) SER

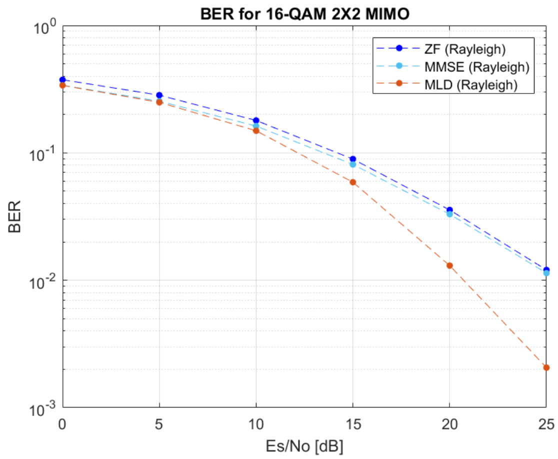


(c) BER

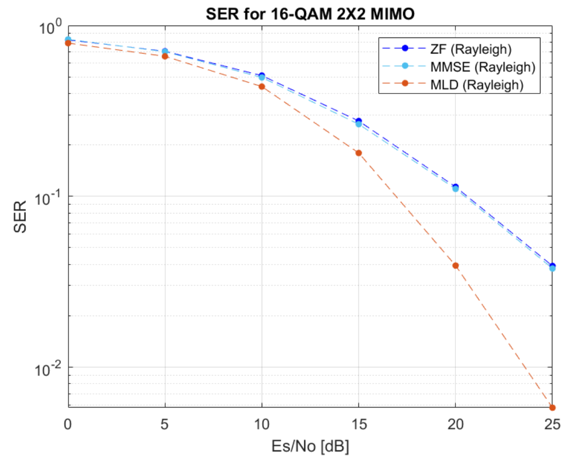


(d) SER

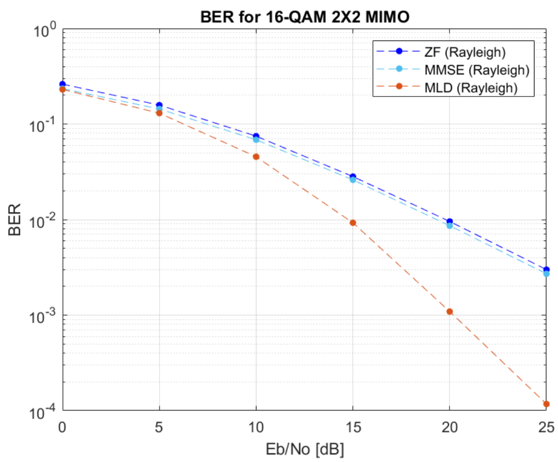
Figure 3: 4-QAM 2×2 MIMO



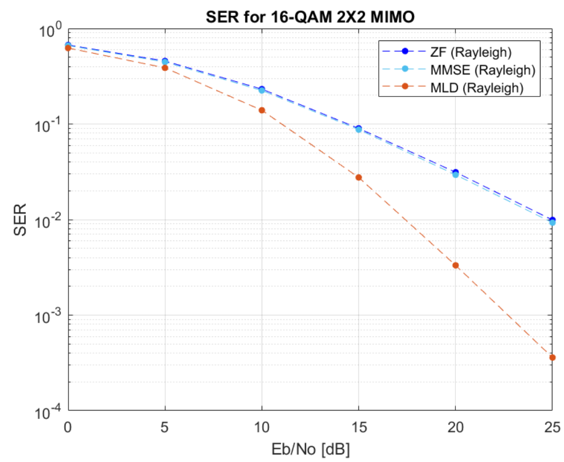
(a) BER



(b) SER

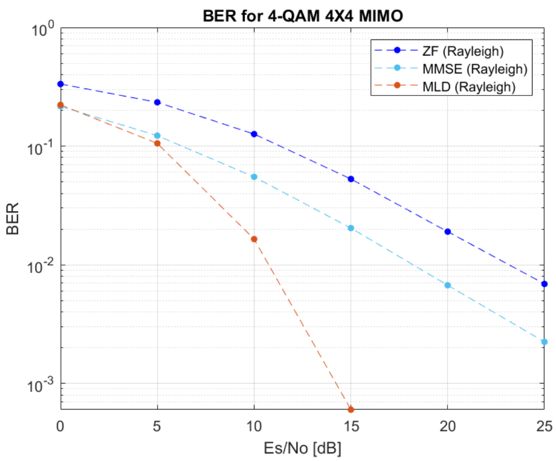


(c) BER

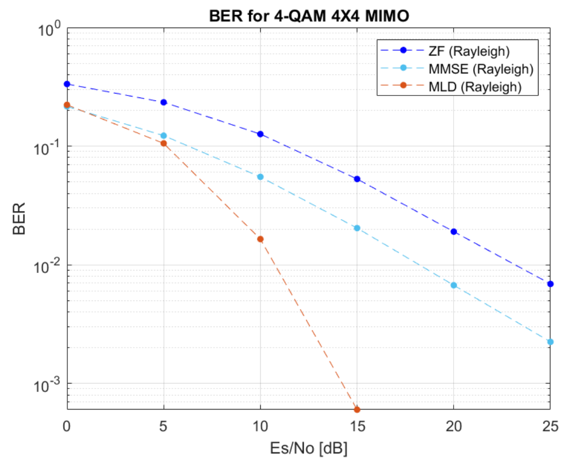


(d) SER

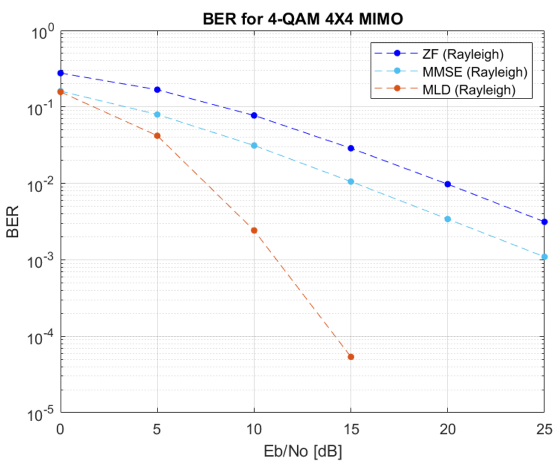
Figure 4: 16-QAM 2×2 MIMO



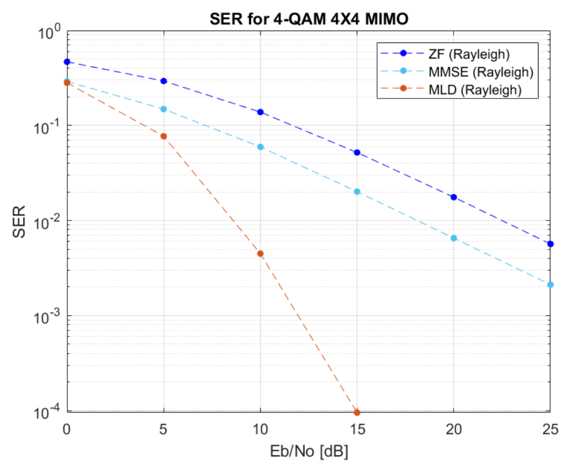
(a) BER



(b) SER

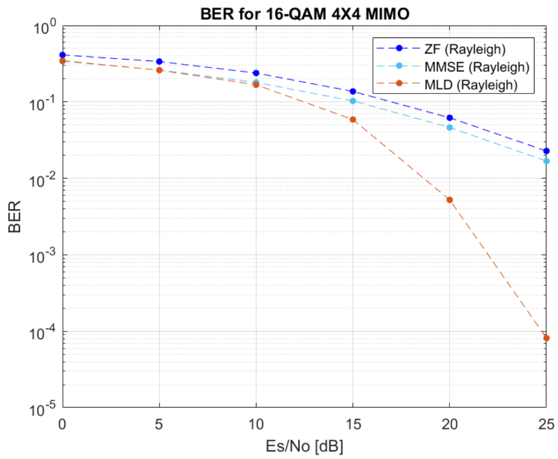


(c) BER

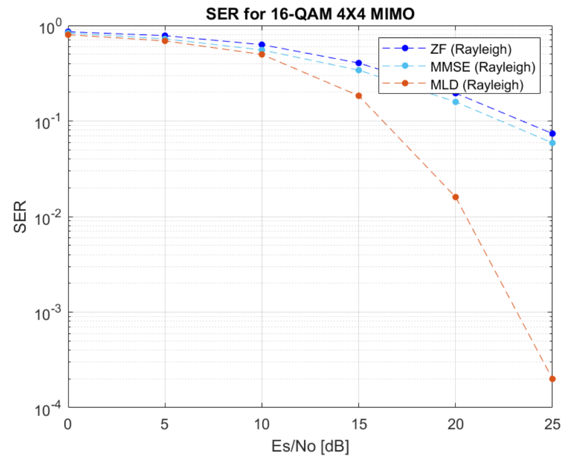


(d) SER

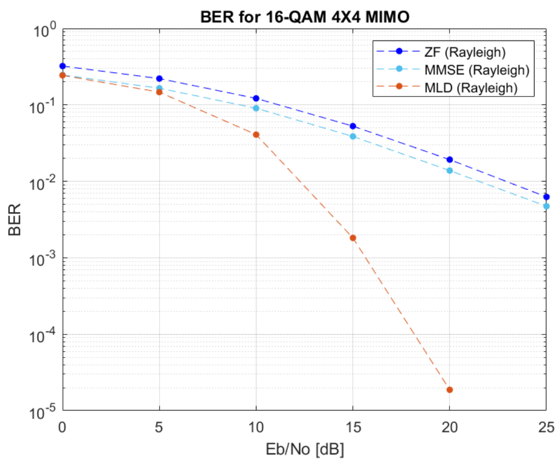
Figure 5: 4-QAM 4×4 MIMO



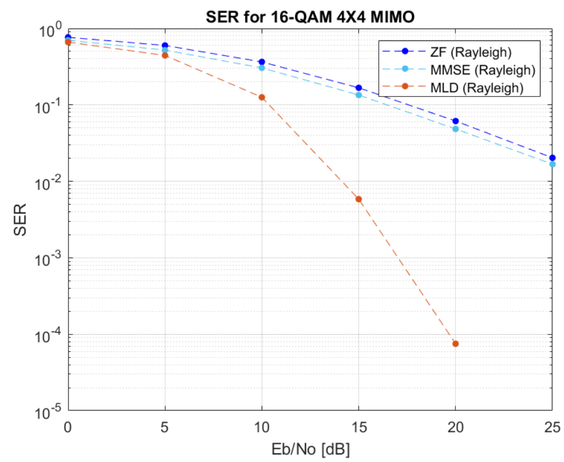
(a) BER



(b) SER

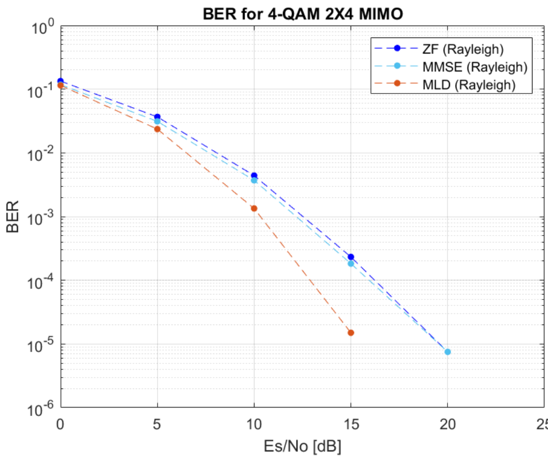


(c) BER

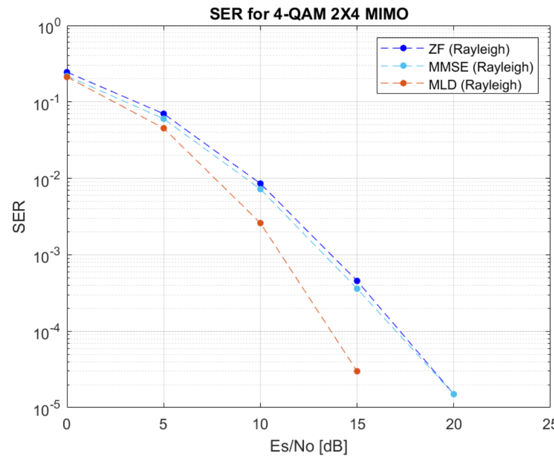


(d) SER

Figure 6: 16-QAM 4×4 MIMO

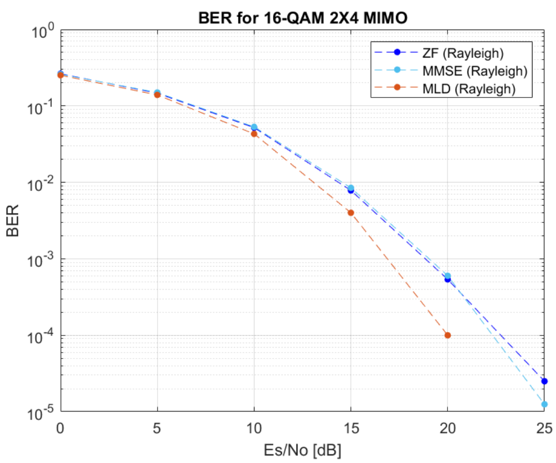


(a) BER

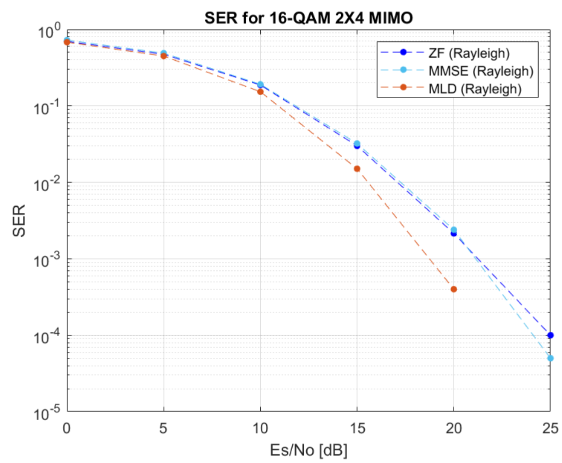


(b) SER

Figure 7: 4-QAM 2×4 MIMO



(a) BER



(b) SER

Figure 8: 16-QAM 2×4 MIMO

3 미해결 & 추가연구 필요 내용

- ZF receiver에서 $W_{ZF} = (H^T H)^{-1} H^T$와 같이 일반 transpose를 사용하여 W_{ZF}를 구한다 하더라도 결과는 동일하다.
  그렇다면 왜 Zero-forcing receiver에서 transpose가 아닌 complex transpose를 사용한건가?
  Complex transpose를 사용함으로써 어떠한 이점이 있는가?

4 Entire Code [1]

$$ReceivedSymbolSequence = \sqrt{EsN0} * H * SymbolSequence + NoiseSequence \tag{10}$$

[1] Uploaded on https://github.com/lightwick/ICS_project/tree/main/MIMO_Rayleigh

main.m

```
close all
clear all
clc

% Environment Varible
M = 4
Nt = 2
Nr = 2
NumberIteration = 10^3;

% Simulation
LengthBitSequence = Nt * log2(M); % log2(M) bits per signal
LengthSignalSequence = Nt;

% EbN0_dB = 0:5:25;
% EbN0 = db2pow(EbN0_dB);
%
% EsN0 = EbN0 * log2(M);
% EsN0_db = pow2db(EsN0);

EsN0_dB = 0:5:25;
EsN0 = db2pow(EsN0_dB);

EbN0 = EsN0 / log2(M);
EbN0_dB = pow2db(EbN0);

BitErrorCount_ZF = zeros(1, length(EsN0_dB));
SignalErrorCount_ZF = zeros(1, length(EsN0_dB));
BitErrorCount_MLD = zeros(1, length(EsN0_dB));
SignalErrorCount_MLD = zeros(1, length(EsN0_dB));

BitErrorCount_MMSE = zeros(1, length(EsN0_dB));
SignalErrorCount_MMSE = zeros(1, length(EsN0_dB));

NormalizationFactor = sqrt(2/3*(M-1)*Nt);

FivePercent = ceil(NumberIteration/20);
for iTotal = 1 : NumberIteration
    if mod(iTotal-100, FivePercent)==0
        tic
    end
    % Bit Generation
    SignalSequence = randi([0 M-1], Nt, 1);
    SignalBinary = de2bi(SignalSequence, log2(M), 'left-msb');
    SymbolSequence = qammod(SignalSequence, M) / NormalizationFactor;

    NoiseSequence = (randn(Nr, 1) + 1j * randn(Nr, 1)) / sqrt(2); % Noise (n) Generation
    H = (randn(Nr, Nt) + 1j * randn(Nr, Nt)) ./ sqrt(2); % Receiver x Transmitter
    for indx_EbN0 = 1 : length(EsN0)
        % Received Signal (y = hs + n) Generation
        ReceivedSymbolSequence = sqrt(EsN0(indx_EbN0)) * H * SymbolSequence +
            NoiseSequence; % log2(M)x1 matrix

        % MLD Receiver
        [BitErrorCount_tmp, SignalErrorCount_tmp] = simulate_mld(ReceivedSymbolSequence,
            SignalSequence, SignalBinary,  M, H, EsN0(indx_EbN0));
        BitErrorCount_MLD(indx_EbN0) = BitErrorCount_MLD(indx_EbN0) + BitErrorCount_tmp;
        SignalErrorCount_MLD(indx_EbN0) = SignalErrorCount_MLD(indx_EbN0) +
            SignalErrorCount_tmp;

        % ZF Receiver
```

```
        [BitErrorCount_tmp, SignalErrorCount_tmp] = simulate_zf(ReceivedSymbolSequence, SignalSequence, SignalBinary, M, H, EsN0(indx_EbN0));
        BitErrorCount_ZF(indx_EbN0) = BitErrorCount_ZF(indx_EbN0) + BitErrorCount_tmp;
        SignalErrorCount_ZF(indx_EbN0) = SignalErrorCount_ZF(indx_EbN0) + SignalErrorCount_tmp;

        % MMSE Receiver
        [BitErrorCount_tmp, SignalErrorCount_tmp] = simulate_mmse(ReceivedSymbolSequence, SignalSequence, SignalBinary, M, H, EsN0(indx_EbN0));
        BitErrorCount_MMSE(indx_EbN0) = BitErrorCount_MMSE(indx_EbN0) + BitErrorCount_tmp;
        SignalErrorCount_MMSE(indx_EbN0) = SignalErrorCount_MMSE(indx_EbN0) + SignalErrorCount_tmp;
    end

    % Progress Keeping
    if mod(iTotal-100, FivePercent)==0
        ElapsedTime = toc;
        EstimatedTime = (NumberIteration-iTotal)*ElapsedTime;
        disp(sprintf("%d%%, estimated wait time %d minutes %d seconds", round(iTotal/NumberIteration*100), floor(EstimatedTime/60), floor(mod(EstimatedTime, 60))))
    end
end

% Error Count to Ratio
SER_MLD = SignalErrorCount_MLD / (LengthSignalSequence * NumberIteration);
BER_MLD = BitErrorCount_MLD / (LengthBitSequence * NumberIteration);

SER_ZF = SignalErrorCount_ZF / (LengthSignalSequence * NumberIteration);
BER_ZF = BitErrorCount_ZF / (LengthBitSequence * NumberIteration);

SER_MMSE = SignalErrorCount_MMSE / (LengthSignalSequence * NumberIteration);
BER_MMSE = BitErrorCount_MMSE / (LengthBitSequence * NumberIteration);

% Plot
figure()

semilogy(EsN0_dB, BER_ZF, 'b.--', 'MarkerSize', 15);
hold on
semilogy(EsN0_dB, BER_MMSE, '.--', 'Color', "#4DBEEE", 'MarkerSize', 15);
semilogy(EsN0_dB, BER_MLD, '.--','Color', '#D95319', 'MarkerSize', 15);
ylabel('BER');
title(sprintf("BER for %d-QAM %dX%d MIMO", M, Nr, Nt));
grid on
legend('ZF (Rayleigh)', 'MMSE (Rayleigh)', 'MLD (Rayleigh)');
xlabel('Es/No [dB]');

figure()
semilogy(EsN0_dB, SER_ZF, 'b.--', 'MarkerSize', 15);
hold on
semilogy(EsN0_dB, SER_MMSE, '.--', 'Color', '#4DBEEE', 'MarkerSize', 15);
semilogy(EsN0_dB, SER_MLD, '.--', 'Color', '#D95319', 'MarkerSize', 15);
ylabel('SER');
title(sprintf("SER for %d-QAM %dX%d MIMO", M, Nr, Nt));
grid on
legend('ZF (Rayleigh)', 'MMSE (Rayleigh)', 'MLD (Rayleigh)');
xlabel('Es/No [dB]');
```

simulation_mld.m

```
function [BitErrorCount, SignalErrorCount] = simulate_mld(ReceivedSymbolSequence,
    SignalSequence, SignalBinary,  M, H, EsN0)
     Nt = size(H,2);
     NormalizationFactor = sqrt(2/3*(M-1)*Nt);

     % Candidates gets processed only once
     persistent Candidates
     if isempty(Candidates)
          Candidates = get_candidates(M, Nt) / NormalizationFactor;
     end

     % 'EuclideanDistance' results in Nt x M^Nt, each column representing each candidate
         symbol combination
     EuclideanDistance = abs(ReceivedSymbolSequence/sqrt(EsN0) * ones(1,M^Nt) - H*
         Candidates).^2;
     [val, idx] = min(sum(EuclideanDistance, 1));

     DetectedBinary_MLD = reshape(de2bi(idx-1, log2(M)*Nt, 'left-msb'),log2(M),[])';
     DetectedSequence_MLD = bi2de(DetectedBinary_MLD, 'left-msb');

     BitErrorCount = sum(SignalBinary~=DetectedBinary_MLD, 'all');
     SignalErrorCount = sum(SignalSequence~=DetectedSequence_MLD, 'all');
end

function Candidates = get_candidates(M, Nt)
     AllNumbers = de2bi([0:M^Nt-1], Nt*log2(M), 'left-msb');
     Candidates = zeros(M^Nt, Nt);
     for ii = 1 : M^Nt
         for jj = 1 : Nt
             Candidates(ii,jj) = bi2de(AllNumbers(ii,log2(M)*(jj-1)+1:log2(M)*jj), 'left-
                 msb');
         end
     end
     Candidates = qammod(Candidates',M);
end
```

simulation_zf.m

```
function [BitErrorCount, SignalErrorCount] = simulate_zf(ReceivedSymbolSequence, SignalSequence, SignalBinary,  M, H, EsN0)
        Nt = size(H,2);
        NormalizationFactor = sqrt(2/3*(M-1)*Nt); % size(H,1) = Nt
        w_zf = (NormalizationFactor) / sqrt(EsN0) * pinv(H); % pinv(H) = inv(H' * H) * H'
        DetectedSymbolSequence_ZF = w_zf * ReceivedSymbolSequence; % Detection (Zero-Forcing: y / h)

        DetectedSignalSequence_ZF = qamdemod(DetectedSymbolSequence_ZF, M); % Detection
        DetectedBinary_ZF = de2bi(DetectedSignalSequence_ZF, log2(M), 'left-msb');

        BitErrorCount = sum(SignalBinary~=DetectedBinary_ZF, 'all');
        SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_ZF, 'all');
end
```

simulation_mmse.m

```
function [BitErrorCount, SignalErrorCount] = simulate_mmse(ReceivedSymbolSequence, SignalSequence, SignalBinary,  M, H, EsN0)
    Nt = size(H,2);
    NormalizationFactor = sqrt(2/3*(M-1) * Nt); % size(H,1) = Nt

    w_mmse = NormalizationFactor / sqrt(EsN0) * inv(H' * H + Nt / EsN0 * eye(Nt)) * H';
    DetectedSymbolSequence_MMSE = w_mmse * ReceivedSymbolSequence; % Detection (Zero-Forcing: y / h)

    DetectedSignalSequence_MMSE = qamdemod(DetectedSymbolSequence_MMSE, M); % Detection
    DetectedBinary_MMSE = de2bi(DetectedSignalSequence_MMSE, log2(M), 'left-msb');

    BitErrorCount = sum(SignalBinary~=DetectedBinary_MMSE, 'all');
    SignalErrorCount = sum(SignalSequence~=DetectedSignalSequence_MMSE, 'all');
end
```