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Automated Reasoning, 2IMF25

older version encoded by 2IW15

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Literature:

S. N. Burris: *Logic for Mathematics and Computer Science*
Prentice Hall, 1998, ISBN 0-13-285974-2

C. Meinel and T. Theobald: *Algorithms and Data Structures in VLSI Design*
Springer, 1998, ISBN 3-540-64486-5

Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein: Introduction to Algorithms
MIT Press, 2009, ISBN 978-0-262-53305-8, only Chapter 29

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Organization

- Course + examination
- Practical assignment
- Each 50 %, each has to be at least 5
- Practical assignment to be done in groups of at most 2
- Deadline for first part of practical assignment:

  December 11, 2017
- Deadline for second part of practical assignment:

  January 11, 2018
- Playing around with the tools Z3 or Yices may start now, and may be have a look at NuSMV

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Example: (free after Lewis Carroll)

1. Good-natured tenured professors are dynamic
2. Grumpy student advisors play slot machines
3. Smokers wearing a cap are phlegmatic
4. Comical student advisors are professors
5. Smoking untenured members are nervous
6. Phlegmatic tenured members wearing caps are comical
7. Student advisors who are not stock market players are scholars
8. Relaxed student advisors are creative
9. Creative scholars who do not play slot machines wear caps
10. Nervous smokers play slot machines
11. Student advisors who play slot machines do not smoke
12. Creative good-natured stock market players wear caps
13. Therefore no student advisor is smoking

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Is it true that claim 13 can be concluded from statements 1 until 12?

We want to determine this fully automatically:

- Translate all statements and the desired conclusion to a formal description
- Apply some computer program with this formal description as input that decides fully automatically whether the conclusion is valid

This is a step further than verifying a given human reasoning as will be studied in the course *Proving with computer assistance*

The first step is giving names to every notion to be formalized

Next all claims (premisses and desired conclusion) should be transformed to formulas containing these names

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name	meaning	opposite
A	good-natured	grumpy
B	tenured	
C	professor	
D	dynamic	phlegmatic
E	wearing a cap	
F	smoke	
G	comical	
H	relaxed	nervous
I	play stock market	
J	scholar	
K	creative	
L	plays slot machine	
M	student advisor	

Using these names all claims can be transformed to a **proposition** over the letters A to M, i.e., an expression composed from these letters and the boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$

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name	meaning	opposite
A	good-natured	grumpy
B	tenured	
C	professor	
D	dynamic	phlegmatic
	$\ldots$	

The claim

> Good-natured tenured professors are dynamic

yields:

$$(A \wedge B \wedge C) \rightarrow D$$

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1. $(A \wedge B \wedge C) \rightarrow D$
2. $(\neg A \wedge M) \rightarrow L$
3. $(F \wedge E) \rightarrow \neg D$
4. $(G \wedge M) \rightarrow C$
5. $(F \wedge \neg B) \rightarrow \neg H$
6. $(\neg D \wedge B \wedge E) \rightarrow G$
7. $(\neg I \wedge M) \rightarrow J$
8. $(H \wedge M) \rightarrow K$
9. $(K \wedge J \wedge \neg L) \rightarrow E$
10. $(\neg H \wedge F) \rightarrow L$
11. $(L \wedge M) \rightarrow \neg F$
12. $(K \wedge A \wedge I) \rightarrow E$
13. Then $\neg(M \wedge F)$

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Hence we want to prove automatically that

$$\begin{array}{l}(((A \wedge B \wedge C) \rightarrow D) \ \wedge \\ ((\neg A \wedge M) \rightarrow L) \ \wedge \\ ((F \wedge E) \rightarrow \neg D) \ \wedge \\ ((G \wedge M) \rightarrow C) \ \wedge \\ ((F \wedge \neg B) \rightarrow \neg H) \ \wedge \\ ((\neg D \wedge B \wedge E) \rightarrow G) \ \wedge \\ ((\neg I \wedge M) \rightarrow J) \ \wedge \\ ((H \wedge M) \rightarrow K) \ \wedge \\ ((K \wedge J \wedge \neg L) \rightarrow E) \ \wedge \\ ((\neg H \wedge F) \rightarrow L) \ \wedge\end{array}$$

$$((L \wedge M) \rightarrow \neg F) \ \wedge$$
$$((K \wedge A \wedge I) \rightarrow E)\ ) \ \rightarrow \ \neg(M \wedge F)$$

is a tautology, meaning that for every valuation of A to M the value of this formula is *true*

How can this be treated?

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Simple method:

Truth tables, that is: compute the result for all 2^n valuations and check whether it is *true*

Here n is the number of variables

In the example we have $n = 13$, hence $2^n = 8192$, by which this approach is feasible

However, in many applications we have $n > 1000$ and need different methods

By putting $\neg$ in front of the formula checking whether a formula yields *false* for all valuations is as difficult as checking whether a formula yields *true* for all valuations

A formula is called **satisfiable** if it yields *true* for some valuations, so is not equivalent to *false*

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A formula composed from boolean variables and the boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ is called a **propositional formula**

The problem to determine whether a given propositional formula is satisfiable is called
SAT(isfiability)

So the method of truth tables is a method for SAT

However, the complexity of this method is always exponential in the number of variables, by which the method is unsuitable for formulas over many variables

Many practical problems can be expressed as SAT-problems over hundreds or thousands of variables

Hence we need other methods than truth tables

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Example: verification of a microprocessor

Here we want that

$\neg$(specified behavior $\leftrightarrow$ actual behavior)

is not satisfiable

This can be expressed as a propositional formula, and proving that this formula is unsatisfiable implies that the microprocessor is correct with respect to its specification

We will see methods for SAT that may be used for huge formulas over many variables, like these

However, all known methods are worst case exponential

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Now we give an example of a formula (the **pigeon hole formula**) that can be concluded to be unsatisfiable, hence is logically equivalent *false*, but for which it is hard to conclude this directly from the formula itself

Choose an integer number $n > 0$ and $n(n+1)$ boolean variables P_{ij} for $i = 1, \ldots, n+1$ and $j = 1, \ldots, n$

Define

$$C_n = \bigwedge_{j=1}^{n+1} (\bigvee_{i=1}^{n} P_{ij})$$

$$R_n = \bigwedge_{i=1,\ldots,n, 1 \leq j < k \leq n+1} (\neg P_{ij} \vee \neg P_{ik})$$

$$PF_n = C_n \wedge R_n$$

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Here

$$\bigwedge_{i=1}^{n} A_i$$

is used as an abbreviation for

$$A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_n$$

and so on

For the report of your practical assignment the use of these notations is strongly recommended

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Put the variables in a matrix as follows

$$\begin{array}{cccc} P_{11} & P_{12} & \cdots & P_{1,n+1} \\ P_{21} & P_{22} & \cdots & P_{2,n+1} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{n,n+1} \end{array}$$

$$C_n = \bigwedge_{j=1}^{n+1} (\bigvee_{i=1}^{n} P_{ij})$$

Validity of C_n means that in every column at least one variable is true

Hence if C_n holds then at least $n+1$ variables are true

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$$\begin{array}{cccc} P_{11} & P_{12} & \cdots & P_{1,n+1} \\ P_{21} & P_{22} & \cdots & P_{2,n+1} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{n,n+1} \end{array}$$

$$R_n = \bigwedge_{i=1,\ldots,n, 1 \le j < k \le n+1} (\neg P_{ij} \vee \neg P_{ik})$$

Validity of R_n means that in every row at most one variable is true:

for any two P's in the same row at least one is false

=

there are no two P's in the row that are both true

Hence if R_n holds then at most n variables are true

Hence C_n and R_n cannot be valid both, hence $PF_n = C_n \wedge R_n$ is unsatisfiable

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This counting argument is closely related to the **pigeon hole principle**:

> if $n+1$ pigeons fly out of a cage having n holes, then there is at least one hole through which at least two pigeons fly

The formula PF_n is called the **pigeon hole formula** for n

The formula is a conjunction of

$$(n+1) + n \times \frac{n(n+1)}{2}$$

disjunctions

The disjunctions are of the shape $\bigvee_{i=1}^{n} P_{ij}$ and $\neg P_{ij} \vee \neg P_{ik}$

If one arbitrary disjunction is removed from the big conjunction, then the resulting formula is always satisfiable

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Summarizing pigeon hole formula:

PF_n is an artificial unsatisfiable formula of size polynomial in n

Minor modifications of PF_n are satisfiable

For most methods proving unsatisfiability of PF_n automatically is hard: it can be done, but for most methods the number of steps is exponential in n

PF_n and modifications are a good testcase for implementations of methods for SAT

Next we consider a type of problem different from satisfiability for wich automated reasoning is desired too

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Consider 12 processes over boolean variables $A, \ldots, J$:

1. if D and E then $D, E :=$ *false*, *false*
2. if F and I then $F, I :=$ *false*, *false*
3. if A and E then $A, E :=$ *false*, *false*
4. if C and D then $C, D :=$ *false*, *false*
5. if D and G then $D, G :=$ *false*, *false*
6. if B and H then $B, H :=$ *false*, *false*
7. if B and I then $B, I :=$ *false*, *false*
8. if A and G then $A, G :=$ *false*, *false*
9. if D and H then $D, H :=$ *false*, *false*
10. if B and C then $B, C :=$ *false*, *false*
11. if B and J then $B, J :=$ *false*, *false*
12. if F and J then $F, J :=$ *false*, *false*

Claim:
From the initial state in which all variables have value *true* never the final state can be reached in which all variables have value *false*

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It is possible to find an invariant proving this claim (try!)

However, in this course we want to treat this kind of questions fully automatically, without human cleverness of choosing an invariant

In this example having 1024 states it is feasible to compute all reachable states, but we want to do this for cases having 10^{10} or 10^{100} reachable states

More general: **model checking**

- finite state space
- simple description of the set of initial states
- simple description of all possible state transitions
- some property has to be proved, for instance about all possible reachable states

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Examples:

- Railways:
  - state transition: if a semaphore shows green then a train may enter the corresponding track
  - property to be proved: trains do not collide
- Telephone network:
  - state transitions:
    - A enters a request for a connection with B
    - a requested connection is made
    - a connection is finished
  - property to be proved: no deadlock occurs
- Program verification
- $\ldots$

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The description of initial state and state transitions comprises the **model**, checking whether in such a model (e.g. in all reachable states) some property holds is called **model checking**

If state spaces and state transitions are described in a symbolic way (e.g. as formulas in propositional logic) then this is called **symbolic model checking**

Crucial is an efficient representation of formulas in propositional logic

This is a step further than SAT: we want efficient representations of all formulas, not only recognizing whether a formula is equivalent to *false*

Basic technique for SAT: resolution, exploited in tools like `Yices` and `Z3`

Basic technique for symbolic model checking: B(inary) D(ecision) D(iagrams), exploited in tools like `NuSMV`

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Not all kind of desired automated reasoning can be described in propositional logic

Other forms:

- Predicate logic: reasoning with $\forall$ and $\exists$

  all men are mortal

  Socrates is a man

  hence Socrates is mortal

- Equational logic: reasoning with equalities

  given: $0 + x = x$ *and*

  $(x + 1) + y = (x + y) + 1$

  conclude:

  $(0+1+1)+(0+1+1) = 0+1+1+1+1$

- Modal logic / temporal logic: reasoning involving a notion of time

  every message sent will eventually be received

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First we will concentrate on propositional logic

To be able to discuss the hardness of SAT we start by some remarks on **complexity** of algorithms

- (time) complexity: the number of steps required for executing an algorithm
- space complexity: the amount of memory required for executing an algorithm

Basic observation:
(time) complexity $\geq$ space complexity

In order to abstract from details we consider orders of magnitude:

$f(n) = O(g(n))$:
f does not grow faster than g

$f(n) = \Omega(g(n))$:
f does not grow slower than g

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More precisely:

$$f(n) = O(g(n))$$

means: there exist $c, n_0 \in \mathbf{N}$ such that

$$f(n) \leq c * g(n)$$

for every $n \geq n_0$

$$f(n) = \Omega(g(n))$$

means: there exist $n_0 \in \mathbf{N}$, $c > 0$ such that

$$f(n) \geq c * g(n)$$

for every $n \geq n_0$

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Often $f(n)$ is the complexity of an algorithm depending on a number n, and g is a well-known function, like

- $g(n) = n$ (linear)
- $g(n) = n^2$ (quadratic)
- $g(n) = n^k$ for some k (polynomial)
- $g(n) = a^n$ for some $a > 1$ (exponential)

Roughly speaking:

- polynomial: still feasible for reasonably big values of n
- exponential: only feasible for very small values of n

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No polynomial algorithm is known for SAT

More precisely: there is no algorithm

- receiving an arbitrary propositional formula as input
- that decides in all cases by execution whether this formula is satisfiable
- that requires no more than $c * n^k$ steps if $n > n_0$, where c, k, n_0 are values independent of the input and n is the size of the input

Even not if huge values are chosen for c, k, n_0

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This can mean two different things:

- such an algorithm exists, but it has not yet been found
- existence of such an algorithm is impossible

Although it has not been proven, the latter is most likely

P is the class of decision problems admitting a polynomial algorithm

So we conjecture that SAT is not in **P**

SAT is in **NP**, i.e.,
there is a notion of **certificate** such that

- if the correct result for SAT is 'no', then no certificate exists
- if the correct result for SAT is 'yes', then a certificate exists, and there is a polynomial algorithm that can decide whether a candidate for a certificate is really a certificate

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For SAT a certificate having these properties is a satisfying sequence of boolean values for the variables

NP is the abbreviation of
non-deterministically polynomial

Basic observation: $\mathbf{P} \subseteq \mathbf{NP}$

Conjecture: $\mathbf{P} \neq \mathbf{NP}$

This is one of the main open problems in theoretical computer science

A decision problem $\mathcal{A}$ in **NP** is called **NP-complete** if from the assumption $\mathcal{A} \in \mathbf{P}$ can be concluded that $\mathbf{P} = \mathbf{NP}$, i.e., every other decision problem in **NP** is in **P** too

So for NP-complete problems the existence of a polynomial algorithm is very unlikely

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It has been proven that SAT is NP-complete (1970), in fact this was the starting point of NP-completeness results

Other NP-complete problems:

- Given a distance table between a set of places and a number n

  Is there a path containing all of these places having a total length $\leq n$?

  (closely related to Traveling Salesman)

- Given an undirected graph

  Is there a cyclic path (hamiltonian circuit) containing every node exactly once?

- Given a number of packages, each having a weight

  Can you divide these packages into two groups each having the same total weight?

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Arithmetic in proposition logic

Binary representation

$$a_1 a_2 \cdots a_n$$

of a number a means that $a_i \in \{0, 1\}$ and

$$a = \sum_{i=1}^{n} a_i * 2^{n-i}$$

Addition

Given a and b, find d satisfying $a + b = d$

We need **carries** $c_0, c_1, \ldots, c_n$

Example: $7 + 21 = 28$:

$c \rightarrow$	0	0	1	1	1	0
$a = 7 \rightarrow$		0	0	1	1	1
$b = 21 \rightarrow$		1	0	1	0	1
$d = 28 \rightarrow$		1	1	1	0	0

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Requirements by which this is a correct computation:

- $d_i = a_i + b_i + c_i \bmod 2$, for $i = 1, \ldots, n$, expressed as a formula:

$$a_i \leftrightarrow b_i \leftrightarrow c_i \leftrightarrow d_i$$

- $c_{i-1} = 1$ if and only if $a_i + b_i + c_i > 1$, for $i = 1, \ldots, n$, expressed as a formula:

$$c_{i-1} \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_i) \vee (b_i \wedge c_i))$$

- $c_0 = 0$ and $c_n = 0$, expressed as a formula:

$$\neg c_0 \wedge \neg c_n$$

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Let ϕ be the conjunction of all of these formulas

Then computation $d = a + b$ follows from the unique satisfying assignment for

$$\phi \ \wedge \ \bigwedge_{i=1}^{n} [\neg] a_i \ \wedge \ \bigwedge_{i=1}^{n} [\neg] b_i$$

In case d does not fit in n digits then c_0 would forced to be 1, by which no satisfying assignment exists, to be solved by adding a leading 0 to a and b

Similarly subtraction: computing $b = d - a$ follows from satisfiability of

$$\phi \ \wedge \ \bigwedge_{i=1}^{n} [\neg] a_i \ \wedge \ \bigwedge_{i=1}^{n} [\neg] d_i$$

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Multiplication

A fast way to do multiplication, closely related to the base school algorithm:

r := 0;

for $i := 1$ to n do
if b_i then $r := 2r + a$ else $r := 2r$

Invariant: $r = [b_1 \cdots b_i] * a$, so at the end $r = b * a$

As a, b are numbers, represented as boolean vectors, we will write $\vec{a} = (a_1, \ldots, a_n)$, and so on

For representing $\vec{b} = 2\vec{a}$ we introduce

$$\mathrm{dup}(\vec{a}, \vec{b}) = \neg a_1 \wedge \neg b_n \wedge \bigwedge_{i=1}^{n-1} (a_{i+1} \leftrightarrow b_i)$$

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Introduce extra boolean variables r_{ij}, s_{ij} for $i = 0, \ldots, n$ and $j = 1, \ldots, n$

where $\vec{r_i} = (r_{i1}, \ldots, r_{in})$ represents the value of r after i steps, and
$\vec{s_i} = (s_{i1}, \ldots, s_{in})$ represents the value of $s = 2r$ after i steps

Then $\vec{r}_n$ will represent the result, the requirement

$$\vec{a} * \vec{b} = \vec{r}_n$$

is described by the formula

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$$\mathrm{mul}(\vec{a}, \vec{b}, \vec{r}_n) \quad =$$

$$\bigwedge_{j=1}^{n} \neg r_{0j} \quad \wedge$$

$$\bigwedge_{i=0}^{n-1} [\mathrm{dup}(\vec{r}_i, \vec{s}_i) \wedge (b_{i+1} \rightarrow \mathrm{plus}(\vec{a}, \vec{s}_i, \vec{r}_{i+1}))$$

$$\wedge(\neg b_{i+1} \rightarrow \bigwedge_{j=1}^{n} (s_{ij} \leftrightarrow r_{i+1,j}))]$$

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In this way we can do all kinds of arithmetic by SAT, for instance factorize a number

Define
$\mathrm{fac}(r) \quad = \quad \mathrm{mul}(a, b, r) \ \wedge \ a > 1 \ \wedge \ b > 1$

r is prime $\iff$ fac(r) is unsatisfiable

If satisfiable, then a, b represent factors

fac(1234567891) is unsatisfiable, so 1234567891 is prime

Found by `minisat` or `Yices` within 1 minute

fac(1234567897) is satisfiable, yielding

$$1234567897 = 1241 \times 994817$$

found by `minisat` or `Yices` within 1 second

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Program correctness by SAT

Basic idea:

- express all integer variables by sequences of boolean variables, in binary notation
- for

  for $j := 1$ to m do $\cdots$

  introduce $m+1$ copies $a_0, \ldots, a_m$ for every boolean variable a, where a_i means: the value of a after i steps
- $a := b$ in step i can be expressed as

  $$(a_{i+1} \leftrightarrow b_i) \wedge \bigwedge_{c} (c_{i+1} \leftrightarrow c_i)$$

  where c runs over all variables $\neq a$

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Required property to be proved = specification of the program

Typically given by a **Hoare triple**:

$$\{P\}S\{Q\}$$

Here

- S is the program

- P is the **precondition**: the property assumed to hold initially
- Q is the **postcondition**: the property that should hold after the program has finished

For proving $\{P\}S\{Q\}$ add the formula

$$P_0 \ \wedge \ \neg Q_m$$

to the formula expressing the semantics of the program and prove that it is unsatisfiable

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Simple example: boolean array $a[1..m]$

CLAIM: After doing
for $j := 1$ to $m-1$ do $a[j+1] := a[j]$

we have

$$\underbrace{a[1] = a[m]}_{\text{postcondition}}$$

Precondition = true, may be ignored

a_{ij} represents value $a[i]$ after j iterations

Semantics of jth iteration:

$$(a_{j+1,j} \leftrightarrow a_{j,j-1}) \wedge \bigwedge_{i \in \{1,\ldots,m\}, i \neq j+1} (a_{ij} \leftrightarrow a_{i,j-1})$$

Negation of postcondition: $\neg(a_{1,m-1} \leftrightarrow a_{m,m-1})$

Prove by a SAT solver that conjunction of all of these claims is unsatisfiable

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Same approach applies for more complicated programs, where for $+$ and $*$ we can use plus$(\cdots)$ and mul$(\cdots)$

Example

CLAIM: After doing

$a := 0$;
for $i := 1$ to m do $a := a + k$

we have $a = m * k$

For fixed m and number n of bits this is proved by proving unsatisfiability of

$$\bigwedge_{j=1}^{n} \neg a_{0,j} \ \wedge \ \bigwedge_{i=0}^{m-1} \text{plus}(\vec{a}_i, \vec{k}, \vec{a}_{i+1}) \ \wedge$$

$$\neg\text{mul}(\vec{m}, \vec{k}, \vec{a}_m)$$

where $\vec{m}$ is the binary encoding of number m

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if b then S_1 else S_2

in step i can be expressed as

$$(b_{i-1} \rightarrow F_1) \wedge (\neg b_{i-1} \rightarrow F_2)$$

where formulas F_1, F_2 express S_1, S_2 in step i

In this way verification of imperative programs having a fixed number of steps can be expressed in SAT

In this way for integers we have to fix a number of bits, and encode all arithmetical operations ourselves

The unfolding of a fixed number of steps to a SAT problem is the basis of **bounded model checking**

Later we will see **SMT: satisfiability modulo theories**, by which we can express (in)equalities on linear expressions like

$$a < 3 * b + c$$

directly

This is supported by the tools `Yices` and `Z3`

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Resolution

This is a method for SAT, being the basis of the best current SAT-solvers

Resolution is only applicable to formulas of a particular shape, namely CNF

A **conjunctive normal form (CNF)** is a conjunction of clauses

A **clause** is a disjunction of literals

A **literal** is either a variable or the negation of a variable

Hence a CNF is of the shape

$$\bigwedge_i (\bigvee_j \ell_{ij})$$

where ℓ_{ij} are literals

For example, the pigeon hole formula PF_n is a CNF

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Arbitrary formulas can be transformed to CNFs in a clever way maintaining satisfiability

This makes resolution applicable to arbitrary formulas

Basic idea of resolution:

Add new clauses in such a way that the conjunction of all clauses remains equivalent to the original CNF

The empty clause $\bot$ is equivalent to *false*

If the clause $\bot$ is created in the resolution process then the conjunction of all clauses is equivalent to *false*, and hence the same holds for the original CNF

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Intuitively:

Clauses are properties that you know to be true

From these clauses you derive new clauses, trying to derive a contradiction: the empty clause

Surprisingly here we need only one rule, the **resolution rule**

This rule states that if there are clauses of the shape $V \vee p$ and $W \vee \neg p$, then the new clause $V \vee W$ may be added

This is correct since

$$(V \vee p) \wedge (W \vee \neg p) \Rightarrow (V \vee W)$$

(apply case analysis p and $\neg p$)

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Order of literals in a clause does not play a role

Double occurrences of literals will be removed

Think of a clause as a **set** of literals

Think of a CNF as a **set** of clauses

Example

We prove that

$$(p \vee q) \wedge (\neg r \vee s) \wedge (\neg q \vee r) \wedge (\neg r \vee \neg s) \wedge (\neg p \vee r)$$

is unsatisfiable

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1	$p \vee q$	
2	$\neg r \vee s$	
3	$\neg q \vee r$	
4	$\neg r \vee \neg s$	
5	$\neg p \vee r$	
6	$p \vee r$	$(1, 3, q)$
7	r	$(5, 6, p)$
8	s	$(2, 7, r)$
9	$\neg r$	$(4, 8, s)$
10	$\bot$	$(7, 9, r)$

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Remarks:

- Lot of freedom in choice

  Other first steps in the example could have been $(3, 4, r)$ or $(2, 4, s)$ or $(1, 5, p)$ or $\ldots$

- Resolution steps in which V contains q and W contains $\neg q$ for some q (or conversely) are allowed but useless

  In that case the new clause $V \vee W$ is of the shape $q \vee \neg q \vee \cdots$ and hence equivalent to *true*, not containing fruitful information

- If a clause consists of a single literal ℓ then by the resolution rule the literal $\neg\ell$ may be removed from every clause containing $\neg\ell$

  This is called **unit resolution**

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This proof system is **sound** due to the correctness of the resolution rule:

> if the empty clause can be derived
> then the original formula is equivalent to *false*

Conversely we will prove that this proof system is **complete**:

> if any formula is equivalent to *false*
> then it is possible to derive the empty clause only by using the resolution rule

Hence it should be possible to solve the problem of the non-smoking student advisor in this way, which we will do now

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Due to $(P \wedge Q) \rightarrow R \equiv \neg P \vee \neg Q \vee R$ this problem is easily transformed to CNF:

1. $\neg A \vee \neg B \vee \neg C \vee D$
2. $A \vee \neg M \vee L$
3. $\neg F \vee \neg E \vee \neg D$
4. $\neg G \vee \neg M \vee C$
5. $\neg F \vee B \vee \neg H$
6. $D \vee \neg B \vee \neg E \vee G$
7. $I \vee \neg M \vee J$
8. $\neg H \vee \neg M \vee K$
9. $\neg K \vee \neg J \vee L \vee E$
10. $H \vee \neg F \vee L$
11. $\neg L \vee \neg M \vee \neg F$
12. $\neg K \vee \neg A \vee \neg I \vee E$
13. M
14. F

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Apply unit resolution on M, F: remove every $\neg M$, $\neg F$

1. $\neg A \vee \neg B \vee \neg C \vee D$
2. $A \vee (\neg M) \vee L$
3. $(\neg F) \vee \neg E \vee \neg D$
4. $\neg G \vee (\neg M) \vee C$
5. $(\neg F) \vee B \vee \neg H$
6. $D \vee \neg B \vee \neg E \vee G$
7. $I \vee (\neg M) \vee J$
8. $\neg H \vee (\neg M) \vee K$
9. $\neg K \vee \neg J \vee L \vee E$
10. $H \vee (\neg F) \vee L$
11. $\neg L \vee (\neg M) \vee (\neg F)$
12. $\neg K \vee \neg A \vee \neg I \vee E$
13. M
14. F

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Apply unit resolution on $\neg L$: remove every L

1. $\neg A \vee \neg B \vee \neg C \vee D$
2. A
3. $\neg E \vee \neg D$
4. $\neg G \vee C$
5. $B \vee \neg H$
6. $D \vee \neg B \vee \neg E \vee G$
7. $I \vee J$
8. $\neg H \vee K$
9. $\neg K \vee \neg J \vee E$
10. H
11. $\neg K \vee \neg A \vee \neg I \vee E$

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Apply unit resolution on A, H: remove every $\neg A$, $\neg H$

1. $\neg B \vee \neg C \vee D$
2. $\neg E \vee \neg D$
3. $\neg G \vee C$
4. B
5. $D \vee \neg B \vee \neg E \vee G$
6. $I \vee J$
7. K
8. $\neg K \vee \neg J \vee E$
9. $\neg K \vee \neg I \vee E$

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Apply unit resolution on B, K: remove every $\neg B$, $\neg K$

1. $\neg C \vee D$
2. $\neg E \vee \neg D$
3. $\neg G \vee C$
4. $D \vee \neg E \vee G$
5. $I \vee J$
6. $\neg J \vee E$
7. $\neg I \vee E$

No unit resolution possible any more

Resolution on I, 5 and 7 yields $J \vee E$

Resolution on J, 6 and $J \vee E$ yields unit clause E

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Apply unit resolution on E: remove every $\neg E$

1. $\neg C \vee D$
2. $\neg D$
3. $\neg G \vee C$
4. $D \vee G$

Unit resolution on $\neg D$ yields $\neg C$ and G, and remaining clause

$$\neg G \vee C$$

Unit resolution on $\neg C$ and G yields empty clause, hence we have proved that the formula is unsatisfiable

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Preferring unit resolution is a good strategy: unit resolution can not cause increase of CNF size

Another good strategy is **subsumption**: ignore or remove clauses V for which a smaller clause W occurs satisfying $W \subset V$

For the rest there is a lot of remaining choice for doing resolution

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A strategy that can always be applied is

Davis-Putnam's procedure (1960):

Repeat until either no clauses are left or the empty clause has been derived:

> Choose a variable p
>
> Apply resolution on every pair of clauses for which the one contains p and the other contains $\neg p$
>
> Remove all clauses containing both q and $\neg q$ for some q
>
> Remove all clauses containing either p or $\neg p$

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Example:

Consider the CNF consisting of the following nine clauses

$$\begin{array}{lll} \neg p \vee \neg s & p \vee r & \neg s \vee t \\ \neg p \vee \neg r & s \vee p & q \vee s \\ \neg q \vee \neg t & r \vee t & \neg r \vee q \end{array}$$

First do all resolution steps w.r.t. p

After removing all clauses containing p, $\neg p$ or the shape $A \vee \neg A$ the following clauses remain:

$$\underbrace{r \vee \neg s,\ \ \neg r \vee s}_{\text{new}},\ \ \neg s \vee t,\ \ q \vee s,\ \ \neg q \vee \neg t,\ \ r \vee t,\ \ \neg r \vee q$$

Next do all resolution steps w.r.t. q

After removing all clauses containing q, $\neg q$ the following clauses remain:

$$r \vee \neg s,\ \ \neg r \vee s,\ \ \neg s \vee t,\ \ r \vee t,\ \ \underbrace{s \vee \neg t,\ \ \neg r \vee \neg t}_{\text{new}}$$

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Next do all resolution steps w.r.t. r

After removing all clauses containing r, $\neg r$ or the shape $A \vee \neg A$ the following clauses remain:

$$\underbrace{\neg t \vee \neg s,\ \ s \vee t}_{\text{new}},\ \ \neg s \vee t,\ \ s \vee \neg t$$

Next do all resolution steps w.r.t. s

After removing all clauses containing s, $\neg s$ or the shape $A \vee \neg A$ the following clauses remain:

$$t,\ \ \neg t$$

Finally we do resolution on t and obtain the empty clause, proving that the original CNF is unsatisfiable

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Theorem:

Davis-Putnam's procedure ends in an empty clause if and only if the original CNF is unsatisfiable

A direct consequence of this theorem is completeness of resolution

Now we will prove the theorem

Soundness of resolution we observed before; we have to prove that

> if the CNF is unsatisfiable then Davis-Putnam's procedure will end in an empty clause

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We assume that Davis-Putnam's procedure does not end in an empty clause; we have to prove that the original CNF is satisfiable

Due to the assumption and the structure of the procedure

> Repeat until either no clauses are left or the empty clause has been derived $\cdots$

the process ends in the empty set of clauses which is trivially satisfiable

The required satisfiability of the original CNF follows from the following property:

> If a CNF X is transformed to X' by a Davis-Putnam step and X' is satisfiable, then X is satisfiable too

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To prove: If a CNF X is transformed to X' by a Davis-Putnam step and X' is satisfiable, then X is satisfiable too

Let U be the set of clauses in X in which p does not occur

Let V be the set of clauses C such that $C \vee p$ occurs in X

Let W be the set of clauses C such that $C \vee \neg p$ occurs in X

Then

$$X : \bigwedge_{C \in U} C \wedge \bigwedge_{C \in V} (C \vee p) \wedge \bigwedge_{C \in W} (C \vee \neg p)$$

and

$$X' : \bigwedge_{C \in U} C \wedge \bigwedge_{C \in V, D \in W} (C \vee D)$$

up to clauses containing the pattern $q \vee \neg q$

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Assume that X' is satisfiable

Consider its part

$$\bigwedge_{C \in V, D \in W} (C \vee D)$$

$\equiv$

$$\bigwedge_{C \in V} (\bigwedge_{D \in W} (C \vee D))$$

$\equiv$ (distributivity)

$$\bigwedge_{C \in V} (C \vee \bigwedge_{D \in W} D)$$

$\equiv$ (distributivity)

$$(\bigwedge_{C \in V} C) \vee (\bigwedge_{D \in W} D)$$

Hence

$$X' \equiv \bigwedge_{C \in U} C \wedge ((\bigwedge_{C \in V} C) \vee (\bigwedge_{D \in W} D))$$

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$$X' \equiv \bigwedge_{C \in U} C \wedge ((\bigwedge_{C \in V} C) \vee (\bigwedge_{D \in W} D))$$

We assume that this is satisfiable

In the corresponding satisfying assignment either $\bigwedge_{C \in V} C$ or $\bigwedge_{C \in W} C$ is true

If $\bigwedge_{C \in V} C$ is true, then we assign the value *false* to the fresh variable p and keep the same assignment for the other variables, by which

$$\bigwedge_{C \in V} (C \vee p) \wedge \bigwedge_{C \in W} (C \vee \neg p)$$

has the value *true*

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Either $\bigwedge_{C \in V} C$ or $\bigwedge_{C \in W} C$ is true

If $\bigwedge_{C \in W} C$ is true, then we assign the value *true* to the variable p and again keep the same assignment for the other variables, by which

$$\bigwedge_{C \in V} (C \vee p) \wedge \bigwedge_{C \in W} (C \vee \neg p)$$

has the value *true*

For both cases we have a satisfying assignment for

$$X : \bigwedge_{C \in U} C \wedge \bigwedge_{C \in V} (C \vee p) \wedge \bigwedge_{C \in W} (C \vee \neg p)$$

End of proof

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Summarizing Davis-Putnam's procedure:

- Procedure to establish satisfiability of any CNF
- Complete: it always ends and always gives the right answer
- One long repeat loop doing at most n steps if there are n variables
- In every step of the loop clauses are added and removed
- Worst case exponential: intermediate CNF may blow up exponentially (and often does in practice $\cdots$)
- By keeping intermediate CNFs in case of satisfiability a satisfying assignment can be constructed from the run of the procedure, as we show by an example

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$p \vee q,\ \neg p \vee \neg q,\ p \vee \neg q,\ \neg p \vee r$

$\downarrow p$

$q \vee r,\ \neg q,\ \neg q \vee r$

$\downarrow q$

r $\qquad r = \mathit{true}$

$\downarrow r$

$\{\}$

find value for the last variable that was removed

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$p \vee q,\ \neg p \vee \neg q,\ p \vee \neg q,\ \neg p \vee r$

$\downarrow p$

$q \vee r,\ \neg q,\ \neg q \vee r$ $\qquad q = \mathit{false}$

$\downarrow q$ $\qquad \uparrow$

r $\qquad r = \mathit{true}$

$\downarrow r$

$\{\}$

evaluate in formula, find next value

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$p \vee q,\ \neg p \vee \neg q,\ p \vee \neg q,\ \neg p \vee r$ $\quad p = \mathit{true}$

$\downarrow p$ $\quad \uparrow$

$q \vee r,\ \neg q,\ \neg q \vee r$ $\quad q = \mathit{false}$

$\downarrow q$ $\quad \uparrow$

r $\quad r = \mathit{true}$

$\downarrow r$

$\{\}$

evaluate all values in formula, find next value, until finished

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The DPLL method

(Davis, Putnam, Logemann, Loveland, 1962)

Recursive procedure for SAT on CNF = set of clauses

DPLL(X):
  $X :=$ unit-resol(X)
  if $X = \emptyset$ then return(satisfiable)
  if $\bot \notin X$ then
    choose variable p in X
    DPLL($X \cup \{p\}$)
    DPLL($X \cup \{\neg p\}$)

unit-resol means: apply unit resolution as long as possible, more precisely:

> As long as a clause occurs consisting of one literal ℓ, remove all clauses containing ℓ and remove $\neg\ell$ from all clauses containing $\neg\ell$

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Execution of DPLL(X) always terminates

(in every recursive call the number of occurring variables is strictly less)

Idea of DPLL:

- First try unit resolution as long as possible
- If you can not proceed by unit resolution or trivial observations then choose a variable p, introduce the cases p and $\neg p$, and for both cases go on recursively
- If all cases in this process end in a contradiction (the empty clause) then the original formula is unsatisfiable
- If one case is found arriving in the empty set of clauses, then a satisfying assignment for the original formula is found by making the consecutive choices for the variables for which case analysis has been done

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Example:

Consider the CNF consisting of the following nine clauses

$\neg p \vee \neg s$	$p \vee r$	$\neg s \vee t$
$\neg p \vee \neg r$	$s \vee p$	$q \vee s$
$\neg q \vee \neg t$	$r \vee t$	$\neg r \vee q$

No unit resolution possible: choose variable p

Add p	Add $\neg p$
Unit resolution:	Unit resolution:
$\neg s$	r
$\neg r$	s
q (use $\neg s$)	q (use r)
t (use $\neg r$)	t (use s)
$\neg t$ (use q)	$\neg t$ (use q)
$\bot$	$\bot$

Both branches yield $\bot$, so original CNF is unsatisfiable

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Example:

Consider the CNF consisting of the following eight clauses

$$\begin{array}{lll} \neg p \vee \neg s & p \vee r & \neg s \vee t \\ \neg p \vee \neg r & s \vee p & q \vee s \\ \neg q \vee \neg t & r \vee t & \end{array}$$

No unit resolution possible: choose variable p

Add p

Unit resolution:	Add $\neg p$
$\neg s$	Unit resolution:
$\neg r$	r
q (use $\neg s$)	s
t (use $\neg r$)	t (use s)
$\neg t$ (use q)	$\neg q$ (use t)

$\bot$

Yields satisfying assignment $p = q = \mathit{false}$, $r = s = t = \mathit{true}$

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In this way DPLL is a complete method for SAT, just like classical Davis-Putnam (DP)

Just like DP, DPLL admits freedom of choice, and this choice may strongly influence the efficiency of the algorithm

Usually DPLL is more efficient than DP; it is still the basis of current strong SAT solvers

Although DPLL is not pure resolution due to the addition of p and $\neg p$, there is a strong relationship between a DPLL proof and resolution:

> A proof of unsatisfiability of a CNF by DPLL can be transformed directly to a resolution proof of the same CNF ending in $\bot$, having the same size

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The key idea is that any resolution derivation from $V \wedge \ell$ to $\bot$ can be transformed to a resolution derivation from V to $\neg\ell$ (or $\bot$), simply by ignoring unit resolution steps on ℓ

Two derivations from $V \wedge p$ to $\bot$ and from $V \wedge \neg p$ to $\bot$ then transform to derivations from V to both p and $\neg p$, yielding $\bot$ in one extra step

Example

Applying unit resolution to the non-smoking student advisors yields

1. $\neg C \vee D$
2. $\neg E \vee \neg D$
3. $\neg G \vee C$
4. $D \vee \neg E \vee G$
5. $I \vee J$
6. $\neg J \vee E$
7. $\neg I \vee E$

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Let us choose p to be E

By adding the clause E unit resolution yields

2	$\neg D$	
4	$D \vee G$	
8	$\neg C$	$(1, 2)$
9	$\neg G$	$(3, 8)$
10	D	$(4, 9)$
11	$\bot$	$(2, 10)$

By adding the clause $\neg E$ unit resolution yields

6	$\neg J$	
7	$\neg I$	
12	I	$(5, 6)$
13	$\bot$	$(7, 12)$

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These two derivations combine to

8	$\neg C \vee \neg E$	$(1, 2)$
9	$\neg G \vee \neg E$	$(3, 8)$
10	$D \vee \neg E$	$(4, 9)$
11	$\neg E$	$(2, 10)$
12	$E \vee I$	$(5, 6)$
13	E	$(7, 12)$
14	$\bot$	$(11, 13)$

The same approach applies for more complicated nested examples

Hence:

DPLL indeed may be (and is) considered as a resolution technique

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Weak point of this version of DPLL:

The (typically very large) CNF is modified by unit resolution in every recursive call

More efficient: keep the same original CNF everywhere, and only remember the list M of literals = unit clauses that are chosen or derived

An original clause C yields a contradiction after a number of unit resolution steps in the DPLL process if and only if it only consists of negations of literals occurring in this list

Notation: $M \models \neg C$

Remember: this means that C conflicts with M

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Now we will reformulate the DPLL process building and modifying a list M of literals and checking for $M \models \neg C$, rather than doing recursion and unit resolution

In this way we mimick the original DPLL program = traversal through the DPLL tree

At every moment the CNF in the original DPLL program corresponds to the combination of

- the literals in M, and
- the original CNF from which all negations of literals from M have been stripped away

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During the process M is a list of literals, where every literal in M may or may not be marked to be a **decision** literal, notation ℓ^d

Idea:

these decision literals originate from a choice in the DPLL algorithm, the other literals in M are derived by unit resolution = unit propagation, or are the negation of a literal that was a decision literal before

Why?

For mimicking backtracking it is essential to recognize these particular decision literals:

> backtracking = go back to last chosen decision literal and continue with its negation

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Starting by M being empty, there are four rules that together mimick the DPLL process:

- **UnitPropagate**: mimicks the generation of a new unit clause
- **Decide**: mimicks the choice p in the DPLL process, only if no unitpropagate is possible
- **Backtrack**: mimicks backtracking to the negation of the last decision in case a branch is unsatisfiable
- **Fail**: mimicks the end of the DPLL process if every branch, and hence the CNF, is unsatisfiable

We say that ℓ is **undefined** in M if neither ℓ nor $\neg\ell$ occurs in M

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The four rules

UnitPropagate:

$$M \Longrightarrow M\ell$$

if ℓ is undefined in M and the CNF contains a clause $C \vee \ell$ satisfying $M \models \neg C$

Decide:

$$M \Longrightarrow M\ell^d$$

if ℓ is undefined in M

Backtrack:

$$M\ell^d N \Longrightarrow M\neg\ell$$

if $M\ell^d N \models \neg C$ for a clause C in the CNF and N contains no decision literals

Fail:

$$M \Longrightarrow \text{fail}$$

if $M \models \neg C$ for a clause C in the CNF and M contains no decision literals

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Observations:

Start with M being empty and apply the rules as long as possible (or stopping when all clauses contain a literal from M) always ends in either

- fail, proving that the CNF is unsatisfiable since the derivation of Fail can be interpreted as a case analysis yielding a contradiction in all cases, or
- a list M yielding a satisfying assignment

In case **UnitPropagate** always gets priority, and **Decide** is only used for ℓ being positive, then such derivations mimick original DPLL computations

However, this derivational framework is much more suitable for describing optimizations as they are used in modern powerful SAT solvers (SATzilla, Picosat, Rsat, Minisat, March, Yices), and we will present

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Example

Let the CNF consist of the four clauses

1. $p \vee q$
2. $p \vee \neg q$
3. $\neg p \vee r$
4. $\neg p \vee \neg r$

We get the following derivation proving unsatisfiability:

$\emptyset$	$\Longrightarrow$	Decide
p^d	$\Longrightarrow$	UnitPropagate, clause 3
$p^d\ r$	$\Longrightarrow$	Backtrack, clause 4
$\neg p$	$\Longrightarrow$	UnitPropagate, clause 1
$\neg p\ q$	$\Longrightarrow$	Fail, clause 2
fail		

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Optimization: Backjump

Example:

Let the CNF consist of the four clauses

1. $\neg p \vee q$
2. $\neg r \vee s$
3. $\neg t \vee \neg u$
4. $\neg q \vee \neg t \vee u$

$\emptyset$	$\Longrightarrow$	Decide
p^d	$\Longrightarrow$	UnitPropagate, clause 1
$p^d\ q$	$\Longrightarrow$	Decide
$p^d\ q\ r^d$	$\Longrightarrow$	UnitPropagate, clause 2
$p^d\ q\ r^d\ s$	$\Longrightarrow$	Decide
$p^d\ q\ r^d\ s\ t^d$	$\Longrightarrow$	UnitPropagate, clause 4
$p^d\ q\ r^d\ s\ t^d\ u$	$\Longrightarrow$	Backtrack, clause 3 ????
$p^d\ q\ r^d\ s\ \neg t$	$\Longrightarrow$	$\cdots$

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For the found contradiction between t^d, u, and clause 3, the decision r^d and the derivation of s does not play a role

If instead of r^d the decision t^d was made directly, we would have had a better backtrack step to $p^d\ q\ \neg t$

However, in large CNFs it is hard to know in advance what will be such a good choice

So we keep the sequence of decisions as it is, but allow this step to $p^d\ q\ \neg t$

Since it does not negate the last decision as in Backtrack, but negates a decision of several steps back, this is called **Backjump**

In order to use this idea in general at every UnitPropagate step we store the corresponding clause and at every contradiction found we investigate which generated literals and corresponding clauses played a role

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A clause conflicting these relevant literals can be obtained by resolution from the original CNF: the **backjump** clause

In our example u was derived using clause 4, and a contradiction was found using clause 3, yielding by resolution the backjump clause $\neg q \vee \neg t$

This clause is of the shape $C' \vee \ell'$, where C' conflicts with the list of literals and ℓ' typically is the negation of the last decision literal

More precisely, the new rule is

Backjump:

$$M\ell^d N \Longrightarrow M\ell'$$

if $M\ell^d N \models \neg C$ for a clause C in the CNF and there is a clause $C' \vee \ell'$ derivable from the CNF such that $M \models \neg C'$ and ℓ' is undefined in M

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Note that this general formulation is a generalization of Backtrack; the improvement is obtained when M is as small as possible

Correctness of this backjump rule is by construction

Implementation not clear by general formulation: how to find the backjump clause $C' \vee \ell'$ derivable from the CNF?

In implementation the choice for this backjump clause is guided by the decisions and unit propagation steps leading to the contradiction causing the backtrack step

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More optimizations: **Learn** and **Forget**

The idea is to modify the CNF during the process of SAT solving:

- **Learn:** new clauses that follow from the original CNF may be added
- **Forget:** clauses that follow from the other clauses may be removed

In modern SAT solvers all backjump clauses that are learned are directly added to the CNF: they may be helpful later on

Subsumption: if the formula contains two clauses $C \subseteq C'$, then one may forget = remove C'

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For all variants the main property remains:

Start with M being empty and apply the rules as long as possible always ends in either

- fail, proving that the CNF is unsatisfiable, or
- a list M yielding a satisfying assignment

Choices made in this process (e.g., which literal to choose for Decide) only influence efficiency of obtaining the result, **not** the validity of the result

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Sometimes a **Restart** $M \Longrightarrow \emptyset$ after learning some clauses is fruitful

Looks counterintuitive, but sometimes it may yield optimal profit of the clauses that have been learned

Combined technology is often called **Conflict-Driven Clause Learning**

Good heuristics for choosing literals for Decide remain crucial

A basic heuristic takes a literal that occurs most often in the relevant clauses

Learning clauses and then restart may cause better choice for decision literals

Lively area of research: every one/two year(s) there is a SAT competition, and at every competition strong improvements show up

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Standard format for SAT competition: **dimacs**

Boolean variables are numbered from 1 to n

For k clauses the file has to start by

`p cnf` n k

(often optional)
followed by k lines each containing a clause

- variable i is denoted by i
- the negation of variable i is denoted by $-i$
- these literals are separated by spaces
- every clause is ended by 0

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Example:

Calling

```
yices-sat -m test.d
```

on the file `test.d` containing

```
p cnf 3 5
1 2 3 0
1 -3 0
-1 2 0
-1 -3 0
2 -3 0
```

yields

```
sat
-1 2 -3 0
```

indeed showing that a satisfying assignment is obtained by making 2 true and 1 and 3 false

---

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As in all versions of `Yices 2` the flag `-m` means 'model': show the model = satisfying assignment in case of satisfiablility

In the older version `Yices 1` it was called `-e`: 'evidence'

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Until now we only considered CNFs, and we are not yet able to apply resolution to arbitrary propositions

We want to be able to apply resolution to establish satisfiability of arbitrary propositions by first transforming the proposition to CNF

Straightforward approach:

Transform the proposition to a logically equivalent CNF

This is always possible:

every 0 in the truth table yields a clause

proposition $\equiv$ conjunction of these clauses

Often it can be done more efficient

Unfortunately:

It often happens that every CNF logically equivalent to a given proposition is unacceptably big

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Example:

$$A : (\cdots((p_1 \leftrightarrow p_2) \leftrightarrow p_3) \cdots \leftrightarrow p_n)$$

This formula yields true if and only if an even number of p_i's has the value *false*

Claim:

Let B be a CNF satisfying $A \equiv B$

Then every clause C in B contains exactly n literals

Proof:

Assume not, then some p_i does not occur in a clause C of B

Then you can give values to the remaining variables such that C is not true, hence neither B, independent of the value for p_i

Then you can give a value to p_i such that A yields *true*

Contradiction to $A \equiv B$ (end of proof of claim)

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The truth table of A contains exactly 2^{n-1} zeroes: half of all entries

Every clause containing n literals yields exactly one zero in the truth table

According to the claim all clauses of B are of this shape

Hence B consists of 2^{n-1} clauses

Conclusion:

Every CNF B equivalent to A has size exponential in the size of A

$\Longrightarrow$ unacceptably large

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Hence we are looking for a way to transform any arbitrary propositional formula A to a CNF B such that

- A is satisfiable if and only if B is satisfiable
- the size of B is linear (or at least polynomial) in the size of A

Note that we weaken the restriction that A and B are equivalent

Such a construction is possible if we allow that B contains a number of fresh variables

---

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For every formule D on at most 3 variables there is a CNF $cnf(D)$ such that $cnf(D) \equiv D$ and $cnf(D)$ contains at most 4 clauses:

$$\begin{aligned} cnf(p \leftrightarrow \neg q) &= (p \vee q) \\ &\quad \wedge(\neg p \vee \neg q) \end{aligned}$$

$$\begin{aligned} cnf(p \leftrightarrow (q \wedge r)) &= (p \vee \neg q \vee \neg r) \\ &\quad \wedge(\neg p \vee q) \\ &\quad \wedge(\neg p \vee r) \end{aligned}$$

$$\begin{aligned} cnf(p \leftrightarrow (q \vee r)) &= (\neg p \vee q \vee r) \\ &\quad \wedge(p \vee \neg q) \\ &\quad \wedge(p \vee \neg r) \end{aligned}$$

$$\begin{aligned} cnf(p \leftrightarrow (q \leftrightarrow r)) &= (p \vee q \vee r) \\ &\quad \wedge(p \vee \neg q \vee \neg r) \\ &\quad \wedge(\neg p \vee q \vee \neg r) \\ &\quad \wedge(\neg p \vee \neg q \vee r) \end{aligned}$$

99

Introduce a new variable for every non-literal subformula of A (including A itself), the **name** of the subformula

For a subformula D of A we define

- $n_D = D$ if D is a literal
- $n_D =$ the name of D, otherwise

The CNF $T(A)$, the **Tseitin transformation** of A, is defined to be the CNF consisting of the clauses:

- n_A
- the clauses of $cnf(q \leftrightarrow \neg n_D)$ for every non-literal subformula of the shape $\neg D$ having name q
- the clauses of $cnf(q \leftrightarrow (n_D \diamond n_E))$ for every subformula of the shape $D \diamond E$ having name q, for $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

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Example: $A : \underbrace{(\underbrace{\neg s \wedge p}_{B}) \leftrightarrow (\underbrace{(\underbrace{q \rightarrow r}_{D}) \vee \neg p}_{C})}$

yields $T(A)$ consisting of the clauses A and

$$\left.\begin{array}{l} A \vee B \vee C \\ A \vee \neg B \vee \neg C \\ \neg A \vee \neg B \vee C \\ \neg A \vee B \vee \neg C \end{array}\right\} cnf(A \leftrightarrow (B \leftrightarrow C))$$

$$\left.\begin{array}{l} B \vee s \vee \neg p \\ \neg B \vee \neg s \\ \neg B \vee p \end{array}\right\} cnf(B \leftrightarrow (\neg s \wedge p))$$

$$\left.\begin{array}{l} \neg C \vee D \vee \neg p \\ C \vee \neg D \\ C \vee p \end{array}\right\} cnf(C \leftrightarrow (D \vee \neg p))$$

$$\left.\begin{array}{l} \neg D \vee r \vee \neg q \\ D \vee \neg r \\ D \vee q \end{array}\right\} cnf(D \leftrightarrow (q \rightarrow r))$$

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Theorem:

For every propositional formula A we have:

A is satisfiable if and only if $T(A)$ is satisfiable

Proof sketch:

- a satisfying assignment for $T(A)$ restricting to the variables from A yields a satisfying assignment for A
- a satisfying assignment for A is extended to a satisfying assignment for $T(A)$ by giving n_D the value of D obtained from the satisfying assignment for A

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Summary of Tseitin transformation:

For every propositional formula A we have:

- A is satisfiable if and only if $T(A)$ is satisfiable
- $T(A)$ contains two types of variables: variables occurring in A and variables representing names of subformulas of A
- The size of $T(A)$ is linear in the size of A
- Every clause in $T(A)$ contains at most 3 literals: $T(A)$ is a **3-CNF**
- A fruitful approach to investigate satisfiability of A is applying a modern CNF based SAT solver on $T(A)$

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There is no need to use a separate tool for transforming an arbitrary propositional formula A to $T(A)$ in dimacs format:

several modern SAT solvers like `Yices` and `Z3` also accept SMT format (satisfiability modulo theories) of which the most basic instance coincides with arbitrary propositional formulas

Internally then first the Tseitin transformation is applied, and then the same approach is followed as by entering dimacs format

Call

`yices-smt -m test.smt` or `z3 test.smt`

where `test.smt` contains the formula

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```
(benchmark test.smt
:logic QF_UF
:extrapreds ((A) (B) (C) (D))
:formula (and
(iff A (and D B))
(implies C B)
(not (or A B (not D)))
(or (and (not A) C) D)
))
```

yields

```
sat
(= A false)
(= B false)
(= D true)
(= C false)
```

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Note that `and` and `or` may have any number of arguments

The tool `yices-smt` is the version of `Yices 2` for the SMT-LIB 1.2 format, a standard format that is accepted by all SMT tools that participate in competitions

It requires the specification of a logic

The simplest logic is `:logic QF_UF`: quantifier free and uninterpreted functions, so not using integers or reals

In the older version `Yices 1` and in `Z3` the logic may be left implicit

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Same example in newer standard SMT-LIB 2 format:

call `z3 -smt2 test` for the file `test` containing

```
(declare-const A Bool)
(declare-const B Bool)
(declare-const C Bool)
(declare-const D Bool)
(assert
(and
(iff A (and D B))
(implies C B)
(not (or A B (not D)))
(or (and (not A) C) D)
))
(check-sat)
(get-model)
```

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Conclusions on resolution for proposition logic:

- Technique to establish satisfiability of CNF
- Tseitin transformation efficiently transforms every propositional formula to 3-CNF, maintaining satisfiability

  In this way resolution is applicable for every propositional formula
- Both DP and DPLL are complete methods for SAT based on resolution, both having freedom of choice

  Often DPLL is more efficient than DP

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Conclusions (continued)

- Modern CNF based SAT solvers, like
  - Minisat
  - Lingeling
  - Yices
  - Z3

essentially use DPLL, extended by Backjump, Learn, Forget and Restart:

Conflict-Driven Clause Learning

- Resolution hardly recognizable in actual algorithms
- Extremely powerful approach for a wide range of problems that seem unrelated to SAT solving

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Amazing example:

Prove termination of the system of three rules

$$aa \to bc, \quad bb \to ac, \quad cc \to ab$$

Solution: interpret a, b, c by matrices over natural numbers in such a way that by doing rewrite steps a particular entry in matrices always strictly decreases

Since this entry is a natural number, this cannot go on forever, proving termination

Restricting to binary encoded numbers of fixed size, and fixing matrix dimension (both ≈ 4) this was encoded in a SAT problem, and the obtained satisfying assignment was transformed to a formal proof of the above shape

Until now all known proofs of this problem are variants of this idea, all found by SAT solving

No human intuition available

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Extension of SAT solving

We have seen how several problems involving e.g. arithmetic or program correctness can be encoded as a SAT problem

Typically, a program is written in which an instance of a problem is entered, and a corresponding SAT problem is produced, after which a plain SAT solver is applied to solve the problem

Apart from SAT solving there are several other formats of **constraint problems** where any solution or an optimal solution has to be found

For instance: **linear optimization** given n real valued variables $x_1, \ldots, x_n$, find the highest (or lowest) value of a linear combination $\sum_{i=1}^{n} a_i x_i$ satisfying a given number of constraints all of the shape $\sum_{i=1}^{n} b_i x_i \leq c$

If the variables are integer valued, this is called **integer optimization**

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For these problems **linear optimization** and **integer optimization** extremely powerful techniques are available, unrelated to SAT solving

In particular these techniques can be used to establish whether a conjunction of inequalities has a solution, and if so, find one, rather than finding an optimal solution

Here we will present the powerful **Simplex method** for linear optimization (following chapter 29 of the algorithms book by Cormen, Leiserson, Rivest and Stein)

This can be combined with techniques for SAT solving, to solve propositional formulas over linear inequalities:

Satisfiability Modulo Theories (SMT)

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The Simplex method

Among all real values $x_1, \ldots, x_n \geq 0$ we want to find the **maximal** value of a linear **goal function**

$$f(x_1, \ldots, x_n) = v + c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

satisfying k linear constraints

$$a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \leq b_i$$

for $i = 1, 2, \ldots, k$

Here v, a_{ij}, c_i and b_i are arbitrary given real values, satisfying $b_i \geq 0$ for $i = 1, 2, \ldots, k$

Choosing $x_i = 0$ for all i satisfies all constraints and yields the value v for the goal function, but probably this is not the maximal value

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More general linear optimization problems can be transformed to this format, for instance

- in an inequality '$\geq$' multiply both sides by -1:

  $x_1 - 2x_2 + 3x_3 \geq -5 \quad \equiv \quad -x_1 + 2x_2 - 3x_3 \leq 5$

- if one wants to minimize, multiply goal function by -1
- if a variable x runs over all reals (positive and negative), replace it by $x_1 - x_2$ for fresh variables x_1, x_2 satisfying $x_i \geq 0$ for $i = 1, 2$
- later we will see how to deal with the situation if not $b_i \geq 0$

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Slack form

For every linear inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

a fresh variable $y_i \geq 0$ is introduced

The linear inequality is replaced by the equality

$$y_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n$$

Together with $y_i \geq 0$ this is equivalent to the original inequality

This format with equalities is called the **slack form**

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Some terminology on a slack form with equations

$$y_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n$$

for $i = 1, 2, \ldots, k$

- The solution $y_i = b_i$ for $i = 1, 2, \ldots, k$ and $x_j = 0$ for $j = 1, 2, \ldots, n$ is called the **basic solution**
- The variables y_i for $i = 1, 2, \ldots, k$ are called **basic**
- The variables x_j for $j = 1, 2, \ldots, n$ are called **non-basic**
- The simplex algorithm consists of a repetition of **pivot**s, in which
- a pivot chooses a basic variable and a non-basic variable, swaps the roles of these two variables, and brings the result in a set of equations in slack form that is equivalent to the original one

We will illustrate this on an example

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Maximize $z = 3 + x_1 + x_3$ satisfying the constraints

$$\begin{array}{rcl} -x_1 + x_2 - x_3 & \leq & 2 \\ x_1 + x_3 & \leq & 3 \\ 2x_1 - x_2 & \leq & 4 \end{array}$$

Slack form:

$$\begin{array}{lcrrr} y_1 & = & 2 & +x_1 & -x_2 & +x_3 \\ y_2 & = & 3 & -x_1 & & -x_3 \\ y_3 & = & 4 & -2x_1 & +x_2 & \end{array}$$

Goal: maximize $z = 3 + x_1 + x_3$

Observation: if we increase x_1, and the other non-basic variables remain 0, then the goal function z will increase

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We want to increase x_1 as much as possible, keeping $x_2 = x_3 = 0$, while in the equations

$$\begin{array}{lcrrrr} y_1 & = & 2 & +x_1 & -x_2 & +x_3 \\ y_2 & = & 3 & -x_1 & & -x_3 \\ y_3 & = & 4 & -2x_1 & +x_2 & \end{array}$$

all variables should be ≥ 0

$y_1 = 2 + x_1 \geq 0$: OK if x_1 increases

$y_2 = 3 - x_1 \geq 0$: only OK if $x_1 \leq 3$

$y_3 = 4 - 2x_1 \geq 0$: only OK if $x_1 \leq 2$

So $y_i \geq 0$ only holds for all i if $x_1 \leq 2$

The highest allowed value for x_1 is 2, and then y_3 will get the value 0

pivot: swap x_1 and y_3: the basic variable y_3 will become non-basic, and the non-basic variable x_1 will become basic

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Swap x_1 and y_3 in

$$\begin{array}{lcrrrr} y_1 & = & 2 & +x_1 & -x_2 & +x_3 \\ y_2 & = & 3 & -x_1 & & -x_3 \\ y_3 & = & 4 & -2x_1 & +x_2 & \end{array}$$

In the equation $y_3 = 4 - 2x_1 + x_2$ move x_1 to the left and y_3 to the right, yielding the equivalent equation

$$x_1 = 2 + \frac{1}{2}x_2 - \frac{1}{2}y_3$$

Next replace every x_1 by $2 + \frac{1}{2}x_2 - \frac{1}{2}y_3$ in the equations for z, y_1, y_2, yielding the following slack form:

maximize $z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$ satisfying

$$\begin{array}{lcrrrr} x_1 & = & 2 & +\frac{1}{2}x_2 & & -\frac{1}{2}y_3 \\ y_1 & = & 4 & -\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}y_3 \\ y_2 & = & 1 & -\frac{1}{2}x_2 & -x_3 & +\frac{1}{2}y_3 \end{array}$$

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This is the end of the first pivot

Note that

- all equations are replaced by equivalent equations, so the new optimization problem is equivalent to the original one
- Now the basic variables are x_1, y_1, y_2 and the non-basic variables are x_2, x_3, y_3
- By construction again we have a slack form with a basic solution in which all non-basic variables are 0, and all basic variables are ≥ 0
- In this new basic solution the goal function

  $$z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$$

  has the value 5, which is an improvement with respect to the original value 3

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In this goal function

$$z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$$

the non-basic variable x_2 has a positive factor, so increasing x_2 would cause a further increase: the starting point for the next pivot

Similar as in the first pivot we try to increase this non-basic variable x_2, while the other non-basic variables remain 0

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$$\begin{array}{lcrcrcr} x_1 & = & 2 & +\frac{1}{2}x_2 & & & -\frac{1}{2}y_3 \\ y_1 & = & 4 & -\frac{1}{2}x_2 & +x_3 & & -\frac{1}{2}y_3 \\ y_2 & = & 1 & -\frac{1}{2}x_2 & -x_3 & & +\frac{1}{2}y_3 \end{array}$$

yields

$x_1 = 2 + \frac{1}{2}x_2 \geq 0$, OK if x_2 increases

$y_1 = 4 - \frac{1}{2}x_2 \geq 0$, only OK if $x_2 \leq 8$

$y_2 = 1 - \frac{1}{2}x_2 \geq 0$, only OK if $x_2 \leq 2$

So the maximal allowed value for x_2 is 2, in which case y_2 will get the value 0

So we will do a pivot swapping x_2 and y_2:
$y_2 = 1 - \frac{1}{2}x_2 - x_3 + \frac{1}{2}y_3$ yields
$x_2 = 2 - 2x_3 - 2y_2 + y_3$, so in the goal function and in the other equations we replace x_2 by $2 - 2x_3 - 2y_2 + y_3$, yielding

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Maximize $z = 6 - y_2$
satisfying

$$\begin{array}{lcrcrcr} x_1 & = & 3 & -x_3 & -y_2 & & \\ x_2 & = & 2 & -2x_3 & -2y_2 & +y_3 & \\ y_1 & = & 3 & +2x_3 & +y_2 & -y_3 & \end{array}$$

Still this optimization problem is equivalent to the original one

Observation: since in

$$z = 6 - y_2$$

y_2 should be ≥ 0, the value of z will always be ≤ 6

On the other hand, in the basic solution with $x_3 = y_2 = y_3 = 0$ and $x_1 = 3$, $x_2 = 2$ and $y_1 = 3$ we obtain $z = 6$, so this basic solution yields the maximal value for z

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Summarizing, starting from a linear optimization problem having a basic solution

- Bring the problem in slack form, that is, maximize $z = v + \sum_{j=1}^{n} c_j x_j$ under a set of constraints of the shape

$$y_i = b_i + \sum_{j=1}^{n} a_{ij} x_j$$

with $b_i \geq 0$, for $i = 1, \ldots, k$

- As long there exists j such that $c_j > 0$ do a **pivot**, that is
  - find the highest value for x_j for which $b_i + a_{ij}x_j \geq 0$ for all i, and $b_i + a_{ij}x_j = 0$ for one particular i
  - swap x_j and y_i and bring the result in slack form

At the end $c_j \leq 0$ for all j, from which can be concluded that the basic solution of this last slack form yields the maximal value for z

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General remarks

- Optimization problems may be unbounded, for instance, there is no maximal value for $3 + x$ satisfying the constraint $x \geq 0$; in this mechanism this will be encountered if all $a_{ij} \geq 0$ for all i for the chosen j, so no equation yields an upper bound on x_j

- A pivot only requires some basic linear algebra of complexity $O(kn)$, still feasible for high values of k, n

- In case $c_j > 0$ for more than one value of j, then the procedure is non-deterministic

- If one always chooses the smallest j with $c_j > 0$, the repetition of pivots can be proved to terminate

- In worst case the number of pivots may be exponential, but in practice this number of pivots is limited, and the simplex method is very efficient

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The method described until now only finds an optimal value when starting by a basic solution

However, in our intended application to SMT the situation is opposite: one is not interested in an optimal solution, only in the question whether there exists a solution

A set of constraints (or a problem) is called **feasible** if it admits a solution

Fortunately, the simplex method presented so far for finding an optimal solution of a feasible set of inequalities, can be applied for deciding feasibility of any given set of inequalities, as we will see now

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For a set of linear inequalities

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, \ldots, k$, and $x_j \geq 0$ for $j = 1, \ldots, n$, we introduce a fresh variable $z \geq 0$, and extend the set of inequalities to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - z \leq b_i$$

for $i = 1, \ldots, k$, and $x_j \geq 0$ for $j = 1, \ldots, n$

This extended problem is always feasible, even if b_i may be negative: choose z very large

---

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original: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$
extended: $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - z \leq b_i$

Theorem

The original problem is feasible if and only if in the extended problem the maximal value of $-z$ is 0

Proof

If there is a solution of the extended problem with $-z = 0$, then it is a solution of the original problem, showing feasibility

Conversely, if the original problem is feasible, then the extended one admits a solution with $-z = 0$

Since we required $z \geq 0$, this is the maximal possible value of $-z$

End of Proof

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In case $b_i \geq 0$ for all i, the original problem is trivially feasible, so assume $b_i < 0$ for some i

Extended problem in slack form:

maximize $-z$ satisfying

$$y_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n + z$$

for $i = 1, \ldots, k$

Since $b_i < 0$ for some i this does not have a basic solution

As the first pivot swap the non-basic variable z with the basic variable y_i for i for which b_i is the most negative

Then after this pivot the resulting equivalent problem in slack form has a basic solution, as is easily proved

Proceed as before; if at the end the value for $-z$ is 0, then the original problem is feasible, otherwise it is not

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Example

Find values $x, y \geq 0$ satisfying

$$x + 3y \geq 12 \ \wedge \ x + y \leq 10 \ \wedge \ x - y \geq 7$$

First step: make '$\leq$':

$$\begin{array}{rrcr} -x & -3y & \leq & -12 \\ x & +y & \leq & 10 \\ -x & +y & \leq & -7 \end{array}$$

Introduce z for maximizing $-z$:

$$\begin{array}{rrrcr} -x & -3y & -z & \leq & -12 \\ x & +y & -z & \leq & 10 \\ -x & +y & -z & \leq & -7 \end{array}$$

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Slack form:

$$\begin{array}{rcrrrr} y_1 & = & -12 & +x & +3y & +z \\ y_2 & = & 10 & -x & -y & +z \\ y_3 & = & -7 & +x & -y & +z \end{array}$$

Indeed $x = y = z = 0$ does not yield a basic solution, since then $y_1 = -12$ and $y_3 = -7$ do not satisfy $y_i \geq 0$

Most negative b_i is $b_1 = -12$, so do pivot on z and y_1

replace every z by $z = 12 - x - 3y + y_1$:

Maximize $-12 + x + 3y - y_1$ satisfying

$$\begin{array}{rcrrrr} z & = & 12 & -x & -3y & +y_1 \\ y_2 & = & 22 & -2x & -4y & +y_1 \\ y_3 & = & 5 & & -4y & +y_1 \end{array}$$

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$$\begin{array}{rcrrrr} z & = & 12 & -x & -3y & +y_1 \\ y_2 & = & 22 & -2x & -4y & +y_1 \\ y_3 & = & 5 & & -4y & +y_1 \end{array}$$

Indeed now we have a basic solution $x = y = y_1 = 0$, $z = 12$, $y_2 = 22$ and $y_3 = 5$, all ≥ 0

This is not by accident, this is always the case

Proceed by simplex algorithm as before

Maximize $-12+x+3y-y_1$, so swap x (or y) with z, y_2 or y_3

z: $12 - x \geq 0$, so $x \leq 12$
y_2: $22 - 2x \geq 0$, so $x \leq 11$
y_3: no bound on x

So next pivot: swap x with y_2

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Replace x by $x = 11 - 2y + \frac{1}{2}y_1 - \frac{1}{2}y_2$ in

Maximize $-12 + x + 3y - y_1$ satisfying

$$\begin{array}{rcrrrr} z & = & 12 & -x & -3y & +y_1 \\ y_2 & = & 22 & -2x & -4y & +y_1 \\ y_3 & = & 5 & & -4y & +y_1 \end{array}$$

yields:
Maximize $-1 + y - \frac{1}{2}y_1 - \frac{1}{2}y_2$ satisfying

$$\begin{array}{rcrrrr} x & = & 11 & -2y & +\frac{1}{2}y_1 & -\frac{1}{2}y_2 \\ z & = & 1 & -y & +\frac{1}{2}y_1 & +\frac{1}{2}y_2 \\ y_3 & = & 5 & -4y & +y_1 & \end{array}$$

Next pivot: swap y and z

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Replace y by $y = 1 - z + \frac{1}{2}y_1 + \frac{1}{2}y_2$ yields:

Maximize $-z$ satisfying

$$\begin{array}{rcrrrr} x & = & 9 & \cdots z & \cdots y_1 & \cdots y_2 \\ y & = & 1 & -z & +\frac{1}{2}y_1 & +\frac{1}{2}y_2 \\ y_3 & = & 1 & \cdots z & \cdots y_1 & \cdots y_2 \end{array}$$

Since the maximization function $-z$ has no positive factors any more, the resulting basic solution

$$z = y_1 = y_2 = 0,\ x = 9,\ y = 1,\ y_3 = 1$$

yields the optimal value $-z = 0$

Since $-z = 0$, the original set of inequalities

$$x + 3y \geq 12 \ \wedge\ x + y \leq 10 \ \wedge\ x - y \geq 7$$

is satisfiable: we found the solution $x = 9$, $y = 1$

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Observations

- Doing this by hand this looks quite elaborative
- However, every step is completely systematic, and easy to implement very efficiently, even for thousands of variables
- Although worst case exponential, in practice this simplex algorithm is the most efficient way to decide whether a conjunction of linear inequalities is satisfiable or not
- The full simplex algorithm serves for deciding whether a solution exists, and if so, find an optimal one
- Linear optimization is also called **linear programming**
- There exists a more complicated **ellipsoid algorithm** for linear programming that is worst case polynomial, but in practice not better than the simplex algorithm

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In linear programming one looks for an optimal solution satisfying a **conjunction** of linear inequalities

In **Satisfiability Modulo Theories** (SMT) this is generalized: one looks for a solution satisfying any propositional formula in which the building blocks may be both boolean variables and linear inequalities

So in SMT we may also use negations and disjunctions

Example:
Can we fit

- a rectangle of width 3 and height 4, and
- a rectangle of width 5 and height 2

in a square of size 6×6, without overlap?

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Expressed in a formula:

Let (X_1, Y_1) be the coordinates of the lower left point of the rectangle of width 3 and height 4

Let (X_2, Y_2) be the coordinates of the lower left point of the rectangle of width 5 and height 2

Fit in square of size 6×6:

$$X_1 \geq 0 \wedge Y_1 \geq 0 \wedge X_2 \geq 0 \wedge Y_2 \geq 0 \wedge$$

$$X_1 \leq 3 \wedge Y_1 \leq 2 \wedge X_2 \leq 1 \wedge Y_2 \leq 4$$

So far only conjunctions, so fit in linear programming format

Still to be done: no overlap

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No overlap
=
either
rectangle 1 is left from rectangle 2: $X_1 + 3 \leq X_2$

or rectangle 1 is right from rectangle 2: $X_2 + 5 \leq X_1$

or rectangle 1 is below rectangle 2: $Y_1 + 4 \leq Y_2$

or rectangle 1 is above rectangle 2: $Y_2 + 2 \leq Y_1$

So the requirement is

$$(X_1 + 3 \leq X_2) \vee (X_2 + 5 \leq X_1) \vee$$

$$(Y_1 + 4 \leq Y_2) \vee (Y_2 + 2 \leq Y_1)$$

which is not allowed in linear programming, but is allowed in SMT

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The SMT tool **Yices** finds a solution on the input

```
(benchmark test.smt
:logic QF_UFLRA
:extrafuns
((X1 Real) (X2 Real)
(Y1 Real) (Y2 Real))
:formula
(and
(>= X1 0) (>= X2 0)
(>= Y1 0) (>= Y2 0)
(<= X1 3) (<= X2 1)
(<= Y1 2) (<= Y2 4)
(or (<= (+ X1 3) X2) (<= (+ X2 5) X1)

(<= (+ Y1 4) Y2) (<= (+ Y2 2) Y1))
))
```

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Here `:logic QF_UFLRA` is the logic for Linear Real Arithmetic; still quantifier free and over uninterpreted functions

Similar formulas can be made and solved for fitting 10 to 15 rectangular components in a big rectangle, as is useful in e.g. chip design

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How does this work?

The SAT solver is extended in such a way that it can deal with the inequalities as atoms, just like boolean variables

So we consider CNFs as before, but now the literals are linear inequalities or their negations, which are inequalities again

In the derivation rules Decide, UnitPropagate, Backtrack and Fail, the only access to the formula is checking whether

$$M \models \neg C$$

for both M being a list of literals and C being a clause

That is, we have to check whether for every literal ℓ in C, the conjunction of ℓ and all literals in M gives rise to a contradiction

Such a conjunction of literals is a conjunction of inequalities, for which the simplex algorithm is called to check whether it is contradictory

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Example

Consider the CNF of the following three clauses

$$\begin{array}{ll} (1) & x \geq y+1 \vee z \geq y+1 \\ (2) & y \geq z \\ (3) & z \geq x+1 \end{array}$$

$(y \geq z)$ (unit propagate on (2))

$(y \geq z)$ $(x \geq y+1)$ (unit propagate on (1) since $y \geq z \wedge z \geq y+1$ is unsatisfiable by simplex)

(fail on (3) since $y \geq z \wedge x \geq y+1 \wedge z \geq x+1$ is unsatisfiable by simplex)

This proves that the given CNF is unsatisfiable

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So SAT solvers are extended to SMT solvers to deal with establishing satisfiability of CNFs defined by

- A CNF is a conjunction of clauses
- A clause is a disjunction of literals
- A literal is a basic formula in some theory or its negation

Here typically a basic formula is a linear inequality, but it also works for other theories

For the theory there should be an efficient and incremental implementation to check whether the conjunction of a given set of literals is satisfiable or not

For linear inequalities the simplex method serves for this goal

By applying the Tseitin transformation this approach works for every propositional formula over basic formulas in such a theory

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Reals or integers?

The simplex method works on linear inequalities over real numbers (LP: linear programming)

Surprisingly, solving linear inequalities over integers (ILP: integer linear programming) is much harder, this is even NP-complete since satisfiability of any propositional CNF can be expressed as an ILP problem

Typically, a set of inequalities may have a solution in reals, but not in integers

Fortunately, often by adding properties of integers (like $x > y \rightarrow x \geq y + 1$) the resulting formula may be unsatisfiable over the reals (to be proved by simplex), by which also unsatisfiability of the ILP problem is proved quickly

Conversely, if a satisfying assignment is found by simplex for which the solution is integer, then also an ILP solution has been found

---

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This has been implemented in SMT solvers like `Yices` and `Z3`

The logic to be used is `:logic QF_UFLIA`

That is: Linear Integer Arithmetic (still quantifier free and over uninterpreted functions)

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Example:

Find positive integer values a, b, c, d such that

$2a > b + c$, $2b > c + d$, $2c > 3d$ and $3d > a + c$

In an SMT solver like `Yices` or `Z3` one can directly enter and solve

```
(benchmark test.smt
:logic QF_UFLIA
:extrafuns ((A Int) (B Int) (C Int)
(D Int))
:formula (and
(> (* 2 A) (+ B C))
(> (* 2 B) (+ C D))
(> (* 2 C) (* 3 D))
(> (* 3 D) (+ A C))
))
```

146

More precisely, if this formula is in file `test.smt`, then calling

`yices-smt -m test.smt` or `z3 test.smt`

yields the output

```
sat
(= A 30)
(= B 27)
(= C 32)
(= D 21)
```

147

Same example in newer SMT-LIB 2 format: call `z3 -smt2 test.smt` for file `test.smt` containing

```
(declare-const A Int)
(declare-const B Int)
(declare-const C Int)
(declare-const D Int)
(assert
(and
(> (* 2 A) (+ B C))
(> (* 2 B) (+ C D))
(> (* 2 C) (* 3 D))
(> (* 3 D) (+ A C))
))
```

```
(check-sat)
(get-model)
```

148

No bound on number size:

```
(benchmark test.smt
:logic QF_UFLIA
:extrafuns ((A Int) (B Int) (C Int))
:formula (and
(= A 98798798987987987987923423879)
(= B 763429999988888888887364578645 87)
(= (+ (* 87 A) (* 93 B)) (+ C C))
))
```

yields the output

```
sat
(= A 98798798987987987987923423879)
(= B 763429999988888888887364578645 87)
(= C 784769725592581081080371995964 2032)
```

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One can also use **functions**:

```
(benchmark test.smt
:logic QF_UFLIA
:extrafuns ((F Int Int))
:formula (and
(> (* 2 (F 1)) (+ (F 2) (F 3)))
(> (* 2 (F 2)) (+ (F 3) (F 4)))
(> (* 2 (F 3)) (* 3 (F 4)))
(> (* 3 (F 4)) (+ (F 1) (F 3)))
))
```

yielding

```
sat
(= (F 1) 30)
(= (F 2) 27)
(= (F 3) 32)
(= (F 4) 21)
```

150

Same example in SMT-LIB 2 format:

```
(declare-fun F (Int) Int)
(assert
(and
(> (* 2 (F 1)) (+ (F 2) (F 3)))
(> (* 2 (F 2)) (+ (F 3) (F 4)))
(> (* 2 (F 3)) (* 3 (F 4)))
(> (* 3 (F 4)) (+ (F 1) (F 3)))
))
(check-sat)
(get-model)
```

151

A convenient operation having boolean arguments that may result in a number is **if-then-else**, abbreviated to `ite`

It has always three arguments:

- the first is the **condition**, which is boolean,
- the second is the result in case this condition is true, and
- the third is the result in case this condition is false

Example:
`(ite (< a b) 13 (* 3 a))`
yields the value 13 in case $a < b$, otherwise it yields $3a$

Example:
`(+ (ite a 1 0) (ite b 1 0) (ite c 1 0))`
yields the number of variables that are true among the boolean variables a, b, c

---

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Some SMT solvers allow **quantifications**
`forall` and `exists`, but typically perform quite weak

As long as quantifications are over a finite domain they can be expanded by `and` and `or`

For instance, $\forall i = 1 \cdots 8 : P(i)$ is expressed by
`(and (P 1) (P 2) (P 3) (P 4)`
`(P 5) (P 6) (P 7) (P 8))`

Typically, formulas are not edited by hand, but are generated by a script, and for solvers it is no problem if the formulas get large

---

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Unique representation
for boolean functions

In "normal" propositional notation equivalent formulas (even in CNF) may appear quite different:

$$(p \vee q) \wedge (q \vee r) \wedge (p \vee \neg q)$$

$$\equiv$$

$$p \wedge (\neg p \vee r \vee q) \wedge (p \vee \neg q)$$

We look for a way to describe boolean functions, in particular given by propositional formulas, such that:

- it is a **unique representation** for boolean functions: equivalent formulas yield the same representation
- for many formulas this representation is efficient

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Why not for **all** formulas?

On n variables a truth table consists of 2^n lines

Hence on n variables there are 2^{2^n} distinct boolean functions

Indeed, there are 2^{64} distinct boolean functions on six variables

If all of these 2^{2^n} distinct boolean functions should have a distinct representation, then **on average** at least 2^n bits are needed for that

Hence for every representation it holds that if some boolean functions have a representation of much less than 2^n bits, then for many others 2^n bits or more are required

This information theoretic argument shows that it is **unavoidable** that most of the boolean functions have **untractable** representation

---

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A computable unique representation immediately implies a method for SAT:

For any formula compute its unique representation

If this representation is equal to the representation of *false*, then the formula is unsatisfiable

Otherwise, the formula is satisfiable

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Unique representation due to Herbrand, 1929

Every boolean function can uniquely be expressed as an exclusive or ($\oplus$) of conjunctions of variables, in which both $\oplus$ and $\wedge$ run over sets, i.e.,

- commutative, associative
- double occurrence is not allowed
- *true* is conjunction over empty set
- *false* is $\oplus$ over empty set

In this way negation is not required as a building block:

$$\neg x \equiv x \oplus \mathit{true}$$

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The unique representation of a propositional

formula can be computed by the rules

$$\begin{array}{rcl} x \leftrightarrow y & \rightarrow & \neg(x \oplus y) \\ x \rightarrow y & \rightarrow & (\neg x) \vee y \\ x \vee y & \rightarrow & (x \wedge y) \oplus x \oplus y \\ \neg x & \rightarrow & x \oplus \mathit{true} \\ x \oplus \mathit{false} & \rightarrow & x \\ x \oplus x & \rightarrow & \mathit{false} \\ x \wedge \mathit{false} & \rightarrow & \mathit{false} \\ x \wedge \mathit{true} & \rightarrow & x \\ x \wedge x & \rightarrow & x \\ x \wedge (y \oplus z) & \rightarrow & (x \wedge y) \oplus (x \wedge z) \end{array}$$

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Example:

$\neg a \vee b$ and $\neg(a \wedge \neg b)$ are equivalent since they yield the same representation $a \wedge b \oplus a \oplus \mathit{true}$:

$$\begin{array}{rcl} \neg a \vee b & \rightarrow & (a \oplus \mathit{true}) \vee b \\ & \rightarrow & ((a \oplus \mathit{true}) \wedge b) \oplus a \oplus \mathit{true} \oplus b \\ & \rightarrow & (a \wedge b) \oplus (\mathit{true} \wedge b) \oplus a \oplus \mathit{true} \oplus b \\ & \rightarrow & (a \wedge b) \oplus b \oplus a \oplus \mathit{true} \oplus b \\ & \rightarrow & (a \wedge b) \oplus a \oplus \mathit{true} \oplus \mathit{false} \\ & \rightarrow & (a \wedge b) \oplus a \oplus \mathit{true} \end{array}$$

$$\begin{array}{rcl} \neg(a \wedge \neg b) & \rightarrow & \neg(a \wedge (b \oplus \mathit{true})) \\ & \rightarrow & \neg((a \wedge b) \oplus (a \wedge \mathit{true})) \\ & \rightarrow & \neg((a \wedge b) \oplus a) \\ & \rightarrow & (a \wedge b) \oplus a \oplus \mathit{true} \end{array}$$

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Although this Herbrand representation is a unique representation it is not of practical use since it is not acceptably efficient for larger formulas

Before introducing B(inary) D(ecision) D(iagrams) as a more efficient unique representation, first we present an important application, to get a feeling for the kind of operations that are required to be efficient

We express the **Reachability problem**

> Given a set of initial states, a transition relation and a set of final states, establish whether any of these final states is reachable from a given initial state

in propositional logic and show how to treat it by manipulation of unique representation

---

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Example: Apply the following statements in any order:

1. if D and E then $D, E := \mathit{false}, \mathit{false}$
2. if F and I then $F, I := \mathit{false}, \mathit{false}$
3. if A and E then $A, E := \mathit{false}, \mathit{false}$
4. if C and D then $C, D := \mathit{false}, \mathit{false}$
5. if D and G then $D, G := \mathit{false}, \mathit{false}$
6. if B and H then $B, H := \mathit{false}, \mathit{false}$
7. if B and I then $B, I := \mathit{false}, \mathit{false}$
8. if A and G then $A, G := \mathit{false}, \mathit{false}$
9. if D and H then $D, H := \mathit{false}, \mathit{false}$
10. if B and C then $B, C := \mathit{false}, \mathit{false}$
11. if B and J then $B, J := \mathit{false}, \mathit{false}$
12. if F and J then $F, J := \mathit{false}, \mathit{false}$

Prove that from the initial state in which all variables have value *true* never the final state can be reached in which all variables have value *false*

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Each of the ingredients should be represented by a formula

For the set I of initial states find a formula Φ_I such that Φ_I yields *true* on a state if and only if it is in I

More precisely, a state is a map π from variables to $\{\mathit{false}, \mathit{true}\}$ and it should hold that $\pi \in I$ if and only if evaluating Φ_I in π yields *true*

For example, in our example we may choose $\Phi_I = A \wedge B \wedge \cdots \wedge J$ expressing the single state in which all variables are true

Similar for the set F of final states, in our example $\Phi_F = \neg A \wedge \neg B \wedge \cdots \wedge \neg J$ expressing the single state in which all variables are false

---

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Describing the transition relation is slightly more involved

A transition relation is not a set of states, but a set of state transitions, where a state transition is a pair of states

In representing this by a formula two kinds of variables occur: the starting point of such a transition and the end point of the transition

Distinguish these kinds by a suffix:

- for the start states the variable names are followed by the symbol 0
- for the end states the variable names are followed by the symbol 1

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A formula Φ_T in this double set of variables describes the following set of transitions:

> there is a transition from π_0 to π_1 if and only if the formula Φ_T yields *true* if for every $i0$ the value $\pi_0(i)$ is evaluated, and for every $i1$ the value $\pi_1(i)$ is evaluated

As an example, consider

$$\mathsf{if}\ \ a\ \ \mathsf{then}\ \ b := c$$

over the four variables a, b, c, d

The transition relation corresponding to this program fragment can be represented in this way by

$$(a0 \leftrightarrow a1)\ \wedge\ (c0 \leftrightarrow c1)\ \wedge\ (d0 \leftrightarrow d1) \wedge$$

$$(a0 \rightarrow (b1 \leftrightarrow c0))\ \wedge\ (\neg a0 \rightarrow (b0 \leftrightarrow b1))$$

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Once we have expressed the initial states and the transition relation in this way we want to compute

$$B_n = \text{set of states reachable in} \leq n \text{ steps from } I$$

for $n = 0, 1, \ldots,$ until $B_n = B_{n-1}$ or until $B_n \cap F \neq \emptyset$

This can be done by

$$B_0 := I$$

$$B_{i+1} := B_i\ \cup\ \{t \mid \exists s : s \in B_i \wedge sTt\ \}$$

We should be able to decompose this full computation into small computable elements

---

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Operations $\cup$ and $\cap$ on sets correspond to $\vee$ and $\wedge$ in propositional notation

In $\exists s : s \in B_i \wedge sTt$ we use variables labelled by 0 for the part $s \in B_i$, and simply use $\wedge$ to compute

$$s \in B_i \wedge sTt$$

This is a boolean function in variables both labelled by 0 and by 1

By $\exists s$ all variables labelled by 0 should be eliminated; this can be done one by one for every variable x:

$$\exists x : \phi\ \equiv\ \underbrace{\phi[x := \mathit{false}] \vee \phi[x := \mathit{true}]}_{\text{computable}}$$

Finally in the resulting representation over variables labelled by 1, every label 1 should be replaced by 0

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Summarizing, the following operations should be efficient:

- Operations $\cup$ and $\cap$ on sets correspond to $\vee$ and $\wedge$ in propositional notation
- Checking $B_n = B_{n-1}$; for this we need a unique representation
- Compute $\phi[x := \mathit{false}]$ and $\phi[x := \mathit{true}]$ from ϕ
- Rename variables (label 1 by label 0)

B(inary) D(ecision) D(iagrams) are a representation in which all of these operations can be done efficiently

---

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The above scheme is followed in the implementation of symbolic model checkers like `NuSMV`, by which such reachability problems can be expressed and solved

`NuSMV` accepts general modal formulas; by
`LTLSPEC G P`
one verifies that a property P will hold globally = forever

So reachability is checked by finding a counterexample for `LTLSPEC G !P`, in which P is the formula representing final states, and `!` is the notation for negation

In the formula describing the transition relation there one writes `a` for $a0$ and `next(a)` for $a1$

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Example

Applying `NuSMV` on

```
MODULE main
VAR
a :  boolean;
b :  boolean;
INIT
a & !b
TRANS
(next(a) = !a) & (next(b) = !b)
LTLSPEC G (a | b)
```

yields true: a final state satisfying $\neg(a \vee b)$ is not reachable

---

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So here $\vee$ and $\wedge$ are written by `|` and `&` in infix notation

One can also use variables of numeric types, and use standard arithmetic operations

Take care that values remain in their range, like the counter c in

```
VAR
c :  0..100;
...
INIT
c = 0 & ···
TRANS
case c < 100 :  next(c) = c+1;
TRUE : next(c) = c; esac
```

where `case` $\cdots$ `esac` describes case analysis: it chooses the first case that yields true

---

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Important remark

`NuSMV` does not allow deadlocks: from every state a step should be possible

This can be obtained by allowing a step in which all variables remain equal in case none of the conditions holds

Typically, this is encoded by a `case` statement in which the last case is `TRUE` and the

statement is `next(x) = x` for all relevant variables `x`

In case this deadlock condition does not hold, `NuSMV` may give a wrong answer without giving any warning

---

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Example

Start with one marble

In every step either one marble is added or the number of marbles is doubled

Find the minimal number of steps to reach exactly 100 marbles

We look for the smallest value of the counter c for which the following module yields a counter example

---

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```
MODULE main
VAR
a :  0..100;
c :  0..20;
INIT
a = 1 & c = 0
TRANS
case c < 20 :  next(c) = c + 1;
TRUE : next(c) = c; esac & (
case a<100 :  next(a) = a + 1;
TRUE : next(a)=a; esac |
case a<=50 :  next(a) = 2*a;
TRUE : next(a)=a; esac)
LTLSPEC G !(a = 100 & c = 8)
```

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Indeed:

```
-> State:  1.1 <-
a = 1
c = 0
-> State:  1.2 <-
a = 2
c = 1
-> State:  1.3 <-
a = 3
c = 2
-> State:  1.4 <-
a = 6
c = 3
-> State:  1.5 <-
a = 12
c = 4
-> State:  1.6 <-
a = 24
c = 5
-> State:  1.7 <-
a = 25
c = 6
-> State:  1.8 <-
a = 50
c = 7
-> State:  1.9 <-
a = 100
c = 8
```

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Example: foxes and rabbits

Three foxes and three rabbits have to cross a river

There is only one boat that can carry at most two animals

When the boat is on the river, at each of the sides the number of foxes should be $\leq$ the number of rabits, otherwise the rabits will be eaten

How to solve this?

Let `f` the number of foxes at the side where the boat is, and let `fb` be the number of foxes that goes into the boat, and similar `r` and `rb`

Let `b` be a boolean expressing where the boat is

The problem is solved by applying `NuSMV` on the following

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```
MODULE main
```

```
VAR
r :  0..3;
rb :  0..2;
f :  0..3;
fb :  0..2;
b :  boolean;
INIT
b & f = 3 & r = 3
TRANS
next(b) = !b &
fb + rb <= 2 &
fb + rb >= 1 &
f - fb <= r - rb &
next(f) = 3 - f + fb &
next(r) = 3 - r + rb
LTLSPEC G !(!b & f = 3 & r = 3)
```

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Features of `NuSMV`

- **LTL** = linear temporal logic, based on X = next and U = until, in which G = global and F = future can be expressed
- **CTL** = computational tree logic, extension of LTL also dealing with quantifiers and branching time
- Default underlying machinery based on BDDs; by calling verbose option `NuSMV -v 3` information on underlying BDDs is shown
- **bounded model checking**: fixing a maximal number k of steps, introduce a_i for every variable a and every $0 \leq i \leq k$ representing the value of a after i steps, and express and solve the problem by SAT

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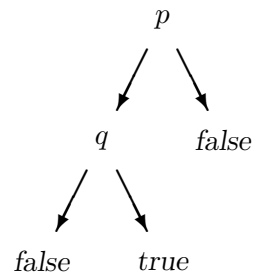
Decision trees

A **decision tree** is a binary tree in which

- nodes are labelled by boolean variables
- leaves are labelled by *false* or *true*



A decision tree describes a boolean function by starting at the root and consider every node as an if–then–else–

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More precisely, if every variable has a boolean value then the corresponding function value is obtained is follows:

- Start at the root
- For a node proceed by its left branch if the corresponding variable is *true*
- For a node proceed by its right branch if the corresponding variable is *false*
- Repeat until a leaf has been reached
- The label of the leaf is the resulting function value

So the example describes the same boolean function as $p \wedge \neg q$

Sometimes the left (*true*) branch is written solid and the right (*false*) branch is written dashed

---

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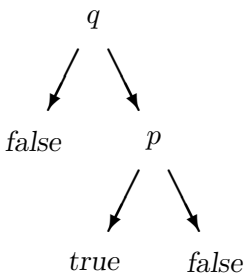
Unfortunately this representation is not yet unique: the same boolean function is described by the decision tree



We may disallow this second representation if we require that below every variable in a tree only **greater** variables are allowed, with respect to some total order $<$ on the variables

Such a decision tree is called **ordered** with respect to $<$

In our example: $p < q$

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Every boolean function on a finite number of variables can be represented as an ordered decision tree:

simply transform its truth table

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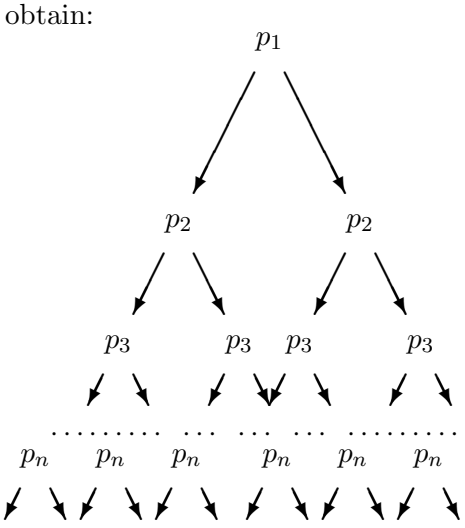
For the order $p_1 < p_2 < p_3 < \cdots < p_n$ we obtain:



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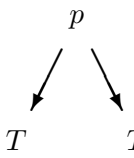
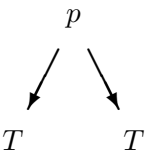
Still the representation is not yet unique, since for every decision tree T and every variable p



describes the same boolean function as T itself

Replacing



by T is called **elimination**

If no elimination is applicable on T then T is called **reduced**

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Now we do have a unique representation:

> Every boolean function can be expressed in exactly one way by a reduced ordered decision tree

Ingredients for the proof:

- Induction on the number of variables
- The smallest variable either occurs only at the top or does not occur at all

Exercise:

Determine the reduced ordered decision trees for the boolean functions described by the following formulas, with respect to the order $p < q < r$

1. $p \vee q \vee r$
2. $r \rightarrow (q \vee (p \wedge q))$
3. $p \leftrightarrow (q \leftrightarrow r)$

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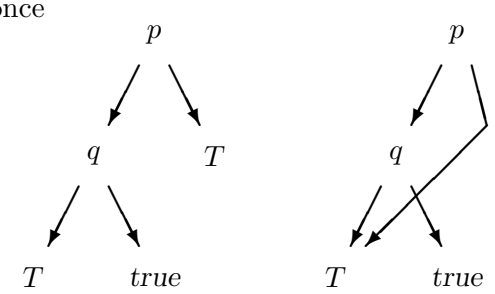
For efficient storage of such decision trees we prefer the following rule

If a subtree of a decision tree has multiple occurrences, we want to store this subtree only once



The right representation is preferred

This is a **D(irected) A(cyclic) G(raph)** rather than a tree

---

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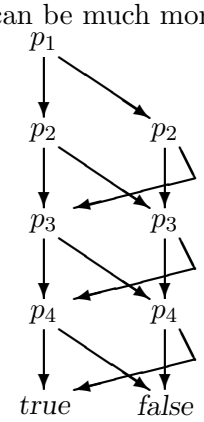
This can be much more efficient



This DAG has 7 nodes, while the tree has 15

Generalized to $p_1, \ldots, p_n$: the DAG has $2n-1$ nodes, while the tree has $2^n - 1$: a ratio of exponential size

---

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For the implementation it hardly makes any difference

In the usual implementation for decision trees (or other binary trees) for every node its label and the pointers to its children are stored

Due to the tree structure to every node there is exactly one pointer

For DAGs the same data structure can be used, only now there may be more pointers to one node

A **B(inary) D(ecision) D(iagram)** is a decision tree in DAG representation

An **O(rdered) BDD** is an ordered decision tree in DAG representation

---

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Both for uniqueness and for efficiency we not only want to admit common reference to the same node, we even want to **force** it

In the example we want to disallow the left representation

The pointer describing the left branch of a node is called the *true*-successor

The pointer describing the right branch of a node is called the *false*-successor

We require that no two nodes in the DAG have the following three properties:

- both are labelled by the same variable
- both have the same *true*-successor
- both have the same *false*-successor

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Every decision tree can be transformed to a DAG satisfying this requirement:

> As long there are two such nodes, remove one of them and redirect every reference to the removed one to the remaining one

This is called **merging**

So both elimination and merging are a way to decrease the size of a DAG without affecting its meaning: it is a kind of **reduction**

A **R(educed) OBDD** is an OBDD on which no elimination and no merging is possible

Main theorem:

For every order on the variables every boolean function has exactly one representation as a ROBDD

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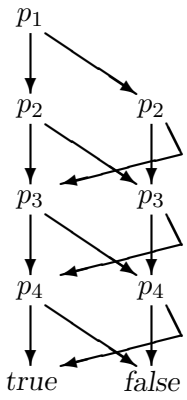
Example:



is the unique ROBDD of the boolean function described by the formula

$$((p_1 \leftrightarrow p_2) \leftrightarrow p_3) \leftrightarrow p_4$$

and the order $p_1 < p_2 < p_3 < p_4$

This formula yields *true* if and only if an even number of the variables has the value *true*

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Proof of the main theorem:

- At least one ROBDD representation:

  Start by the full ordered decision tree representing the truth table

  Apply elimination and merge on this decision tree as long as possible

  This ends since by every step the size decreases

  The resulting BDD is by construction an ROBDD representing the given boolean function

- At most one ROBDD representation:

  Assume ROBDDs T and U represent the same boolean function

  Unwind T to a tree T' and U to a tree U'

  Then T' and U' are reduced ordered decision trees representing the same boolean function

  Earlier result $\implies$ $T' = U'$

  Lemma $\implies$ $T = U$

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Lemma: Every tree has a unique representation as a DAG with maximal sharing, i.e., a DAG on which no merge can be applied

End of proof of main theorem

Since there are 2^{2^n} distinct boolean functions many of them have a very big representation as a ROBDD

The size of the ROBDD strongly depends on the chosen order on the variables

If $p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n$ then the ROBDD of

$$\phi = (p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \cdots \wedge (p_n \vee q_n)$$

only has $2n$ nodes, and shows up for $n = 4$ as follows:

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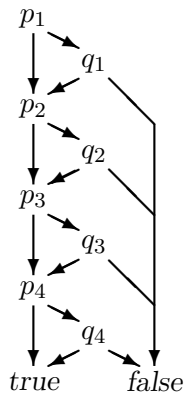

However, if $p_1 < p_2 < \cdots < p_n < q_1 < q_2 < \cdots < q_n$ then the ROBDD of the same formula has over 2^n nodes

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Heuristics for choosing a good order: variables close together in the formula, influencing each other, should be close in the order

Choosing an appropriate order then for many applications the unique ROBDD representation is feasible, even for $n > 1000$

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We still need an algorithm to compute the ROBDD for a given propositional formula and an order on the variables

Such an algorithm can be used for SAT:

> Apply the algorithm to the given formula
>
> If the resulting ROBDD is *false*, then the formula is unsatisfiable, and otherwise it is satisfiable

If the formula is satisfiable, then

- the ROBDD is not equal to *false* and at least one leaf is *true*, otherwise elimination can be applied
- a satisfying assignment is obtained from the ROBDD by following a path from the root to this leaf labelled by *true*

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Algorithm to determine the ROBDD of a formula

Every formula is of the shape:

- *false* or *true*, or
- p for a variable p, or
- $\neg\phi$, or
- $\phi \diamond \psi$ for $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

The ROBDD $ROBDD(\phi)$ of a formula ϕ will be constructed recursively according this recursive structure of the formulas:

- $ROBDD(false) = false$, $ROBDD(true) = true$,
- $ROBDD(p) = p(true, false)$
- $ROBDD(\neg\phi) = ROBDD(\phi \rightarrow false)$
- $ROBDD(\phi \diamond \psi) =$
  $apply(ROBDD(\phi), ROBDD(\psi), \diamond)$

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So it remains to find an algorithm *apply* having two ROBDDs and a binary operation $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$ as input, and having the desired ROBDD as its output

Notation:

- $p(T, U)$ is the BDD having root p for which the left branch is T and the right branch is U
- On OBDDs we define
  $p(T, U)(p := true) = T$
  $p(T, U)(p := false) = U$
- $T(p := true) = T(p := false) = T$
  if p does not occur in T

As the basis of the recursion we define

$apply(T, U, \diamond)$ = value according the truth table of $\diamond$ if $T, U \in \{false, true\}$

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If T, U not both in $\{false, true\}$, hence T and/or U contains at least one variable, then we define

$apply(T, U, \diamond) =$

$$p(\mathit{apply}(T(p := \mathit{true}), U(p := \mathit{true}), \diamond),$$
$$\mathit{apply}(T(p := \mathit{false}), U(p := \mathit{false}), \diamond))$$

where p is the smallest variable occurring in T or U

Writing shortly $\diamond(T, U)$ for $\mathit{apply}(T, U, \diamond)$ this means

$$\diamond(p(T_1, T_2), p(U_1, U_2)) = p(\diamond(T_1, U_1), \diamond(T_2, U_2))$$

$$\diamond(p(T_1, T_2), U)) = p(\diamond(T_1, U), \diamond(T_2, U))$$

if p does not occur in U

$$\diamond(T, p(U_1, U_2)) = p(\diamond(T, U_1), \diamond(T, U_2))$$

if p does not occur in T

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So for two BDDS T, U computing

$$\mathit{apply}(T, U, \diamond) = \diamond(T, U)$$

is done by pushing $\diamond$ downwards, meanwhile combining T and U, until $\diamond$ applied to *true*/*false* has to be computed, which is replaced by its value according to the truth table

In this recursion only calls $\mathit{apply}(T', U', \diamond)$ are done where T' is a node of T and U' is a node of U

This is implemented in such a way that the same call $\mathit{apply}(T', U', \diamond)$ is never executed twice

Keep track for which pairs T', U' this has already been executed, using a hash table

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Hence this algorithm $\mathit{apply}(T, U, \diamond)$ has complexity

$$O(\#T * \#U),$$

where # is the number of nodes of a BDD

Is the resulting BDD $\mathit{apply}(T, U, \diamond)$ always reduced?

NO

This can be repaired by applying elimination and merging in this process for every newly created node

This can be done in such a way that the complexity remains $O(\#T * \#U)$

The resulting algorithm *ROBDD* is essentially the algorithm as it is used in practice, for instance in `NuSMV`

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Exercise

Compute the ROBDD of

$$(p \to r) \wedge (q \leftrightarrow (r \vee p))$$

with respect to the order $p < q < r$ using this algorithm

The algorithm *apply* is polynomial in the size of its input (even quadratic)

However, in general the full algorithm *ROBDD* is not polynomial: intermediate results may have exponential size

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Example:

If $p_1 < p_2 < \cdots < p_n < q_1 < q_2 < \cdots < q_n$ then the ROBDD of

$$\phi = (p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \cdots \wedge (p_n \vee q_n)$$

has more than 2^n nodes

The algorithm *ROBDD* applied on $p \wedge (\phi \wedge \neg p)$ will compute this big ROBDD as an intermediate result, and hence will have exponential complexity, although the final result is simply *false*

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If in a special case all intermediate results are of polynomial size then the full algorithm is polynomial

since the full algorithm consists of a linear number of *apply* calls, each having polynomial complexity

Every formula purely constructed from $\neg, \leftrightarrow$ and variables has an ROBDD of which the size is linear in the size of the formula

Hence for such formulas the algorithm *ROBDD* is always polynomial

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Using resolution the behaviour may be just opposite:

Using only unit resolution it can be established in linear time that any formula of the shape $p \wedge (\phi \wedge \neg p)$ is unsatisfiable

Conversely, there are unsatisfiable formulas purely constructed from $\neg, \leftrightarrow$ and variables of which it can be proved that every resolution proof requires an exponential number of steps

Conclusion:

Resolution and the algorithm *ROBDD* are essentially incomparable

Both techniques are successfully applied to huge formulas

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Typical situation in hardware verification

Two chip designs have to be proven to behave equivalent

More precisely, they should have the same input/output behaviour:

- both have k input nodes, n output nodes and a great number of internal nodes, all representing boolean values, connected by ports
- both chip designs compute the values of all n output nodes whenever the values of the k input nodes are set to boolean values
- if the input nodes are set to the same values for both chip designs, then the resulting values of the output nodes should be the same too, for all possible inputs

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Example:

For a 32-bit multiplier we have $k = 64$ and $n = 32$

Let

- $B_1, \ldots, B_k$ express input nodes of both chip designs
- $A_1, \ldots, A_p$ express internal and output nodes of one chip design, the first $p - n$ being internal and the last n being output
- $C_1, \ldots, C_q$ express internal and output nodes nodes of the other chip design, the first $q - n$ being internal and the last n being output

Typically, k and n are reasonably small and p and q are very big

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We assume both chip designs have no circular behaviour:

the nodes are numbered in such a way that every A_i only depends on

$$B_1, \ldots, B_k, A_1, \ldots, A_{i-1},$$

Similarly for C_i

Let $\phi_1, \ldots, \phi_p$ be the simple formulas reflecting the behaviour of the ports, where ϕ_i describes how A_i depends on

$$B_1, \ldots, B_k, A_1, \ldots, A_{i-1}$$

So $A_i \leftrightarrow \phi_i$ describes how the value of A_i is computed

Similarly $\psi_1, \ldots, \psi_q$ describe $C_1, \ldots, C_q$

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To be proven:

the last n nodes $A_{p-n+1}, \ldots, A_p$ from the one chip design have the same behaviour as the last n nodes $C_{q-n+1}, \ldots, C_q$ from the other chip design

Expressed in a formula:

$$(\bigwedge_{i=1}^{p}(A_i \leftrightarrow \phi_i)) \ \wedge \ (\bigwedge_{i=1}^{q}(C_i \leftrightarrow \psi_i))$$

implies

$$\bigwedge_{i=1}^{n} A_{p-n+i} \leftrightarrow C_{q-n+i}$$

Using resolution these are formulas in $k + p + q$ variables plus auxiliary variables due to Tseitin transformation

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Using BDDs this can be done much more efficient:

- only consider $B_1, \ldots, B_k$ as variables
- subsequently compute ROBDDs for $A_1, \ldots, A_p$ using $\phi_1, \ldots, \phi_p$
- similarly for $C_1, \ldots, C_q$
- check whether the ROBDDs of A_{p-n+i} and C_{q-n+i} coincide for $i = 1, \ldots, n$

In this way in practice BDD technology is used a lot in hardware verification

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BDDs for pseudo boolean constraints
A **pseudo boolean constraint** is a constraint of the shape

$$c_1x_1 + c_2x_2 + c_3x_3 + \cdots + c_nx_n \leq a$$

where c_i, a are given integer values, and x_i are boolean variables

Such pseudo boolean constraints often occur in scheduling problems

Now we show how such a pseudo boolean constraint can be transformed to a CNF by using BDDs

The constraint

$$\sum_{i=1}^{n} c_i x_i \leq a$$

can be seen as a boolean function in $x_1, \ldots, x_n$, yielding true if the constraint holds and yielding false otherwise

Construct the ROBDD of this boolean function

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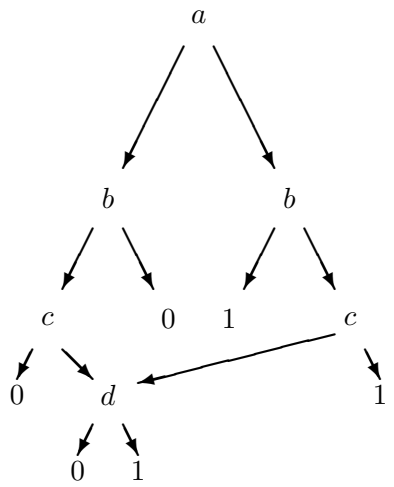
Heuristic: choose variable order in such a way that variables with high absolute coefficients (being influential) are low in the order, so will be tested first

Example:
$3a - 2b + c + d \leq 1$



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How to build such an ROBDD?

- Evaluate values $a, b, \ldots$ in the inequality

  E.g., in the example:
  $a = 1$ yields $-2b + c + d \leq -2$
  $a = 1, b = 1$ yields $c + d \leq 0$

- As soon as an inequality is false, put 0, if it is true, put 1

- Keep track of all inequalities so far

  Every inequality corresponds to a node

  If an inequality is found equivalent to an inequality found before, then point to the corresponding node

In this way sharing is introduced

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The next step is to transform the ROBDD to a CNF

This is done via the Tseitin transformation

For propositional formulas composed from

$$\neg, \vee, \wedge, \rightarrow, \leftrightarrow$$

we discussed the Tseitin transformation to CNF by giving a fresh name to every subformula

Here we consider the nodes of a BDD as an if-then-else, which can be seen as propositional operation with three arguments:

$$\text{if } p \text{ then } q \text{ else } r \quad = \quad ((p \rightarrow q) \wedge (\neg p \rightarrow r))$$

Instead of giving a fresh name to every subformula we give a fresh name to every node in the BDD

This is the same idea as before, with the extra facility of sharing

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The **Tseitin transformation** consists of

- A unit clause consisting of the name of the root

- The CNF

  $$\neg A \vee \neg p \vee B$$
  $$\neg A \vee p \vee C$$
  $$A \vee p \vee \neg C$$
  $$A \vee \neg p \vee \neg B$$
  $$A \vee \neg B \vee \neg C$$

  of $A \leftrightarrow ((p \rightarrow B) \wedge (\neg p \rightarrow C))$ for every node A labelled by p where the *true*-branch points to B and the *false*-branch to C

- Here B, C are either names of nodes or *false* or *true*

In this way the size of the Tseitin transformation is linear in the size of the BDD

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In our example we get the unit clause A together with the CNFs of the following 6 formulas:

$$A \leftrightarrow ((a \rightarrow B_1) \wedge (\neg a \rightarrow B_2))$$

$$B_1 \leftrightarrow ((b \rightarrow C_1) \wedge (\neg b \rightarrow \mathit{false}))$$
$$B_2 \leftrightarrow ((b \rightarrow \mathit{true}) \wedge (\neg b \rightarrow C_2))$$
$$C_1 \leftrightarrow ((c \rightarrow \mathit{false}) \wedge (\neg c \rightarrow D))$$
$$C_2 \leftrightarrow ((c \rightarrow D) \wedge (\neg c \rightarrow \mathit{true}))$$
$$D \leftrightarrow ((d \rightarrow \mathit{false}) \wedge (\neg d \rightarrow \mathit{true}))$$

Note the sharing of D

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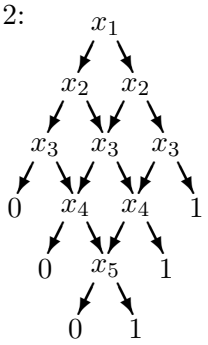
Sharing really helps: for

$$\sum_{i=1}^{2n+1} x_i \leq n$$

the BDD size is $(n+1)^2$, while unshared it would have been exponential in n

For $n = 2$:



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This BDD based technique has been implemented in `minisat+` to solve pseudo boolean constraint problems by means of the SAT solver `minisat`

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Predicate logic

Until now we only reasoned in the world of the domain $\{\mathit{true}, \mathit{false}\}$, together with the usual operations

(except for the excursion to SMT)

Now we want to do automated reasoning in an arbitrary bigger domain D in which we can quantify using $\forall$ and $\exists$, and where we may have **relations** and **functions**

(just like SMT, abstracting from the integers)

This is called **Predicate logic**

Example

- There is a student that is awake during all lectures
- During all boring lectures no student keeps awake
- Then there are no boring lectures

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How can we prove this?

Just like proposition logic: take the conjunction of all given statements and the negation of the conclusion, and try to prove that the resulting formula is unsatisfiable, i.e., equivalent to *false*

In order to express the example by a formula we define relations S, L, B and A:

- $S(x)$: x is a student
- $L(x)$: x is a lecture
- $B(x)$: x is boring
- $A(x, y)$: x is awake during y

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Hence we want to prove that

$$(\exists x(S(x) \wedge \forall y(L(y) \rightarrow A(x, y)))) \wedge$$
$$(\forall x((L(x) \wedge B(x)) \rightarrow \neg \exists y(S(y) \wedge A(y, x)))) \wedge$$
$$\exists x(L(x) \wedge B(x))$$

is unsatisfiable = equivalent to *false*

What does this mean exactly?

In such an expression we may have

- **variables** (here x, y)
- **function symbols** (not here)
- **relation symbols** (here S, L, B, A), also called **predicate symbols**

Function symbols and relation symbols have an **arity** $= 0, 1, 2, 3, \ldots$

A function symbol of arity 0 is also called a **constant**

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We inductively define:

- A **term** is
  - a variable from a set $\mathcal{X}$, or
  - a function symbol of arity n applied on n terms
- A **predicate (formula)** is
  - a relation symbol of arity n applied on n terms, or
  - $\forall x(P)$ for a variable $x \in \mathcal{X}$ and a predicate P, or
  - $\exists x(P)$ for a variable $x \in \mathcal{X}$ and a predicate P, or
  - $\neg P$ for a predicate P, or
  - $P \diamond Q$ for predicates P and Q and $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

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If we give meaning to variables, function symbols and relation symbols in a **model**, then a predicate has a boolean value

More precisely, a **model** is a non-empty set M together with

- $[f] : M^n \rightarrow M$ for every function symbol f of arity n
- $[R] : M^n \rightarrow \{\mathit{false}, \mathit{true}\}$ for every relation symbol R of arity n

For $\alpha : \mathcal{X} \rightarrow M$ we define inductively

- $[x, \alpha] = \alpha(x)$ for $x \in \mathcal{X}$
- $[f(t_1, \ldots, t_n), \alpha] = [f]([t_1, \alpha], \ldots, [t_n, \alpha])$ for every function symbol f of arity n and terms $t_1, \ldots, t_n$
- $[R(t_1, \ldots, t_n), \alpha] = [R]([t_1, \alpha], \ldots, [t_n, \alpha])$ for every function symbol f of arity n and terms $t_1, \ldots, t_n$

So $[f(t_1, \ldots, t_n), \alpha] \in M$ and $[R(t_1, \ldots, t_n), \alpha] \in \{\mathit{false}, \mathit{true}\}$

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In order to define $\forall$ and $\exists$ we need to modify the valuation α

If $\alpha : \mathcal{X} \rightarrow M$, $x \in \mathcal{X}$ and $m \in M$ then we define $\alpha\langle x := m\rangle : \mathcal{X} \rightarrow M$ by

$$\alpha\langle x := m\rangle(x) = m$$

$$\alpha\langle x := m\rangle(y) = \alpha(y)$$

for all $y \in \mathcal{X}$, $y \neq x$

We define

$$[\forall x(P), \alpha] = \bigwedge_{m \in M} [P, \alpha\langle x := m\rangle]$$

$$[\exists x(P), \alpha] = \bigvee_{m \in M} [P, \alpha\langle x := m\rangle]$$

Avoid problems by disallowing $\forall x$ or $\exists x$ to occur inside P for same x: choose fresh x for every quantification

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We define

$$[\neg P, \alpha] = \neg[P, \alpha]$$

and

$$[P \diamond Q, \alpha] = [P, \alpha] \diamond [Q, \alpha]$$

for $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

In this way $[P, \alpha] \in \{\mathit{false}, \mathit{true}\}$ has been defined for every predicate P and every $\alpha : \mathcal{X} \rightarrow M$

A predicate P is **satisfiable** if there is a model M and $\alpha : \mathcal{X} \rightarrow M$ such that $[P, \alpha] = \mathit{true}$

All these definitions are motivated by common knowledge of the usual notions of $\forall$ and $\exists$

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Proposition logic can be seen as a special case of predicate logic in which

- there are no variables,
- there are no function symbols, and
- all relation symbols (corresponding to propositional variables) have arity 0

Predicate logic can be expressed in SMT, but tools like `Yices` and `Z3` are very weak in quantification, e.g.

```
(benchmark test.smt
:extrapreds ((P Int))
:extrafuns ((a Int))
:formula
(and
(forall (?x Int) (P ?x))
(forall (?x Int) (not (P ?x)))
))
```

yields `unknown`

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Now we extend the **resolution method** to be applicable for predicates; this is fully exploited in the tool `Prover9`, being the successor of `Otter`

As before resolution is only defined for conjunctive normal forms (CNF), where

- a CNF is a conjunction of clauses,
- a clause is a disjunction of literals, and
- a literal is an atomic formula or its negation

Here an **atomic formula** is an expression of the shape $P(t_1, \ldots, t_n)$ where P is a relation symbol of arity n and $t_1, \ldots, t_n$ are terms

A clause will be interpreted as being universally quantified over all occurring variables

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As before the only thing to be done by resolution is proving that a CNF is unsatisfiable by deriving the empty clause

As before this will be extended to arbitrary formulas by giving a transformation from an arbitrary formula to CNF

Due to terms and variables occurring in atomic formulas and implicit universal quantification of clauses the resolution rule

$$\frac{P \vee V, \quad \neg P \vee W}{V \vee W}$$

will be slightly more complicated now

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Example:

A special case of the clause

$$P(f(x), y) \vee Q(x, y)$$

is $P(f(x), g(y)) \vee Q(x, g(y))$

A special case of the clause

$$\neg P(x, g(y)) \vee R(x, y)$$

is $\neg P(f(x), g(y)) \vee R(f(x), y)$

On both 'special cases' now resolution yields the new clause $Q(x, g(y)) \vee R(f(x), y)$

Now we want to see

$$\frac{P(f(x), y) \vee Q(x, y), \quad \neg P(x, g(y)) \vee R(x, y)}{Q(x, g(y)) \vee R(f(x), y)}$$

as a valid resolution step

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A **substitution** is a map from variables to terms

A substitution σ can be extended to arbitrary terms and atomic formulas by inductively defining:

$$x\sigma = \sigma(x)$$

for every variable x and

$$F(t_1, \ldots, t_n)\sigma = F(t_1\sigma, \ldots, t_n\sigma)$$

for every function/relation symbol F

So $t\sigma$ is obtained from t by replacing every variable x in t by $\sigma(x)$

For instance, if $\sigma(x) = y$ and $\sigma(y) = g(x)$ then

$$P(f(x), y)\sigma = P(f(y), g(x))$$

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Let X be the set of variables and $T(X)$ the set of terms, then in this way the given substitution

$$\sigma : X \to T(X)$$

is extended to

$$\sigma : T(X) \to T(X)$$

The latter σ is written in postfix notation

If both σ and τ are substitutions we can define the **composition** $\sigma\tau$:

$$x(\sigma\tau) = (x\sigma)\tau$$

for all $x \in X$, by which we have

$$t(\sigma\tau) = (t\sigma)\tau$$

for all $t \in T(X)$

$\sigma\tau$ means: first apply σ, then τ

Here we do not have the usual confusion of composition in prefix notation, where $f \circ g$ means: first apply g, then f

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For a clause $V = P_1 \vee P_2 \vee \cdots \vee P_n$ and a substitution σ we write

$$V\sigma = P_1\sigma \vee P_2\sigma \vee \cdots \vee P_n\sigma$$

For every substitution σ we want to consider the clause $V\sigma$ as a special case of the clause V

Now the general version of the **resolution rule** for predicates reads:

$$\frac{P \vee V, \quad \neg Q \vee W}{V\sigma \vee W\tau}$$

for all substitutions σ, τ satisfying

$$P\sigma = Q\tau$$

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In order to be able to use this rule we need an algorithm to determine whether substitutions σ, τ exist such that $P\sigma = Q\tau$ for given atomic formulas P and Q, and if so, to find them

This is called **unification**

Examples:

- $P(f(x), y)$ and $P(x, g(y))$ unify by choosing
  $\sigma(x) = x, \sigma(y) = g(y)$,
  $\tau(x) = f(x), \tau(y) = y$

- $P(f(x), y)$ and $Q(x, g(y))$ do not unify
- $P(f(x), f(y))$ and $P(x, g(y))$ do not unify
- $P(f(x), x)$ and $P(x, x)$ do not unify

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For unification there is no principal difference between terms and atomic formulas, neither between function symbols and relation symbols

Moreover by renaming of variables we can force that P and Q have no variables in common

Then the behavior of the two substitutions σ, τ can be expressed by one single substitution

Now the general unification problem reads:

> Given two terms P and Q, is there a substitution σ such that $P\sigma = Q\sigma$?
>
> If so, find it

The resulting substitution σ is called a **unifier**

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Simple observation:

If σ is a unifier for (P, Q) and τ is an arbitrary substitution, then $\sigma\tau$ is a unifier too for (P, Q)

Definition:

A unifier σ_0 is called a **most general unifier (mgu)** for (P, Q) if for every unifier σ for (P, Q) a substitution τ exists such that $\sigma = \sigma_0\tau$

If both σ_0 and σ_1 are an mgu for (P, Q), then by definition τ_0, τ_1 exist such that

$$\sigma_1 = \sigma_0\tau_0 \quad \text{and} \quad \sigma_0 = \sigma_1\tau_1$$

From this property one can conclude that σ_0 and σ_1 are equal up to renaming, hence an mgu is unique up to renaming

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Hence we will speak about **the** mgu rather than **an** mgu

We will give an algorithm with two terms as input, and as output:

- whether the terms unify or not, and
- the mgu in case they unify

The existence of an mgu in case two terms unify is a consequence of this property of the algorithm

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Write $v(t)$ for the test whether t is a variable

Write $in(x, t)$ for the test whether the variable x occurs in t, this is called **occur check**

The unification algorithm has the following invariant:

$$\exists\sigma(P\sigma = Q\sigma) \;\equiv\; \exists\sigma(\forall(t, u) \in S(t\sigma = u\sigma))$$

$$\wedge\, (\exists\sigma(P\sigma = Q\sigma) \;\rightarrow\; un)$$

where P, Q are the original terms to be unified

If $S = \emptyset$ then it follows that P and Q unify

If $\neg un$ then it follows that P and Q do not unify

Inspired by the invariant and these observations we arrive at the following unification algorithm:

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$S := \{(P, Q)\}$;
$un := true$;

while $(un \wedge S \neq \emptyset)$ do {

choose $(t, u) \in S$;
$S := S \setminus \{(t, u)\}$;
if $t \neq u$ then

if $v(t) \wedge in(t, u)$ then $un := \mathit{false}$

else if $v(t) \wedge \neg(in(t, u))$ then
$S := S[t := u]$

else if $v(u) \wedge in(u, t)$ then
$un := \mathit{false}$

else if $v(u) \wedge \neg(in(u, t))$ then
$S := S[u := t]$

else if $t = f(t_1, \ldots, t_n) \wedge u = f(u_1, \ldots, u_n)$ then
$S := S \cup \{(t_1, u_1), \ldots, (t_n, u_n)\}$

else if $t = f(\cdots) \wedge u = g(\cdots) \wedge f \neq g$ then
$un := \mathit{false}$

}

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Basic idea of the algorithm:

- S is inspected and decomposed
- un is set to *false* if a reason is found that there is no unification, then the algorithm stops
- if such a reason is not found and the list S of unification requirements is empty, then there is a unifier

This unification algorithm always terminates, since in every step either

- the total number of variables occurring in S strictly decreases, or
- the total number of variables occurring in S remains the same and the total size of S decreases

Here total size of $\{(t_1, u_1), \ldots, (t_n, u_n)\}$ is defined to be

$$\sum_{i=1}^{n} (|t_i| + |u_i|)$$

where $|t|$ is the size of the term t

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After termination the following holds:

- $un = \mathit{false}$ and P and Q are not unifiable, or
- $un = \mathit{true}$ and P and Q are unifiable

If P and Q are unifiable, what about the unifier?

We extend the program by building up the unifier in a variable mgu

As only steps are done that are really forced, the resulting unifier mgu will be a most general unifier of P and Q

We write id for the substitution mapping every variable on itself

For a variable x and a term P we write $[x := P]$ for the substitution mapping x on P and every other variable on itself

This notation will be used for pairs of terms and sets of pairs of terms, meaning that the substitution is applied to every occurring term

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$S := \{(P, Q)\}$;
$un := \mathit{true}$;
$mgu := id$;
while $(un \wedge S \neq \emptyset)$ do {
choose $(t, u) \in S$;
$S := S \setminus \{(t, u)\}$;
if $t \neq u$ then
if $v(t) \wedge in(t, u)$ then $un := \mathit{false}$
else if $v(t) \wedge \neg(in(t, u))$ then
$\{S := S[t := u]$;
$mgu := mgu[t := u]\}$
else if $v(u) \wedge in(u, t)$ then
$un := \mathit{false}$
else if $v(u) \wedge \neg(in(u, t))$ then
$\{S := S[u := t]$;
$mgu := mgu[u := t]\}$

else if $t = f(t_1, \ldots, t_n) \wedge u = f(u_1, \ldots, u_n)$ then

$S := S \cup \{(t_1, u_1), \ldots, (t_n, u_n)\}$

else if $t = f(\cdots) \wedge u = g(\cdots) \wedge f \neq g$ then

$un := \mathit{false}$

}

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Example:

Unify $P(f(x), y)$ and $P(z, g(w))$

Start:
$S = \{(P(f(x), y), P(z, g(w)))\}, mgu = id$

After 1 step:
$S = \{(f(x), z), (y, g(w)))\}, mgu = id$

After 2 steps:
$S = \{(y, g(w))\}, mgu = [z := f(x)]$

After 3 steps:
$S = \emptyset, mgu = [z := f(x)][y := g(w)]$

So the resulting most general unifier σ is given by

$$x\sigma = x,\ y\sigma = g(w),\ z\sigma = f(x),\ w\sigma = w,$$

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Example:

Unify $P(f(x), x)$ and $P(y, g(y))$

Start:
$S = \{(P(f(x), x), P(y, g(y)))\}, mgu = id$

After 1 step:
$S = \{(f(x), y), (x, g(y)))\}, mgu = id$

After 2 steps:
$S = \{(x, g(f(x)))\}, mgu = [y := f(x)]$

After 3 steps:
x occurs in $g(f(x))$, hence no unification

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Remarks:

- If two terms unify, then they have an mgu which is unique up to renaming of variables
- Occur check can be expensive: search for a particular variable in a big term
- There are optimizations of this unification algorithm that are linear
- The unifier can have a size that is exponential in the size of the terms to be unified
- Efficient algorithms use DAG representation for terms

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Example:

Unification of

$$P(x_1, f(x_2, x_2), x_2, f(x_3, x_3), x_3)$$

and

$$P(f(y_1, y_1), y_1, f(y_2, y_2), y_2, f(y_3, y_3))$$

yields an mgu σ in which $x_1\sigma$ is a term containing 32 copies of the same variable and 31 f-symbols

---

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Back to resolution

$$\frac{P \vee V,\ \ \neg Q \vee W}{V\sigma \vee W\tau}$$

for substitutions σ, τ satisfying

$$P\sigma = Q\tau$$

Here we can rename variables by which $P \vee V$ and $\neg Q \vee W$ have no variables in common

Now we restrict this general version of resolution:

instead of allowing the rule for all (infinitely many) unifiers of P and Q we **only** allow the mgu

So the resolution step can be described as follows:

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- Take two (possibly equal) non-empty clauses
- Rename variables such that they do not have variables in common
- Choose a positive literal P in one of the clauses and a negative literal $\neg Q$ in the other
- Unify P and Q
- if they do not unify a corresponding resolution step is not possible
- if they unify then the clause

  $$(V \vee W)\sigma$$

  can be concluded, where
  - $P \vee V$ is the one clause,
  - $\neg Q \vee W$ is the other clause,
  - σ is the most general unifier of P and Q

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As a consequence, given a CNF, there are only finitely many possibilities of doing a resolution step, and these are computable

Theorem
(completeness of resolution)

A predicate in CNF is equivalent to *false* if and only if there is a sequence of resolution steps ending in the empty clause

Theorem
(undecidability of predicate logic)

No algorithm exists that can establish in all cases whether a given predicate in CNF is equivalent to *false*

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These two theorems look contradictory, but they are not:

> After extensive but unsuccessful search for a resolution sequence ending in the empty clause by only using completeness you are not able to conclude that such a resolution sequence does not exist

(compare to halting problem)

We do not prove these important theorems

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Examples of resolution sequences:

1	$P(x, f(y)) \vee \neg P(x, y)$	
2	$\neg P(x, f(f(y)))$	
3	$P(a, g(y))$	
4	$P(a, f(g(y)))$	$(1, 3)$
5	$P(a, f(f(g(y))))$	$(1, 4)$
	$\bot$	$(2, 5)$

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1	$P(x, f(y)) \vee \neg P(x, y)$	
2	$\neg P(x, f(f(y)))$	
3	$P(a, g(y))$	
4	$\neg P(x, f(y))$	$(1, 2)$
5	$\neg P(x, y)$	$(1, 4)$
	$\bot$	$(3, 5)$

In searching for such a resolution proof there is a lot of choice

This choice is more restricted if every clause contains at most one positive literal, making search for resolution much simpler

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A **Horn clause** is a clause containing at most one positive literal

Usually a Horn clause $C \vee \neg C_1 \vee \cdots \vee \neg C_n$ is written as

$$C \leftarrow C_1, \ldots, C_n$$

or as

$$C \texttt{ :- } C_1, \ldots, C_n.$$

The positive literal C is called the **head**

A clause without a head is called a **goal**

A clause consisting only of a head is called a **fact**, it is written as `C.` instead of `C :- .`

A **Prolog program** is a set of Horn clauses in this notation

Prolog is a standard programming language in artificial intelligence

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Example:

```
arrow(a,b).
arrow(a,c).
arrow(b,c).
arrow(c,d).
path(X,Y) :- arrow(X,Y).
path(X,Y) :- arrow(X,Z),path(Z,Y).
```

The first four clauses define a directed graph on four nodes

By the last two clauses the notion of a path in a graph is defined

If we wonder whether a path exists from `a` to `d`, then we add the goal

`:- path(a,d)`

being the clause $\neg$`path(a,d)`

Systematical depth first search for a resolution proof starting from the goal yields:

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1	$\texttt{arrow}(a,b)$	
2	$\texttt{arrow}(a,c)$	
3	$\texttt{arrow}(b,c)$	
4	$\texttt{arrow}(c,d)$	
5	$\texttt{path}(x,y) \vee \neg\texttt{arrow}(x,y)$	
6	$\texttt{path}(x,y) \vee \neg\texttt{arrow}(x,z) \vee \neg\texttt{path}(z,y)$	
7	$\neg\texttt{path}(a,d)$	
	$(5,7)$ no result	
8	$\neg\texttt{arrow}(a,z) \vee \neg\texttt{path}(z,d)$	$(6,7)$
9	$\neg\texttt{path}(b,d)$	$(1,8)$
	$(5,9)$ no result	
10	$\neg\texttt{arrow}(b,z) \vee \neg\texttt{path}(z,d)$	$(6,9)$
11	$\neg\texttt{path}(c,d)$	$(3,10)$
12	$\neg\texttt{arrow}(c,d)$	$(5,11)$
	$\bot$	$(4,12)$

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This kind of search for a resolution proof is called **SLD-resolution**

The found resolution sequence ending in $\bot$ is called a **refutation**

In the example it was proved automatically that a path from a to d exists, while the input is nothing more than

- the definition of a graph
- the definition of the notion 'path'
- the question: 'is there a path from a to d?'

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This mechanism also applies for goals containing variables

For instance, the goal `:- path(a,X)` will yield a refutation, but the goal `:- path(d,X)` will not

This is quite subtle: sometimes search can go on for ever

Prolog is a **declarative programming language**, it is a kind of **logic programming**

In many Prolog implementations occur check is omitted for efficiency reasons, by which

```
p(X,X).
:- p(X,f(X)).
```

yielding the CNF $p(X,X) \wedge \neg p(X, f(X))$ gives rise to an infinite computation

---

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Back to general predicates $\cdots$

Now we shall see how a general predicate can be transformed to CNF maintaining satisfiability

This transformation consists from:

- **prenex normal form:** shift all quantifiers $\forall$ and $\exists$ to the front and write the body as a CNF
- **Skolemization:** removal of $\exists$ by introducing fresh function symbols

---

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Prenex normal form

Starting from an arbitrary predicate we apply the following steps:

- Remove $\leftrightarrow$ by applying

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$$

- Remove $\rightarrow$ by applying

$$A \rightarrow B \equiv (\neg A) \vee B$$

- Remove negations of non-atomic formulas by repetitively applying
  - $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$
  - $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$
  - $\neg(\neg A) \equiv A$
  - $\neg(\forall x(A)) \equiv \exists x(\neg A)$
  - $\neg(\exists x(A)) \equiv \forall x(\neg A)$

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The formula obtained so far is composed from $\vee, \wedge, \exists, \forall$ and literals

- Rename variables until over every variable there is at most one quantification
- Assuming non-empty domain now all quantors may be put at the front, for example

$$B \vee \forall x(A) \equiv \forall x(B \vee A)$$

  where x does not occur in B

  Now the formula consists of a quantor free body on which a number of quantors is applied

- Transform the body to CNF using
  - $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
  - $(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C)$

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The result is a prenex normal form, i.e., a CNF preceded by a number of quantors, being equivalent to the original predicate

If the result is too big then Tseitin's transformation can be applied

In order to reach the desired CNF format with implicit universal quantification it remains to eliminate the $\exists$-symbols

This is done by **Skolemization**, i.e., replace

$$\cdots \exists y \cdots (\cdots y \cdots)$$

by

$$\cdots\cdots (\cdots f(x_1, \ldots, x_n) \cdots)$$

where f is a fresh function symbol and $x_1, \ldots, x_n$ are the universally quantified variables left from y

---

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Example

Skolemization applied to

$$\exists z \forall x \exists y \forall u \forall v \exists w (A(x, y, z, u, v, w)$$

yields

$$\forall x \forall u \forall v (A(x, f(x), c, u, v, g(x, u, v))$$

By Skolemization satisfiability is maintained:

Deriving a contradiction from $\forall x(A(x, f(x)))$ without any knowledge of f coincides with deriving a contradiction from

$$\exists f \forall x (A(x, f(x)))$$

According to the Axiom of Choice we have

$$\exists f \forall x (A(x, f(x))) \equiv \forall x \exists y (A(x, y))$$

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By Skolemization all $\exists$-symbols are removed, introducing a fresh symbol for every removed $\exists$-symbol

The result is a CNF for which

- all variables are implicitly universally quantified, and
- satisfiability is equivalent to satisfiability of the original arbitrary predicate formula

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The example of the boring lectures

$$(\exists x(S(x) \wedge \forall y(L(y) \rightarrow A(x, y))))\wedge$$
$$(\forall x((L(x) \wedge B(x)) \rightarrow \neg\exists y(S(y) \wedge A(y, x))))\wedge$$
$$\exists x(L(x) \wedge B(x))$$

yields the CNF with refutation:

1	$S(c)$	
2	$\neg L(y) \vee A(c, y)$	
3	$\neg L(x) \vee \neg B(x) \vee \neg S(z) \vee \neg A(z, x)$	
4	$L(d)$	
5	$B(d)$	
6	$A(c, d)$	$(2, 4)$
7	$\neg B(d) \vee \neg S(z) \vee \neg A(z, d)$	$(3, 4)$
8	$\neg S(z) \vee \neg A(z, d)$	$(5, 7)$
9	$\neg A(c, d)$	$(1, 8)$
	$\perp$	$(6, 9)$

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This is done automatically by the tool `Prover9`, by calling

```
./prover9 -f stud
```

where `stud` is a file containing

```
formulas(assumptions).
(exists x (S(x) & (L(y) -> A(x,y)))).
L(x) & B(x) ->
-(exists y (S(y) & A(y,x))).
end_of_list.
formulas(goals).
-(exists z (L(z) & B(z))).
end_of_list.
```

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The Prolog graph example can also be done by `Prover9`:

```
formulas(assumptions).
arrow(a,b).
arrow(a,c).
arrow(b,c).
arrow(c,d).
arrow(x,y) -> path(x,y).
(arrow(x,z) & path(z,y)) -> path(x,y).
end_of_list.
formulas(goals).
path(a,d).
end_of_list.
```

By default, in `Prover9` x, y, z, u, v are universally quantified variables, and other names are constants, by which declarations may be omitted

---

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Syntax:

negation: `-`
conjunction: `&`, disjunction: `|`
implication: `->`, bi-implication: `<->`
exists: `exists`, forall: `all`

Terms and variables represent elements of the (unknown) model, the only (but very useful) built-in predicate on this model is equality: `=`

```
formulas(assumptions).
f(x) = g(y).
h(x) = f(a).
end_of_list.
formulas(goals).
h(a) = f(b).
end_of_list.
```

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In case `Prover9` fails, then may be the goal does not follow from the assumptions

One way to prove this is to find a model in which the assumptions hold but the goal does not hold

Automatically searching for a **finite** model with these properties may be done by the tool `Mace4`, accepting the same format

It can be proved that if the goal does not follow from the assumptions, then a model exists in which the assumptions hold but the goal does not hold, but not always a finite model exists

Conversely, it can be the case that the goal follows from the assumptions, but `Prover9` fails to prove this, so if both `Prover9` and `Mace4` fail then no conclusion can be drawn on validity

---

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Equational reasoning

We will give a minimal description of natural numbers in which $2 + 2 = 4$ makes sense and can be proved automatically

Natural numbers:

$$0, s(0), s(s(0)), s(s(s(0))), \ldots$$

These are the closed terms composed from the constant 0 and the unary symbol s

Here a term is called **closed** if it does not contain variables

We want to show that

$$s(s(0)) + s(s(0)) = s(s(s(s(0))))$$

Here $+$ is a binary operator written in infix notation

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This claim only holds if we have some basic rules for $+$:

> $+$ applied to natural numbers should yield a natural number after application of these basic rules

Here natural numbers are defined to be closed terms composed from the constant 0 and the unary symbol s

Hence we need rules by which every closed term containing the symbol $+$ can be rewritten to a closed term not containing $+$

One way to do so is:

$$0 + x = x$$

$$s(x) + y = s(x + y)$$

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What is the meaning of such rules?

- For variables (here: x, y) arbitrary terms may be substituted
- These rules may be applied on any subterm of a term that has to be rewritten

In case the rules are only allowed to be applied from left to right we write an arrow $\rightarrow$ instead of $=$

The rules are called **rewrite rules**

A set of such rewrite rules is called a

term rewrite system (TRS)

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More precisely:

A TRS R is a subset of $T \times T$, where T is the set of terms over a given set of function symbols and variables

An element $(\ell, r) \in R$ is called a **rule** and is usually written as $\ell \rightarrow r$ instead of (ℓ, r)

ℓ is called the left hand side and r is called the right hand side of the rule

The rewrite relation $\rightarrow_R$ is defined to be the smallest relation $\rightarrow_R \subseteq T \times T$ satisfying:

- $\ell\sigma \rightarrow_R r\sigma$ for every $\ell \rightarrow r$ in R and every substitution σ
- if $t_j \rightarrow_R u_j$ and $t_i = u_i$ for every $i \neq j$, then $f(t_1, \ldots, t_n) \rightarrow_R f(u_1, \ldots, u_n)$

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This last property causes that application of rules is allowed on subterms

For instance, we have

$$s(0) + (0 + s(0)) \rightarrow_R s(0) + s(0)$$

If the TRS R consists of the rules

$$0 + x \rightarrow x \quad s(x) + y \rightarrow s(x + y)$$

then indeed $2 + 2 = 4$ holds:

$$\begin{aligned} s(s(0)) + s(s(0)) &\rightarrow_R s(\underbrace{s(0) + s(s(0))}) \\ &\rightarrow_R s(s(\underbrace{0 + s(s(0))})) \\ &\rightarrow_R s(s(s(s(0)))) \end{aligned}$$

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Expressed in Prover9:

```
formulas(assumptions).
R(a(0,x),x).
R(a(s(x),y),s(a(x,y))).
R(x,y) -> R(a(x,z),a(y,z)).
R(x,y) -> R(a(z,x),a(z,y)).
R(x,y) -> R(s(x),s(y)).
RR(x,x).
(RR(x,y) & R(y,z)) -> RR(x,z).
end_of_list.
formulas(goals).
RR(a(s(s(0)),s(s(0))),s(s(s(s(0))))).
end_of_list.
```

`a` = plus operator
`R` = single rewrite step
`RR` = zero or more rewrite steps

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A term t is called a **normal form** if no u exists satisfying $t \rightarrow_R u$

Computation
=
rewrite to normal form
=
apply rewriting as long as possible

So in our example rewriting to normal form of the term $2 + 2$ represented by $(s(s(0)) + s(s(0)))$ yields the term 4 represented by $s(s(s(s(0))))$

A term t is called a **normal form of** a term u if t is a normal form and u rewrites to t in zero or more steps.

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A rewriting sequence is also called a **reduction**; it can be infinite, unfinished, or end in a normal form

Rewriting to normal form is the basic formalism in several kinds of computation

In particular, it is the underlying formalism for both semantics and implementation of **functional programming**, in which the function definitions are interpreted as rewrite rules

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Example

rev(nil) = nil
rev($a : x$) = conc(rev(x),a : nil)
conc(nil,x) = x
conc($a : x$,y) = a : conc(x, y)

Here a, x, y are variables, and = corresponds to $\rightarrow$ in rewrite rules

Then we have a reduction to normal form
rev(1:2:nil) $\rightarrow$
conc(rev(2:nil),1:nil) $\rightarrow$
conc(conc(rev(nil),2:nil),1:nil) $\rightarrow$
conc(conc(nil,2:nil),1:nil) $\rightarrow$
conc(2:nil,1:nil) $\rightarrow$
2:conc(nil,1:nil) $\rightarrow$
2:1:nil

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Without extra requirements a term can have no normal form, or more than one normal form

For instance, with respect to $f(x) \rightarrow f(x)$ the term $f(a)$ does not have a normal form

For instance, with respect to $f(f(x)) \rightarrow a$ the term $f(f(f(a)))$ has two normal forms a and $f(a)$

Now we investigate some nice properties forcing that every term has exactly one normal form

A TRS is called **weakly normalizing** (WN) if every term has at least one normal form

---

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More nice properties:

- R is **terminating** (= strongly normalizing, SN):

  no infinite sequence of terms $t_1, t_2, t_3, \ldots$ exists such that $t_i \rightarrow_R t_{i+1}$ for all i

- R is **confluent** (= Church-Rosser, CR):

  if $t \rightarrow_R^* u$ and $t \rightarrow_R^* v$ then a term w exists satisfying $u \rightarrow_R^* w$ and $v \rightarrow_R^* w$

- R is **locally confluent** (= weak Church-Rosser, WCR):

  if $t \rightarrow_R u$ and $t \rightarrow_R v$ then a term w exists satisfying $u \rightarrow_R^* w$ and $v \rightarrow_R^* w$

Here $\rightarrow_R^*$ is the reflexive transitive closure of $\rightarrow_R$, i.e., $t \rightarrow_R^* u$ if and only if t can be rewritten to u in zero or more steps

---

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Property
If a TRS is terminating, then every term has at least one normal form

Proof: rewriting as long as possible does not go on forever due to termination
So it ends in a normal form

The converse is not true: the TRS over the two constants a, b consisting of the two rules $a \rightarrow a$ and $a \rightarrow b$ is weakly normalizing since

the two terms a and b both have b as a normal form, but it is not terminating due to

$$a \to a \to a \to a \to \cdots$$

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Property

If a TRS is confluent, then every term has at most one normal form

Proof: Assume t has two normal forms u, u'

Then by confluence there is a v such that $u \to_R^* v$ and $u' \to_R^* v$

Since u, u' are normal forms we have $u = v = u'$

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Termination of term rewriting is undecidable, i.e., there is no algorithm that can decide for every finite TRS whether it is terminating

This can be proved by transforming an arbitrary Turing machine to a TRS and prove that the TRS is terminating if and only the Turing machine is halting from every initial configuration

A Turing machine (Q, S, δ) consists of

- a finite set Q of machine states
- a finite set S of tape symbols, including $\square \in S$ representing the blank symbol
- the transition function $\delta : Q \times S \to Q \times S \times \{L, R\}$

Here $\delta(q, s) = (q', s', L)$ means that if the machine is in state q and reads s, this s is replaced by s', the machine shifts to the left, and the new machine state is q'

Similar for $\delta(q, s) = (q', s', R)$: then the machine shifts to the right

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This Turing machine behaviour can be simulated by a TRS: for a Turing machine $M = (Q, S, \delta)$ we define a TRS $R(M)$ over $Q \cup S \cup \{b\}$ where

- symbols from Q are binary
- symbols from S are unary
- b is a constant representing an infinite sequence of blank symbols

The configuration with tape

$$\cdots \square\square\square s'_m s'_{m-1} \cdots s'_1 s_1 s_2 \cdots s_{n-1} s_n \square\square\square \cdots$$

in which the Turing machine is in state q and reads symbol s_1 is represented by the term

$$q(s'_1(s'_2(\cdots(s'_m(b))\cdots)), s_1(s_2(\cdots(s_n(b))\cdots)))$$

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For the Turing machine $M = (Q, S, \delta)$ the TRS $R(M)$ is defined to consist of the rules

$$q(x, s(y)) \to q'(s'(x), y)$$

for all $(q, s) \in Q \times S$ with $\delta(q, s) = (q', s', R)$, and

$$q(t(x), s(y)) \to q'(x, t(s'(y)))$$

for all $(q, s) \in Q \times S$ with $\delta(q, s) = (q', s', L)$, for all $t \in S$

and some extra rules representing b to consist of blank symbols

Theorem

M halts on every configuration if and only if $R(M)$ is terminating

Consequence: TRS termination is undecidable

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Although termination is undecidable, in many special cases termination of a TRS can be proved

General technique:

Find a weight function W from terms to natural numbers in such a way that $W(u) > W(v)$ for all terms u, v satisfying $u \to_R v$

If such a function W exists then R is terminating since an infinite rewriting sequence would give rise to an infinite decreasing sequence of natural numbers which does not exist

---

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In our example

$$+(0, x) \to x$$

$$+(s(x), y) \to s(+(x, y))$$

we find such a weight function W by defining inductively

$W(0) = 1$
$W(s(t)) = W(t) + 1$
$W(t + u) = 2W(t) + W(u)$

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The general idea of weight functions is too general:

It allows arbitrary definitions of weight functions, and we have to prove that $W(t) > W(u)$ for **all** rewrite steps $t \to_R u$, while typically there are infinitely many of them

Now we work out a special case of this idea of weight functions in such a way that for finding a termination proof we only have to

- choose interpretations for the (finitely many) operation symbols rather than for all terms, and
- check $W(\ell) > W(r)$ for the (finitely many) rules $\ell \to r$ rather than for all rewrite steps

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For every symbol f of arity n choose a **monotonic** function $[f] : \mathbf{N}^n \to \mathbf{N}$

Here **monotonic** means:

if for all $a_i, b_i \in \mathbf{N}$ for $i = 1, \ldots, n$ with $a_i > b_i$ for some i and $a_j = b_j$ for all $j \neq i$ then

$$[f](a_1, \ldots, a_n) \; > \; [f](b_1, \ldots, b_n)$$

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Examples

$\lambda x \cdot x$
$\lambda x \cdot x + 1$
$\lambda x \cdot 2x$
$\lambda x, y \cdot x + y$
$\lambda x, y \cdot x + y + 1$
$\lambda x, y \cdot 2x + y$
are monotonic

$\lambda x \cdot 2$
$\lambda x, y \cdot x$
are **not** monotonic

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For a map $\alpha : \mathcal{X} \to \mathbf{N}$ the weight function $[\cdot, \alpha] : T \to \mathbf{N}$ is defined inductively by

$$[x, \alpha] = \alpha(x),$$

$$[f(t_1, \ldots, t_n), \alpha] = [f]([t_1, \alpha], \ldots, [t_n, \alpha])$$

Theorem
Let R be a TRS and let $[f]$ be chosen such that

- $[f]$ is monotonic for every symbol f, and
- $[\ell, \alpha] > [r, \alpha]$ for every $\alpha : \mathcal{X} \to \mathbf{N}$ and every rule $\ell \to r$ in R

Then R is terminating

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Example

For our TRS R consisting of the rules

$$+(0,x) \to x \quad +(s(x),y) \to s(+(x,y))$$

we choose monotonic functions

$$[0] = 1, \quad [s](x) = x + 1$$

$$[+](x,y) = 2x + y$$

Now indeed for every $\alpha : \mathcal{X} \to \mathbf{N}$ we have

$$[+(0,x),\alpha] = 2 + \alpha(x) > \alpha(x) = [x,\alpha]$$

and

$$[+(s(x),y),\alpha] = 2(\alpha(x)+1) + \alpha(y) >$$

$$(2\alpha(x) + \alpha(y)) + 1 = [s(+(x,y)),\alpha]$$

proving termination

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Example

For the TRS R consisting of the single rule

$$f(g(x)) \to g(g(f(x)))$$

we choose monotonic functions

$$[f](x) = 3x$$

$$[g](x) = x + 1$$

Now indeed for every $\alpha : \mathcal{X} \to \mathbf{N}$ we have

$$[f(g(x)),\alpha] = 3(\alpha(x)+1) >$$

$$3\alpha(x) + 1 + 1 = [g(g(f(x))),\alpha]$$

proving termination

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Example

The single rule $f(x) \to g(f(x))$ is not terminating, but by choosing

$$[f](x) = x + 1, \quad [g](x) = 0$$

for every $\alpha : \mathcal{X} \to \mathbf{N}$ we have

$$[f(x),\alpha] = \alpha(x) + 1 > 0 = [g(f(x)),\alpha]$$

Where is the error?

$[g]$ is not monotonic

So monotonicity is essential

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Another technique: **lexicographic path order**

Choose an order $>$ on the set of function symbols

Theorem
If $\ell >_{lpo} r$ for all $\ell \to r$ in R, then R is terminating

Before this makes sense we have to define / characterize $>_{lpo}$

$f(t_1,\ldots,t_n) >_{lpo} u \iff$

- $\exists i : t_i = u \vee t_i >_{lpo} u$, or
- $u = g(u_1,\ldots,u_m)$ and
  $\forall i : f(t_1,\ldots,t_n) >_{lpo} u_i$ and either
  - $f > g$, or
  - $f = g$ and
    $(t_1,\ldots,t_n) >_{lpo}^{lex} (u_1,\ldots,u_m)$

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Lemma
If s is a proper subterm of t, then $t >_{lpo} s$

Easily follows from first bullet

Example
For the rule

$$+(0,x) \to x$$

we have $+(0,x) >_{lpo} x$ by this lemma

For the rule

$$+(s(x),y) \to s(+(x,y))$$

we choose $+ > s$, then by the second item it remains to prove

$$+(s(x), y) >_{lpo} +(x, y)$$

Again using the second item we have to prove

- $+(s(x), y) >_{lpo} x$, follows from lemma
- $+(s(x), y) >_{lpo} y$, follows from lemma
- $(s(x), y) >_{lpo}^{lex} (x, y)$, follows from $s(x) >_{lpo} x$

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Hence $\ell >_{lpo} r$ for all rules $\ell \to r$, proving termination

More interesting example: **Ackermann function**

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m,n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

Looks simple, but $A(4,2)$ turns out to be an integer of 19,729 decimal digits

It is the simplest function that is not **primitive recursive**, and grows much harder than any primitive recursive function

Expressed as TRS:

$$\begin{array}{rcl} A(0,x) & \to & s(x) \\ A(s(x),0) & \to & A(x,s(0)) \\ A(s(x),s(y)) & \to & A(x,A(s(x),y)) \end{array}$$

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$$\begin{array}{rcl} A(0,x) & \to & s(x) \\ A(s(x),0) & \to & A(x,s(0)) \\ A(s(x),s(y)) & \to & A(x,A(s(x),y)) \end{array}$$

Termination can be proved by lexicographic path order, $A > s$

So termination theoretically holds, but the normal form of $A(s(s(s(0)))), s(s(0)))$ is a term containing N symbols s, for N being a number of 19,729 decimal digits, so the normal form does not fit in all computer memory of the world

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Checking termination by lexicographic path order is easy to implement; do not choose $>$ in advance but collect requirements on $>$ during the process of proving $\ell >_{lpo} r$

Several more techniques for proving termination have been developed

Several tools have been developed by which termination of a TRS can be proved fully automatically: AProVE, TTT2

Techniques for proving termination of TRSs also form the basis of several techniques for automatically proving termination of programs

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Back to the other properties

Confluence is strictly stronger than local confluence:

$$a \to b, \quad b \to a, \quad a \to c, \quad b \to d$$

is locally confluent:

if $t \to_R u$ and $t \to_R v$ then either

- $t = a$, then choose $w = c$, or
- $t = b$, then choose $w = d$

In both cases we conclude $u \to_R^* w$ and $v \to_R^* w$

but not confluent:

for $t = a, u = c, v = d$ we have $t \to_R^* u$ and $t \to_R^* v$, but no w exists satisfying $u \to_R^* w$ and $v \to_R^* w$

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Newman's lemma (1942):

Theorem

For terminating TRSs the properties confluence and local confluence are equivalent

For the proof of Newman's lemma we will use the principle of well-founded induction

Note that SN($\to$), CR($\to$) and WCR($\to$) all can be defined for arbitrary binary relations $\to$, in which general setting we will prove Newman's lemma

So SN($\to$) simply means the non-existence of an infinite sequence $t_1 \to t_2 \to t_3 \to \cdots$

We write $\to^+$ for the transitive closure of $\to$: one or more steps

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Principle of well-founded induction

Theorem

Let SN($\to$) and

$\forall t(\underbrace{\forall u(t \to^+ u \Rightarrow P(u))}_{\text{Induction Hypothesis}} \Rightarrow P(t))$

Then $P(t)$ holds for all t

(think of $t \to^+ u$ as $t > u$ as in well-known induction)

Proof of this principle

Assume there exists t such that $\neg P(t)$

Then the induction hypothesis does not hold for this t, so $\neg\forall u(t \to^+ u \Rightarrow P(u))$, yielding u such that $t \to^+ u$ and $\neg P(u)$

Repeat the argument for u, yielding a v, and so on, so yielding an infinite sequence

$$t \to^+ u \to^+ v \to^+ \cdots$$

contradicting SN($\to$) (End of proof)

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Proof of Newman's Lemma

Assume SN($\to$) and WCR($\to$), we have to prove CR($\to$)

We apply the principle of well-founded induction for $P(t)$ being

$$\forall u, v : \text{if } t \to^* u \wedge t \to^* v \text{ then}$$

$$\exists w : u \to^* w \wedge v \to^* w$$

So assume $t \to^* u$ and $t \to^* v$; we have to find w such that $u \to^* w$ and $v \to^* w$

If $t = u$ we may choose $w = v$

if $t = v$ we may choose $w = u$

In the remaining case we have $t \to^+ u$ and $t \to^+ v$

Write $t \to u_1 \to^* u$ and $t \to v_1 \to^* v$

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Using WCR there exists w_1 such that $u_1 \to^* w_1$ and $v_1 \to^* w_1$

Using the induction hypothesis on u_1 there exists w_2 such that $w_1 \to^* w_2$ and $u \to^* w_2$

Now we have $v_1 \to^* w_2$ and $v_1 \to^* v$; using the induction hypothesis on v_1 there exists w such that $w_2 \to^* w$ and $v \to^* w$

t	$\to$	u_1	$\to^*$	u
$\downarrow$	WCR	$\downarrow_*$	IH	$\downarrow_*$
v_1	$\to^*$	w_1	$\to^*$	w_2
$\downarrow_*$		IH		$\downarrow_*$
v		$\to^*$		w

Since $u \to^* w_2$ we have $u \to^* w$, and we are done

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Both confluence and local confluence are undecidable properties

However, for terminating TRSs there is a simple decision procedure for local confluence, and hence for confluence too

> Idea:
>
> analyze overlapping patterns in left hand sides of the rules, yielding **critical pairs**

In our example for addition of natural numbers there is no overlap, hence it is locally confluent

Since we observed it is terminating, by Newman's lemma it is confluent

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Definition of critical pairs

Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ be two (possibly equal) rewrite rules

Rename variables such that ℓ_1, ℓ_2 have no variables in common

Let t be a subterm of ℓ_2, possibly equal to ℓ_2; t is not a variable

Assume t, ℓ_1 unify, with mgu σ: $t\sigma = \ell_1\sigma$

Now $\ell_2\sigma$ can be rewritten in two ways:

- to $r_2\sigma$, and
- to a term u obtained by replacing its subterm $t\sigma = \ell_1\sigma$ to $r_1\sigma$

In the above situation the pair $[u, r_2\sigma]$ is called a **critical pair**

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Example

Assume we have rules for arithmetic including

$$\begin{array}{lcl} x - x & \to & 0 \\ s(x) - y & \to & s(x - y) \end{array}$$

Then $s(x) - s(x)$ can be rewritten in two ways:

- to 0 by the first rule
- to $s(x - s(x))$ by the second rule

Now $[0, s(x - s(x))]$ is a **critical pair**

More precisely, in the above notation we choose

- $\ell_1 \to r_1$ to be the rule $z - z \to 0$
- $\ell_2 \to r_2$ to be the rule $s(x) - y \to s(x - y)$
- $t = \ell_2 = s(x) - y$

Indeed t, ℓ_1 unify, with mgu σ:

$$\sigma(x) = x, \;\; \sigma(y) = \sigma(z) = s(x)$$

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Example

Let R consist of the single rule

$$f(f(x)) \to g(x)$$

By choosing

- $\ell_1 \to r_1$ to be the rule $f(f(x)) \to g(x)$
- $\ell_2 \to r_2$ to be the rule $f(f(y)) \to g(y)$
- $t = f(y)$

we see that t, ℓ_1 unify, with mgu σ:

$$\sigma(x) = x, \;\; \sigma(y) = f(x)$$

yielding the critical pair $[f(g(x)), g(f(x))]$

A critical pair $[t, u]$ is said to **converge** if there is a term v such that $t \to_R^* v$ and $u \to_R^* v$

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Theorem

A TRS R is locally confluent if and only if all critical pairs converge

Example

The single rewrite rule $f(f(x)) \to g(x)$ is not locally confluent, so neither confluent, since

for its critical pair $[f(g(x)), g(f(x))]$ no term v exists such that

$$f(g(x)) \to_R^* v \quad \text{and} \quad g(f(x)) \to_R^* v$$

This is immediate from the observation that both $f(g(x))$ and $g(f(x))$ are normal forms

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For a term t and a TRS R define

$$S(t) = \{v \mid t \to_R^* v\}$$

If R is finite and terminating then $S(t)$ is finite and computable

Using the theorem, for a finite terminating TRS R indeed we have an algorithm to decide whether WCR(R) holds:

- Compute all critical pairs $[t, u]$

  They are found by unification of left hand sides with subterms of left hand sides: there are finitely many of them

- For all critical pairs $[t, u]$ compute

  $$S(t) \cap S(u)$$

- If one of these sets is empty then WCR(R) does not hold
- If all of these sets are non-empty then WCR(R) holds

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A TRS is said to have **no overlap** if there are only **trivial** critical pairs, i.e., the critical pairs obtained by unifying a left hand side with itself

A trivial critical pair always converges since it is of the shape $[t, t]$

As a consequence, every TRS having no overlap is locally confluent

It is not the case that every TRS having no overlap is confluent:

$$\begin{array}{rcl} d(x,x) & \to & b \\ c(x) & \to & d(x, c(x)) \\ a & \to & c(a) \end{array}$$

has no overlap but is not confluent:

$$c(a) \to_R d(a, c(a)) \to_R d(c(a), c(a)) \to_R b$$

$$c(a) \to_R c(c(a)) \to_R^+ c(b)$$

while $[b, c(b)]$ does not converge

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Write $\leftrightarrow_R^*$ for the reflexive symmetric transitive closure of $\to_R$, i.e., $t \leftrightarrow_R^* u$ holds if and only if terms $t_1, \ldots, t_n$ exist for $n \geq 1$ such that

- $t_1 = t$
- $t_n = u$
- For every $i = 1, \ldots, n-1$ either $t_i \to_R t_{i+1}$ or $t_{i+1} \to_R t_i$ holds

A general question is: given R, t, u, does $t \leftrightarrow_R^* u$ hold?

This is called the **word problem**

In general the word problem is undecidable

However, in case R is terminating and confluent then the word problem is decidable and admits a simple algorithm

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A terminating and confluent TRS is called **complete**

Now we give a decision procedure for the word problem for complete TRSs

Rewriting a term t in a terminating TRS as long as possible will always end in a normal form; the result is called a **normal form of** t

Theorem

If R is a complete TRS and t', u' are normal forms of t, u, then $t \leftrightarrow_R^* u$ if and only if $t' = u'$

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For the proof we need a lemma that is easily proved by induction on the length of the path corresponding to $t \leftrightarrow_R^* u$:

> Lemma:
>
> If R is confluent and $t \leftrightarrow_R^* u$ then a term v exists such that $t \rightarrow_R^* v$ and $u \rightarrow_R^* v$

Proof of the theorem:

If $t' = u'$ then $t \rightarrow_R^* t' = u' \leftarrow_R^* u$, hence $t \leftrightarrow_R^* u$

Conversely assume $t \leftrightarrow_R^* u$

Then $t' \leftarrow_R^* t \leftrightarrow_R^* u \rightarrow_R^* u'$, hence $t' \leftrightarrow_R^* u'$

According the lemma a term v exists such that $t' \rightarrow_R^* v$ and $u' \rightarrow_R^* v$

Since t', u' are normal forms we have $t' = v = u'$ End of proof

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The relation $\leftrightarrow_R^*$ is an equivalence relation, and in a complete TRS the normal form is a unique representation for the corresponding equivalence class

According to the theorem there is a very simple decision procedure for the word problem for complete TRSs:

In order to decide whether $t \leftrightarrow_R^* u$, rewrite

- t to a normal form t', en
- u to a normal form u',

Then $t \leftrightarrow_R^* u$ if and only if $t' = u'$

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Example:

R consists of the rule $s(s(s(x))) \rightarrow x$

Does $s^{17}(0) \leftrightarrow_R^* s^{10}(0)$ hold?

We can establish fully automatically that this is not:

- check that R is terminating
- check that R is locally confluent
- compute the normal form $s(s(0))$ of $s^{17}(0)$
- compute the normal form $s(0)$ of $s^{10}(0)$
- these are different, hence the answer is **No**

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Often a TRS R is not complete, but a complete TRS R' satisfying

$$\leftrightarrow_{R'}^* \; = \; \leftrightarrow_R^*$$

can be found in a systematic way

Finding such a complete TRS is called

(Knuth-Bendix) completion

The new complete TRS can be used for the word problem and unique representation of the original TRS

Often the original TRS is only a set of equations

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Idea of Knuth-Bendix completion

Fix a well-founded order $>$ on terms, i.e., SN($>$), that has some closedness properties:

- if $t > u$ then $t\sigma > u\sigma$ for every substitution σ

- if $t > u$ then $f(\ldots, t, \ldots) > f(\ldots, u, \ldots)$ for every symbol f and every position for t

Such an order is called a **reduction order**, and has the property:

> If $\ell > r$ for every rule $\ell \to r$ in R,
> then $\mathrm{SN}(R)$

A typical example of a reduction order is a lexicographic path order

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Starting with a set E of equations and an empty set R of rewrite rules, repeat the following until E is empty:

Remove an equation $t = u$ from E, and

- add $t \to u$ to R if $t > u$
- add $u \to t$ to R if $u > t$
- give up otherwise

After adding any new rule $\ell \to r$ to R compute all critical pairs between this new rule and existing rules of R, or between the new rule and itself

For every such critical pair $[t, u]$

- R-rewrite t to normal form t'
- R-rewrite u to normal form u'
- if $t' \neq u'$, then add $t' = u'$ as an equation to the set E

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What can happen in this Knuth-Bendix procedure?

- it fails due to an equation $t = u$ in E for which neither $t > u$ nor $u > t$ holds
- it fails since the procedure goes on forever: E gets larger and is never empty
- it ends with E being empty

In the last case we really have success: then

- R is terminating since it only contains rule $\ell \to r$ satisfying $\ell > r$
- R is locally confluent since all critical pairs converge, so R is complete
- Convertibility $\leftrightarrow^*_R$ of the resulting R is equivalent to convertibility of the original E since in the whole procedure $\leftrightarrow^*_{R \cup E}$ remains invariant

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Example:

Let E consist of the single equation

$$f(f(x)) = g(x)$$

Choose the lexicographic path order defined by $f > g$

Since

$$f(f(x)) >_{lpo} g(x)$$

we add the rule $f(f(x)) \to g(x)$ to the empty TRS R

Now the critical pair $[f(g(x)), g(f(x))]$ gives rise to the new equation $f(g(x)) = g(f(x))$ in E

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Since

$$f(g(x)) >_{lpo} g(f(x))$$

we add the rule $f(g(x)) \to g(f(x))$ to the TRS R

Together with the older rule $f(f(x)) \to g(x)$ we get the critical pair $[f(g(f(x))), g(g(x))]$

Since $g(g(x))$ is a normal form and

$$f(g(f(x))) \to_R g(f(f(x))) \to_R g(g(x))$$

no new equation is added to E, and E is empty

So we end up in the complete TRS R consisting of the two rules

$$f(f(x)) \to g(x), \quad f(g(x)) \to g(f(x))$$

having the same convertibility relation as the original equation $f(f(x)) = g(x)$

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Example:

For decision trees we consider the set E of equations

$$\begin{array}{rcl} p(x,x) & = & x \\ p(q(x,y),q(z,w)) & = & q(p(x,z),p(y,w)) \\ p(p(x,y),z) & = & p(x,z) \\ p(x,p(y,z)) & = & p(x,z) \end{array}$$

where p, q runs over all boolean variables

It can be proved that two decision trees represent the same boolean function if and only they are equivalent with respect to $\leftrightarrow_E^*$

As a TRS E is not terminating and not confluent

Choose any order $>$ on the boolean variables

Completion yields the TRS R consisting of the rules

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$$\begin{array}{rcll} p(x,x) & \to & x & \text{for all } p \\ p(p(x,y),z) & \to & p(x,z) & \text{for all } p \\ p(x,p(y,z)) & \to & p(x,z) & \text{for all } p \\ p(q(x,y),z) & \to & q(p(x,z),p(y,z)) & \text{for } p > q \\ p(x,q(y,z)) & \to & q(p(x,y),p(x,z)) & \text{for } p > q \end{array}$$

Now rewriting to normal form in R yields the unique representation as an ordered decision tree; unicity is a consequence of completeness of R'

Storing by sharing common subterms yields the ROBDD

Arbitrary formulas are easily transformed to (unordered) BDDs representing the same boolean function; R-rewriting now yields an alternative method for computing ROBDDs

Unfortunately this is not very efficient

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Overview of the course

- Proposition logic
  - SAT by resolution: DP and DPLL and backjump
  - Tseitin transformation
  - (RO)BDDs
  - Symbolic model checking: `NuSMV`
- Extension to SMT
  - Tools: `Yices` and `Z3`
  - Underlying mechanism: linear programming, simplex algorithm
- Predicate logic
  - Resolution, unification
  - Prenex normal form, Skolemization
  - Prolog, `Prover9`
- Equational reasoning by term rewriting
  - termination by monotonic interpretation or lexicographic path order
  - (local) confluence by critical pair analysis
  - Knuth-Bendix completion