Reflections on Prismatic Constructions

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The Calculus of Prismatic Constructions, upon which this platform is based, is an extension of the standard CoC with a mechanism for discriminating inductive constructors.

The pure Calculus of Construction

It is already very well-described elsewhere, so I won't try to provide a full and correct history of the CoC. Suffice to say that it is a logically consistent programming language, that can prove properties within the framework of intuitionistic logic.

At its simplest, it provides five basic constructions :

- universes, of the form Set_n, are the "types of types". Set_{n+1} is the type of Set_n
- products, noted $\forall(x : X), Y\,x$ – or $X \;\rightarrow\; Y$ when Y doesn't depend on x – are the "types of functions". $\mathbb{N} \;\rightarrow\; \mathbb{R}$, for instance is the type of functions from the natural numbers to the real numbers.
- functions or lambdas, noted $\lambda(x : X), Y\,x$, are the "proofs of products". A valid lambda can be interpreted as the proof of a property, quantified over its variable.
- hypotheses, or variables, are the symbols introduced by surrounding quantifiers (λ and $\forall$). In their context, they are valid proofs of their type.

  For example, the identity function can be written $\lambda(A : Set_0), \lambda(a : A), a$, and it is a valid proof of $\forall(A : Set_0), \forall(a : A), A$, since a is a valid proof of A in its context.
- Applications, of the form $f\,x$, where $f : \forall(x : X), Y\,x$ and $x : X$, signify the specialization of a quantified property over an object x.

  For instance, given a proof f of $\forall(x : \mathbb{N}), \exists(y : \mathbb{N}), y = x + 1$, we can prove that $\exists(y : \mathbb{N}), y = 10 + 1$, by applying f to 10 (aka. $f\,10$).

Given these axioms, we can build many theorems and their proofs, in a verifiable manner (i.e. there exists an algorithm to automatically check whether a claim like $x : X$ holds).

However, it's been known for a while that the CoC by itself is not capable of handling a large class of the proofs that modern mathematicians (and even ancient ones) take for granted.

The Limits of Constructions

To illustrate the kind of reasoning that can't be carried out with raw intuitionistic logic, let's take an obvious statement : a boolean is either equal to true or to false.

We'd like to prove this statement using only the tools given by the CoC. For this, we have to define a few concepts, namely our Booleans, $true$ and $false$, and what it means for two things to be equal.

Intuitionistic Booleans

In order for two things to be considered the same, they must at least belong to the same family. In this case, it means that $true$ and $false$ must have the same type. By convention, we'll call the type of "true or false" the $Boolean$ type, in honor of George Boole.

Given a Boolean b, we would like to be able to return different values from a function, depending on whether b is true or false. Otherwise, our Boolean wouldn't be much use in a computation.

With all that in mind, here is the definition I propose the $Boolean$ type :

$$Boolean \equiv \forall (P : Prop)(ptrue : P)(pfalse : P), P$$

That is, a Boolean is a way to produce any P, given two alternatives $ptrue$ and $pfalse$, and nothing else.

There are, intuitively, only two distinct ways to construct a closed Boolean term, given the above definition :

- $true \equiv \lambda (P : Prop)(ptrue : P)(pfalse : P).ptrue$
- $false \equiv \lambda (P : Prop)(ptrue : P)(pfalse : P).pfalse$

Our goal in the following sections will be to try and confirm this intuition, by proving it in the CoC.

Sameness (aka. Identity)

Two values x and y can be said to be the same (when it comes to proving theorems) if everything that can be proven of x can also be proven of y. More formally, given a type A of things, and two values x and y of type A we have :

$$(x \ = \ y) \equiv \forall (P : A \ \rightarrow \ Set_n), P\,x \ \rightarrow \ P\,y$$

We can easily prove some intuitive properties for the = relation, such as :

- reflexivity : $(x = x)$, as proven by $\lambda(P : A \to Set_n)(p : Px).p$
- symmetry : $(x = y) \to (y = x)$, proven by $\lambda(e : x = y)(P : A \to Set_n)(py : Py), e\,(\lambda(a : A).P\,a \to P\,x)\,(\lambda(px : P\,x).px)\,py$
- transitivity : $(x = y) \to (y = z) \to (x = z)$, as proven by $\lambda(e_1 : x = y)(e_2 : y = z)(P : A \to Set_n)(px : Px).e_2\,P\,(e_1\,P\,px)$

Putting it all together

We now have everything we need to prove that every Boolean is either $true$ or $false$. First, let's formally state that property :

$$\forall(b : Boolean), (b = true) \cup (b = false) \equiv \forall(b : Boolean)(P : Set_n), (b = true \to P) \to (b = false \to P)$$

Inductive Types

Inductive types can be described as enumerations of constructors. In Coq (and similarly in other proof assistants), an inductive type must be declared along with its constructors, using a syntax like :

```
Inductive T : forall A..., Type :=
| t0 : forall x0..., T (f0... x0...)
...
| tn : forall xn..., T (fn... xn...)
.
```

Here, we declare the inductive type $T : \forall A..., Type$, and its constructors called t_i $(i \in \{0..n\})$.

As a more concrete example, here is how the type of Booleans can be defined inductively :

```
Inductive Boolean : Type := true : Boolean | false : Boolean.
```

The above definition is essentially a formal statement of the following description of Booleans : a Boolean can have one of two shapes, $true$ or $false$, and cannot be any other thing.

This means that, if we want to prove a property $P\,x$ for some unknown Boolean x, all we need is to prove $P\,true$ and $P\,false$.

This exact information is summed up in what we call the *induction principle* for Booleans. In Coq, it will be given the name `Boolean_rect`, for instance, and have the type $\forall(P : Boolean \to Type),\ P\,true \to P\,false \to \forall(b : Boolean),\ P\,b.$

The μ Combinator : Lambda-constructions with extra steps

$$\mu(x) \equiv \mu_{\varnothing}(x)$$

$$\mu_{\Gamma}(\lambda(x : T).y) \equiv \mu_{\Gamma,(x:T)}(y)$$

$$\mu_{\Gamma}(Hx...) \equiv \lambda^{*}\Gamma^{\uparrow}.\Gamma^{\uparrow}[Hx]...$$

$$\varnothing^{\uparrow} \equiv \varnothing(\Gamma, (x : T_{\Gamma}))^{\uparrow} \equiv \Gamma^{\uparrow}, (x : T_{\Gamma^{\uparrow}})$$