

Applied Statistical Analysis I

Bivariate regression, inference, and prediction

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Quick revision

What is correlation?

- Correlation describes the extent to which two features of the world tend to occur together.
- If higher values of one feature are usually seen with higher values of the other, they are **positively correlated**.
- If the two features show no systematic pattern together, they are **uncorrelated**.
- If higher values of one feature are usually seen with lower values of the other, they are **negatively correlated**.

Measuring correlation

- **Covariance:** the average product of deviations of two quantitative variables from the mean:

$$\text{cov}(X, Y) = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{n}$$

- We only interpret the sign, not the magnitude of the association, given that covariance is scale-dependent.

Measuring correlation

- **Pearson's r** (or *correlation coefficient*): it standardizes average of the product of deviations of two variables from the mean (=standardized covariance)
- We standardize the covariance by dividing by the product of standard deviations of the two variables:

$$r_{xy} = \frac{\text{cov}(X,Y)}{S_x S_y}$$

- Basically, we remove the units and scales from the covariance and get a comparable measure of association.
- It ranges from -1 to 1, with $r = 0$ meaning no correlation.

Inference: Correlation (r vs. ρ)

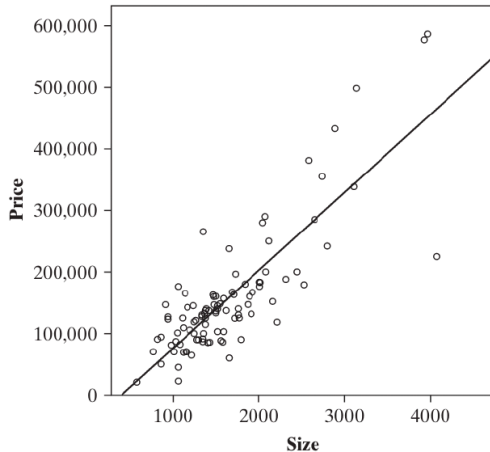
How can we test the statistical significance of correlation?

- Null and alternative hypotheses:
 - H_0 : X and Y are not correlated $\Rightarrow \rho_{xy} = 0$
 - H_a : X and Y are correlated $\Rightarrow \rho_{xy} \neq 0$
- Test statistic:

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

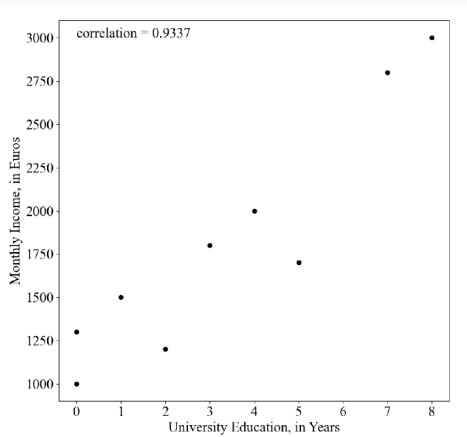
An example

- Is there a correlation between house selling price and house size?
- $r = 0.83378$
- $t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} = \frac{0.834\sqrt{98}}{\sqrt{1-0.834^2}} = \mathbf{14.95}$
- How do we interpret this value?
- It tells us how likely we are to observe data in the sample under the assumption that H_0 is true.



Revision: Bivariate regression

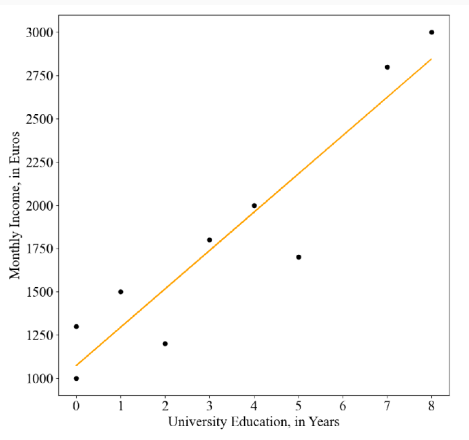
Regression analysis



Just by looking at the plot, can you identify the straight line which best describes the joint variation between X and Y ?

Regression analysis

Find the line with the best fit: $Y_i = \alpha + \beta X_i + \epsilon_i$



Linear regression model:

$$Y_i = \alpha + \beta X_i + \epsilon_i$$

- \hat{Y}_i — Predicted outcome: $\hat{Y}_i = \hat{\alpha} + \hat{\beta}X_i$
- α — Intercept: expected value of Y when $X = 0$
- β — Slope: expected change in Y for a one-unit increase in X
- ϵ_i — Error / residual: difference between the observed and predicted value

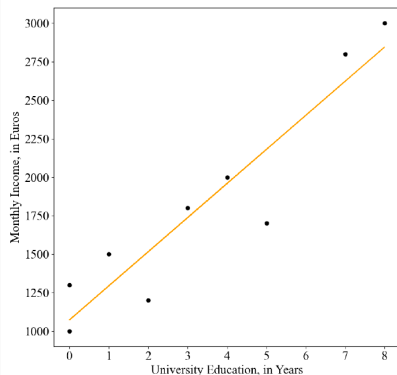
$$\epsilon_i = Y_i - \hat{Y}_i$$

Regression Analysis – Fitted Model

Estimated regression:

$$\widehat{\text{income}} = \hat{\alpha} + \hat{\beta} \cdot \text{education}$$

$$\widehat{\text{income}} = 1072.55 + 221.57 \cdot \text{education}$$



Regression Analysis – Interpreting Coefficients

Intercept $\hat{\alpha} = 1072.549$

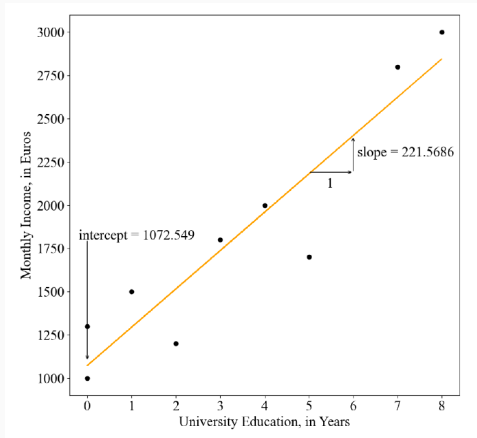
- Expected income when education = 0 years:

$$\begin{aligned}\widehat{\text{income}} &= 1072.549 + 221.5685 \cdot 0 \\ &= 1072.549\end{aligned}$$

Slope $\hat{\beta} = 221.5685$

- Each additional year of university education increases expected income by **221.57** euros, on average:

$$\begin{aligned}\widehat{\text{income}} &= 1072.549 + 221.5685 \cdot 1 \\ &= 1294.1175\end{aligned}$$

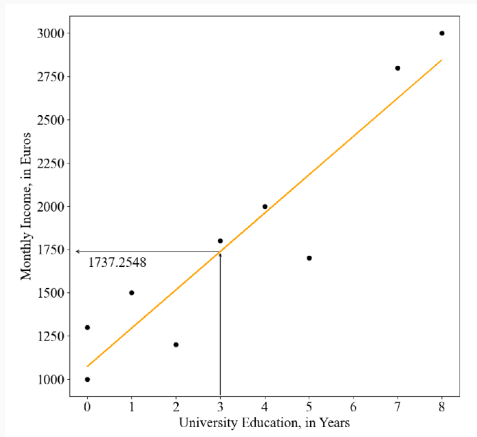


Regression Analysis – Making Predictions

Example: education = 3 years

$$\widehat{\text{income}} = 1072.55 + 221.57 \cdot 3 = 1737.25$$

We can plug any X value (years of education) into the model to predict expected income.



Regression Analysis – Residuals

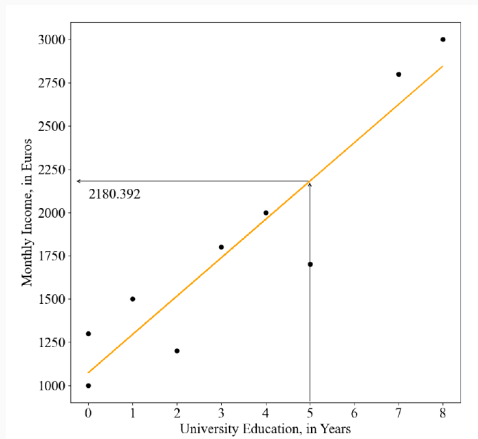
For a person with education = 5 years:

$$\widehat{\text{income}} = 1072.55 + 221.57 \cdot 5 = 2180.39$$

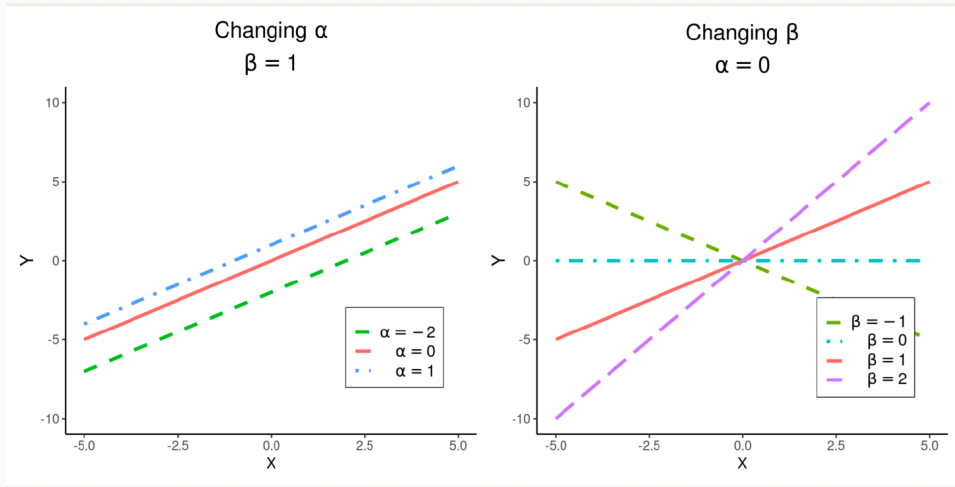
If actual income = 1700:

$$\text{Residual} = Y - \hat{Y} = 1700 - 2180.39 = -480.39$$

Negative residual \rightarrow observed income is below predicted.



Regression Analysis



Varieties of linear relationships

Ordinary Least Squares (OLS)

Ordinary Least Squares (OLS)

- OLS is short for “Ordinary Least Squares”
- The best line is the line that **minimizes** the **sum of squared errors** (SSE)
- The residuals are the vertical deviations from the line (the observed fitting errors):

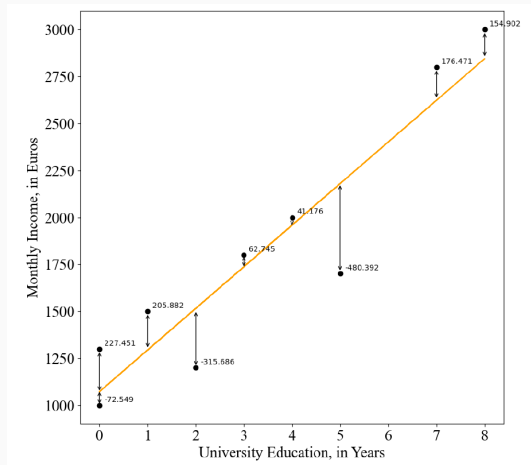
$$e_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i) = y_i - \hat{y}_i$$

- **SSE**: the sum of squared differences between the actual and predicted values of Y.

$$SSE = \sum_{i=1}^n (\hat{e}_i)^2 = \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^n (Y_i - \hat{\alpha} - \hat{\beta} X_i)^2$$

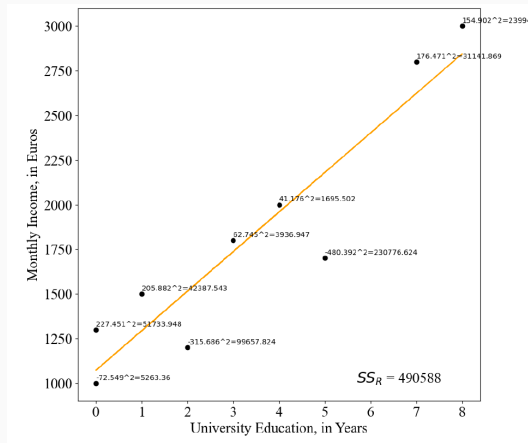
- The goal is to minimize this!

Ordinary Least Squares (OLS)



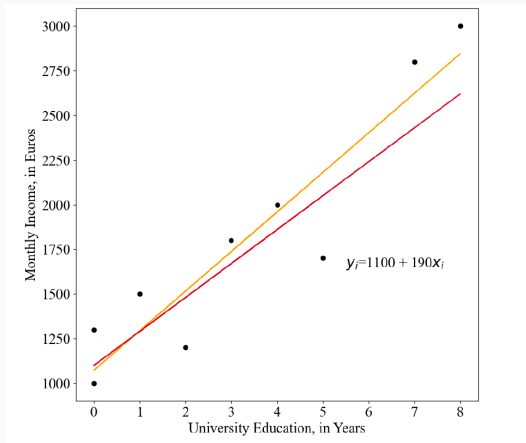
Residuals are the vertical distances between observed points and the regression line.

Ordinary Least Squares (OLS)



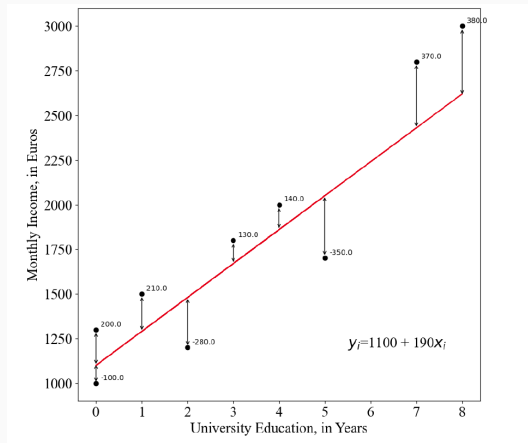
OLS chooses the line that minimizes the squared residuals (errors).

Ordinary Least Squares (OLS)



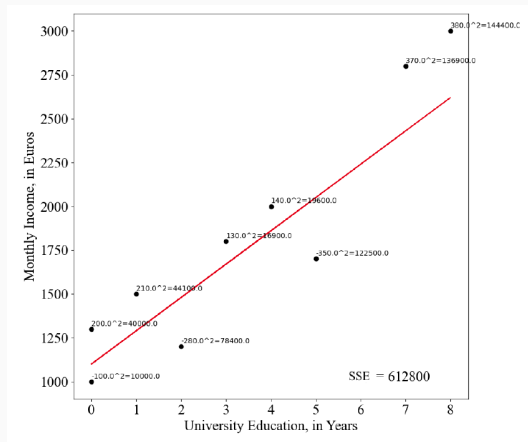
Another possible line — but its total squared residuals (SSE) are larger.

Ordinary Least Squares (OLS)



We are squaring the new residulas to compare lines by their **sum of squared errors (SSE)**.

Ordinary Least Squares (OLS)



$$612,800 > 490,588 \Rightarrow SSE_{\text{red}} > SSE_{\text{orange}}$$

The orange line has the smaller SSE: it is the better fit.

How to pick the best line?

- How to pick the best line? Get the **best slope** and **best intercept** using differential calculus.
- For $\text{Var}(x) \neq 0$, the **slope coefficient** $\hat{\beta}_1$ is given by

$$\hat{\beta}_1 = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2} = \frac{\text{Cov}(x, y)}{\text{Var}(x)}.$$

- The **intercept coefficient** $\hat{\beta}_0$ is given by

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad \text{where} \quad \bar{y} = \sum_{i=1}^n \frac{y_i}{n}, \quad \bar{x} = \sum_{i=1}^n \frac{x_i}{n}.$$

- An estimator is **unbiased** if its expected value is identical to the population value.
- The OLS estimator is **best** in the sense that it has the lowest variance among all unbiased estimators.

Getting the coefficients

In the lecture you saw how least squares (LS) are point estimates for parameters:

$$\hat{\alpha} = \hat{y} - \hat{\beta}\bar{x} \quad \text{and} \quad \hat{\beta}_1 = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

These are the best estimates according to the **Gauss-Markov theorem**.

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- More generally, the **marginal effect** of an infinitesimal change in X on Y, i.e. $\frac{\partial \hat{Y}}{\partial X} = \hat{\beta}$, is constant (independent of X) in case of OLS.

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- Interpreting the **intercept coefficient**: When X is zero, the predicted value for \hat{Y} is $\hat{\alpha}$. Note that it may **not** be a meaningful quantity.

An example

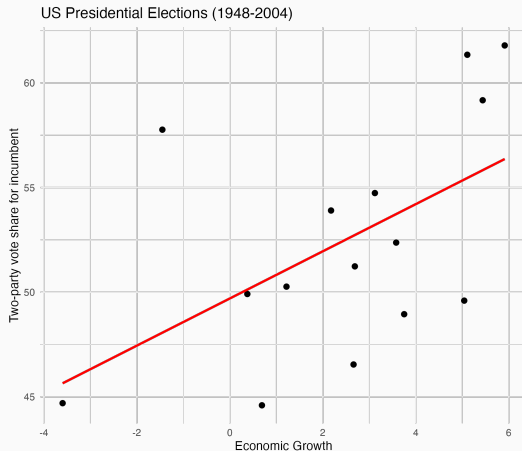
Year	VoteShare	Growth	$(y_i - \bar{y})$	$(x_i - \bar{x})$
1948	52.37	3.579	-0.088	1.131
1952	44.595	.691	-7.863	-1.757
1956	57.764	-1.451	5.306	-3.899
1960	49.913	.377	-2.545	-2.071
1964	61.344	5.109	8.886	2.661
1968	49.596	5.043	-2.862	2.595
1972	61.789	5.914	9.331	3.466
1976	48.948	3.751	-3.510	1.303
1980	44.697	-3.597	-7.761	-6.045
1984	59.17	5.440	6.712	2.992
1988	53.902	2.178	1.444	-0.270
1992	46.545	2.662	-5.913	0.214
1996	54.736	3.121	2.278	0.673
2000	50.265	1.219	-2.193	-1.229
2004	51.233	2.690	-1.225	0.242
	$\bar{y} = 52.4578$	$\bar{x} = 2.4484$		

$$\hat{\beta} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{111.559}{99.0181} = 1.127$$

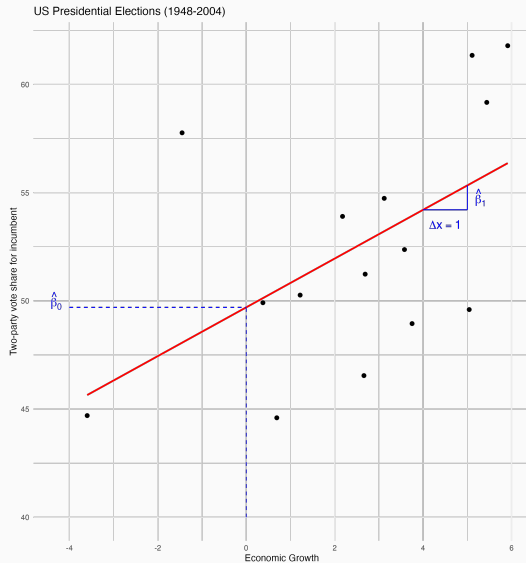
$$\hat{\alpha} = \bar{y} - \hat{\beta}_1 \bar{x} = 52.4578 - 1.127 \times 2.4484 = 49.699$$

An example

- OLS model estimation: $\widehat{\text{VoteShare}}_i = 49.699 + 1.127 \times \text{Growth}_i$
- SSE is minimized at $\sum_{i=1}^n e_i^2 = 311.1486$



An example



Regression Diagnostics

Regression Diagnostics: Residuals

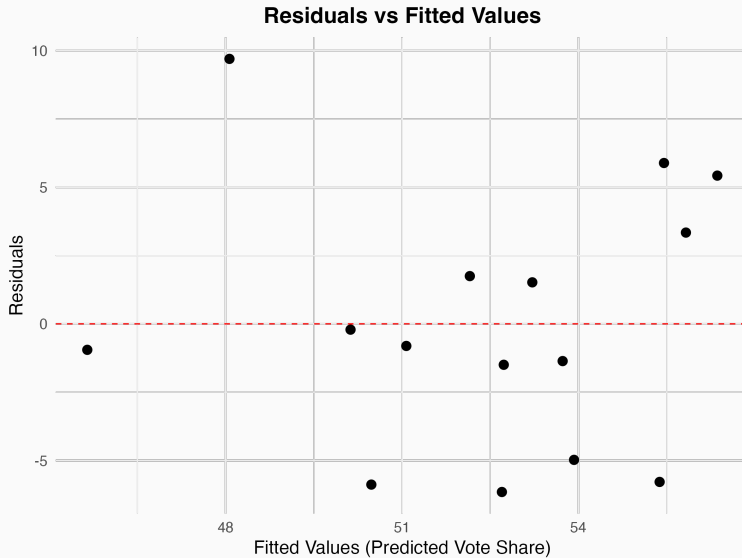
- Patterns in residuals signal that **systematic influences** on Y still have not been captured by our model, or that our model misrepresents the data, or that errors do not have a **constant variance**.
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Regression Diagnostics: Residuals

- Patterns in residuals signal that **systematic influences** on Y still have not been captured by our model, or that our model misrepresents the data, or that errors do not have a **constant variance**.
- The residual plot is a **diagnostic plot** as it helps us to detect patterns in the residuals.
- Residual plot: a *scatterplot of the regression residuals* against the explanatory variable X or the predicted values \hat{Y} .
- Ideally, residual plots should look as if the pattern was generated by pure chance.
- By construction, OLS residuals sum to 0:

$$\sum_{i=1}^n (y_i - \hat{\alpha} - \beta_1 \hat{x}_i) = \sum_{i=1}^n (y_i - \hat{y}_i) = \sum_{i=1}^n e_i = 0$$

Regression Diagnostics: Residuals



Regression inference

Inference about β : the logic

OLS gives us an estimate $\hat{\beta}$, but it's based on a sample. We want to know: does the evidence suggest that the true slope β differs from zero?

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3. Compare $|t|$ to the critical value $t_{n-2, 1-\alpha/2}$ or compute a p -value.

If $|t|$ is large (small p -value), we reject H_0 and conclude that X has a statistically significant linear association with Y .

Standard errors and sampling uncertainty

- The **standard error of $\hat{\beta}$** measures how much $\hat{\beta}$ would vary across repeated random samples.

$$\text{se}(\hat{\beta}) = \frac{\hat{\sigma}}{\sqrt{\sum (x_i - \bar{x})^2}}$$

- The residual standard deviation $\hat{\sigma}$ is:

$$\hat{\sigma} = \sqrt{\frac{\text{SSE}}{n-2}} = \sqrt{\frac{\sum (y_i - \hat{y}_i)^2}{n-2}}$$

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Precision improves when: (1) residuals are smaller (better fit), (2) n is larger, and (3) X has greater variation.

Confidence interval for β

We can express statistical uncertainty using a confidence interval:

$$\hat{\beta} \pm t_{n-2, 1-\alpha/2} \cdot \text{se}(\hat{\beta})$$

- A 95% confidence interval means that if we repeated this sampling process many times, about 95% of those intervals would contain the true value of β .
- If the CI does **not include 0**, the effect is statistically significant at the 5% level.

Example: $\hat{\beta} = 221.6$, $\text{se}(\hat{\beta}) = 35.0 \Rightarrow \text{CI} = [152.2, 291.0] \Rightarrow$ Each additional year of education increases expected income by 152–291 euros.

Interpreting significance and magnitude

- **Statistical significance:** whether the relationship is distinguishable from zero given sampling uncertainty.
- **Substantive magnitude:** whether the size of $\hat{\beta}$ is meaningful in context.
- In large samples, even small effects can be statistically significant.
- In small samples, large but noisy effects may fail to reach significance.

Always report:

- point estimate $\hat{\beta}$
- standard error
- p -value or confidence interval

and interpret them in the context of the research question.

Summary

Putting it all together

1. Estimate coefficients $(\hat{\alpha}, \hat{\beta})$ by OLS.
2. Check assumptions and residual plots.
3. Compute standard errors.
4. Perform hypothesis test:

$$t = \frac{\hat{\beta}}{se(\hat{\beta})}$$

5. Compute and interpret confidence intervals.
6. Evaluate overall model fit (R^2 , adjusted R^2). (We'll talk about it in later lectures).
7. Translate results into substantive conclusions.