VERY BRIEF PROBABILITY THEORY
Note 8.1: Independence

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Definition 1. *Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.*

Definition 2. *Two random variables X and Y are independent if for all $C, D \in \mathcal{R}$,*

$$P(X \in C, Y \in D) = P(X \in C)P(Y \in D),$$

i.e., the events $A = \{X \in C\}$ and $B = \{Y \in D\}$ are independent.

Definition 3. *Two σ-fields $\mathcal{F}$ and $\mathcal{G}$ are independent if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$ the events A and B are independent.*

Theorem 1. *(i) If X and Y are independent, then $\sigma(X)$ and $\sigma(Y)$ are independent.*

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are independent, $X \in \mathcal{F}$, $Y \in \mathcal{G}$, then X and Y are independent.

Proof. (i) If $A \in \sigma(X)$, then $A = \{X \in C\}$ for some $C \in \mathcal{R}$. Likewise if $B \in \sigma(Y)$, then $B = \{Y \in D\}$ for some $D \in \mathcal{R}$. Using the assumption that X and Y are independent, we have

$$P(A \cap B) = P(X \in C, Y \in D) = P(X \in C)P(Y \in D) = P(A)P(B).$$

(ii) If $X \in \mathcal{F}$, $Y \in \mathcal{G}$ and $C, D \in \mathcal{R}$. It follows from the definition of measurable function that $\{X \in C\} \in \mathcal{F}$, $\{Y \in D\} \in \mathcal{G}$. Using the assumption that $\mathcal{F}$ and $\mathcal{G}$ are independent, we have

$$P(X \in C, Y \in D) = P(X \in C)P(Y \in D).$$

$\square$

Theorem 2. *(i) If A and B are independent then so are A^c and B, A and B^c, A^c and B^c.*

(ii) Events A and B are independent iff their indicator functions I_A and I_B are independent.

Proof. (i) If we have $P(A \cap B) = P(A)P(B)$, then

$$\begin{aligned} P(B) - P(A \cap B) &= P(B) - P(A)P(B) \\ P(\Omega \cap B) - P(A \cap B) &= P(B)(1 - P(A)) \\ P(A^c \cap B) &= P(A^c)P(B). \end{aligned}$$

The conclusion of A and B^c are obtained likewise. The case of A^c and B^c is a direct corollary.

(ii) If $C, D \in \mathcal{R}$, then $\{I_A \in C\} \in \{\varnothing, A, A^c, \Omega\}$ and $\{I_B \in D\} \in \{\varnothing, B, B^c, \Omega\}$. The situations involving $\varnothing$ and Ω are trivial to examine. So there are only 4 situations to check, which are all covered in (i).

$\square$

Definition 4. *Events $A_1, \cdots, A_n$ are independent if whenever $I \subset \{1, \cdots, n\}$,*

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i).$$

Definition 5. *Random variables $X_1, \cdots, X_n$ are independent if whenever $B_i \in \mathbb{R}$ for $i = 1, \cdots, n$,*

$$P(\cap_{i=1}^n \{X_i \in B_i\}) = \prod_{i=1}^n P(X_i \in B_i).$$

Definition 6. *σ-fields $\mathcal{F}_1, \cdots, \mathcal{F}_n$ are independent if whenever $A_i \in \mathcal{F}_i$ for $i = 1, \cdots, n$,*

$$P(\cap_{i=1}^n A_i) = \prod_{i=1}^n P(A_i).$$

Theorem 3. *Let $A_1, A_2 \cdots, A_n$ be independent, then*

(i) $A_1^c, A_2 \cdots, A_n$ are independent.

(ii) $I_{A_1}, \cdots, I_{A_n}$ are independent.

Proof. (i) Obviously, we have $P(\cap_{i=2}^n A_i) = \prod_{i=2}^n P(A_i)$, then

$$\begin{aligned}
P(\cap_{i=2}^n A_i) - P(\cap_{i=1}^n A_i) &= \prod_{i=2}^n P(A_i) - \prod_{i=1}^n P(A_i) \\
P(\Omega \cap \cap_{i=2}^n A_i) - P(A_1 \cap \cap_{i=2}^n A_i) &= (1 - P(A_1)) \prod_{i=2}^n P(A_i) \\
P(A_1^c \cap \cap_{i=2}^n A_i) &= P(A_1^c) \prod_{i=2}^n P(A_i).
\end{aligned}$$

(ii) Assume $B_i \in \{A_i, A_i^c\}$ for $i \in \{1, \cdots, n\}$. Then arbitrary $\{B_1, \cdots, B_n\}$ are independent. Thus, for any $D_i \in \mathcal{R}$, $\{I_{A_i} \in D_i\} \in \{\varnothing, A_i, A_i^c, \Omega\}$. Denote $\{I_{A_i} \in D_i\}$ by C_i, then $P(\cap_{i=1}^n C_i) = \prod_{i=1}^n P(C_i)$ hold trivially if some $C_i = \varnothing$. The case that some $C_i = \Omega$ can also be proved by induction easily.

$\square$

Definition 7. *Events $A_1, \cdots, A_n$ are pairwise independent if for $i \neq j \in \{1, \cdots, n\}$,*

$$P(A_i \cap A_j) = P(A_i)P(A_j).$$

Example 1. *Let X_1, X_2, X_3 be independent random variables with $P(X_i = 0) = P(X_i = 1) = 1/2$ for $i = 1, 2, 3$. Let $A_1 = \{X_1 = X_2\}$, $A_2 = \{X_2 = X_3\}$, $A_3 = \{X_3 = X_1\}$. These events are pairwise independent because for $i \neq j$,*

$$P(A_i \cap A_j) = P(X_1 = X_2 = X_3) = 1/4 = P(A_i)P(A_j).$$

However, they are not independent because

$$P(A_1 \cap A_2 \cap A_3) = P(X_1 = X_2 = X_3) = 1/4 \neq 1/8 = P(A_1)P(A_2)P(A_3).$$