Essential techniques of Calculus

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1 Differentiation

1.1 Definition of derivative

The derivative of the function $y = f(x)$, denoted as $f'(x)$ or dy/dx, is defined as the slope of the tangent line to the curve $y = f(x)$ at the point (x, y). This slope is obtained by a limit. First, a line is drawn which passes through the points $(x, f(x))$ and $(x + h, f(x + h))$ (see Fig. 1). The slope of this line is the rise: $f(x+h) - f(x)$ divided by the run: $(x+h) - x = h$. As $h \to 0$, the drawn line approaches the tangent line. The derivative is thus defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \tag{1}$$

Note that taking the limit is not the same as directly substituting $h = 0$, which gives $f'(x) = 0/0$, an undefined value that is not of much use. Commonly, we will need to cancel the h in the denominator against a factor of h in the numerator before setting $h = 0$ and evaluating the derivative.

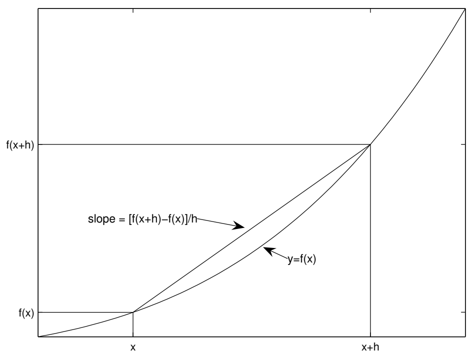

Figure 1: The derivative as the limit of a slope.

The notation dy/dx for the derivative of $y = f(x)$ comes from denoting the change h in the dependent variable x by Δx, pronounced "delta x," and the resulting change in the dependent variable y by Δy, where

$$\Delta y = f(x + \Delta x) - f(x).$$

Therefore, (1) can be rewritten in this new notation as

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \frac{dy}{dx}, \end{aligned}$$

the second line defining $dy/dx = f'(x)$. Note that strictly speaking dy/dx is defined after taking the limit and is a single entity. Sometimes, however, it is expedient to treat dy/dx as a fraction.

Next we derive some general rules that will be useful in evaluating the derivatives of complicated functions. Then we will determine specific rules differentiating some simple functions that can be used to construct complicated functions.

1.2 Derivative of the sum or difference of two functions

Let $h(x) = f(x) + g(x)$. We have

$$\begin{aligned} h'(x) &= \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - (f(x) + g(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= f'(x) + g'(x). \end{aligned}$$

The result can be written as

$$(f + g)' = f' + g'.$$

In a similar fashion,

$$(f - g)' = f' - g'.$$

These simple rules generalize to the sum or difference of an arbitrary number of functions.

1.3 Derivative of the product of two functions

If $h(x) = f(x)g(x)$, then

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x+\Delta x) + f(x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x+\Delta x) + \lim_{\Delta x \to 0} f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}
$$

We write this result as

$$(fg)' = f'g + fg'.$$

Notice that for the proof we add and subtract the same term from the numerator and make use of $\lim_{\Delta x \to 0} g(x + \Delta x) = g(x)$. This important result for the derivative of the product of two functions is known as the *product rule* and should be memorized: "the derivative of the first times the second plus the first times the derivative of the second." It generalizes to the product of an arbitrary number of functions, e.g. for the product of three functions $f(x)$, $g(x)$, $h(x)$:

$$(fgh)' = f'gh + fg'h + fgh'.$$

1.4 Derivative of the quotient of two functions

If $h(x) = f(x)/g(x)$, then

$$
\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x+\Delta x)g(x)}
\end{aligned}
$$

Notice that the factor of $g(x + \Delta x)g(x)$ in the denominator can be set equal to $g(x)^2$ and removed from the limit. We now look to add and subtract the same term from the numerator to form derivatives:

$$
\begin{aligned}
h'(x) &= \frac{1}{g(x)^2} \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x} \\
&= \frac{1}{g(x)^2} \left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x) - \lim_{\Delta x \to 0} f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} \right] \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.
\end{aligned}
$$

We write this result as

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

This result is known as the *quotient rule* and should be memorized: "the derivative of the top times the bottom minus the top times the derivative of the bottom over the bottom squared."

1.5 Derivative of the composition of two functions

Let $h(x)$ be the composition of two functions $f(x)$ and $g(x)$, i.e., $h(x) = f(g(x))$. For example, if $h(x) = \sin(1 + x^2)$, then $h(x) = f(g(x))$, where $g(x) = 1 + x^2$ and $f(x) = \sin x$. We work with the definition of the derivative before the limit is taken. Let $u = g(x)$ and $y = f(u)$. Then,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Now taking the limit $\Delta x \to 0$, (without worrying too much about whether this is valid), we find

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

The translation of this formula is obtained from $h'(x) = dy/dx$, $f'(g(x)) = dy/du$ and $g'(x) = du/dx$. Therefore, the *chain rule* becomes

$$h'(x) = f'(g(x)) \cdot g'(x),$$

or

$$[f(g(x))]' = f'(g(x)) \cdot g'(x).$$

The right-hand-side is to be memorized as "the derivative of the outside times the derivative of the inside." The chain rule generalizes to a large number of compositions, e.g.,

$$[f(g(h(x)))]' = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).$$

We next learn how to take the derivative of the elementary functions represented by power laws, sines, cosines, exponentials and logarithms. These functions may then be combined by addition, subtraction, multiplication, division, and composition to form very complicated functions that in principle you will be able to differentiate.

1.6 Power rule

The derivative of the constant function $f(x) = 1$ is zero since $y = 1$ is the equation of a horizontal line with slope zero and the tangent line to a line is just the line. Similarly, the derivative of the function $f(x) = x$ is 1, since $y = x$ is a line with slope 1.

We now find the derivative of $f(x) = x^2$. From the definition of the derivative

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h \to 0} 2x + h \\ &= 2x. \end{aligned}$$

Notice that the limit can be taken by setting $h = 0$ only after the h in the denominator cancels against a factor of h in the numerator.

Similarly, we can find the derivative of $f(x) = x^n$, where n is a positive integer. Recall the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

where the binomial coefficient is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Here, we will need

$$(x+h)^n = x^n + nhx^{n-1} + \ldots,$$

where the omitted terms in the expansion have factors of h^2, h^3, etc. Therefore,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{nhx^{n-1} + \ldots}{h} \\ &= nx^{n-1}, \end{aligned}$$

where after cancelling the h in the denominator against a factor of h in the numerator, the terms represented by $\ldots$ contain powers of h that go to zero in the limit.

Now, if the exponent is a negative integer, we can also determine the derivative. Suppose $f(x) = x^{-n}$ where n is a positive integer. Recall that this is equivalent to $f(x) = 1/x^n$, and we can make use of the quotient rule. Remembering: "the derivative of the top times the bottom minus the top times the derivative of the bottom over the bottom squared", we have

$$f'(x) = \frac{-nx^{n-1}}{x^{2n}},$$

where the derivative of the top is 0 and the derivative of the bottom is nx^{n-1}. The expression simplifies to

$$f'(x) = -nx^{-n-1},$$

which completes the proof that

$$\frac{d}{dx} x^n = nx^{n-1},$$

if n is any integer (positive, negative or zero). In fact, this relation is true for all real exponents, and this useful result is called the "power rule." The general derivation depends on first defining what is meant by x^p for p any real number.

1.7 Trigonometric functions

When studying Calculus, all arguments of trigonometric functions are assumed to be in radians (which means real numbers). Never use degrees! From trigonometry, you should know the Pythagorean trigonometric identity:

$$\sin^2 x + \cos^2 x = 1,$$

and the addition theorems

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

Furthermore, you should know the small angle approximation for sin: when x is small,

$$\sin(x) \approx x.$$

Also, the values of $\sin x$ and $\cos x$ should be known at $x = 0$, $\pi/2$, π, $3\pi/2$, 2π, the last values being identical to the first because of periodicity. Finally, you should know (and be able to derive) the following symmetry properties:

$$\sin(\pi/2 - x) = \cos x, \quad \cos(\pi/2 - x) = \sin x;$$

and

$$\sin(-x) = -\sin(x), \quad \cos(-x) = \cos(x)$$

Other identities and relations learned in trigonometry may be useful, but could be derived or looked-up as needed.

We now find the derivative of $f(x) = \sin x$. For convenience, we modify the definition of the derivative to write

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \\ &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h}. \end{aligned}$$

As long as the limit results in the slope of the tangent line to the curve of $y = f(x)$ at (x, y) we can use any form of the derivative definition that is most convenient. The above form, before the limit is taken, is called the central difference approximation and is widely used in numerical calculation.

We now apply the addition (and subtraction) theorems to expand the sine functions in the numerator:

$$\begin{aligned} \frac{d}{dx}\sin x &= \lim_{h\to 0} \frac{(\sin x \cos h + \cos x \sin h) - (\sin x \cos h - \cos x \sin h)}{2h} \\ &= \lim_{h\to 0} \frac{2\cos x \sin h}{2h} \\ &= \cos x \left(\lim_{h\to 0} \frac{\sin h}{h} \right) \\ &= \cos x \left(\lim_{h\to 0} \frac{h}{h} \right) \\ &= \cos x, \end{aligned}$$

where in the next-to-last step we have made use of the small angle approximation for $\sin h$. We thus say that "the derivative of sine is cosine," and this should be memorized. To determine the derivative of cosine, we use a symmetry property and the chain rule. Since $f(x) = \cos x = \sin(\pi/2 - x)$, we have using the chain rule ('the derivative of the outside times the derivative of the inside'):

$$\begin{aligned}\frac{d}{dx}\cos x &= \frac{d}{dx}\sin(\pi/2 - x) \\ &= \cos(\pi/2 - x)\cdot\frac{d}{dx}(\pi/2 - x) \\ &= -\sin x.\end{aligned}$$

Therefore, we say that "the derivative of cosine is minus sine," and this needs to be memorized also. Using definitions and the quotient rule, one can proceed to find the derivatives of $\tan x$, $\sec x$, etc.

1.8 Exponential function

We will see that the irrational number $e = 2.71828\ldots$ has special significance in the calculus. The number e can be defined by the limit

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

The exponential function is defined as $\exp(x) = e^x$. Notice that from our definition of e,

$$e^x = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{nx}.$$

If we suppose $x > 0$ and let $m = nx$, then since $m \to \infty$ as $n \to \infty$,

$$\begin{aligned}e^x &= \lim_{m\to\infty}\left(1 + \frac{x}{m}\right)^m \\ &= \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n,\end{aligned}$$

where in the last equation we have just relabelled m back to n. In fact, this result is valid for all values of x.

The derivative of $f(x) = e^x$ can be determined if one assumes it is possible to differentiate inside the limit. By the power rule and the chain rule:

$$\begin{aligned}\frac{d}{dx}e^x &= \frac{d}{dx}\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \\ &= \lim_{n\to\infty}\frac{d}{dx}\left(1 + \frac{x}{n}\right)^n \\ &= \lim_{n\to\infty} n\left(1 + \frac{x}{n}\right)^{n-1}\cdot\frac{d}{dx}\left(1 + \frac{x}{n}\right) \\ &= \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{n-1}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n(1-\frac{1}{n})} \\
&= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} \\
&= e^x.
\end{aligned}
$$

Therefore, the derivative of the exponential function is the exponential function.

1.9 Logarithmic function

The natural logarithm is the inverse function of the exponential function. We have

$$\exp(\ln x) = x.$$

Differentiating both sides with respect to x, and using the chain rule, we have

$$\frac{d}{dx}\exp(\ln x) = \exp(\ln x)\cdot\frac{d}{dx}\ln x = 1.$$

Therefore, we have determined

$$\frac{d}{dx}\ln x = \frac{1}{x};$$

so that the derivative of the natural logarithm is $1/x$.

1.10 Complicated functions

With the addition and subtraction rules, product and quotient rules, and chain rule, together with rules for differentiating power-laws, trigonometric functions, exponentials and logarithms, any function built up from elementary functions can be differentiated. Sometimes the result is messy, but there exists computer algebra programs that can take the derivative (and simplify the result) for you. Nevertheless, it is always useful to be able to take the derivative by hand.

2 Integration

2.1 Definition of integral

The definite integral of a function $f(x) > 0$ from $x = a$ to $x = b$ ($b > a$) is defined as the area bounded by the vertical lines $x = a$, $x = b$, the x-axis and the curve $y = f(x)$ (see Fig. 2). This "area under the curve" is obtained by a limit. First, the area is approximated by a sum of rectangle areas as shown in Fig. 2. Second, the integral is defined to be the limit of the rectangle areas as the width of each individual rectangle goes to zero and the number of rectangles goes to infinity. This resulting infinite sum is called a *Riemann Sum* and we define

$$\int_a^b f(x)dx = \lim_{N\to\infty}\sum_{n=1}^{N} f(x_n)\Delta x, \tag{2}$$

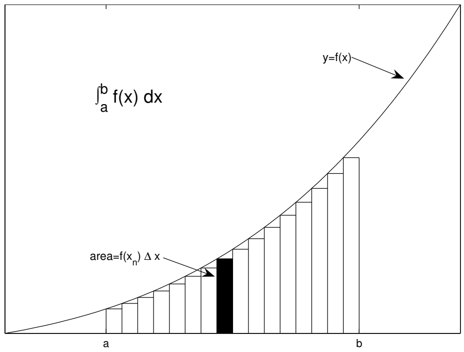

Figure 2: The integral as the area under a curve.

where $x_n = a + (n-1)\Delta x$ and $\Delta x = (b-a)/N$. The symbols on the left-hand-side of (2) are read as "the integral from a to b of f of x dx." The Riemann Sum definition is extended to all values of a and b and for all values of $f(x)$ (positive and negative). From (2),

$$\int_b^a f(x)dx = -\int_a^b f(x)dx,$$

since Δx changes sign, and

$$\int_a^b (-f(x))dx = -\int_a^b f(x)dx.$$

Also, if $a < b < c$, then

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, \tag{3}$$

which states that the total area equals the sum of its parts.

2.2 Fundamental theorem of Calculus

On first glance, there is no obvious relation between derivatives (slope of the tangent line to a curve) and integrals (the area under a curve). Indeed, the determination of area by sums of rectangles was already known by the ancient Greeks who did not yet define a derivative. The connection between derivatives and integrals is so important that the theorem stating the connection is called the fundamental theorem of calculus. We consider the following derivative of an integral:

$$\frac{d}{dx}\int_a^x f(s)ds.$$

Notice that the variable x is the upper limit of the definite integral. Also notice that the integration variable s is what is called a dummy variable. Any symbol can used instead of s, and common choices include x' (where the prime does not mean the derivative) and x (which is not to be confused with the x variable in the upper limit).

We now take the derivative:

$$
\begin{aligned}
\frac{d}{dx}\int_a^x f(s)ds &= \lim_{h\to 0}\frac{\int_a^{x+h} f(s)ds - \int_a^x f(s)ds}{h} \\
&= \lim_{h\to 0}\frac{\int_x^{x+h} f(s)ds}{h} \\
&= \lim_{h\to 0}\frac{hf(x)}{h} \\
&= f(x). \qquad (4)
\end{aligned}
$$

The second line proceeds from the first using (3), and the third line proceeds from the second since the Riemann sum can consist of only a single rectangle whose width goes to zero.

The fundamental theorem of calculus shows us how to integrate functions. Let $F(x)$ be a function such that $F'(x) = f(x)$. We say that $F(x)$ is an antiderivative of $f(x)$. Then from (4), and the fact that the derivative of a constant equals zero,

$$F(x) = \int_a^x f(s)ds + c.$$

Now, $F(a) = c$ and $F(b) = \int_a^b f(s)ds + F(a)$. Therefore, a restatement of the fundamental theorem allows us to integrate a function $f(x)$ provided we can find its antiderivative:

$$\int_a^b f(s)ds = F(b) - F(a). \qquad (5)$$

Unfortunately, finding antiderivatives is much harder than finding derivatives, and indeed, most complicated functions can not be integrated analytically.

2.3 Definite and indefinite integrals

The Riemann sum definition of an integral is called a *definite integral* and its solution is given by (5). It is convenient to also define an indefinite integral by

$$\int f(x)dx = F(x),$$

where F(x) is the antiderivative of $f(x)$. As an example, we know that

$$\frac{d}{dx}(x + c) = 1$$

for any constant c. Therefore, the indefinite integral of 1 is given by

$$\int 1 \cdot dx = x + c.$$

The constant c is always required when computing indefinite integrals because its derivative is zero. The definite integral may be written as

$$\begin{aligned}\int_a^b 1 \cdot dx &= (x + c)|_a^b \\ &= (b + c) - (a + c) \\ &= b - a.\end{aligned}$$

Note that the arbitrary constant c cancels and so is usually excluded in the calculation of definite integrals.

2.4 Simple examples of indefinite integrals

From our known derivatives of elementary functions, we can determine some simple indefinite integrals. The power rule gives us

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1.$$

When $n = -1$, and x is positive, we have

$$\int \frac{1}{x} dx = \ln x + c.$$

If x is negative, using the chain rule we have

$$\frac{d}{dx} \ln(-x) = \frac{1}{x}.$$

Therefore, since

$$|x| = \begin{cases} -x & \text{if } x < 0; \\ x & \text{if } x > 0, \end{cases}$$

we can generalize our indefinite integral to postive and negative x:

$$\int \frac{1}{x} dx = \ln|x| + c.$$

Trigonometric functions can also be integrated:

$$\int \cos x dx = \sin x + c, \quad \int \sin x dx = -\cos x + c.$$

Easily proved identities are an addition rule:

$$\int (f(x) + g(x)) \, dx = \int f(x) dx + \int g(x) dx;$$

and multiplication by a constant:

$$\int A f(x) dx = A \int f(x) dx.$$

This permits integration of functions such as

$$\int (x^2 + 7x + 2)dx = \frac{x^3}{3} + \frac{7x^2}{2} + 2x + c,$$

and

$$\int (5\cos x + \sin x)dx = 5\sin x - \cos x + c.$$

2.5 Substitution

More complicated functions can be integrated using the chain rule. Since

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x),$$

we have

$$\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c.$$

This integration formula is usually implemented by letting $y = g(x)$. Then one writes $dy = g'(x)dx$ to obtain

$$\begin{aligned}\int f'(g(x))g'(x)dx &= \int f'(y)dy \\ &= f(y) + c \\ &= f(g(x)) + c.\end{aligned}$$

As a simple example, we try to integrate

$$\int (1 + 2x)^2 dx.$$

We can do this in two ways. First, expanding the square,

$$\begin{aligned}\int (1 + 2x)^2 dx &= \int (1 + 4x + 4x^2)dx \\ &= x + 2x^2 + \frac{4}{3}x^3 + c.\end{aligned}$$

Second, by substitution, let $y = 1 + 2x$. Then $dy = 2dx$, and we have

$$\begin{aligned}\int (1 + 2x)^2 dx &= \frac{1}{2}\int (1 + 2x)^2 (2dx) \\ &= \frac{1}{2}\int y^2 dy \\ &= \frac{1}{6}y^3 + c = \frac{1}{6}(1 + 2x)^3 + c.\end{aligned}$$

The two results are in fact equivalent (though the constant c is different), since

$$\begin{aligned}\frac{1}{6}(1 + 2x)^3 + c &= \frac{1}{6}(1 + 6x + 12x^2 + 8x^3) + c \\ &= x + 2x^2 + \frac{4}{3}x^3 + (c + \frac{1}{6}).\end{aligned}$$

2.6 Integration by parts

Another integration technique makes use of the product rule for differentiation. Since

$$(fg)' = f'g + fg',$$

we have

$$f'g = (fg)' - fg'.$$

Therefore,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

Commonly, the above integral is done by writing

$$\begin{aligned} u &= g(x) & dv &= f'(x)dx \\ du &= g'(x)dx & v &= f(x). \end{aligned}$$

Then, the formula to be memorized is

$$\int udv = uv - \int vdu.$$

As a simple example, we integrate

$$\int xe^x dx.$$

Since we know how to integrate e^x, we let

$$\begin{aligned} u &= x & dv &= e^x dx \\ du &= dx & v &= e^x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + c \\ &= (x-1)e^x + c. \end{aligned}$$

The result can be checked by differentiation:

$$\begin{aligned} \frac{d}{dx}((x-1)e^x + c) &= e^x + (x-1)e^x \\ &= xe^x. \end{aligned}$$

3 Review problems

Find dy/dx:

1. $y = 5x^4 + x^2 + x$
2. $y = -3x^2 + \frac{1}{2}x - 2$
3. $y = x^3 + \frac{1}{x^3}$
4. $y = (x^2 + 1)(2x + 3)$
5. $y = \frac{x^2+1}{x^2-1}$
6. $y = \sqrt{x}(3x - 2)$
7. $y = (1 + 3x)^4$
8. $y = \sqrt{1 + x^2}$
9. $y = \cos 2x$
10. $y = \cos^2 x$
11. $y = \cos x \sin x$
12. $y = \tan x$
13. $y = \frac{\sin x}{x}$
14. $y = e^{2x}$
15. $y = e^x \cos x + e^{-x} \sin x$
16. $y = e^x \ln x$
17. $y = \ln(xe^x)$

Find $\int f(x)dx$:

18. $\int x^3 dx$
19. $\int \frac{1}{\sqrt{x}} dx$
20. $\int (2\sqrt{x} + 3x^3)dx$
21. $\int \sin 3x dx$
22. $\int x^2 \cos x^3 dx$
23. $\int \sin(1 + x)dx$
24. $\int x \cos(1 + x^2)dx$
25. $\int x \sin x dx$
26. $\int x \ln x dx$
27. $\int x^2 e^x dx$
28. $\int e^x \sin x dx$