1. Definition of Taylor polynomials and Taylor series

What is the 4th Taylor polynomial and Taylor series of $f(x)=e^x$ at $x=e$

$e^x = f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = \cdots,$ $\quad e^e = f(e) = f'(e) = f''(e) = \cdots = f^{(n)}(e) = \cdots$

$$P_4(x) = f(e) + \frac{f'(e)}{1!}(x-e) + \frac{f''(e)}{2!}(x-e)^2 + \frac{f'''(e)}{3!}(x-e)^3 + \frac{f^{(4)}(e)}{4!}(x-e)^4$$

$$= e^e + \frac{e^e}{1!}(x-e) + \frac{e^e}{2!}(x-e)^2 + \frac{e^e}{3!}(x-e)^3 + \frac{e^e}{4!}(x-e)^4$$

Taylor series: $f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(e)}{k!}(x-e)^k = \sum_{k=0}^{\infty} \frac{e^e}{k!}(x-e)^k = e^e + \frac{e^e}{1!}(x-e) + \frac{e^e}{2!}(x-e)^2 +$

$$\cdots + \frac{f^{(n)}(e)}{n!}(x-e)^n + \cdots$$

2. Replacement: What is the Taylor series $\frac{x^7}{1+x^3}$ at $x=0$

Since $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1 + (-x^3) + (-x^3)^2 + (-x^3)^3 + (-x^3)^4 + \cdots$$

$$= 1 - x^3 + x^6 - x^9 + x^{12} \cdots$$

$$\frac{x^7}{1+x^3} = x^7(1 - x^3 + x^6 - x^9 + x^{12} \cdots) = x^7 - x^{10} + x^{13} - x^{16} + x^{19} \cdots$$

3. Integration: Evaluate $\sum_{k=0}^{\infty} \frac{\pi^{2(k+1)}}{2(k+1)\cdot(2k)!} = \frac{\pi^2}{2} - \frac{\pi^4}{4\cdot 2!} + \frac{\pi^6}{6\cdot 4!} - \frac{\pi^8}{8\cdot 6!} + \frac{\pi^{10}}{10\cdot 8!} \cdots$

Hint: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$, try $\int_0^x t\cos t\,dt$

First notice: $\int_0^x t\cos t\,dt = \int_0^x t\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots\right)dt$

$$= \int_0^x \left(t - \frac{t^3}{2!} + \frac{t^5}{4!} - \frac{t^7}{6!} + \frac{t^9}{8!} + \cdots\right)dt$$

$$= \int_0^x t\,dt - \int_0^x \frac{t^3}{2!}dt + \int_0^x \frac{t^5}{4!}dt - \int_0^x \frac{t^7}{6!}dt + \int_0^x \frac{t^9}{8!}dt \cdots$$

$$= \int_0^x t\,dt - \frac{1}{2!}\int_0^x t^3 dt + \frac{1}{4!}\int_0^x t^5 dt - \frac{1}{6!}\int_0^x t^7 dt + \frac{1}{8!}\int_0^x t^9 dt \cdots$$

$$= \frac{t^2}{2}\Big|_0^x - \frac{1}{2!}\frac{t^4}{4}\Big|_0^x + \frac{1}{4!}\frac{t^6}{6}\Big|_0^x - \frac{1}{6!}\frac{t^8}{8}\Big|_0^x + \frac{1}{8!}\frac{t^{10}}{10}\Big|_0^x \cdots$$

$$= \left(\frac{x^2}{2} - \frac{0^2}{2}\right) - \frac{1}{2!}\left(\frac{x^4}{4} - \frac{0^4}{4}\right) + \frac{1}{4!}\left(\frac{x^6}{6} - \frac{0^6}{6}\right) - \frac{1}{6!}\left(\frac{x^8}{8} - \frac{0^8}{8}\right) + \frac{1}{8!}\left(\frac{x^{10}}{10} - \frac{0^{10}}{10}\right) \cdots$$

$$= \frac{x^2}{2} - \frac{1}{2!}\frac{x^4}{4} + \frac{1}{4!}\frac{x^6}{6} - \frac{1}{6!}\frac{x^8}{8} + \frac{1}{8!}\frac{x^{10}}{10} \cdots$$

$$= \frac{x^2}{2} - \frac{x^4}{4\cdot 2!} + \frac{x^6}{6\cdot 4!} - \frac{x^8}{8\cdot 6!} + \frac{x^{10}}{10\cdot 8!} + \cdots$$

Also: $\int_0^x t\cos t\,dt = \int_0^x t(\sin t)'dt = t\sin t\Big|_0^x - \int_0^x \sin t\,(t')\,dt = x\sin x + \cos x\Big|_0^x = x\sin x + \cos x - 1$

Hence we can plug in $x=\pi$, we get

$$\frac{\pi^2}{2} - \frac{\pi^4}{4\cdot 2!} + \frac{\pi^6}{6\cdot 4!} - \frac{\pi^8}{8\cdot 6!} + \frac{\pi^{10}}{10\cdot 8!} \cdots = \int_0^\pi t\cos t\,dt = \pi\sin\pi + \cos\pi - 1 = -2$$

4. Is series $\sum_{k=1}^{\infty} \frac{k+2}{k^2}$ convergent ?

Let $f(x) = \frac{x+2}{x^2}$, which is continuous, positive, decreasing from 1 to $+\infty$

We can use integral test ! Notice: $f(x) = \frac{x+2}{x^2} = \frac{x}{x^2} + \frac{2}{x^2} = \frac{1}{x} + \frac{2}{x^2} = x^{-1} + 2x^{-2}$

$$\int_1^{+\infty} f(x)dx = \lim_{b\to\infty} \int_1^b (x^{-1}+2x^{-2})\,dx = \lim_{b\to+\infty} \left[\int_1^b x^{-1}dx + 2\int_1^b x^{-2}dx\right]$$

$$= \lim_{b\to+\infty} \left[\ \ln x\Big|_1^b + 2\,(-x^{-1})\Big|_1^b\ \right] = \lim_{b\to+\infty}\left[(\ln b - 0) + 2\left(-\frac{1}{b} - \left(-\frac{1}{1}\right)\right)\right]$$

$$= \lim_{b\to+\infty}\left[\ln b - \frac{2}{b} + 2\right] \left(\text{as } b\to+\infty,\ \frac{2}{b}\to 0,\ \ln b \to +\infty\right) \longrightarrow +\infty$$

thus the series is not convergent !

6. Going to a bet, bet A, 10 % probability gaining \$5000 +\$100

20 % probability losing \$2000

remaining 70 % probability gaining or losing nothing

bet B, 28 % probability gaining \$1000 +\$208

72 % probability losing \$100

don't bet, gain \$200

What should you do ?

Let X denote the money you gain

bet A:

X	5000	-2000	0
P	0.1	0.2	0.7

expectation: $EX = 5000 \cdot 0.1 + (-2000)\cdot 0.2 + 0 \cdot 0.7 = 100$

bet B:

X	1000	-100
P	0.28	0.72

expectation: $EX = 1000 \cdot 0.28 + (-100)\cdot 0.72 = 208$

don't bet:

X	200
P	1

meaning definitely happens

expectation: $EX = 200$

this seems more risky

Therefore: bet B $>$ don't bet $>$ bet A

highest gaining in expectation

better than

this seems safer

better than

7. Which rational number has decimal expansion $-3.\overline{14}$?

$-3.\overline{14} = -(3 + 0.\overline{14})$, $\quad 0.\overline{14} = 0.14 + 0.0014 + 0.000014 + 0.00000014 + \cdots$

$$= 0.14 + 0.14(0.01) + 0.14(0.01)^2 + 0.14(0.01)^3 + \cdots$$

$$= \frac{0.14}{1-0.01} = \frac{0.14}{0.99} = \frac{14}{99}$$

thus $-3.\overline{14} = -\left(3 + \frac{14}{99}\right) = -\frac{311}{99}$