0.1 Complex analysis

Definition 0.1.1. A **polydisc** $D(z,r) \subseteq \mathbb{C}^n$ is $D(z_1,r_1)\times\cdots\times D(z_n,r_n)$

Definition 0.1.2 (Wirtinger derivatives)**.**

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

Note.

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\frac{\partial y}{\partial z},\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}}+\frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$
$$dz\wedge d\bar{z}=-2i dx\wedge dy$$

Definition 0.1.3. $f:\Omega\to\mathbb{C}$ is **holomorphic** at $z_0\in\Omega$ if $f'(z)$ exists around z_0. f is **univalent** if f is injective

Theorem 0.1.4 (Cauchy-Riemann equations)**.** If we write $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$, then the existence of $f'(z)$ implies that $\frac{\partial f}{\partial x}=\frac{1}{i}\frac{\partial f}{\partial y}$ which give the **Cauchy-Riemann equations**

$$\begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}\end{cases}$$

If f satisfies Cauchy-Riemann equations around z_0, then f is holomorphic at z_0

Lemma 0.1.5. A univalent map is a biholomorphism to its image

Theorem 0.1.6 (Goursat)**.** If f is holomorphic on $\Omega\subseteq\mathbb{C}$, $\overline{T}\subseteq\Omega$ is a triangle, then $\oint_T f(z)dz=0$

Theorem 0.1.7 (Cauchy's integral theorem)**.** If f is holomorphic on $\Omega\subseteq\mathbb{C}$, $\gamma\subseteq\Omega$ is a piecewise C^1 curve, then $\oint_\gamma f(z)dz=0$

Theorem 0.1.8 (Morera's theorem)**.** $U\subseteq\mathbb{C}$ is open, if $\oint_T f(z)dz=0$ for any triangle $T\subseteq U$, then f is holomorphic on D

Cauchy-Pompeiu formula

Theorem 0.1.9 (Cauchy-Pompeiu formula)**.** f is a complex valued C^1 function on a disc $D\subseteq\mathbb{C}$, then

$$f(\zeta)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)dz}{z-\zeta}-\frac{1}{\pi}\iint_D\frac{\partial f(z)}{\partial\bar{z}}\frac{dx\wedge dy}{z-\zeta}$$

In particular, if f is holomorphic, then

$$f(\zeta)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-\zeta}dz$$

Proof. Denote $D_\epsilon=D-B(0,\epsilon)$, consider

$$\eta=\frac{f(w)dw}{w-z},d\eta=\frac{\partial f(w)}{\partial\bar{w}}\frac{d\bar{w}\wedge dw}{w-z}$$

By Stokes' theorem

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon}\eta=\frac{1}{2\pi i}\int_{D_\epsilon}d\eta$$

As $\epsilon\searrow 0$

$$f(\zeta)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(w)dw}{w-z}+\frac{1}{2\pi i}\iint_D\frac{\partial f(w)}{\partial\bar{w}}\frac{dw\wedge d\bar{w}}{w-z}$$

∎

Osgood's lemma

Lemma 0.1.10 (Osgood's lemma)**.** f is continuous on an open subset $\Omega \subseteq \mathbb{C}^n$ and holomorphic on each variable, then f is holomorphic

Proof. For each $a \in \Omega$, pick $P = D(a, r) \subseteq \Omega$, since $\dfrac{\partial f}{\partial \bar{z}_j} \equiv 0$ on Ω, fix $z_2, \cdots, z_n$, then

$$f(w_1, z_2, \cdots, z_n) = \frac{1}{2\pi i} \int_{|z_1 - a_1| = r_1} \frac{f(z_1, \cdots, z_n)}{z_1 - w_1} dz_1$$

For $w_1 \in D(a_1, r_1)$, iterate and we get

$$f(w_1, \cdots, w_n) = \frac{1}{(2\pi i)^n} \int_{|z_1 - a_1| = r_1} \cdots \int_{|z_n - a_n| = r_n} \frac{f(z_1, \cdots, z_n)}{\prod (z_j - w_j)} dz_1 \cdots dz_n$$

For $w \in P$. Since f is continuous, it is bounded on $\overline{P}$, $\dfrac{1}{z_j - w_j} = \displaystyle\sum_{m=0}^{\infty} \frac{(w_j - a_j)^m}{(z_j - a_j)^{m+1}}$ converges uniformly on compact subsets of $D(a_j, r_j)$. Hence $f(w) = \sum c_\alpha (w - a)^\alpha$, where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|z_1 - a_1| = r_1} \cdots \int_{|z_n - a_n| = r_n} \frac{f(z)}{\prod (z_j - a_j)^{\alpha_j + 1}} dz_1 \cdots dz_n$$

□

Corollary 0.1.11 (Cauchy inequality)**.**

Maximum principle

Theorem 0.1.12 (Maximum principle)**.**

Theorem 0.1.13. $\{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$, f_n are uniformly convergent on each compact subset, then f_n converges to a holomorphic function f, and $D^\alpha f_n \to D^\alpha f$ on each compact subset

Montel's theorem

Theorem 0.1.14 (Montel's theorem)**.** $\mathcal{F} = \{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$ and locally uniformly bounded, i.e. for any $z_0 \in \Omega$, there exists a neighborhood U and M such that $\sup_{z \in K} |f_n| \leq M$, then $\mathcal{F}$ is normal

Schwarz lemma

Lemma 0.1.15 (Schwarz lemma)**.** f is holomorphic on the unit disc $D \subseteq \mathbb{C}$, $f(0) = 0$ and $|f| \leq 1$ on D, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then $f(z) = az$, $a = f'(0)$

Proof. Define $g(z) = \dfrac{f(z)}{z}$, since $f(0) = 0$, 0 is a removable singularity, since $|f(z)| \leq 1$, $|g(z)| \leq 1$ on ∂D, by maximum principle 0.1.12, $|g(z)| \leq 1$ on D, thus $|f(z)| \leq |z|$ on D and $|f'(0)| = |g(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then g attains maximum within D, then $g \equiv a$ for some $|a| = 1$, thus $f(z) = az$ □

Corollary 0.1.16. $D \xrightarrow{f} D$ is a biholomorphic, then $f = e^{i\phi} \dfrac{z - a}{1 - \bar{a}z}$ for some ϕ and $a \in D$

Proof. Denote $\psi_a(z) = \dfrac{z - a}{1 - \bar{a}z}$, ψ_{-a} is the inverse of ψ_a
Assume $f(a) = 0$, consider $g(z) = f \circ \psi_{-a}$, then $g(0) = 0$, by Schwarz lemma 0.1.15, $g = e^{i\phi}$, $f = g \circ \phi_a = e^{i\psi} \dfrac{z - a}{1 - \bar{a}z}$ □

Lemma for Riemann mapping theorem

Lemma 0.1.17. Suppose $0 \in U \subsetneq D$ is a simply connected open set, there exists $U \xrightarrow{f} D$ univalent such that $f(0) = 0$, $|f'(0)| > 1$. Note that this is impossible if $U = D$ due to Schwarz lemma 0.1.15

Proof. Denote $\psi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $\psi_a'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$. Consider $f = \psi_{g(a)} \circ g \circ \psi_{-a}$ with some $\psi_{-a}(U) \xrightarrow{g} D$ univalent, then $f(0) = 0$

$$f'(0) = \frac{1-|g(a)|^2}{(1-|g(a)|^2)^2} g'(a)(1-|a|^2) = \frac{1-|a|^2}{1-|g(a)|^2} g'(a)$$

Since U is simply connected, so is $\psi_{-a}(U)$ given $-a \in D \setminus U$, we can take $g(z) = \sqrt{z}$ to be one branch, since $|a| < 1$, we get

$$|f'(0)| = \frac{1-|a|^2}{1-|a|} \frac{1}{2\sqrt{|a|}} = \frac{1+|a|}{2\sqrt{|a|}} > 1$$

□

Lemma for finding zeros

Lemma 0.1.18. φ is holomorphic on D, f is meromorphic on D and $f \neq 0$ on ∂D, $a_1, \cdots, a_m$ and $b_1, \cdots, b_n$ are the zeros and poles of order $k_1, \cdots, k_m$ and $l_1, \cdots, l_n$ of f in D, then

$$\frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{m} k_i \varphi(a_i) - \sum_{i=1}^{n} l_i \varphi(b_i)$$

Proof. $f(z) = g(z) \prod_{i=1}^{m} (z-z_i)^{q_i}$ with $g \neq 0$ on $\overline{D}$, z_i, q_i could be a_i, k_i or $b_i, -l_i$ depending on whether it is a zero or a pole, hence

$$\begin{aligned}
\frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{g'(z) \prod_{i=1}^{m}(z-z_i) + g(z) \sum_{i=1}^{m} \prod_{j\neq i}(z-z_j)}{g(z) \prod_{i=1}^{m}(z-z_i)} dz \\
&= \frac{1}{2\pi i} \int_{\partial D} \left[\frac{\varphi(z) g'(z)}{g(z)} + \sum_{i=1}^{m} \frac{\varphi(z)}{z-z_i} \right] dz \\
&= \sum_{i=1}^{m} k_i \varphi(a_i) - \sum_{i=1}^{n} l_i \varphi(b_i)
\end{aligned}$$

□

Rouche's theorem

Theorem 0.1.19 (Rouché's theorem)**.**

Hurwitz's theorem

Theorem 0.1.20 (Hurwitz's theorem)**.** $U \subseteq \mathbb{C}$ is open connected, holomorphic functions $\{f_n\}$ converges uniformly to f on compact subsets of U and $f \not\equiv 0$, f has order m at z_0, for r small enough, there exists K such that for any $k \geq K$, f_k has precisely m zeros in $B(z_0, r)$, counting multiplicities, and these zeros converge to z_0 as $k \to \infty$

Remark 0.1.21. $B(z_0, r)$ can't be arbitrarily large. For example, $f_n(z) = z - 1 + \dfrac{1}{n}$ converges uniformly to $f(z) = z - 1$ on compact subsets, f has no zeros in the unit disc D, but f_n all have zeros in D

Proof. For r small enough, f doesn't vanish on $\partial B(z_0, r)$ on which $|f|$ attains minimum, then apply Rouché's theorem 0.1.19 □

Corollary 0.1.22. U is open connected, univalent maps $\{f_n\}$ converges to f on compact subsets, then f is either univalent or constant

Proof. If f is not a constant and $f(z_0) = f(w_0) = \zeta$, then $f(z) - \xi$ has z_0, w_0 as zeros, by Hurwitz's theorem 0.1.20, there exist $\{z_k\}, \{w_k\}$ converging to z_0, w_0 such that $f_{n_k}(z_k) = f_{n_k}(w_k) = \xi$, but f_n's are univalent, hence $z_k = w_k \Rightarrow z_0 = w_0$, i.e. f is univalent □

Riemann mapping theorem

Theorem 0.1.23 (Riemann mapping theorem)**.** $U \subsetneq \mathbb{C}$ is a nonempty simply connected open subset, $z_0 \in U$,then there is a unique biholomorphism f from U to the unit disc such that $f(z_0) = 0$, $f'(z_0) > 0$

Proof of uniqueness. Suppose $U \xrightarrow{f_1, f_2} D$ are biholomorphisms such that $f_i(z_0) = 0$, $f_i'(z_0) > 0$, consider $g = f_2 f_1^{-1}$, $g(0) = 0$, $|g| \leq 1$ on D and $g'(0) = \dfrac{f_2'(z_0)}{f_1'(z_0)} > 0$, by Schwarz lemma 0.1.15, $g(z) = z$, i.e. $f_1 = f_2$ □

Proof of existence. Fix $a \notin U$, $z_0 \in U$. Define

$$\mathcal{F} = \{f \text{ univalent on } U \| f| \leq 1, f(z_0) = 0\}$$

Since U is simply connected, we can pick one branch $h(z) = \sqrt{z - a}$, then $h(U) \cap -h(U) = \varnothing$, $\dfrac{h(z) - h(z_0)}{h(z) + h(z_0)}$ is univalent and bounded, scale to get some $f_0 \in \mathcal{F} \Rightarrow \mathcal{F}$ is nonempty
Let $A = \sup_{f \in \mathcal{F}} |f'(z_0)| > 0$, $f_n'(z_0) \to A$ for some $\{f_n\} \subseteq \mathcal{F}$, by Montel's theorem 0.1.14, f_{n_k} converges to g uniformly on compact subsets, then $|g| \leq 1$, $g(z_0) = 0$ and $0 < A = |g'(z_0)| < \infty$, according to Hurwitz's theorem 0.1.20, g is also univalent, i.e. $g \in \mathcal{F}$ attains maximal derivative at z_0
Suppose $0 \in g(U) \subsetneq D$, if not, by Lemma 0.1.17, there exists univalent map $g(U) \xrightarrow{f} D$ such that $f(0) = 0$, $|f'(0)| > 1$, then $f \circ g \in \mathcal{F}$, but $|(f \circ g)'(z_0)| = |f'(0) g'(z_0)| > |g'(z_0)|$ which is a contradiction □

Remark 0.1.24. Suppose $f_1, f_2 \in \mathcal{F}$ and f_1 is biholomorphic, then $g = f_2 f_1^{-1}$ is a map $D \to D$, with $g(0) = 0$, according to Schwarz lemma 0.1.15, $\dfrac{|f_2'(z_0)|}{|f_1'(z_0)|} = |g'(0)| \leq 1$, and if $|f_2'(z)| = |f_1'(z)|$, $g = e^{i\phi}$, f_2 is also biholomorphic

Example 0.1.25. $U = \mathbb{C} - \{z \geq 0\}$, then $h(z) = \sqrt{z}$ maps U to the upper half plane

Theorem 0.1.26 (Runge's theorem)**.** $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \backslash K$ is the union of its connected components whereas the components are either bounded or not, denote

Hartogs's extension theorem

Theorem 0.1.27 (Hartogs's extension theorem)**.** An isolated singularity is always a removable singularity when $n \geq 2$

Proof. It suffices to consider the case $P = \{|z_1| \leq 1, |z_2| \leq 1\}$ is a polydisc, f is holomorphic on ∂P, then f is holomorphic on P □

Lemma for Remmert-Stein theorem

Lemma 0.1.28. $\Omega \subseteq \mathbb{C}^n$ is connected, $\Omega \xrightarrow{f} \partial B^n$ is holomorphic, then $f \equiv \mathsf{const}$

Proof. If h is holomorphic, then $\dfrac{\partial^2}{\partial z \partial \bar{z}} |h|^2 = |h'|^2$, hence

$$0 = \frac{\partial^2}{\partial z \partial \bar{z}} |f|^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial z \partial \bar{z}} |f_i|^2 = \sum_{i=1}^{n} |f_i'(z)|^2 \Rightarrow f_i'(z) = 0 \Rightarrow f \equiv \mathsf{const}$$

□

Theorem 0.1.29 (Remmert-Stein)**.** $U_1 \subseteq \mathbb{C}^{n_1}, U_2 \subseteq \mathbb{C}^{n_2}$ are nonempty connected open subsets, $B = \{|z| < 1\} \subseteq \mathbb{C}^n$, then there is no proper holomorphic map $U_1 \times U_2 \to B$

Proof. Suppose $f : U_1 \times U_2 \to B$ is a proper holomorphic map. For any $(x, y) \in U_1 \times \partial U_2$, there is a discrete sequence $\{y_\nu\} \subseteq U_2$ converging to y as in Exercise **??**, apply Lemma **??** to $f(x, y) : \{x\} \times U_2 \to B$, $\{f(x, y_\nu)\}$ is discrete, thus there exists a subsequence $\{y_\mu\} \subseteq \{y_\nu\}$ such that $f(x, y_\mu)$ such that $f(x, y) = \lim f(x, y_\mu) \in \partial B$. Then $f(x, y) : U_1 \times \{y\} \to \partial B$ is a holomorphic, by Lemma 0.1.28, $f(x, y)$ is constant on $U_1 \times \{y\}$, hence $U_1 \times \{y\} \subseteq f^{-1}(f(x, y))$ which is noncompact since it has noncompact image under projection to U_1. This contradicts the fact that f is proper ☐

Corollary 0.1.30 (Poincaré)**.** The 2 polydisc $P = \{|z_1| < 1, |z_2| < 1\}$ and the 2 ball $B = \{|z_1|^2 + |z_2|^2 < 1\}$ are not biholomorphic

Theorem 0.1.31 (Weierstrass preparation theorem)**.** f is analytic near 0, $f(0) = 0$, $f(z)$ written as power series around 0 has terms only involve z_1 which can always be achieved by a change of variables as in Exercise **??**, then $f = wh$, where $w(z) = z_1^k + g_{k-1}z^{k-1} + \cdots + g_0$ is a **Weierstrass polynomial**, i.e. $g_i(z)$ are analytic around 0 and $g_i(0) = 0$, $h(z)$ is analytic around 0 and $h(0) \neq 0$

Theorem 0.1.32 (Weierstrass division theorem)**.** Suppose f, g are analytic near 0, g is a Weierstrass polynomial of degree k, then there exist unique h, r such that $f = gh + r$, where r is a polynomial of degree less than k

0.2 Conformal mapping

Definition 0.2.1. A conformal mapping is a map preserves angles and orientation

Note. Antiholomorphic map preserves angles but changes orientation

Definition 0.2.2. **Möbius transformations** are $f(z) = \dfrac{az+b}{cz+d}$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, **Möbius group** acts regularly on $\mathbb{C}P^1$ and preserves cross ratio $(z_0, z_1; z_2, z_3) = \dfrac{(z_2 - z_0)(z_3 - z_1)}{(z_3 - z_0)(z_2 - z_1)}$

Schwarz reflection principle

Lemma 0.2.3 (Schwarz reflection principle)**.** If f is holomorphic on $\{\mathrm{Im} z > 0\}$ and continuous on $\{\mathrm{Im} z \geq 0\}$ with real values on $\mathrm{Im} z = 0$, then it can be extended to $\mathbb{C}$ with $f(\bar{z}) = \overline{f(z)}$ for $\mathrm{z} < 0$

0.3 Weierstrass functions

Definition 0.3.1. $\Lambda \subseteq \mathbb{C}$ is a lattice. *Weierstrass sigma function* associated to lattice Λ is

$$\sigma(z) = z \prod_{\omega \in \Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

Weierstrass zeta function is the logarithmic derivative of σ

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{\omega \in \Lambda^*} \frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2}$$

Weierstrass eta function is

$$\eta(w) = \zeta(z + w) - \zeta(z), w \in \Lambda$$

This is independent of choice of z
Weierstrass elliptic function is

$$\wp(z) = -\zeta'(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2}\right)$$

$$\wp'(z) = -\sum_{\omega \in \Lambda} \frac{2}{(z + \omega)^3}$$

0.4 Zeta function

Theorem 0.4.1 (Euler's reflection formula)**.** $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$, $z \notin \mathbb{Z}$