Theorem 0.0.1. Linear differential equations $y'(t) = A(t)y(t) + b(t)$ with initial condition $y(0) = y_0$, where A, b, y are smooth, then there exists unique local solution

Proof. Define $Ty(t) = \int_0^t A(s)y(s)+b(s)ds+y_0$, note that $\|Ty(t)-Tz(t)\| = \left\| \int_0^t A(s)(y(s) - z(s))ds \right\| \leq |t|\|A\|\|y-z\|$, then there exists $\delta > 0$ such that $|t|\|A\| < 1, \forall |t| \leq \delta$, here we use $\|\cdot\|$ to denote the supremum norm in $|t| \leq \delta$, by Banach fixed point theorem, we have a unique local solution ☐

Example 0.0.2. $v'(t) = Av(t), v(0) = v_0$, $A \in M_n(\mathbb{C})$, the solution is $v(t) = e^{tA}v_0$ since $\frac{d}{dt}A^{tA} = Ae^{tA}$

Picard-Lindelof theorem

Theorem 0.0.3. (**Picard-Lindelöf theorem**) Suppose $f(y,t)$ is uniformly Lipschitz continuous in y and continuous in t, then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a unique solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Remark 0.0.4. $f(y,t)$ is Lipschitz continuous in y and continuous in t would implie loacl uniformly Lipschitz in y and f uniformly continuous
When you have a local solution, you can try to extend it to a maximal length, i.e. $y(t)$ is defined on $(a, b) \supset [-\varepsilon, \varepsilon]$, it is open precisely because of the theorem

Proof. Define $Ty(t) = \int_0^t f(y(s), s)ds$, then
$\|Ty - Tz\| = \left\| \int_0^t f(y(s), s) - f(z(s), s)ds \right\| \leq \left\| C \int_0^t |y(s) - z(s)|ds \right\| \leq C|t|\|y - z\|$, then there exists $\varepsilon > 0$ such that $C|t| < 1$, $\forall t \in [-\varepsilon, \varepsilon]$, then by Banach fixed point theorem, we have a unique local solution ☐

Peano existence theorem

Theorem 0.0.5. (**Peano existence theorem**) Let $f(y,t)$ be a continuous function around $(y_0, 0)$, then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a local solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Proof. Say $|f| \leq M$ around $(y_0, 0)$, Define $\phi_n(t) = \begin{cases} y_0 & , x \leq 0 \\ y_0 + \int_0^t \phi_n \left(s - \frac{\varepsilon}{n} \right) ds & , 0 \leq x \leq \varepsilon \end{cases}$ for $n \geq 1$
By Arzelà-Ascoli theorem, we know that there is a subsequence ϕ_{n_k} converges on $[-\varepsilon, \varepsilon]$, and the limit $\phi(t)$ satisfies $\phi(t) = y_0 + \int_0^t \phi_n(s)\, ds$ which is a local solution to the problem ☐

Remark 0.0.6. The uniqueness may fail without the Lipschitz condition in y, for example, consider $\frac{dy}{dt} = y^{\frac{1}{3}}$, $y(0) = 0$ has solutions $y(t) = 0$ or $y(t) = \pm \left(\frac{2}{3}t\right)^{\frac{3}{2}}$