MATH868C - Several Complex Variables

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1 Review

Definition 1.1. C^1 function $f : \Omega \to \mathbb{C}$ is *holomorphic* if $\bar{\partial} f = 0$. Denote the set of all holomorphic functions on Ω as $A(\Omega)$

Lemma 1.2. If f is holomorphic, then $\displaystyle\int_{\partial\Omega} f dz = 0$

Proof.

$$\int_{\partial\Omega} f dz = \int_{\Omega} d(f dz) = \int_{\Omega} \bar{\partial} f \wedge dz = 0$$

$\square$

Poincare-Lelong formula

Theorem 1.3 (Poincaré-Lelong formula)**.** Since $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$, $dz \wedge d\bar{z} = -2idx \wedge dy = -2id\mu$. In the distributional sense, $-\dfrac{\log r}{2\pi} = -\dfrac{1}{4\pi}\log(x^2+y^2)$ is the fundamental solution of Laplacian equation in dimension 2, i.e. $\Delta \log(x^2+y^2) = 4\pi\delta$, we have

$$\Delta \log |z|^2 dz \wedge d\bar{z} = 4\pi\delta dz \wedge d\bar{z} \Leftrightarrow \bar{\partial}\partial \log |z|^2 = 2\pi i \delta dx \wedge dy$$

Note. $\partial \log |z|^2 = \partial \log(z) + \partial \log(\bar{z}) = \dfrac{dz}{z}$ is integrable around 0

Proof. We prove a slightly general result. For any $\phi \in C_c^\infty(\Omega)$, by definition we have

$$\begin{aligned}
\iint_{\Omega} \phi \bar{\partial}\partial \log |z-w|^2 &= -\iint_{\Omega} \bar{\partial}\phi \wedge \partial \log |z-w|^2 \\
&= -\lim_{\epsilon \to 0} \iint_{|z-w| \geq \epsilon} \bar{\partial}\phi \wedge \partial \log |z-w|^2 \\
&= -\lim_{\epsilon \to 0} \iint_{|z-w| \geq \epsilon} d\left(\phi \partial \log |z-w|^2\right) \\
&= \lim_{\epsilon \to 0} \oint_{|z-w|=\epsilon} \phi \partial \log |z-w|^2 \\
&= \lim_{\epsilon \to 0} \oint_{|z-w|=\epsilon} \frac{\phi}{z-w} dz \\
&= 2\pi i \phi(w)
\end{aligned}$$

$\square$

Cauchy's formula

Theorem 1.4 (Cauchy's formula)**.** If $f \in C^1(\overline{\Omega})$, then

$$f(w) = \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}} f dz \wedge d\bar{z}}{z-w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z-w} dz$$

Proof. By Poincaré-Lelong formula 1.3, we have

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i} \iint_{\Omega} f \bar{\partial}\partial \log |z-w|^2 \\
&= -\frac{1}{2\pi i} \iint_{\Omega} \bar{\partial} f \wedge \partial \log |z-w|^2 + \frac{1}{2\pi i} \int_{\partial\Omega} f \partial \log |z-w|^2 \\
&= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}} f dz \wedge d\bar{z}}{z-w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z-w} dz
\end{aligned}$$

$\square$

Corollary 1.5. If $f \in C^1(\overline{\Omega}) \cap A(\Omega)$, then by Cauchy's formula 1.4, we know

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-w} dz$$

Which is C^∞ in w

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z-w)^{n+1}} dz$$

Cauchy's estimate

Corollary 1.6 (Cauchy's estimate)**.** For $K \subseteq \Omega$ compact, there are constants C_n such that for any $f \in A(\Omega)$

$$\sup_{z\in K} |f^{(n)}(z)| \leq C_n \|f\|_{L^1(\Omega)}$$

Proof. Consider a bump function χ with $\operatorname{supp} \chi \subseteq \Omega$ and $\chi \equiv 1$ on K, then for any $w \in K$

$$\begin{aligned} f(w) &= \frac{1}{2\pi i}\iint_{\Omega} \frac{\partial_{\bar z}(\chi f) dz \wedge d\bar z}{z-w} + \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\chi f}{z-w} dz \\ &= \frac{1}{2\pi i}\iint_{\Omega} \frac{(\partial_{\bar z}\chi) f dz \wedge d\bar z}{z-w} \\ &= \frac{1}{2\pi i}\iint_{\Omega\setminus K} \frac{(\partial_{\bar z}\chi) f dz \wedge d\bar z}{z-w} \end{aligned}$$

$\dfrac{\partial_{\bar z}\chi}{z-w}$ can be bounded on $\Omega \setminus K$ □

Corollary 1.7. $A(\Omega) \subseteq C(\Omega)$ is closed, thus a Fréchet space

Proof. Suppose $\{f_j\} \subseteq A(\Omega)$ converges to f in $C(\Omega)$, but since

$$f_j(w) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f_j(z)}{z-w} dz$$

$f(w) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \frac{f(z)}{z-w} dz$ which implies $\bar\partial f = 0$ □

Montel's theorem

Theorem 1.8 (Montel's theorem)**.** Suppose $\{f_i\} \subseteq A(\Omega)$ are uniformly bounded on each compact subset, then there is a subsequence f_{i_k} uniformly converges on compact subsets

Proof. For $K \subseteq \Omega$ compact, by Cauchy's estimate 1.6, f_j are Lipschitz with the same C_k, by Ascoli-Arzela theorem, f_j are equicontinuous, thus have convergent subsequence, and then use diagonal argument by exhaust Ω with compact subsets K □

Riemann extension theorem

Theorem 1.9 (Riemann extension theorem)**.** $E \subseteq \Omega$ is a discrete subset, $f \in A(\Omega \setminus E)$, and f is bounded around each point in E, then f can be extended to a unique $\tilde f \in A(\Omega)$

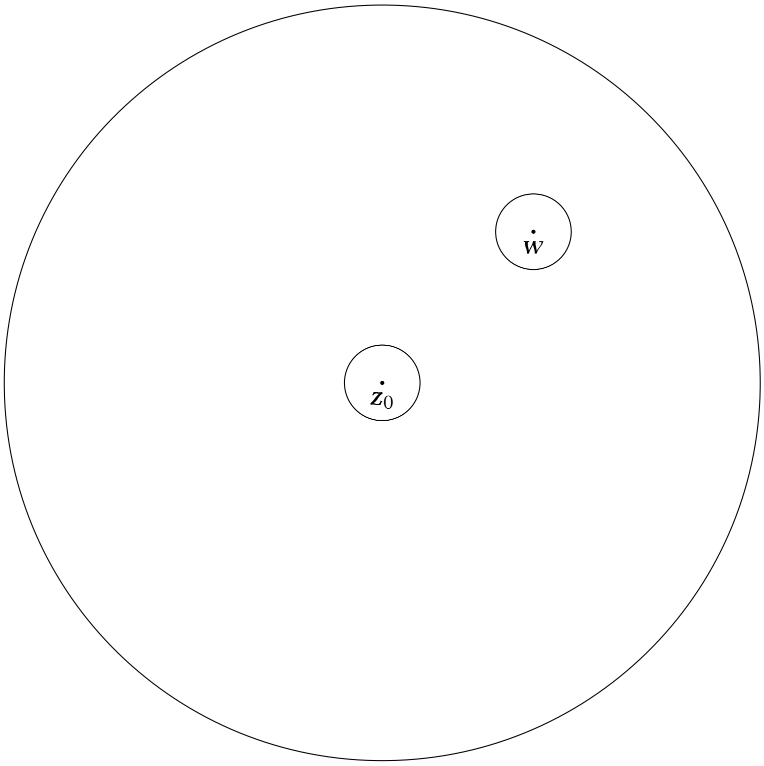
Proof. For $z_0 \in E$, suppose such $\tilde f$ exists, then by Cauchy's formula 1.4, for any $w \in D(z_0, r)$

$$\tilde f(w) = \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(z)}{z-w} dz$$

Thus we just take this as a definition, then

$$\begin{aligned} \tilde f(w) - f(w) &= \frac{1}{2\pi i}\int_{\partial D(z_0,r)} \frac{f(z)}{z-w} dz - \frac{1}{2\pi i}\int_{\partial D(w,\epsilon)} \frac{f(z)}{z-w} dz \\ &= \frac{1}{2\pi i}\int_{\partial D(z_0,\epsilon)} \frac{f(z)}{z-w} dz \end{aligned}$$

Which can be show to arbitrarily small as $\epsilon \to 0$



□

d bar theorem

Theorem 1.10. If $\alpha = g(z)d\bar{z}$ is a smooth $(0,1)$-form on Ω, then there exists $u \in C^\infty(\Omega)$ such that $\bar{\partial} u = \alpha$

Proof. suppose such a u exists, then by Cauchy's formula 1.4

$$u(w) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{z-w} + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{u(z)}{z-w}dz$$

Since $\bar{\partial}\int_{\partial\Omega}\frac{u(z)}{z-w}dz = 0$. This motivates us to first assume α has compact support, and define

$$u(w) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{z-w}$$

Then

$$u(w+\zeta) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{(z-\zeta)-w} = \frac{1}{2\pi i}\iint_\Omega \frac{g(z+\zeta)dz \wedge d\bar{z}}{z-w}$$

Hence

$$\begin{aligned}\partial_{\bar{w}} u(w) &= \frac{1}{2\pi i}\iint_\Omega \frac{\partial_{\bar{z}} g(z)dz \wedge d\bar{z}}{z-w} \\ &= \frac{1}{2\pi i}\iint_\Omega \partial \log|z-w|^2 \wedge \bar{\partial} g \\ &= g(w)\end{aligned}$$

Therefore $\bar{\partial} u = \alpha$. In general, consider a compact exhaustion $\Omega = \bigcup_i K_i$, where $\hat{K}_i = K_i$, $K_i \subset\subset \overset{\circ}{K}_{i+1}$, ensured by Corollary 2.6, let χ_i be a cutoff function such that $\chi_i \equiv 1$ on K_i and $\operatorname{supp}\chi_i \subseteq \overset{\circ}{K}_{i+1}$, then there exists f_i such that $\bar{\partial} f_i = \chi_i \alpha$, by Runge's theorem 2.2, there exists $h_i \in \mathcal{O}(K_i)$ such that $\|f_{i+1} - f_i - h_i\|_{K_i} < \frac{1}{2^i}$. Now define

$$u_N = f_1 + \sum_{k=1}^N (f_{k+1} - f_k - h_k) = f_{N+1} - \sum_{k=1}^N h_N$$

Converges uniformly on compact subsets to u, and $\partial u_N = \alpha$ on K_i for any $i \leq N$ $\square$

2 Runge's theorem

Definition 2.1. $K \subseteq \Omega$ is compact, define

$$\mathcal{O}(K) = \{f|_K : f \text{ is holomorphic in a neighborhood of } K\}$$

Then for we have restriction map $\rho : \mathcal{O}(\Omega) \to \mathcal{O}(K)$, let $\|f\|_K = \max_{z\in K} |f(z)|$ to be the L^∞ norm

Runge's theorem

Theorem 2.2 (Runge's theorem)**.** The following are equivalent

1. The image of ρ is dense
2. No connected component of $\Omega \setminus K$ is relatively compact in Ω
3. If $\xi \in \Omega \setminus K$, then there exists $f \in \mathcal{O}(\Omega)$ such that $|f(\xi)| > \|f\|_K$

Definition 2.3. For $K \subseteq \Omega$ compact, the *holomorphic convex hull* of K relative to Ω is

$$\hat{K} = \hat{K}_\Omega = \{z \in \Omega : |f(z)| \leq \|f\|_K, \forall f \in \mathcal{O}(\Omega)\}$$

Clearly $K \subseteq \hat{K}$

Proposition 2.4.

1. $\hat{K}$ is compact
2. $\|f\|_{\hat{K}} = \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$
3. $\hat{\hat{K}} = \hat{K}$
4. If $\xi \in \Omega \setminus \hat{K}$, then there exists $f \in \mathcal{O}(\Omega)$ such that $|f(\xi)| > \|f\|_K$

Proof.

1. $\hat{K}$ is bounded by considering $f = z$. Suppose $z_i \in \hat{K}$ converges to ξ, if $\xi \in \Omega^c$, then $f = \dfrac{1}{z-\xi}$ will be unbounded on $\hat{K}$, thus $\xi \in \Omega$, but then for any $f \in \mathcal{O}(\Omega)$, $|f(\xi)| = \lim_{i\to\infty} |f(z_i)| \leq \|f\|_K$, thus $\xi \in \hat{K}$
2. By definition, $\|f\|_{\hat{K}} \leq \|f\|_K$, $\forall f \in \mathcal{O}(\Omega)$, and $\|f\|_K \leq \|f\|_{\hat{K}}$, $\forall f \in \mathcal{O}(\Omega)$ is obvious
3. $\hat{\hat{K}} = \{z \in \Omega : |f(z)| \leq \|f\|_{\hat{K}} = \|f\|_K, \forall f \in \mathcal{O}(\Omega)\} = \hat{K}$
4. By definition

$\square$

Example 2.5. K is the unit circle. If Ω is the anulus $\left\{\frac{1}{2} < |z| < 2\right\}$, then $\hat{K} = K$. If Ω is the disc $\{|z| < 2\}$, then $\hat{K} = \{|z| < 1\}$ is the unit disc. Just consider $f = z$ and $f = \dfrac{1}{z}$

Compact exhaustion of a domain

Corollary 2.6. Any domain Ω has an exhaustion by compact sets $\hat{K_i} = K_i$ such that

$$K_i \subset\subset \mathring{K}_{i+1} \subset K_{i+1} \subset\subset \Omega$$

Vanishing theorem

Theorem 2.7. $\mathcal{U} = \{U_i\}$ is an open cover of Ω, then $H^1(\mathcal{U}, \mathcal{O}) = 0$

Proof. Let $\{\phi_i\}$ be a partion of unity. For any cocycle $\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O})$, consider $h_i = \sum_j \phi_j g_{ij}$, then

$$\begin{aligned} h_i - h_j &= \sum_k \phi_k g_{ik} - \sum_k \phi_k g_{jk} \\ &= \sum_k \phi_k (g_{ik} - g_{jk}) \\ &= \sum_k \phi_k g_{ij} \\ &= g_{ij} \end{aligned}$$

Hence $\bar{\partial} h_i - \bar{\partial} h_j = 0$, $\{\bar{\partial} h_i\}$ define a well-defined smooth $(0,1)$ form. By Theorem 1.10, there exist a holomorphic fuction u such that $\bar{\partial} u = \bar{\partial} h_i$, define $f_i = h_i - u$, then $\bar{\partial} f_i = 0$, i.e. $\{f_i\}$'s are holomorphic, and $g_{ij} = f_i - f_j$. In other words, $\{g_{ij}\}$ is the image of $\{f_i\} \in C^1(\mathcal{U}, \mathcal{O})$ under the coboundary map □

Theorem 2.8 (Mittag-Leffler theorem)**.** $\Omega \subseteq \mathbb{C}$ is an open set, $E \subseteq \Omega$ is a discrete subset, then there exists a meromorphic function f with prescribed principal parts on E

Proof. There exists and open cover $\mathcal{U} = \{U_i\}$ and $f_i \in \mathcal{M}(U_i)$ with the prescribed principal parts round each point of E, then $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$ is a coycle, by Theorem 2.7, there exist holomorphic functions $\{g_i\}$ such that $f_i - f_j = g_i - g_j$ on $U_i \cap U_j$, then $f_i - g_i = f_j - g_j$ defines a global meromorphic function f such that $f - f_i = -g_i$ on U_i which is holomorphic □

Weierstrass theorem

Theorem 2.9 (Weierstrass theorem)**.** $E \subseteq \Omega$ is discrete, then

1. There is $f \in \mathcal{M}(\Omega)$ with arbitrary orders precisely at E
2. Any $f \in \mathcal{M}(\Omega)$ can be written as $f = g/h$ for $g, h \in \mathcal{O}(\Omega)$

Proof.

1. First take care of poles, and then multiply by $a_k(z - z_k)^{r_k}$ for each zero z_k, that converges
2.

□

Definition 2.10. Open subset $\Omega \subseteq \mathbb{C}^n$ is called a *domain of holomorphy* if for any $p \in \overline{\Omega} \setminus \Omega$, there is no holomorphic function g defined on an open set $U \ni p$ with $g = f$ on $U \cap \Omega$

Theorem 2.11. For any proper open subset $\Omega \subsetneq \mathbb{C}$ is a domain of holomorphic

Proof. Suppose $p \in \partial\Omega$, $p \in U$ is a neighborhood, $g \in \mathcal{O}(U)$ such that $f = g$ on $\Omega \cap U$, then there exists $\{\xi_n\}$ discrete and converging to p. By Weierstrass theorem 2.9, there exists $f \in \mathcal{O}(\Omega)$ having exactly $\{\xi_i\}$ as zeros, but then g has to be identically zero, so is f which is a contradiction □

Definition 2.12. $\Omega \subseteq \mathbb{C}$ is a domain. $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *upper semicontinuous* if for $y \in \mathbb{R}$ the set $\{u < y\}$ is open

Definition 2.13. An upper semicontinuous function u is *subharmonic* if is not identitcally $-\infty$, and for each $U \subset\subset \Omega$ and harmonic function h on $\overline{U}$ with $u \leq h$ on $\partial\Omega$, we have $u \leq h$ for all $z \in U$

Example 2.14. If $u \in C^2(\Omega)$ and $\Delta u \geq 0$, then u is subharmonic

Example 2.15. If $f_1, \cdots, f_k \in \mathcal{O}(\Omega)$, not all zero, then $u = \log(|f_1|^2 + \cdots + |f_k|^2)$ is subharmonic

References

[1] *An Introduction to Complex Analysis in Several Variables* - Lars Hörmander

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