Section 12.2

November 20, 2019

Motivating example: Randomly picking point on the unit interval
Let $L = [0, 1]$, be the unit interval from 0 to 1, now randomly pick a point(each point has the same "probability" of being picked), what is the probability of $\frac{1}{2}$ being picked? Seems like it should be 0, but then the probability of picking any point is also 0, so the sum of probabilities is 0?
The problem is that probability 0 doesn't mean it is impossible, and since there are continuously many different possible outcomes, we can't sum them to get 1, we have to integrate, notice that we can calculate the probability of picking a number from any sub interval, for example $\left[\frac{1}{4}, \frac{3}{4}\right]$, which is $\frac{\frac{3}{4} - \frac{1}{4}}{1-0} = \frac{1}{2}$, and this is where the mathematical model of using calculus comes in
So we randomly pick a point from L, and let X be the coordinate of that point, X is called a continuous random variable(meaning the possible value/outcome is within a range of continuously many choice), and we define probability density function to help us define probability, in this case, the probability define density function $f(x) = \begin{cases} 1, x \in [0, 1] \\ 0, \text{elsewhere} \end{cases}$, and define the probability of picking a number from interval $[a, b], 0 \leq a \leq b \leq 1$ to be $Pr(a \leq X \leq b) = \frac{b-a}{1-0} = b - a$ which is also the area under $f(x)$ between a and b, notice $Pr\left(X = \frac{1}{2}\right) = Pr\left(\frac{1}{2} \leq X \leq \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2} = 0$
Generally: Suppose X is a continuous random variable taking values within A and B, where A can be $-\infty$ and B can be ∞, we define the probability through a density function $Pr(a \leq X \leq b) = \int_a^b f(x)dx, A \leq a \leq b \leq B$ where f is called the probability density function satisfying $f(x) \geq 0$ for $A \leq x \leq B$(since the probability has to be nonnegative) and $\int_A^B f(x)dx = 1$(since the probability has to integrate to 1) for $A \leq x \leq B$, we want yet another notion: cumulative distribution function $F(x) := Pr(A \leq X \leq x) = \int_A^x f(t)dt$(here use t because x is already taken), then we have $F'(x) = f(x)$ and thus by fundamental theorem of calculus we have $Pr(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$
For example, in the example above, $A = 0$, $B = 1$, $F(x) = \int_0^x f(t)dt = \int_0^x 1dt = t|_0^x = x - 0 = x$ for $0 \leq x \leq 1$

Problems:

1: $f(x) = \frac{1}{50}x, 0 \le x \le 11$, is it a probability density function?

$A = 0,\ B = 11,\quad$ but $\int_A^B f(x)\,dx = \int_0^{11} \frac{1}{50}x\,dx = \frac{x^2}{100}\Big|_0^{11} = \frac{11^2}{100} - \frac{0^2}{100} = \frac{11^2}{10^2} \neq 1$

thus $f(x)$ is not a probability density function

2: $f(x) = kx^2, 0 \le x \le 4$ is a probability density function, what is k

$A=0,\ B=4,\quad$ Since $f(x)$ is a pdf, $\quad 1 = \int_A^B f(x)\,dx = \int_0^4 kx^2\,dx = \frac{kx^3}{3}\Big|_0^4 = \frac{64k}{3}$

thus $1 = \frac{64k}{3} \Rightarrow k = \frac{3}{64}$

3: $f(x) = \frac{1}{10}, 1 \le x \le 11$ is a probability density function, calculate $Pr(X \le 5)$

$A = 1,\ B = 11,\quad Pr(X \le 5) = Pr(1 \le X \le 5) = \int_1^5 f(x)\,dx = \int_1^5 \frac{1}{10}\,dx = \frac{x}{10}\Big|_1^5 = \frac{5}{10} - \frac{1}{10}$

$= \frac{4}{10} = \frac{2}{5}$

4: $F(x) = \frac{1}{2}\sqrt{x-2}, 2 \le x \le 6$ is a cumulative distribution function, what is the probability density function $f(x)$

$A = 2,\ B = 6,\quad f(x) = F'(x) = \left[\frac{1}{2}(x-2)^{\frac{1}{2}}\right]' = \frac{1}{2}\left[(x-2)^{\frac{1}{2}}\right]' = \frac{1}{2}\cdot\frac{1}{2}(x-2)^{-\frac{1}{2}} = \frac{1}{4}(x-2)^{-\frac{1}{2}}$

thus $f(x) = \frac{1}{4}(x-2)^{-\frac{1}{2}},\quad 2 \le x \le 6$

5: $f(x) = \frac{1}{2}(4-x), 2 \le x \le 4$ is a probability density function, what is the cumulative distribution function $F(x)$

$A = 2,\ B = 4,\quad F(x) = \int_A^x f(t)\,dt = \int_2^x \frac{1}{2}(4-t)\,dt = \int_2^x \left(2 - \frac{t}{2}\right)dt$

$= \left(2t - \frac{t^2}{4}\right)\Big|_2^x = \left(2x - \frac{x^2}{4}\right) - \left(2\cdot 2 - \frac{2^2}{4}\right) = 2x - \frac{x^2}{4} - 3$

thus $F(x) = -\frac{x^2}{4} + 2x - 3,\quad 2 \le x \le 4$

6: Suppose that the lifetime X(in hours) of a certain type of flashlight battery is a random variable on the interval $88 \leq x \leq 103$ with probability density function $f(x) = \frac{1}{80}$, find the probability that a battery selected at random will last at least 96 hours

$A = 88,\ B = 103,\quad Pr(X \leqslant 96) = Pr(88 \leqslant X \leqslant 96) = \int_{88}^{96} f(x)dx = \int_{88}^{96} \frac{1}{80} dx = \frac{96-88}{80}$

$= \frac{1}{10}$

7: Let X be a continuous random variable with values between $A = 1$ and $B = \infty$, and with the density function $f(x) =$ [redacted] $3x^{-4}$

a: Find the corresponding cumulative distribution function $F(x)$

b: Compute $Pr(1 \leq X \leq 6)$ and $Pr(X \geq 6)$

a. $F(x) = \int_A^x f(t)dt = \int_1^x 3t^{-4}dt = (-t^{-3})\Big|_1^x = (-x^{-3}) - (-1^{-3}) = 1 - x^{-3}$

b. $Pr(1 \leqslant X \leqslant 6) = Pr(X \leqslant 6) = F(6) = 1 - 6^{-3}$

$Pr(X \geqslant 6) = 1 - Pr(X \leqslant 6) = 1 - (1 - 6^{-3}) = 6^{-3}$

(strictly speaking, $Pr(X \geqslant 6) = 1 - Pr(X < 6)$, but in this case

$Pr(X = 6) = Pr(6 \leqslant X \leqslant 6) = F(6) - F(6) = 0$, thus

$Pr(X \leqslant 6) = Pr(X < 6) + Pr(X = 6) = Pr(X < 6)$)