MATH744 - Lie Groups I

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1 Homeworks

1.1 Homework1

1.
(a)
Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2,\mathbb{R})$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \Rightarrow \begin{cases} a^2+b^2 = 1 \\ ac+bd = 0 \\ c^2+d^2 = 1 \end{cases}$$

we can find $0 \le \theta - \varphi \le 2\pi$ such that $\begin{cases} a = \cos\theta \\ b = \sin\theta \\ c = \cos\varphi \\ d = \sin\varphi \end{cases}$

then $0 = \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos\theta - \varphi$, hence $\theta - \varphi = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$
hence if $\theta - \varphi = \dfrac{\pi}{2}$, $g = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, $\det g = 1$, if $\theta - \varphi = \dfrac{3\pi}{2}$, $g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\det g = 1$, thus $\phi : S^1 \to SO(2,\mathbb{R})$, $\theta \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is an isomorphism
(b)
From the analysis in (a), all matrices of form $\left\{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\right\} = O(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$ is path-connected, and all matrices of form $\left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right\} = SO(2,\mathbb{R})$ is also path-connected, but $O(2,\mathbb{R})$ is not path-connected since $\det : O(2,\mathbb{R}) \to \mathbb{R}$ is a continuous function but the image is $\{\pm 1\}$ is not connected, thus $O(2,\mathbb{R})$ has two connected components

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \neq \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

hence $O(2,\mathbb{R})$ is not abelian
(c)
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ hence $O(2,\mathbb{R}) \setminus SO(2,\mathbb{R})$ is a single conjugacy class
(d)
Notice $\det : O(2,\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1, \text{multiplication}\}$ is a surjective group homomorphism with kernel $SO(2,\mathbb{R})$, hence

$$1 \longrightarrow SO(2,\mathbb{R}) \hookrightarrow O(2,\mathbb{R}) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

is an exact sequence
There is a map $\mu : \mathbb{Z}/2\mathbb{Z} \to O(2,\mathbb{R})$, $\bar{1} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is a homomorphism such that $\det \circ \mu = 1$, hence the exact sequence splits
2.
(a)
Suppose $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(1,1)$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} \Rightarrow \begin{cases} a^2-b^2 = 1 \\ ac-bd = 0 \\ c^2-d^2 = -1 \end{cases}$$

we can find $x, y \in \mathbb{R}$ such that $\begin{cases} b = \sinh x \\ c = \sinh y \end{cases}$, and $\begin{cases} a = \cosh x \\ d = \cosh y \end{cases}$ or $\begin{cases} a = -\cosh x \\ d = \cosh y \end{cases}$ or $\begin{cases} a = \cosh x \\ d = -\cosh y \end{cases}$ or $\begin{cases} a = -\cosh x \\ d = -\cosh y \end{cases}$
then

$$0 = \cosh x \sinh y - \sinh x \cosh y = \sinh(y - x) \iff y = x, \text{or}$$

$$0 = -\cosh x \sinh y - \sinh x \cosh y = -\sinh(x + y) \iff y = -x, \text{or}$$

$$0 = \cosh x \sinh y + \sinh x \cosh y = \sinh(x + y) \iff y = -x, \text{or}$$

$$0 = -\cosh x \sinh y + \sinh x \cosh y = \sinh(x - y) \iff y = x$$

then $\det g = 1, -1, -1, 1$ correspondingly, hence g can only be $\begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$ or $\begin{pmatrix} -\cosh x & \sinh x \\ \sinh x & -\cosh x \end{pmatrix}$, consider $\phi : \mathbb{R}^* \to SO(1,1)$,

$$x \mapsto \begin{cases} \begin{pmatrix} \cosh(\ln x) & \sinh(\ln x) \\ \sinh(\ln x) & \cosh(\ln x) \end{pmatrix}, x > 0 \\ \begin{pmatrix} -\cosh\left(\ln\left(\frac{1}{-x}\right)\right) & \sinh\left(\ln\left(\frac{1}{-x}\right)\right) \\ \sinh\left(\ln\left(\frac{1}{-x}\right)\right) & -\cosh\left(\ln\left(\frac{1}{-x}\right)\right) \end{pmatrix}, x < 0 \end{cases}$$

is an isomorphism easily by checking 4 cases

(b)

Notice $\det : O(1,1) \to \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1, \text{multiplication}\}$ is a surjective group homomorphism with kernel $SO(1,1)$, hence $O(1,1)/SO(1,1) \cong \mathbb{Z}/2\mathbb{Z}$

(c)

Notice ϕ from (a) is continuous, hence $O(1,1)^0 = \left\{ \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} \right\}$, and $\frac{x}{|x|} : \mathbb{R}^* \to \mathbb{Z}/2\mathbb{Z} \cong \{\pm 1, \text{multiplication}\}$ is a surjective group homomorphism with kernel $\mathbb{R}_{>0}$, hence $SO(1,1)/O(1,1)^0 \cong \mathbb{R}^*/\mathbb{R}_{>0}$, $\left|O(1,1)/O(1,1)^0\right| = [O(1,1) : SO(1,1)]\left[SO(1,1) : O(1,1)^0\right] = 4$, hence it is isomorphic to either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, but $-I, J, -J \in O(1,1)/O(1,1)^0$ are of order 2 and $(-I)J = -J, (-I)(-J) = J, J(-J) = -I$, thus the component group $O(1,1)/O(1,1)^0 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

3.

(a)

Here $J \in GL(n,\mathbb{C})$, $G_J = \{g \in GL(n,\mathbb{C}) | g^T J g = J\} = \{g \in GL(n,\mathbb{C}) | g^T = J g^{-1} J^{-1}\}$, $G_I = \{g \in GL(n,\mathbb{C}) | g^T g = I\} = \{g \in GL(n,\mathbb{C}) | g^T = g^{-1}\}$, since J is symmetric, $\exists P \in GL(n,\mathbb{C})$ such that $PJP^T = I$(corresponding to changing of basis), then we have $PG_JP^{-1} = G_I$ since $\forall g \in G_J$, we have $Jg^TJ^{-1} = g^{-1}$, then $(PgP^{-1})^{-1} = Pg^{-1}P^{-1} = PJg^TJ^{-1}P^{-1} = PJP^T(P^T)^{-1}g^TP^T(P^T)^{-1}J^{-1}P^{-1} = (PJP^T)(PgP^{-1})^T(PJP^T)^{-1} = (PgP^{-1})^T$, hence $PgP^{-1} \in G_I$, thus G_J and $O(n,\mathbb{C})$ are conjugate in $GL(n,\mathbb{C})$

(b)

Let $K = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & \\ & b \end{pmatrix} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ which gives $ab = 1$, if n is even, let $J = \begin{pmatrix} K & & \\ & \ddots & \\ & & K \end{pmatrix}$, then the diagonal subgroup of G_J is isomorphic to $\mathbb{C}^{*\frac{n}{2}} = \mathbb{C}^{*\lfloor \frac{n}{2} \rfloor}$,

if n is odd, if $J = \begin{pmatrix} K & & & \\ & \ddots & & \\ & & K & \\ & & & 0 \end{pmatrix}$, then the diagonal subgroup of G_J is isomorphic to $\mathbb{C}^{*\lfloor \frac{n}{2} \rfloor}$, if $J = \begin{pmatrix} K & & & \\ & \ddots & & \\ & & K & \\ & & & 1 \end{pmatrix}$, then the diagonal subgroup of G_J is isomorphic to $\mathbb{C}^{*\lfloor \frac{n}{2} \rfloor} \times \mathbb{Z}/2\mathbb{Z}$ which contains a subgroup isomorphic to $\mathbb{C}^{*\lfloor \frac{n}{2} \rfloor}$

(c)

By (a) and (b), we know that $O(n,\mathbb{C})$ contains a subgroup isomorphic to $\mathbb{C}^{*\lfloor \frac{n}{2} \rfloor}$

(d)

If $A \in \mathrm{Lie}$, then $e^{tA} \in O(n,\mathbb{C})$, $e^{tA^T} = \left(e^{tA}\right)^T = \left(e^{tA}\right)^{-1} = e^{-tA}$, then $A^T = \dfrac{de^{tA^T}}{dt}\,|_{t=0} = \dfrac{de^{-tA}}{dt}\,|_{t=0} = -A$, hence $\mathrm{Lie}\left(\mathrm{O(n,\mathbb{C})}\right) = \{\mathrm{A} \in \mathrm{M_n(\mathbb{C})} \| \mathrm{A^T} = -\mathrm{A}\}$, $\dim \mathrm{Lie}\left(\mathrm{O(n,\mathbb{C})}\right) = \dfrac{n(n-1)}{2}$

4.

(a)

$$1 \longrightarrow SL(n,\mathbb{C}) \hookrightarrow GL(n,\mathbb{C}) \xrightarrow{\det} \mathbb{C}^* \longrightarrow 1$$

is an exact sequence

(b)

$$1 \longrightarrow \mathbb{C}^* \xhookrightarrow{a \mapsto aI} GL(n,\mathbb{C}) \xrightarrow{q} PGL(n,\mathbb{C}) \longrightarrow 1$$

is an exact sequence, where q is the quotient map

(c)

$SL(n,\mathbb{C}) \to GL(n,\mathbb{C})$ pass to $PSL(n,\mathbb{C}) \to PGL(n,\mathbb{C})$ by quotient map, for $AZ \in PGL(n,\mathbb{C})$, $\dfrac{A}{\sqrt[n]{\det A}}Z \mapsto AZ$, hence the map is surjective, if $AZ \in PSL(n,\mathbb{C})$, and $A \in Z(PGL(n,\mathbb{C}))$, then $A = aI$, for some $a \neq 0$, but since $A \in SL(n,\mathbb{C})$, $1 = \det A$, $A \in Z(PSL(n,\mathbb{C}))$, hence the map is also injective

(d)

When n is odd, the previous argument still works, namely $PSL(n,\mathbb{R}) \cong PGL(n,\mathbb{R})$, when $n = 2k$ is even, the injectivity still works, however, for $aI, A \in GL(n,\mathbb{R})$, $\det(aI \cdot A) = a^{2k} \det A$ has the same sign as $\det A$, thus there is an exact sequence

$$1 \longrightarrow PSL(n,\mathbb{R}) \hookrightarrow PGL(n,\mathbb{R}) \xrightarrow{\det} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

5.

(a)

If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix} \in SU(2)$, then

$$I = gg^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\xi} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\bar{\gamma} + \beta\bar{\xi} \\ \bar{\alpha}\gamma + \bar{\beta}\xi & |\gamma|^2 + |\xi|^2 \end{pmatrix}$$

Hence $\begin{cases} |\alpha|^2 + |\beta|^2 = 1 \\ \alpha\bar{\gamma} + \beta\bar{\xi} = 0 \\ |\gamma|^2 + |\xi|^2 \end{cases}$, and also $1 = \det g = \alpha\xi - \beta\gamma$

If $\gamma = 0$, then $|\alpha| = |\xi| = 1$, $\beta = 0$, $\alpha\xi = 1 \Rightarrow \xi = \bar{\alpha}$

If $\gamma = 0$, then $\alpha = -\dfrac{\beta\bar{\xi}}{\bar{\gamma}}$, $\beta = 0$, then $1 = -\dfrac{\beta|\xi|^2}{\bar{\gamma}} - \beta\gamma \Rightarrow -\bar{\gamma} = \beta$, hence $\alpha = \bar{\xi}$

(b)

Since $S^3 \subseteq \mathbb{C}^2 = \{|\alpha|^2 + |\beta|^2 = 1\}$ which is the same zero set, thus $SU(2)$ is topologically equivalent to S^3, and is therefore 3-dimensional, connected and simply connected

(c)

$$1 \longrightarrow SU(2) \hookrightarrow U(2) \xrightarrow{\det} S^1 \longrightarrow 1$$

is an exact sequence, there is a map $\mu : S^1 \to U(2)$, $z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$ which is a homomorphism such that $\det \circ \mu = 1$, hence the exact sequence splits

(d)

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \xi \end{pmatrix} + \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\xi} \end{pmatrix} = \begin{pmatrix} \alpha+\bar{\alpha} & \beta+\bar{\gamma} \\ \gamma+\bar{\beta} & \xi+\bar{\xi} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \alpha+\bar{\alpha} = \xi+\bar{\xi} = \gamma+\bar{\beta} = 0$$

and $0 = trg = \alpha + \xi$, thus $\xi = -\alpha \in i\mathbb{R}$ and $\gamma = -\bar{\beta}$, hence W is a 3-dimensional real vector space of the form $\left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$ and $tr\left(\begin{pmatrix} ix_1 & y_1+iz_1 \\ -y_1+iz_1 & -ix_1 \end{pmatrix} \begin{pmatrix} ix_2 & y_2+iz_2 \\ -y_2+iz_2 & -ix_2 \end{pmatrix} \right) = -2(x_1x_2 + y_1y_2 + z_1z_2)$ is a definite symmetric bilinear form

(e)

$\forall g \in SU(2)$, $g^* = g^{-1}$, $X, Y \in W$, $gXg^{-1} + (gXg^{-1})^* = gXg^{-1} + gX * g^* = g(X + X^*)g^* = 0$, $tr(gXg^{-1}) = tr(X) = 0$, $(gXg^{-1}, gYg^{-1}) = tr(gXg^{-1}gYg^{-1}) = tr(gXYg^{-1}) = tr(XY) = (X, Y)$, hence $SU(2)$ acts on W by conjugation, preserving the form

(f)

By calculation in (d), $\langle X, Y \rangle := \frac{1}{2} tr(XY)$ will be the standard inner product in $\mathbb{R}^3$, and $SU(2)$ still acts on W by conjugation, preserving the form, this can be regarded as a continuous homomorphism $\varphi : SU(2) \to O(3)$, since $SU(2)$ is connected by (b), hence φ is actually a continuous homomorphism from $SU(2)$ to $SO(3)$, now compute its kernel

$$\begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} = ix \begin{pmatrix} |\alpha|^2 - |\beta|^2 & -2\alpha\beta \\ -2\bar{\alpha}\bar{\beta} & |\beta|^2 - |\alpha|^2 \end{pmatrix} \Rightarrow \begin{cases} \beta = 0 \\ |\alpha|^2 = 1 \end{cases}$$

and

$$\begin{pmatrix} & \gamma \\ -\bar{\gamma} & \end{pmatrix} = \begin{pmatrix} \alpha & \\ & \bar{\alpha} \end{pmatrix} \begin{pmatrix} & \gamma \\ -\bar{\gamma} & \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \\ & \alpha \end{pmatrix} = \begin{pmatrix} & \alpha^2\gamma \\ -\bar{\alpha}^2\bar{\gamma} & \end{pmatrix} \Rightarrow \alpha^2 = 1 \Rightarrow \alpha = \pm 1$$

Thus we have the exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow SU(2) \xrightarrow{\varphi} SO(3) \longrightarrow 1$$

6.

(a)

$\forall (x, y) \in X$, if $x = 0$, then $\begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$, if $x \neq 0$, then $\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, hence $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ can move to any point in X by an action of G, hence G acts transitively on X

(b)

If $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ c = 0 \end{cases}$, and since $0 \neq \det g = ad - bc = d$, hence $|H| = q(q-1)$

(c)

Consider $\varphi : G/H \to X$, $gH \mapsto g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is well-defined, and surjectivity follows from (a), injectivity follows from the fact if $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g' \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $g^{-1}g' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow g^{-1}g' \in H$, hence φ is bijective

(d)

Because of (c), $|G| = |G/H||H| = |X||H| = (q^2 - 1)q(q-1) = (q+1)q(q-1)^2$

(e)

Since $Z = \{aI | a \neq 0\}$, $|Z| = q - 1$, $|PGL(2, \mathbb{F}_q)| = |G|/|Z| = (q+1)q(q-1)$, in particular, $|PGL(2, \mathbb{F}_5)| = 120$

(f)

From group action $GL(2, \mathbb{F}_4)$ on $\mathbb{F}_4^2 \backslash \{(0,0)\}$, we can pass to a group action $PGL(2, \mathbb{F}_4)$ on $P\mathbb{F}_4$, this action is faithful since for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $c \neq 0$, $g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \neq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $P\mathbb{F}_4$, if $b \neq 0$, $g \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $P\mathbb{F}_4$, hence $b = c = 0$, if $a \neq d$, $g \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ d \end{pmatrix} \neq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in $P\mathbb{F}_4$, hence $g \in Z$, thus we get a injective group $PGL(2, \mathbb{F}_4) \hookrightarrow S_5$ since $|P\mathbb{F}_4| = 5$, but $|PGL(2, \mathbb{F}_4)| = 60$, $|S_5| = 120$, any subgroup of S_5 of order 60 should be normal, intersection of normal subgroups is again a normal subgroup, but A_5 is a simple group, hence A_5 is the only subgroup of S_5 of order 60, thus $PGL(2, \mathbb{F}_4) \cong A_5$

7.

(a)

By Jordan normal form theorem, $\forall g \in GL(n, \mathbb{C})$, g can be decomposed uniquely as $SU = US$ where S is semisimple and U is unipotent, thus $\log U = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(U-I)^m}{m}$ is actually a finite sum, so is $e^{\log U}$, now consider $U(t) = I + t(U - I)$, when t is small, $e^{\log U(t)} = U(t)$, and since $e^{\log U(t)} - U(t)$ is a matrix with entries polynomials in t, $e^{\log U(t)} = U(t), \forall t$, in particular, $e^{\log U} = U$, the general case follows by dividing g into Jordan blocks

(b)

The image of then exponential map from $M_n(\mathbb{R})$ to $GL(n, \mathbb{R})$ is $\{A \in GL(n, \mathbb{R}) | \det A|U > 0, U \text{ is any A invariant space}\}$, notice that $\det e^{A|U} = e^{trA|U} > 0$, $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp \begin{pmatrix} & \theta \\ -\theta & \end{pmatrix}$, for any $A \in GL(n, \mathbb{R})$ with $\det A > 0$, A can be written in real Jordan canonical form, and then a Jordan block can be written as $SU = US$, U is unipotent, and S is diagonalizable with real numbers or R_θ's, for negative diagonal numbers we can pair them up and notice $R_\pi = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = \exp \begin{pmatrix} & \pi \\ -\pi & \end{pmatrix}$, thus A is in the image of the exponential map

1.2 Homework2

1.

$\ker\phi$ is evidently a vector space, and $\forall X \in \ker\phi, Y \in \mathfrak{g}$, we have $\phi([X,Y]) = [\phi(X),\phi(Y)] = 0 \Rightarrow [X,Y] \in \ker\phi$, thus $\ker\phi$ is an ideal. The converse is not true, since we can easily find an injection ϕ such that it is a vector space homomorphism but not a Lie algebra homomorphism On the other hand, if the converse is true, if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, we can define Lie algebra homomorphism the quotient map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ which have kernel $\mathfrak{h}$

2.

Suppose G is a connected matrix group, $H \leq G$ is a connected subgroup, $\mathfrak{h}, \mathfrak{g}$ are their corresponding Lie algebras, then by Lie subgroup-Lie subalgebra correspondence, we know that G and H consists of elements of the form $\left\{e^{X_1}e^{X_2}\cdots e^{X_n} | X_i \in \mathfrak{g}\right\}$ and $\left\{e^{Y_1}e^{Y_2}\cdots e^{Y_m} | Y_i \in \mathfrak{h}\right\}$, to show $H \trianglelefteq G$, we only need to show $e^Xe^Ye^{-X} \in H, X \in \mathfrak{g}, Y \in \mathfrak{h}$, since then $e^{X_1}\cdots e^{X_n}e^Ye^{-X_n}\cdots e^{-X_1} \in H, Y \in \mathfrak{h}$ and $e^{X_1}\cdots e^{X_n}e^{Y_1}\cdots e^{Y_m}e^{-X_n}\cdots e^{-X_1}, Y_i \in \mathfrak{h}$, notice $e^Xe^Ye^{-X} = e^{e^XYe^{-X}} = e^{Ad_{e^Y}(X)} = e^{e^{ad_Y}(X)}$, where $e^{ad_Y}(X) = X + ad_Y(X) + \dfrac{ad_Y^2}{2}(X) + \cdots \in \mathfrak{h}$ since $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ which is certainly a closed subspace because they are finite dimensional, thus $e^{e^{ad_Y}(X)} \in H$

Conversely, if $H \trianglelefteq G$, then $e^Xe^{tY}e^{-X} = e^{te^XYe^{-X}} \in H, \forall t \in \mathbb{R}, X \in \mathfrak{g}, Y \in \mathfrak{h}$, thus $e^XYe^{-X} \in \mathfrak{h}$, then $e^{tX}Ye^{-tX} \in \mathfrak{h}, \forall t \in \mathbb{R}$, so we have $\left.\dfrac{d}{dt}\right|_{t=0} e^{tX}Ye^{-tX} = [X,Y] \in \mathfrak{h}$, therefore $\mathfrak{h} \leq \mathfrak{g}$ is an ideal

3.

Let $\mathfrak{g}_0 = \mathbb{R}$ be a real Lie algebra of dimension 1 with $[a,b] = 0$ for any $a,b \in \mathbb{R}$, suppose $\mathfrak{g} = \langle\langle\rangle\rangle$ is a real Lie algebra of dimension 1, then $[v,v] = 0$ because of anti-symmetry, hence $\varphi : \mathfrak{g} \to \mathfrak{g}_0, v \mapsto 1$ is a Lie algebra isomorphism, hence Lie algebra of dimension 1 is $\mathfrak{g}_0$ up to isomorphism

Let $\mathfrak{g}_{ab} = \mathbb{R}^2$ be a real Lie algebra of dimension 2 with $[e_1,e_2] = ae_1 + be_2$, suppose $\mathfrak{g} = \langle v\rangle \oplus \langle w\rangle$ is a real Lie algebra of dimension 2, with $[v,w] = av + bw$, hence $\varphi : \mathfrak{g} \to \mathfrak{g}_{ab}, v \mapsto e_1, w \mapsto e_2$ is a Lie algebra isomorphism, suppose $\phi : \mathfrak{g}_{ab} \to \mathfrak{g}_{cd}$ is an isomorphism, $\phi(e_1) = xe_1 + ye_2, \phi(e_2) = ze_1 + we_2$, then we necessarily have $A\begin{pmatrix} a \\ b \end{pmatrix} = \det A\begin{pmatrix} c \\ d \end{pmatrix}$, where $\det A \neq 0$, if $c = d = 0$, then $a = b = 0$, you can just take $A = I$, if c,d are not both zero, then a,b are not both zero and without loss of generality we may assume $d \neq 0$, in that case, if $b \neq 0$, we can take $A = \begin{pmatrix} \frac{b}{d} & \frac{cd-ab}{bd} \\ 0 & \frac{d}{b} \end{pmatrix}$, if $b = 0$, then $a \neq 0$, we can take $A = \begin{pmatrix} \frac{c}{\frac{d}{a}} & -\frac{a}{d} \\ \frac{d}{a} & 0 \end{pmatrix}$, Therefore, there are two classes of real Lie algebras of dimension 2, namely $\mathfrak{g}_{00}$ and $\mathfrak{g}_{01}$

4.

$$\begin{pmatrix} & I \\ -I & \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} & I \\ -I & \end{pmatrix} = \begin{pmatrix} -B & A \\ -D & C \end{pmatrix} \Rightarrow \begin{cases} D = A \\ C = -B \end{cases}$$

Let $J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$, then these are the matrices deonoted as $\mathfrak{g} \leq \mathfrak{gl}(2n,\mathbb{R})$ commuting with J, consider $\varphi : \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{g}, A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ is evidently a linear map, and $[\varphi(A_1 + iB_1), \varphi(A_2 + iB_2)] =$
$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} - \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} [A_1,A_2] - [B_1,B_2] & [A_1,B_2] + [B_1,A_2] \\ -[A_1,B_2] - [B_1,A_2] & [A_1,A_2] - [B_1,B_2] \end{pmatrix} =$
$\varphi([A_1,A_2] - [B_1,B_2] + i([A_1,B_2] + [B_1,A_2])) = \varphi([A_1 + iB_1, A_2 + iB_2])$, thus φ is an isomorphism

5.

(a)

Define group homomorphism $\varphi : \mathbb{H} \to GL(2,\mathbb{C})$, $1 \mapsto \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $i \mapsto \begin{pmatrix} i & \\ & -i \end{pmatrix}$, $j \mapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $k \mapsto \begin{pmatrix} & i \\ i & \end{pmatrix}$, $a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$, i.e. $\lambda + j\mu \mapsto$

$\begin{pmatrix} \lambda & \bar{\mu} \\ -\mu & \bar{\lambda} \end{pmatrix}$ with determinant $|\lambda|^2+|\mu|^2 = |\lambda+j\mu|^2$ which is the norm, and $\begin{pmatrix} \frac{\bar{\lambda}}{|\lambda|^2+|\mu|^2} & \frac{-\bar{\mu}}{|\lambda|^2+|\mu|^2} \\ \frac{\mu}{|\lambda|^2+|\mu|^2} & \frac{\lambda}{|\lambda|^2+|\mu|^2} \end{pmatrix}$ is the left and right inverse, similarly, we can define $\Phi : GL(n,\mathbb{H}) \to GL(2n,\mathbb{C})$, $\begin{pmatrix} \lambda_{11}+j\mu_{11} & \cdots & \lambda_{1n}+j\mu_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1}+j\mu_{n1} & \cdots & \lambda_{nn}+j\mu_{nn} \end{pmatrix} \mapsto \begin{pmatrix} \lambda_{11} & \overline{\mu_{11}} & \cdots & \lambda_{1n} & \overline{\mu_{1n}} \\ -\mu_{11} & \overline{\lambda_{11}} & & -\mu_{1n} & \overline{\lambda_{1n}} \\ \vdots & & \ddots & & \vdots \\ \lambda_{n1} & \overline{\mu_{n1}} & \cdots & \lambda_{nn} & \overline{\mu_{nn}} \\ -\mu_{n1} & \overline{\lambda_{n1}} & & -\mu_{nn} & \overline{\lambda_{nn}} \end{pmatrix}$ which shows that $GL(n,\mathbb{H})$ is a matrix group of complex dimension $2n^2$, it is not compact since the entries are not bounded

(b)

$$\langle \lambda\vec{v}, \vec{w}\rangle = \sum_{i=1}^{n} \overline{\lambda v_i} w_i = \sum_{i=1}^{n} \bar{\lambda}\overline{v_i} w_i = \bar{\lambda}\langle \vec{v}, \vec{w}\rangle$$

$$\langle \vec{v}, \vec{w}\rangle = \sum_{i=1}^{n} \overline{v_i} w_i \lambda = \langle \vec{v}, \vec{w}\rangle\lambda$$

$$\langle \vec{v}, \vec{w}\rangle = \sum_{i=1}^{n} \overline{v_i} w_i = \sum_{i=1}^{n} \overline{\overline{w_i} v_i} = \overline{\sum_{i=1}^{n} \overline{w_i} v_i} = \overline{\langle \vec{w}, \vec{v}\rangle}$$

(c)

$\begin{pmatrix} \lambda & \bar{\mu} \\ -\mu & \bar{\lambda} \end{pmatrix}\begin{pmatrix} \alpha & \bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \lambda\alpha - \bar{\mu}\beta & \lambda\bar{\beta}+\bar{\mu}\bar{\alpha} \\ -\mu\alpha - \bar{\lambda}\beta & -\mu\bar{\beta}+\bar{\lambda}\bar{\alpha} \end{pmatrix}$, we can define $\begin{pmatrix} \lambda & \bar{\mu} \\ -\mu & \bar{\lambda} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha - \bar{\mu}\beta \\ \mu\alpha + \bar{\lambda}\beta \end{pmatrix}$, then the corresponding matrix would be $\begin{pmatrix} \lambda & -\bar{\mu} \\ \mu & \bar{\lambda} \end{pmatrix}$, here we are identifying $\alpha + j\beta$ with $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\begin{pmatrix} \alpha_1 + j\beta_1 \\ \vdots \\ \alpha_n + j\beta_n \end{pmatrix}$ with $\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_n \\ \beta_n \end{pmatrix}$, the we have $\langle \vec{v}, \vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle_1 + j\langle \vec{v}, \vec{w}\rangle_2$ where $\vec{v} = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \vdots \\ \alpha_n \\ \beta_n \end{pmatrix}, \vec{w} = \begin{pmatrix} \gamma_1 \\ \eta_1 \\ \vdots \\ \gamma_n \\ \eta_n \end{pmatrix}$, and $\langle \vec{v}, \vec{w}\rangle_1 = \sum_{i=1}^{n}(\overline{\alpha_i}\gamma_i + \overline{\beta_i}\eta_i)$, $\langle \vec{v}, \vec{w}\rangle_2 = \sum_{i=1}^{n}(-\overline{\beta_i}\gamma_i + \overline{\alpha_i}\eta_i)$, hence $\langle g\vec{v}, g\vec{w}\rangle_1 + j\langle g\vec{v}, g\vec{w}\rangle_2 = \langle g\vec{v}, g\vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle_1 + j\langle \vec{v}, \vec{w}\rangle_2$, $\langle , \rangle_2$ is a symplectic form with respect to vectors of the form $\vec{v} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \vec{w} = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \\ \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$ compare to $\langle , \rangle_1$, thus $g \in U(2n) \cap Sp(2n, \mathbb{C})$

6.

(a)

Suppose $(\vec{v}, t) \in Z$, then $(\vec{v}, t) * (\vec{w}, u) = (\vec{v} + \vec{w}, t + u + \frac{1}{2}\langle \vec{v}, \vec{w}\rangle) = (\vec{v} + \vec{w}, t + u + \frac{1}{2}\langle \vec{w}, \vec{v}\rangle) = (\vec{v}, t) * (\vec{w}, u), \forall (\vec{w}, u) \in H(V)$, then $-\langle \vec{w}, \vec{v}\rangle = \langle \vec{v}, \vec{w}\rangle = \langle \vec{w}, \vec{v}\rangle \Rightarrow \langle \vec{v}, \vec{w}\rangle = 0, \forall \vec{w} \in V$, since $\langle , \rangle$ is non-degenerate, $\vec{v} = 0$, $Z \cong \mathbb{R}$

(b)

Define $\varphi : H(V) \to V, (\vec{v}, t) \mapsto \vec{v}$ which is surjective, then $\ker\varphi = Z$, hence $H(V)/Z \cong V$

(c)

From (b), we can easily see there is an exact sequence $1 \to Z \to H(V) \to V \to 1$, consider $\psi : V \to H(V), \vec{v} \mapsto (\vec{v}, 0)$, we have $\varphi \circ \psi = id_V$, thus the exact sequence splits

(d)

Define $\phi : H(V) \to H, (\vec{v}, \vec{w}, t) \mapsto \begin{pmatrix} 1 & \vec{v}^T & t + \frac{1}{2}\vec{v}^T\vec{w} \\ 0 & I_n & \vec{w} \\ 0 & 0 & 1 \end{pmatrix}$ is an isomorphism where H is Heisenberg group and the non-degenerate symplectic form $\langle , \rangle$ is first isomorphic to $\langle(\vec{v_1}, \vec{w_1}), (\vec{v_2}, \vec{w_2})\rangle = \frac{1}{2}(\vec{v_1}^T\vec{w_2} - \vec{w_1}^T\vec{v_2})$, then

$\phi(\vec{v_1}, \vec{w_1}, t_1)\phi(\vec{v_1}, \vec{w_1}, t_1) = \begin{pmatrix} 1 & \vec{v_1}^T & t_1 + \frac{1}{2}\vec{v_1}^T\vec{w_1} \\ 0 & I_n & \vec{w_1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \vec{v_2}^T & t_2 + \frac{1}{2}\vec{v_2}^T\vec{w_2} \\ 0 & I_n & \vec{w_2} \\ 0 & 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & \vec{v_1}^T + \vec{v_2}^T & t_1 + \frac{1}{2}\vec{v_1}^T\vec{w_1} + t_2 + \frac{1}{2}\vec{v_2}^T\vec{w_2} + \vec{v_1}^T\vec{w_2} \\ 0 & I_n & \vec{w_1} + \vec{w_2} \\ 0 & 0 & 1 \end{pmatrix} =$

$\begin{pmatrix} 1 & \vec{v_1}^T + \vec{v_2}^T & t_1 + t_2 + \frac{1}{2}\langle(\vec{v_1}, \vec{w_1}), (\vec{v_2}, \vec{w_2})\rangle + \frac{1}{2}(\vec{v_1} + \vec{v_2})^T(\vec{w_1} + \vec{w_2}) \\ 0 & I_n & \vec{w_1} + \vec{w_2} \\ 0 & 0 & 1 \end{pmatrix} =$

$\phi(\vec{v_1} + \vec{v_2}, \vec{w_1} + \vec{w_2}, t_1 + t_2 + \frac{1}{2}\langle(\vec{v_1}, \vec{w_1}), (\vec{v_2}, \vec{w_2})\rangle)$

7.

(a)

Let E_{ij} denote the matrix with only the (i, j)-th entry 1, and 0 else where, then $E_{ij}E_{kl} = \delta_{jk}E_{il}$, $[E_{ij}, E_{kl}] = E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{li}E_{kj}$, and $\mathfrak{n}$ is spaned by $\{E_{ij}\}_{j>i}$, it is easy to see that $[\mathfrak{n}, \mathfrak{n}]$ is spaned by $\{E_{ij}\}_{j>i+1}$

(b)

Consider $\{I + aE_{ij}\}_{j>i}$ Notice $(I + aE_{ij})^{-1} = I - aE_{ij}$, and it is easy to calculate that given $j > i, l > k$, we have $(I + aE_{ij})(I + bE_{kl})(I + aE_{ij})^{-1}(I + bE_{kl})^{-1} = I + ab\delta_{jk}E_{il} - ab\delta_{il}E_{kj}$, as the same analysis in (a), we know the commutator subgroup contains all the elements in $I + [\mathfrak{n}, \mathfrak{n}]$, on the other hand, for any $I - P \in N$ with P nilpotent, we have $(I - P)^{-1} = I + P + P^2 + \cdots$, thus $(I - P)(I - Q)(I - P)^{-1}(I - Q)^{-1} = (I - P)(I - Q)(I + P + P^2 + \cdots)(I + Q + Q^2 + \cdots) = (I - P - Q + PQ)(I + P + Q + P^2 + Q^2 + PQ + \cdots) = I + [P, Q] + \cdots$, with the result in (a), we know that $[P, Q] + \cdots \in [\mathfrak{n}, \mathfrak{n}]$, and for $X, Y \in [\mathfrak{n}, \mathfrak{n}]$, $(I + X)(I + Y) = I + (X + Y + XY) \in I + [\mathfrak{n}, \mathfrak{n}]$, thus the commutator subgroup $\{N, N\}$ of N is $I + [\mathfrak{n}, \mathfrak{n}]$, $\log(e^X e^Y) = X + Y + S$, where $S \in [\mathfrak{n}, \mathfrak{n}]$, thus $f(e^X e^Y) = e^{\phi(X+Y+S)} = e^{\phi(X)+\phi(Y)} = f(e^X)f(e^Y)$ meaning f is a homomorphism

(c)

The sufficient and necessary condition needed is $\{N, N\} \leq \ker f$, first notice this map is well defined, that N is the image of $\mathfrak{n}$ under exponential map because you can take $\log$ where the power series has only finitely many terms, and $\mathfrak{n}$ is the Lie algebra of N, if $e^X = e^Y$ where $X, Y \in \mathfrak{n}$, then $X - Y \in [\mathfrak{n}, \mathfrak{n}] \leq \ker \phi$, thus $f(e^X) = e^{\phi(X)} = e^{\phi(Y)} = f(e^Y)$, the condition is obvious necessary, on the other hand, if $\{N, N\} \leq \ker f$, $e^X e^Y = I + X + Y + R$ where $R \in [\mathfrak{n}, \mathfrak{n}]$

8.

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$, suppose $g = kan$ with $k \in SO(2, \mathbb{R}), a = \begin{pmatrix} e^x & \\ & e^{-x} \end{pmatrix}, n = \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix}$, then we would have $\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = g^T g = n^T a^T k^T kan = n^T a^T an = \begin{pmatrix} e^{2x} & ye^{2x} \\ ye^{2x} & y^2e^{2x} + e^{-2x} \end{pmatrix}$ thus we would necessarily have $x = \dfrac{\ln(a^2 + c^2)}{2}, y = \dfrac{ab + cd}{a^2 + c^2}$, and $k = g^{-T}n^T a^T$ this shows the uniqueness, only need to check that k is indeed in $SO(2, \mathbb{R})$ and calculation shows that $k^T k = ang^{-1}g^{-T}n^T a^T = \begin{pmatrix} (ad - bc)^2 & 0 \\ 0 & 1 \end{pmatrix} = I$

9.

(a)

Since $\mathrm{Cent}_G(hgh^{-1}) = h\mathrm{Cent}_G(g)h^{-1}$, up to conjugacy, if $g = rI_3$, then $\mathrm{Cent}_G(g) = SL(3, \mathbb{C})$, if $g = \mathrm{diag}(rI_2, a)$ where $a \neq r$, then $\mathrm{Cent}_G(g) = \{\mathrm{diag}(A, b)\}$, and if $g = \mathrm{diag}(a, b, c)$, where a, b, c are distinct, then $\mathrm{Cent}_G(g) = \mathrm{diag}(\alpha, \beta, \gamma)$, only need to prove $SL(n, \mathbb{C})$ is connected. Notice any element of $SL(n, \mathbb{C})$ can be written as $g = C \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix} C^{-1}$, where $\lambda_1 \cdots \lambda_n = 1$, let $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) + N$, N is nilpotent, consider path $g(t) = C\left(\mathrm{diag}(\lambda_1(t), \cdots, \lambda_{n-1}(t), \lambda_n(t)) + tN\right) C^{-1}$, where $\lambda_j(t) = r_j^t e^{it\theta_j}$, with $\lambda_j = r_j e^{i\theta_j}, 1 \le j \le n-1$, and $\lambda_n(t) = \dfrac{1}{\lambda_1(t) \cdots \lambda_{n-1}(t)}$, then $g(0) = I, g(1) = g$

(b)

Z only consists of three elements $I_3, \omega I_3, \omega^2 I_3$, where $\omega = e^{\frac{2\pi i}{3}}$, up to conjugacy, if $g = rI_3$, then $\mathrm{Cent}_G(g) = PSL(3, \mathbb{C})$ which has trivial component group, if $g = \mathrm{diag}(rI_2, a)$ where $a \neq r$, then $\mathrm{Cent}_G(g) = \{\mathrm{diag}(A, b)\}$ which has trivial component group, if $g = \mathrm{diag}(a, b, c)$, where a, b, c are distinct, suppose $a = b\omega = c\omega^2$(or $a = c\omega = b\omega^2$), then $\mathrm{Cent}_G(g)$ consists of diagonal matrices $\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix}$ and matrices of the form $\begin{pmatrix} & x & \\ & & y \\ z & & \end{pmatrix}$ and $\begin{pmatrix} & & x \\ y & & \\ & z & \end{pmatrix}$, and they form two different connected components other than the diagonal matrices easily seen from the fact that they are the preimages of $\{1\}$ under the maps $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto a_{11}a_{22}a_{33}$, $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto a_{12}a_{23}a_{31}$ and $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \mapsto a_{13}a_{21}a_{32}$, these are all connected since you can find a path connected with $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $\begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$ and $\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$ correspondingly as above, so the component group is $\mathbb{Z}/3\mathbb{Z}$, for other possibilities of a, b, c, $\mathrm{Cent}_G(g) = \mathrm{diag}(\alpha, \beta, \gamma)$ which is connected

1.3 Homework3

1.

We know the following facts:

The Lie algebra a compact Lie group is reductive

If $\mathfrak{g}$ is semisimple Lie algebra with $\dim \mathfrak{g} \leq 4$, then $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$

Suppose $\mathfrak{g}$ is a complex, semisimple Lie algebra, then there exists unique up to isomorphism a compact simply connected Lie group G such that $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$

If G is a 3 dimensional compact connected Lie group, let $\tilde{G}$ be its universal cover, then $\mathrm{Lie}(\tilde{G})_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ or $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, in the first case, $\tilde{G}$ is necessarily isomorphic to $SU(2, \mathbb{C})$ since $\mathfrak{su}(2, \mathbb{C})_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$, and since $Z(SU(2, \mathbb{C})) = \{\pm I\}$, thus G can only be $SU(2, \mathbb{C})$ or $PSU(2, \mathbb{C})$, in the second case, $\tilde{G}$ is $\mathbb{R}^3$, hence $G = \mathbb{R}^3/\Lambda \cong T^3$

If G is a 4 dimensional compact connected Lie group, let $\tilde{G}$ be its universal cover, then since $\mathrm{Lie}(\tilde{G})_{\mathbb{C}}$ is reductive, so it is necessarily $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$, in the first case, $\tilde{G} \cong SU(2, \mathbb{C}) \times \mathbb{R}$ since it has $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathbb{C}$ as its complexified Lie algebra, thus G is $SU(2, \mathbb{C}) \times T^1$ or $PSU(2, \mathbb{C}) \times T^1$, in the second case, $\tilde{G}$ is $\mathbb{R}^4$, $G = T^4$

2.

(a)

Denote $X := (X_1, \cdots, X_n)^T$, $Y := (Y_1, \cdots, Y_n)^T$, suppose $X' = (X'_1, \cdots, X'_n)^T = AX$, $Y' = (Y'_1, \cdots, Y'_n)^T = BY$ such that Y' is also the dual basis of X', where $A = (\alpha_{ij}), B = (\beta_{ij})$, then $\delta_{ij} = (X'_i, Y'_j) = \sum_k \alpha_{ik}\beta_{jk}$, thus $AB^T = I$, which also implies $B^T A = I$, hence

$$\begin{aligned}
\sum_i \pi(X'_i)\pi(Y'_i) &= \sum_{i,k,l} \alpha_{ik}\beta_{il}\pi(X_k)\pi(X_l) \\
&= \sum_{k,l} \left(\sum_i \alpha_{ik}\beta_{il} \right) \pi(X_k)\pi(X_l) \\
&= \sum_{k,l} \delta_{kl}\pi(X_k)\pi(X_l) \\
&= \sum_k \pi(X_k)\pi(X_k)
\end{aligned}$$

Thus Ω is independent of the choice of basis

(b)

Since $(,)$ is nondegenerate, we can choose basis X_i such that $(X_i, X_i) = \varepsilon_i$, where $\varepsilon_i = \begin{cases} 1, 1 \leq i \leq p \\ -1, p+1 \leq i \leq p+q \end{cases}$, and $(X_i, X_j) = 0, i \neq j$, then $Y_i = \varepsilon_i X_i$, $\Omega = \sum_i \pi(X_i)\pi(Y_i) = \sum_i \varepsilon_i \pi(X_i)^2$, let $[X_i, X_j] = \sum_k c_k^{ij} X_k$, c_k^{ij} are the structure constants, notice $\varepsilon_k c_k^{ij} = ([X_i, X_j], X_k) = (X_i, [X_j, X_k]) = \varepsilon_i c_i^{jk}$, so we have

$$c_k^{ij} = -c_k^{ij} \tag{1.1}$$

$$\varepsilon_k c_k^{ij} = \varepsilon_i c_i^{jk} = \varepsilon_j c_j^{ki} \tag{1.2}$$

now let's compute

$$
\begin{aligned}
[\Omega, \pi(X_j)] &= \sum_i \varepsilon_i \pi(X_i)^2 \pi(X_j) - \sum_i \varepsilon_i \pi(X_j)\pi(X_i)^2 \\
&= \sum_i \varepsilon_i \pi(X_i)\left([\pi(X_i), \pi(X_j)] + \pi(X_j)\pi(X_i)\right) + \sum_i \varepsilon_i \left([\pi(X_i), \pi(X_j)] - \pi(X_i)\pi(X_j)\right)\pi(X_i) \\
&= \sum_i \varepsilon_i \pi(X_i)[\pi(X_i), \pi(X_j)] + \sum_i \varepsilon_i [\pi(X_i), \pi(X_j)]\pi(X_i) \\
&= \sum_i \varepsilon_i \pi(X_i)\pi\left([X_i, X_j]\right) + \sum_i \varepsilon_i \pi\left([X_i, X_j]\right)\pi(X_i) \\
&= \sum_i \varepsilon_i \pi(X_i)\pi\left(\sum_k c_k^{ij}\pi(X_k)\right) + \sum_i \varepsilon_i \pi\left(\sum_k c_k^{ij}\pi(X_k)\right)\pi(X_i) \\
&= \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) + \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_k)\pi(X_i) \\
&\overset{(2)}{=\!=} \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) + \sum_{i,k} \varepsilon_i\varepsilon_j\varepsilon_k c_j^{ki}\pi(X_k)\pi(X_i) \\
&= \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) + \sum_{k,i} \varepsilon_i\varepsilon_j\varepsilon_k c_j^{ik}\pi(X_i)\pi(X_k) \\
&\overset{(1)}{=\!=} \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) - \sum_{k,i} \varepsilon_i\varepsilon_j\varepsilon_k c_j^{ki}\pi(X_i)\pi(X_k) \\
&\overset{(2)}{=\!=} \sum_{i,k} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) - \sum_{k,i} \varepsilon_i c_k^{ij}\pi(X_i)\pi(X_k) \\
&= 0
\end{aligned}
$$

Thus Ω commutes with $\pi(X)$ for any $X \in \mathfrak{g}$

(c)

For this part, we assume $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, pick $X_1 = x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_2 = y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X_3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the basis of $\mathfrak{g}$, the its Cartan matrix is $\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$, then we have $Y_1 = \frac{y}{4}, Y_2 = \frac{x}{4}, Y_3 = \frac{h}{8}, \Omega = \frac{1}{4}\pi(x)\pi(y) + \frac{1}{4}\pi(y)\pi(x) + \frac{1}{8}\pi(h)^2$

Since the irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension n is $S^{n-1}(\mathbb{C}^2)$, unique up to isomorphism, let $t_1 = (1, 0)^T, t_2 = (0, 1)^T$ be a basis of $\mathbb{C}^2$, then $t_1^{n-1}, t_1^{n-2}t_2, \cdots, t_1 t_2^{n-2}, t_2^{n-1}$ is a basis of $S^{n-1}(\mathbb{C}^2)$, we have $\Omega t_1 = \frac{3}{8}t_1, \Omega t_2 = \frac{3}{8}t_2$, thus $\Omega t_1^k = \frac{3k}{8}t_1^k$ by induction $\Omega t_1^{k+1} = \Omega\left(t_1^k t_1\right) = t_1^k \Omega t_1 + t_1 \Omega t_k = \frac{3}{8}t_1^{k+1} + \frac{3k}{8}t_1^{k+1} = \frac{3(k+1)}{8}t_1^{k+1}$, hence $\Omega(t_1^k t_2^{m-k}) = t_1^k \Omega t_2^{n-1-k} + t_2^{n-1-k}\Omega t_1^k = \frac{3(n-1)}{8}t_1^k t_2^{n-1-k}$, thus Ω acts by multiplying a scalar $\frac{3(n-1)}{8}$

3.

Let $\Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then $\{X \in SL(2n, \mathbb{C}) | X^T \Omega X = \Omega\}$ is conjugate to $SO(2n, \mathbb{C})$, thus then also induce isomorphic Lie algebra, hence we can identify $\mathfrak{so}(2n, \mathbb{C})$ with $\{X \in M(2n, \mathbb{C}) | \Omega X^T + X\Omega = 0\}$ which is the same as $\left\{\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in M(2n, \mathbb{C}) \middle| B^T = -B, C^T = -C\right\} =: \mathfrak{g}$, then one Cartan subalgebra of $\mathfrak{g}$ will be

$\mathfrak{h} = \left\{\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in M(2n, \mathbb{C}) \middle| D = \mathrm{diag}(d_1, \cdots, d_n)\right\}$, note that

$$
\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}\right] = (d_i - d_j)\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}
$$

$$\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}\right] = (d_i + d_j)\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}$$

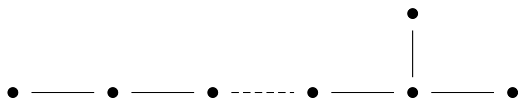
$$\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}\right] = -(d_i + d_j)\begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}$$

Define $e_i \in \mathfrak{h}^*$ with $e_i\left(\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}\right) = d_i$, then the roots are $\Delta = \{\pm(e_i - e_j), \pm(e_i + e_j) | i < j\}$, and a set of simple roots could be $S = \{e_1 - e_2, e_2 - e_3 \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$, we can check that $K\left(\begin{pmatrix} E_{ii} & 0 \\ 0 & -E_{ii} \end{pmatrix}, \begin{pmatrix} E_{jj} & 0 \\ 0 & -E_{jj} \end{pmatrix}\right) = \delta_{ij}3(n-1)$, thus we can treat $\{e_i\}$ as an othonormal basis

Thus $\langle e_{i-1} - e_i, e_i \pm e_{i+1}\rangle\langle e_i \pm e_{i+1}, e_{i-1} - e_i\rangle = \dfrac{4(e_{i-1} - e_i, e_i \pm e_{i+1})^2}{(e_{i-1} - e_i, e_{i-1} - e_i)(e_i \pm e_{i+1}, e_i \pm e_{i+1})} = 1$,

$(e_{i-1} - e_i, e_j \pm e_{j+1}) = 0$ for $i < j$,

and $(e_{n-1} - e_n, e_{n-1} + e_n) = 0$, hence the root system of $\mathfrak{so}(2n, \mathbb{C})$ is of type D_n



By abuse of notation, let $\Omega = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $\left\{X \in SL(2n+1, \mathbb{C}) \middle| X^T \Omega X = \Omega\right\}$ is conjugate to $SO(2n+1, \mathbb{C})$, hence we can identify $\mathfrak{so}(2n+1, \mathbb{C})$ with $\left\{X \in M(2n+1, \mathbb{C}) \middle| \Omega X^T + X\Omega = 0\right\}$ which is the same as

$\left\{\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & -A_{11}^T & A_{23} \\ -A_{23}^T & -A_{13}^T & 0 \end{pmatrix} \in M(2n, \mathbb{C}) \middle| A_{12}^T = -A_{12}, A_{21}^T = -A_{21}\right\} =: \mathfrak{g}$, then one Cartan subalgebra of $\mathfrak{g}$ will be

$\mathfrak{h} = \left\{\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M(2n+1, \mathbb{C}) \middle| D = \mathrm{diag}(d_1, \cdots, d_n)\right\}$, note that

$$\left[\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & -E_{ji} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = (d_i - d_j)\begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & -E_{ji} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = (d_i + d_j)\begin{pmatrix} 0 & E_{ij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ E_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = -(d_i + d_j)\begin{pmatrix} 0 & 0 & 0 \\ E_{ij} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & e_i \\ 0 & 0 & 0 \\ 0 & -e_i^T & 0 \end{pmatrix}\right] = d_i\begin{pmatrix} 0 & 0 & e_i \\ 0 & 0 & 0 \\ 0 & -e_i^T & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 & 0 \\ 0 & -D & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i \\ -e_i^T & 0 & 0 \end{pmatrix}\right] = -d_i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i \\ -e_i^T & 0 & 0 \end{pmatrix}$$

Define $\delta_i \in \mathfrak{h}^*$ with $\delta_i\left(\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}\right) = d_i$, then the roots are $\Delta = \{\pm(\delta_i - \delta_j), \pm(\delta_i + \delta_j) | i < j\} \cup \{\pm\delta_i\}$, and a set of simple roots could be $S = \{\delta_1 - \delta_2, \delta_2 - \delta_3 \cdots, \delta_{n-1} - \delta_n, \delta_n\}$, we can check that

$K\left(\begin{pmatrix} E_{ii} & 0 & 0 \\ 0 & -E_{ii} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} E_{jj} & 0 & 0 \\ 0 & -E_{jj} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \delta_{ij}(3(n-1) + 2)$, thus we can treat $\{\delta_i\}$ as an othonormal basis

Thus $\langle \delta_{i-1}-\delta_i, \delta_i-\delta_{i+1}\rangle\langle \delta_i-\delta_{i+1}, \delta_{i-1}-\delta_i\rangle = \dfrac{4(\delta_{i-1}-\delta_i, \delta_i-\delta_{i+1})^2}{(\delta_{i-1}-\delta_i, \delta_{i-1}-\delta_i)(\delta_i-\delta_{i+1}, \delta_i-\delta_{i+1})} = 1$,
$\langle \delta_{n-1}-\delta_n, \delta_n\rangle\langle \delta_n, \delta_{n-1}-\delta_n\rangle = \dfrac{4(\delta_{n-1}-\delta_n, \delta_n)^2}{(\delta_{n-1}-\delta_n, \delta_{n-1}-\delta_n)(\delta_n, \delta_n)} = 2$, $(\delta_{i-1}-\delta_i, \delta_j-\delta_{j+1}) = 0$ for $i < j$,
and $(\delta_{i-1}-\delta_i, \delta_n) = 0$ for $i < n$, hence the root system of $\mathfrak{so}(2n+1, \mathbb{C})$ is of type B_n

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \dashrightarrow \bullet \longrightarrow \bullet \Longrightarrow \bullet$$

4.

Since the Weyl group acts on Weyl chambers freely, for any regular $\gamma \in V$, $w\gamma \neq \gamma, \forall w \in W$, suppose $w = s_{\gamma_0}$ for some regular γ_0, then H_{γ_0} must contain some regular element, which is a contradiction

Therefore, $w = s_\alpha$ for some $\alpha \in \Delta$

5.

Note that $\alpha^\vee \in V^*$ is defined such that $\langle \beta, \alpha^\vee\rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$, let $S = \{\alpha_1, \cdots, \alpha_n\} \subseteq \Delta$ be a set of simple roots which is also a basis of V, thus the gram matrix $((\alpha_i, \alpha_j))$ will be invertible, for any $f \in V^*$, $i = 1, \cdots, n$, $\langle \alpha_i, f\rangle = \sum_j c_j \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_j \dfrac{2c_j}{(\alpha_j, \alpha_j)}(\alpha_i, \alpha_j)$, so for any choice of $\langle \alpha_i, f\rangle$ has a unique solution $\dfrac{2c_j}{(\alpha_j, \alpha_j)}$, which gives a unique solution c_j, thus $\langle \alpha_i, f\rangle = \sum_j c_j \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \langle \alpha_i, \sum_j c_j \alpha_j^\vee\rangle$, which implies that $f = \sum_j c_j \alpha_j^\vee$, hence $V^* \leq \langle S^\vee\rangle \leq \langle \Delta^\vee\rangle \leq V^* \Rightarrow V^* = \langle \Delta^\vee\rangle$, for any $\alpha \in \Delta$, $i = 1, \cdots, n$, $\langle \alpha_i, (c\alpha)^\vee\rangle = \dfrac{2(\alpha_i, c\alpha)}{(c\alpha, c\alpha)} = \dfrac{1}{c}\dfrac{2(\alpha_i, \alpha)}{(\alpha, \alpha)} = \dfrac{1}{c}\langle \alpha_i, \alpha^\vee\rangle = \langle \alpha_i, \dfrac{\alpha^\vee}{c}\rangle$, thus $(c\alpha)^\vee = \dfrac{\alpha^\vee}{c}$, but the only multiple of α that are in Δ are precisely $\alpha, -\alpha$, hence the only multiple of $\alpha^\vee$ that are in $\Delta^\vee$ are precisely $\alpha^\vee, -\alpha^\vee$

By the definition of the inner prouct on the dual Euclidean space V^*, we know $(\alpha^\vee, \beta^\vee) = \dfrac{4(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)}$, thus $\dfrac{2(\alpha^\vee, \beta^\vee)}{(\alpha^\vee, \alpha^\vee)} = \dfrac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$ and $s_{\alpha^\vee}(\beta^\vee) = \beta^\vee - \dfrac{2(\alpha^\vee, \beta^\vee)}{(\alpha^\vee, \alpha^\vee)}\alpha^\vee = \beta^\vee - \dfrac{2(\alpha, \beta)}{(\beta, \beta)}\alpha^\vee$, then $s_{\alpha^\vee}(\alpha^\vee) = -\alpha^\vee$ and since $(s_\alpha(\beta), s_\alpha(\beta)) = (\beta, \beta)$, we have

$$\begin{aligned}
\left\langle \alpha_i, s_{\alpha^\vee}(\beta^\vee)\right\rangle &= \left\langle \alpha_i, \beta^\vee\right\rangle - \frac{2(\alpha, \beta)}{(\beta, \beta)}\left\langle \alpha_i, \alpha^\vee\right\rangle \\
&= \frac{2(\alpha_i, \beta)}{(\beta, \beta)} - \frac{2(\alpha, \beta)}{(\beta, \beta)}\frac{2(\alpha_i, \alpha)}{(\alpha, \alpha)} \\
&= \frac{2(\alpha_i, s_\alpha(\beta))}{(\beta, \beta)} \\
&= \frac{2(\alpha_i, s_\alpha(\beta))}{(s_\alpha(\beta), s_\alpha(\beta))} \\
&= \left\langle \alpha_i, s_\alpha(\beta)^\vee\right\rangle
\end{aligned}$$

Thus $s_{\alpha^\vee}(\beta^\vee) = s_\alpha(\beta)^\vee \in \Delta^\vee$

Therefore, the dual $(V^*, \Delta^\vee)$ is also a root system

From the calculation above, we also have $\left\langle \beta^\vee, (\alpha^\vee)^\vee\right\rangle = \dfrac{(\alpha, \beta)}{(\beta, \beta)} = \langle \alpha, \beta^\vee\rangle$

Notice that in the construction of a Dynkin diagram, the number of edges between α and β equals $\langle \beta, \alpha^\vee\rangle\langle \alpha, \beta^\vee\rangle$, then the number of edges between $\alpha^\vee$ and $\beta^\vee$ equals $\left\langle \beta^\vee, (\alpha^\vee)^\vee\right\rangle\left\langle \alpha^\vee, (\beta^\vee)^\vee\right\rangle = \langle \alpha, \beta^\vee\rangle\langle \beta, \alpha^\vee\rangle$, but with the arrow reversed, therefore A_n, D_n are the classical root systems that are self-dual, and B_n, C_n are dual to each other

6.

Since

$$(\vec{v},\vec{w}) = \sum_{\alpha\in\Delta}\langle\vec{v},\alpha^\vee\rangle\langle\vec{w},\alpha^\vee\rangle = \sum_{\alpha\in\Delta}\langle\vec{w},\alpha^\vee\rangle\langle\vec{v},\alpha^\vee\rangle = (\vec{w},\vec{v})$$

$$(\vec{v},\vec{v}) = \sum_{\alpha\in\Delta}\langle\vec{v},\alpha^\vee\rangle^2 \geq 0$$

And $(\vec{v},\vec{v}) = 0$ iff $\langle\vec{v},\alpha^\vee\rangle = \dfrac{2(\vec{v},\alpha)}{(\alpha,\alpha)} = 0, \forall\alpha\in\Delta$ iff $\vec{v}=0$, therefore $(,)$ is symmetric and positive definite, finally, we have

$$\begin{aligned}(s_\beta\vec{v},s_\beta\vec{w}) &= \sum_{\alpha\in\Delta}\langle s_\beta\vec{v},\alpha^\vee\rangle\langle s_\beta\vec{w},\alpha^\vee\rangle\\ &= \sum_{\alpha\in\Delta}\frac{4(s_\beta\vec{v},\alpha)(s_\beta\vec{w},\alpha)}{(\alpha,\alpha)}\\ &= \sum_{\alpha\in\Delta}\frac{4(\vec{v},s_\beta\alpha)(\vec{w},s_\beta\alpha)}{(\alpha,\alpha)}\\ &= \sum_{\alpha\in\Delta}\frac{4(\vec{v},s_\beta\alpha)(\vec{w},s_\beta\alpha)}{(s_\beta\alpha,s_\beta\alpha)}\\ &= \sum_{\alpha\in\Delta}\frac{4(\vec{v},\alpha)(\vec{w},\alpha)}{(\alpha,\alpha)}\\ &= (\vec{v},\vec{w})\end{aligned}$$

Thus $(,)$ is also W invariant

7.

Suppose (V,Δ) is a root system, $\tau: V\to V, x\mapsto -x$ is a linear map, since $\alpha\in\Delta\Rightarrow -\alpha\in\Delta$, thus $\tau(\Delta)\subseteq\Delta$, also for any $\alpha,\beta\in\Delta$, $\langle\tau\alpha,(\tau\beta)^\vee\rangle = \dfrac{2(\tau\alpha,\tau\beta)}{(\tau\alpha,\tau\alpha)} = \dfrac{2(-\alpha,-\beta)}{(-\alpha,-\alpha)} = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} = \langle\alpha,\beta^\vee\rangle$, and $\tau^2 = \mathrm{id}$, thus τ is an automorphism (V,Δ)

8.

(a)

Let $e_1 = (1,0), e_2 = (0,1)$ be the standard basis of $\mathbb{R}^2$, suppose the coordinates of $\vec{v_1},\vec{v_2}$ are $(r_1\cos\alpha, r_1,\sin\alpha)$ and $(r_2\cos(\theta+\alpha), r_2\sin(\theta+\alpha))$, since $s_i(\vec{v}) = v - 2\dfrac{\vec{v}\cdot\vec{v_i}}{\vec{v_i}\cdot\vec{v_i}}\vec{v_i}$, we have the matrix of s_1 and s_2 with respect to e_1,e_2 are $\begin{pmatrix}-\cos 2\alpha & -\sin 2\alpha\\ -\sin 2\alpha & \cos 2\alpha\end{pmatrix}$ and $\begin{pmatrix}-\cos 2(\theta+\alpha) & -\sin 2(\theta+\alpha)\\ -\sin 2(\theta+\alpha) & \cos 2(\theta+\alpha)\end{pmatrix}$, thus the matrix for s_1s_2 would be $\begin{pmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{pmatrix}$ which means that s_1s_2 is a rotation by 2θ

(b)

In order for the group $\langle s_1,s_2\rangle$ to be finite, necessarily we need θ to be a rational multiple of π, but for any $\theta = \dfrac{p\pi}{q}$ where $p,q>0$ are relatively prime integers, $2\theta = \dfrac{p}{q}\cdot 2\pi$, if we denote $r = s_1s_2, s = s_1$, then we the relations then we have the relation $s^2 = 1$, $r^q = 1$, and $srsr = 1$, therefore $\langle s_1,s_2\rangle = \langle s,r\rangle = F(s,r)/\langle s^2,r^q,srsr\rangle \cong D_q$, where D_q is the dihedral group, the symmetry group of a regular q-gon

References

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