Definition 0.0.1. The **p-adic intergers** are

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} = \left\{(a_0, a_1, a_2, \cdots) \in \prod \mathbb{Z}/p^n\mathbb{Z} \middle| a_n \equiv a_m \bmod p^m, n \geq m\right\}$$
$$= \left\{b_0 + b_1 p + b_2 p^2 + \cdots\right\}$$

Here $a_n = b_0 + b_1 p + \cdots + b_n p^n$

Example 0.0.2. If $p = 7$, we can write $3 + 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 2 \cdot 7^4 + \cdots$ as

$$\cdots 24163$$

With base 7

Definition 0.0.3. Given $a_0 \in \mathbb{F}_p$, there is a unique solution to $x^p - x = 0$ in $\mathbb{Z}_p$, $\mathbb{F}_p \to \mathbb{Z}_p$ gives the *Teichmüller representative*. Any p adic integer can be uniquely written as $\sum_{i=0}^{\infty} c_i p^i$, c_i's are Teichmüller representatives. To compute additions and multiplications, we introduce *Witt vectors* $(X_0, X_1, \cdots, X_n, \cdots)$, define ghost components or Witt polynomials

$$W_n = X^{(n)} = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}$$

Define addition and multiplication using $X^{(n)}$ by

$$(X+Y)^{(n)} = X^{(n)} + Y^{(n)}, (XY)^{(n)} = X^{(n)} Y^{(n)}$$

Thus we have

$$(X_0, X_1, \cdots) + (Y_0, Y_1, \cdots) = \left(X_0 + Y_0, X_1 + Y_1 + \frac{X_0^p + Y_0^p - (X_0 + Y_0)^p}{p}, \cdots\right)$$
$$(X_0, X_1, \cdots) \times (Y_0, Y_1, \cdots) = (X_0 Y_0, X_0^p Y_1 + X_1 Y_0^p + p X_1 Y_1, \cdots)$$

More generally, we can define universal Witt polynomials

$$W_n = X^{(n)} = \sum_{d|n} d X_d^{\frac{n}{d}}$$

Consider

$$f_X(t) = \prod_{n\geq 1}(1 - X_n t^n) = \sum_{n \geq 0} A_n t^n$$

Where

$$A_n = \sum_I (-1)^{|I|} \prod_{i \in I} X_i$$

I runs over subsets of $\{1, \cdots, n\}$ that add up to n. Then

$$-t\frac{d}{dt}\log f_X(t) = \sum_{d\geq 1} \frac{d X_d t^d}{1 - X_d t^d}$$
$$= \sum_{d \geq 1} d X_d t^d \sum_{i \geq 0} X_d^i t^{di}$$
$$= \sum_{d\geq 1}\sum_{i \geq 1} d X_d^i t^{di}$$
$$= \sum_{n \geq 1} X^{(n)} t^n$$

If $Z = X + Y$, then $f_Z(t) = f_X(t) f_Y(t)$ since

$$\begin{aligned}\sum_{n\geq 1} Z^{(n)} t^n &= -t\frac{d}{dt}\log f_Z(t) \\ &= -t\frac{d}{dt}\log f_X(t) - t\frac{d}{dt}\log f_X(t) \\ &= \sum_{n\geq 1} X^{(n)} t^n + \sum_{n\geq 1} Y^{(n)} t^n\end{aligned}$$

A_j are polynomials in X_i's, B_j are polynomials in Y_i's, we can show that by induction

$$Z_n = C_j - \sum_{I\neq\{n\}} (-1)^{|I|} \prod_{i\in I} Z_i = \sum_{k+l=j} A_k B_l - \sum_I (-1)^{|I|} \prod_{i\in I} Z_i$$

are polynomials in X_i, Y_i's
If $Z = XY$, then . In particular, consider $Y = (r, 0, \cdots)$, $f_Z(t) = f_X(rt)$