

# MATH868C - Several Complex Variables



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# Contents

<b>1</b>	<b>Review</b>	<b>2</b>
<b>2</b>	<b>Runge's theorem</b>	<b>6</b>
<b>3</b>	<b>Subharmonic functions</b>	<b>8</b>
<b>4</b>	<b>Almost complex structure</b>	<b>10</b>
<b>5</b>	<b>Hartogs theorem</b>	<b>14</b>
<b>6</b>	<b>Pseudoconvexity</b>	<b>15</b>
<b>7</b>	<b>Hörmander's <math>L^2</math> estimate</b>	<b>18</b>
<b>8</b>	<b>Kähler metrics</b>	<b>21</b>
	<b>Index</b>	<b>24</b>

# 1 Review

**Definition 1.1.**  $C^1$  function  $f : \Omega \rightarrow \mathbb{C}$  is *holomorphic* if  $\bar{\partial}f = 0$ . Denote the set of all holomorphic functions on  $\Omega$  as  $A(\Omega)$

**Lemma 1.2.** If  $f$  is holomorphic, then  $\int_{\partial\Omega} f dz = 0$

*Proof.*

$$\int_{\partial\Omega} f dz = \int_{\Omega} d(f dz) = \int_{\Omega} \bar{\partial}f \wedge dz = 0$$

Poincare-Lelong formula  $\square$

**Theorem 1.3** (Poincaré-Lelong formula). Since  $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$ ,  $dz \wedge d\bar{z} = -2idz \wedge d\bar{z} = -2id\mu$ . In the distributional sense,  $-\frac{\log r}{2\pi} = -\frac{1}{4\pi} \log(x^2 + y^2)$  is the fundamental solution of Laplacian equation in dimension 2, i.e.  $\Delta \log(x^2 + y^2) = 4\pi\delta$ , we have

$$\Delta \log |z|^2 dz \wedge d\bar{z} = 4\pi\delta dz \wedge d\bar{z} \Leftrightarrow \bar{\partial}\partial \log |z|^2 = 2\pi i \delta dx \wedge dy$$

*Note.*  $\partial \log |z|^2 = \partial \log(z) + \partial \log(\bar{z}) = \frac{dz}{z}$  is integrable around 0

*Proof.* We prove a slightly general result. For any  $\phi \in C_c^\infty(\Omega)$ , by definition we have

$$\begin{aligned} \iint_{\Omega} \phi \bar{\partial}\partial \log |z - w|^2 &= - \iint_{\Omega} \bar{\partial}\phi \wedge \partial \log |z - w|^2 \\ &= - \lim_{\epsilon \rightarrow 0} \iint_{|z-w| \geq \epsilon} \bar{\partial}\phi \wedge \partial \log |z - w|^2 \\ &= - \lim_{\epsilon \rightarrow 0} \iint_{|z-w| \geq \epsilon} d(\phi \partial \log |z - w|^2) \\ &= \lim_{\epsilon \rightarrow 0} \oint_{|z-w|=\epsilon} \phi \partial \log |z - w|^2 \\ &= \lim_{\epsilon \rightarrow 0} \oint_{|z-w|=\epsilon} \frac{\phi}{z - w} dz \\ &= 2\pi i \phi(w) \end{aligned}$$

Cauchy's formula  $\square$

**Theorem 1.4** (Cauchy's formula). If  $f \in C^1(\bar{\Omega})$ , then

$$f(w) = \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_z f dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z - w} dz$$

*Proof.* By Poincaré-Lelong formula 1.3, we have

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \iint_{\Omega} f \bar{\partial}\partial \log |z - w|^2 \\ &= -\frac{1}{2\pi i} \iint_{\Omega} \bar{\partial}f \wedge \partial \log |z - w|^2 + \frac{1}{2\pi i} \int_{\partial\Omega} f \partial \log |z - w|^2 \\ &= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_z f dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z - w} dz \end{aligned}$$

$\square$

**Corollary 1.5.** If  $f \in C^1(\bar{\Omega}) \cap A(\Omega)$ , then by Cauchy's formula 1.4, we know

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - w} dz$$

Which is  $C^\infty$  in  $w$

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - w)^{n+1}} dz$$

**Corollary 1.6** (Cauchy's estimate). For  $K \subseteq \Omega$  compact, there are constants  $C_n$  such that for any  $f \in A(\Omega)$

$$\sup_{z \in K} |f^{(n)}(z)| \leq C_n \|f\|_{L^1(\Omega)}$$

*Proof.* Consider a bump function  $\chi$  with  $\text{supp } \chi \subseteq \Omega$  and  $\chi \equiv 1$  on  $K$ , then for any  $w \in K$

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}}(\chi f) dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\chi f}{z - w} dz \\ &= \frac{1}{2\pi i} \iint_{\Omega} \frac{(\partial_{\bar{z}}\chi) f dz \wedge d\bar{z}}{z - w} \\ &= \frac{1}{2\pi i} \iint_{\Omega \setminus K} \frac{(\partial_{\bar{z}}\chi) f dz \wedge d\bar{z}}{z - w} \end{aligned}$$

$\frac{\partial_{\bar{z}}\chi}{z - w}$  can be bounded on  $\Omega \setminus K$  □

**Corollary 1.7.**  $A(\Omega) \subseteq C(\Omega)$  is closed, thus a Fréchet space

*Proof.* Suppose  $\{f_j\} \subseteq A(\Omega)$  converges to  $f$  in  $C(\Omega)$ , but since

$$f_j(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f_j(z)}{z - w} dz$$

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - w} dz \text{ which implies } \bar{\partial}f = 0$$

□

Montel's theorem

**Theorem 1.8** (Montel's theorem). Suppose  $\{f_i\} \subseteq A(\Omega)$  are uniformly bounded on each compact subset, then there is a subsequence  $f_{i_k}$  uniformly converges on compact subsets

*Proof.* For  $K \subseteq \Omega$  compact, by Cauchy's estimate 1.6,  $f_j$  are Lipschitz with the same  $C_k$ , by Ascoli-Arzelà theorem,  $f_j$  are equicontinuous, thus have convergent subsequence, and then use diagonal argument by exhaust  $\Omega$  with compact subsets  $K$  □

Riemann extension theorem

**Theorem 1.9** (Riemann extension theorem).  $E \subseteq \Omega$  is a discrete subset,  $f \in A(\Omega \setminus E)$ , and  $f$  is bounded around each point in  $E$ , then  $f$  can be extended to a unique  $\tilde{f} \in A(\Omega)$

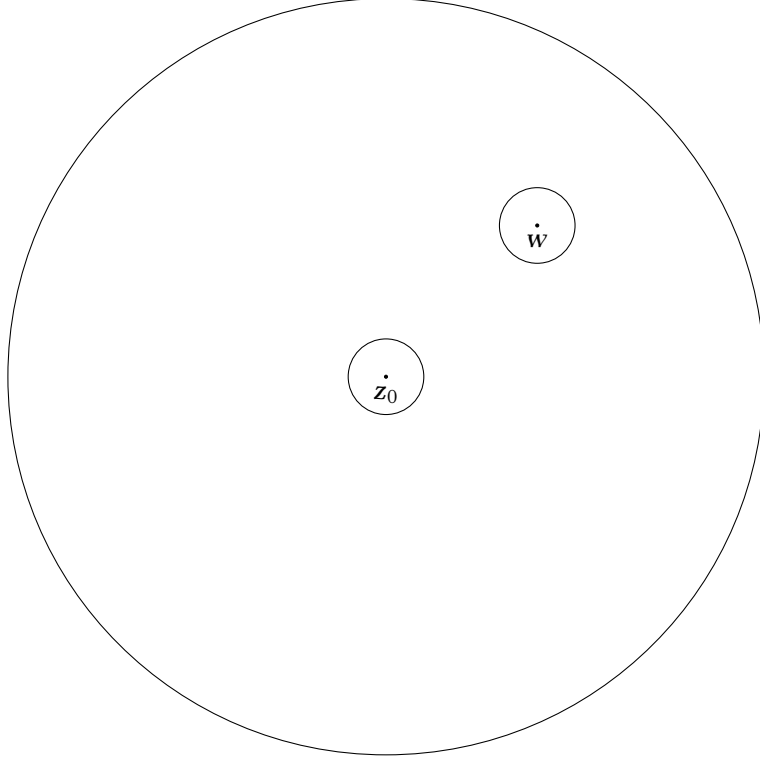
*Proof.* For  $z_0 \in E$ , suppose such  $\tilde{f}$  exists, then by Cauchy's formula 1.4, for any  $w \in D(z_0, r)$

$$\tilde{f}(w) = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - w} dz$$

Thus we just take this as a definition, then

$$\begin{aligned} \tilde{f}(w) - f(w) &= \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - w} dz - \frac{1}{2\pi i} \int_{\partial D(w, \epsilon)} \frac{f(z)}{z - w} dz \\ &= \frac{1}{2\pi i} \int_{\partial D(z_0, \epsilon)} \frac{f(z)}{z - w} dz \end{aligned}$$

Which can be show to arbitrarily small as  $\epsilon \rightarrow 0$



□  
d bar theorem

**Theorem 1.10.** If  $\alpha = g(z)d\bar{z}$  is a smooth  $(0,1)$ -form on  $\Omega$ , then there exists  $u \in C^\infty(\Omega)$  such that  $\bar{\partial}u = \alpha$

*Proof.* suppose such a  $u$  exists, then by Cauchy's formula 1.4

$$u(w) = \frac{1}{2\pi i} \iint_{\Omega} \frac{g(z)dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(z)}{z - w} dz$$

Since  $\bar{\partial} \int_{\partial\Omega} \frac{u(z)}{z - w} dz = 0$ . This motivates us to first assume  $\alpha$  has compact support, and define

$$u(w) = \frac{1}{2\pi i} \iint_{\Omega} \frac{g(z)dz \wedge d\bar{z}}{z - w}$$

Then

$$u(w + \zeta) = \frac{1}{2\pi i} \iint_{\Omega} \frac{g(z)dz \wedge d\bar{z}}{(z - \zeta) - w} = \frac{1}{2\pi i} \iint_{\Omega} \frac{g(z + \zeta)dz \wedge d\bar{z}}{z - w}$$

Hence

$$\begin{aligned} \partial_{\bar{w}} u(w) &= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_z g(z)dz \wedge d\bar{z}}{z - w} \\ &= \frac{1}{2\pi i} \iint_{\Omega} \partial \log |z - w|^2 \wedge \bar{\partial} g \\ &= g(w) \end{aligned}$$

Therefore  $\bar{\partial}u = \alpha$ . In general, consider a compact exhaustion  $\Omega = \bigcup_i K_i$ , where  $\hat{K}_i = K_i$ ,

$K_i \subset \subset K_{i+1}^\circ$ , ensured by Corollary 2.6, let  $\chi_i$  be a cutoff function such that  $\chi_i \equiv 1$  on  $K_i$  and  $\text{supp } \chi_i \subseteq K_{i+1}$ , then there exists  $f_i$  such that  $\bar{\partial}f_i = \chi_i \alpha$ , by Runge's theorem 2.2, there exists  $h_i \in \mathcal{O}(K_i)$  such that  $\|f_{i+1} - f_i - h_i\|_{K_i} < \frac{1}{2^i}$ . Now define

$$u_N = f_1 + \sum_{k=1}^N (f_{k+1} - f_k - h_k) = f_{N+1} - \sum_{k=1}^N h_k$$

Converges uniformly on compact subsets to  $\mathbf{u}$ , and  $\partial \mathbf{u}_N = \alpha$  on  $K_i$  for any  $i \leq N$   $\square$

## 2 Runge's theorem

**Definition 2.1.**  $K \subseteq \Omega$  is compact, define

$$\mathcal{O}(K) = \{f|_K : f \text{ is holomorphic in a neighborhood of } K\}$$

Then for we have restriction map  $\rho : \mathcal{O}(\Omega) \rightarrow \mathcal{O}(K)$ , let  $\|f\|_K = \max_{z \in K} |f(z)|$  to be the  $L^\infty$  norm

Runge's theorem

**Theorem 2.2** (Runge's theorem). The following are equivalent

1. The image of  $\rho$  is dense
2. No connected component of  $\Omega \setminus K$  is relatively compact in  $\Omega$
3. If  $\xi \in \Omega \setminus K$ , then there exists  $f \in \mathcal{O}(\Omega)$  such that  $|f(\xi)| > \|f\|_K$

**Definition 2.3.** For  $K \subseteq \Omega$  compact, the *holomorphic convex hull* of  $K$  relative to  $\Omega$  is

$$\hat{K} = \hat{K}_\Omega = \{z \in \Omega : |f(z)| \leq \|f\|_K, \forall f \in \mathcal{O}(\Omega)\}$$

Clearly  $K \subseteq \hat{K}$

**Proposition 2.4.**

1.  $\hat{K}$  is compact
2.  $\|f\|_{\hat{K}} = \|f\|_K$  for all  $f \in \mathcal{O}(\Omega)$
3.  $\hat{\hat{K}} = \hat{K}$
4. If  $\xi \in \Omega \setminus \hat{K}$ , then there exists  $f \in \mathcal{O}(\Omega)$  such that  $|f(\xi)| > \|f\|_K$

*Proof.*

1.  $\hat{K}$  is bounded by considering  $f = z$ . Suppose  $z_i \in \hat{K}$  converges to  $\xi$ , if  $\xi \in \Omega^c$ , then  $f = \frac{1}{z - \xi}$  will be unbounded on  $\hat{K}$ , thus  $\xi \in \Omega$ , but then for any  $f \in \mathcal{O}(\Omega)$ ,  $|f(\xi)| = \lim_{i \rightarrow \infty} |f(z_i)| \leq \|f\|_K$ , thus  $\xi \in \hat{K}$
2. By definition,  $\|f\|_{\hat{K}} \leq \|f\|_K$ ,  $\forall f \in \mathcal{O}(\Omega)$ , and  $\|f\|_K \leq \|f\|_{\hat{K}}$ ,  $\forall f \in \mathcal{O}(\Omega)$  is obvious
3.  $\hat{\hat{K}} = \{z \in \Omega : |f(z)| \leq \|f\|_{\hat{K}} = \|f\|_K, \forall f \in \mathcal{O}(\Omega)\} = \hat{K}$
4. By definition

□

**Example 2.5.**  $K$  is the unit circle. If  $\Omega$  is the annulus  $\left\{\frac{1}{2} < |z| < 2\right\}$ , then  $\hat{K} = K$ . If  $\Omega$  is the disc  $\{|z| < 2\}$ , then  $\hat{K} = \{|z| < 1\}$  is the unit disc. Just consider  $f = z$  and  $f = \frac{1}{z}$

Compact exhaustion of a domain

**Corollary 2.6.** Any domain  $\Omega$  has an exhaustion by compact sets  $\hat{K}_i = K_i$  such that

$$K_i \subset \subset K_{i+1}^\circ \subset K_{i+1} \subset \subset \Omega$$

Vanishing theorem

**Theorem 2.7.**  $\mathcal{U} = \{U_i\}$  is an open cover of  $\Omega$ , then  $H^1(\mathcal{U}, \mathcal{O}) = 0$

*Proof.* Let  $\{\phi_i\}$  be a partion of unity. For any cocycle  $\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O})$ , consider  $h_i = \sum_j \phi_j g_{ij}$ , then

$$\begin{aligned} h_i - h_j &= \sum_k \phi_k g_{ik} - \sum_k \phi_k g_{jk} \\ &= \sum_k \phi_k (g_{ik} - g_{jk}) \\ &= \sum_k \phi_k g_{ij} \\ &= g_{ij} \end{aligned}$$

Hence  $\bar{\partial}h_i - \bar{\partial}h_j = 0$ ,  $\{\bar{\partial}h_i\}$  define a well-defined smooth  $(0, 1)$  form. By Theorem 1.10, there exist a holomorphic fuction  $u$  such that  $\bar{\partial}u = \bar{\partial}h_i$ , define  $f_i = h_i - u$ , then  $\bar{\partial}f_i = 0$ , i.e.  $\{f_i\}$ 's are holomorphic, and  $g_{ij} = f_i - f_j$ . In other words,  $\{g_{ij}\}$  is the image of  $\{f_i\} \in C^1(\mathcal{U}, \mathcal{O})$  under the coboundary map  $\square$

**Theorem 2.8** (Mittag-Leffler theorem).  $\Omega \subseteq \mathbb{C}$  is an open set,  $E \subseteq \Omega$  is a discrete subset, then there exists a meromorphic function  $f$  with prescribed principal parts on  $E$

*Proof.* There exists an open cover  $\mathcal{U} = \{U_i\}$  and  $f_i \in \mathcal{M}(U_i)$  with the prescribed principal parts round each point of  $E$ , then  $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$  is a coycle, by Theorem 2.7, there exist holomorphic functions  $\{g_i\}$  such that  $f_i - f_j = g_i - g_j$  on  $U_i \cap U_j$ , then  $f_i - g_i = f_j - g_j$  defines a global meromorphic function  $f$  such that  $f - f_i = -g_i$  on  $U_i$  which is holomorphic  $\square$

Weierstrass theorem

**Theorem 2.9** (Weierstrass theorem).  $E \subseteq \Omega$  is discrete, then

1. There is  $f \in \mathcal{M}(\Omega)$  with arbitrary orders precisely at  $E$
2. Any  $f \in \mathcal{M}(\Omega)$  can be written as  $f = g/h$  for  $g, h \in \mathcal{O}(\Omega)$

*Proof.*

1. First take care of poles, and then multiply by  $a_k(z - z_k)^{r_k}$  for each zero  $z_k$ , that converges
2.  $\square$

**Definition 2.10.** Open subset  $\Omega \subseteq \mathbb{C}^n$  is called a *domain of holomorphy* if for any  $p \in \overline{\Omega} \setminus \Omega$ , there is no holomorphic function  $g$  defined on an open set  $U \ni p$  with  $g = f$  on  $U \cap \Omega$

**Theorem 2.11.** For any proper open subset  $\Omega \subseteq \mathbb{C}$  is a domain of holomorphic

*Proof.* Suppose  $p \in \partial\Omega$ ,  $p \in U$  is a neighborhood,  $g \in \mathcal{O}(U)$  such that  $f = g$  on  $\Omega \cap U$ , then there exists  $\{\xi_n\}$  discrete and converging to  $p$ . By Weierstrass theorem 2.9, there exists  $f \in \mathcal{O}(\Omega)$  having exactly  $\{\xi_i\}$  as zeros, but then  $g$  has to be identically zero, so is  $f$  which is a contradiction  $\square$



### 3 Subharmonic functions

**Definition 3.1.**  $\Omega \subseteq \mathbb{C}$  is a domain.  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is *upper semicontinuous* if for  $y \in \mathbb{R}$  the set  $\{u < y\}$  is open

**Definition 3.2.** An upper semicontinuous function  $u$  is *subharmonic* if is not identitically  $-\infty$ , and for each  $U \subset\subset \Omega$  and harmonic function  $h$  on  $\overline{U}$  with  $u \leq h$  on  $\partial U$ , we have  $u \leq h$  for all  $z \in U$

**Example 3.3.** If  $u \in C^2(\Omega)$  and  $\Delta u \geq 0$ , then  $u$  is subharmonic

**Theorem 3.4.** 1. If  $\{u_i\}$  are subharmonic and  $u = \sup u_i$  is finite and upper semicontinuous, then  $u$  is subharmonic

2. If  $u_i \geq u_{i+1}$  are subharmonic, then  $u = \lim u_i$  is subharmonic

*Proof.*

1. By definition

2.  $\{u < y\} = \bigcup \{u_i < y\}$  is open, hence  $u$  is upper semicontinuous. Suppose  $u \leq h$  on  $\partial U$  for some  $U \subset\subset \Omega$  and harmonic function  $h$ . For any  $\epsilon > 0$ , consider

$$F_i = \{x \in \partial U | u_i(x) \geq h(x) + \epsilon\}$$

are compact, thus  $\bigcap F_i = \emptyset$  implies that a finite intersection is empty, hence  $u \leq h + \epsilon$

□

**Fact 3.5.** If  $u$  is subharmonic on  $\Omega$ , then  $u \in L^1_{\text{loc}}(\Omega)$

**Theorem 3.6.** Subharmonic function  $u$  satisfies the sub-mean value property

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta \quad (3.1)$$

For almost all  $r$  sufficiently small

*Proof.*  $u$  is integrable on circle of radius  $r$  about  $z$  for sufficiently small  $r$ , we can find continuous functions  $h_n \geq u_n$  on the circle such that  $h_n \rightarrow u$  in  $L^1$ , extend  $h_n$  to harmonic functions, then

$$u(z) \leq h_n(z) = \frac{1}{2\pi} \int_0^{2\pi} h_n(z + re^{i\theta}) d\theta \rightarrow \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$

□

**Proposition 3.7.** Subharmonic functions satisfies  $\Delta u \geq 0$  in the weak sense

$$\int_{\Omega} u \Delta \phi \geq 0, \forall \phi \in C_c^\infty(\Omega), \phi \geq 0$$

*Proof.* Multiply  $\phi$  on both sides of (3.1) and integrate over  $\Omega$  we get

$$\begin{aligned} \int_{\Omega} 2\pi u(z) \phi(z) d\mu &\leq \int_{\Omega} \phi(z) \int_0^{2\pi} u(z + re^{i\theta}) d\theta d\mu \\ &= \int_{\Omega} u(z) \int_0^{2\pi} \phi(z - re^{i\theta}) d\theta d\mu \end{aligned}$$

Then we get

$$\begin{aligned} 0 &\leq \int_{\Omega} u(z) \int_0^{2\pi} \phi(z - re^{i\theta}) - \phi(z) d\theta d\mu \\ &= \int_{\Omega} u(z) \int_0^{2\pi} -\partial_z \phi(z) r e^{i\theta} - \partial_{\bar{z}} \phi(z) r e^{-i\theta} + \partial_z^2 \phi(z) r^2 e^{2i\theta} + \partial_{\bar{z}}^2 \phi(z) r^2 e^{-2i\theta} + 2\partial_z \partial_{\bar{z}} \phi(z) r^2 + O(r^3) d\theta d\mu \\ &= \int_{\Omega} u(z) \int_0^{2\pi} \frac{1}{2} \Delta \phi(z) r^2 + O(r^3) d\theta d\mu \end{aligned}$$

Divide  $\frac{r^2}{2}$  and let  $r \rightarrow 0$

□

**Proposition 3.8.** Subharmonicity is a local property, i.e. suppose  $u$  is upper semicontinuous on  $\Omega$ , and locally subharmonic, then  $u$  is subharmonic on  $\Omega$

*Proof.* Suppose  $h$  is harmonic,  $U \subset\subset \Omega$ ,  $u \leq h$  on  $\partial\Omega$ , consider  $v = u - h$ , assume  $\sup_U v = M > 0$ , then by the upper semicontinuity, we know that  $F = \{v = M\}$  is compact in  $U$ , there exists  $z_0 \in \partial F$  obtains the least distance from  $\partial U$ , then for any small  $r > 0$ ,  $F$  will miss an arc of positive measure if  $\partial B(z_0, r)$ , hence

$$\frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) d\theta < M$$

But this contradicts sub-mean value property □

**Example 3.9.** If  $f_1, \dots, f_k \in \mathcal{O}(\Omega)$ , not all zero, then  $u = \log(|f_1|^2 + \dots + |f_k|^2)$  is subharmonic since  $\log |f|$  is harmonic and  $\Delta u \geq 0$

## 4 Almost complex structure

**Definition 4.1.**  $V$  is a real vector space, an *almost complex structure* is an endomorphism  $J : V \rightarrow V$  such that  $J^2 = -I$ . Let  $V^{1,0} \oplus V^{0,1} = V_{\mathbb{C}}$  be the  $\pm i$  eigenspaces of  $J$

**Proposition 4.2.** We can find basis such that  $V \cong \mathbb{R}^{2n}$  such that  $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ . For local coordinate  $(x_i, y_i)$  of a complex manifold,  $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}$  is such a basis,  $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$  are the  $\pm i$  eigenvectors. This motivates the definition of a real isomorphism  $\rho : V \rightarrow V^{1,0}$ ,  $v \mapsto \frac{1}{2}(v - iJv)$ , then  $\rho J = i\rho$ . Suppose  $V, W$  both have almost complex structures, given an  $\mathbb{R}$ -linear map  $T : V \rightarrow W$ , let  $\tilde{T} : V^{1,0} \rightarrow W^{1,0}$  be given by the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \rho \downarrow & & \downarrow \rho \\ V^{1,0} & \xrightarrow{\tilde{T}} & W^{1,0} \end{array}$$

$\tilde{T}$  is complex linear if  $TJ = JT \iff \tilde{T}i = i\tilde{T}$ . Alternatively, extend  $T$  to a map  $V_{\mathbb{C}} \rightarrow W_{\mathbb{C}}$ , and this conditions is exactly that this extension preserves  $(1, 0)$  and  $(0, 1)$  subspaces

**Lemma 4.3** (Osgood's lemma). If  $f : \Omega \rightarrow \mathbb{C}$  is continuous and holomorphic in each variable, then it is analytic

*Proof.* Iterate Cauchy's formula and use Fubini's theorem to write

$$f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{w_i \in \Delta(z_i, r_i)} \frac{f(w)dw}{(w_1 - z_1) \cdots (w_n - z_n)}$$

Then

$$\frac{1}{(w_1 - z_1) \cdots (w_n - z_n)} = \sum_I \frac{(z - \xi)^I}{(w - \xi)^I}$$

Then a convergent power series expression follows, with

$$c_I \left( \frac{1}{2\pi i} \right)^n \int_{w \in \Delta(z, r)} \frac{f(w)dw}{(w_1 - z_1)^{I_1+1} \cdots (w_n - z_n)^{I_n+1}}$$

The *total order* of an analytic function  $f$  at  $\xi$  is the smallest value of  $|I|$  for which  $c_I \neq 0$   $\square$

**Definition 4.4.** A set  $E \subseteq \Omega$  is called *thin* if for every  $\xi \in E$  there is a polydisk  $\Delta(\xi, r) \subset \subset \Omega$  and  $g \in A(\Delta(\xi, r))$  such that  $E \cap \Delta(\xi, r) \subseteq Z(g)$ . Note that for  $n = 1$ , this is equivalent to discrete

**Theorem 4.5** (Riemann extension theorem). If  $f \in A(\Omega \setminus E)$  where  $E$  is a thin set, and  $f$  is locally bounded on  $\Omega$ , then there exists  $\tilde{f} \in A(\Omega)$  such that  $\tilde{f} = f$  on the complement of  $E$

*Proof.* Let  $k$  be the total order of  $g$  at  $\xi$ . By an application of Rouché's theorem (and after modifying  $r$  and a change of variables), we can assume that for each  $z_1, \dots, z_{n-1}$  the function  $z_n \mapsto g(z_1, \dots, z_{n-1}, z_n)$  has exactly  $k$  zeros and none on the boundary  $\square$

In higher dimensions, to solve  $\bar{\partial}$  equation, there must be a *integrability condition*. Indeed, if we can solve the equation, then  $0 = \bar{\partial}^2 u = \bar{\partial}\alpha$ , i.e. we require  $\alpha$  to be  $\bar{\partial}$  closed

**Proposition 4.6.** Let  $n \geq 2$ . If  $\alpha$  is a smooth compactly supported  $(0, 1)$  form on  $\mathbb{C}^n$  with  $\bar{\partial}\alpha = 0$ , then there is a  $u \in C_c^\infty$ , with  $\bar{\partial}u = \alpha$

*Proof.*  $\square$

**Corollary 4.7** (Hartogs theorem). Let  $K \subseteq \Omega$  be compact with  $\Omega \setminus K$  connected. If  $f \in A(\Omega \setminus K)$ , there exists  $\tilde{f} \in A(\Omega)$  that is equal to  $f$  on the complement of  $K$

*Proof.* Let  $\phi \in C_c^\infty(\Omega)$  be  $\equiv 1$  in a neighborhood of  $K$ , let  $\alpha = \bar{\partial}((1 - \phi)f)$ . Then  $\alpha$  is  $\bar{\partial}$ -closed and compactly supported. Hence, there is  $u \in C_c^\infty(\mathbb{C}^n)$  with  $\bar{\partial}u = \alpha$ . Then let  $\tilde{f} = (1 - \phi)f - u$ ,  $\tilde{f} \in A(\Omega)$ , since  $u$  is compactly supported,  $\tilde{f} = f$  on  $\Omega \setminus K$   $\square$

*Note.* The assumption that  $\Omega \setminus K$  is connected is necessary. For example, let  $K \subseteq B(0, 1) = \{|z| < 1\}$  be the set where  $|z| = \frac{1}{2}$ , and take

$$f(z) = \begin{cases} z_n & \text{if } 1/2 < |z| < 1 \\ 0 & \text{if } |z| < 1/2 \end{cases}$$

Then there is no holomorphic extension to  $B(0, 1)$

**Proposition 4.8.** If  $\alpha$  is a smooth  $\bar{\partial}$ -closed  $(0, 1)$  form on a polydisc  $\Delta = \Delta(0, r)$ , then  $\alpha = \bar{\partial}u$  for some  $u \in C^\infty(\Delta)$

*Proof.* Just like in the one variable case, exhaust  $\Delta$  by nested closed polydiscs  $K_i$ . Use cut-off functions to find  $u_i, \bar{\partial}u_i$  in a neighborhood of  $K_i$ . Then  $u_{i+1} - u_i$  is holomorphic in a neighborhood of  $K_i$ . Now by the power series expansion, there is a polynomial  $p_i$  such that  $\|u_{i+1} - u_i - p_i\|_{K_i} < 2^{-i}$ . The rest follows as in the proof of the one variable case  $\square$

*Note.* We heavily used the geometric properties of the polydisc

**Corollary 4.9** (Cousin theorem).  $\mathcal{U} = \{u_i\}$  is an open cover of polydisc  $\Delta$ , then  $H^1(\Delta, \mathcal{U}) = 0$

**Theorem 4.10.** If  $\alpha \in C_{(p,q)}^\infty(\Delta)$ ,  $q \geq 1$ ,  $\bar{\partial}\alpha = 0$ . Then  $\alpha = \bar{\partial}u$  for some  $u \in C_{(p,q-1)}^\infty(\Delta)$

**Remark 4.11.** This states that the Dolbeault cohomology groups  $H_{\bar{\partial}}^{p,q}(\Delta) = 0$

*Proof.* Induct on  $k = 1, \dots, n$ , the smallest integer such that  $\alpha$  only involves  $d\bar{z}_1, \dots, d\bar{z}_k$ . If  $k = 1$ , then  $q = 1$  and we have already proven the result. Suppose the result is true for  $k - 1$ . Write  $\alpha = \omega \wedge d\bar{z}_k + \beta$ , where  $\omega$  and  $\beta$  only involve  $d\bar{z}_1, \dots, d\bar{z}_{k-1}$ . We have  $0 = \bar{\partial}\alpha = \bar{\partial}\omega \wedge d\bar{z}_k + \bar{\partial}\beta$ . This implies both  $\omega, \beta$  are holomorphic in the variables  $z_{k+1}, \dots, z_n$ . Apply the one variable solution to find  $\mu, \bar{\partial}\mu = \omega \wedge d\bar{z}_k + \sigma$ , here  $\sigma$  only involves  $d\bar{z}_1, \dots, d\bar{z}_{k-1}$ . Now  $\alpha - \bar{\partial}u = \beta - \sigma$  is  $\bar{\partial}$ -closed. By induction, we can write  $\beta - \sigma = \bar{\partial}v$ , and so we set  $u = v + \mu$   $\square$

**Example 4.12.** Let  $\Omega \subseteq \mathbb{C}^2$  be a domain. For  $\xi \in \Omega$ , let  $\Omega^* = \Omega \setminus \{\xi\}$ . Then  $H_{\bar{\partial}}^{0,1}(\Omega^*) \neq \{0\}$

*Proof.* Without loss of generality assume  $\xi = (0, 0)$ . Consider the  $(0, 1)$ -form

$$\omega = \frac{1}{r^4}(-\bar{z}_2 d\bar{z}_1 + \bar{z}_1 d\bar{z}_2) = \bar{\partial} \left( \frac{\bar{z}_2}{z_1 r^2} \right)$$

Clearly,  $\omega$  is smooth on  $\Omega^*$ , and  $\bar{\partial}\omega = 0$ . Suppose  $\omega = \bar{\partial}u$  for  $u \in C^\infty(\Omega^*)$ . Then  $f(z_1, z_2) = z_1 u - \frac{\bar{z}_2}{r^2}$  is holomorphic on  $\Omega^* \setminus \{z_1 = 0\}$ , and it is locally bounded on  $\Omega^*$ . By Riemann extension, it is holomorphic on  $\Omega^*$ . By Hartogs, it extends to  $\Omega$ . But for  $z_2 \neq 0$  we clearly have  $f(0, z_2) = -\frac{1}{z_2}$ , contradiction  $\square$

**Proposition 4.13.**  $K \subseteq \Omega$  is compact

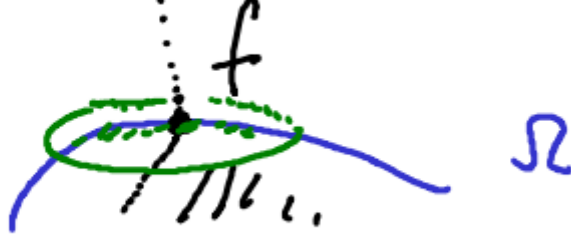
1.  $\hat{K}_\Omega$  is closed in  $\Omega$
2.  $\hat{K}_\Omega$  is not necessarily closed in  $\mathbb{C}^n$ . E.g. if  $n \geq 2$ , let  $\Omega = \mathbb{B}^n \setminus \{0\}$ ,  $K = \{|z| = 1/2\}$ . Then by Hartogs' theorem,  $\hat{K}_\Omega = \mathbb{B}_{1/2}^n \setminus \{0\}$
3.  $\hat{K}_\Omega \subseteq \mathcal{G}(K)$ , the closed convex hull of  $K$ . In particular,  $\hat{K}_\Omega$  is bounded

*Proof.* Let  $w \notin \mathcal{G}(K)$ ,  $z_0 \in \mathcal{G}(K)$  minimizes distance to  $w$ , let  $\xi \in (\mathbb{C}^n)^*$  define a supporting hyperplane for  $\mathcal{G}(K)$  so that  $\mathcal{G}(K) \subseteq \text{Re}\langle \xi, z \rangle \leq 0$  and  $\text{Re}\langle \xi, w \rangle \geq 0$ . Let  $f(z) = \exp\langle \xi, z \rangle$ ,  $|f(z)| = \exp \text{Re}\langle \xi, z \rangle$  which violates the definition, so  $w \notin \hat{K}_\Omega$   $\square$

**Definition 4.14.** A domain  $\Omega \subseteq \mathbb{C}^n$  is *holomorphically convex* if for every compact  $K \subseteq \Omega$ ,  $\hat{K}_\Omega$  is compact. If  $\Omega$  is convex, then it is holomorphically convex. If  $n = 1$ , all domains are holomorphically convex. The previous counter-example shows this is not true if  $n \geq 2$

**Proposition 4.15.**  $\Omega \subseteq \mathbb{C}^n$  is holomorphically convex  $\iff$  every discrete, infinite set  $\{z_j\} \subseteq \Omega$  there is  $f \in A(\Omega)$  with  $|f(z_j)|$  unbounded

*Proof.*  $\Leftarrow$ : If  $\hat{K}_\Omega$  is not compact there is a discrete infinite subset  $\{z_j\} \subseteq \hat{K}$ . But then  $|f(z_j)| \leq \|f\|_K$ ,  $\forall j, f \in A(\Omega)$ . This contradicts the existence of  $f \in A(\Omega)$  where  $|f(z_j)|$  is unbounded



$\Rightarrow$ : Exhaust  $\Omega$  by nested compact sets  $K_j$ ,  $\hat{K}_j = K_j$ . We may assume  $z_j \in K_{j+1} \setminus K_j$ . We can find  $f_j \in A(\Omega)$  such that  $f_j(z_j) = 1$ ,  $\|f_j\|_{K_j} < 1$ , by taking power,  $\|f_j\|_{K_j}$  can actually be arbitrarily small. Let  $g_j \in A(\Omega)$  be such that  $g_j(z_j) = 1$ ,  $g_j(z_i) = 0$  for  $i < j$ . Now define  $\lambda_j$  by

$$\lambda_j = j - \sum_{i=1}^{j-1} \lambda_i g_i f_i(z_j)$$

Assume  $\|\lambda_j g_j f_j\|_{K_j} < 2^{-j}$ . Now let  $f(z) = \sum_{i=1}^{\infty} \lambda_i g_i f_i(z)$ . This converges uniformly on compact sets, and so  $f \in A(\Omega)$ . Finally

$$f(z_j) = \sum_{i=1}^j \lambda_i g_i f_i(z_j) = \lambda_j g_j f_j(z_j) + \sum_{i=1}^{j-1} \lambda_i g_i f_i(z_j) = j$$

□

**Definition 4.16.**  $\Omega \subseteq \mathbb{C}^n$  is called a *domain of holomorphy* if there is  $f \in A(\Omega)$  such that for any  $p \in \overline{\Omega} \setminus \Omega$  and any  $\Omega'$  about  $p$ , there is no  $g \in A(\Omega')$  such that  $g = f$  on  $\Omega' \cap \Omega$

**Theorem 4.17.**  $\Omega \subseteq \mathbb{C}^n$  is holomorphically convex  $\iff$  it is a domain of holomorphy

**Corollary 4.18.** A convex domain in  $\mathbb{C}^n$  is a domain of holomorphy

*Proof.*  $\Rightarrow$  is similar to the one variable case.  $\Leftarrow$  is a theorem of Oka (this will be generalized)  
 $\Rightarrow$ : Fix a polydisc  $\Delta$  about the origin. For  $\xi \in \Omega$ , let  $\Delta_\xi = \xi + r\Delta$ , where  $r$  is the supremum such that  $\xi + r\Delta \subseteq \Omega$ . Let  $E \subseteq \Omega$  be countable dense. Let  $\{\xi_j\}$  be a sequence containing every point of  $E$  infinitely many times. Write  $\Omega = \bigcup K_j$ . Since  $\hat{K}_j \subset \subset \Omega$ ,  $\exists z_j \in \Delta_{\xi_j}$  with  $z_j \notin \hat{K}_j$ . Choose  $f_j \in A(\Omega)$ ,  $f_j(z_j) = 1$ ,  $\|f_j\|_{K_j} < 2^{-j}$ . Set  $f(z) = \prod (1 - f_j)^j$ . Then  $f$  converges uniformly on compact sets, so  $f \in A(\Omega)$ . Now  $f$  has zeros of order  $\geq j$  at  $z_j$ . Any continuation of  $f$  would have a zero of infinite order

$\Rightarrow$ : Let  $d(z) = \sup_{\Delta(z,r) \subseteq \Omega} r$ ,  $d(K) = \inf_{z \in K} d(z)$ . Claim  $d(\hat{K}) = d(K) > 0$ . This will imply  $\hat{K} \subset \subset \Omega$ . Let  $f \in A(\Omega)$  so that the radius of convergence at  $z$  is  $d(z)$ , let  $\delta < d(K)$ ,  $K_\delta = \bigcup_{w \in K} \overline{\Delta(w, \delta)}$ . By Cauchy estimates:  $\|D^I f\|_K \leq \frac{I!}{\delta^{|I|}} \|f\|_{K_\delta}$ . But  $D^I f \in A(\Omega)$ , so for  $z \in \hat{K}$ ,  $|D^I f(z)| \leq \|D^I f\|_K \leq \frac{I!}{\delta^{|I|}} \|f\|_{K_\delta}$ . This implies that the radius of convergence at  $z \in \hat{K}$  is at least  $\delta$ , i.e.  $d(z) \geq \delta$ , and so  $d(\hat{K}) \geq d(K)$ . Since  $K \subseteq \hat{K}$ , the other inequality is trivial □

**Proposition 4.19.** If  $\{\Omega_\alpha\}_{\alpha \in I}$  are domains of holomorphy in  $\mathbb{C}^n$ , then the interior  $\Omega$  of  $\bigcap_{\alpha \in I} \Omega_\alpha$  is also a domain of holomorphy

*Proof.*  $K \subseteq \Omega$  is compact. For each  $\alpha \in I$ ,  $K \subseteq \Omega \subseteq \Omega_\alpha$ , which implies  $\hat{K}_\Omega \subseteq \hat{K}_{\Omega_\alpha}$ . This implies  $d_{\Omega_\alpha}(\hat{K}_{\Omega_\alpha}) \leq d_{\Omega_\alpha}(\hat{K}_\Omega)$ , for all  $\alpha$ . Since  $\Omega_\alpha$  is holomorphically convex,  $d_{\Omega_\alpha}(\hat{K}_{\Omega_\alpha}) = d_{\Omega_\alpha}(K)$ . Hence  $d_\Omega(K) \leq d_{\Omega_\alpha}(K) \leq d_{\Omega_\alpha}(\hat{K}_\Omega)$ . Finally, this implies  $d_\Omega(K) \leq d_\Omega(\hat{K}_\Omega)$ . As before, we conclude that  $d_\Omega(K) = d(\hat{K}_\Omega)$ , and so  $\hat{K}_\Omega$  is compact, so  $\Omega$  is holomorphically convex  $\square$

**Claim 4.20.** Suppose  $\Omega$  is a domain of holomorphy. Let  $f_1, \dots, f_N \in A(\Omega)$ , and define

$$\Omega_c = \{z \in \Omega \mid |f_j(z)| < c, j = 1, \dots, N\}$$

Then  $\Omega_c$  is also a domain of holomorphy

*Proof.* Let  $K \subseteq \Omega_c$ . Let  $z \in \hat{K}_\Omega$ . Then in particular, for any  $j = 1, \dots, N$ ,  $|f_j(z)| \leq \|f_j\|_K < c$ . So  $z \in \Omega_c$ . Now  $\hat{K}_{\Omega_c} \subseteq \hat{K}_\Omega \subseteq \Omega$  and so  $\hat{K}_{\Omega_c}$  is compact  $\square$

**Claim 4.21.** Let  $u : \Omega \subseteq \mathbb{C}^n \rightarrow \mathbb{C}^m$  be holomorphic, with  $\Omega$  a domain of holomorphy. If  $\Omega' \subseteq \mathbb{C}^m$  is a domain of holomorphy, then so is  $\tilde{\Omega} = u^{-1}(\Omega')$

*Proof.* Let  $K \subseteq \tilde{\Omega} \subseteq \Omega$  be compact. Since  $\hat{K}_{\tilde{\Omega}} \subseteq \hat{K}_\Omega \subseteq \Omega$ , it suffices to show  $\hat{K}_{\tilde{\Omega}}$  is closed in  $\Omega$ . Let  $z_j \rightarrow z \in \Omega$ ,  $z_j \in \hat{K}_{\tilde{\Omega}}$ . Notice that  $u(\hat{K}_{\tilde{\Omega}}) \subseteq \widehat{u(K)}_{\Omega'}$ . Hence  $u(z) \in \Omega'$ , and so  $z \in \tilde{\Omega}$   $\square$

**Lemma 4.22.** Let  $\Omega \subseteq \mathbb{C}^n$  be a domain of holomorphy, and  $K \subseteq \Omega$ . Suppose  $f \in A(\Omega)$  is such that  $|f(z)| \leq d(z)$  for all  $z \in K$ , then  $|f(\xi)| \leq d(\xi)$  for all  $\xi \in \hat{K}_\Omega$

*Proof.* We first claim that if  $u \in A(\Omega)$ , then the power series expansion of  $u$  at  $\xi \in \hat{K}_\Omega$  converges on  $\Delta(\xi, |f(\xi)|)$ . This will prove the Lemma, because we can take  $u$  to be the function with no analytic continuation beyond  $\Omega$

Proof of the claim: Let  $0 < \delta < 1$ , as before, the Cauchy estimates provide for some constant  $M$  that

$$|D^I u(z)| \frac{(\delta |f(z)|)^{|I|}}{I!} \leq M, \forall z \in K$$

Now  $D^I u(z) f(z)^{|I|} \in A(\Omega)$ , so the same estimate holds on  $\hat{K}_\Omega$ . This means the radius of convergence at  $\xi \in \hat{K}_\Omega$  is at least  $\delta |f(\xi)|$ . Since  $\delta$  was arbitrary, this proves the claim  $\square$

Fundamental consequence: Let  $D \subset \subset \Omega$  be a 1-dimensional disc

1. Suppose  $f$  is a polynomial in one variable such that  $-\log d(z) \leq \operatorname{Re} f(z)$ , for  $z \in \partial D$
2. Let  $f$  be the restriction of  $F \in A(\Omega)$ . Then  $|e^{-F(z)}| \leq d(z)$ ,  $z \in \partial D$
3. By the maximum principle,  $D \subseteq \widehat{\partial D}_\Omega$
4. From the Lemma, we have  $|e^{-F(z)}| \leq d(z)$ ,  $z \in \partial D$
5. This in turn implies  $-\log d(z) \leq \operatorname{Re} f$  on  $D$

Approximating harmonic functions by polynomials, we conclude that  $u = -\log d$  is subharmonic on any complex line in  $\Omega$

## 5 Hartogs theorem

**Theorem 5.1.**

## 6 Pseudoconvexity

**Definition 6.1.** An upper semicontinuous function  $\phi : \Omega \subseteq \mathbb{C}^n \rightarrow [-\infty, \infty)$  is *plurisubharmonic* if the restriction of  $\phi$  to every complex line  $L \cap \Omega$ ,  $L \cong \mathbb{C}$ , is subharmonic. Let  $P(\Omega)$  be the set of plurisubharmonic (psh) functions on  $\Omega$

**Proposition 6.2.**  $\phi \in C^2(\Omega)$  is psh  $\iff$  for all  $\xi \in \mathbb{C}^n$  and all  $z \in \Omega$ , the complex Hessian is positive semidefinite

$$\sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j \geq 0$$

$\phi$  is strictly psh if  $>$  holds for every  $\xi \neq 0$

**Remark 6.3.** A real  $(1,1)$  form on  $\Omega$  can be written as

$$\omega(z) = i \sum_{i,j=1}^n g_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j$$

where  $g_{i\bar{j}}$  is a Hermitian matrix. We say that  $\omega \geq 0$  (resp.  $\omega > 0$ ) if  $(g_{i\bar{j}}(z))$  is positive semidefinite (resp. positive definite) for every  $z \in \Omega$ . This means that for each  $\xi \in \mathbb{C}^n$ ,  $\xi \neq 0$

$$\sum_{i,j=1}^n g_{i\bar{j}}(z) \xi_i \bar{\xi}_j \geq 0 \text{ (resp. } \omega > 0 \text{)}$$

In the case  $\omega > 0$ ,  $g_{i\bar{j}}$  defines a Hermitian metric on  $\Omega$ , and  $\omega$  is its associate Kähler form

*Proof.* A line  $J : L \hookrightarrow \mathbb{C}^n$  is given by a choice  $\xi \neq 0$  in  $\mathbb{C}^n$ , so that  $J(\tau) = z_0 + \tau \xi$ , then

$$J^*(dz_i) = \xi_i d\tau, J^*(d\bar{z}_i) = \bar{\xi}_i d\bar{\tau}$$

$$\begin{aligned} J^*(i\partial\bar{\partial}\phi) &= \left( \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j \right) i d\tau \wedge d\bar{\tau} \\ &= \left( \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j \right) 2d\mu \end{aligned}$$

On the other hand

$$J^*(i\partial\bar{\partial}\phi) = i\partial_z \bar{\partial}_z(\phi \circ J) = \Delta(\phi \circ J) 2d\mu$$

□

**Definition 6.4.** A domain  $\Omega \subseteq \mathbb{C}^n$  is *pseudoconvex* if there exists a continuous psh exhaustion function  $\phi$ , i.e.

$$\Omega_c = \{z \in \Omega \mid \phi(z) < c\} \subset \subset \Omega$$

For every  $c \in \mathbb{R}$

**Fact 6.5** (Richberg). If  $\Omega$  is pseudoconvex, there is a  $C^\infty$  strictly psh exhaustion function on  $\Omega$  (see Demailly's book)

**Theorem 6.6.**  $\Omega \subseteq \mathbb{C}^n$  is a domain of holomorphy iff it is pseudoconvex

*Proof.* Recall  $d(z) = \sup_{\Delta(z,r) \subseteq \Omega} r$ .  $\implies$ : We have shown that  $-\log d(z)$  is psh. It is also continuous. We claim that  $u(z) = |z|^2 - \log d(z)$  does the job Closedness: If  $z_i \rightarrow w \in \bar{\Omega} \setminus \Omega$ , then  $d(z_i) \rightarrow 0$ , so  $u$  diverges Boundedness: Fix any  $w \in \bar{\Omega} \setminus \Omega$ , then

$$d(z) \leq |z - w| \leq |z| + |w|$$

so for  $|z|$  large

$$\log d(z) \leq 2 \log |z| \leq \frac{1}{2} |z|$$

This means a bound on  $u$  implies a bound on  $|z|$

□



**Example 6.7.** 1. Geometrically convex sets are pseudoconvex(e.g. balls and polydisks)

2. If  $\{\Omega_\alpha\}$  are pseudoconvex, then the interior  $\Omega$  of  $\bigcap \Omega_\alpha$  is pseudoconvex
3. Annuli or punctured domains are not pseudoconvex
4. Let  $\Omega \subseteq \mathbb{C}^n$  be pseudoconvex,  $f_1, \dots, f_k \in A(\Omega)$ , then  $\tilde{\Omega} = \Omega \setminus V(f_1) \cup \dots \cup V(f_k)$  is pseudoconvex. Indeed, if  $\phi$  is the psh exhaustion function on  $\Omega$ , take  $\tilde{\phi} = \phi - \log |f_1| - \dots - \log |f_k|$  on  $\tilde{\Omega}$

**Proposition 6.8.** Suppose  $\Omega \subseteq \mathbb{C}^n$  is pseudoconvex. Then  $-\log d(z)$  is psh

*Proof.*  $D \subset \subset \Omega$  is a disc,  $f$  on  $D$ ,  $F \in A(\Omega)$  restricts to  $f$ , suppose  $-\log d(z) \leq \operatorname{Re} f(z)$ ,  $z \in \partial D$ , or equivalently  $d(z) \geq |e^{-f(z)}|$ ,  $z \in \partial D$ . We want to show this holds in  $D$ . Fix  $w \in \Delta(0, 1)$ . Let

$$K = \{z + \lambda w e^{-f(z)} | z \in \partial D, 0 \leq \lambda \leq 1\}$$

Then  $K \subseteq \Omega$

$$\Lambda = \{\lambda \in [0, 1] | z + \lambda' w e^{-f(z)} \in \Omega, \forall z \in D, 0 \leq \lambda' \leq \lambda\}$$

Notice that  $\Lambda \neq \emptyset$ , since  $0 \in \Lambda$ . We want show that  $\Lambda = [0, 1]$ .  $\Lambda$  is clearly open. Suppose  $\lambda_i \nearrow c$ ,  $\lambda_i \in \Lambda$ , let  $\phi$  be a continuous psh exhasution function on  $\Omega$ , then for each  $j$ ,  $z \in D$ ,  $\phi(z + \lambda_j w e^{-f(z)}) \leq \sup_K \phi$ , but since this is a compact set,  $c \in \Lambda$   $\square$

Pseudoconvexity is a property of the boundary of  $\Omega$

**Proposition 6.9.**  $\Omega \subseteq \mathbb{C}^n$ . Suppose that for every  $\xi \in \bar{\Omega}$  there is an open set  $U$  such that  $U \cap \Omega$  is pseudoconvex. Then  $\Omega$  is a pseudoconvex

*Proof.* Let  $\xi \in \partial\Omega$ , set  $\tilde{\Omega} = U \cap \Omega$ . For  $z$  sufficiently close to  $\xi$ ,  $d(z) = d_\Omega(z) = d_{\tilde{\Omega}}(z)$ , so  $-\log d(z)$  is psh in a neighborhood of  $\partial\Omega$ (say,  $\Omega \setminus F$  for smote closed  $F$ ). Find a smooth proper psh function  $\psi$  on  $\mathbb{C}^n$  such that  $\phi(z) > -\log d(z)$  for  $z \in F$ . Now let  $\phi(z) = \max\{\psi(z), -\log d(z)\}$ . Then  $\phi$  is a continuous psh exhaustion function  $\square$

**Definition 6.10.**  $\Omega \subseteq \mathbb{C}^n$  have a  $C^2$  boundary. In a neighborhood  $U$  of  $z_0 \in \partial\Omega$  we can find a  $C^2$  defining function  $\rho : U \rightarrow \mathbb{R}$ , i.e.

$$\Omega \cap U = \{z \in U | \rho(z) < 0\}, \nabla \rho \neq 0 \text{ on } \partial\Omega \cap U$$

The *Levi form*  $L_{z_0}$  at the point  $z_0$  is the quadratic form  $\operatorname{Hess}(\rho)$  restricted to  $V_{z_0} = T_{z_0} \partial\Omega \cap J(T_{z_0} \partial\Omega)$ . Alternatively, let  $\xi \in \mathbb{C}^n$  satisfy  $\sum_{i=1}^n \frac{\partial \rho}{\partial z_i} \xi_i = 0$ . Then we define

$$L(\xi) = \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_i} (z_0) \xi_i \bar{\xi}_j$$

Here, if  $\xi$  is the vector corresponding to  $v$  then  $L(v) = L(\xi)$

**Lemma 6.11.** Let  $z, w \in \Omega$ ,  $\xi \in \Delta(0, r)$  such that  $z = w + \xi$ . Then  $d(z) \geq d(w) - r$   $d(z) \geq d(w) - r$

*Proof.* Let  $\eta$  be in some polydisk about 0, such that  $z + \eta \in \partial\Omega$ , and  $d(z) = \max |\eta_i|$ . Then  $w + \xi + \eta \in \partial\Omega$ . This implies

$$d(w) \leq \max_j |(\xi + \eta)_j| \leq \max_j |\xi_j| + \max_j |\eta_j| \leq r + d(z)$$

$\square$

**Proposition 6.12.**  $\Omega$  is pseudoconvex  $\iff$  the Levi form is everywhere positive semidefinite on  $\partial\Omega$

*Proof.*  $\Rightarrow$ :  $\rho(z) = \begin{cases} -d_\Omega(z) & z \in \Omega \\ 0 & z \in \partial\Omega, \text{ then } \rho \text{ is } C^2. \text{ The function } \phi = -\log d \text{ is } C^2 \text{ and psh} \\ -d_{\bar{\Omega}^c}(z) & z \in \bar{\Omega}^c \end{cases}$

$$\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} = -\frac{1}{d(z)} \frac{\partial^2 d}{\partial z_i \partial \bar{z}_j} + \frac{1}{d(z)^2} \frac{\partial d(z)}{\partial z_i} \frac{\partial d(z)}{\partial \bar{z}_i}$$

So for  $z \in \Omega$

$$0 \leq \sum_{i,j=1}^n \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j = \sum_{i,j=1}^n \frac{1}{d(z)} \frac{\partial^2 d}{\partial z_i \partial \bar{z}_j}$$

Now let  $z \rightarrow \partial\Omega$

$\Leftarrow$ : Suppose  $c = \frac{\partial^2}{\partial \tau \partial \bar{\tau}} \log d(z_0 + \tau w_0) > 0$ .  $\log d(z_0 + \tau w_0) = \log d(z_0) + \operatorname{Re}(A\tau + B\tau^2) + c|\tau|^2 + o(|\tau|^2)$ . Choose  $\xi_0 \in \partial\Delta(0, d(z_0))$  such that  $z_0 + \xi_0 \in \partial\Omega$ ,  $\max_i |\xi_{0,i}| = d(z_0)$ . Let  $z(\tau) = z_0 + \tau w_0 + \xi_0 \exp(A\tau + B\tau^2)$ . By Lemma 6.11

$$\begin{aligned} d(z(\tau)) &\geq d(z_0 + \tau w_0) - d(z_0) |\exp(A\tau + B\tau^2)| \\ &\geq |\exp(A\tau + B\tau^2)| (e^{c|\tau|^2/2} - 1) \end{aligned}$$

Now  $d(z(0)) = 0$ . The inequality implies

$$\left. \frac{\partial}{\partial \tau} d(z(\tau)) \right|_{\tau=0} = 0, \quad \left. \frac{\partial^2}{\partial \tau \partial \bar{\tau}} d(z(\tau)) \right|_{\tau=0} > 0$$

In other words

$$\sum_{i=1}^n \frac{\partial \rho}{\partial z_i} z'_i(0) = 0, \quad \sum_{i,j=1}^n \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j} z'_i(0) \bar{z}'_j(0) < 0$$

This contradicts  $L_{z(0)} \geq 0$

□

## 7 Hörmander's $L^2$ estimate

**Definition 7.1.**  $H_1, H_2$  are complex Hilbert space,  $T : H_1 \rightarrow H_2$  is an *unbounded operator*, if it is a linear map defined on some linear subspace  $D(T) \leq H_1$  called the domain of  $T$ .  $T$  is *densely defined* if  $D(T)$  is dense in  $H_1$ .  $T$  is *closed* if the graph  $\text{Gr}(T) = \{(x, Tx) \in H_2 \times H_2 | x \in D(T)\}$  is closed.  $T$  has *closed range* if  $R(T) = \{Tx \in H_2 | x \in D(T)\}$  is closed in  $H_2$ . Write  $N(T) = \ker T$

**Definition 7.2.**  $T : H_1 \rightarrow H_2$  is a densely defined unbounded operator, its adjoint  $T^* : H_2 \rightarrow H_1$  is defined as an unbounded operator as follows

- $D(T^*)$  consists of  $y \in H_2$  such that the functional  $\langle T(-), y \rangle : D(T) \rightarrow \mathbb{C}$  is continuous
- By the Hahn-Banach theorem,  $\langle T(-), y \rangle$  extends to a linear functional on  $H_1$
- By the Riesz representation theorem and denseness, there is a vector  $T^*y \in H_1$  such that  $\langle Tx, y \rangle = \langle x, T^*y \rangle$

**Proposition 7.3.** If  $T$  is densely defined, then  $T^*$  is closed

*Proof.* Let  $y_j \in D(T^*)$ ,  $y_j \rightarrow y$ , and  $x_j = T^*y_j \rightarrow x$ . We need to show  $y \in D(T^*)$  and  $x = T^*y$ . Fix  $u \in D(T)$ . Then

$$|u||x| \geq \langle u, x \rangle = \lim_j \langle u, x_j \rangle = \lim_j \langle u, T^*y_j \rangle = \lim_j \langle Tu, y_j \rangle = \langle Tu, y \rangle$$

So the map  $u \mapsto \langle Tu, y \rangle$  is bounded on  $D(T)$  by  $|x|$ . This implies  $y \in D(T^*)$  and  $x = T^*y$   $\square$

**Fact 7.4.** If  $T, T^*$  are densely defined then  $T$  is closed, and  $(T^*)^* = T$

$$\text{Gr}(T^*) = \text{Gr}(-T)^\perp$$

**Lemma 7.5.** If  $T$  is closed and densely defined, then  $\text{Gr}(T^*) = \text{Gr}(-T)^\perp$  in  $H_1 \times H_2$

*Proof.* We have inclusion  $\subseteq$  since

$$\langle (T^*y, y), (x, -Tx) \rangle = \langle T^*y, x \rangle - \langle y, Tx \rangle = 0$$

Now if  $\langle (x, y), (u, -Tu) \rangle = \langle x, u \rangle - \langle y, Tu \rangle = 0$  for all  $u \in D(T)$ , then  $u \mapsto \langle Tu, y \rangle = \langle u, x \rangle$  is bounded on  $D(T)$ , so  $y \in D(T^*)$ , and  $x = T^*y$   $\square$

**Theorem 7.6.** If  $T$  is closed and densely defined, then so is  $T^*$ . Moreover,  $N(T^*) = R(T)^\perp$  and  $N(T) = R(T^*)^\perp$

*Note.*  $(V^\perp)^\perp = \bar{V}$

*Proof.* By Lemma 7.5, any  $(u, v) \in H_1 \times H_2$  can be written as

$$(u, v) = (x, -Tx) + (T^*y, y), x \in D(T), y \in D(T^*)$$

Taking  $u = 0$ , then  $v = y + TT^*y$ . This implies  $\langle v, y \rangle = |y|^2 + |T^*y|^2$ . If  $v \in D(T^*)^\perp$ , then  $y = 0$ , and so  $v = 0$ . Hence  $D(T^*)$  must be dense.  $N(T^*) = R(T)^\perp$  follows from  $\langle Tx, y \rangle = \langle x, T^*y \rangle$   $\square$

$T$  closed, densely defined, equivalent conditions for  $R(T)$  closed

**Proposition 7.7.** Let  $T : H_1 \rightarrow H_2$  be closed and densely defined. The following are equivalent

1.  $R(T)$  is closed
2.  $\exists C$  such that  $|x| \leq C|Tx|$  for all  $x \in D(T) \cap R(T^*)$
3.  $R(T^*)$  is closed
4.  $\exists C$  such that  $|y| \leq C|T^*y|$  for all  $y \in D(T^*) \cap R(T)$

*Proof.* 2. $\Rightarrow$ 1.: Suppose  $Tx_j \rightarrow y$ , then  $x_j$  converges, say to  $x$ ,  $(x_j, Tx_j) \rightarrow (x, y)$

To show 1. $\Rightarrow$ 2., recall  $N(T) = R(T^*)^\perp$ . Hence  $T$  is continuous and 1-1 from  $D(T) \cap R(T^*)$  onto the closed subspace  $R(T)$ . Hence the inverse is continuous by the closed graph theorem. This proves 2.

3.  $\iff$  4.

2. $\Rightarrow$ 4.:

$$|\langle Tx, y \rangle| = |\langle x, T^*y \rangle| \leq |x||T^*y| \leq C|Tx||T^*y|$$

So  $|\langle z, y \rangle| \leq C|T^*y||z|$  for  $z \in R(T)$ ,  $y \in D(T^*)$   $\square$

**Definition 7.8.** Now consider densely defined closed unbounded operators  $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$  satisfying  $S \circ T = 0$ . The *harmonic elements* are

$$\mathcal{H}_2 = N(S) \cap N(T^*)$$

there is an orthogonal decomposition  $H_2 = \mathcal{H}_2 \oplus (N(S) \cap N(T^*))^\perp = \mathcal{H}_2 \oplus \overline{R(T)} \oplus \overline{R(S^*)}$ , so  $N(S) = \mathcal{H}_2 \oplus \overline{R(T)}$

**Theorem 7.9.** There is  $C > 0$  such that for all  $y \in D(S) \cap D(T^*)$

$$|y| \leq C(|Sy| + |T^*y|) \quad (7.1)$$

i.e. the *basic estimate* holds  $\iff \mathcal{H}_2 = 0$  and  $R(T), R(S^*)$  are closed

*Proof.*  $\Rightarrow$ : If  $y \in \mathcal{H}_2$ , then  $|y| \leq C(|Sy| + |T^*y|) = 0$ , hence  $\mathcal{H}_2 = 0$ . If  $y \in R(T) \cap D(T^*)$ , then  $y \in N(S)$  and  $|y| \leq C|T^*y|$ , by Proposition 7.7,  $R(T)$  is closed, similarly,  $R(S^*)$  is closed  $\Leftarrow$ :  $H_2 = R(T) \oplus R(S^*)$  and  $y \in D(S) \cap D(T^*)$ , write  $y = y_1 + y_2$ ,  $y_1 \in R(T) \cap D(T^*)$ ,  $y_2 \in R(S^*) \cap D(S)$ . Apply the previous estimates and the triangle inequality

$$|y| \leq |y_1| + |y_2| \leq C_1|T^*y| + C_2|Sy| \leq C(|Sy| + |T^*y|)$$

□

$\mathcal{D}_{(p,q)}(\Omega) \subseteq L^2_{(p,q)}(\Omega)$  be the smooth  $(p,q)$ -forms with compact support in  $\Omega$ . Consider the unbounded operator  $\bar{\partial} : L^2_{(p,q)}(\Omega) \rightarrow L^2_{(p,q+1)}(\Omega)$  with domain  $D(\bar{\partial}) = \{u \in L^2_{(p,q)}(\Omega) | \bar{\partial}u \in L^2_{(p,q+1)}(\Omega)\}$ , the derivative is in the sense of distributions  $\langle \bar{\partial}u, \alpha \rangle_{L^2} = \langle u, \bar{\partial}^*\alpha \rangle_{L^2}$  for  $\alpha \in \mathcal{D}_{(p,q+1)}(\Omega)$ .  $\bar{\partial}^*$  is called the *formal adjoint* of  $\bar{\partial}$ . Then  $\bar{\partial}u \in L^2$  if there is a constant  $C > 0$  such that  $|\langle u, \bar{\partial}^*\alpha \rangle| \leq C\|\alpha\|_{L^2}$ . In this case, the Hahn-Banach and Riesz representation theorem,  $\langle u, \bar{\partial}^*\alpha \rangle = \langle v, \alpha \rangle$

**Proposition 7.10.**  $\bar{\partial} : L^2_{(p,q)}(\Omega) \rightarrow L^2_{(p,q+1)}(\Omega)$  is a closed operator

*Proof.*  $u_i \rightarrow u$  in  $L^2$ ,  $u_i \in D(\bar{\partial})$ ,  $\bar{\partial}u_i \rightarrow \alpha$ . Let  $\beta \in \mathcal{D}_{(p,q+1)}(\Omega)$ , then

$$\langle u, \bar{\partial}^*\beta \rangle = \lim_{i \rightarrow \infty} \langle u_i, \bar{\partial}^*\beta \rangle = \lim_{i \rightarrow \infty} \langle \bar{\partial}u_i, \beta \rangle = \langle \alpha, \beta \rangle$$

So the map  $\beta \mapsto \langle u, \bar{\partial}^*\beta \rangle$  is bounded and  $\bar{\partial}u = \alpha$ , thus  $\bar{\partial}$  is closed

□

**Example 7.11.** Consider  $T = i \frac{d}{dx}$  on  $L^2([0,1])$ , with  $D(T)$  consisting of  $f, f' \in L^2$ . One can show  $f \in D(T)$  is (absolutely) continuous on  $[0,1]$ . By integration by parts

$$f \mapsto \langle Tf, g \rangle = \langle f, Tg \rangle + i(f(1)\bar{g}(1) - f(0)\bar{g}(0))$$

This is not continuous with respect to the  $L^2$  topology on  $f$ , unless  $g(0) = g(1) = 0$ . Thus  $T^*$  is the same operator  $T$ , but with a different domain of definition

The problem is that while compactly supported functions are dense in  $L^2$  topology, they are not dense in the  $L^2_1$  topology, i.e. the graph norm  $\|u\| + \|\bar{\partial}u\|$

Methods to fix this

1. Hörmander uses a clever choice of (three) weights to prove the basic estimate
2. If  $\Omega$  has sufficiently nice boundary, define boundary conditions (the  $\bar{\partial}$ -Neumann problem)
3. Change the geometry of  $\Omega$  to carry a complete Kähler metric

We will follow the last option (Demially, Gaffney)

Equivalent conditions of completeness and compact exhaustion

**Lemma 7.12.**  $(M, g)$  is a Riemannian manifold, the following are equivalent

1.  $(M, g)$  is complete (Hopf-Rinow theorem)

2.  $\exists$  compact exhaustion function  $\psi$  with  $|d\psi|_g \leq 1$

3.  $\exists$  compact exhaustion  $K_i \subseteq K_{i+1}^\circ$ , and  $0 \leq \psi_i$  supported in  $K_{i+1}$ ,  $\equiv 1$  on  $K_i$ , such that  $|d\psi_i|_g \leq 2^{-i}$

*Proof.* 1. $\Rightarrow$ 2.: Fix  $x_0 \in M$ , Let  $\psi_0(x) = \frac{1}{2}d(x_0, x)$ . Smooth  $\phi_0$  to  $\psi$  with  $|\phi - \phi_0| < 1$ , by convolution with some  $g \in C^\infty$ , compactly supported near 0 and  $\int g = 1$  2. $\Rightarrow$ 3.: Choose a

smooth function  $\rho : \mathbb{R} \rightarrow [0, 1]$  with  $\rho(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2 \end{cases}$  and  $|\rho'(t)| \leq 2$ . Then let  $K_i = \{\psi(x) \leq 2^{i+1}\}$ ,  $\psi_i(x) = \rho(2^{-i-1}\psi(x))$  3. $\Rightarrow$ 2.: Set  $\psi = \sum 2^i(1 - \psi_i)$  2. $\Rightarrow$ 1.:  $|\psi(x) - \psi(y)| \leq d(x, y)$ ,  $\{x_i\}$  is a Cauchy sequence, then  $\{x_i\}$  lies in the set  $\{\psi \leq C\}$  for some  $C$ . Since this is compact, the sequence converges, so  $(M, g)$  is complete  $\square$

**Corollary 7.13.** Let  $\Omega$  have a complete Riemannian metric  $\omega$ . Then  $\mathcal{D}_{(p,q)}(\Omega)$  is dense in graph norm of  $\bar{\partial}$

*Proof.* Set  $u_i = u\phi_i$  as in Lemma 7.12, 3. Then  $u_i \rightarrow u$  in  $L^2$ , and  $\bar{\partial}u_i = \bar{\partial}u\phi_i + u\bar{\partial}\phi_i \rightarrow \bar{\partial}u$  in  $L^2$ . Now choose  $v_i \in \mathcal{D}_{(p,q)}(\Omega)$  so that  $\|v_i - u_i\|_{L^2_i} = \|v_i - u_i\|_{L^2} + \|\bar{\partial}v_i - \bar{\partial}u_i\|_{L^2} \leq 1/i$ , the result follows  $\square$

**Corollary 7.14.**  $(\Omega, \omega)$  is complete,  $\bar{\partial}^*$  with domain

$$D(\bar{\partial}^*) = \left\{ \alpha \in L^2_{(p,q+1)}(\Omega) \mid \bar{\partial}^*\alpha \in L^2_{(p,q)}(\Omega) \right\}$$

is the adjoint of  $T^*$  of  $T = \bar{\partial}$

*Proof.*  $D(T^*) \subseteq D(\bar{\partial}^*)$ . Let  $u \in \mathcal{D}_{(p,q)}(\Omega) \subseteq D(T)$ . If  $\alpha \in D(T^*)$ , then there is a constant  $C > 0$  such that  $|\langle \bar{\partial}u, \alpha \rangle| \leq C|u|$ . But then by definition,  $|\langle u, \bar{\partial}^*\alpha \rangle| \leq C|u|$ . Since  $\mathcal{D}_{(p,q)}(\Omega)$  is dense in  $L^2$ , this implies  $\bar{\partial}^*\alpha \in L^2$

$D(\bar{\partial}^*) \subseteq D(T^*)$ . If  $\bar{\partial}^*\alpha \in L^2$ , there is a constant  $C > 0$  such that  $|\langle u, \bar{\partial}^*\alpha \rangle| \leq C|u|$ . Fix  $u \in D(T)$ . Let  $u_i \in \mathcal{D}_{(p,q)}(\Omega)$  so that  $u_i \rightarrow u$  and  $\bar{\partial}u_i \rightarrow \bar{\partial}u$  in  $L^2$ . Then since  $\langle u_i, \bar{\partial}^*\alpha \rangle = \langle \bar{\partial}u_i, \alpha \rangle$ , we have  $|\langle \bar{\partial}u, \alpha \rangle| \leq C|u|$ , and so  $\alpha \in D(T^*)$   $\square$

## 8 Kahler metrics

**Definition 8.1.**  $\Omega \subseteq \mathbb{C}^n$ . A *hermitian metric* on  $\Omega$  is a positive definite hermitian valued smooth function  $g = (g_{i\bar{j}})$ . The *Kähler form* associated to  $g$  is  $\omega = i \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ , note that  $\bar{\omega} = \omega$ . We assume that  $g$  is tensorial, in the sense that  $\omega$  is a well-defined  $(1,1)$ -form on  $\Omega$ ,  $g_{i\bar{j}} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle$ . This gives a pointwise hermitian inner product on  $(1,0)$ -forms:  $\alpha = \alpha_i dz_i$ ,  $\beta = \beta_j d\bar{z}_j$

$$\langle \alpha, \beta \rangle = \sum_{i,j=1}^n \alpha_i \bar{\beta}_j g^{i\bar{j}}$$

$(g^{i\bar{j}})$  is the inverse matrix of  $(g_{i\bar{j}})$ . Extend this to  $(p,0)$ -forms by

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p, \beta_1 \wedge \cdots \wedge \beta_p \rangle = \det \langle \alpha_i, \beta_j \rangle$$

Extend this to  $(p,q)$ -forms by taking the product of this. This gives a complex anti-linear isometry

$$\bar{*} : \Lambda^{p,q} \rightarrow \Lambda^{n-p,n-q}, \alpha \wedge \bar{*}\beta = \langle \alpha, \beta \rangle \frac{\omega^n}{n!}$$

$\frac{\omega^n}{n!}$  is the volume form. Inducing  $L^2$  inner products

$$\langle \alpha, \beta \rangle = \int_{\Omega} \alpha \wedge \bar{*}\beta$$

Define the *Lefschetz operator*

$$L : \Lambda^{p,q} \rightarrow \Lambda^{p+1,q+1}, \alpha \mapsto \omega \wedge \alpha$$

Let  $\Lambda = L^*$ . Note that  $L^n(1) = \omega^n$ ,  $\Lambda^{n,n} \cong \mathbb{C}$

**Example 8.2.** If  $F$  is of type  $(1,1)$ , write  $F = \sum_{i,j} F_{i\bar{j}} dz_i \wedge d\bar{z}_j$ . Then  $\Lambda F = \sum_{i,j} F_{i\bar{j}} g^{i\bar{j}}$ . If  $\alpha$  is of type  $(1,0)$

$$i\alpha \wedge \bar{\alpha} \wedge \frac{\omega^{n-1}}{(n-1)!} = |\alpha|^2 \frac{\omega^n}{n!}$$

**Definition 8.3.** A hermitian metric is called Kähler if  $d\omega = 0$  ( $\omega$  is closed), equivalently,  $\frac{\partial g_{j\bar{k}}}{\partial z_i} = \frac{\partial g_{i\bar{k}}}{\partial z_j}$

**Proposition 8.4.**  $\omega$  is Kähler iff about any point there are coordinates such that  $g$  is euclidean to order two

*Proof.* We may always choose local coordinate so that  $g_{i\bar{j}}(0) = \delta_{ij}$ ,  $g_{i\bar{j}} = \delta_{ij} + A_{ij}^k z_k + B_{ij}^{\bar{k}} \bar{z}_k + O(|z|^2)$ . The Kähler condition implies:  $A_{ij}^k = A_{kj}^i$ . The fact that  $g_{i\bar{j}}$  is hermitian implies  $B_{ij}^{\bar{k}} = \overline{A_{ji}^k}$ . Define  $w_m = z_m + \frac{1}{2} A_{j\bar{m}}^i z_i \bar{z}_j$ , then  $\frac{\partial w_m}{\partial z_j} = \delta_{mj} + A_{j\bar{m}}^i z_i$ . Now  $\tilde{g}_{m\bar{n}} \frac{\partial w_m}{\partial z_i} \frac{\partial \overline{w_n}}{\partial \bar{z}_i} = g_{i\bar{j}}$ . This implies

$$\begin{aligned} g_{i\bar{j}}(z) &= \delta_{ij} + A_{ij}^k z_k + B_{ij}^{\bar{k}} \bar{z}_k + O(|z|^2) \\ &= \tilde{g}_{i\bar{j}} + \tilde{g}_{m\bar{j}} A_{i\bar{m}}^k z_k + \tilde{g}_{i\bar{k}} \overline{A_{j\bar{n}}^k} \bar{z}_k + O(|z|^2) \end{aligned}$$

□

**Proposition 8.5** (Kähler identities).  $(\Omega, \omega)$  has a Kähler metric. Then the formal  $L^2$  adjoints are given by  $\bar{\partial}^* = -i[\Lambda, \partial]$ ,  $\partial^* = i[\Lambda, \bar{\partial}]$

*Proof.* It suffices to prove these for the euclidean metric. Then is a direct computation

□

**Example 8.6.**  $\Omega \subseteq \mathbb{C}$ ,  $\Lambda(idz \wedge d\bar{z}) = 1$ ,  $f \in \mathcal{D}_{(0,0)}(\Omega)$ ,  $\beta \in \mathcal{D}_{(0,1)}(\Omega)$ ,  $\beta = \beta(z)d\bar{z}$ . Then

$$\begin{aligned}
\langle \bar{\partial}f, \beta \rangle &= \int_{\Omega} \partial_{\bar{z}} f \overline{\beta(z)} idz \wedge d\bar{z} \\
&= - \int_{\Omega} f \overline{\partial z \beta(z)} idz \wedge d\bar{z} \\
&= \int_{\Omega} f i \bar{\partial} \beta \\
&= \int_{\Omega} f \overline{\Lambda(i \partial \beta)} idz \wedge d\bar{z} \\
&= - \int_{\Omega} f \overline{\Lambda(i \partial \beta)} idz \wedge d\bar{z} \\
&= \langle f, -i \Lambda \partial \beta \rangle \\
&= \langle f, \bar{\partial}^* \beta \rangle
\end{aligned}$$

## References

- [1] *An Introduction to Complex Analysis in Several Variables* - Lars Hörmander



## Index

Almost complex structure, 10

Domain of holomorphy, 7

Holomorphic convex hull, 6

holomorphically convex, 12

Kähler form, 21

Lefschetz operator, 21

Levi form, 16

Plurisubharmonic, 15

Pseudoconvex, 15

Subharmonic function, 8

Upper semicontinuous, 8