

My mathematical universe

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Contents

I	Set theory	9
1	Set theory	11
2	Graph theory	13
2.1	Graph	13
II	Abstract Algebra	15
3	Category	17
3.1	Category	17
3.2	Yoneda lemma	21
3.3	Limits	22
3.4	Adjunction	24
3.5	Pushout and pullback	25
3.6	Filtered category	26
3.7	Comma category	27
3.8	Sheaves	28
3.9	Exponential object	30
3.10	Factorization system	31
3.11	Monoidal category	32
4	Group	33
4.1	Groups	33
4.2	Permutation group	35
4.3	Coxeter group	36
4.4	Grothendieck group	37
5	Ring	39
5.1	Rings	39
5.2	Commutative rings	40
6	Module	43
6.1	Modules	43
7	Field	47
7.1	Fields	47
7.2	Number field	48
8	Linear algebra	49
8.1	Vector spaces	49
8.2	Matrices	50
8.3	Eigenspace decomposition	52

9 Lie algebra	53
9.1 Non-associative algebra	53
9.2 Lie algebras	54
9.3 Engel's theorem and Lie's theorem	56
9.4 Killing form	57
9.5 Jordan-Chevalley decomposition	58
9.6 Classification of semisimple Lie algebras	60
9.7 Root system	68
9.8 Dynkin diagram	72
10 Cluster algebra	77
10.1 Cluster algebra	77
10.2 Laurent phenomenon	81
10.3 Y-system	84
10.4 Associahedron	86
10.5 Cluster algebra of geometric type	87
10.6 Rank two case	89
10.7 Cluster algebra of finite type	91
 III Commutative Algebra	 93
11 p-adic numbers	95
 IV Homological Algebra	 97
12 Abelian category	99
13 Spectral sequence	105
 V General topology	 109
14 General topology	111
14.1 General topology	111
14.2 Retract	116
 VI Algebraic Topology	 117
15 Cell structure	119
15.1 CW complexes	119
15.2 Graphs	121
15.3 Simplex category	122
15.4 abstract simplicial complex	126
15.5 CW approximation	127
15.6 Triangulation	128
16 Homology theory	129
16.1 Singular homology	129
16.2 Cellular homology	133

17 Cohomology theory	135
17.1 Singular cohomology	135
17.2 Čech cohomology	137
17.3 Poincaré duality	138
18 Homotopy theory	139
18.1 Homotopy	139
18.2 Model category	141
18.3 Hurewicz fibration	144
18.4 Serre fibration	150
19 Isotopy	151
20 Bundle	153
20.1 Bundles	153
20.2 Vector bundles	155
20.3 Principal bundle	160
20.4 Topological K-theory	161
20.5 Classifying space	162
VII Differential topology	163
21 Smooth manifold	165
VIII Differential geometry	167
22 Riemannian manifold	169
22.1 Differential geometry of surfaces	169
22.2 Curvature	170
22.3 Hyperbolic geometry	172
IX Complex geometry	175
23 Riemann surface	177
24 Complex manifold	179
25 Symplectic manifold	181
X Lie group	183
26 Topological group	185
27 Algebraic group	187
28 Lie group	189
28.1 Lie groups	189
28.2 Exponential map	190

XI Algebraic geometry	193
29 Variety	195
29.1 Affine Varieties	195
29.2 Varieties	197
30 Scheme	199
30.1 Affine schemes	199
30.2 Schemes	200
XII Analysis	201
31 Real analysis	203
32 Complex analysis	205
32.1 Complex analysis	205
32.2 Conformal mapping	210
32.3 Weierstrass functions	211
32.4 Zeta function	212
33 Functional analysis	213
33.1 Topological vector space	213
33.2 Arzela-Ascoli theorem	214
33.3 Baire category theorem	215
33.4 Distribution	216
33.5 Banach algebra	217
33.6 Stone-Weierstrass theorem	220
XIII Differential equations	221
34 Ordinary differential equations	223
35 Classical partial differential equations	225
35.1 Laplace's equation	225
35.2 Heat equation	226
35.3 Wave equation	228
35.4 Euler-Lagrange equation	231
35.5 Energy momentum tensor	232
XIV Mathematical physics	233
36 Special relativity	235
37 Maxwell's equations	237
XV Examples	239
38 Examples in categories	241
39 Examples in algebra	243
40 Examples in algebraic topology	245

<i>CONTENTS</i>	7
41 Examples in geometry	247
42 Examples in Lie groups and Lie algebras	249
43 Examples in algebraic geometry	251
44 Examples in analysis	253
XVI Exercises	255
45 Exercises in combinatorics	257
45.1 Exercises in abstract algebra	258
45.2 Exercises in analysis	259
46 Exercises in category	267
47 Exercises in partial differential equations	271
48 Exercises in algebraic topology	273
49 Exercises in differential topology	275
50 Exercises in bundles	277
51 Exercises in complex geometry	279
52 Exercises in Lie groups and Lie algebras	281
53 Exercises in algebraic geometry	285
54 Exercise in functional analysis	287
XVII Miscellaneous	289
54.1 Hodge structure	291
54.2 Plucker embedding	292
54.3 Graph theory	293
54.4 Moduli space	294
54.5 Teichmüller space	295
54.6 Mapping class group	296
Index	299

Part I

Set theory

Chapter 1

Set theory

Definition 1.0.1. $\{A_i\} \subseteq \mathcal{P}(X)$, $X \xrightarrow{f} Y$ is a map. f **separates** A_i if $\bigcap_i f(A_i) = \emptyset$. f **completely separates** A_i if $f(A_i) = f(a_i)$ for some distinct $a_i \in A_i$. f **perfectly separates** A, B if $A_i = f^{-1}(a_i)$ for some $a_i \in A_i$

Zorn's lemma

Lemma 1.0.2 (Zorn's lemma). P is a nonempty poset and every chain has an upper bound, then P contains a maximal element

Theorem 1.0.3 (Schröder–Bernstein theorem). $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ are injective, then there exists $A \xrightarrow{h} B$ bijective

Inclusion-exclusion principle

Theorem 1.0.4 (Inclusion-exclusion principle). $A_1, \dots, A_n \subseteq S$ are of finite cardinality, then

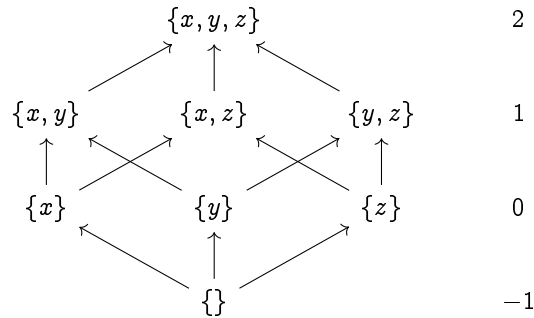
$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^k \sum_{i_1 < \dots < i_k} |A_{i_1} \cap \dots \cap A_{i_k}|$$

Definition 1.0.5. A **lattice** is a partially ordered set in which the supremum and infimum of any two elements exists uniquely

Lemma 1.0.6. Trees are bipartite

Proof. Take some $v \in T$ as the root, and label the nodes that are even distance away from 2 and odd distance away from 1 □

Definition 1.0.7. A **Hasse diagram** is a mathematical diagram used to represent a partially ordered set



Chapter 2

Graph theory

2.1 Graph

Part II

Abstract Algebra

Chapter 3

Category

3.1 Category

Definition 3.1.1. A **category** \mathcal{C} consists of $\text{ob}\mathcal{C}$ class of **objects** and $\text{Hom}\mathcal{C}$ class of **morphisms**, composition $\circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, $(h \circ g) \circ f = h \circ (g \circ f)$, $\text{Hom}(A, A)$ contains **identity** 1_A such that $1_A f = f$, $g 1_A = g$

Identity 1_A is unique since $1'_A = 1_A \circ 1_A = 1_A$. $g \in \text{Hom}(B, A)$ is the **inverse** of $f \in \text{Hom}(A, B)$ if $g \circ f = 1_A$, $f \circ g = 1_B$. Inverse f^{-1} is unique since $g = g \circ f \circ g' = g'$

A **functor** is $F : \mathcal{C} \rightarrow \mathcal{D}$, $\text{ob}\mathcal{C} \rightarrow \text{ob}\mathcal{D}$, $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$, $F(1_A) = 1_{F(A)}$, $F(g \circ f) = F(g) \circ F(f)$

The **dual category** of \mathcal{C} is \mathcal{C}^{op} with the same objects but morphisms reversed, if $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, then $f^{op} \in \text{Hom}_{\mathcal{C}^{op}}(Y, X)$, $(fg)^{op} = g^{op}f^{op}$, 1_X^{op} is still the identity. A **contravariant functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor $F : \mathcal{C}^{op} \rightarrow \mathcal{D}$, equivalently, $\text{Ob}\mathcal{C} \rightarrow \text{Ob}\mathcal{D}$, $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(B), F(A))$, $F(1_A) = 1_{F(A)}$, $F(g \circ f) = F(f) \circ F(g)$. Functors are also called **covariant functors**

Definition 3.1.2. A **semicategory** \mathcal{C} consists of $\text{ob}\mathcal{C}$ class of **objects** and $\text{Hom}\mathcal{C}$ class of **morphisms**, composition $\circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, $(h \circ g) \circ f = h \circ (g \circ f)$. A **semifunctor** is $F : \mathcal{C} \rightarrow \mathcal{D}$, $\text{ob}\mathcal{C} \rightarrow \text{ob}\mathcal{D}$, $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B))$, $F(g \circ f) = F(g) \circ F(f)$

Remark 3.1.3. A category is a semicategory with identities

Definition 3.1.4. $A \xrightarrow{f} B$ is a **monomorphism** if $fg_1 = fg_2 \Rightarrow g_1 = g_2$, is an **epimorphism** if $g_1 f = g_2 f \Rightarrow g_1 = g_2$, is, is a **bimorphism** is both monic and epi, is an **isomorphism** if it is invertible. Monomorphism and epimorphism are dual notions. Isomorphisms are bimorphisms. A category is **balanced** if bimorphisms are isomorphisms

Remark 3.1.5. A bimorphism is not necessary an isomorphism. In the category of rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a bimorphism because $\mathbb{Q} = \mathbb{Z}_{(0)}$ is a localization and the universal property of localization

Definition 3.1.6. A **natural transformation** is a family of morphisms $\eta_A : F(A) \rightarrow G(A)$ making the following diagram commute

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow \eta_A & & \downarrow \eta_B \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

For contravariant functors

$$\begin{array}{ccc} F(B) & \xrightarrow{F(f)} & F(A) \\ \downarrow \eta_B & & \downarrow \eta_A \\ G(B) & \xrightarrow{G(f)} & G(A) \end{array}$$

η is a **natural isomorphism** if η_A are isomorphisms

Definition 3.1.7. \mathcal{C} is a **small category** if $ob(\mathcal{C})$ and $Hom(\mathcal{C})$ are sets, otherwise **large**. \mathcal{C} is a **locally small category** if $Hom(a, b)$ are sets

Definition 3.1.8. A **subcategory** \mathcal{S} is a category consists of subclasses of objects and morphisms with the same composition map

Definition 3.1.9. we say categories \mathcal{C}, \mathcal{D} are **isomorphic** if there are functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F = 1_{\mathcal{C}}$, $F \circ G = 1_{\mathcal{D}}$ and we say \mathcal{C}, \mathcal{D} are **equivalent** if $G \circ F$ is naturally isomorphic to $1_{\mathcal{C}}$ and $F \circ G$ is naturally isomorphic to $1_{\mathcal{D}}$

Definition 3.1.10. Suppose \mathcal{C}, \mathcal{D} are categories, define the **functor category** $[\mathcal{C}, \mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$ has all functors from \mathcal{C} to \mathcal{D} as objects, and natural transformations as morphisms

Definition 3.1.11. $\mathcal{C} \times \mathcal{D}$ is the **product category** with $ob \mathcal{C} \times \mathcal{D} = ob \mathcal{C} \times ob \mathcal{D}$, $Hom_{\mathcal{C} \times \mathcal{D}}(A \times B, C \times D) = Hom_{\mathcal{C}}(A, C) \times Hom_{\mathcal{D}}(B, D)$

Definition 3.1.12. Suppose \mathcal{C}, \mathcal{D} are locally small categories, F is **faithful** if $Hom_{\mathcal{C}}(X, Y) \rightarrow Hom_{\mathcal{D}}(F(X), F(Y))$ is injective, F is **full** if $Hom_{\mathcal{C}}(X, Y) \rightarrow Hom_{\mathcal{D}}(F(X), F(Y))$ is surjective, F is **fully faithful** if $Hom_{\mathcal{C}}(X, Y) \rightarrow Hom_{\mathcal{D}}(F(X), F(Y))$ is bijective, F is **essentially surjective** if $\forall d \in ob \mathcal{D}, \exists c \in ob \mathcal{C}$ such that $Fc \cong d$

A functor F is an equivalence iff it is fully faithful and essentially surjective

Theorem 3.1.13. $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence iff F is fully faithful and essentially surjective

Proof. If F is an equivalence, there exist functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\eta : 1_{\mathcal{C}} \rightarrow GF$, $\xi : 1_{\mathcal{D}} \rightarrow FG$, $\forall d \in \mathcal{C}, \xi_d : d = 1_{\mathcal{D}}(d) \rightarrow FG(d) = F(Gd)$ is an isomorphism, i.e. F is essentially surjective, similarly, so is G

The composition of

$$Hom(c, c') \xrightarrow{F} Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc'), \quad f \mapsto Ff \mapsto GFf$$

Is the same as

$$Hom(c, c') \xrightarrow{\eta} Hom(GFc, GFc'), \quad f \mapsto \eta'_c f \eta_c^{-1}$$

By Exercise 46.0.1, this is bijective, thus $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is injective, i.e. F is faithful. Similarly, consider the composition

$$Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$$

We know $Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$ is surjective, but we also have the following diagram

$$\begin{array}{ccc} Hom(c, c') & \xrightarrow{F} & Hom(Fc, Fc') \\ \eta \downarrow & & \downarrow \xi \\ Hom(GFc, GFc') & \xrightarrow{F} & Hom(FGFc, FGFc') \end{array}$$

Since η, ξ are bijective, $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is surjective, i.e. F is full

Conversely, suppose F is fully faithful and essentially surjective, then for any $d \in \mathcal{D}$, there exists c and an isomorphism $d \xrightarrow{\xi_d} Fc$, denote this c as Gd , we can define a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, $d \mapsto Gd$ (Here we have used the axiom of choice), $d \xrightarrow{f} d' \mapsto c \xrightarrow{Gf} c'$ where $FGf = \xi_d^{-1} f \xi_{d'}$ since F is fully faithful

$$\begin{array}{ccc} d & \xrightarrow{f} & d' \\ \xi_d \downarrow & & \downarrow \xi_{d'} \\ FGd & \xrightarrow{FGf} & FGd' \\ F \uparrow & & \uparrow F \\ Gd & \xrightarrow{Gf} & Gd' \end{array}$$

$\xi : 1_{\mathcal{D}} \rightarrow FG$ is a natural isomorphism

Since F is fully faithful, there are unique $\eta_c : c \rightarrow GFc$, $F(\eta_c) = \xi_{Fc}$

If $f, g : c \rightarrow c'$ such that $\eta_{c'}f = \eta_{c'}g$, then $\xi_{Fc'}Ff = \xi_{Fc'}Fg \Rightarrow Ff = Fg \Rightarrow f = g$

If $f, g : c \rightarrow c'$ such that $f\eta_c = g\eta_c$, then $Ff\xi_{Fc} = Fg\xi_{Fc} \Rightarrow Ff = Fg \Rightarrow f = g$

$$\begin{array}{ccc}
 c & \xrightarrow{\quad} & c' \\
 \eta_c \downarrow & & \downarrow \eta_{c'} \\
 Fc & \xrightarrow{\quad} & Fc' \\
 G \downarrow & & \downarrow G \\
 GFc & \xrightarrow{\quad} & GFc' \\
 F \downarrow & & \downarrow F \\
 FGFc & \xrightarrow{\quad} & FGFc'
 \end{array}
 \begin{array}{l}
 \text{Left side: } \eta_c \text{ (curved arrow from } c \text{ to } Fc), \xi_{Fc} \text{ (curved arrow from } Fc \text{ to } GFc), F \text{ (curved arrow from } GFc \text{ to } FGFc) \\
 \text{Right side: } \eta_{c'} \text{ (curved arrow from } c' \text{ to } Fc'), \xi_{Fc'} \text{ (curved arrow from } Fc' \text{ to } GFc'), F \text{ (curved arrow from } GFc' \text{ to } FGFc')
 \end{array}$$

$\eta : 1_{\mathcal{C}} \rightarrow GF$ is a natural isomorphism □

Definition 3.1.14. The **empty category** is the category with no objects hence morphisms

Definition 3.1.15. $A \xrightarrow{f} B$ is a **constant morphism** if $fg = fh$ for any g, h , f is a **coconstant morphism** if $gf = hf$ for any g, h , f is a **zero morphism** if it is both a constant and a coconstant morphism

Definition 3.1.16. Suppose $u : S \rightarrow A$, $v : T \rightarrow A$ are morphisms, v filter through s means there is a morphism $w : T \rightarrow S$ such that $v = u \circ w$, then mutually filter defines an equivalence relation on monomorphisms(or equivalent by saying that w is an isomorphism), the equivalence classes are called **subobjects** of A , the dual notion is called **quotient objects**

Proposition 3.1.17. Direct limit is an exact functor

Definition 3.1.18. An **injective object** Q is such that for any monomorphism $f : X \rightarrow Y$ and morphism $g : X \rightarrow Q$, there is a morphism $h : Y \rightarrow Q$ such that $g = h \circ f$

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \downarrow g & \swarrow \exists h & \\
 Q & &
 \end{array}$$

the dual notion is called a **projective object** P , such that for any epimorphism $f : X \rightarrow Y$, and morphism $g : P \rightarrow Y$, there is a morphism $h : P \rightarrow X$ such that $g = f \circ h$

$$\begin{array}{ccc}
 & P & \\
 \swarrow \exists h & \downarrow g & \\
 X & \xrightarrow{f} & Y
 \end{array}$$

Definition 3.1.19. A functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is called a **representable** functor if there is an object A in \mathcal{C} such that $\Phi : \mathbf{Hom}(A, -) \rightarrow F$ is a natural isomorphism

Definition 3.1.20. Let \mathcal{C} be a category, we can define **quotient category** by moding out a congruence relation \sim , here \sim is an equivalence relation on $\mathbf{Hom}(X, Y)$ for any X, Y and it respects composition, i.e. suppose $f_1 \sim f_2 : X \rightarrow Y$, $g_1 \sim g_2 : Y \rightarrow Z$, then $g_1 \circ f_1 \sim g_2 \circ f_2$, thus $\mathbf{Hom}_{\mathcal{C}/\sim}(X, Y) = \mathbf{Hom}_{\mathcal{C}}(X, Y) / \sim$

Definition 3.1.21. \mathcal{C} is **concretizable** if there is a faithful functor $F : \mathcal{C} \rightarrow \mathbf{Set}$. A morphism $f : X \rightarrow Y$ is an **embedding** if $F(f)$ is injective, and for any $F(Z) \xrightarrow{\phi} F(X)$, $Z \xrightarrow{h} Y$ such that $F(Z) \xrightarrow{F(h)} F(Y)$, $F(h) = F(f) \circ \phi$, $\phi = F(g)$ for some $Z \xrightarrow{g} X$

Remark 3.1.22. \mathcal{C} may have different concretization

Definition 3.1.23. W is a class of morphisms of \mathcal{C} , the **localization** of \mathcal{C} with respect to W another category, denoted $\mathcal{C}[W^{-1}]$, such that there is a natural localization functor $\mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ such that any functor $F : \mathcal{C} \rightarrow \mathcal{D}$ where F sends morphisms in W to isomorphisms in \mathcal{D} uniquely factor through the localization functor, thus the localization of a category is unique up to isomorphism, one concrete construction is to consider $\mathcal{C}[W^{-1}]$ has the same objects as \mathcal{C} and adding formal inverses to the morphisms in W which is the composition closure of morphisms in W , more concretely, morphisms in $\mathcal{C}[W^{-1}]$ are compositions of morphisms in \mathcal{C} and inverses of morphisms in W

Definition 3.1.24. A **skeleton of a category** \mathcal{D} of \mathcal{C} is a full subcategory such that no two objects in \mathcal{D} are isomorphic and for every object in \mathcal{C} is isomorphic to some object in \mathcal{D} , the functor $\mathcal{D} \hookrightarrow \mathcal{C}$ is an equivalence of categories

Definition 3.1.25. \mathcal{C} is **connected** if there is a finite sequence of morphisms connecting any two objects

Example 3.1.26. $C \leftarrow A \rightarrow B$ is a connected even there is no morphism between B, C

Definition 3.1.27. Suppose \mathcal{C} is a category, a **filtered object** X is an object with a **filtration** of X , a descending filtration

$$\cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X$$

Or an ascending filtration

$$X \rightarrow \cdots \rightarrow X_{n+1} \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots$$

Definition 3.1.28. Suppose \mathcal{C} is a category, $f : X \rightarrow Y$ is a morphism, the **image** of f is a monomorphism $m : I \rightarrow Y$ such that there is a morphism $e : X \rightarrow I$ such that the following diagram commutes and satisfies the universal property

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow e & \nearrow m \\ & I & \\ & \downarrow \exists_1 v & \\ & I' & \end{array} \quad \begin{array}{c} e' \\ \nearrow \\ I' \end{array} \quad \begin{array}{c} m' \\ \nwarrow \\ I' \end{array}$$

Definition 3.1.29. A **quiver** in \mathcal{C} is a functor from $\begin{array}{c} \curvearrowright \bullet \rightrightarrows \bullet \curvearrowleft \end{array}$ to \mathcal{C} . Equivalently, a directed graph allowing multiple arrows and loops

Definition 3.1.30. The **free category** generated by quiver Q has objects vertices in Q and morphisms paths in Q with empty path the identity

Definition 3.1.31. $f \in \text{End}(A)$ is an **involution** if $f^2 = 1_A$

Definition 3.1.32. $A \xrightarrow{i} B$ has **left lifting property** or **LLP** and $X \xrightarrow{p} Y$ has **right lifting property** or **RLP** for each other in this diagram if

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow \exists & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

i, p are **orthogonal** if the lifting is unique

Definition 3.1.33. A class of morphisms \mathbf{M} of \mathcal{C} satisfies **2 out of 3** if any two of $f, g, f \circ g$ are in \mathbf{M} , so is the third. \mathbf{M} is clearly closed under composition

A class of **weak equivalences** is a class of morphisms \mathbf{W} containing isomorphisms and satisfies 2 out of 3. The class of isomorphisms \mathbf{I} is a class of weak equivalences

3.2 Yoneda lemma

Yoneda lemma

Lemma 3.2.1 (Yoneda lemma). \mathcal{C} is locally small

$$\text{Hom}_{\text{Set}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(c, -), F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

$$\text{Hom}_{\text{Set}^{\mathcal{C}^{op}}}(\text{Hom}_{\mathcal{C}}(-, c), F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

If $F = \text{Hom}(-, d)$ or $F = \text{Hom}(d, -)$, then

$$\text{Hom}_{\text{Set}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(c, -), \text{Hom}_{\mathcal{C}}(d, -)) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(d, c)$$

$$\text{Hom}_{\text{Set}^{\mathcal{C}^{op}}}(\text{Hom}_{\mathcal{C}}(-, c), \text{Hom}_{\mathcal{C}}(-, d)) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(c, d)$$

$c \rightarrow \text{Hom}_{\mathcal{C}}(c, -)$ gives an fully faithful embedding of \mathcal{C}^{op} into $\text{Set}^{\mathcal{C}}$, viewing $\{\text{Hom}(c, -)\}$ as a subcategory of $\text{Set}^{\mathcal{C}}$, $c \rightarrow \text{Hom}_{\mathcal{C}}(-, c)$ gives an fully faithful embedding of \mathcal{C} into $\text{Set}^{\mathcal{C}^{op}}$, viewing $\{\text{Hom}(-, c)\}$ as a subcategory of $\text{Set}^{\mathcal{C}^{op}}$

Proof.

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{C}}(c, c) & \xrightarrow{f} & \text{Hom}_{\mathcal{C}}(c, x) \\
 \downarrow \eta_c & & \downarrow \eta_x \\
 & \begin{array}{ccc} 1_c & \xrightarrow{\quad} & f \\ \downarrow & & \downarrow \\ u & \xrightarrow{\quad} & Ff(u) = \eta_x(f) \end{array} & \\
 F(c) & \xrightarrow{Ff} & F(x)
 \end{array}$$

The natural transformation η is determined by the element u in $F(c)$ □

Remark 3.2.2. Functor $\text{Hom}(-, c)$ is called **Yoneda embedding**, here embedding in the sense of a fully faithful functor, which is injective on objects up to isomorphism as in Lemma 46.0.4 Yoneda lemma tells us that if $\text{Hom}(c, -)$ and $\text{Hom}(d, -)$ are naturally isomorphic or $\text{Hom}(-, c)$ and $\text{Hom}(-, d)$ are naturally isomorphic, so are c and d , thus if we know where c goes to or what goes to c , we can determine c up to isomorphism, in other words, an object is determined by the morphisms that interact with it, this explains the uniqueness in universal construction

3.3 Limits

Definition 3.3.1. A **diagram** is a functor $D : J \rightarrow \mathcal{C}$, J is called the **indexed category**, the diagram D can be thought of as indexing a collection of objects and morphisms in \mathcal{C} patterned on J , we say D is a diagram in \mathcal{C} shaped J

Let $F : J \rightarrow \mathcal{C}$ be a diagram in \mathcal{C} , N be an object in \mathcal{C} , then a **cone** from N to F is a family of morphisms ψ_X such that the following diagram commutes, a **cocone** from F to N is a family of morphisms ψ_X such that the following diagram commutes

$$\begin{array}{ccc} & N & \\ \psi_X \swarrow & & \searrow \psi_Y \\ F(X) & \xrightarrow{Ff} & F(Y) \end{array} \quad \begin{array}{ccc} & N & \\ \psi_X \swarrow & & \searrow \psi_Y \\ F(X) & \xrightarrow{Ff} & F(Y) \end{array}$$

A **limit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : N \rightarrow L$ such that the following diagram commutes, a **colimit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : L \rightarrow N$ such that the following diagram commutes

$$\begin{array}{ccc} & N & \\ \psi_X \swarrow & \downarrow \exists_1 u & \searrow \psi_Y \\ & L & \\ \phi_X \swarrow & & \searrow \phi_Y \\ F(X) & \xrightarrow{Ff} & F(Y) \end{array} \quad \begin{array}{ccc} & N & \\ \psi_X \swarrow & \downarrow \exists_1 u & \searrow \psi_Y \\ & L & \\ \phi_X \swarrow & & \searrow \phi_Y \\ F(X) & \xrightarrow{Ff} & F(Y) \end{array}$$

Limits may also be characterized as terminal objects in the category of cones to F , thus unique up to isomorphism, so is colimits, a category contains all limits is called **complete**, and is called **cocomplete** if containing all colimits

The **equaliser** $Eq(f, g)$ is defined to be the limit of the diagram $X \begin{smallmatrix} f \\ \rightrightarrows \\ g \end{smallmatrix} Y$, the **coequaliser** is the colimit

Remark 3.3.2. Direct limit and inverse limit are defined on directed set, thus limit and colimit are more general

Definition 3.3.3. A **directed set** X is a set with a preorder \leq and any pair of elements has an upper bound, i.e., $\forall x, y \in X, \exists z \in X$ such that $x \leq z, y \leq z$

Definition 3.3.4. Given a directed set I , we can define a **direct(inductive) system**, with modules A_i , and functions $f_{ij} : A_i \rightarrow A_j$, $f_{ii} = 1_{A_i}$, $f_{jk} \circ f_{ij} = f_{ik}$, $i \leq j \leq k$, we can also define an **inverse system**, with module A_i , and functions $f_{ij} : A_j \rightarrow A_i$, $f_{ii} = 1_{A_i}$, $f_{ij} \circ f_{jk} = f_{ik}$

We can define morphism between direct and inverse systems and modules

Suppose A_i is a direct system, then a morphism $g_i : A_i \rightarrow B$ is such that $g_j \circ f_{ij} = g_i$, $i \leq j$ or $g_i : B \rightarrow A_i$ is such that $g_j = f_{ij} \circ g_i$, $i \leq j$. Suppose A_i is an inverse system, then a morphism $g_i : A_i \rightarrow B$ is such that $g_i \circ f_{ij} = g_j$, $i \leq j$ or $g_i : B \rightarrow A_i$ is such that $g_i = f_{ij} \circ g_j$, $i \leq j$. We can define morphisms between direct and inverse systems

Suppose A_i, B_i are both direct systems, a morphism $g_i : A_i \rightarrow B_i$ is a family of maps such that $g_j \circ f_{ij} = f_{ij} \circ g_i$, $i \leq j$. Suppose A_i, B_i are both inverse systems, a morphism $g_i : A_i \rightarrow B_i$ is a family of maps such that $g_i \circ f_{ij} = f_{ij} \circ g_j$, $i \leq j$.

Definition 3.3.5. The **direct limit** of a direct system is a module A_∞ and morphisms $\iota_i : A_i \rightarrow A_\infty$ with the universal property: given any morphism $g_i : A_i \rightarrow B$, it induces a unique $g_\infty : A_\infty \rightarrow B$ such that $g \circ \iota_i = g_i$, there is a concrete construction: define the direct limit $\varinjlim A_i = \bigsqcup_{i \in I} A_i / \sim$, where $a_i \sim a_j$, $a_i \in A_i, a_j \in A_j$ if there is an upper bound k such that $f_{ik}(a_i) = f_{jk}(a_j)$, or equivalently, $a_i \sim f_{ij}(a_j)$, $i \leq j$

Definition 3.3.6. The **inverse limit** of an inverse system is a module A_∞ and morphisms $\pi_i : A \rightarrow A_i$ with the universal property: given any morphism $g_i : B \rightarrow A_i$, it induces a unique $g_\infty : B \rightarrow A_\infty$ such that $\pi_i \circ g = g_i$, there is a concrete construction: define the inverse limit

$$\varprojlim A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j), i \leq j \right\},$$

Remark 3.3.7. Direct limit and inverse limit are dual to each other in the categorical sense

Definition 3.3.8. Product, coproduct The **biproducts** $(\bigoplus_i A_i, p_i, \iota_i)$ of A_i is such that $(\bigoplus_i A_i, p_i)$ is the product and $(\bigoplus_i A_i, \iota_i)$ is the coproduct

Definition 3.3.9. An **initial object** \emptyset is for every X , there is a unique $\emptyset \rightarrow X$, a **final object** $*$ is for every X , there is a unique $X \rightarrow *$, a **zero object** is an object which is both initial and final. A **pointed category** is a category with zero object

Remark 3.3.10. The initial and final object are the limit and colimit of empty diagram
In the category of sets, the initial object is \emptyset and a terminal object is $\{*\}$

3.4 Adjunction

Definition 3.4.1. Let $L : \mathcal{D} \rightarrow \mathcal{C}$, $R : \mathcal{C} \rightarrow \mathcal{D}$ be functors, and there is a natural isomorphism $\Phi_{X,Y}$, $X \in \mathcal{C}, Y \in \mathcal{D}$

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(LX, Y) & \xrightarrow{\Phi_{X,Y}} & \text{Hom}_{\mathcal{D}}(X, RY) \\ (Lf, g) \downarrow & & \downarrow (g, Rf) \\ \text{Hom}_{\mathcal{C}}(LX', Y') & \xrightarrow{\Phi_{X',Y'}} & \text{Hom}_{\mathcal{D}}(X', RY') \end{array}$$

Here $f : X' \rightarrow X$, $g : Y \rightarrow Y'$, $\text{Hom}_{\mathcal{C}}(Lf, g)(h) = h \circ g \circ Lf$

We say L is the **left adjoint** of R and R is the **right adjoint** of L

Example 3.4.2. Let $G : \text{Group} \rightarrow \text{Set}$ be the forgetful functor, then the functor $F : \text{Set} \rightarrow \text{Group}$, sending S to $F(S)$ is the left adjoint of G

In the category of R -modules Mod , consider functor $F := - \otimes B$ and functor $G := \text{Hom}(B, -)$, then F is the left adjoint to G , i.e. $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$

3.5 Pushout and pullback

Definition 3.5.1. The **pullback** of $f : X \rightarrow Z$, $g : Y \rightarrow Z$ is $(X \times_Z Y, p_X, p_Y)$ satisfying the universal property

$$\begin{array}{ccccc}
 W & & \xrightarrow{\phi} & & X \\
 \downarrow \psi & \searrow \exists_1 h & & \searrow p_X & \downarrow f \\
 & X \times_Z Y & \xrightarrow{p_X} & X & \\
 & \downarrow p_Y & & & \\
 & Y & \xrightarrow{g} & Z &
 \end{array}$$

p_X is the **base change** of g along f , p_Y is the base change of f along g

If f is an epimorphism, so is p_X

More generally, we can also define the pullback of $f_i : X \rightarrow Y_i$

Definition 3.5.2. The **pushout** of $f : Z \rightarrow X$, $g : Z \rightarrow Y$ is $(X \cup_Z Y, \iota_X, \iota_Y)$ satisfying the universal property

$$\begin{array}{ccc}
 Z & \xrightarrow{f} & X \\
 g \downarrow & & \downarrow \iota_X \\
 Y & \xrightarrow{\iota_Y} & X \cup_Z Y \\
 & \searrow \psi & \searrow \exists_1 h \\
 & & W
 \end{array}$$

ι_X is the **cobase change** of g along f , ι_Y is the cobase change of f along g

If f is a monomorphism, so is ι_X

More generally, we can also define the pushout of $f_i : Z \rightarrow X_i$

Proposition 3.5.3. Pushout preserve epimorphisms and isomorphisms and in the category of sets, pushout preserve injection

Pullback preserve monomorphisms and isomorphisms and in the category of sets, pullback preserve surjection

3.6 Filtered category

Definition 3.6.1. A category J is called **filtered** if it is not empty, and for any two objects $j, j' \in J$, there is an object $k \in J$ and morphisms $f : j \rightarrow k$ and $f' : j' \rightarrow k$, for any two morphisms $u, v : i \rightarrow j$, there is an object $k \in J$ and a morphism $w : j \rightarrow k$ such that $w \circ u = w \circ v$

A filtered colimit is the colimit of a functor $F : J \rightarrow \mathcal{C}$ where J is a filtered category, direct limit is a special case of a filtered colimit

The dual notion is called **cofiltered**

3.7 Comma category

Definition 3.7.1. Consider functors $S : \mathcal{A} \rightarrow \mathcal{C}$, $T : \mathcal{B} \rightarrow \mathcal{C}$ (for source and target), define **comma category** $(S \downarrow T)$ with objects (A, B, h) , $A \in \mathcal{A}, B \in \mathcal{B}$ are objects, $h : S(A) \rightarrow T(B)$ is a morphism, and with morphisms $(f, g) : (A, B, h) \rightarrow (A', B', h')$ where $f : A \rightarrow A', g : B \rightarrow B'$ are morphisms such that the following diagram commutes

$$\begin{array}{ccc} S(A) & \xrightarrow{S(f)} & S(A') \\ h \downarrow & & \downarrow h' \\ T(B) & \xrightarrow{T(g)} & T(B') \end{array}$$

Definition 3.7.2. Consider the comma category of $1_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$, $T : * \rightarrow \mathcal{A}$ which we call **slice category**, sometimes denoted as $(\mathcal{A} \downarrow A_*)$ where $A_* = T(*)$, the objects of the slice category are $A \xrightarrow{\pi_A} A_*$ and morphisms are

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \pi_A \searrow & & \swarrow \pi_{A'} \\ & A_* & \end{array}$$

Its dual notion, the comma category of $S : * \rightarrow \mathcal{B}$, $1_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}$, which we call **coslice category**, sometimes denoted as $(B_* \downarrow \mathcal{B})$ where $B_* = S(*)$, the objects of the slice category are $B_* \xrightarrow{\pi_B} B$ and morphisms are

$$\begin{array}{ccc} & B_* & \\ \pi_B \swarrow & & \searrow \pi_{B'} \\ B & \xrightarrow{g} & B' \end{array}$$

The comma category of $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$, $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ which we call **arrow category**, sometimes denoted as $\mathcal{C}^{\rightarrow}$ the objects of the arrow category are just the morphisms (arrows) $A \xrightarrow{f} A'$, and morphisms are

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ h \downarrow & & \downarrow h' \\ B & \xrightarrow{g} & B' \end{array}$$

Definition 3.7.3. A right inverse are called a **section**, a left inverse is called a **retraction**

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \downarrow g \\ & & X \end{array}$$

f is a section of g , g is a retraction of f

3.8 Sheaves

Definition 3.8.1. \mathcal{A} is an abelian category, open subsets of X form a category τ under inclusion. A **presheaf** is a functor $\tau^{op} \xrightarrow{F} \mathcal{A}$, $F(U \hookrightarrow V) = res_{UV}$ are **restriction maps**. A **morphism of presheaves** $F \xrightarrow{\phi} G$ is a natural transformation, i.e. the following diagram commutes

$$\begin{array}{ccc} F(V) & \xrightarrow{res_{UV}} & F(U) \\ \phi_V \downarrow & & \downarrow \phi_U \\ G(V) & \xrightarrow{res_{UV}} & G(U) \end{array}$$

Definition 3.8.2. $U \subseteq X$ is an open subset, F is a presheaf over X , the **restricted presheaf** $F|_U$ is given by $F|_U(V) = F(U \cap V)$

Definition 3.8.3. $X \xrightarrow{f} Y$ is a continuous map, F is a presheaf over X , the **pushforward presheaf** f_*F of F under f is a presheaf over Y given by $f_*F(V) = F(f^{-1}(V))$

Definition 3.8.4. F is a presheaf, $x \in X$, open subsets containing x is full subcategory $\tau(x)$, the **stalk** F_x is the colimit $\varinjlim_{x \in U} F(U)$, elements in F_x are called **germs**, denote the germ of f at x as f_x

Lemma 3.8.5. $B(f, U) = \{f_x | x \in U, f \in F(U)\}$ form a basis on the **étalé space** $|F| = \bigcup F_x$. The **étalé map** $|F| \rightarrow X$, $f_x \mapsto x$ is a local homeomorphism

Sheaf

Definition 3.8.6. Presheaf F is a **sheaf** if

$$F(U) \xrightarrow{res_{U_i, U}} \prod_i F(U_i) \xrightleftharpoons[res_{U_i \cap U_j, U_j}]{res_{U_i \cap U_j, U_i}} \prod_{i,j} F(U_i \cap U_j)$$

Is an equaliser. Equivalently, F satisfying

1. If $U = \bigcup_i U_i$, $f, g \in F(U)$, $f|_{U_i} = g|_{U_i}$, then $f = g$
2. If $U = \bigcup_i U_i$, $f_i \in F(U_i)$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there exists $f \in F(U)$ such that $f|_{U_i} = f_i$, here f has to be unique because of 1

$Sh(X)$ is the category of sheaves over X

Proposition 3.8.7. $F \xrightarrow{\phi} G$ is a monomorphism or an epimorphism iff $F_x \xrightarrow{\phi_x} G_x$ is injective or surjective on each stalk

Definition 3.8.8 (Sheafification). F is a presheaf over X , the sheaf of sections $X \rightarrow |F|$ is the **sheafification**

Definition 3.8.9. The **constant presheaf** \underline{A} given by $\underline{A}(U) = A$, $res_{UV} = 1_A$

F is a **locally constant sheaf** if for any $x \in X$, there exists $U \ni x$ such that $F|_U$ is a constant sheaf. $F : \Pi_1 X \rightarrow \mathcal{A}$ is a functor. The category of locally constant sheaves is equivalent to the category of covering spaces of X

Definition 3.8.10. Functor $\Gamma : Sh(X) \rightarrow \mathcal{A}$, $F \mapsto F(X)$ is a left exact functor, the **sheaf cohomology** is the right derived functor $R^i \Gamma$, i.e. $R^i \Gamma(F) = H^i(X, F)$

Definition 3.8.11. A **ringed space** (X, \mathcal{O}) is a topological space X and a sheaf of rings over X , \mathcal{O} is the **structure sheaf**. (X, \mathcal{O}) is a **locally ringed space** if each stalk is a local ring

Definition 3.8.12. A morphism between ringed spaces is $(X, \mathcal{O}_X) \xrightarrow{(f, \phi)} (Y, \mathcal{O}_Y)$, $X \xrightarrow{f} Y$ is a continuous map, $\mathcal{O}_Y \xrightarrow{\phi} f_* \mathcal{O}_X$ is a morphism of sheaves. A morphism between locally ringed spaces require ϕ is a local ring homomorphism between stalks

Definition 3.8.13. (X, \mathcal{O}) is a ringed space, a sheaf of \mathcal{O} **modules** F is $F(U)$ which are $\mathcal{O}(U)$ modules such that $res_{UV}(rm) = res_{UV}(r)res_{UV}(m)$

3.9 Exponential object

Definition 3.9.1. Y is an object such that all binary products $X \times Y$ exist, the **exponential object** is Z^Y together with morphism $Z^Y \times Y \xrightarrow{\text{eval}} Z$ satisfying universal property

$$\begin{array}{ccc} X \times Y & & \\ \downarrow \exists_1 f \times 1_Y & \searrow f & \\ Z^Y \times Y & \xrightarrow{\text{eval}} & Z \end{array}$$

Proposition 3.9.2. $\text{Hom}(X \times Y, Z) \rightarrow \text{Hom}(X, Z^Y)$ is an adjunction

3.10 Factorization system

Definition 3.10.1. A **factorization system** (\mathbf{E}, \mathbf{M}) for category \mathcal{C} is two classes of morphisms such that

1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. \mathbf{E}, \mathbf{M} are closed under composition and contain all isomorphisms
3. Factorization is functorial, i.e. for any u, v such that $vme = m'e'u$, there exists a unique w such that the following diagram commutes

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{e} & \bullet & \xrightarrow{m} & \bullet \\
 \downarrow u & & \downarrow \exists_1 w & & \downarrow v \\
 \bullet & \xrightarrow{e'} & \bullet & \xrightarrow{m'} & \bullet
 \end{array}$$

Example 3.10.2. \mathbf{E}, \mathbf{M} being epi and mono in \mathbf{Set} is a factorization system

Definition 3.10.3. A **weak factorization system** (\mathbf{E}, \mathbf{M}) for category \mathcal{C} is two classes of morphisms such that

1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. \mathbf{E} are exactly those morphisms having left lifting property for all morphisms in \mathbf{M}
3. \mathbf{M} are exactly those morphisms having right lifting property for all morphisms in \mathbf{E}

3.11 Monoidal category

Definition 3.11.1. A category \mathcal{C} is **monoidal** if there is a tensor product which is a bifunctor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, a tensor unit 1 , **associator**, **left** and **right unitor** which are natural isomorphisms $(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)$, $1 \otimes x \xrightarrow{\lambda_x} x$, $x \otimes 1 \xrightarrow{\rho_x} x$ such that the following diagrams commute

$$\begin{array}{ccc}
 (x \otimes 1) \otimes y & \xrightarrow{\quad} & x \otimes (1 \otimes y) \\
 & \searrow \quad \nearrow & \\
 & x \otimes y &
 \end{array}$$

$$\begin{array}{ccccc}
 & & (w \otimes x) \otimes (y \otimes z) & & \\
 & \nearrow & & \searrow & \\
 ((w \otimes x) \otimes y) \otimes z & & & & w \otimes (x \otimes (y \otimes z)) \\
 \downarrow & & & & \uparrow \\
 (w \otimes (x \otimes y)) \otimes z & \xrightarrow{\quad} & & & w \otimes ((x \otimes y) \otimes z)
 \end{array}$$

\mathcal{C} is **strictly monoidal** if α, λ, ρ are identities

Chapter 4

Group

4.1 Groups

Semigroup

Definition 4.1.1. A **semigroup** is a semicategory with a single object. A **monoid** M is a category with a single object. A **group** is a monoid with all morphisms invertible. A **groupoid** is a category with all morphisms invertible

A G **set** is a functor from G to the category of sets. Equivalently, a **left group action** is $G \times X \rightarrow X$, $(g, x) \mapsto g \cdot x$ satisfying $1 \cdot x = x$, $g \cdot (h \cdot x) = (gh \cdot x)$, a right G action is functor G^{op} to the category of sets. A G **space** is a functor from G to the category of topological spaces. An **equivariant map** of G spaces is a natural transformation $f : X \rightarrow Y$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Definition 4.1.2. The **center** of G is $Z(G) = \{z \in G | gz = zg, \forall g \in G\}$. $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$

Definition 4.1.3. **Inner automorphism group** of G is $Inn(G) \leq Aut(G)$ consists of conjugations $x \mapsto gxg^{-1}$. **Outer automorphism group** is $Out(G) = Aut(G)/Inn(G)$

Definition 4.1.4. G is **perfect** if $[G, G] = G$

Definition 4.1.5. A group action is **trivial** if $g \cdot x = x$. A group action is **free** if $g \cdot x = h \cdot x$ for some x implies $g = h$, or equivalently. A group action is **transitive** if $G \cdot x = X$. A free and transitive action is also called **regular**. A group action is **faithful** if $\forall x \in X, G \cdot x \neq x$

A **homogeneous space** is a G space with G acting transitively

Torsor

Definition 4.1.6. A **torsor** P is a set with an action $G \times P \rightarrow P, (g, p) \mapsto gp$ such that this action is free and transitive

When chosen an element $e \in P$, automatically you are giving it a group structure with e as identity, notice $e : G \rightarrow P, g \mapsto ge$ is a bijection, suppose $g_p e = p$, then $g_e = 1$, the group structure could be given as follows, $P \times P \rightarrow P, (p, q) \mapsto g_p g_q e$, and the inverse of p is $g_p^{-1} e$. i.e. a torsor is a group forgetting its identity

Definition 4.1.7. Let X, Y be G sets, then we can give Y^X a G set structure by $(gf)(x) = gf(g^{-1}x)$, it is easy to check that a map $f : X \rightarrow Y$ is equivariant iff f is fixed under the G action

Specially, if G acts on Y trivially, we have $(gf)(x) = f(g^{-1}x)$

Definition 4.1.8. $H \leq G$ is a subgroup, $N \trianglelefteq G$ is a normal subgroup, $G = NH = N \rtimes H$ or $HN = H \rtimes N$ is the **semidirect product** and every $g \in G$ can be uniquely written as nh or hn

$$(nh)(n'h') = n(hn'h^{-1})hh' \text{ or } (hn)(h'n') = hh'(h'^{-1}n'h')n'$$

Theorem 4.1.9 (Jordan-Hölder theorem). *Composition series is unique up to reordering*

Definition 4.1.10. A group representation V is a $\mathbb{F}G$ module

Definition 4.1.11. Let (ρ, V) be a group representation of finite group G , $W \leq V$ is called G invariant if $GW \subseteq W$, namely, W is $\mathbb{F}G$ submodule, then we get a subrepresentation on $(\rho|_W, W)$ of G , with $\rho|_W(g) := \rho(g)|_W$, if the only G invariant subspace of W are 0 and V , we say ρ is irreducible

Definition 4.1.12. A group representation (ρ, V) is completely reducible(semisimple) if $V = V_1 \oplus \cdots \oplus V_n$, where $(\rho|_{V_i}, V_i)$ are irreducible subrepresentations, namely, V is the direct sum of simple $\mathbb{F}G$ modules

Definition 4.1.13. Let V be a complex vector space with Hermitian form $(,)$ finite group representation (ρ, V) of G is called unitary if $(\rho(g)v, \rho(g)w) = (v, w), \forall g \in G, v, w \in V$

Proposition 4.1.14. Let (ρ, V) be a unitary representation of G , (ρ, V) is completely reducible

Proof. W is G invariant $\Rightarrow W^\perp$ is G invariant □

Proposition 4.1.15. Let V be a complex vector space, (ρ, V) is a representation, then there exists a positive definite Hermitian form on V such that (ρ, V) become a unitary representation

Corollary 4.1.16. Let V be a complex vector space, a finite group representation (ρ, V) is always completely reducible

Definition 4.1.17. Let (ρ, V) be a representation of G , then the dual representation (ρ^*, V^*) is defined as $\rho^*(g) := \rho(g)^{-T}$

Definition 4.1.18. $H \subseteq G$, (V, π) is a H representation, the **induced representation** is $\mathbb{F}G \otimes_{\mathbb{F}H} V$. Suppose g_1, \dots, g_n is a set of representatives of left cosets in G/H , v_j is a basis for V , then $g_i \otimes v_j$ form a basis for $\mathbb{F}G \otimes_{\mathbb{F}H} V$. If $gg_i = g_j h$ for some $h \in H$, then $g(g_i \otimes v) = gg_i \otimes v = g_j h \otimes v = g_j \otimes hv$

Definition 4.1.19. If X is a right G space, Y is a left G space, $X \times_G Y$ is $X \times Y / \sim$, $(xg, y) \sim (x, gy)$. Equivalently, $X \times_G Y = X \times Y / G$ with left action $g(x, y) = (xg^{-1}, gy)$ or right action $(x, y)g = (xg, g^{-1}y)$

If X, Y are left G spaces, $X \times_G Y = X \times Y / G$ with left action $g(x, y) = (gx, gy)$

If X, Y are right G spaces, $X \times_G Y = X \times Y / G$ with right action $(x, y)g = (xg, yg)$

Sylow's theorem

Theorem 4.1.20 (Sylow's theorem). p is a prime, a p -group is a group consists of elements of order p -th power, a maximal one is a **Sylow p -subgroup** P of G which always exists by Zorn's Lemma 1.0.2

1. If $|G| = p^n m$, $p \nmid m$, then $|P| = p^n$

2. Any subgroup of a Sylow p subgroup is subconjugate to some other Sylow p subgroup

3. $n_p = [G : N_G(P)]$, if the conjugacy class of P is of order $n_p < \infty$, then $n_p \equiv 1 \pmod{p}$

4.2 Permutation group

Definition 4.2.1. Denote $[n] = \{1, \dots, n\}$, a **permutation** $[n] \xrightarrow{\sigma} [n]$ is a bijection, equivalently write $\sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}$. $S_n = \text{Aut}([n])$ is the permutation group

The **length** of $\sigma \in S_n$ is $|\sigma|$ is the number of $\sigma(i) < \sigma(j)$ while $i > j$

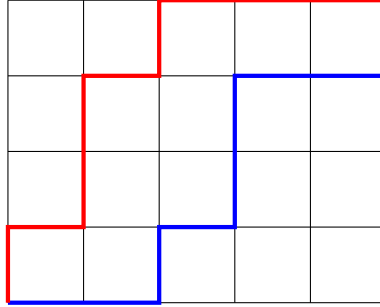
Definition for shuffle

Definition 4.2.2. Just like shuffling a deck of cards, $\sigma \in S_n$ is a (p_1, \dots, p_m) -**shuffle**, $p_1 + \dots + p_m = n$ if $\sigma(i) < \sigma(i+1)$ for $p_1 + \dots + p_k + 1 \leq i \leq p_1 + \dots + p_{k+1}$

A (p, q) shuffle can be represented by a path going only right or up from the lower left corner to the upper right corner in a $(p+1) \times (q+1)$ grid, $|\sigma|$ happen to be the number squares under the path

Example 4.2.3 $((5, 4)$ shuffles in S_9). The red one is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 7 & 8 & 9 & 1 & 3 & 4 & 6 \end{pmatrix}$. The

blue one is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 7 & 8 & 3 & 5 & 6 & 9 \end{pmatrix}$



4.3 Coxeter group

Definition 4.3.1. A **Coxeter group** G are groups with presentations

$$\langle r_1, \dots, r_n | (r_i r_j)^{m_{ij}} = 1 \rangle$$

$m_{ii} = 1$, so we should think of r_i 's as reflections, for $i \neq j$, $m_{ij} = 2$ means r_i, r_j commutes, we should think of r_i, r_j are independent, most commonly, $m_{ij} \geq 3$, if $m_{ij} = \infty$. $M = (m_{ij})$ is the **Coxeter matrix**, the corresponding **Schläfli matrix** is $C_{ij} = -2 \cos(\pi/m_{ij})$

Definition 4.3.2. The **Coxeter element** is the product of all simple reflections s_i 's, different order in multiplication give conjugation, the **Coxeter number** is the order of the Coxeter element

Definition 4.3.3. The **Coxeter diagram** consists of n nodes for r_i 's, there is an edge numbered m_{ij} between i and j if $m_{ij} \geq 3$

4.4 Grothendieck group

Grothendieck group

Definition 4.4.1 (Grothendieck group). The **Grothendieck group** of a commutative monoid M is the abelian group K and $i : M \rightarrow K$ satisfying universal property

$$\begin{array}{ccc} M & & \\ \downarrow i & \searrow f & \\ K & \xrightarrow[\exists_1 g]{} & A \end{array}$$

For any abelian group A

Construction 4.4.2.

$$K = M \times M / \sim, M \xrightarrow{i} K, m \mapsto [m - 0]$$

Abelian group $M \times M \cong \{[m - n] | m, n \in M\}$ is the set of formal differences with addition

$$[m_1 - n_1] + [m_2 - n_2] = [(m_1 + m_2) - (n_1 + n_2)]$$

$[0 - 0]$ is the identity and $[n - m]$ is the inverse to $[m - n]$

$$[m_1 - n_1] \sim [m_2 - n_2] \text{ if } m_1 + n_2 + m = m_2 + n_1 + m \text{ for some } m \in M$$

Construction 4.4.3 (Grothendieck completion).

$$K = F(M) / \sim, m +' n \sim (m + n)$$

$F(M)$ is the free abelian group generated by M with addition $+'$

Definition 4.4.4. The **Grothendieck group** of a semigroup S is a group K and $i : S \rightarrow K$ satisfying the universal property

$$\begin{array}{ccc} S & & \\ \downarrow i & \searrow f & \\ K & \xrightarrow[\exists_1 g]{} & G \end{array}$$

Where G is a group

Construction 4.4.5.

$$K = F(S) / \sim, m *' n \sim (m * n)$$

$F(S)$ is the free group generated by S with multiplication $*'$

Chapter 5

Ring

5.1 Rings

Definition 5.1.1 (Rings). R is an abelian group with addition $+$ and additive identity 0 , a monoid with multiplication \cdot and multiplicative identity 1 , and distributive, $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

Definition 5.1.2. Ring R is **commutative** if $ab = ba$

Definition 5.1.3. u is a **unit** if there exists $v \in R$ such that $uv = vu = 1$. The set of units R^\times is a multiplicative group

Definition 5.1.4. A **semiring** or **rig** is a ring without negatives

Definition 5.1.5. A **rng** is ring without identity

Definition 5.1.6 (Whitehead group). The **Whitehead group** of ring R is an abelian group $K_1(R)$ satisfying universal property

$$\begin{array}{ccc} GL(R) & & \\ \pi \downarrow & \searrow & \\ K_1(R) & \xrightarrow[\exists_1]{} & A \end{array}$$

For any abelian group A

Construction 5.1.7. $K_1(R) = GL(R)/[GL(R), GL(R)]$ is the abelianization of $GL(R)$

Definition 5.1.8. If R is commutative, $SL(R)$ is the kernel of $GL(R) \xrightarrow{\det} R^\times$, $SK_1(R)$ is the kernel of $K_1(R) \xrightarrow{\det} R^\times$, $GL(R) \cong SL(R) \rtimes R^\times$, $K_1(R) \cong SK_1(R) \oplus R^\times$. $K_1(F) = F^\times$

Lemma 5.1.9. Since $GL(R_1 \times R_2) = GL(R_1) \times GL(R_2)$, $K_1(R_1 \times R_2) = K_1(R_1) \oplus K_1(R_2)$

5.2 Commutative rings

Definition 5.2.1. The **determinant** of a matrix is

Definition 5.2.2. $I, J \leq R$ are ideals, the **ideal quotient** $(I : J) = \{r \in R \mid rJ \subseteq I\}$ is also an ideal

Definition 5.2.3. $S \subseteq R$ is **multiplicative closed**, the localization $S^{-1}R$ of R with respect to S is $R \times S / \sim$, $(r, s) \sim (r', s')$ iff there exists $t \in S$ such that $t(rs' - sr) = 0$. $S^{-1}R$ has the universal property that for any $f : R \rightarrow T$ such that maps S to units, then there exists a unique $g : S^{-1}R \rightarrow T$ such that $gi = f$

$$\begin{array}{ccc} R & \xrightarrow{i} & S^{-1}R \\ & \searrow f & \downarrow \exists_1 g \\ & & T \end{array}$$

Definition 5.2.4. Given a ring R and a proper ideal I , we can define an **associated graded ring** $gr_I R := \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$, if M is a left R -module, we can define **associated graded module**

$$gr_I M := \bigoplus_{n=0}^{\infty} I^n M / I^{n+1} M$$

Definition 5.2.5. R is a **local ring** if it has a unique maximal ideal \mathfrak{m} . The **residue field** is $k = R/\mathfrak{m}$

Definition 5.2.6. R is a **semilocal ring** if it has only finitely many maximal ideal

Proposition 5.2.7. Let R be a UFD, f a prime element, then $ht(f) = 1$

Proof. Suppose there exists prime ideal P such that $0 \subsetneq P \subsetneq (f)$, then we can find a prime element in $g \in P$, thus we have $0 \subsetneq (g) \subseteq P \subsetneq (f)$, but then $g = fh$ for some h , but since f is prime, thus $(f) = (g)$ which is a contradiction, such a prime element exists since we can pick any element $0 \neq q = q_1 \cdots q_m \in P$ where q_i 's are prime, but then at least one of them has to be in P \square

Theorem 5.2.8. Let $A \subseteq B$ be finitely generated k -algebras, and A, B are both domains, $0 \neq b \in B \Rightarrow \exists 0 \neq a \in A$ such that for any k -algebra homomorphism $\alpha : A \rightarrow k$ with $\alpha(a) \neq 0$ can be extended to k -algebra homomorphism $\beta : B \rightarrow k$ with $\beta(b) \neq 0$

Definition 5.2.9. Suppose R is a commutative ring with identity, a prime element $p \in R$ is an element which is nonzero nor a unit and $p \mid fg \Rightarrow p \mid f$ or $p \mid g$

Definition 5.2.10. A **graded ring** R is a ring such that $R = \bigoplus_i R_i$ is a direct sum of abelian groups and $R_i R_j \subseteq R_{i+j}$

An ideal is called a **homogeneous ideal** if it consists of only homogeneous elements

Chinese remainder theorem

Theorem 5.2.11 (Chinese remainder theorem). Let R be a commutative ring, and $I_1, \dots, I_n \leq R$ be pairwise coprime ideals, then $R \cong R/I_1 \times \cdots \times R/I_n, r \mapsto (r \bmod I_1, \dots, r \bmod I_n)$

Definition 5.2.12. An **integral domain** is a commutative ring R such that (0) is a prime ideal. Equivalently, $rs \in R \Rightarrow r \in R$ or $s \in R$

Definition 5.2.13. Suppose R is a domain, K is the field of fractions, a **fractional ideal** is an R submodule $I \leq K$ such that $rI \subseteq R$ for some nonzero $r \in R$. I is invertible if $IJ = R$ for some other fractional ideal J

Definition 5.2.14. A **Dedekind domain** is an integral domain such that every proper ideal is a product of prime ideal

Definition 5.2.15. A **discrete valuation ring (DVR)** is a PID with a unique nonzero prime ideal

Definition 5.2.16. A local ring homomorphism $\phi : R \rightarrow S$ between local rings is such that $\phi(m_R) \subseteq m_S$

Definition 5.2.17. $\bigwedge^k A : \bigwedge^k V \rightarrow \bigwedge^k W$ is defined by $\bigwedge^k A(v_1 \wedge \cdots \wedge v_n) := A(v_1) \wedge \cdots \wedge A(v_n)$

Definition 5.2.18. A ring A is an R **algebra** is a ring homomorphism $R \xrightarrow{\phi} A$, $ra = \phi(r)a$
 A is **finite** or ϕ is **finite** if A is a finitely generated R module
 ϕ is of **finite type** if A is **finitely generated** R algebra

Definition 5.2.19. For $p \in \text{Spec} A$, $q \in \text{Spec} B$, $A \subseteq B$, p **lies under** q or q **lies over** p if $q \cap A = p$ $A \subseteq B$ satisfies **lying over property** if every $p \in \text{Spec} A$ lies under some $q \in \text{Spec} B$
 $A \subseteq B$ satisfies the **incomparability property** if different prime ideals q, q' both lie over p , then they don't contain each other

$A \subseteq B$ satisfies **going up property** if for any chain of prime ideals $p_1 \subseteq \cdots \subseteq p_n$, $q_1 \subseteq \cdots \subseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \subseteq \cdots \subseteq q_n$ with q_i lies over p_i

$A \subseteq B$ satisfies **going down property** if for any chain of prime ideals $p_1 \supseteq \cdots \supseteq p_n$, $q_1 \supseteq \cdots \supseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \supseteq \cdots \supseteq q_n$ with q_i lies over p_i

Definition 5.2.20. $R \subseteq S$ are commutative rings, $a \in S$ is **integral** over R if it is a root of some monic polynomial in $R[x]$. The **integral closure** of R in S are the integral elements of S

Going up and Going down theorems

Theorem 5.2.21. B is integral over A , then $A \subseteq B$ satisfies going up property and incomparability property

Definition 5.2.22. The **Krull dimension** of a ring R is $\dim R = \sup_d p_0 \subsetneq \cdots \subsetneq p_d$, p_i are prime ideals

Proposition 5.2.23. A is a integral domain, finitely generated over some subfield k , then $\dim A = \text{trdeg}(\text{Frac} A/k)$

Chapter 6

Module

6.1 Modules

Module

Definition 6.1.1. R is a ring, a **left R module** M is an abelian group with left group action $R \times M \rightarrow M$ such that

- $1m = m$
- $r(m + n) = rm + rn$
- $(r + s)m = rm + sm$
- $(rs)m = r(sm)$

Definition 6.1.2. $X \subseteq M$ is **linearly independent** if for $x_1, \dots, x_n \in X$

$$r_1x_1 + \dots + r_nx_n = 0 \Rightarrow r_i = 0$$

Definition 6.1.3. The submodule generated by $X \subseteq M$ is **Span X** , the **span** of X

Definition 6.1.4. $X \subseteq M$ is a **basis** of M if X is a linearly independent spanning set. M is a free R module on X if X is a basis of M

Note. There is no well-defined dimension for free R modules in general, exemplified in Example 39.0.4

Definition 6.1.5. M is an right R -module, N is a left R -module and G is an abelian group, a map $\phi : M \times N \rightarrow G$ is called an R balanced product if ϕ is bilinear and $\phi(mr, n) = \phi(m, rn)$, we can define tensor product $M \otimes_R N$ is an abelian group satisfying the universal property

$$\begin{array}{ccc} M \times N & \xrightarrow{\otimes} & M \otimes_R N \\ & \searrow f & \downarrow \exists_1 \tilde{f} \\ & & G \end{array}$$

Here f is an R balanced product and \tilde{f} is an abelian group homomorphism
A concrete construction would be $F(M \times N) / \sim$, where $(m + m', n) \sim (m, n) + (m', n)$, $(m, n + n') \sim (m, n) + (m, n')$, $(mr, n) \sim (m, rn)$

Remark 6.1.6. $r(m \otimes n) = mr \otimes n = m \otimes rn$ is called associativity

Definition 6.1.7. Module M is **semisimple** or **completely reducible** if it is the direct sum of simple submodules. Ring R is semisimple if it is a semisimple R module

Theorem 6.1.8. *Tensor product is right exact for R modules*

Definition 6.1.9. Let R be a commutative ring, $S \subseteq R$ is **multiplicatively closed** if $1 = s^0 \in S$ and $rs \in S, \forall r, s \in S$, we can define localization $S^{-1}R$ satisfying universal property

$$\begin{array}{ccc} R & & \\ \downarrow j & \searrow f & \\ S^{-1}R & \xrightarrow[\exists_1 g]{} & T \end{array}$$

Here $f(S) \subseteq T^\times$

Concrete construction: $S^{-1}R := R \times S / \sim$, $(r, s) \sim (r', s')$ if there exists $t \in S$ such that $t(rs' - r's) = 0$

Let M be an R module, we can define localization, $S^{-1}M := M \times S / \sim$, $(m, s) \sim (m', s')$ if there exists $t \in S$ such that $t(sm' - s'm) = 0$

Proof. Suppose $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is a short exact sequence, then it is obvious that $A \otimes D \xrightarrow{f \otimes 1_D} B \otimes D \xrightarrow{g \otimes 1_D} C \otimes D \rightarrow 0$ is a complex and $g \otimes 1_D$ is surjective, now define $\phi : B \otimes D / \ker g \otimes 1_D \rightarrow A \otimes D$, $b \otimes d \mapsto a \otimes d$, where a is the unique element in A such that $g(b - f(a)) = 0$ \square

Definition 6.1.10. Let P_i, A, D be R modules, $\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \rightarrow 0$ be a projective resolution, then we have $0 \rightarrow \text{Hom}_R(A, D) \xrightarrow{\epsilon} \text{Hom}_R(P_0, D) \xrightarrow{d_1} \text{Hom}_R(P_1, D) \xrightarrow{d_2} \text{Hom}_R(P_2, D) \cdots$, define $\text{Ext}_R^n(A, D)$ to be the n -th cohomology group of $0 \rightarrow \text{Hom}_R(P_0, D) \xrightarrow{d_1} \text{Hom}_R(P_1, D) \xrightarrow{d_2} \text{Hom}_R(P_2, D) \cdots$, note that $\text{Ext}_R^0(A, D) \cong \text{Hom}_R(A, D)$

Definition 6.1.11. Let P_i, B, D be R modules, $\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \rightarrow 0$ be a projective resolution, then we have $\cdots D \otimes_R P_2 \xrightarrow{1 \otimes d_2} D \otimes_R P_1 \xrightarrow{1 \otimes d_1} D \otimes_R P_0 \xrightarrow{1 \otimes \epsilon} D \otimes_R B \rightarrow 0$, define $\text{Tor}_n^R(D, B)$ to be the n -th homology group of $\cdots D \otimes_R P_2 \xrightarrow{1 \otimes d_2} D \otimes_R P_1 \xrightarrow{1 \otimes d_1} D \otimes_R P_0 \rightarrow 0$, note that $\text{Tor}_0^R(D, B) \cong D \otimes_R B$

Schur's lemma

Lemma 6.1.12 (Schur's Lemma). R is a ring, M, N are nonzero simple R modules. A homomorphism $\varphi : M \rightarrow N$ is either 0 or an isomorphism. In particular, $\text{End}_R(M)$ is a division ring. Moreover, if $\overline{F} = F$, $\text{Hom}_F(M, N) = \{\lambda \varphi | \lambda \in F\}$ where $M \xrightarrow{\varphi} N$ is an isomorphism (all isomorphisms are scalar multiple of each other), in particular, $\text{Hom}_F(M, M) = \{\lambda 1_M | \lambda \in F\}$

Theorem 6.1.13 (Maschke's theorem). G is a finite group, F is a field, $\text{char } F \nmid |G|$, then FG is a semisimple ring

Theorem 6.1.14 (Artin-Wedderburn theorem). $R = V_1 \oplus \cdots \oplus V_r$ is a semisimple ring, by Schur's lemma 6.1.12, $D_i = \text{End}_R(V_i)$ are division rings, then

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $n_i = \dim_{D_i}(V_i)$. $\sum_{i=1}^r n_i^2 = |G|$, r is the number of conjugacy classes in G

Definition 6.1.15. F is a field, G is a group, the representation ring $R_F(G)$ is the completion of the set of isomorphic classes of representations

Definition 6.1.16. The symmetric k algebra is $S^k(V) \subseteq T^k(V)$ consists of k tensors symmetric under the permutation of S_k . The exterior k algebra is $\bigwedge^k(V) \subseteq T^k(V)$ consists of k tensors antisymmetric under the permutation of S_k . We have

$$\text{Sym} : T^k(V) \rightarrow S^k(V), a_1 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} \sum_{\sigma} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$$

and

$$\text{Alt} : T^k(V) \rightarrow S^k(V), a_1 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} \sum_{\sigma} (-1)^{\text{sgn } \sigma} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$$

For $\alpha \in \bigwedge^k(V)$, $\beta \in \bigwedge^l(V)$, define

$$\alpha\beta = \alpha \odot \beta = \text{Sym}(\alpha \otimes \beta)$$

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \text{Alt}(\alpha \otimes \beta)$$

Which corresponds to determinants

Chapter 7

Field

7.1 Fields

Definition 7.1.1. A **division ring** R is a nonzero ring such that $\mathbb{F}^\times = \mathbb{F} - \{0\}$

A **field** \mathbb{F} is a nonzero commutative ring such that $\mathbb{F}^\times = \mathbb{F} - \{0\}$

Definition 7.1.2. A **character** of G is a group homomorphism $G \rightarrow \mathbb{F}^\times$, and a **cocharacter** is a group homomorphism $\mathbb{F}^\times \rightarrow G$

Lemma 7.1.3. Characters of G , denoted as $ch(G)$ are linear independent on $\mathbb{F}[G]$

Proof. Suppose not, we can find $c_1\chi_1 + \cdots + c_m\chi_m = 0, c_i \in \mathbb{F}^\times$, with minimal terms, since $\chi_1 \neq \chi_m$, there exists $g_0 \in G$ such that $\chi_1(g_0) \neq \chi_m(g_0)$, on the other hand we have $0 = c_1\chi_1(g) + \cdots + c_m\chi_m(g) = c_1\chi_1(g)\chi_m(g_0) + \cdots + c_m\chi_m(g)\chi_m(g_0), \forall g \in G$ and $0 = c_1\chi_1(gg_0) + \cdots + c_m\chi_m(gg_0) = c_1\chi_1(g)\chi_1(g_0) + \cdots + c_m\chi_m(g)\chi_m(g_0), \forall g \in G$, subtract to get $0 = c_1(\chi_m(g_0) - \chi_1(g_0))\chi_1(g) + \cdots + c_{m-1}(\chi_m(g_0) - \chi_{m-1}(g_0))\chi_{m-1}(g)$ with fewer terms which is a contradiction \square

Definition 7.1.4. Suppose E/F is a field extension, we can define **field trace** $Tr_{E/F}(\alpha)$ to be the trace of α as a linear transformation and **field norm** $N_{E/F}(\alpha)$ to be the determinant of α

Definition 7.1.5. \mathbb{F} is a **perfect field** if $\mathbb{F}^p = \mathbb{F}$ if $\text{char}\mathbb{F} = p \neq 0$ or $\text{char}\mathbb{F} = 0$

Definition 7.1.6. E/F is a field extension, $\alpha \in E$ is algebraic over F if α is a zero of some polynomial in $F[x]$. The **algebraic closure** of F in E are the algebraic elements of E

Theorem 7.1.7 (Emil Artin). *Any field F has an algebraically closed extension*

7.2 Number field

Definition 7.2.1. A **number field** is $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{C}$. An **algebraic number field** is $\mathbb{Q} \subseteq \mathbb{F} \subseteq \overline{\mathbb{Q}}$

Definition 7.2.2. E, F are algebraic number fields of finite degree, E/F is finite separable, A, B are corresponding ring of integers, $\{\beta_1, \dots, \beta_n\}$ is an integral basis of B over A . The **discriminant** of E/F with respect to $\{\beta_1, \dots, \beta_n\}$ is $D_{E/F}(\beta_1, \dots, \beta_n) = \det(\text{Tr}(\beta_i \beta_j))$

$$\begin{array}{ccc} B & \hookrightarrow & E \\ \uparrow & & \uparrow \\ A & \hookrightarrow & F \end{array}$$

Lemma 7.2.3. D_K is well defined in $\frac{A}{(A^\times)^2}$

Definition 7.2.4. E, F are algebraic number fields of finite degree, E/F is finite separable, A, B are corresponding ring of integers which are Dedekind domains

$$\begin{array}{ccc} B & \hookrightarrow & E \\ \uparrow & & \uparrow \\ A & \hookrightarrow & F \end{array}$$

$pB = q_1^{e_1} \cdots q_r^{e_r}$ with $e_i > 0$. p is **ramified** if $e_i > 1$ for some i , otherwise unramified. p is **inert** if $r = e = 1$. p **totally split** if $e_i = f_i = 1$

$B/pB \cong \prod_{i=1}^r B/q_i^{e_i}$, $f_i = [k_{q_i} : k_p]$, $[E : F] = \dim_{k_p}(B/pB) = \sum_{i=1}^r e_i f_i$

If E/F is Galois, $G = \text{Aut}(E/F)$ acts transitively on $\{q_1, \dots, q_r\}$, then $n = \sum_{i=1}^r e_i f_i = r e f$

Proof. $B \cong A^n$, $B/pB \cong A^n/pA^n \cong (A/p)^n \cong k_p^n$ □

Example 7.2.5. $2\mathbb{Z}[i] = (1+i)^2$ is ramified, $3\mathbb{Z}[i]$ is inert, $5\mathbb{Z}[i] = (2+i)(2-i)$ totally split

$$\begin{array}{ccc} \mathbb{Z}[i] & \hookrightarrow & \mathbb{Q}[i] \\ \uparrow & & \uparrow \\ \mathbb{Z} & \hookrightarrow & \mathbb{Q} \end{array}$$

Theorem 7.2.6. p ramifies in $O_K \Leftrightarrow p \mid \text{disc}(O_K/\mathbb{Z})$

$$\begin{array}{ccc} O_K & \hookrightarrow & K \\ \uparrow & & \uparrow \\ \mathbb{Z} & \hookrightarrow & \mathbb{Q} \end{array}$$

Proof. $pO_K = \beta_1^{e_1} \cdots \beta_r^{e_r}$, $O_K/pO_K \cong O_K/\beta_i^{e_i}$ is an isomorphism of \mathbb{F}_p algebras. $d_i = \text{disc}((O_K/\beta_i^{e_i})/\mathbb{F}_p)$, $d = \text{disc}((O_K/pO_K)/\mathbb{F}_p)$, thus $d = d_1 \cdots d_r$, since discriminant is functorial, $D = \det(\text{Tr}_{O_K/\mathbb{Z}}()) \mapsto d$, $p \mid D \Leftrightarrow d = 0 \Leftrightarrow d_i = 0$ for some i □

Chapter 8

Linear algebra

8.1 Vector spaces

Definition 8.1.1. A **vector space** V over field F is an F module

Definition 8.1.2. An **affine space** is a vector space witho

Definition 8.1.3. $C \subseteq V$ is **convex** if $tC + (1-t)C \subseteq C$ for $0 \leq t \leq 1$. C is **strictly convex** if $tC + (1-t)C \subsetneq C$ for $0 < t < 1$

Definition 8.1.4. V is a vector space of dimension n , a **q flag** is

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_q = V$$

A complete flag is an n flag

$GL(n, F)$ acts transitively on flags

Lemma 8.1.5. $GL(n, F)$ acts transitively on flags

8.2 Matrices

Definition 8.2.1. E_{ij} is the matrix with 1 on the (i, j) -th entry and otherwise zeros, then $E_{ij}E_{kl} = \delta_{jk}E_{il}$

Elementary matrices are single row operations, i.e.

$$e_{ij}(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & r & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with r on the (i, j) -th entry

$$s_{ij} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & & \ddots & \\ & 1 & & 0 & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

and

$$d_i(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & r & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

We have $e_{ij}(-r) = e_{ij}(r)^{-1}$ and

$$\begin{aligned} e_{ij}(r)e_{ij}(s) &= e_{ij}(r+s) \\ [e_{ij}(r), e_{kl}(s)] &= I + rs\delta_{jk}E_{il} - sr\delta_{li}E_{kj} + \delta_{jk}\delta_{li}(srsE_{kl} - rsrE_{ij}) + rsrs\delta_{jk}\delta_{li}E_{il} \\ &= \begin{cases} I & i \neq l, j \neq k \\ e_{il}(rs) & i = l, j \neq k \\ e_{kj}(-sr) & i \neq l, j = k \\ * & i = l, j = k \end{cases} \end{aligned}$$

Steinberg relations

Definition 8.2.2. $E(n, R) \subseteq SL(n, R)$ is the subgroup generated by elementary matrices of determinant 1. $E(R) = \bigcup E(n, R)$

Lemma 8.2.3. $SL(n, F) = E(n, F)$

$E(n, R)$ is perfect

Lemma 8.2.4. $[E(n, R), E(n, R)] = E(n, R)$ if $n \geq 3$

Proof. For distinct i, j, k , $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$ □

$[GL, GL] = E$

Theorem 8.2.5 (Whitehead). $[GL(R), GL(R)] = E(R)$, hence $K_1(R) = GL(R)/E(R)$

Proof. Since

$$e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

We know

$$[g, h] = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} h & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & \\ & hg \end{pmatrix} \in E(R)$$

□

Definition 8.2.6. The **Kronecker product** of matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ a_{np} & \cdots & b_{np} \end{pmatrix}$$

is

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

8.3 Eigenspace decomposition

Proposition 8.3.1. $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is a linear operator, and $V = \bigoplus_i V_i$, where V_i are T invariant spaces, denote $T|_{V_i}$ as $T - i$, then $ch_T(t) = \prod_i ch_{T_i}(t)$, and $m_T(t) = lcm_i m_{T_i}(t)$

Definition 8.3.2. $T \in \text{Hom}_{\mathbb{F}}(V, V)$. $\lambda \in F$ is an **eigenvalue** if $Tv = \lambda v$ has nontrivial solution, $v \in V$ is a **generalized eigenvector** of rank m of T corresponding to eigenvalue λ if $(T - \lambda 1_V)^m v = 0$, $(T - \lambda 1_V)^{m-1} v \neq 0$ for some $m \geq 1$, and let V_λ be the subspace of all such generalized eigenvectors, called **generalized eigenspace**, notice if V is of finite dimensional, then $V_\lambda = \ker(T - \lambda 1_V)^m$ for some m with m being smallest, suppose $\dim V_\lambda = d$, then the characteristic polynomial of $T|_{V_\lambda}$ is $(t - \lambda)^d$, and the minimal polynomial of $T|_{V_\lambda}$ is $(t - \lambda)^m$

Generalized eigenspace decomposition

Proposition 8.3.3. $\bar{F} = F$, finitely dimensional F vector space V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_{\lambda} V_\lambda$

Definition 8.3.4. $T \in \text{Hom}_{\mathbb{F}}(V, V)$ give V an $F[x]$ module with $x \cdot v = Tv$, $W \leq V$ be a subspace, W is called **T invariant** if $TW \subseteq W$, or rather W is an $F[x]$ submodule

Definition 8.3.5. An linear operator $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is called **semisimple** if V is a semisimple $F[x]$ submodule

Proposition 8.3.6. Let $T \in \text{Hom}_{\mathbb{F}}(V, V)$ be a linear operator with $\bar{F} = F$, then T is semisimple $\Leftrightarrow T$ is diagonalizable

Proof. Since $\bar{F} = F$ and T is semisimple, V can be decomposed as a direct sum of eigenspaces of T , thus T is diagonalizable, conversely, if T is diagonalizable, and $TW \subseteq W$, let V_λ be the eigenspaces of T , denote $W_\lambda = W \cap V_\lambda$, and $W' = \bigoplus_{\lambda} W'_\lambda$, since $T|_{V_\lambda} = \lambda 1_{V_\lambda}$, we can find $W'_\lambda \leq V_\lambda$ such that $V_\lambda = W_\lambda \oplus W'_\lambda$, and of course $TW'_\lambda \subseteq W'_\lambda$ which implies $TW' \subseteq W'$, then we have $V = \bigoplus_{\lambda} V_\lambda = \bigoplus_{\lambda} W_\lambda \oplus W'_\lambda = \bigoplus_{\lambda} W_\lambda \oplus \bigoplus_{\lambda} W'_\lambda = W \oplus W'$ \square

Definition 8.3.7. An linear operator $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is called nilpotent if $T^k = 0$ for some k , T is called unipotent if $T - 1_V$ is nilpotent

Jordan-Chevalley decomposition

Definition 8.3.8 (Jordan-Chevalley decomposition). **Jordan-Chevalley decomposition** of a linear operator $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is $T = T_s + T_n$, where T_s is semisimple, T_n is nilpotent and $[T_s, T_n] = 0$

Existence of Jordan-Chevalley decomposition

Theorem 8.3.9. If V is a finite dimensional F vector space with F being a perfect field, and $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is a linear operator, then Jordan-Chevalley decomposition always exist, additionally, there exist polynomials $p(t), q(t)$ with no constant terms and $T_s = p(T), T_n = q(T)$, moreover, the decomposition is unique

Proof. First consider $\bar{F} = F$, by Proposition 8.3.3, V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_i V_{\lambda_i}$, where $V_{\lambda_i} = \ker(T - \lambda_i 1_V)^{m_i}$ with being m_i being the least and $\dim V_{\lambda_i} = d_i$, define $T_s \in \text{Hom}_{\mathbb{F}}(V, V)$ such that $T_s|_{V_{\lambda_i}} = \lambda_i 1_{V_{\lambda_i}}$ and $T_n = T - T_s$, thus T_s is diagonalizable(semisimple), T_n is nilpotent, $ch_T(t) = \prod_i (t - \lambda_i)^{d_i}$, by Theorem 5.2.11, there exists polynomial $p(t)$ such that $p(t) \equiv 0 \pmod{t}$, $p(t) \equiv \lambda_i \pmod{(t - \lambda_i)^{d_i}}$, and let $q(t) = t - p(t)$, then p, q doesn't have constant terms and $T_s = p(T), T_n = q(T)$. For uniqueness, suppose $T = T_s + T_n = T'_s + T'_n$ are two such decompositions, then $T_s - T'_s = T'_n - T_n$ will be nilpotent which implies $T_s - T'_s = 0$ \square

Chapter 9

Lie algebra

9.1 Non-associative algebra

Definition 9.1.1. A nonassociative \mathbb{F} algebra A is an \mathbb{F} vector space with multiplication \cdot that is distributive $(a + b) \cdot c = a \cdot c + b \cdot c$, $a \cdot (b + c) = a \cdot b + a \cdot c$. A is **unital** if $1 \in A$, A is **symmetric** if $xy = yx$, A is **antisymmetric** if $xy = -yx$, A satisfies **Jacobi identity** if $(xy)z + (yz)x + (zx)y = 0$

A homomorphism $\phi : A \rightarrow B$ is a linear map such that $\phi(xy) = \phi(x)\phi(y)$

Definition 9.1.2. Suppose e_1, \dots, e_n is a basis of A , $e_i e_j = \sum_k c_k^{ij} e_k$, c_k^{ij} are called **structure constants** with respect to e_1, \dots, e_n . If A satisfies Jacobi identity, then

$$\sum_l c_m^{il} c_l^{jk} + \sum_l c_m^{jl} c_l^{ki} + \sum_l c_m^{kl} c_l^{ij} = 0$$

Definition 9.1.3. $B \leq A$ is a **subalgebra** if B is a subspace such that $BB \subseteq B$. $I \leq A$ is a **left ideal** if $AI \subseteq I$. Suppose $I, J \leq A$ are ideals, define **ideal quotients** $(J : I) = \{x \in A | xI \subseteq J\}$ which is an ideal. Homomorphisms preserve ideals

Remark 9.1.4. If A is (anti)symmetric, left ideals are two-sided ideals

Definition 9.1.5. A is **abelian** if $AA = 0$, A is **simple** if it is not abelian and the only ideals are 0 and A , A is **semisimple** if $A = A_1 \oplus \dots \oplus A_n$ is the direct sum of simple subalgebras, A is **reductive** if $A = \mathfrak{s} \oplus \mathfrak{a}$ is a direct sum of a semisimple subalgebra \mathfrak{s} and an abelian subalgebra \mathfrak{a}

Definition 9.1.6. A **derivation** is an endomorphism $D : A \rightarrow A$ such that $D(ab) = D(a)b + aD(b)$. Let $Der_{\mathbb{F}}(A)$ denote all derivations. If $D_1, D_2 \in Der(A)$, then $[D_1, D_2] = D_1 D_2 - D_2 D_1 \in Der(A)$, $Der(A) \leq End(A)$ is a Lie subalgebra

9.2 Lie algebras

Definition 9.2.1. A **Lie algebra** \mathfrak{g} is a antisymmetric nonassociative \mathbb{F} algebra satisfying Jacobi identity, usually with a **Lie bracket** $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ denote the multiplication. If $\text{char} \mathbb{F} = 2$, we also require $[x, x] = 0$

Definition 9.2.2. A **\mathfrak{g} module** V is an abelian group with left group action $\mathfrak{g} \times V \rightarrow V$ such that $1v = v$, $x(v + w) = xv + xw$, $(x + y)v = xv + yv$, $(xy)v = x(yv) - y(xv)$. Equivalently, a **Lie algebra representation** (π, V) is a Lie algebra homomorphism $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, where V is an \mathbb{F} vector space, $xv := \pi(x)v$ give V the \mathfrak{g} module structure

Remark 9.2.3. A \mathfrak{g} module is not a module according to Definition 6.1.1

Definition 9.2.4. A **\mathfrak{g} module homomorphism** $\phi : V \rightarrow W$ between \mathfrak{g} modules is a group homomorphism such that $\phi(xv) = x\phi(v)$. Equivalently, an intertwine map $\phi : V \rightarrow W$ between Lie algebra representations is a linear map such that $\phi(\pi_V(x)v) = \pi_W(x)\phi(v)$, giving the \mathfrak{g} module homomorphism

A subrepresentation (π, W) is a \mathfrak{g} submodule $W \leq V$

Adjoint representation

Definition 9.2.5. The **adjoint endomorphism** associated to x is left multiplication by x , i.e. $ad(x)(y) = [x, y]$, Jacobi identity becomes $ad([x, y]) = [ad(x), ad(y)]$, give a Lie algebra representation (adjoint representation) $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$, $ad(x)$ are called **inner derivations** since $ad(z)[x, y] = [ad(z)x, y] + [x, ad(z)y]$. $ad(\mathfrak{g}) \leq \text{Der}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ are Lie subalgebras
Any Lie algebra homomorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ induce a Lie algebra homomorphism $\phi : ad(\mathfrak{g}) \rightarrow ad(\mathfrak{h})$ by $\phi(ad(x)) = ad(\phi(x))$

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\phi} & \mathfrak{h} \\ \downarrow ad & & \downarrow ad \\ ad(\mathfrak{g}) & \xrightarrow{\phi} & ad(\mathfrak{h}) \end{array}$$

Definition 9.2.6. Centralizer of S is defined to be $C_{\mathfrak{g}}(S) := \{g \in \mathfrak{g} \mid [g, S] = 0\}$, in particular, the center $Z(\mathfrak{g}) := C_{\mathfrak{g}}(\mathfrak{g})$

Normalizer of S is defined to be $N_{\mathfrak{g}}(S) := \{g \in \mathfrak{g} \mid [g, S] \subseteq S\}$

Definition 9.2.7.

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq [[[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]], [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]]] \supseteq \cdots$$

is called the **derived series**, \mathfrak{g} is **solvable** if derived series terminates

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] \supseteq \cdots$$

is called the **lower central series**, \mathfrak{g} is **nilpotent** if lower central series terminates

Example 9.2.8. $[\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$

Proof. Since $\text{Tr}[X, Y] = 0$, thus $[\mathfrak{gl}(V), \mathfrak{gl}(V)] \leq \mathfrak{sl}(V)$, conversely \square

Definition 9.2.9. Let \mathfrak{g} be a Lie algebra, a Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is a nilpotent subalgebra such that $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ (self normalizing or alternatively $(\mathfrak{h} : \mathfrak{h}) = \mathfrak{h}$)

Definition 9.2.10. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a Lie algebra, \mathfrak{g} is called **toral** if \mathfrak{g} consists of semisimple elements

Definition 9.2.11. Let \mathfrak{g} be a Lie algebra, we can show the sum of all solvable ideal is again a solvable ideal, thus \mathfrak{g} has a unique maximal solvable ideal $\text{rad}(\mathfrak{g})$, called the **radical** of \mathfrak{g}

Definition 9.2.12. Let \mathfrak{g} be a complex Lie algebra, \mathfrak{g}_0 is called a **real form** of \mathfrak{g} if $\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$

Example 9.2.13. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be Lie subalgebra, the **tautological representation** (τ, V) is defined by $\tau(x) = x$, then $\tau([x, y]) = [x, y] = [\tau(x), \tau(y)]$

Proposition 9.2.14. Lie algebra \mathfrak{g} is reductive iff its adjoint representation is completely reducible

$$\mathfrak{g} \text{ semisimple, } \phi: \mathfrak{g} \rightarrow \mathfrak{gl}(V) \text{ representation} \Rightarrow \phi(\mathfrak{g}) \leq \mathfrak{sl}(V)$$

Lemma 9.2.15. If \mathfrak{g} is semisimple Lie algebra, and $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra representation, then $\varphi(\mathfrak{g}) \leq \mathfrak{sl}(V)$

Proof. By Proposition 9.6.1, $\varphi(\mathfrak{g}) = \varphi([\mathfrak{g}, \mathfrak{g}]) \leq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ □

Derivations of semisimple Lie algebra are inner derivations

Proposition 9.2.16. If \mathfrak{g} is a semisimple Lie algebra, then $\text{ad}(\mathfrak{g}) = \text{Der}(\mathfrak{g})$

Proof. As an abelian ideal of \mathfrak{g} , $Z(\mathfrak{g}) = 0$, thus $\mathfrak{g} \xrightarrow{\text{ad}} \text{ad}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding. Since $[\delta, \text{ad}(x)] = \text{ad}(\delta(x))$, $\delta \in \text{Der}(\mathfrak{g})$, thus $[\text{Der}(\mathfrak{g}), \text{ad}(\mathfrak{g})] \subseteq \text{ad}(\mathfrak{g})$. Let $K(\cdot, \cdot)$ be the Killing form on $\text{Der}(\mathfrak{g})$, due to Proposition 9.6.6(b) and Proposition 9.4.4(c), $K(\cdot, \cdot)|_{\text{ad}(\mathfrak{g})}$ is nondegenerate, denote $I := \text{ad}(\mathfrak{g})^\perp$ under $K(\cdot, \cdot)$, then $I \cap \text{ad}(\mathfrak{g}) = 0$, otherwise $0 \neq I \cap \text{ad}(\mathfrak{g}) \subseteq \ker K(\cdot, \cdot)|_{\text{ad}(\mathfrak{g})}$, by Exercise 52.0.2, $[I, \text{ad}(\mathfrak{g})] = 0$, thus for any $\delta \in I$, $0 = [\delta, \text{ad}(x)] = \text{ad}(\delta(x))$, since ad is an isomorphism, $\delta(x) = 0$, thus $\delta = 0$, $I = 0$, $\text{ad}(\mathfrak{g}) = \text{Der}(\mathfrak{g})$ □

Remark 9.2.17. When \mathfrak{g} is a semisimple Lie algebra, $\mathfrak{g} \xrightarrow{\text{ad}} \text{ad}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding, we can identify x with $\text{ad}(x)$, by abuse of notations, xy can be defined to be the preimage of $\text{ad}(x)\text{ad}(y) \in \mathfrak{gl}(\mathfrak{g})$

Lemma 9.2.18. Let G be a compact Lie group, we can pick any nonzero k -form α_I at I , and extend it to a k -form on G by $\alpha_A(Y_1, \dots, Y_k) = \alpha_I(Y_1 A^{-1}, \dots, Y_k A^{-1})$, or just $R_A^* \alpha_A = \alpha_I$, then we can define integral $\int_G f(A) \alpha$, then we would have $\int_G f(AB) \alpha = \int_G f(AB) \alpha_A = \int_G f(AB) R_B^* \alpha_{AB} = \int_G f(A) R_B^* \alpha = \int_G f(A) \alpha$, since $R_B^* \alpha = \alpha$, i.e. $(R_B^* \alpha)_A(X_A) = R_B^* \alpha_{AB}(X_A) = \alpha_A(X_A)$, thus this integration is right invariant. Note that this actually gives a right invariant Haar measure

Theorem 9.2.19. Weyl's theorem Weyl's theorem

Let \mathfrak{g} be a semisimple Lie algebra and $\varphi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra representation, then φ is completely reducible, namely, \mathfrak{g} modules are semisimple(completely reducible), thus any irreducible subrepresentation has to be one of the summand in the decomposition

Proof. Weyl's unitary trick □

9.3 Engel's theorem and Lie's theorem

Lemma for Engel's theorem

Lemma 9.3.1. Let $V \neq 0$ be a finite dimensional vector space, suppose $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra consists of nilpotent elements, then there exists $0 \neq v \in V$ such that $\mathfrak{g}v = 0$

Theorem 9.3.2. *Engel's theorem* ^{Engel's theorem} Consider the adjoint representation (ad, \mathfrak{g}) of a finite dimensional Lie algebra \mathfrak{g} , then \mathfrak{g} nilpotent iff $ad(X), X \in \mathfrak{g}$ can be strictly upper triangulized simultaneously iff $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$

\mathfrak{g} nilpotent, I ideal $\Rightarrow I \cap Z(\mathfrak{g})$ is nontrivial

Lemma 9.3.3. Let \mathfrak{g} be a nilpotent Lie algebra, $I \leq \mathfrak{g}$ is a nonzero ideal, then $I \cap Z(\mathfrak{g}) \neq 0$, in particular, if $I = \mathfrak{g}$ then $Z(\mathfrak{g}) \neq 0$ which can also easily being shown from the fact that $Z(\mathfrak{g})$ contains the last nonzero term in the lower central series of \mathfrak{g}

Proof. Consider adjoint map restrict on I , since \mathfrak{g} is nilpotent, $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$, so is $ad(X)|_I$, i.e. $ad(\mathfrak{g})|_I \leq \mathfrak{gl}(I)$ is a Lie subalgebra consists of nilpotent elements, by Lemma 9.3.1, there exists $0 \neq Y \in I$ such that $[X, Y] = 0, \forall X \in \mathfrak{g}$, thus $Y \in I \cap Z(\mathfrak{g})$ \square

Theorem 9.3.4. *Lie's theorem* ^{Lie's theorem}

If (π, V) is a finite representation of a finite dimensional Lie algebra \mathfrak{g} with $\overline{\mathbb{F}} = \mathbb{F}, \text{char} \mathbb{F} = 0$, if \mathfrak{g} is solvable, so is $\pi(\mathfrak{g})$, and $\pi(X), X \in \mathfrak{g}$ can be upper triangulized simultaneously

Remark 9.3.5. If (π, V) is a finite representation of a finite dimensional Lie algebra \mathfrak{g} with $\overline{\mathbb{F}} = \mathbb{F}, \text{char} \mathbb{F} = 0$, if \mathfrak{g} is abelian, so is $\pi(\mathfrak{g})$, but it doesn't imply $\pi(X), X \in \mathfrak{g}$ can be diagonalized simultaneously, for example $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathfrak{gl}(\mathbb{C}^2)$ is abelian, and $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ is not diagonalizable at all

However, due to Proposition 52.0.5, if (π, V) is a finite representation of a finite dimensional Lie algebra \mathfrak{g} with $\overline{\mathbb{F}} = \mathbb{F}$, where \mathfrak{g} is abelian, so is $\pi(\mathfrak{g})$, and suppose $\pi(X), X \in \mathfrak{g}$ are diagonalizable ($\pi(\mathfrak{g})$ is a toral Lie subalgebra), then they can be diagonalized simultaneously

9.4 Killing form

Definition 9.4.1. A bilinear form $(,)$ on Lie algebra \mathfrak{g} is **invariant** or **associative** if the Lie derivative is zero, i.e. $(ad_Y X, Z) + (X, ad_Y Z) = 0$, or equivalently, $([X, Y], Z) = (X, [Y, Z])$

Definition 9.4.2. A **quadratic Lie algebra** is a Lie algebra \mathfrak{g} with an invariant nondegenerate symmetric bilinear form $(,): \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{F}$

Definition 9.4.3. **Killing form** is the bilinear map $K(,): \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{F}$, $(X, Y) \mapsto \text{Tr}(ad(X)ad(Y))$

Some basic properties of Killing form

Proposition 9.4.4.

- (a) The Killing form is symmetric and invariant
- (b) The Killing form on a nilpotent Lie algebra is zero
- (c) Suppose $I \leq \mathfrak{g}$ is an ideal, then Killing form $K_I(,)$ on I is the same as the restriction of Killing form $K(,)$ to I , i.e. $K_I(,) = K(,)|_I$

Proof.

- (a)
- (b) Due to Theorem 9.3.2
- (c) By Exercise 45.1.4, for $X, Y \in I$, we have

$$\begin{aligned}
 K_I(X, Y) &= \text{Tr}(ad(X)|_I ad(Y)|_I) \\
 &= \text{Tr}((ad(X)ad(Y))|_I) \\
 &= \text{Tr}(ad(X)ad(Y)) \\
 &= K(X, Y) \\
 &= K(X, Y)|_I
 \end{aligned}$$

□

Example 9.4.5. The Killing form is a symmetric, bilinear and invariant form, and it is nondegenerate iff \mathfrak{g} is semisimple due to Proposition 9.6.6

nondegenerate, symmetric, bilinear and invariant form is unique up to scalar

Lemma 9.4.6. Any invariant, symmetric and bilinear form on simple Lie algebra \mathfrak{g} is a multiple of the Killing form

Proof. Suppose $(,)$ is an invariant, symmetric and bilinear form, so is $[,]_c = (,) - cK(,)$ for any c . If $(,) \neq 0$, then there exists $x, y \in \mathfrak{g}$ such that $[x, y]_c = 0$ for some c , since the kernel of $[,]_c$ is a nonzero ideal, $[,]_c = 0$ □

9.5 Jordan-Chevalley decomposition

Abstract Jordan-Chevalley decomposition on nonassociative \mathbb{F} -algebras

Lemma 9.5.1. Let \mathfrak{g} be a finite dimensional nonassociative \mathbb{F} algebra (including Lie algebra) with $\overline{\mathbb{F}} = \mathbb{F}$, for any $\delta \in \text{Der}(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$, let $\delta = \delta_s + \delta_n$ be its Jordan-Chevalley decomposition in $\mathfrak{gl}(\mathfrak{g})$, then $\delta_s, \delta_n \in \text{Der}(\mathfrak{g})$

Proof. For any $a \in \mathbb{F}$, define \mathfrak{g}_a be the generalized eigenspace of a , then we have $\mathfrak{g} = \bigoplus_{a \in \mathbb{F}} \mathfrak{g}_a$, and $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, since for any $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, $(\delta - (a+b)1_{\mathfrak{g}})^m([x, y]) = \sum_{k=0}^m \binom{m}{k} [(\delta - a1_{\mathfrak{g}})^{m-k}x, (\delta - b1_{\mathfrak{g}})^k y]$ which can be easily checked by induction. Then we have $\delta_s(x) = ax, \delta_s(y) = by$, and $\delta_s([x, y]) = (a+b)[x, y] = [ax, y] + [x, by] = [\delta_s(x), y] + [x, \delta_s(y)]$, thus $\delta_s \in \text{Der}(\mathfrak{g})$, so does $\delta_n = \delta - \delta_s$. \square

Definition 9.5.2. Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Because Lemma 9.5.1 and Proposition 9.2.16, for any $x \in \mathfrak{g}$, we can identify x with $ad(x)$, and we have Jordan-Chevalley decomposition $ad(x) = ad(x)_s + ad(x)_n = ad(x_s) + ad(x_n)$, where x_s, x_n are defined to be the preimages of $ad(x)_s, ad(x)_n$. Moreover, there exists polynomials $p(t), q(t)$ with no constant terms such that $ad(x_s) = ad(x)_s = p(ad(x))$, $ad(x_n) = ad(x)_n = q(ad(x))$, by abuse of notations, $x_s = p(x)$ and $x_n = q(x)$

Semisimple Lie algebra contains the semisimple and nilpotent parts of its elements

Theorem 9.5.3. Suppose V is a finite dimensional \mathbb{F} vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\text{char} \mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a semisimple Lie algebra, for any $x \in \mathfrak{g}$, $x = x_s + x_n$ is the Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$, moreover, the abstract and usual Jordan-Chevalley decompositions coincide, i.e. $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$

Proof. Define Lie subalgebras $\mathfrak{l}_W := \{y \in \mathfrak{gl}(V) | yW \subseteq W, \text{Tr}(y|_W) = 0\}$ with $W \leq V$ being \mathfrak{g} submodules, and define $\mathfrak{l} = \left(\bigcap_W \mathfrak{l}_W \right) \cap N_{\mathfrak{gl}(V)}(\mathfrak{g})$, for any $x \in \mathfrak{g}$, due to Proposition 45.1.4 and Lemma 9.2.15, $\text{Tr}(x|_W) = \text{Tr}(x) = 0$, $\mathfrak{g} \leq \mathfrak{l}_W \Rightarrow \mathfrak{g} \leq \mathfrak{l}$, thus \mathfrak{l} is a subalgebra of $N_{\mathfrak{gl}(V)}(\mathfrak{g})$ of containing \mathfrak{g} , thus \mathfrak{l} is finite dimensional \mathfrak{g} module, by Theorem 9.2.19, $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{h}$ is a direct sum of \mathfrak{g} modules, since $\mathfrak{l} \leq N_{\mathfrak{gl}(V)}(\mathfrak{g})$, $[\mathfrak{g}, \mathfrak{l}] = 0 \Rightarrow [\mathfrak{g}, \mathfrak{h}] = 0$, i.e. \mathfrak{g} acts trivially on \mathfrak{h} , fix any irreducible \mathfrak{g} submodule W , for any $y \in \mathfrak{h}, x \in \mathfrak{g}, xy - yx = [x, y] = 0, yxv = xyv$ for $v \in W$, i.e. $y \in \text{Hom}_{\mathfrak{g}}(W, W)$, by Lemma 6.1.12, y acts on W as a scalar, but $\text{Tr}(y|_W) = 0$, thus y acts trivially on W , again by Theorem 9.2.19, V can be written as the direct sum of irreducible \mathfrak{g} submodules, thus y acts trivially on $W \Rightarrow y = 0$, therefore $\mathfrak{h}_j = 0 \Rightarrow \mathfrak{g} = \mathfrak{l}$, for any $x \in \mathfrak{g}$, due to Theorem 8.3.9, $x = x_s + x_n$ and $x_s = p(x), x_n = q(x)$ for some polynomials $p(x), q(x)$ with no constant terms, thus if $x \in \mathfrak{l}_W, xW \subseteq W$ and $\text{Tr}(x|_W) = 0$, then $x_s W = p(x)W \subseteq W, \text{Tr}(x_s|_W) = \text{Tr}(p(x)|_W) = 0$, similarly, $x_n W \subseteq W, \text{Tr}(x_n|_W) = 0, x_s, x_n \in \mathfrak{l}_W$, also $x_s = p(x), x_n = q(x) \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$, thus $x_s, x_n \in \mathfrak{l} = \mathfrak{g}$. Since the Jordan-Chevalley decomposition of $ad(x)$ in $\mathfrak{gl}(V)$ is unique and $ad(x_s) + ad(x_n) = ad(x) = ad(x)_s + ad(x)_n$, thus $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$. \square

Corollary 9.5.4. Suppose V is a finite dimensional \mathbb{F} vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\text{char} \mathbb{F} = 0$, \mathfrak{g} is a semisimple Lie algebra, and $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra representation, $x = x_s + x_n$ is the abstract Jordan-Chevalley decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the usual Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$

Proof. Due to Proposition 9.6.2, $\phi(\mathfrak{g})$ is also semisimple

First notice that a linear operator $T \in \text{End}_{\mathbb{F}}(V)$ is diagonalizable (semisimple) iff the dimension of the sum of its eigenspaces is $\dim V$, or equivalently iff all eigenvectors span V

Since $ad(x_s)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 9.2.16, the eigenvectors y_i 's of $ad(x_s)$ spans \mathfrak{g} , say $ad(x_s)(y_i) = \lambda_i y_i$, then as in Definition 9.2.5, $ad(\phi(x_s))(\phi(y_i)) = [\phi(x_s), \phi(y_i)] =$

$\phi([x_s, y_i]) = \phi(ad(x_s)(y_i)) = \phi(\lambda_i y_i) = \lambda_i \phi(y_i)$, thus those $0 \neq \phi(y_i)$ are eigenvectors of $\phi(\mathfrak{g})$, hence $ad(\phi(x_s))$ is semisimple in $\mathfrak{gl}(\mathfrak{gl}(V))$

Since $ad(x_n)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 9.2.16, $ad(x_n)^m = 0$ for some m , then as in Definition 9.2.5 $ad(\phi(x_n))^m = \phi(ad(x_n))^m = \phi(ad(x_n)^m) = 0$, thus $ad(\phi(x_n))$ is also nilpotent in $\mathfrak{gl}(\mathfrak{gl}(V))$

Moreover, as in Definition 9.2.5, $[ad(\phi(x_s)), ad(\phi(x_n))] = [\phi(ad(x_s)), \phi(ad(x_n))] = \phi([ad(x_s), ad(x_n)]) = 0$

Thus $\phi(x) = \phi(x_s) + \phi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\phi(x)$ in $\phi(\mathfrak{g}) \leq \mathfrak{gl}(V)$, by Theorem 9.5.3, this coincide with the usual Jordan-Chevalley decomposition of $\phi(x) = \phi(x)_s + \phi(x)_n$ in $\mathfrak{gl}(V)$, i.e. $\phi(x_s) = \phi(x)_s$, $\phi(x_n) = \phi(x)_n$ \square

9.6 Classification of semisimple Lie algebras

\mathfrak{g} simple implies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$

Proposition 9.6.1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a semisimple Lie algebra, then any ideal of \mathfrak{g} is certain sum of \mathfrak{g}_i 's, and any sum of \mathfrak{g}_i 's is an ideal, in particular, \mathfrak{g}_i 's are ideals, moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

Proof. any ideal $I \leq \mathfrak{g}$ is certain sum of \mathfrak{g}_i 's, because $I \cap \mathfrak{g}_i$ is an ideal of \mathfrak{g}_i which is either 0 or \mathfrak{g}_i itself

If \mathfrak{g} is a simple Lie algebra, then it is not abelian, $[\mathfrak{g}, \mathfrak{g}] \neq 0$, thus $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, generally, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n] = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{g}_n, \mathfrak{g}_n] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$ \square

image of a semisimple Lie algebra is also semisimple

Proposition 9.6.2. If \mathfrak{g} is semisimple, $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is a Lie algebra homomorphism, then $\phi(\mathfrak{g})$ is also semisimple

Proof. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of simple Lie algebras, $\phi(\mathfrak{g}_i)$ are also ideals so $\phi(\mathfrak{g}) = \phi(\mathfrak{g}_1) \oplus \cdots \oplus \phi(\mathfrak{g}_n)$ is a direct sum of ideals, and $[\phi(\mathfrak{g}_i), \phi(\mathfrak{g}_i)] = \phi([\mathfrak{g}_i, \mathfrak{g}_i]) = \phi(\mathfrak{g}_i)$ implies that each $\phi(\mathfrak{g}_i)$ is simple or 0, thus $\phi(\mathfrak{g})$ is also semisimple \square

nilpotent/semisimple implies ad-nilpotent/ad-semisimple

Proposition 9.6.3. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a Lie algebra, $X \in \mathfrak{g}$, then X is nilpotent $\Rightarrow \text{ad}(X)$ is nilpotent, if in addition V is finite dimensional and $\overline{\mathbb{F}} = \mathbb{F}$, then X is semisimple $\Rightarrow \text{ad}(X)$ is semisimple, or rather diagonalizable

Proof. Let L_X, R_X be left and right multiplications by X , then we have $\text{ad}_X = L_X - R_X$ and $[L_X, R_X] = 0$, thus X is nilpotent $\Rightarrow \text{ad}(X)$ is nilpotent

Notice that given $A = (a_{ij})$, $D = \text{diag}(d_1, \dots, d_n)$, $[D, A] = ((d_i - d_j)a_{ij})$, thus $[D, E_{ij}] = (d_i - d_j)E_{ij}$, thus X is diagonalizable $\Rightarrow \text{ad}(X)$ is diagonalizable \square

Cartan's criterion for solvability

Theorem 9.6.4 (Cartan's criterion for solvability). Let V be a finite dimensional \mathbb{F} vector space with $\text{char } \mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra, then \mathfrak{g} is solvable iff $\text{Tr}(XY) = 0$, $\forall X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$

Corollary 9.6.5. Cartan's criterion for semisimplicity Cartan's criterion for semisimplicity

Let \mathfrak{g} is finite dimensional Lie algebra with $\text{char } \mathbb{F} = 0$, then \mathfrak{g} is semisimple iff its Killing form is nondegenerate

Equivalent conditions for semisimplicity

Proposition 9.6.6. The following statements are equivalent

- (a) \mathfrak{g} is semisimple
- (b) The Killing form is nondegenerate
- (c) \mathfrak{g} doesn't have nontrivial abelian ideals
- (d) \mathfrak{g} doesn't have nontrivial solvable ideals
- (e) $\text{rad}(\mathfrak{g}) = 0$

Proof. (a) \Leftrightarrow (b) is due to Corollary 9.6.5 \square

Adjoint representation of $\text{SL}(2, \mathbb{F})$

Example 9.6.7.

Recall $\mathfrak{sl}(2, \mathbb{F}) = \{X \in M(2, \mathbb{F}) \mid \text{Tr}(X) = 0\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$

$\mathfrak{sl}(2, \mathbb{F}) = \langle H, X, Y \rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $\text{ad}_H X = [H, X] = 2X$, $\text{ad}_H Y = [H, Y] = -2Y$, $\text{ad}_X Y = [X, Y] = H$, this is the adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$

Lemma 9.6.8. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, if $V = \mathbb{F}^n$, then there are three $n \times n$ matrices x, y, h such that $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$ due to Example 9.6.7

Lemma 9.6.9. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, $V_\lambda := \{v \in V \mid \pi(H)v = \lambda v\}$, then $\pi(X)V_\lambda \subseteq V_{\lambda+2}$, $\pi(Y)V_\lambda \subseteq V_{\lambda-2}$ and $\pi(H)V_\lambda \subseteq V_\lambda$

Proof. $\pi(H)V_\lambda \subseteq V_\lambda$ is just by definition, suppose $v \in V_\lambda$, $\pi(H)\pi(X)v = 2\pi(X)v + \pi(X)\pi(H)v = (\lambda + 2)\pi(X)v$, $\pi(H)\pi(Y)v = -2\pi(Y)v + \pi(Y)\pi(H)v = (\lambda - 2)\pi(Y)v$, \square

Remark 9.6.10. $\pi(X), \pi(Y)$ are named **raising and lowering operator**

Classification of representations of $\mathfrak{sl}(2, \mathbb{F})$

Theorem 9.6.11. Suppose $\overline{\mathbb{F}} = \mathbb{F}$, $\text{char } \mathbb{F} = 0$, for any integer $m \geq 0$, there is an irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ with dimension $m + 1$

Proof. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$, there exists highest $\lambda \in \mathbb{F}$ such that $V_\lambda \neq 0$, pick $0 \neq u \in V_\lambda$, let $u_k := \pi(Y)^k u \in V_{\lambda-2k}$, then there exists m such that $u_m \neq 0$ but $u_{m+1} = 0$, then u_0, \dots, u_m are independent since they belong distinct eigenspaces, with $\pi(H)u_k = (\lambda - 2k)u_k$, $\pi(X)u_0 = \pi(X)u = 0$ since $u \in V_\lambda$ is of "highest weight", and $\pi(X)u_k = k(\lambda - k + 1)u_{k-1}$, $k > 0$ by induction, $\pi(X)u_1 = \pi(X)\pi(Y)u = ([\pi(X), \pi(Y)] + \pi(Y)\pi(X))u = \pi(H)u + \pi(Y)\pi(X)u = \lambda u = 1(\lambda - 1 + 1)u_0$, $\pi(X)u_{k+1} = \pi(X)\pi(Y)u_k = ([\pi(X), \pi(Y)] + \pi(Y)\pi(X))u_k = \pi(H)u_k + \pi(Y)\pi(X)u_k = (\lambda - 2k)u_k + k(\lambda - k + 1)\pi(Y)u_{k-1} = (k + 1)(\lambda - k)u_k$

Note that since $0 = Xu_{m+1} = (m + 1)(\lambda - m)u_m \Rightarrow \lambda = m$, which implies all possible eigenvalue for $\pi(H)$ has to be integers, when m is even, we call this irreducible representation even, when m is odd, we call this irreducible representation odd

In general, for any finite dimensional representation, we can decompose the representation into irreducible subrepresentations by using this procedure repeatedly

Therefore $0 \neq W := \langle u_0, \dots, u_m \rangle$ is invariant, but π is irreducible, thus $V = W$, and by Lemma 6.1.12, (π, V) is unique up to isomorphism \square

Adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$ is the unique 3 dimensional irreducible representation

Example 9.6.12. $(ad, \mathfrak{sl}(2, \mathbb{F}))$ is the unique irreducible 3 dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$ with $V_0 = \langle H \rangle$, $V_{-2} = \langle Y \rangle$ and $V_2 = \langle X \rangle$, it is irreducible because of Lemma 9.6.15, if we use X, Y, H as basis, then $ad(X), ad(Y), ad(H)$ would have the matrix forms

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$K(X, X) = Tr \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad K(X, Y) = Tr \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 4$$

$$K(X, H) = Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = 0, \quad K(Y, Y) = Tr \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$K(Y, H) = Tr \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 0, \quad K(H, H) = Tr \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8$$

Thus its Cartan matrix is $\Phi = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ which is nondegenerate

Tautological representation of $\mathfrak{sl}(2, \mathbb{F})$

Example 9.6.13. The tautological representation (τ, \mathbb{F}^2) is the unique irreducible 2 dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$ with $V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, $V_{-1} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

Example 9.6.14. Let $S^k(\mathbb{F}^2)$ be the k -th symmetric power of \mathbb{F}^2 which is isomorphic to the set of degree k polynomials in $\mathbb{F}[x, y]$ generated by $\langle x^k, x^{k-1}y, \dots, xy^{k-1}, y^k \rangle$ which is of dimension $k + 1$, with this identification, $(\pi, S^k(\mathbb{F}^2))$ with

$\pi(X)(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, $\pi(X)y = x$, $\pi(Y)x = y$, $\pi(Y)y = 0$, $\pi(H)x = x$, $\pi(H)y = -y$, just as in Example 9.6.13, and define inductively that $\pi(Z)(fg) = g\pi(Z)f + f\pi(Z)g$, this is the unique $k+1$ dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$

Count the number irreducible summand of a representation of $\mathfrak{sl}(2, \mathbb{F})$

Lemma 9.6.15. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$, and V_k be the k -eigenspace of $\pi(H)$, then the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$, whereas $\dim V_0, \dim V_1$ are number of even and odd irreducible summands

Proof. In an even irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are even, so there is a unique 0-eigenvector, in an odd irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are odd, so there is a unique 1-eigenvector, thus the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$ \square

Definition 9.6.16. \mathfrak{g} is a semisimple Lie algebra, \mathfrak{h} is a maximal toral Lie algebra. For $\alpha \in \mathfrak{h}^* = \text{Hom}_{\mathbb{F}}(\mathfrak{h}, \mathbb{F})$, define **root spaces**

$$\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} | \text{ad}_{\mathfrak{h}}(x) = [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

α is a **root** if $\alpha \neq 0$ and $\mathfrak{g}_{\alpha} \neq 0$, denote the set of roots as Δ . $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of \mathfrak{h}
Basic properties of root spaces

Proposition 9.6.17.

- (a) $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] \subseteq \mathfrak{g}_{\alpha+\beta}$
- (b) $\alpha \in \Delta$, any $X \in \mathfrak{g}_{\alpha}$ is nilpotent
- (c) $K(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) = 0$ unless $\alpha + \beta = 0$
- (d) $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

Proof.

- (a) For $X \in \mathfrak{g}_{\alpha}, Y \in \mathfrak{g}_{\beta}, Z \in \mathfrak{h}$, we have

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = \alpha(Z)[X, Y] + \beta(Z)[X, Y] = (\alpha + \beta)(Z)[X, Y]$$

- (b) For $\beta \in \Delta \cup \{0\}, \alpha \in \Delta, X \in \mathfrak{g}_{\alpha}, Y \in \mathfrak{g}_{\beta}, \text{ad}(X)^n(Y) \in \mathfrak{g}_{n\alpha+\beta} = 0$ when n is big enough, thus $\text{ad}(X)$ is nilpotent
- (c) Suppose $\alpha + \beta \neq 0$, then there exists $Z \in \mathfrak{h}$ such that $(\alpha + \beta)(Z) \neq 0$, then for any $X \in \mathfrak{g}_{\alpha}, Y \in \mathfrak{g}_{\beta}$

$$\begin{aligned} (\alpha + \beta)(Z)K(X, Y) &= \alpha(Z)K(X, Y) + \beta(Z)K(X, Y) \\ &= K(\alpha(Z)X, Y) + K(X, \beta(Z)Y) \\ &= K([Z, X], Y) + K(X, [Z, Y]) \\ &= -K([X, Z], Y) + K(X, [Z, Y]) \\ &= 0 \end{aligned}$$

Thus $K(X, Y) = 0$

- (d) Since $K(\mathfrak{g}_{\alpha}, \mathfrak{h}) = 0, \forall \alpha \in \Delta, \ker K(,)|_{\mathfrak{g}_0} \subseteq \ker K(,) = 0$, thus $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

\square

Root space decomposition

Theorem 9.6.18. Semisimple Lie algebra \mathfrak{g} has root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_{\alpha}$

Proof. By Proposition 9.6.17 \square

x, y in $\mathfrak{gl}(V)$ commutes, x nilpotent $\Rightarrow xy$ nilpotent

Lemma 9.6.19. V is an \mathbb{F} vector space, $x, y \in \mathfrak{gl}(V)$ commutes, x is nilpotent, then xy is nilpotent, and $\text{Tr}(xy) = 0$

Proof. $x^m = 0 \Rightarrow (xy)^m = x^m y^m = 0$ □

Maximal toral Lie algebra of semisimple Lie algebra is self centralizing

Proposition 9.6.20. For semisimple Lie algebra \mathfrak{g} with maximal toral Lie subalgebra \mathfrak{h} , $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Proof.

Step I: $C_{\mathfrak{g}}(\mathfrak{h})$ contains semisimple and nilpotent parts of its elements

If $x \in C_{\mathfrak{g}}(\mathfrak{h})$, due to Proposition 9.5.2, there are polynomials $p(t), q(t)$ with no constant terms such that $\text{ad}(x_s) = \text{ad}(x)_s = p(\text{ad}(x))$, $\text{ad}(x_n) = \text{ad}(x)_n = q(\text{ad}(x))$, since $x \in C_{\mathfrak{g}}(\mathfrak{h})$, $\text{ad}(x)|_{\mathfrak{h}} = 0$, $\text{ad}(x_s)|_{\mathfrak{h}} = p(\text{ad}(x))|_{\mathfrak{h}} = 0$, $\text{ad}(x_n)|_{\mathfrak{h}} = q(\text{ad}(x))|_{\mathfrak{h}} = 0$, thus $x_s, x_n \in C_{\mathfrak{g}}(\mathfrak{h})$

Step II: \mathfrak{h} contains all semisimple elements of $C_{\mathfrak{g}}(\mathfrak{h})$

If $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, use Exercise 52.0.4, s and elements of \mathfrak{h} are diagonalizable simultaneously, thus $\mathfrak{h} + \langle s \rangle$ is toral in \mathfrak{g} , then $s \in \mathfrak{h}$ since \mathfrak{h} is maximal

Step III: $K(\cdot, \cdot)|_{\mathfrak{h}}$ is nondegenerate

Suppose there exists $h \in \mathfrak{h}$ such that $K(h, h) = 0$, if $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $\text{ad}(n)$ is nilpotent, and $[n, h] = 0$, thus $[\text{ad}(n), \text{ad}(h)] = \text{ad}([n, h]) = 0$, by Lemma 9.6.19, $\text{Tr}(\text{ad}(n)\text{ad}(h)) = 0$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, thus $K(s, h) = 0$, and according to Step I, $K(h, C_{\mathfrak{g}}(\mathfrak{h})) = 0$ which contradicts Proposition 9.6.17(d) that $K(\cdot, \cdot)|_{C_{\mathfrak{g}}(\mathfrak{h})}$ is nondegenerate

Step IV: $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

If $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $\text{ad}(n)$ is nilpotent, so is $\text{ad}(n)|_{C_{\mathfrak{g}}(\mathfrak{h})}$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, $\text{ad}(s)|_{C_{\mathfrak{g}}(\mathfrak{h})} = 0$, by Theorem 9.3.2, $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

Step V: $\mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] = 0$

Suppose $x \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then $x = \sum [y_i, z_i]$ where $y_i, z_i \in C_{\mathfrak{g}}(\mathfrak{h})$, then $K(x, x) = \sum K(x, [y_i, z_i]) = \sum K([x, y_i], z_i) = 0$, since $K(\cdot, \cdot)$ is nondegenerate on \mathfrak{h} (or \mathfrak{g} or $C_{\mathfrak{g}}(\mathfrak{h})$), thus $x = 0$

Step VI: $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian

Suppose $[C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] \neq 0$, since $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent from Step IV, by Lemma 9.3.3, there exists $0 \neq z \in Z(C_{\mathfrak{g}}(\mathfrak{h})) \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then z can't be semisimple, otherwise $z \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, contradicting Step V, thus its nilpotent part $n \neq 0$, but its semisimple part $s \in \mathfrak{h} \leq Z(C_{\mathfrak{g}}(\mathfrak{h}))$, so is $n = z - s$, but then $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 9.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 9.6.17(d)

Step VII: $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$

Suppose $x \in C_{\mathfrak{g}}(\mathfrak{h}) \setminus \mathfrak{h}$, then it gives a nonzero nilpotent part n , but then since $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian by Step VI, thus $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 9.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 9.6.17(d) □

Remark 9.6.21. $K(\cdot, \cdot)|_{\mathfrak{h}}$ is nondegenerate is not the same as saying that the Killing form of \mathfrak{h} is nondegenerate which obviously violates Proposition 9.6.6, it doesn't contradict Proposition 9.4.4 since $\mathfrak{h} \leq \mathfrak{g}$ is merely a Lie subalgebra but not an ideal, by the nondegeneracy, we can identify \mathfrak{h}^* with \mathfrak{h} by $\mathfrak{h}^* \rightarrow \mathfrak{h}, \alpha \mapsto t_\alpha$, where $K(t_\alpha, x) = \alpha(x)$, and here t behaves like the linear isomorphism $t : \mathfrak{h}^* \rightarrow \mathfrak{h}, \alpha \mapsto t_\alpha$

Some properties about root space decomposition

Proposition 9.6.22.

- (a) Δ spans \mathfrak{h}^*
- (b) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$
- (c) $\alpha \in \Delta, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = K(x, y)t_\alpha$
- (d) $\alpha \in \Delta$, then $0 \neq [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$
- (e) If $\alpha \in \Delta$, then $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$
- (f) If $\alpha \in \Delta, 0 \neq x_\alpha \in \mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $K(x_\alpha, y_\alpha) = \frac{2}{K(t_\alpha, t_\alpha)}$,
define $h_\alpha := [x_\alpha, y_\alpha] = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)}$, then $\mathfrak{s}_\alpha := \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ via
 $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- (g) Given $\alpha \in \Delta, (ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$ will be a representation of \mathfrak{s}_α , thus \mathfrak{g} can be decomposed into irreducible representations of \mathfrak{s}_α , and the highest eigenvectors for this representation are also common highest eigenvectors of $ad(\mathfrak{h})$

Proof.

- (a) Suppose $\langle \Delta \rangle \subsetneq \mathfrak{h}^*$, then there exists $0 \neq h \in \mathfrak{h}$, such that $\forall \alpha \in \Delta, \alpha(h) = 0$, then $\forall x \in \mathfrak{g}_\alpha, [h, x] = \alpha(h)x = 0$, and since \mathfrak{h} is abelian, $[h, \mathfrak{h}] = 0$, thus $[h, \mathfrak{g}] = 0$, but then $h \in Z(\mathfrak{g}) = 0$ which is a contradiction
- (b) If $\alpha \in \Delta$, and $\mathfrak{g}_{-\alpha} = 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 9.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction
- (c) $\forall h \in \mathfrak{h}, K(h, [x, y]) = K([h, x], y) = K(\alpha(h)x, y) = K(t_\alpha, h)K(x, y) = K(h, K(x, y)t_\alpha)$, since $K(\cdot, \cdot)|_{\mathfrak{h}}$ is nondegenerate, $[x, y] = K(x, y)t_\alpha$
- (d) Only need to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$. There exists $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) \neq 0$, otherwise then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 9.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction, thus $[x, y] = K(x, y)t_\alpha \neq 0$ by (c)
- (e) Suppose instead $\alpha(t_\alpha) = 0$, then $[t_\alpha, x] = [t_\alpha, y] = 0, \forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, by nondegeneracy, we can find $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$, then by (c), $[x, y] = K(x, y)t_\alpha = t_\alpha$, thus $\mathfrak{s} = \langle x, y, t_\alpha \rangle \cong ad(\mathfrak{s}) \leq ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is a 3 dimensional solvable Lie algebra, by Theorem 9.3.4, for any $s \in [\mathfrak{s}, \mathfrak{s}]$, $ad(s)$ is nilpotent, thus $ad(t_\alpha)$ is both semisimple and nilpotent, hence $ad(t_\alpha) = 0 \Rightarrow t_\alpha = 0$ which is a contradiction
- (f) $[h_\alpha, x_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, x_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha [h_\alpha, y_\alpha] = \frac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, y_\alpha] = -\frac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)y_\alpha = -2y_\alpha$
- (g) Suppose $x \in \mathfrak{g}$ is a highest eigenvector of representation $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $0 = ad(x_\alpha)(x) = [x_\alpha, x], \forall h \in \mathfrak{h}, [x_\alpha, ad(h)(x)] = [x_\alpha, [h, x]] = [[x_\alpha, h], x] + [h, [x_\alpha, x]] = [-\alpha(h)x_\alpha, x] = 0$

□

Remark 9.6.23. In (f), the choice of x_α is not canonical, however, $h_\alpha = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ is canonical, alpha check is canonical

for example, if we pick $0 \neq x_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $s_{-\alpha} = \langle x_{-\alpha}, y_{-\alpha}, h_{-\alpha} \rangle$, then $h_{-\alpha} = h_\alpha = \frac{2t_{-\alpha}}{K(t_{-\alpha}, t_{-\alpha})} = -h_\alpha$, moreover, according to Lemma 9.4.6, any nondegenerate, symmetric, bilinear and invariant form on \mathfrak{h} is of the form $(,) := cK(,)|_{\mathfrak{h}}$ for some $c \neq 0$, then t'_α the dual of $\alpha \in \mathfrak{h}^*$ given by $(t'_\alpha, x) = \alpha(x), \forall x \in \mathfrak{h}$, then we have $K(t_\alpha, x) = \alpha(x) = (t'_\alpha, x) = cK(t'_\alpha, x) = K(ct'_\alpha, x)$, because of the nondegeneracy of $K(,)|_{\mathfrak{h}}$, $t_\alpha = ct'_\alpha \Rightarrow t'_\alpha = \frac{t_\alpha}{c}$, and $\frac{2t'_\alpha}{(t'_\alpha, t'_\alpha)} = \frac{2\frac{t_\alpha}{c}}{cK(\frac{t_\alpha}{c}, \frac{t_\alpha}{c})} = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)} = h_\alpha$, thus h_α is even canonical defined regardless of the choice of the nondegenerate, symmetric, bilinear and invariant form on \mathfrak{h} , for this reason, we give h_α a new notation α^\vee for later use, whereas $\alpha(\alpha^\vee) = 2$, for any nondegenerate, symmetric, bilinear and invariant form on \mathfrak{h} , note that $\alpha \mapsto \alpha^\vee$ is not linear

Also, according to Lemma 9.4.6, even though $(,)$ is defined up to a scalar, but the orthogonality is always well defined

Due to Proposition 9.6.22(g), if $0 \neq x \in \mathfrak{g}$ is a highest eigenvector for representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $x \in \mathfrak{g}_\beta$, for some $\beta \in \Delta \cup \{0\}$, when $\beta = 0$, these corresponds to the trivial representations, if $\beta \in \Delta$, we can denote one such highest eigenvector $0 \neq x_\beta \in \mathfrak{g}_\beta$, then $ad(h_\alpha)(x_\beta) = [h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta$, thus x_β is a $\beta(h_\alpha) = \frac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}$ -eigenvector, by Proposition 9.6.17(a), $ad(y_\alpha)^j(x_\beta) \in \mathfrak{g}_{\beta-j\alpha}$ are all the nonzero eigenvectors corresponds to eigenvalues $(\beta - j\alpha)(h_\alpha) = 2 \left(\frac{K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)} - j \right)$, and these roots $\beta - j\alpha, j = 0, \dots, k = \beta(h_\alpha)$ form an α -string

One of these irreducible representation is \mathfrak{s}_α itself according to Example 9.6.12

Example 9.6.24. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ which is a semisimple Lie algebra, then

$$\mathfrak{h} = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \middle| \sum z_i = 0 \right\}$$

is a maximal toral Lie subalgebra, denote $\text{diag}(z_1, \dots, z_n)$ as h_z , then $[h_z, E_{ij}] = (z_i - z_j)E_{ij}$, and $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{g} = \mathfrak{h} \bigoplus_{i \neq j} \langle E_{ij} \rangle$, $\Delta = \{e_i - e_j | i \neq j\}$, where $e_i \in \mathfrak{h}^*$ is defined by $e_i(h_z) = z_i$, also

$$\mathfrak{s}_\alpha = \left\{ \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & a & & b \\ & & & \ddots & \\ & & b & & -a \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \right\} \cong \mathfrak{sl}(2, \mathbb{C})$$

where $\alpha = e_i - e_j$

Definition 9.6.25. Define $s_\alpha(\beta) := \beta - \frac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}\alpha, \forall \alpha \in \Delta, \beta \in \mathfrak{h}^*$ is an orthogonal reflection over the hyperplane $H_\alpha = \{\alpha(x) = K(t_\alpha, x) = 0 | x \in \mathfrak{h}\} = \ker \alpha$, more precisely, $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\beta) = \beta, \forall \beta \in H_\alpha$, note here any nondegenerate, symmetric, bilinear and invariant form $(,)$ can be used as the definition in place of the Killing form $K(,)$ thanks to Lemma 9.4.6 Definition of s_alpha

Proposition 9.6.26.

(a) If $\alpha \in \Delta$, then $\dim \mathfrak{g}_\alpha = 1$ alpha string through beta

- (b) If $\alpha \in \Delta$, then $c\alpha \in \Delta \Leftrightarrow c = \pm 1$
 (c) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
 (d) If $\alpha, \beta \in \Delta$, the Cartan integer $\beta(h_\alpha) = \frac{2K(\alpha, \beta)}{K(\alpha, \alpha)} \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Delta$, if $\beta \neq \alpha, -\alpha$, then $\Delta \cap \{\beta + j\alpha | j \in \mathbb{Z}\} = \{\beta + j\alpha | -r \leq j \leq s, j \in \mathbb{Z}\}$ which is an α string through β , and $\beta(h_\alpha) = r - s$

Proof. (a) Let $\mathfrak{m} = \mathfrak{h} \bigoplus_{c\alpha \in \Delta, c \in \mathbb{F}^\times} \mathfrak{g}_{c\alpha}$, then $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is a finite representation of \mathfrak{s}_α because of

Proposition 9.6.17(a), first notice for $x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, thus the 0-eigenspace of $ad(h_\alpha)$ is \mathfrak{h} , but note that for any $h \in \ker \alpha = H_\alpha \leq \mathfrak{h}$ as in Definition 9.6.25, $ad(x_\alpha)(h) = [x_\alpha, h] = \alpha(h)x_\alpha = 0$, $ad(y_\alpha)(h) = [y_\alpha, h] = -\alpha(h)y_\alpha = 0$, $ad(h_\alpha)(h) = [h_\alpha, h] = 0$, thus \mathfrak{s}_α acts trivially on $\ker \alpha$ which is of codimension 1 in \mathfrak{h} which gives $\dim \mathfrak{h} - 1$ copies of trivial representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, since $h_\alpha \notin \ker \alpha$, $\mathfrak{h} = \langle h_\alpha \rangle \oplus \ker \alpha$, by Example 9.6.12 and Lemma 9.6.15, $\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is a 3 dimensional irreducible representation of \mathfrak{s}_α , and $\mathfrak{s}_\alpha \oplus \ker \alpha$ are the only possible even irreducible representations of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, thus $\dim \mathfrak{g}_\alpha = 1$, otherwise, we can choose $0 \neq x'_\alpha \in \mathfrak{g}_\alpha$ linearly independent of x_α , then we have $[h_\alpha, x'_\alpha] = \alpha(h_\alpha)x'_\alpha = 2x'_\alpha$, which gives a contradiction. Moreover, $2\alpha \notin \Delta$, otherwise, we can choose $0 \neq x_{2\alpha} \in \mathfrak{g}_{2\alpha}$, then $[h_\alpha, x_{2\alpha}] = 2\alpha(h_\alpha)x_{2\alpha} = 4x_{2\alpha}$ which also gives a contradiction

(b) Suppose $c\alpha \in \Delta$, for $0 \neq x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, by Theorem 9.6.11, we know that $2c \in \mathbb{Z}$, but by symmetry, if we let $\beta = c\alpha$, then $\alpha = \frac{\beta}{c} \in \Delta$ implies $\frac{2}{c} \in \mathbb{Z}$, thus c can only possibly be $\pm 1, \pm 2, \pm \frac{1}{2}$, but from (a), we know $c \neq 2$, thus $c \neq -2$

thanks to Proposition 9.6.22(b), and by symmetry, $c \neq \pm \frac{1}{2}$, therefore $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker \alpha \oplus \langle h_\alpha \rangle \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker \alpha \oplus \mathfrak{s}_\alpha$

(c) Obviously β can't be α or $-\alpha$, otherwise $\alpha + \beta \notin \Delta$, also $\beta + j\alpha \neq 0, \forall j \in \mathbb{F}$ by (b), for $\beta \in \Delta \setminus \{\alpha, -\alpha\}$, we can consider $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ which is a finite representation of \mathfrak{s}_α with

$\mathfrak{m} = \bigoplus_{\beta+j\alpha \in \Delta, j \in \mathbb{Z}} \mathfrak{g}_{\beta+j\alpha}$, suppose $\beta + j\alpha \in \Delta$, $\dim \mathfrak{g}_{\beta+j\alpha} = 1$ by (a), choose $0 \neq x_{\beta+j\alpha} \in \mathfrak{g}_{\beta+j\alpha}$,

we have $[h_\alpha, x_{\beta+j\alpha}] = (\beta + j\alpha)(h_\alpha)x_{\beta+j\alpha} = (\beta(h_\alpha) + 2j)x_{\beta+j\alpha}$, as j varies in \mathbb{Z} , $\beta(h_\alpha) + 2j$ can't take both 0 and 1, thus the sum of dimension of 0-eigenspace and 1-eigenspace of $ad(h_\alpha)$ on \mathfrak{m} is 1, by Lemma 9.6.15, $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is irreducible, according to Theorem 9.6.11 and Remark 9.6.23, $\mathfrak{m} = \bigoplus_{-r \leq j \leq s} \mathfrak{g}_{\beta+j\alpha}$, for some $r, s \in \mathbb{Z}_{\geq 0}$, and $\beta + j\alpha \in \Delta, \forall -r \leq j \leq s$ which is the α

string through β , note that $ad(x_\alpha)(x_\beta) \neq 0$ as in Theorem 9.6.11, thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

(d) When $\beta = \alpha, -\alpha$, it is trivial. When $\beta \neq \alpha, -\alpha$, as in (c), we know that when j varies from $-r$ to s , $(\beta(h_\alpha) + 2j)$ are all the possible eigenvalues which are integers symmetric over 0, thus $\beta(h_\alpha) + 2s + \beta(h_\alpha) - 2r = 0 \Rightarrow \beta(h_\alpha) = r - s$ is an integer, then $-r \leq -\beta(h_\alpha) \leq s \Rightarrow \beta - \beta(h_\alpha)\alpha \in \Delta$ \square

Connection between roots and root system

Proposition 9.6.27.

(1) Let $V = \mathbb{Q}\Delta$, then $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$

(2) For any $h, k \in \mathfrak{h}$, $K(h, k) = Tr(ad(h)ad(k)) = \sum_{\alpha \in \Delta} \alpha(h)\alpha(k)$

(3) The dual of Killing form restricted on V is rational and positive definite

Proof. Since $K(\cdot, \cdot)|_{\mathfrak{h}}$ is nondegenerate, we define the dual of $K(\cdot, \cdot)|_{\mathfrak{h}}$ on \mathfrak{h}^* as $(\alpha, \beta) := K(t_\alpha, t_\beta)$, which is also nondegenerate

(1) Since $\mathfrak{h}^* = \mathbb{F}\Delta$ by Proposition 9.6.22(a), Pick any basis in Δ , say $\alpha_1, \dots, \alpha_m$, then $\forall \beta \in \Delta$, $\beta = \sum c_j \alpha_j, c_j \in \mathbb{F}$, we have $(\beta, \alpha_k) = \sum c_j (\alpha_j, \alpha_k) \Rightarrow \frac{2(\beta, \alpha_k)}{(\alpha_k, \alpha_k)} = \sum c_j \frac{2(\alpha_j, \alpha_k)}{(\alpha_k, \alpha_k)}$ whereas $\frac{2(\beta, \alpha_k)}{(\alpha_k, \alpha_k)}, \frac{2(\alpha_j, \alpha_k)}{(\alpha_k, \alpha_k)}$ are all integers by Proposition 9.6.26(d), since $(\beta - \sum c_j \alpha_j, \alpha_k) = 0$ and that (\cdot, \cdot) is nondegenerate, meaning c_j 's are the unique solution, hence matrix $((\alpha_j, \alpha_k))$ is nonsingular,

so is matrix $\begin{pmatrix} 2(\alpha_j, \alpha_k) \\ (\alpha_k, \alpha_k) \end{pmatrix}$, thus $c_j \in \mathbb{Q}$, which means $\dim_{\mathbb{Q}} V = \dim_{\mathbb{F}} \mathfrak{h}^*$ and $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$

(2) For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, $ad(h)ad(k)(x_\alpha) = [h, [k, x_\alpha]] = \alpha(k)[h, x_\alpha] = \alpha(h)\alpha(k)x_\alpha$, according to Theorem 9.6.18, we have

$$K(h, k) = Tr(ad(h)ad(k)) = 0 + \sum_{\alpha \in \Delta} Tr(ad(h)|_{\mathfrak{g}_\alpha} ad(k)|_{\mathfrak{g}_\alpha}) = \sum_{\alpha \in \Delta} \alpha(h)\alpha(k)$$

Due to Proposition 9.6.22(b), roots always appears in pairs, if let $\Delta^+ \subseteq \Delta$ consists of exactly one from each pair, then $\sum_{\alpha \in \Delta} \alpha(h)\alpha(k) = 2 \sum_{\alpha \in \Delta^+} \alpha(h)\alpha(k)$ (3) By (2), for any $\lambda, \mu \in \mathfrak{h}^*$

$$\begin{aligned} (\lambda, \mu) &= K(t_\lambda, t_\mu) \\ &= \sum_{\alpha \in \Delta} \alpha(t_\lambda)\alpha(t_\mu) \\ &= \sum_{\alpha \in \Delta} K(t_\alpha, t_\lambda)K(t_\alpha, t_\mu) \\ &= \sum_{\alpha \in \Delta} (\alpha, \lambda)(\alpha, \mu) \\ &= 2 \sum_{\alpha \in \Delta^+} (\alpha, \lambda)(\alpha, \mu) \end{aligned}$$

In particular, for any $\beta \in \Delta$, $(\beta, \beta) = 2 \sum_{\alpha \in \Delta^+} (\alpha, \beta)^2 \Rightarrow \frac{2}{(\beta, \beta)} = \sum_{\alpha \in \Delta^+} \left(\frac{2(\alpha, \beta)}{(\beta, \beta)} \right)^2$, where

$\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}$, thus $0 \leq (\beta, \beta) \in \mathbb{Q}$ and equals 0 iff $(\alpha, \beta) = 0, \forall \alpha \in \Delta \Rightarrow \beta = 0$ by the nondegeneracy of $(,)$, thus $(,)|_V$ is rational and positive definite on a basis, which implies it is rational and positive definite \square

Remark 9.6.28. When $\mathbb{F} = \mathbb{C}$, note that $\mathbb{Q}\Delta < \mathbb{R}\Delta < \mathbb{C}\Delta$, we can view V embedded in the Euclidean space $V_{\mathbb{R}} := \mathbb{R}\Delta = V \otimes_{\mathbb{Q}} \mathbb{R}$ which helps thinking, then we have a root system

Example 9.6.29. Let $\Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then $\{X \in SL(2n, \mathbb{C}) | X^T \Omega X = \Omega\}$ is conjugate to $SO(2n, \mathbb{C})$, thus then also induce isomorphic Lie algebra, hence we can identify $\mathfrak{so}(2n, \mathbb{C})$ with $\{X \in M(2n, \mathbb{C}) | \Omega X^T + X \Omega = 0\}$ which is the same as $\left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in M(2n, \mathbb{C}) \middle| B^T = -B, C^T = -C \right\} =: \mathfrak{g}$, then one Cartan subalgebra of \mathfrak{g} will be $\mathfrak{h} = \left\{ \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in M(2n, \mathbb{C}) \middle| D = \text{diag}(d_1, \dots, d_n) \right\}$, note that

$$\begin{aligned} \left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \right] &= (d_i - d_j) \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix} \\ \left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} \right] &= (d_i + d_j) \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix} \\ \left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix} \right] &= -(d_i + d_j) \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix} \end{aligned}$$

9.7 Root system

Delta is finite spanning set of V , then there is at most one $s: V \rightarrow V$, maps Delta to Delta, $s^2=1$ and $s\alpha = -\alpha$

Lemma 9.7.1. V is a finite dimensional vector space over a field of characteristic 0, $\Delta \subseteq V$ is a finite set such that $V = \text{Span } \Delta$, then for any $\alpha \in \Delta$, there is at most one linear map $s : V \rightarrow V$ such that $s^2 = 1$, $s\alpha = -\alpha$, $s(\Delta) \subseteq \Delta$

Proof. Suppose s, t both satisfy the condition, $sv = v + (s-1)v$, $s(s-1)v = s^2v - sv = v - sv = -(s-1)v$, thus $(s-1)v \in \langle \alpha \rangle \Rightarrow sv = v + f(v)\alpha$, where $f \in V^*$, similarly, $tv = v + g(v)\alpha$, where $g \in V^*$, thus $stv = s(v + g(v)\alpha) = v + g(v)\alpha + f(v + g(v)\alpha)\alpha = v + g(v)\alpha + f(v)\alpha + f(v)g(v)\alpha$, and since $s\alpha = \alpha + f(\alpha)\alpha = -\alpha \Rightarrow f(\alpha) = -2$, thus check $(st)^nv = v + n(f(v) - g(v))\alpha$, but $(st)^n = 1$ for some n because st is just a permutation of Δ , thus $f = g \Rightarrow s = t$ \square

Remark 9.7.2. $s^2 = 1$ and $s(\Delta) \subseteq \Delta$ implies that s is a permutation of Δ , note that this definition doesn't involve inner product on V , you could see this as a more abstract definition of a reflection

Definition of root system

Definition 9.7.3. $V = \mathbb{R}^n$ is the standard Euclidean space with the standard inner product (\cdot, \cdot) , a **root system** Δ is a finite subset of V satisfying

1. $V = \langle \Delta \rangle$
2. If $\alpha \in \Delta$, then the only multiples of α are $\pm\alpha$
3. $s_\alpha(\Delta) \subseteq \Delta$, where

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$$

is the reflection across the hyperplane $H_\alpha = \{\beta \in V \mid (\beta, \alpha) = 0\}$

4. $\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ are called the **Cartan integers**

Remark 9.7.4. $\langle \beta, \alpha \rangle \in \mathbb{Z}$ is the **integrality** condition, and such roots system is called **crystallographic**

$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4 \cos^2 \theta \in \mathbb{Z}$ can only be 0, 1, 2, 3, 4, where θ is the angle between α and β , and it is 4 iff $\alpha = \pm\beta$

$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$	0	1	2	3	4
θ	$\pm \frac{\pi}{2}$	$\pm \frac{\pi}{3}$	$\pm \frac{\pi}{4}$	$\pm \frac{\pi}{6}$	0

$a_{ji} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ gives a Cartan matrix A with $D_{ij} = \frac{\delta_{ij}}{(\alpha_i, \alpha_i)}$, $S_{ij} = 2(\alpha_i, \alpha_j)$

Conversely, given a generalized Cartan matrix, we can find the corresponding Lie algebra, see Kac-Moody algebra

Remark 9.7.5. Thanks to Lemma 9.7.1, s_α is the unique linear map $V \rightarrow V$ such that $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\Delta) \subseteq \Delta$

Definition 9.7.6. Let (V, Δ) be a root system, define the coroot of $\alpha \in \Delta$ to be $\alpha^\vee = \frac{2}{(\alpha, \alpha)} \alpha$, and let $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$, then (V, Δ^\vee) is also a root system

alpha not equal to pm beta are roots, $(\alpha, \beta) > 0 \Rightarrow \alpha - \beta$ is a root

Lemma 9.7.7. $\alpha \neq \pm\beta \in \Delta$. If $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta \in \Delta$, if $\langle \alpha, \beta \rangle < 0$, then $\alpha + \beta \in \Delta$

Proof. Note that $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in \Delta$, $s_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in \Delta$, and $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4 \cos^2 \theta \in \mathbb{Z}$ can only be 0, 1, 2, 3, where θ is the angle between α and β , if $\langle \alpha, \beta \rangle > 0 \Leftrightarrow (\alpha, \beta) > 0$, then $\langle \alpha, \beta \rangle = 1, 2, 3$ or $\langle \alpha, \beta \rangle = 2, 3, \langle \beta, \alpha \rangle = 1$, hence either $\alpha - \beta$ or $\beta - \alpha$ will be in Δ , but then the other will also be in Δ . Similarly, if $\langle \alpha, \beta \rangle < 0 \Leftrightarrow (\alpha, \beta) < 0$, then $\langle \alpha, \beta \rangle = -1, -2, -3$ or $\langle \alpha, \beta \rangle = -2, -3, \langle \beta, \alpha \rangle = -1$, hence $\alpha + \beta \in \Delta$ \square

Definition 9.7.8. $\Delta^+ \subset \Delta$ is a set of **positive roots** if $\alpha \in \Delta \Rightarrow$ precisely one of $\alpha, -\alpha$ is in Δ^+ , and $\alpha, \beta \in \Delta^+ \Rightarrow \alpha + \beta \in \Delta^+$, thus we can correspondingly define negative roots $\Delta^- = \Delta \setminus \Delta^+ = -\Delta^+$. Any hyperplane H that doesn't intersect Δ separates Δ into Δ^+ and Δ^- . $\gamma \in V$ is called **regular** if $(\gamma, \alpha) \neq 0, \forall \alpha \in \Delta$, then $H_\gamma = \{v \in V | (\gamma, v) = 0\}$ separates Δ into $\Delta^+(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) > 0\}$ and $\Delta^-(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) < 0\}$. Conversely, given a hyperplane H that doesn't intersect Δ and separates Δ into Δ^+ and Δ^- , we can find $\gamma \in V$ such that $H = H_\gamma, \Delta^+ = \Delta^+(\gamma), \Delta^- = \Delta^-(\gamma)$

Definition 9.7.9. $\gamma \in V$ is **dominant** if $(\gamma, \alpha) \geq 0, \forall \alpha \in \Delta^+$ and **strictly dominant** if $(\gamma, \alpha) > 0, \forall \alpha \in \Delta^+$

Definition 9.7.10. $\alpha \in \Delta^+$ is **decomposable** if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Delta^+, \alpha$ is a **simple root** of Δ^+ if it is not decomposable. $S = \{\alpha_1, \dots, \alpha_m\}$ is a **base** for Δ^+ if for any $\alpha \in \Delta^+$, there are $c_i \in \mathbb{Z}_{\geq 0}$ such that $\alpha = \sum_i c_i \alpha_i$, which also implies that for any $\alpha \in \Delta^- = -\Delta^+$, there are $c_i \in \mathbb{Z}_{\leq 0}$ such that $\alpha = \sum_i c_i \alpha_i$

v_1, \dots, v_m on one side of a hyperplane, $(v_i, v_j) < 0 \Rightarrow v_i$ linearly independent

Lemma 9.7.11. $S = \{v_1, \dots, v_m\}$ are on one side of a hyperplane H , and $(v_i, v_j) < 0, \forall i \neq j$, then S is linearly independent

Proof. Suppose $\sum_{i=1}^m a_i v_i = 0$ and not all a_i 's are zero, then we can rewrite as $\sum_{k \in K} a_k v_k = \sum_{l \in L} -a_l v_l$, where $a_k > 0, \forall k \in K, a_l < 0, \forall l \in L$, then we have $0 \leq \left(\sum_{k \in K} a_k v_k, \sum_{l \in L} -a_l v_l \right) = \sum_{k \in K, l \in L} -a_k a_l (v_k, v_l) < 0$ which is a contradiction \square

Lemma 9.7.12. S is the set of simple roots of Δ^+ , then S is a base for Δ^+ , and S is linearly independent

Proof. It is obvious that S is a base of Δ^+ by definition. Suppose $\alpha \neq \beta \in S, \alpha \neq -\beta$ is obvious, hence $(\alpha, \beta) \neq 0$. If $(\alpha, \beta) > 0$, then by Lemma 9.7.7, we have $\alpha - \beta \in \Delta$, if $\alpha - \beta \in \Delta^+$, then $\alpha = \beta + (\alpha - \beta)$ gives a contradiction, if $\alpha - \beta \in \Delta^-$, then $\beta - \alpha \in \Delta^+$ and $\beta = \alpha + (\beta - \alpha)$ gives a contradiction, therefore $(\alpha, \beta) < 0$. By lemma 9.7.11, we know S is linearly independent \square

Remark 9.7.13. Given a set of positive roots, there is precisely one base which is the set of simple roots

Conjugacy of roots and Weyl group

Lemma 9.7.14. $\sigma \in \text{GL}(V), \sigma(\Delta) \subseteq \Delta$, then for any $\alpha \in \Delta, \sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$, moreover, $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$

Proof. $\sigma^m = 1$ for some m since there are only finitely many choices of maps $\Delta \rightarrow \Delta$, thus σ is a permutation on Δ , hence $\sigma s_\alpha \sigma^{-1}(\Delta) \subseteq \Delta, (\sigma s_\alpha \sigma^{-1})^2 = 1$ and $\sigma s_\alpha \sigma^{-1} \sigma \alpha = -\sigma \alpha$ implies $\sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$ by Lemma 9.7.1. Compare $s_{\sigma\alpha}(\sigma\beta) = \sigma\beta - \langle \sigma\beta, \sigma\alpha \rangle \sigma\alpha$, and $\sigma s_\alpha \sigma^{-1}(\sigma\beta) = \sigma s_\alpha \beta = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma\beta - \langle \beta, \alpha \rangle \sigma\alpha$, we get $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$ \square

Definition 9.7.15. The **Weyl group** W of Δ is $\langle s_\alpha | \alpha \in \Delta \rangle$. $|W| < \infty$ since $s_\alpha(\Delta) \subseteq \Delta$. W is a subgroup of $O(V)$

Remark 9.7.16. W is a finite Coxeter group

Definition 9.7.17. **Weyl chambers** are connected components of $V \setminus \bigcup_{\alpha \in \Delta} H_\alpha$, as the intersection of open half spaces, Weyl chambers are open convex, each regular $\gamma \in V$ by definition belongs to precisely one Weyl chamber denote as $C(\gamma), C(\gamma) = C(\gamma')$ iff γ, γ' are on the side

for every hyperplane H_α iff $\Delta^+(\gamma) = \Delta^+(\gamma')$, thus Weyl chambers are in one to one correspondence with the sets of positive roots, the fundamental Weyl chamber associated to Δ^+ , or rather, associated to S is $\{\gamma \in V \mid (\gamma, \alpha) > 0, \forall \alpha \in S\}$. For any $\sigma \in W$, since $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$, $\sigma(\Delta^+(\gamma)) = \Delta^+(\sigma\gamma)$, $\sigma(S(\gamma)) = S(\sigma\gamma)$, $\sigma(C(\gamma)) = C(\sigma\gamma)$

For any positive nonsimple root α , there exists simple root β such that $\alpha - \beta$ is a positive root

Lemma 9.7.18. If $\alpha \in \Delta^+ \setminus S$, then there exists $\beta \in S$ such that $\alpha - \beta \in \Delta^+$

Proof. It is obvious that $(\alpha, \beta) \neq 0, \forall \beta \in \Delta^+$, suppose $(\alpha, \beta) < 0, \forall \beta \in S$, then by Lemma 9.7.11, $S \cup \{\alpha\}$ is linearly independent which is impossible, thus $(\alpha, \beta) > 0$ for some $\beta \in \Delta^+$, by Lemma 9.7.7, $\alpha - \beta \in \Delta$, but since $\alpha = \sum_{\alpha_i \in S} c_i \alpha_i$, and $\beta = \alpha_j$ for some j , since some $c_i > 0$, it

necessarily has to be that $\alpha - \beta = (c_j - 1)\beta + \sum_{i \neq j} c_i \alpha_i \in \Delta^+$ □

Lemma 9.7.19. Each $\alpha \in \Delta^+$ can be written as $\alpha_1 + \cdots + \alpha_k$, $\alpha_i \in S$, here α_i may repeat, such that partial sums are all positive roots, i.e. $\alpha_i + \cdots + \alpha_k \in \Delta^+$

Proof. Each $\alpha \in \Delta^+$ can be written uniquely as a sum of simple roots, by induction on the number of summands and Lemma 9.7.18 □

If α is a simple root, then s_α permutes positive roots except α

Lemma 9.7.20. If $\alpha \in S$, then s_α permutes $\Delta^+ \setminus \{\alpha\}$

Proof. For any $\beta \in \Delta^+ \setminus \{\alpha\}$, $\beta = \sum_{\alpha_i \in S} c_i \alpha_i$, $c_j > 0$ for some $\alpha_j \neq \alpha$, then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ still has some positive coefficient, thus it has to be positive, and $s_\alpha(\beta) \neq \alpha = s_\alpha(-\alpha)$ □

$\delta = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$, then for any simple root α , $s_\alpha(\delta) = \delta - \alpha$

Corollary 9.7.21. Let $\delta = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$, then $s_\alpha(\delta) = \delta - \alpha, \forall \alpha \in S$

$s = s_1 \cdots s_q$ is of minimal length $\Rightarrow s_1 \cdots s_q(\alpha_q)$ is a negative root

Lemma 9.7.22. Use $\alpha \prec 0$ and $\alpha \succ 0$ to mean positive and negative roots, suppose $\alpha_i \in S, i = 1, \dots, q$, and denote $s_i := s_{\alpha_i}$, if $s_1 \cdots s_{q-1}(\alpha_q) \prec 0$, then $s_1 \cdots s_q = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1}$. In particular, suppose $s = s_1 \cdots s_q$ is of the smallest length, then $0 \prec s_1 \cdots s_{q-1} \alpha_q = -s_1 \cdots s_q \alpha_q \Rightarrow s_1 \cdots s_q \alpha_q \prec 0$

Proof. Write $\beta_i = s_i \cdots s_{q-1} \alpha_q$, then $\beta_q = \alpha_q \succ 0$, $\beta_1 \prec 0$, thus there must be some $1 \leq p < q$ such that $\beta_{p+1} \succ 0$ but $\beta_p = s_p \beta_{p+1} \prec 0$, by lemma 9.7.20, thus β_{p+1} can only be α_p , hence by Lemma 9.7.14, we have

$$\begin{aligned} s_p &= s_{\alpha_p} = s_{\beta_{p+1}} = s_{s_{p+1} \cdots s_{q-1} \alpha_q} = (s_{p+1} \cdots s_{q-1}) s_q (s_{p+1} \cdots s_{q-1})^{-1} \\ &\Rightarrow s_p \cdots s_{q-1} = s_{p+1} \cdots s_q \\ &\Rightarrow s_1 \cdots s_q = s_1 \cdots s_p s_{p+1} \cdots s_q = s_{p+1} \cdots s_p s_{p+1} \cdots s_{q-1} = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1} \end{aligned}$$

□

Lemma 9.7.23. Denote $n(\sigma)$ the number of positive roots that σ send to negative. $l(\sigma) = n(\sigma)$. Hence there is a unique element w_o such that $w_o(S) = -S$ of maximal length, and $w_o^2 = 1$

Some properties about Weyl group and Weyl chambers

Theorem 9.7.24.

- (a) Let $\gamma \in V$ be regular, then there exists some $\sigma \in W$ such that $\Delta^+(\sigma\gamma) = \Delta^+$, namely, W acts transitively on Weyl chambers
- (b) If S' is another base, then there exists some $\sigma \in W$ such that $\sigma(S') = S$, namely, W acts transitively on bases
- (c) If α is any root, then there exists some $\sigma \in W$ such that $\sigma(\alpha) \in S$
- (d) W is generated by s_α 's for $\alpha \in S$

- (e) If $\sigma \in W$, then $\sigma(S) \subseteq S \Rightarrow \sigma = 1$, namely, W acts freely and transitively(regularly) on bases(and Weyl chambers)

Proof. Let $W' \leq W$ be the subgroup of $O(n)$ generated by s_α 's for $\alpha \in S$

- (a) Let $\delta = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$, choose $\sigma \in W'$ such that $(\sigma\gamma, \delta)$ is the biggest possible, then for any $s_\alpha \in W'$, due to Corollary 9.7.21, we have $(\sigma\gamma, \delta) \geq (s_\alpha \sigma\gamma, \delta) = (\sigma\gamma, s_\alpha \delta) = (\sigma\gamma, \delta - \alpha) = (\sigma\gamma, \delta) - (\sigma\gamma, \alpha) \Rightarrow (\sigma\gamma, \alpha) \geq 0$, since γ is regular, so is $\sigma\gamma$, thus $(\sigma\gamma, \alpha) > 0, \forall \alpha \in S$, $\sigma(C(\gamma)) = C(S)$, i.e. W acts transitively on Weyl chambers
- (b) Directly from (a)
- (c) Thanks to (b), it suffices to prove that each root lies in some base, there exists $\gamma \in H_\alpha \setminus \bigcup_{\beta \in \Delta \setminus \{\pm\alpha\}} H_\beta$, and the perturb γ slightly so that $0 < (\gamma, \alpha) < |(\gamma, \beta)|, \beta \in \Delta \setminus \{\pm\alpha\}$, then $\alpha \in S(\gamma)$
- (d) By (c), if $\alpha \in \Delta, \beta \in S$, then there exists $\sigma \in W'$ such that $\sigma\alpha = \beta$, then $s_\beta = s_{\sigma\alpha} = \sigma s_\alpha \sigma^{-1}$, thus $s_\alpha = \sigma^{-1} s_\beta \sigma \in W'$
- (e) By (d), $\sigma \in W$ can be written as $s = s_{\alpha_1} \cdots s_{\alpha_q}, \alpha_i \in S$ and suppose it is of minimal length, by Lemma 9.7.22, $\sigma(\alpha_q) \prec 0$ contradicting $\sigma(S) \subseteq S$

□

Proposition 9.7.25. The root system is irreducible iff the Lie algebra is simple

9.8 Dynkin diagram

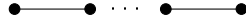
Definition 9.8.1. A **generalized Cartan matrix** A is such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, if $a_{ij} = 0$, then $a_{ji} = 0$, $A = DS$ for some diagonal matrix D and symmetric matrix S , i.e. A is symmetrizable. Note that D would have nonzero diagonal entries, we can pick positive entries, A is a **Cartan matrix** if S is positive definite A is **decomposable** if $a_{ij} = 0$, $i \in I, j \in J$ for some $\{1, \dots, n\} = I \sqcup J$, i.e. A can be diagonalized by blocks

An indecomposable matrix A is of **finite type** if all principal minors are positive, **affine type** if all proper principal minors are positive and $\det A = 0$, **indefinite type** otherwise

Definition 9.8.2. S is a set of positive simple roots, the **Dynkin diagram** is a graph with nodes simple roots, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges between α, β which can only be 0, 1, 2, 3, and an arrow from α to β if $\langle \alpha, \beta \rangle > 1$. The **Coxeter diagram** of the Weyl group W is just the Dynkin diagram without arrows. The **Coxeter graph** of it is the underlying graph

Theorem 9.8.3. We can recover the root system through Dynkin diagram

Definition 9.8.4. Type A_n corresponds to Dynkin diagram



Example 9.8.5. \mathfrak{sl}_{n+1} corresponds to type A_n

Definition 9.8.6. Type B_n corresponds to Dynkin diagram



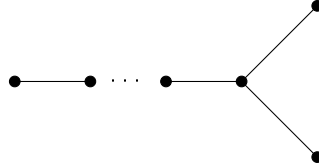
Example 9.8.7. \mathfrak{so}_{2n+1} corresponds to type B_n

Definition 9.8.8. Type C_n corresponds to Dynkin diagram



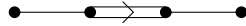
Example 9.8.9. \mathfrak{sp}_{2n} corresponds to type C_n

Definition 9.8.10. Type D_n corresponds to Dynkin diagram



Example 9.8.11. \mathfrak{so}_{2n} corresponds to type D_n

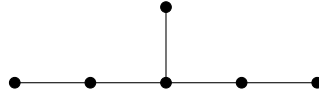
Definition 9.8.12. Type F_4 corresponds to Dynkin diagram



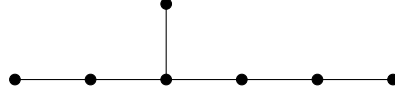
Definition 9.8.13. Type G_4 corresponds to Dynkin diagram



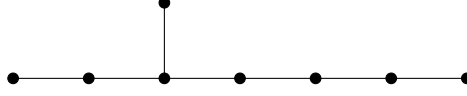
Definition 9.8.14. Type E_6 corresponds to Dynkin diagram



Definition 9.8.15. Type E_7 corresponds to Dynkin diagram



Definition 9.8.16. Type E_8 corresponds to Dynkin diagram



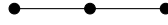
Remark 9.8.17. The number in the subscript is the number of nodes. In particular, we have $A_1 = B_1 = C_1$



$B_2 = C_2$



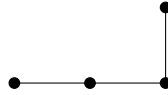
$D_3 = A_3$



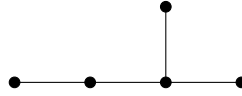
$D_2 = A_1 A_1$

$E_3 = A_2 A_1$

$E_4 = A_4$



$E_5 = D_5$



Cartan-Killing classification

Theorem 9.8.18 (Cartan-Killing classification). *The above Dynkin diagrams classifies simple Lie algebras*

Proof. Consider the **admissible sets** of Euclidean space V , $A = \{v_1, \dots, v_n\}$ of linearly independent unit vectors with $(v_i, v_j) \leq 0$ and $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ if $i \neq j$. Define Coxeter diagram Γ_A for A to have vertices v_1, \dots, v_n , and $d_{ij} = 4(v_i, v_j)^2$ edges between v_i and v_j if $i \neq j$. Assume that Γ_A is connected

a) The number of vertices in Γ_A joined by at least one edge is at most $|A| - 1$

$v = v_1 + \dots + v_n \neq 0$ satisfies $(v, v) = n + 2 \sum_{i < j} (v_i, v_j) > 0$, thus $n > - \sum_{i < j} 2(v_i, v_j) =$

$\sum_{i < j} \sqrt{d_{ij}} \geq N$, where N is the number of pairs v_i, v_j such that $d_{ij} \geq 1$

b) The graph Γ_A contains no cycles

The vectors in a cycle of Γ_A form an admissible set which contradicts a)

c) No vertex in Γ_A has more than 3 edges

Let w be a vertex of Γ_A with adjacent vertices w_1, \dots, w_k , then $(w_i, w_j) = 0$ for $i \neq j$. Let $U = \text{Span}_{\mathbb{R}}(w_1, \dots, w_k, w)$, and extend $\{w_1, \dots, w_k\}$ to an orthonormal basis of U , say by adjoining w_0 . Clearly $(w, w_0) \neq 0$ and $w = \sum_{i=0}^k (w, w_i) w_i$. Hence $1 = (w, w) = \sum_{i=0}^k (w, w_i)^2$,

$\sum_{i=1}^k (w, w_i)^2 < 1$, w has no more than 3 edges

- d) If Γ_A has triple edge, then by c), the Γ_A can only be G_2
- e) Assume Γ_A has a subgraph which is a line along w_1, \dots, w_n , if we replace this subgraph with $w = w_1 + \dots + w_n$, then it is still an admissible set

$$(w, w) = n + 2 \sum_{i=1}^{n-1} (w_i, w_{i+1}) = n - (n-1) = 1, \text{ by d) any vertex } v \text{ has at most edges linked with one such } w_i, \text{ hence } (v, w) = (v, w_i), \text{ this gives an admissible set}$$

- f) A branch point is a vertex having more than 2 adjacent vertices, in this case, exactly 3. Γ_A has only one double edge, or only one branch point, or neither, but not both. Note that if Γ_A has no brach points and double edges corresponds to A_n

If Γ_A has two double edges between w_1, w_2 and v_1, v_2 , then they can be linked through a line, by e), we can collapse it into a single vertex, but this will contradict c)

- g) If Γ_A has a subgraph which is a line through w_1, \dots, w_n , let $w = \sum i w_i$, then $(w, w) = \frac{n(n+1)}{2}$

- h) If Γ_A has a double edge, then Γ_A is F_4 or B_n

By f) we know Γ_A is a line through $v_1, \dots, v_p, w_q, \dots, w_1$, $q \geq p \geq 1$ with single edges except v_p, w_q , let $v = \sum i v_i$, $w = \sum i w_i$, then

$$(v, w)^2 = (p v_p, q w_q)^2 = \frac{p^2 q^2}{2}$$

Since v, w are linearly independent, by Cauchy Schwarz inequality, we have

$$\frac{p^2 q^2}{2} = (v, w)^2 < (v, v)(w, w) = \frac{p(p+1)q(q+1)}{4}$$

Which implies $(p-1)(q-1) < 2$, thus if $p = 1$, then q can be any positive interger, giving B_n , if $p = 2$, then $q = 2$, giving F_4

- i) If Γ_A has a branch point, then Γ_A is D_n or E_6, E_7, E_8

Γ_A is has three branch lines v_1, \dots, v_p, x and w_1, \dots, w_q, x together with z_1, \dots, z_r, x , $p \geq q \geq r$, let $v = \sum i v_i$, $w = \sum i w_i$, $z = \sum i z_i$ which are pairwise orthogonal, $\hat{v}, \hat{w}, \hat{z}$ be normalized vectors of v, w, z , and consider $U = \text{Span}_{\mathbb{R}}(v, w, z, x) = \text{Span}_{\mathbb{R}}(\hat{v}, \hat{w}, \hat{z}, x_0)$, where x_0 is a unit vector orthogonal to v, w, z , then $(x, x_0) \neq 0$

$$1 = (x, x) = (x, \hat{v})^2 + (x, \hat{w})^2 + (x, \hat{z})^2 + (x, x_0)^2$$

Thus by g)

$$\frac{2p^2}{4p(p+1)} + \frac{2q^2}{4q(q+1)} + \frac{2r^2}{4r(r+1)} < 1$$

Hence

$$\frac{1}{1+p} + \frac{1}{1+q} + \frac{1}{1+r} > 1$$

and we know that

$$\frac{1}{1+p} \leq \frac{1}{1+q} \leq \frac{1}{1+r} \leq \frac{1}{2}$$

Hence $r = 1$

$$\frac{1}{1+p} + \frac{1}{1+q} > \frac{1}{2}$$

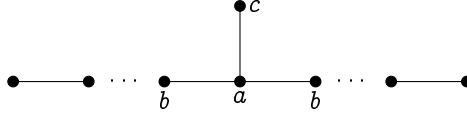
If $q = 1$, then p can be any positive integer, giving D_n , if $q = 2$, then p can only be 2, 3 or 4, giving E_6, E_7, E_8

□

Lemma 9.8.19. (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots and $S = \{\alpha_1, \dots, \alpha_n\}$ be its base, then there exists unique highest root $\gamma \in \Delta$, meaning $\gamma + \alpha_i \in \Delta, \forall \alpha_i \in S$

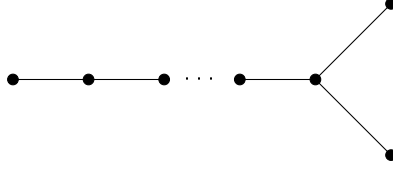
Definition 9.8.20. Let (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots, $S = \{\alpha_1, \dots, \alpha_n\}$ be its base, and γ its unique highest root, the extended Dynkin diagram is the usual Dynkin diagram adding $\alpha_0 = -\gamma$, the number of bonds for each two nodes and direction are still defined as before. Finally, suppose $-\alpha_0 = \sum n_i \alpha_i$, then label γ with Φ , and label node α_i with n_i

Lemma 9.8.21. If the following part of the extended Dynkin diagram

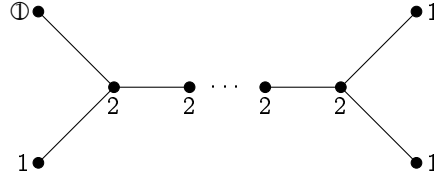


We have $2a = b + c + d$

Example 9.8.22. Consider the classical root system D_n , $\Delta^+ = \{e_i \pm e_j | i < j\}$, $S = \{e_1 - e_2, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}$, then $\gamma = e_1 + e_2$, its usual Dynkin diagram is



So its extended Dynkin diagram is



Chapter 10

Cluster algebra

10.1 Cluster algebra

$\mathbb{Z}\mathbb{P}$ is a UFD

Lemma 10.1.1. \mathbb{P} is a torsion free abelian group written multiplicatively, then the group ring $\mathbb{Z}\mathbb{P}$ is a UFD

Proof. Finitely generated torsion free abelian groups are free □

Definition 10.1.2 (Exchange pattern). $I = \{1, \dots, n\}$, \mathbb{T}_n is the regular n tree, the coefficient group \mathbb{P} is a torsion free abelian group under multiplication, thus the group ring $\mathbb{Z}\mathbb{P}$ is a domain.

Cluster variables are $\mathbf{x}(t) = \{x_i(t)\}_{i \in I}$ for $t \in \mathbb{T}_n$ such that for $\neq j$ and $t \xrightarrow{j} t'$

$$x_i(t) = x_i(t')$$

$\mathcal{M} = \{M_j(t)\}$ are monomials such that

$$M_j(t)(\mathbf{x}) = p_j(t) \prod_i x_i^{b_i}, p_j(t) \in \mathbb{P}, b_i \geq 0$$

and for $t \xrightarrow{j} t'$, b_i 's depend on j and t

$$x_j(t)x_j(t') = M_j(t)(\mathbf{x}(t)) + M_j(t')(\mathbf{x}(t'))$$

satisfying **exchange pattern**

$$(E1) \ x_j \nmid M_j(t)$$

$$(E2) \ x_i \mid M_j(t) \Rightarrow x_i \nmid M_j(t') \text{ for } t \xrightarrow{j} t'$$

$$(E3) \ x_j \mid M_i(t) \Leftrightarrow x_i \mid M_j(t') \text{ for } t \xrightarrow{i} t' \xrightarrow{j} t_1$$

$$(E4) \ \frac{M_i(t_3)}{M_i(t_4)} = \frac{M_i(t_2)}{M_i(t_1)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}} \text{ for } t_1 \xrightarrow{i} t_2 \xrightarrow{j} t_3 \xrightarrow{i} t_4, \ M_0 = (M_j(t_2) + M_j(t_3))|_{x_i=0}$$

Remark 10.1.3. The substitution $x_j \leftarrow \frac{M_0}{x_j}$ is effectively a monomial. Since if $M_j(t_2)$ nor $M_j(t_3)$ contain x_i , then $M_i(t_2)$ nor $M_i(t_3)$ contain x_j which it substitute for nothing

$$\frac{M_i(t_2)}{M_i(t_1)} = \left(\frac{M_i(t_2)}{M_i(t_1)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}} \right) \Big|_{x_j \leftarrow \frac{M_0}{x_j}} = \frac{M_i(t_3)}{M_i(t_4)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Definition 10.1.4. There is an involution between $(\mathbf{x}, \mathcal{M})$ and $(\mathbf{x}', \mathcal{M}')$ where $x'_j(t) = x_j(t')$, $M'_j(t) = M_j(t')$ for every $t \xrightarrow{j} t'$

Definition 10.1.5. Suppose $J \subseteq I$ is a subset of size m , delete sides labeled in $I - J$ in \mathbb{T}_n and choose a connected component which would be \mathbb{T}_m , add to the coefficient group x_k 's $k \in I - J$. This is called a restriction

Definition 10.1.6. Exchange pattern on exponents is a family of $B(t)$ such that for each $t \xrightarrow{\quad} t'$

$$\frac{M_j(t)}{M_j(t')} = \frac{p_j(t)}{p_j(t')} \prod_i x_i^{b_{ij}(t)}$$

Thus

$$M_j(t) = p_j(t) \prod_i x_i^{[b_{ij}(t)]_+}, M_j(t') = p_j(t') \prod_i x_i^{[-b_{ij}(t)]_+}$$

Definition 10.1.7. An $n \times n$ matrix B is **sign-skew-symmetric** if $b_{ii} = 0$ and for $i \neq j$, b_{ij}, b_{ji} are both zeros or of opposite signs. B is **skew-symmetrizable** if there is a diagonal matrix D such that DB is skew symmetric, i.e. $d_i b_{ij} = -d_j b_{ji}$. Skew-symmetrizable matrices are obviously sign-skew-symmetric

Lemma on $(|a|b + a|b|)/2$

Lemma 10.1.8.

$$\frac{|a|b + a|b|}{2} = \begin{cases} ab & a, b > 0 \\ -ab & a, b < 0 \\ 0 & ab < 0 \end{cases} = \begin{cases} \text{sgn}(a)[ab]_+ & \text{sgn}(b)[ab]_+ \\ 0 & ab < 0 \end{cases}$$

Note. $|a| = [a]_+ + [-a]_+$

Definition 10.1.9. A **mutation** on a $m \times n$ ($m > n$) matrix B in direction k denoted by μ_k is given by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

Here $\mu_k(B) = B'$. μ_k is involutive

Theorem 10.1.10. If $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \xrightarrow{k} t'$, then it gives a exchange pattern

Proof. Suppose $B(t)$ is an exchange pattern, then $B(t)$ is obviously sign-skew-symmetric. For $t \xrightarrow{k} t'$, we have

$$\frac{M_k(t)}{M_k(t')} = \frac{p_k(t)}{p_k(t')} \prod_i x_i^{b_{ik}}, \frac{M_k(t')}{M_k(t)} = \frac{p_k(t')}{p_k(t)} \prod_i x_i^{b'_{ik}}$$

Hence $b'_{ik} = -b_{ik}$. Consider $t_1 \xrightarrow{j} t' \xrightarrow{k} t \xrightarrow{j} t_2$

$$\frac{M_j(t')}{M_j(t_1)} = \frac{M_j(t)}{M_j(t_2)} \Big|_{x_k \leftarrow \frac{M_0}{x_k}}$$

becomes

$$\frac{p_j(t')}{p_j(t_1)} \prod_i x_i^{b'_{ij}} = \frac{p_j(t)}{p_j(t_2)} \prod_i x_i^{b_{ij}} \Big|_{x_k \leftarrow \frac{M_0}{x_k}}$$

Where

$$M_0 = \left(p_k(t) \prod_i x_i^{[b_{ik}]_+} + p_k(t') \prod_i x_i^{[-b_{ik}]_+} \right) \Big|_{x_j=0}$$

Case 1: $b_{jk} > 0 \Leftrightarrow b_{kj} < 0$, then $M_0 = p_k(t') \prod_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j, k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j, k} x_i^{[-b_{ik}]_+} \right)^{b_{kj}} = \prod_{i \neq j, k} x_i^{b_{ij} + b_{kj}[-b_{ik}]_+} x_k^{-b_{kj}}$$

Case 2: $b_{jk} < 0 \Leftrightarrow b_{kj} > 0$, then $M_0 = p_k(t') \prod_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j, k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j, k} x_i^{[b_{ik}]_+} \right)^{b_{kj}} = \prod_{i \neq j, k} x_i^{b_{ij} + b_{kj}[b_{ik}]_+} x_k^{-b_{kj}}$$

Case 3: $b_{jk} = 0 \Leftrightarrow b_{kj} = 0$, then

$$\prod_{i \neq j, k} x_i^{b'_{ij}} = \prod_{i \neq j, k} x_i^{b_{ij}}$$

Therefore $b'_{kj} = -b_{kj}$ and $b'_{ij} = b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+$

Conversely, if $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \xrightarrow{k} t'$, take

$$M_k(t) = \prod_i x_i^{[b_{ik}(t)]_+}, M_k(t') = \prod_i x_i^{[-b_{ik}(t)]_+}$$

for $t \xrightarrow{k} t'$, then obviously $x_k \nmid M_k(t)$ since $b_{kk} = 0$ and

$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow -b_{jk} < 0 \Rightarrow x_j \nmid M_k(t')$$

For $t \xrightarrow{k} t' \xrightarrow{j} t_1$

$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow b'_{kj} = -b_{kj} > 0 \Leftrightarrow x_k \mid M_j(t')$$

For $t_1 \xrightarrow{j} t' \xrightarrow{k} t \xrightarrow{j} t_2$, it is the exact argument above by taking $p_j(t) \equiv 1$ □

Mutation of a skew-symmetrizable matrix preserves the skew-symmetrizing matrix

Proposition 10.1.11. Given a skew-symmetrizable matrix B , the all possible mutations $B(t)$ in \mathbb{T}_n are skew-symmetrizable with the same skew-symmetrizing matrix D

Proof. True for each mutation μ_k □

Remark 10.1.12. For cluster algebra of rank $n \leq 2$, the exchange pattern is skew-symmetrizable. If $n = 1$, $B(t) \equiv 0$. If $n = 2$, $B(t_n) = (-1)^n \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$

Definition 10.1.13. Denote $2n$ tuple $\mathbf{p}(t)$ the coefficients $p_j(t), p_j(t')$ for $t \xrightarrow{j} t'$. $\Sigma(t) = (\mathbf{x}(t), \mathbf{p}(t), B(t))$ is a **seed**, $\mathbf{x}(t)$ is the **cluster** of the seed. If we assume $\mathbf{x}(t_0)$ are algebraically independent ($\mathbf{x}(t_0)$ is a cluster of rank n), then so are $\mathbf{x}(t)$ since they are all mutationally equivalent. Denote the collection of all cluster variables \mathcal{X} , the collection of all coefficients \mathcal{P} , the collection of exchange matrices \mathcal{B} , the collection of $M_j(t)$'s \mathcal{M} , the collection of seeds \mathcal{S} . We can take $\mathcal{F} = \mathbb{Z}\mathbb{P}(x_1, \dots, x_n)$ to be the **ambient field**, \mathbf{x} can be some cluster $\mathbf{x}(t_0)$. The **cluster algebra** is the subalgebra $\mathbb{Z}\mathcal{P}[\mathcal{X}]$

Proposition 10.1.14. Given $B(t)$ that give rise to exchange pattern, the coefficients must satisfy

$$p_i(t_1)p_i(t_3)p_i(t_3)^{[b_{ji}(t_3)]_+} = p_i(t_2)p_i(t_4)p_i(t_2)^{[b_{ji}(t_2)]_+} \quad (10.1.1)$$

Proof. For $t_1 \xrightarrow{i} t_2 \xrightarrow{j} t_3 \xrightarrow{i} t_4$

$$\frac{p_i(t_3)}{p_i(t_4)} \prod_k x_k^{b_{ki}(t_3)} = \frac{M_i(t_3)}{M_i(t_4)} = \frac{M_i(t_3)}{M_i(t_4)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}} = \frac{p_i(t_2)}{p_i(t_1)} \prod_k x_k^{b_{ki}(t_2)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Here

$$M_0 = (M_j(t_2) + M_j(t_3))|_{x_i=0} = \left(p_j(t_2) \prod_k x_k^{[b_{kj}(t_2)]_+} + p_j(t_3) \prod_k x_k^{[b_{kj}(t_3)]_+} \right) \Big|_{x_i=0}$$

Take $x_k = 1$ for $k \neq j$, and use the fact that $B(t)$ are sign-skew-symmetric, we get

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)M_0^{b_{ji}(t_2)}$$

With

Case 1: $b_{ij}(t_2) > 0$. Then $b_{ij}(t_3) < 0$, $M_0 = p_j(t_3)$ and

$$p_i(t_1)p_i(t_3)p_i(t_3)^{b_{ji}(t_3)} = p_i(t_2)p_i(t_4)$$

Case 2: $b_{ij}(t_2) < 0$. Then $b_{ij}(t_3) > 0$, $M_0 = p_j(t_2)$ and

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)p_i(t_2)^{b_{ji}(t_2)}$$

Case 3: $b_{ij}(t_2) = 0$. Then $b_{ij}(t_3) = 0$, $M_0 = p_j(t_2) + p_j(t_3)$, but $b_{ji}(t_2) = b_{ji}(t_3) = 0$, hence

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)$$

□

Note. A trivial solution of (10.1.1) is $p_j(t) = 1$

Proposition 10.1.15. The **universal coefficient group** \mathcal{P} of \mathbb{P} is the free abelian group generated by $p_i(t)$ modulo (10.1.1). \mathcal{P} is torsion free, more precisely, it is the free abelian group generated by $p_i(t_0), p_i(t)$ for every $t_0 \xrightarrow{i} t$ and exactly one of $p_i(t), p_i(t')$ for every $t \xrightarrow{i} t'$ where $t, t' \neq t_0$

Definition 10.1.16. Take the field of rational functions of cluster variables $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}\mathcal{P}$ to be the ambient field \mathcal{F} , all other cluster variables $\mathbf{x}(t)$ are also in \mathcal{F} by Theorem 10.2.3. The **universal cluster algebra** \mathcal{A} is the subalgebra generated by all cluster variables with coefficients in $\mathbb{Z}\mathcal{P}$

M-equivalence

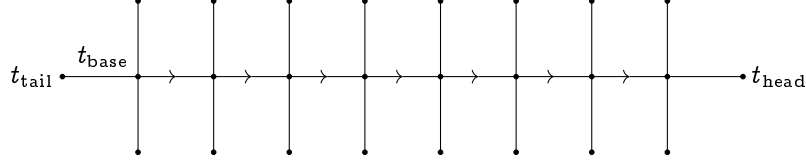
Definition 10.1.17. $t, t' \in \mathbb{T}_n$ are **M-equivalent** if there is a permutation σ of I such that

- $x_{\sigma(i)}(t) = x_i(t')$
- $M_{\sigma(j)}(t)(\mathbf{x}(t)) = M_j(t')(\mathbf{x}(t'))$ and $M_{\sigma(j)}(t_1)(\mathbf{x}(t)) = M_j(t'_1)(\mathbf{x}(t'))$ for $t \xrightarrow{\sigma(j)} t_1$ and $t' \xrightarrow{j} t'_1$

10.2 Laurent phenomenon

Caterpillar lemma

Lemma 10.2.1 (Caterpillar lemma). Define the caterpillar tree $\mathbb{T}_{n,m}$ consists of a spine of $m+2$ nodes, with an orientation from t_{tail} to t_{head} with t_{base} connected to t_{tail} , as illustrated in Figure 10.2.1

Figure 10.2.1: $\mathbb{T}_{4,8}$

T4,8

Let \mathbb{A} be a UFD, exchange polynomial $P \in \mathbb{A}[x_1, \dots, x_n]$ for each edge $t \xrightarrow[j]{P} t'$, denoted $x \xrightarrow[j]{P} t'$ satisfying the generalized exchange pattern

- P doesn't depend on x_j and x_i doesn't divide P
- For $t_0 \xrightarrow[i]{P} t_1 \xrightarrow[j]{Q} t_2$, P, Q_0 are coprime in $\mathbb{A}[x_1, \dots, x_n]$, where $Q_0 = Q|_{x_i=0}$
- For $t_0 \xrightarrow[i]{P} t_1 \xrightarrow[j]{Q} t_2 \xrightarrow[i]{R} t_3$, $LQ_0^b P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$ for some $b \geq 0$ and some Laurent monomial L with coefficients in \mathbb{A} coprime with P

Cluster variables $\mathbf{x}(t) = \{x_i(t)\}$ for $t \in \mathbb{T}_{n,m}$ satisfying for each $t \xrightarrow[i]{P} t'$

- $x_i(t) = x_i(t')$ for any $i \neq j$
- $x_j(t)x_j(t') = P(t)(\mathbf{x}(t))$

Then $\mathbf{x}(t_{\text{head}})$ are Laurent polynomials in $\mathbf{x}(t_0)$ with coefficients in \mathbb{A}

Proof. Write the subring of Laurent polynomials generated by $\mathbf{x}(t)$ as

$$\mathcal{L}(t) = \mathbb{A}[x_1(t)^{\pm 1}, \dots, x_n(t)^{\pm 1}]$$

Make induction on m . If $m = 1$, consider $t_{\text{tail}} = t \xrightarrow[i]{P} t_{\text{base}} = t' \xrightarrow[j]{Q} t_{\text{head}} = t_1$, we have for $k \neq i, j$

$$\begin{aligned} x_k(t_1) &= x_k(t') = x_k(t) \\ x_i(t_1) &= x_i(t') = \frac{P(\mathbf{x}(t))}{x_i(t)} \\ x_j(t_1) &= \frac{Q(\mathbf{x}(t'))}{x_j(t')} = \frac{Q(\mathbf{x}(t))}{x_j(t)} \end{aligned}$$

Now suppose $m \geq 2$, let's show that $X = x_k(t_{\text{head}}) \in \mathcal{L}(t_0)$, by induction, $X \in \mathcal{L}(t_1) \cap \mathcal{L}(t_3)$.

Since $X, x_i(t_1) = \frac{P(\mathbf{x}(t_0))}{x_i(t_0)} \in \mathcal{L}(t_0)$, $X = \frac{f_0}{x_i(t_1)^a}$ for some $f_0 \in \mathcal{L}(t_0)$ and $a \geq 0$, similarly,

$X = \frac{g_0}{x_j(t_2)^b x_i(t_3)^c}$ for some $g_0 \in \mathcal{L}(t_0)$ and $b, c \geq 0$, thanks to Lemma 10.2.2, $X \in \mathcal{L}(t_0)$ \square

Lemma for caterpillar lemma

Lemma 10.2.2. For $t_0 \xrightarrow[i]{P} t_1 \xrightarrow[j]{Q} t_2 \xrightarrow[i]{R} t_3$, $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3) \in \mathcal{L}(t_0)$, and

$$\gcd(x_i(t_1), x_i(t_3)) = \gcd(x_j(t_2), x_i(t_1)) = 1$$

in $\mathcal{L}(t_0)$

Note. $\mathcal{L}(t_0)$ is a UFD, $\mathcal{L}(t_0)^\times$ consists of Laurent monomials with coefficients \mathbb{A}^\times

Proof. Denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, $v = x_i(t_3)$, think of P, Q, R as functions of x_j, x_i, x_j respectively, then

$$\begin{aligned} z &= \frac{P(y)}{x} \\ u &= \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y} \\ v &= \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} + \frac{R\left(\frac{Q(0)}{y}\right)}{z} \end{aligned}$$

$$\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} = R'\left(\frac{Q_0}{y}\right) \frac{Q'(0)}{y} + \frac{1}{2} R''\left(\frac{Q(z)}{y}\right) \Big|_{z=0} z + \dots \equiv R'\left(\frac{Q_0}{y}\right) \frac{Q'(0)}{y} \pmod{z}$$

$$\frac{R\left(\frac{Q_0}{y}\right)}{z} = \frac{L(y)Q_0(y)^b P(y)}{z} = L(y)Q_0(y)^b x$$

Thus $v \in \mathcal{L}(t_0)$

Since $\gcd(P, Q_0) = \gcd(P, L) = 1$

$$\gcd(z, v) = \gcd\left(\frac{P(y)}{x}, L(y)Q_0(y)^b x\right) = \gcd(P(y), L(y)Q_0(y)^b) = 1$$

Since $\frac{Q(z)}{y} \equiv \frac{Q_0}{y} \pmod{z}$

$$\gcd(z, u) = \gcd\left(z, \frac{Q_0}{y}\right) = \gcd(P(y), Q_0) = 1$$

□

Laurent phenomenon

Theorem 10.2.3. *Catepillar lemma 10.2.1 implies that in a cluster algebra, any cluster variable can be expressed as a Laurent polynomial in a given $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$ since there is no subtraction involved*

Proof. $\mathbb{T}_{n,m}$ can be embedded in \mathbb{T}_n . $M_j(t) + M_j(t')$ doesn't depend on x_j and not divisible by x_i for $t \xrightarrow{j} t'$ and any $i \neq j$

For $t_0 \xrightarrow[i]{P} t_1 \xrightarrow[j]{Q} t_2 \xrightarrow[i]{R} t_3$, we have

$$\frac{P}{M_i(t_0)} = 1 + \frac{M_i(t_1)}{M_i(t_0)} = 1 + \frac{M_i(t_2)}{M_i(t_3)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}} = \frac{R}{M_i(t_3)} \Big|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Where $M_0 = (M_j(t_1) + M_j(t_2))|_{x_i=0} = Q_0$, thus

$$\frac{R|_{x_j \leftarrow \frac{Q_0}{x_j}}}{P} = \frac{M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}}}{M_i(t_0)}$$

Note that $M_i(t_0) = p_i(t_0) \prod_k x_k^{[b_{ki}(t_0)]_+}$ and

$$\begin{aligned} M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}} &= p_i(t_3) \prod_k x_k^{[b_{ki}(t_3)]_+} \Big|_{x_j \leftarrow \frac{Q_0}{x_j}} \\ &= p_i(t_3) \left(\frac{Q_0}{x_j} \right)^{[b_{ji}(t_3)]_+} \prod_{k \neq i, j} x_k^{[b_{ki}(t_3)]_+} \\ &= p_i(t_3) Q_0^{[b_{ji}(t_3)]_+} x_j^{-[b_{ji}(t_3)]_+} \prod_{k \neq i, j} x_k^{[b_{ki}(t_3)]_+} \end{aligned}$$

Hence

$$R|_{x_j \leftarrow \frac{Q_0}{x_j}} = \frac{p_i(t_3)}{p_i(t_0)} x_j^{-[b_{ji}(t_3)]_+ - [b_{ji}(t_0)]_+} \prod_{k \neq i, j} x_k^{[b_{ki}(t_3)]_+ - [b_{ki}(t_0)]_+} Q_0^{[b_{ji}(t_3)]_+} P = L Q_0^b P$$

Since the sum of two monomials P doesn't depend on x_i and is not divisible by any x_k for $k \neq i$, Q_0 is a monomial, L is a Laurent monomial, Q_0, P are coprime in $\mathbb{A}[\mathbf{x}]$ and L, P are coprime in $\mathcal{L}[\mathbf{x}]$ \square

10.3 Y-system

A is the Cartan matrix of root system Φ with simple system Π , denote $[\alpha : \alpha_i]$ as the coefficients of $\alpha \in \Phi$, write $\Phi_{\geq -1} = \Phi_+ \cup (-\Pi)$. Since the Coxeter graph is a tree, it is bipartite, up to renaming $I = \{1, \dots, n\} = I_- \sqcup I_+$, $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$ be the indicator. Let $t_\varepsilon = \prod_{i \in I_\varepsilon} s_i$, $t = t_- t_+$ is a Coxeter element, h is the Coxeter number. s_{i_-}, s_{i_+} are reduced words of t_-, t_+ , then

$$w_o = \underbrace{s_{i_-} s_{i_+} \cdots s_{i_\pm}}_{h \text{ times}} = s_{i_o}$$

is the element of longest length

Definition 10.3.1. Suppose Φ is irreducible, then

$$[s_i(\alpha) : \alpha_k] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} [\alpha : \alpha_j] & k = i \\ [\alpha : \alpha_k] & k \neq i \end{cases}$$

Define a piecewise linear modification

$$[\sigma_i(\alpha) : \alpha_k] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} [\alpha : \alpha_j]_+ & k = i \\ [\alpha : \alpha_k] & k \neq i \end{cases}$$

Proposition 10.3.2.

1. σ_i are involutions
2. σ_i, σ_j commutes if i, j are not adjacent in the Coxeter graph
3. σ_i preserves $\Phi_{\geq -1}$

Proposition 10.3.3. Let $\tau_\varepsilon = \prod_{i \in I_\varepsilon} \sigma_i$

1. τ_ε are involutions that preserve $\Phi_{\geq -1}$
2. $\tau_\varepsilon \alpha = t_\varepsilon \alpha$ for $\alpha \in \mathbb{Z}_{\geq 0} \Pi$
3. $\Phi_{\geq -1} \rightarrow \Phi_{\geq -1}^\vee$, $\alpha \mapsto \alpha^\vee$ are τ_ε equivariant, i.e. $(\tau_\varepsilon \alpha)^\vee = \tau_\varepsilon \alpha^\vee$

Definition 10.3.4. A **Y-system** is a family of commuting variables $Y_i(t)$, $i \in I = \{1, \dots, n\}$, $t \in \mathbb{Z}$ such that

$$Y_i(t+1)Y_i(t-1) = \prod_{j \neq i} (1 + Y_j(t))^{-a_{ij}} \quad (10.3.1)$$

Remark 10.3.5. (10.3.1) only really involve those $Y_j(k)$ with $\varepsilon(j) \cdot (-1)^k = \text{const}$, assume $Y_j(k) = Y_j(k+1)$ for $\varepsilon(i) = (-1)^k$, then we have

$$Y_i(k+1) = \begin{cases} \frac{\prod_{j \neq i} (1 + Y_j(k))^{-a_{ij}}}{Y_i(k)} & \varepsilon(i) = (-1)^k + 1 \\ Y_i(k) & \varepsilon(i) = (-1)^k \end{cases}$$

Denote \mathcal{Y} as the collection of all $Y_j(k)$'s, $u_i = Y_i(0)$, define

$$\tau_\varepsilon(u_i) = \begin{cases} \frac{\prod_{j \neq i} (1 + u_j)^{-a_{ij}}}{u_i} & \varepsilon(i) = \varepsilon \\ u_i & \text{otherwise} \end{cases}$$

Theorem 10.3.6 (Zamolodichikov). $Y_i(t)$'s are $2(h+2)$ periodic, i.e. $Y_i(t+2(h+2)) = Y_i(t)$

Theorem 10.3.7. There is a unique family $\{F[\alpha]\}_{\alpha \in \Phi_{\geq -1}}$ of polynomials in u_i such that $F[-\alpha_i] = -1$ and

$$\tau_\varepsilon(F[\alpha]) = \frac{\prod_{\varepsilon(i)=-\varepsilon} (u_i + 1)^{[\alpha^\vee : \alpha_i^\vee]}}{\prod_{\varepsilon(i)=\varepsilon} u_i^{[\alpha^\vee : \alpha_i^\vee]_+}} F[\tau_{-\varepsilon}(\alpha)]$$

Furthermore, $F[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ has constant term 1. Call $F[\alpha]$ Fibonacci polynomials

Any $\alpha \in \Phi_{\geq -1}$ can be written as $\alpha(k, i) = (\tau_- \tau_+)^k(-\alpha_i)$, denote $N[\alpha] = \prod_{j \neq i} F[\alpha(-k, i)]^{-a_{ij}}$,

note that $N[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ also has constant term 1

Theorem 10.3.8. There is a unique bijection $\Phi_{\geq -1} \rightarrow \mathcal{Y}$, $\alpha \mapsto Y[\alpha] = \frac{N[\alpha]}{\mathbf{u}^{\alpha^\vee}}$ such that $Y[-\alpha_i] = u_i$, $\tau_\varepsilon(Y[\alpha]) = Y[\tau_\varepsilon(\alpha)]$

Definition 10.3.9. A **tropical specialization** r_{trop} of a rational expression r is changing the addition $+$ and multiplication \cdot into \oplus and \odot where $a \oplus b = \max(a, b)$, $a \odot b = a + b$

The **compatibility degree** for $\alpha, \beta \in \Phi_{\geq -1}$ is

$$(\alpha || \beta) = (Y[\alpha] + 1)(\beta)_{\text{trop}}$$

Here $(Y[\alpha] + 1)(\beta)$ is evaluation at $\{u_i = [\beta : \alpha_i]\}$. α, β are **compatible** if $(\alpha || \beta) = 0$

$\Delta(\Phi)$ is a simplicial complex with $\Phi_{\geq -1}$ as vertices and mutually compatible subsets of $\Delta(\Phi)$ are simplices, the maximal simplices are **clusters**. The **exchange graph** $E(\Phi)$ is an unoriented graph with clusters as vertices and an edge between clusters which has intersection of cardinality $n - 1$

Remark 10.3.10. $(||)$ is uniquely characterized by

$$\begin{aligned} (-\alpha_i || \beta) &= (Y[-\alpha_i] + 1)(\beta)_{\text{trop}} = (u_i + 1)(\beta)_{\text{trop}} = [\beta : \alpha_i]_+ \\ (\tau_\varepsilon(\alpha) || \tau_\varepsilon(\beta)) &= (Y[\tau_\varepsilon(\alpha)] + 1)(\tau_\varepsilon(\beta))_{\text{trop}} = (\tau_\varepsilon(Y[\alpha]) + 1)(\tau_\varepsilon(\beta))_{\text{trop}} = (\alpha || \beta) \end{aligned}$$

Proposition 10.3.11. Consider perfect bilinear pairing

$$\begin{aligned} \mathbb{Z}\Pi^\vee \times \mathbb{Z}\Pi &\rightarrow \mathbb{Z} \\ (\xi, \gamma) &\mapsto \{\xi, \gamma\} \end{aligned}$$

Where $\{\xi, \gamma\} = \sum \varepsilon(i)[\xi : \alpha_i^\vee][\gamma : \alpha_i]$. Then

$$(\alpha || \beta) = \max(\{\tau_+ \alpha^\vee, \beta\}, \{\alpha^\vee, \tau_+ \beta\}, 0) = \max(-\{\tau_- \alpha^\vee, \beta\}, -\{\alpha^\vee, \tau_- \beta\}, 0)$$

Note. $(||)$ doesn't depend on the choice of the indicator ε

Proposition 10.3.12.

1. $(\alpha || \beta) = (\beta^\vee || \alpha^\vee)$, in particular, if Φ is simply laced, then $(\alpha || \beta) = (\beta || \alpha)$
2. If $(\alpha || \beta) = 0$, then $(\beta || \alpha) = 0$
3. $J \subseteq I$, $\Phi(J) \subseteq \Phi$ is a root subsystem, $(||)$ on $\Phi(J)$ is the same as the restriction

Theorem 10.3.13. $\Delta(\Phi)$ is pure of dimension $n - 1$, and each facet forms a \mathbb{Z} -basis for the root lattice

Theorem 10.3.14. The simplicial cones of all clusters form a complete simplicial fan

Corollary 10.3.15. The geometric realization of $\Delta(\Phi)$ is \mathbb{S}^{n-1}

Conjecture 10.3.16. The simplicial fan of $\Delta(\Phi)$ is the normal fan of some convex polytope $P(\Phi)$

Theorem 10.3.17. $E(\Phi)$ is a regular n tree

Example 10.3.18.

10.4 Associahedron

Definition 10.4.1. Any n regular polygon has $\binom{n}{2} - n = \frac{n(n-3)}{2}$ diagonals, with these as vertices, noncrossing subsets as simplexes, we have given it a abstract simplicial complex structure

10.5 Cluster algebra of geometric type

Semifield is multiplicative torison free

Lemma 10.5.1. Semifield \mathbb{P} is multiplicative torison free

Proof. Suppose $p^m = 1$, then

$$p = \frac{p^m \oplus p^{m-1} \oplus \cdots \oplus p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = \frac{1 \oplus p^{m-1} \oplus \cdots \oplus p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = 1$$

□

Definition 10.5.2. Exchange pattern is **normalized** if \mathbb{P} is a semifield and $p_j(t) \oplus p_j(t') = 1$ for any $t \xrightarrow{j} t'$

Normalized exchange pattern determines the cluster algebra

Proposition 10.5.3. Given p_j, r_j in a semifield \mathbb{P} such that $p_j \oplus r_j = 1$, and exchange matrix $B(t)$ on \mathbb{T}_n , define $p_j(t_0) = p_j, p_j(t) = r_j$ for each $t_0 \xrightarrow{i} t$, this completely determines the cluster algebra

Proof. Define $u_j(t) = \frac{p_j(t)}{p_j(t')}$ for $t \xrightarrow{j} t'$, then

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)}, p_j(t') = \frac{1}{1 \oplus u_j(t)}$$

Then (10.1.1) becomes

$$u_i(t_3)p_j(t_3)^{[b_{ji}(t_3)]_+} = u_i(t_2)p_j(t_2)^{[b_{ji}(t_2)]_+}$$

Case 1: $u_i(t_3)p_j(t_3)^{b_{ji}(t_3)} = u_i(t_2) \Rightarrow u_i(t_3) = u_i(t_2)(1 \oplus u_j(t_2))^{b_{ji}(t_2)}$

Case 2: $u_i(t_3) = u_i(t_2)p_j(t_2)^{b_{ji}(t_2)} = u_i(t_2) \left(\frac{u_j(t_2)}{1 \oplus u_j(t_2)} \right)^{b_{ji}(t_2)}$

Thus for $t \xrightarrow{j} t'$, we have

$$u_i(t') = u_i(t)u_j(t)^{[b_{ji}(t)]_+} (1 \oplus u_j(t))^{-b_{ji}(t)}$$

□

Remark 10.5.4. \mathbf{p} determines \mathbf{u} which in turn determines \mathbf{p}

Fix semifield \mathbb{P} , B is skew-symmetrizable, then (B, \mathbf{p}) determines the cluster algebra $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$ up to isomorphism

Corollary 10.5.5. The exchange graph of a normalized cluster algebra is n -regular

Definition 10.5.6. The **tropical semifield** $(\mathbb{R}, \oplus, \odot)$ is a semifield with multiplication as \odot , min or max as \oplus

The tropical semifield generated by p is the free abelian group generated multiplicatively by p with $p^a \oplus p^b = p^{\min(a,b)}$

Definition 10.5.7. A normalized cluster algebra is of geometric type if \mathbb{P} is the tropical semifield generated by $\{p_i\}_{i \in I'}$ and each $p_j(t)$ is a monomial with nonnegative exponents

Remark 10.5.8. In this particular case, normality just means that for $t \xrightarrow{j} t'$, $p_j(t), p_j(t')$ doesn't have a common variable, or the support doesn't intersect

Proposition 10.5.9. \mathbb{P} is the tropical semifield generated by $p_i, i \in I'$, $B(t)$ is the exchange pattern of exponents, $p_j(t)$ give rise to a cluster algebra of geometric type iff $C(t)$ satisfies the exchange pattern of coefficients, i.e. $p_j(t) = \prod_i p_i^{[c_{ij}(t)]_+}$ and

$$c'_{ij} = \begin{cases} -c_{ij} & j = k \\ c_{ij} + \frac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2} & \text{otherwise} \end{cases}$$

Here the mutation is in direction k

Proof. Suppose $p_j(t)$ give rise to a cluster algebra of geometric type. Define $u_j(t) = \frac{p_j(t)}{p_j(t')} =$

$\prod_{i \in I'} p_i^{c_{ij}(t)}$ for each $t \xrightarrow{j} t'$, then according to Proposition 10.5.3

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)} = \frac{\prod_i p_i^{c_{ij}}}{1 \oplus \prod_i p_i^{c_{ij}}} = \frac{\prod_i p_i^{c_{ij}}}{\prod_i p_i^{-[c_{ij}]_+}} = \prod_i p_i^{[c_{ij}]_+}$$

$$1 = u_k(t)u_k(t') = \prod_i p_i^{c_{ik} + c'_{ik}} \Rightarrow c'_{ik} = -c_{ik}$$

And

$$\begin{aligned} \prod_i p_i^{c'_{ij}} &= \prod_i p_i^{c_{ij}} \left(\prod_i p_i^{c_{ik}} \right)^{[b_{kj}]_+} \left(1 \oplus \prod_i p_i^{c_{ik}} \right)^{-b_{kj}} \\ &= \prod_i p_i^{c_{ij}} \prod_i p_i^{c_{ik}[b_{kj}]_+} \prod_i p_i^{b_{kj}[-c_{ik}]_+} \\ &= \prod_i p_i^{c_{ij} + \frac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2}} \end{aligned}$$

□

Remark 10.5.10. Note if we take $\tilde{B}(t) = (\tilde{b}_{ij})_{i \in I \cup I', j \in I}$ where $\tilde{b}_{ij} = b_{ij}$ for $i, j \in I$ is the principal part of \tilde{B} , $\tilde{b}_{ij} = c_{ij}$ for $i \in I', j \in I$

Corollary 10.5.11. Given \tilde{B}_0 with a skew-symmetrizable principal part B_0 , then there exists a unique exchange pattern of geometric type such that $\tilde{B}(t_0) = \tilde{B}_0$ for $t_0 \in \mathbb{T}_n$

Proof. By Proposition 10.1.11

□

Remark 10.5.12. The class of exchange patterns of geometric type is stable under restriction and direct product

10.6 Rank two case

Cluster algebra of rank 2

Example 10.6.1. If $n = 2$, consider \mathbb{T}_2

$$\overset{1}{\text{---}} t_0 \overset{2}{\text{---}} t_1 \overset{1}{\text{---}} t_2 \overset{2}{\text{---}} t_3 \overset{1}{\text{---}} t_4 \overset{2}{\text{---}} t_5 \overset{1}{\text{---}}$$

The cluster variables are y_i, y_{i+1} for t_i

$$y_{2k+1} = x_1(t_{2k}) = x_1(t_{2k+1}), y_{2k} = x_2(t_{2k-1}) = x_2(t_{2k})$$

$M_2(t_0)$ and $M_2(t_1)$ don't have x_1 and can't both have x_2

If both of them don't have x_2 , then $M_2(t_0), M_2(t_1) \in \mathbb{P}$, thus

$$\cdots x_2 \nmid M_1(t_{-1}) \Leftrightarrow x_1 \nmid M_2(t_0) \Leftrightarrow x_2 \nmid M_1(t_1) \Leftrightarrow x_1 \nmid M_2(t_2) \cdots$$

$$\cdots x_2 \nmid M_1(t_0) \Leftrightarrow x_1 \nmid M_2(t_1) \Leftrightarrow x_2 \nmid M_1(t_2) \Leftrightarrow x_1 \nmid M_2(t_3) \cdots$$

So is every $M_*(t_*) \in \mathbb{P}$, write q_m, r_m as the two monomials of $t_{m-1} \text{---} t_m$, then we have

$$y_{m-1}y_{m+1} = q_m + r_m$$

And for $t_{m-2} \text{---} t_{m-1} \text{---} t_m \text{---} t_{m+1}$ we have

$$\frac{q_{m+1}}{r_{m+1}} = \frac{r_{m-1}}{q_{m-1}} \Leftrightarrow q_{m-1}q_{m+1} = r_{m-1}r_{m+1}$$

If $M_2(t_0) = q_1 x_1^b$, $M_2(t_1) = r_1$ (the other case corresponds to the involution) for some $b > 0$, then $M_1(t_1) = q_2 x_2^c$, $M_1(t_2) = r_2$ for some $c > 0$, we have

$$\frac{M_2(t_2)}{M_2(t_3)} = \frac{M_2(t_1)}{M_2(t_0)} \Big|_{x_1 \leftarrow \frac{M_0}{x_1}} = \frac{r_1}{q_1 x_1^b} \Big|_{x_1 \leftarrow \frac{r_2}{x_1}} = \frac{r_1 x_1^b}{q_1 r_2^b}$$

Since $x_1 \mid M_2(t_3) \Rightarrow x_2 \nmid M_2(t_2)$ gives a contradiction, $x_1 \nmid M_2(t_3) \Rightarrow M_2(t_3) = r_3$, thus $M_2(t_2) = q_3 x_1^b$, periodically, we can conclude

$$M_2(t_{2k}) = q_{2k+1} x_1^b$$

$$M_2(t_{2k+1}) = r_{2k+1}$$

$$M_1(t_{2k-1}) = q_{2k} x_2^c$$

$$M_1(t_{2k}) = r_{2k}$$

Therefore we have

$$y_{2k-1}y_{2k+1} = q_{2k} y_{2k}^c + r_{2k}$$

$$y_{2k}y_{2k+2} = q_{2k+1} y_{2k+1}^b + r_{2k+1}$$

For $t_{2k-1} \text{---} t_{2k} \text{---} t_{2k+1} \text{---} t_{2k+2}$ we have

$$q_{2k} q_{2k+2} r_{2k+1}^c = r_{2k} r_{2k+2}$$

For $t_{2k-2} \text{---} t_{2k-1} \text{---} t_{2k} \text{---} t_{2k+1}$ we have

$$q_{2k-1} q_{2k+1} r_{2k}^b = r_{2k-1} r_{2k+1}$$

The exchange matrices are

$$B(t_m) = (-1)^m \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

Conversely, given such relation, we can always find a corresponding cluster algebra

In particular, consider the coordinate ring of $\text{Gr}_2(5)$, let

$$\begin{aligned} y_m &= [\overline{2m-1}, \overline{2m+1}] \\ q_m &= [\overline{2m-2}, \overline{2m+2}] \\ r_m &= [\overline{2m-2}, \overline{2m-1}] [\overline{2m+1}, \overline{2m+2}] \\ b &= c = 1 \end{aligned}$$

Note. If we denote $m \bmod 2$ as $\langle m \rangle$, then

$$\begin{aligned} p_{\langle m \rangle}(t_m) &= q_{m+1} \\ p_{\langle m+1 \rangle}(t_m) &= r_m \\ x_{\langle m \rangle}(t_m) &= q_m \\ x_{\langle m+1 \rangle}(t_m) &= y_{m+1} \end{aligned}$$

Example 10.6.2. As in Example 10.6.1, by Theorem 10.2.3, we know that

$$y_m = \frac{N_m(y_1, y_2)}{y_1^{d_1(m)} y_2^{d_2(m)}}$$

Where $N_m(y_1, y_2) \in \mathbb{Z}\mathbb{P}[y_1, y_2]$ not divisible by y_1 or y_2

10.7 Cluster algebra of finite type

Definition 10.7.1. Seeds $\Sigma(t), \Sigma(t')$ are **equivalent** if t, t' are \mathcal{M} -equivalent, i.e. there is a permutation σ of I such that $x_{\sigma(i)}(t) = x_i(t')$, $b_{\sigma(i)\sigma(j)}(t') = b_{ij}(t)$, $p_{\sigma(j)}(t) = p_j(t')$. For geometric type, $c_{i\sigma(j)}(t) = c_{ij}(t')$, or rather, $\tilde{b}_{\sigma(i)\sigma(j)}(t') = \tilde{b}_{ij}(t)$ since σ as only a permutation of I fixes I' . By Proposition 10.5.3, if t, t' are equivalent, and $t \xrightarrow{\sigma(j)} t_1$ and $t' \xrightarrow{j} t'_1$, then t_1, t'_1 are equivalent

Cluster algebras $\mathcal{A}(\mathcal{S}), \mathcal{A}'(\mathcal{S}')$ are strongly isomorphic if there is a field isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ that sends seeds in \mathcal{S} to seeds \mathcal{S}' , thus inducing bijection $\mathcal{S} \rightarrow \mathcal{S}'$ and an isomorphism $\mathcal{A} \rightarrow \mathcal{A}'$. $\mathcal{A}(B, -)$ are all the possible normalized cluster algebras. $\mathcal{A}(B), \mathcal{A}(B')$ are strongly isomorphic if there is a one-to-one correspondence between $\mathcal{A}(B, \mathbf{p})$ and $\mathcal{A}(B', \mathbf{p}')$, this is true iff B, B' are mutationally equivalent modulo relabelling rows and columns

\mathcal{A} is of **finite type** if it has finitely many seeds up to equivalences

Definition 10.7.2. The **Cartan counterpart** of B is the **generalized Cartan matrix** $A(B) = (a_{ij})$, $a_{ii} = 2$, $a_{ij} = -|b_{ij}|$ for $i \neq j$, with the same symmetrizing matrix D , i.e. $d_i a_{ij} = d_j a_{ji}$

A Cartan matrix of finite type, $A(B)=A$, $b_{ij}b_{ik}>0$, p normalized, then cluster algebra is of finite type

Theorem 10.7.3. *A is a Cartan matrix of finite type, there is a sign-skew symmetric B_\circ such that $A(B_\circ) = A$ and $b_{ij}b_{ik} > 0$ for all i, j, k , and \mathbf{p}_\circ is normalized, then $\mathcal{A}(B_\circ, \mathbf{p}_\circ)$ is of finite type. Any cluster algebra of finite type is strongly isomorphic to one such data*

Remark 10.7.4. Since the Coxeter graph of A is a tree which is bipartite, thus we can certainly divide them into sinks and sources. Since $b_{ij} > 0$ would be there is a directed edge from i to j , thus we can always find such a B_\circ

Theorem 10.7.5. B, B' sign-skew symmetric, $\mathcal{A}(B), \mathcal{A}(B')$ iff $A(B), A(B')$ are of the same Cartan-Killing type

Theorem 10.7.6. \mathcal{A} is a cluster algebra, the following are equivalent

- (i) \mathcal{A} is of finite type
- (ii) $|\mathcal{X}| < \infty$
- (iii) For every seed $(\mathbf{x}, \mathbf{p}, B)$, $|b_{xy}b_{yx}| < 3$ for $x, y \in \mathbf{x}$
- (iv) $\mathcal{A} = \mathcal{A}(B_\circ, \mathbf{p}_\circ)$ as in Theorem 10.7.3

Theorem 10.7.7. $\mathcal{A}(B)$ consists of cluster algebras all simultaneously of finite type or of infinite type. There is a bijective correspondence between generalized Cartan matrices of finite type and strong isomorphic classes of normalized cluster algebras, through $B \rightarrow A(B)$

Bijection between almost positive roots and X

Theorem 10.7.8. There is a unique bijection $\Phi_{\geq -1} \rightarrow \mathcal{X}$, $\alpha \mapsto x[\alpha] = \frac{P_\alpha(\mathbf{x}_\circ)}{\mathbf{x}^\alpha}$, $P_\alpha \in \mathbb{Z}_{\geq 0}\mathcal{P}$ with nonzero constant term such that $X[-\alpha_i] = x_i$

Theorem 10.7.9. Every seed $(\mathbf{x}, \mathbf{p}, B)$ in \mathcal{A} is uniquely determined by the cluster \mathbf{x} , and for any $\mathbf{x} \in \mathbf{x}$, there is a unique cluster \mathbf{x}' such that $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{\mathbf{x}\}$. The cluster complex $\Delta(\mathcal{A})$ encodes the combinatorics of seed mutations

Theorem 10.7.10. The bijection in Theorem 10.7.8 identifies $\Delta(\mathcal{A})$ and $\Delta(\Phi)$, in particular, the cluster complex doesn't depend on \mathbb{P} nor \mathbf{p}_\circ

Part III

Commutative Algebra

Chapter 11

p-adic numbers

Definition 11.0.1. The p -adic integers are

$$\begin{aligned}\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} &= \left\{ (a_0, a_1, a_2, \dots) \in \prod \mathbb{Z}/p^n\mathbb{Z} \mid a_n \equiv a_m \pmod{p^m}, n \geq m \right\} \\ &= \{b_0 + b_1p + b_2p^2 + \dots\}\end{aligned}$$

Here $a_n = b_0 + b_1p + \dots + b_np^n$

Example 11.0.2. If $p = 7$, we can write $3 + 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 2 \cdot 7^4 + \dots$ as

$$\dots 24163$$

With base 7

Part IV

Homological Algebra

Chapter 12

Abelian category

Definition 12.0.1. Let \mathcal{C} be a locally small category, we say \mathcal{C} is a **preadditive category** if $Hom_{\mathcal{C}}(X, Y)$ are abelian groups, and the addition distributes over composition, i.e. $f \circ (g + h) = f \circ g + f \circ h$, $(f + g) \circ h = f \circ h + g \circ h$

Note that the 0 in the abelian group $Hom_{\mathcal{C}}(X, Y)$ is a zero morphism

A preadditive category is called an **additive category** if any finite set has a biproduct, in particular, it has a zero object, the empty biproduct

An additive category is called a **preabelian category** if every morphism has a kernel and a cokernel, where kernels and cokernels means the equalisers and coequalisers of the morphism $f : X \rightarrow Y$ and the zero morphism $0 : X \rightarrow Y$

A preabelian category is called an **abelian category** if every monomorphisms is normal and every epimorphisms is conormal, a morphism is **normal** if it is a kernel, **conormal** if it is a cokernel and **binormal** if it is both a kernel and a cokernel

Definition 12.0.2. For a morphism $A \xrightarrow{f} B$, define its image $\text{im} f$ by the following commutative diagram

$$\begin{array}{ccccccc} & & A & & & & \\ & & \downarrow \exists_1 & \searrow f & & & \\ 0 & \longrightarrow & \text{im} f & \longrightarrow & B & \twoheadrightarrow & \text{coker} f \longrightarrow 0 \end{array}$$

The image satisfies universal property

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \nearrow \\ & \text{im} f & \\ & \downarrow \exists_1 & \\ & I & \end{array}$$

Example 12.0.3. A ring R can be thought of as a preadditive category with a single object and morphisms $r \in R$. The category of left R modules can be thought of as the functor category $[R, Ab]$, where Ab the category of abelian groups

Proposition 12.0.4. In an abelian category \mathcal{A} , the equaliser of $X \xrightarrow{f} Y$ is isomorphic to the kernel of $f - g$

Definition 12.0.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of preadditive categories, if F is an abelian group homomorphism on $Hom(F(X), F(Y))$ for any X, Y , we say F is an **additive functor**

Definition 12.0.6. Let \mathcal{A} be an abelian category, a $(\mathbb{Z}$ -graded) **chain complex** C_{\bullet} is

$$\cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \rightarrow \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0$, ∂_i are called **boundary maps(differentials)**

We can define chain maps, chain homotopy, boundaries, cycles, and homology groups, and we say the chain complex is exact if each homology groups is zero, the chain complexes form the **category of chain complexes** $Ch\mathcal{A}$

The **homotopy category of chain complexes** often denoted as $K(\mathcal{A})$ is the quotient category with chain maps modulo chain homotopy equivalence as morphisms

a chain map is called a **quasi-isomorphism** if it induces isomorphisms on homology groups

Lemma 12.0.7. An alternative definition of an exact functor F could be that F preserve exactness, i.e. $F(A) \rightarrow F(B) \rightarrow F(C)$ is exact for any short exact sequence $A \rightarrow B \rightarrow C$

Definition 12.0.8. The **direct sum** $(C \oplus D)_\bullet$ of chain complexes C_\bullet, D_\bullet is

$$\cdots \rightarrow C_1 \oplus D_1 \xrightarrow{\partial_1^C \oplus \partial_1^D} C_0 \oplus D_0 \xrightarrow{\partial_0^C \oplus \partial_0^D} C_{-1} \oplus D_{-1} \rightarrow \cdots$$

Definition 12.0.9. A **double complex** $C_{*,*}$ is $\{C_{p,q}\}_{p,q \in \mathbb{Z}}$ two differentials $\partial' : C_{p,q} \rightarrow C_{p-1,q}$, $\partial'' : C_{p,q} \rightarrow C_{p,q-1}$ such that $(\partial')^2 = (\partial'')^2 = 0$ and $\partial'\partial'' + \partial''\partial' = 0$ (∂', ∂'' anticommutes)

The **total chain complexes** are $(Tot^\oplus)_n = \bigoplus_{p+q=n} C_{p,q}$ and $(Tot^\Pi)_n = \prod_{p+q=n} C_{p,q}$ with $\partial = \partial' + \partial''$

Example 12.0.10. $C_* \otimes D_*$ is the total complex of double complex $C_{p,q} := C_p \otimes D_q$, $\partial' := \partial^C \otimes 1$, $\partial'' := (-1)^p 1 \otimes \partial^D$

Definition 12.0.11. A **filtered chain complex** is a filtered object in $Ch\mathcal{A}$

$$\cdots \rightarrow F_{p+1}C_\bullet \rightarrow F_p C_\bullet \rightarrow \cdots \rightarrow C_\bullet$$

Snake lemma

Lemma 12.0.12 (Snake lemma). Given the following commutative diagram with exact rows, then we have an exact sequence

$$\begin{array}{ccccccc} & & \xrightarrow{w_*} & \ker a & \xrightarrow{u_*} & \ker b & \xrightarrow{v_*} & \ker c & \xrightarrow{\quad} & 0 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ D & \longrightarrow & A & \xrightarrow{u} & B & \xrightarrow{v} & C & \longrightarrow & 0 \\ & & \downarrow a & \downarrow b & \downarrow c & & & & & \\ 0 & \longrightarrow & A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \longrightarrow & D' \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\ & & \text{coker } a & \xrightarrow{u'_*} & \text{coker } b & \xrightarrow{v'_*} & \text{coker } c & \longrightarrow & 0 \end{array}$$

Five lemma

Lemma 12.0.13 (Five lemma). If b and d are monic and a is an epi, then c is monic. Dually, if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are iso, then c is also an iso

$$\begin{array}{ccccccccc} A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & D' & \xrightarrow{x'} & E' \\ a \downarrow \cong & & b \downarrow \cong & & c \downarrow & & d \downarrow \cong & & e \downarrow \cong \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & D & \xrightarrow{x} & E \end{array}$$

Horseshoe lemma

Lemma 12.0.14 (Horseshoe lemma). Suppose $P_\bullet \xrightarrow{\varepsilon} M$, $Q_\bullet \xrightarrow{\eta} N$ are projective resolutions, then any exact sequence $0 \rightarrow M \xrightarrow{f} A \xrightarrow{g} N \rightarrow 0$ can be extended into commutative diagram

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & M & \xrightarrow{f} & A & \xrightarrow{g} & N \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

With $(P \oplus Q)_\bullet$ being a projective resolution, every row and column are exact

Proof. Since $A \xrightarrow{g} N$ is epi and Q_0 is projective, we get $Q_0 \xrightarrow{s_0} A$ such that $gs_0 = \partial_0$ which gives us $P_0 \oplus Q_0 \xrightarrow{\begin{pmatrix} f\partial_0 & s_0 \end{pmatrix}} A$, by Lemma 12.0.12, this is epi, and we get an exact sequence $0 \rightarrow Z_0P \rightarrow \ker i_0 \rightarrow Z_0Q \rightarrow 0$, similarly, we can construct $Q_1 \xrightarrow{s_1} \ker i_0$, then $P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} \iota_0\partial_0 & s_1 \end{pmatrix}} \ker i_0$ is again epi by Lemma 12.0.12, inductively, we can construct the commutative diagram

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 \longrightarrow 0 \\
& \downarrow \partial_1 & & \downarrow & \swarrow s_1 & \downarrow \partial_1 & \\
0 & \longrightarrow & Z_0P & \xrightarrow{\iota_0} & \ker i_0 & \longrightarrow & Z_0Q \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 \longrightarrow 0 \\
& \downarrow \partial_0 & & \downarrow & \swarrow s_0 & \downarrow \partial_0 & \\
0 & \longrightarrow & M & \xrightarrow{f} & A & \xrightarrow{g} & N \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

□

Lemma for universal coefficient theorem for cohomology

Lemma 12.0.15. If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence and there is a homomorphism (retraction) $C \xrightarrow{r} B$ such that $rg = 1_B$, then there is an exact sequence $0 \rightarrow \text{coker } f \rightarrow \text{coker}(gf) \rightarrow \text{coker } g \rightarrow 0$

Proof. First observe that we have $0 \rightarrow \text{img}/\text{im}(gf) \rightarrow C/\text{im}(gf) \rightarrow C/\text{img} \rightarrow 0$, $B \rightarrow \text{img}$, $\text{im } f \rightarrow \text{im}(gf)$, thus $B/\text{im } f \rightarrow \text{img}/\text{im}(gf)$, since $rg = 1_B$, $B/\text{im } f \cong \text{img}/\text{im}(gf)$, therefore, $0 \rightarrow B/\text{im } f \rightarrow C/\text{im}(gf) \rightarrow C/\text{img} \rightarrow 0$ □

Lemma 12.0.16. Suppose \mathcal{A} is abelian category, then $\text{im } f = \ker \text{coker } f = \text{coker } \ker f$

Definition 12.0.17. Pick $p \in \mathbb{Z}$, define the **translation** of X by p is $X_\bullet[p]$ where $(X_\bullet[p])_n = X_{p+n}$, differential $X_\bullet[p]_n \rightarrow X_\bullet[p]_{n-1}$ is given by $(-1)^p \partial$ The **translation functor** $T : Ch(\mathcal{A}) \rightarrow Ch(\mathcal{A})$, $X \mapsto X_\bullet[1]$ is an auto morphism of $Ch(\mathcal{A})$

Acyclic model theorem

Theorem 12.0.18 (Acyclic model theorem). ¹ **Model** $\mathcal{M} = \{M_j\}$ is a subclass (possibly with repetition) of objects in \mathcal{C} , $F, G : \mathcal{C} \rightarrow Ch_{\geq 0}$ are functors, $H_n(G(M_j)) = 0$ for any $n \neq 0$, $M_j \in \mathcal{M}$. For any C , there exist $m_j \in F_k M_j$ such that $F_k(C)$ is free with basis $\{F_k(\sigma)(m_j) \mid M_j \xrightarrow{\sigma} C\}$

¹Consult Theorem 9.12 of [1] or <https://amathew.wordpress.com/2010/09/11/the-method-of-acyclic-models/>

Universal coefficient theorem for cohomology

Theorem 12.0.19 (Universal coefficient theorem for cohomology). *There is an exact sequence*

$$0 \rightarrow \text{Ext}^1(H_{n-1}, A) \rightarrow H^n(C; A) \rightarrow \text{Hom}(H_n, A) \rightarrow 0$$

Proof. Since C_n is a free group, so are subgroups B_n, Z_n , exact sequence

$$0 \rightarrow Z_n \hookrightarrow C_n \rightarrow B_{n-1} \rightarrow 0$$

Splits, i.e. we have a splitting homomorphism $B_{n-1} \xrightarrow{s} C_n$, $C_n \cong Z_n \oplus B_{n-1}$, thus exact sequence

$$0 \rightarrow H_n = Z_n/B_n \rightarrow C_n/B_n \rightarrow C_n/Z_n \cong B_{n-1} \rightarrow 0$$

Induces exact sequence

$$\text{Hom}(B_{n-1}, A) \rightarrow \text{Hom}(C_n/B_n, A) \rightarrow \text{Hom}(H_n, A) \rightarrow \text{Ext}^1(B_{n-1}, A) = 0$$

 $\text{Hom}(H_n, A)$ is the cokernel of $\text{Hom}(B_{n-1}, A) \rightarrow \text{Hom}(C_n/B_n, A)$

Note that

$$H^n(C; A) = Z^n(C; A)/B^n(C; A) = \frac{\ker(\text{Hom}(C_n, A) \rightarrow \text{Hom}(C_{n+1}, A))}{\text{im}(\text{Hom}(C_{n-1}, A) \rightarrow \text{Hom}(C_n, A))}$$

 $C_n \xrightarrow{\phi} A \in Z^n(C; A) \Leftrightarrow \phi\partial = 0 \Leftrightarrow \phi \in \text{Hom}(C_n/B_n, A)$, thus $\text{Hom}(C_n/B_n, A) \cong Z^n(C; A)$

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\partial} & C_n \\ & \searrow & \downarrow \phi \\ & & A \end{array}$$

 $C_n \xrightarrow{\psi} A \in B^n(C; A) \Leftrightarrow \psi = \phi\partial$ for some $C_{n-1} \xrightarrow{\phi} A \Leftrightarrow \psi = \phi\partial$ for some $Z_{n-1} \xrightarrow{\phi} A$, and since $B^n(C; A) \subseteq Z^n(C; A) \cong \text{Hom}(C_n/B_n, A)$, we have $B^n(C; A) \cong \text{im}(\text{Hom}(Z_{n-1}, A) \rightarrow \text{Hom}(C_n/B_n, A))$

$$\begin{array}{ccc} C_n & \xrightarrow{\partial} & C_{n-1} \\ & \searrow \psi & \downarrow \phi \\ & & A \end{array}$$

Exact sequence

$$0 \rightarrow B_{n-1} \rightarrow Z_{n-1} \rightarrow B_{n-1}/Z_{n-1} = H_{n-1} \rightarrow 0$$

Induces exact sequence

$$\text{Hom}(Z_{n-1}, A) \rightarrow \text{Hom}(B_{n-1}, A) \rightarrow \text{Ext}^1(H_{n-1}, A) \rightarrow \text{Ext}^1(Z_{n-1}, A) = 0$$

 $\text{Ext}^1(H_{n-1}, A)$ is the cokernel of $\text{Hom}(Z_{n-1}, A) \rightarrow \text{Hom}(B_{n-1}, A)$ Since composition $B_{n-1} \xrightarrow{s} C_n \rightarrow C_n/B_n \xrightarrow{\partial} B_{n-1}$ is identity, we have a homomorphism $r : \text{Hom}(C_n/B_n, A) \rightarrow \text{Hom}(B_{n-1}, A)$ induced by $B_{n-1} \rightarrow C_n/B_n$ such that composition $\text{Hom}(B_{n-1}, A) \rightarrow \text{Hom}(C_n/B_n, A) \xrightarrow{r} \text{Hom}(B_{n-1}, A)$ is identityApply Lemma 12.0.15 to $\text{Hom}(Z_{n-1}, A) \rightarrow \text{Hom}(B_{n-1}, A) \rightarrow \text{Hom}(C_n/B_n, A)$, we get an exact sequence

$$0 \rightarrow \text{Ext}^1(H_{n-1}, A) \rightarrow H^n(C; A) \rightarrow \text{Hom}(H_n, A) \rightarrow 0$$

□

Remark 12.0.20. B_n is not necessarily a direct summand of C_n , a map $B_n \xrightarrow{\phi} A$ may not be possible to extended to $C_n \xrightarrow{\phi} A$, however a map $Z_n \xrightarrow{\phi} A$ can always be extended to $C_n \xrightarrow{\phi} A$

Theorem 12.0.21 (Algebraic Künneth formula). *C, D are free chain complexes, then*

$$0 \rightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \rightarrow H_n(C \otimes D) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(C), H_q(D)) \rightarrow 0$$

Is exact

Proof. If D has trivial differentials, then $H_q(D) = D_q$ is free, hence

$$H_n(C \otimes D) = \bigoplus_{p+q=n} H_p(C \otimes D_q) = \bigoplus_{p+q=n} H_p(C) \otimes D_q = \bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$$

In general, consider exact sequence $0 \rightarrow Z \xrightarrow{i} D \xrightarrow{\partial} B[-1] \rightarrow 0$, $0 \rightarrow B \xrightarrow{i} Z \rightarrow H(D) \rightarrow 0$, then $0 \rightarrow C \otimes Z \rightarrow C \otimes D \rightarrow C \otimes B[-1] \rightarrow 0$ is exact since C_k are free, this gives us long exact sequence

$$\cdots \rightarrow H_n(C \otimes Z) \xrightarrow{1 \otimes i} H_n(C \otimes D) \xrightarrow{1 \otimes \partial} H_n(C \otimes B[-1]) \xrightarrow{1 \otimes i} H_{n-1}(C \otimes Z) \rightarrow \cdots$$

$Z, B[-1]$ have trivial differentials, hence the connecting homomorphism is just

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(B[-1]) = \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(B) \xrightarrow{1 \otimes i} \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(Z)$$

Then we have

$$0 \rightarrow \text{coker}(1 \otimes i) \rightarrow H_n(C \otimes D) \rightarrow \ker(1 \otimes i) \rightarrow 0$$

We also have

$$0 \rightarrow \text{Tor}_1(H_p(C), H_q(D)) \rightarrow H_p(C) \otimes B_q \xrightarrow{1 \otimes i} H_p(C) \otimes Z_q \rightarrow H_p(C) \otimes H_q(D) \rightarrow 0$$

Therefore, we have exact sequence

$$0 \rightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \rightarrow H_n(C \otimes D) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(C), H_q(D)) \rightarrow 0$$

□

Definition 12.0.22. A **composition series** of A is a sequence of subobjects

$$A = A_n \supseteq \cdots \supseteq A_1 \supseteq A_0 = 0$$

With **composition factors** H_{i+1}/H_i simple and **composition length** $\ell(A) = n$

Lemma 12.0.23. $\ell(A)$ is indepent of the composition series

Chapter 13

Spectral sequence

Definition 13.0.1. Suppose \mathcal{A} is an abelian category, a **spectral sequence** consists of objects $\{E_r\}_{r \geq r_0}$ (r_0 is mostly 0), and morphisms $d_r : E_r \rightarrow E_r$ such that $d_r \circ d_r = 0$ and $E_{r+1} \cong H(E_r) = \ker d_r / \text{im} d_r$

Definition 13.0.2. Suppose \mathcal{A} is an abelian category, an **exact couple** is (D, E, i, j, k)

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

Such that it is exact at each term, define differential $d = jk$, then $d^2 = jkj k = j(kj)k = 0$, we can define the **derived couple**

$$\begin{array}{ccc} D' & \xrightarrow{i'} & D' \\ & \swarrow k' & \searrow j' \\ & E' & \end{array}$$

Where $D' = i(D)$, $E' = \ker k / \text{im} j$, $i'(a) = i(a)$, $j'(i(a)) = \overline{j(a)}$, $k'(b) = \overline{k(b)}$, then the derived couple is again an exact couple, thus we can carry this process indefinitely, giving the n -th derived couple $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$

Example 13.0.3. Suppose $\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots$ is a filtration of chain complex C_\bullet (or **filtered chain complex**), exact sequence $0 \rightarrow F_{p-1}C_\bullet \rightarrow F_pC_\bullet \rightarrow (grC_\bullet)_p \rightarrow 0$ give a long exact sequence

$$\cdots \rightarrow H_n(F_{p-1}C_\bullet) \xrightarrow{i_*} H_n(F_pC_\bullet) \xrightarrow{j_*} H_n(F_pC_\bullet / F_{p-1}C_\bullet) \xrightarrow{k_*} H_{n-1}(F_{p-1}C_\bullet) \rightarrow \cdots$$

If we write $D_{p,q}^1 := H_{p+q}(F_pC_\bullet)$, $E_{pq}^1 := H_{p+q}(F_pC_\bullet / F_{p-1}C_\bullet)$, then the long exact sequence become

$$\cdots \rightarrow D_{p,q}^1 \rightarrow D_{p+1,q-1}^1 \rightarrow E_{p,q}^1 \rightarrow D_{p,q-1}^1 \rightarrow \cdots$$

Consider $D^1 = \bigoplus_p D_{p,q}^1$, $E^1 = \bigoplus_{p,q} E_{pq}^1$, then $(D^1, E^1, i_*, j_*, k_*)$ form an exact couple, note that $d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$

Remark 13.0.4. $grC_\bullet = \bigoplus_p F_pC_\bullet / F_{p-1}C_\bullet$ is called the **associated graded complex**
If X is a CW complex, we can take $F_pC_\bullet = C_\bullet(X^p)$, here X^p is the p -th skeleton of X

Definition 13.0.5. A **double cochain complex** $C^{\bullet,\bullet}$ is bigraded with anticommuting differentials d_h, d_v , i.e. $(d_h)^2 = 0$, $(d_v)^2 = 0$, $d_h d_v + d_v d_h = 0$

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{0,1} & \xrightarrow{d_h^{0,1}} & C^{1,1} & \longrightarrow & \cdots \\
& & \uparrow d_v^{0,0} & & \uparrow d_v^{1,0} & & \\
\cdots & \longrightarrow & C^{0,0} & \xrightarrow{d_h^{0,0}} & C^{1,0} & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & &
\end{array}$$

Define the **total cochain complex** to be $C^n = \bigoplus_{p+q=n} C^{p,q}$, with total differential $d = d_h + d_v$,

this is indeed a differential since $d^2 = (d_h + d_v)^2 = (d_h)^2 + d_h d_v + d_v d_h + (d_v)^2 = 0$

We can define the **horizontal filtration** of the total cochain complex $(F_p^h C)^n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C^{k,l}$ and

the **vertical filtration** of the total cochain complex $(F_q^h C)^n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C^{k,l}$

A **double chain complex** $C_{\bullet,\bullet}$ is bigraded with anticommuting differentials ∂^h, ∂^v , i.e. $(\partial^h)^2 = 0$, $(\partial^v)^2 = 0$, $\partial^h \partial^v + \partial^v \partial^h = 0$

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C_{1,1} & \xrightarrow{\partial_{1,1}^h} & C_{0,1} & \longrightarrow & \cdots \\
& & \downarrow \partial_{1,1}^v & & \downarrow \partial_{0,1}^v & & \\
\cdots & \longrightarrow & C_{1,0} & \xrightarrow{\partial_{1,0}^h} & C_{0,0} & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & &
\end{array}$$

Define the **total chain complex** to be $C_n = \bigoplus_{p+q=n} C_{p,q}$, with total differential $\partial = \partial_h + \partial_v$

We can define the **horizontal filtration** of the total chain complex $(F_p^h C)_n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C_{k,l}$ and

the **vertical filtration** of the total chain complex $(F_q^h C)_n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C_{k,l}$

Remark 13.0.6. If d_h, d_v commutes instead of anticommuting, then $C^{\bullet,\bullet}$ can be viewed as a cochain complex of cochain complexes, the total differential becomes $d^n(c) = d_h^p c + (-1)^p d_v^q c$ for any $c \in C^{p,q}$, this is indeed a differential since

$$\begin{aligned}
d^{n+1} d^n(c) &= d^{n+1}(d_h^p c + (-1)^p d_v^q c) \\
&= d^{n+1} d_h^p c + (-1)^p d^{n+1} d_v^q c \\
&= d_h^{p+1} d_h^p c + (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c + (-1)^{2p} d_v^{q+1} d_v^q c \\
&= (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c \\
&= (-1)^p (d_h^p d_v^q - d_v^q d_h^p) c \\
&= 0
\end{aligned}$$

However, these two types of definitions are equivalent

Proposition 13.0.7. Let $E_{p,q}^r$ be the spectral sequence corresponds to the horizontal filtration

- (1) $E_{p,q}^0 \cong C^{p,q}$
- (2) $E_{p,q}^1 \cong H_q(C_{p,\bullet})$
- (3) $E_{p,q}^0 \cong H_p(H_q^v(C))$
- (4) If $C_{p,q}$ vanishes outside the first quadrant, i.e. $C_{p,q} = 0$ for any $p < 0$ or $q < 0$, then the spectral sequence converges to the homology of the total chain complex $E_{p,q}^r \Rightarrow H_{p+q}(C)$, i.e. $E_{p,q}^\infty \cong H_{p+q}(C)$

Proof. (1) By definition $E_{p,q}^0 := (F_p^h C)_{p+q} / (F_{p-1}^h C)_{p+q} \cong C^{p,q}$

(2) $E_{p,q}^1 = H_{p+q}(F_p^h C / F_{p-1}^h C) \cong H_{p+q}(C_{p,\bullet})$

(3)

□

Part V

General topology

Chapter 14

General topology

14.1 General topology

Definition 14.1.1. A **topological space** X is a set with **topology** $\tau \subseteq \mathcal{P}(X)$, such that $\emptyset, X \in \tau$, $U_i \in \tau \Rightarrow \bigcup_i U_i \in \tau$, $U, V \in \tau \Rightarrow U \cap V \in \tau$, elements in τ are **open sets**, complements of open sets are **closed sets**

N is a **neighborhood** of $A \subseteq X$ if $A \subseteq U \subseteq N \subseteq X$ for some open set U

x is a **limit point** of A if any neighborhood of x intersects A . x is a **limit** of $\{x_n\}$ if for any neighborhood U of x , all but finitely many lies in U

A **subspace** is $A \subseteq X$ with **subspace topology** given by $\{U \cap A | U \in \tau\}$

Definition 14.1.2. $X \xrightarrow{f} Y$ is **continuous** at x if for any neighborhood V of $y = f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$. Then f is continuous iff $f^{-1}(V)$ is open for any open set $V \subseteq Y$

Definition 14.1.3. A **base** for τ is $B \subseteq \tau$ such that B covers X and for any $U_1, U_2 \in B$ such that $U_1 \cap U_2 \neq \emptyset$, there exists $U_3 \in B$ such that $U_3 \subseteq U_1 \cap U_2$

A **local base** for τ at x is a collection of neighborhoods $B(x)$ of x such that any neighborhood of x contain an element of $B(x)$

A **subbase** for τ is $B \subseteq \tau$ such that B generates τ , i.e. by arbitrary union of finite intersections, equivalently, τ is the smallest topology containing B . Here empty union and empty intersection are \emptyset and X

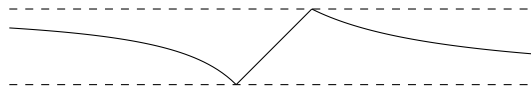
Definition 14.1.4. X is **first countable** if each point has a countable local base

X is **second countable** if it has a countable base

Definition 14.1.5. X is **regular** if any point and a disjoint closed set have disjoint neighborhoods. X is **normal** if disjoint closed sets have disjoint neighborhoods

Definition 14.1.6. $\{A_i\}$ can be **completely separated** if $\{A_i\}$ can be completely separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. Closed subsets $\{A_i\}$ can be **perfectly separated** if $\{A_i\}$ can be perfectly separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. \mathbb{R} can be replaced with I considering

$$\mathbb{R} \rightarrow I, x \mapsto \begin{cases} \frac{x}{x-1} & x \leq 0 \\ x & 0 \leq x \leq 1 \text{ and } I \hookrightarrow \mathbb{R} \\ \frac{2}{x+1} & x \geq 1 \end{cases}$$



Definition 14.1.7 (Kolmogorov classification of topological spaces). X is a T_0 **space** if for any two distinct points in X , at least one of them has a neighborhood which doesn't intersect the other point, i.e. they are **topologically distinguishable**

X is a T_1 **space** if for any two distinct points in X , each of them has a neighborhood which doesn't intersect the other point. $T_1 \Leftrightarrow$ points are closed

X is a T_2 **space** or **Hausdorff space** if any two distinct points have disjoint neighborhoods. Then the limit of $\{x_n\}$ is unique, denotes the limit $x = \lim x_n$

X is a $T_{2\frac{1}{2}}$ **space** or **Urysohn space** if any two distinct points have disjoint closed neighborhoods

X is a T_3 **space** if X is regular Hausdorff

X is a $T_{3\frac{1}{2}}$ **space** if X is completely regular Hausdorff

X is a T_4 **space** if X is normal T_1 space \Leftrightarrow normal Hausdorff

X is a T_5 **space** if X is completely normal Hausdorff

X is a T_6 **space** if X is perfectly normal \Leftrightarrow perfectly normal Hausdorff

Definition 14.1.8. The **box topology** on $\prod_{i \in I} X_i$ has base $\left\{ \prod_{i \in I} U_i \mid U_i \subseteq X_i \text{ open} \right\}$

Lemma 14.1.9. X is Hausdorff iff the diagonal $\{(x, x) \mid x \in X\}$ is closed

Definition 14.1.10. $X \times I \xrightarrow{F} Y$ is a **homotopy** between $X \xrightarrow{f_0, f_1} Y$ if $F(x, 0) = f_0(x)$, $F(x, 1) = f_1(x)$, write $f_t = F(\cdot, t)$. $X \xrightarrow{f} Y$ is a **homotopy equivalence** if there is $Y \xrightarrow{g} X$ such that $gf \simeq 1_X$, $fg \simeq 1_Y$

Definition 14.1.11. $X \xrightarrow{f} Y$ is a **topological embedding** if f is injective and $f : X \rightarrow f(X)$ is a homeomorphism

Definition 14.1.12. $K \subseteq X$ is **compact** if any open cover has a finite subcover. Equivalently, K is disjoint from the intersection of a family of closed sets, then K is disjoint from the intersection of finitely many of them

X is **locally compact** if there is a compact neighborhood for each point

$Y \subseteq X$ is **precompact** if \bar{Y} is compact

Definition 14.1.13. $A \subseteq X$ is **dense** if $\bar{A} = X$

X is **separable** if X has a countable dense subset

Definition 14.1.14. $X_\alpha \subseteq X$, $\{X_\alpha\}$ is **locally finite** if for any $x \in X$, there is a neighborhood of x intersecting only finitely many X_α 's

$\mathcal{U} = \{U_\alpha\}$, $\mathcal{V} = \{V_\beta\}$ are covers of X , \mathcal{V} is a **refinement** of \mathcal{U} if for any V_β , there exists U_α containing V_β

X is **paracompact** if every open cover has a locally finite open refinement

Lemma 14.1.15. Closed subsets of compact space are closed

The image of a compact set is compact

Compact subsets of a Hausdorff space are closed

X compact, Y Hausdorff, injective maps are embeddings

Lemma 14.1.16. X is compact, Y is Hausdorff, an injective map $X \xrightarrow{f} Y$ is a topological embedding

Proof. $f : X \rightarrow f(X)$ is a continuous bijection. If $K \subseteq X$ is closed, K is also compact since X is compact, thus $f(K)$ is compact, $f(K)$ is also closed since Y is Hausdorff \square

Definition 14.1.17. X is called **connected** if it can be written as the union of two open subsets

X is called **locally connected** if for any $x \in X$, there is a local basis that are connected

Proposition 14.1.18. Connected components are closed

Connectedness and local path connectedness implies path connectedness

Remark 14.1.19. Connected components may not be open

Definition 14.1.20. $E \xrightarrow{p} B$ has **lift extension property** for (X, A) if for any $X \xrightarrow{f} B$, a lift $A \xrightarrow{\tilde{f}} E$ can be extended to $\tilde{f} : X \rightarrow E$

$$\begin{array}{ccc} A & \xrightarrow{\tilde{f}} & E \\ \downarrow & \nearrow \tilde{f} & \downarrow p \\ X & \xrightarrow{f} & B \end{array}$$

$E \xrightarrow{p} B$ has **homotopy lifting property** for (X, A) if it has lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$

Proposition 14.1.21. If (X, A) satisfies homotopy extension property, and A is contractible, then the quotient map $X \xrightarrow{q} X/A$ is a homotopy equivalence

Proof. Consider $X \times \{0\} \cup A \times I \rightarrow A \hookrightarrow X$, where $(x, 0) \mapsto x$, $(a, 1) \mapsto *$ can be extended to $f : X \times I \rightarrow X$, $f_0 = 1_X$, $f_1(A) = \{*\}$, thus f_1 induces $r : X/A \rightarrow X$, $f_1 = r q$, $X \times I \xrightarrow{f} X \xrightarrow{q} X/A$ also induce $g : X/A \times I \rightarrow X/A$, where $q f_t = g_t q$, and $g_0 = 1_{X/A}$, $g_1 = q r$ thus $q r \simeq 1_{X/A}$ \square

Definition 14.1.22. $U \subseteq X$ is open if $U \cap K$ is open for any compact subspace $K \subseteq X$ defines a topology. Equivalently, $F \subseteq X$ is closed if $F \cap K$ is closed for any compact subspace $K \subseteq X$. X is **compactly generated** if X has this topology

Definition 14.1.23. A map is **proper** if the preimage of a compact set is compact

A map is **discrete** if the preimage of a discrete set is discrete

Definition 14.1.24. X has **discrete topology** if $\tau = \mathcal{P}(X)$. X has **trivial topology** if $\tau = \{\emptyset, X\}$

Properties of discrete topology

Proposition 14.1.25. Suppose X has discrete topology

(a) Any map $f : Y \rightarrow X$ is continuous iff $f^{-1}(x)$ is open for all $x \in X$

(b) If continuous maps $f, g : X \rightarrow X$ are homotopic, then they are actually the same

Proof.

(a) For any subset $U \subseteq X$, $f^{-1}(U) = \bigcup_{x \in U} f^{-1}(x)$ is open

(b) If $F : X \times I \rightarrow X$ is a homotopy, then the restriction on $\{x\} \times I$ is gives a continuous map $I \rightarrow X$, the image has to be connected, thus the restriction is a constant, thus $f(x) = F(x, 0) = F(x, 1) = g(x)$ \square

Pasting lemma

Lemma 14.1.26. $F_i \subseteq X$ are closed, $\bigcup_i F_i = X$, $f|_{F_i}$ are continuous, then f is continuous

X compact + Y Hausdorff $\Rightarrow f : X \rightarrow Y$ quotient map

Lemma 14.1.27. If X is compact, Y is Hausdorff, a surjective continuous map $f : X \rightarrow Y$ is a quotient map

Proof. Let's use the universal property of quotient space, consider a continuous map $g : X \rightarrow Z$ such that g maps fibers of f to points, thus we have a map $\tilde{g} : Y \rightarrow Z$, $\tilde{g} f = g$, for any closed set F in Z , so is $K = g^{-1}(F) = f^{-1}(\tilde{g}^{-1}(F))$, since X is compact, so is K , hence $f(K) = \tilde{g}^{-1}(F)$ is compact, and since Y is Hausdorff, $\tilde{g}^{-1}(F)$ is closed \square

X locally compact + Hausdorff, F closed iff F intersects K is compact for any K compact

Lemma 14.1.28. X is locally compact, Hausdorff, $F \subseteq X$ is closed iff $F \cap K$ is compact for any compact subset $K \subseteq X$

Proof. F closed $\Rightarrow F \cap K$ closed. Conversely, suppose $F \cap K$ is compact for any compact subsets $K \subseteq X$, for any $x \notin F$, there is a compact set K containing an open neighborhood U of x , $F \cap K$ is compact thus closed, hence $G = U - F \cap K$ is an open neighborhood of x which is disjoint of F , hence F is closed \square

Lemma 14.1.29. X, Y are locally compact, Hausdorff, $p : X \rightarrow Y$ is continuous, proper, then p is closed

Proof. Suppose $F \subseteq X$ is closed, since $p(F \cap p^{-1}(K)) = p(F) \cap K$, by Lemma 14.1.28, we can take any $K \subseteq Y$ compact, hence F is closed \square

Definition 14.1.30. X is noncompact, the **Alexandorff extension** of X is $X^* = X \cup \{\infty\}$ with open sets \emptyset, X^* , open sets in X and complements of closed compact sets of X
 $X \hookrightarrow X^*$ is an open topological embedding

If X is also locally compact Hausdorff, X^* is the **one point compactification** of X which is Hausdorff

X, Y locally compact Hausdorff, $f: X \rightarrow Y$ proper, f send discrete sets to discrete sets

Lemma 14.1.31. X, Y are locally compact Hausdorff, $X \xrightarrow{f} Y$ is proper, then f sends discrete sets to discrete sets

Proof. Suppose $A \subseteq X$ is discrete, $x_0 \in A$, $y_0 = f(x_0) \in Y$, K is a compact neighborhood of y_0 , then $f^{-1}(K)$ is a compact neighborhood of x_0 , thus $f^{-1}(K) \cap A$ is finite, so is $K \cap f(A)$, since Y is Hausdorff, there is a neighborhood U of y_0 such that $U \cap f(A) = y_0$ \square

Lemma 14.1.32. X, Y are locally compact, $X \xrightarrow{p} Y$ is proper and discrete, then $p^{-1}(y)$ is finite, and for any neighborhood V of $p^{-1}(y)$, there is a neighborhood U of y such that $p^{-1}(U) \subseteq V$

Lemma 14.1.33. X, Y are locally compact Hausdorff, $X \xrightarrow{p} Y$ is a proper local homeomorphism, then p is a finite sheeted covering

Definition 14.1.34. The **compact-open topology** on Y^X is given by a subbase $V(K, U) := \{f \in Y^X \mid f(K) \subseteq U\}$, with $K \subseteq X$ compact and $U \subseteq Y$ open

A **normal family** $\{f_i\}$ is a precompact subset of Y^X

Lemma 14.1.35. $\{f_n\}$ converges pointwise on X iff $\{f_n\}$ converges in Y^X with the product topology $\prod_{x \in X} Y$. Hence we call the product topology the **topology of pointwise convergence**

Proof. If f_n converges pointwise on X to f , then for any neighborhood V_i of $f(x_i)$, $i = 1, \dots, k$, V_k contains all but finitely many $f_n(x_i)$, thus for n big enough, $f_n \in V_1 \cap \dots \cap V_k \cap \prod_{x \neq x_0} Y$, i.e. $\{f_n\}$ converges to f in Y^X \square

Theorem 14.1.36. X is compact, Y is a complete metric space, then the topology induced by metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$ is the same as the compact-open topology on Y^X

Theorem 14.1.37. $Y^* \cong Y$

Theorem 14.1.38. The composition $Z^Y \times Y^X \rightarrow Z^X$, $(g, f) \mapsto g \circ f$ is continuous, in particular, if $X = *$, then this becomes the evaluation map $\text{eval} : Z^Y \times Y, (f, y) \mapsto f(y)$

Theorem 14.1.39. $Z^{X \times Y} \cong (Z^Y)^X$

Definition 14.1.40. A topological space X is reducible if $X = X_1 \cup X_2$, X_1, X_2 are proper nonempty closed subsets, $X_1 \not\subseteq X_2$, $X_2 \not\subseteq X_1$, X is **irreducible** if not reducible

Definition 14.1.41. A topological space X is **Noetherian** if $X \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$ terminates, $\dim V = \sup_d (X_0 \supsetneq X_1 \supsetneq \dots \supsetneq X_d)$, V_i 's are closed and irreducible

Tychonoff's theorem

Theorem 14.1.42 (Tychonoff's theorem). $\{K_i\}_{i \in I}$ are compact, so is $\prod_{i \in I} K_i$

Proposition 14.1.43. Connected sets of \mathbb{R} are intervals (a, b) , $[a, b)$, $(a, b]$ or $[a, b]$

Jordan curve theorem

Theorem 14.1.44 (Jordan curve theorem). $S^n \xrightarrow{i} \mathbb{R}^{n+1}$ is injective thus an open embedding by Lemma 14.1.16, denote $X = i(S^n)$, then $Y = \mathbb{R}^{n+1} \setminus X$ consists of exactly two connected components, the interior U which is bounded, and the exterior V which is not. When $n = 1$, U and V are homeomorphic to D and $\mathbb{R}^2 \setminus D$

Definition 14.1.45. A **locally closed set** X is the intersection of an open subset and a closed subset. Equivalently, X is relatively open in \overline{X} . A **constructible set** X if it is a finite union of locally closed sets

Lefschetz fixed point theorem

Theorem 14.1.46 (Lefschetz fixed point theorem). X is a compact triangulable space of dimension n , the **Lefschetz number** of f is $\sum_{k=0}^n \text{tr}(f_*|_{H_k(X; \mathbb{Q})})$. If the Lefschetz number of f is nonzero, then f has fixed points. The converse is not true, i.e. even if the Lefschetz number is zero, then could be fixed points
If $f = \text{id}_X$, then the Lefschetz number is the Euler characteristic χ

Definition 14.1.47. The **join** of X, Y is

$$X * Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

We can also interpret it as all possible paths from X to Y . In general, $\ast_i X_i$ can be thought of as finite sum $\sum_i t_i x_i, t_i \in I, x_i \in X_i$

14.2 Retract

Definition 14.2.1. $A \xrightarrow{i} X$ is inclusion. A **deformation** of A into $B \subseteq X$ in X is a homotopy $A \xrightarrow{f_t} X$ such that $f_0 = i$ and $f_1(A) \subseteq B$, onto if equality holds. $X \xrightarrow{r} A$ is a **retraction** if $ri = 1_A$. r is a **weak retraction** if inclusion $A \xrightarrow{i} X$ has a left homotopy inverse, i.e. $ri \simeq 1_A$. A **deformation retraction** is a deformation $X \xrightarrow{f_t} X$ such that $f_1 = ri$ for some retraction $X \xrightarrow{r} A$. Deformation retraction f_t is **strong** if $f_t|_A = 1_A$. X is **contractible** if X deformation retracts onto a point. (X, A) is a **good pair** if A is a strong neighborhood deformation retract of X .

Some rudimentary lemma about retract and deformation

Lemma 14.2.2. $A \xrightarrow{i} X$ is inclusion

- (1) X is deformable into A iff i is a **weak section**, namely i has a right homotopy inverse, i.e. $ir \simeq 1_X$
- (2) i is a homotopy equivalence iff A is a weak retract of X and X is deformable into A
- (3) If X is deformable into a retract A , then A is a deformation retract of X
- (4) If (X, A) is cofibered, then A is a weak retract of X iff A is a retract of X

Proof.

- (1) If $X \times I \xrightarrow{H} X$ is a homotopy from 1_X to ir , then H is a deformation of X into A since $H_0 = 1_X$, $H_1(X) \subseteq A$. If H is a deformation of X into A , since $H_1(X) \subseteq A$, define $X \xrightarrow{r} A$ such that $ir = H_1$, then r is a right homotopy inverse of i
- (2) i is a homotopy equivalence \Leftrightarrow there exists $X \xrightarrow{r} A$ such that $ri \simeq 1_A$, $1_X \xrightarrow{H} ir \Leftrightarrow r$ is a weak retract, H is a deformation of X into A
- (3) $X \xrightarrow{r} A$ is a retraction, $X \times I \xrightarrow{H} X$ is a deformation of X , then $1_X \simeq ir'$ for some $X \xrightarrow{r'} A$, hence $r \simeq rir' = r' \Rightarrow 1_X \simeq ir \simeq ir$ giving a deformation retract
- (4) $A \times I \xrightarrow{H} A$ is a homotopy from ri to 1_A , since $r(a) = H_0(a)$ and (X, A) is cofibered, we have $X \times I \xrightarrow{F} A$, then $F_0 = r$, $F_1i = 1_A$, i.e. r is homotopic to retraction F_1 □

Definition 14.2.3. \mathcal{C} is a class of topological spaces closed under homeomorphism and closed subsets. X is an **absolute retract** for \mathcal{C} if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow X$ is a retract of Y . X is an **absolute neighborhood retract** for \mathcal{C} if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow X$ is a neighborhood retract of Y

Part VI

Algebraic Topology

Chapter 15

Cell structure

15.1 CW complexes

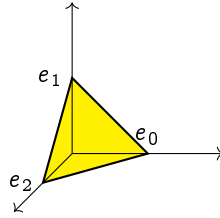
Standard simplex

Definition 15.1.1. With the standard basis $\{e_i\}$ for \mathbb{R}^∞ as vertices, the **standard n -simplex** is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \subseteq \mathbb{R}^\infty \mid \sum t_i = 1, 0 \leq t_i \leq 1 \right\}$$

The **i -th face** of Δ^n is the face opposite to the i -th vertex, i.e. $\{t_i = 0\} \cap \Delta^n$

The **boundary** of Δ^n to be $\partial\Delta^n$ is the union of faces. $\partial\Delta^0 = \emptyset$



$$\Delta^{n-1} \xrightarrow{d_{n,i}} \Delta^n, e_j \mapsto \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases} \text{ is } i\text{-th } \mathbf{face \ map} \text{ attaching } \Delta^{n-1} \text{ to the } i\text{-th face of } \Delta^n$$

$$\Delta^{n+1} \xrightarrow{s_{n,i}} \Delta^n, e_j \mapsto \begin{cases} e_j & j \leq i \\ e_{j-1} & j > i \end{cases} \text{ is the } i\text{-th } \mathbf{degeneracy \ map} \text{ which is a projection}$$

Definition 15.1.2. X has a **cell decomposition** if X can be written as the disjoint union of open n cells, i.e. $X = \bigcup_{n,\alpha} e_\alpha^n$, where cells e_α^n with subspace topology are homeomorphic to open n

disks or open n simplices and disjoint, $X^n = \bigsqcup_{k \leq n, \alpha} e_\alpha^k$ is called the **n -skeleton**, define $X^{-1} = \emptyset$

Suppose X, Y have cell decomposition $X = \bigcup_{n,\alpha} e_\alpha^n, Y = \bigcup_{m,\beta} e_\beta^m$, then $X \times Y$ also has a cell

decomposition $X \times Y = \bigcup_k \bigcup_{\substack{n+m=k \\ \alpha,\beta}} e_\alpha^n \times e_\beta^m$, note that $e_\alpha^n \times e_\beta^m \cong e^{n+m}$

Every topological space has a cell decomposition into points

Definition 15.1.3. A **cellular map** is a map $f : X \rightarrow Y$ between topological spaces with cell decompositions such that $f(X^n) \subseteq Y^n$

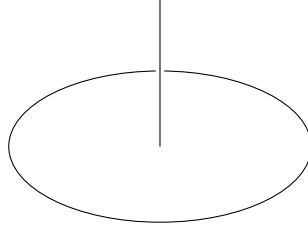
Definition 15.1.4. X is called a **cell complex** if X is a Hausdorff space with cell decomposition $X = \bigcup_{n,\alpha} e_\alpha^n$ and a cell complex structure: a family of characteristic maps $\Phi_\alpha^n : \Delta^n \rightarrow X$ such

that Φ_α^n restricted on $\Delta^n \setminus \partial\Delta^n$ is a homeomorphism onto e_α^n and $\Phi_\alpha^n(\partial\Delta^n) \subseteq X^{n-1}$

Note that in the definition, we could also replace Δ^n with D^n

Remark 15.1.5. Since Δ^n is compact Hausdorff and X is Hausdorff, $\overline{e_\alpha^n} \subseteq \Phi_\alpha^n(\Delta^n)$, $\partial e_\alpha^n \subseteq \Phi_\alpha^n(\partial \Delta^n)$ for $n > 0$, on the other hand, if $\partial e_\alpha^n \subsetneq \Phi_\alpha^n(\partial \Delta^n)$, then there exists $x \in \partial \Delta^n$ such that $y = \Phi_\alpha^n(x) \notin \overline{e_\alpha^n}$, this means there is an open neighborhood U of y disjoint from $\overline{e_\alpha^n}$, but then the preimage of U under Φ_α^n would be a nonempty open subset of Δ^n which intersects $\Delta^n \setminus \partial \Delta^n$ which is impossible, hence $\Phi_\alpha^n(\Delta^n) = \overline{e_\alpha^n}$, $\Phi_\alpha^n(\partial \Delta^n) = \partial e_\alpha^n$ for $n > 0$

A Hausdorff space X with a cell decomposition doesn't immediately give a cell complex structure, for example, consider an open disk union with an open segment right in the middle



Cell complex X can be seen as Δ^n is compact Hausdorff and X is Hausdorff and Lemma 14.1.27

Definition 15.1.6. A cell complex X is called **regular** if all characteristic maps are embeddings

Definition 15.1.7. Let X be a cell complex, it is **closure finite** if $\overline{e_\alpha^n}$ is contained in the union of finitely many cells, and we say X has the **weak topology**, meaning $F \subseteq X$ is closed iff $F \cap \overline{e_\alpha^n}$ is closed in $\overline{e_\alpha^n}$ for any cell, if a cell complex is both closure finite and has the weak topology, we say it is a **CW complex**

Closure finiteness is equivalent of saying $\partial e_\alpha^n \subseteq \bigcup_{k < n, \alpha} e_\alpha^k$ a finite union of cells

Example 15.1.8. Consider $X = D^2$ with a cell complex structure $D^2 \rightarrow D^2$ and $*$ \rightarrow x for each $x \in \partial D^2$, this doesn't satisfy closure finiteness since $\overline{e^2} = X$, but the weak topology is the same as the original one, since if $F \subseteq D^2$ is closed in the weak topology, then $F \cap \overline{e^2} = F$ is closed

Consider $X = S^1$ with a cell complex structure $*$ \rightarrow x for each $x \in S^1$, the weak topology is the discrete topology on S^1 which doesn't match with the original topology on S^1 , but it does satisfy closure finiteness

Remark 15.1.9. Suppose X is a CW complex

X^n is obviously closed due to the weak topology

Since $\overline{e_\alpha^n}$ is contained in the union of finitely many cells, $\overline{e_\alpha^n}$ contains at most finitely many 0 cells, thus any union of 0 cells F is closed because $F \cap \overline{e_\alpha^n}$ is finitely many points which is closed given that X is Hausdorff, therefore X^0 is discrete

Suppose $K \subseteq X$ is a compact subset, then $K \subseteq X = \bigcup e_\alpha^n \subseteq \bigcup \overline{e_\alpha^n} \setminus \partial e_\alpha^n$

contained in finitely many cells, since $K \subseteq \bigcup \overline{e_\alpha^n} \setminus \partial e_\alpha^n$

if $K \cap e_\alpha^n \neq \emptyset$,

, otherwise K intersects infinitely many cells,

Theorem 15.1.10. Another description of CW complexes is as follows:

These two definitions coincides

Proposition 15.1.11. Any compact set of a CW complex is contained in finitely many cells

Proposition 15.1.12. CW complexes are locally contractible, thus they are locally path connected, hence connectedness and path connectedness are equivalent for CW complexes

Theorem 15.1.13. CW complexes are normal, satisfies T_4 axiom

Proposition 15.1.14. If $A \subseteq X$ is a CW subcomplex, then (X, A) is a good pair

Theorem 15.1.15. CW complexes have partitions of unity

Proposition 15.1.16. Covering space of CW complexes are CW complexes

Proposition 15.1.17. The product of two countable CW complexes is again a CW complex

15.2 Graphs

Theorem 15.2.1. *For every group G , there is a connected two dimensional CW complex X with $\pi_1(X) = G$*

Proof. We can always find a surjection from a free group F to G , suppose F is generated by g_α 's, and the kernel K is generated by r_β 's, i.e. F has a group presentation $\langle g_\alpha | r_\beta \rangle$, then define X to be $\bigvee_\alpha S_\alpha^1$ attached with cells e_β^2 's along each word r_β \square

Definition 15.2.2. Cayley graphs, Cayley complexes

Definition 15.2.3. A graph is a one dimensional CW complex, a tree is a contractible graph, a tree is called maximal if it contains all the vertices, note that in a tree, for any two vertices, there is a unique path connecting them

Proposition 15.2.4. Let X be a connected graph, any tree in X is contained in a maximal tree, in particular, X has a maximal tree

Proof. Let's prove more generally any subgraph X_0 is the deformation retraction of subgraph Y which contains all the vertices

Construct $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ as follows, X_{i+1} is obtained by adding the closures of all the edges that connected to X_i , $X = \bigcup_i X_i$, since X is path connected, let $Y_0 = X_0$, and construct $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ as follows, for any vertex in $X_{i+1} - X_i$, choose one edge that connects to Y_i , and add the closure, so we have Y_{i+1} from Y_i , it is easy to see the Y_{i+1} deformation retracts onto Y_i , so $Y = \bigcup_i Y_i$ deformation retracts onto $Y_0 = X_0$

If $X_0 = T$ is a tree, so is Y since Y deformation retracts onto T which is contractible \square

Free basis for connected graphs

Proposition 15.2.5 (Free basis for connected graphs). For a connected graph X with maximal tree T , for any edge $e_\alpha \in X - T$, there is a corresponding loop f_α goes from x_0 to one endpoint of e_α , across e_α to the other, and go back to x_0 , $\pi_1(X, x_0)$ is a free group generated by f_α

Proof. Consider $X \rightarrow X/T$ which is a homotopy equivalence \square

Theorem 15.2.6. *Any subgroup of a free group is also free*

Proof. Let F be a free group, there exists a graph X such that $\pi_1(X) = F$ by just taking the wedge sum of circles at x_0 , let $G \leq F$ be a subgroup, then there exists a covering $Y \xrightarrow{p} X$ such that $p_*(\pi_1(Y, y_0)) = G$, thus $\pi_1(Y, y_0) \cong G$, and since Y is a covering of X , Y is also a graph, by Proposition 15.2.5, $G \cong \pi_1(Y)$ is free \square

15.3 Simplex category

Definition 15.3.1. The **simplex category** $Simp$ has $[n] := \{0, 1, \dots, n\}$ as objects, and order preserving functions as morphisms, there are two special types of morphisms: **Face maps**

$d_{n,i} : [n-1] \rightarrow [n]$, $d_{n,i}(j) = \begin{cases} j & , j < i \\ j+1 & , j \geq i \end{cases}$ and the **degeneracy maps** $s_{n,i} : [n+1] \rightarrow [n]$,

$s_{n,i}(j) = \begin{cases} j & , j \leq i \\ j-1 & , j > i \end{cases}$, they subject to **simplicial identities**:

$$d_j \circ d_i = d_i \circ d_{j-1}, i < j \Leftrightarrow i \leq j-1$$

$$s_j \circ s_i = s_i \circ s_{j+1}, i \leq j \Leftrightarrow i < j+1$$

$$s_j \circ d_i = \begin{cases} d_{i-1} \circ s_j & , j \leq i-2 \Leftrightarrow j < i-1 \\ 1 & , j = i, i-1 \\ d_i \circ s_{j-1} & , j > i \Leftrightarrow j-1 \geq i \end{cases}$$

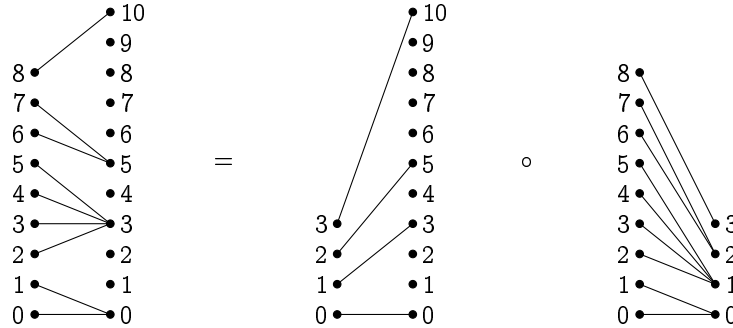
I find it easier to just think about $d_i, s_i : [\infty] \rightarrow [\infty]$

The **semisimple simplex category** is when discarding degeneracy maps, i.e. morphisms are strictly order preserving. The **augmented simplex category** is $Simp \cup \emptyset$. The **unordered simplex category** with the same objects as $Simp$ and all functions as morphisms. The **unordered semisimple simplex category** with the same objects as $Simp$ and all injections as morphisms

Unique decomposition of morphisms in simplex category

Lemma 15.3.2. Thanks to simplicial identities. Any morphism can be uniquely decomposed into a surjection compose with an injection. Any injection can be uniquely decomposed into composition of face maps with index strictly increasing. Any surjection can be uniquely decomposed into composition of degeneracy maps with index nondecreasing

Proof. For example



The right hand side can be written as $d_9 d_8 d_7 d_6 d_4 d_2 d_1 \circ s_2 s_2 s_1 s_1 s_0$

□

Definition 15.3.3. A **simplicial object** in \mathcal{C} is a functor $Simp^{op} \rightarrow \mathcal{C}$, and a **cosimplicial object** is a functor $Simp \rightarrow \mathcal{C}$. If \mathcal{C} is the category of sets, then the simplicial object is called a **simplicial set** $X : Simp^{op} \rightarrow Set$, $X([n]) = X_n$ is a family of sets, the face map $X(d_{n,i})$ sends elements of X_n to its i -th face

Similarly, we have semisimplicial object, augmented simplicial object, unordered simplicial object and unordered semisimplicial object

Example 15.3.4. The standard simplices $\{\Delta^n\}$ in Definition 15.1.1 with face and degeneracy maps is a cosimplicial object Δ in the category of topological spaces

$$\begin{array}{ccc} [n] & \longrightarrow & \Delta^n \\ d_i \downarrow & & \downarrow d_i \\ [n+1] & \longrightarrow & \Delta^{n+1} \end{array} \quad \begin{array}{ccc} [n] & \longrightarrow & \Delta^n \\ s_i \uparrow & & \uparrow s_i \\ [n+1] & \longrightarrow & \Delta^{n+1} \end{array}$$

This functor is faithful and injective on objects, hence we may also just think of standard simplices as the simplex category

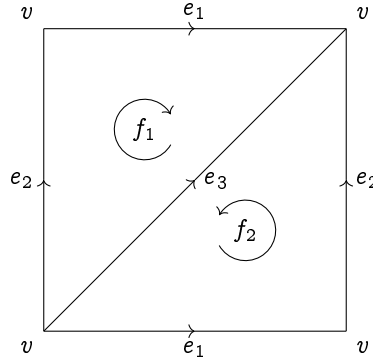
Due to Lemma 15.3.2, any morphism $\Delta^n \rightarrow \Delta^m$ can be uniquely written as an ordered degeneration compose with an ordered inclusion $\Delta^n \twoheadrightarrow \Delta^k \hookrightarrow \Delta^m$

Definition 15.3.5. A Δ -**complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \rightarrow \overline{e}_\alpha^n$ to its i -th face is such that $\Phi_\beta^{n-1} = \Phi_\alpha^n \circ d_{n,i}$ for some Φ_β^{n-1}

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{d_{n,i}} & \Delta^n \\ \Phi_\beta^{n-1} \downarrow & & \downarrow \Phi_\alpha^n \\ \overline{e}_\beta^{n-1} & \hookrightarrow & \overline{e}_\alpha^n \end{array}$$

Remark 15.3.6. A Δ complex X is also called semisimplicial complex because it can be regarded as a semisimple set $X : \mathbf{Simp} \rightarrow \mathbf{Set}$, with $X([n]) = X_n$ being all the n faces, $X(d_{n,i}) : X_n \rightarrow X_{n-1}$ being face maps that map each cell to its i -th face

Example 15.3.7. Consider a Δ complex structure on torus



Definition 15.3.8. An **unordered Δ -complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \rightarrow \overline{e}_\alpha^n$ to any face is such that $\Phi_\beta^{n-1} = \Phi_\alpha^n \circ i$ for some Φ_β^{n-1} , i is an inclusion to that face regardless of order

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{i} & \Delta^n \\ \Phi_\beta^{n-1} \downarrow & & \downarrow \Phi_\alpha^n \\ \overline{e}_\beta^{n-1} & \hookrightarrow & \overline{e}_\alpha^n \end{array}$$

Remark 15.3.9. An unordered Δ complex X can be regarded as an unordered semisimple set $X : \mathbf{Simp} \rightarrow \mathbf{Set}$, with $X([n]) = X_n$ being all the n faces, $X(i) : X_n \rightarrow X_{n-1}$ being face maps that map each cell to the corresponding face

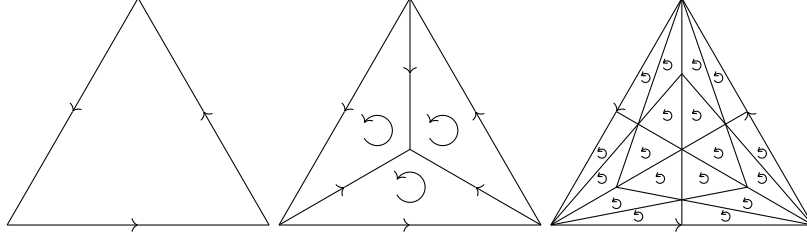
Definition 15.3.10. A regular unordered Δ complex is called a **multicomplex**, prefix multi-means different simplices can have the same faces

A regular unordered Δ complex in which each simplex is uniquely determined by its faces is called a simplicial complex

Definition 15.3.11. A **simplicial map** $f : K \rightarrow L$ is a map such that maps vertices of a simplex of K to the vertices of a simplex of L and linear on each simplex, two simplicial maps f, g are **contiguous** if for any simplex s in K , $f(s), g(s)$ are faces of the same simplex, in particular, f, g are homotopic, just consider $(1-t)f + tg$

Lemma 15.3.12. Any unordered Δ complex can be subdivided once to become a Δ complex, and any Δ complex can be subdivided to be a simplicial complex, therefore, every unordered Δ complex is homeomorphic to a Δ complex and is homeomorphic to a simplicial complex

Example 15.3.13. The one on the left with three edges identified is not a Δ complex, but an unordered Δ complex, the one in the middle is a Δ complex, but not a simplicial complex, the one on the right is a simplicial complex



Definition 15.3.14. A **singular Δ -complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \rightarrow \bar{e}_\alpha^n$ to its i -th face is such that $\Phi_\beta^k \circ q = \Phi_\alpha^n \circ d_{n,i}$ for some $\Phi_\beta^k, k \leq n-1, q : \Delta^{n-1} \rightarrow \Delta^k$ is a degeneration

$$\begin{array}{ccc} \Delta^{n-1} & \xrightarrow{q} & \Delta^k \\ d_{n,i} \downarrow & & \downarrow \Phi_\beta^k \\ \Delta^n & \xrightarrow{\Phi_\alpha^n} & X \end{array}$$

Remark 15.3.15. A singular Δ complex is defined quite like a CW complex, where the characteristic maps are simplicial, instead of cellular

The vertices of a singular Δ complex can be given a partial order which is a total order on each simplex, just start at any point and use Zorn's lemma, in fact, it can be totally ordered

Definition 15.3.16. Suppose $X : \mathbf{Simp} \rightarrow \mathbf{Set}$ is a simplicial set, we can use this combinatorial information to construct its **geometric realization** $|X|$ with $X([n]) = X_n$ represents its n faces and morphisms $X_n \rightarrow X_{n-1}$ represents face maps and morphisms $X_n \rightarrow X_{n+1}$ represents degeneracy maps

The concrete construction is

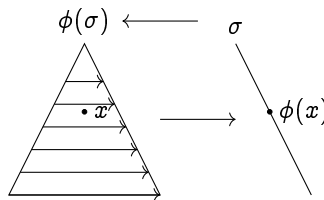
$$|X| := \frac{\bigsqcup_{n \geq 0} \Delta([n]) \times X([n])}{(\Delta\phi(x), \sigma) \sim (x, X\phi(\sigma))} = \frac{\bigsqcup_{n \geq 0} \Delta^n \times X_n}{(\phi(x), \sigma) \sim (x, \phi(\sigma))}$$

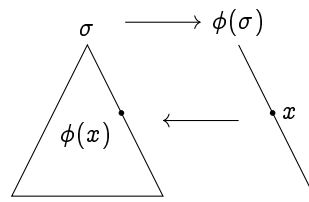
Where X_n is given the discrete topology, $x \in \Delta^n, \sigma \in X_n, \phi$ a morphism ranging over \mathbf{Simp} which is the same as ranging over all face maps and degeneracy maps since every morphism can be decomposed uniquely as a degeneration compose with an inclusion, the difference being taking the transitive closure which is the definition of a quotient space

This basically means that give each n face an n simplex and if there is a face map or a degeneracy map, glue the corresponding simplices according to the map

Proposition 15.3.17. The geometric realization $|X|$ of a simplicial set X is a singular Δ complex, $|-|$ is a functor from the category of simplicial sets to the category of singular Δ complexes, Moreover, if X is a semisimplicial set, then $|X|$ is a Δ complex

Proof. Let's first deal with the degeneration





□

Definition 15.3.18. Given a singular Δ complex (equivalently a simplicial set), its one skeleton with a partial order can be seen as a diagram, consider such a diagram in the category of spaces, we call this a complex of spaces

15.4 abstract simplicial complex

Definition 15.4.1. An **abstract simplicial complex** is $K \subseteq \mathcal{P}(S) \setminus \emptyset$ such that $X \in K \Rightarrow \mathcal{P}(X) \setminus \emptyset \subseteq K$. Finite elements of K are called **faces**. The **dimension** of a face X is $\dim X = |X| - 1$. The d skeleton K^d is the union of faces of dimension no more than d . $\dim K = \sup \dim X$. K^0 are **vertices**. Maximal elements are **facets**. K is **pure** if all facets have dimension $\dim K$. A **simplex** is a subcomplex which contains all its nonempty subsets, for $X \in K$, \overline{X} is the corresponding simplicial complex

Definition 15.4.2. The **closure** \overline{L} of $L \subseteq K$ is smallest subcomplex of K containing L . The **star** of $Y \in K$ is $\text{st } Y = \{X \mid X \cap Y \neq \emptyset\}$, the star of $L \subseteq K$ is $\text{st } L = \bigcup_{Y \in L} \text{st } Y$. The **link** of a face $Y \in K$ is $\text{lk } Y = \{X \mid X \cap Y = \emptyset, X \cup Y \in K\}$. Equivalently, $\text{lk } Y = \overline{\text{st } Y} - \text{st } Y$

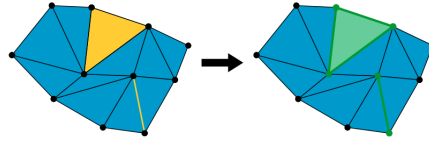


Figure 15.4.1: Two **simplices** and their **closure** Closure of a complex

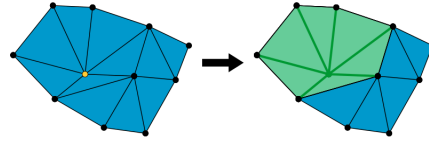


Figure 15.4.2: A **vertex** and its **star** Star of a complex

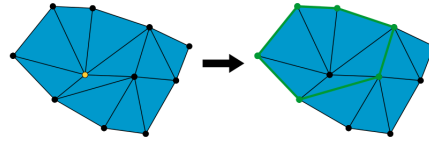


Figure 15.4.3: A **vertex** and its **link** Link of a complex

Note. $\text{lk } \emptyset = K$

Definition 15.4.3. If K, L has disjoint sets of vertices, then $K * L = \{X \sqcup Y \mid X \in K, Y \in L\}$

Definition 15.4.4. $L \subseteq K$, the **deletion** $K \setminus L$ consists of those sets which don't contain sets in L as subsets. The stellar subdivision of $X \in K$ is by introducing a new vertex x , and form $K \setminus X \cup (\overline{x} * \partial \overline{X} * \text{lk } X)$

Definition 15.4.5. A simplicial map $f : K \rightarrow L$ is such that $f(K^d) \subseteq L^d$

15.5 CW approximation

CW approximation

Theorem 15.5.1 (CW approximation). *For any topological space X , there is a CW complex Z and a weak homotopy equivalence $f : Z \rightarrow X$, this is called a CW approximation*

Whitehead's theorem

Theorem 15.5.2 (Whitehead's theorem). *Suppose $f : X \rightarrow Y$ is weak homotopy equivalence between CW complexes, then it is a homotopy equivalence*

15.6 Triangulation

Definition 15.6.1. A **pseudomanifold** M is a pure triangulated space of dimension n such that M is not branching, i.e. any two n simplices have precisely one $(n - 1)$ common face, M is strongly connected, i.e. any two n simplices can be linked with a sequence of simplices having common $(n - 1)$ face pairwise

Note. The dual graph Γ of M is connected and n -regular

Example 15.6.2. The 0 dimensional pseudomanifold is the disjoint union of two points, since the empty set has to be the common face two point. The dual graph is a two points joined by an edge

A 1 dimensional pseudomanifold is a infinite chain or a loop, its dual graph is the same

Definition 15.6.3. D is a nonmaximal simplex, then $\text{lk } D$ is a $n - |D|$ pure dimensional simplicial complex. A pseudomanifold such that $\text{lk } D$ is also pseudomanifolds for any nonmaximal simplex is an **abstract polytope**

Chapter 16

Homology theory

16.1 Singular homology

Definition 16.1.1 (Eilenberg-Steenrod axioms). Top is the category of topological spaces, Ab is the category of abelian groups, \mathcal{T} is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces (X, A) such that $A \subseteq X$, \mathcal{T}_A is the fully faithful subcategory of \mathcal{T} with objects (X, A) , $R: \mathcal{T} \rightarrow Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor

Relative homology are functors $H_n: \mathcal{T} \rightarrow Ab$, then $H_n(-, A)$ define functors $\mathcal{T}_A \rightarrow Ab$, **absolute homology** are functors $H_n(-, \emptyset): Top \rightarrow Ab$, **reduced homology** are $\tilde{H}_n = H_n(-, *)$. $\partial_n: H_n \rightarrow H_{n-1}R$ are natural transformations

$$\begin{array}{ccc} H_n(X, A) & \xrightarrow{H_n(f)} & H_n(Y, B) \\ \downarrow \partial_n & & \downarrow \partial_n \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) \end{array}$$

(H, ∂) is a **homology theory** if it satisfies axioms

Homotopy invariance: $f \simeq g: (X, A) \rightarrow (Y, B)$, then $H_n(f) = H_n(g)$

Additivity: $(X, A) = \bigsqcup_{\alpha} (X_{\alpha}, A_{\alpha})$, then $\bigoplus_{\alpha} H_n(X_{\alpha}, A_{\alpha}) \xrightarrow{\bigoplus_{\alpha} H_n(i_{\alpha})} H_n(X, A)$ is an isomorphism

Exactness:

$$\dots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial_n} \dots$$

Excision: $\bar{Z} \subseteq \overset{\circ}{U}$, then $H_n(X - Z, U - Z) \xrightarrow{H_n(i)} H_n(X, U)$ is an isomorphism

Dimension: $H_n(*) = 0, \forall n \neq 0$, $H_0(*)$ is the **coefficient group**

(H, ∂) is an **extraordinary homology theory** without dimension axiom

Definition 16.1.2. A **singular n -simplex** in X is just a continuous map $\Delta \xrightarrow{\sigma} X$, the free abelian group $C_n(X)$ with singular n -simplices in X as basis consists of n -chains(singular chain) which are finite sums $\sum n_i \sigma_i, n_i \in \mathbb{Z}$, we can tensor $C_n(X)$ with a ring R , $C_n(X; R) := C_n(X) \otimes_{\mathbb{Z}} R$ to be chains with R coefficients, here R could be an abelian group(group ring) or a field Also, if we only consider characteristic maps(for simplicial, Δ , cell complexes), we would get $C_n(X)$ to be simplicial, cellular chains

Remark 16.1.3. Given a topological space, we can form a huge Δ complex $S(X)$

Let $S(X)^0$ be X with discrete topology which can be identified with all the maps $\Delta^0 = * \rightarrow X$, then build on it inductively as a CW complex, suppose $S(X)^n$ is constructed, for each map $\Delta^{n+1} \rightarrow X$, we add an $n+1$ cell by gluing its faces to its restrictions, preserving the order

Similarly, suppose X is a singular Δ complex, we can also construct a Δ complex $\Delta(X)$ by replacing continuous maps with simplicial maps above

The simplicial homology of $S(X), \Delta(X)$ is the same as the singular homology of X

Definition 16.1.4. The boundary map $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_{i=0}^n (-1)^i \sigma [[e_0, \dots, \widehat{e_i}, \dots, e_n]]$$

Where $\sigma : \Delta^n \rightarrow X$ is a singular simplex, we can easily show that $\partial_n \partial_{n+1} = 0$, define cycles $Z_n(X) = \ker \partial_n$ and boundaries $B_n(X) = \text{im} \partial_{n+1}$, and the (singular) homology group $H_n(X) = Z_n(X)/B_n(X)$

Similarly, we can define simplicial cycles, boundaries and homology groups correspondingly. For cell complexes, if $\partial_n \sigma \subseteq X^{n-1}$, σ is called a cellular cycle, and cellular boundary is defined to be the image of some cellular chain, we can therefore define cellular homology

Definition 16.1.5. Define $C_n(X, A)$ to be $C_n(X)/C_n(A)$, $C_\bullet(X, A)$ form a chain complex, $Z_n(X, A)$ can be represented by n -chains with its boundary in A

The cellular homology could also be defined as the homology groups of $\dots \rightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \rightarrow \dots$, where d_{n+1} is induced by $H_{n+1}(X^{n+1}, X^n) \rightarrow H_n(X^n, X^{n-1})$

Definition 16.1.6. Suppose $\mathcal{U} = \{U_j\}$ are a family of subspaces of X and interiors of U_j form an open cover of X , define $C_n^\mathcal{U}(X)$ to be n -chains $\sum n_i \sigma_i$ such that the image of each σ_i is contained in some U_j , $C_n^\mathcal{U}(X, A) := C_n^\mathcal{U}(X)/C_n^\mathcal{U}(A)$

Theorem 16.1.7. The inclusion $C_n^\mathcal{U}(X, A) \rightarrow C_n(X, A)$ is a chain homotopy equivalence

Excision theorem for singular homology

Theorem 16.1.8 (Excision theorem for singular homology). *Singular homology satisfies excision theorem*

Proof. Suppose $\bar{Z} \subseteq \overset{\circ}{U}$, let $A = U, B = X - Z, \mathcal{U} = \{A, B\}$, only need to show $H_n^\mathcal{U}(A \cup B, A) \cong H_n(A \cup B, A) \cong H_n(X, U) \cong H_n(X - Z, U - Z) \cong H_n(B, A \cap B) \cong H_n^\mathcal{U}(B, A \cap B)$
Consider $C_n^\mathcal{U}(B) \hookrightarrow C_n^\mathcal{U}(X) \rightarrow C_n^\mathcal{U}(X)/C_n^\mathcal{U}(A)$ has kernel $C_n^\mathcal{U}(A \cap B)$, thus $C_n^\mathcal{U}(B)/C_n^\mathcal{U}(A \cap B) \cong C_n^\mathcal{U}(X)/C_n^\mathcal{U}(A)$ \square

Definition 16.1.9. (X, A) is called a good pair if A has a neighborhood U deformation retracts onto A

Definition 16.1.10. The reduced singular homology $\tilde{H}_n(X)$ is defined to be the homology group of the chain complex

$$\dots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

Lemma 16.1.11. $\tilde{H}_n(X) \rightarrow H_n(X, *)$ is an isomorphism induced from $C_n(X) \rightarrow C_n(X, *)$

Proof. For $\sum n_i \sigma_i \in C_1(X)$, if $\sum n_i \partial \sigma_i \in C_0(*)$, then $\sum n_i \partial \sigma_i = 0$, thus $H_1(X, *) \cong \tilde{H}_1(X)$
For any $\sum n_i P_i \in C_0(X, *)$ where P_i are points, then $\sum n_i P_i - \sum n_i *$ is a preimage in $Z_0(X)$, a boundary in $Z_0(X)$ certainly maps to a boundary in $C_0(X, *)$, suppose $\sum n_i P_i \in C_0(X, *)$ is a boundary, $\sum n_i P_i - \sum n_i *$ has to be a boundary in $C_0(X)$, thus $H_0(X, *) \cong \tilde{H}_0(X)$ \square

Theorem 16.1.12. If (X, A) is called a good pair, $H_n(X, A) \xrightarrow{q_*} \tilde{H}_n(X/A)$ is an isomorphism

Proof. Consider the quotient map $q : X \rightarrow X/A$ induces $H_n(X, A) \rightarrow H_n(X/A, *) \rightarrow \tilde{H}_n(X/A)$, we show that q_* is an isomorphism, suppose U is a neighborhood of A that deformation retracts onto it, consider the following diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{i_*} & H_n(X, U) & \xleftarrow{i_*} & H_n(X - A, U - A) \\ \downarrow q_* & & \downarrow q_* & & \downarrow q_* \\ H_n(X/A, *) & \xrightarrow{i_*} & H_n(X/A, U/A) & \xleftarrow{i_*} & H_n(X - A/A, U - A/A) \end{array}$$

$H_n(X, A) \xrightarrow{i_*} H_n(X, U)$, $H_n(X/A, *) \xrightarrow{i_*} H_n(X/A, U/A)$ are isomorphisms because of the deformation retraction, $H_n(X - A, U - A) \xrightarrow{i_*} H_n(X, U)$, $H_n(X - A/A, U - A/A) \xrightarrow{i_*} H_n(X/A, U/A)$ are isomorphisms because of the Theorem 16.1.8, $H_n(X - A, U - A) \xrightarrow{q_*} H_n(X - A/A, U - A/A)$ is an isomorphism since $(X - A, U - A) \xrightarrow{q} (X - A/A, U - A/A)$ is a homeomorphism \square

Theorem 16.1.13 (Mayer Vietoris sequence). *Suppose A, B are subspaces of X that the interior of A, B covers X , then we have an exact sequence of homology groups $\cdots \rightarrow H_n(A \cap B) \rightarrow H_n(A) \oplus H_n(B) \rightarrow H_n(A \cup B) \rightarrow \cdots$*

Proof. It is not hard to see there is a short exact sequence $0 \rightarrow C_n^U(A \cap B) \rightarrow C_n^U(A) \oplus C_n^U(B) \rightarrow C_n^U(A \cup B) \rightarrow 0$, $x \mapsto (x, x)$ and $(x, y) \mapsto x - y$ \square

Theorem 16.1.14. *Suppose X has a Δ complex structure, $H_n^\Delta(X) \rightarrow H_n(X)$, $\Phi_\alpha^n \mapsto \Phi_\alpha^n$ is an isomorphism*

Definition 16.1.15. $S^n \xrightarrow{f} S^n$ induces $\mathbb{Z} \cong H_n S^n \xrightarrow{f_*} H_n S^n \cong \mathbb{Z}$, $f_*(1)$ is the **degree** of f

Proposition 16.1.16 (Properties of degrees).

1. $\deg 1 = 1$
2. $\deg(fg) = \deg f \deg g$
3. If f is not surjective, $\deg f = 0$
4. If f is a reflection, $\deg f = -1$
5. Let a be the antipodal map, then $\deg a = (-1)^{n+1}$
6. If f has no fixed points on S^n , then f is homotopic to the antipodal map

Proof.

1. Let Δ_1^n, Δ_2^n maps to the upper and lower hemisphere be a Δ complex structure on S^n , then $\Delta_1^n - \Delta_2^n$ would be a generator, and f maps them to $\Delta_2^n - \Delta_1^n$, thus $\deg f = -1$
2. a is the composition of $n + 1$ reflections
3. Since $f(x) \neq -x$, $\frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$ homotopy f to a

\square

Definition 16.1.17. View Δ^n as $\{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$, we can cut $\Delta^n \times \Delta^m = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \times \{0 \leq x_{n+1} \leq \cdots \leq x_{n+m} \leq 1\}$ into $\binom{n+m}{m}$ simplices

$$\Delta^n \times \Delta^m = \bigcup_{\sigma} \Delta_{\sigma}, \quad \Delta_{\sigma} = \{0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n+m)} \leq 1\}$$

σ runs over (n, m) -shuffles. Each σ can be viewed as a path through a grid as in Definition 4.2.2. Associate a linear map $\ell_{\sigma} : \Delta^{n+m} \rightarrow \Delta_{\sigma} \subseteq \Delta^n \times \Delta^m$, sending the k -th vertex to vertex in the grid. The **cross product**

$$C_n(X) \otimes C_m(Y) \rightarrow C_{n+m}(X \times Y)$$

$$f \otimes g \mapsto f \times g$$

Where

$$f \times g = \sum_{\sigma} (-1)^{|\sigma|} (f \times g) \ell_{\sigma}$$

Here on the right hand side $f \times g : \Delta^n \times \Delta^m \rightarrow X \times Y$, $(a, b) \mapsto (f(a), g(b))$ is different from the left hand side. We have $\partial(f \times g) = \partial f \times g + (-1)^n f \times \partial g$

Eilenberg-Zilber theorem

Theorem 16.1.18 (Eilenberg-Zilber theorem). $C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y)$ is a natural equivalence

Proof. Consider $Top \times Top$ with model $\mathcal{M} = \{(\Delta^n, \Delta^m)\}$, $F, G : Top \times Top \rightarrow Ch_{\geq 0}$, $F(X, Y) = C_*(X \times Y)$, $G(X, Y) = C_*(X) \otimes C_*(Y)$, $H_i(\Delta^n \times \Delta^m) = 0$ for $i \neq 0$,

$F_k(X, Y) = \left\{ \Delta^k \xrightarrow{(\text{id}, \text{id})} \Delta^k \times \Delta^k \xrightarrow{\sigma} X \times Y \right\}$. By Exercise 46.0.7, $H_i(C_*(X) \otimes C_*(Y)) = 0$ for

$i \neq 0$, $C_k(X) = \{ \Delta^k \xrightarrow{\text{id}} \Delta^k \xrightarrow{\sigma} X \}$, $G_k(X, Y) = \left\{ (\sigma \otimes \tau)(\text{id}_{\Delta^p} \otimes \text{id}_{\Delta^q}) \Big| \Delta^p \xrightarrow{\sigma} X, \Delta^q \xrightarrow{\tau} Y \right\}$

There is a natural equivalence $\phi_0 : H_0 F \rightarrow H_0 G$ induced by $\varphi : C_0(X \times Y) = F_0(X, Y) \rightarrow G_0(X, Y) = C_0(X) \otimes C_0(Y)$, $(\sigma, \tau) \mapsto \sigma \otimes \tau$, since $H_0(X \times Y) = C_0(X \times Y)/(x_0, y_0) \sim (x_1, y_1)$, $(x_0, y_0), (x_1, y_1)$ are connected by a path, $H_0(C_*(X) \times C_*(Y)) = C_0(X) \otimes C_0(Y)/(x_0, y_0) \sim (x_1, y_0) \sim (x_1, y_1)$ \square

Cross product and its dual for homology

Remark 16.1.19. We define the **cross product** $C_*(X) \otimes C_*(Y) \xrightarrow{\times} C_*(X \times Y)$ and its dual φ . Define $T : C_*(X \times Y) \rightarrow C_*(Y \times X)$, $(x, y) \mapsto (y, x)$, $\tau : C_*(X) \otimes C_*(Y) \rightarrow C_*(Y) \otimes C_*(X)$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, $T^2 = 1$, $\tau^2 = 1$, $\tau \partial = \partial \tau$

$$\begin{array}{ccc} C_*(X) \otimes C_*(Y) & \xrightarrow{\times} & C_*(X \times Y) \\ \downarrow \tau & & \downarrow T \\ C_*(Y) \otimes C_*(X) & \xrightarrow{\times} & C_*(Y \times X) \end{array}$$

Is not commutative, but \times and $T \circ \times \circ \tau$ are chain homotopic

$$\begin{array}{ccc} C_*(X \times Y) & \xrightarrow{\theta} & C_*(X) \otimes C_*(Y) \\ \downarrow T & & \downarrow \tau \\ C_*(Y \times X) & \xrightarrow{\theta} & C_*(Y) \otimes C_*(X) \end{array}$$

Is not commutative, but θ and $\tau \circ \theta \circ T$ are chain homotopic

Topological Kunneth formula

Theorem 16.1.20 (Topological Kunneth formula).

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \rightarrow H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}_1(H_p(X), H_q(Y)) \rightarrow 0$$

Is exact

Proof. Apply Theorem 16.1.18 and Theorem 12.0.21 \square

16.2 Cellular homology

Chapter 17

Cohomology theory

17.1 Singular cohomology

Definition 17.1.1 (Eilenberg-Steenrod axioms). Top is the category of topological spaces, Ab is the category of abelian groups, \mathcal{T} is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces (X, A) such that $A \subseteq X$, \mathcal{T}_A is the fully faithful subcategory of \mathcal{T} with objects (X, A) , $R: \mathcal{T} \rightarrow Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor

Relative cohomology are contravariant functors $H^n: \mathcal{T} \rightarrow Ab$, then $H^n(-, A)$ define contravariant functors $\mathcal{T}_A \rightarrow Ab$, **absolute cohomology** are contravariant functors $H^n(-, \emptyset): Top \rightarrow Ab$, **reduced cohomology** are $\tilde{H}^n = H^n(-, *)$. $\partial^n: H^n \rightarrow H^{n+1}R$ are natural transformations

$$\begin{array}{ccc} H^n(X, A) & \xrightarrow{H_n(f)} & H^n(Y, B) \\ \downarrow \partial^n & & \downarrow \partial^n \\ H^{n+1}(A) & \xrightarrow{H_{n+1}(f)} & H^{n+1}(B) \end{array}$$

(H, δ) is a **cohomology theory** if it satisfies axioms

Homotopy invariance: $f \simeq g: (X, A) \rightarrow (Y, B)$, then $H^n(f) = H^n(g)$

Additivity: $(X, A) = \bigsqcup_\alpha (X_\alpha, A_\alpha)$, then $\bigoplus_\alpha H^n(X_\alpha, A_\alpha) \xrightarrow{\bigoplus_\alpha H^n(i_\alpha)} H^n(X, A)$ is an isomorphism

Exactness:

$$\dots \xrightarrow{\partial^{n-1}} H^n(X, A) \xrightarrow{H^n(j)} H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial^n} \dots$$

Excision: $\bar{Z} \subseteq \overset{\circ}{U}$, then $H^n(X - Z, U - Z) \xrightarrow{H^n(i)} H^n(X, U)$ is an isomorphism

Dimension: $H^n(*) = 0, \forall n \neq 0$, $H^0(*)$ is the **coefficient group**

(H, δ) is an **extraordinary cohomology theory** without dimension axiom

Definition 17.1.2. Define singular n -cochains to be $C^n(X) = \text{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z})$, if R is a ring, then we can also define cohomology with R coefficients $C^n(X; R) = \text{Hom}_{\mathbb{Z}}(C_n(X), R)$, here R can be abelian groups(group ring) or fields

We can also define simplicial, cellular cochains correspondingly

Remark 17.1.3. Note that $\text{Hom}(A \otimes B, C) \cong \text{Hom}(A, \text{Hom}(B, C))$, $\text{Hom}(C_n(X; R), \mathbb{Z}) = \text{Hom}(C_n(X) \otimes R, \mathbb{Z}) \cong \text{Hom}(C_n(X), \text{Hom}(R, \mathbb{Z})) \not\cong \text{Hom}(C_n(X), R) = C^n(X; R)$

Definition 17.1.4. $\partial_{n+1}: C_{n+1}(X) \rightarrow C_n(X)$ induce the coboundary map $\delta^n: C^n(X) \rightarrow C^{n+1}(X)$, we can define cocycles $Z^n(X) = \ker \delta^n$, coboundaries $B^n = \text{im} \delta^{n-1}$ and cohomology $H^n(X) = Z^n(X)/B^n(X)$

Definition 17.1.5. θ as in Remark 16.1.19, the cross product is composition $\times: C^*(X; R) \otimes C^*(Y; R) \xrightarrow{\theta^*} C^*(X \times Y; R \otimes R) \rightarrow C^*(X \times Y; R)$, here $R \otimes R \rightarrow R$ is the ring multiplication.

$\delta(f \times g) = \delta f \times g + (-1)^{|f|} f \times \delta g$, \times is well defined on cohomology since θ is unique up to natural chain equivalence. If R is commutative, then $f \times g = (-1)^{|f||g|} g \times f$.
 For $[f] \in H^p(X; R)$, $[g] \in H^q(Y; R)$, $[a] \in H_p(X)$, $[b] \in H_q(Y)$, then $([f] \times [g])([a] \times [b]) = f(a)g(b) \in R$

Lemma 17.1.6. If $a \in H^p(Y; R)$, then $1 \times a = p_Y^*(a) \in H^p(X \times Y; R)$

Proof. $C_*(X \times Y) \rightarrow C_*(X) \otimes C_*(Y) \rightarrow C_*(x_0) \otimes C_*(Y) \xrightarrow{\epsilon \otimes 1} \mathbb{Z} \otimes C_*(Y) \cong C_*(Y) \xrightarrow{a} R$ and $C_*(X \times Y) \xrightarrow{p_Y} C_*(Y) \xrightarrow{a} R$ are chain homotopic \square

Definition 17.1.7. $\Delta : X \rightarrow X \times X$ is the diagonal, for $a \in H^p(X; R)$, $b \in H^q(X; R)$, the **cup product** is $a \smile b = \Delta^*(a \times b) \in H^{p+q}(X; R)$, $f^*(a \smile b) = f^*(a) \smile f^*(b)$, if R is commutative, $a \smile b = (-1)^{|a||b|} b \smile a$, $1 \smile a = \Delta^*(1 \times a) = \Delta^*(p_X^*(a)) = (p_X \Delta)^*(a) = 1^*(a) = a$

Proposition 17.1.8. Cross product and cup product determine each other, $a \smile b = \Delta^*(a \times b)$, $a \times b = p_X^*(a) \smile p_Y^*(b)$

Proof. $p_X^*(a) \smile p_Y^*(b) = \Delta^*(p_X^*(a) \times p_Y^*(b)) = \Delta^*(a \times 1 \times 1 \times b) = \Delta^*(1 \times 1 \times a \times b) = (1 \times 1) \smile (a \times b) = 1 \smile (a \times b) = a \times b$ \square

17.2 Čech cohomology

Definition 17.2.1. Given any open cover \mathcal{U} of X , we can define a (abstract) simplicial complex, the nerve $N(\mathcal{U})$, with each U_α a vertex and an n -face if $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}} \neq \emptyset$, and we call $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ the carrier of this face, a cover is called a good cover if each $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ is contractible, in that case, $N(\mathcal{U})$ is homotopic to X

Definition 17.2.2. Suppose \mathcal{V} is a refinement of \mathcal{U} , i.e. every V_β is contained in some U_α , refinement defines a preorder, then inclusion induce a simplicial map $N(\mathcal{V}) \rightarrow N(\mathcal{U})$, different choice of inclusions induce contiguous simplicial maps, thus this is well defined up to homotopy, we can define the direct limit $\varinjlim H^i(N(\mathcal{U}); G)$ to be the Čech cohomology group $H^i(X; G)$

17.3 Poincare duality

Chapter 18

Homotopy theory

18.1 Homotopy

Definition 18.1.1. $\pi_n(X) := [S^n, X]$ are the homotopy groups, the relative homotopy groups $\pi_n(X, A)$ are defined to be all homotopy classes of maps $(I^n, \partial I^n) \rightarrow (X, A)$ or equivalently all homotopy classes of maps $(S^n, s_0) \rightarrow (X, A)$, in particular, if $A = \{x_0\}$, we have $\pi_n(X, x_0) := \langle S^n, X \rangle$ with basepoints x_0, s_0 , furthermore, we also define $\pi_n(X, A, x_0)$ to be all homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ or equivalently all homotopy classes of maps $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$, here $J^{n-1} = \partial I^n - I^{n-1}$
 $\pi_1(X, x_0)$ is called the fundamental group, note that $\pi_n(X, x, x) = \pi_n(X, x)$

Definition 18.1.2. All homotopy classes of paths form the **fundamental groupoid** $\Pi_1(X)$ of X , suppose $A \subseteq X$, we can also define $\Pi_1(X, A)$ to be the subcategory with objects $x \in A$ and morphisms $Hom(x, y), x, y \in A$, $\Pi_1(X, x) = \pi_1(X, x)$, $\Pi_1(X)$ is a connected category if X is pathconnected connected since there is a morphism connecting any two objects, thus $\pi_1(X, x)$ is a skeleton of $\Pi_1(X)$, $\pi_1(X, x) \hookrightarrow \Pi_1(X)$ is an equivalence of categories

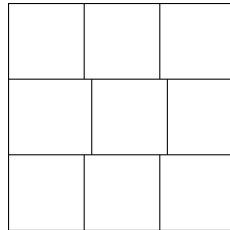
Proposition 18.1.3. $\pi_n(X, x_0)$ are groups, abelian if $n > 1$

Van Kampen's theorem

Theorem 18.1.4 (Van Kampen's theorem). Suppose $X = \bigcup_{\alpha} A_{\alpha}$, interiors of A_{α} cover X , where $X, A_{\alpha}, A_{\alpha} \cap A_{\beta}$ are path connected and $x_0 \in A_{\alpha}$, then the map induced by inclusion $\ast \pi_1(A_{\alpha}, x_0) \rightarrow \pi_1(X, x_0)$ is surjective. Moreover, if $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ are path connected, then kernel is generated by $i_{\alpha}(w)i_{\beta}(w)^{-1}, w \in \pi_1(A_{\alpha} \cap A_{\beta}, x_0)$, where $i_{\alpha} : A_{\alpha} \rightarrow X$ are the inclusions

Proof. Since $A_{\alpha} \cap A_{\beta}$ are path connected, we can cut a loop in X into pieces such that each intermediate point is in $A_{\alpha} \cap A_{\beta}$ for some α, β , thus the map is surjective

Suppose $f_1 \cdots f_n, g_1 \cdots g_m$ are homotopic as loops, suppose F is the homotopy, consider the following diagram, each rectangle is so small that it is inside some A_{α} , then homotopy the path across a cube one at a time, and since $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ are path connected, each vertex lies in $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$ for some α, β, γ , then we can connect it to x_0 through a path in $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$



□

Definition 18.1.5. $X \xrightarrow{f} Y$ is a **weak homotopy equivalence** if $\pi_0(X, x_0) \xrightarrow{f_*} \pi_0(Y, f(x_0))$ is bijective, and on each path connected component, $\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0))$ are isomorphisms

18.2 Model category

Model category

Definition 18.2.1. A **model structure** on \mathcal{C} is three classes of morphisms $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ satisfying

1. \mathbf{W} satisfies 2 out of 3, \mathbf{F}, \mathbf{C} are closed under composition
2. $i \in \mathbf{C}$ has LLP for $p \in \mathbf{F} \cap \mathbf{W}$ and $p \in \mathbf{F}$ has RLP for $i \in \mathbf{C} \cap \mathbf{W}$

$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ i \downarrow & \nearrow & \downarrow p \\ \bullet & \xrightarrow{g} & \bullet \end{array}$$

3. For any morphism f , $f = pi$ with $i \in \mathbf{C} \cap \mathbf{W}$, $p \in \mathbf{F}$. $f = pi$ with $i \in \mathbf{C}$, $p \in \mathbf{F} \cap \mathbf{W}$
4. \mathbf{F}, \mathbf{C} are closed under base change and cobase change. base change of $p \in \mathbf{F} \cap \mathbf{W}$ and cobase change of $i \in \mathbf{C} \cap \mathbf{W}$ are in \mathbf{W} . Isomorphisms $\mathbf{I} \subseteq \mathbf{F} \cap \mathbf{C}$

$(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a **closed model structure** if it satisfying 1,2,3 and

5. $\mathbf{W}, \mathbf{F}, \mathbf{C}$ are closed under retraction, i.e. if f is a retract of g in the arrow category, then f, g belong to the same class

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{j} & X \\ f \downarrow & & \downarrow g & & \downarrow f \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{j} & X' \end{array}$$

By Theorem 18.2.3, a closed model structure is a model structure. Moreover, $\mathbf{F} \cap \mathbf{W}$ is closed under base change, $\mathbf{C} \cap \mathbf{W}$ is closed under cobase change

\mathbf{W} are **weak equivalences**, \mathbf{F} are **fibrations**, \mathbf{C} are **cofibrations**, $\mathbf{F} \cap \mathbf{W}$ are **acyclic fibrations**, $\mathbf{C} \cap \mathbf{W}$ are **acyclic cofibrations**

A **model category** \mathcal{C} is a complete and cocomplete category with a model structure

Remark 18.2.2. \mathcal{C}^{op} is also a model category where fibrations and cofibrations are switched, hence dual of true statements in \mathcal{C} are also true

Equivalence of closed model structure and weak factorization system

Theorem 18.2.3. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure $\Leftrightarrow (\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ are both weak factorization systems

Proof. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure. If $X \xrightarrow{i} Y$ has LLP for all $p \in \mathbf{F} \cap \mathbf{W}$, i can be decomposed as $X \xrightarrow{i'} Y' \xrightarrow{p} Y$ with $i' \in \mathbf{C}, p \in \mathbf{F} \cap \mathbf{W}$, then we have a lift $Y \xrightarrow{f} Y'$ by

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \nearrow f & \downarrow p \\ Y & \xlongequal{\quad} & Y \end{array}$$

Hence $i \in \mathbf{C}$ by

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\ \downarrow i & & \downarrow i' & & \downarrow i \\ Y & \xrightarrow{f} & Y' & \xrightarrow{p} & Y \end{array}$$

Suppose $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ are both weak factorization systems, and $\mathbf{F} \cap \mathbf{W}$ is closed under base change, $\mathbf{C} \cap \mathbf{W}$ is closed under cobase change

For any base cobase change j of $i \in \mathbf{C}$, and any $p \in \mathbf{F} \cap \mathbf{W}$, we get $Y \xrightarrow{h} E$ and then $Y \sqcup Z \xrightarrow{c} e$, hence $j \in \mathbf{C}$

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Z & \xrightarrow{a} & E \\
\downarrow i & & \downarrow j & \nearrow c & \downarrow p \\
Y & \xrightarrow{g} & Y \sqcup Z & \xrightarrow{b} & B
\end{array}$$

(Note: A dashed arrow labeled h also goes from Y to Z .)

The class of isomorphisms $\mathbf{I} \subseteq \mathbf{W} \cap \mathbf{F} \cap \mathbf{C}$ since $1_X \in \mathbf{W} \cap \mathbf{F} \cap \mathbf{C}$ and

$$\begin{array}{ccccc}
X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\
\downarrow f & & \parallel & & \downarrow f \\
Y & \xrightarrow{f^{-1}} & X & \xrightarrow{f} & Y
\end{array}$$

□

Definition 18.2.4. Since \mathcal{C} is complete and cocomplete, \mathcal{C} initial object \emptyset and final object $*$. X is **cofibrant** if $\emptyset \rightarrow X \in \mathbf{C}$, X is **fibrant** if $X \rightarrow * \in \mathbf{F}$

Definition 18.2.5. $A \times I$ is a **cylinder object** for A if the following diagram commutes for some $h \in \mathbf{W}$

$$\begin{array}{ccc}
A \amalg A & \xrightarrow{i} & A \times I \\
& \searrow 1_A + 1_A & \downarrow h \\
& & A
\end{array}$$

$A \times I$ is **good** if $i \in \mathbf{C}$. $A \times I$ is **very good** if $i \in \mathbf{C}$, $h \in \mathbf{F} (\Rightarrow h \in \mathbf{F} \cap \mathbf{W})$. Very good cylinder object always exists by axiom 3 in Definition 18.2.1

Denote $i_0, i_1 : A \rightarrow A \amalg A \rightarrow A \times I$ by going through the first and second factor

A^I is a **path object** for A if the following diagram commutes for some $h \in \mathbf{W}$

$$\begin{array}{ccc}
A & \xrightarrow{h} & A^I \\
& \searrow (1_A, 1_A) & \downarrow p \\
& & A \times A
\end{array}$$

A^I is **good** if $p \in \mathbf{F}$. A^I is **very good** if $p \in \mathbf{F}$, $h \in \mathbf{C} (\Rightarrow h \in \mathbf{C} \cap \mathbf{W})$. Very good path object always exists by axiom 3 in Definition 18.2.1

Denote $p_0, p_1 : A^I \rightarrow A \times A \rightarrow A$ by going to the first and second factor

Lemma 18.2.6. If A is cofibrant and $A \times I$ is good, then $i_0, i_1 \in \mathbf{F} \cap \mathbf{W}$. If A is fibrant and A^I is good, then $p_0, p_1 \in \mathbf{C} \cap \mathbf{W}$

Definition 18.2.7. $f, g : A \rightarrow X$ is **left homotopic**, denoted $f \stackrel{l}{\sim} g$ if there exists a cylinder object $A \times I$ and a left homotopy $A \times I \xrightarrow{H} X$ such that

$$\begin{array}{ccc}
A \amalg A & \xrightarrow{i} & A \times I \\
& \searrow f+g & \downarrow H \\
& & X
\end{array}$$

$f, g : A \rightarrow X$ is **right homotopic**, denoted $f \stackrel{r}{\sim} g$ if there exists a path object A^I and a right homotopy $A \xrightarrow{H} X^I$ such that

$$\begin{array}{ccc}
A & \xrightarrow{H} & X^I \\
& \searrow (f, g) & \downarrow p \\
& & X \times X
\end{array}$$

Example 18.2.8. $\mathcal{C} = Ch_{>-\infty} \mathcal{A}$ is the category of chain complexes bounded below. \mathbf{W} are maps inducing isomorphisms on homologies. \mathbf{F} are epimorphisms in \mathcal{C} . \mathbf{C} are maps that are injective entrywise, and the cokernel is a chain complex of projectives of \mathcal{A} . $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure on \mathcal{C}

The cofibrant objects are those with entries projective, then homotopy category is equivalent to the category with cofibrant objects with chain homotopy classes of maps

Example 18.2.9. \mathcal{C} is the category of semisimplicial sets. \mathbf{W} are morphisms that become homotopies after geometric realization. \mathbf{F} are Kan fibrations. \mathbf{C} are injective morphisms. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure on \mathcal{C}

18.3 Hurewicz fibration

Definition 18.3.1. $E \xrightarrow{p} B$ is a **Hurewicz fibration** if it has homotopy lifting property for any space X

$$\begin{array}{ccc} X & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ X \times I & \xrightarrow{\quad} & B \end{array}$$

$p^{-1}(A) \xrightarrow{p} A$ is also a fibration for any $A \subseteq B$

Definition 18.3.2. $A \xrightarrow{i} X$ is a **Hurewicz cofibration** if for any $X \xrightarrow{f_0} Y$, $A \times I \xrightarrow{g} Y$ such that $g_0 = f_0 i$, there there exists $X \times I \xrightarrow{f} Y$ extends f_0 such that $g = f i$

$$\begin{array}{ccc} Y & \xleftarrow{f_0} & X \\ \uparrow & \nwarrow \text{dashed} & \uparrow i \\ Y^I & \xleftarrow{g} & A \end{array}$$

By Lemma 18.3.4, $A \hookrightarrow X$ is an embedding, (X, A) is a **cofibered pair** satisfying the **homotopy extension property**

Remark 18.3.3. The fiber of a fibration is the kernel in Top . The cofiber X/A of a cofibration is the cokernel in Top

$A \times I$ can be thought of as the "continuous" coproduct $\coprod_i A$, and A^I can be thought of as the

"continuous" product $\prod_i A$

Cofibration is an embedding

Lemma 18.3.4. Cofibration $A \xrightarrow{i} X$ is a topological embedding

Proof. M is the mapping cylinder of $A \xrightarrow{i} X$, $X \times \{0\} \sqcup A \times I \xrightarrow{[]} M$ denote the quotient map, $f_0(x) = [(x, 0)]$, $g_t(a) = [(a, t)]$, then $g_0(a) = [(a, 0)] = [(i(a), 0)] = f_0 i(a)$

$$\begin{array}{ccc} M & \xleftarrow{f_0} & X \\ \uparrow & \nwarrow \text{dashed} & \uparrow i \\ M^I & \xleftarrow{g} & A \end{array}$$

$M|_{A \times \{1\}} \xrightarrow{k} A$, $[(a, 1)] \mapsto a$ is a homeomorphism, $k f_1|_{i(A)}$ is the inverse of i because $k f_1 i(a) = k g_1(a) = k([(a, 1)]) = a$, $i k f_1(i(a)) = i(a)$ \square

Lemma 18.3.5. A surjective fibration $E \xrightarrow{p} B$ with B locally path connected is a quotient map
Mapping cylinder of inclusion has subspace topology if it is a retraction

Lemma 18.3.6. If $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, then the topology of the mapping cylinder of the inclusion $A \hookrightarrow X$ is the same as the subspace topology on $X \times \{0\} \cup A \times I$ induced from $X \times I$. In particular, f is continuous on $X \times \{0\} \cup A \times I$ iff f is continuous on both $X \times \{0\}$ and $A \times I$

Proof. This is trivial if A is closed due to Lemma 14.1.26

Write $Y = X \times \{0\} \cup A \times I$. If $O \subseteq Y$ is open in Y , then obviously $O \cap A \times I$ is open in $A \times I$, $O \cap X$ is open in X

Suppose $O \subseteq Y$ is such that $O \cap A \times I$ is open in $A \times I$, $O \cap X$ is open in X . Define $U_n = \bigcup_V \left\{ V \overset{\text{open}}{\subseteq} X \mid (V \cap A) \times [0, \frac{1}{n}] \subseteq O \right\}$, i.e. U_n is the largest such open subset of X , $U = \bigcup_{n=1}^{\infty} U_n$, then $O \cap A \subseteq U$ since for any $(x, 0) \in O \cap A \subseteq O \cap A \times I$, because $O \cap A \times I$ is open in $A \times I$,

there exists open subset $V \subseteq X$ containing x such that $(V \cap A) \times [0, \frac{1}{n}] \subseteq O \cap A \times I$ for some n , hence $x \in V \subseteq U_n \subseteq U$

$O \cap A \times (0, 1]$ is open in Y , $O \cap (X \setminus \bar{A})$ is open in Y , we only need to show that for any $x \in \bar{A}$, there exists an open neighborhood of $(x, 0)$ contained in O , and it suffices to show that $x \in U$, then $x \in U_n$ for some n , $(U_n \cap O) \times [0, \frac{1}{n}] \cap Y$ is open in Y . Now fix $x \in \bar{A}$

Write the retraction r as (r_1, r_2) , for $t > 0$, $r(x, t) = (r_1(x, t), r_2(x, t))$, since $x \in \bar{A}$, $r(a, t) = (a, t)$, we know $r_1(x, t) \in A$, $r_2(x, t) = t$. We claim: if $r_1(x, t) \in U_n$, then $x \in U_n$. Since U_n is open, there exists open neighborhood V of x such that $r_1(V \times (t - \varepsilon, t + \varepsilon)) \subseteq U_n$ for some $\varepsilon > 0$, in particular $r_1((V \cap A) \times \{t\}) \subseteq U_n$, thus $V \cap A \subseteq U_n \cap A$, by maximality of U_n , $V \subseteq U_n$. Suppose $x \notin U$, then by the claim, $r_1(x, t) \in A \setminus U$ for $t > 0$, then $r_1(x, t) \in A \setminus O$ since $A \cap O \subseteq U$, thus $x = r_1(x, 0) \in \bar{A} \setminus O$ which contradicts the fact $(x, 0) \in A$ \square

$A \rightarrow X$ is a cofibration iff retraction exists

Proposition 18.3.7. (X, A) is cofibered iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ iff $X \times \{0\} \cup A \times I$ is a strong deformation retract of $X \times I$

Proof. If $A \xrightarrow{i} X$ is a cofibration, then $X \times \{0\} \cup A \times I \xrightarrow{1} X \times \{0\} \cup A \times I$ induces a retraction. Conversely, by Lemma 18.3.6, $A \times I \xrightarrow{g_0} Y$, $X \xrightarrow{f_0} Y$ with $g_0 = f_0|_A$ gives a map $X \times \{0\} \cup A \times I \rightarrow Y$, composing with retraction gives $X \times I \rightarrow Y$

A strong deformation retraction is given by $H((x, t), s) = (\text{Pr}_X r(x, st), s \text{Pr}_I r(x, t) + (1-s)t)$ \square

Lemma 18.3.8. If (X, A) is cofibered, so is (X, \bar{A})

Proof. Define $\phi(x) = \inf_{t \in I} \{\text{Pr}_I r(x, t) \neq 0\}$, then there is a retraction $X \times I \xrightarrow{r'} X \times \{0\} \cup \bar{A} \times I$

$$r'(x, t) = \begin{cases} (\text{Pr}_X r(x, t), 0) & t \leq \phi(x) \\ (\text{Pr}_X r(x, \phi(x)), t - \phi(x)) & t \geq \phi(x) \end{cases}$$

\square

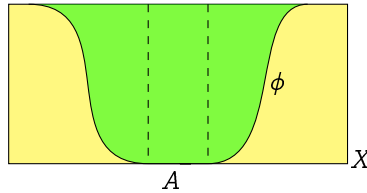
$p: E \rightarrow B$ fibration, $i: A \rightarrow X$ strong deformation retract, A can be perfectly separated $\Rightarrow i$ has LLP

Lemma 18.3.9. $E \xrightarrow{p} B$ is a fibration, A is a strong deformation retract of X and A can be perfectly separated, then

$$\begin{array}{ccc} A & \xrightarrow{f''} & E \\ i \downarrow & \nearrow f & \downarrow p \\ X & \xrightarrow{f'} & B \end{array}$$

f is unique up to homotopy rel A

Proof. $X \xrightarrow{\phi} \mathbb{R}$ is a function such that $\phi^{-1}(0) = A$. $X \xrightarrow{r} A$ is the retract. $X \times I \xrightarrow{D} X$ is a homotopy from ir to 1_X . Define $H(x, t) = \begin{cases} D(x, t/\phi(x)) & t \leq \phi(x) \\ D(x, 1) & t \geq \phi(x) \end{cases}$



Since $E \xrightarrow{p} B$ is a fibration, we have a lift $X \times I \xrightarrow{F} E$ such that $pF = f'H$ and $F_0 = f''r$ because $pF_0 = pf''r = f'ir = f'H_0$. Define f to be the composition $X \xrightarrow{1 \times \phi} X \times I \xrightarrow{F} E$, then we have $fi = F(1 \times \phi) = F_0i = f''$ and $pf = pF(1 \times \phi) = f'H(1 \times \phi) = f'D_1 = f'$

$$\begin{array}{ccccc}
X & \xrightleftharpoons[r]{i} & A & \xrightarrow{f''} & E \\
\downarrow & & \searrow F & & \downarrow p \\
X \times I & \xrightarrow{H} & X & \xrightarrow{f'} & B \\
& \nwarrow 1 \times \phi & & &
\end{array}$$

□

Fibrations have homotopy lifting property for closed cofibrations

Proposition 18.3.10. Fibrations have homotopy lifting property for closed cofibrations

$$\begin{array}{ccc}
X \times \{0\} \cup A \times I & \longrightarrow & E \\
\downarrow i & \nearrow & \downarrow p \\
X \times I & \longrightarrow & B
\end{array}$$

Proposition 18.3.11. If (X, A) , (Y, B) are cofibered, $A \subseteq X$ is closed, then $(X \times Y, X \times B \cup A \times Y)$ is also cofibered. If in addition A or B is a strong deformation retract of X or Y , then $X \times B \cup A \times Y$ is a strong deformation retract of $X \times Y$

Fibers of fibration are homotopy equivalent

Proposition 18.3.12. $E \xrightarrow{p} B$ is a fibration, then the fibers over connected components of B are homotopic. More over, for any path $\gamma : I \rightarrow B$, we can get a lifting $g_t : F_{\gamma(0)} \rightarrow F_{\gamma(t)}$ of $F_{\gamma(0)} \hookrightarrow E$, define $L_\gamma : F_{\gamma(0)} \rightarrow F_{\gamma(1)}$ to be g_1 , if $\gamma \simeq \eta \text{ rel } \partial I : I \rightarrow B$, then $L_\gamma \simeq L_\eta$, and for any $\gamma, \eta : I \rightarrow B$, $\eta(0) = \gamma(1)$, $L_{\gamma\eta} \simeq L_\gamma L_\eta$

Proof. According to homotopy lifting property, lifting up $A \times F_{\gamma(0)} \rightarrow E$ is homeomorphic to $B \times F_{\gamma(0)} \rightarrow E$ □

Remark 18.3.13. We can think of this as an action of $\pi_1(B)$ on $H_*(F)$

Definition 18.3.14. $E_1 \xrightarrow{p_1} B_1$, $E_2 \xrightarrow{p_2} B_2$ are fibrations, $p_1 \xrightarrow{f_0, f_1} p_2$ are **fiber homotopic** if there exists $p_1 \xrightarrow{f_t} p_2$ varying from f_0 to f_1 . p_1, p_2 are **fiber homotopy equivalent** if there are fiber homotopies $p_0 \xrightarrow{f} p_1$ and $p_1 \xrightarrow{g} p_0$ such that fg, gf fiber homotopic to 1

i:A->B cofibration is homotopy equivalence iff A strong deformation retract

Lemma 18.3.15. Cofibration $A \xrightarrow{i} X$ is a homotopy equivalence iff A is a strong deformation retract of X . Fibration $E \xrightarrow{p} B$ is a homotopy equivalence iff there exists a section $B \xrightarrow{s} E$ such that sp is fiber homotopic to 1

Proof. If i is a homotopy equivalence, then there exists $X \xrightarrow{r'} A$ such that $ir' \simeq 1_X$, $r'i \simeq 1_A$, by Lemma 14.2.2, since (X, A) is cofibered, $r' \simeq r$ is a retract and then A is a deformation retract of X . Suppose $X \times I \xrightarrow{F} X$ is a homotopy from 1_X to ir

$\Gamma = X \times \{0\} \cup A \times I \cup X \times \{1\} = X \times \{0\} \cup A \times [0, \frac{1}{2}] \cup A \times [\frac{1}{2}, 1] \cup X \times \{1\}$ is a retract of $X \times I$.

Construct $\Gamma \times I \xrightarrow{G} X$

$$G((x, t), s) = \begin{cases} F(x, (1-s)t) & (x, t) \in X \times \{0\} \cup A \times I \\ F(r(x), 1-s) & (x, t) \in X \times \{1\} \end{cases}$$

G_1 can be extends to $X \times I \xrightarrow{H} X$, then $H_0(x) = G_1(x, 0) = F(x, 0) = x$, $H_1(x) = G_1(x, 1) = F(r(x), 0) = r(x)$, $H_t(a) = G_1(a, t) = F(a, 0) = a$, i.e. H is a strong deformation retraction □

fibration map is a homotopy equivalence iff it is a fiber homotopy equivalence

Lemma 18.3.16. $E \xrightarrow{p} B$ is a fibration, $A \xrightarrow{f} B$ is a map, $A \times_B E = f^*(E)$ is the **pullback fibration**, suppose $f_t : A \rightarrow B$ is a homotopy, then pullback fibrations $f_0^*(E) \rightarrow A$, $f_1^*(E) \rightarrow A$ are fiber homotopy equivalent. In particular, a morphism $p \xrightarrow{f} q$ between two fibrations is a homotopy equivalence iff f is a fiber homotopy equivalence

Proof. $A \times I \xrightarrow{F} B$ is a homotopy, we have the pullback fibration $F^*(E)$, it suffices to show that for any fibration $E \xrightarrow{p} B \times I$, $E_0 := p^{-1}(B \times \{0\}) \simeq p^{-1}(B \times \{1\}) =: E_1$ are fiber homotopy equivalent

Consider the following diagrams

$$\begin{array}{ccc} E_0 & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ E_0 \times I & \xrightarrow{(p,t)} & B \times I \end{array}$$

$$\begin{array}{ccc} E_1 & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ E_1 \times I & \xrightarrow{(p,1-t)} & B \times I \end{array}$$

Then we get fiber preserving maps $f : E_0 \rightarrow E_1$ and $g : E_1 \rightarrow E_0$, and restricts them to each fiber

$$\begin{array}{ccc} p^{-1}(b, 0) & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ p^{-1}(b, 0) \times I & \xrightarrow{(p,t)} & \{b\} \times I \end{array}$$

$$\begin{array}{ccc} p^{-1}(b, 1) & \xhookrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ p^{-1}(b, 1) \times I & \xrightarrow{(p,1-t)} & \{b\} \times I \end{array}$$

We get maps $f|_{p^{-1}(b,0)} : p^{-1}(b,0) \rightarrow p^{-1}(b,1)$, $g|_{p^{-1}(b,1)} : p^{-1}(b,1) \rightarrow p^{-1}(b,0)$, according to Proposition 18.3.12 they are homotopy equivalence and inverses to each other, hence f, g are fiber homotopy equivalences and inverses to each other \square

Corollary 18.3.17. $E \xrightarrow{p} B$ is a fibration, B contractible, then p is fiber homotopy equivalent to $B \times F \rightarrow B$. If B is locally contractible, the fibration is locally homotopy equivalent to a product

Proof. Since B is contractible, identity map is homotopic to a constant map, and the pullback of E under the identity map is E itself, the pullback of E under a constant map is fiber bundle $B \times F$ \square

Definition 18.3.18. (X, x_0) is a pointed space. The **loop space** ΩX consists of all the loops on X starting and ending at x_0 , the constant loop being the basepoint. The **path space** PX consists of all the paths starting at x_0 . $\Omega X \subseteq PX \subseteq X^I$ endowed with the subspace topology

Proposition 18.3.19. $\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$ is an adjunction

Definition 18.3.20. The **mapping path space** P_f is the pullback

$$\begin{array}{ccc} P_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

P_f deformation retracts onto X by shrinking paths. The **homotopy fiber** F_f of f over y is the pullback

$$\begin{array}{ccc} F_f & \longrightarrow & Y^I \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \times Y \end{array}$$

The **mapping cylinder** M_f is the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times I & \longrightarrow & M_f \end{array}$$

M_f deformation retracts onto Y by sliding the cylinder. The mapping cone $M_f/A \times \{0\} \cong C_f$ is the **homotopy cofiber** of f .

Proposition 18.3.21. $X \xrightarrow{f} Y$ can be factorized as $X \hookrightarrow P_f \rightarrow Y$ or $X \hookrightarrow M_f \rightarrow Y$. $P_f \rightarrow Y$, $(x, \gamma) \mapsto \gamma(1)$, $M_f \rightarrow Y$ are Hurewicz fibrations. $X \hookrightarrow P_f$, $X \hookrightarrow M_f$ are closed Hurewicz cofibrations.

Proof.

$$\begin{array}{ccc} X & \longrightarrow & M_f \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & Y \end{array}$$

□

Example 18.3.22. PX is the mapping path space of $* \rightarrow X$, $\Omega X \rightarrow PX \rightarrow X$ is a Hurewicz fibration.

Proposition 18.3.23. Suppose $E \xrightarrow{p} X$ is a fibration, then $E \hookrightarrow E_p$ is a fiber homotopy equivalence, and the restriction on each fiber to the homotopy fiber of p is a homotopy equivalence.

Proposition 18.3.24. If $F \rightarrow E \rightarrow B$ is a fibration, and E is contractible, then F is weakly homotopic to ΩB .

Theorem 18.3.25. $F \rightarrow E \rightarrow B$ is a fibration, the **fibration sequence** is

$$\rightarrow \Omega^2 B \rightarrow \Omega F \rightarrow \Omega E \rightarrow \Omega B \rightarrow F \rightarrow E \rightarrow B$$

$A \rightarrow X \rightarrow X/A$ is a cofibration, the **cofibration (Puppe) sequence** is

$$A \rightarrow X \rightarrow X/A \rightarrow \Sigma A \rightarrow \Sigma X \rightarrow \Sigma(X/A) \rightarrow \Sigma^2 A \rightarrow \dots$$

Theorem 18.3.26. $E \xrightarrow{p} B$ is Serre fibration, fix $x_0 \in p^{-1}(b_0) = F$, $\pi_n(E, F, x_0) \xrightarrow{p^*} \pi_n(B, b_0)$ is an isomorphism, and we have long exact sequence

$$\dots \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \xrightarrow{p^*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \dots \rightarrow \pi_0(E, x_0) \rightarrow 0$$

Theorem 18.3.27. \mathbf{W} consists of all homotopy equivalences, \mathbf{F} consists of all Hurewicz fibrations, \mathbf{C} consists of all closed Hurewicz cofibrations, $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ defines a closed model structure on \mathbf{Top} .

Proof. 2 out of 3 is obvious.

By Lemma 18.3.9 and Lemma 18.3.15, $i \in \mathbf{C} \cap \mathbf{W} \Rightarrow i$ has LLP for any $p \in \mathbf{F}$. Suppose i has LLP for any $p \in \mathbf{F}$, since $Y^I \rightarrow Y \in \mathbf{F}$, $i \in \mathbf{C}$, $A \rightarrow * \in \mathbf{F}$, we get a retraction $X \xrightarrow{r} A$ by

$$\begin{array}{ccc} A & \xlongequal{\quad} & A \\ \downarrow i & \nearrow r & \downarrow \\ X & \longrightarrow & * \end{array}$$

$X^I \rightarrow X \times X \in F, \gamma \mapsto (\gamma(0), \gamma(1))$, then we get a strong deformation retraction

$$\begin{array}{ccc} A & \xrightarrow{\quad} & X^I \\ i \downarrow & \nearrow F & \downarrow \\ X & \xrightarrow{\quad} & X \times X \end{array}$$

Hence $i \in \mathbf{C} \cap \mathbf{W}$ \square

18.4 Serre fibration

Definition 18.4.1. $E \xrightarrow{p} B$ is a **Serre fibration** if it has homotopy lifting property for all $(D^n, \partial D^n)$

Chapter 19

Isotopy

Definition 19.0.1. $f, g : X \rightarrow Y$ are topological embeddings, an **isotopy** from f to g is a homotopy H such that H_t are embeddings

Definition 19.0.2. g, h are embeddings of N in M , an **ambient isotopy** from g to h is $M \times I \xrightarrow{F} M$ such that F_t are homeomorphisms, and $F_0 = 1_M$, $F_1 \circ g = h$

Theorem 19.0.3 (Alexander's trick). $D \subseteq \mathbb{R}^n$ is the unit ball, homeomorphisms of D that are isotopic on ∂D are also isotopic on D

Proof. Suppose $f, g : D \rightarrow D$ are homeomorphisms with $f|_{\partial D}, g|_{\partial D}$ isotopic □

Chapter 20

Bundle

20.1 Bundles

Definition 20.1.1. A bundle is $E \xrightarrow{p} B$, where E is the **total space**, B is the **base space**, and p is the projection, $p^{-1}(b)$ is the **fiber** over b . A **cross section** is $s : B \rightarrow E$, such that $ps = 1_B$. The restriction $p^{-1}(A) \xrightarrow{\pi} A$, $A \subseteq B$ is also a bundle

Definition 20.1.2. Suppose $E \xrightarrow{p} B$ is a bundle, $f : A \rightarrow B$ is a map, then the pullback $f^*(E) = A \times_p E \rightarrow A$ is the **pullback bundle**, the pullback of a section $s : B \rightarrow E$ is defined as $f^*s := s \circ f$, notice $p(f^*s(y)) = p(s(f(y))) = f(y)$

Definition 20.1.3. A **fiber bundle** is a bundle $E \xrightarrow{p} B$ such that there exists an open neighborhood U of b and a homeomorphism ϕ making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times F \\ p \downarrow & \swarrow pr_1 & \\ U & & \end{array}$$

Definition 20.1.4. G is a topological group, a G **fiber bundle** $E \xrightarrow{p} B$ is a fiber bundle and also a morphism of G spaces

Lemma 20.1.5. A fiber bundle is a Serre fibration

Definition 20.1.6. \mathbb{F} is a topological field, a **vector bundle** is a fiber bundle $E \xrightarrow{p} X$ with fiber being \mathbb{F}^n and ϕ restricts on each fiber is an \mathbb{F} isomorphism

Definition 20.1.7. G is a topological group, a **principal G bundle** $p : P \rightarrow B$ is a morphism of G spaces, B with the trivial G action, and for each $b \in B$, there is a local trivialization

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi} & U \times G \\ p \downarrow & \swarrow pr_1 & \\ U & & \end{array}$$

ϕ is an isomorphism

Remark 20.1.8. G action on P preserves fibers, and the action on fiber is free and transitive, each fiber is a G torsor. A morphism of principal G bundles is always an isomorphism. A principal G bundle is trivial iff it has a global section

Proposition 20.1.9. Suppose $P \rightarrow B$ is a principal G bundle, $G \rightarrow G/H$ is a principal H bundle, then $P \rightarrow P/H$ is a principal H bundle

Proof. $P \cong P \times_G G \rightarrow P \times_G (G/H) \cong P/H$ □

Proposition 20.1.10. Suppose $P \rightarrow B$ is a principal G bundle, F is a left G space, $P \times_G F \rightarrow P \times_G * \cong B$ is a G fiber bundle. $X \xrightarrow{f} Y$ is a map, $f^*(P \times_G F) \rightarrow f^*(P) \times_G F$ is a natural homeomorphism

Proposition 20.1.11. $P \xrightarrow{p} B$ is a principal G bundle, X is a right G space, a morphism

$P \xrightarrow{f} X$ induce $P \xrightarrow{\begin{pmatrix} 1 \\ f \end{pmatrix}} P \times X$, $B \cong P/G \rightarrow P \times X/G \cong P \times_G X$ which is a section s_f of $P \times_G X \rightarrow B$, this is a natural bijection

Proposition 20.1.12. $P \rightarrow B \times I$ is principal G bundle, then P and $P_0 \times I$ is an isomorphism, here P_0 is the restriction of P over $B \times \{0\}$

Proof.

$$\begin{array}{ccc} B & \longrightarrow & P \times_G (P_0 \times I) \\ \downarrow & \nearrow & \downarrow \\ B \times I & \longrightarrow & B \times I \end{array}$$

□

20.2 Vector bundles

Proposition 20.2.1. $E \xrightarrow{p} X$ is trivial iff there exist global sections s_1, \dots, s_n that they are linearly independent on each fiber

Definition 20.2.2. Let $E \xrightarrow{p} X$ be a vector bundle, consider two trivializations $\varphi_U : E_U \rightarrow U \times \mathbb{R}^n$ and $\varphi_V : E_V \rightarrow V \times \mathbb{R}^n$ around $x \in X$, then $\varphi_V \circ \varphi_U^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$ is a local isomorphism with inverse $\varphi_U \circ \varphi_V^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$, it is also called a transition function and it can also be regarded as a continuous map $g_{VU} : U \cap V \rightarrow GL(n, \mathbb{R})$ or $g_{VU} \in GL(n, C(U \cap V))$, such that $\varphi_V \circ \varphi_U^{-1}(x, v) = (x, g_{VU}(x)v)$, notice then $g_{UV} = g_{VU}^{-1}$, and g_{VU} 's satisfy the cocycle relation $g_{WV}g_{VU} = g_{WU}$ on $U \cap V \cap W$

Conversely, given $\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{R}^n \times A$ transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R})$ that satisfying cocycle relation $g_{\gamma\beta}g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, mod equivalence relation $(x, v, \alpha) \sim (x, g_{\beta\alpha}(v), \beta)$, $x \in U_\alpha \cap U_\beta$, you will get back the vector bundle

Suppose $s : X \rightarrow E$, is a section, denote $\varphi_i \circ s|_{U_i}(x) = (x, f_i(x))$ over U_i , then $(x, f_j(x)) = \varphi_j \circ s|_{U_j}(x) = \varphi_j \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1} \circ \varphi_i \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1}(x, f_i(x)) = (x, g_{ji}(x)f_i(x))$, $\forall x \in U_i \cap U_j$, thus $f_j = g_{ji}f_i$, conversely, this relation also defines a section

Definition 20.2.3. The pullback of a transition function is defined to be $f^*g_{ij} := g_{ij} \circ f$

Definition 20.2.4. A morphism between vector bundles $\varphi : E \rightarrow F$ is map such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \downarrow p & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

and $\varphi_x : E_x \rightarrow F_{f(x)}$ is a homomorphism between vector spaces

Definition 20.2.5. Let $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ be vector bundles, then direct sum $E \times F \xrightarrow{p \times q} X \times Y$ is also a vector bundle, suppose $\varphi_U : U \rightarrow U \times \mathbb{R}^n$, $\psi_V : V \rightarrow V \times \mathbb{R}^m$ are trivializations, then $\varphi_U \times \psi_V : U \times V \rightarrow U \times \mathbb{R}^n \times V \times \mathbb{R}^m \cong U \times V \times \mathbb{R}^{n+m}$ is also a trivialization

Proposition 20.2.6. Let $E \xrightarrow{p} X$ is a vector bundle, and $f : X \rightarrow Y$ is a homeomorphism, then $E \xrightarrow{f \circ p} Y$ is a vector bundle, suppose $\varphi_U : E_U \rightarrow U \times \mathbb{R}^n$ is a trivialization, then $(f \times 1) \circ \varphi_U =: \psi_{f(U)} : E_U \rightarrow U \times \mathbb{R}^n \rightarrow f(U) \times \mathbb{R}^n$ is a trivialization

Domain is homeomorphic to its graph

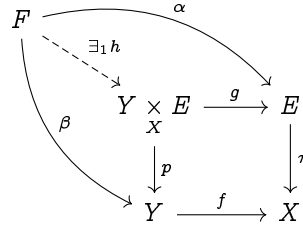
Proposition 20.2.7. $p : \Gamma_f \rightarrow X, (x, f(x)) \mapsto x$ is homeomorphism

Proof. p as a restriction on Γ_f of $X \times Y$ projecting to X is continuous, and define $q : X \rightarrow \Gamma_f, x \mapsto (x, f(x))$, since the composition $X \xrightarrow{q} \Gamma_f \hookrightarrow X \times Y$ which is continuous because $X \xrightarrow{f} Y$, $X \xrightarrow{id} X$ are continuous, q is continuous, and p, q are inverses to each other \square

Definition 20.2.8. $E \xrightarrow{p} X$ is a vector bundle, $f : Y \rightarrow X$ is a continuous map, then we can construct the pullback bundle $f^*E \xrightarrow{p} Y$

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ \downarrow p & & \downarrow \pi \\ Y & \xrightarrow{f} & X \end{array}$$

satisfying universal property



Concrete construction: let $f^*E = Y \times_X E \subseteq Y \times X$ with subspace topology, where $Y \times_X E = \{(y, v) \in Y \times E \mid f(y) = \pi(v)\}$, let's check it is a vector bundle over Y , notice that $Y \times_X E \rightarrow Y$ factor through $Y \times_X E \rightarrow \Gamma_f \rightarrow Y$, $(y, v) \mapsto (y, \pi(v)) = (y, f(y)) \mapsto y$, where Γ_f is the graph of f which is homeomorphic to Y due to Proposition 20.2.7, notice that $Y \times_X E \rightarrow \Gamma_f$ is the restriction of vector bundle $Y \times E \xrightarrow{1 \times \pi} Y \times X$ over Γ_f , thus $Y \times_X E \rightarrow Y$ is a vector bundle, suppose F as in the commutative diagram, then h is simply defined as $h(w) := (\beta(w), \alpha(w))$

Remark 20.2.9. In general, this is a pullback, but it has a vector bundle structure such that it induces an isomorphism on each fiber, now suppose $F \xrightarrow{q} Y$ is another vector bundle such that not only the diagram commutes but also induce isomorphism on each fiber, then $F \cong f^*E$. Use this we have $(fg)^*E \cong g^*(f^*E)$, $f^*(E \oplus F) \cong f^*E \oplus f^*F$, $f^*(E \otimes F) \cong f^*E \otimes f^*F$, $1^*E \cong E$

Definition 20.2.10. Suppose E, F are vector bundles both trivialized over $\{U_\alpha\}$ (this can easily be achieved, just take intersections), suppose the transition functions are $g_{\alpha\beta}, h_{\alpha\beta}$, then define the tensor product of vector bundles $E \otimes F$ by letting its transition functions be $g_{\alpha\beta} \otimes h_{\alpha\beta}$. Similarly, we can define symmetric power and exterior power of vector bundles by specifying its transition function

Does it have universal property also?

Definition 20.2.11. Let $E \xrightarrow{p} X, F \xrightarrow{p} X$ be vector bundles, then the direct sum $E \oplus F \xrightarrow{p} X$ is defined by transition functions $g_{\alpha\beta} \oplus h_{\alpha\beta}$, where $g_{\alpha\beta}, h_{\alpha\beta}$ are transition functions of E, F

Definition 20.2.12. Let $E \rightarrow X$ be a vector bundle, define its dual bundle as follows, if $g_{\alpha\beta}$ is a transition function, the transition function for E^* would be $(g_{\alpha\beta}^{-1})^T$

Definition 20.2.13. quotient bundle, exterior and symmetric power of vector bundle

Proposition 20.2.14. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle, \rangle : E \oplus E \rightarrow \mathbb{R}$ with $\langle, \rangle|_{E_x}$ defines an inner product

Definition 20.2.15. $F \subseteq E$ is called a vector subbundle if F is a subspace of E and $F \xrightarrow{p} X$ is also a vector space

Proposition 20.2.16. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space and $F \subseteq E$ is a vector subbundle, then there exists a vector subbundle $F^\perp \subseteq E$ such that $F_x \oplus F^\perp_x = E|_x$ and $(F|_x)^\perp = F^\perp|_x$

Proof.

□

X compact Hausdorff $\Rightarrow E$ has complement

Theorem 20.2.17. If $E \xrightarrow{p} X$ is vector bundle over a compact Hausdorff space X , then there exists a vector bundle $E' \xrightarrow{p'} X$ such $E \oplus E'$ is a trivial bundle

Proposition 20.2.18. Every Lie group G is parallelizable

Proof. Pick an arbitrary basis e_1, \dots, e_n of T_1G , then $L_g^*(e_i)$ will be a basis of $T_{g^{-1}G}$ since L_g^* is an isomorphism, they form independent global sections of the tangent bundle □

Definition 20.2.19. Tautological bundle

Definition 20.2.20. Let X be a smooth manifold of dimension n (depending on the field), Ω denote the cotangent bundle, then $\omega := \bigwedge^n \Omega$ is called the canonical bundle

Definition 20.2.21. Universal bundle

Theorem 20.2.22. Let X be a paracompact Hausdorff space, there is a bijection $\left[X, \varinjlim Gr_{\mathbb{C}}(n, N)\right] \rightarrow \text{Vect}_{\mathbb{C}}^n(X), [f] \mapsto [f^*(E)]$

Definition 20.2.23. If G is a topological group, then a principal G -bundle P is a fiber bundle with a continuous right G action $P \times G \rightarrow P$, and the action is free and transitive (thus regular), which imply each fiber is a G -torsor, also, $g \mapsto yg$ is a homeomorphism

Definition 20.2.24. Let $E \xrightarrow{p} X$ is a vector bundle, an inner product is a continuous map $\langle, \rangle : E \oplus E \rightarrow \mathbb{R}$ with $\langle, \rangle|_{E_x}$ defines an inner product on E_x

Proposition 20.2.25. Let $E \xrightarrow{p} X$ is a vector bundle with an inner product \langle, \rangle , then we can local trivialization to be isometry on each fiber, i.e. $\langle v, w \rangle = (\varphi_U(v), \varphi_U(w)), v, w \in E_x$, where \langle, \rangle is the standard inner product on $U \times \mathbb{R}^n$

Proposition 20.2.26. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle, \rangle : E \oplus E \rightarrow \mathbb{R}$ with $\langle, \rangle|_{E_x}$ defines an inner product

Definition 20.2.27. let G be a topological group, E, X be G -spaces, then $E \xrightarrow{p} X$ is a G -vector bundle if it is a vector bundle, p is a G map, and for any $x \in X$, $g : E_x \rightarrow E_{gx}$ is a linear map

Definition 20.2.28. Let G be a topological group, H be a closed subgroup, a G vector bundle $\pi : E \rightarrow G/H$ is called a homogeneous vector bundle

Lemma 20.2.29. Let $Y \xrightarrow{f} X, Z \xrightarrow{g} X$ be open surjective continuous maps, then the projection $p_Y : Y \times_X Z \rightarrow Y$ is open surjective

Proof. For surjectivity, if $y \in Y$, since g is surjective, $\exists z \in Z$ such that $g(z) = f(y)$, then $(z, y) \in Y \times_X Z$

To prove p_Y is open, suppose $(z_0, y_0) \in Y \times_X Z$ is in some open set, then $(z_0, y_0) \in U \times V \cap Y \times_X Z$ for some $y_0 \in U, z_0 \in V$ open, since f, g are open, $U' := f(U) \cap g(V)$ is open, let $V' := V \cap f^{-1}(U')$, then we can show V' is in the image of $U \times V \cap Y \times_X Z$, since $\forall y \in V', f(y) \in U' \subseteq g(V)$, thus $f(y) = g(z)$ for some $z \in V$, hence $(y, z) \in U \times V \cap Y \times_X Z$ \square

Proposition 20.2.30. Let $\pi : E \rightarrow G/H$ be a homogeneous vector bundle, E_H be the fiber over the coset H , action $G \times E_H \rightarrow E$ can be regard as $\alpha : G \times_H E_H \rightarrow E$ which is an isomorphism of G vector bundles. Moreover, if H is locally compact, then for a given $\mathbb{R}H$ module E_H , $G \times_H E_H \rightarrow G/H$ is indeed a G vector bundle, hence G vector bundle E is in one to one correspondence with representations of H on E_H , so $K_G(G/H) \cong R(H)$

Proof. E_H is an $\mathbb{R}H$ module, let $G \times_H E_H$ denote the space of orbits of $G \times E_H$ under H by $h \cdot (g, \xi) = (gh^{-1}, h\xi)$, $G \times_H E_H$ is a G space with G action $g \cdot (g', \xi) \mapsto (gg', \xi)$, then the group action can be regarded as $\alpha : G \times_H E_H \rightarrow E, (g, \xi) \mapsto g\xi$, we can find its inverse $\beta : E \rightarrow G \times_H E_H, E_{gH} \ni \xi \mapsto (g, g^{-1}\xi)$, to show that this is continuous, consider $\gamma : G \times E \rightarrow G \times E, (g, \xi) \mapsto (g, g^{-1}\xi)$, then the preimage of $G \times E_H$ will be the pullback $G \times_{G/H} E := \{(g, \xi) \in G \times E | gH = \pi\xi\}$, then $G \times_{G/H} E \rightarrow G \times E_H \rightarrow G \times_H E_H, (g, \xi) \mapsto (g, g^{-1}\xi)$ factors as $G \times_{G/H} E \rightarrow E \xrightarrow{\beta} G \times_H E, (g, \xi) \mapsto \xi \mapsto (g, g^{-1}\xi)$ which open surjective, therefore β is continuous due to the previous Lemma \square

Definition 20.2.31. A clutching function for S^k is $f : S^{k-1} \rightarrow GL(n, \mathbb{C})$, then we can define vector bundle E_f with f being the transition function, conversely, if E is a vector bundle over S^k , since its upper and lower hemispheres are both contractible, $E = E_f$, where f is the transition function, denoting the corresponding matrix T_f

Theorem 20.2.32. $[S^{k-1}, GL(n, \mathbb{C})] \rightarrow \text{Vect}_{\mathbb{C}}^n(S^k), f \mapsto E_f$ is a bijection

Lemma 20.2.33. Suppose $f, g : S^{k-1} \rightarrow GL(n, \mathbb{C})$, then $(E_f \otimes E_g) \oplus \varepsilon^n \cong E_{fg} \oplus \varepsilon^n \cong E_f \oplus E_g$

Proof. Since $GL(n, \mathbb{C})$ is path connected, there is a path $A_t \in GL(2n, \mathbb{C})$ that $A_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, A_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $\begin{pmatrix} T_f & \\ & I \end{pmatrix} A_t \begin{pmatrix} I & \\ & T_g \end{pmatrix} A_t$ is $\begin{pmatrix} T_f & \\ & T_g \end{pmatrix}$ when $t = 0$ and $\begin{pmatrix} T_f T_g & \\ & I \end{pmatrix} = \begin{pmatrix} T_{fg} & \\ & I \end{pmatrix}$ when $t = 1$ \square

Definition 20.2.34. Let $E \xrightarrow{p} X$ be vector bundle of rank n , and there is a inner product over E , we can define the sphere bundle $S(E)$ associated to E to be $S(E) = \bigcup_{x \in X} S(E_x)$ with the subspace topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{array}{ccc} S(E)_U & \xrightarrow{\varphi_U} & U \times S(\mathbb{R}^n) \\ \downarrow & & \downarrow \\ E_U & \xrightarrow{\varphi_U} & U \times \mathbb{R}^n \\ & \searrow p & \downarrow \\ & & U \end{array}$$

Definition 20.2.35. Let $E \xrightarrow{p} X$ be vector bundle of rank n , and there is a inner product over E , we can define the projective bundle $P(E)$ associated to E to be $P(E) = \bigcup_{x \in X} P(E_x)$ with the quotient topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{array}{ccc} S(E)_U & \xrightarrow{\varphi_U} & U \times S(\mathbb{R}^n) \\ \downarrow q & \searrow & \swarrow \\ & U & \\ \uparrow & \swarrow & \searrow \\ P(E)_U & \xrightarrow{\varphi_U} & U \times P(\mathbb{R}^n) \\ \downarrow q & & \downarrow q \end{array}$$

Definition 20.2.36. Let $E \xrightarrow{p} X$ be vector bundle of rank n , and there is a inner product over E , we can define the flag bundle $F(E)$ associated to E to be $F(E) = \bigcup_{x \in X} F(E_x)$ with the subspace topology in $P(E) \times \cdots \times P(E)$

Remark 20.2.37. Consider the pullback of $\pi : F(E) \rightarrow X, \pi^*(E) \subseteq F(E) \times E$, consider its subbundles L_1, \dots, L_n , where L_i is the subbundle that over a point in $F(E)$, it is the i -th factor, then $\pi^*(E) \cong L_1 \oplus \cdots \oplus L_n$

Definition 20.2.38. Let X be a paracompact and Hausdorff space, there exist unique functions $w_1, w_2, \dots, w_i : \text{Vect}_{\mathbb{R}}(X) \rightarrow H^i(X, \mathbb{Z}_2), E \mapsto w_i(E)$, and they only depend on the isomorphism classes of E , satisfying

1. $w_i(f^*(E)) = f^*(w_i(E))$, for pullback bundle $f^*(E)$
2. $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ where $w = 1 + w_1 + w_2 + \cdots \in H^*(X, \mathbb{Z}_2)$
3. $w_i(E) = 0, \forall i > \dim E$
4. If $E \rightarrow \mathbb{R}P^\infty$ is the canonical line bundle, then $w_1(E)$ is the generator of $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ $w_i(E)$ are called the Stiefel-Whitney classes of E

Definition 20.2.39. Let X be a paracompact and Hausdorff space, there exist unique functions $c_1, c_2, \dots, c_i : \text{Vect}_{\mathbb{C}}(X) \rightarrow H^{2i}(X; \mathbb{Z})$, $E \rightarrow c_i(E)$, and they only depend on the isomorphism classes of E , satisfying

1. $c_i(f^*(E)) = f^*(c_i(E))$, for pullback bundle $f^*(E)$
 2. $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ where $c = 1 + c_1 + c_2 + \dots \in H^*(X; \mathbb{Z})$
 3. $c_i(E) = 0, \forall i > \dim E$
 4. If $E \rightarrow \mathbb{C}P^\infty$ is the canonical line bundle, then $c_1(E)$ is a generator of $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[x]$, specify a generator in advance
- $c_i(E)$ are called the Chern classes of E , also we define the Chern polynomial to be $c_t = 1 + c_1 t + c_2 t^2 + \dots$ where t is just a formal variable used to keep tracking of the degree

Lemma 20.2.40. Let L_1, L_2 be line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$

Definition 20.2.41. Suppose L is a line bundle, define the Chern character $ch(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{c_1(L)^2}{2!} + \dots \in H^*(X; \mathbb{Q})$, then we have $ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1) + c_1(L_2)} = e^{c_1(L_1)} e^{c_1(L_2)} = ch(L_1) ch(L_2)$, If we assume $ch(L_1 \oplus L_2) = ch(L_1) + ch(L_2)$, then for $E = L_1 \oplus \dots \oplus L_n$, $ch(E) = ch(L_1) + \dots + ch(L_n) = n + (c_1(L_1) + \dots + c_1(L_n)) + (c_1(L_1)^2 + \dots + c_1(L_n)^2)/2! + \dots$, on the other hand, we have $c(E) = c(L_1) \smile \dots \smile c(L_n) = (1 + c_1(L_1)) \smile \dots \smile (1 + c_1(L_n)) = 1 + c_1(E) + \dots + c_n(E)$, where $c_i(E)$ would just be the i -th elementary symmetric polynomial of $c_1(L_1), \dots, c_1(L_n)$, i.e. $c_i(E) = \sigma_i(c_1(L_1), \dots, c_1(L_n))$, so we can express $c_1(L_1)^k + \dots + c_1(L_n)^k$ in terms of $c_i(E)$, i.e. $c_1(L_1)^k + \dots + c_1(L_n)^k = s_k(c_1(E), \dots, c_n(E))$, thus we have an abstract definition of Chern character, $ch(E) = \dim E + s_1(c_1(E), \dots, c_n(E)) + s_2(c_1(E), \dots, c_n(E))/2! + \dots$

Proposition 20.2.42. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$, $ch(E_1 \otimes E_2) = ch(E_1) ch(E_2)$

20.3 Principal bundle

20.4 Topological K-theory

Definition 20.4.1. Two vector bundles $E \rightarrow X$, $F \rightarrow X$ are stably isomorphic if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^n$, denoted as $E \approx F$, we also denote $E \sim F$ if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^m$ for some n, m

Remark 20.4.2. Here stably isomorphic does not imply isomorphic, for example, $TS^2 \approx_s \varepsilon^2$, since $\varepsilon^3 \approx T^2 \oplus NS^2 \approx T^2 \oplus \varepsilon^1$ whereas TS^2 is not trivial by the hairy ball theorem, and $NS^2 \approx \varepsilon^1$ is trivial because it is very easy to find a nonvanishing global section

Definition 20.4.3. Define the reduced K group to be $\tilde{K}(X)$ which consists of \sim -equivalent classes, and define K group to be the formal difference of isomorphic classes $E - F$, and $E - F = E' - F'$ if $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ for some vector bundle G

Remark 20.4.4. When X is compact Hausdorff, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ is equivalent to $E \oplus F' \oplus \varepsilon^m \cong E' \oplus F \oplus \varepsilon^m$, since we can find G' such that $G \oplus G' \cong \varepsilon^m$ due to Theorem 20.2.17 $K(*) = \{\varepsilon^m - \varepsilon^n\} \cong \mathbb{Z}$, $\tilde{K}(*) = 0$, and when X compact Hausdorff we have an exact sequence $0 \rightarrow K(*) \rightarrow K(X) \rightarrow \tilde{K}(X) \rightarrow 0$, where $K(*) \rightarrow K(X)$ is simply given by $\varepsilon^m - \varepsilon^n \mapsto \varepsilon^m - \varepsilon^n$, $K(X) \rightarrow \tilde{K}(X)$ is defined as follows, given $E - F \in K(X)$, $E - F = E \oplus F' - F \oplus F' = E' - \varepsilon^m$ is mapped to E' , this exact sequence splits since we have map $K(X) \rightarrow K(*)$ given by restriction

Conjecture 20.4.5. Let M be the Möbius line bundle over S^1 , since $M \oplus M \cong \varepsilon^2$, and $M \otimes M \cong \varepsilon^1$, thus real K-theory of S^1 is isomorphic to $\mathbb{Z}[M]/(M^2 - 1, 2M - 2)$

Example 20.4.6. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, TS^n, NS^n be the tangent bundle and normal bundle, then $TS^n \oplus NS^n$ can be seen as the restriction of the trivial bundle $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ on S^n , thus $TS^n \oplus NS^n$ is trivial

Definition 20.4.7. Define external product $K(X) \otimes K(Y) \rightarrow K(X \times Y)$, $a \otimes b \mapsto p_1^*(a)p_2^*(b) =: a \times b$, this is a ring homomorphism

20.5 Classifying space

Definition 20.5.1. Suppose G is a topological group, P_G is the contravariant functor from the category of CW complexes to the category of sets, mapping X to all the principal G bundles over X , a **classifying space** BG is a topological space such that $[-, BG] \rightarrow P_G(-)$ is a natural isomorphism

Lemma 20.5.2. BG is unique up to weak homotopy equivalence

Proof. Suppose $B'G$ is also a classifying space, then $[-, BG] \cong P_G(-) \cong [-, B'G]$ are natural isomorphic, by Theorem 15.5.1, we may assume $BG, B'G$ are both CW complexes, and by Lemma 3.2.1, $X \rightarrow \text{Hom}(-, X)$ is fully faithful functor, thus $BG, B'G$ are homotopic \square

Theorem 20.5.3 (Milnor's construction for classifying space). Define $E^n G$ to be $\overbrace{G * \cdots * G}^{n+1}$ are formal sums $t_0 g_0 + t_1 g_1 + \cdots + t_n g_n$, with $\sum t_i = 1$. $EG := \varinjlim E^n G$ are finite formal sums $\sum t_i g_i$ with $\sum t_i = 1$. $E^n G \rightarrow E^n G/G$, $EG \rightarrow EG/G =: BG$ are principal G bundles, any principal G bundle over X is a pullback bundle of $EG \xrightarrow{p} BG$

Proof. Define G right action on $E^n G, EG$

$$\begin{aligned} E^n G \times G &\rightarrow E^n G, \left(\sum t_i g_i, g \right) \mapsto \sum t_i g_i g \\ EG \times G &\rightarrow EG, \left(\sum t_i g_i, g \right) \mapsto \sum t_i g_i g \end{aligned}$$

Let $U_i = \{p(\sum t_i g_i) | t_i \neq 0\}$, then we would have a equivariant homeomorphism $p^{-1}(U_i) \rightarrow U_i \times G$, $\sum t_i g_i \mapsto (p(\sum t_i g_i), g_i)$ with inverse $U_i \times G \rightarrow p^{-1}(U_i)$, $(p(\sum t_i g_i), g) \mapsto \sum t_j g_j g_i^{-1} g$, this is well defined since $(p(\sum t_i g_i h), g) \mapsto \sum t_j g_j h h^{-1} g_i^{-1} g = \sum t_j g_j g_i^{-1} g$ \square

Definition 20.5.4. A **topological category** \mathcal{C} is a small category where $ob\mathcal{C}, mor\mathcal{C}$ are topological spaces and $i : ob\mathcal{C} \rightarrow mor\mathcal{C}, c \mapsto 1_c, s : mor\mathcal{C} \rightarrow ob\mathcal{C}, c \xrightarrow{f} d \mapsto c, t : mor\mathcal{C} \rightarrow ob\mathcal{C}, c \xrightarrow{f} d \mapsto d, \circ : mor\mathcal{C} \times mor\mathcal{C} \rightarrow mor\mathcal{C}$ are continuous. A **continuous functor** between topological categories is a functor that are continuous on both objects and morphisms

Nerve of a category

Definition 20.5.5. Define **nerve** $N\mathcal{C}$ on category \mathcal{C} which is also a simplicial set, $N\mathcal{C}([n]) := \text{Hom}([n], \mathcal{C})$, the set of all functors from $[n]$ to \mathcal{C} , viewing $[n] = 0 \rightarrow 1 \rightarrow \cdots \rightarrow n$ as a category

Definition 20.5.6 (Segal's construction for classifying space). Define the classifying space of \mathcal{C} to be $B\mathcal{C} := |N\mathcal{C}|$ as in Definition 20.5.5

Part VII

Differential topology

Chapter 21

Smooth manifold

Definition 21.0.1. An n dimensional **manifold** M is a topological space such that for any $p \in M$, there is a neighborhood $U \ni p$ such that U is homeomorphic to \mathbb{R}^n

Definition 21.0.2. A **chart** of $U \subseteq M$ is a homeomorphism $\varphi_U : U \rightarrow \varphi_U(U) \subseteq \mathbb{R}^n$, by abuse of notation, let $x_i : U \rightarrow \mathbb{R}$ be the composition $U \xrightarrow{\varphi_U} \mathbb{R}^n \xrightarrow{x_i} \mathbb{R}$, called **local coordinates**, suppose $\varphi_V : V \rightarrow \varphi_V(V) \subseteq \mathbb{R}^n$ is another chart, and $U \cap V \neq \emptyset$, the **transition map** is $\tau_{VU} = \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \rightarrow \varphi_V(U \cap V)$

M is a **C^k manifold** if transition maps $\{\tau_{VU}\} \subseteq C^k$, if $k = 0$, M is just a topological manifold, if $k = \infty$, M is a **smooth manifold**, if $k = \omega$, M is an **analytic manifold**

If n is even and transition maps are holomorphic, M is a **complex manifold**

Remark 21.0.3. A smooth manifold is locally ringed space locally ringed space with structure sheaf the sheaf of differentiable functions. An analytic manifold is locally ringed space locally ringed space with structure sheaf the sheaf of analytic functions. A complex manifold is locally ringed space with structure sheaf the sheaf of holomorphic functions

Definition 21.0.4. A **smooth map** $f : M \rightarrow N$ is map such that $\psi_V \circ f \circ \varphi_U^{-1}$ is smooth. $C^\infty(M)$ are smooth functions on M . $C_p^\infty(M)$ is the germ at p

Definition 21.0.5. A **submanifold** N is a inclusion and an immersion $i : N \hookrightarrow M$

Definition 21.0.6. The kernel of $C_p^\infty(M) \rightarrow \mathbb{R}, f \mapsto f(p)$ is a maximal ideal m_p , define the **cotangent space** $T_p^*M := \frac{m_p}{m_p^2}$, for $f \in C^\infty(M)$, define $(df)_p = f - f(p) \bmod m_p$,

$(dx_1)_p, \dots, (dx_n)_p$ form a basis of T_p^*M locally, $(df)_p = \frac{\partial f}{\partial x_1}(p)(dx_1)_p + \dots + \frac{\partial f}{\partial x_n}(p)(dx_n)_p$

Definition 21.0.7. The **tangent space** T_pM at p are the derivations $Der(C_p^\infty(M))$

$\left(\frac{\partial}{\partial x_1}\right)_p, \dots, \left(\frac{\partial}{\partial x_n}\right)_p$ form a basis of T_pM locally

Definition 21.0.8. The **pushforward(differential)** of smooth map $\phi : M \rightarrow N$ is $\phi_p : T_pM \rightarrow T_pN$, $\phi_p(X_p)(f) = X_p(f \circ \phi)$

Definition 21.0.9. The **pullback** of smooth map $\phi : M \rightarrow N$ is $\phi^* : C^\infty(N) \rightarrow C^\infty(M)$, $\phi^*(f) = f \circ \phi$

Definition 21.0.10. $(\varphi^*\alpha)_x(X) = \alpha_{\varphi(x)}(d\varphi_x(X)) = \alpha_{\varphi(x)}((\varphi_*)_x(X))$, or in short $\varphi^*\alpha(X) = \alpha(\varphi_*X)$, similarly, for k forms, $\varphi^*\alpha(X_1, \dots, X_k) = \alpha(\varphi_*X_1, \dots, \varphi_*X_k)$

In particular, $\varphi^*(dx) = d(x \circ \varphi)$, pullback is compatible with wedge product, $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$, and pullback is compatible with exterior derivative, $\varphi^*(d\alpha) = d(\varphi^*\alpha)$ The exterior multiplication by α is $\beta \mapsto \alpha \wedge \beta$ The interior multiplication by $v \in TM$ is $v \lrcorner : \omega(-) \mapsto \omega(v, -)$

Definition 21.0.11. $\pi : T^*M \rightarrow M$ is the cotangent bundle, ω is a one form, the **tautological one form** is $\pi^*\omega$

Definition 21.0.12. Let M be a smooth manifold, $X, Y \in C^\infty(M, TM)$ are vector fields, define Lie bracket $[X, Y] \in C^\infty(M, TM)$, $[X, Y](f) := (XY - YX)(f) = X(Y(f)) - Y(X(f))$

Remark 21.0.13. Check from local coordinates, $X(Y(f))$ is not well defined

Definition 21.0.14. Let M, N be smooth manifolds, $f : M \rightarrow N$ is a smooth map, it is called an immersion if df is injective at any point, it is called submersion if df is surjective at any point

Constant rank mapping theorem

Theorem 21.0.15. Suppose M, N are smooth manifolds with dimension m, n , $f : M \rightarrow N$ is a smooth map with constant rank r , then for any $p \in M$, denote $f(p) = q$, there are local charts $(p, U), (q, V)$ such that $\chi_V \circ f \circ \chi_U(x_1, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0)$. Moreover, suppose M is second countable, if f is injective, then f is a immersion, if f is surjective, then f is a submersion, if f is bijective, then f is a diffeomorphism

Proof. If f is surjective but not a submersion, then $r < n$, but then by Theorem 33.3.2, f can't be surjective which is a contradiction □

Constant rank level set theorem

Theorem 21.0.16. Suppose M, N are smooth manifolds with dimension m, n , $f : M \rightarrow N$ is a smooth map with constant rank r , then a level set $S = f^{-1}(c)$ is an embedded submanifold in M of codimension r with $f|_S$ being a proper map

Proposition 21.0.17. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M , for any equivariant map $f : M \rightarrow N$, f has constant rank

Proof. For any $x \in M$, denote $y = f(x)$, it suffices to show $\text{rank}(df)_x = \text{rank}(df)_{gx}$ since G acts transitively on M , note that $f(gx) = gf(x)$, thus $fL_g = L_gf$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\text{rank}(df)_x = \text{rank}(df)_{gx}$ □

Part VIII

Differential geometry

Chapter 22

Riemannian manifold

22.1 Differential geometry of surfaces

Definition 22.1.1. A **differentiable surface** is an embedding $S \hookrightarrow \mathbb{R}^3$

Lemma 22.1.2. $\gamma(t)$ is a geodesic iff $\ddot{\gamma}$ is parallel to the normal \vec{n} , meaning no acceleration in S

A geodesic γ on S has constant speed

The geodesic curvature of a curve γ is the curvature of the projection onto tangent plane, γ is a geodesic iff the geodesic curvature of γ is zero

Proof. $\frac{d}{dt}|\dot{\gamma}|^2 = 2\ddot{\gamma} \cdot \dot{\gamma} = 0$

□

22.2 Curvature

Definition 22.2.1. A **Riemannian manifold** is (M, g) where M is a smooth manifold and **Riemannian metric** $g_p : S^2(T_p M) \rightarrow \mathbb{R}$ is a positive definite

Definition 22.2.2. An **affine connection** is

$$\begin{aligned} \nabla : \Gamma(TM) \otimes \Gamma(TM) &\rightarrow \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

satisfying

- $\nabla_f X = f \nabla_X X$, i.e. ∇ is $C^\infty(M, \mathbb{R})$ linear in the first variable
- $\nabla_X(fY) = XfY + f \nabla_X Y$, i.e. ∇ satisfies Leibniz rule in the second variable

From this we can define covariant derivative ∇ , $\nabla_X f = Xf$, $\nabla_X(\alpha)(Y) = \nabla_X(\alpha(Y)) - \alpha(\nabla_X Y)$, here α is a covector, similarly for any tensor, Write contraction $(\nabla T)(\alpha_1, \dots, \alpha_m, X_1, \dots, X_n, X) = (\nabla_X T)(\alpha_1, \dots, \alpha_m, X_1, \dots, X_n)$, T is a tensor

Note. $\nabla_X(\alpha(Y)) = \nabla_X(\alpha)(Y) + \alpha(\nabla_X Y)$

Definition 22.2.3. ∇ is an affine connection, the **torsion tensor** is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Definition 22.2.4. The Levi-Civita connection ∇ is the one satisfying

- $\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, i.e. $\nabla g = 0$
- $\nabla_X Y - \nabla_Y X = [X, Y]$, i.e. ∇ is torsion free

Definition 22.2.5. ∇ is the Levi-Civita connection, the Riemannian curvature tensor is $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$

Remark 22.2.6. X, Y are commuting vector fields around x_0 , then $\frac{d}{ds} \frac{d}{dt} \tau_{sX}^{-1} \tau_{tY}^{-1} \tau_{sX} \tau_{tY} Z = R_{XY} Z$, τ is the parallel transport

$$\begin{array}{ccc} & \xrightarrow{\tau_{sX}} & \\ \tau_{sY} \uparrow & \square & \downarrow \tau_{sY}^{-1} \\ & \xleftarrow{\tau_{sX}^{-1}} & \end{array}$$

Proposition 22.2.7.

1. $R_{YX} = -R_{XY}$
2. $(R_{XY} Z, W) = -(R_{XY} W, Z)$
3. $R_{XY} Z + R_{YZ} X + R_{ZX} Y = 0$
- 4.

The **second Bianchi identity** follows

$$\nabla_X R_{YZ} + \nabla_Y R_{ZX} + \nabla_Z R_{XY} = 0$$

Remark 22.2.8. If write $(R_{XY} Z, W) = R(X, Y, Z, W)$, then R is antisymmetric about the first two variables and the last two variables, R satisfies Jacobi identity, the first two and the last two variables can switch place

Proof.

- 1.
- 2.
- 3.
4. Follow from above

□

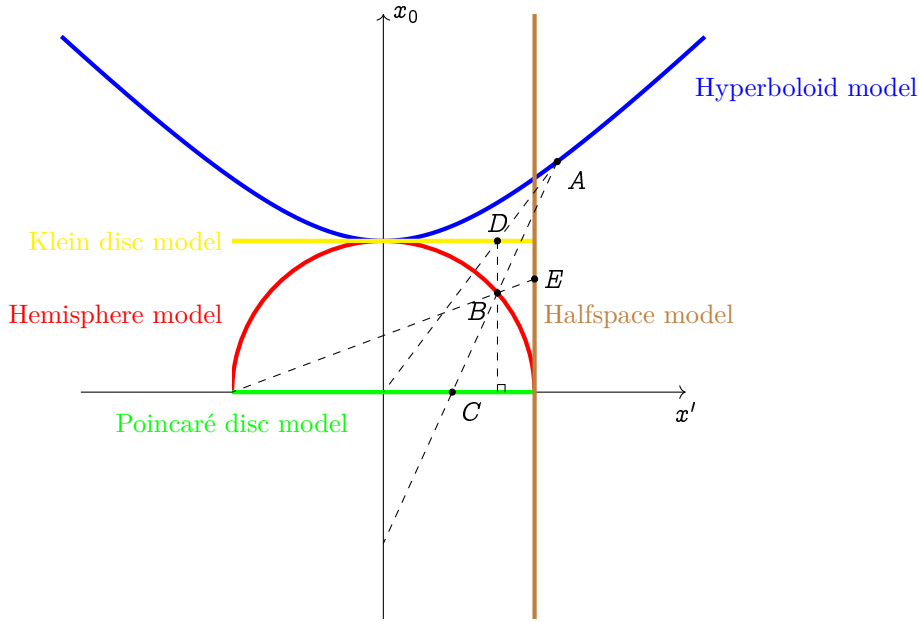
Definition 22.2.9. $\{e_i\}$ is an orthonormal basis, the **Ricci curvature** is $\text{Ric}(X) = \sum R_{X, e_i} e_i$. The **scalar curvature** is $S = \text{Tr Ric} = \sum (\text{Ric}(e_j), e_j) = \sum (R_{e_j, e_i} e_i, e_j)$. The **Einstein curvature** is $G = R - \frac{1}{2}gS$

22.3 Hyperbolic geometry

Definition 22.3.1. \mathbb{R}^{n+1} with metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$ is the **Minkowski space**

The **hyperboloid model** is $\mathbb{H} = \{x_1^2 + \cdots + x_n^2 - x_0^2 = -1, x_0 > 0\}$. The Riemannian metric is the pullback metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$

The geodesics are intersections of \mathbb{H} and two dimensional subspaces of \mathbb{R}^{n+1} $(d \sinh s)^2 + (d \cosh s)^2 = \cosh^2 s ds^2 - \sinh^2 s ds^2 = ds^2$, thus \mathbb{H}^1 is isomorphic to \mathbb{E}^1



$(x', x_n) \mapsto \left(\frac{2x'}{1+x_n}, 1 \right)$, $x' = (x_0, \dots, x_{n-1})$ is the isometry from the hemisphere to the halfspace

$(x', 1) \mapsto \left(\frac{4x'}{4+|x'|^2}, \frac{4-|x'|^2}{4+|x'|^2} \right)$, $x' = (x_0, \dots, x_{n-1})$ is the isometry from the halfspace to the hemisphere

$x \mapsto \left(\frac{x'}{1+x_0} \right)$, $x' = (x_1, \dots, x_n)$ is the isometry from the hemisphere to Poincaré disc

$x \mapsto \left(\frac{2x}{1-|x|^2}, \frac{1+|x|^2}{1-|x|^2} \right)$ is the isometry from Poincaré disc to the hyperboloid

$x \mapsto (1, x')$, $x' = (x_1, \dots, x_n)$ is the isometry from the hemisphere to Klein disc

$x \mapsto \left(\frac{x'}{x_0}, \frac{1}{x_0} \right)$, $x' = (x_1, \dots, x_n)$ is the isometry from the hyperboloid to the hemisphere

$x \mapsto \left(\frac{x'}{x_0}, \frac{1}{x_0} \right)$, $x' = (x_1, \dots, x_n)$ is the isometry from the hemisphere to the hyperboloid

The **hemisphere model** is $\mathbb{H} = \{x_0 > 0\} \cap S^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
\sum_{i=0}^{n-1} \left[d \left(\frac{x_i}{x_n} \right) \right]^2 - \left[d \left(\frac{1}{x_n} \right) \right]^2 &= \sum_{i=0}^{n-1} \left(\frac{x_0 dx_i - x_i dx_0}{x_0^2} \right)^2 - \left(-\frac{dx_0}{x_0^2} \right)^2 \\
&= \sum_{i=0}^{n-1} \frac{x_0^2 dx_i^2 - 2x_i x_0 dx_i dx_0 + x_i^2 dx_0^2}{x_0^4} - \frac{dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(|x'|^2)d(x_0^2)}{2x_0^4} + \frac{|x'|^2 dx_0^2 - dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(1 - x_0^2)d(x_0^2)}{2x_0^4} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2}{x_0^2} + \frac{2dx_0^2}{x_0^2} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2 + dx_0^2}{x_0^2}
\end{aligned}$$

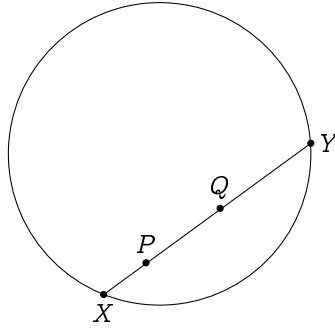
The **half space model** is $\mathbb{H} = \{x_0 > 0\} \cap \{x_n = 1\}$. The Riemannian metric is pullback metric

$$\begin{aligned}
&\frac{\sum_{i=0}^{n-1} d \left(\frac{4x_i}{4 + |x'|^2} \right)^2 + d \left(\frac{4 - |x'|^2}{4 + |x'|^2} \right)^2}{\left(\frac{4x_0}{4 + |x'|^2} \right)^2} \stackrel{X=4+|x'|^2}{=} \frac{\sum_{i=0}^{n-1} d \left(\frac{4x_i}{X} \right)^2 + d \left(\frac{8}{X} - 1 \right)^2}{\left(\frac{4x_0}{X} \right)^2} \\
&= \frac{X^2}{x_0^2} \left(\sum_{i=0}^{n-1} d \left(\frac{x_i}{X} \right)^2 + 4d \left(\frac{1}{X} \right)^2 \right) \\
&= \frac{X^2}{x_0^2} \left(\sum_{i=0}^{n-1} \left(\frac{X dx_i - x_i dX}{X^2} \right)^2 + 4 \frac{dX^2}{X^4} \right) \\
&= \frac{1}{x_0^2} \left(\sum_{i=0}^{n-1} \frac{X^2 dx_i^2 + x_i^2 dX^2 - 2X x_i dX dx_i}{X^2} + 4 \frac{dX^2}{X^2} \right) \\
&= \frac{1}{x_0^2} \left(dx'^2 + \frac{|x'|^2 dX^2}{X^2} + \frac{4dX^2}{X^2} - \frac{dX d(|x'|^2)}{X} \right) \\
&= \frac{1}{x_0^2} \left(dx'^2 + \frac{X dX^2}{X^2} - \frac{dX d(X - 4)}{X} \right) \\
&= \frac{dx'^2}{x_0^2}
\end{aligned}$$

The **Poincaré disc model** is $\mathbb{H} = D^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
 \sum_{i=1}^n d\left(\frac{2x_i}{1-|x|^2}\right)^2 - d\left(\frac{1+|x|^2}{1-|x|^2}\right)^2 &\stackrel{X=1-|x|^2}{=} \sum_{i=1}^n d\left(\frac{2x_i}{X}\right)^2 - d\left(\frac{2}{X} - 1\right)^2 \\
 &= 4 \sum_{i=1}^n \left(\frac{Xdx_i + x_i dX}{X^2}\right)^2 - 4 \left(-\frac{dX}{X^2}\right)^2 \\
 &= 4 \sum_{i=1}^n \frac{X^2 dx_i^2 + x_i^2 dX^2 - 2Xx_i dx_i dX}{X^4} - 4 \frac{dX^2}{X^4} \\
 &= 4 \left(\frac{dx^2}{X^2} + \frac{|x|^2 dX^2}{X^4} - \frac{dX^2}{X^4} - \frac{d(|x|^2)dX}{X^3} \right) \\
 &= 4 \left(\frac{dx^2}{X^2} - \frac{X dX^2}{X^4} - \frac{d(1-X)dX}{X^3} \right) \\
 &= \frac{4dx^2}{X^2} = \frac{4dx^2}{(1-|x|^2)^2}
 \end{aligned}$$

The **Klein disc model** is $\mathbb{H} = D^n$. The distance between P, Q is $\frac{1}{2} \ln \left(\frac{|XQ||PY|}{|XY||PQ|} \right) = \frac{1}{2} \ln(X, P; Q, Y)$, $(X, P; Q, Y)$ is the cross ratio



Theorem 22.3.2. $\text{Isom}(\mathbb{H}^2) = PSL(2, \mathbb{R})$

Proof. An isometry sends half circles and orthogonal lines to half circles or orthogonal lines, by Schwarz reflection principle 32.2.3, it can be regarded as an isometry on \mathbb{CP}^1 sending \mathbb{RP}^1 to \mathbb{RP}^1 , then it necessarily has to be in $PSL(2, \mathbb{R})$ \square

Theorem 22.3.3. $\text{Isom}(\mathbb{H}^3) = PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$

Proof. Since $\partial\mathbb{H}^3$ is the Riemann sphere, every isometry on \mathbb{H}^3 restricts to a conformal map on $\partial\mathbb{H}^3$ because it sends hemispheres and orthogonal planes to hemispheres or orthogonal planes, hence it is a Möbius transformation. On the other hand, Möbius transformations which can all be extended to an isometry on \mathbb{H}^3 , translations $z \mapsto z + \lambda$ can be extended to $(z, x_3) \mapsto (z + \lambda, x_3)$, dilations $z \mapsto \lambda z$ can be extended to $(z, x_3) \mapsto (\lambda z, |\lambda|x_3)$, inversions $z \mapsto -\frac{1}{\bar{z}}$ can be extended to $(z, x_3) \mapsto \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2} \right)$. Therefore the isometry group for \mathbb{H}^3 is $PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$ \square

Part IX

Complex geometry

Chapter 23

Riemann surface

Definition 23.0.1. A **Riemann surface** is a one dimensional complex manifold

Theorem 23.0.2 (Riemann's removable singularity theorem). *f is holomorphic on $X \setminus \{a\}$ and bounded near a , then f is holomorphic on X*

Theorem 23.0.3 (Principle of analytic continuation). *X is connected, $X \xrightarrow{f} Y$ is holomorphic and $f \equiv c$ on some nondiscrete subset of X , then $f \equiv c$ on X*

Remark 23.0.4. This does not apply to higher dimensions, for example, $f(z, w) = z$, but in higher dimensions, we have Theorem 24.0.1

Theorem 23.0.5 (Local behaviour of holomorphic maps). *$X \xrightarrow{f} Y$ is a nonconstant holomorphic map, $a \in X$, $f(a) = b \in Y$. There are local charts $U \xrightarrow{\phi} \mathbb{C}$, $V \xrightarrow{\psi} \mathbb{C}$ of a, b such that $\psi f \phi^{-1} = z^k$ for some $k \geq 1$*

Remark 23.0.6. If the **multiplicity** $k > 1$, a is a **branch point**

Theorem 23.0.7. *$X \xrightarrow{f} Y$ is a proper nonconstant holomorphic map between Riemann surfaces, there exists some n such that f take every value $c \in Y$, counting multiplicities, n times*

Theorem 23.0.8 (Radó's theorem). *A connected Riemann surface is second countable*
Uniformization theorem

Theorem 23.0.9 (Uniformization theorem). *A simply connected Riemann surface is either \mathbb{C} , \mathbb{P} or \mathbb{H}^2*

Chapter 24

Complex manifold

Identity principle

Theorem 24.0.1 (Identity principle). *X is connected, $X \xrightarrow{f} Y$ is holomorphic and $f \equiv c$ on some nonempty open subset of X , then $f \equiv c$ on X*

Definition 24.0.2. M is a smooth manifold, an **almost complex structure** is $J : TM \rightarrow TM$ such that $J^2 = -1_{TM}$

Example 24.0.3. S^4 cannot be given an almost complex structure. S^6 can be given an almost complex structure but not a complex structure

A complex manifold always give an almost complex structure by $J \frac{\partial}{\partial z_i} = i \frac{\partial}{\partial z_i}, J \frac{\partial}{\partial \bar{z}_i} = -i \frac{\partial}{\partial \bar{z}_i}$

Definition 24.0.4. A is a $(1, 1)$ form, the Nijenhuis tensor is

$$N_A(X, Y) = -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY]$$

Theorem 24.0.5 (Newlander-Nirenberg theorem). *J is **integrable** iff $N_J = 0$. Meaning there is a unique complex structure which will give J*

Proposition 24.0.6. Given an almost complex structure, we can find coordinate charts $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ such that $\text{Span} \left\{ \frac{\partial}{\partial z_i} \right\}, \text{Span} \left\{ \frac{\partial}{\partial \bar{z}_i} \right\}$ to be the i and $-i$ eigenspaces of J

Definition 24.0.7. A **Hermitian manifold** M is a complex manifold with a **Hermitian metric** $h = \sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$, where $h_{\alpha\bar{\beta}}$ is a positive definite Hermitian matrix. This gives a Riemannian metric

$$g = \frac{1}{2}(h + \bar{h}) = \frac{1}{2} \left(\sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta + \sum h_{\beta\bar{\alpha}} d\bar{z}_\alpha \otimes dz_\beta \right) = \sum h_{\alpha\bar{\beta}} (dz_\alpha \otimes d\bar{z}_\beta + d\bar{z}_\beta \otimes dz_\alpha)$$

Also gives **associate $(1, 1)$ form**

$$\omega = -\frac{h - \bar{h}}{2i} = \frac{i}{2}(h - \bar{h}) = \frac{i}{2} \sum h_{\alpha\bar{\beta}} (dz_\alpha \otimes d\bar{z}_\beta - d\bar{z}_\beta \otimes dz_\alpha) = \frac{i}{2} \sum h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

Definition 24.0.8. A **Kähler manifold** M is a complex and symplectic manifold with an integrable almost complex structure J with a Riemannian metric $g(u, v) = \omega(u, Jv)$

Chapter 25

Symplectic manifold

Definition 25.0.1. M is a smooth manifold, a **symplectic structure** on M is a 2 form ω that is nondegenerate and anti-symmetric on $T_p M$

Part X

Lie group

Chapter 26

Topological group

Definition 26.0.1. G is a **topological group** if it is a group and a topological space so that the group multiplication $G \times G \rightarrow G$ and the inverse map $G \rightarrow G$ are continuous maps

Definition 26.0.2. $f : G \rightarrow \mathbb{R}/\mathbb{C}$ is a continuous function, $L_y f(x) = f(y^{-1}x)$, $R_y f(x) = f(xy)$, $L_{yz} = L_y L_z$, $R_{yz} = R_y R_z$, f is called left/right uniformly continuous if $\forall \varepsilon > 0$, $\exists V \ni e$ such that $\|L_y f - f\| < \varepsilon / \|R_y f - f\| < \varepsilon$, $\forall y \in V$, $\|\cdot\|$ is the supremum norm

Proposition 26.0.3. If $f \in C_c(G)$, then f is both left and right uniformly continuous

Proof. Easy proof by a very standard analysis argument □

Definition 26.0.4. If f is a Borel measurable function on G , then f factor through G/H , otherwise suppose $f(y) \neq f(z)$, $y, z \in xH$, $f^{-1}(f(y)) \cap xH \subsetneq xH$ is a Borel set which is impossible, because then $x^{-1}f^{-1}(f(y)) \cap H \subsetneq H$ will also be a Borel set, consider $\Gamma = \{S \in \mathcal{P}|H \subseteq H \text{ or } H \cap S = \emptyset\}$, then Γ is a sigma algebra containing all open sets hence Borel algebra, we reached a contradiction

Thus for most purposes one may as well work with G/H which is Hausdorff (L^p spaces for instance, mod almost everywhere vanishing function)

For a locally compact Hausdorff group, A Borel measure μ on G is called left/right invariant if $\mu(xE) = \mu(E)/\mu(Ex) = \mu(E)$, $x \in G, E \in \mathcal{B}(G)$

A linear functional I is left/right invariant if $I(L_x f) = I(f)/I(R_x f) = I(f)$

A left/right Haar measure on G is a left/right invariant Radon measure μ on G , for example, Lebesgue measure on \mathbb{R}^n , counting measure on G with discrete topology

Example 26.0.5. Continuous bijective group homomorphism doesn't imply homeomorphism, which is really obvious, by taking the identity map and a discrete topology on the topological group G

Definition 26.0.6. Let G be a topological group, then a 1-parameter subgroup means a continuous group homomorphism $\varphi : \mathbb{R} \rightarrow G$, $\varphi(s+t) = \varphi(s)\varphi(t)$, in the case of a Lie group, φ is required to be smooth

Definition 26.0.7. Suppose G is a connected, locally pathconnected and (semi-)locally simply connected topological space, then it has a universal cover \tilde{G} which is unique up to an isomorphism, a connected Lie group certain satisfies this

Proposition 26.0.8. Denote $\pi : \tilde{G} \rightarrow G$ as the covering map, let \bar{G} be the set of maps $T : \tilde{G} \rightarrow \tilde{G}$, such that $\pi(Tx) = g(\pi x)$ for some $g \in G$, i.e. the following diagram commutes

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{T} & \tilde{G} \\ \downarrow \pi & & \downarrow \pi \\ G & \xrightarrow{g} & G \end{array}$$

Then \tilde{G} which is a group acts transitively and freely on \tilde{G} , thus we can think of the universal cover \tilde{G} also as a topological group

Proof. Given $x, y \in \tilde{G}$, there is a unique $g \in G$ such that $g(\pi x) = \pi y$, since \tilde{G} is the universal cover, there is a unique lift such that $T(x) = y$, thus the action is free and transitive \square

Chapter 27

Algebraic group

Definition 27.0.1. An **algebraic group** G is a group and an algebraic variety such that multiplication and inverse are morphisms, if G is an affine algebraic variety, then it is called a **linear algebraic group** or **affine algebraic group**. \mathbb{F} group means linear algebraic group over \mathbb{F}

Chapter 28

Lie group

28.1 Lie groups

Definition 28.1.1. A **real Lie group** G is a group and a smooth manifold such that multiplication $G \times G \rightarrow G$ and inverse $G \rightarrow G$ are smooth

A **complex Lie group** is a group and complex manifold such that multiplication and inverse are holomorphic

a Lie subgroup H is a subgroup and an immersed submanifold

Definition 28.1.2. Left multiplication L_g by g is an isomorphism, a vector field X on G is called **left invariant** if $(L_g)_*X = X$, by Exercise 49.0.5, $[X, Y]$ is also left invariant since $(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$

Define **Lie algebra of G** to be left invariant vector fields. Equivalently, T_1G

If $\phi : G \rightarrow H$ is a homomorphism of Lie groups, then $d\phi : \text{Lie}(G) \rightarrow \text{Lie}(H)$ or $(d\phi)_1 : T_1G \rightarrow T_1H$ is an homomorphism of Lie algebras

Suppose $H \leq G$ is a Lie subgroup, then $\text{Lie}(H) = T_1H \leq T_1G$

Proposition 28.1.3. Lie groups are parallelizable

Proof. For any $0 \neq X_1 \in T_1G$, we can define a vector field $X_g = (L_g)_1X_1$, this is a nonvanishing global section of the tangent bundle, G is parallelizable \square

Definition 28.1.4. A Lie group representation (ρ, V) is a Lie group homomorphism $\rho : G \rightarrow GL(V)$

Proposition 28.1.5. Let V be a complex vector space, (π, V) be a Lie group representation of a compact Lie group G , then there exists a positive definite Hermitian form such that (π, V) is unitary

Proof. Choose any positive definite Hermitian form \langle, \rangle , define Hermitian form

$$(v, w) := \int_G \langle \pi(g)v, \pi(g)w \rangle d\mu$$

Where μ is the Haar measure with $\int_G d\mu = 1$, integrals make sense since G is compact, then $(,)$ is G left invariant \square

Definition 28.1.6. Lie group G acts on smooth manifold M , G_p is the stablizer of p . The **isotropy representation** is $G_p \rightarrow GL(T_pM)$, $g \mapsto d_pg$

28.2 Exponential map

Lemma 28.2.1. The exponential map $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is defined on $M_n(\mathbb{C})$ and logarithmic map

$\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(A-I)^k}{k}$ are defined on $|A-I| < 1$ and there inverses to each other locally, moreover, the exponential map is surjective onto $GL(n, \mathbb{C})$

Remark 28.2.2. Note that this also holds for a Banach algebra A

Proof. Just compare the coefficients of multiplication of series □

$$AV \subseteq V \iff e^{tA}V \subseteq V$$

Lemma 28.2.3. Let e^{tA} be a one parameter subgroup, then $V \subseteq \mathbb{R}^n$ is invariant under A iff invariant under e^{tA} , $\forall t$, in particular, $Av = 0$ iff $e^{tA}v = 0, \forall t$

Proof. If $AV \subseteq V$, then $e^{tA}V = \sum_{k=0}^{\infty} t^k \frac{A^k}{k!} V \subseteq V$

If $e^{tA}V \subseteq V, \forall t$, since V is closed, $\left. \frac{d}{dt} \right|_{t=0} e^{tA}V = AV \subseteq V$ □

Proposition 28.2.4. Observe that $v'(t) = Av(t)$ with $v(0) = v_0$ has the solution $v(t) = e^{tA}v_0$. Consider V_m to be the vector space of homogeneous polynomials in n variables of degree m , define group action of $GL(n, \mathbb{C})$ on V_m , $g \cdot f(x) := f(g^{-1}x)$, consider $v(t) = e^{tA} \cdot f := f(e^{-tA}x)$, then $v'(t) = \left. \frac{d}{dt} \right|_{t=0} f(e^{-tA}x) =: D_A f$, where D_A is a linear differential operator $V_m \rightarrow V_m$ by Lemma 28.2.3, then we should have $f(e^{-tA}x) = v(t) = e^{tD_A}f$, therefore we would get $D_A = -A^T$, and it will be easy to check that $D_{[A,B]} = [D_A, D_B]$

Proof. If we denote $g = (g_{ij}) \in GL(n, \mathbb{C})$, $f(x) = \sum_{i_1, \dots, i_n} C_{i_1, \dots, i_n} x_1^{i_1} \cdots x_n^{i_n}$, then $f(g^{-1}x) =$

$\sum_{i_1, \dots, i_n} C_{i_1, \dots, i_n} (g_{11}x_1 + \cdots + g_{1n}x_n)^{i_1} \cdots (g_{n1}x_1 + \cdots + g_{nn}x_n)^{i_n}$ is still a homogeneous polynomial in n variables of degree m

Denote $A = (a_{ij})$,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} f(e^{-tA}x) &= \nabla f(x) \cdot \left. \frac{d}{dt} \right|_{t=0} e^{-tA}x \\ &= -\nabla f(x) \cdot Ax \\ &= -\sum_{i,j} a_{ij} x_j \frac{\partial f}{\partial x_i} \\ &= \left(-\sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i} \right) f \\ &= (-\nabla^T Ax) f \\ &= D_A f \end{aligned}$$

In particular, $D_A x_i = -\sum_{j=1}^n a_{ij} x_j$, thus D_A has matrix $-A^T$ with respect to x_1, \dots, x_n , basis of V_1 □

Example 28.2.5. Consider Lie group $SL(2, \mathbb{C})$ whose Lie algebra is $\mathfrak{sl}(2, \mathbb{C})$, which is generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, thus $D_H = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, $D_X = -x_2 \frac{\partial}{\partial x_1}$, $D_Y = -x_1 \frac{\partial}{\partial x_2}$

Definition 28.2.6. Let G be a (Lie) group, then a 1-parameter subgroup means a (smooth) group homomorphism $\phi : \mathbb{R} \rightarrow G$, $\phi(s+t) = \phi(s)\phi(t)$

Lie group homomorphism induce Lie algebra homomorphism

Proposition 28.2.7. Let $\phi : G \rightarrow H$ be a homomorphism of Lie groups, then $d\phi : \text{Lie}(G) \rightarrow \text{Lie}(H)$ or $(d\phi)_1 : T_1G \rightarrow T_1H$ is an homomorphism of Lie algebras

Proof. Suppose X is a left invariant vector field on G , then $(d\phi)_g X_g = (d\phi)_g(dL_g)_1 X_1(f) = X_1(f \circ \phi \circ L_g) = X_1(f \circ \phi \circ L_g) = (dL_{\phi(g)})_1(d\phi)_1 X_1(f)$ which gives a left invariant vector field, thus using Lemma 49.0.4

$$\begin{aligned} (d\phi)[X, Y](f) &= [X, Y](f \circ \phi) \\ &= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\ &= X(((d\phi Y)f) \circ \phi) - Y(((d\phi X)f) \circ \phi) \\ &= ((d\phi X)(d\phi Y)f) \circ \phi - ((d\phi Y)(d\phi X)f) \circ \phi \\ &= ([d\phi X, d\phi Y]f) \circ \phi \end{aligned}$$

Therefore $(d\phi)[X, Y] = [(d\phi X), (d\phi Y)]$, $d\phi$ is a Lie algebra homomorphism \square

Proposition 28.2.8. One parameter subgroups are precisely the maximal integral curves of the left invariant vector fields starting at 1

Remark 28.2.9. There is a one to one correspondence, $\{\text{One parameter subgroups of } G\} \leftrightarrow \text{Lie}(G) \leftrightarrow T_1G$

Proof. Suppose $\phi : \mathbb{R} \rightarrow G$ is a one parameter subgroup, let $X_1 = \phi'(0)$, then we have a left invariant vector field X on G , think of $\frac{\partial}{\partial t}$ as a left invariant vector field on \mathbb{R} , thus ϕ as Lie group homomorphism induces $(d\phi)\frac{\partial}{\partial t}$ which is also a left invariant vector field and $\phi'(s) = (d\phi)_s \frac{\partial}{\partial t} \Big|_s = X_{\phi(s)}$ as in Proposition 28.2.7

Conversely, if $\phi : \mathbb{R} \rightarrow G$ is the maximal integral curve of some left invariant vector field X , suppose the global flow generated by X is $\varphi : G \times \mathbb{R} \rightarrow G$, then $\varphi(1, t) = \phi(t)$, $\phi(t+s) = \varphi(1, t+s) = \varphi(\varphi(1, t), s) = \varphi(\phi(t), s)$, since $L_{\phi(t)}$ is an isomorphism, thus $L_{\phi(t)} \circ \phi$ is the maximal integral curve starting at $\phi(t)$, thus $\varphi(\phi(t), s) = \phi(t)\phi(s)$ \square

Definition 28.2.10. For any $A \in T_1G$, define the exponential map $\exp A := \phi_A(1)$ where $\phi_A : \mathbb{R} \rightarrow G$ is the one parameter subgroup corresponding to A , also it is easy to see that $\exp tA := \phi_{tA}(1) = \phi_A(t)$ which is a scaling of the integral curve, and $\exp(t+s)A = \exp tA \exp sA$ since $\exp tA$ is a one parameter subgroup, and thus $(\exp A)^{-1} = \exp(-A)$

Proposition 28.2.11. (Properties of exponential map) ^{Properties of exponential map}

Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$

- (a) The exponential map is a smooth map
- (b) $(d\exp)_0 : \mathfrak{g} \cong T_0\mathfrak{g} \rightarrow T_1G \cong \mathfrak{g}$ is the identity map, which implies that the exponential map is a local diffeomorphism around 0
- (c) Suppose $\phi : G \rightarrow H$ is a Lie group homomorphism, then the following diagram commutes

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(d\phi)_1} & \mathfrak{h} \\ \downarrow \exp & & \downarrow \exp \\ G & \xrightarrow{\phi} & H \end{array}$$

Proof.

(a)

(b) For any $A \in \mathfrak{g}$, consider $\gamma : \mathbb{R} \rightarrow G, t \mapsto tA$ which is a one parameter subgroup of \mathfrak{g} , thus $A = \gamma'(0) \in T_0\mathfrak{g}$, and $\exp A = \gamma(1) = A$

(c) Define $\gamma(t) = \phi(\exp tA)$ which is a one parameter subgroup of H since $\gamma(t+s) = \phi(\exp(t+s)A) = \phi(\exp tA \exp sA) = \phi(\exp tA)\phi(\exp sA) = \gamma(t)\gamma(s)$, then $\gamma'(0) = \left. \frac{\partial}{\partial t} \right|_{t=0} \phi(\exp tA) = (d\phi)_1 \left. \frac{\partial}{\partial t} \right|_{t=0} \exp tA = (d\phi)_1 A$, on the other hand, $\exp(t(d\phi)_1 A)$ is one parameter subgroup of H corresponds to $(d\phi)_1 A = \gamma'(0)$, thus $\exp(t(d\phi)_1 A) = \gamma(t) = \phi(\exp tA)$ \square

Proposition 28.2.12. Let G be a Lie group and $H \leq G$ a Lie subgroup, then $\text{Lie}(H) = \{A \in \text{Lie}(G) | \exp tA \in H, \forall t \in \mathbb{R}\}$

Part XI

Algebraic geometry

Chapter 29

Variety

29.1 Affine Varieties

Definition 29.1.1. $V \subseteq \mathbb{A}^n$ is an algebraic set, $f \in k[V]$

$$D(f) = \{(x_1, \dots, x_n) \in V \mid f(x_1, \dots, x_n) \neq 0\} = V(f)^c$$

form a basis for the Zariski topology on V

$D(f)$ can also be thought of as an algebraic set

$$\{(x_1, \dots, x_n, z) \mid zf(x_1, \dots, x_n) = 0\}$$

The coordinate ring can be written as $k[V][\frac{1}{f}] = k[V]_f$, where z is just replaced by $\frac{1}{f}$

Theorem 29.1.2. $\sqrt{I} = \bigcap_{P \supseteq I \text{ prime}} P$

Hilbert Nullstellensatz weak form

Theorem 29.1.3 (Hilbert Nullstellensatz weak form). k is algebraically closed, $\mathfrak{m} < k[x_1, \dots, x_n]$ is a maximal ideal, then $k[x]/\mathfrak{m} \cong k$

Theorem 29.1.4 (Hilbert Nullstellensatz strong form). k is algebraically closed, $I(V(J)) = \sqrt{J}$

Proof. Since $\sqrt{J} = \bigcap_{P \supseteq J \text{ prime}} P$, suppose $f \notin P$ for some $P \supseteq J$, consider $\varphi : k[x] \rightarrow k[x]/P \rightarrow A_{\bar{f}} \rightarrow A_{\bar{f}}/\mathfrak{m}$ which is a field, hence $\ker \varphi$ is a maximal ideal, by Theorem 29.1.3, $B/\mathfrak{m} \cong k[x]/\ker \varphi \cong k$, then $(\varphi(x_1), \dots, \varphi(x_n)) \in V(P) \subseteq V(J)$ but $f(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(f) \neq 0 \Rightarrow f \notin I(V(J))$ \square

Proposition 29.1.5. Morphism $V \xrightarrow{\varphi} W$ induce a ring homomorphism $k[W] \xrightarrow{\varphi^*} k[V]$, $f \mapsto f \circ \varphi$, and if $f(p) = q$, then $(\varphi^*)^{-1}(\mathfrak{m}_q) = \mathfrak{m}_p$, thus conversely, if $\alpha : k[W] \rightarrow k[V]$ is a ring homomorphism, then $\alpha^{-1} : \text{Spm } k[V] \rightarrow \text{Spm } k[W]$ is a morphism which can be identified with $\varphi : V \rightarrow W$, and $\varphi^* = \alpha$

Proposition 29.1.6. A finite morphism $V \xrightarrow{\varphi} W$ between affine varieties is quasifinite

Proof. $\varphi(p) = q \Leftrightarrow (\varphi^*)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_q$, $\mathfrak{m}_p \supseteq \varphi^*(\varphi^*)^{-1}(\mathfrak{m}_p) = \varphi^*(\mathfrak{m}_q)$

$$\varphi^{-1}(q) \leftrightarrow \left\{ \text{maximal ideals of } B = \frac{k[W]}{\langle \varphi^*(\mathfrak{m}_q) \rangle} \right\}$$

Since $k[W]$ is a finite $k[V]$ algebra, so B is finite dimensional over $\frac{k[V]}{\mathfrak{m}_p} \cong k$. By Chinese Remainder theorem 5.2.11, $B \rightarrow B/\mathfrak{m}_1 \times \dots \times B/\mathfrak{m}_s$ is surjective, $\dim B \geq s$, since $\dim B < \infty$, hence $s < \infty$, thus B has only finitely many maximal ideals \square

$W \rightarrow V$ dominant $\Rightarrow k[V] \rightarrow k[W]$ injective

Proposition 29.1.7. $W \xrightarrow{\varphi} V$ is dominant iff $k[V] \xrightarrow{\varphi} k[W]$ is injective

Proof. $f \in \ker \varphi^* \Leftrightarrow f \circ \varphi = 0$, $\text{im} \varphi$ dense $\Rightarrow f = 0$. Conversely, $\overline{\text{im} \varphi} \subsetneq V \Rightarrow 0 \neq f \in I(\overline{\text{im} \varphi})$ \square

Proposition 29.1.8. If $W \xrightarrow{\varphi} V$ is dominant and finite, then φ is surjective

Proof. By Proposition 29.1.7, $k[W]$ is integral over $k[V]$, by Theorem 5.2.21, for any $m_q < k[W]$, there exists maximal ideal $n < k[W]$ such that $n \cap k[V] = m_q$ \square

Corollary 29.1.9. V is an algebraic set, $\dim V = \dim k[V]$. If V is irreducible, then $\dim V = \text{trdeg } k(V)$

Example 29.1.10. $\dim \mathbb{A}^n = \dim k[x_1, \dots, x_n] = \text{trdeg}(k(x_1, \dots, x_n)/k) = n$

Definition 29.1.11. V is an algebraic set, a **regular function** on $U \subseteq V$ is $\frac{f}{g}$, $f, g \in k[V]$ such that g doesn't vanish on U , i.e. a rational function that is regular on U

29.2 Varieties

Definition 29.2.1. A **prevariety** is a locally ringed space (X, \mathcal{O}) such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine variety $(V, \mathcal{O}_{\text{spm}V})$

Definition 29.2.2. A morphism $W \xrightarrow{\varphi} V$ is **dominant** if $\varphi(W)$ is dense

Definition 29.2.3. A morphism $W \xrightarrow{\varphi} V$ is **quasifinite** if $\varphi^{-1}(p)$ is finite for any $p \in V$

Definition 29.2.4. A morphism $W \xrightarrow{\varphi} V$ is **finite** if $k[W]$ is finite $k[V]$ algebra

Proposition 29.2.5. A finite morphism is quasifinite

Proposition 29.2.6. A variety is an integral scheme X over k such that $X \rightarrow \text{Spec } k$ is separated and of finite type

Chapter 30

Scheme

30.1 Affine schemes

Definition 30.1.1. An **affine scheme** is a ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

Lemma 30.1.2. The inclusion of $\operatorname{Spec} k(p) \rightarrow \operatorname{Spec} A$ is given by $A \rightarrow A_p \rightarrow k(p)$

30.2 Schemes

Definition 30.2.1. A **scheme** is a ringed space (X, \mathcal{O}) such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine scheme $(\text{Spec } R, \mathcal{O}_{\text{Spec } R})$

Definition 30.2.2. We say X is a scheme over Y if there is a morphism $X \rightarrow Y$, X is a scheme over R if there is a morphism $X \rightarrow \text{Spec } R$

An R point is a morphism $\text{Spec } R \rightarrow X$, we also write the set of R points as $X(R)$. If S is a commutative R algebra, then the set of S points $X(S)$ consists of morphisms $\text{Spec } S \rightarrow X$ over $\text{Spec } R$

$X(S)$ can also be constructed as the base change X_S

$$\begin{array}{ccc} X_S & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec } S & \longrightarrow & \text{Spec } R \end{array}$$

Definition 30.2.3. A scheme X is **reduced/integral** if $\mathcal{O}(U)$ is reduced/integral for any open subset U

Definition 30.2.4. A morphism $f : X \rightarrow Y$ is **separated** if $\Delta(X)$ is closed, $\Delta : X \rightarrow X \times_Y X$ is the diagonal

Definition 30.2.5. A morphism $f : X \rightarrow Y$ is of **finite type** if Y has an affine open cover Y_i such that there is an affine open cover X_{ij} of $f^{-1}(Y_i)$ such that $f|_{X_{ij}} : X_{ij} \rightarrow Y_i$ are of finite type

Part XII

Analysis

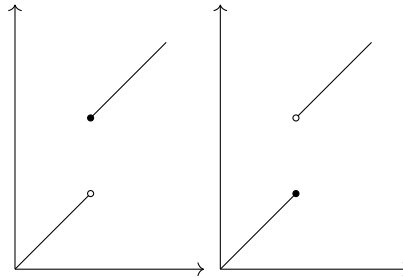
Chapter 31

Real analysis

Definition 31.0.1 (Hyperbolic functions). $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ $\sinh z = -i \sin(iz) = \frac{e^z - e^{-z}}{2}$, $\cosh z = \cos(iz) = \frac{e^z + e^{-z}}{2}$

Definition 31.0.2. X is a convex set, $X \xrightarrow{f} \mathbb{R}$ is **convex** if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for $0 \leq t \leq 1$ and $x, y \in X$, f is **strictly convex** if $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$ for $0 < t < 1$ and $x \neq y \in X$. f is **concave** if $-f$ is convex

Definition 31.0.3. $X \xrightarrow{f} [-\infty, \infty]$ is **upper semicontinuous** at x if for any $y > f(x)$, there exists a neighborhood U of x such that $f(U) < y$, i.e. f can only jump down at x . Thus $X \xrightarrow{f} [-\infty, \infty]$ is upper semicontinuous if $\{f < a\}$ are open. $X \xrightarrow{f} [-\infty, \infty]$ is **lower semicontinuous** if $-f$ is upper semicontinuous, i.e. f can only jump up



Lemma 31.0.4. $\{f_\alpha\}_{\alpha \in A}$ is a family of upper semicontinuous functions, $f = \inf_{\alpha \in A} f_\alpha$ is also upper semicontinuous

Proof.

$$\{f < a\} = \bigcup_{\alpha \in A} \{f_\alpha < a\}$$

□

Lemma 31.0.5. f is upper semicontinuous, K is compact, then f attains maximum over K

Definition 31.0.6. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **harmonic** at $x \in \Omega$ if u is continuous at x and for any ball $B(x, r)$, $u(x) = \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy$. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **subharmonic** at $x \in \Omega$ if u is upper semicontinuous at x and for any ball $B(x, r)$, any continuous v harmonic on $B(x, r)$, $u \leq v$ on $\partial B(x, r) \Rightarrow u \leq v$ on $\overline{B(x, r)}$. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **superharmonic** if $-u$ is subharmonic. Harmonic \Leftrightarrow subharmonic and superharmonic

Lemma 31.0.7. $\Omega \xrightarrow{u} \mathbb{R}$ is subharmonic, $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is convex, then $f \circ u$ is also subharmonic
 f holomorphic $\Rightarrow \log|f|$ subharmonic

Example 31.0.8. If f is holomorphic, then $\log|f|$ is subharmonic

Chapter 32

Complex analysis

32.1 Complex analysis

Definition 32.1.1. A polydisc $D(z, r) \subseteq \mathbb{C}^n$ is $D(z_1, r_1) \times \cdots \times D(z_n, r_n)$

Definition 32.1.2 (Wirtinger derivatives).

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Note.

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \frac{\partial y}{\partial z}, \quad \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y} \frac{\partial y}{\partial \bar{z}} \\ dz \wedge d\bar{z} &= -2i dx \wedge dy \end{aligned}$$

Definition 32.1.3. $f : \Omega \rightarrow \mathbb{C}$ is **holomorphic** at $z_0 \in \Omega$ if $f'(z)$ exists around z_0 . f is **univalent** if f is injective

Theorem 32.1.4 (Cauchy-Riemann equations). If we write $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, then the existence of $f'(z)$ implies that $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$ which give the **Cauchy-Riemann equations**

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

If f satisfies Cauchy-Riemann equations around z_0 , then f is holomorphic at z_0

Lemma 32.1.5. A univalent map is a biholomorphism to its image

Theorem 32.1.6 (Goursat). If f is holomorphic on $\Omega \subseteq \mathbb{C}$, $\bar{T} \subseteq \Omega$ is a triangle, then $\oint_T f(z) dz = 0$

Theorem 32.1.7 (Cauchy's integral theorem). If f is holomorphic on $\Omega \subseteq \mathbb{C}$, $\gamma \subseteq \Omega$ is a piecewise C^1 curve, then $\oint_\gamma f(z) dz = 0$

Theorem 32.1.8 (Morera's theorem). $U \subseteq \mathbb{C}$ is open, if $\oint_T f(z) dz = 0$ for any triangle $T \subseteq U$, then f is holomorphic on D

Cauchy-Pompeiu formula

Theorem 32.1.9 (Cauchy-Pompeiu formula). f is a complex valued C^1 function on a disc $D \subseteq \mathbb{C}$, then

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z) dz}{z - \zeta} - \frac{1}{\pi} \iint_D \frac{\partial f(z)}{\partial \bar{z}} \frac{dx \wedge dy}{z - \zeta}$$

In particular, if f is holomorphic, then

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - \zeta} dz$$

Proof. Denote $D_\epsilon = D - B(0, \epsilon)$, consider

$$\eta = \frac{f(w)dw}{w - z}, d\eta = \frac{\partial f(w)}{\partial \bar{w}} \frac{d\bar{w} \wedge dw}{w - z}$$

By Stokes' theorem

$$\frac{1}{2\pi i} \int_{\partial D_\epsilon} \eta = \frac{1}{2\pi i} \int_{D_\epsilon} d\eta$$

As $\epsilon \searrow 0$

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)dw}{w - z} + \frac{1}{2\pi i} \iint_D \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}$$

□

Osgood's lemma

Lemma 32.1.10 (Osgood's lemma). f is continuous on an open subset $\Omega \subseteq \mathbb{C}^n$ and holomorphic on each variable, then f is holomorphic

Proof. For each $a \in \Omega$, pick $P = D(a, r) \subseteq \Omega$, since $\frac{\partial f}{\partial \bar{z}_j} \equiv 0$ on Ω , fix z_2, \dots, z_n , then

$$f(w_1, z_2, \dots, z_n) = \frac{1}{2\pi i} \int_{|z_1 - a_1| = r_1} \frac{f(z_1, \dots, z_n)}{z_1 - w_1} dz_1$$

For $w_1 \in D(a_1, r_1)$, iterate and we get

$$f(w_1, \dots, w_n) = \frac{1}{(2\pi i)^n} \int_{|z_1 - a_1| = r_1} \dots \int_{|z_n - a_n| = r_n} \frac{f(z_1, \dots, z_n)}{\prod (z_j - w_j)} dz_1 \dots dz_n$$

For $w \in P$. Since f is continuous, it is bounded on \bar{P} , $\frac{1}{z_j - w_j} = \sum_{m=0}^{\infty} \frac{(w_j - a_j)^m}{(z_j - a_j)^{m+1}}$ converges uniformly on compact subsets of $D(a_j, r_j)$. Hence $f(w) = \sum c_\alpha (w - a)^\alpha$, where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|z_1 - a_1| = r_1} \dots \int_{|z_n - a_n| = r_n} \frac{f(z)}{\prod (z_j - a_j)^{\alpha_j + 1}} dz_1 \dots dz_n$$

□

Corollary 32.1.11 (Cauchy inequality).

Maximum principle

Theorem 32.1.12 (Maximum principle).

Theorem 32.1.13. $\{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$, f_n are uniformly convergent on each compact subset, then f_n converges to a holomorphic function f , and $D^\alpha f_n \rightarrow D^\alpha f$ on each compact subset

Montel's theorem

Theorem 32.1.14 (Montel's theorem). $\mathcal{F} = \{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$ and locally uniformly bounded, i.e. for any $z_0 \in \Omega$, there exists a neighborhood U and M such that $\sup_{z \in K} |f_n| \leq M$, then \mathcal{F} is normal

Schwarz lemma

Lemma 32.1.15 (Schwarz lemma). f is holomorphic on the unit disc $D \subseteq \mathbb{C}$, $f(0) = 0$ and $|f| \leq 1$ on D , then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then $f(z) = az$, $a = f'(0)$

Proof. Define $g(z) = \frac{f(z)}{z}$, since $f(0) = 0$, 0 is a removable singularity, since $|f(z)| \leq 1$, $|g(z)| \leq 1$ on ∂D , by maximum principle 32.1.12, $|g(z)| \leq 1$ on D , thus $|f(z)| \leq |z|$ on D and $|f'(0)| = |g(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then g attains maximum within D , then $g \equiv a$ for some $|a| = 1$, thus $f(z) = az$ \square

Corollary 32.1.16. $D \xrightarrow{f} D$ is a biholomorphic, then $f = e^{i\phi} \frac{z-a}{1-\bar{a}z}$ for some ϕ and $a \in D$

Proof. Denote $\psi_a(z) = \frac{z-a}{1-\bar{a}z}$, ψ_{-a} is the inverse of ψ_a

Assume $f(a) = 0$, consider $g(z) = f \circ \psi_{-a}$, then $g(0) = 0$, by Schwarz lemma 32.1.15, $g = e^{i\phi}$, $f = g \circ \phi_a = e^{i\psi} \frac{z-a}{1-\bar{a}z}$ \square

Lemma for Riemann mapping theorem

Lemma 32.1.17. Suppose $0 \in U \subsetneq D$ is a simply connected open set, there exists $U \xrightarrow{f} D$ univalent such that $f(0) = 0$, $|f'(0)| > 1$. Note that this is impossible if $U = D$ due to Schwarz lemma 32.1.15

Proof. Denote $\psi_a(z) = \frac{z-a}{1-\bar{a}z}$, $\psi'_a(z) = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. Consider $f = \psi_{g(a)} \circ g \circ \psi_{-a}$ with some $\psi_{-a}(U) \xrightarrow{g} D$ univalent, then $f(0) = 0$

$$f'(0) = \frac{1-|g(a)|^2}{(1-|g(a)|^2)^2} g'(a)(1-|a|^2) = \frac{1-|a|^2}{1-|g(a)|^2} g'(a)$$

Since U is simply connected, so is $\psi_{-a}(U)$ given $-a \in D \setminus U$, we can take $g(z) = \sqrt{z}$ to be one branch, since $|a| < 1$, we get

$$|f'(0)| = \frac{1-|a|^2}{1-|a|} \frac{1}{2\sqrt{|a|}} = \frac{1+|a|}{2\sqrt{|a|}} > 1$$

\square

Lemma for finding zeros

Lemma 32.1.18. φ is holomorphic on D , f is meromorphic on D and $f \neq 0$ on ∂D , a_1, \dots, a_m and b_1, \dots, b_n are the zeros and poles of order k_1, \dots, k_m and l_1, \dots, l_n of f in D , then

$$\frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{f'(z)}{f(z)} dz = \sum_{i=1}^m k_i \varphi(a_i) - \sum_{i=1}^n l_i \varphi(b_i)$$

Proof. $f(z) = g(z) \prod_{i=1}^m (z-z_i)^{q_i}$ with $g \neq 0$ on \bar{D} , z_i, q_i could be a_i, k_i or $b_i, -l_i$ depending on whether it is a zero or a pole, hence

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{f'(z)}{f(z)} dz &= \frac{1}{2\pi i} \int_{\partial D} \varphi(z) \frac{g'(z) \prod_{i=1}^m (z-z_i) + g(z) \sum_{i=1}^m \prod_{j \neq i} (z-z_j)}{g(z) \prod_{i=1}^m (z-z_i)} dz \\ &= \frac{1}{2\pi i} \int_{\partial D} \left[\frac{\varphi(z)g'(z)}{g(z)} + \sum_{i=1}^m \frac{\varphi(z)}{z-z_i} \right] dz \\ &= \sum_{i=1}^m k_i \varphi(a_i) - \sum_{i=1}^n l_i \varphi(b_i) \end{aligned}$$

\square

Rouché's theorem

Theorem 32.1.19 (Rouché's theorem).

Hurwitz's theorem

Theorem 32.1.20 (Hurwitz's theorem). $U \subseteq \mathbb{C}$ is open connected, holomorphic functions $\{f_n\}$ converges uniformly to f on compact subsets of U and $f \not\equiv 0$, f has order m at z_0 , for r small enough, there exists K such that for any $k \geq K$, f_k has precisely m zeros in $B(z_0, r)$, counting multiplicities, and these zeros converge to z_0 as $k \rightarrow \infty$

Remark 32.1.21. $B(z_0, r)$ can't be arbitrarily large. For example, $f_n(z) = z - 1 + \frac{1}{n}$ converges uniformly to $f(z) = z - 1$ on compact subsets, f has no zeros in the unit disc D , but f_n all have zeros in D

Proof. For r small enough, f doesn't vanish on $\partial B(z_0, r)$ on which $|f|$ attains minimum, then apply Rouché's theorem 32.1.19 \square

Corollary 32.1.22. U is open connected, univalent maps $\{f_n\}$ converges to f on compact subsets, then f is either univalent or constant

Proof. If f is not a constant and $f(z_0) = f(w_0) = \zeta$, then $f(z) - \xi$ has z_0, w_0 as zeros, by Hurwitz's theorem 32.1.20, there exist $\{z_k\}, \{w_k\}$ converging to z_0, w_0 such that $f_{n_k}(z_k) = f_{n_k}(w_k) = \xi$, but f_n 's are univalent, hence $z_k = w_k \Rightarrow z_0 = w_0$, i.e. f is univalent \square

Riemann mapping theorem

Theorem 32.1.23 (Riemann mapping theorem). $U \subsetneq \mathbb{C}$ is a nonempty simply connected open subset, $z_0 \in U$, then there is a unique biholomorphism f from U to the unit disc such that $f(z_0) = 0$, $f'(z_0) > 0$

Proof of uniqueness. Suppose $U \xrightarrow{f_1, f_2} D$ are biholomorphisms such that $f_i(z_0) = 0$, $f'_i(z_0) > 0$, consider $g = f_2 f_1^{-1}$, $g(0) = 0$, $|g| \leq 1$ on D and $g'(0) = \frac{f'_2(z_0)}{f'_1(z_0)} > 0$, by Schwarz lemma 32.1.15, $g(z) = z$, i.e. $f_1 = f_2$ \square

Proof of existence. Fix $a \notin U$, $z_0 \in U$. Define

$$\mathcal{F} = \{f \text{ univalent on } U \mid |f| \leq 1, f(z_0) = 0\}$$

Since U is simply connected, we can pick one branch $h(z) = \sqrt{z - a}$, then $h(U) \cap -h(U) = \emptyset$, $\frac{h(z) - h(z_0)}{h(z) + h(z_0)}$ is univalent and bounded, scale to get some $f_0 \in \mathcal{F} \Rightarrow \mathcal{F}$ is nonempty

Let $A = \sup_{f \in \mathcal{F}} |f'(z_0)| > 0$, $f'_n(z_0) \rightarrow A$ for some $\{f_n\} \subseteq \mathcal{F}$, by Montel's theorem 32.1.14, f_{n_k} converges to g uniformly on compact subsets, then $|g| \leq 1$, $g(z_0) = 0$ and $0 < A = |g'(z_0)| < \infty$, according to Hurwitz's theorem 32.1.20, g is also univalent, i.e. $g \in \mathcal{F}$ attains maximal derivative at z_0

Suppose $0 \in g(U) \subsetneq D$, if not, by Lemma 32.1.17, there exists univalent map $g(U) \xrightarrow{f} D$ such that $f(0) = 0$, $|f'(0)| > 1$, then $f \circ g \in \mathcal{F}$, but $|(f \circ g)'(z_0)| = |f'(0)g'(z_0)| > |g'(z_0)|$ which is a contradiction \square

Remark 32.1.24. Suppose $f_1, f_2 \in \mathcal{F}$ and f_1 is biholomorphic, then $g = f_2 f_1^{-1}$ is a map $D \rightarrow D$, with $g(0) = 0$, according to Schwarz lemma 32.1.15, $\frac{|f'_2(z_0)|}{|f'_1(z_0)|} = |g'(0)| \leq 1$, and if $|f'_2(z)| = |f'_1(z)|$, $g = e^{i\phi}$, f_2 is also biholomorphic

Example 32.1.25. $U = \mathbb{C} - \{z \geq 0\}$, then $h(z) = \sqrt{z}$ maps U to the upper half plane

Theorem 32.1.26 (Runge's theorem). $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ is the union of its connected components whereas the components are either bounded or not, denote

Hartogs's extension theorem

Theorem 32.1.27 (Hartogs's extension theorem). An isolated singularity is always a removable singularity when $n \geq 2$

Proof. It suffices to consider the case $P = \{|z_1| \leq 1, |z_2| \leq 1\}$ is a polydisc, f is holomorphic on ∂P , then f is holomorphic on P \square

Lemma for Remmert-Stein theorem

Lemma 32.1.28. $\Omega \subseteq \mathbb{C}^n$ is connected, $\Omega \xrightarrow{f} \partial B^n$ is holomorphic, then $f \equiv \text{const}$

Proof. If h is holomorphic, then $\frac{\partial^2}{\partial z \partial \bar{z}}|h|^2 = |h'|^2$, hence

$$0 = \frac{\partial^2}{\partial z \partial \bar{z}}|f|^2 = \sum_{i=1}^n \frac{\partial^2}{\partial z \partial \bar{z}}|f_i|^2 = \sum_{i=1}^n |f'_i(z)|^2 \Rightarrow f'_i(z) = 0 \Rightarrow f \equiv \text{const}$$

□

Theorem 32.1.29 (Riemann-Stein). $U_1 \subseteq \mathbb{C}^{n_1}, U_2 \subseteq \mathbb{C}^{n_2}$ are nonempty connected open subsets, $B = \{|z| < 1\} \subseteq \mathbb{C}^n$, then there is no proper holomorphic map $U_1 \times U_2 \rightarrow B$

Proof. Suppose $f : U_1 \times U_2 \rightarrow B$ is a proper holomorphic map. For any $(x, y) \in U_1 \times \partial U_2$, there is a discrete sequence $\{y_\nu\} \subseteq U_2$ converging to y as in Exercise 45.2.1, apply Lemma 14.1.31 to $f(x, y) : \{x\} \times U_2 \rightarrow B$, $\{f(x, y_\nu)\}$ is discrete, thus there exists a subsequence $\{y_\mu\} \subseteq \{y_\nu\}$ such that $f(x, y_\mu)$ such that $f(x, y) = \lim f(x, y_\mu) \in \partial B$. Then $f(x, y) : U_1 \times \{y\} \rightarrow \partial B$ is a holomorphic, by Lemma 32.1.28, $f(x, y)$ is constant on $U_1 \times \{y\}$, hence $U_1 \times \{y\} \subseteq f^{-1}(f(x, y))$ which is noncompact since it has noncompact image under projection to U_1 . This contradicts the fact that f is proper □

Corollary 32.1.30 (Poincaré). The 2 polydisc $P = \{|z_1| < 1, |z_2| < 1\}$ and the 2 ball $B = \{|z_1|^2 + |z_2|^2 < 1\}$ are not biholomorphic

Theorem 32.1.31 (Weierstrass preparation theorem). f is analytic near 0, $f(0) = 0$, $f(z)$ written as power series around 0 has terms only involve z_1 which can always be achieved by a change of variables as in Exercise 45.2.2, then $f = wh$, where $w(z) = z_1^k + g_{k-1}z^{k-1} + \cdots + g_0$ is a **Weierstrass polynomial**, i.e. $g_i(z)$ are analytic around 0 and $g_i(0) = 0$, $h(z)$ is analytic around 0 and $h(0) \neq 0$

Theorem 32.1.32 (Weierstrass division theorem). Suppose f, g are analytic near 0, g is a Weierstrass polynomial of degree k , then there exist unique h, r such that $f = gh + r$, where r is a polynomial of degree less than k

32.2 Conformal mapping

Definition 32.2.1. A conformal mapping is a map preserves angles and orientation

Note. Antiholomorphic map preserves angles but changes orientation

Definition 32.2.2. Möbius transformations are $f(z) = \frac{az + b}{cz + d}$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, Möbius group

acts regularly on \mathbb{CP}^1 and preserves cross ratio $(z_0, z_1; z_2, z_3) = \frac{(z_2 - z_0)(z_3 - z_1)}{(z_3 - z_0)(z_2 - z_1)}$
Schwarz reflection principle

Lemma 32.2.3 (Schwarz reflection principle). If f is holomorphic on $\{\operatorname{Im} z > 0\}$ and continuous on $\{\operatorname{Im} z \geq 0\}$ with real values on $\operatorname{Im} z = 0$, then it can be extended to \mathbb{C} with $f(\bar{z}) = \overline{f(z)}$ for $\operatorname{Im} z < 0$

32.3 Weierstrass functions

Definition 32.3.1. $\Lambda \subseteq \mathbb{C}$ is a lattice

The **Weierstrass σ -function** associated to lattice Λ is

$$\sigma(z) = z \prod_{\omega \in \Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

The **Weierstrass \wp -function** associated to lattice Λ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right)$$

Hence

$$\wp'(z) = - \sum_{\omega \in \Lambda} \frac{2}{(z + \omega)^3}$$

32.4 Zeta function

Theorem 32.4.1 (Euler's reflection formula). $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, $z \notin \mathbb{Z}$

Definition 32.4.2. Polylogarithm is $\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$, $\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} dt$, $\text{Li}_1(z) = -\ln(1-z)$, $\text{Li}_2(z)$ is the **dilogarithm**, $\text{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du$ is the analytic continuation on $\mathbb{C} \setminus \{0, 1\}$, the integration avoid the the cut $[1, \infty]$
 The **Bloch-Wigner function** is $D_2(z) = \text{Im}(\text{Li}_2(z)) + \arg(1-z) \ln |z|$, $z \in \mathbb{C} \setminus \{0, 1\}$

Chapter 33

Functional analysis

33.1 Topological vector space

Definition 33.1.1. A **topological vector space** V over a topological field \mathbb{F} is a topological abelian group such that scalar multiplication $\mathbb{F} \times V \rightarrow V$ is continuous

Definition 33.1.2. A **norm** on a group G is $G \xrightarrow{\|\cdot\|} \mathbb{R}_{\geq 0}$ such that $\|g\| = 0 \Leftrightarrow g = \text{id}$, $\|g^{-1}\| = \|g\|$, $\|gh\| \leq \|g\| \|h\|$

A **norm** on a rng R is a normed abelian group such that $\|rs\| \leq \|r\| \|s\|$

A **norm** on a vector space V over a normed field is a normed abelian group such that $\|kv\| \leq \|k\| \|v\|$

Definition 33.1.3. A **Banach space** is a complete normed vector space

Definition 33.1.4. Y is a topological vector space, T is a set, $\mathcal{G} \subseteq \mathcal{P}(T)$ is a directed set by inclusion, \mathcal{N} is a local base around $0 \in Y$. The **topology of uniform convergence** on sets in \mathcal{G} or \mathcal{G} **topology** is the unique translation invariant topology given by basis

$$U(G, N) = \{f \in Y^T \mid G \in \mathcal{G}, N \in \mathcal{N}, f(G) \subseteq N\}$$

Example 33.1.5. \mathcal{G} is the set of compact subspaces, Y is a metric space

33.2 Arzela-Ascoli theorem

Definition 33.2.1. Let X, Y be a topological spaces, a family of continuous functions $A \subseteq Y^X$ is equicontinuous at $x \in X$, if for any open neighborhood V of $y = f(x)$, there is an open neighborhood U of x such that $f(U) \subseteq V, \forall f \in A$

Definition 33.2.2. A topological space X is called separable if X has a countable dense subset

Arzela-Ascoli theorem

Theorem 33.2.3. Let X be a topological space and Y be a complete metric space, $A \subseteq Y^X$ be a family of equicontinuous functions (meaning pointwise equicontinuous). If X is compact, and $A_x := \{f(x) | f \in A\} \subseteq Y$ is relatively compact for any $x \in X$, then A is relatively compact in Y^X . If X is separable with S being a countable dense subset, and A_x is relatively compact for any $x \in S$, then any sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ converges uniformly on any compact subset of X

33.3 Baire category theorem

Definition 33.3.1. A topological space X is a **Baire space** if for any countable open dense subsets $\{U_i\}$, $\bigcap_{i=1}^{\infty} U_i$ is also dense

Baire category theorem

Theorem 33.3.2 (Baire category theorem). *Every complete metric space X is a Baire space*

Proof. Let $\{U_i\}$ be a countable open dense subsets, suppose $\bigcap_{i=1}^{\infty} U_i$ is not dense, then the complement of its closure is open nonempty, suppose $B(x, r)$ is in the complement of the closure, since U_1 is dense, $U_1 \cap B(x, r) \neq \emptyset$, then there exists $\overline{B(x_1, r_1)} \subseteq U_1 \cap B(x, r)$, similarly, we can find $\overline{B(x_n, r_n)} \subseteq U_n \cap B(x_{n-1}, r_{n-1})$, and we can also assume $r_i \rightarrow 0$, thus $x_i \rightarrow y \in X$ since X is complete, but $y \in B(x, r) \bigcap \bigcap_{i=1}^{\infty} U_i = \emptyset$ which is a contradiction □

33.4 Distribution

Definition 33.4.1. $U \subseteq \mathbb{R}^n$ open, $\mathcal{D}(U) = C_c^\infty(U)$ is the **test function space**, $\{\phi_i\} \subseteq \mathcal{D}(U)$ converges if there exists $K \subseteq U$ compact such that $\text{supp}\phi_i \subseteq K$ and $\partial^\alpha \phi_i$ converges uniformly

33.5 Banach algebra

Definition 33.5.1. A **Banach algebra** is an associative algebra A which is a complete normed ring such that $\|rs\| \leq \|r\|\|s\|$. A is **unital** if A is a ring with identity element having norm 1

Definition 33.5.2. A ***-algebra** is a Banach algebra over \mathbb{C} such that there is an antilinear involution $*$: $A \rightarrow A$, such that $(xy)^* = y^*x^*$. A is a **C^* -algebra** if $\|x^*x\| = \|x\|^2$

Example 33.5.3. X is locally compact, $C_0(X)$ are the continuous functions vanishes at infinity, then $C_0(X)$ is a Banach algebra with the supremum norm, $C_0(X)$ is unital if X is compact with 1 being the identity. $C_0(X)$ is a C^* -algebra with complex conjugation as the involution

Definition 33.5.4. A is a unital Banach algebra over \mathbb{R}, \mathbb{C} , $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ defines the **exponential**

$$\|e^x\| = \left\| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|x\|^k}{k!} = e^{\|x\|}$$

The **logarithm** $\log x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(x-1)^k}{k}$ is defined on $\|x-1\| < 1$

Lemma 33.5.5. e^x and $\log x$ are inverses to each other locally

Proposition 33.5.6. A is a Banach algebra, linear map $D : A \rightarrow A$ is a derivation iff e^{tD} is a group of automorphisms

Lie product formula

Theorem 33.5.7 (Lie product formula). $e^{A+B} = \lim_{n \rightarrow \infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n$

Lie commutator formula

Theorem 33.5.8 (Lie commutator formula). $e^{[A,B]} = \lim_{n \rightarrow \infty} \left[e^{\frac{A}{n}}, e^{\frac{B}{n}} \right]^{n^2}$, the left and right $[\cdot, \cdot]$ are Lie bracket and commutator

Lemma 33.5.9. If $[X, [X, Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$

Proof. Let $A(t) = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}$, $B(t) = e^{t(X+Y)}$, then $A(0) = B(0)$, $B'(t) = B(t)(X+Y)$ and

$$A'(t) = e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} - e^{tX} e^{tY} t[X, Y] e^{-\frac{t^2}{2}[X,Y]}$$

Since $[X, [X, Y]] = 0$, $[Y, [X, Y]] = -[Y, [Y, X]] = 0$

$$e^{-tY} X e^{tY} = \text{Ad}_{e^{-tY}}(X) = e^{ad_{-tY}}(X) = X + t[X, Y]$$

$$A'(t) = e^{tX} e^{tY} (X + Y) e^{-\frac{t^2}{2}[X,Y]} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} (X + Y) = A(t)(X + Y)$$

Thus $A(t), B(t)$ satisfies the same ODE and initial condition, $A(t) = B(t) \Rightarrow e^X e^Y = A(1) = B(1) = e^{X+Y+\frac{1}{2}[X,Y]}$ □

Theorem 33.5.10 (Baker-Campbell-Hausdorff formula). $e^X e^Y = e^Z$ around 0, where $Z =$

$$X + \int_0^1 \psi(e^{ad_X} e^{tad_Y}) dt(Y) \text{ and}$$

$$\begin{aligned} \psi(x) &= \frac{x \log x}{x-1} \\ &= \frac{\frac{y=1-x}{-y} (1-y) \log(1-y)}{-y} \\ &= (1-y) \sum_{n=1}^{\infty} \frac{y^{n-1}}{n} \\ &= \sum_{n=1}^{\infty} \frac{y^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{y^n}{n} \\ &= 1 + \sum_{n=1}^{\infty} \left(\frac{y^n}{n+1} - \frac{y^n}{n} \right) \\ &= 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)} \end{aligned}$$

The first few terms are

$$e^X e^Y = e^{X+Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \dots}$$

Proof. The Riemann sum $\sum_{k=0}^{m-1} \frac{1}{m} e^{-\frac{kx}{m}}$ converges to $\int_0^1 e^{-tx} dt = \frac{1-e^{-x}}{x}$, thus

$$\lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} e^{-\frac{kx}{m}} = \frac{1-e^{-x}}{x}, \text{ we have}$$

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} e^{X+tY} &= \left. \frac{d}{dt} \right|_{t=0} \left(e^{\frac{X}{m}} e^{\frac{tY}{m}} \right)^m \\ &= \lim_{m \rightarrow \infty} \left. \frac{d}{dt} \right|_{t=0} \left(e^{\frac{X}{m}} e^{\frac{tY}{m}} \right)^m \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} e^{\frac{kX}{m}} \frac{Y}{m} e^{\frac{(m-k)X}{m}} \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \frac{1}{m} e^{\frac{kX}{m}} Y e^{-\frac{kX}{m}} e^X \\ &= \left(\lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \frac{1}{m} e^{\frac{kad_X}{m}} \right) (Y) e^X \\ &= \frac{e^{ad_X} - 1}{ad_X} (Y) e^X \end{aligned}$$

$$\begin{aligned} \text{Let } e^{Z(t)} &= e^X e^{tY}, \quad \frac{d}{dt} e^{Z(t)} = \frac{d}{dt} (e^X e^{tY}) = e^X e^{tY} Y = e^{Z(t)} Y, \text{ but } \frac{d}{dt} e^{Z(t)} = \left. \frac{d}{ds} \right|_{s=t} e^{Z(s)} = \\ \left. \frac{d}{ds} \right|_{s=t} e^{Z(t)+Z'(t)(s-t)} &= \frac{e^{ad_{Z(t)}} - 1}{ad_{Z(t)}} (Z'(t)) e^{Z(t)}, \text{ hence } \frac{e^{ad_{Z(t)}} - 1}{ad_{Z(t)}} (Z'(t)) = e^{Z(t)} Y e^{-Z(t)} = \\ Ad_{e^{Z(t)}}(Y) &= e^{ad_{Z(t)}}(Y), \quad Z'(t) = \frac{ad_{Z(t)} e^{ad_{Z(t)}}}{e^{ad_{Z(t)}} - 1} (Y), \text{ since } e^{ad_{Z(t)}} = Ad_{e^{Z(t)}} = Ad_{e^X e^{tY}} = e^{ad_X} e^{tad_Y} \end{aligned}$$

$$\begin{aligned} Z &= Z(1) \\ &= Z(0) + \int_0^1 \frac{ad_{Z(t)} e^{ad_{Z(t)}}}{1 - e^{-ad_{Z(t)}}} (Y) dt \\ &= X + \int_0^1 \frac{e^{ad_X} e^{tad_Y} \log(e^{ad_X} e^{tad_Y})}{e^{ad_X} e^{tad_Y} - 1} dt(Y) \end{aligned}$$



33.6 Stone-Weierstrass theorem

Definition 33.6.1. $\mathcal{F} = \{f_i\}$ is a family of functions on X , \mathcal{F} **separates points** in X if for any $x \neq y \in X$, some f_i separates x, y

Theorem 33.6.2. X is compact Hausdorff, $A \subseteq C(X, \mathbb{R})$ is a unital subalgebra. A is dense in $C(X, \mathbb{R})$ with the topology of uniform convergence iff A separates points
 $S \subseteq C(X, \mathbb{C})$ is a unital $*$ -algebra that separating points, then S is dense in $C(X, \mathbb{C})$

Part XIII

Differential equations

Chapter 34

Ordinary differential equations

Theorem 34.0.1. *Linear differential equations $y'(t) = A(t)y(t) + b(t)$ with initial condition $y(0) = y_0$, where A, b, y are smooth, then there exists unique local solution*

Proof. Define $Ty(t) = \int_0^t A(s)y(s) + b(s)ds + y_0$, note that $\|Ty(t) - Tz(t)\| = \left\| \int_0^t A(s)(y(s) - z(s))ds \right\| \leq |t| \|A\| \|y - z\|$, then there exists $\delta > 0$ such that $|t| \|A\| < 1, \forall |t| \leq \delta$, here we use $\|\cdot\|$ to denote the supremum norm in $|t| \leq \delta$, by Banach fixed point theorem, we have a unique local solution \square

Example 34.0.2. $v'(t) = Av(t), v(0) = v_0, A \in M_n(\mathbb{C})$, the solution is $v(t) = e^{tA}v_0$ since $\frac{d}{dt}A^{tA} = Ae^{tA}$

Theorem 34.0.3. (Picard-Lindelöf theorem) ^{Picard-Lindelöf theorem} Suppose $f(y, t)$ is uniformly Lipschitz continuous in y and continuous in t , then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a unique solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Remark 34.0.4. $f(y, t)$ is Lipschitz continuous in y and continuous in t would imply local uniformly Lipschitz in y and f uniformly continuous

When you have a local solution, you can try to extend it to a maximal length, i.e. $y(t)$ is defined on $(a, b) \supset [-\varepsilon, \varepsilon]$, it is open precisely because of the theorem

Proof. Define $Ty(t) = \int_0^t f(y(s), s)ds$, then $\|Ty - Tz\| = \left\| \int_0^t f(y(s), s) - f(z(s), s)ds \right\| \leq \left\| C \int_0^t |y(s) - z(s)|ds \right\| \leq C|t| \|y - z\|$, then there exists $\varepsilon > 0$ such that $C|t| < 1, \forall t \in [-\varepsilon, \varepsilon]$, then by Banach fixed point theorem, we have a unique local solution \square

Theorem 34.0.5. (Peano existence theorem) ^{Peano existence theorem} Let $f(y, t)$ be a continuous function around $(y_0, 0)$, then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a local solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Proof. Say $|f| \leq M$ around $(y_0, 0)$, Define $\phi_n(t) = \begin{cases} y_0 & , x \leq 0 \\ y_0 + \int_0^t \phi_n \left(s - \frac{\varepsilon}{n} \right) ds & , 0 \leq x \leq \varepsilon \end{cases}$ for $n \geq 1$

By Arzelà-Ascoli theorem, we know that there is a subsequence ϕ_{n_k} converges on $[-\varepsilon, \varepsilon]$, and the limit $\phi(t)$ satisfies $\phi(t) = y_0 + \int_0^t \phi_n(s) ds$ which is a local solution to the problem \square

Remark 34.0.6. The uniqueness may fail without the Lipschitz condition in y , for example, consider $\frac{dy}{dt} = y^{\frac{1}{3}}$, $y(0) = 0$ has solutions $y(t) = 0$ or $y(t) = \pm \left(\frac{2}{3}t\right)^{\frac{3}{2}}$

Chapter 35

Classical partial differential equations

35.1 Laplace's equation

35.2 Heat equation

Definition 35.2.1. The fundamental solution to solution to **heat equation** $u_t - \Delta u = 0$ is

$$E(x, t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Theorem 35.2.2. $U \subseteq \mathbb{R}^n$ is open and bounded, $f \in C_c^1(U \times (0, T])$, then

$$u(x, t) = \int_{\mathbb{R}^{n+1}} E(x - y, t - s) f(s, y) ds dy$$

Satisfies

$$\left(\frac{\partial}{\partial t} - \Delta \right) u(x, t) = f(x, t)$$

Where u is C^1 in t and C^2 in x

Proof. $E(x, t)$ is supported in $t \geq 0$ and $\int_{\mathbb{R}^n} |\nabla_x E(x, t)| dx \leq \frac{C}{\sqrt{t}}$ if $t > 0$, so $\nabla_x E(x, t)$ is integrable near $(0, 0)$

$$\begin{aligned} \nabla_x \int_{\mathbb{R}^{n+1}} E(y, s) f(x - y, t - s) ds dy &= \int_{\mathbb{R}^{n+1}} E(y, s) \nabla_x f(x - y, t - s) ds dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} E(y, s) \nabla_x f(x - y, t - s) ds dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} (\nabla E)(y, s) f(x - y, t - s) ds dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^{n+1}} (\nabla E)(y, s) f(x - y, t - s) ds dy \end{aligned}$$

And

$$\begin{aligned} \Delta \int_{\mathbb{R}^{n+1}} E(y, s) f(x - y, t - s) ds dy &= \int_{\mathbb{R}^{n+1}} (\nabla E)(y, s) \cdot (\nabla f)(x - y, t - s) ds dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} (\nabla E)(y, s) \cdot (\nabla f)(x - y, t - s) ds dy \end{aligned}$$

And

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) \int_{\mathbb{R}^{n+1}} E(y, s) f(x - y, t - s) ds dy &= - \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} (\nabla E)(y, s) \cdot (\nabla f)(x - y, t - s) ds dy \\ &\quad + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} E(y, s) \frac{\partial f}{\partial t}(x - y, t - s) ds dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\infty} \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial s} - \Delta_y \right) E(y, s) f(x - y, t - s) ds dy \\ &\quad + \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} E(y, \varepsilon) f(x - y, t - \varepsilon) ds dy \\ &= f(x, t) \end{aligned}$$

Next, let $u \in C^2(U \times (0, T])$ and $u_t - \Delta u = 0$, $\chi \in C^\infty$, $\chi(x, t) = 1$ if $d((x, t), \Gamma_U) \geq 2$, $\chi(x, t) = 0$ if $d((x, t), \Gamma_U) \leq \varepsilon$ and $(x, t) \in U \times (0, T]$, apply the previous argument to $f(x, t) = \left(\frac{\partial}{\partial t} - \Delta \right) (\chi(x, t) u(x, t)) = \left(\left(\frac{\partial}{\partial t} - \Delta \right) \chi(x, t) \right) u - 2 \nabla \chi \cdot \nabla u \in C_c^1(U \times (0, T])$, we get

$$\left(\frac{\partial}{\partial t} - \Delta \right) \left(\chi(x, t) u(x, t) - \int_{-\infty}^t \int_{\mathbb{R}^n} E(x - y, t - s) f(y, s) ds dy \right) = 0$$

And

$$u(x, t)\chi(x, t) - \int_{-\infty}^t E(x - y, t - s)f(y, s)dsdy = 0$$

if $t = 0$, so if $0 \leq t \leq T$

$$\chi(x, t)u(x, t) = \int_{-\infty}^t \int_{\mathbb{R}^n} E(x - y, t - s) \left(\frac{\partial}{\partial t} - \Delta \right) (\chi(y, s)u(y, s))dsdy$$

□

35.3 Wave equation

Definition 35.3.1. The fundamental solution to **wave equation** $\square u = \left(\frac{\partial^2}{\partial t^2} - \Delta \right) u = 0$ is

$$E(x, t) = \begin{cases} \frac{1}{2\pi^{\frac{n-1}{2}}} \chi_+^{\frac{1-n}{2}}(t^2 - |x|^2) & t > 0 \\ 0 & t < 0 \end{cases}$$

Theorem 35.3.2. $f \in C^2(\mathbb{R}^3)$, $u(x, t) = \frac{1}{4\pi t} \int_{\partial B(x, t)} f(y) dS_y = \frac{t}{4\pi} \int_{S^2} f(x + tw) dS_w$, then $u \in C^2(\mathbb{R}^3 \times [0, \infty))$, $u(x, 0) = 0$, $\frac{\partial}{\partial t} \Big|_{t=0} u(x, t) = f(x)$ and $\square u = 0$ for $t > 0$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \frac{1}{4\pi} \int_{S^2} f(x + tw) dS_w + \frac{t}{4\pi} \int_{S^2} (w \cdot \nabla) f(x + tw) dS_w \\ &= \frac{1}{4\pi} \int_{S^2} f(x + tw) dS_w + \frac{1}{4\pi t} \int_{\partial B(x, t)} n \cdot \nabla f(y) dS_y \\ &= \frac{1}{4\pi} \int_{S^2} f(x + tw) dS_w + \frac{1}{4\pi t} \int_{B(x, t)} \Delta f(y) dy \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x, t) &= \frac{1}{4\pi} \int_{S^2} (w \cdot \nabla) f(x + tw) dS_w - \frac{1}{4\pi t^2} \int_{B(x, t)} \Delta f(y) dy \\ &\quad + \frac{1}{4\pi t} \frac{d}{dt} \int_0^t \int_{S^2} \lambda^2 \Delta f(x + \lambda w) dS_w d\lambda \\ &= \frac{1}{4\pi t^2} \int_{B(x, t)} \Delta f(y) dy - \frac{1}{4\pi t^2} \int_{B(x, t)} \Delta f(y) dy \\ &\quad + \frac{t}{4\pi} \int_{S^2} \Delta f(x + \lambda w) dS_w \\ &= \frac{1}{4\pi t} \int_{\partial B(x, t)} \Delta f(y) dS_y \\ &= \Delta u(x, t) \end{aligned}$$

□

Theorem 35.3.3. $f \in C^2(\mathbb{R}^2)$, then $u(x, t) = \frac{1}{2\pi} \int_{|y| < t} \frac{1}{\sqrt{t^2 - |y|^2}} f(x - y) dy$ solves $\square u = 0$ for $t > 0$, $u(x, 0) = 0$, $u_t(x, 0) = f$

Proof. Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, $f(x_1, x_2, x_3) = f(x_1, x_2)$ is independent of x_3 , then $u(x, t) = \frac{1}{4\pi t} \int_{\partial B(x, t)} f(y) dy = \frac{1}{4\pi t} \int_{\partial B(0, t)} f(x - y) dS_y$

$$\begin{aligned} y_3 &= \pm \sqrt{t^2 - y_1^2 - y_2^2} = \gamma(y), ds = \sqrt{1 + |\nabla \gamma(y)|^2} dy_1 dy_2 = \frac{t}{t^2 - y_1^2 - y_2^2}, \text{upper + lower hemisphere} \\ &= \frac{2}{4\pi t} \int_{|(y_1, y_2)| < t} f(x - y) \frac{t dy_1 dy_2}{\sqrt{t^2 - |(y_1, y_2)|^2}} = \frac{1}{2\pi} \int_{|y| < t} \frac{1}{\sqrt{t^2 - |y|^2}} f(x - y) dy \end{aligned}$$

□

Theorem 35.3.4. $f \in C^\infty(\mathbb{R}^n \times [0, \infty))$, $u(x, t) = \int_0^t E(\cdot, t - s) * f(\cdot, s) ds$, then $\square u = f$, $u(x, 0) = u_t(x, 0) = 0$

Proof. Define $u(x, t, s) = E(\cdot, t - s) * f(\cdot, s) \in C^\infty$ for $t > s$

$$\begin{aligned}\frac{\partial}{\partial t} u(x, t) &= u(x, t, t) + \int_0^t \frac{\partial}{\partial t} u(x, t, s) ds \\ &= \int_0^t \frac{\partial}{\partial t} u(x, t, s) ds\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2} u(x, t) &= \int_0^t \frac{\partial^2}{\partial t^2} u(x, t, s) dx + \frac{\partial}{\partial t} \Big|_{t=s} u(x, t, s) \\ &= f(x, t) + \int_0^t \frac{\partial^2}{\partial t^2} u(x, t, s) dx\end{aligned}$$

Thus $\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x, t) = f(x, t) + \int_0^t \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x, t, s) dx$, the second term is zero for $s < t$

By the same argument, $\square \int_{-\infty}^t E(\cdot, t - s) * f(\cdot, s) ds = f(\cdot, t)$, thus $\Delta E = \delta_{(x, t)}$ is the fundamental solution \square

1 dim wave equation reflection

Lemma 35.3.5. The solution to $\square u = 0$ in $t > 0, x > 0$ with $u(0, t)$ for all $t > 0$, $u(x, 0) = 0$, $u_t(x, 0) = f(x)$, $f \in C^1([0, \infty))$, $f(0) = 0$ is

$$u(x, t) = \frac{1}{2} \int_{|t-x|}^{t+x} f(\lambda) d\lambda$$

Proof. Define $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{f}(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$ which solves $\square \tilde{u} = 0$ for $t > 0, x \in \mathbb{R}$, $\tilde{u}(x, 0) = 0$, $\tilde{u}_t(x, 0) = \tilde{f}$, hence

$$\tilde{u}(x, t) = \frac{1}{2} \int_{x-t}^{x+t} \tilde{f}(\lambda) d\lambda = \frac{1}{2} \int_{|x-t|}^{x+t} f(\lambda) d\lambda$$

\square
Laplacian of a spherical symmetric function

Lemma 35.3.6. $f(x) = f(|x|)$ is spherical symmetric in \mathbb{R}^n , then $(\Delta f)(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r}\right) f$

Proof. Δu is characterized by

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx = - \int_{\mathbb{R}^n} v \Delta u, \forall v \in C_c^\infty(\mathbb{R}^n)$$

If $u(x) = u(|x|)$, $v(x) = v(|x|)$

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx &= \int_{S^{n-1}} \int_0^\infty \frac{\partial u}{\partial r} \frac{\partial v}{\partial r} dr dS_w \\ &= - \int_{S^{n-1}} \int_0^\infty \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u}{\partial r} \right) v(r) r^{n-1} dr dS_w \\ &= - \int_{\mathbb{R}^n} \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u}{\partial r} \right) v(r) dx \\ &= - \int_{\mathbb{R}^n} \left(\frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) u v(r) dx\end{aligned}$$

Note that $\frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial u}{\partial r} \right) = \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$ \square

Theorem 35.3.7. *The solution to $\square u = 0$ in \mathbb{R}^{3+1} with $u(x, 0) = 0$, $u_t(x, 0) = f(x) = f(|x|)$, $f \in C^\infty(\mathbb{R}^3)$ is*

$$u(x, t) = \frac{1}{2|x|} \int_{t-|x|}^{t+|x|} \lambda f(\lambda) d\lambda$$

Proof. By Lemma 35.3.6, when $n = 3$, $\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}\right) u = \frac{1}{\partial r} \frac{\partial^2}{\partial r^2} (ru)$, thus if $\square u = 0$ in \mathbb{R}^{3+1} , $u(x, t) = u(|x|, t)$, then $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right) (ru(r, t)) = 0$ and $ru(r, t) = 0$ if $r = 0$, $\frac{\partial}{\partial t} \Big|_{t=0} (ru(r, t)) = rf(r)$, by Lemma 35.3.5, $ru(r, t) = \frac{1}{2} \int_{|t-r|}^{t+r} \lambda f(\lambda) d\lambda$. We can check $u \in C^1$ \square

Theorem 35.3.8 (Energy estimate version 1). $\square u = 0$ for $t > 0$, then the energy $\frac{1}{2} \int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 35.3.9 (Energy estimate version 2). $\square u = 0$ in $U_T = U \times (0, T]$, $u = 0$ on Γ_U , $u_t(x, 0) = 0$, implicitly, $u_t = 0$ on $\partial U \times [0, T]$, then $\frac{1}{2} \int_U |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 35.3.10 (Energy estimate version 3). $C = \{(x, t) \in \mathbb{R}^{n+1} \mid |x - x_0| \leq |t - t_0|\}$ is the cone, $D_t = \{x \in \mathbb{R}^n \mid |x - x_0| \leq |t - t_0|\}$ is the section at time t , consider the case $t < t_0$, then $\frac{1}{2} \int_{\mathbb{D}_t} |u_t|^2 + |\nabla u|^2 dx$ is decreasing on $0 \leq t \leq t_0$

35.4 Euler-Lagrange equation

35.5 Energy momentum tensor

Definition 35.5.1. ∇ is the gradient, write $\nabla^T \nabla = \nabla \cdot \nabla = \Delta$ is the laplacian, $\nabla \cdot 1 = \text{div}$ is the divergence, $\nabla \nabla^T = D^2$ is the Hessian

Definition 35.5.2. $L(z, q) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is C^∞ , u satisfies Euler-Langrange equation, then

$$\begin{aligned} \nabla_x L(u, \nabla u) &= \frac{\partial L}{\partial z} \nabla u + (\nabla \nabla^T u)(\nabla_q L) \\ &= (\nabla_x \cdot \nabla_q L)(\nabla u) + (\nabla \nabla^T u)(\nabla_q L) \\ &= (\nabla^T u \nabla_q L) \nabla_x \end{aligned}$$

Energy-momentum tensor $T_{\alpha\beta} = \frac{\partial u}{\partial x^\alpha} \frac{\partial L}{\partial q_\beta} - \delta_{\alpha\beta} L$, $T = \nabla^T u \nabla_q L - L1$, then $T \nabla_x = (\nabla^T \nabla_q L) \nabla_x - \nabla_x L = 0$

Example 35.5.3. $u_{tt} - \Delta u + u^3 = 0$, $L(u, \nabla_{x,t} u) = \frac{1}{2}(u_t^2 - |\nabla_x u|^2) - \frac{1}{4}u^4$, $T_{00} = u_t^2 - \left[\frac{1}{2}(u_t^2 - |\nabla u|^2) - \frac{1}{4}u^4 \right] = \frac{1}{2}(u_t^2 + |\nabla u|^2) + \frac{1}{4}u^4$, $T_{0i} = -u_t \frac{\partial u}{\partial x^i}$, thus $0 = (T_{00}, \dots, T_{0n}) \nabla_x = \text{div}(T_{00}, \dots, T_{0n})$

Part XIV

Mathematical physics

Chapter 36

Special relativity

Definition 36.0.1 (Galilean group). The **Galilean group** is the group of **Galilean transformations** generated by rotations in \mathbb{R}^n , translations in \mathbb{R}^{n+1} and **Galilean boosts** $(x, t) \mapsto (x + tv, t)$

$$\begin{pmatrix} R & v & w \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tv + w \\ t + s \\ 1 \end{pmatrix}$$

Definition 36.0.2 (Lorentz group). The **Lorentz group** is the group of **Lorentz transformations** generated by rotations in \mathbb{R}^n and **Lorentz boosts** $(x, t) \mapsto (\sinh sx - \cosh st, \sinh st - \cosh sx)$

Definition 36.0.3 (Poincaré group). The **Galilean group** is the isometry group of the Minkowski space \mathbb{R}^{n+1}

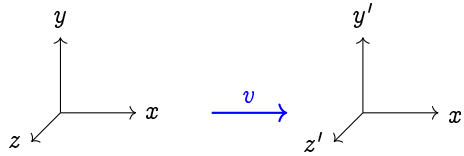
Definition 36.0.4. $(x, ct) \mapsto \left(\gamma(x - vt), \gamma\left(t - \frac{vx}{c^2}\right) \right)$ $\beta = \frac{v}{c}$, $\alpha = \sqrt{1 - \beta^2}$. $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$

is the **Lorentz factor** $\begin{cases} t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ x' = \gamma(x - vt) \end{cases}$, where (x, t) and (x', t') are the coordinates of two

frames, and frame (x', t') is moving towards the positive direction of the x axis with velocity

v , and c is the speed of light, we can find the inverse transformation $\begin{cases} t = \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x = \gamma(x' + vt') \end{cases}$

which makes perfect sense since relatively speaking, frame (x, t) is moving towards the negative direction of the x' axis with velocity v or rather moving towards the positive direction of the x' axis with velocity $-v$



More generally, if we consider $(\vec{x}, t), (\vec{x}', t')$ are the coordinates of two frames, with frame (\vec{x}', t') moving with velocity \vec{v} , then the Lorentz transformation will be

Deduction 36.0.5 (Time dilation). A frame moving (x', t') is at a constant speed v , then $\Delta t = \gamma \Delta t'$. Suppose you are on the train with constant speed v and height h , and let light bouncing up and down perpendicularly, then we have

$$2\sqrt{h^2 + \left(\frac{\Delta t}{2}v\right)^2} = c\Delta t, v\Delta t' = h$$

$$\Rightarrow \Delta t = \gamma \Delta t'$$

Things happen simultaneously in one frame may not be simultaneous in another frame

Deduction 36.0.6 (Length contraction). Suppose a train is moving with speed v , shed a beam light from one end to get to the other end

A in frame (x, t) send a signal when the left end of train passes, B in frame (x', t') on the right end of the train receives and return the signal, suppose the length of the train is l' , and the length appears to be l in frame (x, t) , then it takes time $\frac{l'}{c}$ for B to receive the signal in (x', t') , which takes time $\frac{l'\gamma}{c}$ in (x, t) , when B should be in distance $l + \frac{vl'\gamma}{c}$ from A in (x, t) but distance

$l' + \frac{vl'}{c}$ in (x', t') which take time $\frac{l + \frac{vl'\gamma}{c}}{c}$ and $\frac{l' + \frac{vl'}{c}}{c}$ to get back to A in (x, t) and (x', t') ,

hence we should have $\frac{l + \frac{vl'\gamma}{c}}{c} = \frac{l' + \frac{vl'}{c}}{c}\gamma \Rightarrow l = \gamma l'$

Chapter 37

Maxwell's equations

Theorem 37.0.1 (Maxwell's equations).

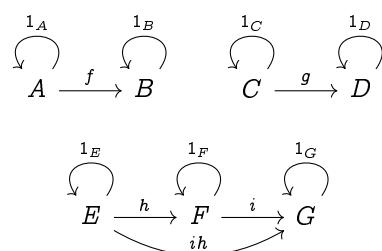
Part XV

Examples

Chapter 38

Examples in categories

Example 38.0.1. The image of a functor is not necessarily a category
Consider the following categories \mathcal{C} and \mathcal{D}



Consider functor $F : \mathcal{C} \rightarrow \mathcal{D}$, $F(A) = E$, $F(B) = F$, $F(C) = F$, $F(D) = G$, $F(f) = h$, $F(g) = i$

Chapter 39

Examples in algebra

Example 39.0.1. Suppose $1 \mapsto k$ is an element in $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, then $m \mid kn \Rightarrow \frac{m}{(n, m)}$ divides $k \frac{n}{(n, m)}$, thus $\frac{m}{(n, m)}$ divides k , thus $k = \frac{im}{(n, m)}, i = 0, \dots, (n, m) - 1$, thus $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$

Consider $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$, then $n(1 \otimes 1) = n \otimes 1 = 0, m(1 \otimes 1) = 1 \otimes m = 0$, thus $(n, m)(1 \otimes 1) = (rn + sm)(1 \otimes 1) = 0$

Apply functor $\text{Hom}(-, \mathbb{Z}/m\mathbb{Z})$ to short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$, we get a left exact sequence $\text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \rightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \rightarrow 0$

Apply functor $- \otimes \mathbb{Z}/m\mathbb{Z}$ to short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \rightarrow 0$, we get a left exact sequence $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \rightarrow 0$

And the kernel and cokernel of $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z}$ are both $\mathbb{Z}/(n, m)\mathbb{Z}$

Example 39.0.2. $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z} \begin{bmatrix} 1 \\ p \end{bmatrix}$

Example 39.0.3. $O(1, 1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}$

Example 39.0.4. F is a field, $R = \text{End}(F^\infty) = \{\text{infinite dimensional matrices}\}$, Consider $R \hookrightarrow R$ by embedding into odd rows and even rows, we have $R^2 \cong R$ as right R modules

Example 39.0.5. $GL(2, \mathbb{F}_2) = SL(2, \mathbb{F}_2) \cong S_3$

Chapter 40

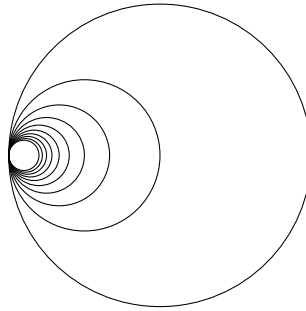
Examples in algebraic topology

Example 40.0.1 (A surjective local homeomorphism may not be a covering). $p : \mathbb{R} \setminus \{0\} \rightarrow S^1$, or n sheeted cover with a point missing, p is discrete but not proper

Example 40.0.2 (Bundle with fiber isomorphic to vector space but not a vector bundle).
 $E := \bigsqcup_{x \in X} \mathbb{R}^n$

Example 40.0.3. $H'_n = H_{k+n}$ also defines a homology theory where the dimension axiom fails Hawaiian earring

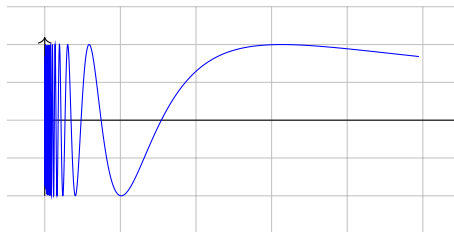
Example 40.0.4 (Hawaiian earring). The **Hawaiian earring** H is the union of circles with radius $\frac{1}{n}$ and centered at $(\frac{1}{n}, 0)$ with subspace topology in \mathbb{R}^2



Proposition 40.0.5. Hawaiian earring is not a CW complex since it is not locally contractible

Example 40.0.6 (Topologist's sine curve). The **topologist's sine curve** is

$$T = \left\{ \left(x, \sin \left(\frac{1}{x} \right) \right) \mid x \in (0, 1] \right\} \cup \{(0, 0)\}$$



Proposition 40.0.7. The topologist's sine curve T is connected but not path connected

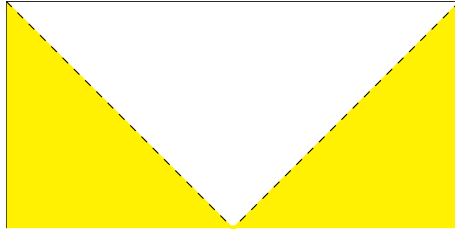
Example 40.0.8 (Warsaw circle). The **Warsaw circle** W is the topologist's sine curve enclosed. Bijective map $W \rightarrow [0, 1)$ is not a homeomorphism, thus not a quotient map. W is weakly homotopic to a point but not homotopic

Example 40.0.9. $X = \mathbb{N}$ with discrete topology, $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ with subspace topology of \mathbb{R} , then $f : X \rightarrow Y, n \mapsto \frac{1}{n}$ is a weak homotopy equivalence, however, X, Y are not homotopy equivalent, otherwise suppose $g : X \rightarrow Y, h : Y \rightarrow X$ such that $hg \simeq 1_X, gh \simeq 1_Y$, suppose $F : Y \times I \rightarrow Y$ is a homotopy, then the restriction of F on $\{y\} \times I$ must be a constant map since the connected components of Y are just points, thus $F(y, 0) = F(y, 1)$, i.e. homotopic maps are in fact the same, for a similar argument on X , we have $hg = 1_X, gh = 1_Y$, thus h is injective which is impossible since $h^{-1}(h(0))$ consists of more than one point

Cofibration counterexample

Example 40.0.10. $D^2 = S^2 \setminus \{N\} \subseteq S^2$ is not a cofibration. $D^2 \setminus \{0\} \subseteq D^2$ is not a cofibration
Mapping cylinder of inclusion may have different topology than induced subspace topology

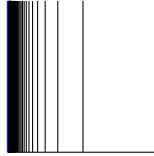
Example 40.0.11. $A = [-1, 0) \cup (0, 1], X = [-1, 1]$, then the mapping cylinder of the inclusion $A \xrightarrow{i} X$ has different topology from the subspace topology $X \times \{0\} \cup A \times I$ induced from $X \times I$



Nonclosed cofibration

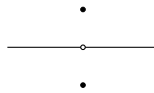
Example 40.0.12. $\{a, b\}$ with trivial topology, $\{a\} \subseteq \{a, b\}$ is a nonclosed cofibration since there is a retraction $I \sqcup I \rightarrow I \sqcup \{0\}, (s, t) \mapsto (s, 0)$

Example 40.0.13. The comb space is $[(0, 0), (1, 0)] \cup \bigcup_{n=1}^{\infty} [(\frac{1}{n}, 0), (\frac{1}{n}, 1)]$



A line with two origins

Example 40.0.14. A topological space with a cell decomposition may not be Hausdorff, consider $(-1, 1)$ with two origins, which has $(-1, 0), (0, 1)$ as 1 cells and two origins as 0 cells



Chapter 41

Examples in geometry

Definition 41.0.1 ($\mathcal{O}(n)$ bundle over Riemann sphere $S^2 \cong \mathbb{CP}^1$). Suppose $(\mathbb{C}, z \mapsto z)$, $(S^2 \setminus 0, z \mapsto \frac{1}{z})$ are the charts coordinate of S^2 with transition map $z \mapsto \frac{1}{z}$ both ways on $\mathbb{C} \setminus 0$, or equivalently

$(U_0, [1, z] \mapsto z)$, $(U_1, [z, 1] \mapsto z)$ are the corresponding charts of \mathbb{CP}^1 with transition map $z \mapsto \frac{1}{z}$ both ways on $U_0 \cap U_1$, namely, $S^2 \rightarrow \mathbb{CP}^1$, $z \mapsto [1, z]$, $\infty \mapsto [0, 1]$ is an isomorphism because of isomorphisms on charts

Now define $\mathcal{O}(n)$ line bundle on S^2 by specifying transition functions $g_{10}(z) = z^{-n}$, $g_{01}(z) = z^n$, $\forall z \in \mathbb{C} \setminus 0 \cong U_0 \cap U_1$

Definition 41.0.2 (Tautological line bundle over Riemann sphere). The tautological bundle is $\mathcal{O}(-1)$, tautological bundle is defined as a subspace E of $\mathbb{CP}^1 \times \mathbb{C}^2$ consists of (l, v) with $v \in l$ projects to the first factor, let's figure out the trivializations!

$\varphi_0 : U_0 \times \mathbb{C}^2 \cap E \rightarrow \mathbb{C} \times \mathbb{C}$, $([1, z], t(1, z)) \mapsto (z, t)$, and $\varphi_1 : U_1 \times \mathbb{C}^2 \cap E \rightarrow \mathbb{C} \times \mathbb{C}$, $([z, 1], t(z, 1)) \mapsto (z, t)$, since $\varphi_1 \circ \varphi_0^{-1} : (U_0 \cap U_1) \times \mathbb{C}^2 \cap E \rightarrow (U_0 \cap U_1) \times \mathbb{C}^2 \cap E$, $(z, t) \mapsto \left(\frac{1}{z}, zt\right)$, the transition function $g_{10}(z) = z$

Remark 41.0.3. $\mathcal{O}(-1)$ doesn't have nonzero global section, suppose s is a global section of $\mathcal{O}(-1)$, then $s(x) = (x, f(x)) \in E \hookrightarrow \mathbb{CP}^1 \times \mathbb{C}^2$ is holomorphic, but then image of s has to be a point, and this point must be zero

Example 41.0.4. We still use U_0, U_1 to denote coordinate charts, φ_0, φ_1 to denote corresponding trivializations

Global sections of $\mathcal{O} = \mathcal{O}(0)$ are exactly holomorphic functions which are just constants, suppose $s : S^2 \rightarrow \mathcal{O}$ is a section, and $\varphi_0 \circ s|_{U_0}(z) = (z, f_0(z))$, $\varphi_1 \circ s|_{U_1}\left(\frac{1}{z}\right) = \left(\frac{1}{z}, f_1\left(\frac{1}{z}\right)\right)$, then we have $(z, f_1(z)) = \varphi_1 \circ s|_{U_1}(z) = \varphi_1 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1} \circ \varphi_0 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1}(z, f_0(z)) = (z, g_{10}(z)f_0(z))$, $\forall z \in U_0 \cap U_1$, thus $f_1(z) = g_{10}(z)f_0(z) = f_0(z)$ which precisely means s corresponds to holomorphic function f over X , $f|_{U_0} = f_0$, $f|_{U_1} = f_1$

Let's show that the canonical bundle (which in the case of a Riemann surface is the same as the cotangent bundle) is $\mathcal{O}(-2)$, since $d\left(\frac{1}{z}\right) = -\frac{1}{z^2}dz$, the transition function would be $g_{10}(z) = -z^2$, but using dz or $-dz$ as the basis element would be isomorphic

Proposition 41.0.5. $H^0(\mathbb{CP}^1, \mathcal{O}(n))$, the vector space of global sections of $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$, $n \geq 0$ generated by homogeneous polynomials $z_0^n, z_0^{n-1}z_1, \dots, z_0z_1^{n-1}, z_1^n$

Proof. $z_0^k z_1^{n-k}$ have the forms z_1^{n-k} and z_0^k in U_0 and U_1 □

Example 41.0.6 (Line bundles on the projective space \mathbb{CP}^n). Suppose $(U_0, [1, z_1, \dots, z_n] \mapsto (z_1, \dots, z_n))$, $(U_n, [z_0, z_1, \dots, z_{n-1}, 1] \mapsto (z_0, \dots, z_{n-1}))$ be coordi-

nate charts of $\mathbb{C}P^n$, with transition map $U_i \cap U_j \rightarrow U_i \cap U_j, \left(\frac{z_0}{z_i}, \dots, \widehat{\frac{z_i}{z_i}}, \dots, \frac{z_n}{z_i} \right) \mapsto \left(\frac{z_0}{z_j}, \dots, \widehat{\frac{z_j}{z_j}}, \dots, \frac{z_n}{z_j} \right)$, which is kind of like multiply by $\frac{z_i}{z_j}$, then the line bundle $\mathcal{O}(m)$ is defined by transition function $g_{ji} = \frac{z_j}{z_i}$ which satisfies the cocycle condition

Similarly, we can check that the tautological bundle $E = \{(l, v) | v \in l\} \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$ projects to $\mathbb{C}P^n$ is $\mathcal{O}(1)$

It is obvious that any degree n polynomial are global section of $\mathcal{O}(n)$

Chapter 42

Examples in Lie groups and Lie algebras

Example 42.0.1. X is topological space, $End(X)$ is a unital nonassociative \mathbb{R} algebra which is not symmetric, antisymmetric, nor does it satisfy Jacobi identity

Example 42.0.2. Consider $C^\infty(M)$ where M is a smooth manifold, then $\mathcal{L}(M) = Der(C^\infty(M))$ consists of vector fields, it is a Lie algebra, hence we can think of derivations as linear differential operator of order 1, then we know that the commutator of two such operators is again a linear differential operator of order 1

Example 42.0.3. Let \mathfrak{g} be a Lie algebra, then ideals of \mathfrak{g} precisely the Lie algebra subrepresentations of the adjoint representation (ad, \mathfrak{g})

Example 42.0.4 (Lie algebra of $M_n(\mathbb{R})$). Suppose $X = \sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}}$ is a left invariant

$$\begin{aligned} X_{kl}(A) &= \sum_{i,j} X_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\ &= X_A(x_{kl}) = (L_A)_0 X_0(x_{kl}) \\ &= X_0(x_{kl} \circ L_A) \\ &= \sum_{i,j} X_{ij}(0) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(0) \\ &= X_{kl}(0) \end{aligned}$$

Thus X_{ij} are constants

$$\begin{aligned} [X, Y] &= \left[\sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} Y_{kl} \frac{\partial}{\partial x_{kl}} \right] \\ &= \sum_{i,j,k,l} X_{ij} Y_{kl} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\ &= \sum_{i,j} X_{ij} Y_{ij} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\ &= 0 \end{aligned}$$

Therefore $Lie(M_n(\mathbb{R})) = 0$

Example 42.0.5 (Lie algebra of $GL(n, \mathbb{R})$). Suppose $X = \sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}$ is a left invariant field

$$\begin{aligned} c_{kl}(A) &= \sum_{i,j} c_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\ &= X_A(x_{kl}) = (L_A)_I X_I(x_{kl}) \\ &= X_I(x_{kl} \circ L_A) \\ &= \sum_{i,j} c_{ij}(I) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(I) \\ &= \sum_i a_{ki} c_{il}(I) \end{aligned}$$

Hence $C(A) = AC(I)$, $\frac{\partial c_{kl}}{\partial x_{ij}} = \delta_{ki} c_{jl}(I)$

$$\begin{aligned} [X, Y] &= \left[\sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} d_{kl} \frac{\partial}{\partial x_{kl}} \right] \\ &= \sum_{i,j,k,l} \left[c_{ij} \frac{\partial}{\partial x_{ij}}, d_{kl} \frac{\partial}{\partial x_{kl}} \right] \\ &= \sum_{i,j,k,l} c_{ij} \frac{\partial}{\partial x_{ij}} \left(d_{kl} \frac{\partial}{\partial x_{kl}} \right) - d_{kl} \frac{\partial}{\partial x_{kl}} \left(c_{ij} \frac{\partial}{\partial x_{ij}} \right) \\ &= \sum_{i,j,k,l} c_{ij} \left(\frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} + d_{kl} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}} \right) - d_{kl} \left(\frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} + c_{ij} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}} \right) \\ &= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\ &= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\ &= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} \\ &= \sum_{i,j,k,l} \left(c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} - d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}} \right) \frac{\partial}{\partial x_{kl}} \\ &= \sum_{j,k,l} (c_{kj} d_{jl} - d_{kj} c_{jl}) \frac{\partial}{\partial x_{kl}} \\ &= \sum_{k,l} \left(\sum_j c_{kj} d_{jl} - d_{kj} c_{jl} \right) \frac{\partial}{\partial x_{kl}} \\ &= \sum_{k,l} b_{kl} \frac{\partial}{\partial x_{kl}} \end{aligned}$$

Here $B = [C, D]$. Therefore $\text{Lie}(GL(n, \mathbb{R})) = \mathfrak{gl}(n, \mathbb{R})$

Example 42.0.6. Consider the $\Phi : GL(n, \mathbb{R}) \rightarrow M_n(\mathbb{R})$, $A \mapsto A^T A$ which is a smooth map, and level set $\Phi^{-1}(I) = O(n, \mathbb{R})$ is the orthogonal group, to show this is a Lie subgroup, thanks to Theorem 21.0.16, it suffices to show Φ is of constant rank, but Φ is equivariant assuming $GL(n, \mathbb{R})$ acts on itself by right multiplication and acts on $M_n(\mathbb{R})$ by $X \cdot A = A^T X A$, $X \in M_n(\mathbb{R})$, $A \in GL(n, \mathbb{R})$, since $\Phi(A) \cdot B = B^T A^T A B = \Phi(AB)$
 $(d\Phi)_I(B) = B^T + B$, and $T_I(O(n, \mathbb{R})) = \ker(d\Phi)_I = \{B \in M(n, \mathbb{R}) | B^T + B = 0\}$

Chapter 43

Examples in algebraic geometry

Example 43.0.1. Suppose $V \subseteq \mathbb{A}^n$ is an affine variety, $m_P \in \text{Spm} k[V]$, $k[V]_{m_P}$ is the stalk of the sheaf of regular functions. Two representatives $\frac{f}{u}, \frac{g}{v}$ are of the same germ $\Leftrightarrow \frac{f}{u} = \frac{g}{v}$ on $D(wuv)$ for some $w(P) \neq 0 \Leftrightarrow w(fv - gu) = 0$

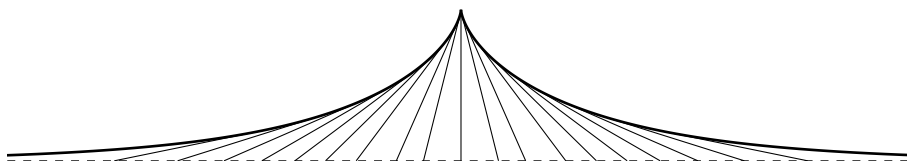
Example 43.0.2.

Chapter 44

Examples in analysis

Example 44.0.1. $D \subseteq \mathbb{C}$ is the unit disc, $f(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{n^2}$ is continuous on \overline{D} and holomorphic on D but not on any point on ∂D

Example 44.0.2 (Tractrix). An interval I with one end point pushed or dragged along the x axis gives a **Tractrix**. The velocity has the same direction as I , i.e. $\frac{dx}{dy} = \pm \frac{\sqrt{a^2 - y^2}}{y}$, which gives solution $x = \pm \left(\ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2} \right)$



Part XVI

Exercises

Chapter 45

Exercises in combinatorics

Exercise 45.0.1. $[n] = \{1, \dots, n\}$, what is the cardinality of $\{f \in \text{Aut}([n]) \mid f(i) \neq i, \forall i \in [n]\}$

Solution. Consider $A_k = \{f \in \text{Aut}([n]) \mid f(k) = k\}$, by Inclusion-exclusion principle 1.0.4, we have

$$\begin{aligned} n! &= \left| \bigcup_{i=1}^n A_i \right| = \sum_{k=1}^n (-1)^k \sum_{i_1 < \dots < i_k} |A_{i_1} \cap \dots \cap A_{i_k}| \\ &= \sum_{k=1}^n (-1)^k \binom{n}{k} (n-k)! \\ &= \sum_{k=1}^n (-1)^k \frac{n!}{k!} \end{aligned}$$

Thus the probability of picking such an auto morphism is $\sum_{i=1}^n \frac{(-1)^i}{i!}$ which approaches e^{-1} as n approaches infinity □

45.1 Exercises in abstract algebra

Exercise 45.1.1. If R is a domain, so is $R[x]$

Solution. Suppose $f = ax^n + \dots$, $g = bx^m + \dots$ for some $a, b \neq 0$, then $fg = abx^{n+m} + \dots \neq 0$ \square

Exercise 45.1.2. If E/F is a Galois extension, then $Tr_{E/F}(\alpha)$ is the sum of all conjugates of α , $N_{E/F}(\alpha)$ is the product of all conjugates of α

Solution. Suppose the minimal polynomial of α is $m(x) = x^n + a_1x^{n-1} + \dots + a_n$ \square

Exercise 45.1.3. If $F \subseteq E \subseteq L$ are field extensions, then $Tr_{L/F} = Tr_{E/F} \circ Tr_{L/E}$

Solution. Suppose x_1, \dots, x_n is a basis for L/E , y_1, \dots, y_m is a basis for E/F \square
 $T: V \rightarrow W \leq W \Rightarrow Tr(T) = Tr(T|_W)$

Exercise 45.1.4. Suppose $T \in \text{Hom}_{\mathbb{F}}(V, V)$ is a linear operator with $T(V) \leq W$, then $Tr(T) = Tr(T|_W)$

Mundane properties of rings

Exercise 45.1.5. R is a ring

1. $0x = 0$, $(-1)x = -x$

Solution.

1.

$$0x = (0 + 0)x = 0x + 0x \Rightarrow 0x = 0$$

$$0x = (1 + (-1))x = 1x + (-1)x = x + (-1)x \Rightarrow (-1)x = -x$$

\square

Exercise 45.1.6. Let R be a commutative ring, and $I_1, \dots, I_n \leq R$ be pairwise coprime ideals, then $I_1 \cdots I_n = I_1 \cap \dots \cap I_n$

Solution. By induction \square

Exercise 45.1.7. Every group G is naturally isomorphic to its opposite G^{op}

Solution. Consider $\phi : G \rightarrow G^{op}$, $g \mapsto g^{-1}$ \square

Exercise 45.1.8. A morphism of G torsors is always an isomorphism

Exercise 45.1.9. X has a left G action and a right H action such that $(gx)h = g(xh)$

1. $X \times_G * \cong X/G$

2. $X \times_G G \cong X$

3. $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$

4. If $H \leq G$, then $X \times_G G \times_Y \cong X \times_H Y$

5. If $H \trianglelefteq G$, $X \times_G (G/H) \cong X/H$

Exercise 45.1.10. $SL(n, F)$ is a perfect group for $n \geq 3$. $SL(2, F)$ is a perfect group if $|k| \geq 4$

Solution. Denote $G_n = SL(n, F)$. Elementary matrices generate G_n and are in $[G_n, G_n]$ \square

45.2 Exercises in analysis

$U \subset \mathbb{R}^n$ open, boundary point is the limit of some discrete sequence

Exercise 45.2.1. $U \subsetneq \mathbb{R}^n$ is a nonempty open set, $x \in \partial U$, then there exists a discrete sequence $\{x_i\} \subseteq U$ converges to x

Solution. x is necessarily an accumulation point since $\partial U \cap U = \emptyset$. Pick $x_0 \in U$, then we can find $\epsilon > 0$ such that $x_0 \notin B(x, \epsilon)$, then pick $x_1 \in B(x, \epsilon/2) \cap U$, and so on \square

f analytic near 0, after change of variables, f has terms only involve one variable

Exercise 45.2.2. f is analytic near 0, by rotation of coordinates, we can always make f has terms only involve one variable

Exercise 45.2.3. Evaluate $\int_0^\infty e^{-s^2 - \frac{1}{s^2}} ds$

Solution. $\left(s - \frac{1}{s}\right)^2 = s^2 + \frac{1}{s^2} - 2$, let $x = s - \frac{1}{s}$ which is increasing on $(0, \infty)$ since $0 < s < \infty$, $-\infty < x < \infty$, then $s = \frac{x + \sqrt{x^2 + 4}}{2}$ and

$$\int_0^\infty e^{-s^2 - \frac{1}{s^2}} ds = e^{-2} \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{1}{2} + \frac{x}{2\sqrt{x^2 + 4}}\right) dx = e^{-2} \int_0^\infty e^{-x^2} dx = \frac{e^{-2}\sqrt{\pi}}{2}$$

\square

Exercise 45.2.4. f is holomorphic on the punctured unit disc, $p > 0$, $\int_D |f(z)|^p dz < \infty$. What can we say about the singularity?

Solution. $|f(z)|^p = e^{p \log |f(z)|}$ is subharmonic by Example 31.0.8, thus essential singularity is impossible

$$|f(z)|^p \leq \frac{4}{\pi|z|^2} \int_{|w-z| < |z|/2} |f(w)|^p dw \leq \frac{C}{|z|^2}$$

Thus $|z|^{\frac{2}{p}} |f(z)| < \infty$

\square

Exercise 45.2.5. $U \subseteq \Omega \subseteq \mathbb{C}$ are open, f is holomorphic on U , \widehat{U}_Ω be the union of U and compact connected components of $\Omega \setminus U$. There exist $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U iff there exists g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$

Solution. Assume $\widehat{U}_\Omega = U \cup K_1 \cup \dots$, where K_i 's are compact

Suppose $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U , by maximum principle, $\{f_n\}$ would be uniformly bounded around K_i , by Montel's theorem 32.1.14, there exists a subsequence of $\{f_n\}$ converges uniformly on K_i , thus converging to g holomorphic on $H(\widehat{U}_\Omega)$, hence $g|_U = f$

Conversely, suppose g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$, \widehat{U}_Ω is simply connected, by Riemann mapping theorem 32.1.23, we can think of \widehat{U}_Ω as the unit disc or \mathbb{C} , by Runge's theorem, there exist $\{f_n\}$ holomorphic on Ω uniformly converging to g on each disc. Thus there exist a subsequence of $\{f_n\}$ converging uniformly to g on compact subsets of \widehat{U}_Ω \square

Exercise 45.2.6. Let Ω be an open subset of \mathbb{C} , $\mathcal{D} = \{D_i\}$ be an open cover of Ω with disks. Given meromorphic functions h_i on D_i , not identically zero. Assume $g_{ij} = \frac{h_i}{h_j}$ are holomorphic on $D_i \cap D_j$, then there exist holomorphic function f_i with no zeros on D_i such that $f_i = g_{ij} f_j$

Solution. It suffices to prove $H^1(\Omega, \mathcal{O}^*) = 0$, since then $H^1(\mathcal{D}, \mathcal{O}^*) = 0$, $(g_{ij}) \in Z^1(\mathcal{D}, \mathcal{O}^*) = B^1(\mathcal{D}, \mathcal{O}^*)$, i.e. there exists $(f_i) \in C^0(\mathcal{D}, \mathcal{O}^*)$ such that $f_i = g_{ij} f_j$

Consider exact sequence of sheaves $0 \rightarrow \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$, then we get a long exact sequence $\dots \rightarrow H^1(\Omega, \mathcal{O}) \rightarrow H^1(\Omega, \mathcal{O}^*) \rightarrow H^2(\Omega, \mathbb{Z}) \rightarrow \dots$, $H^1(\Omega, \mathcal{O}) = 0$ by Mittag-Leffler theorem \square

Exercise 45.2.7. For each real r such that $0 < |r| < 1$, prove that there exists at most one real s with $0 < s < 1$ for which $\Omega := D \setminus \{0, r, s\}$ admits an analytic automorphism different from the identity

Solution. Suppose $\Omega \xrightarrow{\phi} \Omega$ is an analytic automorphism, then $0, r, s$ are all removable singularities, by continuity, ϕ can be extended to $D \xrightarrow{\phi} D$, so is ϕ^{-1} , by continuity, we know ϕ is an automorphism of D , sending $\{0, r, s\}$ to itself bijectively

By Schwarz lemma, we know that an automorphism ϕ of D with $\phi(\alpha) = 0$ iff $\phi = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha}z}$. Now suppose ϕ is an automorphism different from the identity, if $0 < r = s < 1$, then $\phi = -\frac{z - r}{1 - rz}$ is a choice, now we assume $r \neq s$

Case I: $\phi(0) = 0$

$$\phi = e^{i\theta} z, \phi(r) = s, \text{ but } 0 < s < 1, \text{ thus } s = |r|$$

Case II: $\phi(r) = 0$

$$\phi = e^{i\theta} \frac{z - r}{1 - \bar{r}z}, \phi(0) = -re^{i\theta}$$

Case i: $\theta = \pi, \phi(0) = r$, then $s = \phi(s) \Rightarrow \bar{r}s^2 - 2s + r = 0 \Rightarrow s = \frac{1 + \sqrt{1 - |r|^2}}{\bar{r}}$ or $\frac{1 - \sqrt{1 - |r|^2}}{\bar{r}}$, r has to be a positive real number and $s = \frac{1 - \sqrt{1 - r^2}}{r}$

Case ii: $s = \phi(0) = |r|$

Case III: $\phi(s) = 0$

$$\phi = e^{i\theta} \frac{z - s}{1 - sz}, \phi(0) = -se^{i\theta}$$

Case i: $\theta = \pi, \phi(0) = s$, then $r = \phi(r) \Rightarrow sr^2 - 2r + s = 0 \Rightarrow s = \frac{2r}{1 + r^2}$.

Case ii: $r = \phi(0) = -se^{i\theta}, s = \phi(r) \Rightarrow s^2 = 1$ which is impossible

□

Exercise 45.2.8. $F \subseteq \mathbb{C}$ is closed, connected and noncompact, $\Omega = \mathbb{C} \setminus F$, then every $f \in \mathcal{O}(\Omega)$ has a primitive

Solution. It suffices to show that every connected component U of Ω is simply connected. Suppose U is not simply connected, then $\pi_1(U, z_0) \neq 0$, i.e. there is a simple (self non-intersecting) loop $\gamma \subseteq U$ with $\gamma(0) = \gamma(1)$ cannot be deformed to z_0 , by Jordan curve theorem 14.1.44, γ divides \mathbb{C} into the exterior and the interior which is homeomorphic to the unit disc D , suppose $F \cap D$ is empty, then $\bar{D} \subseteq U$, γ can be deformed to z_0 , giving a contradiction, hence $F \cap \bar{D}$ is a compact connected component of F which is also a contradiction. □

Exercise 45.2.9. Consider an open set $\Omega \subseteq \mathbb{C}^2$ such that

$$\{(z, w) \in \mathbb{C}^2 \mid |z| \leq R_1, |w| \leq R_2\} \subseteq \Omega$$

for some positive reals R_1 and R_2 . Let $f \in \mathcal{H}(\Omega)$ be such that $f(z, w) \neq 0$ for every z and w for which $|z| \leq R_1, |w| = R_2$

1. Prove that the number (counted with multiplicities) of zeros of $w \mapsto f(z, w)$ in $D(0, R_2)$ is the same for every $|z| \leq R_1$

2. Let $w_1(z), \dots, w_m(z)$ denote the zeros of $w \mapsto f(z, w)$ (counted with multiplicities). Prove that for each $n \in \mathbb{N}$ the function

$$z \mapsto w_1(z)^n + \dots + w_m(z)^n$$

is holomorphic for $z \in D(0, R_1)$

3. Deduce that n th elementary symmetric function σ_n of $w_1(z), \dots, w_m(z)$ is holomorphic.
4. Prove that there exists a function h that is holomorphic and without any zeros on $\{(z, w) \in \mathbb{C}^2 \mid |z| < R_1, |w| < R_2\}$ such that

$$f(z, w) = h(z, w)[w^m + \sigma_1(z)w^{m-1} + \dots + \sigma_{m-1}(z)w + \sigma_m(z)]$$

for every z and w such that $|z| < R_1$ and $|w| < R_2$

Solution.

1. By Lemma 32.1.18, $\frac{1}{2\pi i} \int_{\partial D(0, R_2)} \frac{f_w(z, w)}{f(z, w)} dw$ is the number of zeros in $D(0, R_2)$ which is continuous, hence the same for every $|z| \leq R_1$
2. By Lemma 32.1.18, $\frac{1}{2\pi i} \int_{\partial D(0, R_2)} w^n \frac{f_w(z, w)}{f(z, w)} dw = w_1(z)^n + \dots + w_m(z)^n$ is holomorphic
3. Directly follows from (2) thanks to Newton's identities
4. Since $\prod_{i=1}^m (w - w_i(z)) = w^m + \sigma_1(z)w^{m-1} + \dots + \sigma_{m-1}(z)w + \sigma_m(z)$ is holomorphic

$$\frac{f(z, w)}{w^m + \sigma_1(z)w^{m-1} + \dots + \sigma_{m-1}(z)w + \sigma_m(z)}$$

has no zeros on D and holomorphic on $\{R_2 - \varepsilon < |w| < R_2\}$, hence by Hartogs's extension theorem 32.1.27, can be extended to a holomorphic function $h(z, w)$, then $f(z, w) = h(z, w)[w^m + \sigma_1(z)w^{m-1} + \dots + \sigma_{m-1}(z)w + \sigma_m(z)]$ on $\{R_2 - \varepsilon < |w| < R_2\}$, by identity theorem, this holds for all $|z| < R_1$ and $|w| < R_2$

□

Exercise 45.2.10. Suppose p_1, \dots, p_n are points on the compact Riemann surface X and $X' = X \setminus \{p_1, \dots, p_n\}$. Suppose $f : X' \rightarrow \mathbb{C}$ is a non-constant holomorphic function. Show that the image of f comes arbitrarily close to every $c \in \mathbb{C}$

Solution. Suppose there exists $c \in \mathbb{C}$ such that $|f - c| \geq \varepsilon$ for some $\varepsilon > 0$, then $\frac{1}{f - c}$ would be a bounded holomorphic function on X' , by Riemann's Removable singularity theorem, $\frac{1}{f - c}$ can be extended to a holomorphic function on X , but since X is compact, $\frac{1}{f - c}$ is a constant which is impossible

□

Exercise 45.2.11. Let X be a compact Riemann surface and let $X \xrightarrow{\sigma} X$ be a biholomorphic map of X onto itself, different from the identity. Let $a \in X$ be a point with $\sigma(a) \neq a$, and suppose that there is a non-constant meromorphic function f on X , holomorphic on $X \setminus \{a\}$, with a pole of order k at a . Prove that σ can have at most $2k$ fixed points on X

Solution. Suppose there are more than $2k$ fixed points of σ , then consider $f - f \circ \sigma^{-1} : X \rightarrow \mathbb{P}^1$ is holomorphic on $X \setminus \{a, \sigma^{-1}(a)\}$ with at least $2k + 1$ zeros and with poles of order k at $a, \sigma^{-1}(a)$, but it should have as many poles as zeros which is a contradiction

□

Exercise 45.2.12. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda' = \mathbb{Z}\omega'_1 + \mathbb{Z}\omega'_2$ are lattices in \mathbb{C} . Show that $\Lambda = \Lambda'$ iff there exists a matrix $A \in GL(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Solution. First note that

$$\Lambda \subseteq \Lambda' \Leftrightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} \text{ for some } A \in M(2, \mathbb{Z})$$

Hence we have

$$\Lambda = \Lambda' \Leftrightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix}, \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = B \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some } A, B \in M(2, \mathbb{Z})$$

Which is equivalent to $A \in GL(2, \mathbb{Z})$ □

Exercise 45.2.13. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\Lambda' = \mathbb{Z}\omega'_1 + \mathbb{Z}\omega'_2$ are lattices in \mathbb{C} and $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$ are the corresponding complex tori

1. Prove that any holomorphic map $X \xrightarrow{f} X'$ is induced by a linear map $\mathbb{C} \xrightarrow{g} \mathbb{C}$ of the form $g(z) = \alpha z + \beta$, where $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda \subseteq \Lambda'$. f is biholomorphic if and only if $\alpha\Lambda = \Lambda'$
2. Show that every torus $X = \mathbb{C}/\Lambda$ is isomorphic to a torus of the form $X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\text{Im}(\tau) > 0$
3. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $\text{Im}(\tau) > 0$. Let $\tau' := \frac{a\tau + b}{c\tau + d}$. Show that the tori $X(\tau)$ and $X(\tau')$ are biholomorphic

Solution.

1. Since \mathbb{C} is the universal cover of \mathbb{C}/Λ' , $f \circ \pi : \mathbb{C} \rightarrow \mathbb{C}/\Lambda'$ has a lift $F : \mathbb{C} \rightarrow \mathbb{C}$, and locally we have $F = \pi'|_V^{-1} \circ f \circ \pi|_U$, thus F is holomorphic

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{F} & \mathbb{C} \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \xrightarrow{f} & \mathbb{C}/\Lambda' \end{array}$$

Fix $\omega \in \Lambda$, since $\pi(z + \omega) = \pi(z)$ for any $z \in \mathbb{C}$, we have $F(z + \omega) - F(z) \in \Lambda'$, hence $F(z + \omega) - F(z)$ is a continuous function of z but Λ' is discrete, thus $F(z + \omega) - F(z) \equiv C_\omega$, where $C_\omega \in \Lambda'$ is a constant. Then $F'(z + \omega) = F'(z)$ which shows $F' : \mathbb{C} \rightarrow \mathbb{C}$ is doubly periodic function, thus induces $G : \mathbb{C}/\Lambda \rightarrow \mathbb{C}$ with $F = G \circ \pi$. Thus G must be a constant, so is F' , therefore F has the form $F(z) = \alpha z + \beta$. Then for any $\omega \in \Lambda$, we have $F(\omega) - F(0) = \alpha\omega \in \Lambda'$, thus $\alpha\Lambda \subseteq \Lambda'$. If f is biholomorphic, then $\pi' \circ F = f \circ \pi \Rightarrow \pi \circ F^{-1} = f^{-1} \circ \pi'$, which implies $\begin{cases} \alpha\Lambda \subseteq \Lambda' \\ \alpha^{-1}\Lambda' \subseteq \Lambda \end{cases} \Rightarrow \alpha\Lambda = \Lambda'$

$$\begin{array}{ccc} \mathbb{C} & \xleftarrow{F^{-1}} & \mathbb{C} \\ \downarrow \pi & & \downarrow \pi' \\ \mathbb{C}/\Lambda & \xleftarrow{f^{-1}} & \mathbb{C}/\Lambda' \end{array}$$

Conversely, if $\alpha\Lambda = \Lambda'$, $\pi \circ F^{-1}$ is doubly periodic and induce f^{-1} , hence f is biholomorphic

2. Suppose $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\text{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$, define $\Lambda' = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \frac{\omega_2}{\omega_1}$, we have $\omega_1\Lambda' = \Lambda$, thus X and $X(\tau)$ are biholomorphic
3. $X(\tau)$ and $X(\tau')$ are biholomorphic iff $\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \alpha A \begin{pmatrix} \tau \\ 1 \end{pmatrix}$, $\alpha \in \mathbb{C} - \{0\}$, $A \in \text{SL}(2, \mathbb{Z})$. If $X(\tau)$ and $X(\tau')$ are biholomorphic, then $\mathbb{Z} + \mathbb{Z}\tau' = \Lambda' = \alpha\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\alpha\tau$ for some $\alpha \in \mathbb{C} - \{0\}$, thus $\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = A \begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \alpha A \begin{pmatrix} \tau \\ 1 \end{pmatrix}$, for some $A \in \text{SL}(2, \mathbb{Z})$, the other direction is easy

□

Exercise 45.2.14. Determine the branch points (or ramification points) of the map $f : \mathbb{C} \rightarrow \mathbb{P}^1$ with

$$f(z) = \frac{1}{2} \left(z + \frac{1}{z} \right)$$

Solution. $f'(z) = \frac{1}{2} \left(1 - \frac{1}{z^2} \right)$ when $z \neq 0$, thus $1, -1$ are branch points.

Consider the chart $(\mathbb{P}^1 - \{0\}, \varphi)$ with $\varphi(z) = \frac{1}{z}$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{P}^1 - \{0\} \\ & \searrow & \downarrow \varphi \\ & & \mathbb{C} \end{array}$$

Thus $g(z) = \varphi \circ f(z) = \frac{z}{2(z^2 + 1)}$, $g'(z) = \frac{1 - z^2}{2(z^2 + 1)}$, hence 0 is not a branch point

□

Exercise 45.2.15. If f and g are two elliptic functions with respect to the same lattice $\Omega \subseteq \mathbb{C}$, prove that there exists an irreducible polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$

Solution. If $f \equiv c$ is a constant, then $P(x, y) = x - c$ is an irreducible polynomial such that $P(f, g) = 0$, so we can assume f, g are not constants. Since $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$, there exists rational functions R_0, \dots, R_n such that $R_0(f) + R_1(f)g + \dots + R_n(f)g^n = 0$, then after multiplying denominators, we get a polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$, since $\mathbb{C}[x, y]$ is a UFD, $P = P_1 \cdots P_k$, where P_i are prime hence irreducible, then $0 = P_1(f, g) \cdots P_k(f, g) \in \mathcal{M}(X)$ which is a field, thus $P_j(f, g) = 0$ for some irreducible polynomial $P_j \in \mathbb{C}[x, y]$

□

Exercise 45.2.16. f is an elliptic function of order $n > 0$, then f' is an elliptic function of order m such that $n + 1 \leq m \leq 2n$. Both bounds can be attained

Solution. f' is elliptic since $f(z + \omega) = f(z) \Rightarrow f'(z + \omega) = f'(z)$ for all $\omega \in \Omega$. Suppose f has poles $[P_1], \dots, [P_k]$ with multiplicities r_1, \dots, r_k , $\sum r_i = n$, then f' also has poles $[P_1], \dots, [P_k]$ with multiplicities $r_1 + 1, \dots, r_k + 1$, $\sum r_i = n + k = m$, since $1 \leq k \leq n$, $n + 1 \leq m \leq 2n$

We can find an elliptic function f of order n which has $[P_1], \dots, [P_{n-m}]$ as its poles with multiplicities $1, \dots, 1, 2n+1-m$, then we get f' is another elliptic function which also has $[P_1], \dots, [P_{n-m}]$ as its poles with multiplicities $2, \dots, 2, 2n+2-m$, thus f' is of order m

□

Exercise 45.2.17. Prove that

$$\wp'(z) = \frac{2\sigma(z - \frac{\omega_1}{2})\sigma(z - \frac{\omega_2}{2})\sigma(z - \frac{\omega_3}{2})}{\sigma(\frac{\omega_1}{2})\sigma(\frac{\omega_2}{2})\sigma(\frac{\omega_3}{2})\sigma(z)^3}.$$

Solution. $\wp'(z)$ has a pole at $z = 0$ of order 3 and $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ as simple roots, thus

$$\wp'(z) = \lambda \frac{\sigma\left(z - \frac{\omega_1}{2}\right) \sigma\left(z - \frac{\omega_2}{2}\right) \sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma(z)^3}$$

for some $\lambda \in \mathbb{C}$, multiply by z^3 on both sides, and let $z \rightarrow 0$, since $\lim_{z \rightarrow 0} \frac{z}{\sigma(z)} = 1$, $\lim_{z \rightarrow 0} z^3 \wp'(z) = -2$, we have

$$-2 = -\lambda \sigma\left(\frac{\omega_1}{2}\right) \sigma\left(\frac{\omega_2}{2}\right) \sigma\left(\frac{\omega_3}{2}\right) \Rightarrow \lambda = \frac{2}{\sigma\left(\frac{\omega_1}{2}\right) \sigma\left(\frac{\omega_2}{2}\right) \sigma\left(\frac{\omega_3}{2}\right)}$$

Hence

$$\wp'(z) = \frac{2\sigma\left(z - \frac{\omega_1}{2}\right) \sigma\left(z - \frac{\omega_2}{2}\right) \sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right) \sigma\left(\frac{\omega_2}{2}\right) \sigma\left(\frac{\omega_3}{2}\right) \sigma(z)^3}$$

□

Let $\Omega \subseteq \mathbb{C}$ be a lattice and $\wp(z)$ the associated Weierstrass \wp -function. We have seen that $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = p(\wp(z))$, where $p(x) = 4x^3 - g_2x - g_3$. The following three problems examine the conditions under which the coefficients g_2 and g_3 of $p(x)$ are real numbers

Exercise 45.2.18. Prove that the following conditions are equivalent

- (i) $g_2, g_3 \in \mathbb{R}$
- (ii) $G_k \in \mathbb{R}$ for all $k \geq 3$
- (iii) $\wp(\bar{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$
- (iv) $\bar{\Omega} = \Omega$ (the last condition says that Ω is a *real lattice*)

Solution. (i) \Rightarrow (ii)

$$g_2 = 60G_4, g_3 = 140G_6 \in \mathbb{R} \Rightarrow G_4, G_6 \in \mathbb{R}$$

Since

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1)G_{2n}z^{2n-2} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + 7G_8z^6 + 9G_{10}z^8 + \cdots \end{aligned}$$

$$\begin{aligned} \wp'(z) &= -\frac{2}{z^3} + \sum_{n=2}^{\infty} (2n-1)(2n-2)G_{2n}z^{2n-3} \\ &= -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + 42G_8z^5 + 72G_{10}z^7 + \cdots \end{aligned}$$

$$\begin{aligned} \wp''(z) &= \frac{6}{z^4} + \sum_{n=2}^{\infty} (2n-1)(2n-2)(2n-3)G_{2n}z^{2n-4} \\ &= \frac{6}{z^4} + 6G_4 + 60G_6z^2 + 210G_8z^4 + 504G_{10}z^6 + \cdots \end{aligned}$$

So we can conclude $\wp''(z) - 6\wp(z)^2 + 30G_4 = z\varphi(z)$, where $\varphi(z)$ is a holomorphic elliptic function, hence $\wp''(z) - 6\wp(z)^2 + 30G_4 = 0$, then the coefficients of z^{2n} ($n \geq 1$) would be $(2n+1)(2n+2)(2n+3)(2n+4)G_{2n+4} - 6(2n+3)G_{2n+4}$ minus terms only involving $G_4, G_6, \dots, G_{2n+2}$ and real numbers, thus by induction, we know $G_{2n+4} \in \mathbb{R}$ ($n \geq 1$)

(ii) \Rightarrow (iii)

Since $\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1)G_{2n}z^{2n-2}$, if $G_k \in \mathbb{R}$ ($k \geq 3$), then $\wp(\bar{z}) = \overline{\wp(z)}$

(iii) \Rightarrow (iv)

The poles of $\overline{\wp(\bar{z})} = \wp(z)$ are exactly $\bar{\Omega}$, thus $\bar{\Omega} = \Omega$

(iv) \Rightarrow (i)

$$g_2 = 60G_4 = 60 \sum_{\omega \in \Omega^*} \frac{1}{\omega^4} = 60 \sum_{\omega \in \bar{\Omega}^*} \frac{1}{\omega^4} = \bar{g}_2 \Rightarrow g_2 \in \mathbb{R}, \text{ similarly, } g_6 \in \mathbb{R} \quad \square$$

Exercise 45.2.19. We say that Ω is *real rectangular* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, and that Ω is *real rhombic* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_2 = \bar{\omega}_1$. Prove that a lattice Ω is real if and only if it is real rectangular or real rhombic

Solution. If Ω is real rectangular or real rhombic, Ω is obviously a real lattice

Conversely, if Ω is a real lattice, suppose $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then there exists $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, otherwise, $\omega_1 \in \mathbb{R}^*, \omega_2 \in i\mathbb{R}^*$ or $\omega_2 \in \mathbb{R}^*, \omega_1 \in i\mathbb{R}^*$, since ω_1, ω_2 are linear independent, but then $\omega = \omega_1 + \omega_2 \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$ which is a contradiction

Since $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \bar{\omega} \in \mathbb{R}^*, \omega - \bar{\omega} \in i\mathbb{R}^*$, thus $\Omega \cap \mathbb{R}^* \neq \emptyset, \Omega \cap i\mathbb{R}^* \neq \emptyset$, let $\eta_1 = \min_{\eta \in \Omega \cap (0, \infty)} \eta$,

then $\Omega \cap \mathbb{R} = \mathbb{Z}\eta_1$, otherwise $\exists \eta \in \mathbb{R} \setminus \mathbb{Z}\eta_1$, then $\eta - \left\lfloor \frac{\eta}{\eta_1} \right\rfloor \eta_1 \in \Omega \cap (0, \infty)$ which is a contradiction

Similarly, $\Omega \cap i\mathbb{R} = \mathbb{Z}\eta_2$ for some $\eta_2 \in i(0, \infty)$. If $\Omega = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$, then Ω is real rectangular, if not, $\exists \gamma \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, such that $|\gamma| = \min_{\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})} |\omega|$, then $\gamma + \bar{\gamma} = \eta_1$ or $-\eta_1$, otherwise

$\gamma + \bar{\gamma} = k\eta_1$ for some $|k| \geq 2$

If $k = 2$, then $\gamma - \eta_1 = \eta_1 - \bar{\gamma} = -(\gamma - \eta_1) \Rightarrow \gamma - \eta_1 \in i\mathbb{R} \Rightarrow \gamma \in \mathbb{Z}\eta_1 + \mathbb{Z}(\gamma - \eta_1) \subseteq \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$

If $k > 2$, then $\gamma - \eta_1 \notin \mathbb{R} \cup i\mathbb{R}$ and $|\gamma - \eta_1| < |\gamma|$, similarly for $k \leq -2$, these are all contradictions

Similarly, we know that $\gamma - \bar{\gamma} = \eta_2$ or $-\eta_2$

Now, for any $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \bar{\omega} = k\eta_1 = k(\gamma + \bar{\gamma})$ for some $k \neq 0$, then $\omega - k\gamma = k\bar{\gamma} - \bar{\omega} = -(\bar{\omega} - k\bar{\gamma}) \Rightarrow \omega - k\gamma \in i\mathbb{R}$, if $\omega \neq k\gamma$, then $\omega - k\gamma = l\eta_2 = l(\gamma - \bar{\gamma}) \Rightarrow \omega \in \mathbb{Z}\gamma + \mathbb{Z}\bar{\gamma}$, therefore, we have $\Omega = \mathbb{Z}\gamma + \mathbb{Z}\bar{\gamma}$, Ω is real rhombic \square

Exercise 45.2.20. Let Ω be a real lattice. Define the real elliptic curve $E_{\mathbb{R}}$ to be the set $\{(x, y) \in \mathbb{R}^2 \mid y^2 = p(x)\}$. Prove that $E_{\mathbb{R}}$ has one or two connected components as Ω is real rhombic or real rectangular, respectively

Solution. The number of connected components of $E_{\mathbb{R}}$ is one or two if $p(x) = 0$ has one real root and two nonreal conjugate complex roots or three distinct real roots correspondingly

Since $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ are simple roots of $\wp'(z)$, the three simple roots of $p(x)$ are $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_3}{2}\right)$, since Ω is a real lattice, $G_k \in \mathbb{R}$ and $\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1)G_{2n}z^{2n-2}$

If Ω is real rectangular, then $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right)$ are both real, thus $E_{\mathbb{R}}$ has two connected components

If Ω is real rhombic, then $\wp\left(\frac{\omega_3}{2}\right)$ is real $\wp\left(\frac{\omega_1}{2}\right) \neq \wp\left(\frac{\omega_2}{2}\right)$ are nonreal conjugate, thus $E_{\mathbb{R}}$ has only one connected component \square

Complex structures on an open annulus

Exercise 45.2.21. $A(r, R) = \{r < |z| < R\}$ is biholomorphic to $\{s < |z| < S\}$ iff $R/r = S/s$, r can be 0, R can be ∞ , but not at the same time

Solution. By scaling or inversion we can assume $r = s = 1$ and $|f(z)| \rightarrow 1$ as $|z| \rightarrow 1$. Suppose $f : A(r, R) \rightarrow A(s, S)$ is a biholomorphism, then consider the Laurent series $f = \sum_{k=-\infty}^{\infty} c_k z^k$, for

$1 < t < R$, by Stokes theorem we have

$$A(t) = \frac{1}{2i} \int_{f(\{|z|=t\})} \bar{z} dz = \frac{1}{2i} \int_{|z|=t} \overline{f(z)} df(z) = \frac{1}{2i} \int_{|z|=t} \overline{f(z)} f'(z) dz = \pi \sum_{k \in \mathbb{Z}} k |c_k|^2 t^{2k}$$

As $t \rightarrow 1$, we have $A(t) \rightarrow \pi \Rightarrow \sum k|c_k|^2 = 1$, thus

$$A(t) - \pi t^2 = \pi t^2 \sum_{k \in \mathbb{Z}} k|c_k|^2 (t^{2k-2} - 1) \geq 0$$

Thus $A(t) \geq \pi t^2$, as $t \rightarrow R$, $A(t) \rightarrow \pi S^2 \geq \pi R^2 \Rightarrow S \geq R$. Therefore we have $S = R$ □

Chapter 46

Exercises in category

X_1, X_2 iso and Y_1, Y_2 iso implies $\text{Hom}(X_1, Y_1), \text{Hom}(X_2, Y_2)$ iso

Exercise 46.0.1. In category \mathcal{C} , if $X \xrightarrow{\phi_X} X'$, $Y \xrightarrow{\phi_Y} Y'$ are isomorphisms, then $\text{Hom}(X, Y)$, $\text{Hom}(X', Y')$ are in bijective correspondence

Solution. Consider $\text{Hom}(X, Y) \rightarrow \text{Hom}(X', Y')$, $f \mapsto \phi_Y f \phi_X^{-1}$ and $\text{Hom}(X', Y') \rightarrow \text{Hom}(X, Y)$, $f' \mapsto \phi_Y^{-1} f' \phi_X$ which are inverses to each other

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi_X \downarrow & & \downarrow \phi_Y \\ X' & \xrightarrow{f'} & Y' \end{array}$$

□

Exercise 46.0.2. Suppose the bottom row of the following commutative diagram is exact, $gf = 0$, then there exists a such that the following diagram commutes

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \exists a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Solution. Since $0 = cgf = g'b f$ and bottom row is exact, we have

$$\begin{array}{ccc} & A & \\ \swarrow \exists a & \downarrow b f & \\ A' & \xrightarrow{f'} & \ker g' \end{array}$$

□

Exercise 46.0.3. $F, G : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ are functors, $F \xrightarrow{\eta} G$ is a natural transformation iff η is natural on each factor

Solution. We have commutative diagram

$$\begin{array}{ccccc} F(A, B) & \xrightarrow{F(f, 1)} & F(A', B) & \xrightarrow{F(1, g)} & F(A', B') \\ \downarrow \eta_{A, B} & & \downarrow \eta_{A', B} & & \downarrow \eta_{A', B'} \\ G(A, B) & \xrightarrow{G(f, 1)} & G(A', B) & \xrightarrow{G(1, g)} & G(A', B') \end{array}$$

□

Fully faithful functor is injective on objects up to isomorphism

Exercise 46.0.4. A fully faithful functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is injective on objects up to isomorphism

Solution. Suppose $F(X) = F(Y) = T$, let $f : X \rightarrow Y$ be the map corresponds to 1_T in $\text{Hom}(F(X), F(Y))$, then $f : X \rightarrow Y$ is an isomorphism because we can also let $g : Y \rightarrow X$ be the map corresponds to 1_T as in $\text{Hom}(F(Y), F(X))$, then $F(g \circ f) = F(g) \circ F(f) = 1_T \circ 1_T = 1_T$, thus $g \circ f$ corresponds to 1_T in $\text{Hom}(F(X), F(X))$, but $F(1_X) = 1_{F(X)} = 1_T$, thus $g \circ f = 1_X$, similarly, $f \circ g = 1_Y$ \square

Exercise 46.0.5. Suppose \mathcal{A} is an abelian category, show \mathcal{A} is balanced. For any $A \xrightarrow{f} B$, $\ker f \xrightarrow{i} A$ is a monomorphism, $B \xrightarrow{\pi} \text{coker } f$ is an epimorphism, and $\text{im } f := \ker \text{coker } f$, $\text{coim } f := \text{coker } \ker f$ are isomorphic

Solution. Suppose $A \xrightarrow{f} B$ is a bimorphism, it is the equaliser of $B \xrightarrow[\pi]{\pi} \text{coker } f$, then $\pi = 0$, $\text{coker } f = 0$, but $A \xrightarrow{1_A} A$ is the kernel of $A \rightarrow 0$, hence A, B are isomorphic $\ker f \xrightarrow{i} A$ is a monomorphism due to the following diagram

$$\begin{array}{ccccc} & C & & & \\ & \downarrow g=0 & \searrow 0 & & \\ \ker f & \xrightarrow{i} & A & \xrightarrow{f} & B \end{array}$$

$B \xrightarrow{\pi} \text{coker } f$ is a monomorphism due to the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{\pi} & \text{coker } f \\ & \searrow 0 & \downarrow g=0 & & \\ & & C & & \end{array}$$

Now let's show coimage and image are isomorphic. $fi = 0$ induces $\text{coker } i \xrightarrow{g} B$, claim that g is monic

Suppose $X \xrightarrow{x} \text{coker } i$ is morphism such that $gx = 0$, it induces $\text{coker } x \xrightarrow{j} B$, since $qpk = 0$, $fk = jqp k = 0$ induces $\ker qp \xrightarrow{l} \ker f$, since qp is epi, $pk = pil = 0$ induces $\text{coker } x \xrightarrow{r} \text{coker } i$, since p is epi, $p = rqp \Rightarrow rq = 1_{\text{coker } i}$, hence q is monic, $qx = 0 \Rightarrow x = 0$

$$\begin{array}{ccccccc} & & \ker qp & & & & \\ & \swarrow l & \downarrow k & & & & \\ \ker f & \xrightarrow{i} & A & \xrightarrow{f} & B & \xrightarrow{\pi} & \text{coker } f \\ & & \downarrow p & \nearrow g & \uparrow j & & \\ X & \xrightarrow{x} & \text{coker } i & \xrightarrow[\leftarrow]{\text{---}r\text{---}} & \text{coker } x & & \end{array}$$

$\pi f = 0$ induces $A \xrightarrow{h} \ker \pi$, claim that h is epi

Suppose $\ker \pi \xrightarrow{y} Y$ is morphism such that $yh = 0$, it induces $A \xrightarrow{p} \ker y$, since $qjk = 0$, $qf = qj k p = 0$ induces $\text{coker } f \xrightarrow{m} \text{coker } j k$, since $j k$ is monic, $qj = m \pi j = 0$ induces $\ker \pi \xrightarrow{s} \ker y$, since j is monic, $j = j k s \Rightarrow ks = 1_{\ker \pi}$, hence k is epi, $yk = 0 \Rightarrow y = 0$

$$\begin{array}{ccccccc} & & \text{coker } j k & & & & \\ & \swarrow m & \uparrow q & & & & \\ \ker f & \xrightarrow{i} & A & \xrightarrow{f} & B & \xrightarrow{\pi} & \text{coker } f \\ & & \downarrow p & \nearrow h & \uparrow j & & \\ & & \ker y & \xrightarrow[\leftarrow]{\text{---}s\text{---}} & \ker \pi & \xrightarrow{y} & Y \end{array}$$

Since $\text{im } f \rightarrow B$ is monic, $A \rightarrow \text{coim } f$ is epi, g, h both induce $\text{coim } f \xrightarrow{\phi} \text{im } f$, then ϕ is monic and epi hence iso

$$\begin{array}{ccccccc}
 \ker f & \xrightarrow{\quad} & A & \xrightarrow{\quad f \quad} & B & \twoheadrightarrow & \operatorname{coker} f \\
 & & \downarrow & & \uparrow & & \\
 & & \operatorname{coim} f & \xrightarrow{\quad \phi \quad} & \operatorname{im} f & &
 \end{array}$$

□

bounded double complex with exact rows or exact columns has exact total complex

Exercise 46.0.6. C is a bounded double complex with exact rows or exact columns, then $\operatorname{Tot}(C)$ is exact

Solution. Without loss of generality, we may assume C is bounded in the first quadrant and has exact rows, use d', d'', d to denote row, column and total differentials

$\operatorname{Tot}(C)$ is exact for all $n < 0$ since $\operatorname{Tot}(C)_n = 0$ for all $n < 0$. now suppose $n \geq 0$,

$d\left(\sum_{k=0}^n x_{k,n-k}\right) = 0$, i.e. $d'x_{k+1,n-k-1} + d''x_{k,n-k} = 0$ for $0 \leq k < n$. Let $x_{0,n+1} = 0$, we can construct $x_{k,n+1-k}$ for $k > 0$ inductively such that $d''x_{k,n-k+1} + d'x_{k+1,n-k} = x_{k,n-k}$ for $0 \leq k \leq n$ as follow:

For $k \geq -1$

$$\begin{aligned}
 d'(x_{k+1,n-k-1} - d''x_{k+1,n-k}) &= d'x_{k+1,n-k-1} - d'd''x_{k+1,n-k} \\
 &= d'x_{k+1,n-k-1} + d''d'x_{k+1,n-k} \\
 &= d'x_{k+1,n-k-1} + d''(d''x_{k,n-k+1} + d'x_{k+1,n-k}) \\
 &= d'x_{k+1,n-k-1} + d''x_{k,n-k} \\
 &= 0
 \end{aligned}$$

By exactness of rows, there exists $x_{k+2,n-k-1}$ such that

$$d'x_{k+2,n-k-1} = x_{k+1,n-k-1} - d''x_{k+1,n-k} \Leftrightarrow d''x_{k+1,n-k} + d'x_{k+2,n-k-1} = x_{k+1,n-k-1}$$

Therefore

$$\begin{aligned}
 d\left(\sum_{k=0}^{n+1} x_{k,n+1-k}\right) &= \sum_{k=1}^{n+1} (d'x_{k,n+1-k} + d''x_{k,n+1-k}) \\
 &= \sum_{k=1}^{n+1} (x_{k-1,n-k+1} - d''x_{k-1,n-k+2} + d''x_{k,n+1-k}) \\
 &= \sum_{k=0}^n (x_{k,n-k} - d''x_{k,n-k+1}) + \sum_{k=1}^{n+1} d''x_{k,n+1-k} \\
 &= \sum_{k=0}^n x_{k,n-k}
 \end{aligned}$$

□

C, D acyclic $\Rightarrow C \otimes D$ acyclic

Exercise 46.0.7. C, D are chain complexes with negative degree terms zeros, $H_n(C) = H_n(D) = 0$ for $n \neq 0$, then so is $C \otimes D$

Solution. Apply Exercise 46.0.6

Exercise 46.0.8. f is a retract of g in the arrow category, if g is an isomorphism, so is f

$$\begin{array}{ccccc}
 X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X'
 \end{array}$$

Proof. $i'g^{-1}r$ is the inverse to f

□

Chapter 47

Exercises in partial differential equations

Exercise 47.0.1. Consider the heat equation with Neumann's boundary condition:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma \times \mathbb{R}^+ \\ u(x, 0) = v(x), & \text{in } \Omega \end{cases}$$

(a) Show that $\overline{u(t)} = \bar{v}$ for $t \geq 0$, where $\bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v(x) dx$ denotes the average of v

(b) Show that $\|u(t) - \bar{v}\| \rightarrow 0$ as $t \rightarrow \infty$

Solution. (a) By divergence theorem, we have

$$0 = \int_{\Omega} u_t - \Delta u = \int_{\Omega} u_t + \nabla 1 \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_{\Omega} u_t = \left(\int_{\Omega} u \right)_t$$

Hence $\int_{\Omega} u = \int_{\Omega} v \Rightarrow \bar{u} = \bar{v}$

(b) By divergence theorem, we have

$$0 = \int_{\Omega} u u_t - u \Delta u = \frac{1}{2} \left(\int_{\Omega} u^2 \right)_t + \int_{\Omega} |\nabla u|^2 \Rightarrow \frac{1}{2} \left(\int_{\Omega} u^2 \right)_t = - \int_{\Omega} |\nabla u|^2 \leq 0$$

Hence $\int_{\Omega} u^2 \leq \int_{\Omega} v^2$. On the other hand, we have

$$\begin{aligned} 0 &= \int_{\Omega} (u_t - \Delta u)^2 \\ &= \int_{\Omega} u_t^2 - 2u_t \Delta u + (\Delta u)^2 \\ &= \int_{\Omega} u_t^2 - 2\nabla u_t \cdot \nabla u + (\Delta u)^2 \\ &= \int_{\Omega} 2(\Delta u)^2 + \left(\int_{\Omega} |\nabla u|^2 \right)_t \end{aligned}$$

Which implies $\int_{\Omega} (\Delta u)^2 = -\frac{1}{2} \left(\int_{\Omega} |\nabla u|^2 \right)_t$, thus

$$\left(\int_{\Omega} |\nabla u|^2 \right)^2 = \left(\int_{\Omega} u \Delta u \right)^2 \leq \int_{\Omega} u^2 \cdot \int_{\Omega} (\Delta u)^2 \leq \int_{\Omega} v^2 \cdot \int_{\Omega} (\Delta u)^2 = -\frac{1}{2} \int_{\Omega} v^2 \cdot \left(\int_{\Omega} |\nabla u|^2 \right)_t$$

Denote $\phi := \int_{\Omega} |\nabla u|^2$ which is a function of t , $C := \frac{1}{2} \int_{\Omega} v^2$, then the above equation becomes

$$\phi^2 \leq -C\phi' \Rightarrow 0 \geq \phi^2 + C\phi' \Rightarrow 0 \geq 1 + C \frac{\phi'}{\phi^2} = \left(t - \frac{C}{\phi}\right)'$$

Which implies

$$t - \frac{C}{\phi(t)} \leq -\frac{C}{\phi(0)} \Rightarrow \frac{C}{\phi(t)} \geq t + \frac{C}{\phi(0)} \geq t \Rightarrow \phi(t) \leq \frac{C}{t}$$

Thus $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$

Now apply Poincaré's lemma, we get

$$\|u - \bar{v}\|_{L^2} = \|u - \bar{u}\|_{L^2} \leq C \|\nabla u\|_{L^2} \rightarrow 0, t \rightarrow \infty$$

□

Exercise 47.0.2.

$$\begin{aligned} \frac{d}{dr} \left(\frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS \right) &= \frac{d}{dr} \left(\frac{1}{|\partial B(0, 1)|} \int_{\partial B(0, 1)} u(x + rz) dS \right) \\ &= \frac{1}{|\partial B(0, 1)|} \int_{\partial B(0, 1)} \frac{d}{dr} u(x + rz) dS \\ &= \frac{1}{|\partial B(0, 1)|} \int_{\partial B(0, 1)} z \cdot \nabla u(x + rz) dS \\ &= \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} \nu \cdot \nabla u(y) dS \\ &= \frac{1}{|\partial B(x, r)|} \int_{B(x, r)} \Delta u(y) dy \\ &= \frac{r}{n} \frac{1}{|B(x, r)|} \int_{B(x, r)} \Delta u(y) dy \end{aligned}$$

$$\begin{aligned} \frac{d}{dr} \left(\int_{B(x, r)} u(y) dy \right) &= \frac{d}{dr} \int_0^r \left(\int_{\partial B(x, s)} u(y) dS \right) ds \\ &= \int_{\partial B(x, r)} u(y) dS \end{aligned}$$

Exercise 47.0.3. $\square u = 0$ in \mathbb{R}^{3+1} , $u(x, 0) = 0$, $u_t(x, 0) = f(x) \in C^2(\mathbb{R}^3)$, show that $\int_0^\infty |u(0, t)|^2 dt \leq C \|f\|_{L^2(\mathbb{R}^3)}$

Proof. Hint: $u(x, t) = \frac{t}{4\pi} \int_{S^2} f(x + tw) dS_w = -\frac{1}{4\pi} \int_{S^2} \int_t^\infty \frac{d}{d\lambda} f(x + \lambda t) d\lambda$

$$u(0, t) = \frac{t}{4\pi} \int_{S^2} f(tw) dS_w, |u(x, t)| \leq \frac{C}{t} \int_{\mathbb{R}^3} |\nabla f| dx$$

□

Chapter 48

Exercises in algebraic topology

Exercise 48.0.1. M is a locally Euclidean, Hausdorff and connected manifold, then paracompactness implies second countable

Proof. An open cover by precompact coordinate charts has a locally finite open refinement $\{U_i\}$, each U_i is precompact and second countable

Define $S_0 = \{U_0\}$ for some U_0 , since $\{U_i\}$ is locally finite, define S_1 to be the union of S_0 and those intersects U_0 , repeating this process, we get S_2, \dots, S_n, \dots , define $S = \bigcup_{n=0}^{\infty} S_n$

M is connected thus path connected, pick any $x_0 \in U_0$, for any $x \in M$, there is a path γ connecting x_0 and x , since γ is compact, it can be covered by S . Hence S is an open cover of M , thus M is second countable \square

Exercise 48.0.2. If G is a discrete group, P is connected, $P \xrightarrow{p} X$ is a principal G bundle iff it is a regular cover with $\text{Aut}(p) = G$

Solution. $P \xrightarrow{p} X$ is a fiber bundle thus a cover, G acts regularly on fibers and $G \leq \text{Aut}(p)$ \square

Exercise 48.0.3. Use Theorem 12.0.18 to prove homotopy invariance of maps on homology

Solution. Suppose $F : X \times I \rightarrow Y$ is a homotopy between f and g , we only need to prove i_0, i_1 are naturally chain homotopic since $F i_0 = f, F i_1 = g$

$$\begin{array}{ccccc} C_{n+1}(X) & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) \\ i_0 \downarrow \downarrow i_1 & & i_0 \downarrow \downarrow i_1 & & i_0 \downarrow \downarrow i_1 \\ C_{n+1}(X \times I) & \longrightarrow & C_n(X \times I) & \longrightarrow & C_{n-1}(X \times I) \\ \downarrow F & & \downarrow F & & \downarrow F \\ C_{n+1}(Y) & \longrightarrow & C_n(Y) & \longrightarrow & C_{n-1}(Y) \end{array}$$

Consider \mathcal{Top} with model $\mathcal{M} = \{\Delta^n\}$, $F, G : \mathcal{Top} \rightarrow Ch_{\geq 0}$, $F(X) = C_*(X)$, $G(X) = C_*(X \times I)$, $H_i(\Delta^n \times I) = 0$ for $i \neq 0$, $F_k(X) = \left\{ \Delta^k \xrightarrow{\text{id}} \Delta^k \xrightarrow{\sigma} X \right\}$, there is an obvious natural equivalence $\phi_0 : H_0 F \rightarrow H_0 G$, then lifts i_0, i_1 are naturally chain homotopic \square

Exercise 48.0.4. K is a CW complex, $X \xrightarrow{f} Y$ is a weak equivalence, then $[K, X] \rightarrow [K, Y]$ is a bijection

Exercise 48.0.5. Quotient map $X \xrightarrow{q} Y$ is a homeomorphism iff q is bijective

Solution. If q is bijective, then for any open subset $U \subseteq X$, $U = q^{-1}(q(U))$, by definition, $q(U)$ is open, i.e. q^{-1} is continuous \square

Cofibration in a Hausdorff space is closed

Exercise 48.0.6. If X is Hausdorff, then cofibration $A \xrightarrow{i} X$ is closed. This is not true if X is not Hausdorff as showed in Example 40.0.12

Solution. Suppose $A \xrightarrow{i} X$ is a not closed, $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I$ is the retraction, pick any $x \in \overline{A} \setminus A$ with x_n converging to x , then $A \times \{1\} \ni r(x, 1) = r(\lim x_n, 1) = \lim r(x_n, 1) = \lim(x_n, 1) = (x, 1)$ which is a contradiction \square

Exercise 48.0.7. $\mathbb{R} \times \mathbb{R} \xrightarrow{\wedge} \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \xrightarrow{\vee} \mathbb{R}$ are continuous

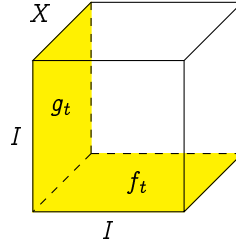
Solution. $x \wedge y = \frac{x + y - |x - y|}{2}$, $x \vee y = \frac{x + y + |x - y|}{2}$ \square

Exercise 48.0.8. $Y^I \rightarrow Y$, $\gamma \mapsto \gamma(0)$ and $Y^I \rightarrow Y \times Y$, $\gamma \mapsto (\gamma(0), \gamma(1))$ are Hurewicz fibrations

Solution. Need $g(x, s) = H(x, 0, s)$, $f(x, t) = H(x, t, 0)$ so that $g(x, 0) = f(x, 0)$

$$\begin{array}{ccc} X & \xrightarrow{g} & Y^I \\ \downarrow & \nearrow H & \downarrow \\ X \times I & \xrightarrow{f_t} & Y \end{array}$$

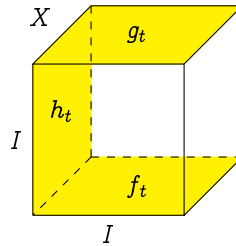
$X \times I^2$ can be deformed onto $X \times I \cup X \times I = X \times (I \cup I)$



Need $h(x, s) = H(x, 0, s)$, $f(x, t) = H(x, t, 0)$, $g(x, t) = H(x, t, 1)$ so that $h(x, 0) = f(x, 0)$, $h(x, 1) = g(x, 0)$

$$\begin{array}{ccc} X & \xrightarrow{h} & Y^I \\ \downarrow & \nearrow H & \downarrow \\ X \times I & \xrightarrow{(f_t, g_t)} & Y \times Y \end{array}$$

$X \times I^2$ can be deformed onto $X \times I \cup X \times I \cup X \times I = X \times (I \cup I \cup I)$



\square

Chapter 49

Exercises in differential topology

$\text{Hom}(V, W) = V^* \otimes W$

Exercise 49.0.1. $\text{Hom}(V, W) \rightarrow V^* \otimes W$, $A \mapsto \sum_{i,j} a_{ji} v_i^* \otimes w_j$ is an isomorphism where $A = (a_{ij})$ is the matrix with respect to basis $\{v_1^*, \dots, v_m^*\}, \{w_1, \dots, w_n\}$

Solution. $A(v_i) = \sum_j a_{ji} w_j$ □

Exercise 49.0.2. Suppose M, N are smooth manifolds of dimension m, n , $f : M \rightarrow N$ is a smooth map, $(x^1, \dots, x^m), (y^1, \dots, y^n)$ are local coordinates around $p \in M, q = f(p) \in N$, then the corresponding matrix of df with respect to basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}), (\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n})$ is $(\frac{\partial y^i}{\partial x^j})$. In particular, this gives the change of coordinates formula

Solution.

$$df \left(\frac{\partial}{\partial x^i} \right) (g) = \frac{\partial (g \circ f)}{\partial x^i} = \sum_j \frac{\partial g}{\partial y^j} \frac{\partial y^j}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} (g)$$

According to Exercise 49.0.1, $df = \sum_{i,j} \frac{\partial y^j}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^j}$, we can define higher differential $d^k f = \sum_{i_1, \dots, i_k, j} \frac{\partial y^j}{\partial x^{i_1} \dots \partial x^{i_k}} dx^{i_1} \dots dx^{i_k} \otimes \frac{\partial}{\partial y^j}$ □

Exterior derivative of one form

Exercise 49.0.3. Suppose $\omega \in \Omega^1(M)$, $X, Y \in TM$, then $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$

Solution. By linearity, we can assume $\omega = u dv$, then

$$\begin{aligned} d\omega(X, Y) &= d(u dv)(X, Y) \\ &= du \wedge dv(X, Y) \\ &= du(X) dv(Y) - du(Y) dv(X) \\ &= XuYv - YuXv \end{aligned}$$

And

$$\begin{aligned} &X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) \\ &= X(u dv(Y)) - Y(u dv(X)) - u dv([X, Y]) \\ &= Xu dv(Y) + uX(dv(Y)) - Yu dv(X) - uY(dv(X)) - u[X, Y]v \\ &= XuYv + uXYv - YuXv - uYXv - uXYv + uYXv \\ &= XuYv - YuXv \end{aligned}$$

□
Pushforward of vector field

Exercise 49.0.4. Suppose $\phi : M \rightarrow N$ is a map of smooth manifolds, then $X(f \circ \phi) = ((\phi_* X) f) \circ \phi$

Solution. $X(f \circ \phi)(p) = X_p(f \circ \phi) = \phi_p X_p(f) = (\phi_* X)_{\phi(p)}(f) = ((\phi_* X)f)(\phi(p))$ \square

Naturality of Lie bracket

Exercise 49.0.5. Suppose X, Y are vector fields on M , $\phi : M \rightarrow N$ is a smooth map, then $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$

Solution. Apply Exercise 49.0.4

$$\begin{aligned}
 \phi_*[X, Y](f) &= [X, Y](f \circ \phi) \\
 &= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
 &= X(((\phi_* Y)f) \circ \phi) - Y(((\phi_* X)f) \circ \phi) \\
 &= ((\phi_* X)((\phi_* Y)f) \circ \phi) - ((\phi_* Y)((\phi_* X)f) \circ \phi) \\
 &= ([\phi_* X, \phi_* Y]f) \circ \phi \\
 &= [\phi_* X, \phi_* Y]f
 \end{aligned}$$

\square

Chapter 50

Exercises in bundles

Exercise 50.0.1. $E \xrightarrow{p} B$ is a Serre fibration, $A \xhookrightarrow{i} X$ is a subcomplex, if either p or i is a weak equivalence, then we have

$$\begin{array}{ccc} A & \xrightarrow{f} & E \\ i \downarrow & \nearrow \exists h & \downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

Solution. If p is a weak equivalence, then fibers are weak contractible
If i is a weak equivalence, then X deformation retracts onto A

□

Chapter 51

Exercises in complex geometry

Chapter 52

Exercises in Lie groups and Lie algebras

Exercise 52.0.1. Suppose \mathfrak{g} is a real semisimple Lie algebra with a negative definite Killing form, then \mathfrak{g} is the Lie algebra of some compact Lie group G

A is the direct sum of ideals \Rightarrow A is the product of these ideals

Exercise 52.0.2. Suppose A is a nonassociative algebra, $A = I_1 \oplus \cdots \oplus I_n$ is a direct sum of ideals, then $I_i I_j \subseteq I_i \cap I_j = 0$, hence $A = I_1 \times \cdots \times I_n$ can be viewed as product of ideals

Remark 52.0.3. If $A = I_1 \oplus \cdots \oplus I_n$ is just a direct sum of subalgebras, Exercise 52.0.2 may not hold true

Pairwise commuting matrices can be diagonalized simultaneously

Exercise 52.0.4. Let $S \subseteq M(n, \mathbb{F})$, ($\overline{\mathbb{F}} = \mathbb{F}$) be a set such that $[X, Y] = 0, \forall X, Y \in S$, then elements in S can be diagonalized simultaneously

Solution. Suppose $0 \neq V_\lambda$ is the λ -eigenspace of $X \in S$, then for any $Y \in S$, $XYv = YXv = \lambda Yv$ for $v \in V_\lambda$, thus $YV_\lambda \subseteq V_\lambda$, thus V_λ is an invariant subspace for all $Y \in \mathfrak{g}$, since Y are all semisimple, by induction we can write V as a direct sum of V_λ 's, and they are all invariant under any $Y \in \mathfrak{g}$, we only need to show that all elements of \mathfrak{g} can be diagonalized simultaneously on each V_λ , if $X|_{V_\lambda} = 1_{V_\lambda}$, then we are done, otherwise we can decompose it into smaller eigenspaces \square

Finite dimensional toral Lie algebra is abelian and its elements can be diagonalized simultaneously

Exercise 52.0.5. Let V be a finite dimensional \mathbb{F} vector space with $\overline{\mathbb{F}} = \mathbb{F}$, and $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a toral Lie algebra, then \mathfrak{g} is abelian. Moreover, $X \in \mathfrak{g}$ can be diagonalized simultaneously

Solution. Suppose $ad(X)|_{\mathfrak{g}} \neq 0$ for some $X \in \mathfrak{g}$, since $\overline{\mathbb{F}} = \mathbb{F}$, there exists $0 \neq Y \in \mathfrak{g}$ and $\lambda \neq 0$ such that $ad(X)(Y) = \lambda Y$, by Proposition 9.6.3, $ad(Y)$ is semisimple, suppose λ_j, X_j are the eigenvalues and linearly independent eigenvectors, then we have $X = \sum c_j X_j$ with $c_j \neq 0$, and $0 = ad(Y)(\lambda Y) = ad(Y)ad(X)(Y) = -ad(Y)^2(X) = -ad(Y)^2(\sum c_j X_j) = -\sum c_j \lambda_j^2 X_j$, thus $c_j \lambda_j^2 = 0 \Rightarrow \lambda_j = 0$, but $0 \neq \lambda Y = ad(X)(Y) = -ad(Y)(X) = -ad(Y)(\sum c_j X_j) = -\sum c_j \lambda_j X_j = 0$ which is a contradiction

Now that we know $[\mathfrak{g}, \mathfrak{g}] = 0$, use Lemma 52.0.4, we know all elements of \mathfrak{g} are diagonalizable simultaneously \square

Lie group homomorphism has constant rank

Exercise 52.0.6. Lie group homomorphism has constant rank

Solution. Let $\phi : G \rightarrow G'$ be a Lie group homomorphism, for any $g \in G$, it suffices to show $\text{rank}(d\phi)_g = \text{rank}(d\phi)_1$, since $\phi(gh) = \phi(g)\phi(h)$, thus $\phi \circ L_g = L_{\phi(g)} \circ \phi$, $(d\phi)_g(dL_g)_1 = d(L_{\phi(g)})_1(d\phi)_1$, and left multiplications are isomorphisms, we have $\text{rank}(d\phi)_g = \text{rank}(d\phi)_1$ \square

Exercise 52.0.7. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M , for any equivariant map $f : M \rightarrow N$, f has constant rank

Solution. For any $x \in M$, denote $y = f(x)$, it suffices to show $\text{rank}(df)_x = \text{rank}(df)_{gx}$ since G acts transitively on M , note that $f(gx) = gf(x)$, thus $f \circ L_g = L_g \circ f$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\text{rank}(df)_x = \text{rank}(df)_{gx}$ \square

Exercise 52.0.8. If $\phi : G \rightarrow H$ be a bijective Lie group homomorphism, then it is an isomorphism

Solution. Apply Exercise 52.0.6 and Theorem 21.0.15 \square

Exercise 52.0.9. Compact semisimple Lie group G has finite center

Solution. Since $\mathfrak{g} = \text{Lie}(G)$ is semisimple, $\text{Lie}(Z(G)) \leq Z(\mathfrak{g}) = 0$, thus $Z(G)$ is discrete, but G is compact, so $Z(G)$ is finite \square

rudimentary facts about topological groups

Exercise 52.0.10. G is a topological group, A is called **symmetric** if $A = A^{-1}$

1. Topology of G is translation invariant, U is open $\Rightarrow xU, Ux$ are open
2. $e \in U$ is a neighborhood, then $e \in V \subseteq U$ a symmetric neighborhood
3. $e \in U$ is a neighborhood, then $e \in V \subseteq VV \subseteq U$ with V being a symmetric neighborhood
4. $H \leq G$ is a subgroup, then so is \bar{H}
5. Open subgroups of G are also closed (closed groups are not necessarily open, consider $\{e\}$)
6. $K_1, K_2 \subseteq G$ are compact sets, so is $K_1 K_2$
7. Suppose G is a connected, U is a neighborhood of 1, then $G = \langle U \rangle$

Solution.

1. Multiplication by x is an isomorphism with x^{-1} being its inverse
2. Take $U \cap U^{-1}$
3. Since the multiplication $G \times G \rightarrow G$ is continuous, consider the preimage of U which contains $V_1 \times V_2$, take $V \subseteq V_1 \cap V_2$ symmetric
4. If $x_\alpha \rightarrow x$, $y_\beta \rightarrow y$, then $x_\alpha^{-1} \rightarrow x^{-1}$, $x_\alpha y_\beta \rightarrow xy$, since these maps are continuous. From this we know that $\bar{H} = \bigcap F$ where F runs over all closed subgroup containing H
5. Suppose $H \leq G$ is open, then $H = G \setminus \bigcup_{x \neq e} xH$ is closed, thus if G is connected, then $H = G$
6. $K_1 K_2$ is the image of $K_1 \times K_2$ under multiplication
7. By b, we there is a symmetric neighborhood $1 \in V \subseteq U$, let V_k be the subset of elements can be written in the product of no more the k elements in V , then $V_1 = V$, $V_k = V_1 V_{k-1}$ is open by induction, $\langle V \rangle = \bigcup_{k=1}^{\infty} V_k$ is also open, by e, G is generated by V hence by U , and if G is not connected, $G_0 = \langle V \rangle$ is called the identity component of G

\square

Exercise 52.0.11. G is a topological group, if G is T_1 , then G is Hausdorff, if G is not T_1 , then $H := \overline{\{e\}}$ is normal subgroup, G/H is a Hausdorff topological group

Solution. If G is T_1 , according to Exercise 52.0.10, for $x \neq y$, $\exists e \in VV \subseteq U$ with V a symmetric neighborhood of e disjoint from $y^{-1}x$, then $xV \cap yV = \emptyset$, suppose $z = xv_1 = yv_2$, then $y^{-1}x = v_2^{-1}v_1 \in VV$ thus reaches a contradiction

According to Proposition 52.0.10, $H = \bigcap H_i$, H_i runs over closed subgroups of G , thus H is the smallest closed subgroup, if H is normal, otherwise $xHx^{-1} \cap H$ is a smaller closed subgroup for some x

In G/H , identity is closed, by invariance of topology under translation, every point is closed, meaning G/H is T_1 thus Hausdorff

Checking G/H is still a topological group: $g \in \bigcup_x xH$ open in G , then $g^{-1} \in (\bigcup_x xH)^{-1} = \bigcup_x H^{-1}x^{-1} = \bigcup_x Hx^{-1} = \bigcup_x x^{-1}H$

If $V \times W \rightarrow VW \subseteq \bigcup_x xH$, then $vw \in \bigcup_x xH$, $\forall v \in V, w \in W$, then $\forall h \in H$, $vhw = vww^{-1}hw \in \bigcup_x xH$, therefore, $VH \times WH \rightarrow VHW H \subseteq \bigcup_x xH$, notice that VH is open as long as V is open since $VH = \bigcup_{h \in H} Vh$ \square

Exercise 52.0.12. $(,)_B$ is the bilinear form given by matrix B , $O(B) = \{X \in GL_n(\mathbb{C}) | X^T B X = 1\}$, the Lie algebra is $\mathfrak{o}(B) = \{X \in M_n(\mathbb{C}) | X^T B + B X = 0\}$

Solution. $\left. \frac{d}{dX} \right|_{X=0} (e^{X^T} B e^X) = X^T B + B X = 0$ \square

Exercise 52.0.13. $T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^\times \right\} \subseteq SL(2, \mathbb{C}) = G$ is the torus, the Weyl group

$W(T) = N_G(T)/Z(T) = N/T \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

Solution. Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, ad - bc = 1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} adx - bcx^{-1} & ab(x^{-1} - x) \\ cd(x - x^{-1}) & adx^{-1} - bcx \end{pmatrix} \in T$$

For any $x \in \mathbb{C}^\times$, which implies that $ab = cd = 0 \Rightarrow a = d = 0$ or $b = c = 0$ and

$$\begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} & -b^{-1} \\ b & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

\square

Chapter 53

Exercises in algebraic geometry

Exercise 53.0.1. $H = V(f)$ is a hypersurface, $f(t_1, \dots, t_n, x) = a_0(t_1, \dots, t_n)x^m + \dots + a_m(t_1, \dots, t_n)$

$$\begin{array}{ccc} H & \hookrightarrow & \mathbb{A}^{n+1} \\ & \searrow \varphi & \downarrow \\ & & \mathbb{A}^n \end{array}$$

φ is finite iff $a_0 \neq 0$ is a constant. φ is quasifinite $\Rightarrow a_0, \dots, a_m$ don't have common zeros

Chapter 54

Exercise in functional analysis

Part XVII

Miscellaneous

54.1 Hodge structure

Theorem 54.1.1 (Stokes theorem). $\langle \partial\Omega, \omega \rangle = \langle \Omega, d\omega \rangle$, here $\langle \Omega, \omega \rangle = \int_{\Omega} \omega$

Theorem 54.1.2 (de Rham's theorem). M is a smooth manifold. $H_{\text{dR}}^p(X; \mathbb{R}) \xrightarrow{\cong} H^p(X; \mathbb{R})$ is an isomorphism

Proof. Since \mathbb{R} is a divisible abelian group, thus an injective \mathbb{Z} module, hence $\text{Ext}_{\mathbb{Z}}^1(A, \mathbb{R})$, thus universal coefficient theorem gives exact sequence

$$0 = \text{Ext}_{\mathbb{Z}}^1(H_{p-1}(X; \mathbb{Z}), \mathbb{R}) \rightarrow H^p(M; \mathbb{R}) \rightarrow \text{Hom}(H_p(X; \mathbb{Z}), \mathbb{R}) \rightarrow 0$$

The isomorphism is given by $H_{\text{dR}}^p(X; \mathbb{R}) \rightarrow \text{Hom}(H_p(X), \mathbb{R})$, $\omega \mapsto \int_{-} \omega$ □

54.2 Plucker embedding

Definition 54.2.1. Consider the Grassmannian $\text{Span}\{v_1, \dots, v_k\} \in \text{Gr}_k(n)$

$$\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{bmatrix}$$

Denote W_{i_1, \dots, i_k} as the minor of i_1, \dots, i_k -th columns

Suppose $i_1 < \dots < i_{k-1}, j_1 < \dots < j_{k+1}$, the **Plücker relations** are

$$\sum_{l=0}^{k+1} W_{i_1, \dots, i_{k-1}, j_l} W_{j_1, \dots, \widehat{j_l}, \dots, j_{k+1}}$$

Example 54.2.2. Consider $\text{Gr}_2(4)$, the only Plücker relation is

$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0$$

54.3 Graph theory

Definition 54.3.1. A graph is

54.4 Moduli space

Consider a parametrized curve $C = \{(t, \mathbf{x}(t))\}_{t \in I}$, $\mathbf{x}(t) \in \mathbb{R}^n$, now we change I to some space X , $\mathbf{x}(t)$ to some algebro-geometric objects, then we have a parametrization of these objects by X

Definition 54.4.1. U is a family of some algebro-geometric objects. A parametrization of U by space X is a map $X \rightarrow U$, attaching some object U_x for each $x \in X$, we can also think of this map as a section of $X \times U \rightarrow X$

We say X is the parametrization space, U is parametrized over X

A moduli functor F is a contravariant functor $Space \rightarrow Set$ that takes a space X to the set of families of objects over X , and take a morphism f to the pullback f^* that taking section s to pullback section $f^*s(y) = (y, \text{Pr}_U s f(y))$

$$\begin{array}{ccc} Y \times U & \longrightarrow & X \times U \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

The category of spaces can be the category of schemes, manifolds, topological spaces, etc.

M is a fine moduli space if F is corepresentable by M , i.e. there is a natural isomorphism $\tau : F \rightarrow \text{Hom}(-, M)$. There is a universal family over M corresponds to $1_M \in \text{Hom}(M, M)$.

Then any family over X is the pullback along some $X \xrightarrow{f} M$ of the universal family. The universal family is essentially unique and "tautological"

M is a coarse moduli space if there is exists a natural transformation $\tau : F \rightarrow \text{Hom}(-, M)$ and universal among these natural transformations, i.e. for any natural transformation $\tau' : F \rightarrow \text{Hom}(-, M')$, there is a morphism $M' \xrightarrow{\phi} M$ such that the following diagram commutes

$$\begin{array}{ccc} & & F \\ & \swarrow \tau' & \downarrow \tau \\ \text{Hom}(-, M') & \xrightarrow{\exists_1 \phi} & \text{Hom}(-, M) \end{array}$$

54.5 Teichmüller space

Definition 54.5.1. S is a orientable surface. A marked Riemann surface is a pair (X, f) where X is a Riemann surface and $S \xrightarrow{f} X$ is a isomorphism, i.e. giving X a complex structure. (X, f) , (Y, g) are equivalent if $gf^{-1} : X \rightarrow Y$ is isotopic to an isomorphism, i.e. X, Y has isotopic complex structures. The Teichmüller space $T(S)$ of S is the the equivalence classes of marked Riemann surfaces. The mapping class group acts on $T(S)$ by $h \cdot (X, f) = (X, fh^{-1})$, then $T(S)$ mod the action is just S

Example 54.5.2. By Uniformization theorem 23.0.9, $T(\mathbb{S}^2)$ is a point corresponds to the Riemann sphere, $T(\mathbb{R}^2)$ is two points corresponds the complex plane and the unit disc. $T(A) = [0, 1)$, where A is the open annulus, and $\lambda \in [0, 1)$ corresponds to $\{\lambda < |z| < \lambda^{-1}\}$ according to Exercise 45.2.21

54.6 Mapping class group

Definition 54.6.1. Suppose $\text{Aut}(X)$ has a natural topology, the mapping class group is $\text{Aut}(X)/\text{Aut}_0(X)$, where $\text{Aut}_0(X)$ is the path connected component of the identity, hence we have exact sequence

$$0 \rightarrow \text{Aut}_0(X) \rightarrow \text{Aut}(X) \rightarrow \text{MCG}(X) \rightarrow 0$$

If X is a space, then a path connecting $f, g \in \text{Aut}(X)$ is an isotopy

Example 54.6.2. $\text{MCG}(S^2) = \mathbb{Z}/2\mathbb{Z}$

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Index

- C^* -algebra, 217
- G set, 33
- G space, 33
- G -vector bundle, 157
- R algebra, 41
- \mathcal{G} topology, 213
- \mathcal{O} modules, 29
- \mathfrak{g} module, 54
- *-algebra, 217
- étalé space, 28
- étale map, 28
- 2 out of 3, 20

- Abelian category, 99
- Absolute neighborhood retract, 116
- Absolute retract, 116
- abstract simplicial complex, 126
- Acyclic cofibration, 141
- Acyclic fibration, 141
- Additive category, 99
- Additive functor, 99
- adjoint endomorphism, 54
- Adjoint functors, 24
- Affine algebraic group, 187
- Affine connection, 170
- Affine scheme, 199
- Affine space, 49
- Alexandorff extension, 114
- Algebraic group, 187
- Algebraic number field, 48
- Almost complex structure, 179
- Ambient isotopy, 151
- Analytic manifold, 165
- Arrow category, 27
- Associate $(1, 1)$ form, 179
- Associated graded complex, 105
- Associated graded module, 40
- Associated graded ring, 40
- Augmented simplex category, 122

- Baire space, 215
- Balanced category, 17

- Banach algebra, 217
- Banach space, 213
- Base change, 25
- Base of a topology, 111
- Basis of a module, 43
- Bimorphism, 17
- Biproduct, 23
- Bloch-Wigner function, 212
- Boundary maps, 100
- Box topology, 112
- Branch point, 177
- Bundle, 153

- Canonical bundle, 157
- Cartan integers, 68
- Cartan's criterion for semisimplicity, 60
- Cartan's criterion for solvability, 60
- Category of chain complexes, 100
- Categroy, 17
- Cauchy-Riemann equations, 205
- Cell complex, 119
- Cell decomposition, 119
- Cellular map, 119
- Center, 33
- Chain complex, 99
- Chart, 165
- Chinese remainder theorem, 40
- Closed set, 111
- Cobase change, 25
- Cocomplete, 22
- Cocone, 22
- Coequaliser, 22
- Cofibered pair, 144
- Cofibrant, 142
- Cofibration sequence, 148
- Cohomology theory, 135
- Colimit, 22
- Comb space, 246
- Comma category, 27
- Compact, 112
- Compact-open topology, 114

- Compactly generated, 113
- Complete, 22
- Completely reducible representation, 34
- Completely separating function, 11
- Complex manifold, 165
- Composition factors, 103
- composition length, 103
- Composition series, 103
- Concave function, 203
- Concrete category, 19
- Cone, 22
- Connected, 112
- Connected category, 20
- Constant morphism, 19
- Constant presheaf, 28
- Constructible set, 115
- Continuous, 111
- Continuous functor, 162
- Contractible, 116
- Contravariant functor, 17
- Convex function, 203
- Convex set, 49
- coroot, 68
- Coslice category, 27
- Cotangent space, 165
- Coxeter diagram, 36
- Coxeter element, 36
- Coxeter group, 36
- Coxeter matrix, 36
- Coxeter number, 36
- Cross product, 131, 132
- Cross section, 153
- Crystallographic root system, 68
- Cup product, 136
- Cylinder object, 142

- Dedekind domain, 40
- Deformation, 116
- Deformation retraction, 116
- Degeneracy map, 122
- Dense, 112
- Derivation, 53
- Derived couple, 105
- Derived series, 54
- determinant, 40
- Diagram, 22
- Dilogarithm, 212
- Direct limit, 22
- Direct sum of chain complexes, 100
- direct sum of vector bundles, 156
- Direct system, 22
- Directed set, 22
- Discrete map, 113
- Discrete topology, 113
- Discrete valuation ring(DVR), 41
- Division ring, 47
- Dominant morphism, 197
- Double cochain complex, 105
- Double complex, 100
- Dual category, 17

- Einstein curvature, 171
- Empty category, 19
- Energy-momentum tensor, 232
- Epimorphism, 17
- Equaliser, 22
- Equicontinuity, 214
- Equivalent category, 18
- Equivariant map, 33
- Essentially surjective, 18
- Exact couple, 105
- Exponential object, 30

- Face map, 122
- Factorization system, 31
- Faithful action, 33
- Faithful functor, 18
- Fiber bundle, 153
- Fiber homotopy, 146
- Fiber homotopy equivalence, 146
- Fibrant, 142
- Fibration sequence, 148
- Field, 47
- Field norm, 47
- Field trace, 47
- Filtered category, 26
- Filtered chain complex, 100, 105
- Filtered colimit, 26
- Filtered object, 20
- Filtration, 20
- Final object, 23
- Finite ring homomorphism, 41
- Finite type ring homomorphism, 41
- First countable, 111
- Flag, 49
- Fractional ideal, 40
- Free action, 33
- Free category, 20
- Full functor, 18
- Functor, 17
- Functor category, 18
- Fundamental groupoid, 139

- Galilean group, 235
- Generalized Cartan matrix, 72
- Generalized eigenspace, 52

- generalized eigenvector, 52
- Geometric realization, 124
- Germ, 28
- Good pair of topological spaces, 116
- Graded ring, 40
- Graph, 293
- Grothendieck completion, 37
- Grothendieck group, 37
- Group, 33
- Group action, 33
- Group character, 47
- Group representation, 34
- Groupoid, 33

- Half space model, 173
- Harmonic, 203
- Hasse diagram, 11
- Hausdorff space, 112
- Hawaiian earring, 245
- Heat equation, 226
- Hemisphere model, 173
- Hermitian manifold, 179
- Hermitian metric, 179
- Holomorphic, 205
- Homogeneous ideal, 40
- Homogeneous space, 33
- Homogeneous vector bundle, 157
- Homology theory, 129
- Homotopy, 112
- Homotopy category of chain complexes, 100
- Homotopy equivalence, 112
- Homotopy extension property, 144
- Homotopy lifting property, 113
- Hurewicz cofibration, 144
- Hurewicz fibration, 144
- Hurwitz's theorem, 207
- Hyperboloid model, 172

- Image of a morphism, 20
- Inclusion-exclusion principle, 11
- Indexed category, 22
- Induced representation, 34
- Inert, 48
- Initial object, 23
- Injective object, 19
- Inner automorphism, 33
- Inner derivations, 54
- Integral domain, 40
- Integral element, 41
- Invariant bilinear form on Lie algebra, 57
- Invariant subspace, 52
- Inverse limit, 23
- Inverse system, 22

- Involution, 20
- Irreducible topological spaces, 114
- Isomorphic category, 18
- Isomorphism, 17
- Isotopy, 151
- Isotropy representation, 189

- Jacobi identity, 53
- Join, 115
- Jordan curve theorem, 115
- Jordan-Chevalley decomposition, 52

- Kähler manifold, 179
- Killing form, 57
- Klein disc model, 174
- Kolmogorov classification of topological spaces, 112
- Kronecker product, 51
- Krull dimension, 41

- Lattice, 11
- Lefschetz fixed point theorem, 115
- Lefschetz number, 115
- Left R module, 43
- Left homotopy, 142
- Left lifting property, 20
- Lie algebra, 54
- Lie algebra of Lie group, 189
- Lie algebra representation, 54
- Lie bracket of vector fields, 166
- Lie group, 189
- Lift extension property, 113
- Limit, 22
- Limit point, 111
- Linear algebraic group, 187
- Linearly independent, 43
- Local base, 111
- Local ring, 40
- Localization of a category, 20
- Locally closed set, 115
- Locally compact, 112
- Locally connected, 112
- Locally constant sheaf, 28
- Locally finite, 112
- Locally ringed space, 28
- Locally small category, 18
- Loop space, 147
- Lorentz boost, 235
- Lorentz factor, 235
- Lorentz group, 235
- Lower central series, 54

- Möbius transformation, 210

- manifold, 165
- Minkowski space, 172
- Model category, 141
- Monoid, 33
- Monoidal category, 32
- Monomorphism, 17
- Montel's theorem, 206
- Morphism, 17
- Multiplicatively closed, 44
- Natural isomorphism, 18
- Natural transformation, 17
- Neighborhood, 111
- Nerve, 162
- Nilpotent Lie algebra, 54
- Noetherian topological space, 114
- Nonassociative \mathbb{F} algebra, 53
- Nonassociative algebra, 53
- Normal family, 114
- Normal topological space, 111
- Normed group, 213
- Normed rng, 213
- Normed vector space, 213
- Object, 17
- One point compactification, 114
- Open set, 111
- Outer automorphism, 33
- Paracompact, 112
- Path object, 142
- path space, 147
- Perfect field, 47
- Perfect group, 33
- perfectly separated, 111
- Perfectly separating function, 11
- Permutation, 35
- Plücker relations, 292
- Poincaré disc model, 174
- Pointed category, 23
- Polydisc, 205
- Polylogarithm, 212
- preabelian category, 99
- Preadditive category, 99
- Precompact, 112
- Presheaf, 28
- Prevariety, 197
- Prime element, 40
- Principal G -bundle, 157
- Product category, 18
- Projective object, 19
- Proper map, 113
- pseudomanifold, 128
- Pullback, 25, 165
- Pullback bundle, 153
- pullback bundle, 155
- Pullback of sections, 153
- Pushforward presheaf, 28
- Pushforward(differential), 165
- Pushout, 25
- Quadratic Lie algebra, 57
- Quasi-isomorphism, 100
- quasifinite morphism, 197
- Quiver, 20
- quotient category, 19
- Quotient object, 19
- Radical ideal of Lie algebra, 54
- Ramified, 48
- Real form, 55
- Refinement, 112
- Regular function, 196
- Regular topological space, 111
- Residue field, 40
- Restricted presheaf, 28
- Restriction map, 28
- Retraction, 27, 116
- Ricci curvature, 171
- Riemann mapping theorem, 208
- Riemann surface, 177
- Riemannian manifold, 170
- Riemannian metric, 170
- Right homotopy, 142
- Right lifting property, 20
- Ring, 39
- Ringed space, 28
- Root, 62
- Root space, 62
- Root system, 68
- Rouche's theorem, 207
- Scalar curvature, 171
- Scheme, 200
- Schur's lemma, 44
- Schwarz lemma, 206
- Second Bianchi identity, 170
- Second countable, 111
- Section, 27
- semicategory, 17
- Semicontinuity, 203
- Semidirect product, 33
- Semifunctor, 17
- Semigroup, 33
- Semilocal ring, 40
- Semisimple linear operator, 52

- Semisimple simplex category, 122
- Separable space, 112
- Separating function, 11
- Serre fibration, 150
- Sheaf, 28
- Sheaf cohomology, 28
- Sheafification, 28
- Shuffle, 35
- Simplex category, 122
- Simplicial identities, 122
- Simplicial map, 123
- Simplicial object, 122
- Skeleton, 119
- Skeleton of a category, 20
- Slice category, 27
- Small category, 18
- Smooth manifold, 165
- Smooth map, 165
- Snake lemma, 100
- Solvable Lie algebra, 54
- span, 43
- Spectral sequence, 105
- Stably isomorphic, 161
- Stalk, 28
- Strictly convex set, 49
- Strong deformation retraction, 116
- Structure sheaf, 28
- Subbase, 111
- Subharmonic, 203
- Submanifold, 165
- Subobject, 19
- Subspace topology, 111
- Superharmonic, 203
- Sylow's theorem, 34
- Symplectic structure, 181
- Tangent space, 165
- Tautological bundle, 247
- Tautological one form, 166
- tautological representation of Lie algebra, 55
- tensor product of vector bundles, 156
- Test function space, 216
- Topological category, 162
- Topological groups, 185
- Topological space, 111
- Topological vector space, 213
- topologist's sine curve, 245
- Topology, 111
- Topology of pointwise convergence, 114
- Topology of uniform convergence, 213
- Toral Lie algebra, 54
- Torsion tensor, 170
- Torsor, 33
- Total chain complex, 100
- Total cochain complex, 106
- Totally split, 48
- Tractrix, 253
- Transition map, 165
- Transitive action, 33
- Translation of a chain complex, 101
- Trivial action, 33
- Trivial topology, 113
- tropical semifield, 87
- Uniformization theorem, 177
- Unit, 39
- Univalent, 205
- Urysohn space, 112
- Vector bundle, 153
- Vector space, 49
- vector subbundle, 156
- Warsaw circle, 245
- Wave equation, 228
- Weak equivalence, 20
- Weak factorization system, 31
- Weak homotopy equivalence, 140
- Weak retraction, 116
- Weierstrass σ -function, 211
- Weierstrass \wp -function, 211
- Weierstrass polynomial, 209
- Weyl chamber, 69
- Weyl group, 69
- Weyl's theorem, 55
- Whitehead group, 39
- Yoneda lemma, 21
- Zero morphism, 19
- Zero object, 23