The capital value of an asset is sometimes defined as the present value of all future net earnings. The capital value of the asset may be written in the form [capital value] $= \int_0^\infty K(t)e^{-rt}dt$, where r is the annual rate of interest, compounded continuously. Find the capital value of an asset that generates income at the rate of \$8000 per year, assuming an interest rate of 8%

Solution: $K(t)$ is the rate of income generated by the asset

In this case $K(t)=8000$, $r=8\%$

hence capital value $= \int_0^\infty 8000e^{-0.08t}dt = \lim_{b\to+\infty}\int_0^b 8000e^{-0.08t}dt$

$\int_0^b 8000e^{-0.08t}dt = \frac{8000}{(-0.08)}\int_0^b(-0.08)e^{-0.08t}dt$ $\quad \underline{\underline{(e^{-0.08t})' = (-0.08)e^{-0.08t}}}$

$\frac{8000}{(-0.08)}e^{-0.08t}\Big|_0^b = -100000\left[e^{-0.08b}-e^0\right] = -100000\left[\frac{1}{e^{0.08b}}-1\right]$

when $b\to+\infty$, $0.08b\to+\infty$, $e^{0.08b}\to+\infty$, $\frac{1}{e^{0.08b}}\to 0$,

$\int_0^b 8000e^{-0.08t}dt = -100000\left[\frac{1}{e^{0.08b}}-1\right] = -100000[0-1] = 100000$

Interestingly, even though you get \$8000 dollars every year, but throughout eternity, you can only get a finite amount of present value

$\int_8^\infty \frac{x}{\sqrt{5+x^2}}dx = \lim_{b\to\infty}\int_8^b\frac{x}{\sqrt{5+x^2}}dx$, $\int_8^b \frac{x}{\sqrt{5+x^2}}dx$ $\quad \frac{u=5+x^2}{du = d(5+x^2) = (5+x^2)'dx = 2x\,dx}$

$\downarrow \div 2$

$\frac{1}{2}du = x\,dx$

$= \int_{5+8^2}^{5+b^2}\frac{1}{\sqrt{u}}\cdot\frac{1}{2}du = \int_{69}^{5+b^2}\frac{1}{2}u^{-\frac{1}{2}}du$

$\underline{\underline{(u^{\frac{1}{2}})' = \frac{1}{2}u^{-\frac{1}{2}}}}$ $\quad u^{\frac{1}{2}}\Big|_{69}^{5+b^2} = \left[(5+b^2)^{\frac{1}{2}} - 69^{\frac{1}{2}}\right] = \sqrt{5+b^2}-\sqrt{69}$

when $b\to+\infty$, $b^2\to+\infty$, $5+b^2\to+\infty$, $\sqrt{5+b^2}\to+\infty$, $\sqrt{5+b^2}-\sqrt{69}\to+\infty$

hence $\int_8^\infty\frac{x}{\sqrt{5+x^2}}dx$ is divergent

$\int_2^\infty \frac{12}{x\ln x}dx = \lim_{b\to\infty}\int_2^b\frac{12}{x\ln x}dx$, $\int_2^b\frac{12}{x\ln x}dx$ $\quad \frac{u=\ln x}{du = d\ln x = \frac{1}{x}dx}$ $\quad \int_{\ln 2}^{\ln b}\frac{12}{u}du$

$\underline{\underline{(\ln u)' = \frac{1}{u}}}$ $\quad 12\ln u\Big|_{\ln 2}^{\ln b} = 12\left[\ln(\ln b) - \ln(\ln 2)\right]$ as $b\to+\infty$,

$\ln b\to+\infty$, $\ln(\ln b)\to+\infty$, $\left[\ln(\ln b)-\ln(\ln 2)\right]\to+\infty$, hence

$12\left[\ln(\ln b)-\ln(\ln 2)\right]\to+\infty$, $\int_2^\infty\frac{12}{x\ln x}dx$ diverges

$$\int_0^{\infty} 2x(3x^2+1)^{-\frac{5}{4}}dx = \lim_{b\to\infty}\int_0^b 2x(3x^2+1)^{-\frac{5}{4}}dx$$

$$\int_0^b 2x(3x^2+1)^{-\frac{5}{4}}dx \xlongequal[du=d(3x^2+1)=(3x^2+1)'dx=6xdx]{u=3x^2+1} \int_{3\cdot 0^2+1}^{3b^2+1} u^{-\frac{5}{4}}\frac{1}{3}du$$

$$\downarrow \div 3$$

$$\frac{1}{3}du = 2xdx$$

$$= \frac{1}{3}\int_1^{3b^2+1} u^{-\frac{5}{4}}du = \frac{\frac{1}{3}}{(-\frac{1}{4})}\int_1^{3b^2+1}(-\frac{1}{4})u^{-\frac{5}{4}}du \xlongequal{(u^{-\frac{1}{4}})'=(-\frac{1}{4})u^{-\frac{5}{4}}} \frac{\frac{1}{3}}{(-\frac{1}{4})}\, u^{-\frac{1}{4}}\Big|_1^{3b^2+1}$$

$$= -\frac{4}{3}\left[(3b^2+1)^{-\frac{1}{4}} - 1^{-\frac{1}{4}}\right] = -\frac{4}{3}\left[\frac{1}{(3b^2+1)^{\frac{1}{4}}} - 1\right]$$

when $b\to+\infty$, $3b^2+1\to+\infty$, $(3b^2+1)^{\frac{1}{4}}\to+\infty$, $\frac{1}{(3b^2+1)^{\frac{1}{4}}}\to 0$,

$$\int_0^b 2x(3x^2+1)^{-\frac{5}{4}}dx = -\frac{4}{3}\left[\frac{1}{(3b^2+1)^{\frac{1}{4}}}-1\right] = -\frac{4}{3}[0-1] = \frac{4}{3}$$

hence $\int_0^{\infty} 2x(3x^2+1)^{-\frac{5}{4}}dx$ converges to $\frac{4}{3}$

Differential equations:

Definition: a differential equation is an equation involving $t, y, y', y'', \cdots$

for example: $y''-4y'-5y=0$, $y'=t^4y+2t^4$

Definition: a solution is a function $y=f(t)$ satisfying the equation, for example,

$y=f(t)=Ce^{5t}$ (C being arbitsary constant) is a solution of equation

$y''-4y'-5y=0$ since $(Ce^{5t})''-4(Ce^{5t})'-5(Ce^{5t}) = 25Ce^{5t}-20Ce^{5t}-5Ce^{5t}=0$