MATH 121 EXAM 1

There will be **50** minutes for you to finish the exam, if you ever get stuck, move on and circle back to it. **No calculators**, one problem per sheet, 100 points in total

1. Convert radians to degrees and degrees to radians
   (a) (5 points) π°

   Solution: $\pi^\circ = \pi \times \frac{\pi}{180}\text{ rad} = \frac{\pi^2}{180}\text{ rad}$

   (b) (5 points) $\frac{5\pi}{12}$ rad

   Solution: $\frac{5\pi}{12}\text{ rad} = \frac{5\pi}{12} \times \frac{180^\circ}{\pi} = 75^\circ$

2. (a) (15 points) Evaluate $\sin\left(\frac{5\pi}{12}\right)$

   Solution:

$$
\begin{aligned}
\sin\left(\frac{5\pi}{12}\right) &= \sin\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) \\
&= \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\
&= \sin\frac{\pi}{6}\cos\frac{\pi}{4} + \cos\frac{\pi}{6}\sin\frac{\pi}{4} \\
&= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{2}\left(\sqrt{3}+1\right)}{4}
\end{aligned}
$$

   (b) (15 points) Find the derivative of $f(t) = te^{\sin(t^2+1)}$

   Solution:

$$
\begin{aligned}
f'(t) &= te^{\sin(t^2+1)} \\
&= t'e^{\sin(t^2+1)} + t\left(e^{\sin(t^2+1)}\right)' \\
&= e^{\sin(t^2+1)} + te^{\sin(t^2+1)}\left(\sin\left(t^2+1\right)\right)' \\
&= e^{\sin(t^2+1)} + te^{\sin(t^2+1)}\cos\left(t^2+1\right)\left(t^2+1\right)' \\
&= e^{\sin(t^2+1)} + te^{\sin(t^2+1)}\cos\left(t^2+1\right)(2t) \\
&= e^{\sin(t^2+1)} + 2t^2\cos\left(t^2+1\right)e^{\sin(t^2+1)}
\end{aligned}
$$

3. (a) (15 points) Compute $\int \frac{3x}{\sqrt{x^2+1}}\mathrm{d}x$

Solution:

$$
\begin{aligned}
\int \frac{3x}{\sqrt{x^2+1}}\mathrm{d}x &= \frac{3}{2}\int \frac{2x\mathrm{d}x}{\sqrt{x^2+1}} \\
&= \frac{3}{2}\int \frac{\mathrm{d}x^2}{\sqrt{x^2+1}} \\
&\xlongequal{u=x^2} \frac{3}{2}\int \frac{\mathrm{d}u}{\sqrt{u+1}} \\
&= \frac{3}{2}\int \frac{\mathrm{d}(u+1)}{\sqrt{u+1}} \\
&\xlongequal{v=u+1} \frac{3}{2}\int \frac{\mathrm{d}v}{\sqrt{v}} \\
&= 3\sqrt{v}+C \\
&= 3\sqrt{(x^2+1)}+C
\end{aligned}
$$

(b) (15 points) Compute $\int \frac{x^2}{e^x}\mathrm{d}x$

Solution:

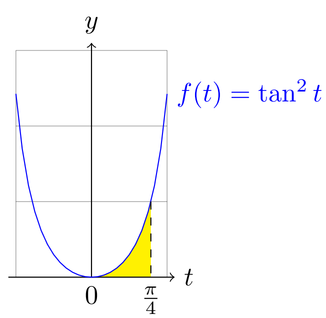
$$
\begin{aligned}
\int \frac{x^2}{e^x}\mathrm{d}x &= \int x^2e^{-x}\mathrm{d}x \\
&= -\int (-x)^2e^{-x}\mathrm{d}(-x) \\
&\xlongequal{u=-x} -\int u^2e^u\mathrm{d}u \\
&= -\int u^2\mathrm{d}e^u \\
&= -u^2e^u+\int e^u\mathrm{d}u^2 \\
&= -u^2e^u+2\int ue^u\mathrm{d}u \\
&= -u^2e^u+2\int u\mathrm{d}e^u \\
&= -u^2e^u+2ue^u-2\int e^u\mathrm{d}u \\
&= -u^2e^u+2ue^u-2e^u+C \\
&= -x^2e^{-x}-2xe^{-x}-2e^{-x}+C
\end{aligned}
$$

4. (a) (15 points) Compute $\int_0^{\sqrt{\pi}} x^3\sin\left(x^2\right)\mathrm{d}x$

Solution:

$$\begin{aligned}
\int_0^{\sqrt{\pi}} x^3 \sin\left(x^2\right)\mathrm{d}x &= \frac{1}{2}\int_0^{\sqrt{\pi}} x^2 \sin\left(x^2\right)2x\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\sqrt{\pi}} x^2 \sin\left(x^2\right)\mathrm{d}x^2 \\
&\overset{u=x^2}{=\!=\!=} \frac{1}{2}\int_0^{\pi} u \sin(u)\mathrm{d}u \\
&= -\frac{1}{2}\int_0^{\pi} u\mathrm{d}\cos u \\
&= -\frac{1}{2}\left[u\cos u\right]\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos u\mathrm{d}u \\
&= -\frac{1}{2}\left[\pi\cos\pi - 0\right] + \frac{1}{2}\sin u\Big|_0^{\pi} \\
&= \frac{\pi}{2}
\end{aligned}$$

(b) (15 points) Below is the graph of function $f(t) = \tan^2 t$, compute the shaded area



Solution:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} f(t)\mathrm{d}t &= \int_0^{\frac{\pi}{4}} \tan^2 t\mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \left(\tan^2 t + 1 - 1\right)\mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \left(\sec^2 t - 1\right)\mathrm{d}t \\
&= \int_0^{\frac{\pi}{4}} \sec^2 t\mathrm{d}t - \int_0^{\frac{\pi}{4}} \mathrm{d}t \\
&= \tan t\Big|_0^{\frac{\pi}{4}} - \frac{\pi}{4} \\
&= \left[\tan\left(\frac{\pi}{4}\right) - 0\right] - \frac{\pi}{4} \\
&= 1 - \frac{\pi}{4}
\end{aligned}$$