Method of integrating factor: $y'+a(t)y=b(t)$ Indentify $a(t)$ and $b(t)$

- find the antiderivative of $a(t)$, $A'(t)=a(t)$, $A(t)=\int a(t)dt$, here you can pick any constant
- multiply the integrating factor $e^{A(t)}$, $[ye^{A(t)}]'=y'e^{A(t)}+y(e^{A(t)})'=$ $y'e^{A(t)}+ye^{A(t)}\cdot A'(t)=y'e^{A(t)}+a(t)ye^{A(t)}=(y'+a(t)y)e^{A(t)}=b(t)e^{A(t)}$
- Take antiderivative on both sides:
  $\int (ye^{A(t)})'dt=\int b(t)e^{A(t)}dt \Rightarrow ye^{A(t)}=\int b(t)e^{A(t)}dt$
- multiply $e^{-A(t)}$ / divide $e^{A(t)}$ on both sides: $y=e^{-A(t)}\int b(t)e^{A(t)}dt$

Solve $y'-2ty=-10t$, $a(t)=-2t$, $b(t)=-10t$, $A(t)=\int -2t\,dt=-t^2$ (+C, choose C=0), $A(t)'=a(t)$, multiply integrating factor $e^{A(t)}=e^{-t^2}$, we get
$[ye^{-t^2}]'=y'e^{-t^2}+ye^{-t^2}(-2t)=y'e^{-t^2}-2tye^{-t^2}=(y'-2ty)e^{-t^2}=-10te^{-t^2}$

Take integral on both sides $ye^{-t^2}=\int [ye^{-t^2}]'dt=\int -10te^{-t^2}dt \xlongequal[du=dt^2=2tdt]{u=t^2}$

$du=dt^2=2tdt \xrightarrow{\div 2} \frac{1}{2}du=tdt$

$=\int -10e^{-u}\cdot\frac{1}{2}du=-5\int e^{-u}du \xlongequal{(-e^{-u})'=e^{-u}} -5\cdot(-e^{-u})+C$
$=5e^{-t^2}+C$

multiply e^{t^2} / divide e^{-t^2} on both sides, get $y=5+Ce^{t^2}$, C is any constant

Solve $(2+t)y'+y=-2$, $(t>0)$, divide $(2+t)$ (since $t>0$, $2+t\neq 0$)
$y'+\frac{1}{2+t}y=-\frac{2}{2+t}$, $a(t)=\frac{1}{2+t}$, $A(t)=\int \frac{1}{2+t}dt=\ln(2+t)$, pick C=0
multiply integrating factor $e^{A(t)}=e^{\ln(2+t)}=2+t$, hence we have
$[y(2+t)]'=y'(2+t)+y(2+t)'=y'(2+t)+y=[y'+\frac{y}{2+t}](2+t)=-\frac{2}{2+t}(2+t)=-2$
$[ye^{\ln(2+t)}]' \xlongequal{\quad} [y'+\frac{y}{2+t}]e^{\ln(2+t)}$
$\Rightarrow y(2+t)=-2t+C \xrightarrow{\div(2+t)} y=\frac{C-2t}{2+t}$, C is an arbitrary constant

Solve initial value problem $y'+12y=1$, $y(0)=1$
$a(t)=12$, $A(t)=12t$, multiply integrating factor e^{12t} on both sides
$[ye^{12t}]'=(y'+12y)e^{12t}=e^{12t} \Rightarrow \int [ye^{12t}]'dt=\int e^{12t}dt \Rightarrow ye^{12t}=\frac{1}{12}e^{12t}+C$
plug in $y(0)=1$, $1=1\times e^{12\times 0}=\frac{1}{12}e^{12\times 0}+C=\frac{1}{12}+C \Rightarrow C=1-\frac{1}{12}=\frac{11}{12}$
$\Rightarrow ye^{12t}=\frac{1}{12}e^{12t}+\frac{11}{12} \xrightarrow{\times e^{-12t}\ /\ \div e^{12t}} y=\frac{1}{12}+\frac{11}{12}e^{-12t}$