# MATH602 - Homological algebra

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## Contents

# 1 Review of category theory

## 1.1 Categories - 1/27/2020

**Definition 1.1.** A **category** $\mathscr{C}$ consists of Ob$\mathscr{C}$ class of **objects** and Hom$\mathscr{C}$ class of **morphisms**, for $f : A \to B$, $g : B \to C$, $\exists g \circ f : A \to C$, the composition is associative $(h \circ g) \circ f = h \circ (g \circ f)$, $\exists 1_A : A \to A$ such that $1_A f = f, \forall f : B \to A$ and $g 1_A = g, \forall g : A \to B$(thus $1_A$ is unique), we denote $Hom_{\mathscr{C}}(A, B)$ to be all the morphisms from $A$ to $B$

**Definition 1.2.** Let $\mathscr{C}$ be a category, $f : A \to B$ is an **isomorphism** if there exists $g : B \to A$ such that $gf = 1_A$, $fg = 1_B$

**Example 1.3.** Let $M$ be a monoid, we can view it as a category $\mathscr{C}_M$, where $Ob\mathscr{C}_M = \{*\}$, $Hom_{\mathscr{C}_M}(*, *) = M$

*Remark.* Book recommandation: Abelian category - Fregd, it defines a category use only morphisms

**Lemma 1.4.** *An isomorphism $f : X \to Y$ has an unique inverse, denoted $f^{-1}$*

*Proof.* □

**Definition 1.5.** A category $\mathscr{C}$ is called a **small category** if $Ob\mathscr{C}$ is a set

**Definition 1.6.** A category $\mathscr{C}$ is called an **essentially small category** if $Ob\mathscr{C}/\sim$ is a set, here $Ob\mathscr{C}/\sim$ is the isomorphic classes of objects

**Example 1.7.** Let $k$ be a field, then the category of finite dimensional $k$ vector fields is not small but essentially small, two $k$ vector spaces are isomorphic iff they have the same dimension

**Example 1.8.** Let $R$ be a commutative ring, the category of $R$ modules, $RMod$ is not essentially small

**Definition 1.9.** Let $P$ be a poset, we can view it as a category $\mathscr{C}_P$, where $Ob\mathscr{C}_p = P$, $Hom_{\mathscr{C}_P}(x, y) = \begin{cases} \{*\}, & x \leq y \\ \varnothing, & \text{else} \end{cases}$

**Exercise 1.10.** *Suppose small category $\mathscr{C}$ satisfies*
$|Hom(x, y)| \leq 1$
$x \neq y \Rightarrow x \not\cong y$
*Then $\mathscr{C}$ is poset*

*Proof.* □

**Definition 1.11.** A category is a **groupoid** if every morphism is an isomorphism, thus a groupoid with only one object is a group

## 1.2 Functors - 1/29/2020

**Definition 1.12.** $\mathscr{C}, \mathscr{D}$ are categories, $F : \mathscr{C} \to \mathscr{D}$ is a **functor** if it is a mapping: $\mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D}$, $\mathscr{C}(A,B) \to \mathscr{D}(F(A),F(B))$, $F(1_A) = 1_{F(A)}$, given $f : A \to B, g : B \to C$, $F(g \circ f) = F(g) \circ F(f) : F(A) \to F(C)$, this kind of functor is called **covariant fucntor**, if $\mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D}$, $\mathscr{C}(A,B) \to \mathscr{D}(F(B),F(A))$, $F(1_A) = 1_{F(A)}$, given $f : A \to B, g : B \to C$, $F(g \circ f) = F(f) \circ F(g) : F(C) \to F(A)$, then this is called a **contravariant functor**
The **dual category** of a category $\mathscr{C}$ is denoted as $\mathscr{C}^{op}$ with the same objects but morphisms reversed, a contravariant functor is just a functor in the dual

**Example 1.13.** **(1):** Let $M, N$ be monoids, a functor $F : \mathscr{C}_M \to \mathscr{C}_N$ is just a homomorphism of monoids
**(2):** Let $M, N$ be groups, a functor $F : \mathscr{C}_M \to \mathscr{C}_N$ is just a homomorphism of groups
**(3):** Let $L/F$ be a field extension, $-\otimes L$ is a functor $Vect_F \to Vect_L$, $V \mapsto V \otimes_F L$, $\phi \mapsto \phi \otimes 1_L$
**(4):** Homology $H_*$ is a functor $Top \to Abgp$, $X \mapsto H_*(X)$
**(5):** Cohomology $H^*$ is a contravariant functor $Top \to Abgp$, $X \mapsto H^*(X)$
**(6):** Let $FinAbgp$ be the category of finite abelian groups, then $D : FinAbgp \to FinAbgp$, $X \mapsto Hom(X, \mathbb{Q}/\mathbb{Z})$ is a contravariant functor, or we could use $Hom(X, \mathbb{C}^\times)$, this is called **Pontrjagin duality**
**(7):** $D : Vect_K \to Vect$, $V \mapsto V^*$ is a contravariant functor

**Notation.** *Suppose $f : X \to Y$ is a morphism in category $\mathscr{C}$, for $Z \in ob\mathscr{C}$, we define*

$$f_* : Hom(Z,X) \to Hom(Z,Y), \quad g \mapsto fg$$

$$f^* : Hom(Y,Z) \to Hom(X,Z), \quad g \mapsto gf$$

**Definition 1.14.** A morphism $f : X \to Y$ is called a monomorphism if $f_*$ is 1-1 for all $Z \in ob\mathscr{C}$
A morphism $f : X \to Y$ is called a epimorphism if $f^*$ is 1-1 for all $Z \in ob\mathscr{C}$

**Definition 1.15.** In category $\mathscr{C}$, an object $X$ is called an **initial object** if $Hom(X,Y)$ consists of exactly one element for all $Y$, $X$ is called a **final object** if $Hom(Y,X)$ consists of exactly one element for all $Y$, $X$ is called a **zero object** if it is both initial and final

**Example 1.16.** **(1):** In the category of sets, $\emptyset$ is an initial object, $\{1\}$ is a final object
**(2):** In the category of abelian groups, 0 is a zero object

**Definition 1.17.** $F, G : \mathscr{C} \to \mathscr{D}$ are covariant functors, $\eta_A : F(A) \to G(A)$ is a family of morphisms such that the following diagram commutes for any $f : A \to B$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

For contravariant functors, we have the following commutative diagram for any $f : A \to B$

$$\begin{tikzcd} F(B) \arrow[r, "F(f)"] \arrow[d, "\eta_B"] & F(A) \arrow[d, "\eta_A"] \\ G(B) \arrow[r, "G(f)"] & G(A) \end{tikzcd}$$

$\eta$ is called a **natural transformation**

If $\eta_A$ are isomorphisms, then $\eta$ is called a **natural isomorphism**, denoted $F \cong G$

## 1.3 Presheaves and Yoneda lemma - 1/31/2020

**Definition 1.18.** Suppose $\mathscr{C}$, $\mathscr{D}$ are categories, we can define the **functor category** $\mathrm{Fun}(\mathscr{C},\mathscr{D}) = \mathscr{D}^{\mathscr{C}}$ with objects functors from $\mathscr{C}$ to $\mathscr{D}$ and morphisms natural transformations

*Remark.* If $I$ is a small category, then $Hom_{\mathscr{C}^I}(F,G)$ is a set

**Definition 1.19.** we say categories $\mathscr{C},\mathscr{D}$ are **isomorphic** if there are functors $F:\mathscr{C}\to\mathscr{D}$ and $G:\mathscr{D}\to\mathscr{C}$ such that $G\circ F = 1_{\mathscr{C}}$, $F\circ G = 1_{\mathscr{D}}$ and we say $\mathscr{C},\mathscr{D}$ are **equivalent** if $G\circ F$ is naturally isomorphic to $1_{\mathscr{C}}$ and $F\circ G$ is naturally isomorphic to $1_{\mathscr{D}}$

**Example 1.20.** Let $\mathscr{C} = Vect_K$ be te category of $K$ vector spaces, define functor $F:\mathscr{C}\to\mathscr{C}$, $V\mapsto V\otimes_K K$ is an equivalence with inverse $G = 1_{\mathscr{C}}$, but this is not an isomorphism, since not every vector space is in the form of a tensor product

**Definition 1.21.** Suppose $\mathscr{C},\mathscr{D}$ are locally small categories, $F:\mathscr{C}\to\mathscr{D}$ is a functor
$F$ is **faithful** if $Hom_{\mathscr{C}}(X,Y)\to Hom_{\mathscr{D}}(F(X),F(Y))$ is injective for any $X,Y$
$F$ is **full** if $Hom_{\mathscr{C}}(X,Y)\to Hom_{\mathscr{D}}(F(X),F(Y))$ is surjective for any $X,Y$
$F$ is **fully faithful** if $Hom_{\mathscr{C}}(X,Y)\to Hom_{\mathscr{D}}(F(X),F(Y))$ is bijective for any $X,Y$
$F$ is **essentially surjective** if $\forall d\in ob\mathscr{D}$, $\exists c\in ob\mathscr{C}$ such that $Fc\cong d$

**Lemma 1.22.** *In category $\mathscr{C}$, if $\phi_X : X\to X'$, $\phi_Y : Y\to Y'$ are isomorphisms, then $Hom(X,Y)$, $Hom(X',Y')$ are in bijective correspondence*

*Proof.* We can define maps $Hom(X,Y)\to Hom(X',Y')$, $f\mapsto \phi_Y f\phi_X^{-1}$ and $Hom(X',Y')\to Hom(X,Y)$, $f'\mapsto \phi_Y^{-1}f'\phi_X$ which are inverses to each other

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

□

**Theorem 1.23.** *A functor $F:\mathscr{C}\to\mathscr{D}$ is an equivalence iff it is fully faithful and essentially surjective*

*Proof.* If $F$ is an equivalence, there exist functor $G:\mathscr{D}\to\mathscr{C}$ and natural isomorphisms $\eta : 1_{\mathscr{C}}\to GF$, $\xi : 1_{\mathscr{D}}\to FG$, $\forall d\in\mathscr{C}$, $\xi_d : d = 1_{\mathscr{D}}(d)\to FG(d) = F(Gd)$ is an isomorphism, i.e. $F$ is essentially surjective, similarly, so is $G$
The composition of

$$Hom(c,c')\xrightarrow{F} Hom(Fc,Fc')\xrightarrow{G} Hom(GFc,GFc'),\quad f\mapsto Ff\mapsto GFf$$

Is the same as

$$Hom(c,c')\xrightarrow{\eta} Hom(GFc,GFc'),\quad f\mapsto \eta_c' f\eta_c^{-1}$$

By Lemma 1.22, this is bijective, thus $Hom(c,c')\xrightarrow{F} Hom(Fc,Fc')$ is injective, i.e. $F$ is faithful. Similarly, consider the composition

$$Hom(Fc,Fc')\xrightarrow{G} Hom(GFc,GFc')\xrightarrow{F} Hom(FGFc,FGFc')$$

We know $Hom(GFc,GFc')\xrightarrow{F} Hom(FGFc,FGFc')$ is surjective, but we also have the following diagram

$$\begin{tikzcd} Hom(c,c') \arrow[r, "F"] \arrow[d, "\eta"'] & Hom(Fc,Fc') \arrow[d, "\xi"] \\ Hom(GFc,GFc') \arrow[r, "F"] & Hom(FGFc,FGFc') \end{tikzcd}$$

Since $\eta, \xi$ are bijective, $Hom(c,c') \xrightarrow{F} Hom(Fc,Fc')$ is surjective, i.e. $F$ is full
Conversely, suppose $F$ is fully faithful and essentially surjective, then for any $d \in \mathscr{D}$, there exists $c$ and an isomorphism $d \xrightarrow{\xi_d} Fc$, denote this $c$ as $Gd$, we can define a functor $G : \mathscr{D} \to \mathscr{C}$, $d \mapsto Gd$(Here we have used the axiom of choice), $d \xrightarrow{f} d' \mapsto c \xrightarrow{Gf} c'$ where $FGf = \xi_d^{-1} f \xi_{d'}$ since $F$ is fully faithful

$$\begin{tikzcd}
d \arrow[r, "f"] \arrow[d, "\xi_d"'] & d' \arrow[d, "\xi_{d'}"] \\
FGd \arrow[r, "FGf"] & FGd' \\
Gd \arrow[r, "Gf"] \arrow[u, "F"] & Gd' \arrow[u, "F"']
\end{tikzcd}$$

$\xi : 1_{\mathscr{D}} \to FG$ is a natural isomorphism
Since $F$ is fully faithful, there are unique $\eta_c : c \to GFc$, $F(\eta_c) = \xi_{Fc}$
If $f, g : c \to c'$ such that $\eta_{c'} f = \eta_{c'} g$, then $\xi_{Fc'} Ff = \xi_{Fc'} Fg \Rightarrow Ff = Fg \Rightarrow f = g$
If $f, g : c \to c'$ such that $f\eta_c = g\eta_c$, then $Ff\xi_{Fc} = Fg\xi_{Fc} \Rightarrow Ff = Fg \Rightarrow f = g$

$$\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dd, bend right, "\eta_c"'] & c' \arrow[d] \arrow[dd, bend left, "\eta_{c'}"] \\
Fc \arrow[r] \arrow[d, "G"'] \arrow[dd, bend right, "\xi_{Fc}"'] & Fc' \arrow[d, "G"] \arrow[dd, bend left, "\xi_{Fc'}"] \\
GFc \arrow[r] \arrow[d, "F"'] & GFc' \arrow[d, "F"] \\
FGFc \arrow[r] & FGFc'
\end{tikzcd}$$

$\eta : 1_{\mathscr{C}} \to GF$ is a natural isomorphism ☐

**Definition 1.24.** $\mathscr{D}^{\mathscr{C}^{op}}$ is the category of **presheaves**. Denote $\mathscr{C}^\vee := Sets^{\mathscr{C}^{op}}$. In particular, if $X$ is a topological space, open subsets with inclusion form a category $\mathscr{C}$, $PreSh(X, \mathscr{D})$ is the category of presheaves on $X$ with values in $\mathscr{D}$

**Lemma 1.25** (Yoneda lemma)**.** ***Yoneda embedding*** $h : \mathscr{C} \to \mathscr{C}^\vee$ *defined as follows is a fully faithful functor*
*For* $X \in ob\mathscr{C}$, $h(X) = Hom_{\mathscr{C}}(-, X)$ *is a contravariant functor* $\mathscr{C} \to Sets$*:*
*For* $Z \in ob\mathscr{C}$, $h(X)(Z) = Hom_{\mathscr{C}}(Z, X)$, *for* $\phi : Z \to W$, $h(X)(\phi) = \phi^* : h(X)(W) = Hom_{\mathscr{C}}(W, X) \to Hom_{\mathscr{C}}(Z, X) = h(X)(Z)$, *hence* $h(X)$ *is an object in* $\mathscr{C}^\vee$
*For* $\psi : X \to Y$, $h(\psi) = \psi_*$ *is a natural transformation* $h(X) \to h(Y)$*:*
*For* $\phi : Z \to W$, *we have the commutative diagram*

$$\begin{tikzcd}
Hom_{\mathscr{C}}(W,X) \arrow[r, "\psi_*"] \arrow[d, "\phi_*"'] & Hom_{\mathscr{C}}(W,Y) \arrow[d, "\phi_*"] & h(X)(W) \arrow[r, "h(\psi)_W"] \arrow[d, "h(Y)(\phi)"'] & h(Y)(W) \arrow[d, "h(Y)(\phi)"] \\
Hom_{\mathscr{C}}(Z,X) \arrow[r, "\psi_*"] & Hom_{\mathscr{C}}(Z,Y) & h(X)(Z) \arrow[r, "h(\psi)_Z"] & h(X)(Z)
\end{tikzcd}$$

*Proof.* ☐

**Definition 1.26.** A **subcategory** $\mathscr{D}$ of $\mathscr{C}$ is a category with objects a subclass of $ob\mathscr{C}$ and morphisms a subclass of $Hom\mathscr{C}$, with the original composition

**Example 1.27.** The image of a functor is not necessarily a category
Consider the following categories $\mathscr{C}$ and $\mathscr{D}$

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] & B \arrow[loop above, "1_B"] & C \arrow[loop above, "1_C"] \arrow[r, "g"] & D \arrow[loop above, "1_D"]
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[loop above, "1_E"] \arrow[r, "h"] \arrow[rr, bend right, "ih"'] & F \arrow[loop above, "1_F"] \arrow[r, "i"] & G \arrow[loop above, "1_G"]
\end{tikzcd}$$

Consider functor $F : \mathscr{C} \to \mathscr{D}$, $F(A) = E$, $F(B) = F$, $F(C) = F$, $F(D) = G$, $F(f) = h$, $F(g) = i$

**Theorem 1.28.** *A functor $F : \mathscr{C} \to \mathscr{D}$ is fully faithful iff it induces an equivalence of categories from $\mathscr{C}$ to a full subcategory of $\mathscr{D}$*

*Proof.* $\square$

## 1.4 Limits - 2/3/2020

**Definition 1.29.** A functor $F$ in $\mathscr{C}^\vee$ is called **representable** if there exists $X \in ob\mathscr{C}$ such that $h(X) \cong F$, here $h$ is the Yoneda embedding. In other words, there exists a natural isomorphism $Hom_{\mathscr{C}}(Y, X) \to F(Y)$, since $h$ is fully faithful, if $F \cong h(X) \cong h(X')$, the natural isomorphism $h(X) \cong h(X')$ comes from an isomorphism $\phi : X \to X'$, hence $X$ is unique to isomorphism

**Definition 1.30.** Let $I$ be a small category, $\mathscr{C}$ be a category, for any $X \in ob\mathscr{C}$, we can define the **constant functor** $K_X : I \to \mathscr{C}$, $i \mapsto X$, $i \xrightarrow{f} j \mapsto 1_X$, hence $K : \mathscr{C} \to \mathscr{C}^I$, $X \mapsto K_X$ is a functor, a natural transformation $f$ between constant functors $K_X \to K_Y$ is just a morphism $f : X \to Y$

**Definition 1.31.** Suppose $F : I \to \mathscr{C}$ is a functor, we get a presheaf $P$, $P(X) = Hom_{\mathscr{C}^I}(K_X, F)$
If $P$ is representable, i.e. $h(L) \cong P$, we write $L = \varprojlim F$ which is called the **limit** of $F$
We also have a functor $F^{op} : I^{op} \to \mathscr{C}^{op}$. The **colimit** is defined to be $\varprojlim F^{op}$

*Remark.* Unravel $Hom_{\mathscr{C}^I}(K_X, F) = P(X) \cong Hom_{\mathscr{C}}(X, L)$
If we take $X = L$, $1_X$ corresponds to a natural transformation $\phi : K_L \to F$, i.e. $\phi_i : L \to F(i)$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\phi_i"] \arrow[dr, "\phi_j"'] & F(i) \arrow[d, "F(i\to j)"] \\ & F(j) \end{tikzcd}$$

Each natural transformation $\psi : K_X \to F$ corresponds to a unique morphism $\widehat{\psi} : X \to L$, due to naturality, the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "\widehat{\psi}"'] \arrow[dr, "\psi_i"] & \\ L \arrow[r, "\phi_i"] & F(i) \end{tikzcd}$$

**Definition 1.32.** A category $I$ is called **discrete** if all morphisms are just identities, it is clear that a discrete category is the same as a class of objects, and a functor $F : I \to \mathscr{C}$ is the same as giving $X_i = F(i)$

**Example 1.33.** Suppose $I$ is a discrete category, $F : I \to \mathscr{C}$ is a functor, we also get functor $F^{op} : I^{op} \to \mathscr{C}^{op}$. The **product** is defined to be the limit $\prod_{i \in I} X_i := \varprojlim F$, and the **coproduct** is $\varprojlim F^{op}$

## 1.5 Equalizers and fiber product - 2/5/2020

**Definition 1.34.** A category $\mathscr{C}$ is **complete** if $\mathscr{C}$ contains all limits, $\mathscr{C}$ is **cocomplete** if $\mathscr{C}$ contains all colimits

**Definition 1.35.** Let $I$ be the category $\bullet \rightrightarrows \bullet$, a functor $F : I \to \mathscr{C}$ is just $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, the limit is defined to be the **equalizer**, the dual notion is called a **coequalizer**

**Theorem 1.36.** *If a category $\mathscr{C}$ contains all products and equalizers, then $\mathscr{C}$ is complete*

*Proof.* The limit of $A \xrightarrow{f} B$ is the same as the equaliser of $A \times B \overset{fp_A}{\underset{p_B}{\rightrightarrows}} B$

$$\begin{tikzcd} & C \arrow[dl, "g"'] \arrow[d, dashed] \arrow[dr, "h"] & \\ & A \times B \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

Then by induction, we can find the limit of $A_i \to \varprojlim_{j \neq i} A_j$ which is $\varprojlim_i A_i$ ☐

**Definition 1.37.** Let $I$ be the category $\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$, a functor $F : I \to \mathscr{C}$ is just $\begin{tikzcd} & Y \arrow[d, "g"'] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, the limit is defined to be the **fiber product(pullback)**, the dual notion is called a **pushforward(pushout)**

**Definition 1.38.** An **Ab-category** $\mathscr{C}$ is a category such that $Hom_{\mathscr{C}}(X, Y)$ are equipped with an abelian group structure, such that $f(g + h) = fg + fh$, $(f + g)h = fh + gh$

*Remark.* An $Ab$-category is also called a **preadditive category**
$End_{\mathscr{C}}(X)$ is a ring, $Aut_{\mathscr{C}}(X) = End_{\mathscr{C}}(X)^{\times}$ is a group

## 1.6 Abelian category - 2/7/2020

**Definition 1.39.** The **biproducts** $(A_1 \oplus \cdots \oplus A_n, p_1, \cdots, p_n, i_1, \cdots, i_n)$ of $A_1, \cdots, A_n$ is such that $(A_1 \oplus \cdots \oplus A_n, p_1, \cdots, p_n)$ is the product of $A_1, \cdots, A_n$ and $(A_1 \oplus \cdots \oplus A_n, i_1, \cdots, i_n)$ is the coproduct of $A_1, \cdots, A_n$

**Lemma 1.40.** *Suppose $\mathscr{A}$ is an $Ab$ category, then for any $A_1, \cdots, A_n$, if the product $\prod A_i$ exists, then it is a biproduct, similarly, if the coproduct $\coprod A_i$ exsits, then it is a biproduct*

*Proof.* Suppose $(A \times B, p_A, p_B)$ is the product of $A, B$, then we can define morphisms $i_A = (1_A, 0) : A \to A \times B$, $i_B = (0, 1_B) : B \to A \times B$

$$\begin{tikzcd} & A \arrow[dl, "1_A"'] \arrow[d, dashed, "i_A"] \arrow[dr, "0"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[dl, "0"'] \arrow[d, dashed, "i_B"] \arrow[dr, "1_B"] & \\ A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B \end{tikzcd}$$

Thus $p_A i_A = 1_A$, $p_B i_A = 0$, $p_A i_B = 0$, $p_B i_B = 1_B$, also if we consider the following commutative diagram

$$\begin{tikzcd} & A \times B \arrow[dl, bend right, "p_A"'] \arrow[d, dashed, "i_A p_A + i_B p_B"] \arrow[dr, bend left, "p_B"] & \\ A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \end{tikzcd}$$

By the uniqueness of the induced map, $i_A p_A + i_B p_B = 1_{A \times B}$, let's show that $(A \times B, i_A, i_B)$ is the coproduct of $A, B$, suppose $h : A \times B \to C$ is a morphism such that $h i_A = f$, $h i_B = g$, then $h = h(i_A p_A + i_B p_B) = h i_A p_A + h i_B p_B = f p_A + g p_B$

$$\begin{tikzcd} & C & \\ A \arrow[ur, "f"] \arrow[r, shift left, "i_A"] & A \times B \arrow[u, dashed, "\exists_1 h"] \arrow[l, shift left, "p_A"] \arrow[r, shift right, "p_B"'] & B \arrow[ul, "g"'] \arrow[l, shift right, "i_B"'] \end{tikzcd}$$

□

**Definition 1.41.** An **additive category** is an $Ab$ category with all finite biproducts, including empty biproduct 0, the zero object

**Definition 1.42.** An **abelian category** $\mathscr{A}$ is an additive category satisfying
**(AB1)** Every map has a kernel and a cokernel
**(AB2)** Every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel

**Example 1.43.** The category of free $\mathbb{Z}$ modules(free abelian groups) is not an abelian category, $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ has 0 as its cokernel but this is not the kernel of $\mathbb{Z} \to 0$

**Example 1.44** (A category with two different $Ab$ structures)**.** Consider rings $\mathbb{Q}[x]$, $\mathbb{Q}[x,y]$ as abelian categories with a single object and morphisms being the elements, multiplication as composition, addition gives an abelian group structure
$\mathbb{Q}[x]$, $\mathbb{Q}[x,y]$ as categories are isomorphic to its underlying monoids, since $\mathbb{Q}[x]$, $\mathbb{Q}[x,y]$ are UFD's, and $\mathbb{Q}[x] = \{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_f \mathbb{N}$, $\mathbb{Q}[x,y] = \{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_g \mathbb{N}$ , where $f, g$ run over all irreducible polynomials of $\mathbb{Q}[x] \setminus \mathbb{Q}$ and $\mathbb{Q}[x,y] \setminus \mathbb{Q}$ which are both countably many, thus as monoids they are both isomorphic to $\{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_{i \in \mathbb{N}} \mathbb{N}$, Where $0 \circ 0 = 0$, $0 \circ (q, (i_0, i_1, \cdots)) = (q, (i_0, i_1, \cdots)) \circ 0 = 0$, $(q, (i_0, i_1, \cdots)) \circ (q', (i'_0, i'_1, \cdots)) = (qq', (i_0 + i'_0, i_1 + i'_1, \cdots))$ with $(1, (0, 0, \cdots))$ as the identity
but $\mathbb{Q}[x]$, $\mathbb{Q}[x,y]$ are not isomorphic as rings

*Remark.* Being an abelian category is purely a property of a category
If all finite products and coproducts are biproducts, i.e. $X \sqcup Y = X \times Y$, with some other exactness properties, then the abelian group structure on $Hom(X, Y)$ comes from this
See Freyd - Abelian category

**Definition 1.45.** Define the **diagonal functor** $\Delta : \mathscr{C} \to \mathscr{C}^I$ mapping $A$ to the constant functor $K_A$

## 1.7 Adjunction - 2/10/2020

**Definition 1.46.** Let $L : \mathscr{D} \to \mathscr{C}$, $R : \mathscr{C} \to \mathscr{D}$ be functors, and there is a natural isomorphism $\Phi_{X,Y}$, $X \in \mathscr{C}, Y \in \mathscr{D}$

$$\begin{array}{ccc} Hom_{\mathscr{C}}(LX, Y) & \xrightarrow{\Phi_{X,Y}} & Hom_{\mathscr{D}}(X, RY) \\ {\scriptstyle (Lf,g)}\downarrow & & \downarrow{\scriptstyle (g,Rf)} \\ Hom_{\mathscr{C}}(LX', Y') & \xrightarrow{\Phi_{X',Y'}} & Hom_{\mathscr{D}}(X', RY') \end{array}$$

Here $f : X' \to X$, $g : Y \to Y'$, $Hom_{\mathscr{C}}(Lf, g)(h) = h \circ g \circ Lf$
We say $L$ is the **left adjoint** of $R$ and $R$ is the **right adjoint** of $L$

**Example 1.47.** Let $G : Group \to Set$ be the forgetful functor, then the functor $F : Set \to Group$, sending $S$ to $F(S)$ is the left adjoint of $G$
In the category of $R$-modules $Mod$, consider functor $F := - \otimes B$ and functor $G := Hom(B, -)$, then $F, G$ are adjoint pairs, i.e. $Hom(A \otimes B, C) \cong Hom(A, Hom(B, C))$

**Theorem 1.48.** *Suppose* $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$ *are a pair of adjoint functors, then there exist natural transformations* $\eta : 1_{\mathscr{A}} \to RL$ *and* $\varepsilon : LR \to 1_{\mathscr{B}}$ *such that the right adjoint of* $LX \xrightarrow{f} Y$ *is* $X \xrightarrow{R(f)\eta_X} RY$ *and left adjoint of* $g : X \to RY$ *is* $LX \xrightarrow{\varepsilon_Y L(g)} Y$*. Moreover, the following composites are identity,* $LX \xrightarrow{L(\eta_X)} LRLX \xrightarrow{\varepsilon_{LX}} LX$, $RY \xrightarrow{\eta_{RY}} RLRY \xrightarrow{R(\varepsilon_Y)} RY$

*Proof.* □

**Theorem 1.49.** *Suppose* $F, G$ *is an adjunction pair, then* $F$ *preserve colimits,* $G$ *preserve limits*

*Proof.* Suppose $\Phi : I \to \mathscr{D}$ is a functor, $L = \varprojlim_{i \in I} \Phi(i)$ exists, applying $G$ to commutative diagram $L \xrightarrow{\varphi_i} \Phi(i)$, we get another commutative diagram $GL \xrightarrow{G\varphi_i} G\Phi(i)$. For any commutative diagram $X \xrightarrow{\psi_i} G\Phi(i)$, by adjunction, we have a commutative diagram $FX \to \Phi(i)$, which induce a map $FX \to L$, by adjunction again, we have $X \to GL$ □

**Definition 1.50.** A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **left exact** if $F$ preserve all finite limits, and **right exact** if $F$ preserve all finite colimits

**Example 1.51.** A left exact functor preserves all equalizers, and all kernels if the category is abelian, a right exact functor preserves all coequalizers, and all cokernels if the category is abelian, left adjoints are right exact, right adjoints are left exact, for example, $- \otimes B$ is right exact and $Hom(B, -)$ is left exact

# 2 Chain complexes

## 2.1 Chain complexes - 2/12/2020

**Definition 2.1.** Let $\mathscr{A}$ be an abelian category, a ($\mathbb{Z}$-graded) **chain complex** $C_\bullet$ is

$$\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0$, $\partial_i$ are called **boundary maps(differentials)**
We can define chain maps, chain homotopy, boundaries, cycles, and homology groups, and we say the chain complex is exact if each homology groups is zero, the chain complexes form the **category of chain complexes** $Ch_\bullet\mathscr{A}$
Similarly, we can also define cochain complex $C^\bullet$

$$\cdots \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \to \cdots$$

Such that $d^{n+1} \circ d^n = 0$, $d^i$ are called **coboundary maps**, cochain complexes form the **category of cochain complexes** $Ch^\bullet\mathscr{A}$

**Lemma 2.2.** $\phi : Ch^\bullet\mathscr{A} \to Ch_\bullet\mathscr{A}$, $(\phi C_\bullet)^n = C_{-n}$, $\phi(d^n) = \partial_n$

*Proof.* □

**Definition 2.3.** Suppose $X_\bullet$ is a chain complex, we can define **cycles** $Z_n(X) := \ker(X_n \xrightarrow{\partial_n} X_{n-1})$, **boundaries** $B_n(X) := \operatorname{im}(X_{n+1} \xrightarrow{\partial_n} X_n)$ and **homology** $H_n(X) := \operatorname{coker}(B_n \to Z_n)$, actually, $Z_n, B_n, H_n$ are functors $Ch\mathscr{A} \to \mathscr{A}$

**Definition 2.4.** $\phi : X_\bullet \to Y_\bullet$ is called a **quasi-isomorphism** if $H_n(\phi) : H_nX \to H_nY$ are isomorphisms

**Example 2.5.** Consider

$$\begin{tikzcd}
X_\bullet : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 5"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{mod}\,5"] & 0 \\
Y_\bullet : & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

**Definition 2.6.** Pick $p \in \mathbb{Z}$, define the **translation** of $X$ by $p$ is $X_\bullet[p]$ where $(X_\bullet[p])_n = X_{p+n}$, differential $X_\bullet[p]_n \to X_\bullet[p]_{n-1}$ is given by $(-1)^p\partial$ The **translation functor** $T : Ch(\mathscr{A}) \to Ch(\mathscr{A})$, $X \mapsto X_\bullet[1]$ is an auto morphism of $Ch(\mathscr{A})$

**Example 2.7.** Suppose $X$ is a topological space, $R$ is a ring, $C_*^{\mathrm{sing}}(X)$ is the singular chain complex, $\Sigma X$ is the suspension of $X$, we have the Freudenthal theorem $H^k(\Sigma X) \cong H^{k-1}(X)$ for $k > 0$

**Definition 2.8.** Pick $p \in \mathbb{Z}$, define the **truncation** of $X$ at $p$ is $\tau_{\geq p}X$, where $(\tau_{\geq p}X)_k = \begin{cases} 0, & k < p \\ Z_pX, & k = p \\ X_k, & k > p \end{cases}$, and define the cokernel of $\tau_{\geq p}X \to X$ to be $\tau_{<p}X$. We get the **truncation functors** $\tau_{\geq p}X \to X$ and $X \to \tau_{<p}X$ Moreover, $H_* : \tau_{\geq p}X \to X$ induce isomorphisms for $k \geq p$ and zero maps for $k < p$, $H_* : X \to \tau_{<p}X$ induce isomorphisms for $k < p$ and zero maps for $k \geq p$

**Example 2.9.** Consider $p = 0$

$$
\begin{tikzcd}
X_2 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & Z_0X \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, two heads] & X_{-1} \arrow[r] \arrow[d, equal] & X_{-2} \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & X_0/Z_0 \arrow[r] & X_{-1} \arrow[r] & X_{-2}
\end{tikzcd}
$$

## 2.2 Arrow category - 2/14/2020

**Definition 2.10.** A chain complex of **amplitude** $[p, q]$ are chains of the following form

$$0 \longrightarrow X_q \xrightarrow{d} \cdots \xrightarrow{d} X_p \longrightarrow 0$$

Let $Ch_{[p,q]}\mathscr{C}$ denote the full subcategory of $Ch\mathscr{C}$ consist of chain complexes of amplitude $[p, q]$

**Definition 2.11.** Suppose $\mathscr{C}$ is a category, we can define the **arrow category** $Ar\mathscr{C}$, where the objects are morphisms in $\mathscr{C}$, and $Hom(X \xrightarrow{f} Y, Z \xrightarrow{g} W)$ consists of commutative diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ Z \arrow[r, "g"] & W \end{tikzcd}$$

Equivalently, $Ar\mathscr{C} = Ch_{[0,1]}\mathscr{C}$

**Lemma 2.12.** *Suppose $\mathscr{A}$ is an abelian category, then* $\ker, \operatorname{coker} : Ar\mathscr{A} \to \mathscr{A}$ *are two functors given by the following diagram*

$$\begin{tikzcd} \ker f \arrow[r, hook] \arrow[d, "u_*"] & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, two heads] \arrow[d, "v"] & \operatorname{coker} f \arrow[d, "v_*"] \\ \ker g \arrow[r, hook] & Z \arrow[r, "g"] & W \arrow[r, two heads] & \operatorname{coker} g \end{tikzcd}$$

*Let $F_1 : \mathscr{A} \to Ar\mathscr{A}$, $X \mapsto 0 \to X \to 0$, where $X$ is of degree $1$, $F_0 : \mathscr{A} \to Ar\mathscr{A}$, $X \mapsto 0 \to X \to 0$, where $X$ is of degree $0$. Then* $\ker$ *is the right adjoint to $F_1$ and* $\operatorname{coker}$ *is the left adjoint to $F_0$*

*Proof.* □

## 2.3 Chain homotopy - 2/17/2020

**Lemma 2.13** (Snake lemma)**.** *Given the following commutative diagram with exact rows, then we have an exact sequence*

$$\begin{tikzcd}
0 \arrow[r, dashed] & \ker a \arrow[r, "u_*"] \arrow[d] & \ker b \arrow[r, "v_*"] \arrow[d] & \ker c \arrow[d] \arrow[dddll, dashed, "\delta"] & \\
0 \arrow[r, dashed] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & 0 \\
& \operatorname{coker} a \arrow[r, "u'_*"] & \operatorname{coker} b \arrow[r, "v'_*"] & \operatorname{coker} c \arrow[r, dashed] & 0
\end{tikzcd}$$

*Proof.* □

**Lemma 2.14.** $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ *is exact iff* $0 \to A_n \to B_n \to C_n \to 0$ *are exact*

*Proof.* □

**Theorem 2.15.** *Suppose* $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ *is exact, then we have* $\partial : H_nC \to H_{n-1}A$ *yielding a long exact sequence*

$$\cdots \to H_nA \to H_nB \to H_nC \xrightarrow{\partial} H_{n-1}A \to H_{n-1}B \to H_{n-1}C \to \cdots$$

*Proof.* Fisrtly, by Lemma 2.13, we have

$$\begin{tikzcd}
0 \arrow[r] & Z_nA \arrow[r] \arrow[d] & Z_nB \arrow[r] \arrow[d] & Z_nC \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
& A_n/\partial A_{n-1} \arrow[r] & B_n/\partial B_{n-1} \arrow[r] & C_n/\partial C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Then apply Lemma 2.13 again, we get

$$\begin{tikzcd}
& H_nA \arrow[r] \arrow[d] & H_nB \arrow[r] \arrow[d] & H_nC \arrow[d] \arrow[dddll, dotted] & \\
& A_n/\partial A_{n+1} \arrow[r] \arrow[d] & B_n/\partial B_{n+1} \arrow[r] \arrow[d] & C_n/\partial C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[d] & \\
& H_{n-1}A \arrow[r] & H_{n-1}B \arrow[r] & H_{n-1}C &
\end{tikzcd}$$

□

**Lemma 2.16** (Five lemma)**.** *If* $b$ *and* $d$ *are monic and* $a$ *is an epi, then* $c$ *is monic. Dually, if* $b$ *and* $d$ *are epis and* $e$ *is monic, then* $c$ *is an epi. In particular, if* $a, b, d$ *and* $e$ *are iso, then* $c$ *is also an iso*

$$\begin{tikzcd} A' \arrow[r, "u'"] \arrow[d, "a"', "\cong"] & B' \arrow[r, "v'"] \arrow[d, "b"', "\cong"] & C' \arrow[r, "w'"] \arrow[d, "c"'] & D' \arrow[r, "x'"] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\ A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & D \arrow[r, "x"] & E \end{tikzcd}$$

**Definition 2.17.** We can define a full subcategory $S(\mathscr{A})$ of short exact sequences, or equivalently just $Ch_{[0,2]}\mathscr{A}$, and define a full subcategory $L(\mathscr{A})$ of long exact sequences

**Lemma 2.18.** *$H$ gives a functor $S(Ch\mathscr{A}) \to L(\mathscr{A})$, sending $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ to its long exact sequence*

*Proof.* □

**Definition 2.19.** Suppose $X_\bullet, Y_\bullet \in Ch\mathscr{A}$, define a complex $Hom_\bullet(X_\bullet, Y_\bullet) \in Ch\mathscr{A}b$ as follows: for each $k \in \mathbb{Z}$, $Hom_k(X_\bullet, Y_\bullet) = \prod_{n\in\mathbb{Z}} Hom(X_n, Y_{k+n})$, and $d(f_n : X_n \to Y_{n+k})_{n\in\mathbb{Z}} = (g_n : X_n \to Y_{n+k-1})_{n\in\mathbb{Z}}$ where $g_n = \partial f_n - (-1)^k f_{n-1}\partial$

$$\begin{tikzcd} X_{n+1} \arrow[r, "\partial"] \arrow[d, "f"] \arrow[dr, "g"] & X_n \arrow[r, "\partial"] \arrow[d, "f_n"] \arrow[dr, "g_n"] & X_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] \arrow[dr, "g"] & X_{n-2} \arrow[d, "f"] \\ Y_{n+k+1} \arrow[r, "\partial"] & Y_{n+k} \arrow[r, "\partial"] & Y_{n+k-1} \arrow[r, "\partial"] & Y_{n+k-2} \end{tikzcd}$$

$Hom_\bullet(X_\bullet, Y_\bullet)$ is a chain complex since for any $f \in Hom_k(X_\bullet, Y_\bullet)$

$$\begin{aligned} d^2 f &= d(\partial f - (-1)^k f\partial) \\ &= \partial(\partial f - (-1)^k f\partial) + (-1)^{k-1}(\partial f - (-1)^k f\partial)\partial \\ &= \partial^2 f - (-1)^k \partial f\partial + (-1)^k \partial f\partial + (-1)^k f\partial^2 \\ &= 0 \end{aligned}$$

If $f \in Hom_0(X_\bullet, Y_\bullet)$, then $df = \partial f - f\partial = 0 \Leftrightarrow f \in Hom(X_\bullet, Y_\bullet)$, i.e. $Hom(X_\bullet, Y_\bullet) = Z_0(Hom_\bullet(X_\bullet, Y_\bullet))$, $f \in B_0 Hom_\bullet(X_\bullet, Y_\bullet) \Leftrightarrow f - 0 = f = ds = \partial s + s\partial$. i.e. $f$ is chain homotopy equivalent to 0. Therefore we define the **chain homotopy** classes of morphisms from $X_\bullet \to Y_\bullet$ to be $H_0(Hom(X_\bullet, Y_\bullet))$

## 2.4 Chain homotopy category - 2/19/2020

**Definition 2.20.** Suppose $\mathscr{A}$ is an abelian category, define $K(\mathscr{A})$ to be the **homotopy category** with $obK(\mathscr{A}) = obCh(\mathscr{A})$, $Hom_{K(\mathscr{A})}(X_\bullet, Y_\bullet) = H_0(Hom_\bullet(X_\bullet, Y_\bullet))$

**Definition 2.21.** Suppose $f : X_* \to Y_*$ is chain map, then the **mapping cone** of $f$ is defined to be the object $C(f)$ in $Ch(\mathscr{A})$ with $C(f)_n = X_{n-1} \oplus Y_n$, $d_{C(f)} = \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix}$, note that $d_{C(f)}^2 = \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix} \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix} = \begin{pmatrix} d_X^2 & 0 \\ fd_X - d_Y f & d_Y^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**Lemma 2.22.** *We have a short exact sequence*

$$(x, y) \longmapsto x$$

$$0 \longrightarrow Y \longrightarrow C(f) \longrightarrow X[-1] \longrightarrow 0$$

$$y \longmapsto (0, y)$$

*Remark.* If $f : X_* \to 0$ is the zero morphism, then $C(f) = X[-1]$

*Proof.* $\square$

**Corollary 2.23.** $f : X_* \to Y_*$ *is a quasi-isomorphism* $\Leftrightarrow$ $C(f)$ *is exact*

## 2.5 Freyd-Mitchell embedding - 2/21/2020

**Theorem 2.24.** *We have a homology functor $H_* : K(\mathscr{A}) \to \mathscr{A}$ such that the following diagram commutes*

$$\begin{tikzcd} Ch(\mathscr{A}) \arrow[r, "H_*"] \arrow[d] & \mathscr{A} \\ K(\mathscr{A}) \arrow[ur, "H_*"] & \end{tikzcd}$$

**Theorem 2.25** (Freyd-Mitchell embedding theorem)**.** *Suppose $\mathscr{A}$ is a small abelian category, then there exists a ring $R$ and a fully faithful embedding $\mathscr{A} \to R\text{-}\mathsf{mod}$, i.e. $\mathscr{A}$ embeds in $R\text{-}\mathsf{mod}$ as a full subcategory. Moreover, the embedding is an exact functor*

**Lemma 2.26.** *Suppose $\mathscr{A}$ is an abelian category, $\mathscr{C}$ is a subcategory, then*
*(1) $\mathscr{C}$ is additive $\Leftrightarrow$ if $\mathscr{C}$ is closed under direct sum, including* 0
*(2) $\mathscr{C}$ is abelian and $\mathscr{C} \hookrightarrow \mathscr{A}$ is exact $\Leftrightarrow$ $\mathscr{C}$ is additive and contain kernels, cokernels*

*Proof.* □

**Definition 2.27.** Suppose $\mathscr{A}, \mathscr{B}$ are abelian categories, a covariant homological **$\delta$ functor** is a family of functors $T_n : \mathscr{A} \to \mathscr{B}$ and for each short exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$, a family of morphisms $\delta_n : T_n C \to T_{n-1} A$ which induces a long exact sequence

$$\cdots \to T_n(A) \xrightarrow{u_n} T_n(B) \xrightarrow{v_n} T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \to \cdots$$

And any chain map

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

The following diagram commutes

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta_n"] \arrow[d] & T_{n-1}(A) \arrow[d] \\ T_n(C') \arrow[r, "\delta_n"] & T_{n-1}(A') \end{tikzcd}$$

Similarly, we can define covariant cohomological $\delta$ functors

**Example 2.28.** $H_n : Ch_\bullet(\mathscr{A}) \to \mathscr{A}$, $C_* \mapsto H_n(C)$, $\delta = \partial$ is a homological $\delta$ functor, $H^n : Ch^\bullet(\mathscr{A}) \to \mathscr{A}$, $C^* \mapsto H^n(C)$, $\delta = d$ is a cohomological $\delta$ functor

**Example 2.29.** Let $\mathscr{A}b$ be the category of abelian groups, define functors $T_1 : \mathscr{A}b \to \mathscr{A}b$, $A \mapsto A_p$, where $A_p$ is $\ker(A \xrightarrow{\times p} A)$ is the $p$ torsion of $A$, and $T_0 : \mathscr{A}b \to \mathscr{A}b$, $A \mapsto A/pA$, where $A/pA = \operatorname{coker}(A \xrightarrow{\times p} A)$. For a short exact sequence $0 \to A \to B \to C \to 0$, by Snake Lemma 2.13, we have long exact sequence

$$0 \to T_1 A \to T_1 B \to T_1 C \xrightarrow{\delta_1} T_0 A \to T_0 B \to T_0 C \to 0$$

**Definition 2.30.** A morphism between delta functors $\{S_i\}$, $\{T_i\}$ is a sequence of natural transformations $\eta_n : S_n \to T_n$ commuting with $\delta$, i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & S_nA \arrow[r, "S_n u"] \arrow[d, "\eta"] & S_nB \arrow[r, "S_n v"] \arrow[d, "\eta"] & S_nC \arrow[r, "\delta_S"] \arrow[d, "\eta"] & S_{n-1}A \arrow[r] \arrow[d, "\eta"] & \cdots \\
\cdots \arrow[r] & T_nA \arrow[r, "T_n u"] & T_nB \arrow[r, "T_n v"] & T_nC \arrow[r, "\delta_T"] & T_{n-1}A \arrow[r] & \cdots
\end{tikzcd}$$

Therefore we get a category of homological $\delta$ functors

**Definition 2.31.** A $\delta$ functor $\{T_n\}$ is called **universal** if for any $\delta$ functor $\{S_n\}$, given a natural transformation $\eta_0 : S_0 \to T_0$, this can be uniquely extended to $\eta_n : S_n \to T_n$ up to isomorphism, in other words, $\{T_n\}$ is a final object in the category of homological $\delta$ functors

## 2.6 Resolutions - 2/24/2020

**Definition 2.32.** Suppose $\mathscr{C}$ is an abelian category, $P$ is **projective** if functor $Hom(P,-):\mathscr{C}\to Sets$ sends epi to epi, or equivalently

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "\exists h"'] \\ X \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

$I$ is **injective** if functor $Hom(-,Q):\mathscr{C}\to Sets$ sends mono to epi, or equivalently

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, dashed, "\exists h"] \\ Q & \end{tikzcd}$$

**Lemma 2.33.** *Coproduct of projective objects is projective, product of injective objects is injective*

*Proof.* Suppose $I_\alpha$ are injective, $A\hookrightarrow B$ is a monomorphism, we have

$$\begin{tikzcd} Hom(B,\prod I_\alpha) \arrow[r] \arrow[d, leftrightarrow] & Hom(A,\prod I_\alpha) \arrow[d, leftrightarrow] \\ \prod Hom(B,I_\alpha) \arrow[r] & \prod Hom(A,I_\alpha) \end{tikzcd}$$

□

**Definition 2.34.** $\mathscr{C}$ has **enough projectives** if for any $X$, there is an epi $P\to X$ from a projective object, $\mathscr{C}$ has **enough injectives** if for any $X$, there is a mono $X\to Q$ to an injective object

**Lemma 2.35.** *Suppose $\mathscr{A}$ is an abelian category, $Hom(P,-)$, $Hom(-,I)$ are left exact. We have*

$$P \text{ is projective} \Leftrightarrow Hom(P,-) \text{ is right exact} \Leftrightarrow Hom(P,-) \text{ is exact}$$

$$I \text{ is injective} \Leftrightarrow Hom(-,I) \text{ is right exact} \Leftrightarrow Hom(-,I) \text{ is exact}$$

*Proof.* □

*Remark.* It is obvious that $A\to B\to C\to 0$ is exact iff $0\to Hom(C,D)\to Hom(B,D)\to Hom(A,D)$ is exact for all $D$, $0\to A\to B\to C$ is exact iff $0\to Hom(D,A)\to Hom(D,B)\to Hom(D,C)$ is exact for all $D$

**Lemma 2.36.** *Functors between abelian categories $F:\mathcal{A}\to\mathcal{B}$, $G:\mathcal{B}\to\mathcal{A}$ is an adjoint pair. If $F$ is left exact, then $G$ preserves injectives. If $G$ is right exact, then $F$ preserves projectives*

*Proof.*

$$Hom_{\mathcal{B}}(-,G(I)) \text{ is right exact} \Leftrightarrow Hom_{\mathcal{A}}(F(-),I) = Hom_{\mathcal{A}}(-,I)\circ F \text{ is right exact}$$

$$Hom_{\mathcal{B}}(F(P),-) \text{ is right exact} \Leftrightarrow Hom_{\mathcal{A}}(P,G(-)) = Hom_{\mathcal{A}}(P,-)\circ G \text{ is right exact}$$

□

**Lemma 2.37.** *An $R$ module $M$ is projective iff $M$ is a direct summand of a free module*

*Proof.* □

**Definition 2.38.** Exact sequence $C_\bullet$ **split** at if there are $s_n : C_n \to C_{n+1}$ such that $\partial_{n+1}s_n\partial_{n+1} = \partial_{n+1}$

**Lemma 2.39.** *$C_\bullet$ split iff $1_C \simeq 0$*

**Lemma 2.40.** *$P_\bullet$ is a projective in $Ch(\mathscr{A})$ iff $P_\bullet$ is a split exact sequence of projectives*

*Proof.* □

**Definition 2.41.** A **left resolution** is morphism $P_\bullet \xrightarrow{\varepsilon} M$ in $Ch_{\geq 0}(\mathscr{A})$, here $M$ means $0 \to M \to 0$ with $M$ at degree 0, then $\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$ is exact. A projective resolution is a left resolution with projectives, an injective resolution is a right resolution with injectives

**Lemma 2.42.** *If $\mathscr{A}$ has enough projectives, then for any $M$, there is a projective resolution $P_\bullet \xrightarrow{\varepsilon} M$*

*Proof.* First there exists exact sequence $0 \to \ker\varepsilon \xrightarrow{i_0} P_0 \xrightarrow{\varepsilon} M \to 0$ where $P_0$ is a projective, then there exists another exact sequence $0 \to \ker i_0 \to P_1 \to \ker\varepsilon \to 0$ where $P_1$ is a projective, then we can splice them to get exact sequence $P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$, inductively we get a projective resolution □

**Theorem 2.43** (Comparison theorem). *Suppose $P_\bullet \xrightarrow{\varepsilon} M$ is a complex with $P_n$ projectives, $Q_\bullet \xrightarrow{\eta} N$ is a left resolution, then for any $M \xrightarrow{f} N$, it can be extend to chain map $f_\bullet : P_\bullet \to Q_\bullet$, and $f_\bullet$ is unique up to homotopy*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

*Proof.* Since $P_0$ is projective, there exists $P_0 \xrightarrow{f_0} Q_0$ such that $f\varepsilon = \eta f_0$, then we have $\eta f_0\partial_1 = f\varepsilon\partial_1 = 0$, $f_0\partial_1 : P_1 \to Z_0Q$, and $Q_1 \xrightarrow{\partial_1} Z_0Q$ is epi, we have $P_1 \xrightarrow{f_1} Q_1$, inductively, we can extend $f$ to a chain map $f_\bullet : P_\bullet \to Q_\bullet$
Suppose $f = 0$, we need to show $f_\bullet \simeq 0$, Write $P_{-1} := M$, $Q_{-1} := N$, $P_n = Q_n = 0, \forall n < -1$, define $s_n : P_n \to Q_{n+1}, \forall n < 0$ to be zero. Since $\eta f_0 = 0$, thus $f_0 : P_0 \to Z_0Q$, and $Q_1 \xrightarrow{\partial_1} Z_0Q$ is epi, we get $P_0 \xrightarrow{s_0} Q_1$ such that $f_0 = \partial_1 s_0 = \partial_1 s_0 + s_{-1}\partial_0$, then since $\partial_1 f_1 = f_0\partial_1 = \partial_1 s_0\partial_1 \Rightarrow f_1 - s_0\partial_1 : P_1 \to Z_1Q$, and $Q_2 \xrightarrow{\partial_2} Z_1Q$ is epi, we get $s_1 : P_1 \to Q_2$ such that $\partial_2 s_1 = f_1 - s_0\partial_1$, inductively we construct a null homotopy $s_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] \arrow[dl, "s_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] \arrow[dl, "s_0"] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "0"] & 0 \arrow[dl, "0"] \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

□

**Lemma 2.44** (Horseshoe lemma). *Suppose $P_\bullet \xrightarrow{\varepsilon} M$, $Q_\bullet \xrightarrow{\eta} N$ are projective resolutions, then any exact sequence $0 \to M \xrightarrow{f} A \xrightarrow{g} N \to 0$ can be extended into commutative diagram*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

*With $(P \oplus Q)_\bullet$ being a projective resolution, every row and column are exact*

*Proof.* Since $A \xrightarrow{g} N$ is epi and $Q_0$ is projective, we get $Q_0 \xrightarrow{s_0} A$ such that $gs_0 = \partial_0$ which gives us $P_0 \oplus Q_0 \xrightarrow{\begin{pmatrix} f\partial_0 & s_0 \end{pmatrix}} A$, by Lemma 2.13, this is epi, and we get an exact sequence $0 \to Z_0P \to \ker i_0 \to Z_0Q \to 0$, similarly, we can construct $Q_1 \xrightarrow{s_1} \ker i_0$, then $P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} \iota_0\partial_0 & s_1 \end{pmatrix}} \ker i_0$ is again epi by Lemma 2.13, inductively, we can construct the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\partial_1"'] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d, "\partial_1"] \arrow[dl, "s_1"] & 0 \\
0 \arrow[r] & Z_0P \arrow[r, "\iota_0"] \arrow[d] & \ker i_0 \arrow[r] \arrow[d] & Z_0Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\partial_0"'] & P_0 \oplus Q_0 \arrow[r] \arrow[d, "i_0"] & Q_0 \arrow[r] \arrow[d, "\partial_0"] \arrow[dl, "s_0"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

## 2.7 Baer's criterion - 2/28/2020

**Theorem 2.45** (Baer's criterion)**.** *A left $R$ module $E$ is injective in the category of left $R$ modules iff every left ideal homomorphism $\phi : I \to E$ can be extended to a homomorphism $R \to E$*

*Proof.* If $E$ is injective, we can certainly extend $\phi : I \to E$

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "\phi"'] & R \arrow[dl, dashed] \\ E & \end{tikzcd}$$

Now suppose extension is always possible, $A \hookrightarrow B$ is a submodule, $\alpha : A \to E$ is a homomorphism, consider poset $\Gamma := \{A \leq C \leq B, \alpha_C : C \to E\}$, $(C, \alpha_C) \leq (D, \alpha_D)$ meaning $C \leq D$ and $\alpha_D|_C = \alpha_C$, by Zorn's lemma, we can pick a maximal element $(C, \alpha)$, suppose $C \lneq B$, there exists $b \in B \setminus C$, consider $J = \{r \in R | rb \in C\}$, we have

$$\begin{tikzcd} J \arrow[r, "\times b"] \arrow[dr, hook] & C \arrow[r, "\alpha"] & E \\ & R \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

define $\beta : C + \langle b \rangle \to E$, $c + rb \mapsto \alpha(c) + f(r)$, contradicting the maximality $\square$

**Definition 2.46.** A left $R$ module $M$ is $r$ **divisible** if $M \xrightarrow{\times r} M$ is surjetive, $M$ is divisible if $M$ is $r$ divisible for any $0 \neq r \in R$

**Corollary 2.47.** *Suppose $R$ is a PID, a left $R$ module $M$ is injective iff $M$ is divisible*

*Proof.* Suppose $M$ is injective, for any $0 \neq r \in R$, $R \xrightarrow{\times r} rR$ is an isomorphism since $R$ is a PID, by Theorem 2.45, we have

$$\begin{tikzcd} R \arrow[r, "\times r"] \arrow[dr, "m"'] & rR \arrow[r, hook] & R \arrow[dl, dashed, "m'"] \\ & M & \end{tikzcd}$$

Here $R \xrightarrow{m} M$, $1 \mapsto m$, thus $m = rm'$
Suppose $M$ is divisible, since $R$ is a PID, for any homomorphism $rR \to M$, $r \mapsto m$, we have $m = rm'$ for some $m'$, giving the extension $R \to M$, $1 \mapsto m'$ $\square$

**Corollary 2.48.** *The category of abelian groups $\mathscr{A}\mathrm{b}$ has enough injectives*

*Proof.* By corollary 2.47, $\mathbb{Q}/\mathbb{Z}$ is injective since $\mathbb{Q}/\mathbb{Z}$ is divisible
Suppose $M$ is an abelian group, define $I = \prod_f \mathbb{Q}/\mathbb{Z}$ which is also injective due to Lemma 2.33, here $f$ runs over $Hom(M, \mathbb{Q}/\mathbb{Z})$, thus we get $h : M \to I$. Suppose $0 \neq m \in \ker h$, then $f(m) = 0$ for any $f \in Hom(M, \mathbb{Q}/\mathbb{Z})$. Define $H = \mathbb{Z}m$ which is isomorphism to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, then we can define $\beta : H \to \mathbb{Q}/\mathbb{Z}$, $m \mapsto \dfrac{1}{n}$, since $\mathbb{Q}/\mathbb{Z}$ is a injective, we can extend to $\alpha : M \to \mathbb{Q}/\mathbb{Z}$, but then $\alpha(m) = \beta(m) \neq 0$ which is a contradiction $\square$

**Theorem 2.49.** *Suppose $\mathscr{A}, \mathscr{B}$ are abelian categories, $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$ are adjoint functors, $L$ is exact, $I \in \mathrm{ob}\mathscr{B}$ is injective, then $R(I)$ is also injective*

*Proof.* For any monomorphism $A \hookrightarrow A'$, $L(A) \hookrightarrow L(A')$ is also monic, we have

$$\begin{tikzcd} Hom(L(A'), I) \arrow[r] \arrow[d, leftrightarrow] & Hom(L(A), I) \arrow[d, leftrightarrow] \\ Hom(A', R(I)) \arrow[r] & Hom(A, R(I)) \end{tikzcd}$$

$\square$

## 2.8 Enough injectives in R-Mod - 3/2/2020

**Lemma 2.50.** *If $M$ is a left $R$ module, $A$ is an abelian group, then $\mathsf{Hom}(M, A)$ is a right $R$ module. Similarly, if $M$ is a ring $R$ module, then $\mathsf{Hom}(A, M)$ is a left $R$ module*

*Proof.* $(fr)(m) = f(rm)$, $(frs)(m) = f(rsm) = (fr)(sm) = ((fr)s)(m)$
$(rf)(m) = f(mr)$, $(rsf)(m) = f(mrs) = (sf)(rm) = (r(sf))(m)$ □

**Proposition 2.51.** *If $M$ is a left $R$ module, $A$ is an abelian group, viewing $R$ as a right $R$ module, then the natural map $\mathsf{Hom}_{\mathscr{A}\mathsf{b}}(M, A) \to \mathsf{Hom}_{\text{R-Mod}}(M, \mathsf{Hom}(R, A))$ is an isomorphism. In other words, $\mathsf{Hom}(R, -)$ is the right adjoint to the forgetful functor $R\text{-}\mathsf{Mod} \to \mathscr{A}\mathsf{b}$, sending a right R module to its underlying abelian group, the forgetfull is clearly an exact functor, thus $\mathsf{Hom}(R, -)$ maps injectives to injectives*

*Proof.* □

**Corollary 2.52.** *R-$\mathsf{Mod}$ has enough injectives*

*Proof.* □

**Definition 2.53.** Suppose $\mathscr{A}, \mathscr{B}$ are abelian category, $F : \mathscr{A} \to \mathscr{B}$ is an **additive funtor** if $F(A \oplus B)$ is naturally isomorphic to $F(A) \oplus F(B)$, and $F$ is **additive**, i.e. $F(f + g) = F(f) + F(g)$, $f, g \in Hom(A, B)$, here $f + g$ is given by the composition $A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus B \xrightarrow{\nabla} Y$, here $\nabla$ is the **codiagonal**

**Example 2.54.** $\bigwedge^2 : \mathrm{Vect}_F \to \mathrm{Vect}_F$ is exact but not additive

## 2.9 Universality of derived functors - 3/6/2020

**Definition 2.55.** The **left derived functor** of $F$ is $L_iF(A) = H_iF(P)$, where $P \to A$ is a projective resolution

**Definition 2.56.** $Y \in \mathrm{ob}\mathscr{A}$ is $F$-**acyclic** if $L_iF(Y) = 0$ for all $i \geq 1$. Projectives are acyclic

**Theorem 2.57.** *$F : \mathscr{A} \to \mathscr{B}$ is right exact, $\mathscr{A}$ has enough projectives, the **left derived functor** $L_iF$ is a universal homological $\delta$ functor*

*Proof.* Suppose $0 \to X \to Y \to Z \to 0$ is exact, $P_X, P_Z$ are projective resolutions of $X, Z$, then $(P_Y)_i = (P_X)_i \oplus (P_Z)_i$ is a projective resolution of $Y$ by Lemma 2.44, since $F(P_Y)_i = F(P_Y)_i \oplus F(P_Z)_i$, $0 \to F(P_X)_i \to F(P_Y)_i \to F(P_Z)_i \to 0$ split, by Lemma 2.13, we have $\cdots \to L_iF(X) \to L_iF(Y) \to L_iF(Z) \xrightarrow{\delta} L_{i-1}F(X) \to \cdots$, i.e. $L_iF$ is a homological $\delta$ functor. Suppose $T_i$ is another homological $\delta$ functor, $\phi_0 : T_0 \to L_0F$ is a natural transformation, since $\mathscr{A}$ has enough projecives, there exists $P \twoheadrightarrow X$ with $P$ projetive, let $K$ be the kernel, we have a short exact sequence $0 \to K \to P \to X \to 0$

$$\begin{tikzcd}
 & T_1X \arrow[r, "\delta"] \arrow[d, dashed, "\exists_1\phi_1"] & T_0K \arrow[r] \arrow[d, "\phi_0"] & T_0P \arrow[r] \arrow[d, "\phi_0"] & T_0X \arrow[d, "\phi_0"] & \\
0 \arrow[r] & L_1FX \arrow[r, "\delta"] & L_0FK \arrow[r] & L_0FP \arrow[r] & L_0FX \arrow[r] & 0
\end{tikzcd}$$

And then inductively for $i > 0$

$$\begin{tikzcd}
 & T_{i+1}X \arrow[r, "\delta"] \arrow[d, dashed, "\exists_1\phi_{i+1}"] & T_iK \arrow[d, "\phi_i"] & \\
0 \arrow[r] & L_{i+1}FX \arrow[r, "\delta"] & L_iFK \arrow[r] & 0
\end{tikzcd}$$

∎

**Corollary 2.58.** *$F : \mathscr{A} \to \mathscr{B}$ is left exact, $\mathscr{A}$ has enough injectives, the **right derived functor** $R^iF$ is a universal cohomological $\delta$ functor*

**Example 2.59.** $F_M : R\text{-}mod \to Ab$, $N \mapsto Hom_R(M, N)$ is left exact, $R^iF_M(N) = Ext^i_R(M, N)$

### 2.10 Filtered category - 3/9/2020

**Lemma 2.60.** *A left adjoint is a right exact functor, a right adjoint is a left exact functor*

*Proof.* Suppose $(L, R)$ are adjoint pair of functors of abelian categories, $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$, $A \to B \to C \to 0$ is exact, then $0 \to Hom(C, RD) \to Hom(B, RD) \to Hom(A, RD)$ is exact, $0 \to Hom(LC, D) \to Hom(LB, D) \to Hom(LA, D)$ is exact, thus $LA \to LB \to LC \to 0$ is exact □

**Proposition 2.61.** *$I$ is small, $\mathscr{C}$ is cocomplete, $K : \mathscr{C} \to \mathscr{C}^I$, $X \mapsto K_X$ is the right adjoint to* $\mathrm{colim} : \mathscr{C}^I \to \mathscr{C}$

*Proof.* □

**Corollary 2.62.** *$I, J$ is small, $\mathscr{C}, \mathscr{C}^I, \mathscr{C}^J$ are cocomplete, $F : I \times J \to \mathscr{C}$ is a bifunctor, which give $F_I : I \to \mathscr{C}^J$, $F_J : J \to \mathscr{C}^I$, then* $\mathrm{colim}F \cong \mathrm{colim}\mathrm{colim}F_I \cong \mathrm{colim}\mathrm{colim}F_J$

*Proof.* □

**Definition 2.63.** $I$ is a **filtered category** if for any $i, j$, there exist $i \to k \leftarrow j$ and for any $i \underset{\psi}{\overset{\phi}{\rightrightarrows}} j$ , there exists $j \xrightarrow{\xi} k$ coequalizes $\phi, \psi$, i.e. $\xi\phi = \xi\psi$

Equivalently, for finitely many $i_\alpha$, there exist $i_\alpha \xrightarrow{\phi_\alpha} j$ and for finitely many $i \xrightarrow{\phi_\alpha} j$, there exists $j \xrightarrow{\psi} k$ coequalizes $\phi_\alpha$

**Lemma 2.64.** *$F : I \to R\text{-}mod$ is a functor, then* $\mathrm{colim}F = \bigoplus_i F(i)/E$*, $E$ is generated by $F(\phi)(a_i) - a_i$, here $i \xrightarrow{\phi} j$*

*Proof.* We have the following diagrams

$$\begin{tikzcd} F(i) \arrow[r, "F(\phi)"] \arrow[dr] & F(j) \arrow[d] \\ & \bigoplus_i F(i)/E \end{tikzcd} \qquad \begin{tikzcd} F(i) \arrow[rr, "F(\phi)"] \arrow[dr] \arrow[ddr] & & F(j) \arrow[dl] \arrow[ddl] \\ & \bigoplus_i F(i)/E \arrow[d, dashed, "\exists_1"] & \\ & M & \end{tikzcd}$$

□

**Lemma 2.65.** *Suppose $I$ is a filtered category, for finitely many $i \to j$, there exist a $k$ and $i \to k$, $j \to k$ such that $i \to k = i \to j \to k$*

*Proof.* Use induction

**(a)** For a single morphism $i \xrightarrow{\phi} j$, there exist $i \xrightarrow{\lambda} k$, $j \xrightarrow{\mu} k$, then there exists $k \xrightarrow{\psi} j$ such that $\psi\lambda = \psi\mu\phi$

$$\begin{tikzcd} i \arrow[r, "\phi"] \arrow[rr, bend right, "\lambda"'] & j \arrow[r, "\mu"] & k \arrow[r, "\psi"] & l \end{tikzcd}$$

**(b)** For $\bullet \xrightarrow{\alpha_d} l$, $i \xrightarrow{\phi} j$, by (a), there exist $i \xrightarrow{\lambda} k$, $j \xrightarrow{\mu} k$ such $\lambda = \mu\phi$, then there exist $l \xrightarrow{\beta} m$, $k \xrightarrow{\gamma} m$

$$\begin{tikzcd} \bullet \arrow[r, "\alpha_d"] & l \arrow[r, "\beta"] & m \\ i \arrow[r, "\phi"] \arrow[rr, bend right, "\lambda"'] & j \arrow[r, "\mu"] & k \arrow[u, "\gamma"'] \end{tikzcd}$$

**(c)** For $\bullet \xrightarrow{\alpha_d} j$, $\bullet \xrightarrow{\beta} i$, there exist $j \xrightarrow{\lambda} k$, $i \xrightarrow{\mu} k$, then there exists $k \xrightarrow{\phi} l$ such that $\phi\lambda\alpha_d = \phi\mu\beta$

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha_d"] \arrow[dr, "\beta"'] & j \arrow[r, "\lambda"] & k \arrow[r, "\phi"] & l \\
& i \arrow[ur, "\mu"'] & &
\end{tikzcd}$$

**(d)** For $\bullet \xrightarrow{\alpha_d} j$, $i \xrightarrow{\beta} \bullet$, there exists $j \xrightarrow{\phi} k$ equalizes $\alpha_d\beta$

$$i \xrightarrow{\beta} \bullet \xrightarrow{\alpha_d} j \xrightarrow{\phi} k$$

**(e)** For $\bullet \xrightarrow{\alpha_d} i$, $\bullet \xrightarrow{\beta} \bullet$, there exists $i \xrightarrow{\phi} j$ equalizes $\alpha_d\beta$

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha_d"] \arrow[loop below, "\beta"] & i \arrow[r, "\phi"] & j
\end{tikzcd}$$

□

**Lemma 2.66.** *Suppose $I$ is a filtered category, $F : I \to R\text{-mod}$ is a functor, then*
***(a)*** *Every element of* colim$F$ *lies in the image of some* $F(i) \to \text{colim}F$
***(b)*** $\ker(F(i) \to \text{colim}F) = \bigcup\limits_{i \xrightarrow{\phi} j} \ker F(\phi)$

*Proof.* **(a)** Any element of colim$F$ is a finite sum $\sum a_i$, since $I$ is filtered, there exist $k$ and $i \xrightarrow{\phi_i} k$
**(b)** $a_i \in \ker(F(i) \to \text{colim}F)$ can be written as finite sum $\sum(F(\phi)(a_j) - a_j)$ with $j \xrightarrow{\phi} k$, by Lemma 2.65, there exist $j \xrightarrow{\psi_j} l$ such that for each $j \xrightarrow{\phi} k$ we have $\psi_j = \psi_k\phi$, then $F(\psi_k)F(\phi) = F(\psi_j)$, hence

$$\begin{aligned}
F(\psi_i)(a_i) &= \left(\sum F(\psi_j)\right)(a_i) \\
&= \left(\sum F(\psi_j)\right)\left(\sum(F(\phi)(a_j) - a_j)\right) \\
&= \sum (F(\psi_k)F(\phi)(a_j) - F(\psi_j)a_j) \\
&= 0
\end{aligned}$$

Therefore $a_i \in \ker F(\psi_i)$ □

**Definition 2.67.** A **sheaf** is a presheaf $F$ such that

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Is an equaliser. $Sh(X)$ is the category of sheaves over $X$

**Proposition 2.68.** *If $\mathscr{D}$ is complete, then $\mathscr{D}^{\mathscr{C}^{op}}$ is complete,* $(\lim F_i)(X) = \lim F_i(C)$. *If $\mathscr{D}$ is cocomplete, then $\mathscr{D}^{\mathscr{C}^{op}}$ is cocomplete,* $(\text{colim}F_i)(X) = \text{colim}F_i(X)$

*Proof.* □

**Corollary 2.69.** $PreSh(X, R\text{-mod})$, $Sh(X, R\text{-mod})$ *are abelian categories*

**Theorem 2.70.** *Inclusion $Sh(X, R\text{-mod}) \to PreSh(X, R\text{-mod})$ is the right adjoint to the sheafification $PreSh(X, R\text{-mod}) \to Sh(X, R\text{-mod})$, hence inclusion is left exact, sheafification is actually exact*

*Proof.* □

**Lemma 2.71.** $0 \to F \to G \to H \to 0$ *is exact iff* $0 \to F_x \to G_x \to H_x \to 0$ *is exact*

*Proof.* □

**Example 2.72.** $X = \mathbb{C} \setminus \{0\}$, $\mathbb{Z}$ is the sheaf of locally integer constant functions, $\mathcal{O}$ is the sheaf of holomorphic functions, $\mathcal{O}^\times$ is the sheaf of nonvanishing holomorphic functions, $0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$ is exact in $\mathit{Sh}(X)$, but not in $\mathit{PreSh}(X)$, since $z \in \mathcal{O}^\times$ is not in $\mathrm{im}(\mathcal{O}(X) \xrightarrow{\exp} \mathcal{O}^\times(X))$

## 2.11 Hom cochain complex

**Definition 2.73.** $P$ is a chain complex with differentials $\partial$, $I$ is a cochain complex with codifferentials $d$, we get a double complex $Hom(P, I) = \{Hom(P_p, I^q)\}$ with horizontal and vertical codifferentials $d', d''$ defined as $d''(f) = f\partial$, $d'(f) = (-1)^{p+q+1}df$. The **Hom cochain complex** is the product total complex $Tot^{\Pi}(Hom(P, I))$

## 2.12 Group homology

**Definition 2.74.** $R$ is a commutative ring, $M$ is right $R[G]$ module, $C_*(G)$ is the tuple complex, equivalently, $B_*(G)$ is the bar complex, $\bar{B}_*(G) = B_*(G) \otimes_{R[G]} R$, thus

$$M \otimes_{R[G]} B_*(G) \cong M \otimes_R R \otimes_{R[G]} B_*(G) \cong M \otimes_R \bar{B}_*(G)$$

Group homology with coefficients in $M$ is

$$\begin{aligned} H_k(G; M) &= H_k(M \otimes_{R[G]} C_*(G)) \\ &= H_k(M \otimes_{R[G]} B_*(G)) \\ &= H_k(M \otimes_R \bar{B}_*(G)) \\ &= Tor_k^{R[G]}(M, R) \end{aligned}$$

The differential of $M \otimes_{\mathbb{Z}[G]} C_*(G)$ is given by

$$\begin{aligned} \partial(m \otimes [g_1|\cdots|g_n]) &= \partial(m \otimes (1, g_1, g_1g_2, \cdots, g_1\cdots g_n)) \\ &= mg_1 \otimes [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i m \otimes [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &\quad + (-1)^n m \otimes [g_1|\cdots|g_{n-1}] \end{aligned}$$

$H_0(G; M) = M \otimes_{R[G]} R = M_G$

# 3 Spectral sequence

## 3.1 Spectral sequence

**Definition 3.1.**

**Lemma 3.2.** *$E \xrightarrow{f} E'$ is a morphism of spectral sequences, and $E^r_{pq} \xrightarrow{f^r_{pq}} E'^r_{pq}$ are isomorphisms for any $p, q$, then $E^s_{pq} \xrightarrow{f^s_{pq}} E'^s_{pq}$ are isomorphisms for any $p, q$, $s \geq r$*

*Proof.* By five lemma 2.16 □

**Definition 3.3.** A spectral sequence $C$ is **bounded** if for all $n, r$, all but finitely many $E^r_{p,n-p}$ vanish

**Definition 3.4.** $H_* \in \mathscr{A}^{\mathbb{Z}}$, $\mathbb{Z}$ is the discrete category, $F_pH_*$ is a filtration of $H_*$. $E$ **weakly converge** to $H_*$ if $E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$. If $F_pH_n$ are Hausdorff and exhaustive, then $E$ **approaches** or **abuts** $H_*$. If $F_pH_n$ are complete, then $E$ **converges** to $H_*$
Bounded spectral sequence $E$ converge to $H_*$ if $F_pH_n$ are bounded, and $E^\infty_{pq} = F_pH_{p+q}/F_{p-1}H_{p+q}$, denote $E^r_{pq} \Rightarrow H_{p+q}$

## 3.2 Spectral sequence of a filtered chain complex

**Definition 3.5.** $C$ is a chain complex, $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$ is a filtration of chain complexes. $FC$ is **exhaustive** if $\bigcup F_pC = C$. $FC$ is **Hausdorff** if $\bigcap F_pC = 0$. $\widehat{C} = \varprojlim C/F_pC$ is the **completion**. $FC$ is **complete** if $\widehat{C} \cong C$, since $C \to \widehat{C}$ has kernel $\bigcap F_pC$, hence completeness implies Hausdorff. $FC$ is **bounded below** if $\forall n$, $F_pC_n = 0$ for $p$ small enough. $FC$ is **bounded above** if $\forall n$, $F_pC_n = C_n$ for $p$ big enough. $FC$ is **bounded** if bounded below and above
$F_pH_n(C) = \mathrm{im}(H_n(F_pC) \to H_n(C))$

**Definition 3.6.** $F_nC$ is a filtered chain complex, $E^0_{pq} = \dfrac{F_pC_{p+q}}{F_{p+1}C_{p+q-1}}$ defines a spectral sequence
$E^1_{pq}$ **converges** to $H_*C$ if $E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}C$

**Definition 3.7.** $E^0_{pq} = F_pC_{p+q}$ defines a spectral sequence

**Theorem 3.8.** $A^r_p = \{x \in F_pC | dx \in F_{p-r}C\}$, $Z^r_p = A^r_p + F_{p-1}C$, $B^r_p = dA^{r-1}_{p+r-1} + F_{p-1}C$, $A^r_p \cap F_{p-1}C = A^{r-1}_{p-1}$

$$E^r_p = \frac{Z^r_p}{B^r_p} = \frac{A^r_p + F_{p-1}C}{dA^{r-1}_{p+r-1} + F_{p-1}C} = \frac{\dfrac{A^r_p + F_{p-1}C}{F_{p-1}C}}{\dfrac{dA^{r-1}_{p+r-1} + F_{p-1}C}{F_{p-1}C}} = \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1}}{dA^{r-1}_{p+r-1} \cap F_{p-1}C}}$$
$$= \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1}}{dA^{r-1}_{p+r-1} \cap A^{r-1}_{p-1}}} = \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1} + A^{r-1}_{p-1}}{A^{r-1}_{p-1}}} = \frac{A^r_p}{dA^{r-1}_{p+r-1} + A^{r-1}_{p-1}}$$

**Lemma 3.9.** *$C$ and $\widehat{C}$ give the same spectral sequence*

**Theorem 3.10.** *If $F_*C$ is bounded, then $E^1_{p,q}$ converges to $H_*C$*
*If $F_*C$ is bounded below and exhaustive, then $E^1_{p,q}$ converges to $H_*C$, the convergence is natural*

**Theorem 3.11.** *$C$ is a complete filtration, then*

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1 H_{n+1}(C/F_pC) \arrow[r] \arrow[d, equal] & H_n(C) \arrow[r] & H_n(C/F_pC) \arrow[r] \arrow[d, equal] & 0 \\
& \bigcap F_pH_n(C) & & \varprojlim H_n(C)/F_pH_n(C) \arrow[d, equal] & \\
& & & H_*(C)/\bigcap F_pH_n(C) &
\end{tikzcd}$$

**Lemma 3.12.** *$F_*C$ is Hausdorff and exhaustive, then*

1. $A^\infty_{pq} = \ker(F_pC_{p+q} \xrightarrow{d} F_pC_{p+q-1})$
2. $F_pH_{p+q}(C) \cong A^\infty / \bigcup dA^r_{p+r,q-r+1}$
3. *The subgroup $e^\infty_{pq} = A^\infty_{pq} + B^\infty_{pq}$ is isomorphic to $F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C)$*

## 3.3 Spectral sequence of a double complex

**Definition 3.13.** $C_{pq}$ is a double complex. Filtration by columns ${}'F_n(Tot(C))$ is the total complex of a truncation of $C$ with $C_{pq} = 0$ for $q > n$

$$\begin{array}{cccccc}
\bullet & \bullet & \bullet & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & 0 & 0 & 0
\end{array}$$

Filtration by rows ${}''F_n(Tot(C))$ is the total complex of a truncation of $C$ with $C_{pq} = 0$ for $p > n$

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet
\end{array}$$

We have

$$'E^0_{pq} = C_{pq}, {}'E^1_{pq} = H_q(C_{p*}), {}'E^2_{pq} = H^h_p H^v_q C$$

$$''E^0_{pq} = C_{qp}, {}''E^1_{pq} = H_q(C_{*p}), {}'E^2_{pq} = H^v_q H^h_p C$$

### 3.4 Hyperhomology

**Definition 3.14.** $\mathcal{A}$ is an abelian category with enough projectives, a **Cartan-Eilenberg resolution** $P_{**}$ of chain complex $A_*$ is an upper half plane double complex consist of projectives and a augmentation $P_{*0} \xrightarrow{\varepsilon} A_*$ such that

1. If $A_p = 0$, then $P_{p,*} = 0$
2. $B_p^h(P) \xrightarrow{B_p(\varepsilon)} B_p(A_*)$, $H_p^h(P) \xrightarrow{H_p(\varepsilon)} H_p(A_*)$ are projective resolutions

**Lemma 3.15.** *Every chain complex $A_*$ has a Cartan-Eilenberg resolution, and $Z_p^h(P) \xrightarrow{Z_p(\varepsilon)} Z_p(A_*)$, $P_{p*} \xrightarrow{\varepsilon_p} A_p$ are projective resolutions*

**Lemma 3.16.** *$f : A \to B$ is a chain map, $P \to A$, $Q \to B$ are Cartan-Eilenberg resolutions, there exists a double complex map $\widetilde{f} : P \to Q$ over $f$*

**Definition 3.17.** $f, g : D \to E$ are maps between double complexes, a chain homotopy from $f$ to $g$ consists of $s^h : D_{pq} \to E_{p+1,q}$ and $s^v : D_{pq} \to E_{p,q+1}$ satisfying

$$f - g = (s^h d^h + d^h s^h) = (s^v d^v + d^v s^v)$$

$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0$$

So that $s^h + s^v : Tot(D)_n \to Tot(E)_{n+1}$ is a chain homotopy between $Tot(f), Tot(g) : Tot^{\oplus}(D) \to Tot^{\oplus}(E)$

**Lemma 3.18.** *$f, g : A \to B$ are chain homotopic, $P \to A$, $Q \to B$ are Cartan-Eilenberg resolutions, $\tilde{f}, \tilde{g} : P \to Q$ are over $f, g$, then $\tilde{f}, \tilde{g}$ are chain homotopic*
*Any two Cartan-Eilenberg resolutions of $P \to A$, $Q \to A$ are chain homotopic. $F$ is an additive functor, then $Tot^{\oplus}(F(P))$, $Tot^{\oplus}(F(Q))$ are chain homotopic*

**Definition 3.19.** $\mathcal{A}, \mathcal{B}$ are abelian categories, $\mathcal{A}$ has enough projectives, $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, $f : A \to B$ is a chain map. Define $\mathbb{L}_i F(A) = H_i(Tot^{\oplus}(F(P)))$, by Lemma 3.18, $\mathbb{L}_i F(A)$ is independent of the choice of $P$, $\mathbb{L}_i F(f) = H_i(Tot(F(f)))$. $\mathbb{L}_i F : \mathbf{Ch}\mathcal{A} \to \mathcal{B}$ is the left hyper-derived functor of $F$

**Lemma 3.20.** *$0 \to A \to B \to C \to 0$ is a short exact sequence of bounded below chain complexes, then we have a long exact sequence*

$$\cdots \to \mathbb{L}_{i+1} F(C) \xrightarrow{\delta} \mathbb{L}_i F(A) \to \mathbb{L}_i F(B) \to \mathbb{L}_i(C) \xrightarrow{\delta} \cdots$$

**Proposition 3.21** (Hyperhomology spectral sequence)**.** *$L_p F(H_q(A)) \Rightarrow \mathbb{L}_{p+q} F(A)$. If $A$ is bounded below, then $H_p(L_q F(A)) \Rightarrow \mathbb{L}_{p+q} F(A)$*

*Proof.* Consider the double complex $P$ of a Cartan-Eilenberg resolution $P \to A$. Since $H_p^h(P) \to H_p A$ is a projective resolution, we have

$$L_p F(H_q(A)) = H_p^v F(H_q^h(P)) = H_p^v H_q^h(F(P)) ={}'' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A)$$

If $A$ is bounded belwo, then

$$H_p(L_q F(A)) = H_p^h H_q^v(F(P)) ={}' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A)$$

□

**Corollary 3.22.**

1. *If $A$ is exact, then $\mathbb{L}_i F(A) = 0$*
2. *If $f : A \to B$ is a quasi-isomorphism, then $\mathbb{L}_* F(f) : \mathbb{L}_* F(A) \to \mathbb{L}_* F(B)$ are isomorphisms*

*3. If $A$ is bounded below and $A_p$ are $F$ acyclic, then $\mathbb{L}_p F(A) = H_p F(A)$*

**Theorem 3.23** (Grothendieck spectral sequence)**.** *$\mathcal{A}, \mathcal{B}$ have enough projectives, $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{A} \to \mathcal{B}$ are right exact functors and $G$ sends projectives to $F$-acyclic objects, then*

$$(L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A)$$

*Proof.* Suppose $P \to A$ is a projective resolution, then by Proposition 3.21, we have

$$(L_p F)(L_q G)(A) \cong L_p F(H_q G(P)) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

$$H_p(L_q F(G(P))) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

Since $G(A)$ is $F$-acyclic, ${}'E_2^{pq} = 0$ for $q \neq 0$ and

$$E_2^{p0} = H_p(FG(P)) = L_p(FG)(A) \cong \mathbb{L}_p(FG)(A)$$

□

**Corollary 3.24** (Hochschild-Serre spectral sequence)**.** *$N \trianglelefteq G$ is a normal subgroup, $A$ is a $\mathbb{Z}G$ module, then*

$$H_p(G/N; H_q(N; A)) \Rightarrow H_{p+q}(G; A)$$

*Proof.* Consider right exact functors

$$F = - \otimes_{\mathbb{Z}[G/N]} \mathbb{Z} : \mathbb{Z}[G/N]\text{-mod} \to \mathbb{Z}\text{-mod}$$

$$G = - \otimes_{\mathbb{Z}[N]} \mathbb{Z} = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N] : \mathbb{Z}[G]\text{-mod} \to \mathbb{Z}[G/N]\text{-mod}$$

The left derived functors of $FG = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is $L_*(FG)(A) = Tor_*^{\mathbb{Z}[G]}(A, \mathbb{Z}) = H_*(G; A)$. For any $\mathbb{Z}[G]$ module $A$ and $\mathbb{Z}[G/N]$ module $B$, we have natural isomorphism

$$Hom_{\mathbb{Z}[G/N]}(A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N], B) \cong Hom_{\mathbb{Z}[G]}(A, B) = Hom_{\mathbb{Z}[G]}(A, U(B))$$

Hence $G$ is left adjoint to forgetful functor $U$ which is exact, by Lemma 2.36, $G$ preserves projectives which are exactly $F$-acyclic objects. Apply Theorem 3.23 we have

$$H_p(G/N; H_q(N; A)) = (L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A) = H_{p+q}(G; A)$$

□

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