MATH730 - Fundamental Concepts of Topology

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1 General topology

Example 1.1. $\mathbb{R}/\mathbb{Q}$ is not Hausdorff

Proposition 1.2. If Y is discrete, then X is connected if every continuous function $f : X \to Y$ is a constant

Proposition 1.3. X is compact, Y is Hausdorff, any continuous function $f : X \to Y$ is closed. In particular, if f is bijective, then f is a homeomorphism

Fact 1.4. X is Hausdorff and locally compact iff X is homeomorhphic to an open subset of a compact Hausdorff space Y through one point compactification

$$\mathtt{Hom}(X \times A, Y) \cong \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$$

as a set. However as topological spaces $\mathtt{Hom}(X \times A, Y) \to \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$ is not surjective. Consider $X = \mathtt{Hom}(A, Y)$

Note. Here $\mathtt{Hom}(A, Y)$ is endowed with compact-open topology

Theorem 1.5. A is locally compact and Hausdorff, then $f : X \times A \to Y$ is continuous iff $f : X \to \mathtt{Hom}(A, Y)$ is continuous. Furthermore, if X is also locally compact and Hausdorff, then

$$\mathtt{Hom}(X \times A, Y) \cong \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$$

as topological spaces

Proposition 1.6. If $g : A \to Y$ is injective, then $\iota_X : X \to X \cup_A Y$ is also injective. If $f : A \to Y$ is surjective, then $\iota_Y : X \to Y \cup_A Y$ is also surjective, moreover, if f is a homeomorphism, so is ι_Y

Proof. Proof of homeomorphism: Show that Y satisfies the universal property of the pushout

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, equal] \arrow[ddr, bend left, "\varphi_2"] & \\
X \arrow[r, "gf"] \arrow[drr, bend right, "\varphi_1"'] & Y \arrow[dr, dashed, "\Phi"] & \\
 & & T
\end{tikzcd}$$

$\forall \varphi_1, \varphi_2$ such that $\varphi_1 f = \varphi_2 g$, $\Phi g f^{-1} = \varphi_1$, thus $\Phi = \varphi_2$ ☐

Definition 1.7 (CW complexes)**.** For $x, y \in X$, define $\varphi : S^0 \to X$ with $\varphi(-1) = x$, $\varphi(1) = y$, Write $X \cup_\varphi D^1$ for pushout

$$\begin{tikzcd}
S^0 \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\
D^1 \arrow[r, "\iota"] & X \cup_\varphi D^1
\end{tikzcd}$$

The image $\iota(\mathtt{Int}(D^1))$ is called a 1-cell, denoted e^1
In general, we have

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\
D^n \arrow[r, "\iota"] & X \cup_\varphi D^n
\end{tikzcd}$$

The image $\iota(\mathtt{Int}(D^n))$ is called an n-cell, denoted e^n. Attaching cells does not disturb the interiors of the cells
A CW complex is built up in the following way

1. Starting with a discrete set X_0, the set of 0-cells or the 0-skeleton
2. Given $(n-1)$-skeleton X_{n-1}, then n-skeleton X_n is obtained by attaching n-cells to X_{n-1}, that is

$$\begin{tikzcd} \coprod_{\alpha\in A_n} S^{n-1}_\alpha \arrow[r, "\phi_\alpha"] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ \coprod_{\alpha\in A_n} D^n_\alpha \arrow[r, "\Phi_\alpha"] & X_n \end{tikzcd}$$

3. The space X is the union of X_n's, topologized by weak topology

The third condition ensures $X_n \hookrightarrow X$ is continuous. As a set, X is the disjoint union of $\Phi_\alpha(\mathrm{Int}(D^n))$, $\Phi_\alpha : D^n \to X_n \to X$ are called characteristic maps, CW complexes are determined by characteristic maps

Definition 1.8. G is a discrete group. $Y \times G \to Y$ is a continuous action, then $q : Y \to Y/G$ is continuous, the action is *properly discontinuous* if $\forall y \in Y$, $\exists U$ neighborhood such that $U \cap Ug = \varnothing, \forall g \neq 1$, this implies that the action is free, then q is a covering map, furthermore, $p : Y/H \to Y/G$ is a covering map for $H \leq G$

Fact 1.9. A finite group which acts freely on a Hausdorff space Y is properly discontinuous

Theorem 1.10 (Monodromy)**.** Let $\tilde{\gamma}_x$ be the lift of γ against p starting at x

$$\begin{aligned} \pi_1(B, b_0) \times F &\to F \\ ([\gamma], x) &\mapsto \tilde{\gamma}_x(1) \end{aligned}$$

specifies a transitive left action of $\pi_1(B, b_0)$ on F, called the monodromy action

Proof. Let c_{b_0} be the constant loop, which is the identity element, by transitivity, path-connectedness and orbit-stabilizer theorem, $F \cong G/G_{x_0}$ ☐

Proposition 1.11. The stabilizer of $x \in F$ under the monodromy action isthe subgroup $p_*(\pi_1(E, x)) \leq \pi_1(B, b_0)$

Corollary 1.12. $\pi_1(B, b_0)/p_*(\pi_1(E, x)) \to F$ is an isomorphism

Proposition 1.13. $\varphi : E_1 \to E_2$ induce map on fibers $F_1 \to F_2$ is $\pi_1(B, b_0)$ equivariant, i.e. $[\gamma] \cdot \varphi(x) = \tilde{\gamma}_{\varphi(x)}(1) = \varphi(\tilde{\gamma}_x)(1) = \varphi(\tilde{\gamma}_x(1)) = \varphi([\gamma] \cdot x)$

Proposition 1.14. $H, K \leq G$, every G equivariant map $\varphi : G/H \to G/K$ is of the form $gH \mapsto g\gamma K$ for some $\gamma \in G$ such that $\gamma H \gamma^{-1} \leq K$, in short, H is subconjugate to K

Proof. An equivariant map is determined by the value at one element, suppose $eH \mapsto \gamma K$, for some $\gamma \in G$. then $gH \mapsto g\gamma K$ which is well-defined should have $ghH = gH$, $gh\gamma H = h\gamma H$, so we need $\gamma^{-1} h \gamma \in K \Rightarrow \gamma^{-1} H \gamma \leq K$ ☐

Corollary 1.15. An equivariant map $\varphi : G/H \to G/K$ exists iff H is subconjugate ot K. The two orbits are isomorphic as G-sets iff H is conjugate to K

Theorem 1.16. There is a bijection of sets

$$\mathrm{Hom}_B(E_1, E_2) \cong \mathrm{Hom}_{\pi_1(B, b_0)}(F_1, F_2)$$

Corollary 1.17.

$$\mathrm{Aut}_B(E) \cong \mathrm{Hom}_G(G/H, G/H) \cong N_G(H)/H = W_G(H)$$

here $H = p_*(\pi_1(E))$

Proof. There exists a surjective homomorhpism $N_G(H) \to \mathrm{Hom}_G(G/H, G/H)$, $\gamma \mapsto gh \mapsto g\gamma H$, thus $eH \mapsto \gamma H \Rightarrow \gamma \in H$, thus $\mathrm{Hom}_G(G/H, G/H) \cong N_G(H)/H$ $\square$

Proposition 1.18. *X is the universal cover of B, $\mathrm{Aut}_B(X) \to F$, $\varphi \mapsto \varphi(x)$, $x \in q^{-1}(b)$ is a bijection as sets*

References

[1] *Algebraic Topology* - Allen Hatcher

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