Review of chapters 8 and 9

December 9, 2019

Problems:

(1) Find the slope of the curve $y = x\sin\left(\pi\sqrt{1-5x^2}\right)$ at $x = 0$

$$y' = x'\sin(\pi\sqrt{1-5x^2}) + x\left[\sin(\pi\sqrt{1-5x^2})\right]'$$

$$= \sin(\pi\sqrt{1-5x^2}) + x\cos(\pi\sqrt{1-5x^2})\cdot(\pi\sqrt{1-5x^2})'$$

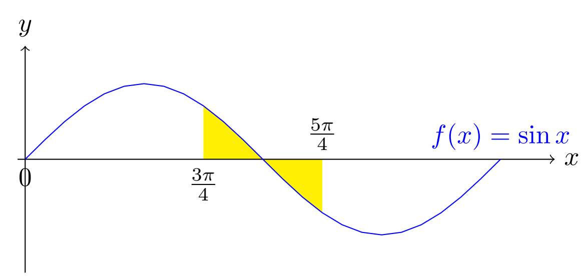
$$= \sin(\pi\sqrt{1-5x^2}) + x\cos(\pi\sqrt{1-5x^2})\,\pi\left((1-5x^2)^{\frac{1}{2}}\right)'$$

$$= \sin(\pi\sqrt{1-5x^2}) + x\cos(\pi\sqrt{1-5x^2})\,\pi\tfrac{1}{2}(1-5x^2)^{-\frac{1}{2}}(1-5x^2)'$$

$$= \sin(\pi\sqrt{1-5x^2}) - x\cos(\pi\sqrt{1-5x^2})\,\pi\tfrac{1}{2}(1-5x^2)^{-\frac{1}{2}}(-10x)$$

$$y'(0) = \sin(\pi\sqrt{1-5\cdot 0^2}) = \sin\pi = 0$$

(2) Find the area of the following



Area = Area above x axis + Area below x axis

$$= \int_{\frac{3\pi}{4}}^{\pi} f(x)dx - \int_{\pi}^{\frac{5\pi}{4}} f(x)dx$$

$$= F(x)\Big|_{\frac{3\pi}{4}}^{\pi} - F(x)\Big|_{\pi}^{\frac{5\pi}{4}}$$

since $f(x) = \sin x$, $F(x)$ could be $-\cos x$

Thus $= \left[(-\cos\pi) - (-\cos\frac{3\pi}{4})\right] - \left[(-\cos\frac{5\pi}{4}) - (-\cos\pi)\right]$

$$= \left[-(-1) + (-\tfrac{\sqrt{2}}{2})\right] - \left[-(-\tfrac{\sqrt{2}}{2}) + (-1)\right] = 2 - \sqrt{2}$$

(3) Let $f(x) = \sin\left(2x + \frac{\pi}{3}\right)$, for what values of $-\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$ does f has a tangent of slope 1

$f'(x) = \cos\left(2x+\frac{\pi}{3}\right)\cdot\left(2x+\frac{\pi}{3}\right)' = 2\cos\left(2x+\frac{\pi}{3}\right) = 1 \Rightarrow \cos\left(2x+\frac{\pi}{3}\right) = \frac{1}{2}$

since $-\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$, $2\cdot\left(-\frac{\pi}{3}\right)+\frac{\pi}{3} \leq 2x+\frac{\pi}{3} \leq 2\cdot\left(\frac{\pi}{2}\right)+\frac{\pi}{3} \Rightarrow -\frac{\pi}{3} \leq 2x+\frac{\pi}{3} \leq \frac{4\pi}{3}$

Hence the only possible solutions are $2x+\frac{\pi}{3} = -\frac{\pi}{3}$ or $2x+\frac{\pi}{3} = \frac{\pi}{3}$

$\Rightarrow x = -\frac{\pi}{3}$ or $x = 0$

(4) What is the third Taylor polynomial of $f(x) = \sin\left(x^2\right)$ centered about $x = 0$

$f'(x) = \cos(x^2)\cdot(x^2)' = 2x\cos(x^2)$, $f''(x) = 2x'\cos(x^2) + 2x\left(\cos(x^2)\right)'$

$= 2\cos(x^2) - 2x\cdot 2x\sin(x^2) = 2\cos(x^2) - 4x^2\sin(x^2)$

$f'''(x) = -2\cdot 2x\sin(x^2) - (4x^2)'\sin(x^2) - 4x^2\cdot 2x\cos(x^2)$

$= -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$, thus $f(0)=0, f'(0)=0, f''(0)=2, f'''(0)=0$

$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = x^2$. Or use Taylor series of $\sin x = x - \frac{x^3}{3!} + \cdots$

(5) Evaluate $\int x^2 \ln\left(x^3\right) \mathrm{d}x$

$\int x^2\ln(x^3)dx \xrightarrow[du=3x^2dx]{u=x^3} \frac{1}{3}\int \ln u\,du = \frac{1}{3}\int \ln u\cdot u'du$

$= \frac{1}{3}\left[u\ln u - \int(\ln u)'u\,du\right] = \frac{1}{3}\left[u\ln u - \int 1\,du\right] = \frac{1}{3}(u\ln u - u) + C$

$= \frac{1}{3}\left(x^3\ln(x^3) - x^3\right) + C$

(6) Evaluate $\int \frac{\cos x}{1+\sin x}\mathrm{d}x$

$\int \frac{\cos x}{1+\sin x}dx \xrightarrow[du=\cos x dx]{u=1+\sin x} \int\frac{1}{u}du = \ln u + C$
$= \ln(1+\sin x) + C$

or $\int\frac{\cos x}{1+\sin x}dx \xrightarrow[dv=\cos x dx]{v=\sin x} \int\frac{1}{1+v}dv \xrightarrow[dw=dv]{w=1+v} \int\frac{1}{w}dw = \ln w + C$

$= \ln(1+v) + C = \ln(1+\sin x) + C$

(7) Evaluate $\int \sin^3 x dx$

$$\int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx = \int \sin x (1 - \cos^2 x) \, dx$$

$$\overset{u = \cos x}{\underset{du = -\sin x \, dx}{=\!=\!=}} -\int (1 - u^2) \, du = -\left(u - \frac{u^3}{3}\right) + C = -u + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C$$