0.1 Affine schemes

Definition 0.1.1. An **affine scheme** is a ringed space $(\mathsf{Spec}\, R, \mathcal{O}_{\mathsf{Spec}\, R})$

Lemma 0.1.2. The inclusion of $\mathsf{Spec}\, k(p) \to \mathsf{Spec}\, A$ is given by $A \to A_p \to k(p)$

0.2 Schemes

Definition 0.2.1. A *scheme* is a ringed space $(X, \mathcal{O})$ such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$

Definition 0.2.2. We say X is a scheme over Y if there is a morphism $X \to Y$, X is a scheme over R if there is a morphism $X \to \operatorname{Spec} R$
An R point is a morphism $\operatorname{Spec} R \to X$, we also write the set of R points as $X(R)$. If S is a commutative R algebra, then the set of S points $X(S)$ consists of morphisms $\operatorname{Spec} S \to X$ over $\operatorname{Spec} R$
$X(S)$ can also be constructed as the base change X_S

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Definition 0.2.3. X is *integral* if $\mathcal{O}(U)$ is integral for any open subset U

Definition 0.2.4. X is *reduced* if $\mathcal{O}(U)$ is reduced for any open subset U

Definition 0.2.5. $f : X \to Y$ is *separated* if $\Delta(X)$ is closed, $\Delta : X \to X \times_Y X$ is the diagonal

Definition 0.2.6. $f : X \to Y$ is of *finite type* if Y has an affine open cover Y_i such that there is an affine open cover X_{ij} of $f^{-1}(Y_i)$ such that $f|_{X_{ij}} : X_{ij} \to Y_i$ are of finite type

Definition 0.2.7. X is a irreducible reduced scheme, $X = \bigcup \operatorname{Spec} A_i$, A_i are integral domains, let B_i be the integral closure of A_i, the *normalization* of X is $Y = \bigcup \operatorname{Spec} B_i$ with the induced finite morphism $Y \to X$

Lemma 0.2.8. Normalizations of dimension 1 schemes are regular, normalizations of dimension 2 schemes only have isolated singularities

Example 0.2.9. $k[x, y]/(x^2 - y^3) \cong k[t^2, t^3]$ with field of fractions $k(t)$ and integral closure $k[t]$, thus the normalization of curve $\operatorname{Spec}\left(\frac{k[x, y]}{(x^2 - y^3)}\right)$ is

$$\begin{aligned} \operatorname{Spec} k[t] &\to \operatorname{Spec}\left(\frac{k[x, y]}{(x^2 - y^3)}\right) \\ t &\mapsto (t^3, t^2) \end{aligned}$$

Definition 0.2.10. $f : X \to S$ is a smooth morphism between schemes if f is locally of finite presentation and flat

Definition 0.2.11. The *direct image functor* of $f : X \to Y$ is $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, $f_*(F)(V) = F(f^{-1}V)$

Definition 0.2.12. The *inverse image functor* of $f : X \to Y$ is $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, the sheafification of

$$U \mapsto \varinjlim_{V \supseteq f(U)} G(V)$$

The pullback sheaf of $y \hookrightarrow Y$ is the stalk $O_{Y,y}$
For $\mathcal{O}_Y$ modules $\mathcal{V}$, we have $f^*\mathcal{V} = f^{-1}\mathcal{V} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$

0.3 coherent sheaf

Definition 0.3.1. A *quasi-coherent sheaf* $\mathcal{F}$ on ringed space X is a sheaf of $\mathcal{O}$ modules that has a local presentation, i.e. for each $x \in X$ there is a neighborhood $U \ni x$ with

$$\mathcal{O}^{\oplus I}\big|_U \to \mathcal{O}^{\oplus J}\big|_U \to \mathcal{F}|_U \to 0$$

exact

Note. Quasi-coherent sheaves are being thought of as ”genearalized vector bundles”

Definition 0.3.2. $\mathcal{F}$ is of *finite type* over X if for each $x \in X$ there is a neighborhood $U \ni x$ such that $\mathcal{O}^n|_U \to \mathcal{F}|_U \to 0$ is exact for some n. Quasi-coherent sheaf $\mathcal{F}$ is a *coherent sheaf* if $\mathcal{F}$ is of finite type over X and for any $\mathcal{O}$-module morphism $\varphi : \mathcal{O}^m|_U \to \mathcal{F}|_U$, $\ker\varphi$ is of finite type over X

Definition 0.3.3. $\mathcal{F}$ is *locally free* if for each $x \in X$ there is a neighborhood $U \ni x$ such that $\mathcal{F}|_U \cong \mathcal{O}^I\big|_U$

Proposition 0.3.4. There is a equivalence of categories between A modules and quasi-coherent sheaves over affine scheme $\operatorname{Spec} A$, sending A module M to the constant sheaf $\underline{M}$, and quasi-coherent sheaf $\mathcal{F}$ to A module of global sections $\mathcal{F}(\operatorname{Spec} A)$

Theorem 0.3.5. Quasi-coherent sheaves over a scheme forms an abelian category