

Definition 0.0.1. *Iterated integral* is defined inductively by

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b f_1(\tau)d\tau \cdots f_{r-1}(\tau) \left(\int_a^\tau f_r(t)dt \right) d\tau$$

If $\alpha : I \rightarrow M$ is a curve, $\alpha^* \omega_i = f_i(t)dt$, then

$$\int_\alpha \omega_1 \cdots \omega_r = \int_0^1 f_1(t)dt \cdots f_r(t)dt$$

is well defined, independent of the parametrization. Set the integral to be 1 if $r = 0$

Proposition 0.0.2.

1. $\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_j \int_\beta \omega_1 \cdots \omega_j \int_\alpha \omega_{j+1} \cdots \omega_r$
2. $\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1$
3. $\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\alpha \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}$, here σ runs over (r, s) shuffles

Lemma 0.0.3. $\omega_i^{(j)}$, $1 \leq i \leq r$, $1 \leq j \leq n$ are closed one forms such that $\sum_j \omega_{i-1}^{(j)} \wedge \omega_i^{(j)} = 0$

for $2 \leq i \leq r$, then $\int_\alpha \sum_j \omega_1^{(j)} \cdots \omega_r^{(j)}$ only depends on the homotopy class of α

Definition 0.0.4. The *Polylogarithms* are

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

Note that

$$\text{Li}_{n+1}(z) = \int_0^z \frac{\text{Li}_n(t)}{t} dt, \quad \text{Li}_1(z) = -\ln(1-z)$$

Hence

$$\text{Li}_n(z) = \int_0^z \left(\frac{dt}{t} \right)^{n-1} \frac{dt}{1-t} = \int_0^1 \left(\frac{dt}{t} \right)^{n-1} \frac{dt}{z^{-1}-t}$$

Dilogarithm $\text{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du$ is the analytic continuation on $\mathbb{C} \setminus \{0, 1\}$, avoiding the the cut $[1, \infty]$

Definition 0.0.5. The *Bloch-Wigner function* is $D_2(z) = \text{Im}(\text{Li}_2(z)) + \arg(1-z) \ln|z|$, $z \in \mathbb{C} \setminus \{0, 1\}$

Definition 0.0.6. The *multiple polylogarithms* are

$$\text{Li}_n(\mathbf{z}) = \sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^n} = \int_0^1 \left(\frac{dt}{t} \right)^{n_1-1} \frac{dt}{a_1-t} \cdots \left(\frac{dt}{t} \right)^{n_d-1} \frac{dt}{a_d-t}$$

Here \mathbf{k} runs over $k_1 > \cdots > k_d \geq 1$, $a_j = a_j(\mathbf{z}) = (z_1 \cdots z_j)^{-1}$, $a_0 = 1$, $a_{n+1} = 0$

Note. For \mathbf{k} runs over $(k_1, \dots, k_d) \in \mathbb{Z}_{\geq 1}^d$

$$\sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^n} = \left(\sum_{k_1} \frac{z_1^{k_1}}{k_1^{n_1}} \right) \cdots \left(\sum_{k_d} \frac{z_d^{k_d}}{k_d^{n_d}} \right) = \text{Li}_{n_1}(z_1) \cdots \text{Li}_{n_d}(z_d)$$

Total differential on \mathcal{L}_n

Lemma 0.0.7. Write $\mathfrak{S}_n = \{0, 1\}^n$, $\mathbf{0} = (0, \dots, 0)$, $\mathbf{1} = (1, \dots, 1)$, $\mathbf{u}_s = (0, \dots, \underset{\substack{\uparrow \\ s\text{-th}}}{1}, \dots, 0)$,

define *multiple logarithm* $\mathcal{L}_n = \text{Li}_1$, $\mathcal{L}_0 = 1$, then

$$\begin{aligned} d_j \mathcal{L}_n(\mathbf{x}) &= \sum_{k_1 > \dots > \widehat{k_j} > \dots > k_n} \frac{x_1^{k_1} \dots \widehat{x_j^{k_j}} \dots x_n^{k_n}}{k_1 \dots \widehat{k_j} \dots k_n} \sum_{k_j = k_{j+1} + 1}^{k_j - 1} x_j^{k_j - 1} dx_j \\ &= \sum_{k_1 > \dots > \widehat{k_j} > \dots > k_n} \frac{x_1^{k_1} \dots \widehat{x_j^{k_j}} \dots x_n^{k_n}}{k_1 \dots \widehat{k_j} \dots k_n} \frac{x_j^{k_j - 1} - x_j^{k_{j+1}}}{x_j - 1} dx_j \end{aligned}$$

Denote $\mathbf{x}_j = (x_1, \dots, x_j x_{j+1}, x_{j+2}, \dots, x_n)$, $\mathbf{x}_n = (x_1, \dots, x_{n-1})$, we have

$$d_j \mathcal{L}_n(\mathbf{x}) = \mathcal{L}_{n-1}(\mathbf{x}_{j-1}) \frac{dx_j}{x_j(x_j - 1)} - \mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_j}{x_j - 1}$$

For $2 \leq j \leq n$, and

$$d_1 \mathcal{L}_n(\mathbf{x}) = -\mathcal{L}_{n-1}(\mathbf{x}_1) \frac{dx_1}{x_1 - 1}$$

Therefore

$$\begin{aligned} d\mathcal{L}_n(\mathbf{x}) &= \sum_{j=1}^n d_j \mathcal{L}_n(\mathbf{x}) \\ &= \sum_{j=1}^{n-1} \left(\mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_j}{1 - x_j} + \mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_{j+1}}{x_{j+1}(x_{j+1} - 1)} \right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{x_n - 1} \\ &= \sum_{j=1}^{n-1} \mathcal{L}_{n-1}(\mathbf{x}_j) \left(-d \ln(1 - x_j) + d \ln \left(\frac{x_{j+1} - 1}{x_{j+1}} \right) \right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{x_n - 1} \\ &= \sum_{j=1}^n \mathcal{L}_{n-1}(\mathbf{x}_j) d \ln \left(\frac{1 - x_{j+1}^{-1}}{1 - x_j} \right) \end{aligned}$$

Here $x_{n+1} = \infty$, $\mathcal{L}_0 = 1$

Suppose $\mathbf{i} \in \mathfrak{S}_n$, $|\mathbf{i}| = k$ and $i_{\tau_1} = \dots = i_{\tau_k} = 1$ for some $1 \leq \tau_1 \leq \dots \leq \tau_k \leq n$, set

$$\mathbf{x}(\mathbf{i}) = \mathbf{y}, \quad y_m = \prod_{j=\tau_{m-1}+1}^{\tau_m} x_j = \frac{a_{\tau_m-1}}{a_{\tau_m}}$$

$$w_j(\mathbf{x}) = d \ln \left(\frac{1 - x_{j+1}^{-1}}{1 - x_j} \right)$$

With $\tau_0 = 0$, $w_0(\mathbf{x}) = 1$. A partial order \preceq on \mathfrak{S}_n is given by $\mathbf{i} \preceq \mathbf{j}$ if $i_k \leq j_k$

Define

$$\begin{aligned} X_n &= \left\{ \prod_{j \leq k} (1 - x_j \dots x_k) = 0 \right\}, \quad S_n = \mathbb{C}^n \setminus X_n \\ X'_n &= \left\{ \prod_i x_i \prod_{j \leq k} (1 - x_j \dots x_k) = 0 \right\}, \quad S'_n = \mathbb{C}^n \setminus X'_n \\ D_n &= \bigcap_j \left\{ \left| x_j - \frac{1}{2} \right| < \frac{1}{2} \right\} \end{aligned}$$

Theorem 0.0.8. Multiple logarithm $\mathcal{L}_n(\mathbf{x})$ is a multi-valued holomorphic function on S_n

$$\mathcal{L}_n(\mathbf{x}) = \sum_{0 \neq \mathbf{j}_1 \prec \dots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2 - \mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1))$$

Here $\mathbf{j} - \mathbf{i} = \begin{cases} s' & j_t = i_t + \delta_{st} \\ 0 & \text{otherwise} \end{cases}$ for $\mathbf{i} \prec \mathbf{j}$, j_s is the s' -th nonzero element in \mathbf{j} , and the integration is taken over $\alpha : I \rightarrow \mathbb{C}^n$

Proof. Use induction and Lemma 0.0.7

$$\begin{aligned} \mathcal{L}_n(\mathbf{x}) &= \int_0^{\mathbf{x}} d\mathcal{L}_n(\mathbf{x}) \\ &= \int_0^{\mathbf{x}} \sum_{k=1}^{n-1} \mathcal{L}_{n-1}(\mathbf{x}_k) d \ln \left(\frac{1 - x_{k+1}^{-1}}{1 - x_k} \right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{1 - x_n} \\ &= \int_0^{\mathbf{x}} \sum_{j=1}^n w_{1-\mathbf{f}_k}(\mathbf{x}) \sum_{0 \neq \mathbf{p}_1 \prec \dots \prec \mathbf{p}_{n-1}} w_{\mathbf{p}_{n-1} - \mathbf{p}_{n-2}}(\mathbf{x}_k(\mathbf{p}_{n-1})) \cdots w_{\mathbf{p}_2 - \mathbf{p}_1}(\mathbf{x}_k(\mathbf{p}_2)) w_1(\mathbf{x}_k(\mathbf{p}_1)) \\ &= \int_0^{\mathbf{x}} \sum_{j=1}^n w_{1-\mathbf{f}_k}(\mathbf{x}) \sum_{0 \neq \mathbf{q}_1 \prec \dots \prec \mathbf{q}_{n-1}} w_{\mathbf{q}_{n-1} - \mathbf{q}_{n-2}}(\mathbf{x}(\mathbf{q}_{n-1})) \cdots w_{\mathbf{q}_2 - \mathbf{q}_1}(\mathbf{x}(\mathbf{q}_2)) w_1(\mathbf{x}(\mathbf{q}_1)) \\ &= \sum_{0 \neq \mathbf{j}_1 \prec \dots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2 - \mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1)) \end{aligned}$$

Here $\mathbf{f}_k = (1, \dots, \underset{\substack{\uparrow \\ k\text{-th}}}{0}, \dots, 1)$, \mathbf{q}_i is \mathbf{p}_i with 0 inserted in as the k -th entry. Note that S_n is given so that $w_i(\mathbf{x}(\mathbf{j}))$ are defined □

Example 0.0.9. When $n = 1$

$$\begin{aligned} \mathcal{L}_1(x_1) &= \int_0^{x_1} w_1(\mathbf{x}(1)) \\ &= \int_0^{x_1} d \ln \left(\frac{1}{1 - x_1} \right) \\ &= \int_0^{x_1} \frac{dx_1}{1 - x_1} \end{aligned}$$

When $n = 2$

$$\begin{aligned} \mathcal{L}_2(\mathbf{x}) &= \int_0^{\mathbf{x}} w_{(1,1)-(1,0)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(1,0)) + w_{(1,1)-(0,1)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(0,1)) \\ &= \int_0^{\mathbf{x}} w_2(\mathbf{x}) w_1(x_1) + w_1(\mathbf{x}) w_1(x_1 x_2) \\ &= \int_0^{\mathbf{x}} d \ln \left(\frac{1}{1 - x_2} \right) d \ln \left(\frac{1}{1 - x_1} \right) + d \ln \left(\frac{1 - x_2^{-1}}{1 - x_1} \right) d \ln \left(\frac{1}{1 - x_1 x_2} \right) \\ &= \int_0^{\mathbf{x}} \frac{dx_2}{1 - x_2} \frac{dx_1}{1 - x_1} + \left(\frac{dx_2}{x_2(x_2 - 1)} + \frac{dx_1}{1 - x_1} \right) \frac{d(x_1 x_2)}{1 - x_1 x_2} \end{aligned}$$

When $n = 3$

$$\begin{aligned}
\mathcal{L}_3(\mathbf{x}) &= \int_0^{\mathbf{x}} w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(1,0,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(1,0,0))+ \\
&\quad w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(0,1,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(0,1,0))+ \\
&\quad w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(1,0,0)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(1,0,0))+ \\
&\quad w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(0,0,1)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(0,0,1))+ \\
&\quad w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,1,0)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,1,0))+ \\
&\quad w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,0,1)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,0,1)) \\
&= \int_0^{\mathbf{x}} w_3(\mathbf{x})w_2(x_1,x_2)w_1(x_1) + w_3(\mathbf{x})w_1(x_1,x_2)w_1(x_1x_2)+ \\
&\quad w_2(\mathbf{x})w_2(x_1,x_2x_3)w_1(x_1) + w_2(\mathbf{x})w_1(x_1,x_2x_3)w_1(x_1x_2x_3)+ \\
&\quad w_1(\mathbf{x})w_2(x_1x_2,x_3)w_1(x_1x_2) + w_1(\mathbf{x})w_1(x_1x_2,x_3)w_1(x_1x_2x_3) \\
&= \int_0^{\mathbf{x}} \frac{dx_3}{1-x_3} \frac{dx_2}{1-x_2} \frac{dx_1}{1-x_1} + \frac{dx_3}{1-x_3} \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1} \right) \frac{d(x_1x_2)}{1-x_1x_2} + \\
&\quad \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2} \right) \frac{d(x_2x_3)}{1-x_2x_3} \frac{dx_1}{1-x_1} + \\
&\quad \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2} \right) \left(\frac{d(x_2x_3)}{x_2x_3(x_2x_3-1)} + \frac{dx_1}{1-x_1} \right) \frac{d(x_1x_2x_3)}{1-x_1x_2x_3} + \\
&\quad \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1} \right) \frac{dx_3}{1-x_3} \frac{d(x_1x_2)}{1-x_1x_2} + \\
&\quad \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1} \right) \left(\frac{dx_3}{x_3(x_3-1)} + \frac{d(x_1x_2)}{1-x_1x_2} \right) \frac{d(x_1x_2x_3)}{1-x_1x_2x_3}
\end{aligned}$$

Definition 0.0.10. $\mathbf{i}, \mathbf{j} \in \mathfrak{S}_n$, $|\mathbf{i}| = k$, $|\mathbf{j}| = l$, the \mathbf{i} -th *retraction* map $\rho_{\mathbf{i}} : \mathfrak{S}_n \rightarrow \mathfrak{S}_k$ is defined by

- If $\mathbf{i} \not\geq \mathbf{j}$, $\rho_{\mathbf{i}}(\mathbf{j}) = 0$
- If $\mathbf{i} \geq \mathbf{j}$, assume τ_1, \dots, τ_k and t_1, \dots, t_l are the nonzero entries in \mathbf{i} and \mathbf{j} , suppose $\tau_{\alpha_r} = t_r$, then $\alpha_1, \dots, \alpha_l$ are the nonzero entries of $\rho_{\mathbf{i}}(\mathbf{j})$

Write $\theta_s = \theta_s(\mathbf{x}) = \frac{dt}{t - a_s}$, the $2^n \times 2^n$ variation matrix $\mathcal{M}_1(\mathbf{x}) = (2\pi i)^l E_{\mathbf{i}, \mathbf{j}}(\mathbf{x})$ associated with $\mathcal{L}_n(\mathbf{x})$ is defined by

$$\begin{aligned}
E_{\mathbf{i}, \mathbf{j}} &= \gamma_{\rho_{\mathbf{i}}(\mathbf{j})}^k(\mathbf{y}) = (-1)^{k-l} \prod_{r=0}^l \int_{a_{\alpha_{r+1}}(\mathbf{y})}^{a_{\alpha_r}(\mathbf{y})} \theta_{\alpha_{r+1}}(\mathbf{y}) \cdots \theta_{\alpha_{r+1}-1}(\mathbf{y}) \\
&= (-1)^{k-l} \prod_{r=0}^l \int_{a_{t_{r+1}}(\mathbf{x})}^{a_{t_r}(\mathbf{x})} \theta_{\tau_{\alpha_{r+1}}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \\
&= (-1)^{k-l} \prod_{r=0}^l \int_{p_r} \theta_{\tau_{\alpha_{r+1}}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x})
\end{aligned}$$

$\tau_{k+1} = t_{l+1} = n + 1$, $\alpha_{l+1} = k + 1$. p_r are independent from \mathbf{i} These thetas are very weird

Proposition 0.0.11.

$$\begin{aligned} E_{i,j} &= \prod_{r=0}^l \mathcal{L}_{\alpha_{r+1}-\alpha_r-1} \left(\frac{a_{t_r}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}, \dots, \frac{a_{\tau_{\alpha_{r+1}-2}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) - a_{t_{r+1}}(\mathbf{x})} \right) \\ &= \mathcal{L}_{k-\alpha_l}(x_{1+t_l} \cdots x_{\tau_{\alpha_l+1}}, \dots, x_{1+\tau_{k-1}} \cdots x_{\tau_k}) \cdot \\ &\quad \prod_{r=0}^{l-1} \mathcal{L}_{\alpha_{r+1}-\alpha_r-1} \left(\frac{1 - x_{1+t_r} \cdots x_{t_{r+1}}}{1 - x_{1+\tau_{\alpha_r+1}} \cdots x_{t_{r+1}}}, \dots, \frac{1 - x_{1+\tau_{\alpha_{r+1}-2}} \cdots x_{t_{r+1}}}{1 - x_{1+\tau_{\alpha_{r+1}-1}} \cdots x_{t_{r+1}}} \right) \end{aligned}$$

Proof.

□

Example 0.0.12.

$$\begin{aligned} E_{1,j}(\mathbf{x}) &= \gamma_j^n(\mathbf{x}) = \prod_{r=0}^l \mathcal{L}_{t_{r+1}-t_r-1} \left(\frac{a_{t_r} - a_{t_{r+1}}}{a_{t_{r+1}} - a_{t_{r+1}}}, \dots, \frac{a_{t_{r+1}-2} - a_{t_{r+1}}}{a_{t_{r+1}-1} - a_{t_{r+1}}} \right) \\ &= \prod_{r=0}^l \mathcal{L}_{t_{r+1}-t_r-1} \left(\frac{1 - x_{1+t_r} \cdots x_{t_{r+1}}}{1 - x_{2+t_r} \cdots x_{t_{r+1}}}, \dots, \frac{1 - x_{t_{r+1}-1} x_{t_{r+1}}}{1 - x_{t_{r+1}}} \right) \\ &= \mathcal{L}_{k-\alpha_l}(x_{1+t_l} \cdots x_{t_l+1}, \dots, x_n) \cdot \\ &\quad \prod_{r=0}^{l-1} \mathcal{L}_{t_{r+1}-t_r-1} \left(\frac{1 - x_{1+t_r} \cdots x_{t_{r+1}}}{1 - x_{2+t_r} \cdots x_{t_{r+1}}}, \dots, \frac{1 - x_{t_{r+1}-1} x_{t_{r+1}}}{1 - x_{t_{r+1}}} \right) \end{aligned}$$

In particular we have

$$E_{1,0}(\mathbf{x}) = \gamma_0^n(\mathbf{x}) = \mathcal{L}_n(\mathbf{x}), \quad E_{1,1}(\mathbf{x}) = \gamma_1^n(\mathbf{x}) = \prod_{r=0}^n \mathcal{L}_0 = 1$$

$E = E(\mathbf{x})$ has columns

$$C_j = \sum_{\mathbf{i} \succeq \mathbf{j}} \gamma_{\rho(\mathbf{j})}^{|\mathbf{i}|}(\mathbf{x}(\mathbf{i})) e_{\mathbf{i}}$$

$e_{\mathbf{i}}$ is the standard unit column vector, using the complete order $<$ on \mathfrak{S}_n : if $|\mathbf{i}| < |\mathbf{j}|$, then $\mathbf{i} < \mathbf{j}$, if $|\mathbf{i}| = |\mathbf{j}|$, then compare the lexicographic order from left to right with $\mathbf{1} < \mathbf{0}$. By definition, the \mathbf{i} -th row of \mathcal{M}_1 is

$$R_i = \sum_{\mathbf{i} \succeq \mathbf{j}} (2\pi i)^{|\mathbf{j}|} \gamma_{\rho(\mathbf{j})}^{|\mathbf{i}|}(\mathbf{x}(\mathbf{i})) e_{\mathbf{j}}^T$$

And the \mathbf{j} -th column of \mathcal{M}_1 is $(2\pi i)^{|\mathbf{j}|} C_j$. Note that $\gamma_{\rho(\mathbf{i})}^{|\mathbf{i}|}(\mathbf{x}(\mathbf{i})) = 1$, and the first entry of R_i is $\mathcal{L}_{|\mathbf{i}|}(\mathbf{x}(\mathbf{i}))$

Example 0.0.13. The variation matrix associated with double logarithm is

$$\begin{aligned} \mathcal{M}_{1,1} &= \begin{bmatrix} \gamma_{\rho(0,0)(0,0)}^0 & (2\pi i) \gamma_{\rho(0,0)(1,0)}^0 & (2\pi i) \gamma_{\rho(0,0)(0,1)}^0 & (2\pi i)^2 \gamma_{\rho(0,0)(1,1)}^0 \\ \gamma_{\rho(1,0)(0,0)(x_1)}^1 & (2\pi i) \gamma_{\rho(1,0)(1,0)(x_1)}^1 & (2\pi i) \gamma_{\rho(1,0)(0,1)(x_1)}^1 & (2\pi i)^2 \gamma_{\rho(1,0)(1,1)(x_1)}^1 \\ \gamma_{\rho(0,1)(0,0)(x_1 x_2)}^1 & (2\pi i) \gamma_{\rho(0,1)(1,0)(x_1 x_2)}^1 & (2\pi i) \gamma_{\rho(0,1)(0,1)(x_1 x_2)}^1 & (2\pi i)^2 \gamma_{\rho(0,1)(1,1)(x_1 x_2)}^1 \\ \gamma_{\rho(1,1)(0,0)(x_1, x_2)}^2 & (2\pi i) \gamma_{\rho(1,1)(1,0)(x_1, x_2)}^2 & (2\pi i) \gamma_{\rho(1,1)(0,1)(x_1, x_2)}^2 & (2\pi i)^2 \gamma_{\rho(1,1)(1,1)(x_1, x_2)}^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & & & \\ \mathcal{L}_1(x_1) & 2\pi i & & \\ \mathcal{L}_1(x_1 x_2) & & 2\pi i & \\ \mathcal{L}_2(x_1, x_2) & (2\pi i) \mathcal{L}_1(x_2) & (2\pi i) \mathcal{L}_1\left(\frac{1-x_1 x_2}{x_2}\right) & (2\pi i)^2 \end{bmatrix} \end{aligned}$$

Variation matrix of multiple logarithm is lower triangular

Lemma 0.0.14. The variation matrix is lower triangular. The principal submatrix of \mathcal{M}_1 with $|\mathbf{i}| = |\mathbf{j}| = k$ is $(2\pi i)^k$ times the $\binom{n}{k} \times \binom{n}{k}$ identity matrix

Proof. By definition □

Proposition 0.0.15. $\mathcal{M}_1(\mathbf{x})$ is the fundamental matrix of linear differential equations

$$\begin{cases} dX_{\mathbf{i}} &= \sum_{|\mathbf{k}|=|\mathbf{i}|-1, \mathbf{k} \prec \mathbf{i}} X_{\mathbf{k}} d\rho_{\mathbf{i}}^{|\mathbf{i}|}(\mathbf{k})(\mathbf{x}(\mathbf{i})) \\ dX_0 &= \mathbf{0} \end{cases}$$

Theorem 0.0.16.

Corollary 0.0.17. The monodromy representation $\rho_{\mathbf{x}} : \pi_1(S_n, \mathbf{x}) \rightarrow \mathrm{GL}_{2^n}(\mathbb{Z})$ is unipotent