0.1 Differential geometry of surfaces

Definition 0.1.1. A **differentiable surface** is an embedding $S \hookrightarrow \mathbb{R}^3$

Lemma 0.1.2. $\gamma(t)$ is a geodesic iff $\ddot{\gamma}$ is parallel to the normal $\vec{n}$, meaning no acceleration in S
A geodesic γ on S has constant speed
The geodesic curvature of a curve γ is the curvature of the projection onto tangent plane, γ is a geodesic iff the geodesic curvature of γ is zero

Proof. $\frac{d}{dt}|\dot{\gamma}|^2 = 2\ddot{\gamma} \cdot \dot{\gamma} = 0$ $\square$

0.2 Curvature

Definition 0.2.1. A **Riemannian manifold** is (M, g) where M is a smooth manifold and **Riemannian metric** $g_p : S^2(T_pM) \to \mathbb{R}$ is a positive definite

Definition 0.2.2. The *volume form* is $\sqrt{|\det g|}dx_i \wedge dx_j$, which happen to be $\star 1$

Definition 0.2.3. *Hodge star* is defined to be $\eta \wedge \star\xi = \langle \eta, \xi \rangle \omega$, ω is the volume form. Consider $(\alpha, \beta) = \int_X \alpha \wedge \star\beta$, $d^* = (-1)^{kl+1} \star d \star$ is the *codifferential* that $(d\alpha, \beta) = (\alpha, d^*\beta)$, $\Delta = dd^* + d^*d$ is the Laplacian

Definition 0.2.4. An **affine connection** is

$$\begin{aligned} \nabla : \Gamma(TM) \otimes \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

satisfying

- $\nabla_{fX} Y = f\nabla_X Y$, i.e. ∇ is $C^\infty(M, \mathbb{R})$ linear in the first variable
- $\nabla_X (fY) = XfY + f\nabla_X Y$, i.e. ∇ satisfies Leibniz rule in the second variable

From this we can define covariant derivative ∇, $\nabla_X f = X_f$, $\nabla_X(\alpha)(Y) = \nabla_X(\alpha(Y)) - \alpha(\nabla_X Y)$, here α is a covector, similarly for any tensor, Write contraction $(\nabla T)(\alpha_1, \cdots, \alpha_m, X_1, \cdots, X_n, X) = (\nabla_X T)(\alpha_1, \cdots, \alpha_m, X_1, \cdots, X_n)$, T is a tensor

Note. $\nabla_X(\alpha(Y)) = \nabla_X(\alpha)(Y) + \alpha(\nabla_X Y)$

Definition 0.2.5. ∇ is an affine connection, the **torsion tensor** is

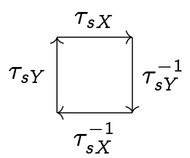
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Definition 0.2.6. The Levi-Civita connection ∇ is the one satisfying

- $\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, i.e. $\nabla g = 0$
- $\nabla_X Y - \nabla_Y X = [X, Y]$, i.e. ∇ is torsion free

Definition 0.2.7. ∇ is the Levi-Civita connection, the Riemannian curvature tensor is $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$

Remark 0.2.8. X, Y are commuting vector fields around x_0, then $\frac{d}{ds}\frac{d}{dt}\tau_{sX}^{-1}\tau_{tY}^{-1}\tau_{sX}\tau_{tY} Z = R_{XY} Z$, τ is the parallel transport



Proposition 0.2.9.

1. $R_{YX} = -R_{XY}$

2. $(R_{XY}Z, W) = -(R_{XY}W, Z)$

3. $R_{XY}Z + R_{YZ}X + R_{ZX}Y = 0$

4.

The **second Bianchi identity** follows

$$\nabla_X R_{YZ} + \nabla_Y R_{ZX} + \nabla_Z R_{XY} = 0$$

Remark 0.2.10. If write $(R_{XY}Z, W) = R(X, Y, Z, W)$, then R is antisymmetric about the first two variables and the last two variables, R satisfies Jacobi identity, the first two and the last two variables can switch place

Proof.

1.

2.

3.

4. Follow from above

□

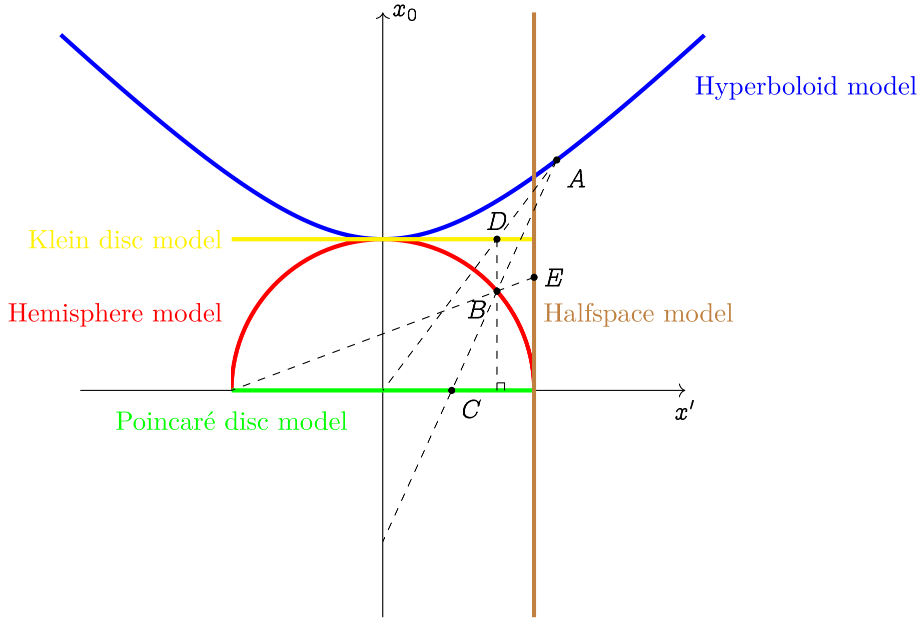
Definition 0.2.11. $\{e_i\}$ is an orthonomal basis, the **Ricci curvature** is $\mathrm{Ric}(X) = \sum R_{X,e_i} e_i$. The **scalar curvature** is $S = \mathrm{Tr}\,\mathrm{Ric} = \sum(\mathrm{Ric}(e_j), e_j) = \sum(R_{e_j,e_i} e_i, e_j)$. The **Einstein curvature** is $G = R - \frac{1}{2} g S$

0.3 Hyperbolic geometry

Definition 0.3.1. $\mathbb{R}^{n+1}$ with metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$ is the **Minkowski space**

The **hyperboloid model** is $\mathbb{H} = \{x_1^2 + \cdots + x_n^2 - x_0^2 = -1, x_0 > 0\}$. The Riemannian metric is the pullback metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$

The geodesics are intersections of $\mathbb{H}$ and two dimensional subspaces of $\mathbb{R}^{n+1}$ $(d\sinh s)^2 + (d\cosh s)^2 = \cosh^2 s ds^2 - \sinh^2 s ds^2 = ds^2$, thus $\mathbb{H}^1$ is isomorphic to $\mathbb{E}^1$



$(x', x_n) \mapsto \left(\dfrac{2x'}{1+x_n}, 1\right)$, $x' = (x_0, \cdots, x_{n-1})$ is the isometry from the hemisphere to the halfspace

$(x', 1) \mapsto \left(\dfrac{4x'}{4+|x'|^2}, \dfrac{4-|x'|^2}{4+|x'|^2}\right)$, $x' = (x_0, \cdots, x_{n-1})$ is the isometry from the halfspace to the hemisphere

$x \mapsto \left(\dfrac{x'}{1+x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to Poincaré disc

$x \mapsto \left(\dfrac{2x}{1-|x|^2}, \dfrac{1+|x|^2}{1-|x|^2}\right)$ is the isometry from Poincaré disc to the hyperboloid

$x \mapsto (1, x')$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to Klein disc

$x \mapsto \left(\dfrac{x'}{x_0}, \dfrac{1}{x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hyperboloid to the hemisphere

$x \mapsto \left(\dfrac{x'}{x_0}, \dfrac{1}{x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to the hyperboloid

The **hemisphere model** is $\mathbb{H} = \{x_0 > 0\} \cap S^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
\sum_{i=0}^{n-1}\left[d\left(\frac{x_i}{x_n}\right)\right]^2 - \left[d\left(\frac{1}{x_n}\right)\right]^2 &= \sum_{i=0}^{n-1}\left(\frac{x_0dx_i - x_idx_0}{x_0^2}\right)^2 - \left(-\frac{dx_0}{x_0^2}\right)^2 \\
&= \sum_{i=0}^{n-1}\frac{x_0^2dx_i^2 - 2x_ix_0dx_idx_0 + x_i^2dx_0^2}{x_0^4} - \frac{dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(|x'|^2)d(x_0^2)}{2x_0^4} + \frac{|x'|^2dx_0^2 - dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(1-x_0^2)d(x_0^2)}{2x_0^4} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2}{x_0^2} + \frac{2dx_0^2}{x_0^2} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2 + dx_0^2}{x_0^2}
\end{aligned}$$

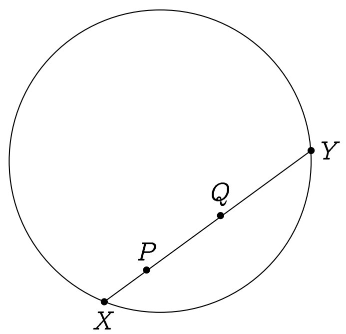
The **half space model** is $\mathbb{H} = \{x_0 > 0\} \cap \{x_n = 1\}$. The Riemannian metric is pullback metric

$$\begin{aligned}
\frac{\sum_{i=0}^{n-1} d\left(\frac{4x_i}{4+|x'|^2}\right)^2 + d\left(\frac{4-|x'|^2}{4+|x'|^2}\right)^2}{\left(\frac{4x_0}{4+|x'|^2}\right)^2} &\underline{\underline{X=4+|x'|^2}} \frac{\sum_{i=0}^{n-1} d\left(\frac{4x_i}{X}\right)^2 + d\left(\frac{8}{X}-1\right)^2}{\left(\frac{4x_0}{X}\right)^2} \\
&= \frac{X^2}{x_0^2}\left(\sum_{i=0}^{n-1} d\left(\frac{x_i}{X}\right)^2 + 4d\left(\frac{1}{X}\right)^2\right) \\
&= \frac{X^2}{x_0^2}\left(\sum_{i=0}^{n-1}\left(\frac{X\,dx_i - x_i\,dX}{X^2}\right)^2 + 4\frac{dX^2}{X^4}\right) \\
&= \frac{1}{x_0^2}\left(\sum_{i=0}^{n-1}\frac{X^2dx_i^2 + x_i^2dX^2 - 2Xx_idXdx_i}{X^2} + 4\frac{dX^2}{X^2}\right) \\
&= \frac{1}{x_0^2}\left(dx'^2 + \frac{|x'|^2dX^2}{X^2} + \frac{4dX^2}{X^2} - \frac{dX\,d(|x'|^2)}{X}\right) \\
&= \frac{1}{x_0^2}\left(dx'^2 + \frac{X\,dX^2}{X^2} - \frac{dX\,d(X-4)}{X}\right) \\
&= \frac{dx'^2}{x_0^2}
\end{aligned}$$

The **Poincaré disc model** is $\mathbb{H} = D^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
\sum_{i=1}^{n} d\left(\frac{2x_i}{1-|x|^2}\right)^2 - d\left(\frac{1+|x|^2}{1-|x|^2}\right)^2 &\xlongequal{X=1-|x'|^2} \sum_{i=1}^{n} d\left(\frac{2x_i}{X}\right)^2 - d\left(\frac{2}{X}-1\right)^2 \\
&= 4\sum_{i=1}^{n}\left(\frac{X dx_i + x_i dX}{X^2}\right)^2 - 4\left(-\frac{dX}{X^2}\right)^2 \\
&= 4\sum_{i=1}^{n}\frac{X^2 dx_i^2 + x_i^2 dX^2 - X x_i dX dx_i}{X^4} - 4\frac{dX^2}{X^4} \\
&= 4\left(\frac{dx^2}{X^2} + \frac{|x|^2 dX^2}{X^4} - \frac{dX^2}{X^4} - \frac{d(|x|^2)dX}{X^3}\right) \\
&= 4\left(\frac{dx^2}{X^2} - \frac{X dX^2}{X^4} - \frac{d(1-X)dX}{X^3}\right) \\
&= \frac{4dx^2}{X^2} = \frac{4dx^2}{(1-|x|^2)^2}
\end{aligned}$$

The **Klein disc model** is $\mathbb{H} = D^n$. The distance between P, Q is $\frac{1}{2}\ln\left(\frac{|XQ||PY|}{|XY||PQ|}\right) = \frac{1}{2}\ln(X, P; Q, Y)$, $(X, P; Q, Y)$ is the cross ratio



Theorem 0.3.2. $\mathrm{Isom}(\mathbb{H}^2) = PSL(2, \mathbb{R})$

Proof. An isometry sends half circles and orthogonal lines to half circles or orthogonal lines, by Schwarz reflection principle **??**, it is can be regard as an isometry on $\mathbb{C}P^1$ sending $\mathbb{R}P^1$ to $\mathbb{R}P^1$, then it necessarily has to be in $PSL(2, \mathbb{R})$ □

Theorem 0.3.3. $\mathrm{Isom}(\mathbb{H}^3) = PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$

Proof. Since $\partial\mathbb{H}^3$ is the Riemann sphere, every isometry on $\mathbb{H}^3$ restricts to a conformal map on $\partial\mathbb{H}^3$ because it sends hemispheres and orthogonal planes to hemispheres or orthogonal planes, hence it is a Möbius transformation. On the other hand, Möbius transformations which can all be extended to an isometry on $\mathbb{H}^3$, translations $z \mapsto z + \lambda$ can be extended to $(z, x_3) \mapsto (z + \lambda, x_3)$, dilations $z \mapsto \lambda z$ can be extended to $(z, x_3) \mapsto (\lambda z, |\lambda| x_3)$, inversions $z \mapsto -\frac{1}{z}$ can be extended to $(z, x_3) \mapsto \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2}\right)$. Therefore the isometry group for $\mathbb{H}^3$ is $PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$ □

0.4 Complex manifold

Identity principle

Theorem 0.4.1 (Identity principle)**.** X is connected, $X \xrightarrow{f} Y$ is holomorphic and $f \equiv c$ on some nonempty open subset of X, then $f \equiv c$ on X

Definition 0.4.2. M is a smooth manifold, an *almost complex structure* is $J : TM \to TM$ such that $J^2 = -1_{TM}$

Example 0.4.3. S^4 cannot be given an almost complex structure. S^6 can be given an almost complex structure but not a complex structure

A complex manifold always give an almost complex structure by $J\dfrac{\partial}{\partial z_i} = i\dfrac{\partial}{\partial z_i}$, $J\dfrac{\partial}{\partial \bar{z}_i} = -i\dfrac{\partial}{\partial \bar{z}_i}$

Definition 0.4.4. A is a $(1,1)$ form, the Nijenhuis tensor is

$$N_A(X,Y) = -A^2[X,Y] + A([AX,Y] + [X,AY]) - [AX,AY]$$

Theorem 0.4.5 (Newslander-Nirenberg theorem)**.** J is *integrable* iff $N_J = 0$. Meaning there is a unique complex structure which will give J

Proposition 0.4.6. Given an almost complex structure, we can find coordinate charts $(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$ such that $\mathtt{Span}\left\{\dfrac{\partial}{\partial z_i}\right\}$, $\mathtt{Span}\left\{\dfrac{\partial}{\partial \bar{z}_i}\right\}$ to be the i and $-i$ eigenspaces of J

Definition 0.4.7. A *Hermitian manifold* M is a complex manifold with a *Hermitian metric* $h = \sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$ on $TM \otimes \mathbb{C}$, where $h_{\alpha\bar{\beta}}$ is a positive definite Hermitian matrix. The real part gives a Riemannian metric

$$\begin{aligned} g &= \frac{1}{2}(h + \bar{h}) \\ &= \frac{1}{2}\left(\sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta + \sum h_{\beta\bar{\alpha}} d\bar{z}_\alpha \otimes dz_\beta\right) \\ &= \frac{1}{2}\sum h_{\alpha\bar{\beta}}(dz_\alpha \otimes d\bar{z}_\beta + d\bar{z}_\beta \otimes dz_\alpha) \\ &= \sum h_{\alpha\bar{\beta}} dz_\alpha d\bar{z}_\beta \end{aligned}$$

Also gives *associate* $(1,1)$ *form*

$$\omega = -\frac{h - \bar{h}}{2i} = \frac{i}{2}(h - \bar{h}) = \frac{i}{2}\sum h_{\alpha\bar{\beta}}(dz_\alpha \otimes d\bar{z}_\beta - d\bar{z}_\beta \otimes dz_\alpha) = \frac{i}{2}\sum h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

Note that the volume form is $\mathtt{vol}_M = \dfrac{\omega^{\wedge n}}{n!}$

Remark 0.4.8. $\omega(u,v) = g(Ju,v)$, $h = g - i\omega$, $g(u,v) = \omega(u,Jv)$. Any one determines the other two. ω corresponds to *Kähler class* in $H^2(M,\mathbb{R})$

Definition 0.4.9. M is a *Kähler manifold* if it satisfies Kähler compatibility condition is $d\omega = 0$, ω is then called a *Kähler form.* We have $\partial_\gamma h_{\alpha\bar{\beta}} = \partial_\alpha h_{\gamma\bar{\beta}}$, $\partial_{\bar{\gamma}} h_{\alpha\bar{\beta}} = \partial_{\bar{\beta}} h_{\alpha\bar{\gamma}}$, this implies at least locally $h_{\alpha\bar{\beta}} = \partial_\alpha f_{\bar{\beta}}$, and then $\partial_\alpha \partial_{\bar{\gamma}} f_{\bar{\beta}} = \partial_{\bar{\gamma}} \partial_\alpha f_{\bar{\beta}} = \partial_{\bar{\beta}} \partial_\alpha f_{\bar{\gamma}} = \partial_\alpha \partial_{\bar{\beta}} f_{\bar{\gamma}}$, hence $h_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \rho$, ρ is called the local *Kähler potential*, ρ is a Kähler potential if $\omega = \dfrac{i}{2}\partial\bar{\partial}\rho$

Definition 0.4.10. Consider $L = \omega \wedge - : H^k(M) \to H^{k+2}(M)$, the *primitive cohomology* is

$$P^{n-k}(M) = \ker\left(H^{n-k}(M) \xrightarrow{L^{k+1}} H^{n+k+2}(M)\right)$$

The *hard Lefschetz theorem* says

$$H^n(M) = \bigoplus L^k P^{n-2k}(M)$$

Theorem 0.4.11 (Serre duality)**.** X is complex manifold of complex dimension n, $E \to X$ is a holomorphic vector bundle, then we have

$$H^i(X, E) \cong H^{n-i}(X, K \otimes E^*)^*$$

Where $K := \bigwedge^n T^*X$ is the canonical bundle
For example, if X is a Riemann surface, $E = \mathcal{O}$, then $H^1(X, \mathcal{O}) \cong H^0(X, K \otimes \mathcal{O})^* \cong H^0(X, \Omega)^* = \Omega(X)^*$

0.5 Symplectic manifold

Definition 0.5.1. M is a smooth manifold, a *symplectic structure* on M is a 2 form ω that is nondegenerate and anti-symmetric on $T_p M$