1. In a certain cell population, cells divide every 10 days, and the age of a cell selected at random is a random variable $X$ with density function $f(x)=2ke^{-kx}$, $0 \le x \le 10$, Find the probability that a cell is at most 5 days old.

Since $f(x)$ is a pdf, $1=\int_0^{10} f(x)dx = \int_0^{10} 2ke^{-kx}dx = 2\int_0^{10} ke^{-kx}dx \overset{(e^{-kx})' = -ke^{-kx}}{=\!=\!=\!=} -2e^{-kx}\Big|_0^{10} = -2(e^{-10k}-1) \Rightarrow e^{-10k}-1=-\frac{1}{2} \Rightarrow e^{-10k}=\frac{1}{2}=2^{-1} \Rightarrow -10k=\ln(2^{-1})=-\ln 2$

$\Rightarrow k=\frac{\ln 2}{10}$

Hence $Pr(X\le 5)=\int_0^5 2ke^{-kx}dx = 2\int_0^5 ke^{-kx}dx \overset{(e^{-kx})' = -ke^{-kx}}{=\!=\!=\!=} -2e^{-kx}\Big|_0^5$

$=-2(e^{-5k}-1)=-2(e^{-5\cdot\frac{\ln 2}{10}}-1)=-2(e^{-\frac{\ln 2}{2}}-1)=2-2e^{-\frac{\ln 2}{2}}$

2. $f(x)=\frac{4}{9}x-\frac{1}{9}x^2$, $1\le X\le 4$ is the pdf of a random variable $X$

(a) What's the cdf $F(x)$

(b) $Pr(0\le X\le 3)$

(a) $F(x)=\int_1^x f(t)dt=\int_1^x\left(\frac{4}{9}t-\frac{1}{9}t^2\right)dt = \frac{4}{9}\int_1^x t\,dt-\frac{1}{9}\int_1^x t^2dt=\frac{4}{9}\frac{t^2}{2}\Big|_1^x-\frac{1}{9}\frac{t^3}{3}\Big|_1^x$

$=\frac{4}{9}\left(\frac{x^2}{2}-\frac{1}{2}\right)-\frac{1}{9}\left(\frac{x^3}{3}-\frac{1}{3}\right)=\left(\frac{2}{9}x^2-\frac{2}{9}\right)-\left(\frac{x^3}{27}-\frac{1}{27}\right)=\frac{2}{9}x^2-\frac{x^3}{27}-\frac{5}{27}$

(b) $Pr(0\le X\le 3)=Pr(1\le X\le 3)=F(3)=\frac{2}{9}\cdot 3^2-\frac{3^3}{27}-\frac{5}{27}=2-1-\frac{5}{27}=\frac{22}{27}$

3. Suppose $X\sim N(-3,4)$ meaning $X$ is a normal variable with expectance $-3$ and variance 4

(a) $Pr(X\ge 1)$

(b) $Pr(X\ge -7)$

We can say $X\sim N(\mu,\sigma^2)$, where $\mu=-3$, $\sigma=\sqrt{4}=2$, and we know $Z=\frac{X-\mu}{\sigma}$ $=\frac{X+3}{2}\sim N(0,1^2)$ is a standard normal distribution

(a) $Pr(X\ge 1)=Pr\left(\frac{X+3}{2}\ge\frac{1+3}{2}\right)=Pr(Z\ge 2)=Pr(0\le Z)-Pr(0\le Z\le 2)$

$=\frac{1}{2}-A(2)=0.5-0.4772=0.0228$

(b) $Pr(X\ge -7)=Pr\left(\frac{X+3}{2}\ge\frac{-7+3}{2}\right)=Pr(Z\ge -2)=1-Pr(Z\le -2)=1-Pr(Z\ge 2)$

$=1-0.0228=0.9772$ or $Pr(Z\ge -2)=Pr(-2\le Z\le 0)+Pr(Z\ge 0)$

$=Pr(0\le Z\le 2)+\frac{1}{2}=A(2)+\frac{1}{2}=0.4772+0.5=0.9772$

4. A person is waiting for taxi, there are two types of taxi, red and blue ones, suppose the number of red taxi is five times as much the number of blue taxi, how many red taxi is he expected to see before the first blue taxi

Let $X$ be the number of red taxi before the first blue taxi shows up, and we know the probability of red taxi shows up is $\frac{5}{6}$, thus $X \sim G(\frac{5}{6})$, $X$ satisfies a geometric distribution with $p = \frac{5}{6}$, therefore the expected value is $E(X) = \frac{p}{1-p} = \frac{\frac{5}{6}}{1-\frac{5}{6}} = 5$