0.1 Vector spaces

Definition 0.1.1. A **vector space** V over field F is an F module

Definition 0.1.2. An **affine space** is a vector space witho

Definition 0.1.3. $C \subseteq V$ is **convex** if $tC + (1-t)C \subseteq C$ for $0 \leq t \leq 1$. C is **strictly convex** if $tC + (1-t)C \subsetneqq C$ for $0 < t < 1$

Definition 0.1.4. V is a vector space of dimension n, a q **flag** is

$$0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_q = V$$

A complete flag is an n flag

GL(n,F) acts transitively on flags

Lemma 0.1.5. $GL(n, F)$ acts transitively on flags

0.2 Matrices

Definition 0.2.1. $I, J \subseteq \{1, \cdots, n\}$, the *submatrix* A_{IJ} of A is the matrix with entries $\{a_{ij} | i \in I, j \in J\}$. The *principal submatrix* are matrices A_{II}

Definition 0.2.2. E_{ij} is the matrix with 1 on the (i,j)-th entry and otherwise zeros, then $E_{ij}E_{kl} = \delta_{jk}E_{il}$
Elementary matrices are single row operations, i.e.

$$e_{ij}(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & r & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with r on the (i,j)-th entry

$$s_{ij} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

and

$$d_i(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & r & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

We have $e_{ij}(-r) = e_{ij}(r)^{-1}$ and

$$\begin{aligned} e_{ij}(r)e_{ij}(s) &= e_{ij}(r+s) \\ [e_{ij}(r), e_{kl}(s)] &= I + rs\delta_{jk}E_{il} - sr\delta_{li}E_{kj} + \delta_{jk}\delta_{li}(srsE_{kl} - rsrE_{ij}) + rsrs\delta_{jk}\delta_{li}E_{il} \\ &= \begin{cases} I & i \neq l, j \neq k \\ e_{il}(rs) & i = l, j \neq k \\ e_{kj}(-sr) & i \neq l, j = k \\ * & i = l, j = k \end{cases} \end{aligned}$$

Steinberg relations

Definition 0.2.3. $E(n,R) \subseteq SL(n,G)$ is the subgroup generated by elementary matrices of determinant 1. $E(R) = \bigcup E(n,R)$

Lemma 0.2.4. $SL(n,F) = E(n,F)$

E(n,R) is perfect

Lemma 0.2.5. $[E(n,R), E(n,R)] = E(n,R)$ if $n \geq 3$

Proof. For distinct i,j,k, $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$ □

Whitehead's lemma

Theorem 0.2.6 (Whitehead's lemma)**.** $[GL(R), GL(R)] = E(R)$, hence $K_1(R) = GL(R)/E(R)$

Proof. Since

$$e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

We know

$$[g,h] = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} h & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & \\ & hg \end{pmatrix} \in E(R)$$

□

Definition 0.2.7. The *Kronecker product* of matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \, B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ a_{np} & \cdots & b_{np} \end{pmatrix}$$

is

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Definition 0.2.8. $\mathtt{tr}(A^*B)$ defines the *Frobenius inner product* over $M(n,\mathbb{C})$

0.3 Eigenspace decomposition

Proposition 0.3.1. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is a linear operator, and $V = \bigoplus_i V_i$, where V_i are T invariant spaces, denote $T|_{V_i}$ as $T-i$, then $ch_T(t) = \prod_i ch_{T_i}(t)$, and $m_T(t) = \underset{i}{lcm}\ m_{T_i}(t)$

Definition 0.3.2. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$. $\lambda \in F$ is an **eigenvalue** if $Tv = \lambda v$ has nontrivial solution, $v \in V$ is a **generalized eigenvector** of rank m of T corresponding to eigenvalue λ if $(T-\lambda 1_V)^m v = 0, (T-\lambda 1_V)^{m-1} v \neq 0$ for some $m \geq 1$, and let V_λ be the subspace of all such generalized eigenvectors, called **generalized eigenspace**, notice if V is of finite dimensional, then $V_\lambda = \ker(T-\lambda 1_V)^m$ for some m with m being smallest, suppose $\dim V_\lambda = d$, then the characteristic polynomial of $T|_{V_\lambda}$ is $(t-\lambda)^d$, and the minimal polynomial of $T|_{V_\lambda}$ is $(t-\lambda)^m$

Generalized eigenspace decomposition

Proposition 0.3.3. $\overline{F} = F$, finitely dimensional F vector space V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_\lambda V_\lambda$

Definition 0.3.4. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ give V an $F[x]$ module with $x \cdot v = Tv$, $W \leq V$ be a subspace, W is called T **invariant** if $TW \subseteq W$, or rather W is an $F[x]$ submodule

Definition 0.3.5. An linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is called **semisimple** if V is a semisimple $\mathbb{F}[x]$ submodule

Proposition 0.3.6. Let $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ be a linear operator with $\overline{\mathbb{F}} = \mathbb{F}$, then T is semisimple $\Leftrightarrow$ T is diagonalizable

Proof. Since $\overline{\mathbb{F}} = \mathbb{F}$ and T is semisimple, V can be decomposed as a direct sum of eigenspaces of T, thus T is diagonalizable, conversely, if T is diagonalizable, and $TW \subseteq W$, let V_λ be the eigenspaces of T, denote $W_\lambda = W \cap V_\lambda$, and $W' = \bigoplus_\lambda W'_\lambda$, since $T|_{V_\lambda} = \lambda 1_{V_\lambda}$, we can find $W'_\lambda \leq V_\lambda$ such that $V_\lambda = W_\lambda \oplus W'_\lambda$, and of course $TW'_\lambda \subseteq W'_\lambda$ which implies $TW' \subseteq W'$, then we have $V = \bigoplus_\lambda V_\lambda = \bigoplus_\lambda W_\lambda \oplus W'_\lambda = \bigoplus_\lambda W_\lambda \bigoplus \bigoplus_\lambda W'_\lambda = W \oplus W'$ □

Definition 0.3.7. An linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is called nilpotent if $T^k = 0$ for some k, T is called unipotent if $T - 1_V$ is nilpotent

Jordan-Chevalley decomposition

Definition 0.3.8 (Jordan-Chevalley decomposition)**. Jordan-Chevalley decomposition** of a linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is $T = T_s + T_n$, where T_s is semisimple, T_n is nilpotent and $[T_s, T_n] = 0$

Existence of Jordan-Chevalley decomposition

Theorem 0.3.9. If V is a finite dimensional $\mathbb{F}$ vector space with $\mathbb{F}$ being a perfect field, and $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is a linear operator, then Jordan-Chevalley decomposition always exist, additionally, there exist polynomials $p(t), q(t)$ with no constant terms and $T_s = p(T), T_n = q(T)$, moreover, the decomposition is unique

Proof. First consider $\overline{\mathbb{F}} = \mathbb{F}$, by Proposition 0.3.3, V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_i V_{\lambda_i}$, where $V_{\lambda_i} = \ker(T - \lambda_i 1_V)^{m_i}$ with being m_i being the least and $\dim V_{\lambda_i} = d_i$, define $T_s \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ such that $T_s|_{V_{\lambda_i}} = \lambda_i 1_{V_{\lambda_i}}$ and $T_n = T - T_s$, thus T_s is diagonalizable(semisimple), T_n is nilpotent, $ch_T(t) = \prod_i (t-\lambda_i)^{d_i}$, by Theorem **??**, there exists polynomial $p(t)$ such that $p(t) \equiv 0 \bmod t$, $p(t) \equiv \lambda_i \bmod (t-\lambda_i)^{d_i}$, and let $q(t) = t - p(t)$, then p, q doesn't have constant terms and $T_s = p(T), T_n = q(T)$. For uniqueness, suppose $T = T_s + T_n = T'_s + T'_n$ are two such decompositions, then $T_s - T'_s = T'_n - T_n$ will be nilpotent which implies $T_s - T'_s = 0$ □

0.4 Bilinear form

Definition 0.4.1. A *symplectic form* ω is bilinear form such that $\omega(u, v) = X^T JY$, here $J = \begin{pmatrix} & -I \\ I & \end{pmatrix}$, in other words, there are $u_1, \cdots, u_n, v_1, \cdots, v_n$ such that $\omega(u_i, v_j) = -\omega(v_j, u_i) = \delta_{ij}$, $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$

Remark 0.4.2. $\omega(x \oplus \xi, y \oplus \eta) = \eta(x) - \xi(y)$ on $V \oplus V^*$ is a symplectic form. Conversely, such a V is called a Lagrangian subspace, a polarization