MATH848G - Index Theory on Manifolds

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1 Pseudodifferential calculus

Definition 1.1. A differential operator P of order m is of the form $p(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$, where $a_\alpha(x)$ are functions and $D_j := \frac{1}{i}\partial_j$, the symbol $p(x,\xi)$ associated with it is defined as $p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, then we necessarily get the estimate, the principal symbol of $p(x,\xi)$ is $\sigma(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$

If a differential operator P of order m is acting on functions on $\mathbb{C}^n \cong \mathbb{R}^{2n}$, then we may consider it as the form $p(x,D) = \sum_{|\alpha+\beta|\le m} a_{\alpha,\beta}(x) D^\alpha \bar{D}^\beta$, and $D_j := \frac{\partial}{\partial z_j}$, $\bar{D}_j := \frac{\partial}{\partial \bar{z}_j}$, the symbol $p(x,\xi)$ associated with it is defined as $p(x,\xi) = \sum_{|\alpha+\beta|\le m} a_{\alpha,\beta}(x)\xi^\alpha\bar{\xi}^\beta$

Definition 1.2. On a manifold M, a differential operator of order m is a linear map $P : C^\infty(M) \to C^\infty(M)$ which in local charts has the above form

More generally, Let $E \to M, F \to M$ be vector bundles, a differential operator $P : \Gamma(E) \to \Gamma(F)$ of order m is a linear operator of the above form in local trivializations

Notice $C^\infty(M) = \Gamma\left(\bigwedge^0 T^*M\right)$, another example is exterior derivative $d : \Gamma\left(\bigwedge^k T^*M\right) \to \Gamma\left(\bigwedge^{k+1} T^*M\right)$ which is a first order differential operator

We can generalize the notion of differential operators(DO) to Pseudodifferential operators(ψDO) via symbols

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^n$ is an open set, the symbols of order m is the space

$$S^m(\Omega) := \left\{ f(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n) \;\middle|\; \begin{array}{l} \forall \alpha,\beta, K \Subset \Omega, \exists C_{K,\alpha,\beta}, \\ \left|D_x^\alpha D_\xi^\beta f(x,\xi)\right| \le C_{K,\alpha,\beta}(1+|\xi|)^{m-|\beta|} \end{array} \right\}$$

Given symbol $p(x,\xi) \in S^m(\Omega)$, it defines a pseudodifferential operator $p(x,D)$ of order m, such that $(p(x,D)f)(x) = \frac{1}{(2\pi)^n}\int p(x,\xi)\widehat{f}(\xi)e^{ix\cdot\xi}d\xi$

All pseudodifferential operators of order m are difined this way, deonted as $\Psi^m(\Omega)$

Remark 1.4. It might be more natural to think of $f \in C^\infty(\Omega \times (\mathbb{R}^n)^*)$ in the manifold setting

Theorem 1.5. Let Ω, Ω' be open subsets of $\mathbb{R}^n$, $\Phi : \Omega \to \Omega'$ is a diffeomorphism, induces a bijection of $S^m(\Omega) \to S^m(\Omega')$

Proof. Since $\Phi : \Omega \to \Omega'$ is a diffeomorphism, $(\Phi_*)_x : T_x\Omega \to T_{\Phi(x)}\Omega'$ and $(\Phi^*)_x : T^*_{\Phi(x)}\Omega' \to T^*_x\Omega$ are isomorphisms ☐

Corollary 1.6. Let M be a smooth manifold, $S^m(M) \subseteq \Gamma(T^*M)$ is well-defined

Definition 1.7. The principal symbol of symbol $p(x,\xi)$ of order m is defined as $\lim_{|\xi|\to\infty} \frac{p(x,\xi)}{|\xi|^m}$

2 K-theory

Grothendieck group

Definition 2.1. For commutative monoid M: there exists an abelian group K and $i : M \to K$ having the universal property
$$\begin{tikzcd} M \arrow[d, hook, "i"] \arrow[dr, "f"] & \\ K \arrow[r, dashed, "g"] & A \end{tikzcd}$$
Where A is an Abelian group

Concrete construction: consider the set of formal differences $M \times M \cong \{m - n | m, n \in M\}$ with addition $(m_1 - n_1) + (m_2 - n_2) := [(m_1 + m_2) - (n_1 + n_2)]$ which is an Abelian group with $0 - 0$ the identity and $n - m$ the inverse to $m - n$, let $K := M \times M/\sim$, $m_1 - n_1 \sim m_2 - n_2$ if $m_1 + n_2 + m = m_2 + n_1 + m$ for some $m \in M$(adding m to make sure it is an equivalence relation), and $i : M \to K, m \mapsto m - 0$

Alternatively, consider the free Abelian group $F(M)$ generated by M, let $K := F(M)/\sim$, $m +' n \sim (m + n)$, or mod the subgroup generated by $m +' n -' (m + n)$, here $+', -'$ are operations in $F(M)$

More generally, for a semigroup S, there exists a group K and $i : S \to K$ having the universal property
$$\begin{tikzcd} S \arrow[d, hook, "i"] \arrow[dr, "f"] & \\ K \arrow[r, dashed, "g"] & G \end{tikzcd}$$
Where G is a group

Concrete construction: consider the free group $F(S)$ generated by S, let $K := F(S)/\sim$, $m *' n \sim (m * n)$, or mod the subgroup generated by $m *' n *' (m * n)^{-1}$, here $*', -^{-1}$ are operations in $F(M)$

Definition 2.2. (Alternative definition of Grothendieck group) Let R be a finite dimensional k algebra(or more generally an Artinian ring), define Grothedieck group $G_0(R)$ to be the set of all finitely generated R modules mod relation, if $0 \to A \to B \to C \to 0$ is exact, then $[B] = [A] + [C]$, namely, $G_0(R) := \{[M]\}$, where $[M]$ are equivalent classes of finitely generated R modules

Definition 2.3. Two vector bundles $E \to X$, $F \to X$ are stably isomorphic if $E \oplus \varepsilon^n \approx F \oplus \varepsilon^n$, denoted as $E \approx_s F$, we also denote $E \sim_s F$ if $E \oplus \varepsilon^n \approx F \oplus \varepsilon^m$ for some n, m

Remark 2.4. Here stably isomorphic does not imply isomorphic, for example, $TS^2 \approx_s \varepsilon^2$, since $\varepsilon^3 \approx T^2 \oplus NS^2 \approx T^2 \oplus \varepsilon^1$ whereas TS^2 is not trivial by the hairy ball theorem, and $NS^2 \approx \varepsilon^1$ is trivial because it is very easy to find a nonvanishing global section

Topologcial \$K\$-theory

Definition 2.5. Let X be a topological space, then the Abelian group $\tilde{K}_{\mathbb{F}}(X)$ is defined to be the $\sim$-equivalent classes of $\mathbb{F}$-vector bundles over X, since all vector bundles over X with $\oplus$ form a commutative monoid, we can define its Grothendieck group $K(X)$ to be the K-theory, as it turns out, $K(X)$ also has a commutative ring structure with $\otimes$, $(E_1 - F_1) \otimes (E_2 - F_2) := E_1 \otimes E_2 \oplus F_1 \otimes F_2 - E_1 \otimes F_2 \oplus F_1 \otimes E_2$

$K_{\mathbb{F}}(X, Y)$ is defined to be $\{E - F \in K_{\mathbb{F}}(X) | \exists \text{isomorphism } E_Y \to F_Y\}$ Note suppose X is compact Hausdorff, we can redefine the equivalence relation for the formal differences to be $E_1 - F_1 = E_2 - F_2$ if $E_1 \oplus F_2 \approx_s E_2 \oplus F_1$, because if $E_1 \oplus F_2 \oplus E \approx E_2 \oplus F_1 \oplus E$, for some E, then there exists E' such that $E \oplus E' \approx \varepsilon^m$ for some m, and any $E - F = (E \oplus F') - \varepsilon^m$ for some F' such that $F \oplus F' \approx \varepsilon^m$, thus we can define a map $K(X) \to \tilde{K}(X), E - \varepsilon^m \mapsto E$ which is obviously surjective, consider $K(X) \to K(*) \cong \mathbb{Z}$ by restricting a vector bundle to a point, and $K(*) \to K(X)$ by extending a vector space into a trivial vector bundle, we then get an exact sequence which splits
$$\begin{tikzcd} 0 \arrow[r] & K(*) \arrow[r] & K(X) \arrow[r] \arrow[l, bend right] & \tilde{K}(X) \arrow[r] & 0 \end{tikzcd}$$
Thus $K(X) \cong \tilde{K}(X) \oplus K(*) \cong \tilde{K}(X) \oplus \mathbb{Z}$

Suppose X is a locally compact Hausdorff space, think of its one point compactification X^* as a pointed space, then $K(X) \cong \ker[K(X^*) \to K(*)] \cong \ker \tilde{K}(X^*) \cong K(X, *)$

From this we have excision, if $Y \subseteq X$ are both compact Hausdorff, $\tilde{K}(X/Y) \cong \tilde{K}((X - Y)^*) \cong K(X - Y) \cong K(X, Y)$

Remark 2.6. Normally $K(X)$ denote $K_{\mathbb{C}}(X)$, $KO(X)$ denote $K_{\mathbb{R}}(X)$

Theorem 2.7. K-theory form an extraordinary cohomology theory

Proof.

□

Definition 2.8. Let X be a locally compact, Hausdorff space, consider all the complexes

$$\cdots \xrightarrow{\alpha_{-2}} E_{-1} \xrightarrow{\alpha_{-1}} E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} E_2 \xrightarrow{\alpha_2} \cdots$$

with only finitely many $E_j \neq 0$, and exact off a compact set, give a semigroup structure by direct sum, then define $K(X) :=$ all such complexes$/\sim$, where a complex is equivalent to the identity if it is chain homotopic to an exact sequence off a compact set, $K(X, A)$ means an extra condition that the complex in consideration are exact over A

Proposition 2.9. The above two definitions of $K(X, A)$ coinside

Proof. If X is compact, the every complex of vector bundles over X will be exact off a compact set and homotopic to a complex with zero maps, then define Euler characteristic map

$$K(X) \to K(X), \left[0 \to E_1 \overset{\alpha_1}{\to} \cdots \to E_n \overset{\alpha_n}{\to} 0\right] = \left[0 \to E_1 \overset{0}{\to} \cdots \to E_n \overset{0}{\to} 0\right] \mapsto \sum_{k=0}^{n} (-1)^k [E_i]$$

which would give an isomorphism
If X is locally compact and Hausdorff, $K(X) \cong \ker\left[K(X^*) \to K(*)\right] \cong K(X)$ $\square$

Theorem 2.10. Let X be a connected compact oriented two dimensional manifold without boundary(oriented closed surface), every complex vector bundle E is a sum of line bundles $[L_1]+\cdots+[L_n]-[L_{n+1}]-\cdots-[L_{n+m}]$, and line bundles are classified by Chern classes in $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, the rank and degree of E are defined as $\mathrm{rank} E = n - m$, $\deg E = \sum_{i \leq n} \deg[L_i] - \sum_{i > n} \deg[L_i]$,
Therefore, $K(X) \xrightarrow{(\mathrm{rank}, \deg)} \mathbb{Z} \oplus \mathbb{Z}$ is an isomorphism

3 Atiyah-Singer index theorem

Atiyah-Singer index theorem

Theorem 3.1. (**Atiyah-Singer index theorem**) M is a compact manifold, $p : T^*M \to M$ is the cotangent bundle, E, F are vector bundles over M, $P : \Gamma(E) \to \Gamma(F)$ is an elliptic ψDO of order m with principal symbol $\sigma(P) : p^*E \to p^*F$, P defines a Fredholm operator $H^s(E) \to H^{s-m}(F)$, from this we get its index(analytic index) $\mathrm{ind}_a(P)$, then
(1): $\mathrm{ind}_a(P)$ is independent of the choice of s
(2): $\sigma(P)$ defines an element of $K(T^*M)$
(3): $\mathrm{ind}_a(P)$ only depends on $[\sigma(P)] \in K(T^*M)$
(4): $\exists$ homomorphism ind_t(topological index): $K(T^*M) \to \mathbb{Z}$ independent of P, E, F such that $\mathrm{ind}_a(P) = \mathrm{ind}_t([\sigma(P)])$

Proof. **(1):** Follows from what we did about ψDOs
(2): Since $\sigma(P)$ is an elliptic operator, at $(x, \xi) \in T^*M$, $\sigma(P)(x, \xi)$ is a linear map $(p^*E)_{(x,\xi)} \to (p^*F)_{(x,\xi)}$ which is invertible except for ξ small, thus $0 \to p^*E \xrightarrow{\sigma(P)} p^*F \to 0$ is exact off a compact set, thus $\left[0 \to p^*E \xrightarrow{\sigma(P)} p^*F \to 0\right]$ defines an element in $K(T^*M)$
(3): The K-theory class of $\sigma(P)$ only depends on $\sigma(P)$ over S^*M, which is compact, then you can extend it to T^*M just by homogeneity, $\sigma(P)(x, r\xi) = r^m \sigma(P)(x, \xi)$, varying this symbol over S^*M continuously, keeping it invertible, gives a homotopy of Fredholm operators,the index doesn't change since the index is locally constant, and such a homotopy preserves the K-theory class
(4): $\square$

Dolbeault theorem

Theorem 3.2. (**Dolbeault theorem**) Consider complex

$$\Omega^{0,0}(E) \xrightarrow{\bar\partial} \Omega^{0,1}(E) \xrightarrow{\bar\partial} \Omega^{0,2}(E) \xrightarrow{\bar\partial} \cdots$$

The j-th cohomology will be $H^j(M, \mathcal{O}_E)$

Proof.

$$0 \longrightarrow \mathcal{O}(E) \hookrightarrow \Omega^{0,0}(E) \xrightarrow{\bar\partial} \Omega^{0,1}(E) \xrightarrow{\bar\partial} \Omega^{0,2}(E) \xrightarrow{\bar\partial} \cdots$$

is a resolution of $\mathcal{O}(E)$ by fine sheaves as defined in Definition 3.3 $\square$

Fine sheaves

Definition 3.3. A fine sheaf $\mathcal{F}$ over X is one with "partitions of unity"; more precisely for any open cover of the space X we can find a family of homomorphisms from the sheaf to itself with sum 1 such that each homomorphism is 0 outside some element of the open cover

Example 3.4. M is compact Riemann surface, $E \to M$ is a holomorphic vector bundle, E is equipped with $\bar\partial$ operator, $\bar\partial : \Gamma(E) \to \Gamma\left(E \otimes \Omega^{0,1}\right)$, $\bar\partial f := \dfrac{\partial f}{\partial \bar z} d\bar z$(Notice when E is trivial this is the normal $\bar\partial$ operator), $\bar\partial$ is an elliptic operator since its principal symbol is given by $\sigma(\bar\partial)(x, \xi) = \bar\xi$, thus $\mathrm{ind}_a(\bar\partial) = \dim H^0(M, E) - \dim H^1(M, E)$, $H^0(M, E)$ is space of all holomorphic sections of E

Lemma 3.5. $E \xrightarrow{p} X$ is a vector bundle, then the pullback bundle $p^*(E) := E \underset{X}{\times} X$ is a trivial bundle

Proof. $\square$

Definition 3.6. Let X be a compact G space(meaning G acts on X), where G is a compact Lie group, a G-vector bundle E is a vector bundle over X which is also a G space, with $g : E_x \to E_{gx}$ a linear map on fibers. In particular, if X is a point, then a G-bundle is just a finite dimensional representation of G, and $K_G(X)$ is the K-theory of such vector bundles, thus $R(G) := K_G(*)$ is the representation ring of G, if X is a trivial G space, then a G-vector bundle is just a continuous family of G representations

Example 3.7. (Examples of representation rings $R(G)$) $R(G) = \mathbb{Z}$

Theorem 3.8. If X is a trivial G space, $K_G(X) \cong K(X) \otimes_{\mathbb{Z}} R(G)$
If X is a free G space, $K_G(X) \cong K(X/G)$

Equivariant Bott periodicity theorem

Theorem 3.9. (**Equivariant Bott periodicity theorem**) Let X be a locally compact G space, where G is a compact Lie group, V is a finite dimensional representation of G, then there is an isomorphism $K_G \to K_G(V \times X)$ defined as follows
Consider the complex

$$0 \longrightarrow \mathbb{C} = \bigwedge^0 V \xrightarrow{\lambda_v} V = \bigwedge^1 V \xrightarrow{\lambda_v} \bigwedge^2 V \xrightarrow{\lambda_v} \cdots \xrightarrow{\lambda_v} \bigwedge^n V \longrightarrow 0$$

Where $\lambda_v := v \wedge -$

Theorem 3.10. (Thom isomorphism theorem in K-theory) Let X be a compact Hausdorff space, $E \to X$ is a complex vector bundle, then multiplication by λ_{E^*} give an isomorphism $K(X) \to K(E)$

Proof. Give an inner product on each fiber varying smoothly, $n = \dim E$, let $G = U(n)$, Y be the fiber bundle over X with fiber over x the stiefel manifold of E_x, then G acts on Y and $\mathbb{C}^n \times Y$ freely, thus $K(X) = K(Y/G) \cong K_G(Y) \xrightarrow{\cong} K_G(\mathbb{C}^n \times Y) \cong K((\mathbb{C}^n \times Y)/G) = K(E)$
Here λ_E is the complex $\bigwedge(p^*E)$ □

4 Clifford algebra

Definition 4.1. Let $V = \mathbb{F}^n$ be a vector space with a quadratic form q, TV be its tensor algebra, let I be the ideal generated by elements of the form $v \otimes v + q(v,v)1$, the Clifford algebra is defined to be $Cl_n = \mathrm{Cliff}(V,q) = TV/I$, in particular, we only consider the case when $V = \mathbb{R}^n$ is the Euclidean space with the standard inner product, if $e_1, \cdots, e_n$ are the standard basis, $e_i \otimes e_i + 1 \in I$, $(e_i + e_j) \otimes (e_i + e_j) + 2 = e_i \otimes e_i + e_j \otimes e_j + e_i \otimes e_j + e_j \otimes e_i + 2 \in I$, we have $e_i^2 = -1$ and $e_i e_j = -e_j e_i, i \neq j$, thus $\dim Cl_n = 2^n$, $Cl_n = Cl_n^{even} \oplus Cl_n^{odd}$, $\dim Cl_n^{even} = \dim Cl_n^{odd} = 2^{n-1}$, Cl_n^{even} is a module over Cl_n^{even}

Example 4.2. If $n = 1$, $Cl_1 \cong \mathbb{R}[e_1]/(e_1^2 + 1) \cong \mathbb{C}$
If $n = 2$, $Cl_2 \cong \mathbb{R}[e_1, e_2]/(e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1) \cong \mathbb{H}$

Proposition 4.3. $Cl_{n+m} \cong Cl_n \otimes Cl_m$, here the tensor product is the graded tensor product

Proof.

□

Proposition 4.4. $Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ is semisimple over $\mathbb{C}$ of dimension 2^n, and

$$Cl_n \otimes_{\mathbb{R}} \mathbb{C} = \begin{cases} M_{2^{\frac{n}{2}}}(\mathbb{C}), \text{if } n \text{ is even} \\ M_{2^{\lfloor n/2 \rfloor}}(\mathbb{C}) \oplus M_{2^{\lfloor n/2 \rfloor}}(\mathbb{C}), \text{if } n \text{ is odd} \end{cases}$$

Proof.

□

5 Spin group

Definition 5.1. $Spin(n)$ is a subgroup of Cl_n^{even} consisting of elements of norm 1

Example 5.2. $Spin(2) = \mathbb{T}$ is the circle group, Since $Cl_3^{even} \cong Cl_2 \cong \mathbb{H}$, $SU(2) \cong Spin(3)$ are the corresponding subgroups

Definition 5.3. $Spin(n)$ naturally acts on $\mathbb{R}^n$ by $g \cdot v = gvg^{-1}$, then it is easy to see that $Spin(n) \to SO(n)$ is a double covering, if M is a Riemannian manifold, it has a principal bundle
$$\begin{tikzcd} SO(n) \arrow[r] & P \arrow[d] \\ & M \end{tikzcd}$$
where P_m are the oriented orthonormal frames on T_mM A spin structure on M is a lifting of P to a principal bundle for $Spin(n)$, note this may or may not exist

Example 5.4. $\mathbb{C}P^{2n}$ doesn't have a spin structure

Proposition 5.5. If spin structures exist, the set of such structures is acted on transitively by $H^1(M, \mathbb{Z}/2\mathbb{Z})$

Proof. Consider the long exact sequence

$$H^1(M, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(M, Spin(n)) \longrightarrow H^1(M, SO(n)) \xrightarrow{w_2} H^2(M, \mathbb{Z}/2\mathbb{Z})$$

$\square$

Proposition 5.6. If n is even and M has a spin structure, fix it, then since $Spin(n) \subseteq Cl_n^{even} \subseteq Cl_n \subseteq Cl_n \otimes \mathbb{C} \cong M^{2^{\frac{n}{2}}}(\mathbb{C})$, and $Cl_n \otimes \mathbb{C}$ has a unique irreducible representation of dimension $2^{\frac{n}{2}}$, called the spin representation Δ, so we get a (comple) spin vector bundle $S = P_{Spin(n)} \times_{Spin(n)} \Delta$

Definition 5.7. There is an elliptic operator acting on S via $\sum_{j=1}^{n} e_j \cdot \nabla e_j$, where $e_1, \cdots, e_n$ is a local orthonormal frame, the symbol is $\sum e_j \xi_j$, this is called the Dirac operator $\not{D}$

Theorem 5.8. Assume M is a closed Riemannian spin manifold of dimension $n = 2l$, $ind\not{D} = \langle \widehat{A}(M), [M] \rangle$, here $\widehat{A}(M)$ is a polynomial in Pontryagin classes defined as follows: suppose $TM \otimes \mathbb{C} \cong L_1 \oplus \cdots \oplus L_l$ where L_j are complex line bundle with Chern class x_j, then $\widehat{A}(M) = \prod_j \dfrac{x_j}{2\sinh \frac{x_j}{2}}$

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