Solution to Exam 2

1. (a) $\int_2^{+\infty} \frac{1}{x\ln x}dx = \lim_{b\to+\infty}\int_2^b \frac{1}{x\ln x}dx = \lim_{b\to+\infty}\int_2^b \frac{1}{\ln x}\cdot\frac{1}{x}dx \xrightarrow[du=d\ln x=(\ln x)'dx=\frac{1}{x}dx]{u=\ln x}$

$\lim_{b\to+\infty}\int_{\ln(2)}^{\ln(b)} \frac{1}{u}du = \lim_{b\to+\infty} \ln u \Big|_{\ln(2)}^{\ln(b)} = \lim_{b\to+\infty}\left[\ln(\ln(b)) - \ln(\ln(2))\right] = +\infty$

since $b\nearrow+\infty \to \ln b\nearrow +\infty \to \ln(\ln b)\nearrow+\infty$, the improper integral is divergent

(b) $\int_e^{+\infty}\frac{1}{x\ln^2 x}dx = \int_e^{+\infty}\frac{1}{x(\ln x)^2}dx = \lim_{b\to+\infty}\int_e^b \frac{1}{x(\ln x)^2}dx = \lim_{b\to+\infty}\int_e^b\frac{1}{(\ln x)^2}\frac{1}{x}dx \xrightarrow[du=d\ln x=\frac{1}{x}dx]{u=\ln x}$

$\lim_{b\to+\infty}\int_{\ln e}^{\ln b}\frac{1}{u^2}du = \lim_{b\to+\infty} -u^{-1}\Big|_1^{\ln b} = \lim_{b\to+\infty}\left[(-(\ln b)^{-1}) - (-1^{-1})\right] = \lim_{b\to+\infty}\left[-\frac{1}{\ln b}+1\right]$

$=\left[-0+1\right]=1$

2. (a) Remember $(A^a)^b = A^{ab}$, $(AB)^a = A^aB^a$, $\left(\frac{A}{B}\right)^a = \frac{A^a}{B^a}$

$\frac{dy}{dt} = y' = \sqrt[3]{\frac{27t^2}{8y}} = \left(\frac{27t^2}{8y}\right)^{\frac13} = \frac{(27t^2)^{\frac13}}{(8y)^{\frac13}} = \frac{27^{\frac13}(t^2)^{\frac13}}{8^{\frac13}y^{\frac13}} = \frac{3t^{\frac23}}{2y^{\frac13}}$

$\Rightarrow 2y^{\frac13}\cdot \cancel{dt}\cdot\frac{dy}{\cancel{dt}} = \frac{3t^{\frac23}}{\cancel{2y^{\frac13}}}\cdot\cancel{2y^{\frac13}}\cdot dt$

$\Rightarrow 2y^{\frac13}dy = 3t^{\frac23}dt \Rightarrow \int 2y^{\frac13}dy = \int 3t^{\frac23}dt \Rightarrow 2\int y^{\frac13}dy = 3\int t^{\frac23}dt$

$\Rightarrow 2\frac{1}{\frac13+1}y^{\frac13+1} = 3\frac{1}{\frac23+1}t^{\frac23+1}+C \Rightarrow \frac32 y^{\frac43} = \frac95 t^{\frac53}+C$

plug in $y(0)=-8$, then $\frac32(-8)^{\frac43} = \frac95 0^{\frac53}+C \Rightarrow C = \frac32(-8)^{\frac43} = \frac32\left((-8)^{\frac13}\right)^4$

$=\frac32(-2)^4 = \frac32\cdot16=24$, thus $\frac32 y^{\frac43} = \frac95 t^{\frac53}+24 \Rightarrow$

$y^{\frac43} = \frac23\cdot\frac32 y^{\frac43} = \frac23\left(\frac95 t^{\frac53}+24\right) = \frac65 t^{\frac53}+16$

$\Rightarrow y^4 = \left(y^{\frac43}\right)^3 = \left(\frac65 t^{\frac53}+16\right)^3 \Rightarrow y = \sqrt[4]{\left(\frac65 t^{\frac53}+16\right)^3}$ (discarded since $y(0)=-8$)

or $y = -\sqrt[4]{\left(\frac65 t^{\frac53}+16\right)^3}$