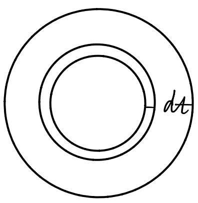
7th problem on Ch09.1–5 Review test

A volcano erupts and spreads lava in all directions, the density of the deposits at a distance t kilometers from the center is $D(t)$ thousand tons per square kilometers and is determined by the following formula

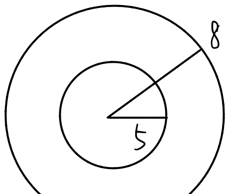
$$D(t) = 11\,(t^2+14)^{-2}$$

Find the tonnage of lava deposited between the distances of 5 and 8 kilometers from the center

Solution: consider the tonnage of lava deposited in an infinitesimally small circular band which should be



$$D(t)\cdot 2\pi t\,dt \quad (\text{density} \times \text{circumference} \times \text{width})$$
$$\|$$
$$11\,(t^2+14)^{-2}\cdot 2\pi t\,dt$$



Hence the tonnage of lava deposited between the distance of 5 and 8 kilometers from the center is to "sum" (integrate) over all these lava deposited in each infinitesimally small circular band

$$\int_5^8 11\,(t^2+14)^{-2}\cdot 2\pi t\,dt \;\overset{dt^2 = 2t\,dt}{=\!=}\; \int_5^8 11\,(t^2+14)^{-2}\cdot \pi\,dt^2$$

$$\overset{u=t^2}{=\!=}\; \int_{5^2}^{8^2} 11\,(u+14)^{-2}\,\pi\,du = 11\pi\int_{25}^{64} (u+14)^{-2}\,du \;\overset{d(u+14)=du}{=\!=}$$

$$11\pi\int_{25}^{64} (u+14)^{-2}\,d(u+14) \;\overset{v=u+14}{=\!=}\; 11\pi\int_{25+14}^{64+14} v^{-2}\,dv$$

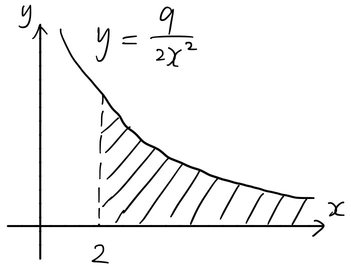
$$= 11\pi\int_{39}^{78} v^{-2}\,dv = \frac{11\pi}{(-1)}\int_{39}^{78} (-1)\,v^{-2}\,dv \;\overset{(v^{-1})'=(-1)v^{-2}}{=\!=}$$

$$\frac{11\pi}{(-1)}\, v^{-1}\Big|_{39}^{78} = \frac{11\pi}{(-1)}\left[78^{-1} - 39^{-1}\right]$$

$$= -11\pi\left[\frac{1}{78} - \frac{1}{39}\right] = \frac{11\pi}{78}$$

Section 9.6 Improper integral

Find the area under the graph of $y = \frac{9}{2x^2}$ for $x \geq 2$



The area is $\int_2^\infty \frac{9}{2x^2}\,dx = \lim\limits_{b\to\infty} \int_2^b \frac{9}{2x^2}\,dx$

here $\lim\limits_{b\to\infty} \int_2^b \frac{9}{2x^2}\,dx$ by definition is a number that $\int_2^b \frac{9}{2x^2}\,dx$ converges to as $b \to +\infty$

$$\int_2^b \frac{9}{2x^2}\,dx = \frac{9}{2}\int_2^b x^{-2}\,dx = \frac{9}{2\cdot(-1)}\int_2^b (-1)x^{-2}\,dx$$

$$\xlongequal{(x^{-1})'=(-1)x^{-2}} \frac{9}{2\cdot(-1)}\, x^{-1}\Big|_2^b = -\frac{9}{2}\left[b^{-1} - 2^{-1}\right]$$

$= -\frac{9}{2}\left[\frac{1}{b} - \frac{1}{2}\right]$, as $b \to \infty$, $\frac{1}{b} \to 0$, hence

$$-\frac{9}{2}\left[\frac{1}{b} - \frac{1}{2}\right] \to -\frac{9}{2}\left[0 - \frac{1}{2}\right] = \frac{9}{4}$$

We say limit $\lim\limits_{b\to\infty} \int_2^b \frac{9}{2x^2}\,dx$ excist and $\int_2^\infty \frac{9}{2x^2}\,dx$ is convergent

How about $\int_1^\infty \frac{x^6}{x^7+5}\,dx$?

$\int_1^\infty \frac{x^6}{x^7+5}\,dx = \lim\limits_{b\to\infty} \int_1^b \frac{x^6}{x^7+5}\,dx$ is by definition $\int_1^b \frac{x^6}{x^7+5}\,dx \to$? as $b \to \infty$

$$\int_1^b \frac{x^6}{x^7+5}\,dx = \frac{1}{7}\int_1^b \frac{7x^6}{x^7+5}\,dx \xlongequal{dx^7 = 7x^6dx} \frac{1}{7}\int_1^b \frac{dx^7}{x^7+5} \xlongequal{u=x^7}$$

$$\frac{1}{7}\int_{1^7}^{b^7} \frac{du}{u+5} = \frac{1}{7}\int_1^{b^7} \frac{du}{u+5} \xlongequal{d(u+5)=du} \frac{1}{7}\int_1^{b^7} \frac{d(u+5)}{u+5} \xlongequal{v=u+5} \frac{1}{7}\int_{1+5}^{b^7+5} \frac{dv}{v}$$

$$= \frac{1}{7}\int_6^{b^7+5} \frac{1}{v}\,dv \xlongequal{(\ln v)' = \frac{1}{v}} \frac{1}{7}\ln v\Big|_6^{b^7+5} = \frac{1}{7}\left[\ln(b^7+5) - \ln(6)\right]$$ but

as $b \to \infty$, $b^7 \to \infty$, $b^7+5 \to \infty$, $\ln(b^7+5) \to \infty$, $\ln(b^7+5) - \ln(6) \to \infty$,

$\frac{1}{7}\left[\ln(b^7+5) - \ln(6)\right] \to \infty$

We say $\lim\limits_{b\to\infty} \int_1^b \frac{x^6}{x^7+5}\,dx$ doesn't exist and $\int_1^\infty \frac{x^6}{x^7+5}\,dx$ is divergent