0.1 Topological vector space

Definition 0.1.1. A **topological vector space** V over a topological field $\mathbb{F}$ is a topological abelian group such that scalar multiplication $\mathbb{F} \times V \to V$ is continuous

Definition 0.1.2. A **norm** on a group G is $G \xrightarrow{\|\|} \mathbb{R}_{\geq 0}$ such that $\|g\| = 0 \Leftrightarrow g = \mathrm{id}$, $\|g^{-1}\| = \|g\|$, $\|gh\| \leq \|g\|\|h\|$
A **norm** on a rng R is a normed abelian group such that $\|rs\| \leq \|r\|\|s\|$
A **norm** on a vector space V over a normed field is a normed abelian group such that $\|kv\| \leq |k|\|v\|$

Definition 0.1.3. A **Banach space** is a complete normed vector space

Definition 0.1.4. Y is a topological vector space, T is a set, $\mathcal{G} \subseteq \mathcal{P}(T)$ is a directed set by inclusion, $\mathcal{N}$ is a local base around $0 \in Y$. The **topology of uniform convergence** on sets in $\mathcal{G}$ or $\mathcal{G}$ **topology** is the unique translation invariant topology given by basis

$$U(G, N) = \left\{ f \in Y^T \middle| G \in \mathcal{G}, N \in \mathcal{N}, f(G) \subseteq N \right\}$$

Example 0.1.5. $\mathcal{G}$ is the set of compact subspaces, Y is a metric space

0.2 Arzela-Ascoli theorem

Definition 0.2.1. Let X, Y be a topological spaces, a family of continuous functions $A \subseteq Y^X$ is equicontinuous at $x \in X$, if for any open neighborhood V of $y = f(x)$, there is an open neighborhood U of x such that $f(U) \subseteq V, \forall f \in A$

Definition 0.2.2. A topological space X is called separable if X has a countable dense subset

Arzela-Ascoli theorem

Theorem 0.2.3. Let X be a topological space and Y be a complete metric space, $A \subseteq Y^X$ be a family of equicontinuous functions(meaning pointwise equicontinuous). If X is compact, and $A_x := \{f(x) | f \in A\} \subseteq Y$ is relatively compact for any $x \in A$, then A is relatively compact in Y^X. If X is separable with S being a countable dense subset, and A_x is relatively compact for any $x \in S$, then any sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ converges uniformly on any compact subset of X

0.3 Baire category theorem

Definition 0.3.1. A topological space X is a **Baire space** if for any countable open dense subsets $\{U_i\}$, $\bigcap_{i=1}^{\infty} U_i$ is also dense

Baire category theorem

Theorem 0.3.2 (Baire category theorem)**.** Every complete metric space X is a Baire space

Proof. Let $\{U_i\}$ be a countable open dense subsets, suppose $\bigcap_{i=1}^{\infty} U_i$ is not dense, then the complement of its closure is open nonempty, suppose $B(x,r)$ is in the complement of the closure, since U_1 is dense, $U_1 \cap B(x,r) \neq \emptyset$, then there exists $\overline{B(x_1,r_1)} \subseteq U_1 \cap B(x,r)$, similarly, we can find $\overline{B(x_n,r_n)} \subseteq U_n \cap B(x_{n-1},r_{n-1})$, and we can also assume $r_i \to 0$, thus $x_i \to y \in X$ since X is complete, but $y \in B(x,r) \bigcap \bigcap_{i=1}^{\infty} U_i = \emptyset$ which is a contradiction $\square$

0.4 Distribution

Definition 0.4.1. $U \subseteq \mathbb{R}^n$ open, $\mathscr{D}(U) = C_c^\infty(U)$ is the **test function space**, $\{\phi_i\} \subseteq \mathscr{D}(U)$ converges if there exists $K \subseteq U$ compact such that $\mathrm{supp}\phi_i \subseteq K$ and $\partial^\alpha \phi_i$ converges uniformly

0.5 Banach algebra

Definition 0.5.1. A **Banach algebra** is an associative algebra A which is a complete normed rng such that $\|rs\| \leq \|r\|\|s\|$. A is **unital** if A is a ring with identity element having norm 1

Definition 0.5.2. A ***-algebra** is a Banach algebra over $\mathbb{C}$ such that there is an antilinear involution $* : A \to A$, such that $(xy)^* = y^* x^*$. A is a C^***-algebra** if $\|x^* x\| = \|x^*\|\|x\|$

Example 0.5.3. X is locally compact, $C_0(X)$ are the continuous functions vanishes at infinity, then $C_0(X)$ is a Banach algebra with the supremum norm, $C_0(X)$ is unital if X is compact with 1 being the identity. $C_0(X)$ is a C^*-algebra with complex conjugation as the involution

Definition 0.5.4. A is a unital Banach algebra over $\mathbb{R}, \mathbb{C}$, $e^x = \displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ defines the **exponential**

$$\|e^x\| = \left\| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|x\|^k}{k!} = e^{\|x\|}$$

The **logarithm** $\log x = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{x+1}(x-1)^k}{k}$ is defined on $\|x-1\| < 1$

Lemma 0.5.5. e^x and $\log x$ are inverses to each other locally

Proposition 0.5.6. A is a Banach algebra, linear map $D : A \to A$ is a derivation iff e^{tD} is a group of automorphisms

Lie product formula

Theorem 0.5.7 (Lie product formula)**.** $e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n$

Lie commutator formula

Theorem 0.5.8 (Lie commutator formula)**.** $e^{[A,B]} = \lim_{n\to\infty} \left[e^{\frac{A}{n}}, e^{\frac{B}{n}} \right]^{n^2}$, the left and right $[,]$ are Lie bracket and commutator

Lemma 0.5.9. If $[X,[X,Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$

Proof. Let $A(t) = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}$, $B(t) = e^{t(X+Y)}$, then $A(0) = B(0)$, $B'(t) = B(t)(X+Y)$ and

$$A'(t) = e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} - e^{tX} e^{tY} t[X,Y] e^{-\frac{t^2}{2}[X,Y]}$$

Since $[X,[X,Y]] = 0$, $[Y,[X,Y]] = -[Y,[Y,X]] = 0$

$$e^{-tY} X e^{tY} = Ad_{e^{-tY}}(X) = e^{ad_{-tY}}(X) = X + t[X,Y]$$

$$A'(t) = e^{tX} e^{tY} (X+Y) e^{-\frac{t^2}{2}[X,Y]} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} (X+Y) = A(t)(X+Y)$$

Thus $A(t), B(t)$ satisfies the same ODE and initial condition, $A(t) = B(t) \Rightarrow e^X e^Y = A(1) = B(1) = e^{X+Y+\frac{1}{2}[X,Y]}$ □

Theorem 0.5.10 (Backer-Campbell-Hausdorff formula)**.** $e^X e^Y = e^Z$ around 0, where $Z =$

$X + \int_0^1 \psi(e^{ad_X} e^{tad_Y}) dt(Y)$ and

$$\begin{aligned}
\psi(x) &= \frac{x \log x}{x-1} \\
&\overset{y=1-x}{=\!=\!=} \frac{(1-y)\log(1-y)}{-y} \\
&= (1-y)\sum_{n=1}^{\infty} \frac{y^{n-1}}{n} \\
&= \sum_{n=1}^{\infty} \frac{y^{n-1}}{n} - \sum_{n=1}^{\infty} \frac{y^n}{n} \\
&= 1 + \sum_{n=1}^{\infty} \left(\frac{y^n}{n+1} - \frac{y^n}{n} \right) \\
&= 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}
\end{aligned}$$

The first few terms are

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\cdots}$$

Proof. The Riemann sum $\sum_{k=0}^{m-1} \frac{1}{m} e^{-\frac{kx}{m}}$ converges to $\int_0^1 e^{-tx} dt = \frac{1-e^{-x}}{x}$, thus $\lim_{m\to\infty} \sum_{k=0}^{m-1} e^{-\frac{kX}{m}} = \frac{1-e^{-X}}{X}$, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} &= \left.\frac{d}{dt}\right|_{t=0} \left(e^{\frac{X}{m}} e^{\frac{tY}{m}} \right)^m \\
&= \lim_{m\to\infty} \left.\frac{d}{dt}\right|_{t=0} \left(e^{\frac{X}{m}} e^{\frac{tY}{m}} \right)^m \\
&= \lim_{m\to\infty} \sum_{k=0}^{m-1} e^{\frac{kX}{m}} \frac{Y}{m} e^{\frac{(m-k)X}{m}} \\
&= \lim_{m\to\infty} \sum_{k=0}^{m-1} \frac{1}{m} e^{\frac{kX}{m}} Y e^{-\frac{kX}{m}} e^X \\
&= \left(\lim_{m\to\infty} \sum_{k=0}^{m-1} \frac{1}{m} e^{\frac{kad_X}{m}} \right)(Y) e^X \\
&= \frac{e^{ad_X}-1}{ad_X}(Y)e^X
\end{aligned}$$

Let $e^{Z(t)} = e^X e^{tY}$, $\frac{d}{dt}e^{Z(t)} = \frac{d}{dt}(e^X e^{tY}) = e^X e^{tY} Y = e^{Z(t)}Y$, but $\frac{d}{dt}e^{Z(t)} = \left.\frac{d}{ds}\right|_{s=t} e^{Z(s)} = \left.\frac{d}{ds}\right|_{s=t} e^{Z(t)+Z'(t)(s-t)} = \frac{e^{ad_{Z(t)}}-1}{ad_{Z(t)}}(Z'(t))e^{Z(t)}$, hence $\frac{e^{ad_{Z(t)}}-1}{ad_{Z(t)}}(Z'(t)) = e^{Z(t)}Ye^{-Z(t)} = Ad_{e^{Z(t)}}(Y) = e^{ad_{Z(t)}}(Y)$, $Z'(t) = \frac{ad_{Z(t)}e^{ad_{Z(t)}}}{e^{ad_{Z(t)}}-1}(Y)$, since $e^{ad_{Z(t)}} = Ad_{e^{Z(t)}} = Ad_{e^X e^{tY}} = e^{ad_X} e^{tad_Y}$

$$\begin{aligned}
Z &= Z(1) \\
&= Z(0) + \int_0^1 \frac{ad_{Z(t)}e^{ad_{Z(t)}}}{1-e^{-ad_{Z(t)}}}(Y)dt \\
&= X + \int_0^1 \frac{e^{ad_X} e^{tad_Y} \log(e^{ad_X} e^{tad_Y})}{e^{ad_X} e^{tad_Y} - 1} dt(Y)
\end{aligned}$$

□

0.6 Stone-Weierstrass theorem

Definition 0.6.1. $\mathcal{F} = \{f_i\}$ is a family of functions on X, $\mathcal{F}$ **separates points** in X if for any $x \neq y \in X$, some f_i separates x, y

Theorem 0.6.2. X is compact Hausdorff, $A \subseteq C(X, \mathbb{R})$ is a unital subalgebra. A is dense in $C(X, \mathbb{R})$ with the topology of uniform convergence iff A separates points
$S \subseteq C(X, \mathbb{C})$ is a unital *-algebra that separating points, then S is dense in $C(X, \mathbb{C})$