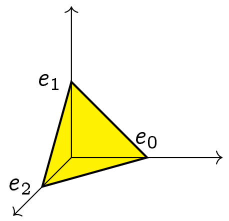
0.1 CW complexes

Standard simplex

Definition 0.1.1. With the standard basis $\{e_i\}$ for $\mathbb{R}^\infty$ as vertices, the **standard n-simplice** is

$$\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \subseteq \mathbb{R}^\infty \middle| \sum t_i = 1, 0 \leq t_i \leq 1\right\}$$

The **i-th face** of Δ^n is the face oppose to the i-th vertex, i.e. $\{t_i = 0\} \cap \Delta^n$
The **boundary** of Δ^n to be $\partial\Delta^n$ is the union of faces. $\partial\Delta^0 = \varnothing$



$\Delta^{n-1} \xrightarrow{d_{n,i}} \Delta^n$, $e_j \mapsto \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$ is i-th **face map** attaching Δ^{n-1} to the i-th face of Δ^n

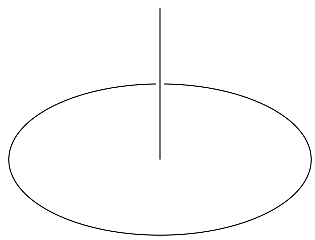
$\Delta^{n+1} \xrightarrow{s_{n,i}} \Delta^n$, $e_j \mapsto \begin{cases} e_j & j \leq i \\ e_{j-1} & j > i \end{cases}$ is the i-th **degeneracy map** which is a projection

Definition 0.1.2. X has a **cell decomposition** if X can be written as the disjoint union of open n cells, i.e. $X = \bigcup\limits_{n,\alpha} e_\alpha^n$, where cells e_α^n with subspace topology are homeomorphic to open n disks or open n simplices and disjoint, $X^n = \bigsqcup\limits_{k \leq n, \alpha} e_\alpha^k$ is called the **n-skeleton**, define $X^{-1} = \emptyset$
Suppose X, Y have cell decomposition $X = \bigcup\limits_{n,\alpha} e_\alpha^n$, $Y = \bigcup\limits_{m,\beta} e_\beta^m$, then $X \times Y$ also has a cell decomposition $X \times Y = \bigcup\limits_{k} \bigcup\limits_{\substack{n+m=k \\ \alpha,\beta}} e_\alpha^n \times e_\beta^m$, note that $e_\alpha^n \times e_\beta^m \cong e^{n+m}$
Every topological space has a cell decomposition into points

Definition 0.1.3. A **cellular map** is a map $f : X \to Y$ between topological spaces with cell decompositions such that $f(X^n) \subseteq Y^n$

Definition 0.1.4. X is called a **cell complex** if X is a Hausdorff space with cell decomposition $X = \bigcup\limits_{n,\alpha} e_\alpha^n$ and a cell complex structure: a family of characteristic maps $\Phi_\alpha^n : \Delta^n \to X$ such that Φ_α^n restricted on $\Delta^n \setminus \partial\Delta^n$ is a homeomorphism onto e_α^n and $\Phi_\alpha^n(\partial\Delta^n) \subseteq X^{n-1}$
Note that in the definition, we could also replace Δ^n with D^n

Remark 0.1.5. Since Δ^n is compact Hausdorff and X is Hausdorff, $\overline{e_\alpha^n} \subseteq \Phi_\alpha^n(\Delta^n)$, $\partial e_\alpha^n \subseteq \Phi_\alpha^n(\partial\Delta^n)$ for $n > 0$, on the other hand, if $\partial e_\alpha^n \subsetneq \Phi_\alpha^n(\partial\Delta^n)$, then there exists $x \in \partial\Delta^n$ such that $y = \Phi_\alpha^n(x) \notin \overline{e_\alpha^n}$, this means there is an open neighborhood U of y disjoint from $\overline{e_\alpha^n}$, but then the preimage of U under Φ_α^n would be a nonempty open subset of Δ^n which intersects $\Delta^n \setminus \partial\Delta^n$ which is impossible, hence $\Phi_\alpha^n(\Delta^n) = \overline{e_\alpha^n}$, $\Phi_\alpha^n(\partial\Delta^n) = \partial e_\alpha^n$ for $n > 0$
A Hausdorff space X with a cell decomposition doesn't immediately give a cell complex structure, for example, consider an open disk union with an open segment right in the middle



Cell complex X can be seen as Δ^n is compact Hausdorff and X is Hausdorff and Lemma **??**

Definition 0.1.6. A cell complex X is called **regular** if all characteristic maps are embeddings

Definition 0.1.7. Let X be a cell complex, it is **closure finite** if $\overline{e^n_\alpha}$ is contained in the union of finitely many cells, and we say X has the **weak topology**, meaning $F \subseteq X$ is closed iff $F \cap \overline{e^n_\alpha}$ is closed in $\overline{e^n_\alpha}$ for any cell, if a cell complex is both closure finite and has the weak topology, we say it is a **CW complex**
Closure finiteness is equivalent of saying $\partial e^n_\alpha \subseteq \bigcup\limits_{k<n,\alpha} e^k_\alpha$ a finite union of cells

Example 0.1.8. Consider $X = D^2$ with a cell complex structure $D^2 \to D^2$ and $* \to x$ for each $x \in \partial D^2$, this doesn't satisfy closure finiteness since $\overline{e^2} = X$, but the weak topology is the same as the original one, since if $F \subseteq D^2$ is closed in the weak topology, then $F \cap \overline{e^2} = F$ is closed
Consider $X = S^1$ with a cell complex structure $* \to x$ for each $x \in S^1$, the weak topology is the discrete topology on S^1 which doesn't match with the original topology on S^1, but it does satisfy closure finiteness

Remark 0.1.9. Suppose X is a CW complex
X^n is obviously closed due to the weak topology
Since $\overline{e^n_\alpha}$ is contained in the union of finitely many cells, $\overline{e^n_\alpha}$ contains at most finitely many 0 cells, thus any union of 0 cells F is closed because $F \cap \overline{e^n_\alpha}$ is finitely many points which is closed given that X is Hausdorff, therefore X^0 is discrete
Suppose $K \subseteq X$ is a compact subset, then $K \subseteq X = \bigcup e^n_\alpha \subseteq \bigcup \overline{e^n_\alpha} \setminus \partial e^n_\alpha$
contained in finitely many cells, since $K \subseteq \bigcup \overline{e^n_\alpha} \setminus \partial e^n_\alpha$
if $K \cap e^n_\alpha \neq \emptyset$,
, otherwise K intersects infinitely many cells,

Theorem 0.1.10. Another description of CW complexes is as follows:
These two definitions coincides

Proposition 0.1.11. Any compact set of a CW complex is contained in finitely many cells

Proposition 0.1.12. CW complexes are locally contractible, thus they are locally path connected, hence connectedness and path connectedness are equivalent for CW complexes

Theorem 0.1.13. CW complexes are normal, satisfies T_4 axiom

Proposition 0.1.14. If $A \subseteq X$ is a CW subcomplex, then (X, A) is a good pair

Theorem 0.1.15. CW complexes have partitions of unity

Proposition 0.1.16. Covering space of CW complexes are CW complexes

Proposition 0.1.17. The product of two countable CW complexes is again a CW complex

0.2 Graphs

Theorem 0.2.1. For every group G, there is a connected two dimensional CW complex X with $\pi_1(X) = G$

Proof. We can always find a surjection from a free group F to G, suppose F is generated by g_α's, and the kernel K is generated by r_β's, i.e. F has a group presentation $\langle g_\alpha | r_\beta \rangle$, then define X to be $\bigvee_\alpha S^1_\alpha$ attached with cells e^2_β's along each word r_β □

Definition 0.2.2. Cayley graphs, Cayley complexes

Definition 0.2.3. A **graph** G is a one dimensional CW complex, a **tree** T is a contractible graph, $T \subseteq G$ is maximal if T contains all vertices, note that in a tree there is a unique path between two vertices

Proposition 0.2.4. Let X be a connected graph, any tree in X is contained in a maximal tree, in particular, X has a maximal tree

Proof. Let's prove more generally any subgraph X_0 is the deformation retraction of subgraph Y which contains all the vertices

Construct $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ as follows, X_{i+1} is obtained by adding the closures of all the edges that connected to X_i, $X = \bigcup_i X_i$, since X is path connected, let $Y_0 = X_0$, and construct $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ as follows, for any vertex in $X_{i+1} - X_i$, choose one edge that connects to Y_i, and add the closure, so we have Y_{i+1} from Y_i, it is easy to see the Y_{i+1} deformation retracts onto Y_i, so $Y = \bigcup_i Y_i$ deformation retracts onto $Y_0 = X_0$

If $X_0 = T$ is a tree, so is Y since Y deformation retracts onto T which is contractible □

Free basis for connected graphs

Proposition 0.2.5 (Free basis for connected graphs)**.** For a connected graph X with maximal tree T, for any edge $e_\alpha \in X - T$, there is a corresponding loop f_α goes from x_0 to one endpoint of e_α, across e_α to the other, and go back to x_0, $\pi_1(X, x_0)$ is a free group generated by f_α

Proof. Consider $X \to X/T$ which is a homotopy equivalence □

Theorem 0.2.6. Any subgroup of a free group is also free

Proof. Let F be a free group, there exists a graph X such that $\pi_1(X) = F$ by just taking the wedge sum of circles at x_0, let $G \leq F$ be a subgroup, then there exists a covering $Y \xrightarrow{p} X$ such that $p_*(\pi_1(Y, y_0)) = G$, thus $\pi_1(Y, y_0) \cong G$, and since Y is a covering of X, Y is also a graph, by Proposition 0.2.5, $G \cong \pi_1(Y)$ is free □

0.3 Simplex category

Definition 0.3.1. The **simplex category** Simp has $[n] := \{0, 1, \cdots, n\}$ as objects, and order preserving functions as morphisms, there are two special types of morphisms: **Face maps** $d_{n,i} : [n-1] \to [n]$, $d_{n,i}(j) = \begin{cases} j & , j < i \\ j+1 & , j \geq i \end{cases}$ and the **degeneracy maps** $s_{n,i} : [n+1] \to [n]$, $s_{n,i}(j) = \begin{cases} j & , j \leq i \\ j-1 & , j > i \end{cases}$, they subject to **simplicial identities**:

$$d_j \circ d_i = d_i \circ d_{j-1}, i < j \Leftrightarrow i \leq j-1$$

$$s_j \circ s_i = s_i \circ s_{j+1}, i \leq j \Leftrightarrow i < j+1$$

$$s_j \circ d_i = \begin{cases} d_{i-1} \circ s_j & , j \leq i-2 \Leftrightarrow j < i-1 \\ 1 & , j = i, i-1 \\ d_i \circ s_{j-1} & , j > i \Leftrightarrow j-1 \geq i \end{cases}$$

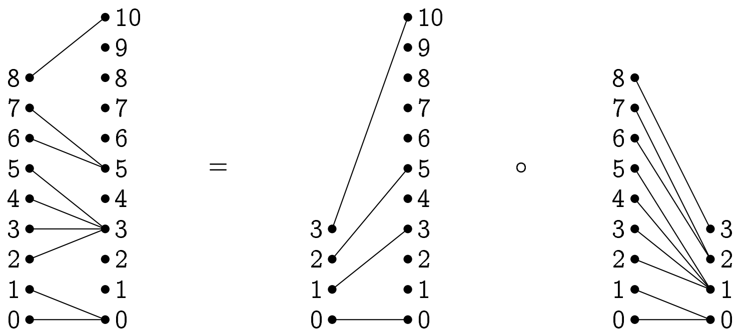
I find it easier to just think about $d_i, s_i : [\infty] \to [\infty]$

The **semisimple simplex category** is when discarding degeneracy maps, i.e. morphisms are strictly order preserving. The **augmented simplex category** is $\mathit{Simp} \cup \varnothing$. The **unordered simplex category** with the same objects as Simp and all functions as morphisms. The **unordered semisimplex category** with the same objects as Simp and all injections as morphisms

Unique decomposition of morphisms in simplex category

Lemma 0.3.2. Thanks to simplicial identities. Any morphism can be uniquely decomposed into a surjection compose with an injection. Any injection can be uniquely decomposed into composition of face maps with index strictly increasing. Any surjection can be uniquely decomposed into composition of degeneracy maps with index nondecreasing

Proof. For example



The right hand side can be written as $d_9 d_8 d_7 d_6 d_4 d_2 d_1 \circ s_2 s_2 s_1 s_1 s_1 s_0$ ☐

Definition 0.3.3. A **simplicial object** in $\mathscr{C}$ is a functor $\mathit{Simp}^{op} \to \mathscr{C}$, and a cosimplicial object is a functor $\mathit{Simp} \to \mathscr{C}$ If $\mathscr{C}$ is the category of sets, then the simplicial object is called a **simplicial set** $X : \mathit{Simp}^{op} \to \mathit{Set}$, $X([n]) = X_n$ is a family of sets, the face map $X(d_{n,i})$ sends elements of X_n to its i-th face

Similarly, we have semisimplicial object, augmented simplicial object, unordered simplicial object and unordered semisimplicial object

Example 0.3.4. The standard simplices $\{\Delta^n\}$ in Definition 0.1.1 with face and degeneracy maps is a cosimplicial object Δ in the category of topological spaces

$$\begin{tikzcd} {[n]} \arrow[r, mapsto] \arrow[d, "d_i"'] & \Delta^n \arrow[d, "d_i"] & {[n]} \arrow[r, mapsto] & \Delta^n \\ {[n+1]} \arrow[r, mapsto] & \Delta^{n+1} & {[n+1]} \arrow[r, mapsto] \arrow[u, "s_i"] & \Delta^{n+1} \arrow[u, "s_i"'] \end{tikzcd}$$

This functor is faithful and injective on objects, hence we may also just think of standard simplices as the simplex category

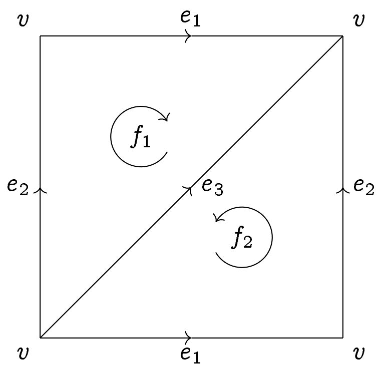
Due to Lemma 0.3.2, any morphism $\Delta^n \to \Delta^m$ can be uniquely written as an ordered degeneration compose with an ordered inclusion $\Delta^n \twoheadrightarrow \Delta^k \hookrightarrow \Delta^m$

Definition 0.3.5. A **Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \to \overline{e_\alpha^n}$ to its i-th face is such that $\Phi_\beta^{n-1} = \Phi_\alpha^n \circ d_{n,i}$ for some Φ_β^{n-1}

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, hook, "d_{n,i}"] \arrow[d, "\Phi_\beta^{n-1}"'] & \Delta^n \arrow[d, "\Phi_\alpha^n"] \\
\overline{e_\beta^{n-1}} \arrow[r, hook] & \overline{e_\alpha^n}
\end{tikzcd}$$

Remark 0.3.6. A Δ complex X is also called semisimplicial complex because it can be regarded as a semisimple set $X : Simp \to Set$, with $X([n]) = X_n$ being all the n faces, $X(d_{n,i}) : X_n \to X_{n-1}$ being face maps that map each cell to its i-th face

Example 0.3.7. Consider a Δ complex structure on torus



Definition 0.3.8. An **unordered Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \to \overline{e_\alpha^n}$ to any face is such that $\Phi_\beta^{n-1} = \Phi_\alpha^n \circ i$ for some Φ_β^{n-1}, i is an inclusion to that face regardless of order

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, hook, "i"] \arrow[d, "\Phi_\beta^{n-1}"'] & \Delta^n \arrow[d, "\Phi_\alpha^n"] \\
\overline{e_\beta^{n-1}} \arrow[r, hook] & \overline{e_\alpha^n}
\end{tikzcd}$$

Remark 0.3.9. An unordered Δ complex X can be regarded as an unordered semisimple set $X : Simp \to Set$, with $X([n]) = X_n$ being all the n faces, $X(i) : X_n \to X_{n-1}$ being face maps that map each cell to the corresponding face

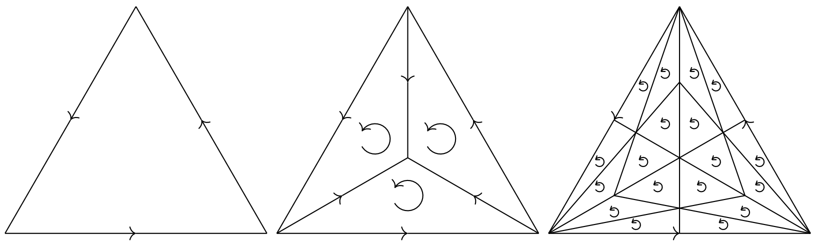
Definition 0.3.10. A regular unordered Δ complex is called a **multicomplex**, prefix multi- means different simplices can have the same faces

A regular unordered Δ complex in which each simplex is uniquely determined by its faces is called a simplicial complex

Definition 0.3.11. A **simplicial map** $f : K \to L$ is a map such that maps vertices of a simplex of K to the vertices of a simplex of L and linear on each simplex, two simplicial maps f, g are **contiguous** if for any simplex s in K, $f(s), g(s)$ are faces of the same simplex, in particular, f, g are homotopic, just consider $(1 - t)f + tg$

Lemma 0.3.12. Any unordered Δ complex can be subdivided once to become a Δ complex, and any Δ complex can be subdivided to be a simplicial complex, therefore, every unordered Δ complex is homeomorphic to a Δ complex and is homeomorphic to a simplicial complex

Example 0.3.13. The one on the left with three edges identified is not a Δ complex, but an unordered Δ complex, the one in the middle is a Δ complex, but not a simplicial complex, the one on the right is a simplicial complex



Definition 0.3.14. A **singular Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi^n_\alpha : \Delta^n \to \overline{e^n_\alpha}$ to its i-th face is such that $\Phi^k_\beta \circ q = \Phi^n_\alpha \circ d_{n,i}$ for some $\Phi^k_\beta, k \leq n-1$, $q : \Delta^{n-1} \to \Delta^k$ is a degeneration

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, two heads, "q"] \arrow[d, hook, "d_{n,i}"'] & \Delta^k \arrow[d, "\Phi^k_\beta"] \\ \Delta^n \arrow[r, "\Phi^n_\alpha"] & X \end{tikzcd}$$

Remark 0.3.15. A singular Δ complex is defined quite like a CW complex, where the characteristic maps are simplicial, instead of cellular
The vertices of a singular Δ complex can be given a partial order which is a total order on each simplex, just start at any point and use Zorn's lemma, in fact, it can be totally ordered

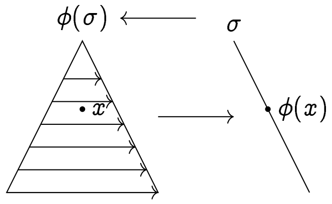
Definition 0.3.16. Suppose $X : Simp \to Set$ is a simplicial set, we can use this combinatorial information to construct its **geometric realization** $|X|$ with $X([n]) = X_n$ represents its n faces and morphisms $X_n \to X_{n-1}$ represents face maps and morphisms $X_n \to X_{n+1}$ represents degeneracy maps
The concrete construction is

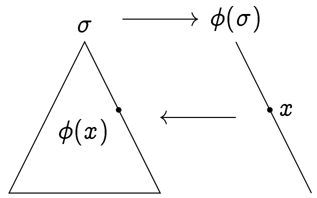
$$|X| := \frac{\bigsqcup_{n\geq 0} \Delta([n]) \times X([n])}{(\Delta\phi(x), \sigma) \sim (x, X\phi(\sigma))} = \frac{\bigsqcup_{n\geq 0} \Delta^n \times X_n}{(\phi(x), \sigma) \sim (x, \phi(\sigma))}$$

Where X_n is given the discrete topology, $x \in \Delta^n$, $\sigma \in X_n$, ϕ a morphism ranging over $Simp$ which is the same as ranging over all face maps and degeneracy maps since every morphism can be decomposed uniquely as a degeneration compose with an inclusion, the difference being taking the transitive closure which is the definition of a quotient space
This basically means that give each n face an n simplex and if there is a face map or a degeneracy map, glue the corresponding simplices according to the map

Proposition 0.3.17. The geometric realization $|X|$ of a simplicial set X is a singular Δ complex, $|-|$ is a functor from the cateogory of simplicial sets to the category of singular Δ complexes, Moreover, if X is a semisimplicial set, then $|X|$ is a Δ complex

Proof. Let's first deal with the degeneration





□

Definition 0.3.18. Given a singular Δ complex(equivalently a simplicial set), its one skeleton with a partial order can be seen as a diagram, consider such a diagram in the category of spaces, we call this a complex of spaces

0.4 abstract simplicial complex

Definition 0.4.1. An **abstract simplicial complex** is $K \subseteq \mathscr{P}(S) \setminus \varnothing$ such that $X \in K \Rightarrow \mathscr{P}(X) \setminus \varnothing \subseteq K$. Finite elements of K are called **faces**. The **dimension** of a face X is $\dim X = |X| - 1$. The d skeleton K^d is the union of faces of dimension no more than d. $\dim K = \sup_{X \in K} \dim X$. K^0 are **vertices**. Maximal elements are **facets**. K is **pure** if all facets have dimension $\dim K$. A **simplex** is a subcomplex which contains all its nonempty subsets, for $X \in K$, $\overline{X}$ is the corresponding simplicial complex

Definition 0.4.2. The **closure** $\overline{L}$ of $L \subseteq K$ is smallest subcomplex of K containing L. The **star** of $Y \in K$ is $\operatorname{st} L = \{X | X \cap Y \neq 0\}$, the star of $L \subseteq K$ is $\operatorname{st} L = \bigcup_{Y \in L} \operatorname{st} Y$. The **link** of a face $Y \in K$ is or $\operatorname{lk} Y = \{X | X \cap Y = \varnothing, X \cup Y \in X\}$. Equivalently, $\operatorname{lk} Y = \overline{\operatorname{st} Y} - \operatorname{st} \overline{Y}$

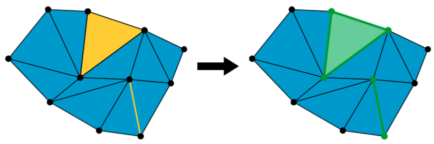


Figure 0.4.1: Two simplices and their closure

Closure of a complex

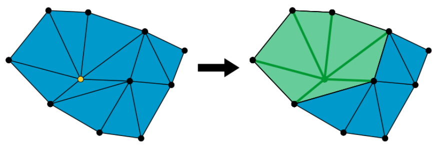


Figure 0.4.2: A vertex and its star

Star of a complex

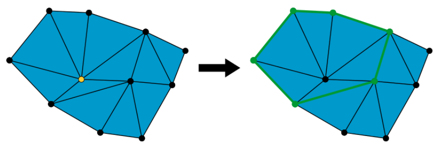


Figure 0.4.3: A vertex and its link

Link of a complex

Note. $\operatorname{lk} \varnothing = K$

Definition 0.4.3. If K, L has disjoint sets of vertices, then $K * L = \{X \sqcup Y | X \in K, Y \in L\}$

Definition 0.4.4. $L \subseteq K$, the **deletion** $K \setminus L$ consists of those sets which don't contain sets in L as subsets. The stellar subdivision of $X \in K$ is by introducing a new vertex x, and form $K \setminus X \cup (\overline{x} * \partial \overline{X} * \operatorname{lk} X)$

Definition 0.4.5. A simplicial map $f : K \to L$ is such that $f(K^d) \subseteq L^d$

0.5 CW approximation

CW approximation

Theorem 0.5.1 (CW approximation)**.** For any topological space X, there is a CW complex Z and a weak homotopy equivalence $f : Z \to X$, this is called a CW approximation

Whitehead's theorem

Theorem 0.5.2 (Whitehead's theorem)**.** Suppose $f : X \to Y$ is weak homotopy equivalence between CW complexes, then it is a homotopy equivalence

0.6 Triangulation

Definition 0.6.1. A **pseudomanifold** M is a pure triangulated space of dimension n such that M is not branching, i.e. any two n simplices have precisely one $(n-1)$ common face, M is strongly connected, i.e. any two n simplices can be linked with a sequence of simplices having common $(n-1)$ face pairwise

Note. The dual graph Γ of M is connected and n-regular

Example 0.6.2. The 0 dimensional pseudomanifold is the disjoint union of two points, since the empty set has to be the common face two point. The dual graph is a two points joined by an edge, this example is weird
A 1 dimensional pseudomanifold is a infinite chain or a loop, its dual graph is the same

Definition 0.6.3. D is a nonmaximal simplex, then $\operatorname{lk} D$ is a $n-|D|$ pure dimensional simplicial complex. A pseudomanifold such that $\operatorname{lk} D$ is also pseudomanifolds for any nonmaximal simplex is an **abstract polytope**