Definition 0.1. G is a Lie group, $K \leq G$ is a closed subgroup, $X = G/K$ is then a homogeneous space with transitive left G-action, $\Gamma \leq G$ is a discrete subgroup. The so called *automorphic functions* are $\mathbb{C}$-valued functions f on X such that

$$f(\gamma \cdot x) = f(x), \quad \forall x \in X, \gamma \in \Gamma \tag{0.1}$$

Loosely speaking, *automorphic forms* (for Γ) on X are automorphic functions that are also eigenfunctions for invariant differential operators on X (+ some technical growth conditions when necessary)

Question 0.2. How to decompose automorphic functions into sums (or integrals) of automorphic forms

Example 0.3. $\Gamma = \mathbb{Z}$, $X = G = \mathbb{R}$, automorphic functions are functions on $\mathbb{R}/\mathbb{Z} = \mathbb{T}$, automorphic forms are $e^{2\pi inx}$, $n \in \mathbb{Z}$. Fourier analysis tells us $L^2(\mathbb{R}/\mathbb{Z}) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}e^{2\pi inx}$

Example 0.4. $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup, $G/K = \mathcal{H} = \{\operatorname{Im} z > 0\}$ is the Poincaré upper half plane. G-invariant differential operators on $\mathcal{H}$ are polynomials with constant coefficients of the hyperbolic Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, Examples of automorphic forms in this setting: Maass forms. Γ are sometimes called "modular groups", the corresponding automorphic forms on $\mathcal{H}$ are called *modular forms*

Note. $\mathcal{H}$ has the structure of a complex manifold, it is natural to look for holomorphic automorphic forms

Example 0.5.

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

Where $q = e^{2\pi iz}$, $z \in \mathcal{H}$, is invariant under $\mathrm{SL}_2(\mathbb{Z})$, hence a modular form

Definition 0.6. G induces a right action on $\mathbb{C}(X)$ by $(f \cdot g)(x) = f(gx)$, (0.1) becomes $f \cdot \gamma = f$, $\forall \gamma \in \Gamma$. More generally, we can allow a nontrivial *automorphy factor* $(f \cdot_c g) = c_g(x)f(gx)$, $\forall g \in G$, here $c_g : X \to \mathbb{C}^\times$

Exercise 0.7. For the action to be well-defined, the family of functions c_g must satisfy $c_{g_1g_2}(x) = c_{g_2}(x)c_{g_1}(g_2x)$, so called cocycle condition, $\forall g_1, g_2 \in G$, $x \in X$

Theorem 0.8.