0.1 Affine Varieties

Definition 0.1.1. $V \subseteq \mathbb{A}^n$ is an algebraic set, $f \in k[V]$

$$D(f) = \{(x_1, \cdots, x_n) \in V | f(x_1, \cdots, x_n) \neq 0\} = V(f)^c$$

form a basis for the Zariski topology on V
$D(f)$ can also be thought of as an algebraic set

$$\{(x_1, \cdots, x_n, z) | zf(x_1, \cdots, x_n) = 0\}$$

The coordinate ring can be written as $k[V][\frac{1}{f}] = k[V]_f$, where z is just replaced by $\frac{1}{f}$

Theorem 0.1.2. $\sqrt{I} = \bigcap_{P \supseteq I \text{ prime}} P$

Hilbert Nullstellensatz weak form

Theorem 0.1.3 (Hilbert Nullstellensatz weak form)**.** k is algebraically closed, $m < k[x_1, \cdots, x_n]$ is a maximal ideal, then $k[x]/m \cong k$

Theorem 0.1.4 (Hilbert Nullstellensatz strong form)**.** k is algebraically closed, $I(V(J)) = \sqrt{J}$

Proof. Since $\sqrt{J} = \bigcap_{P \supseteq J \text{ prime}} P$, suppose $f \notin P$ for some $P \supseteq J$, consider $\varphi : k[x] \to k[x]/P \to A_{\overline{f}} \to A_{\overline{f}}/m$ which is a field, hence $\ker \varphi$ is a maximal ideal, by Theorem 0.1.3, $B/m \cong k[x]/\ker \varphi \cong k$, then $(\varphi(x_1), \cdots, \varphi(x_n)) \in V(P) \subseteq V(J)$ but $f(\varphi(x_1), \cdots, \varphi(x_n)) = \varphi(f) \neq 0 \Rightarrow f \notin I(V(J))$ □

Proposition 0.1.5. Morphism $V \xrightarrow{\varphi} W$ induce a ring homomorphism $k[W] \xrightarrow{\varphi^*} k[V], f \mapsto f \circ \varphi$, and if $f(p) = q$, then $(\varphi^*)^{-1}(m_q) = m_p$, thus conversely, if $\alpha : k[W] \to k[V]$ is a ring homomorphism, then $\alpha^{-1} : Spmk[V] \to Spm[W]$ is a morphism which can be identified with $\varphi : V \to W$, and $\varphi^* = \alpha$

Proposition 0.1.6. A finite morphism $V \xrightarrow{\varphi} W$ between affine varieties is quasifinite

Proof. $\varphi(p) = q \Leftrightarrow (\varphi^*)^{-1}(m_p) = q$, $m_p \supseteq \varphi^*(\varphi^*)^{-1}(m_p) = \varphi^*(m_q)$

$$\varphi^{-1}(q) \leftrightarrow \left\{ \text{maximal ideals of } B = \frac{k[W]}{\langle \varphi^*(m_q) \rangle} \right\}$$

Since $k[W]$ is a finite $k[V]$ algerba, so B is finite dimensional over $\dfrac{k[V]}{m_p} \cong k$ By Chinese Remainder theorem **??**, $B \to B/m_1 \times \cdots \times B/m_s$ is surjective, $\dim B \geq s$, since $\dim B < \infty$, hence $s < \infty$, thus B has only finitely many maximal ideals □

W->V dominant => k[V]->k[W] injective

Proposition 0.1.7. $W \xrightarrow{\varphi} V$ is dominant iff $k[V] \xrightarrow{\varphi} k[W]$ is injective

Proof. $f \in \ker \varphi^* \Leftrightarrow f \circ \varphi = 0$, $\mathrm{im}\varphi$ dense $\Rightarrow f = 0$. Conversely, $\overline{\mathrm{im}\varphi} \subsetneq V \Rightarrow 0 \neq f \in I(\overline{\mathrm{im}\varphi})$ □

Proposition 0.1.8. If $W \xrightarrow{\varphi} V$ is dominant and finite, then φ is surjective

Proof. By Proposition 0.1.7, $k[W]$ is integral over $k[V]$, by Theorem **??**, for any $m_q < k[V]$, there exists maximal ideal $n < k[W]$ such that $n \cap k[V] = m_q$ □

Corollary 0.1.9. V is an algebraic set, $\dim V = \dim k[V]$. If V is irreducible, then $\dim V = \mathrm{trdeg}\, k(V)$

Example 0.1.10. $\dim \mathbb{A}^n = \dim k[x_1, \cdots, k_n] = \mathrm{trdeg}(k(x_1, \cdots, x_n)/k) = n$

Definition 0.1.11. V is an algebraic set, a **regular function** on $U \subseteq V$ is $\dfrac{f}{g}$, $f, g \in k[V]$ such that g doesn't vanish on U, i.e. a rational function that is regular on U

Noether's normalization lemma

Lemma 0.1.12 (Noether's normalization lemma)**.** Every affine k scheme is finite over some affine n space

0.2 Varieties

Definition 0.2.1. A *prevariety* is a locally ringed space $(X, \mathcal{O})$ such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine variety $(V, \mathcal{O}_{\operatorname{Spm} V})$

Definition 0.2.2. A morphism $W \xrightarrow{\varphi} V$ is *dominant* if $\varphi(W)$ is dense

Definition 0.2.3. A morphism $W \xrightarrow{\varphi} V$ is *quasifinite* if $\varphi^{-1}(p)$ is finite for any $p \in V$

Definition 0.2.4. A morphism $W \xrightarrow{\varphi} V$ is *finite* if $k[W]$ is finite $k[V]$ algebra

Proposition 0.2.5. A finite morphism is quasifinite

Proposition 0.2.6. A variety is an integral scheme X over k such that $X \to \operatorname{Spec} k$ is separated and of finite type

Definition 0.2.7. The *canonical bundle* of an algebraic variety X of dimension n is $K = \bigwedge^n \Omega$, Ω is the cotangent bundle

Definition 0.2.8. The *Picard group* is $H^1(X, \mathcal{O}^*)$

0.3 Blowing up

Definition 0.3.1. The blow up of the origin in $\mathbb{A}^n$ is

$$Bl_0\mathbb{A}^n = \{(x_1, \cdots, x_n) \times [y_1, \cdots, y_n] \in \mathbb{A}^n \times \mathbb{P}^{n-1} | x_i y_j = x_j y_i\}$$

Let $\varphi : Bl_0\mathbb{A}^n \to \mathbb{A}^n$ be the projection to the first factor, then $Bl_0\mathbb{A}^n$ is covered by n open affine charts $U_i = \{y_i \neq 0\} \cap Bl_0\mathbb{A}^n$, where $k[U_i] = k\left[x_i, \frac{y_1}{y_i}, \cdots, \frac{y_n}{y_i}\right]$, so $U_i \cong \mathbb{A}^n$, with $\varphi|_{U_i} : U_i \to \mathbb{A}^n$ given by
$k[x_1, \cdots, x_n] \xrightarrow{\alpha_i} k\left[x_i, \frac{y_1}{y_i}, \cdots, \frac{y_n}{y_i}\right], x_j \mapsto x_i\frac{y_j}{y_i}$
$\forall i$, $\varphi|_{U_i}|_{D(x_i)} : D(x_i) \to D(x_i)$ is an isomorphism, $\varphi|_{U_i}^{-1}(0) = V(\alpha_i(x_1, \cdots, x_n)) = V(x_i) \cong Spmk\left[\frac{y_1}{y_i}, \cdots, \frac{y_n}{y_i}\right] \cong \mathbb{A}^{n-1}$, and these $V(x_i)$'s glue to give $\varphi^{-1}(0) \cong \mathbb{P}^{n-1}$ which called the exceptional divisor

Proposition 0.3.2. There is a bijection between points on $\varphi^{-1}(0)$ and the line in $\mathbb{A}^n$ passing 0

Proof. Let $L = \bigcap_{i=1}^{n}\{x_i = a_i t\}$, not all a_i's are zero be a line, then $\varphi|_{U_i}^{-1}(L\backslash 0) = \{x_i = a_i t, t \neq 0, a_i y_j = a_j y_i\}$, and $\overline{\varphi|_{U_i}^{-1}(L \setminus 0)} = \overline{\{x_i = a_i t, a_i y_j = a_j y_i\}}$, so this line corresponds to $[a_1, \cdots, a_n] \in \mathbb{P}^{n-1} \cong \varphi^{-1}(0)$, thus if $L' \neq L$, $\overline{\varphi|_{U_i}^{-1}(L \setminus 0)} \cap \overline{\varphi|_{U_i}^{-1}(L' \setminus 0)} = \emptyset$
$Bl_0\mathbb{A}^n$ is nonsingular since it is covered by affine spaces $\mathbb{A}^n$, $Bl_0\mathbb{A}^n$ is irreducible since $Bl_0\mathbb{A}^n \setminus \varphi^{-1}(0) \cong \mathbb{A} \setminus 0$ is irreducible, and each point of $\varphi^{-1}(0)$ is in the closure of some line L in $Bl_0\mathbb{A}^n \setminus \varphi^{-1}(0)$, so $Bl_0\mathbb{A}^n \setminus \varphi^{-1}(0)$ is dense in $Bl_0\mathbb{A}^n$ ☐

Definition 0.3.3. If $V \subseteq \mathbb{A}^n$ is a closed subvariety containing 0, then the blow up of the origin $Bl_0V := \overline{\varphi^{-1}(V \setminus 0)}$, from this, we get an birational isomorphism $\varphi : Bl_0V \to V$ which is an isomorphism away from 0