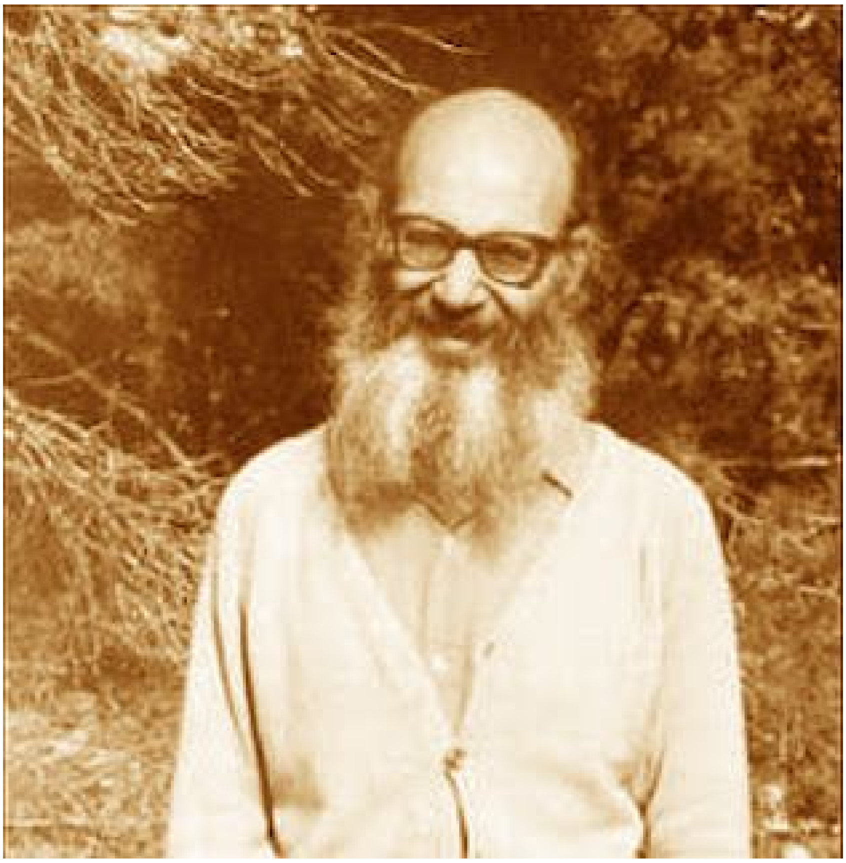
MATH808K - Algebraic K-Theory



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1 Projective modules

K-theory is the study of categories of vector bundles or similar objects. A vector bundle is a parametrized family of vector spaces: $p : E \to X$ is a vector bundle, X is a topological space. For each $x \in X$, $E_x = p^{-1}(x)$ is a vector space depending "continuously" on $x \in X$. K-theory deals with parametrized linear algebra. Often we don't deal directly with geometry, but with rings

Swan-Serre Theorem

Theorem 1.1 (Swan-Serre)**.** There is an equivalence of cateogories between vector bundles over X and finitely generated projective modules over an associated ring of functions on X. Here are 3 categories in which this works

1. X compact Hausdorff, $R = C(X)$ the continuous function on X
2. X affine variety over a field k, $R = \mathcal{O}(X)$ is the ring of regular functions. If X is projective, it is more complicated
3. X stein manifold(holomorphic submanifold of $\mathbb{C}^n$), $R = \mathcal{O}(X)$ holomorphic functions on X, category of vector bundles is holomorphic category

Review of projective modules
In this course, a ring almost always have units but not necessarily commutative

Definition 1.2. R is a ring with unit. A free R-module is one isomorphic to R^I, I is some index set. A finitely generated free R-module is one isomorphic to R^n. R is said to have the invariant basis property if $R^n \cong R^m \Rightarrow n = m$. Note that this is always true if R is commutative(reason: true for fields, and if R is commutative, $k = R/m$ is field, $R \otimes k$ is a vector space over k)

Example 1.3 (Counter-example)**.** k is a field, $R = \mathrm{End}_k(k^\infty)$ doesn't have the invariant basis property, as $R \cong R^2$. Idea: $k^\infty \oplus k^\infty \cong k^\infty$

Theorem 1.4. R is ring, P is an R-module. The following are equivalent

1. P is a direct summand in a free R-module, i.e. $F \cong P \oplus Q$ for some free R-module F
2. $\mathrm{Hom}_R(P, -)$ is an exact functor
3. P has the property that if $\phi : M \to N$ is a surjective R-module map and we are given $\alpha : P \to N$, there exists $\beta : P \to M$ such that $\alpha = \phi \circ \beta$

An R-module with these 3 equivalent conditions is called *projective*

Proof. 2⇒3 is due to the fact that $\mathrm{Hom}_R(P, -)$ can only fail to be exact on the right, i.e. given $0 \to M' \to M \to N \to 0$ ☐

The first invariant of K-theory is $K_0(R)$, then Grothendieck group of finitely generated projective modules over R. If P and Q are finitely generated projecive R modules, we can "add" by taking direct sum, but not subtract. $K_0(R)$ is the group with generators $[P]$, P finitely generated R-module with relations $[P] = [Q]$ if $P \cong Q$. We build in the relation $[P] + [Q] = [P \oplus Q]$. Note that every element of $K_0(R)$ is of the form $[P] - [Q]$ for some P, Q, $[P] - [Q] = [P'] - [Q'] \iff P \oplus Q' \oplus S \cong R' \oplus Q \oplus S$ for some S. In general, "addition" of projective modules does not have the cancellation property, just as addition of vector bundles does not

Example 1.5. TS^2 is not free since not trivial, Euler characteristic

Fact 1.6 (reference: Hatcher's K-theory book)**.**
1. Any vector bundle (by definition) is locally trivial, then rank $X \to \mathbb{N}$ is continuous, hence locally constant
2. Any vector bundle can be equipped with a metric, i.e. a family of inner products varying continuously with $x \in X$. (Construction: use local triviality and patch with partition of unity)

3. Any vector bundle can be embedded into a trivial vector bundle $X \times \mathbb{F}^n$ for n large enough. (Use local triviality and partition of unity)

4. 2+3 $\Rightarrow$ Any vector bundle is a direct summand in a trivial bundle

proof of Theorem 1.1. Send $p : E \to X$ to the set of sections $\Gamma(E)$, then $\Gamma(E)$ is a $\mathcal{O}(X)$-module, from above, $\Gamma(E)$ is finitely generated and projective. The rest is formal $\square$

Example 1.7. Observation: Any vector bundle over $S^n, n \geq 1$ is obtained by gluing("clutching"): two trivial vector bundles over the upper and lower hemispheres via a map $S^{n-1} \to \mathrm{GL}(k, \mathbb{F})$. This is because any vector bundle over a contractible space is trivial, so

$$Vect^k_{\mathbb{F}}(S^n) \cong [S^{n-1}, \mathrm{GL}(k, \mathbb{F})] \cong \pi_{n-1}(O(k), U(k), Sp(k)) for \mathbb{R}, \mathbb{C}, \mathbb{H}$$

$X = S^2$, $\mathbb{F} = \mathbb{R}$, what is the classification of rank n vector bundles over X? We see that rank k vector bundles over S^2 are classified by $\pi_1(O(k))$, since S^1 is connected, any map $S^1 \to O(k)$ lies in a single component, both isomorphic to $\mathsf{SO}(k)$, for $k \geq 3$, $\mathsf{SO}(k)$ is a simple Lie group and $\pi_1(\mathsf{SO}(k)) \cong \mathbb{Z}/2$(except $SO(4)$ is only semi-simple with two cover?)

Implication for K-theory: The stable isomorphic classes of vector bundles E over S^2 is characterized by

$$\begin{cases} \mathsf{rank} \in \mathbb{N} \\ Stiefel-Whitneynumber = \langle w_2(E), [S^2] \rangle \in \mathbb{Z}/2 \end{cases}$$

Similar analysis holds for S^n

$$\begin{cases} \mathsf{rank} \in \mathbb{N} \\ \text{something in } \pi_{n-1}(\mathsf{SO}) \end{cases}$$

Here $\pi_{n-1}(\mathsf{SO}) = \pi_{n-1}(SO(\infty)) = \varinjlim_k \pi_{n-1}(\mathsf{SO}(k))$, $\pi_{n-1}(\mathsf{SO}(k))$ stablizes as $k \to \infty$

Theorem 1.8 (Bott periodicity theomem)**.**

$$\pi_{n-1}(\mathsf{SO}) = \begin{cases} \mathbb{Z}, & n \text{ is a multiple of } 4 \\ \mathbb{Z}/2, & n \equiv 1, 2 \bmod 8 \\ 0, & \text{otherwise} \end{cases}$$

Lessons form this example: stable classification is much easier than the unstable classification. A stably trivial bundle need not to be trivial. These lessons carry over to the purely algebraic setting of projective modules over a ring. To get a corresponding example with projective modules over a Noetherian commutative ring, take $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, $\mathsf{Spec}\, R$ is an "algebraic model" for S^2. Our non-trivial but stably trivial vector bundle can be constructed as $\{(x, y, z, u, v, w) | x^2 + y^2 + z^2 = 1, xu + yv + zw = 0\}$

2 Homotopy invariance

theorem1 - 1/29/2021

Theorem 2.1. The calssification of the otpological vector budnels over a compact Haudsorrfff spce X is homotopy invariant. In other words, if $f, g : X \to Y$ are maps of compact spaces and E is an $\mathbb{F}$ bundle over Y, then $f \simeq g \Rightarrow f^*(e) \cong g^*(E)$

Corollary 2.2. Every vector bundle over a contractible space is trivial

theorem2 - 1/29/2021

Theorem 2.3. A is a unital Banach algebra(For application, $A = C(X, M_n(\mathbb{F}))$). Let $IdemA$ be the set of idempotents in $A(x^2 = x)$. If $e, f \in IdemA$ lies in the same component, then they are conjugate under $\mathrm{GL}_1(A)$

Proof. It's enough to show that if $e, f \in IdemA$ are sufficiently close in norm, then e, f are conjugate. Suppose e, f are close and let $a = e + f - 1 \in A$, then a^2 is close to $(2e - 1)^2 = 1$, so a^2 is invertible, thus a is invertible. $ae = fa$ since $ae = (e+f-1)e = fe$, $fa = f(e+f-1) = fe$, thus $aea^{-1} = f$ □

proof Theorem 2.1. Embed E as a direct summand in a tribial bundle of rank n, then $f^*(E), g^*(E)$ are obtained by projecting down from $X \times \mathbb{F}^n$ via homotopic idempotents in $C(X, M_n(\mathbb{F}))$ □

Projective modules over a local ring

Definition 2.4. R is a ring with unit, R is called local if the non-invertible elements in R constitute a 2-sided ideal m. Obviously m is the unique maximal 2-sided ideal

Caution: In the non-commutative case, having a unique maximal 2-sided ideal is not good enough! Since $M_n(\mathbb{F})$ has this property for $\mathbb{F}$ a field, and this ring is not local
Note: If R is local and $x \in R$ has a left inverse, then it also has a right inverse. Suppose $ax = 1$, then $ax \notin m$, so $x \notin m$, so x is invertible

Fact 2.5.

1. If R is local with maximal ideal m and $x \in m$, then $1 + x$ is invertible. If not, then $1 + x \in m \Rightarrow 1 \in m$ which is a contradiction

2. (Nakayama's lemma) R is a local ring with maximal ideal m, and let m be a finitely generated R-module, if $mM = M$, then $M = 0$ Proof: Let $M = Rx1 + \cdots Rx_n$ such that n is minimal, then since $mM = M$, $x_n = r1x_1 + \cdots + r_n x_n$ with $r_j \in m$, now $(1 - r_n)x_n = r_1 x_1 + \cdots + r_{n-1}x_{n-1}$, but $1 - r_n$ is invertible, we can divide to get $x_n = \cdots$, contradicting the minimality unless $n = 0$, i.e. $M = 0$

Theorem 2.6. Let R be a local ring, M a finitely generated projective R-module, then M is free

Proof. $M \oplus N \cong R^n$. $m(M \oplus N) = m^n$, so $(R/m)M$ is a direct summand in $(R/m)^n$, but R/m is a division ring, so $(R/m)M \cong (R/m)^k$ for some $0 \leq k \leq n$, and $(R/m)N = (R/m)^{n-k}$. Let $\dot{x}_1, \cdots, \dot{x}_k$ be a free basis for $(R/m)M \cong M/nM$ and extend it to a free basis by adding $\dot{x}_{k+1}, \cdots, \dot{x}_n$ for $(R/m)N$, pull these back to $x_1, \cdots, x_k \in M$ and $x_{k+1}, \cdots, x_n \in N$. $M = Rx_1 \cdots Rx_k$ by Nakayama's lemma $x_1, \cdots, x_n$ is another generating set for R^n with n elements, writing the the matrix of x_i's and e_i's gives the linear independence □

Corollary 2.7. $K_0(R) = \mathbb{Z}$, with the class of a projective module given by its rank(this is only stable case, note that the theorem actually prove the non-stable case, which is more general)

3 Homeworks

References

[1] *The K-Book* - Charles Weibel

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