0.1 Singular cohomology

Definition 0.1.1 (Eilenberg-Steenrod axioms)**.** Top is the category of topological spaces, Ab is the category of abelian groups , $\mathcal{T}$ is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces (X, A) such that $A \subseteq X$, $\mathcal{T}_A$ is the fully faithful subcategory of $\mathcal{T}$ with objects (X, A), $R : \mathcal{T} \to Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor
Relative cohomology are contravariant functors $H^n : \mathcal{T} \to Ab$, then $H^n(-, A)$ define contravariant functors $\mathcal{T}_A \to Ab$, **absolute cohomology** are contravariant functors $H^n(-, \varnothing) : Top \to Ab$, **reduced cohomology** are $\widetilde{H}^n = H^n(-, *)$. $\partial^n : H^n \to H^{n+1}R$ are natural transformations

$$\begin{array}{ccc} H^n(X,A) & \xrightarrow{H_n(f)} & H^n(Y,B) \\ \downarrow{\scriptstyle \partial^n} & & \downarrow{\scriptstyle \partial^n} \\ H^{n+1}(A) & \xrightarrow{H_{n-1}(f)} & H^{n+1}(B) \end{array}$$

(H, δ) is a **cohomology theory** if it satisfies axioms
Homotopy invariance: $f \simeq g : (X, A) \to (Y, B)$, then $H^n(f) = H^n(g)$
Additivity: $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$, then $\bigoplus_\alpha H^n(X_\alpha, A_\alpha) \xrightarrow{\bigoplus_\alpha H^n(i_\alpha)} H^n(X, A)$ is an isomorphism
Exactness:

$$\cdots \xrightarrow{\partial^{n-1}} H^n(X,A) \xrightarrow{H^n(j)} H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial^n} \cdots$$

Exicision: $\bar{Z} \subseteq \overset{\circ}{U}$, then $H^n(X - Z, U - Z) \xrightarrow{H^n(i)} H^n(X, U)$ is an isomorphism
Dimension: $H^n(*) = 0, \forall n \neq 0$, $H^0(*)$ is the **coefficient group**
(H, δ) is an **extraordinary cohomology theory** without dimension axiom

Definition 0.1.2. Define singular n-cochains to be $C^n(X) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z})$, if R is a ring, then we can also define cohomology with R coefficients $C^n(X; R) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), R)$, here R can be abelian groups(group ring) or fields
We can also define simplicial, cellular cochains correspondingly

Remark 0.1.3. Note that $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$, $\mathrm{Hom}(C_n(X; R), \mathbb{Z}) = \mathrm{Hom}(C_n(X) \otimes R, \mathbb{Z}) \cong \mathrm{Hom}(C_n(X), \mathrm{Hom}(R, \mathbb{Z})) \ncong \mathrm{Hom}(C_n(X), R) = C^n(X; R)$

Definition 0.1.4. $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$ induce the coboundary map $\delta^n : C^n(X) \to C^{n+1}(X)$, we can define cocycles $Z^n(X) = \ker \delta^n$, coboundaries $B^n = \mathrm{im}\delta^{n-1}$ and cohomology $H^n(X) = Z^n(X)/B^n(X)$

Definition 0.1.5. θ as in Remark **??**, the cross product is composition $\times : C^*(X; R) \otimes C^*(Y; R) \xrightarrow{\theta^*} C^*(X \times Y; R \otimes R) \to C^*(X \times Y; R)$, here $R \otimes R \to R$ is the ring multiplication. $\delta(f \times g) = \delta f \times g + (-1)^{|f|} f \times \delta g$, $\times$ is well defined on cohomology since θ is unique up to natural chain equivalence. If R is commutative, then $f \times g = (-1)^{|f||g|} g \times f$
For $[f] \in H^p(X; R)$, $[g] \in H^q(Y; R)$, $[a] \in H_p(X)$, $[b] \in H_q(Y)$, then $([f] \times [g])([a] \times [b]) = f(a)g(b) \in R$

Lemma 0.1.6. If $a \in H^p(Y; R)$, then $1 \times a = p_Y^*(a) \in H^p(X \times Y; R)$

Proof. $C_*(X \times Y) \to C_*(X) \otimes C_*(Y) \to C_*(x_0) \otimes C_*(Y) \xrightarrow{\epsilon \otimes 1} \mathbb{Z} \otimes C_*(Y) \cong C_*(Y) \xrightarrow{a} R$ and $C_*(X \times Y) \xrightarrow{p_Y} C_*(Y) \xrightarrow{a} R$ are chain homotopic □

Definition 0.1.7. $\Delta : X \to X \times X$ is the diagonal, for $a \in H^p(X; R)$, $b \in H^q(X; R)$, the **cup product** is $a \smile b = \Delta^*(a \times b) \in H^{p+q}(X; R)$, $f^*(a \smile b) = f^*(a) \smile f^*(b)$, if R is commutative, $a \smile b = (-1)^{|a||b|} b \smile a$, $1 \smile a = \Delta^*(1 \times a) = \Delta^*(p_X^*(a)) = (p_X \Delta)^*(a) = 1^*(a) = a$

Proposition 0.1.8. Cross product and cup product determine each other, $a \smile b = \Delta^*(a \times b)$, $a \times b = p_X^*(a) \smile p_Y^*(b)$

Proof. $p_X^*(a) \smile p_Y^*(b) = \Delta^*(p_X^*(a) \times p_Y^*(b)) = \Delta^*(a \times 1 \times 1 \times b) = \Delta^*(1 \times 1 \times a \times b) = (1 \times 1) \smile (a \times b) = 1 \smile (a \times b) = a \times b$ $\square$

0.2 Cech cohomology

Definition 0.2.1. Given any open cover $\mathcal{U}$ of X, we can define a (abstract) simplicial complex, the nerve $N(\mathcal{U})$, with each U_α a vertex and an n-face if $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}} \neq \emptyset$, and we call $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ the carrier of this face, a cover is called a good cover if each $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ is contractible, in that case, $N(\mathcal{U})$ is homotopic to X

Definition 0.2.2. Suppose $\mathcal{V}$ is a refinement of $\mathcal{U}$, i.e. every V_β is contained in some U_α, refinement defines a preorder, then inclusion induce a simplicial map $N(\mathcal{V}) \to N(\mathcal{U})$, different choice of inclusions induce contiguous simplicial maps, thus this is well defined up to homotopy, we can define the direct limit $\varinjlim H^i(N(\mathcal{U}); G)$ to be the Čech cohomology group $H^i(X; G)$

0.3 Poincare duality