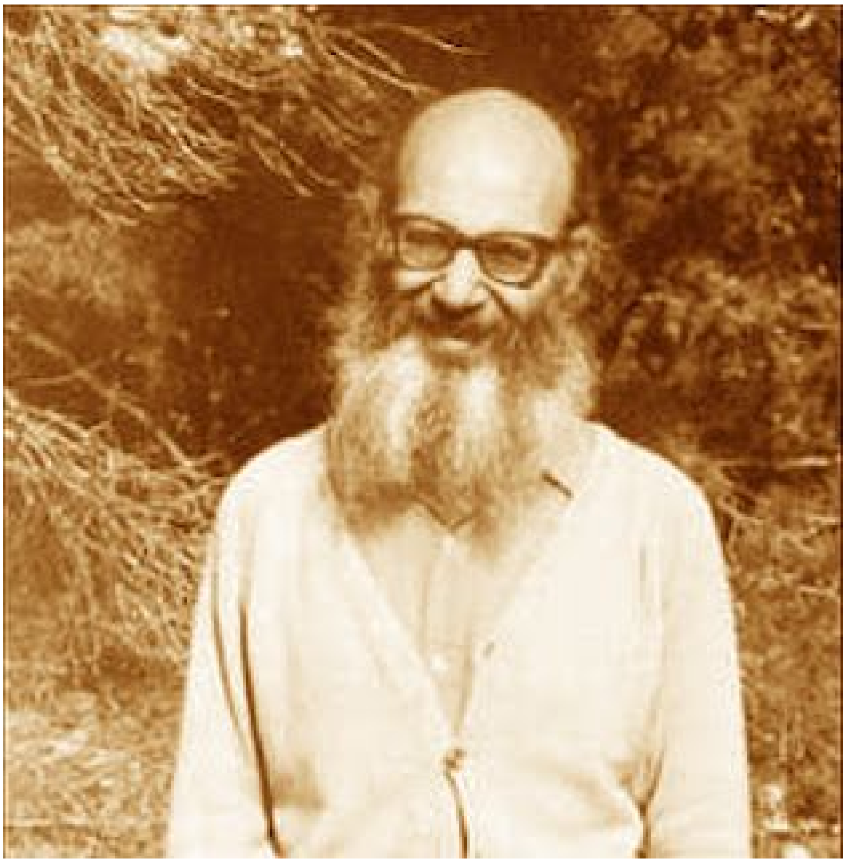
MATH808K - Algebraic K-Theory



Taught by Jonathan Rosenberg
Notes taken by Haoran Li
2021 Spring

Department of Mathematics
University of Maryland

Contents

1 Projective modules

K-theory is the study of categories of vector bundles or similar objects. A vector bundle is a parametrized family of vector spaces: $p : E \to X$ is a vector bundle, X is a topological space. For each $x \in X$, $E_x = p^{-1}(x)$ is a vector space depending "continuously" on $x \in X$. K-theory deals with parametrized linear algebra. Often we don't deal directly with geometry, but with rings

Swan-Serre Theorem

Theorem 1.1 (Swan-Serre)**.** There is an equivalence of cateogories between vector bundles over X and finitely generated projective modules over an associated ring of functions on X. Here are 3 categories in which this works

1. X compact Hausdorff, $R = C(X)$ the continuous function on X
2. X affine variety over a field k, $R = \mathcal{O}(X)$ is the ring of regular functions. If X is projective, it is more complicated
3. X stein manifold(holomorphic submanifold of $\mathbb{C}^n$), $R = \mathcal{O}(X)$ holomorphic functions on X, category of vector bundles is holomorphic category

Review of projective modules
In this course, a ring almost always have units but not necessarily commutative

Definition 1.2. R is a ring with unit. A free R-module is one isomorphic to R^I, I is some index set. A finitely generated free R-module is one isomorphic to R^n. R is said to have the invariant basis property if $R^n \cong R^m \Rightarrow n = m$. Note that this is always true if R is commutative(reason: true for fields, and if R is commutative, $k = R/m$ is field, $R \otimes k$ is a vector space over k)

Example 1.3 (Counter-example)**.** k is a field, $R = \mathtt{End}_k(k^\infty)$ doesn't have the invariant basis property, as $R \cong R^2$. Idea: $k^\infty \oplus k^\infty \cong k^\infty$

Solution. R corresponds to a matrix with rows and columns ranging in $\mathbb{N}$, and each column has all but finitely many non-zero elements, and we can decompose any matrix to the direct sum of two matrices with even columns or odd columns

$$\begin{bmatrix} \circ & \times & \circ & \times & \circ & \times & \cdots \\ \circ & \times & \circ & \times & \circ & \times & \cdots \\ \circ & \times & \circ & \times & \circ & \times & \cdots \\ \circ & \times & \circ & \times & \circ & \times & \cdots \\ \circ & \times & \circ & \times & \circ & \times & \cdots \\ \circ & \times & \circ & \times & \circ & \times & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

□

Theorem 1.4. R is ring, P is an R-module. The following are equivalent

1. P is a direct summand in a free R-module, i.e. $F \cong P \oplus Q$ for some free R-module F
2. $\mathtt{Hom}_R(P, -)$ is an exact functor
3. P has the property that if $\phi : M \to N$ is a surjective R-module map and we are given $\alpha : P \to N$, there exists $\beta : P \to M$ such that $\alpha = \phi \circ \beta$

An R-module with these 3 equivalent conditions is called *projective*

Proof. 2⇒3 is due to the fact that $\mathtt{Hom}_R(P, -)$ can only fail to be exact on the right, i.e. given $0 \to M' \to M \to N \to 0$ □

The first invariant of K-theory is $K_0(R)$, then Grothendieck group of finitely generated projective modules over R. If P and Q are finitely generated projecive R modules, we can "add" by taking direct sum, but not subtract. $K_0(R)$ is the group with generators $[P]$, P finitely generated R-module with relations $[P] = [Q]$ if $P \cong Q$. We build in the relation $[P] + [Q] = [P \oplus Q]$. Note that every element of $K_0(R)$ is of the form $[P] - [Q]$ for some P, Q, $[P] - [Q] = [P'] - [Q'] \iff P \oplus Q' \oplus S \cong R' \oplus Q \oplus S$ for some S. In general, "addition" of projective modules does not have the cancellation property, just as addition of vector bundles does not

Example 1.5. TS^2 is not free since $\chi(S^2) = 2 \neq 0$

Fact 1.6 (reference: Hatcher's K-theory book)**.**

1. Any vector bundle (by definition) is locally trivial, then rank $X \to \mathbb{N}$ is continuous, hence locally constant
2. Any vector bundle can be equipped with a metric, i.e. a family of inner products varying continuously with $x \in X$. (Construction: use local triviality and patch with partition of unity)
3. Any vector bundle can be embedded into a trivial vector bundle $X \times \mathbb{F}^n$ for n large enough. (Use local triviality and partition of unity)
4. 2+3 $\Rightarrow$ Any vector bundle is a direct summand in a trivial bundle

proof of Theorem 1.1. Send $p : E \to X$ to the set of sections $\Gamma(E)$, then $\Gamma(E)$ is a $\mathcal{O}(X)$-module, from above, $\Gamma(E)$ is finitely generated and projective. The rest is formal ☐

Example 1.7. Observation: Any vector bundle over $S^n, n \geq 1$ is obtained by gluing("clutching"): two trivial vector bundles over the upper and lower hemispheres via a map $S^{n-1} \to \mathrm{GL}(k, \mathbb{F})$. This is because any vector bundle over a contractible space is trivial, so

$$\mathrm{Vect}^k_{\mathbb{F}}(S^n) \cong [S^{n-1}, \mathrm{GL}(k, \mathbb{F})] \cong \begin{cases} \pi_{n-1}(O(k)) & \mathbb{F} = \mathbb{R} \\ \pi_{n-1}(U(k)) & \mathbb{F} = \mathbb{C} \\ \pi_{n-1}(Sp(k)) & \mathbb{F} = \mathbb{H} \end{cases}$$

$X = S^2$, $\mathbb{F} = \mathbb{R}$, what is the classification of rank n vector bundles over X? We see that rank k vector bundles over S^2 are classified by $\pi_1(O(k))$, since S^1 is connected, any map $S^1 \to O(k)$ lies in a single component, both isomorphic to $\mathrm{SO}(k)$, for $k \geq 3$, $\mathrm{SO}(k)$ is a simple Lie group and $\pi_1(\mathrm{SO}(k)) \cong \mathbb{Z}/2$ for $k \geq 3$(except $SO(4)$ is only semi-simple with two cover?)

Note. $SU(2)$ as quaternions acts on S^3, gives a double cover of $SO(3)$

Implication for K-theory: The stable isomorphic classes of vector bundles E over S^2 is characterized by

$$\begin{cases} \text{rank} \in \mathbb{N} \\ \text{Stiefel-Whitney number} = \langle w_2(E), [S^2] \rangle \in \mathbb{Z}/2 \end{cases}$$

Similar analysis holds for S^n

$$\begin{cases} \text{rank} \in \mathbb{N} \\ \text{something in } \pi_{n-1}(\mathrm{SO}) \end{cases}$$

Here $\pi_{n-1}(\mathrm{SO}) = \pi_{n-1}(SO(\infty)) = \varinjlim_k \pi_{n-1}(\mathrm{SO}(k))$, $\pi_{n-1}(\mathrm{SO}(k))$ stablizes as $k \to \infty$

Theorem 1.8 (Bott periodicity theomem)**.**

$$\pi_{n-1}(\mathrm{SO}) = \begin{cases} \mathbb{Z}, & n \text{ is a multiple of } 4 \\ \mathbb{Z}/2, & n \equiv 1, 2 \bmod 8 \\ 0, & \text{otherwise} \end{cases}$$

Lessons form this example: stable classification is much easier than the unstable classification. A stably trivial bundle need not to be trivial. These lessons carry over to the purely algebraic setting of projective modules over a ring. To get a corresponding example with projective modules over a Noetherian commutative ring, take $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, $\mathsf{Spec}\, R$ is an "algebraic model" for S^2. Our non-trivial but stably trivial vector bundle can be constructed as $\{(x, y, z, u, v, w) | x^2 + y^2 + z^2 = 1, xu + yv + zw = 0\}$

2 Homotopy invariance

theorem1 - 1/29/2021

Theorem 2.1. The calssification of the otpological vector budnels over a compact Haudsorrfff spce X is homotopy invariant. In other words, if $f, g : X \to Y$ are maps of compact spaces and E is an $\mathbb{F}$ bundle over Y, then $f \simeq g \Rightarrow f^*(E) \cong g^*(E)$

Corollary 2.2. Every vector bundle over a contractible space is trivial

theorem2 - 1/29/2021

Theorem 2.3. A is a unital Banach algebra(For application, $A = C(X, M_n(\mathbb{F}))$). Let $\mathtt{Idem}\, A$ be the set of idempotents in $A(x^2 = x)$. If $e, f \in \mathtt{Idem}\, A$ lies in the same component, then they are conjugate under $\mathrm{GL}_1(A)$

Proof. It's enough to show that if $e, f \in \mathtt{Idem}\, A$ are sufficiently close in norm, then e, f are conjugate. Suppose e, f are close and let $a = e + f - 1 \in A$, then a^2 is close to $(2e-1)^2 = 1$, so a^2 is invertible, thus a is invertible. $ae = fa$ since $ae = (e+f-1)e = fe$, $fa = f(e+f-1) = fe$, thus $aea^{-1} = f$ ☐

Proof Theorem 2.1. Embed E as a direct summand in a trivial bundle of rank n, then $f^*(E), g^*(E)$ are obtained by projecting down from $X \times \mathbb{F}^n$ via homotopic idempotents in $C(X, M_n(\mathbb{F}))$ ☐

Projective modules over a local ring

Definition 2.4. R is a ring with unit, R is called local if the non-invertible elements in R constitute a 2-sided ideal m. Obviously m is the unique maximal 2-sided ideal

Caution: In the non-commutative case, having a unique maximal 2-sided ideal is not good enough! Since $M_n(\mathbb{F})$ has this property for $\mathbb{F}$ a field, and this ring is not local
Note: If R is local and $x \in R$ has a left inverse, then it also has a right inverse. Suppose $ax = 1$, then $ax \notin m$, so $x \notin m$, so x is invertible

Fact 2.5.

1. If R is local with maximal ideal m and $x \in m$, then $1 + x$ is invertible. If not, then $1 + x \in m \Rightarrow 1 \in m$ which is a contradiction

2. (Nakayama's lemma) R is a local ring with maximal ideal m, and let m be a finitely generated R-module, if $mM = M$, then $M = 0$ Proof: Let $M = Rx1 + \cdots Rx_n$ such that n is minimal, then since $mM = M$, $x_n = r1x_1 + \cdots + r_n x_n$ with $r_j \in m$, now $(1 - r_n)x_n = r_1 x_1 + \cdots + r_{n-1}x_{n-1}$, but $1 - r_n$ is invertible, we can divide to get $x_n = \cdots$, contradicting the minimality unless $n = 0$, i.e. $M = 0$

Theorem 2.6. Let R be a local ring, M a finitely generated projective R-module, then M is free

Proof. $M \oplus N \cong R^n$. $m(M \oplus N) = m^n$, so $(R/m)M$ is a direct summand in $(R/m)^n$, but R/m is a division ring, so $(R/m)M \cong (R/m)^k$ for some $0 \le k \le n$, and $(R/m)N = (R/m)^{n-k}$. Let $\dot{x}_1, \cdots, \dot{x}_k$ be a free basis for $(R/m)M \cong M/nM$ and extend it to a free basis by adding $\dot{x}_{k+1}, \cdots, \dot{x}_n$ for $(R/m)N$, pull these back to $x_1, \cdots, x_k \in M$ and $x_{k+1}, \cdots, x_n \in N$. $M = Rx_1 \cdots Rx_k$ by Nakayama's lemma $x_1, \cdots, x_n$ is another generating set for R^n with n elements, writing the the matrix of x_i's and e_i's gives the linear independence ☐

Corollary 2.7. For R is a local ring, $K_0(R) = \mathbb{Z}$, with the class of a projective module given by its rank(this is only stable case, note that the theorem actually prove the non-stable case, which is more general)

theorem 1,2021-2-1

Theorem 2.8 (Lemma 2.4, [1])**.** R is a commutative ring, M is a finitely generated R-module. The following are equivalent

1. M is projective

2. M is locally free, i.e. for each prime ideal p, $\exists s \in R \setminus p$ such that $M[1/s]$ is free of finite rangk over $R[1/s]$

3. For each prime ideal p of M, M_p is a free R_p-module, and M is finitely presented, i.e. there is an exact sequence $R^m \to R^n \to M \to 0$(If R is Noetherian, then finitely generated implies finitely presented)

Proof. 1⇒2,3: Assume M is projective, $R^n \cong M \oplus P$, then M is finitely presented, i.e. $R^n \to R^n \to M \to 0$. Also $R_p^n \cong M_p \oplus P_p$ over teh local ring R_p, by Theorem 2.8, M_p is finitely generated and free over R_p. So after inverting things in $R \setminus p$, we get a free module. But we really only have to invert finitely many elements since M is finitely presented, which is the same as inverting their product, which is a single $s \in M \setminus p$, so $M[1/s]$ is free of finite rangk over $R[1/s]$
3⇒1: Since M is finitely presented, we have an exact sequence $R^m \to R^n \xrightarrow{\epsilon} M \to 0$. To show M is projective, it suffices to show that $\epsilon^* : \mathtt{Hom}(M, R^n) \to \mathtt{Hom}(M, M)$ is surjective, since $\mathtt{id}_M$ then has a preimage. Given a prime ideal p, M_p is free over R_p, so ϵ_p^* is split surjective, so is ϵ^*, since $\mathtt{Hom}(M, M)/\mathtt{Hom}(M, R^n)$ is 0 after localizing at any prime ideal, hence 0 ☐

So at least over a Noetherian commutative ring, M finitely generated projective $\iff$ M_p finitely generated free for each $p \in \mathtt{Spec}\, R$. This suggests(and we used this to study vector bundles over spheres) that to construct interesting projective modules, we get them by patching(clutching) finitely generated free modules over different open sets

Example 2.9 (Milnor square, his book on algebraic K-theory)**.** Suppose $f : R \to S$ is a map of rings, $I \cong f(I)$, then R is the puback of the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{f} & S/I \end{array}$$

Think of S as R/J, so we are patching over I and J

Example 2.10 (Topological example)**.** $R = C(S^2)$, I is the ideal of functions which vanish on the open upper hemisphere and tend to zero along the equator. Then $R/I \cong C(D_-^2)$, continuous functions on the closed lower hemisphere. Let S be the continuous functions on the closed upper hemisphere $C(D_+^2)$, $R \to S$ is just restriction S and R/I are both of the form $C(D^2)$, and since D^2 is contractible, every vector bundle on D^2 is trivial, and so every finitely generated projective S-module or R/I-module is free. But S^2 is contractible, we want to describe fintiely generated projective R-modules in terms of patching of trivial bundles along the equator

Theorem 2.11 (Milnor patching theorem)**.** In a Milnor square

1. An R-moudle P obtained by patching a finitely generated projective S-module P_1 and a finitely generated projective R/I-module P_2 is finitely genereated and projective

2. In the situation of 1, $S \otimes_R P \cong P_1$ and $P/IP \cong P \otimes_R R/I \cong P_2$

3. (important)Every finitely generated projective R-mod arises this way

4. Suppose P is obtained by patching S^n and $(R/I)^n$ via $g \in \mathrm{GL}_n(S/I)$, and Q is obtained similarly via $g^{-1} \in \mathrm{GL}_n(S/I)$, Then $P \oplus Q \cong R^{2n}$

5. More generally, if P is obtained by patching S^n and $(R/I)^n$ via $g \in \mathrm{GL}_n(S/I)$, and Q is obtained by patching S^n and $(R/I)^n$ via h, then $P \oplus Q \cong$ patching of S^n and $(R/I)^n$ via $gh \oplus R^n$

Proof. Translate to algebra the idea of gluing vector bundles when $X = X_1 \cup_Y X_2$, here X_1, X_2 are closed subsets of a compact Hausdorff sopace X_1 and $Y = X_1 \cap X_2$. Alternatively, the Milnor patching theorem is a kind of "algebraic Mayer-Vietoris" proof of 3: Recall R is embedded in $S \oplus R/I$. Let P be a finitely generated projective R-module Let $P_1 = S \otimes_R P$ which is finitely generated projective S-module, let $P_2 = P/IP$ is a finitely generated projective R/I-module. We want to show that P is obtained by patching P_1 and P_2. We have an exact sequence of R-modules

$$0 \to R \to S \oplus R/I \to S/I \to 0$$

$\mathrm{Hom}_R(P, -)$ and $- \otimes_R P$ is exact since P is projective, so we have exact sequence

$$0 \to P \to P_1 \oplus P_2 \to S/I \otimes_R P$$

which says that P is patched from P_1 and P_2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ through homotopy $\begin{bmatrix} 1-t & -t \\ t & 1-t \end{bmatrix}$ which has determinant $(1-t)^2 + t^2 > 0$, hence $\begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & gh \end{bmatrix}$, here $\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ through homotopy $\begin{bmatrix} 1 & tg \\ 0 & 1 \end{bmatrix}$ □

3 Line bundles and Picard groups

We assume in this section R is commutative. We showed that a finitely generated R-module P is projective exactly when it's locally free. We call it(by analogy with topology) a line bundle if P is locally free of rank 1. Commutativity of R enables you to tensor two R-modules together to get a new one. This preserves the property of being a line bundle. So we can define the Picard group $\mathrm{Pic}(R)$ of R, to be the abelian group of line bundles over R up to isomorphism with tensor product as the group operation and $[R]$ as the identity element. The dual line bundle $P^\vee = \mathrm{Hom}_R(P, R)$, and $P \otimes_R P^\vee \to R$ is an isomorphism, so $[P^\vee] = [P]^{-1}$. For topological line dunles over $\mathbb{C}$, the dual line bundle can also be identified with the complex conjuate bundle. The dual of vector bundle $E \to X$ is the bundle whose fibers are the dual vector spaces. Over $\mathbb{C}$, $\mathbb{C}^* \cong \overline{\mathbb{C}}$ via the usual inner product

Example 3.1.

1. $R = C^{\mathbb{R}}(X)$, X compact Hausdorff. $\mathrm{Pic}(R) \cong H^1(X, \mathbb{Z}/2)$
2. $R = C^{\mathbb{C}}(X)$, X compact Hausdorff. $\mathrm{Pic}(R) \cong H^1(X, \mathbb{Z})$

Theorem 3.2. If M is a manifold of dimension n and E is a vector bundle of rank k which is stably trivial, then E is trivial provided $\begin{cases} k > n & \mathbb{F} = \mathbb{R} \\ k > (n+1)/2 & \mathbb{F} = \mathbb{C} \end{cases}$

Example 3.3. The tangent bundle of S^2 is stably trivial but not trivial, in this case, $n = 2, k = 2, \mathbb{F} = \mathbb{R}$. But every stably trivial 3 plane bundle over S^2 is trivial

Proof. There is a fibration $O(n-1) \to O(n) \to S^{n-1}$, we have $\pi_{j+1}(S^{n-1}) \to \pi_j(O(n)) \to \pi_j(O(n-1)) \to \pi_j(S^{n-1})$. The map $\pi_j(O(n-1)) \to \pi_j(O(n))$ is sujective if $j < n-1$. an iso if $j < n-2$. and we can decompose M as (n-1)-skeleton$\cup D^n$ via gluing along the (n-1)-skeleton Similarly with $U(n-1) \to U(n) \to S^{2n-1}$, which gives $\pi_j(U(n-1)) \to \pi_j(U(n))$ surjective if $j < 2n-1$, an iso if $j < 2n-2$ □

The rank function: R is a commutative ring, we get a notion of local rank function of a finitely generated projective R-mod P, $\mathrm{Spec}(R) \to N, p \mapsto \mathrm{rank}(P_p)$. Claim: The rank funciton is continuous in Zariski topology

Proof of claim. p is contained in a maximal ideal m, $\mathrm{rank}\,\mathrm{Spec}\,R[1/s]\,\mathrm{rank}(P_p) = \mathrm{rank}(P_m)$. If $\mathrm{rank}(P_p) \neq \mathrm{rank}(P_q) = \mathrm{rank}\,\mathrm{Spec}\,R[1/s']$ □

Theorem 3.4 (Stable rank theorem, Bass-Serre, Bass's book on algebraic K theory, not easy)**.** R commutative and Noetherian of dimension d. If R has a finitely projective module P of constant rank k. then if $k > d$

1. $P = P_0 \oplus R^{k-d}$ for some projective module P_0 of rank d
2. P is stably isomorphic to P' of same rank $\Rightarrow P \cong P'$
3. P stably free $\Rightarrow P$ is free

Corollary 3.5. If R is a commutative ring with no idempotents other than 0 and 1, then every finitely generated projective R-module has constant rank

Proof. $\mathrm{Spec}\,R$ is connected iff R has no non-trivial idempotent. And an integral domain is connected(if x is an idempotent, so is $1-x$, and $x(1-x) = 0$) □

Back to line bundles
Let R be a commutative ring, P finitely generated projective R-module. We can form new modules $\bigwedge^j P$. If P has constant rank k, then $\bigwedge^j P$ is projective of rank $\binom{k}{j}$. So we can define the determinant of P, $\det(P) = \bigwedge^k P, k = \mathrm{rank}\,P$. If R has no non-trivial idempotents, the determinant module is defined for all finitely generated projective modules

1. $\det P$ is a line bundle
2. $\det(P \oplus Q) = \det P \otimes \det Q$

From these facts we deduce

Theorem 3.6. *R is a commutative ring, if two line bundles are stably isomorphic iff isomorphic, i.e. define the same class in $\mathrm{Pic}(R)$*

Proof. Assume $P \oplus R^n \cong P' \oplus R^n$, then $P \cong \det P \otimes R = \det(P \oplus R^n) \cong \det(P' \oplus R^n) = \det P' \otimes R \cong P'$ □

Theorem 3.7. *R commutative Noetherian of dimension 1(in particular, Dedekind domain: Noetherian integral domain which is integrally closed in the field of fractions). Then every finitely generated projective R-module of constant rank is of the form $P \oplus R^{k-1}$ for some k and for some line bundle P, namely the determinant bundle of the module. Thus finitely generated projective R-module are completely classified by rank and determinant*

Proof. Just use stable rank theorem □

Corollary 3.8. *If R is a commutative Noetherian integral domain of dimension 1, then $K_0(R) \cong \mathbb{Z} \oplus \mathrm{Pic}(R)$, $[P] \mapsto (\mathrm{rank}\, P, \det P)$*

Note $\det(P \oplus Q) = \det P \otimes \det Q$ says it is a homomorphism in Pic, rank adding says it is a homomorphism in $\mathbb{Z}$

4 Cartier divisor

R commutative integral domain, F its field of fractions. A fractional ideal of R is a non-zero R-submodule I of F, is such that $xI \subsetneq R$ for some $x \in F^\times$(i.e. we can clear denominators). I is invertible if $IJ = R$ for some other fraction ideal J. Principal fractional ideals are Rx for $x \in F^\times$. Principal fractional ideals are invertible. The invertible fractional ideals form the Cartier group $\mathtt{Cart}(R)$ of Cartier divisors under multiplication

Proposition 4.1. The line bundles over R are the same as invertible ideals up to isomorphism. If I, J are fractional ideals and I is invertible, then $I \otimes_R J \cong IJ$. There is an exact sequence

$$1 \to R^\times \to F^\times \xrightarrow{\mathrm{div}} \mathtt{Cart}(R) \to \mathtt{Pic}(R) \to 0$$

Proof. Suppose I is an invertible fractional ideal, $IJ = R$, then there exist $x_i \in I, y_i \in J$, $\sum_{i=1}^n x_i y_i = 1$, $\{x_i\}$ define homomorphisms $R^n \to I$, $\{y_i\}$'s define homomorphisms $I \to R^n$. $I \to R^n \to I$ is the identity, so I is projective, and has rank one because $I \otimes_R F = F$, so I is a line bundle. Conversely, suppose L is a line bundle, we get an embedding $L = R \otimes_R L \subsetneq F \otimes_R L = F$, L is isomorphic to a nonzero R submodule of F, i.e. a fractional ideal. Since I is finitely generated and projective, we have $R \cong L \otimes_R L^\vee$, hence L is invertible $\square$

Equivalent definition for Dedekind domain: Integral domain for which every fractional ideal is invertible

Theorem 4.2. *R Dedekind domain, every finitely generated torsion-free R-module M is projective and isomorphic to $I \otimes R^{n-1}$ for some fractional ideal I*

Proof. Induction on $\mathtt{rank}\, M = \dim_F(F \otimes_R M)$. M torsion-free means M embeds into $F \otimes_R M$, so if $\mathtt{rank}\, M = 0$, $M = 0$. Suppose $\mathtt{rank}\, M = n$, M embeds into F^n and then projects to the last factor F, image can't be zero, so we have $0 \to M_0 \to M \to I \to 0$ with I fractional ideal projective, so $M = M_0 \oplus I$, by induction, $M = I \oplus J \oplus R^{n-2}$. $I \oplus J = IJ \oplus R$ since they have the same determinant $\square$

Applications: R is a PID: $1 \to R^\times \to F^\times \to \mathtt{Cart}(R) \to 1$ since $\mathtt{Pic}(R) = 0$

Example 4.3. $R = \mathbb{Z}, F = \mathbb{Q}$, get

$$1 \to \mathbb{Z}^\times \to \mathbb{Q}^\times \to \mathbb{Q}^\times/\{\pm 1\} \to 1$$

$R = \mathbb{Z}[i], F = \mathbb{Q}[i]$, $R^\times = \{\pm 1, \pm i\}$
$R = k[t], F = k(t)$, $R^\times = k^\times$

F be a number field, $R = \mathcal{O}_F$. Then R is a Dedekind domain, the $\mathtt{Pic}(R)$ is finite, and $R^\times$ is finitely generated, in fact, $\mathtt{rank}(R^\times) = r_1 + r_2 - 1$. Most of the time, R is not a PID
k is a field, C is a smooth affine curve over k, R is the ring of regular functions on C, then R is a Dedekind domain. In the case $k = \mathbb{C}$, $\mathtt{Pic}(R)$ is the group of algebraic line bundles. $\mathtt{Cart}(R)$ indeed corresponds to the group of divisors(finite linear combinations of points)

5 Homeworks

6 Mayer-Vietoris for Picard groups

Theorem 6.1 (In Weibel I.3)**.** In a Milnor square, we get an eaxt sequence

$$1 \to R^\times \xrightarrow{\Delta} S^\times \times (R/I)^\times \xrightarrow{ratio} (S/I)^\times \to \mathrm{Pic}(R) \to \mathrm{Pic}(S) \times \mathrm{Pic}(R/I) \to \mathrm{Pic}(S/I)$$

The connecting homomorphism is given by the glueing map the ratio map is first/second

Classification of topological vector bundles and characteristic classes
Let X be a compact Hausdorff space, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $E \to X$ be a rank k vector bundle. Recall that E embeds in a trivial rank n bundle for some n sufficiently large, then at each point $x \in X$, we have an associated fiber $E_x \subseteq \mathbb{F}^n$. thus we get a continuous map $X \to \mathrm{Gr}_{\mathbb{F}}(k, n)$

$$\mathrm{Gr}_{\mathbb{F}}(k,n) = \begin{cases} O(n)/O(k) \times O(n-k) & \mathbb{F} = \mathbb{R} \\ U(n)/U(k) \times U(n-k) & \mathbb{F} = \mathbb{C} \end{cases}$$

Over $\mathrm{Gr}_{\mathbb{F}}(k,n)$, we have a canonical or universal bundle of rank k. Now we get classifying space $BO(k) = \varinjlim_{n\to\infty} \mathrm{Gr}_{\mathbb{R}}(k,n)$ and $BU(k) = \varinjlim_{n\to\infty} \mathrm{Gr}_{\mathbb{C}}(k,n)$, $BO(1) = \mathbb{RP}^\infty$, $BU(1) = \mathbb{CP}^\infty$

Theorem 6.2 (Classification theorem)**.** X is a compact Hausdorff space, there is a natural bijection

$$\{\text{isomorphism class of rank } k \text{ real vector bundle over } X\} \leftrightarrow [X, BO(k)]$$

$$\{\text{isomorphism class of rank } k \text{ complex vector bundle over } X\} \leftrightarrow [X, BU(k)]$$

Given a map $X \to BO(k)$, the associated vector bundle is just the pullback. Note that since X is compact, the image lands in $\mathrm{Gr}(k,n)$ for some$n$

Proof. We saw the every vector bundle arises this way, we also showed that homotopic maps give isomorphic vector bundles. It remains to show that isomorphic vector bundles come from homotopic maps. Suppose $E \cong F$ are rank k vector bundles, both embedded in a trivial n-dimensional vector bundle. Taking n large enough we can construct ϕ as a map $X \to \mathrm{GL}_n(\mathbb{F})$, $\phi(x) : E_x \to F_x$, sending E_x and $E_x^\perp$ to F_x and $F_x^\perp$ isomorphically. We want to show that maps classifying E and F are homotopic. Trick: double n, and replace ϕ by $\begin{bmatrix} \phi & \\ & \phi^{-1} \end{bmatrix}$ This map can be homotoped to the identity. This shows the classifying maps to $\mathrm{Gr}_{\mathbb{F}}(k, 2n)$ are homotopic □

Characteristic classes
It turns out that $H^*(BO(k), \mathbb{Z}/2)$ and $H^*(BU(k), \mathbb{Z})$ are polynomial rings on k variables(prove this by induction on k using the fibrations $O(k-1) \to O(k) \to S^{k-1}$, $U(k-1) \to U(k) \to S^{2k-1}$). From this, one gets the following result

Theorem 6.3 (Existence and uniqueness theorem for characteristic classes, ref: see Hatcher on vector bundles and K theory)**.** There are unique classes $w_j \in H^j(BO(k), \mathbb{Z}/2)$ with the following properties

1. $H^*(BO(k), \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, \cdots, w_k]$
2. If L is a real line bundle over X, w_1 pulls back to the element of $H^1(X, \mathbb{Z}/2) = \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2)$ which classifies the line bundles, as $\mathrm{Hom}(\pi_1(X), \mathbb{Z}/2)$ corresponds to the two to one coverings $\tilde{X}$, and $L = \tilde{X} \times_{\pi_1(X)} \mathbb{R}$
3. For arbitrary real vector bundles E of rank k over X, we define $w(E)$, called the total Stiefel-Whitney class of E, to be $1 + f^*(w_1) + f^*(w_2) + \ldots$ which has the property that $w(E) = w(F)$ if $E \cong F$, $w(E)w(F) = w(E \oplus F)$, $w(\text{trivial bundle}) = 1$. These imply that w is a stable invariant

There are unique classes $c_j \in H^j(BU(k), \mathbb{Z})$ with the following properties

1. $H^*(BU(k), \mathbb{Z}) \cong \mathbb{Z}[c_1, \cdots, c_k]$
2. If L is a complex line bundle over X, $c_1(L)$ under usual isomorphism $\mathrm{Pic}(X) \cong H^2(X, \mathbb{Z})$
3. For arbitrary complex vector bundles E of rank k over X, we define $c(E)$, called the total Chern class of E, to be $1 + f^*(c_1) + f^*(c_2) + \dots$ which has the property that $c(E) = c(F)$ if $E \cong F$, $c(E)c(F) = c(E \oplus F)$, $c(\text{trivial bundle}) = 1$

Note. These facts follow from above:

1. If E is a rank k complex vector bundle over X, we can think of E as a real vector bundle(realification) of rank $2k$, then $w_{2j}(E) = c_j(E) \bmod 2$, $w_{2j+1}(E) = 0$
2. $w_1(L_1 \otimes L_2) = w_1(L_1) + w_1(L_2)$, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$

Theorem 6.4 (Lerary-Hirsch theorem)**.** $F \xrightarrow{\iota} E \xrightarrow{p} B$ is a fiber bundle, $H^*(F, R)$ is finitely generated free R-module, and $\exists c_j \in H^{k_j}(E, R)$ such that $\iota^*(c_j)$ form a basis for $H^*(F, R)$. Then $H^*(B, R) \otimes_R H^*(F, R) \to H^*(E, R)$, $\sum b_i \otimes \iota^*(c_j) \mapsto p^*(b_i) \smile c_j$ is an isomorphism

Theorem 6.5 (Splitting principle)**.** Informal statement: For formulas involving characteristic classes, you can pretend that all vector bundles are direct sum of line bundles
Formal statement: Let X be a compact Hausdorff space, $E \to X$ a rank k vector bundle(over $\mathbb{R}$ or $\mathbb{C}$), then there is another space Y_E with a surjective projection $p : Y_E \to X$ such that $p^* : H^*(X) \to H^*(Y_E)$ is injective, and $p^*(E)$ splits as a direct sum of line bundles

Proof. Idea: Let Y_E be the fiber bundle over X whose fiber over $x \in X$ is the flag manifold of E_x which can be identified with $U(k)/U(1)^k$ or $O(k)/O(1)^k$. By Leray-Hirsch theorem, p^* is injective on cohomology(with $\mathbb{Z}$ coefficients in the complex case and $\mathbb{Z}/2$ in the real case)
We get the formula for behavior under tensor products: if $E = L_1 \oplus \cdots \oplus L_k$, $F = M_1 \oplus \cdots \oplus M_l$, then $E \otimes F \cong \oplus_{j,m} L_j \otimes M_m$, so $C(E \otimes F) = \prod_{j,m}(1 + c_1(L_j) + c_1(M_m))$, $c(E) = \prod_j (1 + c_1(L_j))$, we can express c_k using elementary symmetric functions $\square$

What is this good for? K theory is about stable classification of vector bundles. Sometimes this is hard to compute, but we have a calculational tool: E is stably isomorphic to F implies that E and F has the same characteristic classes

Example 6.6. Given line bundles L, L', $c_1(L) = x, c_1(L') = y$, $c(L \otimes L') = 1 + (x + y) + xy$, so if a rank 2 bundle E is to be stably isomorphic to $L \oplus L'$, we need to have $c_1(E) = x + y$ and $c_2(E) = xy$, this is necessary but not sufficient

Example 6.7. Over S^2, we have complex vector bundles of rank 1, $L_n, n \in \mathbb{Z}$ with $c_1(L_n) = nx$, x is the usual generator of $H^2(S^2)$. If we realify L_n, there are only two stable isomorphism classes, since L_n is stably isomorphic to L_{n+2}

7 Algebraic and analytic vector bundles

A vector bundle of rank k on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules which is locally free of rank k. In the cases of affine schemes and Stein manifolds, a vector bundle $\mathcal{F}$ over X is equivalent to a finitely generated projective module over $R = \mathcal{O}_X(X)$. However, this fails over projective schemes and compact complex manifold, since R is to small(consists only of constant functions)
Classifying vector bundles
Suppose $\mathcal{F}$ is a vector bundle of rank k over X, choose $\{U_i\}$ such that $\mathcal{F}(U_i) = \mathcal{O}_X(U_i)^k$, it give a Cech cocycle in $\mathrm{GL}_k(\mathcal{O}_X)$, so $H^1(X, \mathrm{GL}_k(O_X))$ classify vector bundles of rank k. This is somewhat mysterious unless $k = 1$, when it is the cohomology of a multiplicative group sheaf $\mathbb{G}_m$ or $O_X^\times$. So we have $\mathsf{Pic}(X, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X^\times)$
Chern classes of line bundles
Suppose X is an analytic manifold, a Stein manifold, a compact complex manifold, or the underlying analytic space of a projective variety over $\mathbb{C}$, forgetting analytic or algebraic structures, we have a topological space X_{top}
There is a natural map(forget analytic or algebraic structures) $VB(X, O_X) \to VB(X_{top})$ The forget map need not be either injective nor surjective, i.e. many algebraic vector bundles can give rise to the same topological vector bundle, and some topological vector bundles can't be made algebraic Then we can define Chern classes
Another useful construction: The exponential sequence This works in the analytic but not in the algebraic category. Say X is a connected complex analytic space or a complex manifold, $\mathcal{O}$ is the sheaf of holomorphic functions, then we have an exact sequence of sheaves:

$$0 \to \mathbb{Z} \xhookrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 1$$

This gives a long exact sequence in sheaf cohomology

$$0 \to \mathbb{Z} \to H^0(X, O) \to H^0(X, O^\times) \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, O^\times) = \mathsf{Pic}(X, O) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

There are cases where $c_1 : \mathsf{Pic}(X, O) \to H^2(X, \mathbb{Z})$ is an isomorphism, the the classification of analytic and topological line bundles is the same

Example 7.1. $X = \mathbb{CP}^1$, every analytic vector bundle is determined by its Chern class, and all Chern classes can be realized. So the analytic line bundles are all of the form $\mathcal{O}(n)$, $n \in \mathbb{Z}$

Theorem 7.2 (Highly non-trivial, GAGA)**.** For complex projective varieties, classifications of algebraic and analytic vector bundles coincide

Example 7.3. $\mathsf{Pic}(\mathbb{P}^1_k) \cong \mathbb{Z}$ with line budnles denoted by $O(m)$, $m \in \mathbb{Z}$, whose local sections are homogeneous Laurent polynomials of degree m. If $k = \mathbb{C}$, use exponential sequence we have

$$0 \to H^1(\mathbb{P}^n, \mathcal{O}) \to H^1(\mathbb{P}^n, \mathcal{O}^\times) = \mathsf{Pic}(\mathbb{P}^n) \xrightarrow{c_1} H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathcal{O}) = 0$$

Similarities and differences between algebraic and topological categories
We have a classification theorem for line bundles , the forgetful map from the algebraic to topological category is an isomorphism
The same topological bundles may admit inequivalent algebraic structures
Exact sequence of algebraic vector bundles do not split in general, this is different from the projective modules over a ring

Example 7.4. E Elliptic curve over $\mathbb{C}$

$$0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}* \to H^1(E, \mathbb{Z})$$

$H^1(E, \mathbb{Z}) = \mathbb{Z}^2$, $H^2(E, \mathbb{Z}) = \mathbb{Z}$, we have $H^1(E, \mathcal{O}) = C$, $H^2(E, \mathcal{O}) = 0$. So we get the short exact seq

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}/\mathbb{Z} \times \mathbb{Z} \arrow[r] & \mathsf{Pic}(E) \arrow[r] & \mathbb{Z} \arrow[l, bend right] \arrow[r] & 0 \end{tikzcd}$$

and $\mathsf{Pic}(E) = \mathbb{T}^2 \times \mathbb{Z}$

$X = P_k^1$. every vector bundle over X splits as a direct sum of line bundles
But unlike the case of complex top bundles over $S^2 = P_C^1$, not every rank two vector bundle splits as $V = \mathsf{det}(V) \oplus$ trivial vector bundle. For example, $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O}(1) \oplus \mathcal{O}(-1)$ are both top trivial, but diff algebraically
Even on $\mathbb{P}^1$, we have exact sequence

$$0 \to \mathcal{O}(-2) \to\xrightarrow{\alpha} \mathcal{O}(-1) \oplus \mathcal{O}(-1) \xrightarrow{\beta} \mathcal{O} \to 0$$

doesn't split. Since there are no morphisms from $\mathcal{O}(p) \to \mathcal{O}(q)$, $p > q$ x_0, x_1 homogeneous coordinates, $\beta(f, g) = x_0 f + x_1 g$, $\alpha(h) = (x_1 h, x_0 h)$
Invertible ideal sheaves and divisors
X is integral scheme, there is a common field of fractions for all $\mathcal{O}_X(U)$, called $k(X)$

Example 7.5. $X = \mathbb{P}_k^n$, $\mathcal{K}$ is the constant sheaf $k(X)$. A fractional ideal sheaf $\mathcal{I}$ is a $\mathcal{O}$-submodule of $\mathcal{K}$ containded in $f\mathcal{O}$ for some $f \in k(X)$, and invertible if $\mathcal{I}\mathcal{J} = \mathcal{O}$ for some other fractional ideal sheaf $\mathcal{J}$

Proposition 7.6. X integral scheme, then

$$1 \to H^0(X, \mathcal{O}^\times) \to k(X)^\times \to \mathsf{Cart}(X) \to \mathsf{Pic}(X) \to 1$$

In case X is integral, separrated, and locally factroian(localrings are all UFD), there is an iso between Cartier and Weil div(integral linear com of closed integral subschemes of codimension 1)

8 Grothendieck group

Weibel chapter II
M is an abelian monoid, F is the forgetful functor, the group completion functor G is the left adjoint of F, i.e.

$$Hom(M, F(A)) \cong Hom(G(M), A)$$

Construction of $G(M)$

Example 8.1. X compact Hausdorff, $M = VB(X)$, either over $\mathbb{R}$ or $\mathbb{C}$, $G(M) = K^0(X)$

Example 8.2. X scheme, $M = VB(X)$, $G(M) = K^0(X)$(may differ from the topological case, even with the same notation)

When X is a projective variety over $\mathbb{C}$

References

[1] *The K-Book* - Charles Weibel

Index