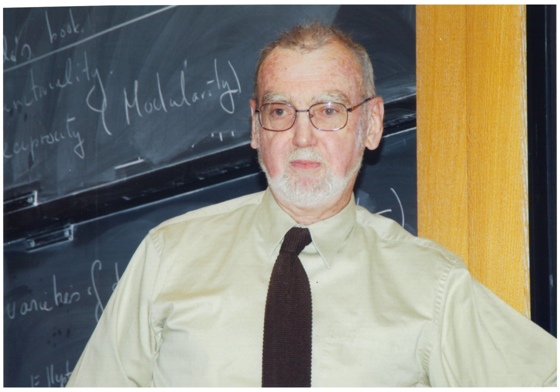
MATH808F - Modular Forms



Taught by Jingren Chi
Notes taken by Haoran Li

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Department of Mathematics
University of Maryland

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1 Overview

Definition 1.1. G is a Lie group, $K \leq G$ is a closed subgroup, $X = G/K$ is then a homogeneous space with transitive left G-action, $\Gamma \leq G$ is a discrete subgroup. The so called *automorphic functions* are $\mathbb{C}$-valued functions f on X such that

$$f(\gamma \cdot x) = f(x), \quad \forall x \in X, \gamma \in \Gamma \tag{1.1}$$

Loosely speaking, *automorphic forms* (for Γ) on X are automorphic functions that are also eigenfunctions for invariant differential operators on X (+ some technical growth conditions when necessary)

Question 1.2. How to decompose automorphic functions into sums (or integrals) of automorphic forms

Example 1.3. $\Gamma = \mathbb{Z}$, $X = G = \mathbb{R}$, automorphic functions are functions on $\mathbb{R}/\mathbb{Z} = \mathbb{T}$, automorphic forms are $e^{2\pi inx}$, $n \in \mathbb{Z}$. Fourier analysis tells us $L^2(\mathbb{R}/\mathbb{Z}) = \widehat{\bigoplus}_{n\in\mathbb{Z}} \mathbb{C}e^{2\pi inx}$

Example 1.4. $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup, $G/K = \mathcal{H} = \{\operatorname{Im} z > 0\}$ is the Poincaré upper half plane. G-invariant differential operators on $\mathcal{H}$ are polynomials with constant coefficients of the hyperbolic Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$, Examples of automorphic forms in this setting: Maass forms. Γ are sometimes called "modular groups", the corresponding automorphic forms on $\mathcal{H}$ are called *modular forms*

Note. $\mathcal{H}$ has the structure of a complex manifold, it is natural to look for holomorphic automorphic forms

Example 1.5.

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

Where $q = e^{2\pi iz}$, $z \in \mathcal{H}$, is invariant under $\mathrm{SL}_2(\mathbb{Z})$, hence a modular form

Definition 1.6. G induces a right action on $\mathbb{C}(X)$ by $(f \cdot g)(x) = f(gx)$, (1.1) becomes $f \cdot \gamma = f$, $\forall \gamma \in \Gamma$. More generally, we can allow a nontrivial *automorphy factor* $(f \cdot_c g) = c_g(x)f(gx)$, $\forall g \in G$, here $c_g : X \to \mathbb{C}^\times$

Exercise 1.7. For the action to be well-defined, the family of functions c_g must satisfy $c_{g_1g_2}(x) = c_{g_2}(x)c_{g_1}(g_2x)$, so called cocycle condition, $\forall g_1, g_2 \in G$, $x \in X$

Exercise 1.8. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, denote $j(g, z) = cz + d$, $G = \mathrm{SL}(2, \mathbb{R})$ acting on $\mathcal{H}$ by $g \cdot z = \frac{az+b}{cz+d}$. For $k \in \mathbb{Z}$, we consider the automorphy factor $c_g(z) = (cz + d)^{-k}$. Show c_g satisfies the cocycle condition

Definition 1.9. Then we get an action $(f \cdot_k g)(z) = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$, $z \in \mathcal{H}$. For a modular group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, holomorphic function f on $\mathcal{H}$ is called a *holomorphic modular form* of *weight* k and *level* Γ (one may also need to add some boundness condition) if $f \cdot_k \gamma = f$, $\forall \gamma \in \Gamma$ which is equivalent to $f\left(\frac{az+b}{cz+d}\right) = (cz + d)^k f(z)$

Remark 1.10. To unify these examples for $G = \mathrm{SL}_2(\mathbb{R})$ to acts on $\mathcal{H}$ and "get rid of" the automorphy factors, it is better to consider $\Gamma\backslash G$. The advantage is $\Gamma\backslash G$ has a large symmetry group coming from right multiplication of G, (whereas $\Gamma\backslash\mathcal{H}$ does not have so many automorphisms). The invariant differential operators on $\Gamma\backslash\mathcal{H}$ come from $Z(\mathfrak{g})$, then center of the universal enveloping algebra of $\mathrm{Lie}(G)$. The automorphic forms in the above examples all correspond to certain C^∞ functions on $\Gamma\backslash G$, their automorphy factors are determined by their behavior under right $K = \mathrm{SO}(2, \mathbb{R})$ action

Example 1.11. Classical Maass forms on $\Gamma\backslash\mathcal{H}$ correspond to certain right K-invariant functions on $\Gamma\backslash G$. The basic problem of decomposing automorphic functions motivates the more refined problem of decomposiing the right regular representation of G on $L^2(\Gamma\backslash G)$

Theorem 1.12. Assume $\Gamma\backslash G$ is compact (equivalently, $\Gamma\backslash\mathcal{H}$ is compact. Modular groups which unfortunately do not satisfy this assumption, is one of the difficulty of the subject), then

$$\begin{aligned} L^2(\Gamma\backslash G) &= \bigoplus_{\pi} \pi \otimes \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) \\ &= \bigoplus_{\pi} \pi^{\oplus m_\pi} \end{aligned}$$

π run over irreducible representations of G, $m_\pi = \dim \mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G)) < \infty$. Each multiplicity space $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ can be identified with a space of certain automorphic forms (The automorphy factors, eigenvalues of Laplacian are determined by the G-representations π)

Remark 1.13. In general we only assume that $\Gamma\backslash\mathcal{H}$ has finite volume, then we still have a decomposition of a subspae of $L^2(\Gamma\backslash G)$ (the discrete spectrum) whose orthogonal complement (the continuous spectrum) can be analyzed using theory of Eisenstein series. This is not the end of the story! Now comes the (arguably) more interesting part: when $\Gamma \leq G$ is arithmetic (e.g. modular groups, groups coming from indefinite quaternion algebras over $\mathbb{Q}$), then we can decompse each multiplicity space $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ further under the action of a big algebra on $L^2(\Gamma\backslash G)$ commuting with the right regular G-representation, this is the so-called "Hecke algebra". Where does this extra symmetry come from? Let $N_G(\Gamma) = \{g \in G | g\Gamma g^{-1} = \Gamma\}$ be the normarlizer, then $N_G(\Gamma)$ acts on $\Gamma\backslash G$ by left multiplication (so obviously commute with right G-action). This action factors through the quotient group $\Gamma\backslash N_G(\Gamma)$ and also induces automorphisms of $\Gamma\backslash\mathcal{H}$. Thus we ge4t an action of $\Gamma\backslash N_G(\Gamma)$ on $L^2(\Gamma\backslash G)$ that commutes with right G-regular representations. So $\Gamma\backslash N_G(\Gamma)$ acts on the multiplicity spaces $\mathrm{Hom}_G(\pi, L^2(\Gamma\backslash G))$ and decompose it further. The group $\Gamma\backslash N_G(\Gamma)$ is small (finite if $\Gamma\backslash G$ is compact, not sure if only finite volume), so the resulting decomposition is not so interesting. However, the action of $\Gamma\backslash N_G(\Gamma)$ on $\Gamma\backslash\mathcal{H}$ (and $\Gamma\backslash G$) can be extended to certain correspondences on $\Gamma\backslash\mathcal{H}$ (and $\Gamma\backslash G$)

Definition 1.14. Two discrete subgroups Γ_1, Γ_2 of G are *commensurable*, denoted $\Gamma_1 \approx \Gamma_2$, if their intersection $\Gamma_1 \cap \Gamma_2$ has fintie index in both of them. For $\Gamma \leq G$, let $\tilde{\Gamma} = \{g \in G | g\Gamma g^{-1} = \Gamma\}$ be the *commensurator* of Γ (this generalizes normalizer), elements in $\tilde{\Gamma}$ define correspondences on $\Gamma\backslash\mathcal{H}$ (and $\Gamma\backslash G$), which induces action of the convolution algebra $\mathbb{C}[\Gamma\backslash\tilde{\Gamma}/\Gamma]$ on $L^2(\Gamma\backslash G)$, and also on the cohomology of $\Gamma\backslash\mathcal{H}$. For modular groups Γ, we have $\tilde{\Gamma} = \mathrm{SL}_2(\mathbb{Q})$ which is large. For non-arithmetic groups Γ, $\tilde{\Gamma}/\Gamma$ is finite (This dichotomy between arithmetic and non-arithmetic cofinte volume subgroups follows from a general result of Margulis)

Remark 1.15. We will be mainly interested in congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, i.e. subgroups that contain $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$. In particular, such groups are modular, hence arithmetic. For each congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$, we have G and $H_\Gamma = \mathbb{C}[\Gamma\backslash\tilde{\Gamma}/\Gamma]$ left and right acting on $L^2(\Gamma\backslash G)$, $\tilde{\Gamma} = \mathrm{SL}_2(\mathbb{Q})$. Put all these together (for the various congruence subgroups), $G = \mathrm{SL}_2(\mathbb{R})$ and $\varprojlim_\Gamma H_\Gamma = C_c^\infty(\mathrm{SL}_2(\mathbb{A}_f))$ left and right act on $\varinjlim_\Gamma L^2(\Gamma\backslash G) = L^2(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}))$ ($\varprojlim_\Gamma \Gamma\backslash G = \mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})$), decompose $L^2(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}))$ into $\mathrm{SL}_2(\mathbb{A})$ representations, the irreducible summands are L^2-automorphic representations (Actually, we'll work with GL_2 instead, which is technically simpler). For (nice) irreducible representaion π of $\mathrm{GL}_2(\mathbb{A})$, Jacquet-Langlands associate an Euler prduct $L(s,\pi) = \prod_p L_p(s,\pi)$, (at least formally, may have convergence issues). This is done using tensor product theorem, which says roughly $\pi = \bigotimes_p \pi_p$ (restricted tensor product), π_p is the irreducible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, $L_p(s,\pi)$ is defined using only the factor π_p. Whether π occurs in decomposition of $L^2(\mathrm{GL}_2(\mathbb{Q}) \cdot Z(\mathbb{A}))\backslash\mathrm{GL}_2(\mathbb{A}))$ can be determinded by analytic properties of $L(s,\pi)$. This is basically the converse theorem. If π occurs as a direct summand, then $\dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A})}(\pi, L^2(\mathrm{GL}_2(\mathbb{Q}) \cdot Z(\mathbb{A}))\backslash\mathrm{GL}_2(\mathbb{A}))) = 1$ (Multiplicity one theorem)

2 Upper half plane

Definition 2.1. $\mathcal{H} = \mathcal{H}^+ = \{\operatorname{Im}(z) > 0\}$, $\mathcal{H}^- = \{\operatorname{Im}(z) < 0\}$ are the upper and lower half planes, $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R} = \mathbb{CP}^1 - \mathbb{RP}^1$

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/Z = \mathrm{GL}_2^+(\mathbb{R})/Z \leq \mathrm{GL}_2(\mathbb{R})/Z = \mathrm{PGL}_2(\mathbb{R})$$

is a subgroup of index 2, $\mathrm{PGL}_2(\mathbb{R})$ has two connected components, $\mathrm{PSL}_2(\mathbb{R})$ is its identity component

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/Z = \mathrm{GL}_2(\mathbb{C})/Z = \mathrm{PGL}_2(\mathbb{C})$$

Definition 2.2. Consider natural projection $\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1$, $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \mapsto \frac{z_1}{z_2}$, the standard action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2 - \{0\}$ by matrix multiplication, which induces an action on $\mathbb{CP}^1$ by *fractional linear transformation*. Since scalar matrices act trivially, this induces an action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{CP}^1$

Fact 2.3.
1. Under this action, $\mathrm{PGL}_2(\mathbb{C})$ is identified with the holomorphic automorphism group of $\mathbb{CP}^1$, also algebraic automorphism group
2. For any three distinct points $z_1, z_2, z_3 \in \mathbb{CP}^1$, there exists a unique $g \in \mathrm{PGL}_2(\mathbb{C})$ such that $gz_1 = 0$, $gz_2 = 1$, $gz_3 = \infty$. So any non scalar matrix has at most two fixed points on $\mathbb{CP}^1$

Lemma 2.4.
1. $\mathrm{PSL}_2(\mathbb{R})$ has three orbits on $\mathbb{CP}^1$: $\mathcal{H}, \mathcal{H}^-, \mathbb{RP}^1$
2. $\mathrm{PGL}_2(\mathbb{R})$ has two orbits on $\mathbb{CP}^1$: $\mathcal{H}^\pm, \mathbb{RP}^1$
3. $\mathrm{PSL}_2(\mathbb{R})$ is the group of holomorphic automorphisms of $\mathcal{H}$

Proof. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$, then

$$\begin{aligned}
\operatorname{Im}(gz) &= \operatorname{Im} \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\
&= \operatorname{Im} \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2} \\
&= \frac{(ad-bc)\operatorname{Im} z}{|cz+d|^2} \\
&= \frac{\det(g)}{|cz+d|^2} \operatorname{Im} z
\end{aligned}$$

So $\mathrm{PSL}_2(\mathbb{R})$ preserves $\mathcal{H}, \mathcal{H}^-, \mathbb{RP}^1$. While $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{R})$ interchanges $\mathcal{H}$ and $\mathcal{H}^-$

$$\begin{bmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{bmatrix} \cdot i = x + yi$$

For any $(x, y) \in \mathcal{H}$, thus $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathcal{H}$. For 3. first use Cayley transformation $\begin{bmatrix} 0 & -i \\ 1 & i \end{bmatrix}$ which induces an isomorphism $\mathcal{H} \to \mathbb{D}$, then use Schwartz lemma to determine $\mathrm{Aut}(\mathbb{D})$, and then translate back to $\mathcal{H}$ ☐

Exercise 2.5. The stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$ is

$$\mathrm{SO}(2) = \left\{ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \middle| \theta \in \mathbb{R} \right\}$$

So the stabilier of i in $\mathrm{PSL}_2(\mathbb{R})$ is $\mathrm{SO}(2)/\{\pm I\} \cong \mathrm{SO}(2)$. $\mathcal{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \cong \mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}(2)$ is a homogeneous space

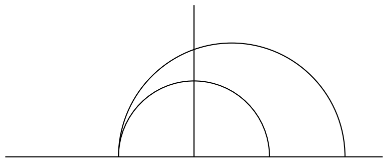
Exercise 2.6. $g^*(dx^2+dy^2) = |cz+d|^{-4}(dx^2+dy^2)$. Hence $g^*(y^{-2}(dx^2+dy^2)) = (y\circ g)^2 g^*(dx^2+dy^2) = \mathrm{Im}(gz)^{-2}|cz+d|^{-4}(dx^2+dy^2) = y^{-2}(dx^2+dy^2)$

Definition 2.7. The *hyperbolic metric* on $\mathcal{H}$ is $\dfrac{dx^2+dy^2}{y^2}$. Then $\mathcal{H}$ becomes a model of hyperbolic plane: a two dimensional simply connected Riemannian manifold with constant Gaussian curvature -1. $\mathrm{PSL}_2(\mathbb{R})$ are isometries on $\mathcal{H}$

Proposition 2.8. $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathcal{H})$, the group of orientation preserving isometries. The group of isometries $\mathrm{Isom}(\mathcal{H})$ is genenrated by $\mathrm{Isom}^+(\mathcal{H})$ and reflection $z \mapsto -\bar{z}$

Proof. We have already seen $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Hol}(\mathcal{H})$, the group of holomorphic automorphisms and $\mathrm{PSL}_2(\mathbb{R}) \leq \mathrm{Isom}^+(\mathcal{H})$, but $\mathrm{Isom}^+(\mathcal{H}) \leq \mathrm{Hol}(\mathcal{H})$ since orientation preserving conformal maps are holomorphic ☐

Fact 2.9. The geodesics on $\mathcal{H}$ are semi-circles othogonal to the real axis and half-lines orthogonal to the real axis, see [Miyake, Lemma 1.4.1]



The hyperbolic metric induces a volume form $d\mu = \dfrac{dx \wedge dy}{y^2}$, and the hyperbolic Laplace operator $\Delta = -y^{-2}\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)$

Exercise 2.10. $d\mu, \Delta$ are invariant under $\mathrm{PSL}_2(\mathbb{R})$ action (since the action preserves the metric)

Theorem 2.11 (Classification of motions)**.** $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $g \neq \pm I$, then g has one or two fixed points on $\mathbb{CP}^1$

Proof.

Case 1: $c = 0$

case i: $a = d = \pm 1$, so $b \neq 0$, g is a translation on $\mathbb{CP}^1$, ∞ is the only fixed point

case ii: $a \neq d$, g is a linear function on $\mathbb{CP}^1$, $\infty, \dfrac{b}{d-a} \in \mathbb{R}$ are the two fixed points

Case 2: $c \neq 0$, then $\infty \mapsto \dfrac{a}{c}$ is not fixed

$$\frac{az+b}{cz+d} = z \Rightarrow z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c}$$

case i: $|a+d| = 2$, the only one fixed point is $\dfrac{a-d}{2c} \in \mathbb{R}$

case ii: $|a+d| > 2$, there are two fixed points

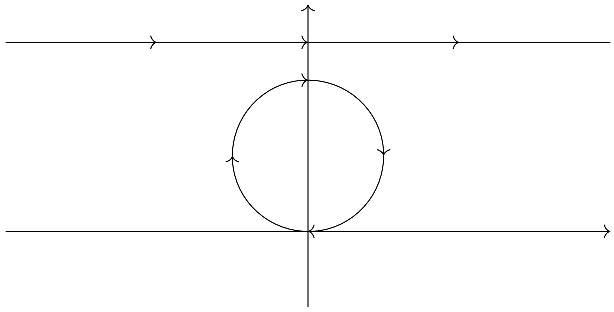
case iii: $|a+d| < 2$, there are two fixed points in $\mathcal{H}, \mathcal{H}^-$ and are conjugate of each other

In summary, there are three kinds of non-identity fractional linear transformation

1. Parabolic: When $|\operatorname{tr} g| = 2$, only one fixed point, which is on $\mathbb{RP}^1$
2. Hyperbolic: When $|\operatorname{tr} g| > 2$, two fixed points, both in $\mathbb{RP}^1$
3. Elliptic: When $|\operatorname{tr} g| < 2$, two fixed points, one in $\mathcal{H}$, the other one in $\mathcal{H}^-$

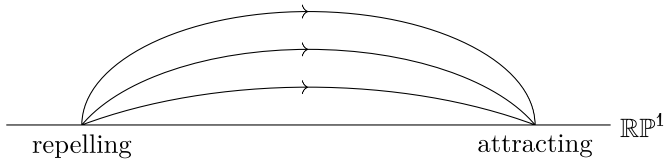
$\square$

Example 2.12. Translation $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} : z \mapsto z + b$ is a parabolic motion. In general, parabolic elements move points along *horocycles*, i.e. horiontal lines or circles tangent to the x-axis

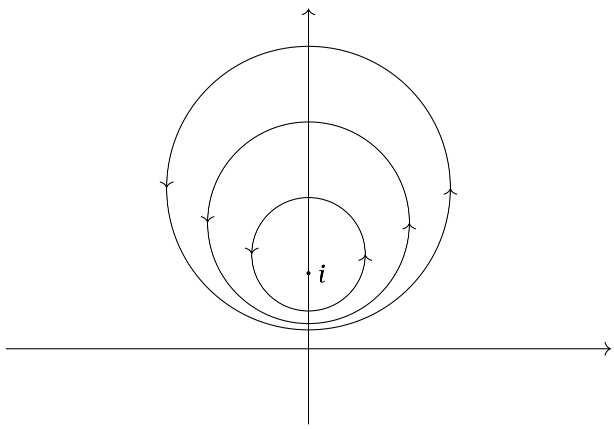


One can view horocycles as circles in $\mathbb{CP}^1 = S^2$ that are tangent to $\mathbb{RP}^1 = S^1$. $\mathrm{PSL}_2(\mathbb{R})$ action takes horocycles to horocycles and acts transitiviely on the set of horocycles. For any horizontal horocycle (say $\operatorname{Im} z = 1$), its stabilizer is $U = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$, identified with its image in $\mathrm{PSL}_2(\mathbb{R})$. Hence the set of horocycles can be identified with $\mathrm{PSL}_2(\mathbb{R})/U \cong (\mathbb{R}^2 - \{0\})/\{\pm I\}$ (note that $SL_2(\mathbb{R})/U \cong \mathbb{R}^2 - \{0\}$)

Example 2.13. $g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} : z \mapsto a^2 z$ is a hyperbolic motion, fixing $0, \infty$. In general, hyperbolic element moves points along *hypercycles*, i.e. intersections of circles in $\mathbb{CP}^1$ passing through the fixed points on $\mathbb{RP}^1$ with $\mathcal{H}$



Example 2.14. $g = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a elliptic motion, moving points along circles with *hyperbolic center* i, fixes i, induces counter-clockwise rotation of angle 2θ on the tangent space at i



Remark 2.15. Elliptic motions may have finite order ($\theta = \frac{\pi}{n}, n \in \mathbb{Z}$), parabolic and hyperbolic motions have infinite orders

3 Actions of Lie groups and discrete subgroups

Definition 3.1. Topological group G is acting on topological space X, G_x deonte the stabilizer of x. If X is Hausdorff, then G_x is closed. The set of orbits $G\backslash X$ is equipped with the quotient topology

Lemma 3.2. The quotient map $\pi : X \to G\backslash X$ is open. Moreover, if X is second countable, then so is $G\backslash X$

Proof. If $U \subseteq X$ is open, then $\pi(U) = \bigcup_{x\in U} Gx$ is a union of open subsets, hence also open. A countable basis will be mapped to a countable basis of $G\backslash X$ by π □

Lemma 3.3. If $H \subseteq G$ is a closed subgroup, then G/H is Hausdorff

Proof. $\{0\}$ is closed, and the topology is translational invariant □

Theorem 3.4. Suppose G is a second countable, locally compact topological group, acting transitively and continually on a locally compact Hausdorff space X, then fro any $x \in X$, the orbit map $G/G_x \to X$, $gG_x \mapsto gx$ is a homeomorphism

Proof. Consider the following diagram, we know ϕ is bijective and continuous, it suffices to show that ϕ is open

$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ G/G_x \arrow[r, "\phi"] & X \end{tikzcd}$$

Since G is second countable, there exists a dense subset $\{g_i\} \subseteq G$, Suppose $U \subseteq G$ is open, we need to show Ux is open. Fix $g \in U$, consider map $G \times G \to G$, $(a, b) \mapsto gab$, there exists a compact neighborhood K such that $K^{-1} = K$, $gK^2 \subseteq U$. Denote $W_n = g_n Kx$, if $\mathring{W}_n \neq \varnothing$, then $(Kx)^\circ \neq \varnothing$, for $gx \in Ux$, $gx \in (gK^{-1}Kx)^\circ = (gK^2x)^\circ \subseteq (Ux)^\circ$. Hence it suffice to show that $\mathring{W}_n$ for some n, which is guaranteed by Baire category theorem: Locally compact Hausdorff spaces are Baire spaces, suppose $\mathring{W}_n = \varnothing$, then W_n^c will be dense, so will $\bigcap W_n^c = (\bigcup W_n)^c = X^c = \varnothing$ which is a contradiction □

Theorem 3.5.

1. Let G be a Lie group and $H \subseteq G$ a closed subgroup. Then there exists a unique smooth manifold structure on G/H such that the quotient map $G \to G/H$ is a C^∞ submersion

2. Let G be a Lie group acting transitively on a smooth manifold M. Then for any $x \in M$, then map $G/G_x \to M$ is a diffeomorphism

Proof. Warner: Foundations of differentiable manifolds and Lie groups, Thm 3.58, 3.62 □

Example 3.6. Orbit map at $i \in \mathcal{H}$ induces diffeomorphisms $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2) \to \mathcal{H}$, $\mathrm{PSL}_2(\mathbb{R})/\mathrm{SO}(2) \to \mathcal{H}$

Definition 3.7. G is a topological group. A subgroup $\Gamma \subseteq G$ is a *discrete* if the induced topology is discrete

Lemma 3.8. A discrete subgroup Γ of a Hausdorff topological group G is closed

Proof. Since Γ is discrete, there exists an open neighborhood $U \ni 1$ such that $U \cap \Gamma = \{1\}$, there exists an open neighborhood $V \ni 1$ such that $V^{-1}V \subseteq U$, suppose g is in the closure of Γ, then $V^{-1}g \cap \Gamma$ is not empty, assume $\alpha, \beta \in V^{-1}g \cap \Gamma$, then $\alpha\beta^{-1} \in V^{-1}V \cap \Gamma \subseteq U \cap \Gamma = \{1\}$, thus $\alpha = \beta$, i.e. $V^{-1}g \cap \Gamma = \{\alpha\}$. If $g \neq \alpha$, then there exists an open neighborhood $g \in W \subseteq V^{-1}g$ which doesn't contain α since G is Hausdorff, but this contradicts the fact that g is in the closure of Γ, thus $g = \alpha \in \Gamma$ □

Lemma 3.9. G is a locally compact group and $K \subsetneq G$ is a compact subgroup. Then the natural map $G \xrightarrow{\pi} G/K$ is proper

Proof. Cover G by open subsets V_i with compact closure. For any $A \subseteq G/K$ compact, thus closed, $A \subseteq \bigcup_i \pi(V_i)$ by finitely may open sets, then closed set $\pi^{-1}(A) \subseteq \bigcup_i \overline{V_i}K$ which is compact, so is $\pi^{-1}(A)$ □

Definition 3.10. A group Γ is acting continuously on a topological space X. We say it acts *properly* if for any compact subsets $A, B \subseteq X$

$$\#\{\gamma \in \Gamma | \gamma A \cap B \neq \varnothing\} < \infty$$

Note that this implies that the stabilizers are finite

Proposition 3.11. G is a locally compact group $K \subseteq G$ is a compact subgroup. For any subgroup $\Gamma \subseteq G$, the following are equivalent

1. Γ is discrete
2. Γ acts properly on G/K on the left

Proof. $1 \Rightarrow 2$: Suppose $A, B \subseteq G/K$ are closed, by Lemma 3.9, $C = \pi^{-1}(A)$, $D = \pi^{-1}(B)$ are also compact, so is DC^{-1}, then

$$\{g \in \Gamma | \gamma A \cap B \neq \varnothing\} \subseteq \{g \in \Gamma | \gamma C \cap D \neq \varnothing\} = \Gamma \cap DC^{-1}$$

is discrete and compact, hence finite
$2 \Rightarrow 1$: Let V be a neighborhood of 1 with $\overline{V}$ compact, then

$$\Gamma \cap V \subseteq \{g \in \Gamma | \pi(g) \cap \pi(V) \neq \varnothing\} \subseteq \{g \in \Gamma | g\pi(1) \cap \pi(\overline{V}) \neq \varnothing\}$$

should be finite, by shrinking V, we get $\Gamma \cap V = \{1\}$, i.e. Γ is discrete □

Example 3.12. $\mathrm{SL}_2(\mathbb{Z})$ and its finite index subgroups are discrete in $\mathrm{SL}_2(\mathbb{R})$ since $\mathrm{SL}_2(\mathbb{Z}) = M_2(\mathbb{Z}) \cap \mathrm{SL}_2(\mathbb{R})$

Example 3.13. $\mathrm{SL}_2(\mathbb{Q})$ is note discrete in $\mathrm{SL}_2(\mathbb{R})$, the stabilizer of $i \in \mathcal{H}$ in $\mathrm{SL}_2(\mathbb{Q})$ is

$$\mathrm{SL}_2(\mathbb{Q}) \cap \mathrm{SO}(2) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a, b \in \mathbb{Q}, a^2 + b^2 = 1 \right\}$$

are in 1 to 1 correspondence with $\mathbb{QP}^1$ which is infinite, so $\mathrm{SL}_2(\mathbb{Q})$ does not act properly on $\mathcal{H}$

Remark 3.14. A discrete group Γ acts properly on X $\iff$ the map $\Gamma \times X \to X \times X$, $(g, x) \mapsto (x, gx)$ is proper

Proposition 3.15. G is a locally compact group $K \subseteq G$ is a compact subgroup. $\Gamma \subseteq G$ is a discrete subgroup. Then $\forall z \in G/K$, there exists a neighborhood U of z such that

$$\{g \in \Gamma | g(U) \cap U \neq \varnothing\} = \{g \in \Gamma | gz = z\}$$

Proposition 3.16. G is a locally compact group $K \subseteq G$ is a compact subgroup. $\Gamma \subseteq G$ is a discrete subgroup. Then $\Gamma \backslash G/K$ is Hausdorff

Proof. Shimura, proposition 7.1.8 □

Example 3.17. $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, then $\Gamma \backslash \mathcal{H}$ is Hausdorff, second countable

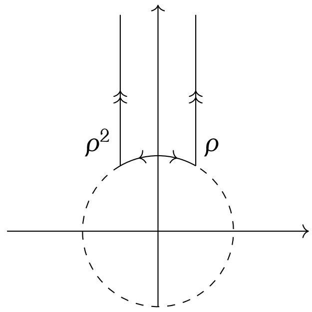
4 Quotients of upper half plane

Definition 4.1. When $z \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ is a fixed point of an elliptic/parabolic/hyperbolic element in Γ, we say z is an elliptic/parabolic/hyperbolic point of Γ

Exercise 4.2. $\mathrm{SL}_2(\mathbb{Z})$ is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, the same is true for $\mathrm{PSL}_2(\mathbb{Z})$

Hint. Use Euclid's algorithm

Let $D = \{z \in \mathcal{H} \mid |z| \geq 1, |\operatorname{Re}(z)| \leq \frac{1}{2}\}$



Theorem 4.3. For any $z \in \mathcal{H}$, there exists $\gamma z \in D$

Theorem 4.4. If $z, z' \in D$, $z \neq z'$ are in the same Γ orbit, then either $\operatorname{Re}(z) = \pm\frac{1}{2}$, $z = z' \pm 1$ or $|z| = 1$, $z' = -\frac{1}{z}$

Theorem 4.5. The stabilizer of $z \in D$ in $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ is

$$\overline{\Gamma}_z = \begin{cases} \langle S \rangle & \text{order 2} & , z = i \\ \langle TS \rangle & \text{order 3} & , z = \rho = e^{\pi i/3} \\ \langle ST \rangle & \text{order 3} & , z = \rho^2 = e^{2\pi i/3} \\ \{1\} & \text{order 1} & , \text{otherwise} \end{cases}$$

$\Gamma\backslash\mathcal{H} \cong D/\sim\, \cong \mathbb{C}$ is a homeomorphism. Need to pay attension to elliptic points $\rho \sim \rho^2$, i, near i, $\mathcal{H} \to \Gamma\backslash\mathcal{H}$ looks like $z \mapsto z^2$, near ρ or ρ^2, $\mathcal{H} \to \Gamma\backslash\mathcal{H}$ looks like $z \mapsto z^3$, $\Rightarrow$ locally around the elliptic points i, ρ, ρ^2, $\Gamma\backslash\mathcal{H}$ is homeomorphic to the quotient of unit disc by finite order rotation automorphisms. The quotients are still unit discs and have natural complex structure
Around non-elliptic points, $\Gamma\backslash\mathcal{H}$ is homeomorphic to a neighborhood in $\mathcal{H}$ and inherits complex structure. This way we get a complex structure on $\Gamma\backslash\mathcal{H}$. Since it is homeomorphic to $\mathbb{C}$, uniformization theorem $\Rightarrow$ isomorphic to either $\mathbb{C}$ or $\mathbb{D}$ as a complex manifold. There are no non-constant bounded Γ-invariant holomorphic function on $\mathcal{H}$, so $\Gamma\backslash\mathcal{H}$ is not isomorphic to $\mathbb{D}$. Thus $\Gamma\backslash\mathcal{H} \cong \mathbb{C}$ as Riemann surfaces

Definition 4.6. $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup. A connected subset $F \subseteq \mathcal{H}$ is a *fundamental domain* for Γ if it satisfies

1. $\mathcal{H} = \bigcup_{\gamma \in \Gamma} \gamma F$
2. $F = \overline{F^\circ}$
3. $\gamma F^\circ \cap F^\circ = \varnothing$, $\forall \gamma \in \Gamma - \{\pm I\}$

A fundamental domain F for Γ is *locally finite* if for any compact $K \subseteq \mathcal{H}$, $\{\gamma \in \Gamma \mid K \cap \gamma F \neq \varnothing\}$ is finite. It is *convex* if $\forall z, w \in F$, the (hyperbolic) geodesic segment joining z, w lies in F

Define an equivalence relation on F by $z \sim w$ if $\exists \gamma \in \Gamma$ such that $\gamma z = w$, note that $\sim$ is only nontrivial on the boundary ∂F, we have natural map $F/\sim\, \xrightarrow{\theta} \Gamma\backslash\mathcal{H}$

Proposition 4.7. θ is continuous, bijective. It is a homeomorphism iff F is locally finite

Beardon: The geometry of discrete groups, 9.2.2,9.2.4. ☐

One can construct nice fundamental domains as follows: choose $z_0 \in \mathcal{H}$ non-elliptic point for Γ, for any $\gamma \in \Gamma - Z_\Gamma$, denote

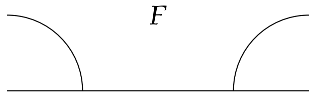
$$\begin{aligned} F_\gamma &= \{z \in \mathcal{H} | d(z, z_0) \leq d(z, \gamma z_0)\} \\ U_\gamma &= \{z \in \mathcal{H} | d(z, z_0) < d(z, \gamma z_0)\} \\ C_\gamma &= \{z \in \mathcal{H} | d(z, z_0) = d(z, \gamma z_0)\} \end{aligned}$$

Let $F(z_0) = \bigcap_{\gamma \in \Gamma - Z_\Gamma} F_\gamma$, $U(z_0) = \bigcap_{\gamma \in \Gamma - Z_\Gamma} U_\gamma$

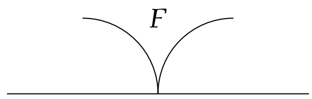
Proposition 4.8. $F(z_0)$ is a locally finite convex fundamental domain for Γ, $U(z_0)$ is the interior of $F(z_0)$

Miyake §1.6. ☐

The boundary of $F = F(z_0)$ consists of geodesic segments of the form $L_\gamma = F \cap \gamma F \subseteq C_\gamma$, see [Miyake 1.6.2] for the inclusion. Some L_γ may have infinite length, in that case it extends to some point on $\mathbb{R} \cup \{\infty\}$, called *ends* of F. Two kinds of ends:



open ends



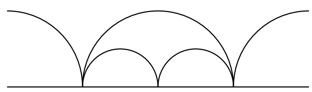
cusps

Theorem 4.9. $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup, $z_0 \in \mathcal{H}$ is a non-elliptic point of Γ, $F = F(z_0)$ as above. The following are equivalent

1. F has finitely many sides and all ends on $\mathbb{R} \cup \{\infty\}$ are cusps
2. $\mathrm{Vol}(F)$ is finite

Note. The sides of F are the segments L_γ of nonzero length

Proof. 1.→2.: Follows from Lemma 4.10
2.→1.: Finiteness follows from Lemma 4.10. If there are open ends, then there will be infinitely many geodesic triangles in F with vertices on $\mathbb{R} \cup \{\infty\}$, each such triangle has area π by Lemma 4.10, so $\mathrm{Vol}(F) = \infty$ which is a contradiction



☐

Lemma from Gauss-Bonet

Lemma 4.10. Let P be a polygon on $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ whose sides consists of N geodesics. Let $\alpha_1, \cdots, \alpha_N$ be the interior angle at each vertex (we allow the vertex to be on $\mathbb{R} \cup \{\infty\}$), so the angle at such a vertex is 0), then $\mathrm{Vol}(P) = (N-2)\pi - \sum_{i=1}^N \alpha_i$. In particular, if $N = 3$ and all 3 vertices are all on $\mathbb{R} \cup \{\infty\}$, then $\mathrm{Vol}(P) = \pi$

Proof. If all vertices are in $\mathcal{H}$, Gauss-Bonet says

$$\int_P k d\mu + \sum_{i=1}^N (\pi - \alpha_i) = 2\pi\chi(P)$$

In this particular setting, $k \equiv -1$ is the constant curvature, $\chi(P) = 1$ is the Euler characteristic. If there are cusps, truncate and take limit ☐

Theorem 4.11. Let Γ, z_0, $F = F(z_0)$ be as above. Suppose $\mathrm{Vol}(F) < \infty$, then

1. Each of the (finitely many) cusps is a parabolic point for Γ, and not a hyperbolic point. Its stabilizer in $\overline{\Gamma}$ is isomorphic to $\mathbb{Z}$
2. There are finitely many elliptic points in F, all lying on ∂F
3. Each Γ-orbit of parabolic points of Γ contains at least one cusp of F

Proof.

i a) A cusp cannot be a hyperbolic point: Suppose not, then we may assume the cusp is at 0 and $\exists \gamma = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \Gamma - \{\pm I\}$. Then γ fixes the geodesic $(0, 1\infty)$ and acts by fixes the geodesic $(0, i\infty)$ and acts by translation by a fixed hyperbolic distance on $(0, i\infty)$. Then any point on $(0, i\infty)$ can be moved to fixed segment S on $(0, i\infty)$ by applying some power of γ. Let z be an interior of F. Then the geodesic from z to 0 lie in the interior of F, choose a sequence of points z_n converging to 0 on this geodesic. Then we can find a sequence w_n on $(0, i\infty)$ such that $d(z_n, w_n) \to 0$ as $n \to \infty$. Apply some power of γ to each w_n to move them in S, then z_n will be moved to accumulate near S because γ is an isometry. This contradicts local finiteness

b) Any cusp is a parabolic point ($\Rightarrow$ stabilizer in $\overline{\Gamma}$ isomorphic to $\mathbb{Z}$). Observation: If F and $\gamma(F)$ have a common cusp S, then $\gamma^{-1}(S)$ is also a cusp of F. $\forall$ cusp S of F, there are infinitely many $\gamma \in \Gamma$ such that S is also a cusp of $\gamma(F)$. If stabilizer of s in Γ is trivial, then there would be infinitely many cusps of F by the above observation. This contradicts Theorem **??**. So stabilizer of S in $\overline{\Gamma}$ is nontrivial and by 1a), S is a parabolic point

ii Elliptic points cannot lie in F° since $\gamma(F^\circ) \cap F^\circ = \varnothing$ for any non-scalar $\gamma \in \Gamma$. They are either a vertex of F or mid-point of a side, hence finitely many

iii Assertion on parabolic points of Γ will be proved later. It essentially boils down to Hausdorffness of the compactification of $\Gamma\backslash\mathcal{H} \cong F/\sim$ by adding cusps

□

Remark 4.12. In fact part 3. suggests how to define compactification of $\Gamma\backslash\mathcal{H}$ without using any specific fundamental domain

Compactifying $\Gamma\backslash\mathcal{H}$: Fix $\Gamma \le \mathrm{SL}_2(\mathbb{R})$ discrete subgroup. Let P_Γ be the set of parabolic points of Γ on $\mathbb{R} \cup \{\infty\}$($P_\Gamma = \varnothing$ if no parabolic points). Let $\mathcal{H}^* = \mathcal{H}^*_\Gamma = \mathcal{H} \cup P_\Gamma$. Our goal is to put a topology on $\mathcal{H}^*$, show that when $\mathrm{Vol}(\Gamma\backslash\mathcal{H}) < \infty$, $\Gamma\backslash\mathcal{H}^*$ is a nice compactification of $\Gamma\backslash\mathcal{H}$ and can be identified with a quotient of $F^* = F \cup \{\text{cusps of } F\}$ where F is a fundamental domain for Γ as above
For $l > 0$, let $U_\infty(l) = \{z \in \mathcal{H} | \operatorname{Im} z > l\}$ and $U^*_\infty(l) = U_\infty(l) \cup \{\infty\}$, for $t \in \mathbb{R}$, let $U_t(l) = \sigma U_\infty(l)$, $U^*_t(l) = \sigma U^*_\infty(l) = U_t(l) \cup \{t\}$, where $\sigma \in \mathrm{PSL}_2(\mathbb{R})$ is chosen so that $\sigma\infty = t$. The boundaries of $U_t(l)$ are horocycles at $t \in \mathbb{R} \cup \{\infty\}$. Define a topology on $\mathcal{H}^*$ so that $\mathcal{H}$ is an open subset, and $U^*_t(l)$ form a system of neighborhoods of $t \in \mathbb{R} \cup \{\infty\}$, then $\mathcal{H}^*$ is a second countable(since Γ and hence P_Γ is countable and Hausdorff), but not locally compact. Γ acts continuously on $\mathcal{H}^*$, but not properly: stabilizes at $t \in P_\Gamma$ are isomorphic to $\mathbb{Z}$, infinite

Example 4.13. When $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $P_\Gamma = \mathbb{Q} \cup \{\infty\}$. Γ acts transitively on P_Γ (an easy check, also follows from Theorem **??** and Theorem **??**, noticing that the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp ∞). $\overline{\Gamma}_\infty = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$, $U^*_\infty(l)/\overline{\Gamma}_\infty$ is homeomorphic to an open disc

Lemma 4.14. For any $t \in P_\Gamma$, $\overline{\Gamma}_t \cong \mathbb{Z}$ and a generator has the form $\sigma \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \sigma^{-1}$ where $h > 0$, $\sigma \in \mathrm{SL}_2(\mathbb{R})$, $\sigma\infty = t$

Proof. We may assume $t = \infty$, then

$$\Gamma \subseteq \left\{ \pm \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$$

Since $\infty \in P_\Gamma$, $\exists x > 0$ such that $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in \Gamma_\infty \cdot \{\pm 1\}$. If $\exists \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \in \Gamma_\infty$, may assume $|a| \leq 1$, then

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^n \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}^{-n} = \begin{bmatrix} 1 & a^{2n}x \\ 0 & 1 \end{bmatrix} \in \Gamma$$

Γ is discrete $\Rightarrow a = \pm 1$. $\Rightarrow \overline{\Gamma}_\infty$ is a discrete subgroup of $\left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\} \cong \mathbb{R} \Rightarrow \overline{\Gamma}_\infty \cong \mathbb{Z}$ □

Lemma 4.15. Suppose $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \in \overline{\Gamma}$ for some $h \neq 0$, let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. If $|hc| < 1$, then $c = 0$

Miyake 1.7.3. Define inductively $\gamma_n \in \Gamma \cdot \{\pm 1\}$, $\gamma_0 = \gamma$, $\gamma_{n+1} = \gamma_n \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \gamma_n^{-1}$, $|hc| < 1$ implies $\gamma_n \xrightarrow{n\to\infty} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$. Γ discrete $\Rightarrow c = 0$ □

Lemma 4.16. For any compact subset $K \subseteq \mathcal{H}$, for any $s \in P_\Gamma$, $\exists l > 0$ such that $K \cap \gamma U_s(l) = \varnothing$, $\forall \gamma \in \Gamma$ (Or equivalently, $\gamma(K) \cap U_s(l) = \varnothing$, $\forall \gamma \in \Gamma$)

Proof. Let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = s$, since K is compact, $\exists 0 < l_1 < l_2$ such that

$$\sigma^{-1}(K) \subseteq \{z \in \mathcal{H} | l_1 < \mathrm{Im}(z) < l_2\}$$

Since s is a parabolic point, $\exists h \neq 0$ scuh that $\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \in \sigma^{-1}\Gamma \cdot \{\pm 1\}\sigma$. Let $l = \max\{h^2/l_1, l_2\}$. Let $\gamma \in \Gamma$ and denote $\delta = \sigma^{-1}\gamma\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $c = 0$, then $\delta U_\infty(l) \cap \sigma^{-1}K = U_\infty(l) \cap \sigma^{-1}K = \varnothing$. If $c \neq 0$, then by Lemma ??, $|hc| \geq 1 \Rightarrow z \in U_\infty(l)$

$$\mathrm{Im}(\delta z) = \frac{\mathrm{Im}\, z}{|cz + d|^2} \leq \frac{1}{c^2 \,\mathrm{Im}\, z} < \frac{1}{c^2 l} \leq \frac{h^2}{l} \leq l_1$$

$\Rightarrow \delta U_\infty(l) \cap \sigma^{-1}K = \varnothing$. Thus $\gamma U_s(l) \cap K = \gamma\sigma U_\infty(l) \cap K = \sigma(\delta U_\infty(l) \cap \sigma^{-1}K) = \varnothing$ □

Lemma 4.17. Let $s, t \in P_\Gamma$, then $\forall l > 0$, $\exists l' > 0$ such that $\forall \gamma \in \Gamma$, if $\gamma s \neq t$, then $\gamma U_s(l) \cap U_t(l') = \varnothing$

Proof. Let $\sigma \in \mathrm{SL}_2(\mathbb{R})$ with $\sigma\infty = s$, since $s \in P_\Gamma$, $\exists h \neq 0$ such that

$$\delta = \sigma \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \sigma^{-1} \in \Gamma_s \cdot \{\pm 1\} \subseteq \Gamma \cdot \{\pm 1\}$$

Let $K = \{z \in \mathcal{H} | \,\mathrm{Im}\, z = l, 0 \leq \mathrm{Re}\, z \leq |h|\}$, by Lemma ??, $\exists l' > 0$ such that $\gamma\sigma(K) \cap U_t(l') = \varnothing$, $\forall \gamma \in \Gamma$. Let $\gamma \in \Gamma$ with $\gamma s = t$. Suppose $\gamma U_s(l) \cap U_t(l') \neq \varnothing$, then $\gamma(\partial U_s(l) - \{s\}) \cap U_t(l') \neq \varnothing$ since $\gamma s \neq t$. $\Rightarrow$ for some $n \in \mathbb{Z}$, $\gamma\delta^n\sigma(K) \cap U_t(l') \neq \varnothing$ is a contradiction □

Corollary 4.18. $\forall s \in P_\Gamma$, $\exists C > 0$ such that $\overline{\Gamma}\backslash U_s^*(l) \to \Gamma\backslash\mathcal{H}^*$ is an open embedding for any $l > C$

Proof. Take $s = t$ in Lemma ??, we see that for $l >> 0$, $\{\gamma \in \overline{\Gamma} | \gamma U_s(l) \cap U_s(l) \neq \varnothing\} = \overline{\Gamma}_s$ □

Corollary 4.19. $\Gamma\backslash\mathcal{H}^*$ is locally compact Hausdorff

Proof. Corollary ?? $\Rightarrow$ locally compact since $\overline{\Gamma}_s\backslash\overline{U_s^*(l)}$ is compact. (Exercise: check this and also check that $\overline{U_s^*(l)}$ is not compact). From previous note, $\Gamma\backslash\mathcal{H}$ is Hausdorff, the rest follows from Lemma ?? and Lemma ?? □

Finally we can finish the proof of Theorem ?? 3. Suppose $s \in P_\Gamma$ and the orbit Γs does not contain any cusp of F. Fix a neighborhood $U = U_s^*(l)$ of s. The hypothesis $\mathrm{Vol}(F) < \infty$ implies that F has only finitely many cusps: $\{s_1, \cdots, s_n\}$ (by Theorem ??). By Lemma ??, there exist neighborhoods U_i of s_i such that $\gamma U \cap U_i = \varnothing$, $\forall \gamma \in \Gamma$, $\forall 1 \leq i \leq n$. By Lemma ??, we can shrink U so that it does not intersect the compact set $K = F - \bigcup_{i=1}^n U_i$. Then $\gamma U \cap F = \varnothing$, $\forall \gamma \in \Gamma$, contradicting the definition of fundamental domain. Thus Γs contains some cusps of F

Remark 4.20. Suppose $\mathrm{Vol}(F) < \infty$. Let F^* be the closure of F in $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$, then $F^* = F \cup \{\text{cusps}\}$ and $F^*/\sim$ is homeomorphic to $\Gamma \backslash \mathcal{H}^*$

Definition 4.21 (Riemann surface structure on $\Gamma\backslash\mathcal{H}^*$). $\forall z \in \mathcal{H}^* = \mathcal{H} \cup P_\Gamma$, let U_z be an open neighborhood of z such that $\{\gamma \in \Gamma | \gamma U_z \cap U_z \neq \varnothing\} = \Gamma_z$. Existence of U_z follows from Proposition ?? in previous note, when $z \in \mathcal{H}$, and Lemma ?? (or corollary ??) when $z \in P_\Gamma$. Then $\Gamma_z \backslash U_z \to \Gamma\backslash\mathcal{H}^*$ is an open embedding for any $z \in \mathcal{H}^*$. We use $\{\Gamma_z / U_z, \phi_z\}_{z \in \mathcal{H}^*}$ as coordinate charts, ϕ_z is to be defined

1. If $z \in \mathcal{H}$ is a non-elliptic point, then $\overline{\Gamma}_z = \{1\}$, let $\phi_z : \Gamma_z \backslash U_z \to U_z$ be the natural homeomorphism
2. If $z \in \mathcal{H}$ is elliptic, then $\overline{\Gamma}_z$ is a cyclic group of order $n > 1$. Let $\lambda : \mathcal{H} \to \mathbb{D}$ be an isomorphism of complex manifold such that $\lambda(z) = 0$. By Schwarz lemma, $\lambda \overline{\Gamma}_z \lambda^{-1}$ is the group generated by $\frac{2\pi}{n}$ rotation. Define $\phi_z : \Gamma_z \backslash U_z \to \mathbb{C}$ by $\phi_z(w) = \lambda(w)^n$

$$\begin{tikzcd} U_z \arrow[r, hook] \arrow[d] & \mathcal{H} \arrow[r, "\lambda"] & \mathbb{D} \arrow[d, "u \mapsto u^n"] \\ \Gamma_z \backslash U_z \arrow[rr, "\phi_z"] & & \mathbb{D} \end{tikzcd}$$

3. If $s \in P_\Gamma$ is parabolic, then by Lemma ??,

$$\sigma^{-1} \overline{\Gamma}_s \sigma = \left\langle \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \right\rangle \cong \mathbb{Z}$$

Where $h > 0$, here $\sigma \in \mathrm{SL}_2(\mathbb{R})$, $\sigma\infty = s$, define $\phi_s(w) = \exp(\frac{2\pi i}{h} \sigma^{-1}(w))$

$$\begin{tikzcd} U_s \arrow[r, "\sigma^{-1}"] \arrow[d] & \mathcal{H} \cup \{\infty\} \arrow[d, "\exp(\frac{2\pi i}{h} z)"] \\ \Gamma_s \backslash U_s \arrow[r, "\phi_s"] & \mathbb{D} \end{tikzcd}$$

Let's write $X(\Gamma) = \Gamma\backslash\mathcal{H}^*$, $Y(\Gamma) = \Gamma\backslash\mathcal{H}$. Riemann surfaces with complex structures defined above. $X(\Gamma) - Y(\Gamma)$ is a discrete set of cusps of $X(\Gamma)$

Theorem 4.22 (Siegel). $X(\Gamma)$ is compact $\iff$ $Y(\Gamma)$ has finite volume

Proof. $\Rightarrow$: $X(\Gamma)$ compact $\Rightarrow$ finitely many cusps, a neighborhood of each cusp has finite volume

$$\int_l^\infty \int_0^h \frac{dxdy}{y^2} = h \int_l^\infty < \infty$$

$\Leftarrow$: Let F be the fundamental domain as in Theorem ??, then $\mathrm{Vol}(F) = \mathrm{Vol}(Y(\Gamma)) < \infty \Rightarrow X(\Gamma) \approx F^*/\sim$ is compact by Theorem ??. Here $F^* = F \cup \{\text{cusps}\}$ is viewed as a closed subset of $\mathbb{CP}^1$, hence compact $\square$

5 Holomorphic modular forms

$\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup such that $\mathrm{Vol}(\Gamma\backslash\mathcal{H}) < \infty$. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, let $j(g,z) = cz + d$. For a function f on $\mathcal{H}$, $k \in \mathbb{Z}$, define

$$(f \cdot_k g)(z) = \det(g)^{\frac{k}{2}} j(g,z)^{-k} f(gz)$$

Suppose f is holomorphic on $\mathcal{H}$ and $f \cdot_k \gamma = f$, $\forall \gamma \in \Gamma$. Let $t = \sigma\infty \in P_\Gamma$, $\sigma \in \mathrm{SL}_2(\mathbb{R})$(parabolic point), then $\sigma^{-1}\Gamma_t\sigma \cap \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & h\mathbb{Z} \\ 0 & 1 \end{bmatrix}$, for some $h > 0$, $\Rightarrow f \cdot_k \sigma \cap \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \sigma^{-1} = f \Rightarrow (f \cdot_k \sigma) \cdot_k \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} = f \cdot_k \sigma \Rightarrow (f \cdot_k \sigma)(z + h) = (f \cdot_k \sigma)(z)$. The Fourier expansion is

$$f \cdot_k \sigma = \sum_{n\in\mathbb{Z}} a_n e^{\frac{2\pi i n z}{h}}$$

Definition 5.1. f is meromorphic/holomorphic/vanishes at $t \in P_\Gamma$ if $a_n = 0, \forall n << 0/n < 0/n \leq 0$

Definition 5.2. A holomorphic function f on $\mathcal{H}$ is a holomorphic/meromorphic modular form of weight k and level Γ if it satisfies

1. $f \cdot_k \gamma = f, \forall \gamma \in \Gamma$
2. f is holomorphic/meromorphic at all $t \in P_\Gamma$

It is a cusp form if furthermore it vanishes at all $t \in P_\Gamma$. Let $A_k(\Gamma)$ denote the set of meromorphic forms of weight k, level Γ, $M_k(\Gamma)$ denote the set of holomorphic forms of weight k, level Γ, $S_k(\Gamma)$ denote the set of cusp forms of weight k, level Γ

Remark 5.3. Since $f \cdot_k \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = (-1)^k f$, if $-I \in \Gamma$, then for any odd k, $A_k(\Gamma) = 0$. $A_0(\Gamma)$ is the field of rational functions on $X(\Gamma) = \Gamma\backslash\mathcal{H}^*$, $M_0(\Gamma) = \mathbb{C}$

$$A(\Gamma) = \bigoplus_{k\in\mathbb{Z}} A_k(\Gamma) \supseteq M(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M_k(\Gamma)$$

are graded rings, $S(\Gamma) = \bigoplus_{k\in\mathbb{Z}} S_k(\Gamma)$ is a graded ideal in $M(\Gamma)$

Example 5.4. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \langle T, S \rangle$, $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then condition is equivalent to $f(z+1) = f(z)$, $f(-\frac{1}{z}) = z^k f(z)$. Using this, one can show that *Ramanujan's function*

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, q = e^{2\pi i z}$$

is a cusp form of weight 12, level $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. See [Bump §1.3] for details

Definition 5.5 (Holomorphic Eisenstein series)**.** $k > 2$ is an even integer

$$E_k(z) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} (mz + n)^{-k} = \zeta(k) G_k(z), z \in \mathcal{H}$$

where

$$G_k(z) = \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} (cz + d)^{-k} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k}$$

Use $j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z)$ to deduce $G_k \cdot_k \gamma = G_k, \forall \gamma \in \Gamma$
Generalizing this construction: Suppose $\forall \gamma \in \Gamma$, we have a function $\phi_\gamma(z)$ on $\mathcal{H}$ satisfying

1. $\phi_{\delta\gamma}(z) = \phi_\delta(\gamma z) j(\gamma, z)^{-k}, \forall \gamma, \delta \in \Gamma$, $z \in \mathcal{H}$
2. $\phi_{u\gamma} = \phi_\gamma$, $\forall \gamma \in \Gamma, u \in \Gamma_\infty$

Consider the formal sum $\Phi(z) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma} \phi_\delta(z)$, then $\Phi \cdot_k \gamma = \Phi, \forall \gamma \in \Gamma$ if the sum converges absolutely. Take $\phi_\gamma(z) = j(\gamma, z)^{-k}$, get G_k as above. Take $\phi_\gamma(z) = j(\gamma, z)^{-k} e^{2\pi i m \gamma z}, m \in \mathbb{Z}_{\geq 0}$, get Poincaré series $P_m(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e^{2\pi i m \gamma z}$, absolutely converges when $k > 2$ is even. When $m > 0$, P_m is a cusp form of weight k, level $\mathrm{SL}_2(\mathbb{Z})$. $P_0(z) = G_k(z)$ is not cusp form. Fourier expansion

$$E_k(z) = \zeta(k) + \frac{(2\pi)^k (-1)^{\frac{k}{2}}}{(k-1)!} \sum \sigma_{k-1}(n) q^n$$

where $\sigma_r(n) = \sum_{d|n} d^r, q = e^{2\pi i n z}$
Method:

1. Direct computation: $f(z) = \sum_{n \in \mathbb{Z}} a_n q^n, q = e^{2\pi i z}$, then

$$a_n = \int_0^1 f(x+iy) e^{-2\pi i n (x+iy)} dx$$

explicit formula for Fourier coefficients of $P_m(z)$

2. Faster trick for $E_k(z)$: (see [Shimura §2.2]) use the identity

$$\pi \cot(\pi z) = z^{-1} + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} - \frac{1}{z-m} \right)$$

Fact 5.6. $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is spanned by $\{P_m(z), m \in \mathbb{Z}_{>0}\}$ when $k > 2$, later we will see $S_k(\Gamma)$ is finite dimensional

Define $j(z) = \dfrac{G_4(z)^3}{\Delta(z)}, \forall z \in \mathcal{H}$. Then $j \in A_0(\mathrm{SL}_2(\mathbb{Z}))$ induces isomorphism $j : \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}^* \to \mathbb{CP}^1$, thus $A_0(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}(j)$. Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

Clearly $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[G_4, G_6]$ as a graded ring, e.g. $\Delta = \dfrac{1}{1728}(G_4^3 - G_6^2)$, see [Bump 1.3.3, 1.3.4] for simple proof

f is a cusp form iff f(z)Im(z)^k/2 is bounded on H

Lemma 5.7. Let $f \in A_k(\Gamma)$, then $f \in S_k(\Gamma) \iff f(z)\,\mathrm{Im}(z)^{\frac{k}{2}}$ is bounded on $\mathcal{H}$

Proof. Let $t = \sigma\infty \in P_\Gamma$, $\sigma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$, Fourier expansion at t, $(f \cdot_k \sigma)(z) = \sum_n a_n e^{2\pi i n z/h}$

$$|f(\sigma z)\,\mathrm{Im}(\sigma z)^{\frac{k}{2}}| = |(cz+d)^k (f \cdot_k \sigma)(z) |cz+d|^{-k}\,\mathrm{Im}(z)^{\frac{k}{2}}| = |\sum_n a_n e^{2\pi i n z/h}|\,\mathrm{Im}(z)^{\frac{k}{2}}$$

Bounded when $\mathrm{Im}(z) \to \infty \iff a_n = 0, \forall n \leq 0$ □

Suppose $f_1, f_2 \in M_k(\Gamma)$, at least one in $S_k(\Gamma)$, then $f_1 f_2 \in S_{2k}(\Gamma)$, so $f_1 f_2\,\mathrm{Im}(z)^k$ is bounded by Lemma 5.7. Also $f_1(\gamma z)\overline{f_2(\gamma z)}\,\mathrm{Im}(\gamma z)^k = f_1(z)\overline{f_2(z)}\,\mathrm{Im}(z)^k, \forall \gamma \in \Gamma$. So we have a well-defined integral

$$(f_1, f_2) = \int_{\Gamma \backslash \mathcal{H}} f_1(z)\overline{f_2(z)}\,\mathrm{Im}(z)^k \frac{dxdy}{y^2}$$

Which is called Peterson inner product

Exercise 5.8. $(f, E_k) = 0$, $\forall f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, for $k > 2$ is even

In particular, $\forall f\in S_k(\Gamma)$, the function $\tilde f(z)=f(z)\,\mathrm{Im}(z)^{\frac{k}{2}}$ satisfies $|\tilde f(\gamma z)|=|\tilde f(z)|$, $\forall\gamma\in\Gamma$ and $\displaystyle\int_{\Gamma\backslash\mathcal{H}}|\tilde f(z)|^2d\mu<\infty$. $\tilde f(z)$ is almost in $L^2(\Gamma\backslash\mathcal{H})$ but not quite, since $\tilde f(\gamma z)=e(\gamma,z)^k\tilde f(z)$, $\forall\gamma\in\Gamma$ where $e(\gamma,z)=\dfrac{j(\gamma,z)}{|j(\gamma,z)|}$. $\tilde f$ is an example of a Maass (cusp) form of weight k, in particular, it is eigenfunction of $\Delta_k=-y^2\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)+iky\dfrac{\partial}{\partial x}$, f is holomorphic $\iff L_k\tilde f=0$, where $L_k=-(z-\bar z)\dfrac{\partial}{\partial\bar z}-\dfrac{k}{2}$ is the Maass lowering operator. To better understand these, consider $\Gamma\backslash\mathrm{GL}_2(\mathbb{R})^+$ instead of $\Gamma\backslash\mathcal{H}$. Let $g=\begin{bmatrix}a&b\\c&d\end{bmatrix}\in\mathrm{GL}_2(\mathbb{R})^+$ with $gi=z$, $e(\gamma,z)=e(\gamma,gi)=\dfrac{e(\gamma g,i)}{e(g,i)}$. Define $\phi_f(g)=\tilde f(gi)e(g,i)^{-k}=f(gi)\det(g)^{\frac{k}{2}}j(g,i)^{-k}=f(\frac{ai+b}{ci+d})(ad-bc)^{\frac{k}{2}}(ci+d)^{-k}$. Recall $f\in S_k(\Gamma)$, then we get

1. $\phi_f(\gamma g)=\phi_f(g),\forall\gamma\in\Gamma$
2. $\phi_f\left(g\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}\right)=e^{ik\theta}\phi_f(g),\forall\theta$
3. $\displaystyle\int_{\Gamma\backslash\mathcal{H}}|\tilde f(z)|^2d\mu<\infty\Rightarrow\phi_f\in L^2(\Gamma\backslash\mathrm{GL}_2(\mathbb{R})^+/Z^+)=L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{R}))$, here $Z^+=\left\{\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}\middle|\lambda>0\right\}$, $\mathrm{GL}_2(\mathbb{R})^+/Z^+=\mathrm{SL}_2(\mathbb{R})$

Haar measure on $\mathrm{GL}_2(\mathbb{R})^+$: Each element in $\mathrm{GL}_2(\mathbb{R})^+$ can be written uniquely as

$$g=\lambda\begin{bmatrix}1&x\\0&1\end{bmatrix}\begin{bmatrix}y^{\frac12}&0\\0&y^{-\frac12}\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$$

Here $\lambda>0$, $y>0$, $x\in\mathbb{R}$, $\theta\in[0,2\pi)$, this is the *Iwasawa decomposition* $\mathrm{SL}_2(\mathbb{R})=NAK$, $dg=\dfrac{d\lambda}{\lambda}\dfrac{dxdy}{y^2}d\theta$

$\phi_f\in C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbb{R})^+)$ is a eigenfunction of $Z(\mathfrak{gl}_2)$(inducing Δ_k), annihilated by certain nilpotent element in $\mathfrak{gl}_2$(inducing L_k)

f is a cusp form $\iff\displaystyle\int_{U\cap\Gamma\backslash U}\phi_f(Ug)du=0,\forall g$, for all unipotent subgroup $U\subseteq\mathrm{SL}_2(\mathbb{R})$ such that $U\cap\Gamma=\{1\}$

6 Automorphic forms on $\mathrm{GL}(2,\mathbb{R})$

Let $G = \mathrm{GL}_2(\mathbb{R})^+$, $G_1 = \mathrm{SL}_2(\mathbb{R}) \supseteq K = \mathrm{SO}(2)$, $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{gl}_2(\mathbb{R})$, $\mathfrak{g}_\mathbb{C} = \mathfrak{gl}_2(\mathbb{C})$, $\mathbb{R}^\times \cong Z_G \supseteq Z_G^+ \cong \mathbb{R}_{>0}$ are the centers. A standard basis for $\mathfrak{g}$ is

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Their relations are $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$
$r : G \to \mathrm{End}_\mathbb{C}(C^\infty(G))$ defines a right regular representation

$$(r_g f)(x) = f(xg), \forall g, x \in G, f \in C^\infty(G)$$

This gives a representation of $\mathfrak{g}_\mathbb{C}$

$$(Xf)(g) = \frac{d}{dt} f(ge^{tX}), \forall \mathrm{X} \in \mathfrak{g}$$

The universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$ can be identified with left-invariant differential operators on G, $Z_\mathfrak{g}$ is the center of $U(\mathfrak{g}_\mathbb{C})$, which can be identified with bi-invariant differential operators on G
The Harish-Chandra isomorphism $Z_\mathfrak{g} \cong \mathbb{C}[Z,\Delta]$, where $\Delta = -\frac{1}{4}(H^2 + 2EF + 2FE)$ is the casimir element. In particular, $Z_\mathfrak{g}$ induces G-invariant differential operator on $\mathcal{H} = G_1/K$, Δ induces $-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

Definition 6.1. $f \in C^\infty(G)$ is $Z_\mathfrak{g}$-finite if the span of $\{Df, D \in Z_\mathfrak{g}\}$ is finite dimensional, it is (right) K-finite if the span of $\{r_k f, k \in K\}$ is finite dimensional. Let $\omega : Z_G \to \mathbb{C}^\times$ be a unitary character

$$C^\infty(G,\omega) = \{f \in C^\infty(G) | f(zg) = \omega(z)f(g), \forall z \in Z_G, g \in G\}$$

f is Z_g finite and K finite, then f is real analytic

Lemma 6.2. If $f \in C^\infty(G)$ is $Z_\mathfrak{g}$-finite and K-finite, then f is real analytic

Proof. We may assume f is an eigenfunction of K and $Z_\mathfrak{g}$ finite $\Rightarrow P(\tilde{\Delta})f = 0$ for some polynomial P with constant coefficient, where

$$\begin{aligned} \tilde{\Delta} &= -\frac{1}{4}(H^2 + 2EF + 2FE + Z^2) \\ &= \underbrace{-\frac{1}{4}(H^2 + 2E^2 + 2F^2 + Z^2)}_{\text{elliptic differential operator}} + \underbrace{\frac{1}{2}(E-F)^2}_{\text{scalar on } f} \end{aligned}$$

$\mathrm{Lie}\, K = \mathbb{R} \cdot (E - F)$. Hence $P(\tilde{\Delta})$ has the same effect as an elliptic differential operator with analytic coefficient, f is analytic □

Remark 6.3. Lemma 6.2 is true under weaker assumption that f is locally integrable

Definition 6.4. Let $\Gamma \leq G_1$ be a discrete subgroup with $\mathrm{Vol}(\Gamma\backslash G_1) < \infty (\Leftrightarrow \mathrm{Vol}(\Gamma\backslash\mathcal{H}) < \infty)$. Let $\chi : \Gamma \to \mathbb{C}^\times$, $\omega : Z_G \to \mathbb{C}^\times$ be unitary characters that agree on $Z_\Gamma = \Gamma \cap Z_G$. An automrophic form on G with character χ, ω is a function $\phi \in C^\infty(G,\omega)$ satisfying

1. $\phi(\gamma g) = \chi(\gamma)\phi(g), \forall \gamma \in \Gamma, g \in G$
2. ϕ is (right) K-finite
3. ϕ is $Z_\mathfrak{g}$-finite
4. ϕ is of moderarte growth

Denote $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ the space of such functions. Moderate growth means $\exists C, N > 0$ such that $|\phi(g)| < C\|g\|^N, \forall g \in G$, $\|g\|$ is the Euclidean norm of $(g, \det g^{-1}) \in M_2(\mathbb{R}) \oplus \mathbb{R} \cong \mathbb{R}^5$

Definition 6.5. $\phi \in \mathcal{A}(\Gamma\backslash G, \chi, \omega)$ is a *cusp form* if

$$\int_{U\cap\Gamma\backslash U} \phi(ug)du = 0, \forall g, \forall \text{ unipotent subgroup } U \subseteq G \text{ with } U \cap \Gamma \neq \{1\}$$

Deonte $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ the space of cusp forms

Example 6.6. $\phi_f(g) = f(gi)\det(g)^{\frac{k}{2}} j(g,i)^{-k}, \forall g \in G$

$$\begin{array}{ccc} M_k(\Gamma) & \hookrightarrow & \mathcal{A}(\Gamma\backslash G/Z_G^+) \\ \uparrow & & \uparrow \\ S_k(\Gamma) & \hookrightarrow & \mathcal{A}_0(\Gamma\backslash G/Z_G^+) \end{array}$$

Remark 6.7. Since r_g commutes with $Z_{\mathfrak{g}}, \forall g \in G$, $Z_{\mathfrak{g}}$-finite condition is preserved under r_g, condition 1,4 are also preserved. But condition 3 may not in general: If f is K-finite, then $r_g f$ is gKg^{-1}-finite. At least we get a representation of K on $\mathcal{A}, \mathcal{A}_0$, so they split as direct sum of K-eigenspaces (weight spaces)

Theorem 6.8. $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ and $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ are stable under the action of $U(\mathfrak{g}_{\mathbb{C}})$ (induced from the right regular representation on $C^\infty(G)$)

Proof. Easy to see: automorphy, $Z_{\mathfrak{g}}$-finite, cuspidal conditions are preserved by $\mathfrak{g}$, K-finiteness is also note hard. The tricky part is $\mathfrak{g}$ preserves growth condition (even the exponent N)
$\mathfrak{g}$ preserves K-finiteness: Consider another basis of $\mathfrak{sl}_2(\mathbb{C})$, $W = C^{-1}HC = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $R = C^{-1}EC$, $L = C^{-1}FC$, here $C = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$ is the Cayley transform. Then $\mathrm{Lie}(K)_{\mathbb{C}} = \mathbb{C}\cdot W$. If $\phi \in C^\infty(G)$ has weight m, i.e.

$$\phi\left(g\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = e^{im\theta}\phi(g), \forall g \in G$$

Then $W\phi = m\phi$ since $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = e^{i\theta W}$. Since $[W,R] = 2R$, $[W,L] = -2L$, we see that R raises weight by 2, L lowers weight by 2. So K-finiteness is preserved by $\mathfrak{g}$
$\mathfrak{g}$ preserves growth condition: By Harish-Chandra theorem 6.9, $\exists \alpha \in C_c^\infty(G)$ such that $\phi = \phi * \alpha$ where $(\phi * \alpha)(g) = \int_G \phi(x)\alpha(x^{-1}g)dx$. Then $\forall \mathrm{X} \in \mathfrak{g}$, $X\phi = X(\phi*\alpha) = \phi * (X\alpha)$ satisfies growth condition with same exponent as ϕ ☐

Harish-Chandra theorem

Theorem 6.9 (Harish-Chandra)**.** Let $f \in C^\infty(G)$ be $Z_{\mathfrak{g}}$-finite and K-finite. Let U be a neighborhood of 1 in G. Then $\exists \alpha \in I_c^\infty(G)$ with support in U such that $f = f * \alpha$. Here

$$I_c^\infty(G) = \{\alpha \in C_c^\infty(G), \alpha(kxk^{-1}) = \alpha(x), \forall k \in K, x \in G\}$$

Recall: Frechet topology on $C^\infty(G)$: $X \subseteq G$ compact subset, $N \in \mathbb{Z}_{\geq 0}$. Semi-norm

$$\|f\|_{X,N} = \max\{|Df(x)||D \in U(\mathfrak{g}_{\mathbb{C}}) \text{ of order } \leq N, x \in X\}$$

A neighborhood base at 0: $\{f \in C^\infty(G)|\|f\|_{X,N} < \epsilon\}$. In this topology, $f_n \to f$ if $\forall D \in U(\mathfrak{g}_{\mathbb{C}})$, $Df_n \to Df$ uniformly on any compact subset of G

Proof. Use Proposition 6.12 ☐

(g,k)-modules

Definition 6.10. A $(\mathfrak{g}, K)$-module is a $\mathbb{C}$ vector space V with a representation of $\mathfrak{g}_{\mathbb{C}}$ and K such that

1. Every $v \in V$ is K-finite
2. $\forall \mathrm{X} \in \mathrm{Lie}(K)$, $\left.\dfrac{d}{dt}\right|_{t=0} (e^{tX}\cdot v) = X \cdot v$, $\forall v \in V$

3. $\forall X \in \mathfrak{g}, k \in K, v \in V$, $k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$

Denote

$$V(n) = \{v \in V, k \cdot v = \chi_n(k)v, \forall k \in K\}$$

Where $\chi_n\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = e^{in\theta}$, $\forall \theta \in \mathbb{R}$. A $(\mathfrak{g}, K)$-module V is *admissible* if $\dim V(n) < \infty, \forall n \in \mathbb{Z}$

Remark 6.11. 1. $\iff V = \bigoplus_{n\in\mathbb{Z}} V(n) \iff V$ is a the direct sum of finite dimensional K-invariant subspaces (Zorn's Lemma $\Rightarrow V(n)$ has a basis consisting of K-eigenvectors)
1. $\Rightarrow t \mapsto e^{tX} \cdot v$ is differentiable $\forall X \in \mathrm{Lie}(K)$, $v \in V$, so 2. makes sense
Proposition 6.12 $\Rightarrow U(\mathfrak{g}_{\mathbb{C}})f$ is an admissible $(\mathfrak{g}, K)$-module for any K-finite, $Z_{\mathfrak{g}}$-finite $f \in C^\infty(G)$

f Z_g finite, K finite, decompose U(g_C)f

Proposition 6.12. Let $f \in C^\infty(G)$ be $Z_{\mathfrak{g}}$-finite and K-finite. Let V be the closure of $U(\mathfrak{g}_{\mathbb{C}}) \cdot f$ in $C^\infty(G)$. Then V is (right) G-invariant. Moreover, each K-weight space $V(n)$ of V is finite dimensional and $U(\mathfrak{g}_{\mathbb{C}}) \cdot f = \bigoplus_{n\in\mathbb{Z}} V(n)$

Proof. Let $V_0 = U(\mathfrak{g}_{\mathbb{C}})f$, so $V = \overline{V_0}$
Step 1: Show that V is G-stable
Let $\tilde{V} \subseteq C^\infty(G)$ be the smallest closed G-invariant subspace containing V_0. Then $V \subseteq \tilde{V}$. Suppose $V \neq \tilde{V}$. Then $\exists$ continuous nonzero linear functional λ on $\tilde{V}$ such that $\lambda(V) = 0$ by Hahn-Banach. Consider function $\phi(g) = \lambda(r_g f)$. Easy to check: f is $Z_{\mathfrak{g}}$-finite, K-finite $\Rightarrow$ same for ϕ. By Lemma 6.2, ϕ is analytic. On the other hand, $\forall D \in U(\mathfrak{g}_{\mathbb{C}})$, $D\phi = \lambda(Df) = 0$ since $Df \in V_0 \Rightarrow \phi = 0 \Rightarrow \lambda$ vanish on the dense subspace $G \cdot f$ in $\tilde{V} \Rightarrow \lambda = 0$, contradiction
Step 2: Let $V_0(n) = \{v \in V | Wv = nv\} = V_0 \cap V(n)$, we claim that $V_0 = \bigoplus_{n\in\mathbb{Z}} V(n)$. $\forall n$, consider the projector $E_n : C^\infty(G) \to C^\infty(G)$

$$(E_n\phi)(g) = \int_K \phi(gk^{-1})\chi_n(k)dk, \mathrm{Vol}(K, dk) = 1$$

E_n is continuous, identity on $V(n)$, $E_nV = V(n)$, need to show $E_nV_0 \subseteq V_0$. $\forall v \in V_0$, $v = \sum_{n=-M}^{M} E_n v$, fix $m \in [-M, M]$, let P be a polynomial such that $P(m) = 1$ and $P(n) = 0$ for any $n \in [-M, M], n \neq m$, then

$$V_0 \ni P(W)v = \sum_{n=-M}^{M} P(n)E_n v = E_m v \Rightarrow E_m v \in V_0(m)$$

Thus $V_0(n) = E_nV$, so it is dense in $V(n) = E_nV$ and $V_0 = \bigoplus V_0(n)$
Step 3: Remains to show $\dim V_0(n) < \infty$ (Then by density, $V_0(n) = V(n)$ and hence $V_0 = \oplus V(n)$). $f = \sum_{n=-M}^{M} E_n f$, $Z_{\mathfrak{g}}E_n f = E_n Z_{\mathfrak{g}} f$ is finite dimensional $\forall n \Rightarrow E_n f$ is $Z_{\mathfrak{g}}$-finite $\forall -M \le n \le M$. So we may assume $f \in V(n_0)$ for some $n_0 \in \mathbb{Z}$. By PBW, $\{R^a L^b W^c\}_{a,b,c\ge 0}$ form a $\mathbb{C}$-basis of $U(\mathfrak{g}_{\mathbb{C}})$
Recall

$$-4\Delta = W^2 + 2W + 4LR = W^2 - 2W + 4RL \Rightarrow U(\mathfrak{g}_{\mathbb{C}}) = \sum_{i>0} R^i A + \sum_{i\ge 0} L^i A$$

Where A is the subalgebra generated by $Z_{\mathfrak{g}}$ and W. Let $\{f_1, \cdots, f_r\} \subseteq V(n_0)$ be a basis of $Z_{\mathfrak{g}}f$, then

$$V_0 = \sum_{\alpha=1}^{r}\left(\sum_{i>0} \mathbb{C}R^i f_\alpha + \sum_{i\ge 0} \mathbb{C}L^i f_\alpha\right)$$
$$\Rightarrow V_0(n) = \sum_{\alpha=1}^{r}\left(\sum_{i>0} \mathbb{C}R^{\frac{n-n_0}{2}} f_\alpha + \sum_{i\ge 0} \mathbb{C}L^{\frac{n-n_0}{2}} f_\alpha\right)$$

is of finite dimensional $\square$

Summary: if $f \in C^\infty(G)$ is $Z_{\mathfrak{g}}$ finite and K-finite, we have

1. $(U\mathfrak{g}_{\mathbb{C}} f)$ is an admissible $(\mathfrak{g}, K)$-module

2. If f has moderate growth with exponent $N > 0$, then the same is true for Df, $\forall D \in U\mathfrak{g}_{\mathbb{C}}$

In particular, $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega) \subseteq \mathcal{A}(\Gamma\backslash G, \chi, \omega)$ are $(\mathfrak{g}, K)$-modules
The $(\mathfrak{g}, K)$-module structure on $\mathcal{A}(\Gamma\backslash G, \chi, \omega)$ is complicated: $Z_{\mathfrak{g}}$ does not act semi-simply if Γ has cusps. Simpler on $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$, decompose into direct sum of irreducible admissible $(\mathfrak{g}, K)$-modules with finite multiplicity. We will prove this by L^2 theory

Schur's lemma

Lemma 6.13 (Schur)**.** $Z_{\mathfrak{g}}$ acts by a character on any irreducible admissible $(\mathfrak{g}, K)$-module (infinitesimal character)

Proof. Let $n \in \mathbb{Z}$ with $V(n) \neq 0$. $Z_{\mathfrak{g}}$ commutes with K action by 3. in Definition 6.10. Admissible $\Rightarrow \dim V(n) < \infty \Rightarrow Z_{\mathfrak{g}}$ acts by a character $\eta : Z_{\mathfrak{g}} \to \mathbb{C}$ on $V(n)$. $U\mathfrak{g}_{\mathbb{C}} V(n)$ is K-stable by 3. Irreducible $\Rightarrow V = U(\mathfrak{g}_{\mathbb{C}})V(n) \Rightarrow Z_{\mathfrak{g}}$ acts by η on V □

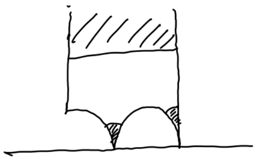
Another theorem of Harish-Chandra

Theorem 6.14 (Harish-Chandra)**.** Let $J \subseteq Z_{\mathfrak{g}}$ be an ideal of finite codimension. Then $\mathcal{A}(\Gamma\backslash G, \chi, \omega)[J]$ (subspace annihilated by J) is an admissible $(\mathfrak{g}, K)$-module. We will only prove the special case for $\mathcal{A}_0$

7 Growth conditions and fundamental estimates

Reference: [Borel, Automorphic forms on $\mathrm{SL}_2(\mathbb{R})$]
$N = \left\{\begin{bmatrix}1 & * \\ 0 & 1\end{bmatrix}\right\} \subseteq B = \left\{\begin{bmatrix}* & * \\ 0 & *\end{bmatrix}\right\} \subseteq G = \mathrm{GL}_2(\mathbb{R})^+ \supseteq G_1 = \mathrm{SL}_2(\mathbb{R}) \supseteq K = \mathrm{SO}(2)$, $A = \left\{\begin{bmatrix}* & 0 \\ 0 & *\end{bmatrix}\right\}$, $B = NA$, Iwasawa decomposition: $G = NAK$. $\Gamma \leq G_1$ is a discrete subgroup, $\mathrm{Vol}(\Gamma\backslash G_1) < \infty$. α is a obvious map $\mathrm{GL}_2(\mathbb{R}) \to \mathbb{CP}^1$

Definition 7.1 (Siegel set)**.** For $a = \begin{bmatrix}a_1 & 0 \\ 0 & a_2\end{bmatrix} \in A$, $\alpha(a) = \frac{a_1}{a_2} = \mathrm{Im}(a \cdot i)$, $\forall t > 0$, let $A_t = \{a \in A | \alpha(a) > t\}$. Suppose $\infty \in P_\Gamma$ is a cusp of Γ, then $\Gamma \cap N = \left\langle \begin{bmatrix}1 & h \\ 0 & 1\end{bmatrix} \right\rangle \cong \mathbb{Z}$, for some $h > 0$. Siegel sets at ∞ are of the form $S_t = \Omega A_t K$, $t > 0$, $\Omega \subseteq N$ is a compact subset containing an interval of length h. In general, for a cusp $s = \sigma\infty$, Siegel sets at s are $S_t = \sigma\Omega A_t \sigma^{-1} K$



Let $\Gamma\backslash P_\Gamma = \{s_1, \cdots, s_n\}$, One can find Siegel sets S_i at each cusp s_i and a compact mod center set $C \subseteq G$ suc that $C \cup \bigcup_{i=1}^n S_i$ is a fundamental domain for $\Gamma\backslash G$

Definition 7.2 (Growth conditions)**.** Suppose $\infty \in P_\Gamma$, a function $\phi : \Gamma\backslash G \to \mathbb{C}$ has moderate growth at ∞ if $\exists$ Siegel set $S_t = \Omega A_t K$ and $\lambda \in \mathbb{R}$ such that $\phi(g) << \mathrm{Im}(g \cdot i)^\lambda$, $\forall g \in S_t$. If this holds for all $\lambda \in \mathbb{R}$, we say ϕ has rapid decay at ∞. In general, for a cusp $s = \sigma\infty \in P_\Gamma$, we say ϕ has moderate growth/repid decay at s if the function $\phi_\sigma(g) = \phi(\sigma g)$ is so at $\infty \in P_{\sigma^{-1}\Gamma\sigma}$

phi moderate growth at all cusps <=> phi moderate growth on G

Proposition 7.3. ϕ has moderate growth at all cusps of $\Gamma \iff \phi$ has moderate growth on G(as defined last time), see [Borel-5.11]

Definition 7.4. $\omega : Z_G \to \mathbb{C}^\times$ is a unitary character, define

$$L^2(\Gamma\backslash G, \omega) = \{f : \Gamma\backslash G \to \mathbb{C} | f(zg) = \omega(z)f(g), \forall z \in Z_G, \|f\|_2 = \int_{\Gamma\backslash G/Z_G} |f|^2 < \infty\}$$

|f*phi(g)|«_phi Im(gi)|f|_2

Proposition 7.5. Suppose $\infty \in P_\Gamma$, let $\phi \in C_c(G)$, $t > 0$, then $|f * \phi(g)| <<_\phi \mathrm{Im}(gi)\|f\|_2$, $\forall g \in NA_tK$, $f \in L^2(\Gamma\backslash G, \omega)$

Proof. Let $C \subseteq G$ be a compact subset such that $C^{-1} = \mathrm{Supp}\,\phi$, then

$$|f * \phi(g)| = |\int_G f(gx)\phi(x^{-1})dx| \leq \|\phi\|_\infty \int_{gC} |f|$$

Let $g \in \lambda \begin{bmatrix}1 & x \\ 0 & 1\end{bmatrix} \begin{bmatrix}y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}}\end{bmatrix} K$, $y = \mathrm{Im}(gi) > t$. Let $C_N \subseteq N$, $C_A \subset A$ closed intervals such that $KC \subseteq C_N C_A K$ Then $gC \subseteq \lambda \begin{bmatrix}1 & x \\ 1 & 1\end{bmatrix} y^{-\frac{1}{2}} \begin{bmatrix}y & 0 \\ 0 & 1\end{bmatrix} C_N C_A K = \lambda y^{-\frac{1}{2}} \begin{bmatrix}1 & x \\ 1 & 1\end{bmatrix} (ad \begin{bmatrix}y & 0 \\ 0 & 1\end{bmatrix} C_N) \begin{bmatrix}y & \\ 0 & 1\end{bmatrix} C_A K$. $ad \begin{bmatrix}y & \\ 0 & 1\end{bmatrix} C_N$ scales C_N by y, Need about $O(g)$ fundemantal domains to cover gC, $\int_{gC} |f| << y \int_C |f| << y\|f\|_1 << y\|f\|_2$

Constant term: Suppose $\infty \in P_\Gamma$, Let $\Gamma_N = \Gamma \cap N = \left\langle \begin{bmatrix}1 & h \\ 0 & 1\end{bmatrix} \right\rangle \cong \mathbb{Z}$, Let ϕ be a left Γ_N invariant function on G

$$\phi_B(g) = \int_{\Gamma_N\backslash N} \phi(ng)dn$$

Then $\phi_B : N\backslash G \to \mathbb{C}$. For other cusps corresponding to conjugates $U \subseteq P$ of $N \subseteq B$, similarly, define $\phi_P : U\backslash G \to \mathbb{C}$ $\square$

Fundamental estimate [Borel-7.4]

Lemma for fundamental estimate

Lemma 7.6. Let $f \in C^\infty(G)$ be left Γ_N invariant, assume $\infty \in P_\Gamma$, let $X_1, \cdots, X_4 \in \mathfrak{g}$ be an $\mathbb{R}$ basis, then $\exists c > 0$ independent of f

$$|f(g) - f_B(g)| \le c|\operatorname{Im}(gi)|^{-1} \sum_{j=1}^{4} |X_j f|_B(g), \forall g \in G$$

A_0 subseteq L^2

Corollary 7.7. Any cusp form $\phi \in \mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ has rapid decay at all cusps. In particular, $|\phi|$ is bounded on G and $\phi \in L^2(\Gamma\backslash G, \chi, \omega)$

Proof. Reduce to show rapid decay at $\infty \in P_\Gamma$, by assumption and Proposition 7.3, $\phi(g) << \operatorname{Im}(gi)^\lambda$ near ∞ for some exponent $\lambda \in \mathbb{R}$. Theorem 6.9 $\Rightarrow D\phi(g) << \operatorname{Im}(gi)^\lambda$ near ∞, $\forall D \in U(\mathfrak{g}_\mathbb{C})$, same bound for $|D\phi|_B$. Take $D = X_j$ in Lemma 7.6, recall $\phi_B = 0$, get $\phi(g) << \operatorname{Im}(gi)^{\lambda-1} \Rightarrow$ same for $D\phi$ and $|D\phi|_B$ Repeatedly apply Lemma 7.6 and Theorem 6.9, we get

$$\phi(g) << \operatorname{Im}(gi)^{\lambda-2}, \cdots \operatorname{Im}(gi)^{\lambda-m}, \forall m > 0$$

near ∞, thus rapid decay ∎

Definition 7.8 (L^2 cusp forms)**.**

$$L_0^2(\Gamma\backslash G, \chi, \omega) = \left\{ \phi \in L^2(\Gamma\backslash G, \chi, \omega) \middle| \int_{\Gamma\cap U\backslash U} \phi(ug)du = 0, \forall \text{ a.e. } g, \text{ unipotent } U \subseteq G \text{ such that } \Gamma \cap U \neq \{1\} \right\}$$

f in L_0^2, |f*g(g)|<=c|f|_2

Corollary 7.9. Let $\phi \in C_c^\infty(G)$, then $\exists c > 0$ such that forall $f \in L_0^2(\Gamma\backslash G, \chi, \omega)$, $|(f*\phi)(g)| \le c\|f\|_2$, for all $g \in G$. Moreover, $f * \phi$ is rapidly decreasing near cusps

Proof. Suffices to prove this near all cusps, reduce to $\infty \in P_\Gamma$, by Proposition 7.5, $|f * \phi(g)| << \operatorname{Im}(gi)\|f\|_2$ for g near ∞

$$|D(f * \phi)(g)| = |f * D\phi(g)| << \operatorname{Im}(gi)\|f\|_2, \forall D \in U(\mathfrak{g}_\mathbb{C})$$

Same for $|D(f * \phi)(g)|_B$. By assumption $(f * \phi)_B = f_B * \phi = 0$. Apply Lemma 7.6 repeatedly $\Rightarrow f * \phi$ decrease rapidly at ∞ ∎

proof of lemma 7.6. Fix $g \in G$, let

$$\phi(t) = f(g) - f\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g\right)$$

then $\phi \in C^\infty(\mathbb{R})$, $\phi(0) = 0$, $\int_0^1 \phi dt = f(g) - f_B(g)$, $|\phi(t)| \le \int_0^1 |\phi'|ds$, $|f(g) - f_B(g)| \le \int_0^1 |\phi|dt \le \int_0^1 |\phi'|dt$ Write $g = \lambda \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix} k$, $k \in K$

$$\begin{aligned} \begin{bmatrix} 1 & t+h \\ 0 & 1 \end{bmatrix} g &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g g^{-1} \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} g \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g k^{-1} \begin{bmatrix} 1 & h/y \\ 0 & 1 \end{bmatrix} k \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g \exp(h \operatorname{ad}(k)^{-1}(y^{-1}E)) \end{aligned}$$

$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Write $\operatorname{ad}(k)^{-1}E = \sum_{j=1}^4 a_j(k)X_j$, $a_j : K \to \mathbb{R}$ are continuous and bounded

$$\begin{aligned} |\phi'(t)| &= \left| y^{-1} \sum_{j=1}^{4} a_j(k)(X_j f)\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g\right) \right| \\ &\le cy^{-1} \sum_{j=1}^{4} \left| X_j f\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} g\right) \right| \end{aligned}$$

$c = \max_{j,k} |a_j(k)|$ is independent of f

$$|f(g) - f_B(g)| \leq \int_0^1 |\phi'(t)| dt \leq c y^{-1} \sum_{j=1}^{4} |\mathrm{X}_f|_B(g)$$

$\square$

8 Cuspidal spectrum

$\chi : \Gamma \to \mathbb{C}^\times$, $\omega : Z_G \to \mathbb{C}^\times$ are unitary charaterss that agree on $Z_\Gamma = \Gamma \cap Z_G$

Proposition 8.1. $L^2_0(\Gamma\backslash G, \chi, \omega)$ is closed in $L^2(\Gamma\backslash G, \chi, \omega)$ [Borel, 8.2]

Proof. Suppose $\infty \in P_\Gamma$, $\forall \phi \in C_c^\infty(N\backslash G)$, let

$$\lambda_{B,\phi}(f) = \int_{\Gamma_N\backslash G} f(x)\phi(x)dx$$

Then

$$\lambda_{B,\phi}(f) = \int_{N\backslash G} \phi(x) \left(\int_{\Gamma_N\backslash} f(nx)dx \right) dx = \int_{N\backslash G} \phi(x) f_B(x)dx$$

$f_B = 0 \iff \lambda_{B,\phi}(f) = 0$, $\forall \phi \Rightarrow L^2_0 = \bigcap_{P,\phi} \ker \lambda_{P,\phi}$, it suffices to show $\lambda_{P,\phi}$ is continuous. $\exists$ compact set $E \subseteq G$ such that $\mathrm{Supp}(\phi) \subseteq \Gamma_N E \Rightarrow |\lambda_{P,\phi}| \le \|\phi\|_\infty \int_E |f| << \|f\|_2$ ☐

So $L^2_0(\Gamma\backslash G, \chi, \omega)$ is a Hilbert space. Corollary 7.7 $\Rightarrow \mathcal{A}_0(\Gamma\backslash G, \chi, \omega) \subseteq L^2_0(\Gamma\backslash G, \chi, \omega)$

Definition 8.2 (convolution operator)**.** $\phi \in C_c^\infty(G)$, $f \in L^2(\Gamma\backslash G, \chi, \omega)$, $(r_\phi f)(g) = \int_G f(gh)\phi(h)dh = f * \phi^\vee$, $\phi^\vee(x) = \phi(x^{-1})$. $r_\phi f \in C^\infty \cap L^2$. If $f \in L^2_0$, then $r_\phi f \in C^\infty \cap L^2_0$ since

$$\int_{\Gamma\cap U\backslash U} (r_\phi f)(ug)du = \int_G \int_{\Gamma\cap U\backslash\cap U} f(ugh)\phi(h)dudh = \int_G f_B(gh)\phi(h)dh = 0$$

So r_ϕ preserves the subspace L^2_0, moreover, Corollary 7.9 $\Rightarrow r_\phi f$ has rapid decay at all cusps if $f \in L^2_0$

Recall: L is a bounded linear operator on Hilbert space H

1. L is *compact* if maps bounded sets to precompact sets. L is compact $\iff$ L is the limit of finite rank operators
2. L is *Hilbert-Schmidt* if H is separable and for any orthonormal basis $\{e_i\}$, $\sum(Le_i.Le_i)$ is finite. Hilbert-Schmidt operators are compact
3. L is self-adjoint if $(Lv, w) = (v, Lw), \forall v, w \in H$

r_phi:L_0^2->L_0^2 is Hilbert-Schmidt

Proposition 8.3. For any $\phi \in C_c^\infty(G)$, $r_\phi : L^2_0 \to L^2_0$ is a Hilbert-Schmidt operator, hence a compact operator

Borel 9.5. , Corollary 7.9, $|(r_\phi f)(x)| \le c\|f\|_2, \forall x \in G$, $f \mapsto r_\phi f$ is a continuous linear functional on L^2_0

$$r_\phi f(x) = (f, K_x) = \int_{\Gamma\backslash G/Z_G} f(y)\overline{K_x(y)}dy$$

for some $K_x \in L^2_0$, $(K_x, K_x) = r_\phi K_x(x) \le c\|K_x\|_2$, thus $\|K_x\|_2 \le c$. Let $K(x,y) = \overline{K_x(y)}$, then $|K(x,y)| \in L^2(\Gamma\backslash G/Z_G \times \Gamma\backslash G/Z_G)$ and $\Rightarrow r_\phi$ is Hilbert-Schmidt [Bump Theorem 2.3.2] ☐

Remark 8.4. Combined with Dixmier-Malliavin theorem (any $\phi \in C_c^\infty(G)$ is a finite linear combination of convolutions $\alpha * \beta$, $\alpha, \beta \in C_c^\infty(G)$), one deduce that r_ϕ is of trace class on L^2_0, so $\mathrm{Tr}(r_\phi) = \int K(x,x)dx < \infty$. However, its kernel $K(x,y)$ is not explicit in general. When $\Gamma\backslash\mathcal{H}$ is compact, $L^2_0 = L^2$ and $K(x,y)$ coincides with the explicit naive kernel (see [Bump prop 2.3.1])

$$K^{\mathrm{naive}}(x,y) = \int_{Z_G} \sum_{\gamma\in\Gamma} \chi(\gamma)\phi(x^{-1}\gamma yu)\omega(u)du$$

Then $\mathrm{Tr}(r_\phi) = \int_{\Gamma\backslash G/Z_G} K^{\mathrm{naive}}(x,x)dx$ = explicit expression involving conjugacy classes of Γ. In general, when there are cusps, the naive kernel is not L^2, so r_ϕ is not compact on L^2, $r_\phi|_{L^2_0}$ suffices for the purpose of spectral decomposition of L^2_0. To get formula for $\mathrm{Tr}(r_\phi|L^2_0)$, need to truncate the explicit naive kernel. As comparison, nonzero convolution operators on $L^2(\mathbb{R})$ are never compact. $L^2(\mathbb{R})$ does not have irreducible subrepresentations of $\mathbb{R}$, but decompose into direct integrals of $\mathbb{R}$-irreducible representations

Theorem 8.5. Let T be a compact self-adjoint operator on a separable Hilbert space H, then H has an orthonomral basis $\{\phi_i\}$ consisting of eigenvectors of T, $T\phi_i = \lambda_i\phi_i$. If $\dim H = \infty$, then $\lambda_i \to 0$. In particular, if $\lambda \neq 0$ is an eigenvalue of T, then the λ-eigentsapce is finite dimensional. See [Bump Theorem 2.3.1]

Lemma 8.6. $\phi \in C_c^\infty(G)$

1. If $\phi(g) = \overline{r(g^{-1})}$, $\forall g \in G$, then r_ϕ is self-adjoint
2. If $\phi(k_\theta g) = e^{im\theta}\phi(g)$, then $r_\phi(L^2) \subseteq C^\infty(, m) = \{f \in C^\infty | f(gk_\theta) = e^{im\theta}f(g)\}$

Proof. □

Definition 8.7. A representation of G on a Hilbert space H is a homomorphism $\rho : G \to \mathrm{End}(H)$ such that $G \times H \to H$ is continuous ρ is irreducible if there is no nonzero proper closed G-invariant subspace of H

$$H(m) = \{v \in H | \rho(k)v = \chi_m(k)v, \forall k \in K\}$$

Stone-Weierstrass $\Rightarrow H = \widehat{\bigoplus_{m\in\mathbb{Z}}} H(m)$, see [Bump Exercise 2.1.5] ρ is admissible if $\dim H(m) < \infty, \forall m \in \mathbb{Z}$. $\phi \in C_c(G)$, $\rho(\phi)v = \int_G \phi(g)\rho(g)v dg$, $\forall v \in H$, this is the unique element in H such that

$$(\rho(\phi)v, w) = \int_G \phi(g)(\rho(g), w)dg$$

If $\phi(g) = \overline{\phi(g^{-1})}$, then $\rho(\phi)$ is self-adjoint

rho(phi)v=v

Lemma 8.8. ρ is a unitary representation of G on a Hilbert space H, let $0 \neq v \in H$, $\epsilon > 0$, then $\exists \phi \in C_c^\infty(G)$ such that $\rho(\phi)$ is self-adjoint and $|\rho(\phi)v - v| < \epsilon$. In particular, if $|v| > \epsilon$, this implies $\rho(\phi)v \neq 0$. Moreover, if $v \in H(m)$, we may choose $\phi \in C_c^\infty(K\backslash G/K, m)$, i.e. $\phi(k_\theta g) = \phi(gk_\theta) = e^{-im\theta}\phi(g)$, $\forall g \in G, \theta \in \mathbb{R}$. If $\dim H(m) < \infty$, then $\exists \phi \in C_c^\infty(K\backslash G/K, m)$ such that $\rho(\phi)v = v$

Bump Lemma 2.3.2. . Use continuity of the map $G \to H$, $g \mapsto \rho(g)v$ to find ϕ, replace ϕ by $\phi(g) + \overline{\phi(g^{-1})}$ to make it self-adjoint. Suppose $v \in H(m)$. May assume ϕ is K-conjugation invariant by averaging over K. Let

$$\tilde{\phi}(g) = \int_0^{2\pi} e^{im\theta}\phi(k_\theta g)\frac{d\theta}{2\pi}$$

ϕ is self-adjoint, K-conjugation invariant $\Rightarrow \tilde{\phi} \in C_c^\infty(K\backslash G/K, m)$ is self-adjoint and $\rho(\tilde{\phi})v = \rho(\phi)v$. Have shown $v \in \overline{\rho(C_c^\infty(K\backslash G/K, m)) \cdot v}$. If $\dim H(m) < \infty$, then deduce $v \in \rho(C_c^\infty(K\backslash G/K, m)) \cdot v$ □

Theorem 8.9. The right regular representation of g on L_0^2 decompose into Hilbert space direact sum of irreducible representations of G

Proof. By Zorn's lemma, suffice to show that for any closed G invariant subspace $0 \neq H \leq L_0^2$, H contains an irreducible subrepresentation of G. Let $0 \neq f \in H$, $\exists \phi \in C_c^\infty(G)$ such that r_ϕ is self-adjoint and $r_\phi f \neq 0$, by Proposition 8.3, r_ϕ is compact, by Theorem 8.5, r_ϕ has a nonzero eigenvaule α with finite dim eigenspace $L \leq H$. Let L_0 be a minimal element of the set

$$\{0 \neq L \cap W | W \leq H \text{ closed } G \text{ invaraint}\}$$

$V = \bigcap_{W \leq H, W\cap L = L_0} W$, show V is irreducible. Supoose $V = V_1 \oplus V_2$, $0 \neq f_0 \in L_0 \subseteq V$, write $f_0 = f_1 + f_2$, supose $f_1 \neq 0$

$$(r_\phi f_1 - \lambda f_1) + (r_\phi f_2 - \lambda f_2) = (r_\phi f_0 - \lambda f_0) = 0$$

Thus $r_\phi f_1 = \lambda f_i$, $0 \neq f_1 \in L \cap V_1 \leq L_0$, by the minimality of L_0, $L \cap V_1 = V_0$, $V_1 = V$ by the definition of V □

Next analyze structure of irreducible representations in L_0^2. $\forall m \in \mathbb{Z}$, let $E_m = \chi_m(k)^{-1}dk$, $\chi\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = e^{im\theta}$, $dk = \dfrac{d\theta}{2\pi}$ is the Haar measure on K

For any representation of G on Hilbert space H, $E_m v = \int_K \chi_m(k)^{-1} k \cdot v dk$. $E_m(H) = H(m)$ is the idempotent projection to $H(m)$. $\forall \phi \in C_c^\infty(G)$,

$$\phi * E_m(g) = \int_K \phi(gk)\chi_m(k)dk$$

$$E_m * \phi(g) = \int_K \chi_m(k)\phi(kg)dk$$

Hence $E_m * C_c^\infty(G) * E_m = C_c^\infty(K\backslash G/K, m) = \{\phi \in C_c^\infty(G), \phi(kg) = \phi(gk) = \chi_m(k)^{-1}\phi(g)\}$

C_c^infty(K G /K,m) is commutative

Lemma 8.10. The convolution algebra $C_c^\infty(K\backslash G/K, m)$ is commutative

Proof. (Gelfond trick): Consider involution $\iota : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}$, then ι acts as identity on K, A. Gauss decomposition: $G = KAK \Rightarrow \iota$ acts by identity on $C_c^\infty(K\backslash G/K, m) \Rightarrow \phi_1 * \phi_2 = \iota(\phi_1 * \phi_2) = \iota(\phi_2) * \iota(\phi_1) = \phi_2 * \phi_1$ ☐

H<=L_0^2 irreducible, then dim H(m)<=1

Proposition 8.11. Let $H \subseteq L_0^2(\Gamma\backslash G, \chi, \omega)$ be an irreducible G-subrepresentation, then $\dim H(m) \leq 1$, $\forall m \in \mathbb{Z}$. In particular, H is admissible

Proof. Suppose $0 \neq v \in H(m)$. By Lemma 8.8, $\exists \phi \in C_c^\infty(K\backslash G/K, m)$ self-adjoint, $r_\phi(v) \neq 0$. Moreover, r_ϕ is compact by Proposition 8.3. So by Theorem 8.5, r_ϕ has an eigenspace $V \subseteq H(m)$ with eigenvalue $\lambda > 0$, $V \neq 0$ and $\dim V < \infty$. Since $C_c^\infty(K\backslash G/K, m)$ is commutative by Lemma 8.10, $\exists$ one dimension subspace $L \leq V$ preserved by $C_c^\infty(K\backslash G/K, m)$. If $L \neq H(m)$, let $w \in H(m)$ be orthogonal to L, $W = \overline{C_c^\infty(G)w}$, W is G-invariant closed subspace of H (easy to see). $\forall \phi \in C_c^\infty(G)$, $v \in L$, we have

$$\langle r_\phi w, v\rangle = \langle r_\phi w, E_m v\rangle = \langle w, r_{\tilde\phi * E_m} v\rangle = \langle w, r_{E_m * \tilde\phi * E_m} v\rangle$$

where $\tilde\phi(g) = \overline{\phi(g^{-1})}$ and we used that

1. $v, w \in H(m)$
2. E_m is self-adjoint
3. $r_{\tilde\phi}$ is the adjoint of r_ϕ

Since $E_m\tilde\phi * E_m \in C_c^\infty(K\backslash G/K, m)$, $r_{E_m*\tilde\phi*E_m}(v) \in L \Rightarrow \langle r_\phi w, v\rangle = 0 \Rightarrow W \perp L \Rightarrow 0 \neq W \leq H$ is a proper subrepresentation, contradiction. Hene $L = H(m)$ and $\dim H(m) = 1$ ☐

Denote $H_{\text{fin}} = \bigoplus_{m\in\mathbb{Z}} H(m)$, the space of K-finite vectors. H_{fin} is dense in H by Stone-Weierstrass

Theorem 8.12. Let $0 \neq H \subseteq L_0^2(\Gamma\backslash G, \chi, \omega)$ be an irreducible G-representation. Then H_{fin} is an irreducible admissible $(\mathfrak{g}, K)$-submodule of $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$. Moreover, the multiplicity space $\mathrm{Hom}_G(H, L_0^2)$ is finite dimensional

Proof. Let $0 \neq f \in H(m)$, then $\dim H(m) = 1$ by Proposition 8.11. By Lemma 8.8, $\exists \phi \in C_c^\infty(K\backslash G/K, m)$ such that $r_\phi f = f \Rightarrow f \in C^\infty(\Gamma\backslash G, \chi, \omega) \cap H(m)$ has moderate growth by Proposition 7.3 and Corollary 7.9. So

$$H_{\text{fin}} = \bigoplus_{m\in\mathbb{Z}} H(m) \subseteq C^\infty(\Gamma\backslash, \chi, \omega)$$

In particular, H_{fin} is a $(\mathfrak{g}, K)$-module. It is irreducible since H is irreducible G-representation. By Lemma 6.13, $Z_{\mathfrak{g}}$ acts as scalar on $H_{\text{fin}} \Rightarrow H_{\text{fin}} \subseteq \mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$. $\forall T \in \mathrm{Hom}_G(H, L_0^2)$, $r_\phi Tf = Tr_\phi f = Tf \Rightarrow Tf$ lies in 1-eigenspace of the compact operator r_ϕ, by Theorem 8.5, this eigenspace is finite dimensional. Since H is irreducible, T is determined by Tf, we get $\dim \mathrm{Hom}_G(H, L_0^2) < \infty$ ☐

Lemma 8.13. For all irreducible G subrepresentation $0 \neq H \subseteq L_0^2(\Gamma\backslash G, \chi, \omega)$, we have $\mathrm{Hom}_G(H, L_0^2) = \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\mathsf{fin}}, \mathcal{A}_0)$

Proof. Let $0 \neq T \in \mathrm{Hom}_{(\mathfrak{g},K)}(H_{\mathsf{fin}}, \mathcal{A}_0)$, Lemma 6.13 $\Rightarrow (Tv, Tw) = c(v, w)$ for some $c > 0 \Rightarrow$ T can be extended to a bounded linear operator $\tilde{T} : H \to L_0^2(\Gamma\backslash, \chi, \omega)$. Remains to show $\tilde{T}$ is G equivariant. Suffices to show $\forall v \in H_{\mathsf{fin}}$, $f \in L_0^2$, $(\tilde{T}gv, f) = (r_g\tilde{T}v, f)$. Both sides are $Z_{\mathfrak{g}}$-finite, K-finite functions on G. By Lemma 6.2, they are analytic. Their derivatives agree since T is $U\mathfrak{g}_{\mathbb{C}}$ equivariant. They agree at $g = 1 \Rightarrow$ they agree on G since G is connected □

In particular, if H_1, H_2 are irreducible summands of L_0^2 and $H_{1,\mathsf{fin}} \cong H_{2,\mathsf{fin}}$ as $(\mathfrak{g}, K)$-module, then $H_1 \cong H_2$ (This is true for any irreducible unitary representation of G)

Corollary 8.14. $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)$ decomposes into direct sum of irreducible admissible (unitary) $(\mathfrak{g}, K)$-modules with finite multiplicities. Each irreducible summand has the form H_{fin} for some irreducible G subrepresentation $H \subseteq L_0^2(\Gamma\backslash G, \chi, \omega)$ and $\dim \mathrm{Hom}_{\mathfrak{g},K}(H_{\mathsf{fin}}, \mathcal{A}_0) = \dim_G(H, L_0^2) < \infty$

Fact 8.15. There are finitely many non-isomorphic irreducible $(\mathfrak{g}, K)$-module with given infinitesimal and central character

Granting this, we get

Corollary 8.16. Let $I \subseteq Z_{\mathfrak{g}}$ be an ideal of finite codimension. Then $\mathcal{A}_0(\Gamma\backslash G, \chi, \omega)[I]$ is admissible. (This proves the special case of Theorem 6.14)

Summary:

$$\bigoplus_{H} \bigoplus_{m\in\mathbb{Z}} H(m) = \mathcal{A}_0(\Gamma\backslash G, \chi, \omega) \underset{\text{dense}}{\subseteq} L_0^2(\Gamma\backslash G, \chi, \omega) = \widehat{\bigoplus}_{0\neq H\leq L_0^2 \text{ irreducible } G \text{ rep}} \widehat{\bigoplus} H(m)$$

Thus first sum in $\mathcal{A}_0$ is algebraic by $Z_{\mathfrak{g}}$-finiteness, the finiteness of multiplicities and the fact stated above

9 Representations of $\mathrm{GL}(2,\mathbb{R})$

$G = \mathrm{GL}_2(\mathbb{R})^+$, $K = \mathrm{SO}(2)$, $\mathfrak{g} = \mathfrak{gl}_2$, $\{W, R, L, Z\}$ is a basis for $\mathfrak{g}_{\mathbb{C}}$, V is an irreducible admissible $(\mathfrak{g}, K)$-module, then $V = \bigoplus_{m\in\mathbb{Z}} V(m)$

$$V(m) = \{v \in V | W\cdot v = mv\} = \{v \in V | k\cdot v = \chi_m(k)v, \forall k \in K\}$$

$\Sigma = \{m \in \mathbb{Z} | V(m) \neq 0\}$ is the set of K-types

Recall

1. $\Delta = -\frac{1}{4}(W^2 + 2RL + 2LR) = -LR - \frac{W}{2}(1 + \frac{W}{2}) = -RL + \frac{W}{2}(1 - \frac{W}{2})$
2. $RV(m) \subseteq V(m+2)$, $LV(m) \subseteq V(m-2)$
3. $U(\mathfrak{g}_{\mathbb{C}}) = \bigoplus_{i>0} R^i A \oplus \bigoplus_{i\geq 0} L^i A$, A is the subalgebra generated by W and $Z_{\mathfrak{g}} = \mathbb{C}[Z, \Delta]$

We deduce

1. $\forall 0 \neq x \in V(m)$, $V = \mathbb{C}x \oplus \bigoplus_{n>0} \mathbb{C}R^n x \oplus \bigoplus_{n>0} \mathbb{C}L^n x$
2. $\dim V(m) \leq 1$, Σ has same parity
3. Let λ be the eigenvalue of Δ on V, then $\forall x \in V(m)$, $LRx = (-\lambda - \frac{m}{2}(1 + \frac{m}{2}))x$, $RLx = (-\lambda + \frac{m}{2}(1 - \frac{m}{2}))x$. If $x \neq 0, Rx = 0$, then $\lambda = -\frac{m}{2}(1 + \frac{m}{2})$, If $x \neq 0, Lx = 0$, then $\lambda = \frac{m}{2}(1 - \frac{m}{2})$
4. Suppose $\lambda = \frac{n}{2}(1 - \frac{n}{2})$, $n \in \mathbb{Z}$, $0 \neq x \in V(m)$. If $Rx = 0$, then $\frac{n}{2}(1 - \frac{n}{2}) = -\frac{m}{2}(1 + \frac{m}{2}) \Rightarrow m = -n$ or $m = n - 2$ If $Lx = 0$, then $\frac{n}{2}(1 - \frac{n}{2}) = \frac{m}{2}(1 - \frac{m}{2}) \Rightarrow m = n$ or $m = 2 - n$

Consequence: irreducible admissible (g, K) moduels are uniquely dtetrmined by infinitesimal character and K types [Bump, Thm2.5.1,2.5.2]
Classification: Fix eigenvalue λ of Δ, μ of Z, parity $\epsilon \in \{0, 1\}$

1. If $\lambda \notin \{\frac{n}{2}(1 - \frac{n}{2}) | n \equiv \epsilon \bmod 2\}$, then $\Sigma = \epsilon + 2\mathbb{Z}$
2. If $\lambda = \frac{n}{2}(1 - \frac{n}{2})$ for $n \in \mathbb{Z}_{n\geq 1}$, $n \equiv \epsilon \bmod 2$. Then there are 3 possibilities: $\Sigma^+(n) = n + 2\mathbb{Z}_{\geq 0}$, $\Sigma^-(n) = -n - 2\mathbb{Z}_{\geq 0}$, $\Sigma^0(n) = \{n-2, \cdots, 2-n\}$

Parabolic induction: $B = NA \subseteq G$ standard Borel, $\chi : A \to \mathbb{C}^\times$ be a quasi character(a group homomorphism), also viewed as $B \to \mathbb{C}^\times$

$$\mathrm{Ind}_B^G(\chi) = \{f \in C^\infty(G) | f(bg) = \chi(b)f(g)), \forall b \in B, g \in G\}$$

inner product $\langle f_1, f_2\rangle = \int_K f_1(k)\overline{f_2(k)}dk$. Let H_χ be the Hilbert space completion of $\mathrm{Ind}_B^G \chi$, this a representation of G induced by right regular action on $C^\infty(G)$. (Have to check $\langle r_g f, r_g f\rangle \leq c\langle f, f\rangle$, $\forall f \in \mathrm{Ind}_B^G \chi$, so r_g is a bounded operator on H_χ). We have a G-equivariant map

$$\mathrm{Ind}_B^G \chi_1 \times \mathrm{Ind}_B^G \chi_2 \to \mathrm{Ind}_B^G \chi_1\chi_2$$
$$(f_1, f_2) \mapsto f_1 f_2$$

Let $\delta : A \to \mathbb{C}^\times$, $\delta\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\right) = \dfrac{a_1}{a_2}$, there exists a nonzero G-equivariant bounded linear functional

$$\mathrm{Ind}_B^G \delta \to \mathbb{C}, f \mapsto \int_K f(k)dk$$

Inducing G-equivariant pairing $\mathrm{Ind}_B^G \chi\delta^{\frac{1}{2}} \times \mathrm{Ind}_B^G \chi^{-1}\delta^{\frac{1}{2}} \to \mathbb{C}$

Definition 9.1. normalized induction $i_B^G\chi = \mathrm{Ind}_B^G \chi\delta^{\frac{1}{2}}$. So $i_B^G\chi$ is unitary with G-invariant product. When χ is unitary, $\chi^{-1} = \bar{\chi}$, so $i_B^G\chi$ is unitary with G-invariant inner product $\langle f_1, f_2\rangle = \int_K f_1(k)\bar{f_2}(k)dk$

If $s_1, s_2 \in \mathbb{C}$, $\epsilon \in \{0,1\}$, and χ is defined by $\chi\left(\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\right) = |a_1|^{s_1}|a_2|^{s_2}\,\mathrm{sgn}(a_1)^{\epsilon}$, recall $a_1 a_2 > 0$, deonte $H^\infty(s_1, s_2, \epsilon) = i_B^G\chi$, $H(s_1, s_2, \epsilon)$ its Hilbert space completion. Denote $s = \frac{1}{2}(s_1 - s_2 + 1)$. Note $\forall f \in H^\infty(s_1, s_2, \epsilon)$, $f(-g) = (-1)^\epsilon f(g)$. $\forall m \in \epsilon + 2\mathbb{Z}$, there is a unique $f_m \in H^\infty(s_1, s_2, \epsilon)$ such that

$$f_m\left(u\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\right) = u^{s_1+s_2}y^s e^{im\theta}$$

$H(s_1, s_2, \epsilon)_{\mathrm{fin}} = \bigoplus_{m+\in\epsilon+2\mathbb{Z}} \mathbb{C}f_m$, $(\mathfrak{g}, K)$-modules of $H(s_1, s_2, \epsilon)$
Use formulas in HW1, one calculates

$$Wf_m = m \cdot f_m, Rf_m = (s + \frac{m}{2})f_{m+2}, Lf_m = (s - \frac{m}{2})f_{m-2}$$

$$\Delta f_m = \lambda f_m, Zf_m = \mu f_m$$

where $\lambda = s(1-s)$, $\mu = s_1 + s_2$, $s = \frac{1}{2}(s_1 - s_2 + 1)$
From this we get

1. If $s \notin \frac{\epsilon}{2} + \mathbb{Z}$, then $H(s_1, s_2, \epsilon)$ is irreducible. Denote its $(\mathfrak{g}, K)$-module by $P_\mu(\lambda, \epsilon)$, principal series
2. If $s = \frac{n}{2}$, $n \in \epsilon + 2\mathbb{Z}$, $n \geq 1$, then $H(s_1, s_2, \epsilon)$ has two irreducible subrepresentations

   $$H_+ = \widehat{\bigoplus_{m\geq n}}\mathbb{C}f_m, H_- = \widehat{\bigoplus_{m\leq -n}}\mathbb{C}f_m$$

   denote these $(\mathfrak{g}, K)$-modules by $D_\mu^+(n)$, $D_\mu^-(n)$, and the set of K-types are $\Sigma^+(n) = n + 2\mathbb{Z}_{\geq 0}$, $\Sigma^-(n) = -n - 2\mathbb{Z}_{\geq 0}$. The quotient $H/H^+ \oplus H^-$ is a finite dimensional G representation (0 if $n = 1$) with K-types $\Sigma^0(n) = \{m \in n + 2\mathbb{Z} | 2 - n \leq m \leq n - 2\}$
3. If $s = 1 - \frac{n}{2}$, $n \in \epsilon + 2\mathbb{Z}$, $n > 1$, then $H(s_1, s_2, \epsilon)$ has a finite dimensional irreducible subrepresentation $H^0 = \bigoplus_{2-n\leq m\leq n-2, m\equiv n \bmod 2}\mathbb{C}f_m$ with set of K-type $\Sigma^0(n)$. $H/H^0 = H^+ \oplus H^-$ with $(\mathfrak{g}, K)$-module $D_\mu^+(n) \oplus D_\mu^-(n)$

$D_\mu^\pm(n), n \geq 2$ are called *discrete series*. The limit of discrete series $D_\mu^\pm(1), D_\mu^\pm(2)$ are called *Steinberg representations*

$$(\mathrm{Ind}_B^G 1)_{\mathrm{fin}}/\mathbb{C} \cong D^+(2) \oplus D^-(2)$$

where $\mu = 0$
Irreducible unitary representations

1. Unitary principal series: $P_\mu(\lambda, \epsilon) = H(s_1, s_2, \epsilon)$, $s_1, s_2 \in i\mathbb{R}$, $s = \frac{1}{2}(s_1 - s_2 + 1) \in \frac{1}{2} + i\mathbb{R}$, $\mu = s_1 + s_2 \in i\mathbb{R}$, $\lambda = s(1-s) \in [\frac{1}{4}, +\infty)$, $\epsilon \in \{0, 1\}$. And if $s = \frac{1}{2}$, we require $\epsilon = 0$
2. Complementary series: $P_\mu(\lambda, 0)$, $\mu \in i\mathbb{R}$, $0 < \lambda < \frac{1}{4}$, $s \in (0, 1)$
3. Discrete series: $D_\mu^\pm(n)$, $n \in \mathbb{Z}_{\geq 2}$, $\mu \in i\mathbb{R}$, $\lambda = \frac{n}{2}(1 - \frac{n}{2})$
4. Limit of discrete series: $D_\mu^\pm(1)$, $n \in \mathbb{Z}_{\geq 2}$, $\mu \in i\mathbb{R}$, $s = \frac{1}{2}$, $\lambda = \frac{1}{4}$, unitary since $D_\mu^+(1) \oplus D_\mu^-(1) \cong H(\frac{\mu}{2}, \frac{\mu}{2}, 1)$
5. One dimensional representations: $g \mapsto (\det g)^{\frac{\mu}{2}}$, $\mu \in i\mathbb{R}$

Among these, 1,3,4 are tempered. 3 is square integrable. Recall tempered/square integrable means one (equivalently any) nonzero matrix coefficient belong to $L^{2+\epsilon}(G/Z_G, \omega)$, $\forall \epsilon > 0/L^2(G/Z_G, \omega)$, ω is an unitary central character. In particular, discrete series are subrepresentations of right regular G representation on $L^2(G/Z_G, \omega)$
Alternative realization of $H(s_1, s_2, \epsilon)$ (restricted to $\mathrm{SL}_2(\mathbb{R}) = G_1$). Recall $N \backslash \mathrm{SL}_2(\mathbb{R}) \cong \mathbb{R}^2 \setminus \{(0,0)\}$, $N \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto (c, d)$

$$H^\infty(s_1, s_2, \epsilon) \cong \{f \in C^\infty(\mathbb{R}^2 - \{(0,0)\}) | f(\lambda x_1, \lambda x_2) = \mathrm{sgn}(\lambda)^\epsilon |\lambda|^{-2s} f(x_1, x_2)\}$$

G_1 action on RHS: $(g \cdots f)(x_1, x_2) = f((x_1, x_2)g)$, inner product

$$\langle f_1, f_2 \rangle = \frac{1}{2\pi} \int_1^{2\pi} f_1 \overline{f_2}(-\sin\theta, \cos\theta) d\theta$$

Suppose $-2s = n \in \mathbb{Z}_{\geq 0}$, $\epsilon \equiv n \bmod 2$, then get

$$H_n^\infty = \{\{f \in C^\infty(\mathbb{R}^2 - \{(0,0)\}) | f(\lambda x_1, \lambda x_2) = \lambda^n f(x_1, x_2)\}$$

With completion H_n, a representation of $G_1 = \mathrm{SL}_2(\mathbb{R})$. Let

$$f_{m,n}(x_1, x_2) = (x_1 + ix_2)^{\frac{m+n}{2}} (x_1 - ix_2)^{\frac{n-m}{2}}, \forall m \in n + 2\mathbb{Z}$$

$H_n^{\mathrm{fin}} = \bigoplus_{m \in n+2\mathbb{Z}} \mathbb{C} f_{m,n}$ space of K-finite vectors in H_n
Let $U_+ = \mathbb{CP}^1 \setminus \{-i\} \subseteq \mathcal{H}$, $U_- = \mathbb{CP}^1 \setminus \{i\} \subseteq \mathcal{H}^-$, $U = U_+ \cap U_- \supseteq \mathbb{RP}^1$, then

$$0 \to \Gamma(\mathbb{CP}^1, \mathcal{O}(n)) \to \Gamma(U, \mathcal{O}(n)) \to H^1_{\{\pm i\}}(\mathbb{CP}^1, \mathcal{O}(n)) \to H^1(\mathbb{CP}^1, \mathcal{O}(n)) = 0$$

$\Gamma(\mathbb{CP}^1, O(n)) = \bigoplus_{-n \leq m \leq n, m \equiv n \bmod 2} \mathbb{C} f_{m,n}$, finite dimensional representations of G_1, $\Gamma(U, O(n)) = H_n^{\mathrm{fin}}$, $H^1_{\{\pm i\}}(\mathbb{CP}^1, \mathcal{O}(n)) = D^+(n+2) \oplus D^-(n+2)$, $D^+(n+2) = H^1_{\{i\}}(\mathbb{CP}^1, \mathcal{O}(n)) = \bigoplus_{m \geq n+2, m \equiv n \bmod 2} \mathbb{C} f_{m,n}$, $D^-(n+2) = H^1_{\{-i\}}(\mathbb{CP}^1, \mathcal{O}(n)) = \bigoplus_{m \leq -n-2, m \equiv n \bmod 2} \mathbb{C} f_{m,n}$. All these are $(\mathfrak{sl}_2, K)$-modules
Now consider the case of H_{-n}, $n \in \mathbb{Z}_{>0}$, $\epsilon \equiv n \bmod 2$, H_{-n}^{fin} has basis $f_{m,-n}$. Cech sequence for $O(-n)$:

$$0 = \Gamma(\mathbb{CP}^1, O(-n)) \to \Gamma(U_+, O(-n)) \oplus \Gamma(U_-, O(-n)) \to \Gamma(U, O(-n)) \to H^1(\mathbb{CP}^1, O(-n)) \to 0$$

$$\Gamma(U_+, O(-n)) = \bigoplus_{m \leq -n, m \equiv n \bmod 2} \mathbb{C} f_{m,-n} = D^-(n) \to \Gamma(U, O(-n)) = H_n^{\mathrm{fin}}$$

$$\Gamma(U_-, O(-n)) = \bigoplus_{m \geq n, m \equiv n \bmod 2} \mathbb{C} f_{m,-n} = D^+(n) \to \Gamma(U, O(-n)) = H_n^{\mathrm{fin}}$$

By Serre duality

$$H^1(\mathbb{CP}^1, O(-n)) \cong H^0(\mathbb{CP}^1, O(n-2))^\vee = \bigoplus_{2-n \leq m \leq n-2, m \equiv n \bmod 2} \mathbb{C} f_{m,-n}$$

Which is 0 if $n = 1$
Using this description, get embedding

$$D^-(n) \to L^2_{\mathrm{hol}}(\mathcal{H}, n) = \left\{ f \text{ holomorphic on } \mathcal{H} \middle| \int_{\mathcal{H}} |f(z)|^2 (\operatorname{Im} z)^k \frac{dxdy}{y^2} < \infty \right\}$$

$f_{m,-n} \mapsto (z+i)^{\frac{m-n}{2}} (z-i)^{\frac{-m-n}{2}}$ $L^2_{\mathrm{hol}}(\mathcal{H}, n)$ is a unitary representation of $G_1 = \mathrm{SL}_2(\mathbb{R})$, $(g \cdot f)(z) = (f|_n g^t)(z) = j(g^t, z)^n f(g^t, z)$ Isometric embedding $L^2_{\mathrm{hol}}(\mathcal{H}, n) \to L^2(G_1)$, $f \mapsto \phi_f$, $\phi_f(g) = f(g^t i) j(g^t, i)^{-n}$
The case of $\tilde{G} = \mathrm{GL}_2(\mathbb{R})$, $\tilde{K} = O(2) = K \sqcup K\eta$, $\eta = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathrm{ad}(\eta)(k) = k^{-1}$, $\forall k \in K$, $\eta^2 = 1$, $\mathrm{ad}(\eta)(W) = -W$, $\mathrm{ad}(\eta)(R) = L$, $\mathrm{ad}(\eta)(L) = R$. Let ρ be a representation of G, define another G-representation $\eta(\rho)$ by $\eta(\rho)(g) = \rho(\mathrm{ad}(\eta) g)$, then $\mathrm{Ind}_G^{\tilde{G}} \rho \cong \rho \oplus \eta(\rho)$
Suppose ρ is irreducible, the following are equivalent

1. ρ extends to an irreducible representation of $\tilde{G}$
2. $\mathrm{Ind}_G^{\tilde{G}} \rho$ is reducible $\tilde{G}$ representation
3. $\rho \cong \eta(\rho)$ as G representation

e.g. $\mathrm{Ind}_G^{\tilde{G}}\,\mathsf{triv} \cong \mathsf{triv} \oplus \mathsf{sgn}$ as $\tilde{G}$ representations
If these are satisfied, then there are 2 isomorphic classes of $\tilde{G}$ representation that extends ρ: $\tilde{\rho}$ and $\tilde{\rho} \otimes \mathsf{sgn}$
If the irreducible representation ρ does not extend to $\tilde{G}$ representation, then $\mathrm{Ind}_G^{\tilde{G}}\rho \cong (\mathrm{Ind}_G^{\tilde{G}}\rho) \otimes \mathsf{sgn}$ is an irreducible representation of $\tilde{G}$ that restricts to the G representation $\rho \oplus \eta(\rho)$
Note: An explicit $\tilde{G}$ isomorphism

$$\iota : \mathrm{Ind}_G^{\tilde{G}}\rho \xrightarrow{\cong} (\mathrm{Ind}_G^{\tilde{G}}\rho) \otimes \mathsf{sgn}$$
$$(f : \tilde{G} \to V) \mapsto \iota f(g) = \begin{cases} f(g) & \text{if } g \in G \\ -f(g) & \text{if } g \in \eta G \end{cases}$$

Conversely, let σ be an irreducible representation of $\tilde{G}$. If $\sigma|_G$ is irreducible, then $\sigma \cong \sigma \otimes \mathsf{sgn}$. If $\sigma|_G$ is reducible, then $\sigma \cong \sigma \otimes \mathsf{sgn}$ and $\sigma|_G \cong \rho \oplus \eta(\rho)$ for some irreducible representation ρ of G

Example 9.2. Irreducible representations of $\tilde{K}$: triv, sgn, $\mathrm{Ind}_K^{\tilde{K}}\chi_m$, $0 \neq m \in \mathbb{Z}$. For $\tilde{G} = \mathrm{GL}_2(\mathbb{R})$, the principal series $P_\mu(\lambda, \epsilon)$ and finite dimensional representations of G has 2 non-isomorphic extensions to $\tilde{G}$ representation and $(\mathfrak{g}, K)$-modules

$D_\mu^\pm(n)$, $(n \geq 1)$ cannot be extended to $(\mathfrak{g}, \tilde{K})$-module, or $\tilde{G}$ representation, $D_\mu^+(n) \oplus D_\mu^-(n)$ are irreducible $(\mathfrak{g}, \tilde{K})$-modules, irreducible $\tilde{G}$ representations. Note that $D_\mu^-(n) = \eta(D_\mu^+(n))$

Remark 9.3. η acts on K-types by negation, so K-type of $(\mathfrak{g}, \tilde{K})$ modules are symmetric under negation

References

[1] *A First Course in Modular Forms* - Fred Diamond, Jerry Shurman

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