0.1 Laplace's equation

0.2 Heat equation

Definition 0.2.1. The fundamental solution to solution to **heat equation** $u_t - \Delta u = 0$ is

$$E(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Theorem 0.2.2. $U \subseteq \mathbb{R}^n$ is open and bounded, $f \in C_c^1(U \times (0,T])$, then

$$u(x,t) = \int_{\mathbb{R}^{n+1}} E(x-y, t-s) f(s,y) ds dy$$

Satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = f(x,t)$$

Where u is C^1 in t and C^2 in x

Proof. $E(x,t)$ is supported in $t \geq 0$ and $\int_{\mathbb{R}^n} |\nabla_x E(x,t)| dx \leq \frac{C}{\sqrt{t}}$ if $t > 0$, so $\nabla_x E(x,t)$ is integrable near $(0,0)$

$$\begin{aligned} \nabla_x \int_{\mathbb{R}^{n+1}} E(y,s) f(x-y, t-s) ds dy &= \int_{\mathbb{R}^{n+1}} E(y,s) \nabla_x f(x-y,t-s) ds dy \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} E(y,s) \nabla_x f(x-y,t-s) ds dy \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\nabla E)(y,s) f(x-y,t-s) ds dy \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n+1}} (\nabla E)(y,s) f(x-y,t-s) ds dy \end{aligned}$$

And

$$\begin{aligned} \Delta \int_{\mathbb{R}^{n+1}} E(y,s) f(x-y,t-s) ds dy &= \int_{\mathbb{R}^{n+1}} (\nabla E)(y,s) \cdot (\nabla f)(x-y,t-s) ds dy \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty (\nabla E)(y,s) \cdot (\nabla f)(x-y,t-s) ds dy \end{aligned}$$

And

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \int_{\mathbb{R}^{n+1}} E(y,s) f(x-y,t-s) ds dy &= -\lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\nabla E)(y,s) \cdot (\nabla f)(x-y,t-s) ds dy \\ &\quad + \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} E(y,s) \frac{\partial f}{\partial x}(x-y,t-s) ds dy \\ &= \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial s} - \Delta_y\right) E(y,s) f(x-y,t-s) ds dy \\ &\quad + \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} E(y,\varepsilon) f(x-y,t-\varepsilon) ds dy \\ &= f(x,t) \end{aligned}$$

Next, let $u \in C^2(U \times (0,T])$ and $u_t - \Delta u = 0$, $\chi \in C^\infty$, $\chi(x,t) = 1$ if $d((x,t), \Gamma_U) \geq 2$, $\chi(x,t) = 0$ if $d((x,t),\Gamma_U) \leq \varepsilon$ and $(x,t) \in U \times (0,T]$, apply the previous argument to $f(x,t) = \left(\frac{\partial}{\partial t} - \Delta\right)(\chi(x,t)u(x,t)) = \left(\left(\frac{\partial}{\partial t} - \Delta\right)\chi(x,t)\right) u - 2\nabla\chi \cdot \nabla u \in C_c^1(U \times (0,T])$, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\chi(x,t)u(x,t) - \int_{-\infty}^t \int_{\mathbb{R}^n} E(x-y,t-s) f(y,s)\right) = 0$$

And

$$u(x,t)\chi(x,t) - \int_{-\infty}^{t} E(x-y,t-s)f(y,s)dsdy = 0$$

if $t = 0$, so if $0 \leq t \leq T$

$$\chi(x,t)u(x,t) = \int_{-\infty}^{t}\int_{\mathbb{R}^n} E(x-y,t-s)\left(\frac{\partial}{\partial t} - \Delta\right)(\chi(y,s)u(y,s))dsdy$$

$\square$

0.3 Wave equation

Definition 0.3.1. The fundamental solution to **wave equation** $\square u = \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$ is

$$E(x,t) = \begin{cases} \frac{1}{2\pi^{\frac{n-1}{2}}} \chi_+^{\frac{1-n}{2}}(t^2 - |x|^2) & t > 0 \\ 0 & t < 0 \end{cases}$$

Theorem 0.3.2. $f \in C^2(\mathbb{R}^3)$, $u(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y)dS_y = \frac{t}{4\pi}\int_{S^2} f(x+tw)dS_w$, then $u \in C^2\left(\mathbb{R}^3 \times [0,\infty)\right)$, $u(x,0) = 0$, $\left.\frac{\partial}{\partial t}\right|_{t=0} u(x,t) = f(x)$ and $\square u = 0$ for $t > 0$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= \frac{1}{4\pi}\int_{S^2} f(x+tw)dS_w + \frac{t}{4\pi}\int_{S^2}(w\cdot\nabla)f(x+tw)dS_w \\ &= \frac{1}{4\pi}\int_{S^2} f(x+tw)dS_w + \frac{1}{4\pi t}\int_{\partial B(x,t)} n\cdot\nabla f(y)dS_y \\ &= \frac{1}{4\pi}\int_{S^2} f(x+tw)dS_w + \frac{1}{4\pi t}\int_{B(x,t)} \Delta f(y)dy\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \frac{1}{4\pi}\int_{S^2}(w\cdot\nabla)f(x+tw)dS_w - \frac{1}{4\pi t^2}\int_{B(x,t)}\Delta f(y)dy \\ &\quad + \frac{1}{4\pi t}\frac{d}{dt}\int_0^t\int_{S^2}\lambda^2\Delta f(x+\lambda w)dS_w d\lambda \\ &= \frac{1}{4\pi t^2}\int_{B(x,t)}\Delta f(y)dy - \frac{1}{4\pi t^2}\int_{B(x,t)}\Delta f(y)dy \\ &\quad + \frac{t}{4\pi}\int_{S^2}\Delta f(x+\lambda w)dS_w \\ &= \frac{1}{4\pi t}\int_{\partial B(x,t)}\Delta f(y)dS_y \\ &= \Delta u(x,t)\end{aligned}$$

$\square$

Theorem 0.3.3. $f \in C^2(\mathbb{R}^2)$, then $u(x,t) = \frac{1}{2\pi}\int_{|y|<t}\frac{1}{\sqrt{t^2-|y|^2}}f(x-y)dy$ solves $\square u = 0$ for $t > 0$, $u(x,0) = 0$, $u_t(x,0) = f$

Proof. Consider $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x_1,x_2,x_3) = f(x_1,x_2)$ is independent of x_3, then $u(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y)dy = \frac{1}{4\pi t}\int_{\partial B(0,t)} f(x-y)dS_y$

$y_3 = \pm\sqrt{t^2 - y_1^2 - y_2^2} = \gamma(y), ds = \sqrt{1+|\nabla\gamma(y)|^2}dy_1dy_2 = \frac{t}{t^2 - y_1^2 - y_2^2}$, upper + lower hemisphere

$= \frac{2}{4\pi t}\int_{|(y_1,y_2)|<t} f(x-y)\frac{tdy_1dy_2}{\sqrt{t^2-|(y_1,y_2)|^2}} = \frac{1}{2\pi}\int_{|y|<t}\frac{1}{\sqrt{t^2-|y|^2}}f(x-y)dy$ $\square$

Theorem 0.3.4. $f \in C^\infty(\mathbb{R}^n \times [0,\infty))$, $u(x,t) = \int_0^t E(\cdot,t-s) * f(\cdot,s)ds$, then $\square u = f$, $u(x,0) = u_t(x,0) = 0$

Proof. Define $u(x,t,s) = E(\cdot, t-s) * f(\cdot, s) \in C^\infty$ for $t > s$

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= u(x,t,t) + \int_0^t \frac{\partial}{\partial t}u(x,t,s)ds \\ &= \int_0^t \frac{\partial}{\partial t}u(x,t,s)ds\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \int_0^t \frac{\partial^2}{\partial t^2}u(x,t,s)dx + \left.\frac{\partial}{\partial t}\right|_{t=s} u(x,t,s) \\ &= f(x,t) + \int_0^t \frac{\partial^2}{\partial t^2}u(x,t,s)dx\end{aligned}$$

Thus $\left(\frac{\partial^2}{\partial t^2} - \Delta\right)u(x,t) = f(x,t) + \int_0^t \left(\frac{\partial^2}{\partial t^2} - \Delta\right)u(x,t,s)dx$, the second term is zero for $s < t$

By the same argument, $\Box \int_{-\infty}^t E(\cdot, t-s) * f(\cdot, s)ds = f(\cdot, t)$, thus $\Delta E = \delta_{(x,t)}$ is the fundamental solution $\Box$

1 dim wave equation reflection

Lemma 0.3.5. The solution to $\Box u = 0$ in $t > 0, x > 0$ with $u(0,t)$ for all $t > 0$, $u(x,0) = 0$, $u_t(x,0) = f(x)$, $f \in C^1([0,\infty))$, $f(0) = 0$ is

$$u(x,t) = \frac{1}{2}\int_{|t-x|}^{t+x} f(\lambda)d\lambda$$

Proof. Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$ which solves $\Box\tilde{u} = 0$ for $t > 0, x \in \mathbb{R}$, $\tilde{u}(x,0) = 0$, $\tilde{u}_t(x,0) = \tilde{f}$, hence

$$\tilde{u}(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \tilde{f}(\lambda)d\lambda = \frac{1}{2}\int_{|x-t|}^{x+t} f(\lambda)d\lambda$$

$\Box$

Laplacian of a spherical symmetric function

Lemma 0.3.6. $f(x) = f(|x|)$ is spherical symmetric in $\mathbb{R}^n$, then $(\Delta f)(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) f$

Proof. Δu is characterized by

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx = -\int_{\mathbb{R}^n} v\Delta u, \forall v \in C_c^\infty(\mathbb{R}^n)$$

If $u(x) = u(|x|)$, $v(x) = v(|x|)$

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx &= \int_{S^{n-1}}\int_0^\infty \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} dr dS_w \\ &= -\int_{S^{n-1}}\int_0^\infty \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) v(r) r^{n-1} dr dS_w \\ &= -\int_{\mathbb{R}^n} \frac{1}{r^{n-}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) v(r)dx \\ &= -\int_{\mathbb{R}^n}\left(\frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) uv(r)dx\end{aligned}$$

Note that $\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) = \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$ $\Box$

Theorem 0.3.7. The solution to $\Box u = 0$ in $\mathbb{R}^{3+1}$ with $u(x,0) = 0$, $u_t(x,0) = f(x) = f(|x|)$, $f \in C^\infty(\mathbb{R}^3)$ is

$$u(x,t) = \frac{1}{2|x|} \int_{t-|x|}^{t+|x|} \lambda f(\lambda) d\lambda$$

Proof. By Lemma 0.3.6, when $n = 3$, $\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) u = \frac{1}{\partial r}\frac{\partial^2}{\partial r^2}(ru)$, thus if $\Box u = 0$ in $\mathbb{R}^{3+1}$, $u(x,t) = u(|x|,t)$, then $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(ru(r,t)) = 0$ and $ru(r,t) = 0$ if $r = 0$, $\left.\frac{\partial}{\partial t}\right|_{t=0}(ru(r,t)) = rf(r)$, by Lemma 0.3.5, $ru(r,t) = \frac{1}{2}\int_{|t-r|}^{t+r} \lambda f(\lambda) d\lambda$. We can check $u \in C^1$ $\square$

Theorem 0.3.8 (Energy estimate version 1)**.** $\Box u = 0$ for $t > 0$, then the energy $\frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 0.3.9 (Energy estimate version 2)**.** $\Box u = 0$ in $U_T = U \times (0,T]$, $u = 0$ on Γ_U, $u_t(x,0) = 0$, implicitly, $u_t = 0$ on $\partial U \times [0,T]$, then $\frac{1}{2}\int_{\mathbb{U}} |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 0.3.10 (Energy estimate version 3)**.** $C = \left\{(x,t) \in \mathbb{R}^{n+1} \middle| |x - x_0| \leq |t - t_0|\right\}$ is the cone, $D_t = \{x \in \mathbb{R}^n | |x - x_0| \leq |t - t_0|\}$ is the section at time t, considet the case $t < t_0$, then $\frac{1}{2}\int_{\mathbb{D}_t} |u_t|^2 + |\nabla u|^2 dx$ is decreasing on $0 \leq t \leq t_0$

0.4 Euler-Lagrange equation

0.5 Energy momentum tensor

Definition 0.5.1. ∇ is the gradient, write $\nabla^T\nabla = \nabla\cdot\nabla = \Delta$ is the laplacian, $\nabla\cdot 1 = \mathtt{div}$ is the divergence, $\nabla\nabla^T = D^2$ is the Hessian

Definition 0.5.2. $L(z,q):\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ is C^∞, u satisfies Euler-Langrange equation, then

$$\begin{aligned}\nabla_x L(u,\nabla u) &= \frac{\partial L}{\partial z}\nabla u + (\nabla\nabla^T u)(\nabla_q L)\\ &= (\nabla_x\cdot\nabla_q L)(\nabla u) + (\nabla\nabla^T u)(\nabla_q L)\\ &= (\nabla^T u\nabla_q L)\nabla_x\end{aligned}$$

Energy-momentum tensor $T_{\alpha\beta} = \dfrac{\partial u}{\partial x^\alpha}\dfrac{\partial L}{\partial q_\beta} - \delta_{\alpha\beta}L$, $T = \nabla^T u\nabla_q L - L1$, then $T\nabla_x = (\nabla^T\nabla_q L)\nabla_x - \nabla_x L = 0$

Example 0.5.3. $u_{tt} - \Delta u + u^3 = 0$, $L(u,\nabla_{x,t}u) = \frac{1}{2}(u_t^2 - |\nabla_x u|^2) - \frac{1}{4}u^4$, $T_{00} = u_t^2 - \left[\frac{1}{2}(u_t^2 - |\nabla u|^2) - \frac{1}{4}u^4\right] = \frac{1}{2}(u_t^2 + |\nabla u|^2) + \frac{1}{4}u^4$, $T_{0i} = -u_t\dfrac{\partial u}{\partial x^i}$, thus $0 = (T_{00},\cdots,T_{0n})\nabla_x = \mathtt{div}(T_{00},\cdots,T_{0n})$