0.1 Non-associative algebra

Definition 0.1.1. A *nonassociative* $\mathbb{F}$ *algebra* A is an $\mathbb{F}$ vector space with multiplication $\cdot$ that is distributive $(a+b)\cdot c = a\cdot c+b\cdot c$, $a\cdot(b+c) = a\cdot b+a\cdot c$. A is *unital* if $1 \in A$, A is *symmetric* if $xy = yx$, A is *antisymmetric* if $xy = -yx$, A satisfies *Jacobi identity* if $(xy)z+(yz)x+(zx)y = 0$ A homomorphism $\phi : A \to B$ is a linear map such that $\phi(xy) = \phi(x)\phi(y)$

Definition 0.1.2. Suppose $e_1, \cdots, e_n$ is a basis of A, $e_ie_j = \sum_i c_k^{ij} e_k$, c_k^{ij} are called *structure constants* with respect to $e_1, \cdots, e_n$. If A satisfies Jacobi identity, then

$$\sum_l c_m^{il} c_l^{jk} + \sum_l c_m^{jl} c_l^{ki} + \sum_l c_m^{kl} c_l^{ij} = 0$$

Definition 0.1.3. $B \leq A$ is a *subalgebra* if B is a subspace such that $BB \subseteq B$. $I \leq A$ is a *left ideal* if $AI \subseteq I$. Suppose $I, J \leq A$ are ideals, define *ideal quotients* $(J : I) = \{x \in A | xI \subseteq J\}$ which is an ideal. Homomorphisms preserve ideals

Remark 0.1.4. If A is (anti)symmetric, left ideals are two-sided ideals

Definition 0.1.5. A is *abelian* if $AA = 0$, A is *simple* if it is not abelian and the only ideals are 0 and A, A is *semisimple* if $A = A_1 \oplus \cdots \oplus A_n$ is the direct sum of simple subalgebras, A is *reductive* if $A = \mathfrak{s} \oplus \mathfrak{a}$ is a direct sum of a semisimple subalgebra $\mathfrak{s}$ and an abelian subalgebra $\mathfrak{a}$

Definition 0.1.6. A *derivation* is an endomorphism $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$. Let $\mathrm{Der}_{\mathbb{F}}(A)$ denote all derivations. If $D_1, D_2 \in \mathrm{Der}(A)$, then $[D_1, D_2] = D_1D_2 - D_2D_1 \in \mathrm{Der}(A)$, $\mathrm{Der}(A) \leq \mathrm{End}(A)$ is a Lie subalgebra

Definition 0.1.7. $R \xrightarrow{\varphi} S$ is a ring homomorphism between commutative rings. The module of *Kähler differentials* is $\Omega_{S/R}$ satisfies the universal property that any derivation uniquely factors through $d_{S/R} : R \to \Omega_{S/R}$, i.e. $\mathrm{Hom}_S(\Omega_{S/R}, M) \cong \mathrm{Der}_R(S, M)$. $\Omega_{S/R}$ can be constructed as

$$\frac{\{ds : s \in S\}}{dr = 0, d(s + t) = ds + dt, d(st) = sdt + tds}$$

Another construction: define I to be the kernel of $S \otimes_R S \to S$, $s \otimes t \mapsto st$, and $\Omega_{S/R} = I/I^2$, with $ds = 1 \otimes s - s \otimes 1$, the free S module generated by ds consists of $t \otimes s - ts \otimes 1$, which is the kernel of $S \otimes_R S \to S \otimes_R R$, $t \otimes s \mapsto ts \otimes 1$, moding I^2 is precisely the Leibniz rule because $d(st)-sdt-tds = (1\otimes st-st\otimes 1)-(s\otimes t-st\otimes 1)-(t\otimes s-ts\otimes 1) = (1\otimes s-s\otimes 1)(1\otimes t-t\otimes 1) \in I^2$

Definition 0.1.8. $f : X \to Y$ is a morphism of schemes, consider the diagonal $\Delta : Y \to X \times_Y X$, let $I = \ker \Delta^*$, the *cotangent sheaf* is $\Omega_{X/Y} = I/I^2$, and a derivation $d : \mathcal{O}_X \to \Omega_{X/Y}$

0.2 Lie algebras

Definition 0.2.1. A **Lie algebra** $\mathfrak{g}$ is a antisymmetric nonassociative $\mathbb{F}$ algebra satisfying Jacobi identity, usually with a **Lie bracket** $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ denote the multiplication. If $\mathtt{char}\mathbb{F}=2$, we also require $[x,x]=0$

Definition 0.2.2. A $\mathfrak{g}$ **module** V is an abelian group with left group action $\mathfrak{g}\times V\to V$ such that $1v=v$, $x(v+w)=xv+xw$, $(x+y)v=xv+yv$, $(xy)v=x(yv)-y(xv)$. Equivalently, a **Lie algebra representation** (π,V) is a Lie algebra homomorphism $\pi:\mathfrak{g}\to\mathfrak{gl}(V)$, where V is an $\mathbb{F}$ vector space, $xv:=\pi(x)v$ give V the $\mathfrak{g}$ module structure

Remark 0.2.3. A $\mathfrak{g}$ module is not a module according to Definition **??**

Definition 0.2.4. A $\mathfrak{g}$ **module homomorphism** $\phi:V\to W$ between $\mathfrak{g}$ modules is a group homomorphism such that $\phi(xv)=x\phi(v)$. Equivalently, an intertwine map $\phi:V\to W$ between Lie algebra representations is a linear map such that $\phi(\pi_V(x)v)=\pi_W(x)\phi(v)$, giving the $\mathfrak{g}$ module homomorphism
A subrepresentation (π,W) is a $\mathfrak{g}$ submodule $W\leq V$

Adjoint representation

Definition 0.2.5. The **adjoint endomorphism** associated to x is left multiplication by x, i.e. $ad(x)(y)=[x,y]$, Jacobi identity becomes $ad([x,y])=[ad(x),ad(y)]$, give a Lie algebra representation(adjoint representation) $ad:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$, $ad(x)$ are called **inner derivations** since $ad(z)[x,y]=[ad(z)x,y]+[x,ad(z)y]$. $ad(\mathfrak{g})\leq\mathrm{Der}(\mathfrak{g})\leq\mathfrak{gl}(\mathfrak{g})$ are Lie subalgebras
Any Lie algebra homomorphism $\phi:\mathfrak{g}\to\mathfrak{h}$ induce a Lie algebra homomorphism $\phi:ad(\mathfrak{g})\to ad(\mathfrak{h})$ by $\phi(ad(x))=ad(\phi(x))$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi"] \arrow[d, "ad"] & \mathfrak{h} \arrow[d, "ad"] \\
ad(\mathfrak{g}) \arrow[r, "\phi"] & ad(\mathfrak{h})
\end{tikzcd}$$

Definition 0.2.6. Centralizer of S is defined to be $C_{\mathfrak{g}}(S):=\{g\in\mathfrak{g}|[g,S]=0\}$, in particular, the center $Z(\mathfrak{g}):=C_{\mathfrak{g}}(\mathfrak{g})$
Normalizer of S is defined to be $N_{\mathfrak{g}}(S):=\{g\in\mathfrak{g}|[g,S]\subseteq S\}$

Definition 0.2.7.

$$\mathfrak{g}\supseteq[\mathfrak{g},\mathfrak{g}]\supseteq[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]\supseteq[[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]]\supseteq\cdots$$

is called the **derived series**, $\mathfrak{g}$ is **solvable** if derived series terminates

$$\mathfrak{g}\supseteq[\mathfrak{g},\mathfrak{g}]\supseteq[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]\supseteq[\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]]\supseteq\cdots$$

is called the **lower central series**, $\mathfrak{g}$ is **nilpotent** if lower central series terminates

Example 0.2.8. $[\mathfrak{gl}(V),\mathfrak{gl}(V)]=\mathfrak{sl}(V)$

Proof. Since $\mathrm{Tr}[X,Y]=0$, thus $[\mathfrak{gl}(V),\mathfrak{gl}(V)]\leq\mathfrak{sl}(V)$, conversely ☐

Definition 0.2.9. Let $\mathfrak{g}$ be a Lie algebra, a Cartan subalgebra $\mathfrak{h}\leq\mathfrak{g}$ is a nilpotent subalgebra such that $N_{\mathfrak{g}}(\mathfrak{h})=\mathfrak{h}$(self normalizing or alternatively $(\mathfrak{h}:\mathfrak{h})=\mathfrak{h}$)

Definition 0.2.10. Let $\mathfrak{g}\leq\mathfrak{gl}(V)$ be a Lie algebra, $\mathfrak{g}$ is called **toral** if $\mathfrak{g}$ consists of semisimple elements

Definition 0.2.11. Let $\mathfrak{g}$ be a Lie algebra, we can show the sum of all solvable ideal is again a solvable ideal, thus $\mathfrak{g}$ has a unique maximal solvable ideal $\mathtt{rad}(\mathfrak{g})$, called the **radical** of $\mathfrak{g}$

Definition 0.2.12. Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{g}_0$ is called a **real form** of $\mathfrak{g}$ if $\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$

Example 0.2.13. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be Lie subalgebra, the **tautological representation** (τ, V) is defined by $\tau(x) = x$, then $\tau([x, y]) = [x, y] = [\tau(x), \tau(y)]$

Proposition 0.2.14. Lie algebra $\mathfrak{g}$ is reductive iff its adjoint representation is completely reducible

g semisimple, phi:g->gl(V) representation => phi(g)<=sl(V)

Lemma 0.2.15. If $\mathfrak{g}$ is semisimple Lie algebra, and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, then $\varphi(\mathfrak{g}) \leq \mathfrak{sl}(V)$

Proof. By Proposition 0.6.1, $\varphi(\mathfrak{g}) = \varphi([\mathfrak{g}, \mathfrak{g}]) \leq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ □

Derivations of semisimple Lie algebra are inner derivations

Proposition 0.2.16. If $\mathfrak{g}$ is a semisimple Lie algebra, then $ad(\mathfrak{g}) = Der(\mathfrak{g})$

Proof. As an abelian ideal of $\mathfrak{g}$, $Z(\mathfrak{g}) = 0$, thus $\mathfrak{g} \xrightarrow{ad} ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding. Since $[\delta, ad(x)] = ad(\delta(x)), \delta \in Der(\mathfrak{g})$, thus $[Der(\mathfrak{g}), ad(\mathfrak{g})] \subseteq ad(\mathfrak{g})$, Let $K(,)$ be the Killing form on $Der(\mathfrak{g})$, due to Proposition 0.6.6(b) and Proposition 0.4.4(c), $K(,)|_{ad(g)}$ is nondegenerate, denote $I := ad(\mathfrak{g})^{\perp}$ under $K(,)$, then $I \cap ad(\mathfrak{g}) = 0$, otherwise $0 \neq I \cap ad(\mathfrak{g}) \subseteq \ker K(,)|_{ad(\mathfrak{g})}$, by Exercise ??, $[I, ad(\mathfrak{g})] = 0$, thus for any $\delta \in I$, $0 = [\delta, ad(x)] = ad(\delta(x))$, since ad is an isomorphism, $\delta(x) = 0$, thus $\delta = 0$, $I = 0$, $ad(\mathfrak{g}) = Der(\mathfrak{g})$ □

Remark 0.2.17. When $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{g} \xrightarrow{ad} ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding, we can identify x with $ad(x)$, by abuse of notations, xy can defined to be the preimage of $ad(x)ad(y) \in \mathfrak{gl}(\mathfrak{g})$

Lemma 0.2.18. Let G be a compact Lie group, we can pick any nonzero k-form α_I at I, and extend it to a k-form on G by $\alpha_A(Y_1, \cdots, Y_k) = \alpha_I(Y_1 A^{-1}, \cdots, Y_k A^{-1})$, or just $R_A^* \alpha_A = \alpha_I$, then we can define integral $\int_G f(A)\alpha$, then we would have $\int_G f(AB)\alpha = \int_G f(AB)\alpha_A = \int_G f(AB) R_B^* \alpha_{AB} = \int_G f(A) R_B^* \alpha = \int_G f(A)\alpha$, since $R_B^* \alpha = \alpha$, i.e. $(R_B^* \alpha)_A(X_A) = R_B^* \alpha_{AB}(X_A) = \alpha_A(X_A)$, thus this integration is right invariant. Note that this actually gives a right invariant Haar measure

Weyl's theorem

Theorem 0.2.19. Weyl's theorem
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, then φ is completely reducible, namely, $\mathfrak{g}$ modules are semisimple(completely reducible), thus any irreducible subrepresentation has to be one of the summand in the decomposition

Proof. Weyl's unitary trick □

0.3 Engel's theorem and Lie's theorem

Lemma for Engel's theorem

Lemma 0.3.1. Let $V \neq 0$ be a finite dimensional vector space, suppose $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra consists of nilpotent elements, then there exists $0 \neq v \in V$ such that $\mathfrak{g}v = 0$

Engel's theorem

Theorem 0.3.2. Engel's theorem Consider the adjoint representation $(ad, \mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ nilpotent iff $ad(X), X \in \mathfrak{g}$ can be strictly upper triangulized simultaneously iff $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$

g nilpotent, I ideal => I intersect Z(g) is nontrivial

Lemma 0.3.3. Let $\mathfrak{g}$ be a nilpotent Lie algebra, $I \leq \mathfrak{g}$ is a nonzero ideal, then $I \cap Z(\mathfrak{g}) \neq 0$, in particular, if $I = \mathfrak{g}$ then $Z(\mathfrak{g}) \neq 0$ which can also easily being shown from the fact that $Z(\mathfrak{g})$ contains the last nonzero term in the lower central series of $\mathfrak{g}$

Proof. Consider adjoint map restrict on I, since $\mathfrak{g}$ is nilpotent, $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$, so is $ad(X)|_I$, i.e. $ad(\mathfrak{g})_I \leq \mathfrak{gl}(I)$ is a Lie subalgebra consists of nilpotent elements, by Lemma 0.3.1, there exists $0 \neq Y \in I$ such that $[X, Y] = 0, \forall X \in \mathfrak{g}$, thus $Y \in I \cap Z(\mathfrak{g})$ □

Lie's theorem

Theorem 0.3.4. Lie's theorem
If (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}, \mathrm{char}\mathbb{F} = 0$, if $\mathfrak{g}$ is solvable, so is $\pi(\mathfrak{g})$, and $\pi(X), X \in \mathfrak{g}$ can be upper triangulized simultaneously

Remark 0.3.5. If (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}, \mathrm{char}\mathbb{F} = 0$, if $\mathfrak{g}$ is abelian, so is $\pi(\mathfrak{g})$, but it doesn't imply $\pi(X), X \in \mathfrak{g}$ can be diagonalized simultaneously, for example $\left\langle \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle \subseteq \mathfrak{gl}(\mathbb{C}^2)$ is abelian, and $\left\langle \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle$ is not diagonalizable at all
However, due to Proposition ??, if (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}$, where $\mathfrak{g}$ is abelian, so is $\pi(\mathfrak{g})$, and suppose $\pi(X), X \in \mathfrak{g}$ are diagonalizable($\pi(\mathfrak{g})$ is a toral Lie subalgebra), then they can be diagonalized simultaneously

0.4 Killing form

Definition 0.4.1. A bilinear form $(,)$ on Lie algebra $\mathfrak{g}$ is **invariant** or **associative** if the Lie derivative is zero, i.e. $(ad_Y X, Z) + (X, ad_Y Z) = 0$, or equivalently, $([X,Y], Z) = (X, [Y,Z])$

Definition 0.4.2. A **quadratic Lie algebra** is a Lie algebra $\mathfrak{g}$ with an invariant nondegenerate symmetric bilinear form $(,) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F}$

Definition 0.4.3. **Killing form** is the bilinear map $K(,) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto \mathtt{Tr}\,(ad(X)ad(Y))$

Some basic properties of Killing form

Proposition 0.4.4.

(a) The Killing form is symmetric and invariant

(b) The Killing form on a nilpotent Lie algebra is zero

(c) Suppose $I \leq \mathfrak{g}$ is an ideal, then Killing form $K_I(,)$ on I is the same as the restriction of Killing form $K(,)$ to I, i.e. $K_I(,) = K(,)|_I$

Proof.

(a)

(b) Due to Theorem 0.3.2

(c) By Exercise **??**, for $X, Y \in I$, we have

$$\begin{aligned} K_I(X,Y) &= Tr(ad(X)|_I ad(Y)|_I) \\ &= Tr\left((ad(X)ad(Y))|_I\right) \\ &= Tr(ad(X)ad(Y)) \\ &= K(X,Y) \\ &= K(X,Y)|_I \end{aligned}$$

□

Example 0.4.5. The Killing form is a symmetric, bilinear and invariant form, and it is nondegenerate iff $\mathfrak{g}$ is semisimple due to Proposition 0.6.6

nondegenerate, symmetric, bilinear and invariant form is unique up to scalar

Lemma 0.4.6. Any invariant, symmetric and bilinear form on simple Lie algebra $\mathfrak{g}$ is a multiple of the Killing form

Proof. Suppose $(,)$ is an invariant, symmetric and bilinear form, so is $[,]_c = (,) - cK(,)$ for any c. If $(,) \neq 0$, then there exists $x, y \in \mathfrak{g}$ such that $[x,y]_c = 0$ for some c, since the kernel of $[,]_c$ is a nonzero ideal, $[,]_c = 0$ □

0.5 Jordan-Chevalley decomposition

Abstract Jordan-Chevalley decomposition on nonassociative F-algebras

Lemma 0.5.1. Let $\mathfrak{g}$ be a finite dimensional nonassociative $\mathbb{F}$ algebra(including Lie algebra) with $\overline{\mathbb{F}} = \mathbb{F}$, for any $\delta \in Der(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$, let $\delta = \delta_s + \delta_n$ be its Jordan-Chevalley decomposition in $\mathfrak{gl}(\mathfrak{g})$, then $\delta_s, \delta_n \in Der(\mathfrak{g})$

Proof. For any $a \in \mathbb{F}$, define $\mathfrak{g}_a$ be the generalized eigenspace of a, then we have $\mathfrak{g} = \bigoplus_{a\in\mathbb{F}} \mathfrak{g}_a$, and $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, since for any $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, $(\delta-(a+b)1_{\mathfrak{g}})^m([x, y]) = \sum_{k=0}^{m} \binom{m}{k} [(\delta - a1_{\mathfrak{g}})^{m-k}x, (\delta - b1_{\mathfrak{g}})^k y]$ which can be easily checked by induction. Then we have $\delta_s(x) = ax, \delta_s(y) = by$, and $\delta_s([x, y]) = (a + b)[x, y] = [ax, y] + [x, by] = [\delta_s(x), y] + [x, \delta_s y]$, thus $\delta_s \in Der(\mathfrak{g})$, so does $\delta_n = \delta - \delta_s$ $\square$

Definition 0.5.2. Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Because Lemma 0.5.1 and Proposition 0.2.16, for any $x \in \mathfrak{g}$, we can identify x with $ad(x)$, and we have Jordan-Chevalley decomposition $ad(x) = ad(x)_s + ad(x)_n = ad(x_s) + ad(x_n)$, where x_s, x_n are defined to be the preimages of $ad(x)_s, ad(x)_n$. Moreover, there exists polynomials $p(t), q(t)$ with no constant terms such that $ad(x_s) = ad(x)_s = p(ad(x))$, $ad(x_n) = ad(x)_n = q(ad(x))$, by abuse of notations, $x_s = p(x)$ and $x_n = q(x)$

Semisimple Lie algebra contains the semisimple and nilpotent parts of its elements

Theorem 0.5.3. Suppose V is a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a semisimple Lie algebra, for any $x \in \mathfrak{g}$, $x = x_s + x_n$ is the Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$, moreover, the abstract and usual Jordan-Chevalley decompositions coincide, i.e. $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$

Proof. Define lie subalgebras $\mathfrak{l}_W := \{y \in \mathfrak{gl}(V) | yW \subseteq W, Tr(y|_W) = 0\}$ with $W \leq V$ being $\mathfrak{g}$ submodules, and define $\mathfrak{l} = \left(\bigcap_W \mathfrak{l}_W\right) \bigcap N_{\mathfrak{gl}(V)}(\mathfrak{g})$, for any $x \in \mathfrak{g}$, due to Proposition **??** and Lemma 0.2.15, $Tr(x|_W) = Tr(x) = 0$, $\mathfrak{g} \leq \mathfrak{l}_W \Rightarrow \mathfrak{g} \leq \mathfrak{l}$, thus $\mathfrak{l}$ is a subalgebra of $N_{\mathfrak{gl}(V)}(\mathfrak{g})$ of containing $\mathfrak{g}$, thus $\mathfrak{l}$ is finite dimensional $\mathfrak{g}$ module, by Theorem 0.2.19, $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{h}$ is a direct sum of $\mathfrak{g}$ modules, since $\mathfrak{l} \leq N_{\mathfrak{gl}(V)}(\mathfrak{g})$, $[\mathfrak{g}, \mathfrak{l}] = 0 \Rightarrow [\mathfrak{g}, \mathfrak{h}] = 0$, i.e. $\mathfrak{g}$ acts trivially on $\mathfrak{h}$, fix any irreducible $\mathfrak{g}$ submodule W, for any $y \in \mathfrak{h}$, $x \in \mathfrak{g}$, $xy - yx = [x, y] = 0$, $yxv = xyv$ for $v \in W$, i.e. $y \in \mathrm{Hom}_{\mathfrak{g}}(W, W)$, by Lemma **??**, y acts on W as a scalar, but $Tr(y|_W) = 0$, thus y acts trivially on W, again by Theorem 0.2.19, V can written as the direct sum of irreducible $\mathfrak{g}$ submodules, thus y acts trivially on $W \Rightarrow y = 0$, therefore $\mathfrak{h}_j = 0 \Rightarrow \mathfrak{g} = \mathfrak{l}$, for any $x \in \mathfrak{g}$, due to Theorem **??**, $x = x_s + x_n$ and $x_s = p(x), x_n = q(x)$ for some polynomials $p(x), q(x)$ with no constant terms, thus if $x \in \mathfrak{l}_W$, $xW \subseteq W$ and $Tr(x|_W) = 0$, then $x_s W = p(x)W \subseteq W$, $Tr(x_s|_W) = Tr(p(x|_W)) = 0$, similarly, $x_n W \subseteq W$, $Tr(x_n|_W) = 0$, $x_s, x_n \in \mathfrak{l}_W$, also $x_s = p(x), x_n = q(x) \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$, thus $x_s, x_n \in \mathfrak{l} = \mathfrak{g}$

Since the Jordan-Chevalley decomposition of $ad(x)$ in $\mathfrak{gl}(V)$ is unique and $ad(x_s) + ad(x_n) = ad(x) = ad(x)_s + ad(x)_n$, thus $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$ $\square$

Corollary 0.5.4. Suppose V is a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$, $\mathfrak{g}$ is a semisimple Lie algebra, and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, $x = x_s + x_n$ is the abstract Jordan-Chevalley decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the usual Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$

Proof. Due to Proposition 0.6.2, $\phi(\mathfrak{g})$ is also semisimple

First notice that a linear operator $T \in \mathrm{End}_{\mathbb{F}}(V)$ is diagonalizable(semisimple) iff the dimension of the sum of its eigenspaces is $\dim V$, or equivalently iff all eigenvectors span V

Since $ad(x_s)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 0.2.16, the eigenvectors y_i's of $ad(x_s)$ spans $\mathfrak{g}$, say $ad(x_s)(y_i) = \lambda_i y_i$, then as in Definition 0.2.5, $ad(\phi(x_s))(\phi(y_i)) = [\phi(x_s), \phi(y_i)] = \phi([x_s, y_i]) = \phi(ad(x_s)(y_i)) = \phi(\lambda_i y_i) = \lambda_i \phi(y_i)$, thus those $0 \neq \phi(y_i)$ are eigenvectors of $\phi(\mathfrak{g})$,

hence $ad(\phi(x_s))$ is semisimple in $\mathfrak{gl}\,(\mathfrak{gl}(V))$
Since $ad(x_n)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 0.2.16, $ad(x_n)^m = 0$ for some m, then as in Definition 0.2.5 $ad(\phi(x_n))^m = \phi(ad(x_n))^m = \phi(ad(x_n)^m) = 0$, thus $ad(\phi(x_n))$ is also nilpotent in $\mathfrak{gl}\,(\mathfrak{gl}(V))$
Moreover, as in Definition 0.2.5, $[ad(\phi(x_s)), ad(\phi(x_n))] = [\phi(ad(x_s)), \phi(ad(x_n))] = \phi([ad(x_s), ad(x_n)]) = 0$
Thus $\phi(x) = \phi(x_s) + \phi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\phi(x)$ in $\phi(\mathfrak{g}) \leq \mathfrak{gl}(V)$, by Theorem 0.5.3, this coincide with the usual Jordan-Chevalley decomposition of $\phi(x) = \phi(x)_s + \phi(x)_n$ in $\mathfrak{gl}(V)$, i.e. $\phi(x_s) = \phi(x)_s$, $\phi(x_n) = \phi(x)_n$ $\square$

0.6 Classification of semisimple Lie algebras

g simple implies [g,g]=g

Proposition 0.6.1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a semisimple Lie algebra, then any ideal of $\mathfrak{g}$ is certain sum of $\mathfrak{g}_i$'s, and any sum of $\mathfrak{g}_i$'s is an ideal, in particular, $\mathfrak{g}_i$'s are ideals, moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

Proof. any ideal $I \leq \mathfrak{g}$ is certain sum of $\mathfrak{g}_i$'s, because $I \cap \mathfrak{g}_i$ is an ideal of $\mathfrak{g}_i$ which is either 0 or $\mathfrak{g}_i$ itself
If $\mathfrak{g}$ is a simple Lie algebra, then it is not abelian, $[\mathfrak{g}, \mathfrak{g}] \neq 0$, thus $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, generally, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n] = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{g}_n, \mathfrak{g}_n] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$ □

image of a semisimple Lie algebra is also semisimple

Proposition 0.6.2. If $\mathfrak{g}$ is semisimple, $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $\phi(\mathfrak{g})$ is also semisimple

Proof. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of simple Lie algebras, $\phi(\mathfrak{g}_i)$ are also ideals so $\phi(\mathfrak{g}) = \phi(\mathfrak{g}_1) \oplus \cdots \oplus \phi(\mathfrak{g}_n)$ is a direct sum of ideals, and $[\phi(\mathfrak{g}_i), \phi(\mathfrak{g}_i)] = \phi([\mathfrak{g}_i, \mathfrak{g}_i]) = \phi(\mathfrak{g}_i)$ implies that each $\phi(\mathfrak{g}_i)$ is simple or 0, thus $\phi(\mathfrak{g})$ is also semisimple □

nilpotent/semisimple implies ad-nilpotent/ad-semisimple

Proposition 0.6.3. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a Lie algebra, $X \in \mathfrak{g}$, then X is nilpotent $\Rightarrow ad(X)$ is nilpotent, if in addition V is finite dimensional and $\overline{\mathbb{F}} = \mathbb{F}$, then X is semisimple $\Rightarrow ad(X)$ is semisimple, or rather diagonalizable

Proof. Let L_X, R_X be left and right multiplications by X, then we have $ad_X = L_X - R_X$ and $[L_X, R_X] = 0$, thus X is nilpotent $\Rightarrow ad(X)$ is nilpotent
Notice that given $A = (a_{ij})$, $D = \mathtt{diag}(d_1, \cdots, d_n)$, $[D, A] = ((d_i - d_j)a_{ij})$, thus $[D, E_{ij}] = (d_i - d_j)E_{ij}$, thus X is diagonalizble $\Rightarrow ad(X)$ is diagonalizable □

Cartan's criterion for solvability

Theorem 0.6.4 (Cartan's criterion for solvability)**.** Let V be a finite dimensional $\mathbb{F}$ vector space with $\mathtt{char}\,\mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra, then $\mathfrak{g}$ is solvable iff $\mathrm{Tr}(XY) = 0$, $\forall X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$

Cartan's criterion for semisimplicity

Corollary 0.6.5. Cartan's criterion for semisimplicity
Let $\mathfrak{g}$ is finite dimensional Lie algebra with $\mathtt{char}\,\mathbb{F} = 0$, then $\mathfrak{g}$ is semisimple iff its Killing form is nondegenerate

Equivalent conditions for semisimplicity

Proposition 0.6.6. The following statements are equivalent
(a) $\mathfrak{g}$ is semisimple
(b) The Killing form is nondegenerate
(c) $\mathfrak{g}$ doesn't have have nontrivial abelian ideals
(d) $\mathfrak{g}$ doesn't have have nontrivial solvable ideals
(e) $\mathtt{rad}(\mathfrak{g}) = 0$

Proof. $(a) \Leftrightarrow (b)$ is due to Corollary 0.6.5 □

Adjoint representation of SL(2,F)

Example 0.6.7.
Recall $\mathfrak{sl}(2, \mathbb{F}) = \{X \in M(2, \mathbb{F}) | \mathrm{Tr}(X) = 0\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{F} \right\}$
$\mathfrak{sl}(2, \mathbb{F}) = \langle H, X, Y \rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $ad_H X = [H, X] = 2X, ad_H Y = [H, Y] = -2Y, ad_X Y = [X, Y] = H$, this is the adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$

Lemma 0.6.8. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, if $V = \mathbb{F}^n$, then there are theree $n \times n$ matrices x, y, h such that $[h, x] = 2x, [h, y] = -2y, [x, y] = h$ due to Example 0.6.7

Lemma 0.6.9. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, $V_\lambda := \{v \in V | \pi(H)v = \lambda v\}$, then $\pi(X)V_\lambda \subseteq V_{\lambda+2}$, $\pi(Y)V_\lambda \subseteq V_{\lambda-2}$ and $\pi(H)V_\lambda \subseteq V_\lambda$

Proof. $\pi(H)V_\lambda \subseteq V_\lambda$ is just by definition, suppose $v \in V_\lambda$, $\pi(H)\pi(X)v = 2\pi(X)v + \pi(X)\pi(H)v = (\lambda+2)\pi(X)v$, $\pi(H)\pi(Y)v = -2\pi(Y)v + \pi(Y)\pi(H)v = (\lambda-2)\pi(Y)v$, □

Remark 0.6.10. $\pi(X), \pi(Y)$ are named **raising and lowering operator**

Classification of representations of sl(2,F)

Theorem 0.6.11. Suppose $\overline{\mathbb{F}} = \mathbb{F}, \operatorname{char}\mathbb{F} = 0$, for any integer $m \geq 0$, there is an irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$ with dimension $m+1$

Proof. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$, there exists highest $\lambda \in \mathbb{F}$ such that $V_\lambda \neq 0$, pick $0 \neq u \in V_\lambda$, let $u_k := \pi(Y)^k u \in V_{\lambda-2k}$, then there exists m such that $u_m \neq 0$ but $u_{m+1} = 0$, then $u_0, \cdots, u_m$ are independent since they belong distinct eigenspaces, with $\pi(H)u_k = (\lambda - 2k)u_k$, $\pi(X)u_0 = \pi(X)u = 0$ since $u \in V_\lambda$ is of "highest weight", and $\pi(X)u_k = k(\lambda - k + 1)u_{k-1}, k > 0$ by induction, $\pi(X)u_1 = \pi(X)\pi(Y)u = ([\pi(X), \pi(Y))] + \pi(Y)\pi(X))u = \pi(H)u + \pi(Y)\pi(X)u = \lambda u = 1(\lambda - 1 + 1)u_0$, $\pi(X)u_{k+1} = \pi(X)\pi(Y)u_k = ([\pi(X), \pi(Y))] + \pi(Y)\pi(X))u_k = \pi(H)u_k + \pi(Y)\pi(X)u_k = (\lambda - 2k)u_k + k(\lambda - k + 1)\pi(Y)u_{k-1} = (k+1)(\lambda - k)u_k$

Note that since $0 = Xu_{m+1} = (m+1)(\lambda - m)u_m \Rightarrow \lambda = m$, which implies all possible eigenvalue for $\pi(H)$ has to be integers, when m is even, we call this irreducible representation even, when m is odd, we call this irreducible representation odd

In general, for any finite dimensional representation, we can decompose the representation into irreducible subrepresentations by using this procedure repeatedly

Therefore $0 \neq W := \langle u_0, \cdots, u_m \rangle$ is invariant, but π is irreducible, thus $V = W$, and by Lemma **??**, (π, V) is unique up to isomorphism □

Adjoint representation of sl(2,F) is the unique 3 dimensional irreducible representation

Example 0.6.12. $(ad, \mathfrak{sl}(2,\mathbb{F}))$ is the unique irreducible 3 dimensional representation of $\mathfrak{sl}(2,\mathbb{F})$ with $V_0 = \langle H \rangle$, $V_{-2} = \langle Y \rangle$ and $V_2 = \langle X \rangle$, it is irreducible because of Lemma 0.6.15, if we use X, Y, H as basis, then $ad(X), ad(Y), ad(H)$ would have the matrix forms

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$K(X,X) = Tr\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad K(X,Y) = Tr\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 4$$

$$K(X,H) = Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = 0, \quad K(Y,Y) = Tr\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$K(Y,H) = Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 0, \quad K(H,H) = Tr\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8$$

Thus its Cartan matrix is $\Phi = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ which is nondegenerate

Tautological representation of sl(2,F)

Example 0.6.13. The tautological representation $(\tau, \mathbb{F}^2)$ is the unique irreducible 2 dimensional representation of $\mathfrak{sl}(2,\mathbb{F})$ with $V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, $V_{-1} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

Example 0.6.14. Let $S^k\left(\mathbb{F}^2\right)$ be the k-th symmetric power of $\mathbb{F}^2$ which is isomorphic to the the set of degree k polynomials in $\mathbb{F}[x,y]$ generated by

$\left\langle x^k, x^{k-1}y, \cdots, xy^{k-1}, y^k \right\rangle$ which is of dimension $k+1$, with this identification, $\left(\pi, S^k\left(\mathbb{F}^2\right)\right)$ with

$\pi(X)(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, $\pi(X)y = x$, $\pi(Y)x = y$, $\pi(Y)y = 0$, $\pi(H)x = x$, $\pi(H)y = -y$, just as in Example 0.6.13, and define inductively that $\pi(Z)(fg) = g\pi(Z)f + f\pi(Z)g$, this is the unique $k+1$ dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$

Count the number irreducible summand of a representation of sl(2,F)

Lemma 0.6.15. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$, and V_k be the k-eigenspace of $\pi(H)$, then the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$, whereas $\dim V_0, \dim V_1$ are number of even and odd irreducible summands

Proof. In an even irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are even, so there is a unique 0-eigenvector, in an odd irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are odd, so there is a unique 1-eigenvector, thus the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$ □

Definition 0.6.16. $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{h}$ is a maximal toral Lie algebra. For $\alpha \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{h}, \mathbb{F})$, define **root spaces**

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | ad_h(x) = [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

α is a **root** if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, denote the set of roots as Δ. $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of $\mathfrak{h}$

Basic properties of root spaces

Proposition 0.6.17.

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

(b) $\alpha \in \Delta$, any $X \in \mathfrak{g}_\alpha$ is nilpotent

(c) $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$

(d) $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

Proof.

(a) For $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $Z \in \mathfrak{h}$, we have

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = \alpha(Z)[X, Y] + \beta(Z)[X, Y] = (\alpha + \beta)(Z)[X, Y]$$

(b) For $\beta \in \Delta \cup \{0\}$, $\alpha \in \Delta$, $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$, $ad(X)^n(Y) \in \mathfrak{g}_{n\alpha+\beta} = 0$ when n is big enough, thus $ad(X)$ is nilpotent

(c) Supppose $\alpha + \beta \neq 0$, then there exists $Z \in \mathfrak{h}$ such that $(\alpha + \beta)(Z) \neq 0$, then for any $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$

$$\begin{aligned} (\alpha + \beta)(Z)K(X, Y) &= \alpha(Z)K(X, Y) + \beta(Z)K(X, Y) \\ &= K(\alpha(Z)X, Y) + K(X, \beta(Z)Y) \\ &= K([Z, X], Y) + K(X, [Z, Y]) \\ &= -K([X, Z], Y) + K(X, [Z, Y]) \\ &= 0 \end{aligned}$$

Thus $K(X, Y) = 0$

(d) Since $K(\mathfrak{g}_\alpha, \mathfrak{h}) = 0, \forall \alpha \in \Delta$, $\ker K(,)|_{\mathfrak{g}_0} \subseteq \ker K(,) = 0$, thus $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

□

Root space decomposition

Theorem 0.6.18. Semisimple Lie algebra $\mathfrak{g}$ has root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$

Proof. By Proposition 0.6.17 □

x,y in gl(V) commutes, x nilpotent => xy nilpotent

Lemma 0.6.19. V is an $\mathbb{F}$ vector space, $x, y \in \mathfrak{gl}(V)$ commutes, x is nilpotent, then xy is nilpotent, and $Tr(xy) = 0$

Proof. $x^m = 0 \Rightarrow (xy)^m = x^m y^m = 0$ □

Maximal toral Lie algebra of semisimple Lie algebra is self centralizing

Proposition 0.6.20. For semisimple Lie algebra $\mathfrak{g}$ with maximal toral Lie subalgebra $\mathfrak{h}$, $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Proof.

Step I: $C_{\mathfrak{g}}(\mathfrak{h})$ contains semisimple and nilpotent parts of its elements

If $x \in C_{\mathfrak{g}}(\mathfrak{h})$, due to Proposition 0.5.2, there are polynomials $p(t), q(t)$ with no constant terms such that $ad(x_s) = ad(x)_s = p(ad(x)), ad(x_n) = ad(x)_n = q(ad(x))$, since $x \in C_{\mathfrak{g}}(\mathfrak{h})$, $ad(x)|_{\mathfrak{h}} = 0$, , $ad(x_s)|_{\mathfrak{h}} = p(ad(x))|_{\mathfrak{h}} = 0$, $ad(x_n)|_{\mathfrak{h}} = q(ad(x))|_{\mathfrak{h}} = 0$, thus $x_s, x_n \in C_{\mathfrak{g}}(\mathfrak{h})$

Step II: $\mathfrak{h}$ contains all semisimple elements of $C_{\mathfrak{g}}(\mathfrak{h})$

If $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, use Exercise **??**, s and elements of $\mathfrak{h}$ are diagonalizable simultaneously, thus $\mathfrak{h} + \langle s \rangle$ is toral in $\mathfrak{g}$, then $s \in \mathfrak{h}$ since $\mathfrak{h}$ is maximal

Step III: $K(,)|_{\mathfrak{h}}$ is nondegenerate

Suppose there exists $h \in \mathfrak{h}$ such that $K(h, \mathfrak{h}) = 0$, if $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $ad(n)$ is nilpotent, and $[n, \mathfrak{h}] = 0$, thus $[ad(n), ad(h)] = ad([n, h]) = 0$, by Lemma 0.6.19, $Tr(ad(n)ad(h)) = 0$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, thus $K(s, h) = 0$, and according to Step I, $K(h, C_{\mathfrak{g}}(\mathfrak{h})) = 0$ which contradicts Proposition 0.6.17(d) that $K(,)|_{C_{\mathfrak{g}}(\mathfrak{h})}$ is nondegenerate

Step IV: $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

If $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $ad(n)$ is nilpotent, so is $ad(n)|_{C_{\mathfrak{g}}(\mathfrak{h})}$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, $ad(s)|_{C_{\mathfrak{g}}(\mathfrak{h})} = 0$, by Theorem 0.3.2, $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

Step V: $\mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] = 0$

Suppose $x \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then $x = \sum [y_i, z_i]$ where $y_i, z_i \in C_{\mathfrak{g}}(\mathfrak{h})$, then $K(x, x) = \sum K(x, [y_i, z_i]) = \sum K([x, y_i], z_i) = 0$, since $K(,)$ is nondegenerate on $\mathfrak{h}$(or $\mathfrak{g}$ or $C_{\mathfrak{g}}(\mathfrak{h})$), thus $x = 0$

Step VI: $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian

Suppose $[C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] \neq 0$, since $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent from Step IV, by Lemma 0.3.3, there exists $0 \neq z \in Z(C_{\mathfrak{g}}(\mathfrak{h})) \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then z can't be semisimple, otherwise $z \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, contradicting Step V, thus its nilpotent part $n \neq 0$, but its semisimple part $s \in \mathfrak{h} \leq Z(C_{\mathfrak{g}}(\mathfrak{h}))$, so is $n = z - s$, but then $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 0.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 0.6.17(d)

Step VII: $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$

Suppose $x \in C_{\mathfrak{g}}(\mathfrak{h}) \backslash \mathfrak{h}$, then it gives a nonzero nilpotent part n, but then since $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian by Step VI, thus $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 0.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 0.6.17(d)

□

Remark 0.6.21. $K(,)|_{\mathfrak{h}}$ is nondegenerate is not the same as saying that the Killing form of $\mathfrak{h}$ is nondegenerate which obviously violates Proposition 0.6.6, it doesn't contradict Proposition 0.4.4 since $\mathfrak{h} \leq \mathfrak{g}$ is merely a Lie subalgebra but not an ideal, by the nondegeneracy, we can identity $\mathfrak{h}^*$ with $\mathfrak{h}$ by $\mathfrak{h}^* \to \mathfrak{h}, \alpha \mapsto t_\alpha$, where $K(t_\alpha, x) = \alpha(x)$, and here t behaves like the linear isomorphism $t : \mathfrak{h}^* \to \mathfrak{h}, \alpha \mapsto t_\alpha$

Some properties about root space decomposition

Proposition 0.6.22.

(a) Δ spans $\mathfrak{h}^*$

(b) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$

(c) $\alpha \in \Delta$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = K(x, y)t_\alpha$

(d) $\alpha \in \Delta$, then $0 \neq [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$

(e) If $\alpha \in \Delta$, then $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$

(f) If $\alpha \in \Delta$, $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $K(x_\alpha, y_\alpha) = \dfrac{2}{K(t_\alpha, t_\alpha)}$, define $h_\alpha := [x_\alpha, y_\alpha] = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$, then $\mathfrak{s}_\alpha := \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ via $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(g) Given $\alpha \in \Delta$, $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$ will be a representation of $\mathfrak{s}_\alpha$, thus $\mathfrak{g}$ can be decomposed into irreducible representations of $\mathfrak{s}_\alpha$, and the highest eigenvectors for this representation are also common highest eigenvectors of $ad(\mathfrak{h})$

Proof.

(a) Suppose $\langle \Delta \rangle \subsetneq \mathfrak{h}^*$, then there exists $0 \neq h \in \mathfrak{h}$, such that $\forall \alpha \in \Delta, \alpha(h) = 0$, then $\forall x \in \mathfrak{g}_\alpha, [h, x] = \alpha(h)x = 0$, and since $\mathfrak{h}$ is abelian, $[h, \mathfrak{h}] = 0$, thus $[h, \mathfrak{g}] = 0$, but then $h \in Z(\mathfrak{g}) = 0$ which is a contradiction

(b) If $\alpha \in \Delta$, and $\mathfrak{g}_{-\alpha} = 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 0.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction

(c) $\forall h \in \mathfrak{h}, K(h, [x, y]) = K([h, x], y) = K(\alpha(h)x, y) = K(t_\alpha, h)K(x, y) = K(h, K(x, y)t_\alpha)$, since $K(,)|_{\mathfrak{h}}$ is nondegenerate, $[x, y] = K(x, y)t_\alpha$

(d) Only need to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$. There exists $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) \neq 0$, otherwise then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 0.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction, thus $[x, y] = K(x, y)t_\alpha \neq 0$ by (c)

(e) Suppose instead $\alpha(t_\alpha) = 0$, then $[t_\alpha, x] = [t_\alpha, y] = 0, \forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, by nondegeneracy, we can find $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$, then by (c), $[x, y] = K(x, y)t_\alpha = t_\alpha$, thus $\mathfrak{s} = \langle x, y, t_\alpha \rangle \cong ad(\mathfrak{s}) \leq ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is a 3 dimensional solvable Lie algebra, by Theorem 0.3.4, for any $s \in [\mathfrak{s}, \mathfrak{s}]$, ad(s) is nilpotent, thus $ad(t_\alpha)$ is both semisimple and nilpotent, hence $ad(t_\alpha) = 0 \Rightarrow t_\alpha = 0$ which is a contradiction

(f) $[h_\alpha, x_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, x_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha$ $[h_\alpha, y_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, y_\alpha] = -\dfrac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)y_\alpha = -2y_\alpha$

(g) Suppose $x \in \mathfrak{g}$ is a highest eigenvector of representation $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $0 = ad(x_\alpha)(x) = [x_\alpha, x]$, $\forall h \in \mathfrak{h}, [x_\alpha, ad(h)(x)] = [x_\alpha, [h, x]] = [[x_\alpha, h], x] + [h, [x_\alpha, x]] = [-\alpha(h)x_\alpha, x] = 0$

□

alpha check is canonical

Remark 0.6.23. In (f), the choice of x_α is not canonical, however, $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ is canonical, for example, if we pick $0 \neq x_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $\mathfrak{s}_{-\alpha} = \langle x_{-\alpha}, y_{-\alpha}, h_{-\alpha}\rangle$, then $h_{-\alpha} = h_\alpha = \dfrac{2t_{-\alpha}}{K(t_{-\alpha}, t_{-\alpha})} = -h_\alpha$, moreover, according to Lemma 0.4.6, any nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$ is of the form $(,) := cK(,)|_\mathfrak{h}$ for some $c \neq 0$, then t'_α the dual of $\alpha \in \mathfrak{h}^*$ given by $(t'_\alpha, x) = \alpha(x), \forall x \in \mathfrak{h}$, then we have $K(t_\alpha, x) = \alpha(x) = (t'_\alpha, x) = cK(t'_\alpha, x) = K(ct'_\alpha, x)$, because of the nondegeneracy of $K(,)|_\mathfrak{h}$, $t_\alpha = ct'_\alpha \Rightarrow t'_\alpha = \dfrac{t_\alpha}{c}$, and $\dfrac{2t'_\alpha}{(t'_\alpha, t'_\alpha)} = \dfrac{2\frac{t_\alpha}{c}}{cK\left(\frac{t_\alpha}{c}, \frac{t_\alpha}{c}\right)} = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)} = h_\alpha$, thus h_α is even canonical defined regardless of the choice of the nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$, for this reason, we give h_α a new notation $\alpha^\vee$ for later use, whereas $\alpha(\alpha^\vee) = 2$, for any nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$, note that $\alpha \mapsto \alpha^\vee$ is not linear

Also, according to Lemma 0.4.6, even though $(,)$ is defined up to a scalar, but the orthogonality is always well defined

Due to Proposition 0.6.22(g), if $0 \neq x \in \mathfrak{g}$ is a highest eigenvector for representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $x \in \mathfrak{g}_\beta$, for some $\beta \in \Delta \cup \{0\}$, when $\beta = 0$, these corresponds to the trivial representations, if $\beta \in \Delta$, we can denote one such highest eigenvector $0 \neq x_\beta \in \mathfrak{g}_\beta$, then $ad(h_\alpha)(x_\beta) = [h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta$, thus x_β is a $\beta(h_\alpha) = \dfrac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}$-eigenvector, by Proposition 0.6.17(a), $ad(y_\alpha)^j(x_\beta) \in \mathfrak{g}_{\beta - j\alpha}$ are all the nonzero eigenvectors corresponds to eigenvalues $(\beta - j\alpha)(h_\alpha) = 2\left(\dfrac{K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)} - j\right)$, and these roots $\beta - j\alpha, j = 0, \cdots, k = \beta(h_\alpha)$ form an α-string

One of these irreducible representation is $\mathfrak{s}_\alpha$ itself according to Example 0.6.12

Example 0.6.24. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ which is a semisimple Lie algebra, then

$$\mathfrak{h} = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \middle| \sum z_i = 0 \right\}$$

is a maximal toral Lie subalgebra, denote $\mathrm{diag}(z_1, \cdots, z_n)$ as h_z, then $[h_z, E_{ij}] = (z_i - z_j)E_{ij}$, and $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{g} = \mathfrak{h} \bigoplus_{i \neq j} \langle E_{ij}\rangle$, $\Delta = \{e_i - e_j | i \neq j\}$, where $e_i \in \mathfrak{h}^*$ is defined by $e_i(h_z) = z_i$, also

$$\mathfrak{s}_\alpha = \left\{ \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & a & & b & & \\ & & & \ddots & & & \\ & & b & & -a & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \right\} \cong \mathfrak{sl}(2, \mathbb{C})$$

where $\alpha = e_i - e_j$

Definition of s_alpha

Definition 0.6.25. Define $s_\alpha(\beta) := \beta - \dfrac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}\alpha, \forall \alpha \in \Delta, \beta \in \mathfrak{h}^*$ is an orthogonal reflection over the hyperplane $H_\alpha = \{\alpha(x) = K(t_\alpha, x) = 0 | x \in \mathfrak{h}\} = \ker \alpha$, more precisely, $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\beta) = \beta, \forall \beta \in H_\alpha$, note here any nondegenerate, symmetric, bilinear and invariant form $(,)$ can be used as the definition in place of the Killing form $K(,)$ thanks to Lemma 0.4.6

alpha string through beta

Proposition 0.6.26.
(a) If $\alpha \in \Delta$, then $\dim \mathfrak{g}_\alpha = 1$

(b) If $\alpha \in \Delta$, then $c\alpha \in \Delta \Leftrightarrow c = \pm 1$
(c) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
(d) If $\alpha, \beta \in \Delta$, the Cartan integer $\beta(h_\alpha) = \dfrac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Delta$, if $\beta \neq \alpha, -\alpha$, then $\Delta \bigcap \{\beta + j\alpha | j \in \mathbb{Z}\} = \{\beta + j\alpha | -r \leq j \leq s, j \in \mathbb{Z}\}$ which is an α string through β, and $\beta(h_\alpha) = r - s$

Proof. **(a)** Let $\mathfrak{m} = \mathfrak{h} \bigoplus_{c\alpha \in \Delta, c \in \mathbb{F}^\times} \mathfrak{g}_{c\alpha}$, then $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is a finite representation of $\mathfrak{s}_\alpha$ because of Proppsition 0.6.17(a), first notice for $x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, thus the 0-eigenspace of $ad(h_\alpha)$ is $\mathfrak{h}$, but note that for any $h \in \ker\alpha = H_\alpha \leq \mathfrak{h}$ as in Definition 0.6.25, $ad(x_\alpha)(h) = [x_\alpha, h] = \alpha(h)x_\alpha = 0$, $ad(y_\alpha)(h) = [y_\alpha, h] = -\alpha(h)y_\alpha = 0$, $ad(h_\alpha)(h) = [h_\alpha, h] = 0$, thus $\mathfrak{s}_\alpha$ acts trivially on $\ker\alpha$ which is of codimension 1 in $\mathfrak{h}$ which gives $\dim\mathfrak{h} - 1$ copies of trivial representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, since $h_\alpha \notin \ker\alpha$, $\mathfrak{h} = \langle h_\alpha \rangle \oplus \ker\alpha$, by Example 0.6.12 and Lemma 0.6.15, $\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is a 3 dimensional irreducible representation of $\mathfrak{s}_\alpha$, and $\mathfrak{s}_\alpha \oplus \ker\alpha$ are the only possible even irreducible representations of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, thus $\dim\mathfrak{g}_\alpha = 1$, otherwise, we can choose $0 \neq x'_\alpha \in \mathfrak{g}_\alpha$ linearly independent of x_α, then we have $[h_\alpha, x'_\alpha] = \alpha(h_\alpha)x'_\alpha = 2x'_\alpha$, which gives a contradiction. Moreover, $2\alpha \notin \Delta$, otherwise, we can choose $0 \neq x_{2\alpha} \in \mathfrak{g}_{2\alpha}$, then $[h_\alpha, x_{2\alpha}] = 2\alpha(h_\alpha)x_{2\alpha} = 4x_{2\alpha}$which also gives a contradiction
(b) Suppose $c\alpha \in \Delta$, for $0 \neq x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, by Theorem 0.6.11, we know that $2c \in \mathbb{Z}$, but by symmetry, if we let $\beta = c\alpha$, then $\alpha = \dfrac{\beta}{c} \in \Delta$ implies $\dfrac{2}{c} \in \mathbb{Z}$, thus c can only possibly be $\pm 1, \pm 2, \pm\dfrac{1}{2}$, but from (a), we know $c \neq 2$, thus $c \neq -2$ thanks to Proposition 0.6.22(b), and by symmetry, $c \neq \pm\dfrac{1}{2}$, therefore $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker\alpha \oplus \langle h_\alpha \rangle \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker\alpha \oplus \mathfrak{s}_\alpha$
(c) Obiviously β can't be α or α, otherwise $\alpha + \beta \notin \Delta$, also $\beta + j\alpha \neq 0, \forall j \in \mathbb{F}$ by (b), for $\beta \in \Delta \setminus \{\alpha, -\alpha\}$, we can consider $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ which is a finite representation of $\mathfrak{s}_\alpha$ with $\mathfrak{m} = \bigoplus_{\beta + j\alpha \in \Delta, j \in \mathbb{Z}} \mathfrak{g}_{\beta + j\alpha}$, suppose $\beta + j\alpha \in \Delta$, $\dim\mathfrak{g}_{\beta+j\alpha} = 1$ by (a), choose $0 \neq x_{\beta+j\alpha} \in \mathfrak{g}_{\beta+j\alpha}$, we have $[h_\alpha, x_{\beta+j\alpha}] = (\beta + j\alpha)(h_\alpha)x_{\beta+j\alpha} = (\beta(h_\alpha) + 2j)\, x_{\beta+j\alpha}$, as j varies in $\mathbb{Z}$, $\beta(h_\alpha) + 2j$ can't take both 0 and 1, thus the sum of dimension of 0-eigenspace and 1-eigenspace of $ad(h_\alpha)$ on $\mathfrak{m}$ is 1, by Lemma 0.6.15, $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is irreducible, according to Theorem 0.6.11 and Remark 0.6.23, $\mathfrak{m} = \bigoplus_{-r \leq j \leq s} \mathfrak{g}_{\beta+j\alpha}$, for some $r, s \in \mathbb{Z}_{\geq 0}$, and $\beta + j\alpha \in \Delta, \forall -r \leq j \leq s$ which is the α string through β, note that $ad(x_\alpha)(x_\beta) \neq 0$ as in Theorem 0.6.11, thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
(d) When $\beta = \alpha, -\alpha$, it is trivial. When $\beta \neq \alpha, -\alpha$, as in (c), we know that when j varies from $-r$ to s, $(\beta(h_\alpha) + 2j)$ are all the possible eigenvalues which are integers symmetric over 0, thus $\beta(h_\alpha) + 2s + \beta(h_\alpha) - 2r = 0 \Rightarrow \beta(h_\alpha) = r - s$ is an integer, then $-r \leq -\beta(h_\alpha) \leq s \Rightarrow \beta - \beta(h_\alpha)\alpha \in \Delta$ □

Connection between roots and root system

Proposition 0.6.27.
(1) Let $V = \mathbb{Q}\Delta$, then $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$
(2) For any $h, k \in \mathfrak{h}$, $K(h,k) = Tr(ad(h)ad(k)) = \sum_{\alpha \in \Delta} \alpha(h)\alpha(k)$
(3) The dual of Killing form restricted on V is rational and positive definite

Proof. Since $K(,)|_{\mathfrak{h}}$ is nondegenerate, we define the dual of $K(,)|_{\mathfrak{h}}$ on $\mathfrak{h}^*$ as $(\alpha, \beta) := K(t_\alpha, t_\beta)$, which is also nondegenerate
(1) Since $\mathfrak{h}^* = \mathbb{F}\Delta$ by Proposition 0.6.22(a), Pick any basis in Δ, say $\alpha_1, \cdots, \alpha_m$, then $\forall \beta \in \Delta$, $\beta = \sum c_j\alpha_j, c_j \in \mathbb{F}$, we have $(\beta, \alpha_k) = \sum c_j(\alpha_j, \alpha_k) \Rightarrow \dfrac{2(\beta,\alpha_k)}{(\alpha_k,\alpha_k)} = \sum c_j \dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}$ whereas $\dfrac{2(\beta,\alpha_k)}{(\alpha_k,\alpha_k)}, \dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}$ are all integers by Proposition 0.6.26(d), since $(\beta - \sum c_j\alpha_j, \alpha_k) = 0$ and that $(,)$ is nondegenerate, meaning c_j's are the unique solution, hence matrix $((\alpha_j, \alpha_k))$ is nonsingular,

so is matrix $\left(\dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}\right)$, thus $c_j \in \mathbb{Q}$, which means $\dim_{\mathbb{Q}} V = \dim_{\mathbb{F}} \mathfrak{h}^*$ and $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$

(2) For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, $ad(h)ad(k)(x_\alpha) = [h,[k,x_\alpha]] = \alpha(k)[h,x_\alpha] = \alpha(h)\alpha(k)x_\alpha$, according to Theorem 0.6.18, we have

$$K(h,k) = Tr(ad(h)ad(k)) = 0 + \sum_{\alpha\in\Delta} Tr\left(ad(h)|_{\mathfrak{g}_\alpha} ad(k)|_{\mathfrak{g}_\alpha}\right) = \sum_{\alpha\in\Delta} \alpha(h)\alpha(k)$$

Due to Proposition 0.6.22(b), roots always appears in pairs, if let $\Delta^+ \subseteq \Delta$ consists of exactly one from each pair, then $\sum_{\alpha\in\Delta} \alpha(h)\alpha(k) = 2\sum_{\alpha\in\Delta^+} \alpha(h)\alpha(k)$ **(3)** By (2), for any $\lambda, \mu \in \mathfrak{h}^*$

$$\begin{aligned}
(\lambda,\mu) &= K(t_\lambda, t_\mu) \\
&= \sum_{\alpha\in\Delta} \alpha(t_\lambda)\alpha(t_\beta) \\
&= \sum_{\alpha\in\Delta} K(t_\alpha,t_\lambda)K(t_\alpha,t_\mu) \\
&= \sum_{\alpha\in\Delta} (\alpha,\lambda)(\alpha,\mu) \\
&= 2\sum_{\alpha\in\Delta^+} (\alpha,\lambda)(\alpha,\mu)
\end{aligned}$$

In particular, for any $\beta \in \Delta$, $(\beta,\beta) = 2\sum_{\alpha\in\Delta^+}(\alpha,\beta)^2 \Rightarrow \dfrac{2}{(\beta,\beta)} = \sum_{\alpha\in\Delta^+}\left(\dfrac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2$, where $\dfrac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, thus $0 \leq (\beta,\beta) \in \mathbb{Q}$ and equals 0 iff $(\alpha,\beta) = 0, \forall \alpha \in \Delta \Rightarrow \beta = 0$ by the nondegeneracy of $(,)$, thus $(,)|_V$ is rational and positive definite on a basis, which implies it is rational and positive definite $\square$

Remark 0.6.28. When $\mathbb{F} = \mathbb{C}$, note that $\mathbb{Q}\Delta < \mathbb{R}\Delta < \mathbb{C}\Delta$, we can view V embedded in the Euclidean space $V_{\mathbb{R}} := \mathbb{R}\Delta = V \otimes_{\mathbb{Q}} \mathbb{R}$ which helps thinking, then we have a root system

Example 0.6.29. Let $\Omega = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, then $\left\{X \in SL(2n,\mathbb{C}) \middle| X^T\Omega X = \Omega\right\}$ is conjugate to $SO(2n,\mathbb{C})$, thus then also induce isomorphic Lie algebra, hence we can identify $\mathfrak{so}(2n,\mathbb{C})$ with $\left\{X \in M(2n,\mathbb{C}) \middle| \Omega X^T + X\Omega = 0\right\}$ which is the same as $\left\{\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in M(2n,\mathbb{C}) \middle| B^T = -B, C^T = -C\right\} =: \mathfrak{g}$, then one Cartan subalgebra of $\mathfrak{g}$ will be $\mathfrak{h} = \left\{\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in M(2n,\mathbb{C}) \middle| D = \mathrm{diag}(d_1,\cdots,d_n)\right\}$, note that

$$\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}\right] = (d_i - d_j)\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}\right] = (d_i + d_j)\begin{pmatrix} 0 & E_{ij} \\ 0 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}\right] = -(d_i + d_j)\begin{pmatrix} 0 & 0 \\ E_{ij} & 0 \end{pmatrix}$$

0.7 Root system

Delta is finite spaning set of V, then there is at most one s:V->V, maps Delta to Delta, s^2=1 and alpha to -alpha

Lemma 0.7.1. V is a finite dimensional vector space over a field of characteristic 0, $\Delta \subseteq V$ is a finite set such that $V = \mathsf{Span}\,\Delta$, then for any $\alpha \in \Delta$, there is at most one linear map $s : V \to V$ such that $s^2 = 1$, $s\alpha = -\alpha$, $s(\Delta) \subseteq \Delta$

Proof. Suppose s, t both satisfy the condition, $sv = v + (s-1)v$, $s(s-1)v = s^2v - sv = v - sv = -(s-1)v$, thus $(s-1)v \in \langle\alpha\rangle \Rightarrow sv = v + f(v)\alpha$, where $f \in V^*$, similarly, $tv = v + g(v)\alpha$, where $g \in V^*$, thus $stv = s(v + g(v)\alpha) = v + g(v)\alpha + f(v + g(v)\alpha)\alpha = v + g(v)\alpha + f(v)\alpha + f(v)g(v)\alpha$, and since $s\alpha = \alpha + f(\alpha)\alpha = -\alpha \Rightarrow f(\alpha) = -2$, thus check $(st)^n v = v + n(f(v) - g(v))\alpha$, but $(st)^n = 1$ for some n because st is just a permutation of Δ, thus $f = g \Rightarrow s = t$ ☐

Remark 0.7.2. $s^2 = 1$ and $s(\Delta) \subseteq \Delta$ implies that s is a permutation of Δ, note that this definition doesn't involve inner product on V, you could see this as a more abstract definition of a reflection

Definition of root system

Definition 0.7.3. $V = \mathbb{R}^n$ is the standard Euclidean space with the standard inner product $(,)$, a **root system** Δ is a finite subset of V satisfying

1. $V = \langle\Delta\rangle$

2. If $\alpha \in \Delta$, then the only multiples of α are $\pm\alpha$

3. $s_\alpha(\Delta) \subseteq \Delta$, where

$$s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

is the reflection across the hyperplane $H_\alpha = \{\beta \in V | (\beta,\alpha) = 0\}$

4. $\langle\beta,\alpha\rangle = \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ are called the **Cartan integers**

Remark 0.7.4. $\langle\beta,\alpha\rangle \in \mathbb{Z}$ is the **integrality** condition, and such roots system is called **crystallographic**

$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\dfrac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta \in \mathbb{Z}$ can only be $0, 1, 2, 3, 4$, where θ is the angle between α and β, and it is 4 iff $\alpha = \pm\beta$

$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$	0	1	2	3	4
θ	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{3}$	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{6}$	0

$a_{ji} = \dfrac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ gives a Cartan matrix A with $D_{ij} = \dfrac{\delta_{ij}}{(\alpha_i,\alpha_i)}$, $S_{ij} = 2(\alpha_i,\alpha_j)$
Conversely, given a generalized Cartan matrix, we can find the corresponding Lie algebra, see Kac-Moody algebra

Remark 0.7.5. Thanks to Lemma 0.7.1, s_α is the unique linear map $V \to V$ such that $s_\alpha(\alpha) = -\alpha, s_\alpha(\Delta) \subseteq \Delta$

Definition 0.7.6. Let (V,Δ) be a root system, define the coroot of $\alpha \in \Delta$ to be $\alpha^\vee = \dfrac{2}{(\alpha,\alpha)}\alpha$, and let $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$, then $(V,\Delta^\vee)$ is also a root system

alpha not equal to pm beta are roots, (alpha,beta)>0 => alpha-beta is a root

Lemma 0.7.7. $\alpha \neq \pm\beta \in \Delta$. If $\langle\alpha,\beta\rangle > 0$, then $\alpha - \beta \in \Delta$, if $\langle\alpha,\beta\rangle < 0$, then $\alpha + \beta \in \Delta$

Proof. Note that $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha \in \Delta$, $s_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta \in \Delta$, and $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\dfrac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta \in \mathbb{Z}$ can only be $0, 1, 2, 3$, where θ is the angle between α and β, if $\langle\alpha,\beta\rangle > 0 \Leftrightarrow (\alpha,\beta) > 0$, then $\langle\alpha,\beta\rangle = 1, \langle\beta,\alpha\rangle = 1, 2, 3$ or $\langle\alpha,\beta\rangle = 2, 3, \langle\beta,\alpha\rangle = 1$, hence either $\alpha - \beta$ or $\beta - \alpha$ will be in Δ, but then the other will also be in Δ. Similarly, if $\langle\alpha,\beta\rangle < 0 \Leftrightarrow (\alpha,\beta) < 0$, then $\langle\alpha,\beta\rangle = -1, \langle\beta,\alpha\rangle = -1, -2, -3$ or $\langle\alpha,\beta\rangle = -2, -3, \langle\beta,\alpha\rangle = -1$, hence $\alpha + \beta \in \Delta$ ☐

Definition 0.7.8. $\Delta^+ \subset \Delta$ is a set of **positive roots** if $\alpha \in \Delta \Rightarrow$ precisely one of $\alpha, -\alpha$ is in Δ^+, and $\alpha, \beta \in \Delta^+ \Rightarrow \alpha + \beta \in \Delta^+$, thus we can correspondingly define negative roots $\Delta^- = \Delta \setminus \Delta^+ = -\Delta^+$. Any hyperplane H that doesn't intersect Δ separates Δ into Δ^+ and Δ^-. $\gamma \in V$ is called **regular** if $(\gamma, \alpha) \neq 0, \forall \alpha \in \Delta$, then $H_\gamma = \{v \in V | (\gamma, v) = 0\}$ separates Δ into $\Delta^+(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) > 0\}$ and $\Delta^-(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) < 0\}$. Conversely, given a hyperplane H that doesn't intersect Δ and separates Δ into Δ^+ and Δ^-, we can find $\gamma \in V$ such that $H = H_\gamma$, $\Delta^+ = \Delta^+(\gamma)$, $\Delta^- = \Delta^-(\gamma)$

Definition 0.7.9. $\gamma \in V$ is **dominant** if $(\gamma, \alpha) \geq 0, \forall \alpha \in \Delta^+$ and **strictly dominant** if $(\gamma, \alpha) > 0, \forall \alpha \in \Delta^+$

Definition 0.7.10. $\alpha \in \Delta^+$ is **decomposable** if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Delta^+$, α is a **simple root** of Δ^+ if it is not decomposable. $S = \{\alpha_1, \cdots, \alpha_m\}$ is a **base** for Δ^+ if for any $\alpha \in \Delta^+$, there are $c_i \in \mathbb{Z}_{\geq 0}$ such that $\alpha = \sum_i c_i \alpha_i$, which also implies that for any $\alpha \in \Delta^- = -\Delta^+$, there are $c_i \in \mathbb{Z}_{\leq 0}$ such that $\alpha = \sum_i c_i \alpha_i$

v_1,...,v_m on one side of a hyperplane, (v_i,v_j)<0 => v_i linearly independent

Lemma 0.7.11. $S = \{v_1, \cdots, v_m\}$ are on one side of a hyperplane H, and $(v_i, v_j) < 0, \forall i \neq j$, then S is linearly independent

Proof. Suppose $\sum_{i=1}^m a_i v_i = 0$ and not all a_i's are zero, then we can rewrite as $\sum_{k \in K} a_k v_k = \sum_{l \in L} -a_l v_l$, where $a_k > 0, \forall k \in K, a_l < 0, \forall l \in L$, then we have $0 \leq \left(\sum_{k \in K} a_k v_k, \sum_{l \in L} -a_l v_l \right) = \sum_{k \in K, l \in L} -a_k a_l (v_k, v_l) < 0$ which is a contradiction □

Lemma 0.7.12. S is the set of simple roots of Δ^+, then S is a base for Δ^+, and S is linearly independent

Proof. It is obvious that S is a base of Δ^+ by definition. Suppose $\alpha \neq \beta \in S$, $\alpha \neq -\beta$ is obvious, hence $(\alpha, \beta) \neq 0$. If $(\alpha, \beta) > 0$, then by Lemma 0.7.7, we have $\alpha - \beta \in \Delta$, if $\alpha - \beta \in \Delta^+$, then $\alpha = \beta + (\alpha - \beta)$ gives a contradiction, if $\alpha - \beta \in \Delta^-$, then $\beta - \alpha \in \Delta^+$ and $\beta = \alpha + (\beta - \alpha)$ gives a contradiction, therefore $(\alpha, \beta) < 0$. By lemma 0.7.11, we know S is linearly independent □

Remark 0.7.13. Given a set of positive roots, there is precisely one base which is the set of simple roots

Conjugacy of roots and Weyl group

Lemma 0.7.14. $\sigma \in \mathrm{GL}(V)$, $\sigma(\Delta) \subseteq \Delta$, then for any $\alpha \in \Delta$, $\sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$, moreover, $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$

Proof. $\sigma^m = 1$ for some m since there are only finitely many choices of maps $\Delta \to \Delta$, thus σ is a permutation on Δ, hence $\sigma s_\alpha \sigma^{-1}(\Delta) \subseteq \Delta$, $\left(\sigma s_\alpha \sigma^{-1}\right)^2 = 1$ and $\sigma s_\alpha \sigma^{-1} \sigma\alpha = -\sigma\alpha$ implies $\sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$ by Lemma 0.7.1. Compare $s_{\sigma\alpha}(\sigma\beta) = \sigma\beta - \langle \sigma\beta, \sigma\alpha \rangle \sigma\alpha$, and $\sigma s_\alpha \sigma^{-1}(\sigma\beta) = \sigma s_\alpha \beta = \sigma\left(\beta - \langle \beta, \alpha \rangle \alpha\right) = \sigma\beta - \langle \beta, \alpha \rangle \sigma\alpha$, we get $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$ □

Definition 0.7.15. The **Weyl group** W of Δ is $\langle s_\alpha | \alpha \in \Delta \rangle$. $|W| < \infty$ since $s_\alpha(\Delta) \subseteq \Delta$. W is a subgroup of $O(V)$

Remark 0.7.16. W is a finite Coxeter group

Definition 0.7.17. **Weyl chambers** are connected components of $V \setminus \bigcup_{\alpha \in \Delta} H_\alpha$, as the intersection of open half spaces, Weyl chambers are open convex, each regular $\gamma \in V$ by definition belongs to precisely one Weyl chamber denote as $C(\gamma)$, $C(\gamma) = C(\gamma')$ iff γ, γ' are on the side for every hyperplane H_α iff $\Delta^+(\gamma) = \Delta^+(\gamma')$, thus Weyl chambers are in one to one correspondence with the sets of positive roots, the fundamental Weyl chamber associated to Δ^+, or rather, associated to S is $\{\gamma \in V | (\gamma, \alpha) > 0, \forall \alpha \in S\}$. For any $\sigma \in W$, since $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$, $\sigma(\Delta^+(\gamma)) = \Delta^+(\sigma\gamma)$, $\sigma(S(\gamma)) = S(\sigma\gamma)$, $\sigma(C(\gamma)) = C(\sigma\gamma)$

For any positive nonsimple root alpha, there exists simple root beta such that alpha-beta is a positive root

Lemma 0.7.18. If $\alpha \in \Delta^+ \setminus S$,then there exists $\beta \in S$ such that $\alpha - \beta \in \Delta^+$

Proof. It is obvious that $(\alpha, \beta) \neq 0, \forall \beta \in \Delta^+$, suppose $(\alpha, \beta) < 0, \forall \beta \in S$, then by Lemma 0.7.11, $S \cup \{\alpha\}$ is linearly independent which is impossible, thus $(\alpha, \beta) > 0$ for some $\beta \in \Delta^+$, by Lemma 0.7.7, $\alpha - \beta \in \Delta$, but since $\alpha = \sum\limits_{\alpha_i \in S} c_i \alpha_i$, and $\beta = \alpha_j$ for some j, since some $c_i > 0$, it necessarily has to be that $\alpha - \beta = (c_j - 1)\beta + \sum\limits_{i \neq j} c_i \alpha_i \in \Delta^+$ □

Lemma 0.7.19. Each $\alpha \in \Delta^+$ can be written as $\alpha_1 + \cdots + \alpha_k$, $\alpha_i \in S$, here α_i may repeat, such that partial sums are all positive roots, i.e. $\alpha_i + \cdots + \alpha_i \in \Delta^+$

Proof. Each $\alpha \in \Delta^+$ can be written uniquely as a sum of simple roots, by induction on the number of summands and Lemma 0.7.18 □

If alpha is a simple root, then s_alpha permutes postive roots except alpha

Lemma 0.7.20. If $\alpha \in S$, then s_α permutes $\Delta^+ \setminus \{\alpha\}$

Proof. For any $\beta \in \Delta^+ \setminus \{\alpha\}$, $\beta = \sum\limits_{\alpha_i \in S} c_i \alpha_i$, $c_j > 0$ for some $\alpha_j \neq \alpha$, then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ still has some positive coefficient, thus it has to be positive, and $s_\alpha(\beta) \neq \alpha = s_\alpha(-\alpha)$ □

delta=half of sum of positive roots, then for any simple root alpha, s_alpha(delta)=delta-alpha

Corollary 0.7.21. Let $\delta = \dfrac{1}{2} \sum\limits_{\beta \in \Delta^+} \beta$, then $s_\alpha(\delta) = \delta - \alpha, \forall \alpha \in S$

s=s1...sq is of minimal length => s1...sq(alpha_q) is a negative root

Lemma 0.7.22. Use $\alpha \prec 0$ and $\alpha \succ 0$ to mean positive and negative roots, suppose $\alpha_i \in S, i = 1, \cdots, q$, and denote $s_i := s_{\alpha_i}$, if $s_1 \cdots s_{q-1}(\alpha_q) \prec 0$, then $s_1 \cdots s_q = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1}$. In particular, suppose $s = s_1 \cdots s_q$ is of the smallest length, then $0 \prec s_1 \cdots s_{q-1}\alpha_q = -s_1 \cdots s_q \alpha_q \Rightarrow s_1 \cdots s_q \alpha_q \prec 0$

Proof. Write $\beta_i = s_i \cdots s_{q-1}\alpha_q$, then $\beta_q = \alpha_q \succ 0$, $\beta_1 \prec 0$, thus there must be some $1 \leq p < q$ such that $\beta_{p+1} \succ 0$ but $\beta_p = s_p \beta_{p+1} \prec 0$, by lemma 0.7.20, thus β_{p+1} can only be α_p, hence by Lemma 0.7.14, we have

$$
\begin{aligned}
& s_p = s_{\alpha_p} = s_{\beta_{p+1}} = s_{s_{p+1} \cdots s_{q-1}\alpha_q} = (s_{p+1} \cdots s_{q-1}) s_q (s_{p+1} \cdots s_{q-1})^{-1} \\
& \Rightarrow s_p \cdots s_{q-1} = s_{p+1} \cdots s_q \\
& \Rightarrow s_1 \cdots s_q = s_1 \cdots s_p s_{p+1} \cdots s_q = s_{p1} \cdots s_p s_p \cdots s_{q-1} = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1}
\end{aligned}
$$

□

Lemma 0.7.23. Denote $n(\sigma)$ the number of positive roots that σ send to negative. $l(\sigma) = n(\sigma)$. Hence there is a unique element $w_\circ$ such that $w_\circ(S) = -S$ of maximal length, and $w_\circ^2 = 1$

Some properties about Weyl group and Weyl chambers

Theorem 0.7.24.

(a) Let $\gamma \in V$ be regular, then there exists some $\sigma \in W$ such that $\Delta^+(\sigma\gamma) = \Delta^+$, namely, W acts trasitively on Weyl chambers

(b) If S' is a another base, then there exists some $\sigma \in W$ such that $\sigma(S') = S$, namely, W acts trasitively on bases

(c) If α is any root, then there exists some $\sigma \in W$ such that $\sigma(\alpha) \in S$

(d) W is generated by s_α's for $\alpha \in S$

(e) If $\sigma \in W$, then $\sigma(S) \subseteq S \Rightarrow \sigma = 1$, namely, W acts freely and transitively(regularly) on bases(and Weyl chambers)

Proof. Let $W' \leq W$ be the subgroup of $O(n)$ generated by s_α's for $\alpha \in S$

(a) Let $\delta = \dfrac{1}{2} \sum_{\beta \in \Delta^+} \beta$, choose $\sigma \in W'$ such that $(\sigma\gamma, \delta)$ is the biggest possible, then for any $s_\alpha \in W'$, due to Corollary 0.7.21, we have $(\sigma\gamma, \delta) \geq (s_\alpha \sigma\gamma, \delta) = (\sigma\gamma, s_\alpha \delta) = (\sigma\gamma, \delta - \alpha) = (\sigma\gamma, \delta) - (\sigma\gamma, \alpha) \Rightarrow (\sigma\gamma, \alpha) \geq 0$, since γ is regular, so is $\sigma\gamma$, thus $(\sigma\gamma, \alpha) > 0, \forall \alpha \in S$, $\sigma(C(\gamma)) = C(S)$, i.e. W acts transitively on Weyl chambers

(b) Directly from (a)

(c) Thanks to (b), it suffices to prove that each root lies in some base, there exists $\gamma \in H_\alpha \setminus \bigcup_{\beta \in \Delta \setminus \{\pm\alpha\}} H_\beta$, and the perturb γ slightly so that $0 < (\gamma, \alpha) < |(\gamma, \beta)|, \beta \in \Delta \setminus \{\pm\alpha\}$, then $\alpha \in S(\gamma)$

(d) By (c), if $\alpha \in \Delta$, $\beta \in S$, then there exists $\sigma \in W'$ such that $\sigma\alpha = \beta$, then $s_\beta = s_{\sigma\alpha} = \sigma s_\alpha \sigma^{-1}$, thus $s_\alpha = \sigma^{-1} s_\beta \sigma \in W'$

(e) By (d), $\sigma \in W$ can be written as $s = s_{\alpha_1} \cdots s_{\alpha_q}, \alpha_i \in S$ and suppose it is of minimal length, by Lemma 0.7.22, $\sigma(\alpha_q) \prec 0$ contradicting $\sigma(S) \subseteq S$

□

Proposition 0.7.25. The root system is irreducible iff the Lie algebra is simple

0.8 Dynkin diagram

Definition 0.8.1. A **generalized Cartan matrix** A is such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, if $a_{ij} = 0$, then $a_{ji} = 0$, $A = DS$ for some diagonal matrix D and symmetric matrix S, i.e. A is symmetrizable. Note that D would have nonzero diagonal entries, we can pick positive entries, A is a **Cartan matrix** if S is positive definite A is **decomposable** if $a_{ij} = 0$, $i \in I, j \in J$ for some $\{1, \cdots, n\} = I \sqcup J$, i.e. A can be diagonalized by blocks
An indecomposable matrix A is of **finite type** if all principal minors are positive, **affine type** if all proper principal minors are positive and $\det A = 0$, **indefinite type** otherwise

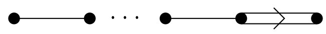
Definition 0.8.2. S is a set of positive simple roots, the **Dynkin diagram** is a graph with nodes simple roots, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges between α, β which can only be $0, 1, 2, 3$, and an arrow from α to β if $\langle \alpha, \beta \rangle > 1$. The **Coxter diagram** of the Weyl group W is just the Dynkin diagram without arrows. The **Coxeter graph** of it is the underlying graph

Theorem 0.8.3. We can recover the root system through Dynkin diagram

Definition 0.8.4. Type A_n corresponds to Dynkin diagram

Example 0.8.5. $\mathfrak{sl}_{n+1}$ corresponds to type A_n

Definition 0.8.6. Type B_n corresponds to Dynkin diagram

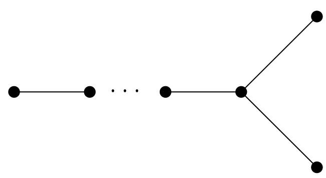


Example 0.8.7. $\mathfrak{so}_{2n+1}$ corresponds to type B_n

Definition 0.8.8. Type C_n corresponds to Dynkin diagram

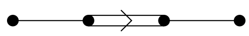
Example 0.8.9. $\mathfrak{sp}_{2n}$ corresponds to type C_n

Definition 0.8.10. Type D_n corresponds to Dynkin diagram

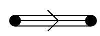


Example 0.8.11. $\mathfrak{so}_{2n}$ corresponds to type D_n

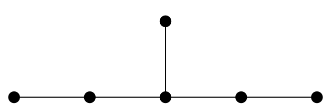
Definition 0.8.12. Type F_4 corresponds to Dynkin diagram



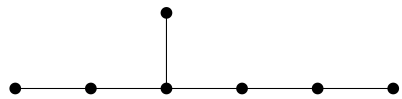
Definition 0.8.13. Type G_4 corresponds to Dynkin diagram



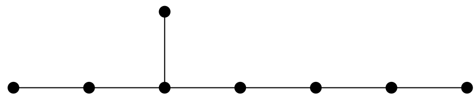
Definition 0.8.14. Type E_6 corresponds to Dynkin diagram



Definition 0.8.15. Type E_7 corresponds to Dynkin diagram

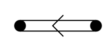


Definition 0.8.16. Type E_8 corresponds to Dynkin diagram



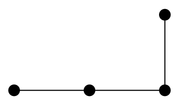
Remark 0.8.17. The number in the subscript is the number of nodes. In particular, we have $A_1 = B_1 = C_1$

$B_2 = C_2$

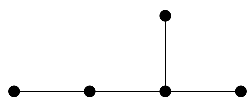


$D_3 = A_3$

$D_2 = A_1A_1$
$E_3 = A_2A_1$
$E_4 = A_4$



$E_5 = D_5$



Cartan-Killing classification

Theorem 0.8.18 (Cartan-Killing classification)**.** The above Dynkin diagrams classifies simple Lie algebras

Proof. Consider the **admissible sets** of Euclidean space V, $A = \{v_1, \cdots, v_n\}$ of linearly independent unit vectors with $(v_i, v_j) \leq 0$ and $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ if $i \neq j$. Define Coxeter diagram Γ_A for A to have vertices $v_1, \cdots, v_n$, and $d_{ij} = 4(v_i, v_j)^2$ edges between v_i and v_j if $i \neq j$. Assume that Γ_A is connected

a) The number of vertices in Γ_A joined by at least one edge is at most $|A| - 1$

$v = v_1 + \cdots + v_n \neq 0$ satisfies $(v, v) = n + 2\sum_{i<j}(v_i, v_j) > 0$, thus $n > -\sum_{i<j} 2(v_i, v_j) = \sum_{i<j} \sqrt{d_{ij}} \geq N$, where N is the number of pairs v_i, v_j such that $d_{ij} \geq 1$

b) The graph Γ_A contains no cycles

The vectors in a cycle of Γ_A form an admissible set which a contradicts a)

c) No vertex in Γ_A has more than 3 edges

Let w be a vertex of Γ_A with adjacent vertices $w_1, \cdots, w_k$, then $(w_i, w_j) = 0$ for $i \neq j$. Let $U = \mathrm{Span}_{\mathbb{R}}(w_1, \cdots, w_k, w)$, and extend $\{w_1, \cdots, w_k\}$ to an orthonormal basis of U, say by adjoining w_0. Clearly $(w, w_0) \neq 0$ and $w = \sum_{i=0}^{k}(w, w_i)w_i$. Hence $1 = (w, w) = \sum_{i=0}^{k}(w, w_i)^2$, $\sum_{i=1}^{k}(w, w_i)^2 < 1$, w has no more than 3 edges

d) If Γ_A has triple edge, then by c), the Γ_A can only be G_2

e) Assume Γ_A has a subgraph which is a line along $w_1, \cdots, w_n$, if we replace this subgraph with $w = w_1 + \cdots + w_n$, then it is still an admissible set

$(w, w) = n + 2\sum_{i=1}^{n-1}(w_i, w_{i+1}) = n - (n-1) = 1$, by d) any vertex v has at most edges linked with one such w_i, hence $(v, w) = (v, w_i)$, this gives an admissible set

f) A branch point is a vertex having more than 2 adjacent vertices, in this case, exactly 3. Γ_A has only one double edge, or only one branch point, or neither, but not both. Note that if Γ_A has no brach points and double edges corresponds to A_n

If Γ_A has two double edges between w_1, w_2 and v_1, v_2, then they can be linked through a line, by e), we can collapse it into a single vertex, but this will contradict c)

g) If Γ_A has a subgraph which is a line through $w_1, \cdots, w_n$, let $w = \sum iw_i$, then $(w, w) = \dfrac{n(n+1)}{2}$

h) If Γ_A has a double edge, then Γ_A is F_4 or B_n

By f) we know Γ_A is a line through $v_1, \cdots, v_p, w_q, \cdots, w_1$, $q \geq p \geq 1$ with single edges except v_p, w_q, let $v = \sum iv_i$, $w = \sum iw_i$, then

$$(v, w)^2 = (pv_p, qw_q)^2 = \frac{p^2q^2}{2}$$

Since v, w are linearly independent, by Cauchy Schwarz inequality, we have

$$\frac{p^2q^2}{2} = (v, w)^2 < (v, v)(w, w) = \frac{p(p+1)q(q+1)}{4}$$

Which implies $(p-1)(q-1) < 2$, thus if $p = 1$, then q can be any positive interger, giving B_n, if $p = 2$, then $q = 2$, giving F_4

i) If Γ_A has a branch point, then Γ_A is D_n or E_6, E_7, E_8

Γ_A is has three branch lines $v_1, \cdots, v_p, x$ and $w_1, \cdots, w_q, x$ together with $z_1, \cdots, z_r, x$, $p \geq q \geq r$, let $v = \sum iv_i$, $w = \sum iw_i$, $z = \sum iz_i$ which are pairwise orthogonal, $\hat{v}, \hat{w}, \hat{z}$ be normalized vectors of v, w, z, and consider $U = \mathrm{Span}_{\mathbb{R}}(v, w, z, x) = \mathrm{Span}_{\mathbb{R}}(\hat{v}, \hat{w}, \hat{z}, x_0)$, where x_0 is a unit vector orthogonal to v, w, z, then $(x, x_0) \neq 0$

$$1 = (x, x) = (x, \hat{v})^2 + (x, \hat{w})^2 + (x, \hat{z})^2 + (x, x_0)^2$$

Thus by g)

$$\frac{2p^2}{4p(p+1)} + \frac{2q^2}{4q(q+1)} + \frac{2r^2}{4r(r+1)} < 1$$

Hence

$$\frac{1}{1+p} + \frac{1}{1+q} + \frac{1}{1+r} > 1$$

and we know that

$$\frac{1}{1+p} \leq \frac{1}{1+q} \leq \frac{1}{1+r} \leq \frac{1}{2}$$

Hence $r = 1$

$$\frac{1}{1+p} + \frac{1}{1+q} > \frac{1}{2}$$

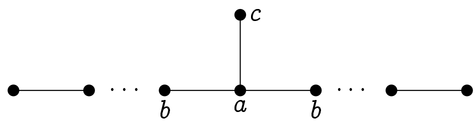
If $q = 1$, then p can be any positive integer, giving D_n, if $q = 2$, then p can only be $2, 3$ or 4, giving E_6, E_7, E_8

$\square$

Lemma 0.8.19. (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots and $S = \{\alpha_1, \cdots, \alpha_n\}$ be its base, then there exists unique highest root $\gamma \in \Delta$, meaning $\gamma + \alpha_i \in \Delta, \forall \alpha_i \in S$

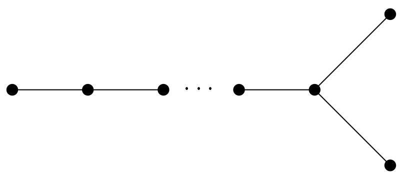
Definition 0.8.20. Let (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots, $S = \{\alpha_1, \cdots, \alpha_n\}$ be its base, and γ its unique highest root, the extended Dynkin diagram is the usual Dynkin diagram adding $\alpha_0 = -\gamma$, the number of bonds for each two nodes and direction are still defined as before. Finally, suppose $-\alpha_0 = \sum n_i \alpha_i$, then lebel γ with ①, and label node α_i with n_i

Lemma 0.8.21. If the following part of the extended Dynkin diagram



We have $2a = b + c + d$

Example 0.8.22. Consider the classical root system D_n, $\Delta^+ = \{e_i \pm e_j | i < j\}$, $S = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$, then $\gamma = e_1 + e_2$, its usual Dynkin diagram is



So its extended Dynkin diagram is