Section 12.5

December 3, 2019

Definition: Let X be a discrete random variable that takes value $0, 1, 2, 3, \cdots$, and let $p_n = Pr(X = n)$, then we necessarily have $p_i \geq 0$ and $\sum_{i=0}^{\infty} p_i = p_0 + p_1 + p_2 + p_3 + \cdots = 1$, define the expected value to be $\overline{X} = E(X) = \sum_{i=0}^{\infty} ip_i = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + \cdots$, variance to be $Var(X) = \sum_{i=0}^{\infty}(i - \overline{X})^2 p_i = (0 - \overline{X})^2 \cdot p_0 + (1 - \overline{X})^2 \cdot p_1 + (2 - \overline{X})^2 \cdot p_2 + (3 - \overline{X})^2 \cdot p_3 + \cdots$, and standard deviation to be $\sigma(X) = \sqrt{Var(X)}$

We use notation $X \sim P(\lambda)$ to mean X satisfies Poisson distribution(X is a Poisson random variable, etc), concretely means that $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$, we can compute $E(X) = Var(X) = \lambda$

We use notation $X \sim G(p)$ to mean X satisfies geometric distribution(X is a geometric random variable, etc), concretely means that $p_n = p^n(1-p)$, we can compute $E(X) = \frac{p}{1-p}, Var(X) = \frac{p}{(1-p)^2}$

Modeling based on Poisson and Geometric distribution:

Suppose a company runs by waiting for phone calls, let X be the number of phone calls in a minute, and suppose the average of phone calls in a minute is λ, then $X \sim P(\lambda)$, and $Pr(X = n) = \frac{\lambda^n}{n!}e^{-\lambda}$

Suppose you have a uneven coin with probability of showing heads p, and therefore with probability of showing tails $1 - p$, let X be the number of heads before the first tail, then $X \sim G(p)$, $Pr(x = n) = p^n(1-p)$

Problems:

(1) The monthly number of fire insurance claims filed with the Firebug Insurance Company is Poisson distributed with $\lambda = 13$

(a) What is the probability that in a given month no claims are filed?

(b) What is the probability that in a given month no more than two claims are filed?

(c) What is the probability that in a given month three or more claims are filed?

Let X = # fire insurance claims filed in a month, $X \sim P(13)$

(a) $Pr(X=0) = p_0 = \frac{13^0}{0!}e^{-13} = e^{-13}$

(b) $Pr(X \leq 2) = Pr(X=0) + Pr(X=1) + Pr(X=2) = p_0 + p_1 + p_2 = e^{-13} + \frac{13^1}{1!}e^{-13} + \frac{13^2}{2!}e^{-13} = \left(1 + 13 + \frac{13^2}{2}\right)e^{-13}$

(c) $Pr(X \geq 3) = 1 - Pr(X \leq 2)$

(2) A bakery makes gourmet cookies. For a batch of 6000 oatmeal and raisin cookies, how many raisins should be used so that the probability of a cookie having no raisins is 0.03? Assume the number of raisins in a random cookie has a Poisson distribution

Let $X = \#$ of raisins in a cookie, $X \sim P(\lambda)$ for some λ

since $0.03 = Pr(X=0) = P_0 = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-\lambda} \Rightarrow -\lambda = \ln(0.03) \Rightarrow \lambda = -\ln(0.03)$

hence the average (expected value) of X is $E(X) = \lambda = \ln(0.03^{-1}) = \ln(\frac{100}{3})$

thus the total # of rasins would be $6000\lambda = 6000\ln(\frac{100}{3})$

(3) In a certain town, there are two competing taxicab companies, Red Cab and Blue Cab. The taxis mix with downtown traffic in a random manner. There are five times as many Red taxis as Blue taxis. Suppose you stand on a downtown street and count the number X of Red taxis before the first Blue taxi appears

(a) Determine the formula for $Pr(X = n)$

(b) What is the likelihood of observing at least four Red taxis before the first Blue taxi?

(c) What is the average number of consecutive Red taxis prior to the appearance of a Blue taxi?

$X \sim G(\frac{5}{6})$ (since $\frac{5}{6} \div \frac{1}{6} = 5$, $\frac{5}{6} + \frac{1}{6} = 1$)

(a) $Pr(X=n) = P_n = (\frac{5}{6})^n \frac{1}{6}$

(b) $Pr(X \geq 4) = 1 - Pr(X \leq 3) = 1 - P_0 - P_1 - P_2 - P_3 = 1 - \frac{1}{6} - (\frac{5}{6})\frac{1}{6} - (\frac{5}{6})^2\frac{1}{6} - (\frac{5}{6})^3\frac{1}{6}$

(c) $E(X) = \dfrac{\frac{5}{6}}{1-\frac{5}{6}} = 5$

(4) Suppose that a large number of persons become infected by a particular strain of a bacteria that is present in food served by a restaurant and that the germ usually produces a certain symptom in 12% of the persons infected. What is the probability that when customers are examined the first person to have the symptom is the fifth customer examined?

Let $X = \#$ persons before the first having the symptom,

$X \sim G(88\%) = G(0.88)$

$Pr(X = 4) = 0.88^4 0.12$