Use $H_{\mathbb{F}}$ or $H(\mathbb{F})$ to indicate coefficients in $\mathbb{F}$

Definition 0.0.1. A *pure Hodge structure* of weight n on $H_{\mathbb{Z}}$ is a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$. Equivalently, $H_{\mathbb{C}} = F^p \oplus \overline{F^{n+1-p}}$ by introducing the decreasing *Hodge filtration* $F^p = \bigoplus_{i \geq p} H^{i,n-i}$, then $\overline{F^q} = \bigoplus_{j \leq p} H^{j,n-j}$, $H^{p,q} = F^p \cap \overline{F^q}$, $F^p \cap \overline{F^{n+1-p}} = 0$

Example 0.0.2. X is a complex manifold, $H_{\mathbb{Z}} = H^n(X;\mathbb{Z})$, then

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q} = \bigoplus_{p+q=n} H^p(X;\mathbb{C}) \wedge \overline{H^p(X;\mathbb{C})}$$

Example 0.0.3. *Tate structure* $\mathbb{Z}(-k)$ is of weight $2k$ given by $H_{\mathbb{Z}} = \mathbb{Z}$ with filtration $F^k = \begin{cases} H_{\mathbb{C}} = \mathbb{C} & k \leq p \\ 0 & k > p \end{cases}$

Definition 0.0.4. A *polarization* over $\mathbb{Q}$ of a Hodge structure over $\mathbb{Q}$ of weight k is a $(-1)^k$ symmetric nondegenerate flat bilinear map $\beta : \mathbb{V}_{\mathbb{Q}} \times \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}$ such that the Hermitian form $\beta_x(C_x v, \bar{w})$ on each fiber $\mathcal{V}_x$ is positive definite, here C_x is the *Weil operator*, given as the direct sum of multiplication i^{p-q} on $H_x^{p,q}$

Definition 0.0.5. A *mixed Hodge structure* on $H_{\mathbb{Z}}$ consists of an increasing *weight filtration* $W_\bullet$ on $H_{\mathbb{Q}}$ and a decreasing filtration $F^\bullet$ that are compatible, i.e.

$$F^p(\operatorname{gr}_k W)_{\mathbb{C}} = F^p\left(\frac{W_{k+1}}{W_k}\right)_{\mathbb{C}} = \frac{F^p \cap W_{k+1}(\mathbb{C})}{W_k(\mathbb{C})} = \frac{F^p \cap W_{k+1}(\mathbb{C}) + W_k(\mathbb{C})}{W_k(\mathbb{C})}$$

is a pure Hodge structure of weight k of $\operatorname{gr}_k W$

Definition 0.0.6. A *variation* of Hodge structure of weight k over $\mathbb{Q}$ and a complex manifold X is $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^\bullet)$, $\mathbb{V}_{\mathbb{Q}}$ is a locally constant sheaf of $\mathbb{Q}$ vector spaces, $\mathcal{F}^\bullet$ is a decreasing filtration of holomorphic subbundles of the locally free sheaf $\mathcal{V} = \mathcal{O}_X \otimes \mathbb{V}_{\mathbb{Q}}$ such that

- $(\mathcal{V}_x, \mathcal{F}_x^\bullet)$ has a pure Hodge structure of weight k, i.e. $\mathcal{V}_x = \mathcal{F}^p \oplus \overline{\mathcal{F}^{k+1-p}}$
- (Griffiths transversality) $\nabla \mathcal{F}^p \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$

Definition 0.0.7. A variation of mixed Hodge structure over $\mathbb{Q}$ and a complex manifold X is $(\mathbb{V}_{\mathbb{Q}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$, $\mathcal{W}_\bullet$ is an increasing filtration of $\mathbb{V}_{\mathbb{Q}}$ by locally constant subsheaves such that

- $(\mathcal{V}_x, (\mathcal{W}_\bullet)_x, \mathcal{F}_x^\bullet)$ has a mixed Hodge structure, i.e. $()$ is a pure Hodge structure of weight k
- $\nabla \mathcal{F}^p \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$

Remark 0.0.8. Given a locally constant sheaf is equivalent to given a monodromy representation $\rho_{\mathrm{x}} : \pi_1(X, \mathrm{x}) \to \operatorname{Aut}_{\mathbb{Q}}(\mathcal{V}_x)$. A variation is *unipotent* if the the monodromy representation is unipotent

Theorem 0.0.9 (Deligne)**.** $\tilde{X}$ is a normalization of X, $(\mathbb{V}_{\mathbb{Q}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ is a unipotent variation of mixed Hodge structure of weight k, then there is a unique extension $\tilde{\mathcal{V}}$ over $\tilde{X}$ such that

- Inside every section of $\tilde{\mathcal{V}}$, flat sections increase at most at the rate of $O(\log(\|x\|^k))$ on each compact set of $\tilde{X} - X$
- Every flat section of $\mathcal{V}^\vee$ increases at most at the rate of $O(\log(\|x\|^k))$

These conditions are equivalent to

- In a local basis of $\tilde{\mathcal{V}}$, the connection matrix ω of $\mathcal{V}$ has at most logarithmic singularities along $\tilde{X} - X$
- The residue of ω along any irreducible component of $\tilde{X} - X$ is nilpotent