0.1 Iterated integral

Definition 0.1.1. Chen's *Iterated integral* is defined inductively by

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b \left(\int_a^t f_1(\tau)d\tau \cdots f_{r-1}(\tau)d\tau \right) f_r(t)dt$$

If $\alpha : I \to M$ is a curve, $\alpha^*\omega_i = f_i(t)dt$, then

$$\int_\alpha \omega_1 \cdots \omega_r = \int_0^1 f_1(t)dt \cdots f_r(t)dt$$

is well defined, independent of the parametrization. Set the integral to be 1 if $r = 0$. Iterated integral can also be written as

$$\int_0^1 f_1(t)dt \cdots f_r(t)dt = \int_{0 \le t_1 \le \cdots \le t_n \le 1} f_1(t_1)dt_1 \wedge \cdots \wedge f_n(t_n)dt_n$$

Proposition 0.1.2.

1. $\displaystyle\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{j=0}^{r} \int_\alpha \omega_1 \cdots \omega_j \int_\beta \omega_{j+1} \cdots \omega_r$, here $\beta(0) = \alpha(1)$
2. $\displaystyle\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1$
3. $\displaystyle\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\alpha \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}$, here σ runs over (r, s)-shuffles

Lemma 0.1.3. $\omega_i^{(j)}$, $1 \le i \le r$, $1 \le j \le n$ are closed one forms such that $\displaystyle\sum_j \omega_{i-1}^{(j)} \wedge \omega_i^{(j)} = 0$ for $2 \le i \le r$, then $\displaystyle\int_\alpha \sum_j \omega_1^{(j)} \cdots \omega_r^{(j)}$ only depends on the homotopy class of α

$$\int_a^b df_1(t)df_2(t) = [f_1(b) - f_1(a)][f_2(b) - f_2(a)] - \int_a^b df_2(t)df_1(t)$$

0.2 Polylogarithm

Definition 0.2.1. The *Polylogarithms* are

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

Note that

$$\mathrm{Li}_{n+1}(z) = \int_0^z \frac{\mathrm{Li}_n(t)}{t} dt, \quad \mathrm{Li}_1(z) = -\ln(1-z)$$

Hence

$$\mathrm{Li}_n(z) = \int_0^z \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{1-t} = \int_0^1 \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{z^{-1}-t}$$

Dilogarithm $\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du$ is the analytic continuation on $\mathbb{C} \setminus \{0,1\}$, avoiding the the cut $[1, \infty]$

Lemma 0.2.2. $\mathrm{Li}_k(z)$ satisfies differential equation

$$\left[(1-z)\frac{d}{dz}\right]\left(z\frac{d}{dz}\right)^{k-1} y = 1$$

Other solutions are $\frac{\ln^j z}{j!}$, $1 \le j \le k-1$

To compute the monodromy around $x = 0$, take $q(\epsilon)$ to be the loop $x = \epsilon e^{it}$, we get 0.
To compute the monodromy around $x = 1$, take $q(\epsilon)$ to be the composition of $x = (1-t)\epsilon + t(1-\epsilon)$, $x = 1 - \epsilon e^{it}$ and $x = (1-t)(1-\epsilon) + t\epsilon$, we get $-\frac{2\pi i}{(n-1)!}\log^{n-1} x$

The variation matrix is

$$\Lambda = \begin{bmatrix} 1 & & & & & \\ \mathrm{Li}_1(x) & 1 & & & & \\ \mathrm{Li}_2(x) & \log x & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \mathrm{Li}_{n-1}(x) & \frac{\log^{n-2} x}{(n-2)!} & \cdots & \log x & 1 & \\ \mathrm{Li}_n(x) & \frac{\log^{n-1} x}{(n-1)!} & \cdots & \frac{\log^2 x}{2!} & \log x & 1 \end{bmatrix} \tau_n(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & & & \\ -dv & 0 & & & & \\ 0 & du & 0 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & \cdots & du & 0 & \\ 0 & 0 & \cdots & 0 & du & 0 \end{bmatrix}$$

The monodromy representation ρ is as follows
For monodromy around $x = 0$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & \vdots & \vdots & \ddots & \\ & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & 1 \end{bmatrix}$$

For monodromy around $x = 1$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

0.3 Multiple polylogarithm

Definition 0.3.1. The *multiple polylogarithms* are

$$\mathrm{Li}_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \int_0^1 \frac{dt}{a_1 - t}\left(\frac{dt}{t}\right)^{n_1-1} \cdots \frac{dt}{a_d - t}\left(\frac{dt}{t}\right)^{n_d-1}$$

Here $\mathbf{k}$ runs over $0 < k_1 < \cdots < k_d$, $a_j = a_j(\mathbf{x}) = (x_j \cdots x_d)^{-1}$

Define $\mathrm{Li}_0(x) = \dfrac{x}{1-x}$

Note. For $\mathbf{k}$ runs over $(k_1, \cdots, k_d) \in \mathbb{Z}^d_{\geq 1}$

$$\sum_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \left(\sum_{k_1} \frac{x_1^{k_1}}{k_1^{n_1}}\right) \cdots \left(\sum_{k_d} \frac{x_d^{k_d}}{k_d^{n_d}}\right) = \mathrm{Li}_{n_1}(x_1) \cdots \mathrm{Li}_{n_d}(x_d)$$

Can be written in terms of multiple polylogarithms

Note.

$$\begin{aligned}
\mathrm{Li}_{n_1,\cdots,n_{i-1},0,n_{i+1},\cdots,n_d}(x_1,\cdots,x_d) &= \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k-1}\cdots x_d^{k_d}}{k_1^{n_1}\cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \\
&= \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k-1}\cdots x_{i-1}^{k_{i-1}} x_{i+1}^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1}\cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{x_i^{k_{i-1}+1} - x_i^{k_{i+1}}}{1-x_i} \\
&= \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k-1}(\cdots x_{i-1}x_i)^{k_{i-1}} x_{i+1}^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1}\cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{x_i}{1-x_i} \\
&- \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k-1}\cdots x_{i-1}^{k_{i-1}} (x_i x_{i+1})^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1}\cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{1}{1-x_i} \\
&= \mathrm{Li}_{n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_d}(x_1,\cdots,x_{i-1}x_i, x_{i+1},\cdots,x_d)\frac{x_i}{1-x_i} \\
&- \mathrm{Li}_{n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_d}(x_1,\cdots,x_{i-1}, x_i x_{i+1},\cdots,x_d)\frac{1}{1-x_i}
\end{aligned}$$

$$\mathrm{Li}_{n_1,\cdots,n_{d-1},0}(x_1,\cdots,x_d) = \mathrm{Li}_{n_1,\cdots,n_{d-1}}(x_1,\cdots,x_{d-1}x_d)\frac{x_d}{1-x_d}$$

$$\mathrm{Li}_{0,n_2,\cdots,n_d}(x_1,\cdots,x_d) = \mathrm{Li}_{n_2,\cdots,n_{d-1}}(x_2,\cdots,x_d)\frac{x_1}{1-x_1} - \mathrm{Li}_{n_2,\cdots,n_{d-1}}(x_1x_2,\cdots,x_d)\frac{1}{1-x_1}$$

Exercise 0.3.2 (Derivatives of polylogarithms)**.** Observe the following

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(\sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}} x_i^{k_i} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}} k_i^{n_i} k_{i+1}^{n_{i+1}} \cdots}\right) &= \sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}} x_i^{k_i-1} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}} k_i^{n_i-1} k_{i+1}^{n_{i+1}} \cdots} \\
&= \sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}} x_i^{k_i} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}} k_i^{n_i-1} k_{i+1}^{n_{i+1}} \cdots} \frac{1}{x_i}
\end{aligned}$$

Write $u_i = \log(x_i)$, $v_i = \log(1-x_i)$, $u_{ij} = \log(x_i \cdots x_j)$, $v_{ij} = \log(1 - x_i \cdots x_j)$. If $m_i > 1$, then

$$d_i \,\mathrm{Li}_{n_1,\cdots,n_d}(z_1,\cdots,z_d) = \mathrm{Li}_{n_1,\cdots,n_i-1,\cdots,n_d}(z_1,\cdots,z_d)\frac{dx_i}{x_i}$$

$$d_d \,\mathrm{Li}_{n_1,\cdots,n_{d-1},1}(z_1,\cdots,z_d) = \mathrm{Li}_{n_1,\cdots,n_{d-1}}(z_1,\cdots,z_{d-1}z_d)\frac{dx_d}{1-x_d}$$

$$d_1 \operatorname{Li}_{1,n_2,\cdots,n_d}(z_1,\cdots,z_d) = \operatorname{Li}_{n_2,\cdots,n_d}(z_2,\cdots,z_d)\frac{dx_1}{1-x_1} - \operatorname{Li}_{n_2,\cdots,n_d}(z_1z_2,\cdots,z_d)\frac{dx_1}{x_1(1-x_1)}$$

$$d_i \operatorname{Li}_{n_1,\cdots,n_{i-1},1,n_{i+1},\cdots,n_d}(z_1,\cdots,z_d) = \operatorname{Li}_{n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_d}(z_1,\cdots,z_{i-1}z_i,z_{i+1},\cdots,z_d)\frac{dx_i}{1-x_i} - \operatorname{Li}_{n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_d}(z_1,\cdots,z_{i-1},z_iz_{i+1},\cdots,z_d)\frac{dx_i}{x_i(1-x_i)}$$

Remark 0.3.3.

Theorem 0.3.4. $\operatorname{Li}_n(x) + \operatorname{Li}_n(x^{-1}) =$

Proof. $\operatorname{Li}_0(x)+\operatorname{Li}_0(x^{-1}) = -1$, $\operatorname{Li}_1(x)-\operatorname{Li}_1(x^{-1}) = \pi i - \log x$, $\operatorname{Li}_2(x)+\operatorname{Li}_2(x^{-1}) = -\frac{\pi^2}{6} - \frac{\log^2(-x)}{2}$

$$d(\operatorname{Li}_n(x) + (-1)^n \operatorname{Li}_n(x^{-1})) = (\operatorname{Li}_{n-1}(x) + (-1)^{n-1}\operatorname{Li}_{n-1}(x^{-1}))\frac{dx}{x}$$

$\square$

0.4 $\mathrm{Li}_{1,1}$

$$\begin{aligned}
\mathrm{Li}_{1,1}(x,y) &= \int \frac{dy}{1-y}\frac{dx}{1-x} + \frac{d(xy)}{1-xy}\left(\frac{dy}{1-y} - \frac{dx}{x(1-x)}\right) \\
&= \int d\log(1-y)d\log(1-x) + d\log(1-xy)d\log\frac{x(1-y)}{1-x} \\
&= \int dv_2 dv_1 + dv_{12}dw_1
\end{aligned}$$

To compute the monodromy around $x=0$, take $q(\epsilon)$ to be the loop $(x=\epsilon e^{it}, y=\epsilon)$, we get 0.
To compute the monodromy around $y=0$, take $q(\epsilon)$ to be the loop $(x=\epsilon, y=\epsilon e^{it})$, we get 0.
To compute the monodromy around $x=1$, take $q(\epsilon)$ to be the composition of $(x=(1-t)\epsilon + t(1-\epsilon), y=\epsilon)$, $(x=1-\epsilon e^{it}, y=\epsilon)$ and $(x=(1-t)(1-\epsilon)+t\epsilon, y=\epsilon)$, we get 0.
To compute the monodromy around $y=1$, take $q(\epsilon)$ to be the composition of $(x=\epsilon, y=(1-t)\epsilon + t(1-\epsilon))$, $(x=\epsilon, y=1-\epsilon e^{it})$ and $(x=\epsilon, y=(1-t)(1-\epsilon)+t\epsilon)$, we get $-2\pi i\,\mathrm{Li}_1(x)$.
To compute the monodromy around $xy=1$, take q to be the loop ($x=x^0$, y such that $\int_q d\log(1-xy)=2\pi i$), we get $-2\pi i\,\mathrm{Li}_1(\frac{1-xy}{1-x})$
The variation matrix is

$$\Lambda = \begin{bmatrix} 1 & & & \\ \mathrm{Li}_1(y) & 1 & & \\ \mathrm{Li}_1(xy) & & 1 & \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 \end{bmatrix} \tau_{1,1}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & \\ -dv_2 & 0 & & \\ -dv_{12} & 0 & 0 & \\ 0 & -dv_1 & -dw_1 & 0 \end{bmatrix}$$

Note that $\mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) = -\log\left(\frac{x(y-1)}{1-x}\right) = \mathrm{Li}_1(y) - \mathrm{Li}_1(x^{-1}) = \mathrm{Li}_1(y) - \mathrm{Li}_1(x) - \log x - i\pi$
The monodromy representation ρ is as follows
For monodromy around $x=0$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}$$

For monodromy around $y=0$, identity.
For monodromy around $x=1$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}$$

For monodromy around $y=1$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}$$

For monodromy around $xy=1$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ -1 & & 1 & \\ & & & 1 \end{bmatrix}$$

0.5 $\mathrm{Li}_{1,2}$

$$\mathrm{Li}_{2,1} =$$

0.6 $\mathrm{Li}_{2,1}$

$$\begin{aligned}\mathrm{Li}_{2,1}(x,y) &= \int \frac{dy}{1-y}\frac{dx}{1-x}\frac{dx}{x} + \frac{d(xy)}{1-xy}\left(\frac{dy}{1-y} - \frac{dx}{x(1-x)}\right)\frac{dx}{x} + \frac{d(xy)}{1-xy}\frac{d(xy)}{xy}\frac{dy}{1-y} \\ &= \int dv_2 dv_1 du_1 + dv_{12} dw_1 du_1 + dv_{12} du_{12} dv_2\end{aligned}$$

To compute the monodromy around $x = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon e^{it}, y = \epsilon)$, we get 0
To compute the monodromy around $y = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon, y = \epsilon e^{it})$, we get 0
To compute the monodromy around $y = 1$, take $q(\epsilon)$ to be the composition of $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$, $(x = \epsilon, y = 1 - \epsilon e^{it})$ and $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$, we get $-2\pi i \,\mathrm{Li}_2(x)$
To compute the monodromy around $x = 1$, take $q(\epsilon)$ to be the composition of $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$, $(x = 1 - \epsilon e^{it}, y = \epsilon)$ and $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$, we get 0
To compute the monodromy around $xy = 1$, take q to be the loop ($x = x^0, y$ such that $\int_q d\log(1-xy) = 2\pi i$), we get $2\pi i(\mathrm{Li}_2(y) - \mathrm{Li}_2(x^{-1}))$
The variation matrix is

$$\Lambda = \begin{bmatrix} 1 &&&&& \\ \mathrm{Li}_1(y) & 1 &&&& \\ \mathrm{Li}_1(xy) && 1 &&& \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 && \\ \mathrm{Li}_2(xy) && \log(xy) && 1 & \\ \mathrm{Li}_{2,1}(x,y) & \mathrm{Li}_2(x) & g(x,y) & \log x & \mathrm{Li}_1(y) & 1 \end{bmatrix} \tau_{2,1}(2\pi i)$$

Where

$$\begin{aligned} g(x,y) &= -I(x^{-1}y^{-1}; 0, y^{-1}; 1) \\ &= -\int_{x^{-1}y^{-1}}^{1} d\log t d\log(t - y^{-1}) \\ &= -\int_{x^{-1}y^{-1}}^{1} d\log yt d\log(yt - 1) \\ &= -\int_{x^{-1}}^{y} d\log t d\log(t-1) \\ &= -\int_{y}^{x^{-1}} d\log(t-1) d\log t \\ &= -\int_{y}^{x^{-1}} d\log(t-1) d\log t \\ &= \mathrm{Li}_2(y) - \mathrm{Li}_2(x^{-1}) + \log(xy)\,\mathrm{Li}_1(y) \end{aligned}$$

$$\omega = \begin{bmatrix} 0 &&&&& \\ -dv_2 & 0 &&&& \\ -dv_{12} & 0 & 0 &&& \\ 0 & -dv_1 & -w_1 & 0 && \\ 0 & 0 & du_{12} & 0 & 0 & \\ 0 & 0 & 0 & du_1 & -dv_2 & 0 \end{bmatrix}$$

The monodromy representation ρ is as follows
For monodromy around $x = 0$

$$\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ && -1 & 1 && \\ && 1 && 1 & \\ &&& 1 && 1 \end{bmatrix}$$

For monodromy around $y = 0$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & 1 & & 1 & \\ & & & & & 1 \end{bmatrix}$$

For monodromy around $x = 1$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & -1 & 1 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

For monodromy around $y = 1$

$$\begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & & 1 & & & \\ & & -1 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

For monodromy around $xy = 1$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ -1 & & 1 & & & \\ & & & 1 & & \\ -\frac{\log(xy)}{2\pi i} & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

0.7 $\mathrm{Li}_{1,1,1}$

$$\mathrm{Li}_{2,1} =$$

0.8 $\mathrm{Li}_{3,1}$

$$
\begin{aligned}
\mathrm{Li}_{3,1}(x,y) &= \int \frac{dy}{1-y}\frac{dx}{1-x}\left(\frac{dx}{x}\right)^2 + \frac{d(xy)}{1-xy}\left(\frac{dy}{1-y} - \frac{dx}{x(1-x)}\right)\left(\frac{dx}{x}\right)^2 \\
&+ \frac{d(xy)}{1-xy}\frac{d(xy)}{xy}\frac{dy}{1-y}\frac{dx}{x} + \frac{d(xy)}{1-xy}\left(\frac{d(xy)}{xy}\right)^2\frac{dy}{1-y} \\
&= \int dv_2 dv_1 (du_1)^2 + dv_{12} dw_1 (du_1)^2 + dv_{12} du_{12} dv_2 du_1 + dv_{12}(du_{12})^2 dv_2
\end{aligned}
$$

Here we write $w_1 = u_1 + v_2 - v_1 = \log\frac{x(1-y)}{1-x}$
To compute the monodromy around $x = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon e^{it}, y = \epsilon)$, we get 0.
To compute the monodromy around $y = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon, y = \epsilon e^{it})$, we get 0.
To compute the monodromy around $x = 1$, take $q(\epsilon)$ to be the composition of $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$, $(x = 1-\epsilon e^{it}, y = \epsilon)$ and $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$, we get 0.
To compute the monodromy around $y = 1$, take $q(\epsilon)$ to be the composition of $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$, $(x = \epsilon, y = 1-\epsilon e^{it})$ and $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$, we get $-2\pi i\,\mathrm{Li}_3(x)$.
To compute the monodromy around $xy = 1$, take q to be the loop $(x = x^0, y$ such that $\int_q d\log(1-xy) = 2\pi i)$, we get $-2\pi i\,\mathrm{Li}_1(\frac{1-xy}{1-x})$
The variation matrix is

$$
\Lambda = \begin{bmatrix}
1 &&&&&&& \\
\mathrm{Li}_1(y) & 1 &&&&&& \\
\mathrm{Li}_1(xy) && 1 &&&&& \\
\mathrm{Li}_{1,1}(x,y) &&& 1 &&&& \\
\mathrm{Li}_2(xy) &&&& 1 &&& \\
\mathrm{Li}_{2,1}(x,y) &&&&& 1 && \\
\mathrm{Li}_3(xy) &&&&&& 1 & \\
\mathrm{Li}_{3,1}(x,y) &&&&&&& 1
\end{bmatrix} \tau_{1,1}(2\pi i)
$$

$$
\omega = \begin{bmatrix}
0 &&&&&&& \\
-dv_2 & 0 &&&&&& \\
-dv_{12} & 0 & 0 &&&&& \\
& -dv_1 & -dw_1 & 0 &&&& \\
&& du_{12} & 0 & 0 &&& \\
&&& du_1 & -dv_2 & 0 && \\
&&&& du_{12} & 0 & 0 & \\
&&&&& du_1 & -dv_2 & 0
\end{bmatrix}
$$

The monodromy representation ρ is as follows
For monodromy around $x = 0$ For monodromy around $y = 0$, identity.
For monodromy around $x = 1$ For monodromy around $y = 1$ For monodromy around $xy = 1$

0.9 $\mathrm{Li}_{2,2}$

$$\mathrm{Li}_{2,1} =$$

0.10 $\mathrm{Li}_{4,1}$

$$\mathrm{Li}_{n,1}(x,y) = \int (dv_2 dv_1 + dv_{12} dw_1)(du_1)^{n-1} + dv_{12}\left(\sum_{k=1}^{n-1} du_{12}^k dv_2 (du_1)^{n-1-k}\right)$$

Here $w_1 = u_1 + v_2 - v_1$
To compute the monodromy around $x = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon e^{it}, y = \epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around $y = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon, y = \epsilon e^{it})$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 \cdots (du_1)^{n-1} = 0$$

To compute the monodromy around $x = 1$, take $q(\epsilon)$ to be the composition of $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$, $(x = 1 - \epsilon e^{it}, y = \epsilon)$ and $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around $y = 1$, take $q(\epsilon)$ to be the composition of $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$, $(x = \epsilon, y = 1 - \epsilon e^{it})$ and $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 (du_1)^{n-1} = -2\pi i\, \mathrm{Li}_n(x)$$

To compute the monodromy around $xy = 1$, take q to be the loop $(x = x^0, y$ such that $\int_q d\log(1-xy) = 2\pi i)$

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{pq} - \int_p &= \sum_{k=0}^{n-1} \int_p dv_{12} (du_{12})^k \int_q (du_{12})^{n-1-k} dv_2 + \int_q dv_{12} du_{12}^{n-1} dv_2 \\
&= (-1)^{n+1} \int_{q^{-1}} dv_2 du_2^{n-1} dv_{12} \\
&= (-1)^{n+1} \int_{q^{-1}} \frac{\mathrm{Li}_n(y) - \mathrm{Li}_n(y^0)}{y - 1/x^0} \\
&= (-1)^n 2\pi i (\mathrm{Li}_n(y^0) - \mathrm{Li}_n(1/x^0))
\end{aligned}$$

The variation matrix is

$$\Lambda = \begin{bmatrix} 1 \\ \mathrm{Li}_1(y) \\ \mathrm{Li}_1(xy) \\ \mathrm{Li}_{1,1}(x,y) \\ \mathrm{Li}_2(xy) \\ \mathrm{Li}_{2,1}(x,y) \\ \mathrm{Li}_3(xy) \\ \mathrm{Li}_{3,1}(x,y) \\ \mathrm{Li}_4(xy) \\ \mathrm{Li}_{4,1}(x,y) \end{bmatrix} \tau_{1,1}(2\pi i)$$

$$\omega = \begin{bmatrix}
0 & & & & & & & & & \\
dv_2 & 0 & & & & & & & & \\
-dv_{12} & & 0 & & & & & & & \\
 & -dv_2 & d(-u_1+v_1-v_2) & 0 & & & & & & \\
 & & & d(u_1+u_2) & 0 & & & & & \\
 & & & du_1 & -dv_2 & 0 & & & & \\
 & & & & d(u_1+u_2) & & 0 & & & \\
 & & & & & du_1 & -dv_2 & 0 & & \\
 & & & & & & d(u_1+u_2) & & 0 & \\
 & & & & & & & du_1 & -dv_2 & 0
\end{bmatrix}$$

The monodromy representation ρ is as follows

0.11 $\mathrm{Li}_{n,1}$

$$\mathrm{Li}_{n,1}(x,y) = \int (dv_2 dv_1 + dv_{12} dw_1)(du_1)^{n-1} + dv_{12}\left(\sum_{k=1}^{n-1} du_{12}^k dv_2 (du_1)^{n-1-k}\right)$$

Here $w_1 = u_1 + v_2 - v_1$
To compute the monodromy around $x = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon e^{it}, y = \epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around $y = 0$, take $q(\epsilon)$ to be the loop $(x = \epsilon, y = \epsilon e^{it})$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 \cdots (du_1)^{n-1} = 0$$

To compute the monodromy around $x = 1$, take $q(\epsilon)$ to be the composition of $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$, $(x = 1 - \epsilon e^{it}, y = \epsilon)$ and $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around $y = 1$, take $q(\epsilon)$ to be the composition of $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$, $(x = \epsilon, y = 1 - \epsilon e^{it})$ and $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$

$$\lim_{\epsilon\to 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 (du_1)^{n-1} = -2\pi i\, \mathrm{Li}_n(x)$$

To compute the monodromy around $xy = 1$, take q to be the loop $(x = x^0, y$ such that $\int_q d\log(1-xy) = 2\pi i)$

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{pq} - \int_p &= \sum_{k=0}^{n-1} \int_p dv_{12}(du_{12})^k \int_q (du_{12})^{n-1-k} dv_2 + \int_q dv_{12} du_{12}^{n-1} dv_2 \\
&= (-1)^{n+1} \int_{q^{-1}} dv_2 du_2^{n-1} dv_{12} \\
&= (-1)^{n+1} \int_{q^{-1}} \frac{\mathrm{Li}_n(y) - \mathrm{Li}_n(y^0)}{y - 1/x^0} \\
&= (-1)^n 2\pi i(\mathrm{Li}_n(y^0) - \mathrm{Li}_n(1/x^0))
\end{aligned}$$

The variation matrix is

$$\Lambda = \begin{bmatrix} 1 \\ \mathrm{Li}_1(y) \\ \mathrm{Li}_1(xy) \\ \mathrm{Li}_{1,1}(x,y) \\ \vdots \\ \mathrm{Li}_n(xy) \\ \mathrm{Li}_{n,1}(x,y) \end{bmatrix} \tau_{n,1}(2\pi i)$$

$$\omega = \begin{bmatrix}
0 \\
-dv_2 & 0 \\
-dv_{12} & 0 & 0 \\
& -dv_2 & -dw_1 & 0 \\
& & du_{12} & 0 & 0 \\
& & & du_1 & -dv_2 & 0 \\
& & & & \ddots & \ddots & \ddots \\
& & & & & du_{12} & 0 & 0 \\
& & & & & & du_1 & -dv_2 & 0
\end{bmatrix}$$

The monodromy representation ρ is as follows

0.12 Variation matrix

Theorem 0.12.1. Λ is the fundamental solution of the system of linear differential equations

$$d\Lambda = \omega\Lambda$$

Example 0.12.2. For

$$\begin{aligned}\mathrm{Li}_{1,1}(x,y) &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 + dv_{12} d(u_1 - v_1 + v_2) \\ &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 + dv_{12} du_1 - dv_{12} dv_1 + dv_{12} dv_2\end{aligned}$$

$(0,0) < (0,1) < (1,0) < (1,1)$ in $\mathfrak{S}(1,1)$

$$\Lambda = \begin{bmatrix} 1 & & & \\ \mathrm{Li}_1(y) & 1 & & \\ \mathrm{Li}_1(xy) & & 1 & \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \mathrm{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 \end{bmatrix} \tau_{1,1}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & \\ -dv_2 & 0 & & \\ -dv_{12} & 0 & 0 & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 \end{bmatrix}$$

Example 0.12.3. For

$$\begin{aligned}\mathrm{Li}_{2,1}(x,y) &= \int_{(0,0)}^{(x,y)} (dv_1 dv_2 + dv_{12} d(u_1 - v_1 + v_2)) du_1 + dv_{12} d(u_1 + u_2) dv_2 \\ &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 du_1 + dv_{12} du_1 du_1 - dv_{12} dv_1 du_1 \\ &+ dv_{12} dv_2 du_1 + dv_{12} du_1 dv_2 + dv_{12} u_2 dv_2\end{aligned}$$

$(0,0) < (0,1) < (1,0) < (1,1) < (2,0) < (2,1)$ in $\mathfrak{S}(2,1)$

$$\Lambda = \begin{bmatrix} 1 & & & & & \\ \mathrm{Li}_1(y) & 1 & & & & \\ \mathrm{Li}_1(xy) & & 1 & & & \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \log\frac{1-x}{(1-y)x} & 1 & & \\ \mathrm{Li}_2(xy) & & \log(xy) & & 1 & \\ \mathrm{Li}_{2,1}(x,y) & \mathrm{Li}_2(x) & g(x,y) & \log x & \mathrm{Li}_1(y) & 1 \end{bmatrix} \tau_{2,1}(2\pi i)$$

Where $dg = \log\dfrac{1-x}{(1-y)x}\dfrac{dx}{x} + \log(xy)\dfrac{dy}{1-y}$

$$\omega = \begin{bmatrix} 0 & & & & & \\ -dv_2 & 0 & & & & \\ -dv_{12} & 0 & 0 & & & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 & & \\ 0 & 0 & d(u_1 + u_2) & 0 & 0 & \\ 0 & 0 & 0 & du_1 & -dv_2 & 0 \end{bmatrix}$$

Example 0.12.4. For

$$\begin{aligned}\mathrm{Li}_{1,1,1}(x,y,z) &= \int_{(0,0,0)}^{(x,y,z)} \\ &= \int_{(0,0,0)}^{(x,y,z)}\end{aligned}$$

$(0,0,0)<(0,0,1)<(0,1,0)<(1,0,0)<(0,1,1)<(1,0,1)<(1,1,0)<(1,1,1)$ in $\mathfrak{S}(1,1,1)$

$$\Lambda=\begin{bmatrix} 1 \\ \mathrm{Li}_1(z) & 1 \\ \mathrm{Li}_1(yz) & & 1 \\ \mathrm{Li}_1(xyz) & & & 1 \\ \mathrm{Li}_{1,1}(y,z) & \mathrm{Li}_1(y) & \log\frac{1-y}{(1-z)y} & & 1 \\ \mathrm{Li}_{1,1}(xy,z) & \mathrm{Li}_1(xy) & & \log\frac{1-xy}{(1-z)xy} & & 1 \\ \mathrm{Li}_{1,1}(x,yz) & & \mathrm{Li}_1(x) & \log\frac{1-x}{(1-yz)x} & & & 1 \\ \mathrm{Li}_{1,1,1}(x,y,z) & g(x,y) & \mathrm{Li}_1(x)\log\frac{1-y}{(1-z)y} & h(x,y) & \mathrm{Li}_1(x) & \log\frac{1-y}{(1-x)x} & \log\frac{1-z}{(1-y)y} & 1 \end{bmatrix}\tau_{1,1,1}(2\pi i)$$

Where

$$\omega=\begin{bmatrix} 0 \\ -dv_3 & 0 \\ -dv_{23} & 0 & 0 \\ -dv_{13} & 0 & 0 & 0 \\ 0 & -dv_2 & d(v_2-u_2-v_3) & 0 & 0 \\ 0 & -dv_{12} & 0 & d(v_{12}-u_1-u_2-v_3) & 0 & 0 \\ 0 & 0 & -dv_1 & d(v_1-u_1-v_{23}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -dv_1 & d(v_1-u_1-v_2) & d(v_2-u_2-v_3) & 0 \end{bmatrix}$$

Example 0.12.5. For

$$\begin{aligned}\mathrm{Li}_{1,2}(x,y) &= dv_{12}d(u_1+u_2)d(u_1-v_2)+(dv_1dv_2+dv_{12}d(u_1-v_1+v_2))du_2 \\ &= dv_{12}du_1du_1-dv_{12}du_1dv_2+dv_{12}du_2du_1-dv_{12}du_2dv_2 \\ &+dv_1dv_2du_2+dv_{12}du_1du_2-dv_{12}dv_1du_2+dv_{12}dv_2du_2\end{aligned}$$

$(0,0)<(0,1)<(1,0)<(1,1)<(0,2)<(1,2)$ in $\mathfrak{S}(1,2)$

$$\Lambda=\begin{bmatrix} 1 \\ \mathrm{Li}_1(y) & 1 \\ \mathrm{Li}_1(xy) & 0 & 1 \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \log\frac{1-x}{(1-y)x} & 1 \\ \mathrm{Li}_2(y) & \log y & & & 1 \\ \mathrm{Li}_2(xy) & 0 & \log(xy) & & & 1 \\ \mathrm{Li}_{1,2}(x,y) & \mathrm{Li}_1(x)\log y & g(x,y) & \log y & \mathrm{Li}_1(x) & -\mathrm{Li}_1(x^{-1}) & 1 \end{bmatrix}\tau_{1,2}(2\pi i)$$

Where $g(x,y)=-I((xy)^{-1};y^{-1},0;1)$

$$\omega=\begin{bmatrix} 0 \\ -dv_2 & 0 \\ -dv_{12} & 0 & 0 \\ 0 & -dv_1 & d(-u_1+v_1-v_2) & 0 \\ 0 & du_2 & 0 & 0 & 0 \\ 0 & 0 & d(u_1+u_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & du_2 & -dv_1 & d(v_1-u_1) & 0 \end{bmatrix}$$

Example 0.12.6. For

$$\begin{aligned}\mathrm{Li}_{2,2}(x,y) &= (dv_{12}du(u_1+u_2)d(u_1-v_2)+(dv_1dv_2+dv_{12}d(u_1-v_1+v_2))du_2)du_1 \\ &+((dv_1dv_2+dv_{12}d(u_1-v_1+v_2))du_1+dv_{12}d(u_1+u_2)dv_2)du_2 \\ &= dv_{12}du_1du_1du_1-dv_{12}du_1dv_2du_1+dv_{12}du_2du_1du_1-dv_{12}du_2dv_2du_1 \\ &+dv_1dv_2du_2du_1+dv_{12}du_1du_2du_1-dv_{12}dv_1du_2du_1+dv_{12}dv_2du_2du_1 \\ &+dv_1dv_2du_1du_2+dv_{12}du_1du_1du_2-dv_{12}dv_1du_1du_2 \\ &+dv_{12}dv_2du_1du_2+dv_{12}du_1dv_2du_2+dv_{12}u_2dv_2du_2\end{aligned}$$

$(0,0) < (0,1) < (1,0) < (1,1) < (0,2) < (2,0) < (1,2) < (2,1) < (2,2)$ in $\mathfrak{S}(2,2)$

$$\Lambda = \begin{bmatrix} 1 & & & & & & & & \\ \mathrm{Li}_1(y) & 1 & & & & & & & \\ \mathrm{Li}_1(xy) & & 1 & & & & & & \\ \mathrm{Li}_{1,1}(x,y) & \mathrm{Li}_1(x) & \log\frac{1-x}{(1-y)x} & 1 & & & & & \\ \mathrm{Li}_2(y) & \log y & & & 1 & & & & \\ \mathrm{Li}_2(xy) & & \log(xy) & & & 1 & & & \\ \mathrm{Li}_{1,2}(x,y) & \mathrm{Li}_1(x)\log y & g(x,y) & \log y & \mathrm{Li}_1(x) & \log\frac{1-x}{x} & 1 & & \\ \mathrm{Li}_{2,1}(x,y) & \mathrm{Li}_2(x) & h(x,y) & \log x & & \mathrm{Li}_1(y) & & 1 & \\ \mathrm{Li}_{2,2}(x,y) & \mathrm{Li}_2(x)\log y & i(x,y) & \log x\log y & \mathrm{Li}_2(x) & \mathrm{Li}_2(y)-\mathrm{Li}_2(x)-\frac{1}{2}\log^2 x & \log x & \log y & 1 \end{bmatrix} \tau_{2,2}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & & & & & & \\ -dv_2 & 0 & & & & & & & \\ -dv_{12} & 0 & 0 & & & & & & \\ 0 & -dv_1 & d(-u_1+v_1-v_2) & 0 & & & & & \\ 0 & du_2 & 0 & 0 & 0 & & & & \\ 0 & 0 & d(u_1+u_2) & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & du_2 & -dv_1 & d(v_1-u_1) & 0 & & \\ 0 & 0 & 0 & du_1 & 0 & -dv_2 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & du_1 & du_2 & 0 \end{bmatrix}$$

0.13 Bloch-Wigner polylogarithm

Definition 0.13.1. The Bloch-Wigner polylogarithm is defined as

$$\mathcal{L}_n(z) = \Re_n\left(\sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \operatorname{Li}_{n-r}(z) \log^r |z|\right)$$

Here $\Re_n$ is Re if n is odd and $\mathtt{Im}$ if n is even. B_n are Bernoulli numbers. For instance, $\mathcal{L}_1(z) = 1$, $\mathcal{L}_2(z) = \operatorname{Im}(\operatorname{Li}_2(z)) + \operatorname{Im}(\log(1-z)) \log |z|$

Lemma 0.13.2. $\mathcal{L}_n(z) + (-1)^n \mathcal{L}_n(z^{-1}) = 0$. $\mathcal{L}_3(z) + \mathcal{L}_3\left(\frac{1}{1-z}\right) + \mathcal{L}_3(1 - z^{-1}) = \zeta(3)$. $\mathcal{L}_2(z) - \mathcal{L}_2\left(\frac{1}{1-z}\right) = 0$

Proof.

□

0.14 Hopf algebra structure

Definition 0.14.1. Iterated integrals form a Hopf algebra H with coproduct

$$\Delta I(a_0; a_1, \cdots; a_n; a_{n+1}) = \sum_{0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1} I(a_{i_0}; a_{i_1}, \cdots, a_{i_k}; a_{ik+1}) \otimes \prod_{p=1}^{k} I(a_{i_p}; a_{i_p+1}, \cdots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

The product is the just shuffle product, $\Delta_{i_1,\cdots,i_k}$ means those in grading $(i_1, \cdots, i_k)$. $\Delta'(x) = \Delta(x) - 1 \otimes x - x \otimes 1$ is the reduced coproduct. The space of indecomposables $Q(H) = H/(H_{>0} \cdot H_{>0})$ is mod products. The projection $\frac{1}{n}R = P : H \to Q(H)$, where R is defined inductively as $R(x) = nx - \mu(1 \otimes R)\Delta'(x)$, μ is multiplication. The cobracket is defined as $\delta(x) = (P \otimes P)(1 - \tau)\Delta(x)$, $\tau(x \otimes y) = y \otimes x$
Symbol of a multiple polylogarithm is defined to be $\Delta_{1,\cdots,1}(x)$, and omit log sign