Section 12.3

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Definition: Let X be a continuous random variable whose possible values lie between A and B, and let $f(x)$ be the pdf(probability density function) for X. Then the **expected value**(or **mean**) of X is defined to be $E(X) := \int_A^B xf(x)dx$

The **Variance** of X is defined to be $Var(X) := \int_A^B [x - E(X)]^2 f(x)dx$

Useful simplification: $Var(X) = \int_A^B x^2 f(x)dx - E(X)^2$

If we give a more general definition: $E(g(X)) := \int_A^B g(x)f(x)dx$, notice expected value corresponds to when $g(x) = x$, we have $Var(X) = E(X^2) - E(X)^2$, similar formula holds true for discrete random variable also

Problems:

1: A newspaper publisher estimates that the proportion X of space devoted to news on a given day is a random variable with the beta probability density $f(x) = 30x^2(1-x)^2, 0 \le x \le 1$

(a) Find the cdf(cumulative distribution function) $F(x)$ for X

(b) Find the probability that less than 25% of the newspaper's space on a given day contains news

(c) Find expected value $E(X)$

(d) Compute variance $Var(X)$

(a) $F(x) = \int_A^x f(t)\,dt = \int_0^x 30t^2(1-t)^2 dt = 30\int_0^x t^2(1-2t+t^2)\,dt$

$= 30\int_0^x (t^2-2t^2+t^4)\,dt = 30\left(\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^5}{5}\right)\Big|_0^x$

$= 30\left(\frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right) = 10x^3 - 15x^4 + 6x^5$

(b) $Pr(X \le 25\%) = Pr(0 \leqslant X \leqslant \frac{1}{4}) = F(\frac{1}{4}) - F(0) = F(\frac{1}{4}) = \frac{10}{4^3} - \frac{15}{4^4} + \frac{6}{4^5} = \frac{53}{512}$

(c) $E(X) = \int_A^B xf(x)dx = \int_0^1 x \cdot 30x^2(1-x)^2 dx = 30\int_0^1 x^3(1-2x+x^2)dx$

$= 30\int_0^1 (x^3-2x^4+x^5)dx = 30\left(\frac{x^4}{4} - \frac{2x^5}{5} + \frac{x^6}{6}\right)\Big|_0^1 = 30\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = \frac{1}{2}$

(d) $Var(X) = \int_A^B x^2 f(x)dx - E(X)^2 = \int_0^1 x^2 \cdot 30x^2(1-x)^2 dx - \left(\frac{1}{2}\right)^2 = 30\int_0^1 x^4(1-2x+x^2)dx - \frac{1}{4}$

$= 30\int_0^1 (x^4-2x^5+x^6)\,dx - \frac{1}{4} = 30\left(\frac{x^5}{5} - \frac{x^6}{3} + \frac{x^7}{7}\right)\Big|_0^1 - \frac{1}{4} = 30\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) - \frac{1}{4} = \frac{1}{28}$

2: The useful life (in hundreds of hours) of a certain machine component is a random variable X with the cumulative distribution function $F(x) = \frac{1}{1521}x^2, 0 \le x \le 39$

(a) Find expected value $E(X)$

(b) Compute variance $Var(X)$

Remember $F(x)$ is a antiderivative of $f(x)$, $f(x) = F'(x) = \frac{2}{1521}x$

(a) $E(X) = \int_A^B x f(x) dx = \int_0^{39} x \cdot \frac{2x}{1521} dx = \int_0^{39} \frac{2x^2}{1521} dx = \frac{2}{1521} \int_0^{39} x^2 dx = \frac{2}{1521} \frac{x^3}{3} \Big|_0^{39}$

$= \frac{2}{1521} \cdot \frac{39^3}{3} = 26$

(b) $Var(X) = \int_A^B x^2 f(x) dx - E(X)^2 = \int_0^{39} x^2 \cdot \frac{2x}{1521} dx - 26^2 = \frac{2}{1521} \int_0^{39} x^3 dx - 26^2$

$= \frac{2}{1521} \frac{x^4}{4} \Big|_0^{39} - 26^2 = \frac{2}{1521} \frac{39^4}{4} - 26^2 = \frac{169}{2}$

3: The amount of time (in minutes) that a person spends reading the editorial page of the newspaper is a random variable with the density function, $f(x) = \frac{1}{50}x, 0 \le x \le 10$. Find the average time spent reading the editorial page

Finding the average time spent reading the editorial page means finding the expected value $E(X)$, and

$$E(X) = \int_A^B x f(x) dx = \int_0^{10} x \cdot \frac{1}{50} x dx = \frac{1}{50} \int_0^{10} x^2 dx = \frac{1}{50} \frac{x^3}{3} \Big|_0^{10} = \frac{1}{50} \frac{10^3}{3} = \frac{20}{3}$$

A bit more for the review test of chapter 10(involving section 10.2):

Problem 3: Use three repetitions of the Newton-Raphson algorithm to approximate the zero of $e^x + 8x - 3$ near $x_0 = 0$

Solution:

Step I: Identify your equation with a corresponding function, in this particular case, $f(x) = e^x + 8x - 3$

Step II: Iteration by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$, $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$, $x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)}$, $\cdots$, $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$, in this particular case, $f'(x) = e^x + 8$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.2222222222$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.2193433075$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.2193427483$$

Thus the zero of $e^x + 8x - 3$ near $x_0 = 0$ is approximately 0.21934

Problem 4: Suppose that an investment of \$590 yields returns of \$100, \$200, and \$300 at the end of the first, second, and third months, respectively. Determine the internal rate of return on this investment

Solution:

Step I: Suppose the internal rate of return is i, then we should have

$$\begin{aligned} 590 &= 100(1+i)^{-1} + 200(1+i)^{-2} + 300(1+i)^{-3} \\ \xRightarrow{\div 10} 59 &= 10(1+i)^{-1} + 20(1+i)^{-2} + 30(1+i)^{-3} \\ \xRightarrow{\times(1+i)^3} 59(1+i)^3 &= 10(1+i)^2 + 20(1+i) + 30 \\ \Rightarrow 59(1+i)^3 &- 10(1+i)^2 - 20(1+i) - 30 = 0 \end{aligned}$$

Step II: Construct a problem for Newton-Raphson algorithm, in this particular case, we use the Newton-Raphson algorithm to approximate the zero of $59x^3 - 10x^2 - 20x - 30$ near $x_0 = 1$

Step III: Using Newton-Raphson algorithm, $f(x) = 59x^3 - 10x^2 - 20x - 30$, $f'(x) = 177x^2 - 20x - 20$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.00729927$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.007235299$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.007235294$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.007235294$$

Thus $1 + i \approx 1.0072 \Rightarrow i \approx 0.0072 = 0.72\%$