Separation of variables

$\frac{dy}{dt} = y' = 3y$, we separate $y$ and $t$ on two sides by $\div y$ and $\times dt$ on both sides, then we get $\frac{dy}{y} = 3dt$, integrate on both sides

$\Rightarrow \int \frac{dy}{y} = \int 3dt \Rightarrow \ln|y| = 3t + C \xrightarrow{\text{take exponential}} |y| = e^{3t+C}$

$\Rightarrow y = e^{3t+C}$ or $-e^{3t+C}$ $C$ is a constant

Note that in dividing $y$, we missed a solution $y=0$

$\frac{dy}{dt} = y' = 12te^{-2y} - 19e^{-2y}$, $y(0) = 8$ DE with IC

$\frac{dy}{dt} = (12t-19)e^{-2y} \xrightarrow{\times e^{2y}dt} e^{2y}dy = (12t-19)dt$

$\Rightarrow \int e^{2y}dy = \int (12t-19)dt \Rightarrow \frac{1}{2}e^{2y} = 6t^2 - 19t + C$

plug in $y(0)=8$, $\frac{1}{2}e^{16} = \frac{1}{2}e^{2\cdot 8} = \frac{1}{2}e^{2\cdot y(0)} = 6\cdot 0^2 - 19\cdot 0 + C = C$, hence

$\frac{1}{2}e^{16} = C$, thus $\frac{1}{2}e^{2y} = 6t^2 - 19t + \frac{1}{2}e^{16} \xrightarrow{\times 2} e^{2y} = 12t^2 - 38t + e^{16}$

$\xrightarrow{\text{take } \ln} 2y = \ln(12t^2 - 38t + e^{16}) \xrightarrow{\div 2} y = \frac{1}{2}\ln(12t^2 - 38t + e^{16})$

Solve $y^5 y' = t\cos t$, $y(0) = 2$

$y^5 \frac{dy}{dt} = t\cos t \xrightarrow{\times dt} y^5 dy = t\cos t\, dt \Rightarrow \int y^5 dy = \int t\cos t\, dt$

$\int t\cos t\, dt = \int t(\sin t)' dt = \int [(t\sin t)' - t'\sin t] dt = \int (t\sin t)' dt - \int \sin t\, dt$

$= t\sin t + \cos t + C$

$\Rightarrow \frac{1}{6}y^6 = t\sin t + \cos t + C$, plug in $y(0) = 2$, $\frac{32}{3} = \frac{1}{6}\times 2^6 = 0\times\sin(0) + \cos(0) + C$

$= 1 + C \Rightarrow C = 1 - \frac{32}{3} = \frac{29}{3}$, hence $\frac{1}{6}y^6 = t\sin t + \cos t + \frac{29}{3} \xrightarrow{\times 6} y^6 = 6t\sin t + 6\cos t + 54$

$y = \sqrt[6]{6t\sin t + 6\cos t + 54}$, not choosing the negative part because $y(0) = 2 > 0$

10.3 consider DE $y' = 6y + t$, can't really separate the variables

rewrite as $y' - 6y = t$ key observation: $[ye^{-6t}]' = y'e^{-6t} + y(e^{-6t})' = y'e^{-6t} - 6ye^{-6t} = (y' - 6y)e^{-6t}$, thus we $\times e^{-6t}$ on both sides to get $[ye^{-6t}]' = (y - y')e^{-6t} = te^{-6t}$, then $\times dt$ and integrate $\int [ye^{-6t}]' dt = \int te^{-6t} dt$

$\Rightarrow ye^{-6t} = -\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C$, $C$ is constant, then $\div e^{-6t}$/or $\times e^{6t}$ on both sides, we have $y = -\frac{1}{6}t - \frac{1}{36} + Ce^{6t}$

Generalization: Consider DE $y' + a(t)y = b(t)$

Step I: Find an antiderivative of $a(t)$, denote as $A(t)$, $A'(t) = a(t)$, notice there is choice of constant $C$ from $\int a(t)dt$, you can pick any $C$

In correspondence, $a(t) = -6$, $A(t) = -6t$, $A'(t) = a(t)$

key observation is $[ye^{A(t)}]' = y'e^{A(t)} + y(e^{A(t)})' = y'e^{A(t)} + ye^{A(t)}A'(t)$

$= y'e^{A(t)} + a(t)ye^{A(t)} = (y' + a(t)y)e^{A(t)}$

Step II: multiplying integrating factor $e^{A(t)}$ on both sides, get

In correspondence, multiply $e^{-6t}$ on both sides

$[ye^{A(t)}]' = (y' + a(t)y)e^{A(t)} = b(t)e^{A(t)}$

Step III: $\times dt$ and the integrate, $\int [ye^{A(t)}]' dt = \int b(t)e^{A(t)} dt$

$\Rightarrow ye^{A(t)} = \int b(t)e^{A(t)} dt$

In correspondence, $ye^{-6t} = \int te^{-6t} dt = -\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C$

Step IV: $\div e^{A(t)}$, $y = e^{-A(t)} \int b(t)e^{A(t)} dt$

In correspondence, $y = \dfrac{(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C)}{e^{-6t}} = \dfrac{(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C)}{\frac{1}{e^{6t}}}$

$= e^{6t}(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C) = -\frac{1}{6}t - \frac{1}{36} + Ce^{6t}$