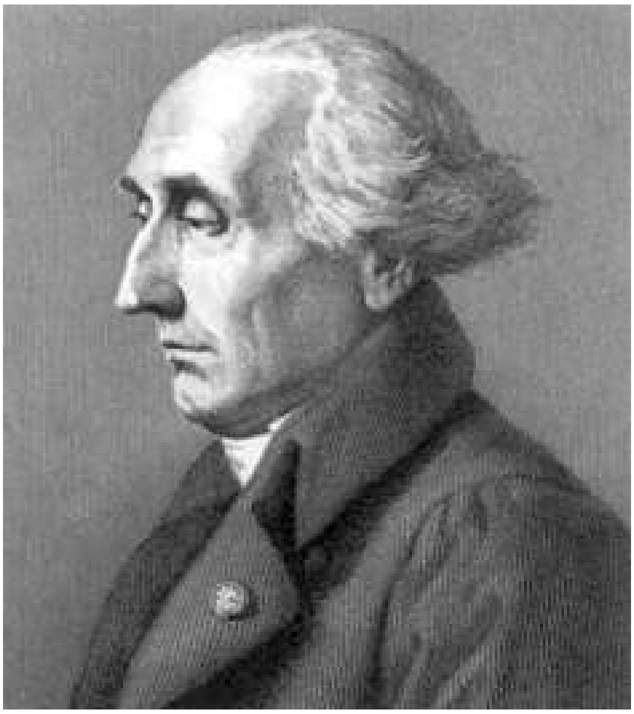
# MATH673 - Partial Differential equations I



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# 1 Homeworks

## 1.1 Homework1

**Problem .** 2.5.3 Define

$$\phi(r) := \frac{1}{|\partial B(0,r)|}\int_{\partial B(0,r)} u(x)dS + \int_{B(0,r)} \Phi(x)f(x)dx - \Phi(r)\int_{B(0,r)} f(x)dx$$

Where $\Phi(x) = \dfrac{1}{n(n-2)|B(0,1)|}\dfrac{1}{|x|^{n-2}}$, then

$$\begin{aligned}
\phi'(r) &= \frac{r}{n}\frac{1}{|B(0,r)|}\int_{B(0,r)} \Delta u(x)dx + \int_{\partial B(0,r)} \Phi(x)f(x)dS \\
&\quad - \Phi'(r)\int_{B(0,r)} f(x)dx - \Phi(r)\int_{\partial B(0,r)} f(x)dS \\
&= -\frac{r}{n}\frac{1}{|B(0,r)|}\int_{B(0,r)} f(x)dx + \frac{r}{n}\frac{1}{|B(0,r)|}\int_{B(0,r)} f(x)dx \\
&\quad + \int_{\partial B(0,r)} (\Phi(x) - \Phi(r))\, f(x)dS \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\phi(s) &= u(0) + \frac{1}{|\partial B(0,s)|}\int_{\partial B(0,s)} (u(x) - u(0))\, dS + \int_{B(0,s)} \Phi(x)f(x)dx - \Phi(s)\int_{B(0,s)} f(x)dx \\
&:= u(0) + I_1 + I_2 + I_3
\end{aligned}$$

$$|I_1| \le \frac{1}{|\partial B(0,s)|}\int_{\partial B(0,s)} |u(x) - u(0)|\, dS \to 0,\ s \to 0$$

By the continuity of $u$

$$|I_2| \le \|f\|_{L^\infty}\int_{B(0,s)} \Phi(x)dx = \|f\|_{L^\infty}\int_{B(0,s)} \Phi(x)dx = \frac{s^2\,\|f\|_{L^\infty}}{2(n-2)} \to 0,\ s \to 0$$

$$|I_3| \le \Phi(s)\,\|f\|_{L^\infty}\int_{B(0,s)} dx = \|f\|_{L^\infty}\int_{B(0,s)} \Phi(x)dx = \frac{s^2\,\|f\|_{L^\infty}}{n(n-2)} \to 0,\ s \to 0$$

Thus

$$\begin{aligned}
u(0) &= \lim_{s\to 0}\phi(s) = \lim_{s\to r}\phi(s) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(0,r)} g(x)dS + \int_{B(0,r)} (\Phi(x) - \Phi(r))\, f(x)dx
\end{aligned}$$

**Problem .** 2.5.4 **a)**
Since

$$\frac{d}{dr}\left\{\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} v(y)dS\right\} = \frac{r}{n}\frac{1}{|B(x,r)|}\int_{B(x,r)} \Delta v(y)dy \ge 0$$

Thus

$$v(x) \le \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} v(y)dS$$

and

$$\begin{aligned}
\frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)dy &= \frac{1}{|B(x,r)|}\int_0^r \left(\int_{\partial B(x,s)} v(y)dS\right) ds \\
&\ge v(x)\frac{1}{|B(x,r)|}\int_0^r |\partial B(x,s)|ds \\
&= v(x)
\end{aligned}$$

**b)**
Suppose $v$ attains at $x_0 \in U$ its maximum $\max_{\overline{U}} v > \max_{\partial U} v$, consider the connected component $U_0$ which contains $x_0$, then by $a)$ we know that the set $\{x \in U_0 \; v(x) = \max_{\overline{U}} v\}$ is both open and closed, thus $\max_{\partial U_0} v = \max_{\overline{U}} v$ which is a contradiction.
**c)**
Since $\phi$ is convex, $\phi''(x) \geq 0$, thus

$$\Delta v(x) = \Delta \phi(u(x)) = \phi'(u(x))\Delta u + \phi''(u(x))|\nabla u|^2 = \phi''(u(x))|\nabla u|^2 \geq 0$$

Hence $v$ is subharmonic.
**d)**

$$\Delta v = \Delta |\nabla u|^2 = 2\sum_{i,j} u_{x_i x_j}^2 + 2\nabla u \cdot \nabla(\Delta u) \geq 0$$

By **a)** we know $v$ is subharmonic

**Problem .** 2.5.5 Define $M := \max_{B(0,1)} |f|$ and $v := u + \frac{M}{2n}|x|^2$, then we have $\Delta v = \Delta u + M = M - f \geq 0$ in $B^0(0,1)$, thus $v$ is subharmonic, maximum principle still holds, hence

$$\max_{B(0,1)} u \leq \max_{B(0,1)} v = \max_{\partial B(0,1)} v \leq \max_{\partial B(0,1)} |g| + \frac{M}{2n} = \max_{B(0,1)} |v| \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|$$

Similary, we could consider $\begin{cases} -\Delta(-u) = -f \\ -u = -g \end{cases}$, then we would have

$$-\min_{B(0,1)} u = \max_{B(0,1)} (-u) \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f|$$

Therefore, we have

$$\max_{B(0,1)} |u| \leq \max_{\partial B(0,1)} |g| + \frac{1}{2n} \max_{B(0,1)} |f| \leq \max_{\partial B(0,1)} |g| + \max_{B(0,1)} |f|$$

**Problem .** 2.5.6 Using Poisson's formula and the fact that $u$ is harmonic, we have

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dS \\ &\leq \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \int_{\partial B(0,r)} \frac{u(y)}{(r - |x|)^n} dS \\ &= \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \frac{1}{(r-|x|)^n} \int_{\partial B(0,r)} u(y) dS \\ &= \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0) \end{aligned}$$

Similarly, we have

$$\begin{aligned} u(x) &= \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dS \\ &\geq \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \int_{\partial B(0,r)} \frac{u(y)}{(r + |x|)^n} dS \\ &= \frac{r^2 - |x|^2}{|\partial B(0,1)|r} \frac{1}{(r+|x|)^n} \int_{\partial B(0,r)} u(y) dS \\ &= \frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \end{aligned}$$

Thus $\frac{r^{n-2}(r - |x|)}{(r + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{r^{n-2}(r + |x|)}{(r - |x|)^{n-1}} u(0)$

**Problem .** 2.5.7 Since $K(x,y) \in C^{\infty}(B^0(0,1))$, thus

$$u(x) := \int_{\partial B(0,r)} K(x,y)g(y)dS(y) \in C^{\infty}(B^0(0,1))$$

Also,

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y)g(y)dS(y) = 0$$

Since $\Delta_x K(x,y) = 0$, when $y \in \partial B(0,r)$
By taking $u \equiv 1$ we get $\int_{\partial B(0,r)} K(x,y)dS(y) = 1$
Since $g \in C(\partial B(0,r))$, $|g| \le M$ for some $M > 0$, and $\forall \epsilon > 0, \exists \delta > 0$, such that $\left|g(y) - g(x^0)\right| \le \epsilon, \forall y \in \partial B(0,r) \cap B(x^0,\delta)$, hence, when $|x - x^0| < \frac{\delta}{2}$, we have $|x-y| \ge |x^0 - y| - |x - x^0| > \frac{\delta}{2}, \forall y \in \partial B(0,r) - B(x^0,\delta)$, hence

$$\begin{aligned}
\left|u(x) - g(x^0)\right| &= \left|\int_{\partial B(0,r)} K(x,y)\left(g(y) - g(x^0)\right)dS(y)\right| \\
&\le \left|\int_{\partial B(0,r)\cap B(x^0,\delta)} K(x,y)\left(g(y) - g(x^0)\right)dS(y)\right| \\
&\quad + \left|\int_{\partial B(0,r)- B(x^0,\delta)} K(x,y)\left(g(y) - g(x^0)\right)dS(y)\right| \\
&\le \epsilon \int_{\partial B(0,r)\cap B(x^0,\delta)} K(x,y)dS(y) + 2M\int_{\partial B(0,r)- B(x^0,\delta)} K(x,y)dS(y) \\
&\le \epsilon \int_{\partial B(0,r)} K(x,y)dS(y) + \frac{2M\left(r^2 - |x|^2\right)}{|\partial B(0,1)|r}\int_{\partial B(0,r)} \frac{2^n}{\delta^n}dS(y) \\
&= \epsilon + \frac{2^{n+1}Mr^{n-2}\left(r^2 - |x|^2\right)}{\delta^n}
\end{aligned}$$

Thus $\lim_{\substack{x\to x^0 \\ x\in B^0(0,r)}} u(x) = g(x^0)$

**Problem .** 1[January 2010] **a)**
Suppose $u, u_1$ are two bounded solution, then $\forall \epsilon > 0, \exists R > 0$, such that $\forall r > R$

$$u(x) - \epsilon \ln|x| \le u_1(x) \le u(x) + \epsilon\ln|x|$$

on $\partial B(0,r)$, using maximum principle for $u(x) - \epsilon\ln|x| - u_1(x)$ and $u_1(x) - u(x) - \epsilon\ln|x|$ on $B(0,r) - B^0(0,1)$
Thus the inequality above holds for any $\epsilon > 0$ and $|x| > 1$, let $\epsilon \to 0$, we have $u = u_1$
**b)**
$u \equiv 1$ and $u(x) = \frac{1}{|x|}$ are both bounded solutions with $f \equiv 1$
One additional condition that ascertain the uniqueness of the solution could be $\lim_{x\to\infty} u(x) = 0$, in this case we would have

$$u(x) - \epsilon \le u_1(x) \le u(x) + \epsilon$$

On $\partial B(0,r)$ for sufficiently large $r$

**Problem .** 1[January 2005] **Lemma:** If $g = \sum_{k=0}^{\infty} a_k z^k$ is a holomorphic fuction on $\mathbb{C}$, then

$$\int\int_{B(0,R)} |f'(z)|^2 dxdy = \sum_{k=0}^{\infty} |a_k|^2 \frac{\pi R^{2k+2}}{k+1}$$

**Proof:** Using the fact that

$$\int\int_{B(0,R)} z^k \overline{z}^l dxdy = \begin{cases} \dfrac{\pi R^{2k+2}}{k+1}, & k = l \\ 0, & k \neq l \end{cases}$$

**Method 1:**
Since $u$ is harmonic on $\mathbb{R}^2$, there exists a holomorphic function $f$ on $\mathbb{C}$, such that $\mathrm{Re} f = u$, then $|\nabla u| = |f'|$, hence we have $\int\int_{\mathbb{R}^2} |\nabla u|^2 dxdy = \int\int_{\mathbb{C}} |f'(z)|^2 dxdy < \infty$, according to the lemma above, we have $f' \equiv 0$, hence $f$ is a constant, so is $u$
**Method 2:**
According to **2.5.4 d)**, we know that $|\nabla u|^2$ is subharmonic, thus

$$|\nabla u(x)|^2 \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} |\nabla u(y)|^2 dy \leq \frac{1}{|B(x,r)|} \int_{\mathbb{R}^2} |\nabla u(y)|^2 dy$$

Which tends to $0$ as $r$ tends to $0$, thus $\nabla u \equiv 0$, $u$ is a constant

**Problem .** 1[August 2004] Assume $f \not\equiv 0$, since $f \in C^1(\mathbb{R}^n)$, there is some $B(x,\epsilon)$, such that $f$ is positive(negative) on it, consider $v \equiv 1$, then we have

$$\begin{aligned} 0 &= \int_{\partial B(x,\epsilon)} v \frac{\partial u}{\partial n} dS = \int_{B(x,\epsilon)} v \Delta u dx + \int_{B(x,\epsilon)} \nabla v \cdot \nabla u dx \\ &= \int_{B(x,\epsilon)} \nabla v \cdot \nabla u dx - \int_{B(x,\epsilon)} v f dx \\ &= -\int_{B(x,\epsilon)} f dx < (>) 0 \end{aligned}$$

Which is a contradiction, Thus $f \equiv 0$

## 1.2 Homework2

**Problem .** 2.5.13 **(a)**
We have $u_t = -\frac{x}{2t^{\frac{3}{2}}}v'$ and $u_{xx} = \frac{1}{t}v''$, thus $u_t - u_{xx} = 0 \Leftrightarrow -\frac{x}{2t^{\frac{3}{2}}}v' - \frac{1}{t}v'' = 0 \Leftrightarrow v'' + \frac{z}{2}v' = 0$

Multiply $e^{\frac{z^2}{4}}$ on both sides, we get $e^{\frac{z^2}{4}}v'' + \frac{ze^{\frac{z^2}{4}}}{2}v' = \left(e^{\frac{z^2}{4}}v'\right)' = 0$, Thus $e^{\frac{z^2}{4}}v' = c$ for some constant $c$, $v' = ce^{-\frac{z^2}{4}} \Rightarrow v = c\int_0^z e^{-\frac{s^2}{4}}ds + d$ for some other constant $d$

**(b)**

According to (a), we have $u(x,t) = c\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4}}ds + d$, thus $u_x = \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4t}}$, which again solve the heat equation, in order to obtain a fundamental solution, we also need $\lim_{t\downarrow 0} u_x(x,t) = u_x(x,0) = \delta_0$, thus we should have $1 = \int_{\mathbb{R}} u_x(x,0)dx = \int_{\mathbb{R}} \lim_{t\downarrow 0} u_x(x,t)dx = \lim_{t\downarrow 0}\int_{\mathbb{R}} u_x(x,t)dx = \lim_{t\downarrow 0}\int_{\mathbb{R}} \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4t}}dx = 2c\sqrt{\pi} \Rightarrow\ c = \frac{1}{\sqrt{4\pi}}$

**Problem .** 2.5.14 Consider $v = ue^{ct}$, we have $v_t - \Delta v = e^{ct}(u_t - \Delta u + cu)$, thus the initial value problem becomes

$$\begin{cases} v_t - \Delta v = fe^{ct}, \text{ in } \mathbb{R}^n \times (0,\infty) \\ v = g, \text{ on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Solve it to get

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)e^{cs}dyds$$

Thus

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)e^{-ct}dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)e^{-c(t-s)}dyds$$

**Problem .** January 2013, Problem 4 **(a)**
Since $u(x,t) = \int_0^x u_x(y,t)dy + u(0,t) = \int_0^x u_x(y,t)dy$

$$|u(x,t)|^2 = \left|\int_0^x u_x(y,t)dy\right|^2 \le \left(\int_0^1 |u_x(y,t)|dy\right)^2 \le \left(\int_0^1 |u_x(y,t)|^2dy\right)\left(\int_0^1 1^2dy\right) = \int_0^1 |u_x(y,t)|^2dy$$

Hence $\sup_x |u(x,t)|^2 \le \int_0^1 |u_x(y,t)|^2dy$
Then we have

$$\begin{aligned} \int_0^1 u^3dx &\le \int_0^1 |u^3|dx \le \left(\int_0^1 |u_x|^2dx\right)\int_0^1 |u|dx \\ &\le \left(\int_0^1 |u_x|^2dx\right)\left(\int_0^1 |u|^2dx\right)^{\frac{1}{2}}\left(\int_0^1 1^2dy\right)^{\frac{1}{2}} \\ &= \left(\int_0^1 |u_x|^2dx\right)\left(\int_0^1 |u|^2dx\right)^{\frac{1}{2}} \end{aligned}$$

Since $u$ is smooth, we have

$$\begin{aligned}\frac{d}{dt}\int_0^1 |u|^2dx &= \int_0^1 \frac{d}{dt}u^2dx = \int_0^1 2uu_t dx = 2\int_0^1 u(u_{xx}+cu^2)dx \\ &= 2uu_{xx}|_0^1 - 2\int_0^1 |u_x|^2dx + 2c\int_0^1 u^3dx \\ &\le -2\int_0^1 |u_x|^2dx + 2c\left(\int_0^1 |u_x|^2dx\right)\left(\int_0^1 |u|^2dx\right)^{\frac{1}{2}} \\ &= -2\left(\int_0^1 |u_x|^2dx\right)\left(1-c\left(\int_0^1 |u|^2dx\right)^{\frac{1}{2}}\right)\end{aligned}$$

**(b)**

Since $u$ is smooth, so is $F(t) := \left(\int_0^1 |u|^2dx\right)^{\frac{1}{2}}$, suppose $\exists t_0 \in (0,\infty)$ such that $F(t_0) \ge \frac{1}{c^2}$, then $\varnothing \ne S := \left\{t \in [0,t_0] \mid F(t) = \frac{1}{c^2}\right\}$ is closed, let $0 < t_1 = \min_{t\in S} t$, then we have $F(t_1) = \frac{1}{c^2}$ and $F(t) < \frac{1}{c^2}$ for $t \in [0,t_1)$, using intermediate value theorem, we have $0 < F(t_1) - F(0) = F'(\xi)t_1$, where $\xi \in (0,t_1)$, but according to (a), we have $F'(\xi) \le -2\left(\int_0^1 |u_x|^2dx\right)\left(1-cF(\xi)^{\frac{1}{2}}\right) \le 0$ which leads to a contradiction

**Problem .** January 2012, Problem 2 **(a)**
This is equivalent to solve ODE $\begin{cases} w' + w^3 = 0,\ t > 0 \\ w(0) = c\end{cases}$, thus $w(t) = \dfrac{1}{\sqrt{2t+\frac{1}{c^2}}}$

**(b)**
Assume otherwise, then $\exists (t_0,x_0) \in (0,\infty)\times(0,1)$ such that $(w-u)(t_0,x_0) < 0$, suppose $w-u$ must attain the minimum of $[0,t_0]\times[0,1]$ at $(t_1,x_1)$, then $(w-u)(t_1,x_1) < 0$, thus $0 \le w(t_1.x_1) < u(t_1.x_1)$, hence $w^3(t_1.x_1) < u^3(t_1.x_1)$, then we would have $0 < u^3(t_1.x_1) - w^3(t_1.x_1) = (\frac{\partial}{\partial t} - \Delta)(w-u)(t_1,x_1) \ge 0$ which is a contradiction, hence $u(t,x) \le \dfrac{1}{\sqrt{2t+\frac{1}{c^2}}}$ on $[0,\infty)\times[0,1]$

**Problem .** August 2011, Problem 3 **(a)**
Assume $\exists (x_0,t_0) \in \Omega\times(0,\infty)$ such that $u(x_0,t_0) < 0$, suppose $u(x_1,t_1) = \min_{\bar{\Omega}_{t_0}} u < 0$, since $|f'| \le K$, $f(x) = \int_0^x f'dy \le \int_0^x K dy = Kx$ if $x \ge 0$, or $f(x) \ge Kx$ if $x < 0$
let $K_1 > K$, then $0 \ge \left(\frac{\partial}{\partial t} - \Delta\right)\left(ue^{-K_1t}\right) = e^{-K_1t}\left[\left(\frac{\partial}{\partial t}-\Delta\right)u - K_1u\right] = e^{-K_1t}\left[f(u) - K_1u\right] = e^{-K_1t}\int_u^0 \left[K_1 - f'\right]dx > 0$ at $(x_1,t_1)$ which is a contradiction
**(b)**
Assume $\exists (x_0,t_0) \in \Omega\times(0,\infty)$ such that $u(x_0,t_0) > Me^{Kt_0} \Rightarrow u(x_0,t_0)e^{-K_1t_0} > M$ for some $K_1 > K$, suppose $u(x_1,t_1)e^{-K_1t_1} = \max_{\bar{\Omega}_{t_0}} ue^{-K_1t} > 0$
$0 \le \left(\frac{\partial}{\partial t} - \Delta\right)\left(ue^{-K_1t}\right) = e^{-K_1t}\left[f(u) - K_1u\right] = e^{-K_1t}\int_0^u \left[f' - K_1\right]dx < 0$ at $(x_1,t_1)$ which is a contradiction

**Problem .** August 2005, Problem 6 Suppose $|f'| < K$, $u,w$ are both solutions, then $(u-w)e^{Kt} = 0$ on $\Omega\times\{0\}\cup\partial\Omega\times(0,\infty)$, Assume $\exists (x_0,t_0) \in \Omega\times(0,\infty)$ such that $(u-w)(x_0,t_0)e^{Kt_0} < 0$, suppose $(u-w)(x_1,t_1)e^{-Kt_1} = \min_{\bar{\Omega}_{t_0}}(u-w)e^{Kt} < 0$, the we have $0 \ge \left(\frac{\partial}{\partial t} - \Delta\right)\left((u-w)e^{Kt}\right) = e^{-Kt}\left[(f(w) - f(u)) + K(u-w)\right] = e^{-Kt}\int_u^w \left[f' + K\right]dx > 0$ at $(x_1,t_1)$, which is a contradiction

**Problem .** 2.5.15 Define $v(x,t) = u(x,t) - g(t)$ on $x \geq 0$, extend $v$ to $\{x < 0\}$ by odd reflection, then we have $v(x,t) = -v(-x,t)$ on $x < 0$, thus the initial boundary problem becomes

$$\begin{cases} v_t - v_{xx} = -g'(t), \text{ in } \mathbb{R}_+ \times (0,\infty) \\ v_t - v_{xx} = g'(t), \text{ in } \mathbb{R}_- \times (0,\infty) \\ v = 0, \text{ on } \mathbb{R}_+ \times \{t = 0\} \\ v = 0, \text{ on } \{x = 0\} \times [0,\infty) \end{cases}$$

Define $f(x,t) = \begin{cases} -g'(t), \ x \geq 0 \\ g'(t), \ x < 0 \end{cases}$, we have

$$\begin{aligned}
v(x,t) &= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4(t-s)}} f(y,s) dy ds \\
&= \int_0^t \frac{1}{\sqrt{4\pi(t-s)}} \left( \int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}} g'(s) dy - \int_0^\infty e^{-\frac{(x-y)^2}{4(t-s)}} g'(s) dy \right) ds \\
&= \frac{1}{\sqrt{\pi}} \int_0^t \left( \int_{-\infty}^{-\frac{x}{\sqrt{4(t-s)}}} e^{-y^2} dy - \int_{-\frac{x}{\sqrt{4(t-s)}}}^\infty e^{-y^2} dy \right) g'(s) ds \\
&= \frac{1}{\sqrt{\pi}} g(s) \left( \int_{-\infty}^{-\frac{x}{\sqrt{4(t-s)}}} e^{-y^2} dy - \int_{-\frac{x}{\sqrt{4(t-s)}}}^\infty e^{-y^2} dy \right) \Bigg|_0^t + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{((t-s))^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) ds \\
&= -\operatorname{sgn}(x) \frac{1}{\sqrt{\pi}} g(t) \int_{-\infty}^\infty e^{-y^2} dy - \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{((t-s))^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) ds \\
&= -\operatorname{sgn}(x) g(t) - \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{((t-s))^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) ds
\end{aligned}$$

Where $\operatorname{sgn}(x) = \begin{cases} -1, \ x < 0 \\ 0, \ x = 0 \\ 1, \ x > 0 \end{cases}$, thus $u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{((t-s))^{\frac{3}{2}}} e^{-\frac{x^2}{4(t-s)}} g(s) ds$ is the solution

**Problem .** 2.5.17 **a)**
Define $\phi(r) := \frac{1}{4r^n} \int\int_{E(0,0;r)} v(y,s) \frac{|y|^2}{s^2} dy ds$, then

$$\begin{aligned}
\phi'(r) &= \frac{1}{r^{n+1}} \int\int_{E(0,0;r)} -n v_s \psi - \frac{n}{2s} \sum_{i=1}^n v_{y_i} y_i dy ds \\
&\geq \frac{1}{r^{n+1}} \int\int_{E(0,0;r)} -n \Delta v \psi - \frac{n}{2s} \sum_{i=1}^n v_{y_i} y_i dy ds \\
&= 0
\end{aligned}$$

Since $\phi(r) = v(0,0) + \frac{1}{4r^n} \int\int_{E(0,0;r)} (v(y,s) - v(0,0)) \frac{|y|^2}{s^2} dy ds \to v(0,0)$, $r \to 0$, $v(0,0) = \lim_{s\to 0} \phi(r) \leq \lim_{s \to r} \phi(s) = \phi(r)$
Thus $v(x,t) \leq \frac{1}{4r^n} \int\int_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dy ds$
**b)**
Define $v_\epsilon(x,t) = v(x,t) + \epsilon|x|^2$, then $\left(\frac{\partial}{\partial t} - \Delta\right) v_\epsilon = \left(\frac{\partial}{\partial t} - \Delta\right) v - 2n\epsilon < 0$, $v_\epsilon$ can't attain its maximum at $(x_0,t_0) \in \overline{U_T} \backslash \Gamma_T$, otherwise $\frac{\partial}{\partial t} v_\epsilon(x_0,t_0) \geq 0, \Delta v_\epsilon(x_0,t_0) \leq 0$, then $\Delta v_\epsilon(x_0,t_0) \geq 0$,

thus $\max\limits_{\overline{U_T}} \left(v + \epsilon|x|^2\right) \leq \max\limits_{\Gamma_T} \left(v + \epsilon|x|^2\right) \leq \max\limits_{\Gamma_T} v + C\epsilon$ since $U$ is bounded, hence $\max\limits_{\overline{U_T}} v = \max\limits_{\Gamma_T} v$

**c)**

Since $\phi$ is convex, $\phi''(x) \geq 0$

$$\left(\frac{\partial}{\partial t} - \Delta\right) v(x) = \left(\frac{\partial}{\partial t} - \Delta\right) \phi\left(v(x,t)\right) = \phi'(v) \left(\frac{\partial v}{\partial t} - \Delta\right) v - \phi''(v)|\nabla v|^2 \leq 0$$

Thus $v$ is a subsolution

**d)**

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) v &= \left(\frac{\partial}{\partial t} - \Delta\right) \left(|\nabla u|^2 + u_t^2\right) \\
&= 2\nabla u \cdot \nabla u_t + 2u_t u_{tt} - 2\sum_{i,j} u_{x_i x_j}^2 - 2\nabla u \cdot \nabla(\Delta u) - 2|\nabla u_t|^2 - 2u_t(\Delta u)_t \\
&= 2\nabla u \cdot \nabla(u_t - \Delta u) + 2u_t(u_t - \Delta u)_t - 2\sum_{i,j} u_{x_i x_j}^2 - 2|\nabla u_t|^2 \\
&\leq 0
\end{aligned}$$

Thus $v$ is a subsolution

## 1.3 Homework3

**Problem .** 2.5.24 Since the solution is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}[g(x+t) - g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

and $g, h$ are compactly supported, $u(x,t)$ is also compactly supported for any fixed $t$, hence $k(t), p(t)$ make sense
**(a)**

$$\frac{d}{dt}(p(t)+k(t)) = \int_{-\infty}^{+\infty} u_t u_{tt} + u_x u_{xt} dx = \int_{-\infty}^{+\infty} u_t u_{xx} + u_x u_{xt} dx = \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}(u_t u_x)\, dx = u_t u_x \big|_{-\infty}^{+\infty} = 0$$

Thus $k(t) + p(t)$ is a constant in $t$
**(b)**

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}[g'(x+t) - g'(x-t)] + \frac{1}{2}[h(x+t) + h(x-t)] \\ u_x(x,t) &= \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t) - h(x-t)] \end{aligned}$$

Suppose $\operatorname{supp} g \cup \operatorname{supp} h \subset D(0,T)$, then consider $t > T$, one of $g'(x+t), g'(x-t)$ must be zero and one of $h(x+t), h(x-t)$ must be zero, thus $u_t^2 = u_x^2$, $p(t) = k(t)$
**Additional Problem**

$$\begin{aligned} |u(x,t)| &= \left| \frac{1}{4\pi t}\int_{\partial B(x,t)} f(y) dS_y \right| \\ &= \left| \frac{t}{4\pi}\int_{S^2} f(x+t\omega) dS_\omega \right| \\ &= \left| -\frac{t}{4\pi}\int_{S^2}\int_t^{-\infty} \frac{d}{d\lambda} f(x+\lambda\omega) d\lambda dS_\omega \right| \\ &= \left| -\frac{t}{4\pi}\int_{S^2}\int_t^{-\infty} (\nabla f \cdot \omega)(x+\lambda\omega) d\lambda dS_\omega \right| \\ &\le \frac{t}{4\pi}\int_{S^2}\int_t^{-\infty} |\nabla f|\,|\omega|(x+\lambda\omega) d\lambda dS_\omega \\ &= \frac{t}{4\pi}\int_{S^2}\int_t^{-\infty} |\nabla f|\,(x+\lambda\omega) d\lambda dS_\omega \\ &= \frac{t}{4\pi}\int_{\partial B(x,\lambda)}\int_t^{-\infty} |\nabla f|\,(y) d\lambda dS_y \\ &\le \frac{1}{4\pi t}\int_{\mathbb{R}^3} |\nabla f|\, dy \end{aligned}$$

**Problem .** January 2007, Problem 2 **(a)**
Define

$$E(t) = \frac{1}{2}\int_{\partial\Omega} u^2 dS + \frac{1}{2}\int_\Omega \left\{u_t^2 + |\nabla u|^2 + V(x)u^2\right\} dx$$

Then we have

$$\begin{aligned}
E'(t) &= \int_{\partial\Omega} u u_t dS + \int_{\Omega} \{u_t u_{tt} + \nabla u \cdot \nabla u_t + V(x) u u_t\}\, dx \\
&= \int_{\partial\Omega} u u_t dS + \int_{\Omega} \{u_t (\Delta u - V(x)u) + \nabla u \cdot \nabla u_t + V(x) u u_t\}\, dx \\
&= \int_{\partial\Omega} u u_t dS + \int_{\Omega} \{u_t \Delta u + \nabla u \cdot \nabla u_t\}\, dx \\
&= \int_{\partial\Omega} u u_t dS + \int_{\partial\Omega} u_t \frac{\partial u}{\partial n} dS \\
&= \int_{\partial\Omega} u_t \left(u + \frac{\partial u}{\partial n}\right) dS \\
&= 0
\end{aligned}$$

**(b)**

Uniqueness theorem: Suppose $u, v$ are two solutions to the equation, then $u = v$

Consider $w = u - v$ which satisfies equation $w_{tt} - \Delta w + V(x)w = 0$, $x \in \Omega, t > 0$, with initial conditions $w(x, 0) = 0, w_t(x, 0) = 0$ and boundary conditions $w + \dfrac{\partial w}{\partial n} = 0$

Since $E(0) = \dfrac{1}{2}\displaystyle\int_{\partial\Omega} w^2(x, 0) dS + \dfrac{1}{2}\displaystyle\int_{\Omega} \left\{w_t^2(x, 0) + |\nabla w|^2(x, 0) + V(x)w^2(x, 0)\right\} dx = 0$ and $E'(t) = 0$, hence $E(t) = 0$, since $V(x) \geq 0$, $u = 0$

**(c)**

As for $h \neq 0$, the result is the same as (b)

**Problem .** August 2003, Problem 4 Suppose there are two solutions $u, v$, define $w = u - v$, which would satisfies equation

$$w_{tt} - w_{xx} - w_{yyt} = 0,\ (x, y) \in \Omega,\ t > 0$$

$$w(x, y, 0) = 0,\ w_t(x, y, 0) = 0$$

$$w(0, y, t) = w(1, y, t) = 0,\ w_y(x, 0, t) = w_y(x, 1, t) = 0$$

Consider $E(t) = \dfrac{1}{2}\displaystyle\int_{\Omega} (w_t^2 + w_x^2) dx dy$, since

$$w(0, y, t) = w(1, y, t) = 0 \Rightarrow w_t(0, y, t) = w_t(1, y, t) = 0$$

$$w_y(x, 0, t) = w_y(x, 1, t) = 0 \Rightarrow w_{yt}(x, 0, t) = w_{yt}(x, 1, t) = 0$$

Thus we have

$$\begin{aligned}
E'(t) &= \int_{\Omega} (w_t w_{tt} + w_x w_{xt}) dx dy \\
&= \int_{\Omega} (w_t w_{xx} + w_t w_{yyt} + w_x w_{xt}) dx dy \\
&= \int_0^1 \int_0^1 (w_t w_x)_x dx dy + \int_0^1 \int_0^1 (w_t w_{yt})_y dy dx - \int_{\Omega} w_{yt}^2 dx dy \\
&\leq \int_0^1 [w_t w_x(1, y, t) - w_t w_x(0, y, t)] dy + \int_0^1 [w_t w_{yt}(x, 1, t) - w_t w_{yt}(x, 0, t)] dx \\
&= 0
\end{aligned}$$

Thus $u_x = 0$, but $u(0, y, t) = 0$, hence $u = 0$

**Problem .** August 2003, Problem 5 Since

$$\begin{aligned}
u(0,t)^2 &= \left(\frac{1}{4\pi t}\int_{\partial B(0,t)} g(y)dS_y\right)^2 \\
&\le \frac{1}{8\pi^2 t^2}\left(\int_{\partial B(0,t)} g(y)^2 dS_y\right)\left(\int_{\partial B(0,t)} 1^2 dS_y\right) \\
&= \frac{1}{4\pi}\int_{S^2} g(y)^2 dS_y
\end{aligned}$$

Thus

$$\int_0^\infty u(0,t)^2 dt \le \int_0^\infty \frac{1}{4\pi}\int_{\partial B(0,t)} g(y)^2 dS_y dt = \frac{1}{4\pi}\int_{\mathbb{R}^3} g(x)^2 dx$$

**Problem .** August 2006, Problem 3 **(a)**
Direct check to find $u_{tt} - \Delta u = 0$ when $r \neq 0$, if $t < 1$, then $\psi(t+r) = 0$ on a neighborhood of $r = 0$, thus $u(x,t) = 0$ on this neighborhood, hence $u_{tt} - \Delta u = 0$, if $t < 1$
**(b)**
The extended formula should be given by $u(x,t) = \dfrac{\psi(t+r) - \psi(t-r)}{r} = \displaystyle\int_{-1}^{1} \psi'(t + \lambda r) d\lambda$
**(c)**
Since $\psi \in C^k$ by the formula of $\psi$ we showed that $u \in C^{k-1}$
**(d)**
Since $\psi$ is compactly supported, so is $u$
Define Energy to be $E(t) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^3} \left(u_t^2 + |\nabla u|^2\right) dx$, then

$$\begin{aligned}
E'(t) &= \int_{\mathbb{R}^3} \left(u_t u_{tt} + \nabla u \cdot \nabla u_t\right) dx \\
&= \int_{\mathbb{R}^3} \left(u_t \Delta u + \nabla u \cdot \nabla u_t\right) dx \\
&= \int_{B(0,R)} \left(u_t \Delta u + \nabla u \cdot \nabla u_t\right) dx \\
&= \int_{\partial B(0,R)} u_t \frac{\partial u}{\partial n} dS \\
&= 0
\end{aligned}$$

Hence the energy is conserved

**Problem .** August 2001, Problem 1 **(a)**

$$u(x,t) = \frac{1}{4\pi t}\int_{\partial B(x,t)} g(y) dS_y = \frac{t}{4\pi}\int_{S^2} g(x + t\omega) dS_\omega$$

but $|x + t\omega| \ge t|\omega| - |x| = t - |x| \ge a$, $g(x + t\omega) = 0$, hence $u(x,t) = 0$
**(b)**
First we prove $g = 0$, assume $g(x_1) > 0$, then $u(x_0, |x_1 - x_0|) = \dfrac{|x_1 - x_0|}{4\pi}\displaystyle\int_{S^2} g(x_0 + |x_1 - x_0|\omega) dS_\omega > 0$, since $g \in C(\mathbb{R}^3), g \ge 0$, that is a contradiction, thus $g = 0$, $u(x,t) = 0$

## 1.4 Homework4

**Problem .** August 2006, Problem 1 Define $F(x,t,p,q,z) = q + xp - z^2$
The characteristics are

$$\begin{cases} \dot{x} = x \\ \dot{t} = 1 \\ \dot{z} = xp + q = z^2 \end{cases}$$

With initial condition

$$\begin{cases} x(0) = x^0 \\ t(0) = 0 \\ z(0) = f(x^0) \end{cases}$$

Thus we get

$$\begin{cases} x = x^0 e^s \\ t = s \\ z(0) = \dfrac{f(x^0)}{1 - sf(x^0)} \end{cases}$$

which is defined on $\{tf(xe^{-t}) \neq 1\}$, thus $u(x,t) = \dfrac{f(xe^{-t})}{1 - tf(xe^{-t})}$

**Problem .** January 2005, Problem 2(b) Define $F(x,t,p,q,z) = q + (2z+1)p$
The characteristics are

$$\begin{cases} \dot{x} = 2z + 1 \\ \dot{t} = 1 \\ \dot{z} = 0 \end{cases}$$

With initial condition

$$\begin{cases} x(0) = x^0 \\ t(0) = 0 \\ z(0) = u(x^0, 0) \end{cases}$$

Thus we get

$$\begin{cases} x = x^0 + \left(2u(x^0,0) + 1\right) s \\ t = s \\ z(0) = u(x^0, 0) \end{cases}$$

By Rankine-Hugoniot condition, we have $0 = F(u_l) - F(u_r) = \dot{s}(t)(u_l - u_r) = 3\dot{s}(t)$
Thus we have a piecewise smooth solution

$$u(x,t) = \begin{cases} 1, \ x < 0 \\ -2, \ x > 0 \end{cases}$$

**Problem .** January 2000, Problem 2 Consider the initial condition to be

$$g(x) = \begin{cases} 0, \ x < 0 \\ 1, \ 0 < x < 1 \\ 0, \ x > 1 \end{cases}$$

Then

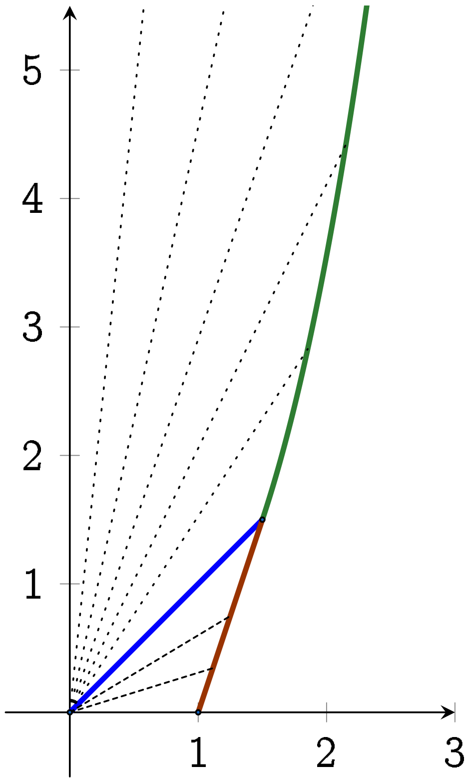
$$X(t) = \begin{cases} 1 + \frac{t}{3}, \ 0 < t < \frac{3}{2} \\ \sqrt[3]{\frac{9}{4}t}, \ t > \frac{3}{2} \end{cases} \qquad f\left(\frac{x}{t}\right) = \begin{cases} \sqrt{\frac{x}{t}}, \ 0 < x < X(t), \frac{x}{t} < 1 \\ 1, \ 0 < x < X(t), \frac{x}{t} > 1 \end{cases}$$

$\forall t \in \left[0, \frac{3}{2}\right]$, $\displaystyle\int_0^\infty u(x,t)dx = \int_0^t \sqrt{\frac{x}{t}}dx + \int_t^{1+\frac{t}{3}} 1dx = \frac{2}{3}t + \left(1 - \frac{2}{3}t\right) = 1$

$\forall t \in \left(\frac{3}{2}, \infty\right)$, $\displaystyle\int_0^\infty u(x,t)dx = \int_0^{\sqrt[3]{\frac{9}{4}t}} \sqrt{\frac{x}{t}}dx = 1$

Along $x = t, 0 \leq t \leq \frac{3}{2}$, $u_l = 1 = u_r$
Along $x = 1 + \frac{t}{3}, 0 \leq t \leq \frac{3}{2}$, $1 = u_l > \frac{1}{3} = \dot{X}(t) > 0 = u_r$

Along $x = \sqrt[3]{\frac{9}{4}t}, t > \frac{3}{2}$, $u_l = t^{-\frac{1}{3}}\sqrt[3]{\frac{3}{2}} > \frac{1}{3}t^{-\frac{2}{3}}\sqrt[3]{\frac{9}{4}} = \dot{X}(t) > 0 = u_r$
Which shows that $u$ satisfies the entropy condition, and along $x = t, 0 \le t \le \frac{3}{2}$, $u$ is a rarefaction wave, along $X(t)$, $u$ is a shock wave




**Problem .** August 1998, Problem 5 $0 = u_t + uu_x = u_t + \left(\frac{1}{2}u^2\right)_x$, since $u$ and $\xi(t)$ are both continuous, so is $u(\xi(t), t)$, there is a $C^1$ function $u_L$ in a neighborhood of $C_l$ which agrees with $u$ on $C_l$ and curve $C$, and there is a $C^1$ function $u_R$ in a neighborhood of $C_r$ which agrees with $u$ on $C_r$ and curve $C$, $\dfrac{d}{dt}u(\xi(t), t) = \dfrac{d}{dt}u_L(\xi(t), t)$ is continuous since $u_L$ and $\xi$ are both $C^1$ function, then we have $\dfrac{d}{dt}u_L(\xi(t), t) = \dfrac{d}{dt}u_R(\xi(t), t) \Rightarrow 0 = \dfrac{d}{dt}u_L(\xi(t), t) - \dfrac{d}{dt}u_R(\xi(t), t) = \left(u_x^- - u_x^+\right)\xi' + \left(u_t^- - u_t^+\right)$
On the other hand, since $u$ is continuous, $0 = u_t^- + u^- u_x^- = u_t^- + uu_x^- = u_t^+ + u^+ u_x^+ = u_t^+ + uu_x^+$
$\Rightarrow 0 = \left(u_t^- + uu_x^-\right) - \left(u_t^+ + uu_x^+\right) = \left(u_x^- - u_x^+\right)u + \left(u_t^- - u_t^+\right)$
But $u_x$ has jump discontinuity on the curve, hence $u_x^- - u_x^+ \neq 0$, compare to get $\xi'(t) = u$

## 1.5 Homework5

**Problem .** August 1996, Problem 1a Assume $u$ attains its maximum $\max_{\partial\Omega} u$ at $x^0 \in \partial\Omega$, since $f \geq 0$, $U$ is a domain which is bounded, maximum principle applies. Suppose $u(x) < u(x^0), \forall x \in \Omega$, then according to Hopf lemma, $\frac{\partial u}{\partial n}(x^0) > 0$ since $\Omega$ has a smooth boundary, which contradicts $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Thus $\exists y \in \Omega$, such that $u(y) = u(x^0)$, by strong maximum principle, $u$ is a constant and $f = \Delta u + a(x) \cdot \nabla u = 0$

**6.**

Since $f$ is bounded, assume $|f| < C$, consider $v := u - Cw$, we have $v(x^0) = 0$, $v \leq 0$ on $\partial U$ and $Lv = Lu - CLw \leq Lu - C < 0$, by maximum principle, we know $v(x) < v(x^0), \forall x \in U$, by Hopf lemma, $\frac{\partial u}{\partial \nu}(x^0) - C\frac{\partial w}{\partial \nu}(x^0) = \frac{\partial v}{\partial \nu}(x^0) > 0$, similarly, if we consider $-u - Cw$ and $-w$, we have $-\frac{\partial u}{\partial \nu}(x^0) - C\frac{\partial w}{\partial \nu}(x^0) > 0$ and $-\frac{\partial w}{\partial \nu}(x^0) > 0$. Also, since $u = 0$ on $\partial U$, $\left|Du(x^0)\right| = \left|\frac{\partial u}{\partial \nu}(x^0)\right|$, hence we have

$$\begin{cases} -C\frac{\partial w}{\partial \nu}(x^0) > -\frac{\partial u}{\partial \nu}(x^0) \\ -C\frac{\partial w}{\partial \nu}(x^0) > \frac{\partial u}{\partial \nu}(x^0) \end{cases} \Rightarrow -C\frac{\partial w}{\partial \nu}(x^0) > \left|\frac{\partial u}{\partial \nu}(x^0)\right| \Rightarrow C\left|\frac{\partial w}{\partial \nu}(x^0)\right| > \left|Du(x^0)\right|$$

**Problem .** 7. **(a)**

$$0 = \int_U u\Delta u = \frac{1}{2}\int_{\partial U} u\frac{\partial u}{\partial \nu} - \frac{1}{2}\int_U |\nabla u|^2 \Rightarrow \int_U |\nabla u|^2 = 0 \Rightarrow |\nabla u| \equiv 0 \Rightarrow u \equiv \text{const}$$

**(b)**

Using maximum principle, let $x^0 \in \partial U$ be a maximizer of $u$, suppose $u(x) < u(x^0), \forall x \in U$, by Hopf lemma, $\frac{\partial u}{\partial \nu}(x^0) > 0$ which contradicts $\frac{\partial u}{\partial \nu} = 0$ on $\partial U$, thus $\exists y \in U$, such that $u(y) = u(x^0)$, hence $u$ is a constant by strong maximum principle

# References

[1] *Partial Differential Equations* - Lawrence C. Evans