MATH868C - Several Complex Variables

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Contents

1 Review

Definition 1.1. C^1 function $f : \Omega \to \mathbb{C}$ is *holomorphic* if $\bar{\partial} f = 0$. Denote the set of all holomorphic functions on Ω as $A(\Omega)$

Lemma 1.2. If f is holomorphic, then $\displaystyle\int_{\partial\Omega} f dz = 0$

Proof.

$$\int_{\partial\Omega} f dz = \int_{\Omega} d(f dz) = \int_{\Omega} \bar{\partial} f \wedge dz = 0$$

□

Poincare-Lelong formula

Theorem 1.3 (Poincaré-Lelong formula)**.** Since $\Delta = \partial_x^2 + \partial_y^2 = 4\partial_z \partial_{\bar{z}} = 4\partial_{\bar{z}}\partial_z$, $dz \wedge d\bar{z} = -2i dx \wedge dy = -2i d\mu$. In the distributional sense, $-\dfrac{\log r}{2\pi} = -\dfrac{1}{4\pi}\log(x^2+y^2)$ is the fundamental solution of Laplacian equation in dimension 2, i.e. $\Delta \log(x^2 + y^2) = 4\pi\delta$, we have

$$\Delta \log |z|^2 dz \wedge d\bar{z} = 4\pi\delta dz \wedge d\bar{z} \Leftrightarrow \bar{\partial}\partial \log |z|^2 = 2\pi i \delta dx \wedge dy$$

Note. $\partial \log |z|^2 = \partial \log(z) + \partial \log(\bar{z}) = \dfrac{dz}{z}$ is integrable around 0

Proof. We prove a slightly general result. For any $\phi \in C_c^\infty(\Omega)$, by definition we have

$$\begin{aligned}
\iint_{\Omega} \phi \bar{\partial}\partial \log |z - w|^2 &= -\iint_{\Omega} \bar{\partial}\phi \wedge \partial \log |z - w|^2 \\
&= -\lim_{\epsilon \to 0} \iint_{|z-w| \geq \epsilon} \bar{\partial}\phi \wedge \partial \log |z - w|^2 \\
&= -\lim_{\epsilon \to 0} \iint_{|z-w| \geq \epsilon} d\left(\phi \partial \log |z - w|^2\right) \\
&= \lim_{\epsilon \to 0} \oint_{|z-w| = \epsilon} \phi \partial \log |z - w|^2 \\
&= \lim_{\epsilon \to 0} \oint_{|z-w| = \epsilon} \frac{\phi}{z - w} dz \\
&= 2\pi i \phi(w)
\end{aligned}$$

□

Cauchy's formula

Theorem 1.4 (Cauchy's formula)**.** If $f \in C^1(\overline{\Omega})$, then

$$f(w) = \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}} f dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z - w} dz$$

Proof. By Poincaré-Lelong formula 1.3, we have

$$\begin{aligned}
f(w) &= \frac{1}{2\pi i} \iint_{\Omega} f \bar{\partial}\partial \log |z - w|^2 \\
&= -\frac{1}{2\pi i} \iint_{\Omega} \bar{\partial} f \wedge \partial \log |z - w|^2 + \frac{1}{2\pi i} \int_{\partial\Omega} f \partial \log |z - w|^2 \\
&= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}} f dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f}{z - w} dz
\end{aligned}$$

□

Corollary 1.5. If $f \in C^1(\overline{\Omega}) \cap A(\Omega)$, then by Cauchy's formula 1.4, we know

$$f(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z - w} dz$$

Which is C^∞ in w

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{(z - w)^{n+1}} dz$$

Cauchy's estimate

Corollary 1.6 (Cauchy's estimate)**.** For $K \subseteq \Omega$ compact, there are constants C_n such that for any $f \in A(\Omega)$
$$\sup_{z \in K} |f^{(n)}(z)| \leq C_n \|f\|_{L^1(\Omega)}$$

Proof. Consider a bump function χ with $\operatorname{supp} \chi \subseteq \Omega$ and $\chi \equiv 1$ on K, then for any $w \in K$

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \iint_{\Omega} \frac{\partial_{\bar{z}}(\chi f) dz \wedge d\bar{z}}{z - w} + \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\chi f}{z - w} dz \\ &= \frac{1}{2\pi i} \iint_{\Omega} \frac{(\partial_{\bar{z}} \chi) f dz \wedge d\bar{z}}{z - w} \\ &= \frac{1}{2\pi i} \iint_{\Omega \setminus K} \frac{(\partial_{\bar{z}} \chi) f dz \wedge d\bar{z}}{z - w} \end{aligned}$$

$\dfrac{\partial_{\bar{z}} \chi}{z - w}$ can be bounded on $\Omega \setminus K$ ☐

Corollary 1.7. $A(\Omega) \subseteq C(\Omega)$ is closed, thus a Fréchet space

Proof. Suppose $\{f_j\} \subseteq A(\Omega)$ converges to f in $C(\Omega)$, but since

$$f_j(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f_j(z)}{z - w} dz$$

$f(w) = \dfrac{1}{2\pi i} \displaystyle\int_{\partial\Omega} \dfrac{f(z)}{z - w} dz$ which implies $\bar{\partial} f = 0$ ☐

Montel's theorem

Theorem 1.8 (Montel's theorem)**.** Suppose $\{f_i\} \subseteq A(\Omega)$ are uniformly bounded on each compact subset, then there is a subsequence f_{i_k} uniformly converges on compact subsets

Proof. For $K \subseteq \Omega$ compact, by Cauchy's estimate 1.6, f_j are Lipschitz with the same C_k, by Ascoli-Arzela theorem, f_j are equicontinuous, thus have convergent subsequence, and then use diagonal argument by exhaust Ω with compact subsets K ☐

Riemann extension theorem

Theorem 1.9 (Riemann extension theorem)**.** $E \subseteq \Omega$ is a discrete subset, $f \in A(\Omega \setminus E)$, and f is bounded around each point in E, then f can be extended to a unique $\tilde{f} \in A(\Omega)$

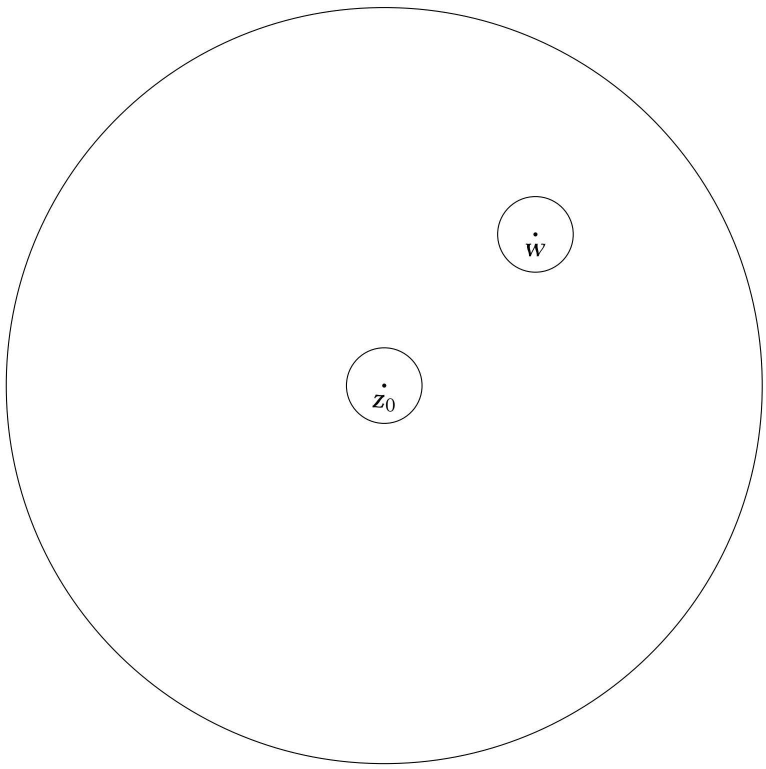
Proof. For $z_0 \in E$, suppose such $\tilde{f}$ exists, then by Cauchy's formula 1.4, for any $w \in D(z_0, r)$

$$\tilde{f}(w) = \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - w} dz$$

Thus we just take this as a definition, then

$$\begin{aligned} \tilde{f}(w) - f(w) &= \frac{1}{2\pi i} \int_{\partial D(z_0, r)} \frac{f(z)}{z - w} dz - \frac{1}{2\pi i} \int_{\partial D(w, \epsilon)} \frac{f(z)}{z - w} dz \\ &= \frac{1}{2\pi i} \int_{\partial D(z_0, \epsilon)} \frac{f(z)}{z - w} dz \end{aligned}$$

Which can be show to arbitrarily small as $\epsilon \to 0$



∎

d bar theorem

Theorem 1.10. If $\alpha = g(z)d\bar{z}$ is a smooth $(0,1)$-form on Ω, then there exists $u \in C^\infty(\Omega)$ such that $\bar{\partial}u = \alpha$

Proof. suppose such a u exists, then by Cauchy's formula 1.4

$$u(w) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{z-w} + \frac{1}{2\pi i}\int_{\partial\Omega}\frac{u(z)}{z-w}dz$$

Since $\bar{\partial}\int_{\partial\Omega}\frac{u(z)}{z-w}dz = 0$. This motivates us to first assume α has compact support, and define

$$u(w) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{z-w}$$

Then

$$u(w+\zeta) = \frac{1}{2\pi i}\iint_\Omega \frac{g(z)dz \wedge d\bar{z}}{(z-\zeta)-w} = \frac{1}{2\pi i}\iint_\Omega \frac{g(z+\zeta)dz \wedge d\bar{z}}{z-w}$$

Hence

$$\begin{aligned}\partial_{\bar{w}}u(w) &= \frac{1}{2\pi i}\iint_\Omega \frac{\partial_{\bar{z}}g(z)dz \wedge d\bar{z}}{z-w}\\ &= \frac{1}{2\pi i}\iint_\Omega \partial \log|z-w|^2 \wedge \bar{\partial}g\\ &= g(w)\end{aligned}$$

Therefore $\bar{\partial}u = \alpha$. In general, consider a compact exhaustion $\Omega = \bigcup_i K_i$, where $\hat{K}_i = K_i$, $K_i \subset\subset \mathring{K}_{i+1}$, ensured by Corollary 2.6, let χ_i be a cutoff function such that $\chi_i \equiv 1$ on K_i and $\operatorname{supp}\chi_i \subseteq \mathring{K}_{i+1}$, then there exists f_i such that $\bar{\partial}f_i = \chi_i\alpha$, by Runge's theorem 2.2, there exists $h_i \in \mathcal{O}(K_i)$ such that $\|f_{i+1} - f_i - h_i\|_{K_i} < \frac{1}{2^i}$. Now define

$$u_N = f_1 + \sum_{k=1}^{N}(f_{k+1} - f_k - h_k) = f_{N+1} - \sum_{k=1}^{N} h_N$$

Converges uniformly on compact subsets to u, and $\partial u_N = \alpha$ on K_i for any $i \leq N$ $\square$

2 Runge's theorem

Definition 2.1. $K \subseteq \Omega$ is compact, define

$$\mathcal{O}(K) = \{f|_K : f \text{ is holomorphic in a neighborhood of } K\}$$

Then for we have restriction map $\rho : \mathcal{O}(\Omega) \to \mathcal{O}(K)$, let $\|f\|_K = \max_{z\in K} |f(z)|$ to be the L^∞ norm

Runge's theorem

Theorem 2.2 (Runge's theorem)**.** The following are equivalent

1. The image of ρ is dense
2. No connected component of $\Omega \setminus K$ is relatively compact in Ω
3. If $\xi \in \Omega \setminus K$, then there exists $f \in \mathcal{O}(\Omega)$ such that $|f(\xi)| > \|f\|_K$

Definition 2.3. For $K \subseteq \Omega$ compact, the *holomorphic convex hull* of K relative to Ω is

$$\hat{K} = \hat{K}_\Omega = \{z \in \Omega : |f(z)| \le \|f\|_K, \forall f \in \mathcal{O}(\Omega)\}$$

Clearly $K \subseteq \hat{K}$

Proposition 2.4.

1. $\hat{K}$ is compact
2. $\|f\|_{\hat{K}} = \|f\|_K$ for all $f \in \mathcal{O}(\Omega)$
3. $\hat{\hat{K}} = \hat{K}$
4. If $\xi \in \Omega \setminus \hat{K}$, then there exists $f \in \mathcal{O}(\Omega)$ such that $|f(\xi)| > \|f\|_K$

Proof.

1. $\hat{K}$ is bounded by considering $f = z$. Suppose $z_i \in \hat{K}$ converges to ξ, if $\xi \in \Omega^c$, then $f = \dfrac{1}{z-\xi}$ will be unbounded on $\hat{K}$, thus $\xi \in \Omega$, but then for any $f \in \mathcal{O}(\Omega)$, $|f(\xi)| = \lim_{i\to\infty} |f(z_i)| \le \|f\|_K$, thus $\xi \in \hat{K}$
2. By definition, $\|f\|_{\hat{K}} \le \|f\|_K$, $\forall f \in \mathcal{O}(\Omega)$, and $\|f\|_K \le \|f\|_{\hat{K}}$, $\forall f \in \mathcal{O}(\Omega)$ is obvious
3. $\hat{\hat{K}} = \{z \in \Omega : |f(z)| \le \|f\|_{\hat{K}} = \|f\|_K, \forall f \in \mathcal{O}(\Omega)\} = \hat{K}$
4. By definition

□

Example 2.5. K is the unit circle. If Ω is the anulus $\left\{\frac{1}{2} < |z| < 2\right\}$, then $\hat{K} = K$. If Ω is the disc $\{|z| < 2\}$, then $\hat{K} = \{|z| < 1\}$ is the unit disc. Just consider $f = z$ and $f = \dfrac{1}{z}$

Compact exhaustion of a domain

Corollary 2.6. Any domain Ω has an exhaustion by compact sets $\hat{K_i} = K_i$ such that

$$K_i \subset\subset \overset{\circ}{K}_{i+1} \subset K_{i+1} \subset\subset \Omega$$

Vanishing theorem

Theorem 2.7. $\mathcal{U} = \{U_i\}$ is an open cover of Ω, then $H^1(\mathcal{U}, \mathcal{O}) = 0$

Proof. Let $\{\phi_i\}$ be a partion of unity. For any cocycle $\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{O})$, consider $h_i = \sum_j \phi_j g_{ij}$, then

$$\begin{aligned} h_i - h_j &= \sum_k \phi_k g_{ik} - \sum_k \phi_k g_{jk} \\ &= \sum_k \phi_k (g_{ik} - g_{jk}) \\ &= \sum_k \phi_k g_{ij} \\ &= g_{ij} \end{aligned}$$

Hence $\bar{\partial} h_i - \bar{\partial} h_j = 0$, $\{\bar{\partial} h_i\}$ define a well-defined smooth $(0,1)$ form. By Theorem 1.10, there exist a holomorphic fuction u such that $\bar{\partial} u = \bar{\partial} h_i$, define $f_i = h_i - u$, then $\bar{\partial} f_i = 0$, i.e. $\{f_i\}$'s are holomorphic, and $g_{ij} = f_i - f_j$. In other words, $\{g_{ij}\}$ is the image of $\{f_i\} \in C^1(\mathcal{U}, \mathcal{O})$ under the coboundary map □

Theorem 2.8 (Mittag-Leffler theorem)**.** $\Omega \subseteq \mathbb{C}$ is an open set, $E \subseteq \Omega$ is a discrete subset, then there exists a meromorphic function f with prescribed principal parts on E

Proof. There exists and open cover $\mathcal{U} = \{U_i\}$ and $f_i \in \mathcal{M}(U_i)$ with the prescribed principal parts round each point of E, then $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$ is a coycle, by Theorem 2.7, there exist holomorphic functions $\{g_i\}$ such that $f_i - f_j = g_i - g_j$ on $U_i \cap U_j$, then $f_i - g_i = f_j - g_j$ defines a global meromorphic function f such that $f - f_i = -g_i$ on U_i which is holomorphic □

Weierstrass theorem

Theorem 2.9 (Weierstrass theorem)**.** $E \subseteq \Omega$ is discrete, then

1. There is $f \in \mathcal{M}(\Omega)$ with arbitrary orders precisely at E
2. Any $f \in \mathcal{M}(\Omega)$ can be written as $f = g/h$ for $g, h \in \mathcal{O}(\Omega)$

Proof.

1. First take care of poles, and then multiply by $a_k(z - z_k)^{r_k}$ for each zero z_k, that converges
2.

□

Definition 2.10. Open subset $\Omega \subseteq \mathbb{C}^n$ is called a *domain of holomorphy* if for any $p \in \overline{\Omega} \setminus \Omega$, there is no holomorphic function g defined on an open set $U \ni p$ with $g = f$ on $U \cap \Omega$

Theorem 2.11. For any proper open subset $\Omega \subsetneq \mathbb{C}$ is a domain of holomorphic

Proof. Suppose $p \in \partial\Omega$, $p \in U$ is a neighborhood, $g \in \mathcal{O}(U)$ such that $f = g$ on $\Omega \cap U$, then there exists $\{\xi_n\}$ discrete and converging to p. By Weierstrass theorem 2.9, there exists $f \in \mathcal{O}(\Omega)$ having exactly $\{\xi_i\}$ as zeros, but then g has to be identically zero, so is f which is a contradiction □

3 Subharmonic functions

Definition 3.1. $\Omega \subseteq \mathbb{C}$ is a domain. $u : \Omega \to \mathbb{R} \cup \{-\infty\}$ is *upper semicontinuous* if for $y \in \mathbb{R}$ the set $\{u < y\}$ is open

Definition 3.2. An upper semicontinuous function u is *subharmonic* if is not identitcally $-\infty$, and for each $U \subset\subset \Omega$ and harmonic function h on $\overline{U}$ with $u \leq h$ on ∂U, we have $u \leq h$ for all $z \in U$

Example 3.3. If $u \in C^2(\Omega)$ and $\Delta u \geq 0$, then u is subharmonic

Theorem 3.4.

1. If $\{u_i\}$ are subharmonic and $u = \sup u_i$ is finite and upper semicontinuous, then u is subharmonic
2. If $u_i \geq u_{i+1}$ are subharmonic, then $u = \lim u_i$ is subharmonic

Proof.

1. By definition
2. $\{u < y\} = \bigcup\{u_i < y\}$ is open, hence u is upper semicontinuous. Suppose $u \leq h$ on ∂U for some $U \subset\subset \Omega$ and harmonic function h. For any $\epsilon > 0$, consider

$$F_i = \{x \in \partial U | u_i(x) \geq h(x) + \epsilon\}$$

are compact, thus $\bigcap F_i = \varnothing$ implies that a finite intersection is empty, hence $u \leq h + \epsilon$

□

Fact 3.5. If u is subharmonic on Ω, then $u \in L^1_{\text{loc}}(\Omega)$

Theorem 3.6. Subharmonic function u satisfies the sub-mean value property

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta \tag{3.1}$$

For almost all r sufficiently small

Proof. u is integrable on circle of radius r about z for sufficiently small r, we can find continuous functions $h_n \geq u_n$ on the circle such that $h_n \to u$ in L^1, extend h_n to harmonic functions, then

$$u(z) \leq h_n(z) = \frac{1}{2\pi} \int_0^{2\pi} h_n(z + re^{i\theta}) d\theta \to \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta$$

□

Proposition 3.7. Subharminoc functions satisfies $\Delta u \geq 0$ in the weak sense

$$\int_\Omega u \Delta \phi \geq 0, \forall \phi \in C_c^\infty(\Omega), \phi \geq 0$$

Proof. Multiply ϕ on both sides of (3.1) and integrate over Ω we get

$$\begin{aligned}
\int_\Omega 2\pi u(z)\phi(z) d\mu &\leq \int_\Omega \phi(z) \int_0^{2\pi} u(z + re^{i\theta}) d\theta d\mu \\
&= \int_\Omega u(z) \int_0^{2\pi} \phi(z - re^{i\theta}) d\theta d\mu
\end{aligned}$$

Then we get

$$\begin{aligned}
0 &\leq \int_\Omega u(z) \int_0^{2\pi} \phi(z - re^{i\theta}) - \phi(z) d\theta d\mu \\
&= \int_\Omega u(z) \int_0^{2\pi} -\partial_z \phi(z) re^{i\theta} - \partial_{\bar{z}} \phi(z) re^{-i\theta} + \partial_z^2 \phi(z) r^2 e^{2i\theta} + \partial_{\bar{z}}^2 \phi(z) r^2 e^{-2i\theta} + 2\partial_z \partial_{\bar{z}} \phi(z) r^2 + O(r^3) d\theta d\mu \\
&= \int_\Omega u(z) \int_0^{2\pi} \frac{1}{2} \Delta \phi(z) r^2 + O(r^3) d\theta d\mu
\end{aligned}$$

Divide $\frac{r^2}{2}$ and let $r \to 0$ □

Proposition 3.8. Subharmonicity is a local property, i.e. suppose u is upper semicontinuous on Ω, and locally subharmonic, then u is subharmonic on Ω

Proof. Suppose h is harmonic, $U \subset\subset \Omega$, $u \leq h$ on $\partial\Omega$, consider $v = u - h$, assume $\sup_U v = M > 0$, then by the upper semicontinuity, we know that $F = \{v = M\}$ is compact in U, there exists $z_0 \in \partial F$ obtains the least distance from ∂U, then for any small $r > 0$, F will miss an arc of positive measure if $\partial B(z_0, r)$, hence

$$\frac{1}{2\pi} v(z_0 + re^{i\theta}) d\theta < M$$

But this contradicts sub-mean value property □

Example 3.9. If $f_1, \cdots, f_k \in \mathcal{O}(\Omega)$, not all zero, then $u = \log(|f_1|^2 + \cdots + |f_k|^2)$ is subharmonic since $\log|f|$ is harmonic and $\Delta u \geq 0$

4 Almost complex structure

Definition 4.1. V is a real vector space, an *almost complex structure* is an endomorphism $J : V \to V$ such that $J^2 = -I$. Let $V^{1,0} \oplus V^{0,1} = V_{\mathbb{C}}$ be the $\pm i$ eigenspaces of J

Proposition 4.2. We can find basis such that $V \cong \mathbb{R}^{2n}$ such that $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. For local coordinate (x_i, y_i) of a complex manifold, $\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}$ is such a basis, $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ are the $\pm i$ eigenvectors. This motivates the definition of a real isomorphism $\rho : V \to V^{1,0}$, $v \mapsto \frac{1}{2}(v - iJv)$, then $\rho J = i\rho$. Suppose V, W both have almost complex structures, given an $\mathbb{R}$-linear map $T : V \to W$, let $\tilde{T} : V^{1,0} \to W^{1,0}$ be given by the commutative diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho"'] & W \arrow[d, "\rho"] \\ V^{1,0} \arrow[r, "\tilde{T}"] & W^{1,0} \end{tikzcd}$$

$\tilde{T}$ is complex linear if $TJ = JT \iff \tilde{T}i = i\tilde{T}$. Alternatively, extend T to a map $V_{\mathbb{C}} \to W_{\mathbb{C}}$, and this conditions is exactly that this extension preserves $(1,0)$ and $(0,1)$ subspaces

Lemma 4.3 (Osgood's lemma)**.** If $f : \Omega \to \mathbb{C}$ is continuous and holomorphic in each variable, then it is analytic

Proof. Iterate Cauchy's formula and use Fubini's theorem to write

$$f(z) = \left(\frac{1}{2\pi i} \right)^n \int_{w_i \in \Delta(z_i, r_i)} \frac{f(w)dw}{(w_1 - z_1) \cdots (w_n - z_n)}$$

Then

$$\frac{1}{(w_1 - z_1) \cdots (w_n - z_n)} = \sum_I \frac{(z - \xi)^I}{(w - \xi)^I}$$

Then a convergent power series expression follows, with

$$c_I \left(\frac{1}{2\pi i} \right)^n \int_{w \in \Delta(z,r)} \frac{f(w)dw}{(w_1 - z_1)^{i_1+1} \cdots (w_n - z_n)^{i_n+1}}$$

The *total order* of an analytic function f at ξ is the smallest value of $|I|$ for which $c_I \neq 0$ □

Definition 4.4. A set $E \subseteq \Omega$ is called *thin* if for every $\xi \in E$ there is a polydisk $\Delta(\xi, r) \subset\subset \Omega$ and $g \in A(\Delta(\xi, r))$ such that $E \cap \Delta(\xi, r) \subseteq Z(g)$. Note that for $n = 1$, this is equivalent to discrete

Theorem 4.5 (Riemann extension theorem)**.** If $f \in A(\Omega \setminus E)$ where E is a thin set, and f is locally bounded on Ω, then there exists $\tilde{f} \in A(\Omega)$ such that $\tilde{f} = f$ on the complement of E

Proof. Let k be the total order of g at ξ. By an application of Rouché's theorem (and after modifying r and a change of variables), we can assume that for each $z_1, \cdots, z_{n-1}$ the function $z_n \mapsto g(z_1, \cdots, z_{n-1}, z_n)$ has exactly k zeros and none on the boundary □

In higher dimensions, to solve $\bar{\partial}$ equation, there must be a *integrability condition*. Indeed, if we can solve the equation, then $0 = \bar{\partial}^2 u = \bar{\partial}\alpha$, i.e. we require α to be $\bar{\partial}$ closed

Proposition 4.6. Let $n \geq 2$. If α is a smooth compactly supported $(0,1)$ form on $\mathbb{C}^n$ with $\bar{\partial}\alpha = 0$, then there is a $u \in C_c^\infty$, with $\bar{\partial}u = \alpha$

Proof.

□

Corollary 4.7 (Hartogs theorem)**.** Let $K \subseteq \Omega$ be compact with $\Omega \setminus K$ connected. If $f \in A(\Omega \setminus K)$, there exists $\tilde{f} \in A(\Omega)$ that is equal to f on the complement of K

Proof. Let $\phi \in C_c^\infty(\Omega)$ be $\equiv 1$ in a neighborhood of K, let $\alpha = \bar\partial((1-\phi)f)$. Then α is $\bar\partial$-closed and compactly supported. Hence, there is $u \in C_c^\infty(\mathbb{C}^n)$ with $\bar\partial u = \alpha$. Then let $\tilde f = (1-\phi)f - u$, $\tilde f \in A(\Omega)$, since u is compactly supported, $\tilde f = f$ on $\Omega \setminus K$ ☐

Note. The assumption that $\Omega \setminus K$ is connected is necessary. For example, let $K \subseteq B(0,1) = \{|z| < 1\}$ be the set where $|z| = \frac{1}{2}$, and take

$$f(z) = \begin{cases} z_n & \text{if } 1/2 < |z| < 1 \\ 0 & \text{if } |z| < 1/2 \end{cases}$$

Then there is no holomorphic extension to $B(0,1)$

Proposition 4.8. If α is a smooth $\bar\partial$-closed $(0,1)$ form on a polydisk $\Delta = \Delta(0,r)$, then $\alpha = \bar\partial u$ for some $u \in C^\infty(\Delta)$

Proof. Just like in the one variable case, exhaust Δ by nested closed polydiscs K_i. Use cut-off functions to find u_i, $\bar\partial u_i$ in a neighborhood of K_i. Then $u_{i+1} - u_i$ is holomorphic in a neighborhood of K_i. Now by the power series expansion, there is a polynomial p_i such that $\|u_{i+1} - u_i - p_i\|_{K_i} < 2^{-i}$. The rest follows as in the proof of the one variable case ☐

Note. We heavily used the geometric properties of the polydisc

Corollary 4.9 (Cousin theorem)**.** $\mathcal{U} = \{u_i\}$ is an open cover of polydisc Δ, then $H^1(\Delta, \mathcal{U}) = 0$

Theorem 4.10. If $\alpha \in C^\infty_{(p,q)}(\Delta)$, $q \geq 1$, $\bar\partial\alpha = 0$. Then $\alpha = \bar\partial u$ for some $u \in C^\infty_{(p,q-1)}(\Delta)$

Remark 4.11. This states that the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(\Delta) = 0$

Proof. Induct on $k = 1, \cdots, n$, the smallest integer such that α only involves $d\bar z_1, \cdots, d\bar z_k$. If $k = 1$, then $q = 1$ and we have already proven the result. Suppose the result is true for $k-1$. Write $\alpha = \omega \wedge d\bar z_k + \beta$, where ω and β only involve $d\bar z_1, \cdots, d\bar z_{k-1}$. We have $0 = \bar\partial\alpha = \bar\partial\omega \wedge d\bar z_k + \bar\partial\beta$. This implies both ω, β are holomorphic in the variables $z_{k+1}, \cdots, z_n$. Apply the one variable solution to find μ, $\bar\partial\mu = \omega \wedge d\bar z_k + \sigma$, here σ only involves $d\bar z_1, \cdots, d\bar z_{k-1}$. Now $\alpha - \bar\partial u = \beta - \sigma$ is $\bar\partial$-closed. By induction, we can write $\beta - \sigma = \bar\partial v$, and so we set $u = v + \mu$ ☐

Example 4.12. Let $\Omega \subseteq \mathbb{C}^2$ be a domain. For $\xi \in \Omega$, let $\Omega^* = \Omega \setminus \{\xi\}$. Then $H^{0,1}_{\bar\partial}(\Omega^*) \neq \{0\}$

Proof. Without loss of generality assume $\xi = (0,0)$. Consider the $(0,1)$-form

$$\omega = \frac{1}{r^4}(-\bar z_2 d\bar z_1 + \bar z_1 d\bar z_2) = \bar\partial\left(\frac{\bar z_2}{z_1 r^2}\right)$$

Clearly, ω is smooth on Ω^*, and $\bar\partial\omega = 0$. Suppose $\omega = \bar\partial u$ for $u \in C^\infty(\Omega^*)$. Then $f(z_1, z_2) = z_1 u - \dfrac{\bar z_2}{r^2}$ is holomorphic on $\Omega^* \setminus \{z_1 = 0\}$, and it is locally bounded on Ω^*. By Riemann extension, it is holomorphic on Ω^*. By Hartogs, it extends to Ω. But for $z_2 \neq 0$ we clearly have $f(0, z_2) = -\dfrac{1}{z_2}$, contradiction ☐

Proposition 4.13. $K \subseteq \Omega$ is compact

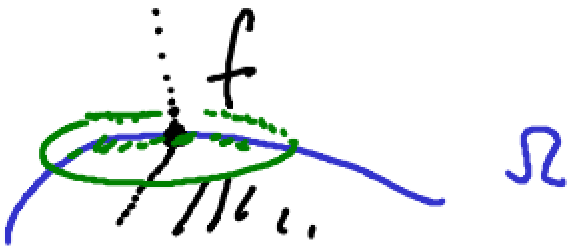
1. $\hat K_\Omega$ is closed in Ω
2. $\hat K_\Omega$ is not necessarily closed in $\mathbb{C}^n$. E.g. if $n \geq 2$, let $\Omega = \mathbb{B}^n \setminus \{0\}$, $K = \{|z| = 1/2\}$. Then by Hartogs' theorem, $\hat K_\Omega = \mathbb{B}^n_{1/2} \setminus \{0\}$
3. $\hat K_\Omega \subseteq \mathcal{C}(K)$, the closed convex hull of K. In particular, $\hat K_\Omega$ is bounded

Proof. Let $w \notin \mathcal{C}(K)$, $z_0 \in \mathcal{C}(K)$ minimizes distance to w, let $\xi \in (\mathbb{C}^n)^*$ define a supporting hyperplane for $\mathcal{C}(K)$ so that $\mathcal{C}(K) \subseteq \operatorname{Re}\langle \xi, z\rangle \leq 0$ and $\operatorname{Re}\langle \xi, w\rangle \geq 0$. Let $f(z) = \exp\langle \xi, z\rangle$, $|f(z)| = \exp \operatorname{Re}\langle \xi, z\rangle$ which violates the definition, so $w \notin \hat K_\Omega$ ☐

Definition 4.14. A domain $\Omega \subseteq \mathbb{C}^n$ is *holomorphically convex* if for every compact $K \subseteq \Omega$, $\hat{K}_\Omega$ is compact. If Ω is convex, then it is holomorphically convex. If $n = 1$, all domains are holomorphically convex. The previous counter-example shows this is not true if $n \geq 2$

Proposition 4.15. $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex $\iff$ every discrete, infinite set $\{z_j\} \subseteq \Omega$ there is $f \in A(\Omega)$ with $|f(z_j)|$ unbounded

Proof. $\Leftarrow$: If $\hat{K}_\Omega$ is not compact there is a discrete infinite subset $\{z_j\} \subseteq \hat{K}$. But then $|f(z_j)| \leq \|f\|_K$, $\forall j, f \in A(\Omega)$. This contradicts the existence of $f \in A(\Omega)$ where $|f(z_j)|$ is unbounded



$\Rightarrow$: Exhaust Ω by nested compact sets K_j, $\hat{K}_j = K_j$. We may assume $z_j \in K_{j+1} \setminus K_j$. We can find $f_j \in A(\Omega)$ such that $f_j(z_j) = 1$, $\|f_j\|_{K_j} < 1$, by taking power, $\|f_j\|_{K_j}$ can actually be arbitrarily small. Let $g_j \in A(\Omega)$ be such that $g_j(z_j) = 1$, $g_j(z_j) = 0$ for $i < j$. Now define λ_j by

$$\lambda_j = j - \sum_{i=1}^{j-1} \lambda_i g_i f_i(z_j)$$

Assume $\|\lambda_j g_j f_j\|_{K_j} < 2^{-j}$. Now let $f(z) = \sum_{i=1}^{\infty} \lambda_i g_i f_i(z)$. This converges uniformly on compact sets, and so $f \in A(\Omega)$. Finally

$$f(z_j) = \sum_{i=1}^{j} \lambda_i g_i f_i(z_j) = \lambda_j g_j f_j(z_j) + \sum_{i=1}^{j-1} \lambda_i g_i f_i(z_j) = j$$

□

Definition 4.16. $\Omega \subseteq \mathbb{C}^n$ is called a *domain of holomorphy* if there is $f \in A(\Omega)$ such that for any $p \in \overline{\Omega} \setminus \Omega$ and any Ω' about p, there is no $g \in A(\Omega')$ such that $g = f$ on $\Omega' \cap \Omega$

Theorem 4.17. $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex $\iff$ it is a domain of holomorphy

Corollary 4.18. A convex domain in $\mathbb{C}^n$ is a domain of holomorphy

Proof. $\Rightarrow$ is similar to the one variable case. $\Leftarrow$ is a theorem of Oka (this will be generalized)
$\Rightarrow$: Fix a polydisc Δ about the origin. For $\xi \in \Omega$, let $\Delta_\xi = \xi + r\Delta$, where r is the supremum such that $\xi + r\Delta \subseteq \Omega$. Let $E \subseteq \Omega$ be countable dense. Let $\{\xi_j\}$ be a sequence containing every point of E infinitely may times. Write $\Omega = \bigcup K_j$. Since $\hat{K}_j \subset\subset \Omega$, $\exists z_j \in \Delta_{\xi_j}$ with $z_j \notin \hat{K}_j$. Choose $f_j \in A(\Omega)$, $f_j(z_j) = 1$, $\|f_j\|_{K_j} < 2^{-j}$. Set $f(z) = \prod (1 - f_j)^j$. Then f converges uniformly on compact sets, so $f \in A(\Omega)$. Now f has zeros of order $\geq j$ at z_j. Any continuation of f would have a zero of infinite order
$\Rightarrow$: Let $d(z) = \sup_{\Delta(z,r) \subseteq \Omega} r$, $d(K) = \inf_{z \in K} d(z)$. Claim $d(\hat{K}) = d(K) > 0$. This will imply $\hat{K} \subset\subset \Omega$. Let $f \in A(\Omega)$ so that the radius of convergence at z is $d(z)$, let $\delta < d(K)$, $K_\delta = \bigcup_{w \in K} \overline{\Delta(w, \delta)}$. By Cauchy estimates: $\|D^I f\|_K \leq \frac{I!}{\delta^{|I|}} \|f\|_{K_\delta}$. But $D^I f \in A(\Omega)$, so for $z \in \hat{K}$, $|D^I f(z)| \leq \|D^I f\|_K \leq \frac{I!}{\delta^{|I|}} \|f\|_{K_\delta}$. This implies that the radius of convergence at $z \in \hat{K}$ is at least δ, i.e. $d(z) \geq \delta$, and so $d(\hat{K}) \geq d(K)$. Since $K \subseteq \hat{K}$, the other inequality is trivial □

Proposition 4.19. If $\{\Omega_\alpha\}_{\alpha\in I}$ are domains of holomorphy in $\mathbb{C}^n$, then the interior Ω of $\bigcap_{\alpha\in I}\Omega_\alpha$ is also a domain of holomorphy

Proof. $K \subseteq \Omega$ is compact. For each $\alpha \in I$, $K \subseteq \Omega \subseteq \Omega_\alpha$, which implies $\hat{K}_\Omega \subseteq \hat{K}_{\Omega_\alpha}$. This implies $d_{\Omega_\alpha}(\hat{K}_{\Omega_\alpha}) \leq d_{\Omega_\alpha}(\hat{K}_\Omega)$, for all α. Since Ω_α is holomorphically convex, $d_{\Omega_\alpha}(\hat{K}_{\Omega_\alpha}) = d_{\Omega_\alpha}(K)$. Hence $d_\Omega(K) \leq d_{\Omega_\alpha}(K) \leq d_{\Omega_\alpha}(\hat{K}_\Omega)$. Finally, this implies $d_\Omega(K) \leq d_\Omega(\hat{K}_\Omega)$. As before, we conclude that $d_\Omega(K) = d(\hat{K}_\Omega)$, and so $\hat{K}_\Omega$ is compact, so Ω is holomorphically convex ☐

Claim. Suppose Ω is a domain of holomorphy. Let $f_1, \cdots, f_N \in A(\Omega)$, and define

$$\Omega_c = \{z \in \Omega \| f_j(z)| < c, j = 1, \cdots, N\}$$

Then Ω_c is also a domain of holomorphy

Proof. Let $K \subseteq \Omega_c$. Let $z \in \hat{K}_\Omega$. Then in particular, for any $j = 1, \cdots, N$, $|f_j(z)| \leq \|f_j\|_K < c$. So $z \in \Omega_c$. Now $\hat{K}_{\Omega_c} \subseteq \hat{K}_\Omega \subseteq \Omega$ and so $\hat{K}_{\Omega_c}$ is compact ☐

Claim. Let $u : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$ be holomorphic, with Ω a domain of holomorphy. If $\Omega' \subseteq \mathbb{C}^m$ is a domain of holomorphy, then so is $\tilde{\Omega} = u^{-1}(\Omega')$

Proof. Let $K \subseteq \tilde{\Omega} \subseteq \Omega$ be compact. Since $\hat{K}_{\tilde{\Omega}} \subseteq \hat{K}_\Omega \subseteq \Omega$, it suffices to show $\hat{K}_{\tilde{\Omega}}$ is closed in Ω. Let $z_j \to z \in \Omega$, $z_j \hat{K}_{\tilde{\Omega}}$. Notice that $u(\hat{K}_{\tilde{\Omega}}) \subseteq \widehat{u(K)}_{\Omega'}$. Hence $u(z) \in \Omega'$, and so $z \in \tilde{\Omega}$ ☐

Lemma 4.20. Let $\Omega \subseteq \mathbb{C}^n$ be a domain of holomorphy, and $K \subseteq \Omega$. Suppose $f \in A(\Omega)$ is such that $|f(z)| \leq d(z)$ for all $z \in K$, then $|f(\xi)| \leq d(\xi)$ for all $\xi \in \hat{K}_\Omega$

Proof. We first claim that if $u \in A(\Omega)$, then the power series expansion of u at $\xi \in \hat{K}_\Omega$ converges on $\Delta(\xi, |f(\xi)|)$. This will prove the Lemma, because we can take u to be teh function with no analytic continuation beyond Ω
Proof of the claim: Let $0 < \delta < 1$, as before, the Cauchy estimates provide for some constant M that

$$|D^I u(z)| \frac{(\delta |f(z)|)^{|I|}}{I!} \leq M, \forall z \in K$$

Now $D^I u(z) f(z)^{|I|} \in A(\Omega)$, so the same estimate holds on $\hat{K}_\Omega$. This means the radius of convergence at $\xi \in \hat{K}_\Omega$ is a t least $\delta |f(\xi)|$. Since δ was arbitrary, this proves the claim ☐

Fundamental consequence: Let $D \subset\subset \Omega$ be a 1-dimensional disc

1. Suppose f is a polynomial in one variable such that $-\log d(z) \leq \operatorname{Re} f(z)$, for $z \in \partial D$
2. Let f be the restriction of $F \in A(\Omega)$. Then $|e^{-F(z)}| \leq d(z)$, $z \in \partial D$
3. By the maximum principle, $D \subseteq \widehat{\partial D}_\Omega$
4. From the Lemma, we have $|e^{-F(z)}| \leq d(z)$, $z \in \partial D$
5. This in turn implies $-\log d(z) \leq \operatorname{Re} f$ on D

Approximating harmonic functions by polynomials, we conclude that $u = -\log d$ is subharmonic on any complex line in Ω

5 Hartogs theorem

Theorem 5.1.

6 Pseudoconvexity

Definition 6.1. An upper semicontinuous function $\phi : \Omega \subseteq \mathbb{C}^n \to [-\infty, \infty)$ is *plurisubharmonic* if the restriction of ϕ to every complex line $L \cap \Omega$, $L \cong \mathbb{C}$, is subharmonic. Let $P(\Omega)$ be the set of plurisubharmonic (psh) functions on Ω

Proposition 6.2. $\phi \in C^2(\Omega)$ is psh $\iff$ for all $\xi \in \mathbb{C}^n$ and all $z \in \Omega$, the complex Hessian is positive semidefinite

$$\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\xi}_j \geq 0$$

ϕ is strictly psh if $>$ holds for every $\xi \neq 0$

Remark 6.3. A real $(1, 1)$ form on Ω can be written as

$$\omega(z) = i \sum_{i,j=1}^{n} g_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j$$

where $g_{i\bar{j}}$ is a Hermitian matrix. We say that $\omega \geq 0$(resp. $\omega > 0$) if $(g_{i\bar{j}}(z))$ is positive semidefinite(resp. positive definite) for every $z \in \Omega$. This means that for each $\xi \in \mathbb{C}^n$, $\xi \neq 0$

$$\sum_{i,j=1}^{n} g_{i\bar{j}}(z)\xi_i \bar{\xi}_j \geq 0 (\text{resp. } \omega > 0)$$

In the case $\omega > 0$, $g_{i\bar{j}}$ defines a Hermitian metric on Ω, and ω is its associate Kähler form

Proof. A line $\jmath : L \hookrightarrow \mathbb{C}^n$ is given by a choice $\xi \neq 0$ in $\mathbb{C}^n$, so that $\jmath(\tau) = z_0 + \tau\xi$, then

$$\jmath^*(dz_i) = \xi_i d\tau, \jmath^*(d\bar{z}_i) = \bar{\xi}_i d\bar{\tau}$$

$$\begin{aligned} \jmath^*(i\partial\bar{\partial}\phi) &= \left(\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\xi}_j \right) i d\tau \wedge d\bar{\tau} \\ &= \left(\sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z)\xi_i \bar{\xi}_j \right) 2d\mu \end{aligned}$$

On the other hand

$$\jmath^*(i\partial\bar{\partial}\phi) = i\partial_z \bar{\partial}_z (\phi \circ \jmath) = \Delta(\phi \circ \jmath) 2d\mu$$

□

Definition 6.4. A domain $\Omega \subseteq \mathbb{C}^n$ is *pseudoconvex* if there exists a continuous psh exhaustion function ϕ, i.e.

$$\Omega_c = \{z \in \Omega | \phi(z) < c\} \subset\subset \Omega$$

For every $c \in \mathbb{R}$

Fact 6.5 (Richberg)**.** If Ω is pseudoconvex, there is a C^∞ strictly psh exhaustion function on Ω (see Demailly's book)

Theorem 6.6. $\Omega \subseteq \Omega$ is a domain of holomorphy iff it is pseudoconvex

Proof. Recall $d(z) = \sup_{\Delta(z,r) \subseteq \Omega} r$. $\Rightarrow$: We have shown that $-\log d(z)$ is psh. It is also continuous. We claim that $u(z) = |z|^2 - \log d(z)$ does the job Closedness: If $z_i \to w \in \overline{\Omega} \setminus \Omega$, then $d(z_i) \to 0$, so u diverges Boundedness: Fix any $w \in \overline{\Omega} \setminus \Omega$, then

$$d(z) \leq |z - w| \leq |z| + |w|$$

so for $|z|$ large

$$\log d(z) \leq 2 \log |z| \leq \frac{1}{2}|z|$$

This means a bound on u implies a bound on $|z|$ □

Example 6.7. 1. Geometrically convex sets are pseudoconvex(e.g. balls and polydisks)

2. If $\{\Omega_\alpha\}$ are pseudoconvex, then the interior Ω of $\bigcap \Omega_\alpha$ is pseudoconvex
3. Annuli or punctured domains are not pseudoconvex
4. Let $\Omega \subseteq \mathbb{C}^n$ be pseudoconvex, $f_1, \cdots, f_k \in A(\Omega)$, then $\tilde{\Omega} = \Omega \setminus V(f_1) \cup \cdots \cup V(f_k)$ is pseudoconvex. Indeed, if ϕ is the psh exhaustion function on Ω, take $\tilde{\phi} = \phi - \log|f_1| - \cdots - \log|f_k|$ on $\tilde{\Omega}$

Proposition 6.8. Suppose $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex. Then $-\log d(z)$ is psh

Proof. $D \subset\subset \Omega$ is a disc, f on D, $F \in A(\Omega)$ restricts to f, suppose $-\log d(z) \leq \operatorname{Re} f(z)$, $z \in \partial D$, or equivalently $d(z) \geq |e^{-f(z)}|$, $z \in \partial D$. We want to show this holds in D. Fix $w \in \Delta(0,1)$. Let

$$K = \{z + \lambda w e^{-f(z)} | z \in \partial D, 0 \leq \lambda \leq 1\}$$

Then $K \subseteq \Omega$

$$\Lambda = \{\lambda \in [0,1] | z + \lambda' w e^{-f(z)} \in \Omega, \forall z \in D, 0 \leq \lambda' \leq \lambda\}$$

Notice that $\Lambda \neq \varnothing$, since $0 \in \Lambda$. We want show that $\Lambda = [0,1]$. Λ is clearly open. Suppose $\lambda_i \nearrow c$, $\lambda_i \in \Lambda$, let ϕ be a continuous psh exhasution function on Ω, then for each j, $z \in D$, $\phi(z + \lambda_j w e^{-f(z)}) \leq \sup_K \phi$, but since this is a compact set, $c \in \Lambda$ □

Pseudoconvexity is a property of the boundary of Ω

Proposition 6.9. $\Omega \subseteq \mathbb{C}^n$. Suppose that for every $\xi \in \overline{\Omega}$ there is an open set $\xi \in U$ such that $U \cap \Omega$ is pseudoconvex. Then Ω is a pseudoconex

Proof. Let $\xi \in \partial\Omega$, set $\tilde{\Omega} = U \cap \Omega$. For z sufficiently close to ξ, $d(z) = d_\Omega(z) = d_{\tilde{\Omega}}(z)$, so $-\log d(z)$ is psh in a neighborhood of $\partial\Omega$(say, $\Omega \setminus F$ for smote closed F).Find a smooth proper psh function ψ on $\mathbb{C}^n$ such that $\phi(z) > -\log d(z)$ for $z \in F$. Now let $\phi(z) = \max\{\psi(z), -\log d(z)\}$. Then ϕ is a continuous psh exhaustion function □

Definition 6.10. $\Omega \subseteq \mathbb{C}^n$ have a C^2 boundary. In a neighborhood U of $z_0 \in \partial\Omega$ we can find a C^2 defining function $\rho : U \to \mathbb{R}$, i.e.

$$\Omega \cap U = \{z \in U | \rho(z) < 0\}, \nabla\rho \neq 0 \text{ on } \partial\Omega \cap U$$

The *Levi form* L_{z_0} at the point z_0 is the quadratic form $\operatorname{Hess}(\rho)$ restricted to $V_{z_0} = T_{z_0}\partial\Omega \cap J(T_{z_0}\partial\Omega)$. Alternatively, let $\xi \in \mathbb{C}^n$ satisfy $\sum_{i=1}^n \frac{\partial\rho}{\partial z_i}\xi_i = 0$. Then we define

$$L(\xi) = \sum_{i,j=1}^n \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_j}(z_0)\xi_i\bar{\xi}_j$$

Here, if ξ is the vector corresponding to v then $L(v) = L(\xi)$

d(z)>=d(w)-r

Lemma 6.11. Let $z, w \in \Omega$, $\xi \in \Delta(0,r)$ such that $z = w + \xi$. Then $d(z) \geq d(w) - r$

Proof. Let η be in some polydisk about 0, such that $z + \eta \in \partial\Omega$, and $d(z) = \max|\eta_i|$. Then $w + \xi + \eta \in \partial\Omega$. This implies

$$d(w) \leq \max_j |(\xi + \eta)_j| \leq \max_j |\xi_j| + \max_j |\eta_j| \leq r + d(z)$$

□

Proposition 6.12. Ω is pseudoconvex $\iff$ the Levi form is everywhere positive semidefinite on $\partial\Omega$

Proof. $\Rightarrow$: $\rho(z) = \begin{cases} -d_{\Omega}(z) & z \in \Omega \\ 0 & z \in \partial\Omega \\ -d_{\overline{\Omega}^c}(z) & z \in \overline{\Omega}^c \end{cases}$, then ρ is C^2. The function $\phi = -\log d$ is C^2 and psh

$$\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j} = -\frac{1}{d(z)} \frac{\partial^2 d}{\partial z_i \partial \bar{z}_j} + \frac{1}{d(z)^2} \frac{\partial d(z)}{\partial z_i} \frac{\partial d(z)}{\partial \bar{z}_i}$$

So for $z \in \Omega$

$$0 \leq \sum_{i,j=1}^{n} \frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}(z) \xi_i \bar{\xi}_j = \sum_{i,j=1}^{n} \frac{1}{d(z)} \frac{\partial^2 d}{\partial z_i \partial \bar{z}_j}$$

Now let $z \to \partial\Omega$

$\Leftarrow$: Suppose $c = \frac{\partial^2}{\partial\tau\partial\bar{\tau}} \log d(z_0 + \tau w_0) > 0$. $\log d(z_0 + \tau w_0) = \log d(z_0) + \mathrm{Re}(A\tau + B\tau^2) + c|\tau|^2 + o(|\tau|^2)$. Choose $\xi_0 \in \partial\Delta(0, d(z_0))$ such that $z_0 + \xi_0 \in \partial\Omega$, $\max_i |\xi_{0,i}| = d(z_0)$. Let $z(\tau) = z_0 + \tau w_0 + \xi_0 \exp(A\tau + B\tau^2)$. By Lemma 6.11

$$\begin{aligned} d(z(\tau)) &\geq d(z_0 + \tau w_0) - d(z_0)|\exp(A\tau + B\tau^2)| \\ &\geq |\exp(A\tau + B\tau^2)|(e^{c|\tau|^2/2} - 1) \end{aligned}$$

Now $d(z(0)) = 0$. The inequalitity implies

$$\frac{\partial}{\partial\tau} d(z(\tau))\bigg|_{\tau=0} = 0, \ \frac{\partial^2}{\partial\tau\partial\bar{\tau}} d(z(\tau))\bigg|_{\tau=0} > 0$$

In other words

$$\sum_{i=1}^{n} \frac{\partial\rho}{\partial z_i} z_i'(0) = 0, \sum_{i,j=1}^{n} \frac{\partial^2\rho}{\partial z_i \partial \bar{z}_j} z_i'(0) \bar{z}_j'(0) < 0$$

This contradicts $L_{z(0)} \geq 0$ $\square$

7 Hörmander's L^2 estimate

Definition 7.1. H_1, H_2 are complex Hilbert space, $T : H_1 \to H_2$ is an *unbounded operator*, if it is a linear map defined on some linear subspace $D(T) \leq H_1$ called the domain of T. T is *densely defined* if $D(T)$ is dense in H_1. T is *closed* if the graph $\mathrm{Gr}(T) = \{(x, Tx) \in H_2 \times H_2 | x \in D(T)\}$ is closed. T has *closed range* if $R(T) = \{Tx \in H_2 | x \in D(T)\}$ is closed in H_2. Write $N(T) = \ker T$

Definition 7.2. $T : H_1 \to H_2$ is a densely defined unbounded operator, its adjoint $T^* : H_2 \to H_1$ is defined as an unbounded operator as follows

- $D(T^*)$ consists of $y \in H_2$ such that the funcitonal $\langle T(-), y\rangle : D(T) \to \mathbb{C}$ is continuous
- By the Hahn-Banach theorem, $\langle T(-), y\rangle$ extends to a linear functional on H_1
- By the Riesz representation theorem and denseness, there is a vector $T^*y \in H_1$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$

Proposition 7.3. If T is densely defined, then T^* is closed

Proof. Let $y_j \in D(T^*)$, $y_j \to y$, and $x_j = T^*y_j \to x$. We need to show $y \in D(T^*)$ and $x = T^*y$. Fix $u \in D(T)$. Then

$$|u||x| \geq \langle u, x\rangle = \lim_j \langle u, x_j\rangle = \lim_j \langle u, T^*y_j\rangle = \lim_j \langle Tu, y_j\rangle = \langle Tu, y\rangle$$

So the map $u \mapsto \langle Tu, y\rangle$ is bounded on $D(T)$ by $|x|$. This implies $y \in D(T^*)$ and $x = T^*y$ □

Fact 7.4. If T, T^* are densely defined then T is closed, and $(T^*)^* = T$

Gr(T*)=Gr(-T)^perp

Lemma 7.5. If T is closed and densely defined, then $\mathrm{Gr}(T^*) = \mathrm{Gr}(-T)^\perp$ in $H_1 \times H_2$

Proof. We have inclusion $\subseteq$ since

$$\langle (T^*y, y), (x, -Tx)\rangle = \langle T^*y, x\rangle - \langle y, Tx\rangle = 0$$

Now if $\langle (x, y), (u, -Tu)\rangle = \langle x, u\rangle - \langle y, Tu\rangle = 0$ for all $u \in D(T)$, then $u \mapsto \langle Tu, y\rangle = \langle u, x\rangle$ is bounded on $D(T)$, so $y \in D(T^*)$, and $x = T^*y$ □

Theorem 7.6. If T is closed and densely defined, then so is T^*. Moreover, $N(T^*) = R(T)^\perp$ and $N(T) = \overline{R(T^*)^\perp}$

Note. $(V^\perp)^\perp = \overline{V}$

Proof. By Lemma 7.5, any $(u, v) \in H_1 \times H_2$ can be written as

$$(u, v) = (x, -Tx) + (T^*y, y), x \in D(T), y \in D(T^*)$$

Taking $u = 0$, then $v = y + TT^*y$. This implies $\langle v, y\rangle = |y|^2 + |T^*y|^2$. If $v \in D(T^*)^\perp$, then $y = 0$, and so $v = 0$. Hence $D(T^*)$ must be dense. $N(T^*) = R(T)^\perp$ follows form $\langle Tx, y\rangle = \langle x, T^*y\rangle$ □

T closed, densely defined, equivalent conditions for R(T) closed

Proposition 7.7. Let $T : H_1 \to H_2$ be closed and densely defined. The following are equivalent

1. $R(T)$ is closed
2. $\exists C$ such that $|x| \leq C|Tx|$ for all $x \in D(T) \cap R(T^*)$
3. $R(T^*)$ is closed
4. $\exists C$ such that $|y| \leq C|T^*y|$ for all $y \in D(T^*) \cap R(T)$

Proof. 2.⇒1.: Suppose $Tx_j \to y$, then x_j converges, say to x, $(x_j, Tx_j) \to (x, y)$
To show 1.⇒2., recall $N(T) = R(T^*)^\perp$. Hence T is continuous and 1-1 from $D(T) \cap R(T^*)$ onto the closed subspace $R(T)$. Hence the inverse is continuous by the closed graph theorem. This proves 2.
3. ⟺ 4.
2.⇒4.:

$$|\langle Tx, y\rangle| = |\langle x, T^*y\rangle| \leq |x||T^*y| \leq C|Tx||T^*y|$$

So $|\langle z, y\rangle| \leq C|T^*y||z|$ for $z \in R(T)$, $y \in D(T^*)$ □

Definition 7.8. Now consider densely defined closed unbounded operators $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ satisfying $S \circ T = 0$. The *harmonic elements* are

$$\mathcal{H}_2 = N(S) \cap N(T^*)$$

there is an orthogonal decomposition $H_2 = \mathcal{H}_2 \oplus (N(S) \cap N(T^*))^\perp = \mathcal{H}_2 \oplus \overline{R(T)} \oplus \overline{R(S^*)}$, so $N(S) = \mathcal{H}_2 \oplus \overline{R(T)}$

Theorem 7.9. There is $C > 0$ such that for all $y \in D(S) \cap D(T^*)$

$$|y| \leq C(|Sy| + |T^*y|) \tag{7.1}$$

i.e. the *basic estimate* holds $\iff$ $\mathcal{H}_2 = 0$ and $R(T), R(S^*)$ are closed

Proof. $\Rightarrow$: If $y \in \mathcal{H}_2$, then $|y| \leq C(|Sy| + |T^*y|) = 0$, hence $\mathcal{H}_2 = 0$. If $y \in R(T) \cap D(T^*)$, then $y \in N(S)$ and $|y| \leq C|T^*y|$, by Proposition 7.7, $R(T)$ is closed, similarly, $R(S^*)$ is closed $\Leftarrow$: $H_2 = R(T) \oplus R(S^*)$ and $y \in D(S) \cap D(T^*)$, write $y = y_1 + y_2$, $y_1 \in R(T) \cap D(T^*)$, $y_2 \in R(S^*) \cap D(S)$. Apply the previous estimates and the triangle inequality

$$|y| \leq |y_1| + |y_2| \leq C_1|T^*y| + C_2|Sy| \leq C(|Sy| + |T^*y|)$$

□

$\mathcal{D}_{(p,q)}(\Omega) \subseteq L^2_{(p,q)}(\Omega)$ be the smooth (p, q)-forms with compact support in Ω. Consider the unbounded operator $\bar{\partial} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q+1)}(\Omega)$ with domain $D(\bar{\partial}) = \{u \in L^2_{(p,q)}(\Omega) | \bar{\partial} u \in L^2_{(p,q+1)}(\Omega)\}$, the derivative is in the sense of distributions $\langle \bar{\partial} u, \alpha \rangle_{L^2} = \langle u, \bar{\partial}^* \alpha \rangle_{L^2}$ for $\alpha \in \mathcal{D}_{(p,q+1)}(\Omega)$. $\bar{\partial}^*$ is called the *formal adjoint* of $\bar{\partial}$. Then $\bar{\partial} u \in L^2$ if there is a constant $C > 0$ such that $|\langle u, \bar{\partial}^* \alpha \rangle| \leq C \|\alpha\|_{L^2}$. In this case, the Hahn-Banach and Riesz representation theorem, $\langle u, \bar{\partial}^* \alpha \rangle = \langle v, \alpha \rangle$

Proposition 7.10. $\bar{\partial} : L^2_{(p,q)}(\Omega) \to L^2_{(p,q+1)}(\Omega)$ is a closed operator

Proof. $u_i \to u$ in L^2, $u_i \in D(\bar{\partial})$, $\bar{\partial} u_i \to \alpha$. Let $\beta \in \mathcal{D}_{(p,q+1)}(\Omega)$, then

$$\langle u, \bar{\partial}^* \beta \rangle = \lim_{i \to \infty} \langle u_i, \bar{\partial}^* \beta \rangle = \lim_{i \to \infty} \langle \bar{\partial} u_i, \beta \rangle = \langle \alpha, \beta \rangle$$

So the map $\beta \mapsto \langle u, \bar{\partial}^* \beta \rangle$ is bounded and $\bar{\partial} u = \alpha$, thus $\bar{\partial}$ is closed □

Example 7.11. Consider $T = i \frac{d}{dx}$ on $L^2([0, 1])$, with $D(T)$ consisting of $f, f' \in L^2$. One can show $f \in D(T)$ is (absolutely) continuous on $[0, 1]$. By integration by parts

$$f \mapsto \langle Tf, g \rangle = \langle f, Tg \rangle + i(f(1)\bar{g}(1) - f(0)\bar{g}(0))$$

This is not continuous with respect to the L^2 topology on f, unless $g(0) = g(1) = 0$. Thus T^* is the same operator T, but with a different domain of definition

The problem is that while compactly supported functions are dense in L^2 topology, they are not dense in the L^2_1 topology, i.e. the graph norm $\|u\| + \|\bar{\partial} u\|$
Methods to fix this

1. Hömander uses a clever choice of (three) weights to prove the basic estimate
2. If Ω has sufficiently nice boundary, define boundary conditions (the $\bar{\partial}$-Neumann problem)
3. Change the geometry of Ω to carry a complete Kähler metric

We will follow the last option (Demially, Gaffney)

Equivalent conditions of completeness and compact exhaustion

Lemma 7.12. (M, g) is a Riemannian manifold, the following are equivalent

1. (M, g) is complete (Hopf-Rinow theorem)

2. $\exists$ compact exhaustion function ψ with $|d\psi|_g \leq 1$

3. $\exists$ compact exhaustion $K_i \subseteq K_{i+1}^\circ$, and $0 \leq \psi_i$ supported in K_{i+1}, $\equiv 1$ on K_i, such that $|d\psi_i|_g \leq 2^{-i}$

Proof. 1.$\Rightarrow$2.: Fix $x_0 \in M$, Let $\psi_0(x) = \frac{1}{2}d(x_0, x)$. Smooth ϕ_0 to ψ with $|\phi - \phi_0| < 1$, by convolution with some $g \in C^\infty$, compactly supported near 0 and $\int g = 1$ 2.$\Rightarrow$3.: Choose a smooth function $\rho : \mathbb{R} \to [0,1]$ with $\rho(t) = \begin{cases} 1 & t \leq 1 \\ 0 & t \geq 2 \end{cases}$ and $|\rho'(t)| \leq 2$. Then let $K_i = \{\psi(x) \leq 2^{i+1}\}$, $\psi_i(x) = \rho(2^{-i-1}\psi(x))$ 3.$\Rightarrow$2.: Set $\psi = \sum 2^i(1 - \psi_i)$ 2.$\Rightarrow$1.: $|\psi(x) - \psi(y)| \leq d(x,y)$, $\{x_i\}$ is a Cauchy sequence, then $\{x_i\}$ lies in the set $\{\psi \leq C\}$ for some C. Since this is compact, the sequence converges, so (M, g) is complete □

Corollary 7.13. Let Ω have a complete Riemannian metric ω. Then $\mathcal{D}_{(p,q)}(\Omega)$ is dense in graph norm of $\bar{\partial}$

Proof. Set $u_i = u\phi_i$ as in Lemma 7.12, 3. Then $u_i \to u$ in L^2, and $\bar{\partial}u_i = \bar{\partial}u\psi_i + u\bar{\partial}\psi_i \to \bar{\partial}u$ in L^2. Now choose $v_i \in \mathcal{D}_{(p,q)}(\Omega)$ so that $\|v_i - u_i\|_{L^2_1} = \|v_i - u_i\|_{L^2} + \|\bar{\partial}v_i - \bar{\partial}u_i\|_{L^2} \leq 1/i$, the result follows □

Corollary 7.14. (Ω, ω) is complete, $\bar{\partial}^*$ with domain

$$D(\bar{\partial}^*) = \left\{ \alpha \in L^2_{(p,q+1)}(\Omega) \middle| \bar{\partial}^*\alpha \in L^2_{(p,q)}(\Omega) \right\}$$

is the adjoint of T^* of $T = \bar{\partial}$

Proof. $D(T^*) \subseteq D(\bar{\partial}^*)$. Let $u \in \mathcal{D}_{(p,q)}(\Omega) \subseteq D(T)$. If $\alpha \in D(T^*)$, then there is a constant $C > 0$ such that $|\langle \bar{\partial}u, \alpha \rangle| \leq C|u|$. But then by definition, $|\langle u, \bar{\partial}^*\alpha \rangle| \leq C|u|$. Since $\mathcal{D}_{(p,q)}(\Omega)$ is dense in L^2, this implies $\bar{\partial}^*\alpha \in L^2$
$D(\bar{\partial}^*) \subseteq D(T^*)$. If $\bar{\partial}^*\alpha \in L^2$, there is a constant $C > 0$ such that $|\langle u, \bar{\partial}^*\alpha \rangle| \leq C|u|$. Fix $u \in D(T)$. Let $u_i \in \mathcal{D}_{(p,q)}(\Omega)$ so that $u_i \to u$ and $\bar{\partial}u_i \to \bar{\partial}u$ in L^2. Then since $\langle u_i, \bar{\partial}^*\alpha \rangle = \langle \bar{\partial}u_i, \alpha \rangle$, we have $|\langle \bar{\partial}u, \alpha \rangle| \leq C|u|$, and so $\alpha \in D(T^*)$ □

8 Kahler metrics

Definition 8.1. $\Omega \subseteq \mathbb{C}^n$. A *hermitian metric* on Ω is a positive definite hermitian valued smooth function $g = (g_{i\bar{j}})$. The *Kähler form* associated to g is $\omega = i\sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$, note that $\bar{\omega} = \omega$. We assume that g is tensorial, in the sense that ω is a well-defined $(1,1)$-form on Ω, $g_{i\bar{j}} = \left\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right\rangle$. This gives a pointwise hermitian inner product on $(1,0)$-forms: $\alpha = \alpha_i dz_i$, $\beta = \beta_j dz_j$

$$\langle \alpha, \beta \rangle = \sum_{i,j=1}^n \alpha_i \overline{\beta_j} g^{i\bar{j}}$$

$(g^{i\bar{j}})$ is the inverse matrix of (g_{ij}). Extend this to $(p,0)$-forms by

$$\langle \alpha_1 \wedge \cdots \wedge \alpha_p, \beta_1 \wedge \cdots \beta_p \rangle = \det\langle \alpha_i, \beta_j \rangle$$

Extend this to (p,q)-forms by taking the product of this. This gives a complex anti-linear isometry

$$\bar{*} : \Lambda^{p,q} \to \Lambda^{n-p,n-q}, \alpha \wedge \bar{*}\beta = \langle \alpha, \beta \rangle \frac{\omega^n}{n!}$$

$\frac{\omega^n}{n!}$ is the volume form. Inducing L^2 inner products

$$\langle \alpha, \beta \rangle = \int_\Omega \alpha \wedge \bar{*}\beta$$

Define the *Lefschetz operator*

$$L : \Lambda^{p,q} \to \Lambda^{p+1,q+1}, \alpha \mapsto \omega \wedge \alpha$$

Let $\Lambda = L^*$. Note that $L^n(1) = \omega^n$, $\Lambda^{n,n} \cong \mathbb{C}$

Example 8.2. If F is of type $(1,1)$, write $F = \sum_{i,j} F_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Then $\Lambda F = \sum_{i,j} F_{i\bar{j}} g^{i\bar{j}}$. If α is of type $(1,0)$

$$i\alpha \wedge \bar{\alpha} \wedge \frac{\omega^{n-1}}{(n-1)!} = |\alpha|^2 \frac{\omega^n}{n!}$$

Definition 8.3. A hermitian metric is called Kähler if $d\omega = 0$ (ω is closed), equivalently, $\frac{\partial g_{j\bar{k}}}{\partial z_i} = \frac{\partial g_{i\bar{k}}}{\partial z_j}$

Proposition 8.4. ω is Kähler iff about any point there are coordinates such that g is euclidean to order two

Proof. We may always choose local coordinate so that $g_{i\bar{j}}(0) = \delta_{ij}$, $g_{i\bar{j}} = \delta_{ij} + A^k_{i\bar{j}} z_k + B^{\bar{k}}_{i\bar{j}} \bar{z}_k + O(|z|^2)$. The Kähler condition implies: $A^k_{i\bar{j}} = A^i_{k\bar{j}}$. The fact that $g_{i\bar{j}}$ is hermitian implies $B^{\bar{k}}_{i\bar{j}} = \overline{A^k_{j\bar{i}}}$. Define $w_m = z_m + \frac{1}{2} A^i_{j\bar{m}} z_i z_j$, then $\frac{\partial w_m}{\partial z_j} = \delta_{mi} + A^i_{j\bar{m}} z_i$. Now $\tilde{g}_{m\bar{n}} \frac{\partial w_m}{\partial z_i} \overline{\frac{\partial w_m}{\partial z_i}} = g_{i\bar{j}}$. This implies

$$\begin{aligned} g_{i\bar{j}}(z) &= \delta_{ij} + A^k_{i\bar{j}} z_k + B^{\bar{k}}_{i\bar{j}} \bar{z}_k + O(|z|^2) \\ &= \tilde{g}_{i\bar{j}} + \tilde{g}_{m\bar{j}} A^k_{i\bar{m}} z_k + \tilde{g}_{i\bar{k}} \overline{A^k_{j\bar{n}} z_k} + O(|z|^2) \end{aligned}$$

□

Proposition 8.5 (Kähler identities)**.** (Ω, ω) has a Kähler metric. Then the formal L^2 adjoints are given by $\bar{\partial}^* = -i[\Lambda, \partial], \partial^* = i[\Lambda, \bar{\partial}]$

Proof. It suffices to prove these for the euclidean metric. Then is a direct computation □

Example 8.6. $\Omega \subseteq \mathbb{C}$, $\Lambda(idz \wedge d\bar{z}) = 1$, $f \in \mathscr{D}_{(0,0)}(\Omega)$, $\beta \in \mathscr{D}_{(0,1)}(\Omega)$, $\beta = \beta(z)d\bar{z}$. Then

$$\begin{aligned}
\langle \bar{\partial} f, \beta \rangle &= \int_\Omega \partial_{\bar{z}} f \overline{\beta(z)} idz \wedge d\bar{z} \\
&= -\int_\Omega f \overline{\partial z \beta(z)} idz \wedge d\bar{z} \\
&= \int_\Omega f \overline{i \partial \beta} \\
&= \int_\Omega f \overline{\Lambda(i\partial\beta) idz \wedge d\bar{z}} \\
&= -\int_\Omega f \overline{\Lambda(i\partial\beta)} idz \wedge d\bar{z} \\
&= \langle f, -i\Lambda\partial\beta \rangle \\
&= \langle f, \bar{\partial}^* \beta \rangle
\end{aligned}$$

9 Solving $\bar\partial$ equation

Theorem 9.1. If $\Omega \subseteq \mathbb{C}^n$ is pseudoconnvex, then there is a complete Kähler metric on Ω

Proof. From Richberg's lemma, there is a smooth strictly plurisubharmonic exhaustion function ψ on Ω. By adding a constant, we can assume $\psi > 0$. Let ω_0 denote the euclidean Kähler form on Ω, and consider: $\omega = \omega_0 + i\partial\bar\partial\psi^2$, $i\partial\bar\partial\psi^2$ is semi-positive definite

$$\omega = \omega_0 + i\partial\psi \wedge \bar\partial\psi + i\psi\partial\bar\partial\psi \geq \omega_0 + i\partial\psi \wedge \bar\partial\psi$$

$$\omega^n \geq \omega_0 \wedge \omega^{n-1} + i\partial\psi \wedge \bar\partial\psi \wedge \omega^{n-1} \geq i\partial\psi \wedge \bar\partial\psi \wedge \omega^{n-1}$$

$$\frac{\omega^n}{n!} \geq \frac{2}{n} i\partial\psi \wedge \bar\partial\psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{2}{n}|\partial\psi|^2_\omega \frac{\omega^n}{n!}$$

So $|d\psi|_\omega = \sqrt{2}|\partial\psi|_\omega \leq C$. Completeness follows from Lemma 7.12 $\square$

Definition 9.2. Let $d\mu = \omega^n/n!$ be a Kähler metric on Ω, $\phi \in C^0(\Omega)$ be an exhaustion function. Define $L^2_{(p,q)}(\Omega, \phi)$ to be the completion of smooth (p, q)-forms with respect to the norm

$$\|\alpha\|^2_\phi = \int_\Omega |\alpha|^2_\omega e^{-\phi} d\mu$$

The same density theorems apply as for unweighted spaces
For compactly supported α

$$\int_\Omega \langle \bar\partial u, \alpha\rangle e^{-\phi} d\mu = \int_\Omega \langle u, e^\phi \bar\partial^*(e^{-\phi}\alpha)\rangle e^{-\phi} d\mu$$

The new adjoint is $-i[\Lambda, \partial_\phi]$, $\partial_\phi = e^\phi \partial e^{-\phi} = \partial - \partial\phi$. Moreover

$$\int_\Omega \langle \partial_\phi u, \alpha\rangle e^{-\phi} d\mu = \int_\Omega \langle \partial(e^{-\phi}u), \alpha\rangle d\mu = \int_\Omega \langle u, \bar\partial^*\alpha\rangle e^{-\phi} d\mu$$

So $\partial^*_\phi = i[\Lambda, \bar\partial]$

Definition 9.3. The Laplacian is $\Delta = d^*d + dd^*$. The Dolbeault laplacians are

$$\square_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*, \square_\partial = \partial^*_\phi\partial_\phi + \partial_\phi\partial^*_\phi$$

The curvature is the pure imaginary $(1, 1)$ form

$$F_\phi = \bar\partial\partial_\phi + \partial_\phi\bar\partial = \partial\bar\partial\phi$$

Lemma 9.4. $\square_{\bar\partial} - \square_\partial = [iF_\phi, \Lambda]$

Proof.

$$\bar\partial^*\bar\partial = -i[\Lambda, \partial_\phi]\bar\partial = -i\Lambda\partial_\phi\bar\partial + i\partial_\phi\Lambda\bar\partial$$

$$\bar\partial\bar\partial^* = \bar\partial(-i[\Lambda, \partial_\phi]) = -i\bar\partial\Lambda\partial_\phi + i\bar\partial\partial_\phi\Lambda$$

$$-\partial^*_\phi\partial_\phi = -i[\Lambda, \bar\partial]\partial_\phi = -i\Lambda\bar\partial\partial_\phi + i\bar\partial\Lambda\partial_\phi$$

$$-\partial_\phi\partial^*_\phi = -\partial_\phi(i[\Lambda, \bar\partial]) = -i\partial_\phi\Lambda\bar\partial + i\partial_\phi\bar\partial\Lambda$$

$\square$

Corollary 9.5. For $\alpha \in D(\bar\partial) \cap D(\bar\partial^*)$

$$\|\bar\partial\alpha\|^2 + \|\bar\partial^*\alpha\|^2 \geq \langle [iF_\phi, \Lambda]\alpha, \alpha\rangle$$

Proof. $\langle \partial_\phi\alpha, \alpha\rangle \geq 0$ can be throw away, and $\langle \square_{\bar\partial}\alpha, \alpha\rangle$ give the left hand side by integration by parts $\square$

Lemma 9.6. Write $iF_\phi = f_{i\bar{j}} dz_i \wedge d\bar{z}_j$. If there is $C_0 > 0$ such that $\sum f_{i\bar{j}}(z)\xi_i\bar{\xi}_j \geq C_0|\xi|^2$ for all $\xi \in \mathbb{C}^n$ and all $z \in \Omega$, then there is $C_1 > 0$ such that

$$\langle [iF_\phi, \Lambda]\alpha, \alpha\rangle \geq C_1\|\alpha\|^2$$

for all $\alpha \in L^2_{(n,q)}(\Omega, \omega), q \geq 1$

Note that $[iF_\phi, \Lambda] = 0$ on $(n, 0)$ forms. In particular, the condition that $q \geq 1$ is necessary. The applications of theis result extends to (p, q) froms, $q \geq 1$. We will only prove a couple of special cases. For the general result see Demailly

Example 9.7. Consider an $(n, 1)$ form α. Let θ_i denote an orthonormal frame for $T^{1,0}\Omega$. Write $\omega = i\sum_j \theta_j \wedge \bar{\theta}_j$, and

$$\alpha = i\sum_{i=1}^{n} \alpha_i(z)\theta_1 \wedge \cdots \wedge \theta_n \wedge \bar{\theta}_i$$

Write $iF_\phi = \sum f_{i\bar{j}}\theta_i \wedge \bar{\theta}_j$

Write $\sum_{i,j} \frac{\partial^2\phi_0}{\partial z_i \partial\bar{z}_j}(z)\xi_i\bar{\xi}_j \geq m(z)|\xi|^2$, where m is a continuous function, $m(z) > 0$ for all $z \in \Omega$. We will replace a given ϕ_0 with $\phi = \chi \circ \phi_0$, for an appropriate increasing convex function χ
Set $M(t) = (\min_{\phi_0(z)\geq t} m(z))^{-1}$. Then $M(t)$ is a positive, continuous, increasing function of t. Note that $M(\phi_0(z))m(z) \geq 1$

Claim. We can find a smooth $\tilde{M} \geq M$ that is increasing

Assuming the claim, set $\chi(t) = \int^t \tilde{M}(\tau)d\tau$. Then χ is convex, and $\chi'(t) \geq M(t)$. Set $\phi = \chi \circ \phi_0$. Then ϕ is psh exhaustion function, and

$$\sum_{i,j} \frac{\partial^2\phi}{\partial z_i \partial\bar{z}_j}(z)\xi_i\bar{\xi}_j \geq \sum_{i,j} \chi' \circ \phi_0 \frac{\partial^2\phi_0}{\partial z_i \partial\bar{z}_j}(z)\xi_i\bar{\xi}_j \geq M(\phi_0(z))m(z)|\xi|^2 \geq |\xi|^2$$

Proof of claim. This is probably obvious, by here is one idea: Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth with compact support in $(-1, 1)$, $0 \leq \psi \leq 1$ and $\int_{\mathbb{R}} \psi = 1$. Set

$$\tilde{M}(t) = \int_{\mathbb{R}} M(s)\psi(s - t - 1)ds = \int_{\mathbb{R}} M(\tau + t + 1)\psi(\tau)d\tau$$

The first equality proves that $\tilde{M}$ is smooth. By the second equality

$$\tilde{M}(t) \geq \int_{\mathbb{R}} M(t)\psi(\tau)d\tau = M(t)$$

Also by the second equality, for $\delta > 0$

$$\tilde{M}(t + \delta) - M(t) = \int_{\mathbb{R}} (M(\tau + t + 1) - M(\tau + t + 1))\psi(\tau)d\tau \geq 0$$

So $\tilde{M}$ is increasing ☐

Summary: $\Omega \subseteq \mathbb{C}^n$ pseudoconvex, ω complete. For an (n, q) form α, $q \geq 1$, and $\alpha \in D(\bar\partial) \cap D(\bar\partial^*)$, we have proved the basic estimate

$$\|\bar\partial\alpha\|_\phi + \|\bar\partial^*\alpha\|_\phi \geq C\|\alpha\|_\phi$$

This implies that $\bar\partial$ has closed range and that the harmonics $\mathcal{H}^{n,q} = \{0\}$, i.e. $\ker\bar\partial = \operatorname{im}\bar\partial$

Theorem 9.8. If $\alpha \in L^2_{(n,q)}(\Omega, \phi)$, $q \geq 1$, and $\bar\partial\alpha = 0$, then there is $u \in L^2_{(n,q-1)}(\Omega, \phi)$ such that $\bar\partial u = \alpha$. Moreover, $\|u\| \leq C\|\alpha\|$

Note. In applications, having the estimate on the norms is crucial. Notice that $\|\|$ depends on the Kähler metric ω and the weight ϕ. We can get rid of the dependence on ω, i.e. prove the same result for the euclidean metric, by taking a limit of solutions for metrics $\omega_\epsilon = \omega_0 + \epsilon\omega$ as $\epsilon \to 0$ (see Demailly). On the other hand, ϕ is necessary. Optimizing the choice of ϕ is an important issue we will skip for now. Finally, the result holds more generally for (p, q) forms: $\bigwedge^p T^*\Omega \cong \bigwedge^{n-p} T\Omega \otimes \bigwedge T^*\Omega$, so trivializing $\bigwedge^{n-p} T\Omega$ reduces the problem to this case. More importantly. This is not exactly what we want, since we are trying to solve $\bar{\partial}$ for all smooth α, not just those that are integrable. Let $L^2_{(p,q)}(\Omega, \mathrm{loc})$ denote the (p, q) forms that are locally L^2. Notice that here the metric ω and weight ϕ are irrelevant

Theorem 9.9. If $\alpha \in L^2_{(p,q)}(\Omega, \mathrm{loc})$, $q \geq 1$, and $\bar{\partial}\alpha = 0$, then there is $u \in L^2_{(p,q-1)}(\Omega, \mathrm{loc})$ such that $\bar{\partial}u = \alpha$

Proof. The idea is to find some weight $\tilde{\phi}$ such that $\alpha \in L^2_{(p,q)}(\Omega, \tilde{\phi})$, and then apply the previous result. We again choose the form $\tilde{\phi} = \chi \circ \phi$, where χ is increasing and convex. To do this, let $K_i = \{z \in \Omega | \phi(z) \leq i\}$, and let $\ell_i = \|\alpha\|^2_{L^2(K_i)}$. Now choose χ so that $e^{-\chi(i)}(\ell_{i+1} - \ell_i) \leq 2^{-i}$, then

$$\int_{K_{i+1}\setminus K_i} |\alpha|^2 e^{-\tilde{\phi}} d\mu \leq e^{-\chi(i)} \int_{K_{i+1}\setminus K_i} |\alpha|^2 d\mu \leq e^{-\chi(i)}(\ell_{i+1} - \ell_i) \leq 2^{-i}$$

Hence

$$\int_\Omega |\alpha|^2 e^{-\tilde{\phi}} d\mu = \sum_i \int_{K_{i+1}\setminus K_i} |\alpha|^2 e^{-\tilde{\phi}} d\mu \leq \sum_i 2^{-i} < \infty$$

□

Elliptic regularity: The result applies, in particular, to the case where α is smooth. However, the theorem just concludes that the solution $\bar{\partial}u = \alpha$ is locally in L^2. We want to improve this

Theorem 9.10. If α is a smooth (p, q)-form on Ω, $q \geq 1$, with $\bar{\partial}\alpha = 0$, then there is a smooth $(p, q-1)$-form on Ω such that $\bar{\partial}u = \alpha$

Proof. First consider the case $q = 1$, i.e. u is a function. We know that $\partial^*\partial = \bar{\partial}^*\bar{\partial}$, so $\|\partial u\| \leq \|\bar{\partial}u\|$. In particular, if $\alpha \in L^2(\Omega, \mathrm{loc})$, then u is in $L^2_1(\Omega, \mathrm{loc})$ (one distributional derivative in L^2). Now differentiate the equation $\bar{\partial}u = \alpha$ k times to conclude that $u \in L^2_k(\Omega, \mathrm{loc})$ for arbitrary k. On the other hand, the Sobolev embedding theorem implies $L^2_k \hookrightarrow C^j$ for $k \geq n + j$. Hence, if α is smooth, so is u
If $q \geq 2$, then we have

$$L^2_{(p,q-1)}(\Omega, \phi) = R(\bar{\partial}) \oplus R(\bar{\partial}^*)$$

Moreover, $N(\bar{\partial})^\perp = R(\bar{\partial}^*)$. Hence, u may be taken to be in the range of $\bar{\partial}^*$. Since $(\bar{\partial}^*)^2 = 0$, we have $\Box_{\bar{\partial}}u = \bar{\partial}^*\bar{\partial}u = \bar{\partial}^*\alpha$. Let $\Delta = dd^* + d^*d$ be the ordinary or de Rham Laplacian. Then in standard basis and euclidean metric, Δ acts on the coefficients of (p, q)-forms □

Claim. $\Delta = 2\Box_{\bar{\partial}}$

Proof. Write $d = \partial + \bar{\partial}$, $d^* = \partial^* + \bar{\partial}^*$, then

$$\Delta = \Box_{\bar{\partial}} + \Box_\partial + \bar{\partial}\partial^* + \partial^*\bar{\partial} + \partial\bar{\partial}^* + \bar{\partial}^*\partial$$

But the cross terms vanish: e.g. since $\bar{\partial}^2 = 0$

$$\bar{\partial}\partial^* = i\bar{\partial}[\Lambda, \bar{\partial}] = i\bar{\partial}\Lambda\bar{\partial}$$

$$\partial^*\bar{\partial} = i[\Lambda, \bar{\partial}]\bar{\partial} = -i\bar{\partial}\Lambda\bar{\partial}$$

The result now follows, since $\Box_\partial = \Box_{\bar{\partial}}$ □

It follows that if $\alpha \in L^2_1(\Omega, \text{loc})$, then $\Delta u = 2\bar{\partial}^*\alpha$ is in $L^2(\Omega, \text{loc})$. We now appeal to the following interior estimate: if $U \subset\subset U' \subset\subset \Omega$, then there is a constant $C > 0$ such that

$$\|u\|_{L^2_{k+1}(U)} \le C\left(\|\|_{L^2_k(U')} + \|\Delta u\|_{L^2_k(U')}\right)$$

for all smooth (p,q)-forms. By "bootstrapping" the equation, we conclude that if $\bar{\partial}u = \alpha$, $\bar{\partial}^*u = 0$, for α smooth, then u is in $L^2_k(\Omega, \text{loc})$ for any k, and hence is smooth by Sobolev embedding

Theorem 9.11. If $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex, then $H^k_{\text{dR}}(\Omega) = 0$ for $k > n$

Remark 9.12. This is sharp, i.e. it is possible that $H^n_{\text{dR}}(\Omega) = 0$. For example $A_i = \{\frac{1}{2} < |z_i| < 2\}$, $\Omega = A_1 \times \cdots \times A_n$, let $Y = S^1 \times \cdots \times S^1 \subset \Omega$, and

$$\beta = \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}$$

Then $d\beta = \bar{\partial}\beta = 0$, and

$$\int_Y \beta = \int_{|z_1|=1} \frac{dz_1}{z_1} \cdots \int_{|z_n|=1} \frac{dz_n}{z_n} = (2\pi i)^n$$

Hence $0 \neq [\beta] \in H^n_{\text{dR}}(\Omega)$

Claim. Suppose β is a k-form such that $d\beta$ is of type $(k+1, 0)$. Then β is cohomologous to a $(k, 0)$-form

Proof. Write $\beta = \sum_{q=0}^{l} \beta_{(k-q,q)}$. Proof by induction on l. The case $l = 0$ is trivial. Assume $l \ge 1$ and that the result holds for $l - 1$. We have

$$d\beta = \partial\beta + \bar{\partial}\beta = \sum_{q=0}^{l} \partial\beta_{(k-q,q)} + \bar{\partial}\beta_{(k-q,q)}$$

Thus $\bar{\partial}\beta_{(k-l,l)} = 0$, $\beta_{(k-l,l)} = \bar{\partial}u$, for a smooth $(k-l, l-1)$ form u. Let $\tilde{\beta} = \beta - du$. Then $\tilde{\beta} = \sum_{q=0}^{l-1} \tilde{\beta}_{(k-q,q)}$, $d\beta = d\tilde{\beta}$, ☐

Remark 9.13. Another proof use Morse theory: ϕ is a C^∞ psh exhaustion function, Morse function on Ω. The complex Hessian $\dfrac{\partial^2}{\partial z_j \partial \bar{z}_j}\phi > 0$ bound the number of negative eigenvalues of the real Hessian
Lefschetz hyperplane theorem:

Solution to the Levi problem

Theorem 9.14. $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy iff it is pseudoconvex. We have already proved $\Rightarrow$, by showing that $-\log d(z)$ is a psh exhaustion function. We also showed that being a domain of holomorphy is equivalent to being holomorphically convex

Theorem 9.15. If we can solve $\bar{\partial}$ on $\Omega \subseteq \mathbb{C}^n$, then it is a domain of holomorphy

Proof. Let $\tilde{\alpha} \in C^\infty_{(p,q)}(\tilde{\Omega})$, $q \ge 1$, and $\bar{\partial}\tilde{\alpha} = 0$. Set $\beta = z_n^{-1}\bar{\partial}\phi \wedge \pi^*\tilde{\alpha}$. Then $\beta \in C^\infty_{(p,q+1)}(\Omega)$. Moreover, $\bar{\partial}\beta = 0$. By the hypothesis of the theorem, there is $u \in C^\infty_{(p,q)}(\Omega)$ such that $\bar{\partial}u = \beta$. Consider $\alpha = \phi\pi^*\tilde{\alpha} - z_n u$. Then $\bar{\partial}\alpha = 0$. Hence, there is $v \in C^\infty_{(p,q-1)}(\Omega)$ such that $\bar{\partial}v = \alpha$. Finally, since z_n vanishes and $\phi \equiv 1$ on $j(\tilde{\Omega})$ This proves the claim. By the induction hypothesis, $\tilde{\Omega}$ is a domain of holomorphy. Hence, there is $\tilde{f} \in A(\tilde{\Omega})$ that blows up at ξ. As in the first part of the argument above, there is $u \in C^\infty(\Omega)$ such that $f = \phi\pi^*\tilde{f} - z_n u \in A(\Omega)$, and $j^*f = \tilde{f}$. Hence, f blows up at ξ ☐

10 Proper mapping theroems

Theorem 10.1 (Remmert theorem)**.** If $f : X \to Y$ is proper and $V \subseteq X$ is an analytic subvariety of X, then $f(V) \subseteq Y$ is an analytic subvariety of Y

Theorem 10.2 (Remmert-Stein theorem)**.** $V \subseteq X$ is an analytic subvariety, $W \subseteq X \setminus V$ is an irreducible analytic subvariety. If $\dim V < \dim W$, then the $\overline{W} \subseteq X$ is an irreducible analytic subvariety

Remark 10.3. The assumption on dimension is necessary. For example, take the graph in $\mathbb{C}^2 \subseteq \mathbb{CP}^2$ of an entire function f. Suppose f has essential singularity at infinity

Corollary 10.4. X is an analytic space, V is an analytic subvariety, $W \subseteq X \setminus V$ is an analytic subvariety. If $\dim V$ is less than the dimension of any irreducible components of W, then $\overline{W} \subseteq X$ is an analytic subvariety

Theorem 10.5 (Chow's theorem)**.** $V \subseteq \mathbb{CP}^n$ is an analytic subvariety, then V is a projective algebraic variety

Remark 10.6. X is a Kahler manifold. X is a Moisezon manifold(meaning the algebraic of dimension of $X = \dim X$). A theorem of Moisezon says Kahler + Moisezon $\Rightarrow$ projective. There is an equivalence between Hodge manifolds(Kahler manifolds with integral Kahler class) and projective algberaic manifolds

Corollary 10.7. Every compact Riemann surface is an algebraic curve A holomorphic map between nonsingular projective algebraic varieties is a morphism of varieties Every memomorphic function on $\mathbb{CP}^n$ is rational

Definition 10.8. $\phi : X \to Y$ is holomorphic. $\mathcal{E} \to X$, $\mathcal{F} \to Y$ are sheaves of $\mathcal{O}_X$, $\mathcal{O}_Y$ modules. The sheaf $\phi^{-1}\mathcal{F}$ has presheaf is intuitively $\phi^{-1}\mathcal{F}(U) = \mathcal{F}(\phi(U))$. Alternatively, the espace étalé $X \times_Y \mathcal{F}$, at the level of stalks, $(\phi^{-1}\mathcal{F})_p = \mathcal{F}_{\phi(p)}$. By composition with ϕ defines a sheaf map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, making $\mathcal{O}_X, \mathcal{E}$ into $\phi^{-1}\mathcal{O}_Y$ modules

$$\phi^*\mathcal{F} = \mathcal{O}_X \otimes_{\phi^{-1}\mathcal{O}_Y} \phi^{-1}\mathcal{F}$$

There are canonical maps $\phi^*(\phi_*\mathcal{E}) \to \mathcal{E}$, $\mathcal{F} \to \phi_*(\phi^*\mathcal{F})$ through

$$\mathcal{E}(\phi(\phi^{-1}(U))) \to \mathcal{E}(U), \mathcal{F}(U) \to \mathcal{F}(\phi^{-1}(\phi(U)))$$

More generally, there is a canonical bijection $\mathrm{Hom}_{\mathcal{O}_X}(\phi^*\mathcal{F}, \mathcal{E}) \leftrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \phi_*\mathcal{E})$

Fact 10.9. ϕ_* is left exact. ϕ^* is right exact. $\phi^*\mathcal{O}_Y = \mathcal{O}_X$. If $\mathcal{F}$ is coherent, then $\phi^*\mathcal{F}$ is coherent. $\phi_*\mathcal{E}$ is not necessarily coherent even if $\mathcal{E}$ is. For example, let $\phi : \mathbb{C} \to \mathbb{C}^*$ be the exponential map, then $\phi_*\mathcal{O}_{\mathbb{C}}$ is infinitely generated

Theorem 10.10 (Grauert-Remmert, Direct image theorem)**.** $\phi : X \to Y$ is finite, $\mathcal{E} \to X$ is coherent, then $\phi_*\mathcal{E}$ is coherent. Moreover, ϕ_* is right exact

Example 10.11. $X = \mathbb{C}^2 \setminus \{0\}$, $Y = \mathbb{C}$, $\phi(z_1, z_2) = z_1$. Let $D = \{(0, z_2) \in X\} \cong \mathbb{C}^*$. $\phi(D) = \{0\}$, let $\mathcal{I}_D$ be the ideal sheaf, $\mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_D \to 0$, let $f = 1/z_2$ is holomorphic on D, but is not in the image of $\phi_*\mathcal{O}_X \to \phi_*(\mathcal{O}_X/\mathcal{I}_D)$ by Hartogs's theorem. Thus ϕ_* is not right exact

Example 10.12. $X = \{0\}$, $Y = \mathbb{C}$, ϕ is the inclusion. Consider injection $0 \to \mathcal{O}_Y \xrightarrow{\times z} \mathcal{O}_Y$. The induced map $\phi^*\mathcal{O}_Y \to \phi^*\mathcal{O}_Y$ is zero, hence ϕ^* is not left exact

$R^i\phi_*\mathcal{E}(V) = H^i(\phi^{-1}(V), \mathcal{E})$
Problem: $\mathcal{E} \to X$ is coherent, $R^i\phi_*\mathcal{E} \to Y$ might not be coherent

Theorem 10.13 (Grauert's proper coherence theorem)**.** X, Y are complex spaces. $\phi : X \to Y$ is proper, $\mathcal{E} \to X$ is coherent, then $R^i\phi_*\mathcal{E} \to Y$ is coherent for all i

Note. If Y is a point, then X is compact, this gives the Cartan-Serre theorem

References

[1] *An Introduction to Complex Analysis in Several Variables* - Lars Hörmander

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