MATH868C - Several Complex Variables

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Contents

1 Subharmonic functions

Definition 1.1. $\Omega \subseteq \mathbb{C}$ is an open set, $h \in C^2(\Omega)$ is *harmonic* if $\Delta h = \dfrac{4\partial^2}{\partial z \partial \bar{z}} h = 0$, denote the set of harmonic functions $H(\Omega)$

Definition 1.2. $u : \Omega \to [-\infty, +\infty)$ is subharmonic, denoted $u \in SH(\Omega)$ if

- u is upper semi-continuous, i.e. $\{u < r\}$ is open
- For any compact $K \subseteq \Omega$, and $h \in H(\operatorname{Int} K) \cap C(K)$ such that $u \leq h$ on ∂K, then $u \leq h$ on K

Theorem 1.3. $\{u_j\} \subseteq SH(\Omega)$, $v = \sup\limits_j u_j$. If v is upper semi-continuous, then $v \in SH(\Omega)$, $u = \inf\limits_j u_j$ is upper semi-continuous generally doesn't imply $u \in SH(\Omega)$, but if $\{u_j\}$ is decreasing, then $u \in SH(\Omega)$

Theorem 1.4. $u : \Omega \to [-\infty, +\infty)$ is upper semi-continuous. The following are equivalent

1. $u \in SH(\Omega)$
2. For any $\overline{D} \subseteq \Omega$, and any polynomial $f(z)$, if $u \leq \operatorname{Re} f$ on ∂D, then $u \leq \operatorname{Re} f$
3. $\Omega_\delta = \{z \in \Omega | \operatorname{dist}(z, \partial\Omega) > \delta\} \subseteq \Omega$, for $z \in \Omega_\delta$
$$2\pi u(z) \int_0^\delta d\mu(r) \leq \int_0^\delta \int_0^{2\pi} u(z + re^{i\theta}) d\theta d\mu(r)$$
here $d\mu$ is any measure on $[0, \delta]$, take $d\mu(r) = rdr$, the average of the disk, take $d\mu(r)$ to be Dirac measure, the average of the circle

Proof. 1. $\Rightarrow$ 2. is by definition. 2. $\Rightarrow$ 3.

- If $p(z) = \sum_{j=0}^k a_j z^j$, then $2\pi \operatorname{Re} p(z) \int_0^\delta d\mu(r) = \int_0^\delta \int_0^{2\pi} \operatorname{Re} p(z + re^{i\theta} d\theta d\mu(r))$
- $\varphi \in C(\partial D(z, r))$, $r \in [0, \delta]$ such that $u \leq \varphi$ on $\partial D(z, r)$. Fourier: $\exists p_k = \sum_{j=0}^l a_j^k z^j$ such that $\varphi \leq \operatorname{Re} p_k \leq \varphi + \frac{1}{k}$ (Rudin). $u \leq \operatorname{Re} p_k$ on $\partial D(z, r)$, by 2. $u(z) \leq \operatorname{Re} p_k(z)$, then $2\pi u(z) \leq 2\pi \operatorname{Re} p_k(z) = \int_0^{2\pi} \operatorname{Re} p_k(z + re^{i\theta}) d\theta \to \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta$ as $k \to \infty$
- $u : X \to [-\infty, \infty)$ is upper semi-continuous and bounded above, $\{f_j\} \subseteq C(X)$ such that $f_j \searrow u$, then there exists $\{\varphi_j\} \subseteq C(\partial D(z, r))$ such that $\varphi_j \searrow u$ on $\partial D(z, r)$, then $2\pi u(z) \leq \int_0^{2\pi} \varphi_j(z + re^{i\theta}) d\theta \to \int_0^{2\pi} u(z + re^{i\theta}) d\theta$, integrate this over $[0, \delta]$ of $d\mu$

3. $\Rightarrow$ 1. Assume 1. doesn't hold, $\exists K \subseteq \Omega$ compact, $h \in C(K) \cap H(\operatorname{Int} K)$ such that $u \leq h$ on ∂K but $u(z) > h(z)$ for some $z \in K$, define $F = \{z \in K | u(z) = \max_K (u - h)\} \neq \varnothing$ and closed, compact, thus $\exists x \in F$ such that $\operatorname{dist}(x, \partial K)$ is a minimizer. For some r, an open part of $\partial D(z, r)$ lies outside F, $\int_0^{2\pi} (u - h)(x + re^{i\theta}) d\theta < (u - h)(x)$ which is a contradiction ☐

Corollary 1.5. $f \in \mathcal{O}(\Omega) \Rightarrow \log |f| \in SH(\Omega)$, if $f = 0$, $\log |f| = -\infty$

Proof. $\overline{D} \subseteq \Omega$, $p = \sum_{j=0}^k a_j z^j$, if $\log |f| \leq \operatorname{Re} p$ on ∂D, then $|f| \leq e^{\operatorname{Re} p} \Leftrightarrow |f| \leq |e^p|$ on $\partial D \Rightarrow |\frac{f}{e^p}| \leq 1$ on $\partial D \Rightarrow \cdots$ ☐

Corollary 1.6. $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and increasing, $u \in SH(\Omega)$, then $\varphi \circ u \in SH(\Omega)$

Proof. Sub-mean value inequality + Jensen inequality ☐

Theorem 1.7. $u \in SH(\Omega)$ $u \not\equiv -\infty$ on a component of Ω, then $u \in L^1_{\text{loc}}(\Omega)$. $\Delta u \geq 0$ as a distribution, i.e. $\int_\Omega u \Delta v \geq 0$ $\forall v \in C_0^2(\Omega), v \geq 0$

Proof. $r = \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$, $x \in \Omega_r$, $2\pi u(x) \le \int_0^{2\pi} u(x + \delta e^{i\theta}) d\theta$, $\delta \in [0, r]$. $2\pi \int_\Omega u(x)v(x)d\lambda \le \int_\Omega \int_0^{2\pi} u(x + \delta e^{i\theta})v(x)d\theta d\lambda(x) \Rightarrow 0 \le \int_\Omega u(x) \int_0^{2\pi} u(x + \delta e^{i\theta}) \frac{v(x-\delta e i\theta)-v(x)}{\delta^2} d\theta d\lambda$ as $\delta \to 0$, $0 \le \int_\Omega u(x) 2\pi \Delta v(x) d\lambda$ $\square$

Theorem 1.8 (Implicit function theorem)**.** $(w, z) = (w_1, \cdots, w_m, z_1, \cdots, z_n)$, $f_j(w, z)$ are analytic in a neighborhood of $(w^0, z^0) \in \mathbb{C}^{m+n}$, suppose $f_j(w^0, z^0) = 0$, $\det(\frac{\partial f_j}{\partial w_k}) \ne 0$ at (w^0, z^0), then $\exists w(z)$ analytic in a neighborhood of z^0 with $w(z^0) = w^0$, $F(w(z), z) = 0$

2 Cauchy's formula

Definition 2.1. $D = D(x_1, r_1) \times \cdots \times D(x_n, r_n)$ is called a *polydisc*. $\partial_0 D = \partial D_1 \times \cdots \times \partial D_n \subsetneq \partial D$ is the *distinguished boundary* of D

Theorem 2.2. $u \in C(\overline{D}) \cap \mathcal{O}(D)$, then

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n, \forall z \in D$$

Proof.

$$u(z_1, \cdots, z_{n-1}, z) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{u(z_1, \cdots, z_{n-1}, \xi)}{\xi_n - z_n} d\xi$$

$z_{n-1} \mapsto u(z_1, \cdots, z_{n-2}, z_{n-1}, z_n^k) \rightrightarrows z_{n-1} \mapsto u(z_1, \cdots, z_{n-2}, z_{n-1}, \xi_n)$ is uniform convergence □

Theorem 2.3. $K \subseteq K$ is compact, $\exists C_{K,\alpha} > 0$ such that

$$\sup_K |\partial^\alpha u| \leq C_{K,\alpha} \sup_K |u|$$

Proof. If $K = D_1 \times \cdots \times D_n$, in general, cover K with a finite number of polydisks □

Corollary 2.4. $\{u_k\} \subseteq \mathcal{O}(\Omega)$

1. (Montel) $\{u_k\}$ uniformly bounded on every compact $K \subseteq \Omega$, then $\exists u \in \mathcal{O}(\Omega)$, $k_j \in \mathbb{N}$ such that $u_{k_j} \rightrightarrows u$ uniformly on compact subsets
2. If $u_j \rightrightarrows u$, then $u \in \mathcal{O}(\Omega)$

Proof.

1. $\{\partial^\alpha u_j\}$ are equicontinuous(Arzela-Ascoli), $\{\partial^\alpha u_j\}$ is relatively compact w.r.t. uniform convergence. To finish, exhaust Ω by compact subsets, and take a diagonal process to assure relative compactness for all partial derivatives, Cauchy-Riemann conditions is satisfied for the limit

□

Theorem 2.5 (Cauchy's estimates)**.** If $|u(z)| \leq M$ on D, $|\partial^j u(0)| \leq M j_1! \cdots j_m! \frac{1}{r_1^{j_1}} \cdots \frac{1}{r_m^{j_m}}$

Theorem 2.6 (Hartogs' theorem)**.** $f : \Omega \to \mathbb{C}^n$, f is holomorphic in every variable separately, then $f \in \mathcal{O}(\Omega)$

Example 2.7. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} \\ 0 \end{cases}$, this can't be a counterexample in complex variables since $z_1^2 + z_2^2 = 0$ at points other than $(0, 0)$

Theorem 2.8 (Cauchy-Pompeiu formula)**.** $f \in C^1(\overline{U})$, $\int_{\partial U} f dz = \int_U d(f dz) = \int_U \bar{\partial} f \wedge dz = \int_U \frac{\partial f}{\partial \bar{z}}$

Theorem 2.9. $f \in C_0^\infty(\mathbb{C})$, $\frac{\partial u}{\partial \bar{z}} = f$ always has a solution $u \in C^\infty(\mathbb{C})$

Proof.

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}$$

□

Theorem 2.10. $f = \sum f_j d\bar{z}_j$, $f_j \in C_0^\infty(\mathbb{C}^n)$, $\bar{\partial} f = 0$, then there exists unique $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = f$

Proof.

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\tau - z_1} f_1(\tau, z-2, \cdots, z_n) d\tau \wedge d\bar{\tau}$$

$$\bar{\partial} f = 0 \Leftrightarrow \frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}$$

Need to show $\frac{\partial u}{\partial \bar{z}_k} = f_k$. $k = 1$, Cauchy-Pompeiu implies $\frac{\partial u}{\partial \bar{z}_1} f_1(z_1, \cdots, z_n)$, $k > 1$, $\frac{\partial u}{\partial \bar{z}_k} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\tau}\frac{\partial f_k}{\partial \bar{z}_1}(z_1 - \tau, z_2, \cdots, z_n) d\tau \wedge d\bar{\tau} = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{1}{\tau - z_1}\frac{\partial f_k}{\partial \bar{z}_1}(z_1 - \tau, z_2, \cdots, z_n) d\tau \wedge d\bar{\tau}$ Why is u compactly supported $\square$

3 Hartogs phenomenon

Theorem 3.1 (Hartogs phenomenon)**.** $\Omega \subseteq \mathbb{C}^n$ is open, $n > 1$, $K \subseteq \Omega$ is compact, $f \in \mathcal{O}(\Omega \setminus K)$, then there exists $g \in \mathcal{O}(\Omega)$ such that $f \equiv g$ on $\Omega \setminus K$

Remark 3.2. Holomorphic functions with more than one variable don't have isolated poles. The zero set of a holomorphic function with more than one variable is not contained in a compact set

Proof. $\varphi \in C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on a neighborhood of K, $v = \bar{\partial}(1-\varphi)f = -f\bar{\partial}\varphi \in (C_0^\infty)_{0,1}(\mathbb{C}^n)$, $\bar{\partial}v = \bar{\partial}(-f\bar{\partial}\varphi) = -f\bar{\partial}\bar{\partial}\varphi = 0 \Rightarrow \exists_1 u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial}g = -f\bar{\partial}\varphi - \bar{\partial}u = 0 \Rightarrow g \in \mathcal{O}(\Omega)$, on $\partial\Omega$, $1 - \varphi \equiv 1$, $u \equiv 0$, then use the identity theorem ☐

Definition 3.3. Ω is a domain of holomorphy if there exists $f \in \mathcal{O}(\Omega)$ such that f doesn't extend to Ω' where $\Omega \subsetneq \Omega' \subseteq \mathbb{C}^n$

Question 3.4. Characterize the domains of holomorphy

Consider $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$, D is the domain of convergence satisfies

1. D is *polycircular*, i.e. $z \in D \Rightarrow (e^{i\theta_1}z_1, \cdots, e^{i\theta_n}z_n) \in D$
2. D is *log-convex*, i.e. if $z^1, z^2 \in D$, then $(z^1)^\beta (z^2)^{1-\beta} \in D$, $0 < \beta < 1$, here $z^\beta = (z_1^\beta, \cdots, z_n^\beta)$, note that $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b$ is Young's inequality

Definition 3.5. D is *Reinhardt* if D is polycircular. D is *log-convex Reinhardt* if D also satisfies

- $D^* = \log|D| = \{(t_1, \cdots, t_n) \in \mathbb{R}^n | (e^{t_1}, \cdots, e^{t_n}) \in D\}$ is a convex set of $\mathbb{R}^n$ and $D^* + (\mathbb{R}_-)^n \subseteq D^*$

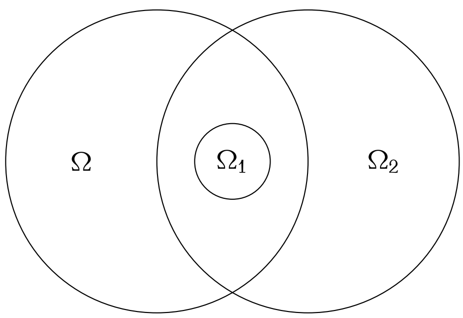
If D is Reinhardt, then $\hat{D} = \cap\{\tilde{D}$ Reinhardt and log convex, $D \subseteq \tilde{D}\}$ is the log-convex cover of D

Theorem 3.6. If $0 \in D \subseteq \mathbb{C}^m$ is connected Reinhardt, $f \in \mathcal{O}(D)$, then there exists $g \in \mathcal{O}(\hat{D})$ such that $g|_D = f$

Proof. $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$ in $D(0,\delta)^m$ extends to D, use the fact that the domain of absolute convergence of $\sum_{\alpha \geq 0} a_\alpha z^\alpha$ is a log convex Reinhardt domain containing D ☐

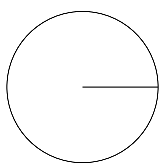
Definition 3.7. Domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy if there are no open sets Ω_1, Ω_2 such that

- $\varnothing \neq \Omega_1 \subseteq \Omega_2 \cap \Omega$



- For any $f \in \mathcal{O}(\Omega)$, there exists $g \in \mathcal{O}(\Omega_2)$ such that $f|_{\Omega_1} = g|_{\Omega_1}$

Remark 3.8. Local property of $\partial\Omega$, to rule out anomalies from slit type domain



Definition 3.9. $K \subseteq \Omega$ is compact, its holomorphically convex hull is

$$\hat{K}_\Omega = \left\{ z \in \Omega \middle| |f(z)| \leq \sup_K |f|, \forall f \in \mathcal{O}(\Omega) \right\}$$

Note. $K \subseteq \hat{K}_\Omega$

Remark 3.10. $\hat{K}_\Omega$ is contained in the convex hull of K

Proof. Let $L(\mathbb{C}^n)$ be the set of complex linear functionals of $\mathbb{C}^n$, $L_\mathbb{R}(\mathbb{C}^n)$ be the set of real linear funcitonals of $\mathbb{C}^n$, $\operatorname{Re} L(\mathbb{C}^n) = L_\mathbb{R}(\mathbb{C}^n)$

$$\begin{aligned}
\mathrm{convex}(K) &\xlongequal{\text{Hahn-Banach}} \{z \in \Omega | \Lambda(z) \leq \sup_K \Lambda, \Lambda \in L_\mathbb{R}(\mathbb{C}^n)\} \\
&= \{z \in \Omega | e^{\Lambda(z)} \leq e^{\sup_K \Lambda(z)}, \Lambda \in L_\mathbb{R}(\mathbb{C}^n)\} \\
&= \{z \in \Omega | e^{\operatorname{Re}\beta(z)} \leq e^{\sup_K \operatorname{Re}\beta(z)}, \beta \in L(\mathbb{C}^n)\} \\
&= \{z \in \Omega || e^{\beta(z)}| \leq \sup_K |e^\beta|, \beta \in L(\mathbb{C}^n)\}
\end{aligned}$$

□

Remark 3.11. $K \subseteq \Omega$ is compact and convex, then $\widehat{\partial K_\Omega} = K$. Maximal principle: $K \subseteq \widehat{K_\Omega} = \widehat{\partial K_\Omega}$

Proof. $\widehat{\partial K_\Omega} \subseteq \mathrm{convex}(\partial K) \xlongequal{\text{Milman-Kreim}} K$ □

Lemma 2.5.3

Lemma 3.12. Let D be a polydisk centered at $0 \in \mathbb{C}^n$, $\Delta^D_\Omega(z) = \sup\{r > 0, s.t. z + rD \subseteq \Omega\}$ is the "weighted" L^∞ distance of z to $\partial\Omega$. $r = \inf_{z\in K} \Delta^D_\Omega(z) > 0$, $K \subseteq \Omega$ is compact. Let $\xi \in \hat{K}_\Omega$, if $u \in \mathcal{O}(\Omega)$, then the power series expansion of u at ξ

$$f(z) = \sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha u(\xi)}{\partial \xi^\alpha} (z - \xi)^\alpha$$

is convergent for any $z \in \xi + rD$

Proof. $\sup_K u(z) \leq M$, then $|\frac{\partial^\alpha u(z)}{\partial z^\alpha}| \leq \frac{\alpha! M}{(r-\epsilon)^{|\alpha|}}$ for any $z \in K$, $\Rightarrow |\frac{\partial^\alpha u(\xi)}{\partial \xi^\alpha}| \leq \frac{\alpha! M}{(r-\epsilon)^{|\alpha|}}$, then $|u(z)| = |\sum \frac{1}{\alpha!} \frac{\partial^\alpha u(\xi)}{\partial \xi^\alpha} (z - \xi)^\alpha| \leq \sum_\alpha \frac{M}{|r-\epsilon|^\alpha} (z - \xi)^\alpha \xrightarrow{\text{Abel's criterion}}$ absolutely convergent for $z \in \xi + (r - \epsilon)D$, then let $\epsilon \searrow 0$ □

Definition 3.13. $\Omega \subseteq \mathbb{C}^n$ is holomorphically convex if for any $K \subseteq \Omega$ compact, $\hat{K}_\Omega$ is also compact

Theorem 3.14. $\Omega \subseteq \mathbb{C}^n$ is open, the following are equivalent

(i) Ω is a domain of holomorphy

(ii) $K \subseteq \Omega$ compact $\Rightarrow \hat{K}_\Omega \subseteq \Omega$ is compact, i.e. $K \subseteq \Omega$ is holomophically compact

(iii) $\exists f \in \mathcal{O}(\Omega)$ such that doesn't extend over any point $\in \partial\Omega$ holomorphically

Proof. (i)$\Rightarrow$(ii): $r = \inf_{z\in K} \Delta^D_\Omega(z) > 0 \xrightarrow{Lemma 3.12} \hat{K}_\Omega + rD \subseteq \Omega$
(iii)$\Rightarrow$(i): Trivial
(ii)$\Rightarrow$(iii): Let M be a dense, countable set in Ω. Let $\xi, \cdots, \xi_n, \cdots$ be a sequence containing each element of M infinitely many times. Let $K = K^1 \subseteq K^2 \subseteq \cdots \subseteq \Omega$ is another compact exhaustion of Ω, by (ii), we have $\hat{K}^1_\Omega \subseteq \hat{K}^2_\Omega \subseteq \cdots \subseteq \Omega$ is another compact exhaustion of Ω. $D_\xi = \xi + \Delta^D_\Omega(\xi) D \subseteq \Omega$. For any ξ_j, $\exists z_j \in D_{\xi_j}$ such that $z_j \in \hat{K}^j_\Omega$ because it's compact. $\exists f \in \mathcal{O}(\Omega)$ such that $\begin{cases} |f_j(z_j)| = 1 \\ \sup_{z\in K_j} |f_j(z)| < 1 - \epsilon \end{cases} \xRightarrow{\text{manipulate power}} \begin{cases} f_j(z_j) = 1 \\ \sup_{z\in K_j} |f_j(z)| < \frac{1}{2^j} \end{cases}$. $f(z) = \prod_{j=1}^\infty (1 - f_j)^j$.
Claim: $f \in \mathcal{O}(\Omega)$. $z \in \Omega \Rightarrow z + \epsilon D \subseteq K_j$ for all $j \geq j_0$. $\log \prod_{j\geq j_0} (1 - f_j)^j = \sum_{j\geq j_0} j \log(1 - f_j)$, thus $|\log(1 - f_j(z))| \leq C|f_j(z)|$, then $\sum_{j\geq j_0} j|\log(1 - f_j)| \leq \sum_{j\geq j_0} jC|f_j| \leq C \sum_{j\geq j_0} \frac{j}{2^j} < \infty$ □

Exercise 3.15. f is not identically zero, but $f(z_j) = 0$, $\frac{\partial^\alpha f}{\partial z^\alpha}(z_j) = 0$, for all $|\alpha| \leq j$. Assume f extends across $\partial\Omega$, contradiction

Example 3.16. $\Omega \subseteq \mathbb{C}^n$ convex and open $\Rightarrow$ Ω is a domain of holomorphy. Pf: Only need to show $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$. $\hat{K}_\Omega \subset \mathsf{convex}(K) \subseteq \Omega \Rightarrow \hat{K}_\Omega$ is closed and bounded $\Rightarrow \hat{K}_\Omega$ is compact

Reinhardt domains: $0 \in \Omega$ is a connected Reinhardt domain, $\Omega^* = \{\xi \in \mathbb{R}^n | (e^{\xi_1}, \cdots, e^{\xi_n}) \in \Omega\}$. Ω is log convex $\Leftrightarrow$ Ω^* is convex, and for any $\xi \in \Omega^*$, $\eta_j \leq \xi_j \Rightarrow \eta \in \Omega^*$

Theorem 3.17. Ω is a Reinhardt domain, tthen the following are equivalent

(i) Ω is a domain of holomorphy

(ii) Ω is log-convex

Proof. (i)$\Rightarrow$(ii): $\exists f \in \mathcal{O}(\Omega)$ that does not extend through any boundary point $\Rightarrow$ power series expansion of f at 0 absolutely converges exactly on $D \Rightarrow D$ is log-convex

(ii)$\Rightarrow$(i): Take $K \subset\subset \Omega$, $\exists \xi^j$ such that $K \subseteq \cup_{j=1}^k D(0, \xi_1^j) \times \cdots \times D(0, \xi_n^j)$, $z \in K \Rightarrow$ $|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \leq \sup_{1\leq j\leq k} |\xi_1^j|^{\alpha_1} \cdots |\xi_n^j|^{\alpha_n}$ which holds for $\alpha \in \mathbb{Q}_+^n$, by density, it also holds for $\alpha\mathbb{R}_+^n$, thus also holds for $z \in \hat{K}_\Omega$, assume that $z_1, \cdots, z_j \neq 0$, $z_{j+1} = \cdots = z_n = 0$. Take $\alpha = (\alpha_1, \cdots, \alpha_j, 0, \cdots, 0)$, $|z_1|^{\alpha_1} \cdots |z_j|^{\alpha_j} \leq \sup_{1\leq l\leq k}(\alpha_1 \log|\xi_1^l| + \cdots + \alpha_j \log|\xi_j^l|)$. claim (HW): $(\log|z_1|, \cdots, \log|z_j|)$ is in the convex hull of vectors $(\log\eta_1, \cdots, \log\eta_j)$ such that $\eta_j \leq \xi_j^l$ for some $l \in \{1, \cdots, k\}$ $\square$

Lemma 3.18. $f \in \mathcal{O}(\Omega)$ such that $|f(z)| \leq \Delta_\Omega^D(z) = \sup\{r > 0 | z + rD \subseteq \Omega\}$, $z \in K$, $\xi \in \hat{K}_\Omega$, $u \in \mathcal{O}(\Omega)$, then the power series expansion of u near ξ absolutely converges in $\xi + |f(\xi)|D$

4 Plurisubharmonicity

Definition 4.1. $\Omega \subseteq \mathbb{C}^n$, $u : \Omega \to [-\infty, \infty)$ is plurishbharmonic if

1. u is upper semi-continuous on Ω, $\limsup_{z \to z_0} u(z) \leq u(z_0)$
2. $w \in \mathbb{C}^n$, $z \in \Omega$, then $\tau \mapsto u(z + \tau w)$ is subharmonic, for all $\tau \in \mathbb{C}$ such that $z + \tau w \in \Omega$

Exercise 4.2. $f \in \mathcal{O}(\Omega) \Rightarrow \log |f| \in PSH(\Omega)$

Theorem 4.3. $u \in C^2(\Omega)$, then $u \in PSH(\Omega)$ iff $z \in \Omega$, $D^2 u(z)$ is semi-positive definite $\Rightarrow$ Pseudo-convexity

Proof. Restrict on one line, $\dfrac{\partial^2}{\partial \tau \partial \bar{\tau}} u \geq 0$ □

Theorem 4.4. $u \in PSH(\Omega)$, $\varphi \in C_0^\infty(\mathbb{C}^n)$, $\varphi(z) = \varphi(|z|)$, $\operatorname{supp} \varphi \subseteq \overline{B(0,1)}$. $\int_{\mathbb{C}^n} \varphi d\lambda = 1$. $u_\epsilon(z) = \dfrac{1}{\epsilon^{2n}} \int_{\mathbb{C}} u(z - \eta) \varphi(\frac{\eta}{\epsilon}) d\lambda(\eta)$, $z \in \Omega_\epsilon = \{z \in \Omega | d(z, \Omega^c) > \epsilon\}$, $u_\epsilon \in PSH(\Omega_\epsilon) \cap C^\infty(\Omega_\epsilon)$, $\forall z \in \Omega$, $u_\epsilon(z) \searrow u(z)$ as $\epsilon \searrow 0$

Proposition 4.5. $u_j \in PSH$ decreasing, then $v(x) = \lim_j u_j(x) \in PSH$

Theorem 4.6. $\Omega \subseteq \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^m$, $f : \Omega \to \Omega'$ is holomorphic, $u \in PSH(\Omega')$, then $u \in PSH(\Omega)$(subharmonicity define on complex manifold)

Proof. Check $u \in PSH(\Omega') \cap C^2(\Omega')$, $\dfrac{\partial^2}{\partial z_j \partial \bar{z}_k}(u \circ f(z)) = \dfrac{\partial^2 u}{\partial \xi_p \partial \bar{\xi}_q} \dfrac{\partial f_p}{\partial z_j} \overline{\dfrac{\partial f_q}{\partial z_k}} = \operatorname{Hess}_{\mathbb{C}}(u)(\partial f, \bar{\partial} f) \geq 0$. $K \subseteq \Omega$ is compact, $\hat{K}_\Omega^p = \{z \in \Omega | u(z) \leq \sup_{y \in K} u(y), \forall u \in PSH(\Omega)\} \Rightarrow \hat{K}_\Omega^p \subseteq \hat{K}_\Omega$ (PSH-hull). Ω is pseudo-convex if $\forall K \subseteq \Omega$ compact, $\hat{K}_\Omega^p \subseteq \Omega$ is also compact. Ω convex $\Rightarrow$ Ω pseudoconvex $\Rightarrow$ Ω is holomorphically convex □

Theorem 4.7. $\Omega \subseteq \mathbb{C}^n$ is open, the following are equivalent

(i) $z \mapsto -\log \delta(z, \Omega^c) \in PSH(\Omega)$

(ii) $\exists u \in PSH(\Omega) \cap C(\Omega)$ such that $\Omega_c^u = \{z \in \Omega | u(z) \leq c\} \subseteq \Omega$ compact

(iii) $K \subseteq \Omega$ compact $\Rightarrow \hat{K}_\Omega^p \subseteq \Omega$ compact

Remark 4.8. If u satisfies (ii), then ti is called a PSH exhaustion of Ω

Proof. (ii)$\Rightarrow$(iii): $K \subseteq \Omega$ compact, $\sup_{y \in K} u(y) = M < \infty$, $\hat{K}_\Omega^p \subseteq \{z \in \Omega | u(z) \leq M\} = \Omega_M^u \overset{\text{compact}}{\subseteq} \Omega$
(i)$\Rightarrow$(ii): $u(z) = -\log \delta(z, \Omega^c)$, check that $z \mapsto \delta(z, \Omega^c)$ is continuous, take $F(\tau)$ to be a polynomial, such that $-\log(z + \tau w, \Omega^c) \leq \operatorname{Re} F(\tau)$ on $\partial\Omega$, then there exists polynomial f such that $f(z + \tau w) = F(\tau)$, then $\delta(z + \tau w, \Omega^c) \geq |e^{-f(z+\tau w)}|$ on ∂D, by Theorem 2.5.4, Inequality holds on $\widehat{\partial D} \supseteq \Omega$ □

Theorem 4.9. $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex, $K \subseteq \Omega$ is compact, $z \in \widehat{H_\Omega^p}^c$ ($\Rightarrow \widehat{K_\Omega^p} \subseteq \Omega$ compact), then there exists $u \in C^\infty(\Omega)$ such that

(a) $u \in PSH(\Omega)$ and $i\partial\bar{\partial} u > \delta \sum_{j=1}^n dz_j \wedge d\bar{z}_j$

(b) $\Omega_c = \{u \leq c\} \subseteq \Omega$ is compact for any $c \in \mathbb{R}$

(c) $|u| < 0$ on K and $u(z) > 0$

Remark 4.10. You can find $v \in PSH(\Omega) \cap C(\Omega)$ such that (b) and (c) holds for v (to make v smooth)

Proof. $\Omega_c\{v < c\}$, $v_j(z) = \int_{\Omega_{j+1}} v(\xi)\frac{1}{\epsilon^{2n}}\varphi(\frac{z-\xi}{\epsilon})d\lambda(\xi) + \epsilon|z|^2$, ϵ small enough such that $v_j \in PSH(\Omega_j)$. Pick smooth function $\chi : \mathbb{R} \to \mathbb{R}$ non-decreasing and convex, $\chi(t) = 0$, $\forall t \leq 0$, $\chi(t) > 0$, $\forall t > 0$, $\chi'(t) > 0$, $\forall t > 0$. Recall: $\phi \in PSH \Rightarrow \chi \circ \phi \in PSH$ (Section 1.6). $\chi(v_j(z) - j + 1) = 0$, $\forall z \in \Omega_{j-2}$, $u_k(z) = v_0(z) + a_1\chi(v_1(z)) + a_2\chi(v_2(z) - 1) + \cdots + a_k\chi(v_k(z) - j + 1)$, here $v_0(z)$ is PSH in Ω_{-1}, $a_1\chi(v_1(z))$ is PSH on Ω_0 provided a_1 is big enough, $\cdots$, thus $u_k \xrightarrow[C^\infty]{} u \in C^\infty(\Omega) \cap PSH(\Omega)$ ☐

Example 4.11. Ω_1, Ω_2 pseudoconvex $\Rightarrow \Omega_1 \cap \Omega_2$ pseudoconvex

Proof. u_1, u_2 are PSH exhaustions for Ω_1, Ω_2, $v = \max(u_1|_{\Omega_1\cap\Omega_2}, u_2|_{\Omega_1\cap\Omega_2}) \in PSH(\Omega_1 \cap \Omega_2)$ is an exhaustion of $\Omega_1 \cap \Omega_2$ ☐

Theorem 4.12. $\Omega \subseteq \mathbb{C}^n$ is open bounded, pseudoconvex. $\forall z \in \bar{\Omega}$, $\exists B \ni z$ open ball such that $\Omega \cap B$ is pseudoconvex (HW: Show that balls are pseudoconvex)

Proof. $\Rightarrow$ is trivial. $\Leftarrow$: $\partial\Omega$ is compact, $\exists \tilde{B}_j \subset\subset B_j$ such that $\delta(y, (B_j \cap \Omega)^c) = \delta(y, \Omega^c)$ for all $y \in \tilde{B}_j$, $B_j \cap \Omega$ pseudoconvex $\Rightarrow -\log\delta(z, B_j^c)$ is PSH, thus $-\log(z, \Omega^c)$ is PSH on $\tilde{B}_j$, finite cover $\tilde{B}_1, \cdots, \tilde{B}_k$, $\exists F \subseteq \Omega$ closed such that $-\log(z, \Omega^c) \in PSH(\Omega \setminus F)$, $\exists M > -\log\delta(z, \Omega^c)$ on F, $\max(-\log\delta(z, \Omega^c), M) \in PSH(\Omega)$ ☐

Definition 4.13. A *densely defined operator* on Hilbert spaces H_1 is $T : \operatorname{Dom} T \leq H_1 \to H_2$ linear, $\operatorname{Dom} T$ dense in H_1, graph of T is closed

Theorem 4.14. $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$, the following hold

- $H_2 = (\ker S \cap \ker T^*) \oplus \overline{\operatorname{im} T} \oplus \overline{\operatorname{im} S^*}$
- $\ker S = (\ker S \cap \ker T^*) \oplus \overline{\operatorname{im} T}$
- If $\|T^*x\|_1^2 + \|Sx\|_3^2 \geq C\|x\|_2^2$, $\forall x \in \operatorname{Dom} S \cap \operatorname{Dom} T^* \leq H_2$. then $\operatorname{im} T = \ker S$, moreover, $\forall v \in \ker S$, $\exists u \in T$, such that $Tu = v$ with $\|u\|_1^2 \leq \frac{1}{C}\|v\|_2^2$

5 Homeworks

5.1 Homework1

Problem 5.1. Let (X, d) be a metric space

1. Let $f_j : X \to [-\infty, \infty)$ be a decreasing sequence of upper semi-continuous functions. Show that $f_j := \lim f_j$ is also upper semi-continuous
2. Let $f : X \to [-\infty, \infty)$ be an upper semi-continuous function such that $f(x) \leq M \in \mathbb{R}$ for all $x \in X$. Show that there exist a decreasing sequence of continuous functions such that $f_j \searrow f$ pointwise everywhere on X. [Hint: Show that the functions $f_j(x) := \sup_{y \in X}(f(y) - jd(y, x))$ satisfy the requirements]

Solution.

1. The infinimum of a family of upper semicontinuous functions is again upper semicontinuous
2. Consider $f_n(x) = \sup_{y \in X}(f(y) - nd(y, x))$ which surely is monotone decreasing, for any fixed x, it is obvious $f(x) \leq f_n(x)$, suppose $\lim_{n\to\infty} f_n(x) > f(x)$, then $\exists y_n, f(y_n) - nd(y_n, x) - f(x) > \eta$ for some $\eta > 0$, hence $d(y_n, x) < \dfrac{f(y_n) - f(x) - \eta}{n} \leq \dfrac{M - f(x) - \eta}{n}$, thus $\lim_{n\to\infty} y_n = x$, since f is semicontinuous, $\exists \delta > 0$, such that$f(y) < f(x) + \eta, \forall y \in B(x, \delta)$, thus $\exists N$, such that $f(y_n) < f(x) + \eta, \forall n > N$, but then $\eta > f(y_n) - f(x) \geq f(y_n) - nd(y_n, x) - f(x) > \eta$ which is a contradiction. Therefore, $\lim_{n\to\infty} f_n(x) = f(x)$. Next we will prove that f_n is indeed continuous, since $f_n(x)$ could be seen as the supremum of a family of continuous functions in x over the family $\{f(y) - nd(y, x)\}_{y \in X}$, it is lower semicontinuous. To show that f_n is also upper semicontinuous, we only need to show that, $\forall x \in \{f_n < a\}, \exists \delta > 0$, such that $B(x, \delta) \in \{f_n < a\}$. we have $f(z) - nd(z, y) \leq f(z) - nd(x, z) + nd(y, x) \leq f_n(x) + nd(y, x) < a \Rightarrow f_n(y) < a$, as long as δ is small enough

□

Problem 5.2. Let $\Omega \subset \mathbb{R}^n$ and $f \in C^2(\Omega)$ a real valued. If $x \in \Omega$ show that

$$\lim_{r\to 0} \frac{\int_{\mathbb{S}^{n-1}} f(x + r\xi) d\xi - f(x)}{r^2 \mu(\mathbb{S}^{n-1})} = \frac{1}{n} \Delta f(x) := \frac{1}{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_i} f(x),$$

where $d\xi$ is the surface measure of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $\mu(\mathbb{S}^{n-1})$ is the surface area of the unit sphere. [Hint: Use Taylor's formula and some linear algebra wisdom. Also, it was pointed out to me that the constant $\frac{1}{n}$ may need to adjusted in front of $\Delta f(x)$ on the right hand side. I leave it up to you to find the correct constant, which your precise calculations should naturally yield]

Solution.

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} f(x + r\xi) \mathrm{d}\xi - \mu(\mathbb{S}^{n-1}) f(x) &= \int_{\mathbb{S}^{n-1}} (f(x + r\xi) - f(x)) \\
&= \int_{\mathbb{S}^{n-1}} \left(r\xi^T Df(x) + \frac{r^2}{2} \xi^T D^2 f(\eta) \xi \right) \\
&= \int_{\mathbb{S}^{n-1}} \frac{r^2}{2} \xi^T D^2 f(\eta) \xi
\end{aligned}$$

Where $\eta = x + \theta r\xi, 0 < \theta < 1$ depends on $r\xi$. Then we have

$$
\begin{aligned}
\lim_{r\to 0}\frac{\int_{\mathbb{S}^{n-1}} f(x+r\xi)\mathrm{d}\xi-\mu(\mathbb{S}^{n-1})f(x)}{r^2\mu(\mathbb{S}^{n-1})} &= \frac{1}{2\mu(\mathbb{S}^{n-1})}\lim_{r\to 0}\int_{\mathbb{S}^{n-1}}\xi^T D^2 f(\eta)\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\xi^T D^2 f(x)\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\xi^T P^T \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n^2}(x)\end{pmatrix} P\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\zeta^T \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n^2}(x)\end{pmatrix} \zeta \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\frac{\partial^2 f}{\partial x_1^2}(x)\zeta_1^2+\cdots+\frac{\partial^2 f}{\partial x_n^2}(x)\zeta_n^2 \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} V\cdot\zeta \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{B}^{n}} \mathrm{div}V \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{B}^{n}} \Delta f(x) \\
&= \frac{1}{2n}\Delta f(x)
\end{aligned}
$$

Where $P\in O(n)$, $\zeta := P\xi$, $V=\left(\frac{\partial^2 f}{\partial x_1^2}(x)\zeta_1,\cdots,\frac{\partial^2 f}{\partial x_n^2}(x)\zeta_n\right)^T$ □

5.2 Homework2

Problem 5.3 (Unique analytic continuation)**.** Let $\Omega \subset \mathbb{C}^n$ open and connected, and $f, g \in A(\Omega)$. If $f = g$ on an open subset of Ω show that $f = g$ everywhere on Ω

Problem 5.4. Show that the function $u \in C_0^k(\mathbb{C}^n)$ constructed in Theorem 2.3.1 is unique

5.3 Homework3

Problem 5.5. Let $\Omega \subset \mathbb{C}^n$ be open and let P be a polydisk, whose closure is contained in Ω. Show that $\widehat{\partial_0 P}_\Omega = \overline{P}$, where $\partial_0 P$ is the distinguished boundary of P

Problem 5.6. Argue precisely why the function f constructed in the proof of Theorem 2.5.5 can not be identically zero!

Problem 5.7. $\mathbb{C}^n$ can be viewed as a $2n$-dimensional real vector space and an n-dimensional complex vector space. Show that any $\mathbb{R}$-linear functional on $\mathbb{C}^n$ is the real part of a $\mathbb{C}$-linear functional on $\mathbb{C}^n$

5.4 Homework4

Problem 5.8. Let $\lambda := (\lambda_1, \ldots, \lambda_j)$, $z := (z_1, \ldots, z_j)$ and $\xi := (\xi_1, \ldots, \xi_j)$ be as in the proof of Corollary 2.5.8. Show that

$$\sum_{i=1}^{j} \lambda_i \log |z_i| \leq \sup_{\xi \in k} \sum_{i=1}^{j} \lambda_j \log |\xi_i|, \ \forall \lambda \in \mathbb{R}_+^n \quad \text{with } \ \lambda_1 + \ldots + \lambda_j = 1$$

is equivalent with $(\log |z_1|, \ldots, \log |z_j|)$ being in the convex hull of the set of all points $(\eta_1, \ldots, \eta_j)$ such that $\eta_i \leq \log |\xi_i|$ for $1 \leq i \leq j$. [Hint: One direction is easy. For show that being in the convex hull implies the inequality use the fact that a closed convex set is always the intersection of half spaces]

Problem 5.9. Let δ as defined by Hörmander on page 37. Show that $z \to \delta(z, \Omega^c)$ is a continuous on $\mathbb{C}^n$, where Ω is an open subset of $\mathbb{C}^n$

5.5 Homework5

Problem 5.10. In Theorem 1.1 of Chapter VIII.1 (Demailly's textbook): argue carefully that $T^{**} = T$ and $Ker\ T^{\perp} = \overline{Im\ T^{*}}$

5.6 Homework6

Problem 5.11. Given a Hermitian metric $h := \sum_{j,k} h_{j,\bar{k}} dz_j \wedge \overline{dz_k}$ on a complex manifold Ω, show that it is possible to define a Hermitian metric on the vector bundle of $(p,\bar{q})$-forms on Ω

References

[1] *An Introduction to Complex Analysis in Several Variables* - Lars Hörmander

[2] *Complex Analytic and Differential Geometry* - Jean-Pierre Demailly

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