Definition 0.0.1. An n dimensional *manifold* M is a topological space such that for any $p \in M$, there is a neighborhood $U \ni p$ such that U is homeomorphic to $\mathbb{R}^n$

Definition 0.0.2. A *chart* of $U \subseteq M$ is a homeomorphism $\varphi_U : U \to \varphi_U(U) \subseteq \mathbb{R}^n$, by abuse of notation, let $x_i : U \to \mathbb{R}$ be the composition $U \xrightarrow{\varphi_U} \mathbb{R}^n \xrightarrow{x_i} \mathbb{R}$, called *local coordinates*, suppose $\varphi_V : V \to \varphi_V(V) \subseteq \mathbb{R}^n$ is another chart, and $U \cap V \neq \varnothing$, the *transition map* is $\tau_{VU} = \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$
M is a $\mathbf{C^k}$ *manifold* if transition maps $\{\tau_{VU}\} \subseteq C^k$, if $k = 0$, M is just a topological manifold, if $k = \infty$, M is a *smooth manifold*, if $k = \omega$, M is an *analytic manifold*
If n is even and transition maps are holomorphic, M is a *complex manifold*

Remark 0.0.3. A smooth manifold is locally ringed space locally ringed space with structure sheaf the sheaf of differentiable functions. An analytic manifold is locally ringed space locally ringed space with structure sheaf the sheaf of analytic functions. A complex manifold is locally ringed space with structure sheaf the sheaf of holomorphic functions

Definition 0.0.4. A *smooth map* $f : M \to N$ is map such that $\psi_V \circ f \circ \varphi_U^{-1}$ is smooth. $C^\infty(M)$ are smooth functions on M. $C_p^\infty(M)$ is the germ at p

Definition 0.0.5. A *submanifold* N is a inclusion and an immersion $i : N \hookrightarrow M$

Definition 0.0.6. The kernel of $C_p^\infty(M) \to \mathbb{R}$, $f \mapsto f(p)$ is a maximal ideal m_p, define the *cotangent space* $T_p^*M := \dfrac{m_p}{m_p^2}$, for $f \in C^\infty(M)$, define $(df)_p = f - f(p) \operatorname{mod} m_p$, $(dx_1)_p, \cdots, (dx_n)_p$ form a basis of T_p^*M locally, $(df)_p = \dfrac{\partial f}{\partial x_1}(p)(dx_1)_p + \cdots + \dfrac{\partial f}{\partial x_n}(p)(dx_n)_p$

Definition 0.0.7. The *tangent space* T_pM at p are the derivations $Der(C_p^\infty(M))$
$\left(\dfrac{\partial}{\partial x_1}\right)_p, \cdots, \left(\dfrac{\partial}{\partial x_n}\right)_p$ form a basis of T_pM locally

Definition 0.0.8. The *pushforward(differential)* of smooth map $\phi : M \to N$ is $\phi_p : T_pM \to T_pN$, $\phi_p(X_p)(f) = X_p(f \circ \phi)$

Definition 0.0.9. The *pullback* of smooth map $\phi : M \to N$ is $\phi^* : C^\infty(N) \to C^\infty(N)$, $\phi^*(f) = f \circ \phi$

Definition 0.0.10. $(\varphi^*\alpha)_x(X) = \alpha_{\varphi(x)}(d\varphi_x(X)) = \alpha_{\varphi(x)}((\varphi_*)_x(X))$, or in short $\varphi^*\alpha(X) = \alpha(\varphi_*X)$, similarly, for k forms, $\varphi^*\alpha(X_1, \cdots, X_k) = \alpha(\varphi_*X_1, \cdots, \varphi_*X_k)$
In particular, $\varphi^*(dx) = d(x \circ \varphi)$, pullback is compatible with wedge product, $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$, and pullback is compatible with exterior derivative, $\varphi^*(d\alpha) = d(\varphi^*\alpha)$ The exterior multiplication by α is $\beta \mapsto \alpha \wedge \beta$ The interior multiplication by $v \in TM$ is $v \lrcorner : \omega(-) \mapsto \omega(v, -)$

Definition 0.0.11. $\pi : T^*M \to M$ is the cotangent bundle, ω is a one form, the *tautological one form* is $\pi^*\omega$

Definition 0.0.12. Let M be a smooth manifold, $X, Y \in C^\infty(M, TM)$ are vector fields, define Lie bracket $[X, Y] \in C^\infty(M, TM)$, $[X, Y](f) := (XY - YX)(f) = X(Y(f)) - Y(X(f))$

Remark 0.0.13. Check from local coordinates, $X(Y(f))$ is not well defined

Definition 0.0.14. Let M, N be smooth manifolds, $f : M \to N$ is a smooth map, it is called an immersion if df is injective at any point, it is called submersion if df is surjective at any point

Constant rank mapping theorem

Theorem 0.0.15. Suppose M, N are smooth manifolds with dimension m, n, $f : M \to N$ is a smooth map with constant rank r, then for any $p \in M$, denote $f(p) = q$, there are local charts (p, U), (q, V) such that $\chi_V \circ f \circ \chi_U(x_1, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots, 0)$. Moreover, suppose M is second countable, if f is injective, then f is a immersion, if f is surjective, then f is a submersion, if f is bijective, then f is a diffeomorphism

Proof. If f is surjective but not a submersion, then $r < n$, but then by Theorem **??**, f can't be surjective which is a contradiction $\square$

Constant rank level set theorem

Theorem 0.0.16. Suppose M, N are smooth manifolds with dimension m, n, $f : M \to N$ is a smooth map with constant rank r, then a level set $S = f^{-1}(c)$ is an embedded submanifold in M of codimension r with $f|_S$ being a proper map

Proposition 0.0.17. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M, for any equivariant map $f : M \to N$, f has constant rank

Proof. For any $x \in M$, denote $y = f(x)$, it suffices to show $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ since G acts transitively on M, note that $f(gx) = gf(x)$, thus $fL_g = L_g f$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ $\square$

Stokes' theorem

Theorem 0.0.18 (Stokes' theorem)**.** $\langle \partial\Omega, \omega\rangle = \langle \Omega, d\omega\rangle$, here $\langle \Omega, \omega\rangle = \int_\Omega \omega$

Theorem 0.0.19 (de Rham's theorem)**.** M is a smooth manifold. $H^p_{\mathtt{dR}}(X;\mathbb{R}) \xrightarrow{\cong} H^p(X;\mathbb{R})$ is an isomorphism

Proof. Since $\mathbb{R}$ is a divisible abelian group, thus an injective $\mathbb{Z}$ module, hence $\mathtt{Ext}^1_{\mathbb{Z}}(A, \mathbb{R})$, thus universal coefficient theorem gives exact sequence

$$0 = \mathtt{Ext}^1_{\mathbb{Z}}(H_{p-1}(X;\mathbb{Z}), \mathbb{R}) \to H^p(M;\mathbb{R}) \to \mathtt{Hom}(H_p(X;\mathbb{Z}), \mathbb{R}) \to 0$$

The isomorphism is given by $H^p_{\mathtt{dR}}(X;\mathbb{R}) \to \mathtt{Hom}(H_p(X), \mathbb{R})$, $\omega \mapsto \int_- \omega$ $\square$

Definition 0.0.20. $E \to X$ is a vector bundle, a connection on E is an $\mathbb{R}$ linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E)$$

satisfying Leibniz rule $\nabla(f\sigma) = f \otimes \nabla\sigma + df \otimes \sigma$

Lemma 0.0.21. $\nabla_X(\sigma) = (\nabla\sigma)(X)$ defines a covariant derivative, conversely, every covariant derivative is defined this way