My mathematical universe

Haoran Li

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Contents

I Set theory 9

1 Set theory 11

2 Graph theory 13
2.1 Graph . . . 13

II Abstract Algebra 15

3 Category 17
3.1 Category . . . 17
3.2 Yoneda lemma . . . 21
3.3 Limits . . . 22
3.4 Adjunction . . . 24
3.5 Pushout and pullback . . . 25
3.6 Filtered category . . . 26
3.7 Comma category . . . 27
3.8 Sheaves . . . 28
3.9 Exponential object . . . 30
3.10 Factorization system . . . 31
3.11 Monoidal category . . . 32

4 Group 33
4.1 Groups . . . 33
4.2 Permutation group . . . 35
4.3 Coxeter group . . . 36
4.4 Grothendieck group . . . 37

5 Ring 39
5.1 Rings . . . 39
5.2 Commutative rings . . . 40

6 Module 43
6.1 Modules . . . 43

7 Field 47
7.1 Fields . . . 47
7.2 Number field . . . 48

8 Linear algebra 49
8.1 Vector spaces . . . 49
8.2 Matrices . . . 50
8.3 Eigenspace decomposition . . . 52

9 Lie algebra **53**
9.1 Non-associative algebra . . . 53
9.2 Lie algebras . . . 54
9.3 Engel's theorem and Lie's theorem . . . 56
9.4 Killing form . . . 57
9.5 Jordan-Chevalley decomposition . . . 58
9.6 Classification of semisimple Lie algebras . . . 60
9.7 Root system . . . 68
9.8 Dynkin diagram . . . 72

10 Cluster algebra **77**
10.1 Cluster algebra . . . 77
10.2 Laurent phenomenon . . . 81
10.3 Y-system . . . 84
10.4 Associahedron . . . 86
10.5 Cluster algebra of geometric type . . . 87
10.6 Rank two case . . . 89
10.7 Cluster algebra of finite type . . . 91

11 Polylogarithm **93**
11.1 Polylogarithm . . . 93

III Commutative Algebra **99**

12 p-adic numbers **101**

IV Homological Algebra **103**

13 Abelian category **105**

14 Spectral sequence **111**

V General topology **115**

15 General topology **117**
15.1 General topology . . . 117
15.2 Retract . . . 122

VI Algebraic Topology **123**

16 Cell structure **125**
16.1 CW complexes . . . 125
16.2 Graphs . . . 127
16.3 Simplex category . . . 128
16.4 abstract simplicial complex . . . 132
16.5 CW approximation . . . 133
16.6 Triangulation . . . 134

17 Homology theory **135**
17.1 Singular homology . . . 135
17.2 Cellular homology . . . 139

18 Cohomology theory **141**
18.1 Singular cohomology 141
18.2 Cech cohomology 143
18.3 Poincare duality 144

19 Homotopy theory **145**
19.1 Homotopy 145
19.2 Model category 147
19.3 Hurewicz fibration 150
19.4 Serre fibration 156

20 Isotopy **157**

21 Bundle **159**
21.1 Bundles 159
21.2 Vector bundles 161
21.3 Principal bundle 166
21.4 Topological K-theory 167
21.5 Classifying space 168

VII Differential topology **169**

22 Smooth manifold **171**

VIII Differential geometry **173**

23 Riemannian manifold **175**
23.1 Differential geometry of surfaces 175
23.2 Curvature 176
23.3 Hyperbolic geometry 178

IX Complex geometry **181**

24 Riemann surface **183**

25 Complex manifold **185**

26 Symplectic manifold **187**

X Lie group **189**

27 Topological group **191**

28 Algebraic group **193**

29 Lie group **195**
29.1 Lie groups 195
29.2 Exponential map 196

XI Algebraic geometry 199

30 Variety 201
30.1 Affine Varieties 201
30.2 Varieties 203

31 Scheme 205
31.1 Affine schemes 205
31.2 Schemes 206

XII Analysis 207

32 Real analysis 209

33 Complex analysis 211
33.1 Complex analysis 211
33.2 Conformal mapping 216
33.3 Weierstrass functions 217
33.4 Zeta function 218

34 Functional analysis 219
34.1 Topological vector space 219
34.2 Arzela-Ascoli theorem 220
34.3 Baire category theorem 221
34.4 Distribution 222
34.5 Banach algebra 223
34.6 Stone-Weierstrass theorem 226

XIII Differential equations 227

35 Ordinary differential equations 229

36 Classical partial differential equations 231
36.1 Laplace's equation 231
36.2 Heat equation 232
36.3 Wave equation 234
36.4 Euler-Lagrange equation 237
36.5 Energy momentum tensor 238

XIV Mathematical physics 239

37 Special relativity 241

38 Maxwell's equations 243

XV Examples 245

39 Examples in categories 247

40 Examples in algebra 249

41 Examples in algebraic topology 251

42 Examples in geometry **253**

43 Examples in Lie groups and Lie algebras **255**

44 Examples in algebraic geometry **257**

45 Examples in analysis **259**

XVI Exercises 261

46 Exercises in combinatorics **263**
46.1 Exercises in abstract algebra 264
46.2 Exercises in analysis 265

47 Exercises in category **273**

48 Exercises in partial differential equations **277**

49 Exercises in algebraic topology **279**

50 Exercises in differential topology **281**

51 Exercises in bundles **283**

52 Exercises in complex geometry **285**

53 Exercises in Lie groups and Lie algebras **287**

54 Exercises in algebraic geometry **291**

55 Exercise in functional analysis **293**

XVII Miscellaneous 295

56 Hodge structure **297**

57 Plucker embedding **299**

58 Graph theory **301**

59 Moduli space **303**

60 Teichmüller space **305**

61 Mapping class group **307**

Index **311**

Part I

Set theory

Chapter 1

Set theory

Definition 1.0.1. $\{A_i\} \subseteq \mathcal{P}(X)$, $X \xrightarrow{f} Y$ is a map. f **separates** A_i if $\bigcap_i f(A_i) = \varnothing$. f **completely separates** A_i if $f(A_i) = f(a_i)$ for some distinct $a_i \in A_i$. f **perfectly separates** A, B if $A_i = f^{-1}(a_i)$ for some $a_i \in A_i$

Zorn's lemma

Lemma 1.0.2 (Zorn's lemma)**.** P is a nonempty poset and every chain has an upper bound, then P contains a maximal element

Theorem 1.0.3 (Schröder–Bernstein theorem)**.** $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ are injective, then there exists $A \xrightarrow{h} B$ bijective

Inclusion-exclusion principle

Theorem 1.0.4 (Inclusion-exclusion principle)**.** $A_1, \cdots, A_n \subseteq S$ are of finite cardinality, then

$$\left|\bigcup_{i=1}^{n} A_n\right| = \sum_{k=1}^{n}(-1)^k \sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

Definition 1.0.5. A **lattice** is a partially ordered set in which the supremum and infinimum of any two elements exists uniquely

Lemma 1.0.6. Trees are bipartite

Proof. Take some $v \in T$ as the root, and label the nodes that are even distance away from 2 and odd distance away from 1 $\square$

Definition 1.0.7. A **Hasse diagram** is a mathematical diagram used to represent a partially ordered set

$$\begin{tikzcd}
 & \{x,y,z\} & & 2 \\
\{x,y\} \arrow[ur] & \{x,z\} \arrow[u] & \{y,z\} \arrow[ul] & 1 \\
\{x\} \arrow[u] \arrow[ur] & \{y\} \arrow[ul] \arrow[ur] & \{z\} \arrow[u] \arrow[ul] & 0 \\
 & \{\} \arrow[ul] \arrow[u] \arrow[ur] & & -1
\end{tikzcd}$$

Chapter 2

Graph theory

2.1 Graph

Definition 2.1.1. A **complete graph** K_n consists of n vertices and all possible edges **Com-**

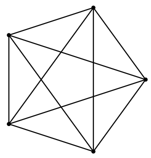


Figure 2.1.1: K_5

K5 complete graph

plete bipartite graphs are $K_{n,m}$ with n, m vertices on each side and all possible edges between them

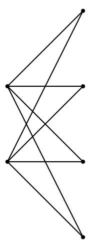


Figure 2.1.2: $K_{2,4}$

K2,4 complete bipartite graph

Definition 2.1.2. G is k vertex connected if G has more than k vertices and remain connected when removing less than k vertices

Proposition 2.1.3. Every convex polytope can be represented by a 3 vertex connected planar graph

Remark 2.1.4. A graph embedded in the plane through different ways may have different dual graphs However, if the graph is 3 vertex connected, then the dual graph will be canonical

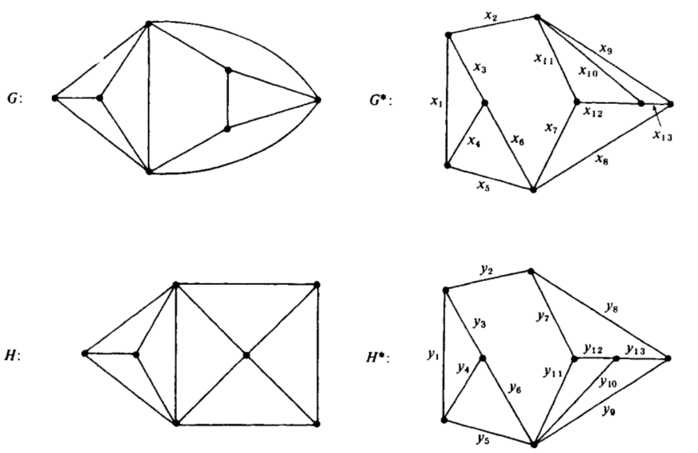


Figure 2.1.3: Different dual graphs for different planar embeddings of the same graph

Different dual graphs

Part II

Abstract Algebra

Chapter 3

Category

3.1 Category

Definition 3.1.1. A **category** $\mathscr{C}$ consists of $\mathrm{ob}\mathscr{C}$ class of **objects** and $\mathrm{Hom}\mathscr{C}$ class of **morphisms**, composition $\circ : \mathtt{Hom}(A,B) \times \mathtt{Hom}(B,C) \to \mathtt{Hom}(A,C)$, $(h \circ g) \circ f = h \circ (g \circ f)$, $\mathtt{Hom}(A,A)$ contains **identity** 1_A such that $1_A f = f$, $g 1_A = g$
Identity 1_A is unique since $1'_A = 1_A \circ 1_A = 1_A$. $g \in \mathrm{Hom}(B,A)$ is the **inverse** of $f \in \mathrm{Hom}(A,B)$ if $g \circ f = 1_A$, $f \circ g = 1_B$. Inverse f^{-1} is unique since $g = g \circ f \circ g' = g'$
A **functor** is $F : \mathscr{C} \to \mathscr{D}$, $\mathtt{ob}\mathscr{C} \to \mathtt{ob}\mathscr{D}$, $\mathtt{Hom}_{\mathscr{C}}(A,B) \to \mathtt{Hom}_{\mathscr{D}}(F(A),F(B))$, $F(1_A) = 1_{F(A)}$, $F(g \circ f) = F(g) \circ F(f)$
The **dual category** of $\mathscr{C}$ is $\mathscr{C}^{op}$ with the same objects but morphisms reversed, if $f \in Hom_{\mathscr{C}}(X,Y)$, then $f^{op} \in Hom_{\mathscr{C}^{op}}(Y,X)$, $(fg)^{op} = g^{op} f^{op}$, 1_X^{op} is still the identity. A **contravariant functor** $F : \mathscr{C} \to \mathscr{D}$ is a functor $F : \mathscr{C}^{op} \to \mathscr{D}$, equivalently, $\mathsf{Ob}\mathscr{C} \to \mathsf{Ob}\mathscr{D}$, $\mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(F(B),F(A))$, $F(1_A) = 1_{F(A)}$, $F(g \circ f) = F(f) \circ F(g)$. Functors are also called **covariant fucntors**

Definition 3.1.2. A **semicategory** $\mathscr{C}$ consists of $\mathtt{ob}\mathscr{C}$ class of **objects** and $\mathrm{Hom}\mathscr{C}$ class of **morphisms**, composition $\circ : \mathtt{Hom}(A,B) \times \mathtt{Hom}(B,C) \to \mathtt{Hom}(A,C)$, $(h \circ g) \circ f = h \circ (g \circ f)$.
A **semifunctor** is $F : \mathscr{C} \to \mathscr{D}$, $\mathtt{ob}\mathscr{C} \to \mathtt{ob}\mathscr{D}$, $\mathtt{Hom}_{\mathscr{C}}(A,B) \to \mathtt{Hom}_{\mathscr{D}}(F(A),F(B))$, $F(g \circ f) = F(g) \circ F(f)$

Remark 3.1.3. A category is a semicategory with identities

Definition 3.1.4. $A \xrightarrow{f} B$ is a **monomorphism** if $fg_1 = fg_2 \Rightarrow g_1 = g_2$, is an **epimorphism** if $g_1 f = g_2 f \Rightarrow g_1 = g_2$, is, is a **bimorphism** is both monic and epi, is an **isomorphism** if it is invertible. Monomorphism and epimorphism are dual notions. Isomorphisms are bimorphisms. A category is **balanced** if bimorphisms are isomorphisms

Remark 3.1.5. A bimorphism is not necessary an isomorphism. In the category of rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a bimorphism because $\mathbb{Q} = \mathbb{Z}_{(0)}$ is a localization and the universal property of localization

Definition 3.1.6. A **natural transformation** is a family of morphisms $\eta_A : F(A) \to G(A)$ making the following diagram commute

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle \eta_A} & & \downarrow{\scriptstyle \eta_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

For contravariant functors

$$\begin{array}{ccc} F(B) & \xrightarrow{F(f)} & F(A) \\ \downarrow{\scriptstyle \eta_B} & & \downarrow{\scriptstyle \eta_A} \\ G(B) & \xrightarrow{G(f)} & G(A) \end{array}$$

η is a **natural isomorphism** if η_A are isomorphisms

Definition 3.1.7. $\mathscr{C}$ is a **small categroy** if $ob(\mathscr{C})$ and $Hom(\mathscr{C})$ are sets, otherwise **large**. $\mathscr{C}$ is a **locally small category** if $Hom(a, b)$ are sets

Definition 3.1.8. A **subcategory** $\mathscr{S}$ is a category consists of subclasses of objects and morphisms with the same composition map

Definition 3.1.9. we say categories $\mathscr{C}, \mathscr{D}$ are **isomorphic** if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F = 1_{\mathscr{C}}$, $F \circ G = 1_{\mathscr{D}}$ and we say $\mathscr{C}, \mathscr{D}$ are **equivalent** if $G \circ F$ is naturally isomorphic to $1_{\mathscr{C}}$ and $F \circ G$ is naturally isomorphic to $1_{\mathscr{D}}$

Definition 3.1.10. Suppose $\mathscr{C}, \mathscr{D}$ are categories, define the **functor category** $[\mathscr{C}, \mathscr{D}]$ or $\mathscr{D}^{\mathscr{C}}$ has all functors from $\mathscr{C}$ to $\mathscr{D}$ as objects, and natural transformations as morphisms

Definition 3.1.11. $\mathscr{C} \times \mathscr{D}$ is the **product category** with $ob\mathscr{C} \times D = ob\mathscr{C} \times ob\mathscr{D}$, $Hom_{\mathscr{C}\times\mathscr{D}}(A \times B, C \times D) = Hom_{\mathscr{C}}(A, C) \times Hom_{\mathscr{D}}(B, D)$

Definition 3.1.12. Suppose $\mathscr{C}, \mathscr{D}$ are locally small categories, F is **faithful** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is injective, F is **full** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is surjective, F is **fully faithful** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is bijective, F is **essentially surjective** if $\forall d \in ob\mathscr{D}$, $\exists c \in ob\mathscr{C}$ such that $Fc \cong d$

A functor F is an equivalence iff it is fully faithful and essentially surjective

Theorem 3.1.13. $F : \mathscr{C} \to \mathscr{D}$ is an equivalence iff F is fully faithful and essentially surjective

Proof. If F is an equivalence, there exist functor $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : 1_{\mathscr{C}} \to GF$, $\xi : 1_{\mathscr{D}} \to FG$, $\forall d \in \mathscr{C}$, $\xi_d : d = 1_{\mathscr{D}}(d) \to FG(d) = F(Gd)$ is an isomorphism, i.e. F is essentially surjective, similarly, so is G
The composition of

$$Hom(c, c') \xrightarrow{F} Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc'), \quad f \mapsto Ff \mapsto GFf$$

Is the same as

$$Hom(c, c') \xrightarrow{\eta} Hom(GFc, GFc'), \quad f \mapsto \eta'_c f \eta_c^{-1}$$

By Exercise 47.0.1, this is bijective, thus $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is injective, i.e. F is faithful. Similarly, consider the composition

$$Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$$

We know $Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$ is surjective, but we also have the following diagram

$$\begin{tikzcd}
Hom(c, c') \arrow[r, "F"] \arrow[d, "\eta"'] & Hom(Fc, Fc') \arrow[d, "\xi"] \\
Hom(GFc, GFc') \arrow[r, "F"] & Hom(FGFc, FGFc')
\end{tikzcd}$$

Since η, ξ are bijective, $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is surjective, i.e. F is full
Conversely, suppose F is fully faithful and essentially surjective, then for any $d \in \mathscr{D}$, there exists c and an isomorphism $d \xrightarrow{\xi_d} Fc$, denote this c as Gd, we can define a functor $G : \mathscr{D} \to \mathscr{C}$, $d \mapsto Gd$(Here we have used the axiom of choice), $d \xrightarrow{f} d' \mapsto c \xrightarrow{Gf} c'$ where $FGf = \xi_d^{-1} f \xi_{d'}$ since F is fully faithful

$$\begin{tikzcd}
d \arrow[r, "f"] \arrow[d, "\xi_d"'] & d' \arrow[d, "\xi_{d'}"] \\
FGd \arrow[r, "FGf"] & FGd' \\
Gd \arrow[r, "Gf"] \arrow[u, "F"] & Gd' \arrow[u, "F"']
\end{tikzcd}$$

$\xi : 1_{\mathscr{D}} \to FG$ is a natural isomorphism
Since F is fully faithful, there are unique $\eta_c : c \to GFc$, $F(\eta_c) = \xi_{Fc}$
If $f, g : c \to c'$ such that $\eta_{c'} f = \eta_{c'} g$, then $\xi_{Fc'} Ff = \xi_{Fc'} Fg \Rightarrow Ff = Fg \Rightarrow f = g$
If $f, g : c \to c'$ such that $f\eta_c = g\eta_c$, then $Ff\xi_{Fc} = Fg\xi_{Fc} \Rightarrow Ff = Fg \Rightarrow f = g$

$$\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dd, bend right, "\eta_c"'] & c' \arrow[d] \arrow[dd, bend left, "\eta_{c'}"] \\
Fc \arrow[r] \arrow[d, "G"] \arrow[dd, bend right, "\xi_{Fc}"'] & Fc' \arrow[d, "G"'] \arrow[dd, bend left, "\xi_{Fc'}"] \\
GFc \arrow[r] \arrow[d, "F"] & GFc' \arrow[d, "F"'] \\
FGFc \arrow[r] & FGFc'
\end{tikzcd}$$

$\eta : 1_{\mathscr{C}} \to GF$ is a natural isomorphism $\square$

Definition 3.1.14. The **empty category** is the category with no objects hence morphisms

Definition 3.1.15. $A \xrightarrow{f} B$ is a **constant morphism** if $fg = fh$ for any g, h, f is a **coconstant morphism** if $gf = hf$ for any g, h, f is a **zero morphism** if it is both a constant and a coconstant morphism

Definition 3.1.16. Suppose $u : S \to A$, $v : T \to A$ are morphisms, v filter through s means there is a morphism $w : T \to S$ such that $v = u \circ w$, then mutually filter defines an equivalence relation on monomorphisms(or equivalent by saying that w is an isomorphism), the equivalence classes are called **subobjects** of A, the dual notion is called **quotient objects**

Proposition 3.1.17. Direct limit is an exact functor

Definition 3.1.18. An **injective object** Q is such that for any monomorphism $f : X \to Y$ and morphism $g : X \to Q$, there is a morphism $h : Y \to Q$ such that $g = h \circ f$

$$\begin{tikzcd}
X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, dashed, "\exists h"] \\
Q &
\end{tikzcd}$$

the dual notion is called a **projective object** P, such that for any epimorphism $f : X \to Y$, and morphism $g : P \to Y$, there is a morphism $h : P \to X$ such that $g = f \circ h$

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] \\
X \arrow[r, two heads, "f"] & Y
\end{tikzcd}$$

Definition 3.1.19. A functor $F : \mathscr{C} \to Set$ is called a **representable** functor if there is an object A in $\mathscr{C}$ such that $\Phi : Hom(A, -) \to F$ is a natural isomorphism

Definition 3.1.20. Let $\mathscr{C}$ be a category, we can define **quotient category** by moding out a congruence relation $\sim$, here $\sim$ is an equivalence relation on $Hom(X, Y)$ for any X, Y and it respects composition, i.e. suppose $f_1 \sim f_2 : X \to Y$, $g_1 \sim g_2 : Y \to Z$, then $g_1 \circ f_1 \sim g_2 \circ f_2$, thus $Hom_{\mathscr{C}/\sim}(X, Y) = Hom_{\mathscr{C}}(X, Y)/\sim$

Definition 3.1.21. $\mathscr{C}$ is **concretizable** if there is a faithful functor $F : \mathscr{C} \to Set$. A morphism $f : X \to Y$ is an **embedding** if $F(f)$ is injective, and for any $F(Z) \xrightarrow{\phi} F(X)$, $Z \xrightarrow{h} Y$ such that $F(Z) \xrightarrow{F(h)} F(Y)$, $F(h) = F(f) \circ \phi$, $\phi = F(g)$ for some $Z \xrightarrow{g} X$

Remark 3.1.22. $\mathscr{C}$ may have different concretization

Definition 3.1.23. W is a class of morphisms of $\mathscr{C}$, the **localization** of $\mathscr{C}$ with respect to W another category, denoted $\mathscr{C}[W^{-1}]$, such that there is a natural localization functor $\mathscr{C} \to \mathscr{C}[W^{-1}]$ such that any functor $F : \mathscr{C} \to \mathscr{D}$ where F sends morphisms in W to isomorphisms in $\mathscr{D}$ uniquely factor through the localization functor, thus the localization of a category is unique up to isomorphism, one concrete construction is to consider $\mathscr{C}[W^{-1}]$ has the same objects as $\mathscr{C}$ and adding formal inverses to the morphisms in $\overline{W}$ which is the composition closure of morphisms in W, more concretely, morphisms in $\mathscr{C}[W^{-1}]$ are compositions of morphisms in $\mathscr{C}$ and inverses of morphisms in W

Definition 3.1.24. A **skeleton of a category** $\mathscr{D}$ of $\mathscr{C}$ is a full subcategory such that no two objects in $\mathscr{D}$ are isomorphic and for every object in $\mathscr{C}$ is isomorphic to some object in $\mathscr{D}$, the functor $\mathscr{D} \hookrightarrow \mathscr{C}$ is an equivalence of categories

Definition 3.1.25. $\mathscr{C}$ is **connected** if there is a finite sequence of morphisms connecting any two objects

Example 3.1.26. $C \leftarrow A \rightarrow B$ is a connected even there is no morphism between B, C

Definition 3.1.27. Suppose $\mathscr{C}$ is a category, a **filtered object** X is an object with a **filtration** of X, a descending filtration

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X$$

Or an ascending filtration

$$X \to \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

Definition 3.1.28. Suppose $\mathscr{C}$ is a category, $f : X \to Y$ is a morphism, the **image** of f is a monomorphism $m : I \to Y$ such that there is a morphism $e : X \to I$ such that the following diagram commutes and satisfies the universal property

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "e'"'] & & Y \\
& I \arrow[ur, hook, "m"] \arrow[d, dashed, "\exists_1 v"] & \\
& I' \arrow[uur, hook, bend right, "m'"'] &
\end{tikzcd}$$

Definition 3.1.29. A **quiver** in $\mathscr{C}$ is a functor from $\circlearrowleft \bullet \rightrightarrows \bullet \circlearrowright$ to $\mathscr{C}$. Equivalently, a directed graph allowing multiple arrows and loops

Definition 3.1.30. The **free category** generated by quiver Q has objects vertices in Q and morphisms paths in Q with empty path the identity

Definition 3.1.31. $f \in End(A)$ is an **involution** if $f^2 = 1_A$

Definition 3.1.32. $A \xrightarrow{i} B$ has **left lifting property** or **LLP** and $X \xrightarrow{p} Y$ has **right lifting property** or **RLP** for each other in this diagram if

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & Y
\end{tikzcd}$$

i, p are **orthogonal** if the lifting is unique

Definition 3.1.33. A class of morphisms $\mathbf{M}$ of $\mathscr{C}$ satisfies **2 out of 3** if any two of $f, g, f \circ g$ are in $\mathbf{M}$, so is the third. $\mathbf{M}$ is clearly closed under composition
A class of **weak equivalences** is a class of morphisms $\mathbf{W}$ containing isomorphisms and satisfies 2 out of 3. The class of isomorphisms $\mathbf{I}$ is a class of weak equivalences

3.2 Yoneda lemma

Yoneda lemma

Lemma 3.2.1 (Yoneda lemma)**.** $\mathscr{C}$ is locally small

$$Hom_{Set^{\mathscr{C}}}(Hom_{\mathscr{C}}(c,-),F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

$$Hom_{Set^{\mathscr{C}^{op}}}(Hom_{\mathscr{C}}(-,c),F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

If $F = Hom(-,d)$ or $F = Hom(d,-)$, then

$$Hom_{Set^{\mathscr{C}}}(Hom_{\mathscr{C}}(c,-),Hom_{\mathscr{C}}(d,-)) \xrightarrow{\cong} Hom_{\mathscr{C}}(d,c)$$

$$Hom_{Set^{\mathscr{C}^{op}}}(Hom_{\mathscr{C}}(-,c),Hom_{\mathscr{C}}(-,d)) \xrightarrow{\cong} Hom_{\mathscr{C}}(c,d)$$

$c \to Hom_{\mathscr{C}}(c,-)$ gives an fully faithful embedding of $\mathscr{C}^{op}$ into $Set^{\mathscr{C}}$, viewing $\{Hom(c,-)\}$ as a subcategory of $Set^{\mathscr{C}}$, $c \to Hom_{\mathscr{C}}(-,c)$ gives an fully faithful embedding of $\mathscr{C}$ into $Set^{\mathscr{C}^{op}}$, viewing $\{Hom(-,c)\}$ as a subcategory of $Set^{\mathscr{C}^{op}}$

Proof.

$$\begin{tikzcd}
Hom_{\mathscr{C}}(c,c) \arrow[rrr, "f"] \arrow[ddd, "\eta_c"'] & & & Hom_{\mathscr{C}}(c,x) \arrow[ddd, "\eta_x"] \\
& 1_c \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & Ff(u) = \eta_x(f) & \\
F(c) \arrow[rrr, "Ff"'] & & & F(x)
\end{tikzcd}$$

The natural transformation η is determined by the element u in $F(c)$ ☐

Remark 3.2.2. Functor $Hom(-,c)$ is called **Yoneda embedding**, here embedding in the sense of a fully faithful functor, which is injective on objects up to isomorphism as in Lemma 47.0.4 Yoneda lemma tells us that if $Hom(c,-)$ and $Hom(d,-)$ are naturally isomorphic or $Hom(-,c)$ and $Hom(-,d)$ are naturally isomorphic, so are c and d, thus if we know where c goes to or what goes to c, we can determine c up to isomorphism, in other words, an object is determined by the morphisms that interact with it, this explains the uniqueness in universal construction

3.3 Limits

Definition 3.3.1. A **diagram** is a functor $D : J \to \mathscr{C}$, J is called the **indexed category**, the diagram D can be thought of as indexing a collection of objects and morphisms in $\mathscr{C}$ patterned on J, we say D is a diagram in $\mathscr{C}$ shaped J
Let $F : J \to \mathscr{C}$ be a diagram in $\mathscr{C}$, N be an object in $\mathscr{C}$, then a **cone** from N to F is a family of morphisms ψ_X such that the following diagram commutes, a **cocone** from F to N is a family of morphisms ψ_X such that the following diagram commutes

$$\begin{tikzcd} & N \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\ F(X) \arrow[rr, "Ff"] & & F(Y) \end{tikzcd} \qquad \begin{tikzcd} & N & \\ F(X) \arrow[ur, "\psi_X"] \arrow[rr, "Ff"] & & F(Y) \arrow[ul, "\psi_Y"'] \end{tikzcd}$$

A **limit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : N \to L$ such that the following diagram commutes, a **colimit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : L \to N$ such that the following diagram commutes

$$\begin{tikzcd} & N \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] \arrow[d, dotted, "\exists_1 u"] & \\ & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\ F(X) \arrow[rr, "Ff"] & & F(Y) \end{tikzcd} \qquad \begin{tikzcd} & N & \\ & L \arrow[u, dotted, "\exists_1 u"'] & \\ F(X) \arrow[uur, bend left, "\psi_X"] \arrow[ur, "\phi_X"] \arrow[rr, "Ff"] & & F(Y) \arrow[uul, bend right, "\psi_Y"'] \arrow[ul, "\phi_Y"'] \end{tikzcd}$$

Limits may also be characterized as terminal objects in the category of cones to F, thus unique up to isomorphism, so is colimits, a category contains all limits is called **complete**, and is called **cocomplete** if containing all colimits

The **equaliser** $Eq(f, g)$ is defined to be the limit of the diagram $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$, the **coequaliser** is the colimit

Remark 3.3.2. Direct limit and inverse limit are defined on directed set, thus limit and colimit are more general

Definition 3.3.3. A **directed set** X is a set with a preorder $\leq$ and any pair of elements has an upper bound, i.e., $\forall x, y \in X, \exists z \in X$ such that $x \leq z, y \leq z$

Definition 3.3.4. Given a directed set I, we can define a **direct(inductive) system**, with modules A_i, and functions $f_{ij} : A_i \to A_j$, $f_{ii} = 1_{A_i}$, $f_{jk} \circ f_{ij} = f_{ik}, i \leq j \leq k$, we can also define an **inverse system**, with module A_i, and functions $f_{ij} : A_j \to A_i$, $f_{ii} = 1_{A_i}$, $f_{ij} \circ f_{jk} = f_{ik}$
We can define morphism betweem direct and inverse systems and modules
Suppose A_i is a direct system, then a morphism $g_i : A_i \to B$ is such that $g_j \circ f_{ij} = g_i, i \leq j$ or $g_i : B \to A_i$ is such that $g_j = f_{ij} \circ g_i, i \leq j$. Suppose A_i is an inverse system, then a morphism $g_i : A_i \to B$ is such that $g_i \circ f_{ij} = g_j, i \leq j$ or $g_i : B \to A_i$ is such that $g_i = f_{ij} \circ g_j, i \leq j$, We can define morphisms between direct and inverse systems
Suppose A_i, B_i are both direct systems, a morphism $g_i : A_i \to B_i$ is a family of maps such that $g_j \circ f_{ij} = f_{ij} \circ g_i, i \leq j$. Suppose A_i, B_i are both inverse systems, a morphism $g_i : A_i \to B_i$ is a family of maps such that $g_i \circ f_{ij} = f_{ij} \circ g_j, i \leq j$.

Definition 3.3.5. The **direct limit** of a direct system is a module A_∞ and morphisms $\iota_i : A_i \to A$ with the universal property: given any morphism $g_i : A_i \to B$, it induces a unique $g_\infty : A_\infty \to B$ such that $g \circ \iota_i = g_i$, there is a concrete construction: define the direct limit $\varinjlim A_i = \bigsqcup\limits_{i \in I} A_i / \sim$, where $a_i \sim a_j, a_i \in A_i, a_j \in A_j$ if there is an upper bound k such that $f_{ik}(a_i) = f_{jk}(a_k)$, or equivalently, $a_i \sim f_{ij}(a_i), i \leq j$

Definition 3.3.6. The **inverse limit** of an inverse system is a module A_∞ and morphisms $\pi_i : A \to A_i$ with the universal property: given any morphism $g_i : B \to A_i$, it induces a unique $g_\infty : B \to A_\infty$ such that $\pi_i \circ g = g_i$, there is a concrete construction: define the inverse limit $\varprojlim A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \middle| a_i = f_{ij}(a_j), i \leq j \right\}$,

Remark 3.3.7. Direct limit and inverse limit are dual to each other in the categorical sense

Definition 3.3.8. Product, coproduct The **biproducts** $(\bigoplus_i A_i, p_i, \iota_i)$ of A_i is such that $(\bigoplus_i A_i, p_i)$ is the product and $(\bigoplus_i A_i, \iota_i)$ is the coproduct

Definition 3.3.9. An **initial object** $\varnothing$ is for every X, there is a unique $\varnothing \to X$, a **final object** $*$ is for every X, there is a unique $X \to *$, a **zero object** is an object which is both initial and final. A **pointed category** is a category with zero object

Remark 3.3.10. The initial and final object are the limit and colimit of empty diagram
In the category of sets, the initial object is $\varnothing$ and a terminal object is $\{*\}$

3.4 Adjunction

Definition 3.4.1. Let $L : \mathscr{D} \to \mathscr{C}$, $R : \mathscr{C} \to \mathscr{D}$ be functors, and there is a natural isomorphism $\Phi_{X,Y}$, $X \in \mathscr{C}, Y \in \mathscr{D}$

$$\begin{array}{ccc} Hom_{\mathscr{C}}(LX, Y) & \xrightarrow{\Phi_{X,Y}} & Hom_{\mathscr{D}}(X, RY) \\ {\scriptstyle (Lf,g)}\downarrow & & \downarrow{\scriptstyle (g,Rf)} \\ Hom_{\mathscr{C}}(LX', Y') & \xrightarrow{\Phi_{X',Y'}} & Hom_{\mathscr{D}}(X', RY') \end{array}$$

Here $f : X' \to X$, $g : Y \to Y'$, $Hom_{\mathscr{C}}(Lf, g)(h) = h \circ g \circ Lf$
We say L is the **left adjoint** of R and R is the **right adjoint** of L

Example 3.4.2. Let $G : Group \to Set$ be the forgetful functor, then the functor $F : Set \to Group$, sending S to $F(S)$ is the left adjoint of G
In the category of R-modules Mod, consider functor $F := - \otimes B$ and functor $G := Hom(B, -)$, then F is the left adjoint to G, i.e. $Hom(A \otimes B, C) \cong Hom(A, Hom(B, C))$

3.5 Pushout and pullback

Definition 3.5.1. The **pullback** of $f : X \to Z$, $g : Y \to Z$ is $(X \times_Z Y, p_X, p_Y)$ satisfying the universal property

$$\begin{tikzcd}
W \arrow[rrd, bend left, "\phi"] \arrow[rd, dashed, "\exists_1 h"] \arrow[rdd, bend right, "\psi"'] & & \\
& X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

p_X is the **base change** of g along f, p_Y is the base change of f along g
If f is an epimorphism, so is p_X
More generally, we can also define the puullback of $f_i : X \to Y_i$

Definition 3.5.2. The **pushout** of $f : Z \to X$, $g : Z \to Y$ is $(X \cup_Z Y, \iota_X, \iota_Y)$ satisfying the universal property

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\iota_X"] \arrow[rdd, bend left, "\phi"] & \\
Y \arrow[r, "\iota_Y"'] \arrow[rrd, bend right, "\psi"'] & X \cup_Z Y \arrow[rd, dashed, "\exists_1 h"] & \\
& & W
\end{tikzcd}$$

ι_X is the **cobase change** of g along f, ι_Y is the cobase change of f along g
If f is a monomorphism, so is ι_X
More generally, we can also define the pushout of $f_i : Z \to X_i$

Proposition 3.5.3. Pushout preserve epimorphisms and isomorphisms and in the category of sets, pushout preserve injection
Pullback preserve monomorphisms and isomorphisms and in the category of sets, pullback preserve surjection

3.6 Filtered category

Definition 3.6.1. A category J is called **filtered** if it is not empty, and for any two objects $j, j' \in J$, there is an object $k \in J$ and morphisms $f : j \to k$ and $f' : j' \to k$, for any two morphisms $u, v : i \to j$, there is an object $k \in J$ and a morphism $w : j \to k$ such that $w \circ u = w \circ v$

A filtered colimit is the colimit of a functor $F : J \to \mathscr{C}$ where J is a filtered category, direct limit is a special case of a filtered colimit

The dual notion is called **cofiltered**

3.7 Comma category

Definition 3.7.1. Consider functors $S : \mathscr{A} \to \mathscr{C}$, $T : \mathscr{B} \to \mathscr{C}$(for source and target), define **comma category** $(S \downarrow T)$ with objects (A, B, h), $A \in \mathscr{A}, B \in \mathscr{B}$ are objects, $h : S(A) \to T(B)$ is a morphism, and with morphisms $(f, g) : (A, B, h) \to (A', B', h')$ where $f : A \to A', g : B \to B'$ are morphisms such that the following diagram commutes

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"'] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"] & T(B') \end{tikzcd}$$

Definition 3.7.2. Consider the comma category of $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$, $T : * \to \mathscr{A}$ which we call **slice category**, sometimes denoted as $(\mathscr{A} \downarrow A_*)$ where $A_* = T(*)$, the objects of the slice category are $A \xrightarrow{\pi_A} A_*$ and morphisms are

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\pi_A"'] & & A' \arrow[dl, "\pi_{A'}"] \\ & A_* & \end{tikzcd}$$

Its dual notion, the comma category of $S : * \to \mathscr{B}$ $1_{\mathscr{B}} : \mathscr{B} \to \mathscr{B}$, which we call **coslice category**, sometimes denoted as $(B_* \downarrow \mathscr{B})$ where $B_* = S(*)$, the objects of the slice category are $B_* \xrightarrow{\pi_B} B$ and morphisms are

$$\begin{tikzcd} & B_* \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_{B'}"] & \\ B \arrow[rr, "g"] & & B' \end{tikzcd}$$

The comma category of $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ which we call **arrow category**, sometimes denoted as $\mathscr{C}^{\to}$ the objects of the arrow category are just the morphisms(arrows) $A \xrightarrow{f} A'$, and morphisms are

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & A' \arrow[d, "h'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

Definition 3.7.3. A right inverse are called a **section**, a left inverse is called a **retraction**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, equal] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

f is a section of g, g is a retraction of f

3.8 Sheaves

Definition 3.8.1. $\mathscr{A}$ is an abelian category, open subsets of X form a category τ under inclusion. A **presheaf** is a functor $\tau^{op} \xrightarrow{F} \mathscr{A}$, $F(U \hookrightarrow V) = res_{UV}$ are **restriction maps**
A **morphism of presheaves** $F \xrightarrow{\phi} G$ is a natural transformation, i.e. the following diagram commutes

$$\begin{tikzcd} F(V) \arrow[r, "res_{UV}"] \arrow[d, "\phi_V"'] & F(U) \arrow[d, "\phi_U"] \\ G(V) \arrow[r, "res_{UV}"] & G(U) \end{tikzcd}$$

Definition 3.8.2. $U \subseteq X$ is an open subset, F is a presheaf over X, the **restricted presheaf** $F|_U$ is given by $F|_U(V) = F(U \cap V)$

Definition 3.8.3. $X \xrightarrow{f} Y$ is a continuous map, F is a presheaf over X, the **pushforward presheaf** f_*F of F under f is a presheaf over Y given by $f_*F(V) = F(f^{-1}(V))$

Definition 3.8.4. F is a presheaf, $x \in X$, open subsets containing x is full subcategory $\tau(x)$, the **stalk** F_x is the colimit $\varinjlim_{x \in U} F(U)$, elements in F_x are called **germs**, denote the germ of f at x as f_x

Lemma 3.8.5. $B(f, U) = \{f_x | x \in U, f \in F(U)\}$ form a basis on the **étalé space** $|F| = \bigcup F_x$. The **étale map** $|F| \to X$, $f_x \mapsto x$ is a local homeomorphism

Sheaf

Definition 3.8.6. Presheaf F is a **sheaf** if

$$\begin{tikzcd} F(U) \arrow[r, "res_{U_i, U}"] & \prod_i F(U_i) \arrow[r, shift left, "res_{U_i \cap U_j, U_i}"] \arrow[r, shift right, "res_{U_i \cap U_j, U_j}"'] & \prod_{i,j} F(U_i \cap U_j) \end{tikzcd}$$

Is an equaliser. Equivalently, F satisfying

1. If $U = \bigcup_i U_i$, $f, g \in F(U)$, $f|_{U_i} = g|_{U_i}$, then $f = g$
2. If $U = \bigcup_i U_i$, $f_i \in F(U_i)$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there exists $f \in F(U)$ such that $f|_{U_i} = f_i$, here f has to be unique because of 1

$Sh(X)$ is the category of sheaves over X

Proposition 3.8.7. $F \xrightarrow{\phi} G$ is a monomorphism or a epimorphism iff $F_x \xrightarrow{\phi_x} G_x$ is injective or surjective on each stalk

Definition 3.8.8 (Sheafification)**.** F is a presheaf over X, the sheaf of sections $X \to |F|$ is the **sheafification**

Definition 3.8.9. The **constant presheaf** $\underline{A}$ given by $\underline{A}(U) = A$, $res_{UV} = 1_A$
F is a **locally constant sheaf** if for any $x \in X$, there exists $U \ni x$ such that $F|_U$ is a constant sheaf. $F : \Pi_1 X \to \mathscr{A}$ is a functor. The category of locally constant sheaves is equivalent to the category of covering spaces of X

Definition 3.8.10. Functor $\Gamma : Sh(X) \to \mathscr{A}$, $F \mapsto F(X)$ is a left exact functor, the **sheaf cohomology** is the right derived functor $R^i\Gamma$, i.e. $R^i\Gamma(F) = H^i(X, F)$

Definition 3.8.11. A **ringed space** $(X, \mathcal{O})$ is a topological space X and a sheaf of rings over X, $\mathcal{O}$ is the **structure sheaf**. $(X, \mathcal{O})$ is a **locally ringed space** if each stalk is a local ring

Definition 3.8.12. A morphism between ringed spaces is $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$, $X \xrightarrow{f} Y$ is a continuous map, $\mathcal{O}_Y \xrightarrow{\phi} f_*\mathcal{O}_X$ is a morphism of sheaves. A morphism between locally ringed spaces require ϕ is a local ring homomorphism between stalks

Definition 3.8.13. $(X, \mathcal{O})$ is a ringed space, a sheaf of $\mathcal{O}$ **modules** F is $F(U)$ which are $\mathcal{O}(U)$ modules such that $res_{UV}(rm) = res_{UV}(r)res_{UV}(m)$

3.9 Exponential object

Definition 3.9.1. Y is an object such that all binary products $X \times Y$ exist, the **exponential object** is Z^Y together with morphism $Z^Y \times Y \xrightarrow{\text{eval}} Z$ satisfying universal property

$$\begin{tikzcd}
X \times Y \arrow[d, dashed, "\exists_1 \tilde{f} \times 1_Y"'] \arrow[dr, "f"] & \\
Z^Y \times Y \arrow[r, "\text{eval}"'] & Z
\end{tikzcd}$$

Proposition 3.9.2. $Hom(X \times Y, Z) \to Hom(X, Z^Y)$ is an adjunction

3.10 Factorization system

Definition 3.10.1. A **factorization system** $(\mathbf{E}, \mathbf{M})$ for category $\mathscr{C}$ is two classes of morphisms such that
1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. $\mathbf{E}, \mathbf{M}$ are closed under composition and contain all isomorphisms
3. Factorization is functorial, i.e. for any u, v such that $vme = m'e'u$, there exists a unique w such that the following diagram commutes

$$\begin{tikzcd}
\bullet \arrow[r, "e"] \arrow[d, "u"] & \bullet \arrow[r, "m"] \arrow[d, dashed, "\exists_1 w"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "e'"] & \bullet \arrow[r, "m'"] & \bullet
\end{tikzcd}$$

Example 3.10.2. $\mathbf{E}, \mathbf{M}$ being epi and mono in Set is a factorization system

Definition 3.10.3. A **weak factorization system** $(\mathbf{E}, \mathbf{M})$ for category $\mathscr{C}$ is two classes of morphisms such that
1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. $\mathbf{E}$ are exactly those morphisms having left lifting property for all morphisms in $\mathbf{M}$
3. $\mathbf{M}$ are exactly those morphisms having right lifting property for all morphisms in $\mathbf{E}$

3.11 Monoidal category

Definition 3.11.1. A category $\mathscr{C}$ is **monoidal** if there is a tensor product which is a bifunctor $\mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$, a tensor unit 1, **associator**, **left** and **right unitor** which are natural isomorphisms $(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)$, $1 \otimes x \xrightarrow{\lambda_x} x$, $x \otimes 1 \xrightarrow{\rho_x} x$ such that the following diagrams commute

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr] \arrow[dr] & & x \otimes (1 \otimes y) \\
& x \otimes y \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (w \otimes x) \otimes (y \otimes z) \arrow[dr] & \\
((w \otimes x) \otimes y) \otimes z \arrow[ur] \arrow[d] & & w \otimes (x \otimes (y \otimes z)) \\
(w \otimes (x \otimes y)) \otimes z \arrow[rr] & & w \otimes ((x \otimes y) \otimes z) \arrow[u]
\end{tikzcd}$$

$\mathscr{C}$ is **strictly monoidal** if α, λ, ρ are identities

Chapter 4

Group

4.1 Groups

Semigroup

Definition 4.1.1. A **semigroup** is a semicategory with a single object. A **monoid** M is a category with a single object. A **group** is a monoid with all morphisms invertible. A **groupoid** is a category with all morphisms invertible

A G **set** is a functor from G to the category of sets. Equivalently, a **left group action** is $G \times X \to X$, $(g, x) \mapsto g \cdot x$ satisfying $1 \cdot x = x$, $g \cdot (h \cdot x) = (gh \cdot x)$, a right G action is functor G^{op} to the category of sets. A G **space** is a functor from G to the category of topological spaces

An **equivariant map** of G spaces is a natural transformation $f : X \to Y$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Definition 4.1.2. The **center** of G is $Z(G) = \{z \in G | gz = zg, \forall g \in G\}$. $Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$

Definition 4.1.3. **Inner automorphism group** of G is $Inn(G) \leq Aut(G)$ consists of conjugations $x \mapsto gxg^{-1}$. **Outer automorphism group** is $Out(G) = Aut(G)/Inn(G)$

Definition 4.1.4. G is **perfect** if $[G, G] = G$

Definition 4.1.5. A group action is **trivial** if $g \cdot x = x$. A group action is **free** if $g \cdot x = h \cdot x$ for some x implies $g = h$, or equivalently. A group action is **transitive** if $G \cdot x = X$. A free and transitive action is also called **regular**. A group action is **faithful** if $\forall x \in X, G \cdot x \neq x$

A **homogeneous space** is a G space with G acting transitively

Torsor

Definition 4.1.6. A **torsor** P is a set with an action $G \times P \to P, (g, p) \mapsto gp$ such that this action is free and transitive

When chosen an element $e \in P$, automatically you are giving it a group structure with e as identity, notice $e : G \to P, g \mapsto ge$ is a bijection, suppose $g_p e = p$, then $g_e = 1$, the group structure could be given as follows, $P \times P \to P, (p, q) \mapsto g_p g_q e$, and the inverse of p is $g_p^{-1} e$. i.e. a torsor is a group forgetting its identity

Definition 4.1.7. Let X, Y be G sets, the we can give Y^X a G set structure by $(gf)(x) = gf(g^{-1}x)$, it is easy to check that a map $f : X \to Y$ is equivariant iff f is fixed under the G action

Specially, if G acts on Y trivially, we have $(gf)(x) = f(g^{-1}x)$

Definition 4.1.8. $H \leq G$ is a subgroup, $N \trianglelefteq G$ is a normal subgroup, $G = NH = N \rtimes H$ or $HN = H \ltimes N$ is the **semidirect product** and every $g \in G$ can be uniquely written as nh or hn

$$(nh)(n'h') = n(hn'h^{-1})hh' \text{ or } (hn)(h'n') = hh'(h'^{-1}n'h')n'$$

Theorem 4.1.9 (Jordan-Hölder theorem)**.** Composition series is unique up to reordering

Definition 4.1.10. A **group representation** V is a $\mathbb{F}G$ module

Definition 4.1.11. Let (ρ, V) be a group representation of finite group G, $W \leq V$ is called G invariant if $GW \subseteq W$, namely, W is $\mathbb{F}G$ submodule, then we get a subrepresentation on $(\rho|_W, W)$ of G, with $\rho|_W(g) := \rho(g)|_W$, if the only G invariant subspace of W are 0 and V, we say ρ is irreducible

Definition 4.1.12. A group representation (ρ, V) is completely reducible(semisimple) if $V = V_1 \oplus \cdots \oplus V_n$, where $(\rho|_{V_i}, V_i)$ are irreducible subrepresentations, namely, V is the direct sum of simple $\mathbb{F}G$ modules

Definition 4.1.13. Let V be a complex vector space with Hermitian form $(,)$ finite group representation (ρ, V) of G is called unitary if $(\rho(g)v, \rho(g)w) = (v, w), \forall g \in G, v, w \in V$

Proposition 4.1.14. Let (ρ, V) be a unitary representation of G, (ρ, V) is completely reducible

Proof. W is G invariant $\Rightarrow W^\perp$ is G invariant $\square$

Proposition 4.1.15. Let V be a complex vector space, (ρ, V) is a representation, then there exists a positive definite Hermitian form on V such that (ρ, V) become a unitary representation

Corollary 4.1.16. Let V be a complex vector space, a finite group representation (ρ, V) is always completely reducible

Definition 4.1.17. Let (ρ, V) be a representation of G, then the dual representation (ρ^*, V^*) is defined as $\rho^*(g) := \rho(g)^{-T}$

Definition 4.1.18. $H \subseteq G$, (V, π) is a H representation, the **induced representation** is $\mathbb{F}G \otimes_{\mathbb{F}H} V$. Suppose $g_1, \cdots, g_n$ is a set of representatives of left cosets in G/H, v_j is a basis for V, then $g_i \otimes v_j$ form a basis for $\mathbb{F}G \otimes_{\mathbb{F}H} V$. If $gg_i = g_j h$ for some $h \in H$, then $g(g_i \otimes v) = gg_i \otimes v = g_j h \otimes v = g_j \otimes hv$

Definition 4.1.19. If X is a right G space, Y is a left G space, $X \times_G Y$ is $X \times Y/\sim$, $(xg, y) \sim (x, gy)$. Equivalently, $X \times_G Y = X \times Y/G$ with left action $g(x, y) = (xg^{-1}, gy)$ or right action $(x, y)g = (xg, g^{-1}y)$
If X, Y are left G spaces, $X \times_G Y = X \times Y/G$ with left action $g(x, y) = (gx, gy)$
If X, Y are right G spaces, $X \times_G Y = X \times Y/G$ with right action $(x, y)g = (xg, yg)$

Sylow's theorem

Theorem 4.1.20 (Sylow's theorem)**.** p is a prime, a p-group is a group consists of elements of order p-th power, a maximal one is a **Sylow p-subgroup** P of G which always exists by Zorn's Lemma 1.0.2

1. If $|G| = p^n m$, $p \nmid m$, then $|P| = p^n$

2. Any subgroup of a Sylow p subgroup is subconjugate to some other Sylow p subgroup

3. $n_p = [G : N_G(P)]$, if the cojugacy class of P is of order $n_p < \infty$, then $n_p \equiv 1 \bmod p$

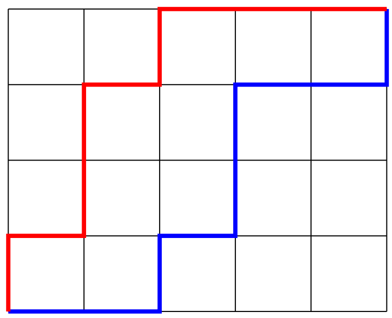
4.2 Permutation group

Definition 4.2.1. Denote $[n] = \{1, \cdots, n\}$, a **permutation** $[n] \xrightarrow{\sigma} [n]$ is a bijection, equivalently write $\sigma = \begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(n) \end{pmatrix}$. $S_n = \mathrm{Aut}([n])$ is the permutation group
The **length** of $\sigma \in S_n$ is $|\sigma|$ is the number of $\sigma(i) < \sigma(j)$ while $i > j$

Definition for shuffle

Definition 4.2.2. Just like shuffling a deck of cards, $\sigma \in S_n$ is a $(p_1, \cdots, p_m)$**-shuffle**, $p_1 + \cdots + p_m = n$ if $\sigma(i) < \sigma(i+1)$ for $p_1 + \cdots + p_k + 1 \leq i \leq p_1 + \cdots + p_{k+1}$
A (p, q) shuffle can be represented by a path going only right or up from the lower left corner to the upper right corner in a $(p+1) \times (q+1)$ grid, $|\sigma|$ happen to be the number squares under the path

Example 4.2.3 ($(5,4)$ shuffles in S_9)**.** The red one is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 7 & 8 & 9 & 1 & 3 & 4 & 6 \end{pmatrix}$. The blue one is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 7 & 8 & 3 & 5 & 6 & 9 \end{pmatrix}$



4.3 Coxeter group

Definition 4.3.1. A **Coxeter group** G are groups with presentations

$$\langle r_1, \cdots, r_n | (r_i r_j)^{m_{ij}} = 1 \rangle$$

$m_{ii} = 1$, so we should think of r_i's as reflections, for $i \neq j$, $m_{ij} = 2$ means r_i, r_j commutes, we should think of r_i, r_j are independent, most commonly, $m_{ij} \geq 3$, if $m_{ij} = \infty$. $M = (m_{ij})$ is the **Coxeter matrix**, the corresponding **Schläfli matrix** is $C_{ij} = -2\cos(\pi/m_{ij})$

Definition 4.3.2. The **Coxeter element** is the product of all simple reflections s_i's, different order in multiplication give conjugation, the **Coxeter number** is the order of the Coxeter element

Definition 4.3.3. The **Coxeter diagram** consists of n nodes for r_i's, there is an edge numbered m_{ij} between i and j if $m_{ij} \geq 3$

4.4 Grothendieck group

Grothendieck group

Definition 4.4.1 (Grothendieck group)**.** The **Grothendieck group** of a commutative monoid M is the abelian group K and $i : M \to K$ satisfying universal property

$$\begin{tikzcd} M \arrow[d, hook, "i"] \arrow[dr, "f"] & \\ K \arrow[r, dashed, "\exists_1 g"'] & A \end{tikzcd}$$

For any abelian group A

Construction 4.4.2.

$$K = M \times M/\sim,\ M \xrightarrow{i} K, m \mapsto [m - 0]$$

Abelian group $M \times M \cong \{[m - n] | m, n \in M\}$ is the set of formal differences with addition

$$[m_1 - n_1] + [m_2 - n_2] = [(m_1 + m_2) - (n_1 + n_2)]$$

$[0 - 0]$ is the identity and $[n - m]$ is the inverse to $[m - n]$

$$[m_1 - n_1] \sim [m_2 - n_2] \text{ if } m_1 + n_2 + m = m_2 + n_1 + m \text{ for some } m \in M$$

Construction 4.4.3 (Grothedieck completion)**.**

$$K = F(M)/\sim,\ m +' n \sim (m + n)$$

$F(M)$ is the free abelian group generated by M with addition $+'$

Definition 4.4.4. The **Grothendieck group** of a semigroup S is a group K and $i : S \to K$ satisfying the universal property

$$\begin{tikzcd} S \arrow[d, hook, "i"] \arrow[dr, "f"] & \\ K \arrow[r, dashed, "\exists_1 g"'] & G \end{tikzcd}$$

Where G is a group

Construction 4.4.5.

$$K = F(S)/\sim,\ m *' n \sim (m * n)$$

$F(S)$ is the free group generated by S with multiplication $*'$

Chapter 5

Ring

5.1 Rings

Definition 5.1.1 (**Rings**)**.** R is an abelian group with addition $+$ and additive identity 0, a monoid with multiplication $\cdot$ and multiplicative identity 1, and distributive, $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$

Definition 5.1.2. Ring R is **commutative** if $ab = ba$

Definition 5.1.3. u is a **unit** if there exists $v \in R$ such that $uv = vu = 1$. The set of units $R^\times$ is a multiplicative group

Definition 5.1.4. A **semiring** or **rig** is a ring without negatives

Definition 5.1.5. A **rng** is ring without identity

Definition 5.1.6 (Whitehead group)**.** The **Whitehead group** of ring R is an abelian group $K_1(R)$ satisfying universal property

$$\begin{tikzcd} GL(R) \arrow[d, "\pi"'] \arrow[dr] & \\ K_1(R) \arrow[r, dashed, "\exists_1"'] & A \end{tikzcd}$$

For any abelian group A

Construction 5.1.7. $K_1(R) = GL(R)/[GL(R), GL(R)]$ is the abelianization of $GL(R)$

Definition 5.1.8. If R is commutative, $SL(R)$ is the kernel of $GL(R) \xrightarrow{\det} R^\times$, $SK_1(R)$ is the kernel of $K_1(R) \xrightarrow{\det} R^\times$, $GL(R) \cong SL(R) \rtimes R^\times$, $K_1(R) \cong SK_1(R) \oplus R^\times$. $K_1(F) = F^\times$

Lemma 5.1.9. Since $GL(R_1 \times R_2) = GL(R_1) \times GL(R_2)$, $K_1(R_1 \times R_2) = K_1(R_1) \oplus K_1(R_2)$

5.2 Commutative rings

Definition 5.2.1. The **determinant** of a matrix is

Definition 5.2.2. $I, J \leq R$ are ideals, the **ideal quotient** $(I : J) = \{r \in R | rJ \subseteq I\}$ is also an ideal

Definition 5.2.3. $S \subseteq R$ is **multiplictive closed**, the localization $S^{-1}R$ of R with respect to S is $R \times S/\sim$, $(r, s) \sim (r', s')$ iff there exists $t \in S$ such that $t(rs' - sr) = 0$. $S^{-1}R$ has the universal property that for any $f : R \to T$ such that maps S to units, then there exists a unique $g : S^{-1}R \to T$ such that $gi = f$

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "\exists_1 g"] \\ & T \end{tikzcd}$$

Definition 5.2.4. Given a ring R and a proper ideal I, we can define an **associated graded ring** $gr_I R := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$, if M is a left R-module, we can define **associated graded module** $gr_I M := \bigoplus_{n=0}^{\infty} I^n M/I^{n+1}M$

Definition 5.2.5. R is a **local ring** if it has a unique maximal ideal m. The **residue field** is $k = R/m$

Definition 5.2.6. R is a **semilocal ring** if it has only finitely many maximal ideal

Proposition 5.2.7. Let R be a UFD, f a prime element, then $ht(f) = 1$

Proof. Suppose there exists prime ideal P such that $0 \subsetneq P \subsetneq (f)$, then we can find a prime element in $g \in P$, thus we have $0 \subsetneq (g) \subseteq P \subsetneq (f)$, but then $g = fh$ for some h, but since f is prime, thus $(f) = (g)$ which is a contradiction, such a prime element exists since we can pick any element $0 \neq q = q_1 \cdots q_m \in P$ where q_i's are prime, but then at least one of them has to be in P $\square$

Theorem 5.2.8. Let $A \subseteq B$ be finitely generated k-algebras, and A, B are both domains, $0 \neq b \in B \Rightarrow \exists 0 \neq a \in A$ such that for any k-algebra homomorphism $\alpha : A \to k$ with $\alpha(a) \neq 0$ can be extended to k-algebra homomorphism $\beta : B \to k$ with $\beta(b) \neq 0$

Definition 5.2.9. Suppose R is a commutative ring with identity, a prime element $p \in R$ is an element which is nonzero nor a unit and $p|fg \Rightarrow p|f$ or $p|g$

Definition 5.2.10. A **graded ring** R is a ring such that $R = \bigoplus_i R_i$ is a direct sum of abelian groups and $R_i R_j \subseteq R_{i+j}$
An ideal is called a **homogeneous ideal** if it consists of only homogeneous elements

Chinese remainder theorem

Theorem 5.2.11 (Chinese remainder theorem)**.** Let R be a commutative ring, and $I_1, \cdots, I_n \leq R$ be pairwise coprime ideals, then $R \cong R/I_1 \times \cdots R/I_n, r \mapsto (r \bmod I_1, \cdots, r \bmod I_n)$

Definition 5.2.12. An **integral domain** is a commutative ring R such that (0) is a prime ideal. Equivalently, $rs \in R \Rightarrow r \in R$ or $s \in R$

Definition 5.2.13. Suppose R is a domain, K is the field of fractions, a **fractional ideal** is an R submodule $I \leq K$ such that $rI \subseteq R$ for some nonzero $r \in R$. I is invertible if $IJ = R$ for some other fractional ideal J

Definition 5.2.14. A **Dedekind domain** is an integral domain such that every proper ideal is a product of prime ideal

Definition 5.2.15. A **discrete valuation ring(DVR)** is a PID with a unique nonzero prime ideal

Definition 5.2.16. A local ring homomorphism $\phi : R \to S$ between local rings is such that $\phi(m_R) \subseteq m_S$

Definition 5.2.17. $\bigwedge^k A : \bigwedge^k V \to \bigwedge^k W$ is defined by $\bigwedge^k A(v_1 \wedge \cdots \wedge v_n) := A(v_1) \wedge \cdots \wedge A(v_n)$

Definition 5.2.18. A ring A is an R **algebra** is a ring homomorphism $R \xrightarrow{\phi} A$, $ra = \phi(r)a$
A is **finite** or ϕ is **finite** if A is a finitely generated R module
ϕ is of **finite type** if A is **finitely generated** R algebra

Definition 5.2.19. For $p \in SpecA$, $q \in SpecB$, $A \subseteq B$, p **lies under** q or q **lies over** p if $q \cap A = p$ $A \subseteq B$ satisfies **lying over property** if every $p \in SpecA$ lies under some $q \in SpecB$
$A \subseteq B$ satisfies the **incomparability property** if different prime ideals q, q' both lie over p, then they don't contain each other
$A \subseteq B$ satisfies **going up property** if for any chain of prime ideals $p_1 \subseteq \cdots \subseteq p_n$, $q_1 \subseteq \cdots \subseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \subseteq \cdots \subseteq q_n$ with q_i lies over p_i
$A \subseteq B$ satisfies **going down property** if for any chain of prime ideals $p_1 \supseteq \cdots \supseteq p_n$, $q_1 \supseteq \cdots \supseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \supseteq \cdots \supseteq q_n$ with q_i lies over p_i

Definition 5.2.20. $R \subseteq S$ are commutative rings, $a \in S$ is **integral** over R if it is a root of some monic polynomial in $R[x]$. The **integral closure** of R in S are the integral elements of S

Going up and Going down theorems

Theorem 5.2.21. *B is integral over A, then $A \subseteq B$ satisfies going up property and incomparability property*

Definition 5.2.22. The **Krull dimension** of a ring R is $\dim R = \sup_d p_0 \subsetneqq \cdots \subsetneqq p_d$, p_i are prime ideals

Proposition 5.2.23. *A is a integral domain, finitely generated over some subfield k, then $\dim A = \mathrm{trdeg}(\mathrm{Frac}A/k)$*

Chapter 6

Module

6.1 Modules

Module

Definition 6.1.1. R is a ring, a **left** R **module** M is an abelian group with left group action $R \times M \to M$ such that

- $1m = m$
- $r(m+n) = rm + rn$
- $(r+s)m = rm + sm$
- $(rs)m = r(sm)$

Definition 6.1.2. $X \subseteq M$ is **linearly independent** if for $x_1, \cdots, x_n \in X$

$$r_1 x_1 + \cdots + r_n x_n = 0 \Rightarrow r_i = 0$$

Definition 6.1.3. The submodule generated by $X \subseteq M$ is $\operatorname{Span} X$, the **span** of X

Definition 6.1.4. $X \subseteq M$ is a **basis** of M if X is a linearly independent spanning set. M is a free R module on X if X is a basis of M

Note. There is no well-defined dimension for free R modules in general, examplified in Example 40.0.4

Definition 6.1.5. M is an right R-module, N is a left R-module and G is an abelian group, a map $\phi : M \times N \to G$ is called an R balanced product if ϕ is bilinear and $\phi(mr, n) = \phi(m, rn)$, we can define tensor product $M \otimes_R N$ is an abelian group satisfying the universal property

$$\begin{tikzcd} M \times N \arrow[r, "\otimes"] \arrow[dr, "f"'] & M \otimes_R N \arrow[d, dashed, "\exists_1 \tilde{f}"] \\ & G \end{tikzcd}$$

Here f is an R balanced product and $\tilde{f}$ is an abelian group homomorphism
A concrete construction would be $F(M \times N)/\sim$, where $(m+m', n) \sim (m,n) + (m', n)$, $(m, n+n') \sim (m,n) + (m, n')$, $(mr, n) \sim (m, rn)$

Remark 6.1.6. $r(m \otimes n) = mr \otimes n = m \otimes rn$ is called associativity

Definition 6.1.7. Module M is **semisimple** or **completely reducible** if it is the direct sum of simple submodules. Ring R is semisimple if it is a semisimple R module

Theorem 6.1.8. Tensor product is right exact for R modules

Definition 6.1.9. Let R be a commutative ring, $S \subseteq R$ is **multiplicatively closed** if $1 = s^0 \in S$ and $rs \in S, \forall r, s \in S$, we can define localization $S^{-1}R$ satisfying universal property

$$\begin{tikzcd} R \arrow[d, "j"'] \arrow[dr, "f"] & \\ S^{-1}R \arrow[r, dashed, "\exists_1 g"'] & T \end{tikzcd}$$

Here $f(S) \subseteq T^{\times}$
Concrete construction: $S^{-1}R := R \times S/\sim$, $(r, s) \sim (r', s')$ if there exists $t \in S$ such that $t(rs' - r's) = 0$
Let M be an R module, we can define localization, $S^{-1}M := M \times S/\sim$, $(m, s) \sim (m', s')$ if there exists $t \in S$ such that $t(s'm - sm') = 0$

Proof. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence, then it is obvious that $A \otimes D \xrightarrow{f \otimes 1_D} B \otimes D \xrightarrow{g \otimes 1_D} C \otimes D \to 0$ is a complex and $g \otimes 1_D$ is surjective, now define $\phi : B \otimes D / \ker g \otimes 1_D \to A \otimes D$, $b \otimes d \mapsto a \otimes d$, where a is the unique element in A such that $g(b - f(a)) = 0$ $\square$

Definition 6.1.10. Let P_i, A, D be R modules, $\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ be a projective resolution, then we have $0 \to Hom_R(A, D) \xrightarrow{\epsilon} Hom_R(P_0, D) \xrightarrow{d_1} Hom_R(P_1, D) \xrightarrow{d_2} Hom_R(P_2, D) \cdots$, define $Ext^n_R(A, D)$ to be the n-th cohomology group of $0 \to Hom_R(P_0, D) \xrightarrow{d_1} Hom_R(P_1, D) \xrightarrow{d_2} Hom_R(P_2, D) \cdots$, note that $Ext^0_R(A, D) \cong Hom_R(A, D)$

Definition 6.1.11. Let P_i, B, D be R modules, $\cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \to 0$ be a projective resolution, then we have$\cdots D \otimes_R P_2 \xrightarrow{1 \otimes d_2} D \otimes_R P_1 \xrightarrow{1 \otimes d_1} D \otimes_R P_0 \xrightarrow{1 \otimes \epsilon} D \otimes_R B \to 0$, define $Tor^R_n(D, B)$ to be the n-th homology group of $\cdots D \otimes_R P_2 \xrightarrow{1 \otimes d_2} D \otimes_R P_1 \xrightarrow{1 \otimes d_1} D \otimes_R P_0 \to 0$, note that $Tor^R_0(D, B) \cong D \otimes_R B$

Schur's lemma

Lemma 6.1.12 (Schur's Lemma)**.** *R is a ring, M, N are nonzero simple R modules. A homomorphism $\varphi : M \to N$ is either 0 or an isomorphism. In particular, $\mathrm{End}_R(M)$ is a division ring. Moreover, if $\overline{F} = F$, $\mathrm{Hom}_F(M, N) = \{\lambda\varphi | \lambda \in F\}$ where $M \xrightarrow{\varphi} N$ is an isomorphism(all isomorphisms are scalar multiple of each other), in particular, $\mathrm{Hom}_F(M, M) = \{\lambda 1_M | \lambda \in F\}$*

Theorem 6.1.13 (Maschke's theorem)**.** *G is a finite group, F is a field, $\mathrm{char} F \nmid |G|$, then FG is a semisimple ring*

Theorem 6.1.14 (Artin-Wedderburn theorem)**.** *$R = V_1 \oplus \cdots \oplus V_r$ is a semisimple ring, by Schur's lemma 6.1.12, $D_i = \mathrm{End}_R(V_i)$ are division rings, then*

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

where $n_i = \dim_{D_i}(V_i)$. $\sum_{i=1}^{r} n_i^2 = |G|$, r is the number of conjugacy classes in G

Definition 6.1.15. F is a field, G is a group, the representation ring $R_F(G)$ is the completion of the set of isomorphic classes of representations

Definition 6.1.16. The symmetric k algebra is $S^k(V) \subseteq T^k(V)$ consists of k tensors symmetric under the permutation of S_k. The exterior k algebra is $\bigwedge^k(V) \subseteq T^k(V)$ consists of k tensors antisymmetric under the permutation of S_k. We have

$$\mathrm{Sym} : T^k(V) \to S^k(V),\ a_1 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} \sum_{\sigma} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

and

$$\mathrm{Alt} : T^k(V) \to S^k(V),\ a_1 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} \sum_{\sigma} (-1)^{\operatorname{sgn} \sigma} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$$

For $\alpha \in \bigwedge^k(V)$, $\beta \in \bigwedge^l(V)$, define

$$\alpha\beta = \alpha \odot \beta = \mathrm{Sym}(\alpha \otimes \beta)$$

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

Which corresponds to determinants

Chapter 7

Field

7.1 Fields

Definition 7.1.1. A **division ring** R is a nonzero ring such that $\mathbb{F}^\times = \mathbb{F} - \{0\}$
A **field** $\mathbb{F}$ is a nonzero commutative ring such that $\mathbb{F}^\times = \mathbb{F} - \{0\}$

Definition 7.1.2. A **character** is of G is a group homomorphism $G \to \mathbb{F}^\times$, and a **cocharacter** is a group homomorphism $\mathbb{F}^\times \to G$

Lemma 7.1.3. Characters of G, denoted as $ch(G)$ are linear independent on $\mathbb{F}[G]$

Proof. Suppose not, we can find $c_1\chi_1 + \cdots + c_m\chi_m = 0, c_i \in \mathbb{F}^\times$, with minimal terms, since $\chi_1 \neq \chi_m$, there exists $g_0 \in G$ such that $\chi_1(g_0) \neq \chi_m(g_0)$, on the other hand we have $0 = c_1\chi_1(g) + \cdots + c_m\chi_m(g) = c_1\chi_1(g)\chi_m(g_0) + \cdots + c_m\chi_m(g)\chi_m(g_0), \forall g \in G$ and $0 = c_1\chi_1(gg_0) + \cdots + c_m\chi_m(gg_0) = c_1\chi_1(g)\chi_1(g_0) + \cdots + c_m\chi_m(g)\chi_m(g_0), \forall g \in G$, subtract to get $0 = c_1(\chi_m(g_0) - \chi_1(g_0))\chi_1(g) + \cdots + c_{m-1}(\chi_m(g_0) - \chi_{m-1}(g_0))\chi_{m-1}(g)$ with fewer terms which is a contradiction □

Definition 7.1.4. Suppose E/F is a field extension, we can define **field trace** $Tr_{E/F}(\alpha)$ to be the trace of α as a linear transformation and **field norm** $N_{E/F}(\alpha)$ to be the determinant of α

Definition 7.1.5. $\mathbb{F}$ is a **perfect field** if $\mathbb{F}^p = \mathbb{F}$ if $\mathrm{char}\mathbb{F} = p \neq 0$ or $\mathrm{char}\mathbb{F} = 0$

Definition 7.1.6. E/F is a field extension, $\alpha \in E$ is algebraic over F if α is a zero of some polynomial in $F[x]$. The **algebraic closure** of F in E are the algebraic elements of E

Theorem 7.1.7 (Emil Artin)**.** Any field F has an algebraically closed extension

7.2 Number field

Definition 7.2.1. A **number field** is $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{C}$. An **algebraic number field** is $\mathbb{Q} \subseteq \mathbb{F} \subseteq \overline{\mathbb{Q}}$

Definition 7.2.2. E, F are algebraic number fields of finite degree, E/F is finite separable, A, B are corresponding ring of integers, $\{\beta_1, \cdots, \beta_n\}$ is an integral basis of B over A. The **discriminant** of E/F with respect to $\{\beta_1, \cdots, \beta_n\}$ is $D_{E/F}(\beta_1, \cdots, \beta_n) = \det(Tr(\beta_i\beta_j))$

$$\begin{tikzcd} B \arrow[r, hook] & E \\ A \arrow[u, hook] \arrow[r, hook] & F \arrow[u, hook] \end{tikzcd}$$

Lemma 7.2.3. D_K is well defined in $\dfrac{A}{(A^\times)^2}$

Definition 7.2.4. E, F are algebraic number fields of finite degree, E/F is finite separable, A, B are corresponding ring of integers which are Dedekind domains

$$\begin{tikzcd} B \arrow[r, hook] & E \\ A \arrow[u, hook] \arrow[r, hook] & F \arrow[u, hook] \end{tikzcd}$$

$pB = q_1^{e_1} \cdots q_r^{e_r}$ with $e_i > 0$. p is **ramified** if $e_i > 1$ for some i, otherwise unramified. p is **inert** if $r = e = 1$. p **totally split** if $e_i = f_i = 1$
$B/pB \cong \prod_{i=1}^r B/q_i^{e_i}$, $f_i = [k_{q_i} : k_p]$, $[E : F] = \dim_{k_p}(B/pB) = \sum_{i=1}^r e_i f_i$
If E/F is Galois, $G = Aut(E/F)$ acts transitively on $\{q_1, \cdots, q_r\}$, then $n = \sum_{i=1}^r e_i f_i = ref$

Proof. $B \cong A^n$, $B/pB \cong A^n/pA^n \cong (A/p)^n \cong k_p^n$ □

Example 7.2.5. $2\mathbb{Z}[i] = (1+i)^2$ is ramified, $3\mathbb{Z}[i]$ is inert, $5\mathbb{Z}[i] = (2+i)(2-i)$ totally split

$$\begin{tikzcd} \mathbb{Z}[i] \arrow[r, hook] & \mathbb{Q}[i] \\ \mathbb{Z} \arrow[u, hook] \arrow[r, hook] & \mathbb{Q} \arrow[u, hook] \end{tikzcd}$$

Theorem 7.2.6. p ramifies in $O_K \Leftrightarrow p \mid \mathrm{disc}(O_K/\mathbb{Z})$

$$\begin{tikzcd} O_K \arrow[r, hook] & K \\ \mathbb{Z} \arrow[u, hook] \arrow[r, hook] & \mathbb{Q} \arrow[u, hook] \end{tikzcd}$$

Proof. $pO_K = \beta_1^{e_1} \cdots \beta_r^{e_r}$, $O_K/pO_K \cong O_K/\beta_i^{e_i}$ is an isomorphism of $\mathbb{F}_p$ algebras. $d_i = \mathrm{disc}((O_K/\beta_i^{e_i})/\mathbb{F}_p))$, $d = \mathrm{disc}((O_K/pO_K)/\mathbb{F}_p))$, thus $d = d_1 \cdots d_r$, since discriminant is functorial, $D = \det(Tr_{O_K/\mathbb{Z}}()) \mapsto d$, $p|D \Leftrightarrow d = 0 \Leftrightarrow d_i = 0$ for some i □

Chapter 8

Linear algebra

8.1 Vector spaces

Definition 8.1.1. A **vector space** V over field F is an F module

Definition 8.1.2. An **affine space** is a vector space witho

Definition 8.1.3. $C \subseteq V$ is **convex** if $tC + (1-t)C \subseteq C$ for $0 \leq t \leq 1$. C is **strictly convex** if $tC + (1-t)C \subsetneqq C$ for $0 < t < 1$

Definition 8.1.4. V is a vector space of dimension n, a q **flag** is

$$0 = V_0 \subsetneqq V_1 \subsetneqq \cdots \subsetneqq V_q = V$$

A complete flag is an n flag

GL(n,F) acts transitively on flags

Lemma 8.1.5. $GL(n, F)$ acts transitively on flags

8.2 Matrices

Definition 8.2.1. E_{ij} is the matrix with 1 on the (i, j)-th entry and otherwise zeros, then $E_{ij}E_{kl} = \delta_{jk}E_{il}$
Elementary matrices are single row operations, i.e.

$$e_{ij}(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & r & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

with r on the (i, j)-th entry

$$s_{ij} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

and

$$d_i(r) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & r & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

We have $e_{ij}(-r) = e_{ij}(r)^{-1}$ and

$$\begin{aligned} e_{ij}(r)e_{ij}(s) &= e_{ij}(r+s) \\ [e_{ij}(r), e_{kl}(s)] &= I + rs\delta_{jk}E_{il} - sr\delta_{li}E_{kj} + \delta_{jk}\delta_{li}(srsE_{kl} - rsrE_{ij}) + rsrs\delta_{jk}\delta_{li}E_{il} \\ &= \begin{cases} I & i \neq l, j \neq k \\ e_{il}(rs) & i = l, j \neq k \\ e_{kj}(-sr) & i \neq l, j = k \\ * & i = l, j = k \end{cases} \end{aligned}$$

Steinberg relations

Definition 8.2.2. $E(n, R) \subseteq SL(n, G)$ is the subgroup generated by elementary matrices of determinant 1. $E(R) = \bigcup E(n, R)$

Lemma 8.2.3. $SL(n, F) = E(n, F)$

E(n,R) is perfect

Lemma 8.2.4. $[E(n, R), E(n, R)] = E(n, R)$ if $n \geq 3$

Proof. For distinct i, j, k, $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$ □

[GL,GL]=E

Theorem 8.2.5 (Whitehead)**.** $[GL(R), GL(R)] = E(R)$, hence $K_1(R) = GL(R)/E(R)$

Proof. Since

$$e_{12}(1)e_{21}(-1)e_{12}(1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

We know

$$[g,h] = \begin{pmatrix} g & \\ & g^{-1} \end{pmatrix} \begin{pmatrix} h & \\ & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & \\ & hg \end{pmatrix} \in E(R)$$

□

Definition 8.2.6. The **Kronecker product** of matrices

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \ B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ a_{np} & \cdots & b_{np} \end{pmatrix}$$

is

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

8.3 Eigenspace decomposition

Proposition 8.3.1. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is a linear operator, and $V = \bigoplus_i V_i$, where V_i are T invariant spaces, denote $T|_{V_i}$ as $T-i$, then $ch_T(t) = \prod_i ch_{T_i}(t)$, and $m_T(t) = \underset{i}{lcm}\ m_{T_i}(t)$

Definition 8.3.2. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$. $\lambda \in F$ is an **eigenvalue** if $Tv = \lambda v$ has nontrivial solution, $v \in V$ is a **generalized eigenvector** of rank m of T corresponding to eigenvalue λ if $(T-\lambda 1_V)^m v = 0, (T-\lambda 1_V)^{m-1} v \neq 0$ for some $m \geq 1$, and let V_λ be the subspace of all such generalized eigenvectors, called **generalized eigenspace**, notice if V is of finite dimensional, then $V_\lambda = \ker(T-\lambda 1_V)^m$ for some m with m being smallest, suppose $\dim V_\lambda = d$, then the characteristic polynomial of $T|_{V_\lambda}$ is $(t-\lambda)^d$, and the minimal polynomial of $T|_{V_\lambda}$ is $(t-\lambda)^m$

Generalized eigenspace decomposition

Proposition 8.3.3. $\overline{F} = F$, finitely dimensional F vector space V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_\lambda V_\lambda$

Definition 8.3.4. $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ give V an $F[x]$ module with $x \cdot v = Tv$, $W \leq V$ be a subspace, W is called T **invariant** if $TW \subseteq W$, or rather W is an $F[x]$ submodule

Definition 8.3.5. An linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is called **semisimple** if V is a semisimple $\mathbb{F}[x]$ submodule

Proposition 8.3.6. Let $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ be a linear operator with $\overline{\mathbb{F}} = \mathbb{F}$, then T is semisimple $\Leftrightarrow$ T is diagonalizable

Proof. Since $\overline{\mathbb{F}} = \mathbb{F}$ and T is semisimple, V can be decomposed as a direct sum of eigenspaces of T, thus T is diagonalizable, conversely, if T is diagonalizable, and $TW \subseteq W$, let V_λ be the eigenspaces of T, denote $W_\lambda = W \cap V_\lambda$, and $W' = \bigoplus_\lambda W'_\lambda$, since $T|_{V_\lambda} = \lambda 1_{V_\lambda}$, we can find $W'_\lambda \leq V_\lambda$ such that $V_\lambda = W_\lambda \oplus W'_\lambda$, and of course $TW'_\lambda \subseteq W'_\lambda$ which implies $TW' \subseteq W'$, then we have $V = \bigoplus_\lambda V_\lambda = \bigoplus_\lambda W_\lambda \oplus W'_\lambda = \bigoplus_\lambda W_\lambda \bigoplus \bigoplus_\lambda W'_\lambda = W \oplus W'$ ☐

Definition 8.3.7. An linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is called nilpotent if $T^k = 0$ for some k, T is called unipotent if $T - 1_V$ is nilpotent

Jordan-Chevalley decomposition

Definition 8.3.8 (Jordan-Chevalley decomposition)**. Jordan-Chevalley decomposition** of a linear operator $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is $T = T_s + T_n$, where T_s is semisimple, T_n is nilpotent and $[T_s, T_n] = 0$

Existence of Jordan-Chevalley decomposition

Theorem 8.3.9. If V is a finite dimensional $\mathbb{F}$ vector space with $\mathbb{F}$ being a perfect field, and $T \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ is a linear operator, then Jordan-Chevalley decomposition always exist, additionally, there exist polynomials $p(t), q(t)$ with no constant terms and $T_s = p(T), T_n = q(T)$, moreover, the decomposition is unique

Proof. First consider $\overline{\mathbb{F}} = \mathbb{F}$, by Proposition 8.3.3, V can be decomposed into the direct sum of generalized eigenspaces $V = \bigoplus_i V_{\lambda_i}$, where $V_{\lambda_i} = \ker(T - \lambda_i 1_V)^{m_i}$ with being m_i being the least and $\dim V_{\lambda_i} = d_i$, define $T_s \in \mathrm{Hom}_{\mathbb{F}}(V,V)$ such that $T_s|_{V_{\lambda_i}} = \lambda_i 1_{V_{\lambda_i}}$ and $T_n = T - T_s$, thus T_s is diagonalizable(semisimple), T_n is nilpotent, $ch_T(t) = \prod_i (t-\lambda_i)^{d_i}$, by Theorem 5.2.11, there exists polynomial $p(t)$ such that $p(t) \equiv 0 \bmod t$, $p(t) \equiv \lambda_i \bmod (t-\lambda_i)^{d_i}$, and let $q(t) = t - p(t)$, then p, q doesn't have constant terms and $T_s = p(T), T_n = q(T)$. For uniqueness, suppose $T = T_s + T_n = T'_s + T'_n$ are two such decompositions, then $T_s - T'_s = T'_n - T_n$ will be nilpotent which implies $T_s - T'_s = 0$ ☐

Chapter 9

Lie algebra

9.1 Non-associative algebra

Definition 9.1.1. A **nonassociative** $\mathbb{F}$ **algebra** A is an $\mathbb{F}$ vector space with multiplication $\cdot$ that is distributive $(a+b)\cdot c = a\cdot c + b\cdot c$, $a\cdot(b+c) = a\cdot b + a\cdot c$. A is **unital** if $1 \in A$, A is **symmetric** if $xy = yx$, A is **antisymmetric** if $xy = -yx$, A satisfies **Jacobi identity** if $(xy)z + (yz)x + (zx)y = 0$
A homomorphism $\phi : A \to B$ is a linear map such that $\phi(xy) = \phi(x)\phi(y)$

Definition 9.1.2. Suppose $e_1, \cdots, e_n$ is a basis of A, $e_ie_j = \sum_i c_k^{ij} e_k$, c_k^{ij} are called **structure constants** with respect to $e_1, \cdots, e_n$. If A satisfies Jacobi identity, then

$$\sum_l c_m^{il} c_l^{jk} + \sum_l c_m^{jl} c_l^{ki} + \sum_l c_m^{kl} c_l^{ij} = 0$$

Definition 9.1.3. $B \leq A$ is a **subalgebra** if B is a subspace such that $BB \subseteq B$. $I \leq A$ is a **left ideal** if $AI \subseteq I$. Suppose $I, J \leq A$ are ideals, define **ideal quotients** $(J : I) = \{x \in A | xI \subseteq J\}$ which is an ideal. Homomorphisms preserve ideals

Remark 9.1.4. If A is (anti)symmetric, left ideals are two-sided ideals

Definition 9.1.5. A is **abelian** if $AA = 0$, A is **simple** if it is not abelian and the only ideals are 0 and A, A is **semisimple** if $A = A_1 \oplus \cdots \oplus A_n$ is the direct sum of simple subalgebras, A is **reductive** if $A = \mathfrak{s} \oplus \mathfrak{a}$ is a direct sum of a semisimple subalgebra $\mathfrak{s}$ and an abelian subalgebra $\mathfrak{a}$

Definition 9.1.6. A **derivation** is an endomorphism $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$. Let $Der_{\mathbb{F}}(A)$ denote all derivations. If $D_1, D_2 \in Der(A)$, then $[D_1, D_2] = D_1D_2 - D_2D_1 \in Der(A)$, $Der(A) \leq End(A)$ is a Lie subalgebra

9.2 Lie algebras

Definition 9.2.1. A **Lie algebra** $\mathfrak{g}$ is a antisymmetric nonassociative $\mathbb{F}$ algebra satisfying Jacobi identity, usually with a **Lie bracket** $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ denote the multiplication. If $\mathtt{char}\mathbb{F}=2$, we also require $[x,x]=0$

Definition 9.2.2. A $\mathfrak{g}$ **module** V is an abelian group with left group action $\mathfrak{g}\times V\to V$ such that $1v=v$, $x(v+w)=xv+xw$, $(x+y)v=xv+yv$, $(xy)v=x(yv)-y(xv)$. Equivalently, a **Lie algebra representation** (π,V) is a Lie algebra homomorphism $\pi:\mathfrak{g}\to\mathfrak{gl}(V)$, where V is an $\mathbb{F}$ vector space, $xv:=\pi(x)v$ give V the $\mathfrak{g}$ module structure

Remark 9.2.3. A $\mathfrak{g}$ module is not a module according to Definition 6.1.1

Definition 9.2.4. A $\mathfrak{g}$ **module homomorphism** $\phi:V\to W$ between $\mathfrak{g}$ modules is a group homomorphism such that $\phi(xv)=x\phi(v)$. Equivalently, an intertwine map $\phi:V\to W$ between Lie algebra representations is a linear map such that $\phi(\pi_V(x)v)=\pi_W(x)\phi(v)$, giving the $\mathfrak{g}$ module homomorphism
A subrepresentation (π,W) is a $\mathfrak{g}$ submodule $W\leq V$

Adjoint representation

Definition 9.2.5. The **adjoint endomorphism** associated to x is left multiplication by x, i.e. $ad(x)(y)=[x,y]$, Jacobi identity becomes $ad([x,y])=[ad(x),ad(y)]$, give a Lie algebra representation(adjoint representation) $ad:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$, $ad(x)$ are called **inner derivations** since $ad(z)[x,y]=[ad(z)x,y]+[x,ad(z)y]$. $ad(\mathfrak{g})\leq\mathrm{Der}(\mathfrak{g})\leq\mathfrak{gl}(\mathfrak{g})$ are Lie subalgebras
Any Lie algebra homomorphism $\phi:\mathfrak{g}\to\mathfrak{h}$ induce a Lie algebra homomorphism $\phi:ad(\mathfrak{g})\to ad(\mathfrak{h})$ by $\phi(ad(x))=ad(\phi(x))$

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\phi"] \arrow[d, "ad"] & \mathfrak{h} \arrow[d, "ad"] \\
ad(\mathfrak{g}) \arrow[r, "\phi"] & ad(\mathfrak{h})
\end{tikzcd}$$

Definition 9.2.6. Centralizer of S is defined to be $C_{\mathfrak{g}}(S):=\{g\in\mathfrak{g}|[g,S]=0\}$, in particular, the center $Z(\mathfrak{g}):=C_{\mathfrak{g}}(\mathfrak{g})$
Normalizer of S is defined to be $N_{\mathfrak{g}}(S):=\{g\in\mathfrak{g}|[g,S]\subseteq S\}$

Definition 9.2.7.

$$\mathfrak{g}\supseteq[\mathfrak{g},\mathfrak{g}]\supseteq[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]\supseteq[[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]],[[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]]]\supseteq\cdots$$

is called the **derived series**, $\mathfrak{g}$ is **solvable** if derived series terminates

$$\mathfrak{g}\supseteq[\mathfrak{g},\mathfrak{g}]\supseteq[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]\supseteq[\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]]\supseteq\cdots$$

is called the **lower central series**, $\mathfrak{g}$ is **nilpotent** if lower central series terminates

Example 9.2.8. $[\mathfrak{gl}(V),\mathfrak{gl}(V)]=\mathfrak{sl}(V)$

Proof. Since $\mathrm{Tr}[X,Y]=0$, thus $[\mathfrak{gl}(V),\mathfrak{gl}(V)]\leq\mathfrak{sl}(V)$, conversely □

Definition 9.2.9. Let $\mathfrak{g}$ be a Lie algebra, a Cartan subalgebra $\mathfrak{h}\leq\mathfrak{g}$ is a nilpotent subalgebra such that $N_{\mathfrak{g}}(\mathfrak{h})=\mathfrak{h}$(self normalizing or alternatively $(\mathfrak{h}:\mathfrak{h})=\mathfrak{h}$)

Definition 9.2.10. Let $\mathfrak{g}\leq\mathfrak{gl}(V)$ be a Lie algebra, $\mathfrak{g}$ is called **toral** if $\mathfrak{g}$ consists of semisimple elements

Definition 9.2.11. Let $\mathfrak{g}$ be a Lie algebra, we can show the sum of all solvable ideal is again a solvable ideal, thus $\mathfrak{g}$ has a unique maximal solvable ideal $\mathtt{rad}(\mathfrak{g})$, called the **radical** of $\mathfrak{g}$

Definition 9.2.12. Let $\mathfrak{g}$ be a complex Lie algebra, $\mathfrak{g}_0$ is called a **real form** of $\mathfrak{g}$ if $\mathfrak{g} \cong \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$

Example 9.2.13. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be Lie subalgebra, the **tautological representation** (τ, V) is defined by $\tau(x) = x$, then $\tau([x, y]) = [x, y] = [\tau(x), \tau(y)]$

Proposition 9.2.14. Lie algebra $\mathfrak{g}$ is reductive iff its adjoint representation is completely reducible

g semisimple, phi:g->gl(V) representation => phi(g)<=sl(V)

Lemma 9.2.15. If $\mathfrak{g}$ is semisimple Lie algebra, and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, then $\varphi(\mathfrak{g}) \leq \mathfrak{sl}(V)$

Proof. By Proposition 9.6.1, $\varphi(\mathfrak{g}) = \varphi([\mathfrak{g}, \mathfrak{g}]) \leq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$ ☐

Derivations of semisimple Lie algebra are inner derivations

Proposition 9.2.16. If $\mathfrak{g}$ is a semisimple Lie algebra, then $ad(\mathfrak{g}) = Der(\mathfrak{g})$

Proof. As an abelian ideal of $\mathfrak{g}$, $Z(\mathfrak{g}) = 0$, thus $\mathfrak{g} \xrightarrow{ad} ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding. Since $[\delta, ad(x)] = ad(\delta(x)), \delta \in Der(\mathfrak{g})$, thus $[Der(\mathfrak{g}), ad(\mathfrak{g})] \subseteq ad(\mathfrak{g})$, Let $K(,)$ be the Killing form on $Der(\mathfrak{g})$, due to Proposition 9.6.6(b) and Proposition 9.4.4(c), $K(,)|_{ad(g)}$ is nondegenerate, denote $I := ad(\mathfrak{g})^{\perp}$ under $K(,)$, then $I \cap ad(\mathfrak{g}) = 0$, otherwise $0 \neq I \cap ad(\mathfrak{g}) \subseteq \ker K(,)|_{ad(\mathfrak{g})}$, by Exercise 53.0.2, $[I, ad(\mathfrak{g})] = 0$, thus for any $\delta \in I$, $0 = [\delta, ad(x)] = ad(\delta(x))$, since ad is an isomorphism, $\delta(x) = 0$, thus $\delta = 0$, $I = 0$, $ad(\mathfrak{g}) = Der(\mathfrak{g})$ ☐

Remark 9.2.17. When $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{g} \xrightarrow{ad} ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is an embedding, we can identify x with $ad(x)$, by abuse of notations, xy can defined to be the preimage of $ad(x)ad(y) \in \mathfrak{gl}(\mathfrak{g})$

Lemma 9.2.18. Let G be a compact Lie group, we can pick any nonzero k-form α_I at I, and extend it to a k-form on G by $\alpha_A(Y_1, \cdots, Y_k) = \alpha_I(Y_1 A^{-1}, \cdots, Y_k A^{-1})$, or just $R_A^* \alpha_A = \alpha_I$, then we can define integral $\int_G f(A)\alpha$, then we would have $\int_G f(AB)\alpha = \int_G f(AB)\alpha_A = \int_G f(AB) R_B^* \alpha_{AB} = \int_G f(A) R_B^* \alpha = \int_G f(A)\alpha$, since $R_B^* \alpha = \alpha$, i.e. $(R_B^* \alpha)_A(X_A) = R_B^* \alpha_{AB}(X_A) = \alpha_A(X_A)$, thus this integration is right invariant. Note that this actually gives a right invariant Haar measure

Weyl's theorem

Theorem 9.2.19. Weyl's theorem
Let $\mathfrak{g}$ be a semisimple Lie algebra and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, then φ is completely reducible, namely, $\mathfrak{g}$ modules are semisimple(completely reducible), thus any irreducible subrepresentation has to be one of the summand in the decomposition

Proof. Weyl's unitary trick ☐

9.3 Engel's theorem and Lie's theorem

Lemma for Engel's theorem

Lemma 9.3.1. Let $V \neq 0$ be a finite dimensional vector space, suppose $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra consists of nilpotent elements, then there exists $0 \neq v \in V$ such that $\mathfrak{g}v = 0$

Engel's theorem

Theorem 9.3.2. Engel's theorem Consider the adjoint representation $(ad, \mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ nilpotent iff $ad(X), X \in \mathfrak{g}$ can be strictly upper triangulized simultaneously iff $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$

g nilpotent, I ideal => I intersect Z(g) is nontrivial

Lemma 9.3.3. Let $\mathfrak{g}$ be a nilpotent Lie algebra, $I \leq \mathfrak{g}$ is a nonzero ideal, then $I \cap Z(\mathfrak{g}) \neq 0$, in particular, if $I = \mathfrak{g}$ then $Z(\mathfrak{g}) \neq 0$ which can also easily being shown from the fact that $Z(\mathfrak{g})$ contains the last nonzero term in the lower central series of $\mathfrak{g}$

Proof. Consider adjoint map restrict on I, since $\mathfrak{g}$ is nilpotent, $ad(X)$ is nilpotent for any $X \in \mathfrak{g}$, so is $ad(X)|_I$, i.e. $ad(\mathfrak{g})_I \leq \mathfrak{gl}(I)$ is a Lie subalgebra consists of nilpotent elements, by Lemma 9.3.1, there exists $0 \neq Y \in I$ such that $[X, Y] = 0, \forall X \in \mathfrak{g}$, thus $Y \in I \cap Z(\mathfrak{g})$ ☐

Lie's theorem

Theorem 9.3.4. Lie's theorem
If (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}, \mathrm{char}\mathbb{F} = 0$, if $\mathfrak{g}$ is solvable, so is $\pi(\mathfrak{g})$, and $\pi(X), X \in \mathfrak{g}$ can be upper triangulized simultaneously

Remark 9.3.5. If (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}, \mathrm{char}\mathbb{F} = 0$, if $\mathfrak{g}$ is abelian, so is $\pi(\mathfrak{g})$, but it doesn't imply $\pi(X), X \in \mathfrak{g}$ can be diagonalized simultaneously, for example $\left\langle \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle \subseteq \mathfrak{gl}(\mathbb{C}^2)$ is abelian, and $\left\langle \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \right\rangle$ is not diagonalizable at all
However, due to Proposition 53.0.5, if (π, V) is a finite representation of a finite dimensional Lie algebra $\mathfrak{g}$ with $\overline{\mathbb{F}} = \mathbb{F}$, where $\mathfrak{g}$ is abelian, so is $\pi(\mathfrak{g})$, and suppose $\pi(X), X \in \mathfrak{g}$ are diagonalizable($\pi(\mathfrak{g})$ is a toral Lie subalgebra), then they can be diagonalized simultaneously

9.4 Killing form

Definition 9.4.1. A bilinear form $(,)$ on Lie algebra $\mathfrak{g}$ is **invariant** or **associative** if the Lie derivative is zero, i.e. $(ad_Y X, Z) + (X, ad_Y Z) = 0$, or equivalently, $([X,Y],Z) = (X,[Y,Z])$

Definition 9.4.2. A **quadratic Lie algebra** is a Lie algebra $\mathfrak{g}$ with an invariant nondegenerate symmetric bilinear form $(,) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F}$

Definition 9.4.3. Killing form is the bilinear map $K(,) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X,Y) \mapsto \mathrm{Tr}\,(ad(X)ad(Y))$

Some basic properties of Killing form

Proposition 9.4.4.

(a) The Killing form is symmetric and invariant

(b) The Killing form on a nilpotent Lie algebra is zero

(c) Suppose $I \leq \mathfrak{g}$ is an ideal, then Killing form $K_I(,)$ on I is the same as the restriction of Killing form $K(,)$ to I, i.e. $K_I(,) = K(,)|_I$

Proof.

(a)

(b) Due to Theorem 9.3.2

(c) By Exercise 46.1.4, for $X, Y \in I$, we have

$$\begin{aligned} K_I(X,Y) &= Tr(ad(X)|_I ad(Y)|_I) \\ &= Tr\left((ad(X)ad(Y))|_I\right) \\ &= Tr(ad(X)ad(Y)) \\ &= K(X,Y) \\ &= K(X,Y)|_I \end{aligned}$$

□

Example 9.4.5. The Killing form is a symmetric, bilinear and invariant form, and it is nondegenerate iff $\mathfrak{g}$ is semisimple due to Proposition 9.6.6

nondegenerate, symmetric, bilinear and invariant form is unique up to scalar

Lemma 9.4.6. Any invariant, symmetric and bilinear form on simple Lie algebra $\mathfrak{g}$ is a multiple of the Killing form

Proof. Suppose $(,)$ is an invariant, symmetric and bilinear form, so is $[,]_c = (,) - cK(,)$ for any c. If $(,) \neq 0$, then there exists $x, y \in \mathfrak{g}$ such that $[x,y]_c = 0$ for some c, since the kernel of $[,]_c$ is a nonzero ideal, $[,]_c = 0$ □

9.5 Jordan-Chevalley decomposition

Abstract Jordan-Chevalley decomposition on nonassociative F-algebras

Lemma 9.5.1. Let $\mathfrak{g}$ be a finite dimensional nonassociative $\mathbb{F}$ algebra(including Lie algebra) with $\overline{\mathbb{F}} = \mathbb{F}$, for any $\delta \in Der(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$, let $\delta = \delta_s + \delta_n$ be its Jordan-Chevalley decomposition in $\mathfrak{gl}(\mathfrak{g})$, then $\delta_s, \delta_n \in Der(\mathfrak{g})$

Proof. For any $a \in \mathbb{F}$, define $\mathfrak{g}_a$ be the generalized eigenspace of a, then we have $\mathfrak{g} = \bigoplus_{a\in\mathbb{F}} \mathfrak{g}_a$, and $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, since for any $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, $(\delta - (a+b)1_{\mathfrak{g}})^m([x,y]) = \sum_{k=0}^{m} \binom{m}{k} \left[(\delta - a1_{\mathfrak{g}})^{m-k}x, (\delta - b1_{\mathfrak{g}})^k y\right]$ which can be easily checked by induction. Then we have $\delta_s(x) = ax, \delta_s(y) = by$, and $\delta_s([x,y]) = (a+b)[x,y] = [ax,y] + [x,by] = [\delta_s(x), y] + [x, \delta_s y]$, thus $\delta_s \in Der(\mathfrak{g})$, so does $\delta_n = \delta - \delta_s$ ☐

Definition 9.5.2. Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Abstract Jordan-Chevalley decomposition on semisimple Lie algebras

Because Lemma 9.5.1 and Proposition 9.2.16, for any $x \in \mathfrak{g}$, we can identify x with $ad(x)$, and we have Jordan-Chevalley decomposition $ad(x) = ad(x)_s + ad(x)_n = ad(x_s) + ad(x_n)$, where x_s, x_n are defined to be the preimages of $ad(x)_s, ad(x)_n$. Moreover, there exists polynomials $p(t), q(t)$ with no constant terms such that $ad(x_s) = ad(x)_s = p(ad(x))$, $ad(x_n) = ad(x)_n = q(ad(x))$, by abuse of notations, $x_s = p(x)$ and $x_n = q(x)$

Semisimple Lie algebra contains the semisimple and nilpotent parts of its elements

Theorem 9.5.3. Suppose V is a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a semisimple Lie algebra, for any $x \in \mathfrak{g}$, $x = x_s + x_n$ is the Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$, moreover, the abstract and usual Jordan-Chevalley decompositions coincide, i.e. $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$

Proof. Define lie subalgebras $\mathfrak{l}_W := \{y \in \mathfrak{gl}(V) | yW \subseteq W, Tr(y|_W) = 0\}$ with $W \leq V$ being $\mathfrak{g}$ submodules, and define $\mathfrak{l} = \left(\bigcap_W \mathfrak{l}_W\right) \bigcap N_{\mathfrak{gl}(V)}(\mathfrak{g})$, for any $x \in \mathfrak{g}$, due to Proposition 46.1.4 and Lemma 9.2.15, $Tr(x|_W) = Tr(x) = 0$, $\mathfrak{g} \leq \mathfrak{l}_W \Rightarrow \mathfrak{g} \leq \mathfrak{l}$, thus $\mathfrak{l}$ is a subalgebra of $N_{\mathfrak{gl}(V)}(\mathfrak{g})$ of containing $\mathfrak{g}$, thus $\mathfrak{l}$ is finite dimensional $\mathfrak{g}$ module, by Theorem 9.2.19, $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{h}$ is a direct sum of $\mathfrak{g}$ modules, since $\mathfrak{l} \leq N_{\mathfrak{gl}(V)}(\mathfrak{g})$, $[\mathfrak{g}, \mathfrak{l}] = 0 \Rightarrow [\mathfrak{g}, \mathfrak{h}] = 0$, i.e. $\mathfrak{g}$ acts trivially on $\mathfrak{h}$, fix any irreducible $\mathfrak{g}$ submodule W, for any $y \in \mathfrak{h}$, $x \in \mathfrak{g}$, $xy - yx = [x,y] = 0$, $yxv = xyv$ for $v \in W$, i.e. $y \in \mathrm{Hom}_{\mathfrak{g}}(W, W)$, by Lemma 6.1.12, y acts on W as a scalar, but $Tr(y|_W) = 0$, thus y acts trivially on W, again by Theorem 9.2.19, V can written as the direct sum of irreducible $\mathfrak{g}$ submodules, thus y acts trivially on $W \Rightarrow y = 0$, therefore $\mathfrak{h}_j = 0 \Rightarrow \mathfrak{g} = \mathfrak{l}$, for any $x \in \mathfrak{g}$, due to Theorem 8.3.9, $x = x_s + x_n$ and $x_s = p(x), x_n = q(x)$ for some polynomials $p(x), q(x)$ with no constant terms, thus if $x \in \mathfrak{l}_W$, $xW \subseteq W$ and $Tr(x|_W) = 0$, then $x_s W = p(x)W \subseteq W$, $Tr(x_s|_W) = Tr(p(x|_W)) = 0$, similarly, $x_n W \subseteq W$, $Tr(x_n|_W) = 0$, $x_s, x_n \in \mathfrak{l}_W$, also $x_s = p(x), x_n = q(x) \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$, thus $x_s, x_n \in \mathfrak{l} = \mathfrak{g}$

Since the Jordan-Chevalley decomposition of $ad(x)$ in $\mathfrak{gl}(V)$ is unique and $ad(x_s) + ad(x_n) = ad(x) = ad(x)_s + ad(x)_n$, thus $ad(x_s) = ad(x)_s, ad(x_n) = ad(x)_n$ ☐

Corollary 9.5.4. Suppose V is a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, $\mathrm{char}\mathbb{F} = 0$, $\mathfrak{g}$ is a semisimple Lie algebra, and $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra representation, $x = x_s + x_n$ is the abstract Jordan-Chevalley decomposition, then $\phi(x) = \phi(x_s) + \phi(x_n)$ is the usual Jordan-Chevalley decomposition in $\mathfrak{gl}(V)$

Proof. Due to Proposition 9.6.2, $\phi(\mathfrak{g})$ is also semisimple

First notice that a linear operator $T \in \mathrm{End}_{\mathbb{F}}(V)$ is diagonalizable(semisimple) iff the dimension of the sum of its eigenspaces is $\dim V$, or equivalently iff all eigenvectors span V

Since $ad(x_s)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 9.2.16, the eigenvectors y_i's of $ad(x_s)$ spans $\mathfrak{g}$, say $ad(x_s)(y_i) = \lambda_i y_i$, then as in Definition 9.2.5, $ad(\phi(x_s))(\phi(y_i)) = [\phi(x_s), \phi(y_i)] =$

$\phi([x_s, y_i]) = \phi(ad(x_s)(y_i)) = \phi(\lambda_i y_i) = \lambda_i \phi(y_i)$, thus those $0 \neq \phi(y_i)$ are eigenvectors of $\phi(\mathfrak{g})$, hence $ad(\phi(x_s))$ is semisimple in $\mathfrak{gl}(\mathfrak{gl}(V))$
Since $ad(x_n)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$ as in Proposition 9.2.16, $ad(x_n)^m = 0$ for some m, then as in Definition 9.2.5 $ad(\phi(x_n))^m = \phi(ad(x_n))^m = \phi(ad(x_n)^m) = 0$, thus $ad(\phi(x_n))$ is also nilpotent in $\mathfrak{gl}(\mathfrak{gl}(V))$
Moreover, as in Definition 9.2.5, $[ad(\phi(x_s)), ad(\phi(x_n))] = [\phi(ad(x_s)), \phi(ad(x_n))] = \phi([ad(x_s), ad(x_n)]) = 0$
Thus $\phi(x) = \phi(x_s) + \phi(x_n)$ is the abstract Jordan-Chevalley decomposition of $\phi(x)$ in $\phi(\mathfrak{g}) \leq \mathfrak{gl}(V)$, by Theorem 9.5.3, this coincide with the usual Jordan-Chevalley decomposition of $\phi(x) = \phi(x)_s + \phi(x)_n$ in $\mathfrak{gl}(V)$, i.e. $\phi(x_s) = \phi(x)_s$, $\phi(x_n) = \phi(x)_n$ $\square$

9.6 Classification of semisimple Lie algebras

g simple implies [g,g]=g

Proposition 9.6.1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a semisimple Lie algebra, then any ideal of $\mathfrak{g}$ is certain sum of $\mathfrak{g}_i$'s, and any sum of $\mathfrak{g}_i$'s is an ideal, in particular, $\mathfrak{g}_i$'s are ideals, moreover, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$

Proof. any ideal $I \leq \mathfrak{g}$ is certain sum of $\mathfrak{g}_i$'s, because $I \cap \mathfrak{g}_i$ is an ideal of $\mathfrak{g}_i$ which is either 0 or $\mathfrak{g}_i$ itself
If $\mathfrak{g}$ is a simple Lie algebra, then it is not abelian, $[\mathfrak{g}, \mathfrak{g}] \neq 0$, thus $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, generally, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n, \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n] = [\mathfrak{g}_1, \mathfrak{g}_1] \oplus \cdots \oplus [\mathfrak{g}_n, \mathfrak{g}_n] = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n = \mathfrak{g}$ ☐

image of a semisimple Lie algebra is also semisimple

Proposition 9.6.2. If $\mathfrak{g}$ is semisimple, $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then $\phi(\mathfrak{g})$ is also semisimple

Proof. Suppose $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is a direct sum of simple Lie algebras, $\phi(\mathfrak{g}_i)$ are also ideals so $\phi(\mathfrak{g}) = \phi(\mathfrak{g}_1) \oplus \cdots \oplus \phi(\mathfrak{g}_n)$ is a direct sum of ideals, and $[\phi(\mathfrak{g}_i), \phi(\mathfrak{g}_i)] = \phi([\mathfrak{g}_i, \mathfrak{g}_i]) = \phi(\mathfrak{g}_i)$ implies that each $\phi(\mathfrak{g}_i)$ is simple or 0, thus $\phi(\mathfrak{g})$ is also semisimple ☐

nilpotent/semisimple implies ad-nilpotent/ad-semisimple

Proposition 9.6.3. Let $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a Lie algebra, $X \in \mathfrak{g}$, then X is nilpotent $\Rightarrow ad(X)$ is nilpotent, if in addition V is finite dimensional and $\overline{\mathbb{F}} = \mathbb{F}$, then X is semisimple $\Rightarrow ad(X)$ is semisimple, or rather diagonalizable

Proof. Let L_X, R_X be left and right multiplications by X, then we have $ad_X = L_X - R_X$ and $[L_X, R_X] = 0$, thus X is nilpotent $\Rightarrow ad(X)$ is nilpotent
Notice that given $A = (a_{ij})$, $D = \mathrm{diag}(d_1, \cdots, d_n)$, $[D, A] = ((d_i - d_j)a_{ij})$, thus $[D, E_{ij}] = (d_i - d_j)E_{ij}$, thus X is diagonalizble $\Rightarrow ad(X)$ is diagonalizable ☐

Cartan's criterion for solvability

Theorem 9.6.4 (Cartan's criterion for solvability)**.** Let V be a finite dimensional $\mathbb{F}$ vector space with $\mathrm{char}\,\mathbb{F} = 0$, $\mathfrak{g} \leq \mathfrak{gl}(V)$ is a Lie subalgebra, then $\mathfrak{g}$ is solvable iff $\mathrm{Tr}(XY) = 0$, $\forall X \in \mathfrak{g}, Y \in [\mathfrak{g}, \mathfrak{g}]$

Cartan's criterion for semisimplicity

Corollary 9.6.5. Cartan's criterion for semisimplicity
Let $\mathfrak{g}$ is finite dimensional Lie algebra with $\mathrm{char}\,\mathbb{F} = 0$, then $\mathfrak{g}$ is semisimple iff its Killing form is nondegenerate

Equivalent conditions for semisimplicity

Proposition 9.6.6. The following statements are equivalent
(a) $\mathfrak{g}$ is semisimple
(b) The Killing form is nondegenerate
(c) $\mathfrak{g}$ doesn't have have nontrivial abelian ideals
(d) $\mathfrak{g}$ doesn't have have nontrivial solvable ideals
(e) $\mathrm{rad}(\mathfrak{g}) = 0$

Proof. $(a) \Leftrightarrow (b)$ is due to Corollary 9.6.5 ☐

Adjoint representation of SL(2,F)

Example 9.6.7.
Recall $\mathfrak{sl}(2, \mathbb{F}) = \{X \in M(2, \mathbb{F}) | \mathrm{Tr}(X) = 0\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a, b, c \in \mathbb{F} \right\}$
$\mathfrak{sl}(2, \mathbb{F}) = \langle H, X, Y \rangle$, where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $ad_H X = [H, X] = 2X$, $ad_H Y = [H, Y] = -2Y$, $ad_X Y = [X, Y] = H$, this is the adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$

Lemma 9.6.8. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, if $V = \mathbb{F}^n$, then there are theree $n \times n$ matrices x, y, h such that $[h, x] = 2x, [h, y] = -2y, [x, y] = h$ due to Example 9.6.7

Lemma 9.6.9. Let (π, V) be a finite dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, $V_\lambda := \{v \in V | \pi(H)v = \lambda v\}$, then $\pi(X)V_\lambda \subseteq V_{\lambda+2}$, $\pi(Y)V_\lambda \subseteq V_{\lambda-2}$ and $\pi(H)V_\lambda \subseteq V_\lambda$

Proof. $\pi(H)V_\lambda \subseteq V_\lambda$ is just by definition, suppose $v \in V_\lambda$, $\pi(H)\pi(X)v = 2\pi(X)v+\pi(X)\pi(H)v = (\lambda+2)\pi(X)v$, $\pi(H)\pi(Y)v = -2\pi(Y)v + \pi(Y)\pi(H)v = (\lambda-2)\pi(Y)v$, □

Remark 9.6.10. $\pi(X), \pi(Y)$ are named **raising and lowering operator**

Classification of representations of sl(2,F)

Theorem 9.6.11. Suppose $\overline{\mathbb{F}} = \mathbb{F}, \operatorname{char}\mathbb{F} = 0$, for any integer $m \geq 0$, there is an irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$ with dimension $m+1$

Proof. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{F})$, there exists highest $\lambda \in \mathbb{F}$ such that $V_\lambda \neq 0$, pick $0 \neq u \in V_\lambda$, let $u_k := \pi(Y)^k u \in V_{\lambda-2k}$, then there exists m such that $u_m \neq 0$ but $u_{m+1} = 0$, then $u_0, \cdots, u_m$ are independent since they belong distinct eigenspaces, with $\pi(H)u_k = (\lambda-2k)u_k$, $\pi(X)u_0 = \pi(X)u = 0$ since $u \in V_\lambda$ is of "highest weight", and $\pi(X)u_k = k(\lambda-k+1)u_{k-1}, k > 0$ by induction, $\pi(X)u_1 = \pi(X)\pi(Y)u = ([\pi(X),\pi(Y))] + \pi(Y)\pi(X))u = \pi(H)u + \pi(Y)\pi(X)u = \lambda u = 1(\lambda-1+1)u_0$, $\pi(X)u_{k+1} = \pi(X)\pi(Y)u_k = ([\pi(X),\pi(Y))]+\pi(Y)\pi(X))u_k = \pi(H)u_k+\pi(Y)\pi(X)u_k = (\lambda-2k)u_k+k(\lambda-k+1)\pi(Y)u_{k-1} = (k+1)(\lambda-k)u_k$

Note that since $0 = Xu_{m+1} = (m+1)(\lambda-m)u_m \Rightarrow \lambda = m$, which implies all possible eigenvalue for $\pi(H)$ has to be integers, when m is even, we call this irreducible representation even, when m is odd, we call this irreducible representation odd

In general, for any finite dimensional representation, we can decompose the representation into irreducible subrepresentations by using this procedure repeatedly

Therefore $0 \neq W := \langle u_0, \cdots, u_m\rangle$ is invariant, but π is irreducible, thus $V = W$, and by Lemma 6.1.12, (π, V) is unique up to isomorphism □

Adjoint representation of sl(2,F) is the unique 3 dimensional irreducible representation

Example 9.6.12. $(ad, \mathfrak{sl}(2,\mathbb{F}))$ is the unique irreducible 3 dimensional representation of $\mathfrak{sl}(2,\mathbb{F})$ with $V_0 = \langle H\rangle$, $V_{-2} = \langle Y\rangle$ and $V_2 = \langle X\rangle$, it is irreducible because of Lemma 9.6.15, if we use X, Y, H as basis, then $ad(X), ad(Y), ad(H)$ would have the matrix forms

$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus

$$K(X,X) = Tr\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0, \quad K(X,Y) = Tr\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 4$$

$$K(X,H) = Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = 0, \quad K(Y,Y) = Tr\begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$K(Y,H) = Tr\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} = 0, \quad K(H,H) = Tr\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 8$$

Thus its Cartan matrix is $\Phi = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ which is nondegenerate

Tautological representation of sl(2,F)

Example 9.6.13. The tautological representation $(\tau, \mathbb{F}^2)$ is the unique irreducible 2 dimensional representation of $\mathfrak{sl}(2,\mathbb{F})$ with $V_1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, $V_{-1} = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$

Example 9.6.14. Let $S^k(\mathbb{F}^2)$ be the k-th symmetric power of $\mathbb{F}^2$ which is isomorphic to the the set of degree k polynomials in $\mathbb{F}[x,y]$ generated by
$\langle x^k, x^{k-1}y, \cdots, xy^{k-1}, y^k\rangle$ which is of dimension $k+1$, with this identification, $(\pi, S^k(\mathbb{F}^2))$ with

$\pi(X)(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, $\pi(X)y = x$, $\pi(Y)x = y$, $\pi(Y)y = 0$, $\pi(H)x = x$, $\pi(H)y = -y$, just as in Example 9.6.13, and define inductively that $\pi(Z)(fg) = g\pi(Z)f + f\pi(Z)g$, this is the unique $k+1$ dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$

Count the number irreducible summand of a representation of sl(2,F)

Lemma 9.6.15. Let (π, V) be a finite dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$, and V_k be the k-eigenspace of $\pi(H)$, then the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$, whereas $\dim V_0, \dim V_1$ are number of even and odd irreducible summands

Proof. In an even irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are even, so there is a unique 0-eigenvector, in an odd irreducible representation of $\mathfrak{sl}(2, \mathbb{F})$ all the eigenvalues are odd, so there is a unique 1-eigenvector, thus the number of irreducible summand of (π, V) is $\dim V_0 + \dim V_1$ □

Definition 9.6.16. $\mathfrak{g}$ is a semisimple Lie algebra, $\mathfrak{h}$ is a maximal toral Lie algebra. For $\alpha \in \mathfrak{h}^* = \mathrm{Hom}_{\mathbb{F}}(\mathfrak{h}, \mathbb{F})$, define **root spaces**

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | ad_h(x) = [h, x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$$

α is a **root** if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, denote the set of roots as Δ. $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of $\mathfrak{h}$

Basic properties of root spaces

Proposition 9.6.17.

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

(b) $\alpha \in \Delta$, any $X \in \mathfrak{g}_\alpha$ is nilpotent

(c) $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ unless $\alpha + \beta = 0$

(d) $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

Proof.

(a) For $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $Z \in \mathfrak{h}$, we have

$$[Z, [X, Y]] = [[Z, X], Y] + [X, [Z, Y]] = \alpha(Z)[X, Y] + \beta(Z)[X, Y] = (\alpha + \beta)(Z)[X, Y]$$

(b) For $\beta \in \Delta \cup \{0\}$, $\alpha \in \Delta$, $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$, $ad(X)^n(Y) \in \mathfrak{g}_{n\alpha+\beta} = 0$ when n is big enough, thus $ad(X)$ is nilpotent

(c) Supppose $\alpha + \beta \neq 0$, then there exists $Z \in \mathfrak{h}$ such that $(\alpha + \beta)(Z) \neq 0$, then for any $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$

$$\begin{aligned}(\alpha + \beta)(Z)K(X, Y) &= \alpha(Z)K(X, Y) + \beta(Z)K(X, Y) \\ &= K(\alpha(Z)X, Y) + K(X, \beta(Z)Y) \\ &= K([Z, X], Y) + K(X, [Z, Y]) \\ &= -K([X, Z], Y) + K(X, [Z, Y]) \\ &= 0\end{aligned}$$

Thus $K(X, Y) = 0$

(d) Since $K(\mathfrak{g}_\alpha, \mathfrak{h}) = 0, \forall \alpha \in \Delta$, $\ker K(,)|_{\mathfrak{g}_0} \subseteq \ker K(,) = 0$, thus $K(,)|_{\mathfrak{g}_0}$ is nondegenerate

□

Root space decomposition

Theorem 9.6.18. Semisimple Lie algebra $\mathfrak{g}$ has root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$

Proof. By Proposition 9.6.17 □

x,y in gl(V) commutes, x nilpotent => xy nilpotent

Lemma 9.6.19. *V is an $\mathbb{F}$ vector space, $x, y \in \mathfrak{gl}(V)$ commutes, x is nilpotent, then xy is nilpotent, and $Tr(xy) = 0$*

Proof. $x^m = 0 \Rightarrow (xy)^m = x^m y^m = 0$ □

Maximal toral Lie algebra of semisimple Lie algebra is self centralizing

Proposition 9.6.20. For semisimple Lie algebra $\mathfrak{g}$ with maximal toral Lie subalgebra $\mathfrak{h}$, $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$

Proof.

Step I: $C_{\mathfrak{g}}(\mathfrak{h})$ contains semisimple and nilpotent parts of its elements

If $x \in C_{\mathfrak{g}}(\mathfrak{h})$, due to Proposition 9.5.2, there are polynomials $p(t), q(t)$ with no constant terms such that $ad(x_s) = ad(x)_s = p(ad(x)), ad(x_n) = ad(x)_n = q(ad(x))$, since $x \in C_{\mathfrak{g}}(\mathfrak{h})$, $ad(x)|_{\mathfrak{h}} = 0$, , $ad(x_s)|_{\mathfrak{h}} = p(ad(x))|_{\mathfrak{h}} = 0$, $ad(x_n)|_{\mathfrak{h}} = q(ad(x))|_{\mathfrak{h}} = 0$, thus $x_s, x_n \in C_{\mathfrak{g}}(\mathfrak{h})$

Step II: $\mathfrak{h}$ contains all semisimple elements of $C_{\mathfrak{g}}(\mathfrak{h})$

If $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, use Exercise 53.0.4, s and elements of $\mathfrak{h}$ are diagonalizable simultaneously, thus $\mathfrak{h} + \langle s \rangle$ is toral in $\mathfrak{g}$, then $s \in \mathfrak{h}$ since $\mathfrak{h}$ is maximal

Step III: $K(,)|_{\mathfrak{h}}$ is nondegenerate

Suppose there exists $h \in \mathfrak{h}$ such that $K(h, \mathfrak{h}) = 0$, if $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $ad(n)$ is nilpotent, and $[n, \mathfrak{h}] = 0$, thus $[ad(n), ad(h)] = ad([n, h]) = 0$, by Lemma 9.6.19, $Tr(ad(n)ad(h)) = 0$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, thus $K(s, h) = 0$, and according to Step I, $K(h, C_{\mathfrak{g}}(\mathfrak{h})) = 0$ which contradicts Proposition 9.6.17(d) that $K(,)|_{C_{\mathfrak{g}}(\mathfrak{h})}$ is nondegenerate

Step IV: $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

If $n \in C_{\mathfrak{g}}(\mathfrak{h})$ be a nilpotent element, then $ad(n)$ is nilpotent, so is $ad(n)|_{C_{\mathfrak{g}}(\mathfrak{h})}$, if $s \in C_{\mathfrak{g}}(\mathfrak{h})$ be a semisimple element, according to Step II, $s \in \mathfrak{h}$, $ad(s)|_{C_{\mathfrak{g}}(\mathfrak{h})} = 0$, by Theorem 9.3.2, $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent

Step V: $\mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] = 0$

Suppose $x \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then $x = \sum [y_i, z_i]$ where $y_i, z_i \in C_{\mathfrak{g}}(\mathfrak{h})$, then $K(x, x) = \sum K(x, [y_i, z_i]) = \sum K([x, y_i], z_i) = 0$, since $K(,)$ is nondegenerate on $\mathfrak{h}$(or $\mathfrak{g}$ or $C_{\mathfrak{g}}(\mathfrak{h})$), thus $x = 0$

Step VI: $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian

Suppose $[C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})] \neq 0$, since $C_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent from Step IV, by Lemma 9.3.3, there exists $0 \neq z \in Z(C_{\mathfrak{g}}(\mathfrak{h})) \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, then z can't be semisimple, otherwise $z \in \mathfrak{h} \cap [C_{\mathfrak{g}}(\mathfrak{h}), C_{\mathfrak{g}}(\mathfrak{h})]$, contradicting Step V, thus its nilpotent part $n \neq 0$, but its semisimple part $s \in \mathfrak{h} \leq Z(C_{\mathfrak{g}}(\mathfrak{h}))$, so is $n = z - s$, but then $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 9.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 9.6.17(d)

Step VII: $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$

Suppose $x \in C_{\mathfrak{g}}(\mathfrak{h}) \backslash \mathfrak{h}$, then it gives a nonzero nilpotent part n, but then since $C_{\mathfrak{g}}(\mathfrak{h})$ is abelian by Step VI, thus $[n, C_{\mathfrak{g}}(\mathfrak{h})] = 0$, by Lemma 9.6.19, $K(n, C_{\mathfrak{g}}(\mathfrak{h})) = 0$, contradicting Proposition 9.6.17(d)

□

Remark 9.6.21. $K(,)|_{\mathfrak{h}}$ is nondegenerate is not the same as saying that the Killing form of $\mathfrak{h}$ is nondegenerate which obviously violates Proposition 9.6.6, it doesn't contradict Proposition 9.4.4 since $\mathfrak{h} \leq \mathfrak{g}$ is merely a Lie subalgebra but not an ideal, by the nondegeneracy, we can identity $\mathfrak{h}^*$ with $\mathfrak{h}$ by $\mathfrak{h}^* \to \mathfrak{h}, \alpha \mapsto t_\alpha$, where $K(t_\alpha, x) = \alpha(x)$, and here t behaves like the linear isomorphism $t : \mathfrak{h}^* \to \mathfrak{h}, \alpha \mapsto t_\alpha$

Some properties about root space decomposition

Proposition 9.6.22.

(a) Δ spans $\mathfrak{h}^*$

(b) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$

(c) $\alpha \in \Delta$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = K(x, y)t_\alpha$

(d) $\alpha \in \Delta$, then $0 \neq [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \langle t_\alpha \rangle$

(e) If $\alpha \in \Delta$, then $\alpha(t_\alpha) = K(t_\alpha, t_\alpha) \neq 0$

(f) If $\alpha \in \Delta$, $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, then there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $K(x_\alpha, y_\alpha) = \dfrac{2}{K(t_\alpha, t_\alpha)}$, define $h_\alpha := [x_\alpha, y_\alpha] = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$, then $\mathfrak{s}_\alpha := \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$ via $x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_\alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(g) Given $\alpha \in \Delta$, $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$ will be a representation of $\mathfrak{s}_\alpha$, thus $\mathfrak{g}$ can be decomposed into irreducible representations of $\mathfrak{s}_\alpha$, and the highest eigenvectors for this representation are also common highest eigenvectors of $ad(\mathfrak{h})$

Proof.

(a) Suppose $\langle \Delta \rangle \subsetneq \mathfrak{h}^*$, then there exists $0 \neq h \in \mathfrak{h}$, such that $\forall \alpha \in \Delta, \alpha(h) = 0$, then $\forall x \in \mathfrak{g}_\alpha, [h, x] = \alpha(h)x = 0$, and since $\mathfrak{h}$ is abelian, $[h, \mathfrak{h}] = 0$, thus $[h, \mathfrak{g}] = 0$, but then $h \in Z(\mathfrak{g}) = 0$ which is a contradiction

(b) If $\alpha \in \Delta$, and $\mathfrak{g}_{-\alpha} = 0$, then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 9.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction

(c) $\forall h \in \mathfrak{h}, K(h, [x, y]) = K([h, x], y) = K(\alpha(h)x, y) = K(t_\alpha, h)K(x, y) = K(h, K(x, y)t_\alpha)$, since $K(,)|_{\mathfrak{h}}$ is nondegenerate, $[x, y] = K(x, y)t_\alpha$

(d) Only need to show that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0$. There exists $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) \neq 0$, otherwise then $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0, \forall \beta$ by Proposition 9.6.17(c), then $\mathfrak{g}_\alpha = 0$ which is a contradiction, thus $[x, y] = K(x, y)t_\alpha \neq 0$ by (c)

(e) Suppose instead $\alpha(t_\alpha) = 0$, then $[t_\alpha, x] = [t_\alpha, y] = 0, \forall x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$, by nondegeneracy, we can find $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ such that $K(x, y) = 1$, then by (c), $[x, y] = K(x, y)t_\alpha = t_\alpha$, thus $\mathfrak{s} = \langle x, y, t_\alpha \rangle \cong ad(\mathfrak{s}) \leq ad(\mathfrak{g}) \leq \mathfrak{gl}(\mathfrak{g})$ is a 3 dimensional solvable Lie algebra, by Theorem 9.3.4, for any $s \in [\mathfrak{s}, \mathfrak{s}]$, ad(s) is nilpotent, thus $ad(t_\alpha)$ is both semisimple and nilpotent, hence $ad(t_\alpha) = 0 \Rightarrow t_\alpha = 0$ which is a contradiction

(f) $[h_\alpha, x_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, x_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)x_\alpha = 2x_\alpha$ $[h_\alpha, y_\alpha] = \dfrac{2}{K(t_\alpha, t_\alpha)}[t_\alpha, y_\alpha] = -\dfrac{2}{K(t_\alpha, t_\alpha)}\alpha(t_\alpha)y_\alpha = -2y_\alpha$

(g) Suppose $x \in \mathfrak{g}$ is a highest eigenvector of representation $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $0 = ad(x_\alpha)(x) = [x_\alpha, x]$, $\forall h \in \mathfrak{h}, [x_\alpha, ad(h)(x)] = [x_\alpha, [h, x]] = [[x_\alpha, h], x] + [h, [x_\alpha, x]] = [-\alpha(h)x_\alpha, x] = 0$

□

alpha check is canonical

Remark 9.6.23. In (f), the choice of x_α is not canonical, however, $h_\alpha = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)}$ is canonical, for example, if we pick $0 \neq x_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $\mathfrak{s}_{-\alpha} = \langle x_{-\alpha}, y_{-\alpha}, h_{-\alpha}\rangle$, then $h_{-\alpha} = h_\alpha = \dfrac{2t_{-\alpha}}{K(t_{-\alpha}, t_{-\alpha})} = -h_\alpha$, moreover, according to Lemma 9.4.6, any nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$ is of the form $(,) := cK(,)|_{\mathfrak{h}}$ for some $c \neq 0$, then t'_α the dual of $\alpha \in \mathfrak{h}^*$ given by $(t'_\alpha, x) = \alpha(x), \forall x \in \mathfrak{h}$, then we have $K(t_\alpha, x) = \alpha(x) = (t'_\alpha, x) = cK(t'_\alpha, x) = K(ct'_\alpha, x)$, because of the nondegeneracy of $K(,)|_{\mathfrak{h}}$, $t_\alpha = ct'_\alpha \Rightarrow t'_\alpha = \dfrac{t_\alpha}{c}$, and $\dfrac{2t'_\alpha}{(t'_\alpha, t'_\alpha)} = \dfrac{2\frac{t_\alpha}{c}}{cK\left(\frac{t_\alpha}{c}, \frac{t_\alpha}{c}\right)} = \dfrac{2t_\alpha}{K(t_\alpha, t_\alpha)} = h_\alpha$, thus h_α is even canonical defined regardless of the choice of the nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$, for this reason, we give h_α a new notation $\alpha^\vee$ for later use, whereas $\alpha(\alpha^\vee) = 2$, for any nondegenerate, symmetric, bilinear and invariant form on $\mathfrak{h}$, note that $\alpha \mapsto \alpha^\vee$ is not linear

Also, according to Lemma 9.4.6, even though $(,)$ is defined up to a scalar, but the orthogonality is always well defined

Due to Proposition 9.6.22(g), if $0 \neq x \in \mathfrak{g}$ is a highest eigenvector for representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{g})$, then $x \in \mathfrak{g}_\beta$, for some $\beta \in \Delta \cup \{0\}$, when $\beta = 0$, these corresponds to the trivial representations, if $\beta \in \Delta$, we can denote one such highest eigenvector $0 \neq x_\beta \in \mathfrak{g}_\beta$, then $ad(h_\alpha)(x_\beta) = [h_\alpha, x_\beta] = \beta(h_\alpha)x_\beta$, thus x_β is a $\beta(h_\alpha) = \dfrac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}$-eigenvector, by Proposition 9.6.17(a), $ad(y_\alpha)^j(x_\beta) \in \mathfrak{g}_{\beta - j\alpha}$ are all the nonzero eigenvectors corresponds to eigenvalues $(\beta - j\alpha)(h_\alpha) = 2\left(\dfrac{K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)} - j\right)$, and these roots $\beta - j\alpha, j = 0, \cdots, k = \beta(h_\alpha)$ form an α-string

One of these irreducible representation is $\mathfrak{s}_\alpha$ itself according to Example 9.6.12

Example 9.6.24. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ which is a semisimple Lie algebra, then

$$\mathfrak{h} = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \middle| \sum z_i = 0 \right\}$$

is a maximal toral Lie subalgebra, denote $\mathrm{diag}(z_1, \cdots, z_n)$ as h_z, then $[h_z, E_{ij}] = (z_i - z_j)E_{ij}$, and $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{g} = \mathfrak{h} \bigoplus_{i \neq j} \langle E_{ij}\rangle$, $\Delta = \{e_i - e_j | i \neq j\}$, where $e_i \in \mathfrak{h}^*$ is defined by $e_i(h_z) = z_i$, also

$$\mathfrak{s}_\alpha = \left\{ \begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & a & & b & & \\ & & & \ddots & & & \\ & & b & & -a & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \right\} \cong \mathfrak{sl}(2, \mathbb{C})$$

where $\alpha = e_i - e_j$

Definition of s_alpha

Definition 9.6.25. Define $s_\alpha(\beta) := \beta - \dfrac{2K(t_\alpha, t_\beta)}{K(t_\alpha, t_\alpha)}\alpha, \forall \alpha \in \Delta, \beta \in \mathfrak{h}^*$ is an orthogonal reflection over the hyperplane $H_\alpha = \{\alpha(x) = K(t_\alpha, x) = 0 | x \in \mathfrak{h}\} = \ker \alpha$, more precisely, $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\beta) = \beta, \forall \beta \in H_\alpha$, note here any nondegenerate, symmetric, bilinear and invariant form $(,)$ can be used as the definition in place of the Killing form $K(,)$ thanks to Lemma 9.4.6

alpha string through beta

Proposition 9.6.26.

(a) If $\alpha \in \Delta$, then $\dim \mathfrak{g}_\alpha = 1$

(b) If $\alpha \in \Delta$, then $c\alpha \in \Delta \Leftrightarrow c = \pm 1$
(c) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
(d) If $\alpha, \beta \in \Delta$, the Cartan integer $\beta(h_\alpha) = \dfrac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Delta$, if $\beta \neq \alpha, -\alpha$, then $\Delta \bigcap \{\beta + j\alpha | j \in \mathbb{Z}\} = \{\beta + j\alpha | -r \leq j \leq s, j \in \mathbb{Z}\}$ which is an α string through β, and $\beta(h_\alpha) = r - s$

Proof. **(a)** Let $\mathfrak{m} = \mathfrak{h} \bigoplus_{c\alpha \in \Delta, c \in \mathbb{F}^\times} \mathfrak{g}_{c\alpha}$, then $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is a finite representation of $\mathfrak{s}_\alpha$ because of Proppsition 9.6.17(a), first notice for $x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, thus the 0-eigenspace of $ad(h_\alpha)$ is $\mathfrak{h}$, but note that for any $h \in \ker\alpha = H_\alpha \leq \mathfrak{h}$ as in Definition 9.6.25, $ad(x_\alpha)(h) = [x_\alpha, h] = \alpha(h)x_\alpha = 0$, $ad(y_\alpha)(h) = [y_\alpha, h] = -\alpha(h)y_\alpha = 0$, $ad(h_\alpha)(h) = [h_\alpha, h] = 0$, thus $\mathfrak{s}_\alpha$ acts trivially on $\ker\alpha$ which is of codimension 1 in $\mathfrak{h}$ which gives $\dim\mathfrak{h} - 1$ copies of trivial representation of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, since $h_\alpha \notin \ker\alpha$, $\mathfrak{h} = \langle h_\alpha \rangle \oplus \ker\alpha$, by Example 9.6.12 and Lemma 9.6.15, $\mathfrak{s}_\alpha = \langle x_\alpha, y_\alpha, h_\alpha \rangle$ is a 3 dimensional irreducible representation of $\mathfrak{s}_\alpha$, and $\mathfrak{s}_\alpha \oplus \ker\alpha$ are the only possible even irreducible representations of $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$, thus $\dim\mathfrak{g}_\alpha = 1$, otherwise, we can choose $0 \neq x'_\alpha \in \mathfrak{g}_\alpha$ linearly independent of x_α, then we have $[h_\alpha, x'_\alpha] = \alpha(h_\alpha)x'_\alpha = 2x'_\alpha$, which gives a contradiction. Moreover, $2\alpha \notin \Delta$, otherwise, we can choose $0 \neq x_{2\alpha} \in \mathfrak{g}_{2\alpha}$, then $[h_\alpha, x_{2\alpha}] = 2\alpha(h_\alpha)x_{2\alpha} = 4x_{2\alpha}$which also gives a contradiction
(b) Suppose $c\alpha \in \Delta$, for $0 \neq x \in \mathfrak{g}_{c\alpha}$, we have $ad(h_\alpha)(x) = [h_\alpha, x] = c\alpha(h_\alpha)x = 2cx$, by Theorem 9.6.11, we know that $2c \in \mathbb{Z}$, but by symmetry, if we let $\beta = c\alpha$, then $\alpha = \dfrac{\beta}{c} \in \Delta$ implies $\dfrac{2}{c} \in \mathbb{Z}$, thus c can only possibly be $\pm 1, \pm 2, \pm\dfrac{1}{2}$, but from (a), we know $c \neq 2$, thus $c \neq -2$ thanks to Proposition 9.6.22(b), and by symmetry, $c \neq \pm\dfrac{1}{2}$, therefore $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker\alpha \oplus \langle h_\alpha \rangle \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \ker\alpha \oplus \mathfrak{s}_\alpha$
(c) Obiviously β can't be α or α, otherwise $\alpha + \beta \notin \Delta$, also $\beta + j\alpha \neq 0, \forall j \in \mathbb{F}$ by (b), for $\beta \in \Delta \setminus \{\alpha, -\alpha\}$, we can consider $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ which is a finite representation of $\mathfrak{s}_\alpha$ with $\mathfrak{m} = \bigoplus_{\beta + j\alpha \in \Delta, j \in \mathbb{Z}} \mathfrak{g}_{\beta+j\alpha}$, suppose $\beta + j\alpha \in \Delta$, $\dim\mathfrak{g}_{\beta+j\alpha} = 1$ by (a), choose $0 \neq x_{\beta+j\alpha} \in \mathfrak{g}_{\beta+j\alpha}$, we have $[h_\alpha, x_{\beta+j\alpha}] = (\beta + j\alpha)(h_\alpha)x_{\beta+j\alpha} = (\beta(h_\alpha) + 2j)\, x_{\beta+j\alpha}$, as j varies in $\mathbb{Z}$, $\beta(h_\alpha) + 2j$ can't take both 0 and 1, thus the sum of dimension of 0-eigenspace and 1-eigenspace of $ad(h_\alpha)$ on $\mathfrak{m}$ is 1, by Lemma 9.6.15, $(ad|_{\mathfrak{s}_\alpha}, \mathfrak{m})$ is irreducible, according to Theorem 9.6.11 and Remark 9.6.23, $\mathfrak{m} = \bigoplus_{-r \leq j \leq s} \mathfrak{g}_{\beta+j\alpha}$, for some $r, s \in \mathbb{Z}_{\geq 0}$, and $\beta + j\alpha \in \Delta, \forall -r \leq j \leq s$ which is the α string through β, note that $ad(x_\alpha)(x_\beta) \neq 0$ as in Theorem 9.6.11, thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$
(d) When $\beta = \alpha, -\alpha$, it is trivial. When $\beta \neq \alpha, -\alpha$, as in (c), we know that when j varies from $-r$ to s, $(\beta(h_\alpha) + 2j)$ are all the possible eigenvalues which are integers symmetric over 0, thus $\beta(h_\alpha) + 2s + \beta(h_\alpha) - 2r = 0 \Rightarrow \beta(h_\alpha) = r - s$ is an integer, then $-r \leq -\beta(h_\alpha) \leq s \Rightarrow \beta - \beta(h_\alpha)\alpha \in \Delta$ □

Connection between roots and root system

Proposition 9.6.27.
(1) Let $V = \mathbb{Q}\Delta$, then $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$
(2) For any $h, k \in \mathfrak{h}$, $K(h,k) = Tr(ad(h)ad(k)) = \sum_{\alpha \in \Delta} \alpha(h)\alpha(k)$
(3) The dual of Killing form restricted on V is rational and positive definite

Proof. Since $K(,)|_{\mathfrak{h}}$ is nondegenerate, we define the dual of $K(,)|_{\mathfrak{h}}$ on $\mathfrak{h}^*$ as $(\alpha, \beta) := K(t_\alpha, t_\beta)$, which is also nondegenerate
(1) Since $\mathfrak{h}^* = \mathbb{F}\Delta$ by Proposition 9.6.22(a), Pick any basis in Δ, say $\alpha_1, \cdots, \alpha_m$, then $\forall \beta \in \Delta$, $\beta = \sum c_j\alpha_j, c_j \in \mathbb{F}$, we have $(\beta, \alpha_k) = \sum c_j(\alpha_j, \alpha_k) \Rightarrow \dfrac{2(\beta,\alpha_k)}{(\alpha_k,\alpha_k)} = \sum c_j \dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}$ whereas $\dfrac{2(\beta,\alpha_k)}{(\alpha_k,\alpha_k)}, \dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}$ are all integers by Proposition 9.6.26(d), since $(\beta - \sum c_j\alpha_j, \alpha_k) = 0$ and that $(,)$ is nondegenerate, meaning c_j's are the unique solution, hence matrix $((\alpha_j, \alpha_k))$ is nonsingular,

so is matrix $\left(\dfrac{2(\alpha_j,\alpha_k)}{(\alpha_k,\alpha_k)}\right)$, thus $c_j \in \mathbb{Q}$, which means $\dim_{\mathbb{Q}} V = \dim_{\mathbb{F}} \mathfrak{h}^*$ and $\mathfrak{h}^* = V \otimes_{\mathbb{Q}} \mathbb{F}$
(2) For $0 \neq x_\alpha \in \mathfrak{g}_\alpha$, $ad(h)ad(k)(x_\alpha) = [h,[k,x_\alpha]] = \alpha(k)[h,x_\alpha] = \alpha(h)\alpha(k)x_\alpha$, according to Theorem 9.6.18, we have

$$K(h,k) = Tr(ad(h)ad(k)) = 0 + \sum_{\alpha\in\Delta} Tr\left(ad(h)|_{\mathfrak{g}_\alpha} ad(k)|_{\mathfrak{g}_\alpha}\right) = \sum_{\alpha\in\Delta} \alpha(h)\alpha(k)$$

Due to Proposition 9.6.22(b), roots always appears in pairs, if let $\Delta^+ \subseteq \Delta$ consists of exactly one from each pair, then $\sum\limits_{\alpha\in\Delta} \alpha(h)\alpha(k) = 2\sum\limits_{\alpha\in\Delta^+} \alpha(h)\alpha(k)$ **(3)** By (2), for any $\lambda,\mu \in \mathfrak{h}^*$

$$\begin{aligned}(\lambda,\mu) &= K(t_\lambda,t_\mu)\\ &= \sum_{\alpha\in\Delta} \alpha(t_\lambda)\alpha(t_\beta)\\ &= \sum_{\alpha\in\Delta} K(t_\alpha,t_\lambda)K(t_\alpha,t_\mu)\\ &= \sum_{\alpha\in\Delta} (\alpha,\lambda)(\alpha,\mu)\\ &= 2\sum_{\alpha\in\Delta^+} (\alpha,\lambda)(\alpha,\mu)\end{aligned}$$

In particular, for any $\beta \in \Delta$, $(\beta,\beta) = 2\sum\limits_{\alpha\in\Delta^+} (\alpha,\beta)^2 \Rightarrow \dfrac{2}{(\beta,\beta)} = \sum\limits_{\alpha\in\Delta^+} \left(\dfrac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2$, where $\dfrac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, thus $0 \leq (\beta,\beta) \in \mathbb{Q}$ and equals 0 iff $(\alpha,\beta) = 0, \forall\alpha \in \Delta \Rightarrow \beta = 0$ by the nondegeneracy of $(,)$, thus $(,)|_V$ is rational and positive definite on a basis, which implies it is rational and positive definite ☐

Remark 9.6.28. When $\mathbb{F} = \mathbb{C}$, note that $\mathbb{Q}\Delta < \mathbb{R}\Delta < \mathbb{C}\Delta$, we can view V embedded in the Euclidean space $V_{\mathbb{R}} := \mathbb{R}\Delta = V \otimes_{\mathbb{Q}} \mathbb{R}$ which helps thinking, then we have a root system

Example 9.6.29. Let $\Omega = \begin{pmatrix} 0 & I\\ I & 0\end{pmatrix}$, then $\left\{X \in SL(2n,\mathbb{C}) \middle| X^T\Omega X = \Omega\right\}$ is conjugate to $SO(2n,\mathbb{C})$, thus then also induce isomorphic Lie algebra, hence we can identify $\mathfrak{so}(2n,\mathbb{C})$ with $\left\{X \in M(2n,\mathbb{C}) \middle| \Omega X^T + X\Omega = 0\right\}$ which is the same as $\left\{\begin{pmatrix} A & B\\ C & -A^T\end{pmatrix} \in M(2n,\mathbb{C}) \middle| B^T = -B, C^T = -C\right\} =: \mathfrak{g}$, then one Cartan subalgebra of $\mathfrak{g}$ will be $\mathfrak{h} = \left\{\begin{pmatrix} D & 0\\ 0 & -D\end{pmatrix} \in M(2n,\mathbb{C}) \middle| D = \mathrm{diag}(d_1,\cdots,d_n)\right\}$, note that

$$\left[\begin{pmatrix} D & 0\\ 0 & -D\end{pmatrix}, \begin{pmatrix} E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}\right] = (d_i - d_j)\begin{pmatrix} E_{ij} & 0\\ 0 & -E_{ji}\end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0\\ 0 & -D\end{pmatrix}, \begin{pmatrix} 0 & E_{ij}\\ 0 & 0\end{pmatrix}\right] = (d_i + d_j)\begin{pmatrix} 0 & E_{ij}\\ 0 & 0\end{pmatrix}$$

$$\left[\begin{pmatrix} D & 0\\ 0 & -D\end{pmatrix}, \begin{pmatrix} 0 & 0\\ E_{ij} & 0\end{pmatrix}\right] = -(d_i + d_j)\begin{pmatrix} 0 & 0\\ E_{ij} & 0\end{pmatrix}$$

9.7 Root system

Delta is finite spaning set of V, then there is at most one s:V->V, maps Delta to Delta, s^2=1 and alpha to -alpha

Lemma 9.7.1. V is a finite dimensional vector space over a field of characteristic 0, $\Delta \subseteq V$ is a finite set such that $V = \operatorname{Span}\Delta$, then for any $\alpha \in \Delta$, there is at most one linear map $s : V \to V$ such that $s^2 = 1$, $s\alpha = -\alpha$, $s(\Delta) \subseteq \Delta$

Proof. Suppose s, t both satisfy the condition, $sv = v + (s-1)v$, $s(s-1)v = s^2v - sv = v - sv = -(s-1)v$, thus $(s-1)v \in \langle\alpha\rangle \Rightarrow sv = v + f(v)\alpha$, where $f \in V^*$, similarly, $tv = v + g(v)\alpha$, where $g \in V^*$, thus $stv = s(v + g(v)\alpha) = v + g(v)\alpha + f(v + g(v)\alpha)\alpha = v + g(v)\alpha + f(v)\alpha + f(v)g(v)\alpha$, and since $s\alpha = \alpha + f(\alpha)\alpha = -\alpha \Rightarrow f(\alpha) = -2$, thus check $(st)^n v = v + n(f(v) - g(v))\alpha$, but $(st)^n = 1$ for some n because st is just a permutation of Δ, thus $f = g \Rightarrow s = t$ □

Remark 9.7.2. $s^2 = 1$ and $s(\Delta) \subseteq \Delta$ implies that s is a permutation of Δ, note that this definition doesn't involve inner product on V, you could see this as a more abstract definition of a reflection

Definition of root system

Definition 9.7.3. $V = \mathbb{R}^n$ is the standard Euclidean space with the standard inner product $(,)$, a **root system** Δ is a finite subset of V satisfying

1. $V = \langle\Delta\rangle$

2. If $\alpha \in \Delta$, then the only multiples of α are $\pm\alpha$

3. $s_\alpha(\Delta) \subseteq \Delta$, where

$$s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

is the reflection across the hyperplane $H_\alpha = \{\beta \in V | (\beta,\alpha) = 0\}$

4. $\langle\beta,\alpha\rangle = \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ are called the **Cartan integers**

Remark 9.7.4. $\langle\beta,\alpha\rangle \in \mathbb{Z}$ is the **integrality** condition, and such roots system is called **crystallographic**

$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\dfrac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta \in \mathbb{Z}$ can only be $0, 1, 2, 3, 4$, where θ is the angle between α and β, and it is 4 iff $\alpha = \pm\beta$

$\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$	0	1	2	3	4
θ	$\pm\frac{\pi}{2}$	$\pm\frac{\pi}{3}$	$\pm\frac{\pi}{4}$	$\pm\frac{\pi}{6}$	0

$a_{ji} = \dfrac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ gives a Cartan matrix A with $D_{ij} = \dfrac{\delta_{ij}}{(\alpha_i,\alpha_i)}$, $S_{ij} = 2(\alpha_i,\alpha_j)$
Conversely, given a generalized Cartan matrix, we can find the corresponding Lie algebra, see Kac-Moody algebra

Remark 9.7.5. Thanks to Lemma 9.7.1, s_α is the unique linear map $V \to V$ such that $s_\alpha(\alpha) = -\alpha$, $s_\alpha(\Delta) \subseteq \Delta$

Definition 9.7.6. Let (V,Δ) be a root system, define the coroot of $\alpha \in \Delta$ to be $\alpha^\vee = \dfrac{2}{(\alpha,\alpha)}\alpha$, and let $\Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$, then $(V,\Delta^\vee)$ is also a root system

alpha not equal to pm beta are roots, (alpha,beta)>0 => alpha-beta is a root

Lemma 9.7.7. $\alpha \neq \pm\beta \in \Delta$. If $\langle\alpha,\beta\rangle > 0$, then $\alpha - \beta \in \Delta$, if $\langle\alpha,\beta\rangle < 0$, then $\alpha + \beta \in \Delta$

Proof. Note that $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha \in \Delta$, $s_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta \in \Delta$, and $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\dfrac{(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = 4\cos^2\theta \in \mathbb{Z}$ can only be $0, 1, 2, 3$, where θ is the angle between α and β, if $\langle\alpha,\beta\rangle > 0 \Leftrightarrow (\alpha,\beta) > 0$, then $\langle\alpha,\beta\rangle = 1, \langle\beta,\alpha\rangle = 1, 2, 3$ or $\langle\alpha,\beta\rangle = 2, 3, \langle\beta,\alpha\rangle = 1$, hence either $\alpha - \beta$ or $\beta - \alpha$ will be in Δ, but then the other will also be in Δ. Similarly, if $\langle\alpha,\beta\rangle < 0 \Leftrightarrow (\alpha,\beta) < 0$, then $\langle\alpha,\beta\rangle = -1, \langle\beta,\alpha\rangle = -1, -2, -3$ or $\langle\alpha,\beta\rangle = -2, -3, \langle\beta,\alpha\rangle = -1$, hence $\alpha + \beta \in \Delta$ □

Definition 9.7.8. $\Delta^+ \subset \Delta$ is a set of **positive roots** if $\alpha \in \Delta \Rightarrow$ precisely one of $\alpha, -\alpha$ is in Δ^+, and $\alpha, \beta \in \Delta^+ \Rightarrow \alpha + \beta \in \Delta^+$, thus we can correspondingly define negative roots $\Delta^- = \Delta \setminus \Delta^+ = -\Delta^+$. Any hyperplane H that doesn't intersect Δ separates Δ into Δ^+ and Δ^-. $\gamma \in V$ is called **regular** if $(\gamma, \alpha) \neq 0, \forall \alpha \in \Delta$, then $H_\gamma = \{v \in V | (\gamma, v) = 0\}$ separates Δ into $\Delta^+(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) > 0\}$ and $\Delta^-(\gamma) = \{\alpha \in \Delta | (\gamma, \alpha) < 0\}$. Conversely, given a hyperplane H that doesn't intersect Δ and separates Δ into Δ^+ and Δ^-, we can find $\gamma \in V$ such that $H = H_\gamma$, $\Delta^+ = \Delta^+(\gamma)$, $\Delta^- = \Delta^-(\gamma)$

Definition 9.7.9. $\gamma \in V$ is **dominant** if $(\gamma, \alpha) \geq 0, \forall \alpha \in \Delta^+$ and **strictly dominant** if $(\gamma, \alpha) > 0, \forall \alpha \in \Delta^+$

Definition 9.7.10. $\alpha \in \Delta^+$ is **decomposable** if $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Delta^+$, α is a **simple root** of Δ^+ if it is not decomposable. $S = \{\alpha_1, \cdots, \alpha_m\}$ is a **base** for Δ^+ if for any $\alpha \in \Delta^+$, there are $c_i \in \mathbb{Z}_{\geq 0}$ such that $\alpha = \sum_i c_i \alpha_i$, which also implies that for any $\alpha \in \Delta^- = -\Delta^+$, there are $c_i \in \mathbb{Z}_{\leq 0}$ such that $\alpha = \sum_i c_i \alpha_i$

v_1,...,v_m on one side of a hyperplane, (v_i,v_j)<0 => v_i linearly independent

Lemma 9.7.11. $S = \{v_1, \cdots, v_m\}$ are on one side of a hyperplane H, and $(v_i, v_j) < 0, \forall i \neq j$, then S is linearly independent

Proof. Suppose $\sum_{i=1}^m a_i v_i = 0$ and not all a_i's are zero, then we can rewrite as $\sum_{k \in K} a_k v_k = \sum_{l \in L} -a_l v_l$, where $a_k > 0, \forall k \in K, a_l < 0, \forall l \in L$, then we have $0 \leq \left(\sum_{k \in K} a_k v_k, \sum_{l \in L} -a_l v_l \right) = \sum_{k \in K, l \in L} -a_k a_l (v_k, v_l) < 0$ which is a contradiction $\square$

Lemma 9.7.12. S is the set of simple roots of Δ^+, then S is a base for Δ^+, and S is linearly independent

Proof. It is obvious that S is a base of Δ^+ by definition. Suppose $\alpha \neq \beta \in S$, $\alpha \neq -\beta$ is obvious, hence $(\alpha, \beta) \neq 0$. If $(\alpha, \beta) > 0$, then by Lemma 9.7.7, we have $\alpha - \beta \in \Delta$, if $\alpha - \beta \in \Delta^+$, then $\alpha = \beta + (\alpha - \beta)$ gives a contradiction, if $\alpha - \beta \in \Delta^-$, then $\beta - \alpha \in \Delta^+$ and $\beta = \alpha + (\beta - \alpha)$ gives a contradiction, therefore $(\alpha, \beta) < 0$. By lemma 9.7.11, we know S is linearly independent $\square$

Remark 9.7.13. Given a set of positive roots, there is precisely one base which is the set of simple roots

Conjugacy of roots and Weyl group

Lemma 9.7.14. $\sigma \in \mathrm{GL}(V)$, $\sigma(\Delta) \subseteq \Delta$, then for any $\alpha \in \Delta$, $\sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$, moreover, $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$

Proof. $\sigma^m = 1$ for some m since there are only finitely many choices of maps $\Delta \to \Delta$, thus σ is a permutation on Δ, hence $\sigma s_\alpha \sigma^{-1}(\Delta) \subseteq \Delta$, $\left(\sigma s_\alpha \sigma^{-1}\right)^2 = 1$ and $\sigma s_\alpha \sigma^{-1} \sigma\alpha = -\sigma\alpha$ implies $\sigma s_\alpha \sigma^{-1} = s_{\sigma\alpha}$ by Lemma 9.7.1. Compare $s_{\sigma\alpha}(\sigma\beta) = \sigma\beta - \langle \sigma\beta, \sigma\alpha \rangle \sigma\alpha$, and $\sigma s_\alpha \sigma^{-1}(\sigma\beta) = \sigma s_\alpha \beta = \sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma\beta - \langle \beta, \alpha \rangle \sigma\alpha$, we get $\langle \beta, \alpha \rangle = \langle \sigma\beta, \sigma\alpha \rangle$ $\square$

Definition 9.7.15. The **Weyl group** W of Δ is $\langle s_\alpha | \alpha \in \Delta \rangle$. $|W| < \infty$ since $s_\alpha(\Delta) \subseteq \Delta$. W is a subgroup of $O(V)$

Remark 9.7.16. W is a finite Coxeter group

Definition 9.7.17. **Weyl chambers** are connected components of $V \setminus \bigcup_{\alpha \in \Delta} H_\alpha$, as the intersection of open half spaces, Weyl chambers are open convex, each regular $\gamma \in V$ by definition belongs to precisely one Weyl chamber denote as $C(\gamma)$, $C(\gamma) = C(\gamma')$ iff γ, γ' are on the side for every hyperplane H_α iff $\Delta^+(\gamma) = \Delta^+(\gamma')$, thus Weyl chambers are in one to one correspondence with the sets of positive roots, the fundamental Weyl chamber associated to Δ^+, or rather, associated to S is $\{\gamma \in V | (\gamma, \alpha) > 0, \forall \alpha \in S\}$. For any $\sigma \in W$, since $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$, $\sigma(\Delta^+(\gamma)) = \Delta^+(\sigma\gamma)$, $\sigma(S(\gamma)) = S(\sigma\gamma)$, $\sigma(C(\gamma)) = C(\sigma\gamma)$

For any positive nonsimple root alpha, there exists simple root beta such that alpha-beta is a positive root

Lemma 9.7.18. If $\alpha \in \Delta^+ \setminus S$,then there exists $\beta \in S$ such that $\alpha - \beta \in \Delta^+$

Proof. It is obvious that $(\alpha, \beta) \neq 0, \forall \beta \in \Delta^+$, suppose $(\alpha, \beta) < 0, \forall \beta \in S$, then by Lemma 9.7.11, $S \cup \{\alpha\}$ is linearly independent which is impossible, thus $(\alpha, \beta) > 0$ for some $\beta \in \Delta^+$, by Lemma 9.7.7, $\alpha - \beta \in \Delta$, but since $\alpha = \sum\limits_{\alpha_i \in S} c_i \alpha_i$, and $\beta = \alpha_j$ for some j, since some $c_i > 0$, it necessarily has to be that $\alpha - \beta = (c_j - 1)\beta + \sum\limits_{i \neq j} c_i \alpha_i \in \Delta^+$ □

Lemma 9.7.19. Each $\alpha \in \Delta^+$ can be written as $\alpha_1 + \cdots + \alpha_k$, $\alpha_i \in S$, here α_i may repeat, such that partial sums are all positive roots, i.e. $\alpha_i + \cdots + \alpha_i \in \Delta^+$

Proof. Each $\alpha \in \Delta^+$ can be written uniquely as a sum of simple roots, by induction on the number of summands and Lemma 9.7.18 □

If alpha is a simple root, then s_alpha permutes postive roots except alpha

Lemma 9.7.20. If $\alpha \in S$, then s_α permutes $\Delta^+ \setminus \{\alpha\}$

Proof. For any $\beta \in \Delta^+ \setminus \{\alpha\}$, $\beta = \sum\limits_{\alpha_i \in S} c_i \alpha_i$, $c_j > 0$ for some $\alpha_j \neq \alpha$, then $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ still has some positive coefficient, thus it has to be positive, and $s_\alpha(\beta) \neq \alpha = s_\alpha(-\alpha)$ □

delta=half of sum of positive roots, then for any simple root alpha, s_alpha(delta)=delta-alpha

Corollary 9.7.21. Let $\delta = \frac{1}{2} \sum\limits_{\beta \in \Delta^+} \beta$, then $s_\alpha(\delta) = \delta - \alpha, \forall \alpha \in S$

s=s1...sq is of minimal length => s1...sq(alpha_q) is a negative root

Lemma 9.7.22. Use $\alpha \prec 0$ and $\alpha \succ 0$ to mean positive and negative roots, suppose $\alpha_i \in S, i = 1, \cdots, q$, and denote $s_i := s_{\alpha_i}$, if $s_1 \cdots s_{q-1}(\alpha_q) \prec 0$, then $s_1 \cdots s_q = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1}$. In particular, suppose $s = s_1 \cdots s_q$ is of the smallest length, then $0 \prec s_1 \cdots s_{q-1}\alpha_q = -s_1 \cdots s_q \alpha_q \Rightarrow s_1 \cdots s_q \alpha_q \prec 0$

Proof. Write $\beta_i = s_i \cdots s_{q-1}\alpha_q$, then $\beta_q = \alpha_q \succ 0$, $\beta_1 \prec 0$, thus there must be some $1 \leq p < q$ such that $\beta_{p+1} \succ 0$ but $\beta_p = s_p \beta_{p+1} \prec 0$, by lemma 9.7.20, thus β_{p+1} can only be α_p, hence by Lemma 9.7.14, we have

$$
\begin{aligned}
& s_p = s_{\alpha_p} = s_{\beta_{p+1}} = s_{s_{p+1} \cdots s_{q-1}\alpha_q} = (s_{p+1} \cdots s_{q-1}) s_q (s_{p+1} \cdots s_{q-1})^{-1} \\
& \Rightarrow s_p \cdots s_{q-1} = s_{p+1} \cdots s_q \\
& \Rightarrow s_1 \cdots s_q = s_1 \cdots s_p s_{p+1} \cdots s_q = s_{p1} \cdots s_p s_p \cdots s_{q-1} = s_1 \cdots s_{p-1} s_{p+1} \cdots s_{q-1}
\end{aligned}
$$

□

Lemma 9.7.23. Denote $n(\sigma)$ the number of positive roots that σ send to negative. $l(\sigma) = n(\sigma)$. Hence there is a unique element $w_\circ$ such that $w_\circ(S) = -S$ of maximal length, and $w_\circ^2 = 1$

Some properties about Weyl group and Weyl chambers

Theorem 9.7.24.

(a) Let $\gamma \in V$ be regular, then there exists some $\sigma \in W$ such that $\Delta^+(\sigma\gamma) = \Delta^+$, namely, W acts trasitively on Weyl chambers

(b) If S' is a another base, then there exists some $\sigma \in W$ such that $\sigma(S') = S$, namely, W acts trasitively on bases

(c) If α is any root, then there exists some $\sigma \in W$ such that $\sigma(\alpha) \in S$

(d) W is generated by s_α's for $\alpha \in S$

(e) If $\sigma \in W$, then $\sigma(S) \subseteq S \Rightarrow \sigma = 1$, namely, W acts freely and transitively(regularly) on bases(and Weyl chambers)

Proof. Let $W' \leq W$ be the subgroup of $O(n)$ generated by s_α's for $\alpha \in S$

(a) Let $\delta = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta$, choose $\sigma \in W'$ such that $(\sigma\gamma, \delta)$ is the biggest possible, then for any $s_\alpha \in W'$, due to Corollary 9.7.21, we have $(\sigma\gamma, \delta) \geq (s_\alpha \sigma\gamma, \delta) = (\sigma\gamma, s_\alpha \delta) = (\sigma\gamma, \delta - \alpha) = (\sigma\gamma, \delta) - (\sigma\gamma, \alpha) \Rightarrow (\sigma\gamma, \alpha) \geq 0$, since γ is regular, so is $\sigma\gamma$, thus $(\sigma\gamma, \alpha) > 0, \forall \alpha \in S$, $\sigma(C(\gamma)) = C(S)$, i.e. W acts transitively on Weyl chambers

(b) Directly from (a)

(c) Thanks to (b), it suffices to prove that each root lies in some base, there exists $\gamma \in H_\alpha \setminus \bigcup_{\beta \in \Delta \setminus \{\pm\alpha\}} H_\beta$, and the perturb γ slightly so that $0 < (\gamma, \alpha) < |(\gamma, \beta)|, \beta \in \Delta \setminus \{\pm\alpha\}$, then $\alpha \in S(\gamma)$

(d) By (c), if $\alpha \in \Delta$, $\beta \in S$, then there exists $\sigma \in W'$ such that $\sigma\alpha = \beta$, then $s_\beta = s_{\sigma\alpha} = \sigma s_\alpha \sigma^{-1}$, thus $s_\alpha = \sigma^{-1} s_\beta \sigma \in W'$

(e) By (d), $\sigma \in W$ can be written as $s = s_{\alpha_1} \cdots s_{\alpha_q}, \alpha_i \in S$ and suppose it is of minimal length, by Lemma 9.7.22, $\sigma(\alpha_q) \prec 0$ contradicting $\sigma(S) \subseteq S$

□

Proposition 9.7.25. The root system is irreducible iff the Lie algebra is simple

9.8 Dynkin diagram

Definition 9.8.1. A **generalized Cartan matrix** A is such that $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, if $a_{ij} = 0$, then $a_{ji} = 0$, $A = DS$ for some diagonal matrix D and symmetric matrix S, i.e. A is symmetrizable. Note that D would have nonzero diagonal entries, we can pick positive entries, A is a **Cartan matrix** if S is positive definite A is **decomposable** if $a_{ij} = 0$, $i \in I, j \in J$ for some $\{1, \cdots, n\} = I \sqcup J$, i.e. A can be diagonalized by blocks
An indecomposable matrix A is of **finite type** if all principal minors are positive, **affine type** if all proper principal minors are positive and $\det A = 0$, **indefinite type** otherwise

Definition 9.8.2. S is a set of positive simple roots, the **Dynkin diagram** is a graph with nodes simple roots, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges between α, β which can only be $0, 1, 2, 3$, and an arrow from α to β if $\langle \alpha, \beta \rangle > 1$. The **Coxter diagram** of the Weyl group W is just the Dynkin diagram without arrows. The **Coxeter graph** of it is the underlying graph

Theorem 9.8.3. We can recover the root system through Dynkin diagram

Definition 9.8.4. Type A_n corresponds to Dynkin diagram

Example 9.8.5. $\mathfrak{sl}_{n+1}$ corresponds to type A_n

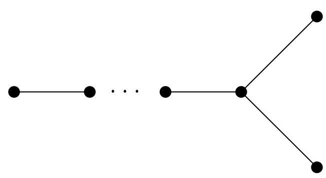
Definition 9.8.6. Type B_n corresponds to Dynkin diagram

Example 9.8.7. $\mathfrak{so}_{2n+1}$ corresponds to type B_n

Definition 9.8.8. Type C_n corresponds to Dynkin diagram

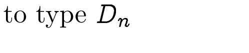
Example 9.8.9. $\mathfrak{sp}_{2n}$ corresponds to type C_n

Definition 9.8.10. Type D_n corresponds to Dynkin diagram



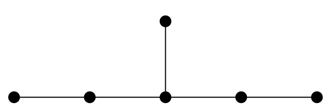
Example 9.8.11. $\mathfrak{so}_{2n}$ corresponds to type D_n

Definition 9.8.12. Type F_4 corresponds to Dynkin diagram

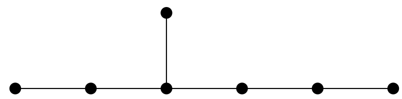


Definition 9.8.13. Type G_4 corresponds to Dynkin diagram

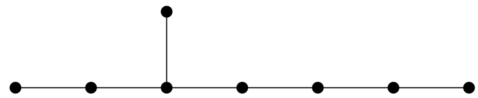
Definition 9.8.14. Type E_6 corresponds to Dynkin diagram



Definition 9.8.15. Type E_7 corresponds to Dynkin diagram

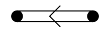


Definition 9.8.16. Type E_8 corresponds to Dynkin diagram



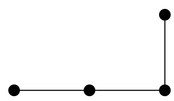
Remark 9.8.17. The number in the subscript is the number of nodes. In particular, we have $A_1 = B_1 = C_1$

$B_2 = C_2$

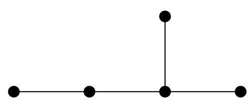


$D_3 = A_3$

$D_2 = A_1A_1$
$E_3 = A_2A_1$
$E_4 = A_4$



$E_5 = D_5$



Cartan-Killing classification

Theorem 9.8.18 (Cartan-Killing classification)**.** The above Dynkin diagrams classifies simple Lie algebras

Proof. Consider the **admissible sets** of Euclidean space V, $A = \{v_1, \cdots, v_n\}$ of linearly independent unit vectors with $(v_i, v_j) \leq 0$ and $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ if $i \neq j$. Define Coxeter diagram Γ_A for A to have vertices $v_1, \cdots, v_n$, and $d_{ij} = 4(v_i, v_j)^2$ edges between v_i and v_j if $i \neq j$. Assume that Γ_A is connected

a) The number of vertices in Γ_A joined by at least one edge is at most $|A| - 1$

$v = v_1 + \cdots + v_n \neq 0$ satisfies $(v, v) = n + 2\sum_{i<j}(v_i, v_j) > 0$, thus $n > -\sum_{i<j} 2(v_i, v_j) = \sum_{i<j} \sqrt{d_{ij}} \geq N$, where N is the number of pairs v_i, v_j such that $d_{ij} \geq 1$

b) The graph Γ_A contains no cycles

The vectors in a cycle of Γ_A form an admissible set which a contradicts a)

c) No vertex in Γ_A has more than 3 edges

Let w be a vertex of Γ_A with adjacent vertices $w_1, \cdots, w_k$, then $(w_i, w_j) = 0$ for $i \neq j$. Let $U = \mathrm{Span}_{\mathbb{R}}(w_1, \cdots, w_k, w)$, and extend $\{w_1, \cdots, w_k\}$ to an orthonormal basis of U, say by adjoining w_0. Clearly $(w, w_0) \neq 0$ and $w = \sum_{i=0}^{k}(w, w_i)w_i$. Hence $1 = (w, w) = \sum_{i=0}^{k}(w, w_i)^2$, $\sum_{i=1}^{k}(w, w_i)^2 < 1$, w has no more than 3 edges

d) If Γ_A has triple edge, then by c), the Γ_A can only be G_2

e) Assume Γ_A has a subgraph which is a line along $w_1, \cdots, w_n$, if we replace this subgraph with $w = w_1 + \cdots + w_n$, then it is still an admissible set

$(w, w) = n + 2\sum_{i=1}^{n-1}(w_i, w_{i+1}) = n - (n-1) = 1$, by d) any vertex v has at most edges linked with one such w_i, hence $(v, w) = (v, w_i)$, this gives an admissible set

f) A branch point is a vertex having more than 2 adjacent vertices, in this case, exactly 3. Γ_A has only one double edge, or only one branch point, or neither, but not both. Note that if Γ_A has no brach points and double edges corresponds to A_n

If Γ_A has two double edges between w_1, w_2 and v_1, v_2, then they can be linked through a line, by e), we can collapse it into a single vertex, but this will contradict c)

g) If Γ_A has a subgraph which is a line through $w_1, \cdots, w_n$, let $w = \sum iw_i$, then $(w, w) = \dfrac{n(n+1)}{2}$

h) If Γ_A has a double edge, then Γ_A is F_4 or B_n

By f) we know Γ_A is a line through $v_1, \cdots, v_p, w_q, \cdots, w_1$, $q \geq p \geq 1$ with single edges except v_p, w_q, let $v = \sum iv_i$, $w = \sum iw_i$, then

$$(v, w)^2 = (pv_p, qw_q)^2 = \frac{p^2q^2}{2}$$

Since v, w are linearly independent, by Cauchy Schwarz inequality, we have

$$\frac{p^2q^2}{2} = (v, w)^2 < (v, v)(w, w) = \frac{p(p+1)q(q+1)}{4}$$

Which implies $(p-1)(q-1) < 2$, thus if $p = 1$, then q can be any positive interger, giving B_n, if $p = 2$, then $q = 2$, giving F_4

i) If Γ_A has a branch point, then Γ_A is D_n or E_6, E_7, E_8

Γ_A is has three branch lines $v_1, \cdots, v_p, x$ and $w_1, \cdots, w_q, x$ together with $z_1, \cdots, z_r, x$, $p \geq q \geq r$, let $v = \sum iv_i$, $w = \sum iw_i$, $z = \sum iz_i$ which are pairwise orthogonal, $\hat{v}, \hat{w}, \hat{z}$ be normalized vectors of v, w, z, and consider $U = \mathrm{Span}_{\mathbb{R}}(v, w, z, x) = \mathrm{Span}_{\mathbb{R}}(\hat{v}, \hat{w}, \hat{z}, x_0)$, where x_0 is a unit vector orthogonal to v, w, z, then $(x, x_0) \neq 0$

$$1 = (x, x) = (x, \hat{v})^2 + (x, \hat{w})^2 + (x, \hat{z})^2 + (x, x_0)^2$$

Thus by g)

$$\frac{2p^2}{4p(p+1)} + \frac{2q^2}{4q(q+1)} + \frac{2r^2}{4r(r+1)} < 1$$

Hence

$$\frac{1}{1+p} + \frac{1}{1+q} + \frac{1}{1+r} > 1$$

and we know that

$$\frac{1}{1+p} \leq \frac{1}{1+q} \leq \frac{1}{1+r} \leq \frac{1}{2}$$

Hence $r = 1$

$$\frac{1}{1+p} + \frac{1}{1+q} > \frac{1}{2}$$

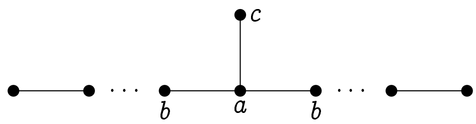
If $q = 1$, then p can be any positive integer, giving D_n, if $q = 2$, then p can only be $2, 3$ or 4, giving E_6, E_7, E_8

□

Lemma 9.8.19. (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots and $S = \{\alpha_1, \cdots, \alpha_n\}$ be its base, then there exists unique highest root $\gamma \in \Delta$, meaning $\gamma + \alpha_i \in \Delta, \forall \alpha_i \in S$

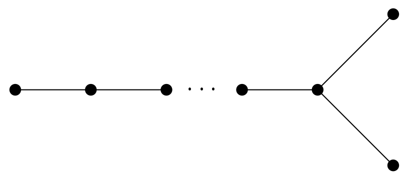
Definition 9.8.20. Let (V, Δ) be a irreducible root system, Δ^+ be a set of positive roots, $S = \{\alpha_1, \cdots, \alpha_n\}$ be its base, and γ its unique highest root, the extended Dynkin diagram is the usual Dynkin diagram adding $\alpha_0 = -\gamma$, the number of bonds for each two nodes and direction are still defined as before. Finally, suppose $-\alpha_0 = \sum n_i \alpha_i$, then lebel γ with ①, and label node α_i with n_i

Lemma 9.8.21. If the following part of the extended Dynkin diagram

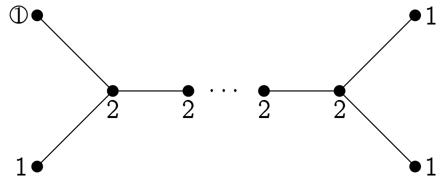


We have $2a = b + c + d$

Example 9.8.22. Consider the classical root system D_n, $\Delta^+ = \{e_i \pm e_j | i < j\}$, $S = \{e_1 - e_2, \cdots, e_{n-1} - e_n, e_{n-1} + e_n\}$, then $\gamma = e_1 + e_2$, its usual Dynkin diagram is



So its extended Dynkin diagram is



Chapter 10

Cluster algebra

10.1 Cluster algebra

ZP is a UFD

Lemma 10.1.1. $\mathbb{P}$ is a torsion free abelian group written multiplicatively, then the group ring $\mathbb{ZP}$ is a UFD

Proof. Finitely generated torsion free abelian groups are free $\square$

Definition 10.1.2 (Exchange pattern)**.** $I = \{1, \cdots, n\}$, $\mathbb{T}_n$ is the regular n tree, the coefficient group $\mathbb{P}$ is a torsion free abelian group under multiplication, thus the group ring $\mathbb{ZP}$ is a domain. Cluster variables are $\mathbf{x}(t) = \{x_i(t)\}_{i \in I}$ for $t \in \mathbb{T}_n$ such that for $\neq j$ and $t \overset{j}{\text{———}} t'$

$$x_i(t) = x_i(t')$$

$\mathcal{M} = \{M_j(t)\}$ are monomials such that

$$M_j(t)(\mathbf{x}) = p_j(t) \prod_i x_i^{b_i}, p_j(t) \in \mathbb{P}, b_i \geq 0$$

and for $t \overset{j}{\text{———}} t'$, b_i's depend on j and t

$$x_j(t)x_j(t') = M_j(t)(\mathbf{x}(t)) + M_j(t')(\mathbf{x}(t'))$$

satisfying **exchange pattern**

(E1) $x_j \nmid M_j(t)$

(E2) $x_i \mid M_j(t) \Rightarrow x_i \nmid M_j(t')$ for $t \overset{j}{\text{———}} t'$

(E3) $x_j \mid M_i(t) \Leftrightarrow x_i \mid M_j(t')$ for $t \overset{i}{\text{———}} t' \overset{j}{\text{———}} t_1$

(E4) $\dfrac{M_i(t_3)}{M_i(t_4)} = \left.\dfrac{M_i(t_2)}{M_i(t_1)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$ for $t_1 \overset{i}{\text{———}} t_2 \overset{j}{\text{———}} t_3 \overset{i}{\text{———}} t_4$, $M_0 = (M_j(t_2) + M_j(t_3))|_{x_i = 0}$

Remark 10.1.3. The substitution $x_j \leftarrow \dfrac{M_0}{x_j}$ is effectively a monomial. Since if $M_j(t_2)$ nor $M_j(t_3)$ contain x_i, then $M_i(t_2)$ nor $M_i(t_3)$ contain x_j which it substitute for nothing

$$\frac{M_i(t_2)}{M_i(t_1)} = \left(\left.\frac{M_i(t_2)}{M_i(t_1)}\right|_{x_j \leftarrow \frac{M_0}{x_j}} \right)\Bigg|_{x_j \leftarrow \frac{M_0}{x_j}} = \left.\frac{M_i(t_3)}{M_i(t_4)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Definition 10.1.4. There is an involution between $(\mathbf{x}, \mathcal{M})$ and $(\mathbf{x}', \mathcal{M}')$ where $x'_j(t) = x_j(t')$, $M'_j(t) = M_j(t')$ for every $t \overset{j}{\text{———}} t'$

Definition 10.1.5. Suppose $J \subseteq I$ is a subset of size m, delete sides labeled in $I - J$ in $\mathbb{T}_n$ and choose choose a connected component which would be $\mathbb{T}_m$, add to the coefficient group x_k's $k \in I - J$. This is called a restriction

Definition 10.1.6. Exchange pattern on exponents is a family of $B(t)$ such that for each t ——— t'

$$\frac{M_j(t)}{M_j(t')} = \frac{p_j(t)}{p_j(t')} \prod_i x_i^{b_{ij}(t)}$$

Thus

$$M_j(t) = p_j(t) \prod_i x_i^{[b_{ij}(t)]_+}, M_j(t') = p_j(t') \prod_i x_i^{[-b_{ij}(t)]_+}$$

Definition 10.1.7. An $n \times n$ matrix B is **sign-skew-symmetric** if $b_{ii} = 0$ and for $i \neq j$, b_{ij}, b_{ji} are both zeros or of opposite signs. B is **skew-symmetrizable** if there is a diagonal matrix D such that DB is skew symmetric, i.e. $d_i b_{ij} = -d_j b_{ji}$. Skew-symmetrizable matrices are obviously sign-skew-symmetric

Lemma on (|a|b+a|b|)/2

Lemma 10.1.8.

$$\frac{|a|b + a|b|}{2} = \begin{cases} ab & a, b > 0 \\ -ab & a, b < 0 \\ 0 & ab < 0 \end{cases} = \operatorname{sgn}(a)[ab]_+ = \operatorname{sgn}(b)[ab]_+$$

Note. $|a| = [a]_+ + [-a]_+$

Definition 10.1.9. A **mutation** on a $m \times n$ $(m > n)$ matrix B in direction k denoted by μ_k is given by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

Here $\mu_k(B) = B'$. μ_k is involutive

Theorem 10.1.10. If $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \overset{k}{\text{———}} t'$, then it gives a exchange pattern

Proof. Suppose $B(t)$ is an exchange pattern, then $B(t)$ is obviously sign-skew-symmetric. For $t \overset{k}{\text{———}} t'$, we have

$$\frac{M_k(t)}{M_k(t')} = \frac{p_k(t)}{p_k(t')} \prod_i x_i^{b_{ik}}, \frac{M_k(t')}{M_k(t)} = \frac{p_k(t')}{p_k(t)} \prod_i x_i^{b'_{ik}}$$

Hence $b'_{ik} = -b_{ik}$. Consider $t_1 \overset{j}{\text{———}} t' \overset{k}{\text{———}} t \overset{j}{\text{———}} t_2$

$$\frac{M_j(t')}{M_j(t_1)} = \left.\frac{M_j(t)}{M_j(t_2)}\right|_{x_k \leftarrow \frac{M_0}{x_k}}$$

becomes

$$\frac{p_j(t')}{p_j(t_1)} \prod_i x_i^{b'_{ij}} = \left.\frac{p_j(t)}{p_j(t_2)} \prod_i x_i^{b_{ij}}\right|_{x_k \leftarrow \frac{M_0}{x_k}}$$

Where

$$M_0 = \left.\left(p_k(t) \prod_i x_i^{[b_{ik}]_+} + p_k(t') \prod_i x_i^{[-b_{ik}]_+}\right)\right|_{x_j = 0}$$

Case 1: $b_{jk} > 0 \Leftrightarrow b_{kj} < 0$, then $M_0 = p_k(t') \prod\limits_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j,k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j,k} x_i^{[-b_{ik}]_+} \right)^{b_{kj}} = \prod_{i \neq j,k} x_i^{b_{ij} + b_{kj}[-b_{ik}]_+} x_k^{-b_{kj}}$$

Case 2: $b_{jk} < 0 \Leftrightarrow b_{kj} > 0$, then $M_0 = p_k(t') \prod\limits_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j,k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j,k} x_i^{[b_{ik}]_+} \right)^{b_{kj}} = \prod_{i \neq j,k} x_i^{b_{ij} + b_{kj}[b_{ik}]_+} x_k^{-b_{kj}}$$

Case 3: $b_{jk} = 0 \Leftrightarrow b_{kj} = 0$, then

$$\prod_{i \neq j,k} x_i^{b'_{ij}} = \prod_{i \neq j,k} x_i^{b_{ij}}$$

Therefore $b'_{kj} = -b_{kj}$ and $b'_{ij} = b_{ij} + \mathtt{sgn}(b_{ik})[b_{ik}b_{kj}]_+$

Conversely, if $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \overset{k}{\text{———}} t'$, take

$$M_k(t) = \prod_i x_i^{[b_{ik}(t)]_+}, M_k(t') = \prod_i x_i^{[-b_{ik}(t)]_+}$$

for $t \overset{k}{\text{———}} t'$, then obviously $x_k \nmid M_k(t)$ since $b_{kk} = 0$ and

$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow -b_{jk} < 0 \Rightarrow x_j \nmid M_k(t')$$

For $t \overset{k}{\text{———}} t' \overset{j}{\text{———}} t_1$

$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow b'_{kj} = -b_{kj} > 0 \Leftrightarrow x_k \mid M_j(t')$$

For $t_1 \overset{j}{\text{———}} t' \overset{k}{\text{———}} t \overset{j}{\text{———}} t_2$, it is the exact argument above by taking $p_j(t) \equiv 1$ ☐

Mutation of a skew-symmetrizable matrix preserves the skew-symmetrizing matrix

Proposition 10.1.11. Given a skew-symmetrizable matrix B, the all possible mutations $B(t)$ in $\mathbb{T}_n$ are skew-symmetrizable with the same skew-symmetrizing matrix D

Proof. True for each mutation μ_k ☐

Remark 10.1.12. For cluster algebra of rank $n \leq 2$, the exchange pattern is skew-symmetrizable. If $n = 1$, $B(t) \equiv 0$. If $n = 2$, $B(t_n) = (-1)^n \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$

Definition 10.1.13. Denote $2n$ tuple $\mathbf{p}(t)$ the coefficients $p_j(t), p_j(t')$ for $t \overset{j}{\text{———}} t'$. $\Sigma(t) = (\mathbf{x}(t), \mathbf{p}(t), B(t))$ is a **seed**, $\mathbf{x}(t)$ is the **cluster** of the seed. If we assume $\mathbf{x}(t_0)$ are algebraically independent ($\mathbf{x}(t_0)$ is a cluster of rank n), then so are $\mathbf{x}(t)$ since they are all mutationally equivalent. Denote the collection of all cluster variables $\mathcal{X}$, the collection of all coefficients $\mathcal{P}$, the collection of exchange matrices $\mathcal{B}$, the collection of $M_j(t)$'s $\mathcal{M}$, the collection of seeds $\mathcal{S}$
We can take $\mathcal{F} = \mathbb{ZP}(x_1, \cdots, x_n)$ to be the **ambient field**, $\mathbf{x}$ can be some cluster $\mathbf{x}(t_0)$. The **cluster algebra** is the subalgebra $\mathbb{Z}\mathcal{P}[\mathcal{X}]$

Proposition 10.1.14. Given $B(t)$ that give rise to exchange pattern, the coefficients must satisfy

$$p_i(t_1)p_i(t_3)p_i(t_3)^{[b_{ji}(t_3)]_+} = p_i(t_2)p_i(t_4)p_i(t_2)^{[b_{ji}(t_2)]_+} \tag{10.1.1}$$

Proof. For $t_1 \overset{i}{\text{———}} t_2 \overset{j}{\text{———}} t_3 \overset{i}{\text{———}} t_4$

$$\frac{p_i(t_3)}{p_i(t_4)} \prod_k x_k^{b_{ki}(t_3)} = \frac{M_i(t_3)}{M_i(t_4)} = \left.\frac{M_i(t_3)}{M_i(t_4)}\right|_{x_j \leftarrow \frac{M_0}{x_j}} = \left.\frac{p_i(t_2)}{p_i(t_1)} \prod_k x_k^{b_{ki}(t_2)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Here

$$M_0 = \left.(M_j(t_2) + M_j(t_3))\right|_{x_i=0} = \left.\left(p_j(t_2) \prod_k x_k^{[b_{kj}(t_2)]_+} + p_j(t_3) \prod_k x_k^{[b_{kj}(t_3)]_+}\right)\right|_{x_i=0}$$

Take $x_k = 1$ for $k \neq j$, and use the fact that $B(t)$ are sign-skew-symmetric, we get

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)M_0^{b_{ji}(t_2)}$$

With

Case 1: $b_{ij}(t_2) > 0$. Then $b_{ij}(t_3) < 0$, $M_0 = p_j(t_3)$ and

$$p_i(t_1)p_i(t_3)p_i(t_3)^{b_{ji}(t_3)} = p_i(t_2)p_i(t_4)$$

Case 2: $b_{ij}(t_2) < 0$. Then $b_{ij}(t_3) > 0$, $M_0 = p_j(t_2)$ and

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)p_i(t_2)^{b_{ji}(t_2)}$$

Case 3: $b_{ij}(t_2) = 0$. Then $b_{ij}(t_3) = 0$, $M_0 = p_j(t_2) + p_j(t_3)$, but $b_{ji}(t_2) = b_{ji}(t_3) = 0$, hence

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)$$

□

Note. A trivial solution of (10.1.1) is $p_j(t) = 1$

Proposition 10.1.15. The **universal coefficient group** $\mathcal{P}$ of $\mathbb{P}$ is the free abelian group generated by $p_i(t)$ modulo (10.1.1). $\mathcal{P}$ is torsion free, more precisely, it is the free abelian group generated by $p_i(t_0), p_i(t)$ for every $t_0 \overset{i}{\text{———}} t$ and exactly one of $p_i(t), p_i(t')$ for every $t \overset{i}{\text{———}} t'$ where $t, t' \neq t_0$

Definition 10.1.16. Take the field of rational functions of cluster variables $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}\mathcal{P}$ to be the ambient field $\mathcal{F}$, all other cluster variables $\mathbf{x}(t)$ are also in $\mathcal{F}$ by Theorem 10.2.3. The **universal cluster algebra** $\mathcal{A}$ is the subalgebra generated by all cluster variables with coefficients in $\mathbb{Z}\mathcal{P}$

M-equivalence

Definition 10.1.17. $t, t' \in \mathbb{T}_n$ are $\mathcal{M}$**-equivalent** if there is a permutation σ of I such that

- $x_{\sigma(i)}(t) = x_i(t')$
- $M_{\sigma(j)}(t)(\mathbf{x}(t)) = M_j(t')(\mathbf{x}(t'))$ and $M_{\sigma(j)}(t_1)(\mathbf{x}(t)) = M_j(t_1')(\mathbf{x}(t'))$ for $t \overset{\sigma(j)}{\text{———}} t_1$ and $t' \overset{j}{\text{———}} t_1'$

10.2 Laurent phenomenon

Caterpillar lemma

Lemma 10.2.1 (Caterpillar lemma)**.** Define the caterpillar tree $\mathbb{T}_{n,m}$ consists of a spine of $m+2$ nodes, with an orientation from t_{tail} to t_{head} with t_{base} connected to t_{tail}, as illustrated in Figure 10.2.1

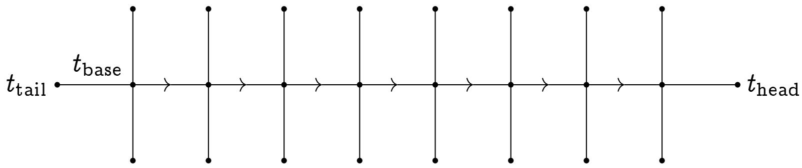


Figure 10.2.1: $\mathbb{T}_{4,8}$

T4,8

Let $\mathbb{A}$ be a UFD, exchange polynomial $P \in \mathbb{A}[x_1, \cdots, x_n]$ for each edge $t \overset{j}{\text{——}} t'$, denoted $x \underset{P}{\overset{j}{\text{——}}} t'$ satisfying the generalized exchange pattern

- P doesn't depend on x_j and x_i doesn't divide P
- For $t_0 \underset{P}{\overset{i}{\text{——}}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2$, P, Q_0 are coprime in $\mathbb{A}[x_1, \cdots, x_n]$, where $Q_0 = Q|_{x_i=0}$
- For $t_0 \underset{P}{\overset{i}{\text{——}}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2 \underset{R}{\overset{i}{\text{——}}} t_3$, $LQ_0^bP = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$ for some $b \geq 0$ and some Laurent monomial L with coefficients in $\mathbb{A}$ coprime with P

Cluster variables $\mathbf{x}(t) = \{x_i(t)\}$ for $t \in \mathbb{T}_{n,m}$ satisfying for each $t \underset{P}{\overset{i}{\text{——}}} t'$

- $x_i(t) = x_i(t')$ for any $i \neq j$
- $x_j(t)x_j(t') = P(t)(\mathbf{x}(t))$

Then $\mathbf{x}(t_{\mathsf{head}})$ are Laurent polynomials in $\mathbf{x}(t_0)$ with coefficients in $\mathbb{A}$

Proof. Write the subring of Laurent polynomials generated by $\mathbf{x}(t)$ as

$$\mathcal{L}(t) = \mathbb{A}[x_1(t)^{\pm 1}, \cdots, x_n(t)^{\pm 1}]$$

Make induction on m. If $m = 1$, consider $t_{\mathsf{tail}} = t \underset{P}{\overset{i}{\text{——}}} t_{\mathsf{base}} = t' \underset{Q}{\overset{j}{\text{——}}} t_{\mathsf{head}} = t_1$, we have for $k \neq i, j$

$$\begin{aligned} x_k(t_1) &= x_k(t') = x_k(t) \\ x_i(t_1) &= x_i(t') = \frac{P(\mathbf{x}(t))}{x_i(t)} \\ x_j(t_1) &= \frac{Q(\mathbf{x}(t'))}{x_j(t')} = \frac{Q(\mathbf{x}(t'))}{x_j(t)} \end{aligned}$$

Now suppose $m \geq 2$, let's show that $X = x_k(t_{\mathsf{head}}) \in \mathcal{L}(t_0)$, by induction, $X \in \mathcal{L}(t_1) \cap \mathcal{L}(t_3)$. Since $X, x_i(t_1) = \dfrac{P(\mathbf{x}(t_0))}{x_i(t_0)} \in \mathcal{L}(t_0)$, $X = \dfrac{f_0}{x_i(t_1)^a}$ for some $f_0 \in \mathcal{L}(t_0)$ and $a \geq 0$, similarly, $X = \dfrac{g_0}{x_j(t_2)^b x_i(t_3)^c}$ for some $g_0 \in \mathcal{L}(t_0)$ and $b, c \geq 0$, thanks to Lemma 10.2.2, $X \in \mathcal{L}(t_0)$ $\square$

Lemma for caterpillar lemma

Lemma 10.2.2. For $t_0 \underset{P}{\overset{i}{\text{——}}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2 \underset{R}{\overset{i}{\text{——}}} t_3$, $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3) \in \mathcal{L}(t_0)$, and

$$\mathsf{gcd}(x_i(t_1), x_i(t_3)) = \mathsf{gcd}(x_j(t_2), x_i(t_1)) = 1$$

in $\mathcal{L}(t_0)$

Note. $\mathcal{L}(t_0)$ is a UFD, $\mathcal{L}(t_0)^\times$ consists of Laurent monomials with coefficients $\mathbb{A}^\times$

Proof. Denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, $v = x_i(t_3)$, think of P, Q, R as functions of x_j, x_i, x_j respectively, then

$$\begin{aligned} z &= \frac{P(y)}{x} \\ u &= \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y} \\ v &= \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} + \frac{R\left(\frac{Q(0)}{y}\right)}{z} \end{aligned}$$

$$\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} = R'\left(\frac{Q_0}{y}\right)\frac{Q'(0)}{y} + \frac{1}{2}\,R\left(\frac{Q(z)}{y}\right)''\Bigg|_{z=0} z + \cdots \equiv R'\left(\frac{Q_0}{y}\right)\frac{Q'(0)}{y} \bmod z$$

$$\frac{R\left(\frac{Q_0}{y}\right)}{z} = \frac{L(y)Q_0(y)^b P(y)}{z} = L(y)Q_0(y)^b x$$

Thus $v \in \mathcal{L}(t_0)$
Since $\mathsf{gcd}(P, Q_0) = \mathsf{gcd}(P, L) = 1$

$$\mathsf{gcd}(z, v) = \mathsf{gcd}\left(\frac{P(y)}{x}, L(y)Q_0(y)^b x\right) = \mathsf{gcd}\left(P(y), L(y)Q_0(y)^b\right) = 1$$

Since $\dfrac{Q(z)}{y} \equiv \dfrac{Q_0}{y} \bmod z$

$$\mathsf{gcd}(z, u) = \mathsf{gcd}\left(z, \frac{Q_0}{y}\right) = \mathsf{gcd}\left(P(y), Q_0\right) = 1$$

□

Laurent phenonmenon

Theorem 10.2.3. Catepillar lemma 10.2.1 implies that in a cluster algebra, any cluster variable can be expressed as a Laurent polynomial in a given $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$ since there is no subtraction involved

Proof. $\mathbb{T}_{n,m}$ can be embedded in $\mathbb{T}_n$. $M_j(t) + M_j(t')$ doesn't depend on x_j and not divisible by x_i for $t \overset{j}{\text{———}} t'$ and any $i \neq j$
For $t_0 \overset{i}{\underset{P}{\text{———}}} t_1 \xrightarrow[Q]{j} t_2 \overset{i}{\underset{R}{\text{———}}} t_3$, we have

$$\frac{P}{M_i(t_0)} = 1 + \frac{M_i(t_1)}{M_i(t_0)} = 1 + \frac{M_i(t_2)}{M_i(t_3)}\Bigg|_{x_j \leftarrow \frac{M_0}{x_j}} = \frac{R}{M_i(t_3)}\Bigg|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Where $M_0 = \left.(M_j(t_1) + M_j(t_2))\right|_{x_i = 0} = Q_0$, thus

$$\frac{R|_{x_j \leftarrow \frac{Q_0}{x_j}}}{P} = \frac{M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}}}{M_i(t_0)}$$

Note that $M_i(t_0) = p_i(t_0)\prod_k x_k^{[b_{ki}(t_0)]_+}$ and

$$\begin{aligned}
M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}} &= p_i(t_3)\prod_k x_k^{[b_{ki}(t_3)]_+}\Bigg|_{x_j \leftarrow \frac{Q_0}{x_j}} \\
&= p_i(t_3)\left(\frac{Q_0}{x_j}\right)^{[b_{ji}(t_3)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+} \\
&= p_i(t_3) Q_0^{[b_{ji}(t_3)]_+} x_j^{-[b_{ji}(t_3)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+}
\end{aligned}$$

Hence

$$R|_{x_j \leftarrow \frac{Q_0}{x_j}} = \frac{p_i(t_3)}{p_i(t_0)} x_j^{-[b_{ji}(t_3)]_+ - [b_{ji}(t_0)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+ - [b_{ki}(t_0)]_+} Q_0^{[b_{ji}(t_3)]_+} P = LQ_0^b P$$

Since the sum of two monomials P doesn't depend on x_i and is not divisible by any x_k for $k \neq i$, Q_0 is a monomial, L is a Laurent monomial, Q_0, P are coprime in $\mathbb{A}[\mathbf{x}]$ and L, P are coprime in $\mathcal{L}[\mathbf{x}]$ $\square$

10.3 Y-system

A is the Cartan matrix of root system Φ with simple system Π, denote $[\alpha:\alpha_i]$ as the coefficients of $\alpha\in\Phi$, write $\Phi_{\geq -1}=\Phi_+\cup(-\Pi)$. Since the Coxeter graph is a tree, it is bipartite, up to renaming $I=\{1,\cdots,n\}=I_-\sqcup I_+$, $\varepsilon(i)=\varepsilon$ for $i\in I_\varepsilon$ be the indicator. Let $t_\varepsilon=\prod\limits_{i\in I_\varepsilon} s_i$, $t=t_-t_+$ is a Coxeter element, h is the Coxeter number. $s_{\mathbf{i}_-},s_{\mathbf{i}_+}$ are reduced words of t_-,t_+, then

$$w_\circ=\underbrace{s_{\mathbf{i}_-}s_{\mathbf{i}_+}\cdots s_{\mathbf{i}_\pm}}_{h\text{ times}}=s_{\mathbf{i}_\circ}$$

is the element of longest length

Definition 10.3.1. Suppose Φ is irreducible, then

$$[s_i(\alpha):\alpha_k]=\begin{cases}-[\alpha:\alpha_i]-\sum\limits_{j\neq i}a_{ij}[\alpha:\alpha_j] & k=i\\ [\alpha:\alpha_k] & k\neq i\end{cases}$$

Define a piecewise linear modification

$$[\sigma_i(\alpha):\alpha_k]=\begin{cases}-[\alpha:\alpha_i]-\sum\limits_{j\neq i}a_{ij}[\alpha:\alpha_j]_+ & k=i\\ [\alpha:\alpha_k] & k\neq i\end{cases}$$

Proposition 10.3.2.

1. σ_i are involutions

2. σ_i,σ_j commutes if i,j are not adjacent in the Coxeter graph

3. σ_i preserves $\Phi_{\geq -1}$

Proposition 10.3.3. Let $\tau_\varepsilon=\prod\limits_{i\in I_\varepsilon}\sigma_i$

1. τ_ε are involutions that preserve $\Phi_{\geq -1}$

2. $\tau_\varepsilon\alpha=t_\varepsilon\alpha$ for $\alpha\in\mathbb{Z}_{\geq 0}\Pi$

3. $\Phi_{\geq -1}\to\Phi_{\geq -1}^\vee$, $\alpha\mapsto\alpha^\vee$ are τ_ε equivariant, i.e. $(\tau_\varepsilon\alpha)^\vee=\tau_\varepsilon\alpha^\vee$

Definition 10.3.4. A **Y-system** is a family of commuting variables $Y_i(t)$, $i\in I=\{1,\cdots,n\}$, $t\in\mathbb{Z}$ such that

$$Y_i(t+1)Y_i(t-1)=\prod_{j\neq i}(1+Y_i(t))^{-a_{ij}} \tag{10.3.1}$$

Remark 10.3.5. (10.3.1) only really involve those $Y_j(k)$ with $\varepsilon(j)\cdot(-1)^k=\mathtt{const}$, assume $Y_j(k)=Y_j(k+1)$ for $\varepsilon(i)=(-1)^k$, then we have

$$Y_i(k+1)=\begin{cases}\dfrac{\prod_{j\neq i}(1+Y_i(k))^{-a_{ij}}}{Y_i(k)} & \varepsilon(i)=(-1)^k+1\\ Y_i(k) & \varepsilon(i)=(-1)^k\end{cases}$$

Denote $\mathcal{Y}$ as the collection of all $Y_j(k)$'s, $u_i=Y_i(0)$, define

$$\tau_\varepsilon(u_i)=\begin{cases}\dfrac{\prod_{j\neq i}(1+u_i)^{-a_{ij}}}{u_i} & \varepsilon(i)=\varepsilon\\ u_i & \text{otherwise}\end{cases}$$

Theorem 10.3.6 (Zamolodichikov)**.** $Y_i(t)$'s are $2(h+2)$ periodic, i.e. $Y_i(t+2(h+2)) = Y_i(t)$

Theorem 10.3.7. There is a unique family $\{F[\alpha]\}_{\alpha\in\Phi_{\geq -1}}$ of polynomials in u_i such that $F[-\alpha_i] = -1$ and

$$\tau_\varepsilon(F[\alpha]) = \frac{\prod\limits_{\varepsilon(i)=-\varepsilon}(u_i+1)^{[\alpha^\vee:\alpha_i^\vee]}}{\prod\limits_{\varepsilon(i)=\varepsilon} u_i^{[\alpha^\vee:\alpha_i^\vee]_+}} F[\tau_{-\varepsilon}(\alpha)]$$

Furthermore, $F[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ has constant term 1. Call $F[\alpha]$ Fibonacci polynomials

Any $\alpha \in \Phi_{\geq -1}$ can be written as $\alpha(k,i) = (\tau_-\tau_+)^k(-\alpha_i)$, denote $N[\alpha] = \prod\limits_{j\neq i} F[\alpha(-k,i)]^{-a_{ij}}$, note that $N[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ also has constant term 1

Theorem 10.3.8. There is a unique bijection $\Phi_{\geq} - 1 \to \mathcal{Y}$, $\alpha \mapsto Y[\alpha] = \dfrac{N[\alpha]}{\mathbf{u}^{\alpha^\vee}}$ such that $Y[-\alpha_i] = u_i$, $\tau_\varepsilon(Y[\alpha]) = Y[\tau_\varepsilon(\alpha)]$

Definition 10.3.9. A **tropical specialization** r_{trop} of a rational expression r is changing the addition $+$ and multiplication $\cdot$ into $\oplus$ and $\odot$ where $a \oplus b = \max(a,b)$, $a \odot b = a + b$

The **compatibility degree** for $\alpha, \beta \in \Phi_{\geq -1}$ is

$$(\alpha||\beta) = (Y[\alpha]+1)(\beta)_{\mathrm{trop}}$$

Here $(Y[\alpha]+1)(\beta)$ is evaluation at $\{u_i = [\beta : \alpha_i]\}$. α, β are **compatible** if $(\alpha||\beta) = 0$

$\Delta(\Phi)$ is a simplicial complex with $\Phi_{\geq -1}$ as vertices and mutually compatible subsets of $\Delta(\Phi)$ are simplices, the maximal simplices are **clusters**. The **exchange graph** $E(\Phi)$ is an unoriented graph with clusters as vertices and an edge between clusters which has intersection of cardinality $n-1$

Remark 10.3.10. $(||)$ is uniquely characterized by

$$(-\alpha_i||\beta) = (Y[-\alpha_i]+1)(\beta)_{\mathrm{trop}} = (u_i+1)(\beta)_{\mathrm{trop}} = [\beta : \alpha_i]_+$$

$$(\tau_\varepsilon(\alpha)||\tau_\varepsilon(\beta)) = (Y[\tau_\varepsilon(\alpha)]+1)(\tau_\varepsilon(\beta))_{\mathrm{trop}} = (\tau_\varepsilon(Y[\alpha])+1)(\tau_\varepsilon(\beta))_{\mathrm{trop}} = (\alpha||\beta)$$

Proposition 10.3.11. Consider perfect bilinear pairing

$$\begin{aligned} \mathbb{Z}\Pi^\vee \times \mathbb{Z}\Pi &\to \mathbb{Z} \\ (\xi,\gamma) &\mapsto \{\xi,\gamma\} \end{aligned}$$

Where $\{\xi,\gamma\} = \sum \varepsilon(i)[\xi : \alpha_i^\vee][\gamma : \alpha_i]$. Then

$$(\alpha||\beta) = \max(\{\tau_+\alpha^\vee, \beta\}, \{\alpha^\vee, \tau_+\beta\}, 0) = \max(-\{\tau_-\alpha^\vee, \beta\}, -\{\alpha^\vee, \tau_-\beta\}, 0)$$

Note. $(||)$ doesn't depend on the choice of the indicator ε

Proposition 10.3.12.

1. $(\alpha||\beta) = (\beta^\vee||\alpha^\vee)$, in particular, if Φ is simply laced, then $(\alpha||\beta) = (\beta||\alpha)$

2. If $(\alpha||\beta) = 0$, then $(\beta||\alpha) = 0$

3. $J \subseteq I$, $\Phi(J) \subseteq \Phi$ is a root subsystem, $(||)$ on $\Phi(J)$ is the same as the restriction

Theorem 10.3.13. $\Delta(\Phi)$ is pure of dimension $n-1$, and each facet forms a $\mathbb{Z}$-basis for the root lattice

Theorem 10.3.14. The simplicial cones of all clusters form a complete simplicial fan

Corollary 10.3.15. The geometric realization of $\Delta(\Phi)$ is $\mathbb{S}^{n-1}$

Conjecture 10.3.16. The simplicial fan of $\Delta(\Phi)$ is the normal fan of some convex polytope $P(\Phi)$

Theorem 10.3.17. $E(\Phi)$ is a regular n tree

Example 10.3.18.

10.4 Associahedron

Definition 10.4.1. Any n regular polygon has $\binom{n}{2} - n = \frac{n(n-3)}{2}$ diagonals, with these as vertices, noncrossing subsets as simplexes, we have given it a abstract simplicial complex structure

10.5 Cluster algebra of geometric type

Semifield is multiplicative torison free

Lemma 10.5.1. Semifield $\mathbb{P}$ is multiplicative torison free

Proof. Suppose $p^m = 1$, then

$$p = \frac{p^m \oplus p^{m-1} \oplus \cdots + p}{p^{m-1} \oplus p^{m-2} \oplus \cdots + 1} = \frac{1 \oplus p^{m-1} \oplus \cdots + p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = 1$$

□

Definition 10.5.2. Exchange pattern is **normalized** if $\mathbb{P}$ is a semifield and $p_j(t) \oplus p_j(t') = 1$ for any $t \overset{j}{\text{———}} t'$

Normalized exchange pattern determines the cluster algebra

Proposition 10.5.3. Given p_j, r_j in a semifield $\mathbb{P}$ such that $p_j \oplus r_j = 1$, and exchange matrix $B(t)$ on $\mathbb{T}_n$, define $p_j(t_0) = p_j$, $p_j(t) = r_j$ for each $t_0 \overset{i}{\text{———}} t$, this completely determines the cluster algebra

Proof. Define $u_j(t) = \dfrac{p_j(t)}{p_j(t')}$ for $t \overset{j}{\text{———}} t'$, then

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)}, p_j(t') = \frac{1}{1 \oplus u_j(t)}$$

Then (10.1.1) becomes

$$u_i(t_3)p_j(t_3)^{[b_{ji}(t_3)]_+} = u_i(t_2)p_j(t_2)^{[b_{ji}(t_2)]_+}$$

Case 1: $u_i(t_3)p_j(t_3)^{b_{ji}(t_3)} = u_i(t_2) \Rightarrow u_i(t_3) = u_i(t_2)(1 \oplus u_j(t_2))^{b_{ji}(t_2)}$

Case 2: $u_i(t_3) = u_i(t_2)p_j(t_2)^{b_{ji}(t_2)} = u_i(t_2)\left(\dfrac{u_j(t_2)}{1 \oplus u_j(t_2)}\right)^{b_{ji}(t_2)}$

Thus for $t \overset{j}{\text{———}} t'$, we have

$$u_i(t') = u_i(t)u_j(t)^{[b_{ji}(t)]_+}(1 \oplus u_j(t))^{-b_{ji}(t)}$$

□

Remark 10.5.4. $\mathbf{p}$ determines $\mathbf{u}$ which in turn determines $\mathbf{p}$
Fix semifield $\mathbb{P}$, B is skew-symmetrizable, then $(B, \mathbf{p})$ determines the cluster algebra $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$ up to isomorphism

Corollary 10.5.5. The exchange graph of a normalized cluster algebra is n-regular

Definition 10.5.6. The **tropical semifield** $(\mathbb{R}, \oplus, \odot)$ is a semifield with multiplication as $\odot$, $\min$ or $\max$ as $\oplus$
The tropical semifield generated by p is the free abelian group generated multiplicatively by p with $p^a \oplus p^b = p^{\min(a,b)}$

Definition 10.5.7. A normalized cluster algebra is of geometric type if $\mathbb{P}$ is the tropical semifield generated by $\{p_i\}_{i \in I'}$ and each $p_j(t)$ is a monomial with nonnegative exponents

Remark 10.5.8. In this particular case, normality just means that for $t \overset{j}{\text{———}} t'$, $p_j(t), p_j(t')$ doesn't have a common variable, or the support doesn't intersect

Proposition 10.5.9. $\mathbb{P}$ is the tropical semifield generated by $p_i, i \in I'$, $B(t)$ is the exchange pattern of exponents, $p_j(t)$ give rise to a cluster algebra of geometric type iff $C(t)$ satisfies the exchange pattern of coefficients, i.e. $p_j(t) = \prod_i p_i^{[c_{ij}(t)]_+}$ and

$$c'_{ij} = \begin{cases} -c_{ij} & j = k \\ c_{ij} + \dfrac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2} & \text{otherwise} \end{cases}$$

Here the mutation is in direction k

Proof. Suppose $p_j(t)$ give rise to a cluster algebra of geometric type. Define $u_j(t) = \dfrac{p_j(t)}{p_j(t')} = \prod_{i \in I'} p_i^{c_{ij}(t)}$ for each $t \overset{j}{\text{———}} t'$, then according to Proposition 10.5.3

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)} = \frac{\prod_i p_i^{c_{ij}}}{1 \oplus \prod_i p_i^{c_{ij}}} = \frac{\prod_i p_i^{c_{ij}}}{\prod_i p_i^{-[-c_{ij}]_+}} = \prod_i p_i^{[c_{ij}]_+}$$

$$1 = u_k(t)u_k(t') = \prod_i p_i^{c_{ik}+c'_{ik}} \Rightarrow c'_{ik} = -c_{ik}$$

And

$$\begin{aligned} \prod_i p_i^{c'_{ij}} &= \prod_i p_i^{c_{ij}} \left(\prod_i p_i^{c_{ik}} \right)^{[b_{kj}]_+} \left(1 \oplus \prod_i p_i^{c_{ik}} \right)^{-b_{kj}} \\ &= \prod_i p_i^{c_{ij}} \prod_i p_i^{c_{ik}[b_{kj}]_+} \prod_i p_i^{b_{kj}[-c_{ik}]_+} \\ &= \prod_i p_i^{c_{ij} + \frac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2}} \end{aligned}$$

□

Remark 10.5.10. Note if we take $\tilde{B}(t) = (\tilde{b}_{ij})_{i \in I \cup I', j \in I}$ where $\tilde{b}_{ij} = b_{ij}$ for $i, j \in I$ is the principal part of $\tilde{B}$, $\tilde{b}_{ij} = c_{ij}$ for $i \in I', j \in I$

Corollary 10.5.11. Given $\tilde{B}_0$ with a skew-symmetrizable principal part B_0, then there exists a unique exchange pattern of geometric type such that $\tilde{B}(t_0) = \tilde{B}_0$ for $t_0 \in \mathbb{T}_n$

Proof. By Proposition 10.1.11 □

Remark 10.5.12. The class of exchange patterns of geometric type is stable under restriction and direct product

10.6 Rank two case

Cluster algebra of rank 2

Example 10.6.1. If $n = 2$, consider $\mathbb{T}_2$

$$\xrightarrow{\ 1\ } t_0 \xrightarrow{\ 2\ } t_1 \xrightarrow{\ 1\ } t_2 \xrightarrow{\ 2\ } t_3 \xrightarrow{\ 1\ } t_4 \xrightarrow{\ 2\ } t_5 \xrightarrow{\ 1\ }$$

The cluster variables are y_i, y_{i+1} for t_i

$$y_{2k+1} = x_1(t_{2k}) = x_1(t_{2k+1}), y_{2k} = x_2(t_{2k-1}) = x_2(t_{2k})$$

$M_2(t_0)$ and $M_2(t_1)$ don't have x_1 and can't both have x_2
If both of them don't have x_2, then $M_2(t_0), M_2(t_1) \in \mathbb{P}$, thus

$$\cdots x_2 \nmid M_1(t_{-1}) \Leftrightarrow x_1 \nmid M_2(t_0) \Leftrightarrow x_2 \nmid M_1(t_1) \Leftrightarrow x_1 \nmid M_2(t_2) \cdots$$

$$\cdots x_2 \nmid M_1(t_0) \Leftrightarrow x_1 \nmid M_2(t_1) \Leftrightarrow x_2 \nmid M_1(t_2) \Leftrightarrow x_1 \nmid M_2(t_3) \cdots$$

So is every $M_*(t_*) \in \mathbb{P}$, write q_m, r_m as the two monomials of t_{m-1} —— t_m, then we have

$$y_{m-1}y_{m+1} = q_m + r_m$$

And for t_{m-2} —— t_{m-1} —— t_m —— t_{m+1} we have

$$\frac{q_{m+1}}{r_{m+1}} = \frac{r_{m-1}}{q_{m-1}} \Leftrightarrow q_{m-1}q_{m+1} = r_{m-1}r_{m+1}$$

If $M_2(t_0) = q_1x_1^b$, $M_2(t_1) = r_1$(the other case corresponds to the involution) for some $b > 0$, then $M_1(t_1) = q_2x_2^c$, $M_1(t_2) = r_2$ for some $c > 0$, we have

$$\frac{M_2(t_2)}{M_2(t_3)} = \left.\frac{M_2(t_1)}{M_2(t_0)}\right|_{x_1 \leftarrow \frac{M_0}{x_1}} = \left.\frac{r_1}{q_1x_1^b}\right|_{x_1 \leftarrow \frac{r_2}{x_1}} = \frac{r_1x_1^b}{q_1r_2^b}$$

Since $x_1 \mid M_2(t_3) \Rightarrow x_2 \nmid M_2(t_2)$ gives a contradiction, $x_1 \nmid M_2(t_3) \Rightarrow M_2(t_3) = r_3$, thus $M_2(t_2) = q_3x_1^b$, periodically, we can conclude

$$\begin{aligned} M_2(t_{2k}) &= q_{2k+1}x_1^b \\ M_2(t_{2k+1}) &= r_{2k+1} \\ M_1(t_{2k-1}) &= q_{2k}x_2^c \\ M_1(t_{2k}) &= r_{2k} \end{aligned}$$

Therefore we have

$$\begin{aligned} y_{2k-1}y_{2k+1} &= q_{2k}y_{2k}^c + r_{2k} \\ y_{2k}y_{2k+2} &= q_{2k+1}y_{2k+1}^b + r_{2k+1} \end{aligned}$$

For $t_{2k-1} \xrightarrow{\ 1\ } t_{2k} \xrightarrow{\ 2\ } t_{2k+1} \xrightarrow{\ 1\ } t_{2k+2}$ we have

$$q_{2k}q_{2k+2}r_{2k+1}^c = r_{2k}r_{2k+2}$$

For $t_{2k-2} \xrightarrow{\ 2\ } t_{2k-1} \xrightarrow{\ 1\ } t_{2k} \xrightarrow{\ 2\ } t_{2k+1}$ we have

$$q_{2k-1}q_{2k+1}r_{2k}^b = r_{2k-1}r_{2k+1}$$

The exchange matrices are

$$B(t_m) = (-1)^m \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

Conversely, given such relation, we can always find a corresponding cluster algebra

In particular, consider the coordinate ring of $\mathrm{Gr}_2(5)$, let

$$\begin{aligned} y_m &= [\overline{2m-1}, \overline{2m+1}] \\ q_m &= [\overline{2m-2}, \overline{2m+2}] \\ r_m &= [\overline{2m-2}, \overline{2m-1}][\overline{2m+1}, \overline{2m+2}] \\ b &= c = 1 \end{aligned}$$

Note. If we denote $m \bmod 2$ as $\langle m \rangle$, then

$$\begin{aligned} p_{\langle m \rangle}(t_m) &= q_{m+1} \\ p_{\langle m+1 \rangle}(t_m) &= r_m \\ x_{\langle m \rangle}(t_m) &= q_m \\ x_{\langle m+1 \rangle}(t_m) &= y_{m+1} \end{aligned}$$

Example 10.6.2. As in Example 10.6.1, by Theorem 10.2.3, we know that

$$y_m = \frac{N_m(y_1, y_2)}{y_1^{d_1(m)} y_2^{d_2(m)}}$$

Where $N_m(y_1, y_2) \in \mathbb{ZP}[y_1, y_2]$ not divisible by y_1 or y_2

10.7 Cluster algebra of finite type

Definition 10.7.1. Seeds $\Sigma(t), \Sigma(t')$ are **equivalent** if t, t' are $\mathcal{M}$-equivalent, i.e. there is a permutation σ of I such that $x_{\sigma(i)(t)} = x_i(t')$, $b_{\sigma(i)\sigma(j)}(t') = b_{ij}(t)$, $p_{\sigma(j)}(t) = p_j(t')$. For geometric type, $c_{i\sigma(j)}(t) = c_{ij}(t')$, or rather, $\tilde{b}_{\sigma(i)\sigma(j)}(t') = \tilde{b}_{ij}(t)$ since σ as only a permutation of I fixes I'. By Proposition 10.5.3, if t, t' are equivalent, and $t \xrightarrow{\sigma(j)} t_1$ and $t' \xrightarrow{j} t'_1$, then t_1, t'_1 are equivalent
Cluster algebras $\mathcal{A}(\mathcal{S})$, $\mathcal{A}'(\mathcal{S}')$ are strongly isomorphic if there is a field isomorphism $\mathcal{F} \to \mathcal{F}'$ that sends seeds in $\mathcal{S}$ to seeds $\mathcal{S}'$, thus inducing bijection $\mathcal{S} \to \mathcal{S}'$ and an isomorphism $\mathcal{A} \to \mathcal{A}'$
$\mathcal{A}(B, -)$ are all the possible normalized cluster algebras. $\mathcal{A}(B), \mathcal{A}(B')$ are strongly isomorphic if there is a one-to-one correspondence between $\mathcal{A}(B, \mathbf{p})$ and $\mathcal{A}(B', \mathbf{p}')$, this is true iff B, B' are mutationally equivalent modulo relabelling rows and columns
$\mathcal{A}$ is of **finite type** if it has finitely many seeds up to equivalences

Definition 10.7.2. The **Cartan counterpart** of B is the **generalized Cartan matrix** $A(B) = (a_{ij})$, $a_{ii} = 2$, $a_{ij} = -|b_{ij}|$ for $i \neq j$, with the same symmetrizing matrix D, i.e. $d_i a_{ij} = d_j a_{ji}$

A Cartan matrix of finite type, A(B)=A, bijbik>0, p normalized, then cluster algebra is of finite type

Theorem 10.7.3. A is a Cartan matrix of finite type, there is a sign-skew symmetric $B_\circ$ such that $A(B_\circ) = A$ and $b_{ij}b_{ik} > 0$ for all i, j, k, and $\mathbf{p}_\circ$ is normalized, then $\mathcal{A}(B_\circ, \mathbf{p}_\circ)$ is of finite type. Any cluster algebra of finite type is strongly isomorphic to one such data

Remark 10.7.4. Since the Coxeter graph of A is a tree which is bipartite, thus we can certainly divide them into sinks and sources. Since $b_{ij} > 0$ would be there is a directed edge from i to j, thus we can always find such a $B_\circ$

Theorem 10.7.5. B, B' sign-skew symmetric, $\mathcal{A}(B), \mathcal{A}(B')$ iff $A(B), A(B')$ are of the same Cartan-Killing type

Theorem 10.7.6. $\mathcal{A}$ is a cluster algebra, the following are equivalent

(i) $\mathcal{A}$ is of finite type

(ii) $|\mathcal{X}| < \infty$

(iii) For every seed $(\mathbf{x}, \mathbf{p}, B)$, $|b_{xy}b_{yx}| < 3$ for $x, y \in \mathbf{x}$

(iv) $\mathcal{A} = \mathcal{A}(B_\circ, p_\circ)$ as in Theorem 10.7.3

Theorem 10.7.7. $\mathcal{A}(B)$ consists of cluster algebras all simultaneously of finite type or of infinite type. There is a bijective correspondence between generalized Cartan matrices of finite type and strong isomorphic classes of normalized cluster algebras, through $B \to A(B)$

Bijection between almost positive roots and X

Theorem 10.7.8. There is a unique bijection $\Phi_{\geq -1} \to \mathcal{X}$, $\alpha \mapsto x[\alpha] = \dfrac{P_\alpha(\mathbf{x}_\circ)}{\mathbf{x}^\alpha}$, $P_\alpha \in \mathbb{Z}_{\geq 0}\mathcal{P}$ with nonzero constant term such that $X[-\alpha_i] = x_i$

Theorem 10.7.9. Every seed $(\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{A}$ is uniquely determined by the cluster $\mathbf{x}$, and for any $x \in \mathbf{x}$, there is a unique cluster $\mathbf{x}'$ such that $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{x\}$. The the cluster complex $\Delta(\mathcal{A})$ encodes the combinatorics of seed mutations

Theorem 10.7.10. The bijection in Theorem 10.7.8 identifies $\Delta(\mathcal{A})$ and $\Delta(\Phi)$, in particular, the cluster complex doesn't depend on $\mathbb{P}$ nor $\mathbf{p}_\circ$

Chapter 11

Polylogarithm

11.1 Polylogarithm

Definition 11.1.1. *Iterated integral* is defined inductively by

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b f_1(\tau)d\tau \cdots f_{r-1}(\tau)\left(\int_a^\tau f_r(t)dt\right)d\tau$$

If $\alpha : I \to M$ is a curve, $\alpha^*\omega_i = f_i(t)dt$, then

$$\int_\alpha \omega_1 \cdots \omega_r = \int_0^1 f_1(t)dt \cdots f_r(t)dt$$

is well defined, independent of the parametrization. Set the integral to be 1 if $r = 0$

Proposition 11.1.2.

1. $\displaystyle\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_j \int_\beta \omega_1 \cdots \omega_j \int_\alpha \omega_{j+1} \cdots \omega_r$

2. $\displaystyle\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1$

3. $\displaystyle\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\alpha \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}$, here σ runs over (r, s) shuffles

Lemma 11.1.3. $\omega_i^{(j)}$, $1 \le i \le r$, $1 \le j \le n$ are closed one forms such that $\displaystyle\sum_j \omega_{i-1}^{(j)} \wedge \omega_i^{(j)} = 0$ for $2 \le i \le r$, then $\displaystyle\int_\alpha \sum_j \omega_1^{(j)} \cdots \omega_r^{(j)}$ only depends on the homotopy class of α

Definition 11.1.4. The *Polylogarithms* are

$$\mathrm{Li}_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}$$

Note that

$$\mathrm{Li}_{n+1}(z) = \int_0^z \frac{\mathrm{Li}_n(t)}{t}dt, \quad \mathrm{Li}_1(z) = -\ln(1-z)$$

Hence

$$\mathrm{Li}_n(z) = \int_0^z \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{1-t} = \int_0^1 \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{z^{-1}-t}$$

Dilogarithm $\displaystyle\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u}du$ is the analytic continuation on $\mathbb{C} \setminus \{0, 1\}$, avoiding the the cut $[1, \infty]$

Definition 11.1.5. The *Bloch-Wigner function* is $D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\ln|z|$, $z \in \mathbb{C}\setminus\{0,1\}$

Definition 11.1.6. The *multiple polylogarithms* are

$$\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) = \sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \int_0^1 \left(\frac{dt}{t}\right)^{n_1-1} \frac{dt}{a_1 - t} \cdots \left(\frac{dt}{t}\right)^{n_d-1} \frac{dt}{a_d - t}$$

Here $\mathbf{k}$ runs over $k_1 > \cdots > k_d \geq 1$, $a_j = a_j(\mathbf{z}) = (z_1 \cdots z_j)^{-1}$, $a_0 = 1$, $a_{n+1} = 0$

Note. For $\mathbf{k}$ runs over $(k_1, \cdots, k_d) \in \mathbb{Z}_{\geq 1}^d$

$$\sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \left(\sum_{k_1} \frac{z_1^{k_1}}{k_1^{n_1}}\right) \cdots \left(\sum_{k_d} \frac{z_d^{k_d}}{k_d^{n_d}}\right) = \mathrm{Li}_{n_1}(z_1) \cdots \mathrm{Li}_{n_d}(z_d)$$

Total differential on L_n

Lemma 11.1.7. Write $\mathfrak{S}_n = \{0,1\}^n$, $\mathbf{0} = (0, \cdots, 0)$, $\mathbf{1} = (1, \cdots, 1)$, define *multiple logarithm* $\mathcal{L}_n = \mathrm{Li}_{\mathbf{1}}$, $\mathcal{L}_0 = 1$, then

$$\begin{aligned} d_j \mathcal{L}_n(\mathbf{x}) &= \sum_{k_1 > \cdots \widehat{k_j} > \cdots k_n} \frac{x_1^{k_1} \cdots \widehat{x_j^{k_j}} \cdots x_n^{k_n}}{k_1 \cdots \widehat{k_j} \cdots k_n} \sum_{k_j = k_{j+1}+1}^{k_{j-1}-1} x_j^{k_j - 1} dx_j \\ &= \sum_{k_1 > \cdots \widehat{k_j} > \cdots k_n} \frac{x_1^{k_1} \cdots \widehat{x_j^{k_j}} \cdots x_n^{k_n}}{k_1 \cdots \widehat{k_j} \cdots k_n} \frac{x_j^{k_{j-1}-1} - x_j^{k_{j+1}}}{x_j - 1} dx_j \end{aligned}$$

Denote $\mathbf{x}_j = (x_1, \cdots, x_j x_{j+1}, x_{j+2}, \cdots, x_n)$, $\mathbf{x}_n = (x_1, \cdots, x_{n-1})$, we have

$$d_j \mathcal{L}_n(\mathbf{x}) = \mathcal{L}_{n-1}(\mathbf{x}_{j-1}) \frac{dx_j}{x_j(x_j - 1)} - \mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_j}{x_j - 1}$$

For $2 \leq j \leq n$, and

$$d_1 \mathcal{L}_n(\mathbf{x}) = -\mathcal{L}_{n-1}(\mathbf{x}_1) \frac{dx_1}{x_1 - 1}$$

Therefore

$$\begin{aligned} d\mathcal{L}_n(\mathbf{x}) &= \sum_{j=1}^n d_j \mathcal{L}_n(\mathbf{x}) \\ &= \sum_{j=1}^{n-1} \left(\mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_j}{1 - x_j} + \mathcal{L}_{n-1}(\mathbf{x}_j) \frac{dx_{j+1}}{x_{j+1}(x_{j+1} - 1)} \right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{x_n - 1} \\ &= \sum_{j=1}^{n-1} \mathcal{L}_{n-1}(\mathbf{x}_j) \left(-d\ln(1 - x_j) + d\ln\left(\frac{x_{j+1} - 1}{x_{j+1}}\right)\right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{x_n - 1} \\ &= \sum_{j=1}^n \mathcal{L}_{n-1}(\mathbf{x}_j) d\ln\left(\frac{1 - x_{j+1}^{-1}}{1 - x_j}\right) \end{aligned}$$

Here $x_{n+1} = \infty$, $\mathcal{L}_0 = 1$

Suppose $\mathbf{i} \in \mathfrak{S}_n$, $|\mathbf{i}| = k$ and $i_{\tau_1} = \cdots = i_{\tau_k} = 1$ for some $1 \leq \tau_1 \leq \cdots \leq \tau_k \leq n$, set

$$\mathbf{x}(\mathbf{i}) = \mathbf{y}, \quad y_m = \prod_{j = \tau_{m-1}+1}^{\tau_m} x_j = \frac{a_{\tau_{m-1}}}{a_{\tau_m}}$$

$$w_j(\mathbf{x}) = d\ln\left(\frac{1 - x_{j+1}^{-1}}{1 - x_j}\right)$$

With $\tau_0 = 0$, $w_0(\mathbf{x}) = 1$. A partial order $\preceq$ on $\mathfrak{S}_n$ is given by $\mathbf{i} \preceq \mathbf{j}$ if $i_k \leq j_k$

Theorem 11.1.8. Multiple logarithm $\mathcal{L}_n(\mathbf{x})$ is a multi-valued holomorphic function on

$$\mathbb{C}^n \setminus \left\{ \prod_{j\le k}(1 - x_j \cdots x_k) = 0 \right\}$$

And

$$\mathcal{L}_n(\mathbf{x}) = \sum_{0\neq \mathbf{j}_1 \prec \cdots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2-\mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1))$$

Here $\mathbf{j} - \mathbf{i} = \begin{cases} s' & j_t = i_t + \delta_{st} \\ 0 & \text{otherwise} \end{cases}$ for $\mathbf{i} \prec \mathbf{j}$, j_s is the s'-th nonzero element in $\mathbf{j}$, and the integration is taken over $\alpha : I \to \mathbb{C}^n$

Proof. Use induction and Lemma 11.1.7

$$\begin{aligned}
\mathcal{L}_n(\mathbf{x}) &= \int_{\mathbf{0}}^{\mathbf{x}} d\mathcal{L}_n(\mathbf{x}) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{k=1}^{n-1} \mathcal{L}_{n-1}(\mathrm{x}_k) d\ln\left(\frac{1 - x_{k+1}^{-1}}{1 - x_k}\right) + \mathcal{L}_{n-1}(\mathrm{x}_n)\frac{dx_n}{1 - x_n} \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{j=1}^{n} w_{\mathbf{1}-\mathbf{f}_k}(\mathbf{x}) \sum_{0\neq \mathbf{p}_1 \prec \cdots \prec \mathbf{p}_{n-1}} w_{\mathbf{p}_{n-1}-\mathbf{p}_{n-2}}(\mathrm{x}_k(\mathbf{p}_{n-1})) \cdots w_{\mathbf{p}_2-\mathbf{p}_1}(\mathrm{x}_k(\mathbf{p}_2)) w_1(\mathrm{x}_k(\mathbf{p}_1)) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{j=1}^{n} w_{\mathbf{1}-\mathbf{f}_k}(\mathbf{x}) \sum_{0\neq \mathbf{q}_1 \prec \cdots \prec \mathbf{q}_{n-1}} w_{\mathbf{q}_{n-1}-\mathbf{q}_{n-2}}(\mathbf{x}(\mathbf{q}_{n-1})) \cdots w_{\mathbf{q}_2-\mathbf{q}_1}(\mathbf{x}(\mathbf{q}_2)) w_1(\mathbf{x}(\mathbf{q}_1)) \\
&= \sum_{0\neq \mathbf{j}_1 \prec \cdots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2-\mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1))
\end{aligned}$$

Here $\mathbf{f}_k = (1, \cdots, \underset{k\text{-th}}{\underset{\uparrow}{0}}, \cdots, 1)$, $\mathbf{q}_i$ is $\mathbf{p}_i$ with 0 inserted in as the k-th entry. Note that the domain is given so that $w_i(\mathbf{x}(\mathbf{j}))$ are defined ☐

Example 11.1.9. When $n = 1$

$$\begin{aligned}
\mathcal{L}_1(x_1) &= \int_0^{x_1} w_1(\mathbf{x}(1)) \\
&= \int_0^{x_1} d\ln\left(\frac{1}{1 - x_1}\right) \\
&= \int_0^{x_1} \frac{dx_1}{1 - x_1}
\end{aligned}$$

When $n = 2$

$$\begin{aligned}
\mathcal{L}_2(\mathbf{x}) &= \int_{\mathbf{0}}^{\mathbf{x}} w_{(1,1)-(1,0)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(1, 0)) + w_{(1,1)-(0,1)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(0, 1)) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} w_2(\mathbf{x}) w_1(x_1) + w_1(\mathbf{x}) w_1(x_1x_2) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} d\ln\left(\frac{1}{1 - x_2}\right) d\ln\left(\frac{1}{1 - x_1}\right) + d\ln\left(\frac{1 - x_2^{-1}}{1 - x_1}\right) d\ln\left(\frac{1}{1 - x_1x_2}\right) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \frac{dx_2}{1 - x_2}\frac{dx_1}{1 - x_1} + \left(\frac{dx_2}{x_2(x_2 - 1)} + \frac{dx_1}{1 - x_1}\right)\frac{d(x_1x_2)}{1 - x_1x_2}
\end{aligned}$$

When $n = 3$

$$\begin{aligned}
\mathcal{L}_3(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} & w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(1,0,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(1,0,0))+ \\
& w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(0,1,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(0,1,0))+ \\
& w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(1,0,0)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(1,0,0))+ \\
& w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(0,0,1)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(0,0,1))+ \\
& w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,1,0)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,1,0))+ \\
& w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,0,1)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,0,1)) \\
= \int_{\mathbf{0}}^{\mathbf{x}} & w_3(\mathbf{x})w_2(x_1,x_2)w_1(x_1) + w_3(\mathbf{x})w_1(x_1,x_2)w_1(x_1x_2)+ \\
& w_2(\mathbf{x})w_2(x_1,x_2x_3)w_1(x_1) + w_2(\mathbf{x})w_1(x_1,x_2x_3)w_1(x_1x_2x_3)+ \\
& w_1(\mathbf{x})w_2(x_1x_2,x_3)w_1(x_1x_2) + w_1(\mathbf{x})w_1(x_1x_2,x_3)w_1(x_1x_2x_3) \\
= \int_{\mathbf{0}}^{\mathbf{x}} & \frac{dx_3}{1-x_3}\frac{dx_2}{1-x_2}\frac{dx_1}{1-x_1} + \frac{dx_3}{1-x_3}\left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\frac{d(x_1x_2)}{1-x_1x_2}+ \\
& \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2}\right)\frac{d(x_2x_3)}{1-x_2x_3}\frac{dx_1}{1-x_1}+ \\
& \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2}\right)\left(\frac{d(x_2x_3)}{x_2x_3(x_2x_3-1)} + \frac{dx_1}{1-x_1}\right)\frac{d(x_1x_2x_3)}{1-x_1x_2x_3}+ \\
& \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\frac{dx_3}{1-x_3}\frac{d(x_1x_2)}{1-x_1x_2}+ \\
& \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\left(\frac{dx_3}{x_3(x_3-1)} + \frac{d(x_1x_2)}{1-x_1x_2}\right)\frac{d(x_1x_2x_3)}{1-x_1x_2x_3}
\end{aligned}$$

Definition 11.1.10. $\mathbf{i}, \mathbf{j} \in \mathfrak{S}_n$, $|\mathbf{i}| = k$, $|\mathbf{j}| = l$, the $\mathbf{i}$-th *retraction* map $\rho_{\mathbf{i}} : \mathfrak{S}_n \to \mathfrak{S}_k$ is defined by

- If $\mathbf{i} \not\succeq \mathbf{j}$, $\rho_{\mathbf{i}}(\mathbf{j}) = \mathbf{0}$
- If $\mathbf{i} \succeq \mathbf{j}$, assume $\tau_1, \cdots, \tau_k$ and $t_1, \cdots, t_l$ are the nonzero entries in $\mathbf{i}$ and $\mathbf{j}$, suppose $\tau_{\alpha_r} = t_r$, then $\alpha_1, \cdots, \alpha_l$ are the nonzero entries of $\rho_{\mathbf{i}}(\mathbf{j})$

Write $\theta_s = \theta_s(\mathbf{x}) = \dfrac{dt}{t - a_s}$, the $2^n \times 2^n$ *variation matrix* $\mathcal{M}_{[n]}(\mathbf{x}) = (2\pi i)^l E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ associated with $\mathcal{L}_n(\mathbf{x})$ is defined by

$$\begin{aligned}
E_{\mathbf{i},\mathbf{j}} = \gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{y}) &= (-1)^{k-l}\prod_{r=0}^{l}\int_{a_{\alpha_{r+1}}(\mathbf{y})}^{a_{\alpha_r}(\mathbf{y})} \theta_{\alpha_r+1}(\mathbf{y})\cdots\theta_{\alpha_{r+1}-1}(\mathbf{y}) \\
&= (-1)^{k-l}\prod_{r=0}^{l}\int_{a_{t_{r+1}}}^{a_{t_r}} \theta_{\tau_{\alpha_r+1}}(\mathbf{x})\cdots\theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \\
&= (-1)^{k-l}\prod_{r=0}^{l}\int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x})\cdots\theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x})
\end{aligned}$$

$\tau_{k+1} = t_{l+1} = n + 1$, $\alpha_{l+1} = k + 1$. p_r are independent from $\mathbf{i}$ These thetas are very weird

Proposition 11.1.11.

$$
\begin{aligned}
E_{\mathbf{i},\mathbf{j}} &= \prod_{r=0}^{l} \mathcal{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{a_{t_r}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})},\cdots,\frac{a_{\tau_{\alpha_{r+1}-2}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}\right)\\
&= \mathcal{L}_{k-\alpha_l}(x_{1+t_l}\cdots x_{\tau_{\alpha_l+1}},\cdots,x_{1+\tau_{k-1}}\cdots x_{\tau_k})\cdot\\
&\quad \prod_{r=0}^{l-1} \mathcal{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{1+\tau_{\alpha_r+1}}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{1+\tau_{\alpha_{r+1}-2}}\cdots x_{t_{r+1}}}{1-x_{1+\tau_{\alpha_{r+1}-1}}\cdots x_{t_{r+1}}}\right)
\end{aligned}
$$

Proof. $\square$

Example 11.1.12.

$$
\begin{aligned}
E_{\mathbf{1},\mathbf{j}}(\mathbf{x}) = \gamma_{\mathbf{j}}^{n}(\mathbf{x}) &= \prod_{r=0}^{l} \mathcal{L}_{t_{r+1}-t_r-1}\left(\frac{a_{t_r}-a_{t_{r+1}}}{a_{t_r+1}-a_{t_{r+1}}},\cdots,\frac{a_{t_{r+1}-2}-a_{t_{r+1}}}{a_{t_{r+1}-1}-a_{t_{r+1}}}\right)\\
&= \prod_{r=0}^{l} \mathcal{L}_{t_{r+1}-t_r-1}\left(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{2+t_r}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{t_{r+1}-1}x_{t_{r+1}}}{1-x_{t_{r+1}}}\right)
\end{aligned}
$$

In particular we have

$$
\begin{aligned}
E_{\mathbf{1},\mathbf{0}}(\mathbf{x}) &= \gamma_{\mathbf{0}}^{n}(\mathbf{x})\\
&= \mathcal{L}_n\left(\frac{1-x_1\cdots x_{n+1}}{1-x_2\cdots x_{n+1}},\cdots,\frac{1-x_nx_{n+1}}{1-x_{n+1}}\right)\\
&= \mathcal{L}_n\left(\frac{x_1\cdots x_n-\frac{1}{x_{n+1}}}{x_2\cdots x_n-\frac{1}{x_{n+1}}},\cdots,\frac{x_n-\frac{1}{x_{n+1}}}{1-\frac{1}{x_{n+1}}}\right)\\
&= \mathcal{L}_n(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
E_{\mathbf{1},\mathbf{1}}(\mathbf{x}) &= \gamma_{\mathbf{1}}^{n}(\mathbf{x})\\
&= \prod_{r=0}^{n} \mathcal{L}_0\\
&= 1
\end{aligned}
$$

Define matrix $L_n = L_n(\mathbf{x})$ with columns

$$
C_{\mathbf{j}} = \sum_{\mathbf{i}\succeq\mathbf{j}} \gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i}))e_{\mathbf{i}}
$$

$e_{\mathbf{i}}$ is the standard unit column vector, using the complete order $<$ on $\mathfrak{S}_n$: if $|\mathbf{i}| < |\mathbf{j}|$, then $\mathbf{i} < \mathbf{j}$, if $|\mathbf{i}| = |\mathbf{j}|$, then compare the lexicographic order from left to right with $1 < 0$. By definition, the $\mathbf{i}$-th row of L_n is

$$
R_{\mathbf{i}} = \sum_{\mathbf{i}\succeq\mathbf{j}} (2\pi i)^{|\mathbf{j}|}\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i}))e_{\mathbf{j}}^{T}
$$

Note that $\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{i})}(\mathbf{x}(\mathbf{i})) = 1$, and the first entry of $R_{\mathbf{i}}$ is $\mathcal{L}_{|\mathbf{i}|}(\mathbf{x}(\mathbf{i}))$

Example 11.1.13. The variation matrix associated with double logarithm is

$$
\mathcal{M}_{1,1} = [1]
$$

Variation matrix is lower triangular

Lemma 11.1.14. The variation matrix is lower triangular

Proof. $\square$

Part III

Commutative Algebra

Chapter 12

p-adic numbers

Definition 12.0.1. The **p-adic intergers** are

$$\begin{aligned}\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z} &= \left\{(a_0, a_1, a_2, \cdots) \in \prod \mathbb{Z}/p^n\mathbb{Z} \middle| a_n \equiv a_m \bmod p^m, n \geq m\right\} \\ &= \left\{b_0 + b_1 p + b_2 p^2 + \cdots\right\}\end{aligned}$$

Here $a_n = b_0 + b_1 p + \cdots + b_n p^n$

Example 12.0.2. If $p = 7$, we can write $3 + 6 \cdot 7 + 7^2 + 4 \cdot 7^3 + 2 \cdot 7^4 + \cdots$ as

$$\cdots 24163$$

With base 7

Part IV

Homological Algebra

Chapter 13

Abelian category

Definition 13.0.1. Let $\mathscr{C}$ be a locally small category, we say $\mathscr{C}$ is a **preadditive category** if $Hom_{\mathscr{C}}(X, Y)$ are abelian groups, and the addition distributes over composition, i.e. $f \circ (g+h) = f \circ g + f \circ h$, $(f+g) \circ h = f \circ h + g \circ h$
Note that the 0 in the abelian group $Hom_{\mathscr{C}}(X, Y)$ is a zeor morphism
A preadditive category is called an **additive category**if any finite set has a biproduct, in particular, it has a zero object, the empty biproduct
An additive category is called a **preabelian category** if every morphism has a kernel and a cokernel, where kernels and cokernels means the equalisers and coequalisers of the morphism $f : X \to Y$ and the zero morphism $0 : X \to Y$
A preabelian category is called an **abelian category** if every monomorphisms is normal and every epimorphisms is conormal, a morphism is **normal** if it is a kernel, **conormal** if it is a cokernel and **binormal** if it is both a kernel and a cokernel

Definition 13.0.2. For a morphism $A \xrightarrow{f} B$, define its image $\mathrm{im} f$ by the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[d, dashed, "\exists_1"'] \arrow[dr, "f"] & & & \\
0 \arrow[r] & \mathrm{im} f \arrow[r, hook] & B \arrow[r, two heads] & \mathrm{coker} f \arrow[r] & 0
\end{tikzcd}$$

The image satisfies universal property

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] \arrow[ddr, bend right] & & B \\
 & \mathrm{im} f \arrow[ur, hook] \arrow[d, dashed, "\exists_1"] & \\
 & I \arrow[uur, hook, bend right] &
\end{tikzcd}$$

Example 13.0.3. A ring R can be thought of as a preadditive category with a single object and morphisms $r \in R$ The category of left R modules can be thought of as the functor category $[R, Ab]$, where Ab the category of abelian groups

Proposition 13.0.4. In an abelian category $\mathscr{A}$, the equaliser of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the isomorphic to the kernel of $f - g$

Definition 13.0.5. Let $F : \mathscr{C} \to \mathscr{D}$ be a funtor of preadditive cateogories, if F is an abelian group homomorphism on $Hom(F(X), F(Y))$ for any X, Y, we say F is an **additive funtor**

Definition 13.0.6. Let $\mathscr{A}$ be an abelian category, a ($\mathbb{Z}$-graded) **chain complex** $C_\bullet$ is

$$\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0$, ∂_i are called **boundary maps(differentials)**
We can define chain maps, chain homotopy, boundaries, cycles, and homology groups, and we say the chain complex is exact if each homology groups is zero, the chain complexes form the **category of chain complexes** $Ch\mathscr{A}$
The **homotopy category of chain complexes** often denoted as $K(\mathscr{A})$ is the quotient category with chain maps modulo chain homotopy equivalence as morphisms
a chain map is called a **quasi-isomorphism** if it induces isomorphisms on homology groups

Lemma 13.0.7. An alternative definition of an exact functor F could be that F preserve exactness, i.e. $F(A) \to F(B) \to F(C)$ is exact for any short exact sequence $A \to B \to C$

Definition 13.0.8. The **direct sum** $(C \oplus D)_\bullet$ of chain complexes $C_\bullet, D_\bullet$ is

$$\cdots \to C_1 \oplus D_1 \xrightarrow{\partial_1^C \oplus \partial_1^D} C_0 \oplus D_0 \xrightarrow{\partial_0^C \oplus \partial_0^D} C_{-1} \oplus D_{-1} \to \cdots$$

Definition 13.0.9. A **double complex** $C_{*,*}$ is $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ two differentials $\partial' : C_{p,q} \to C_{p-1,q}$, $\partial'' : C_{p,q} \to C_{p,q-1}$ such that $(\partial')^2 = (\partial'')^2 = 0$ and $\partial'\partial'' + \partial''\partial' = 0$($\partial', \partial''$ anticommutes)
The **total chain complexes** are $(Tot^{\oplus})_n = \bigoplus\limits_{p+q=n} C_{p,q}$ and $(Tot^{\Pi})_n = \prod\limits_{p+q=n} C_{p,q}$ with $\partial = \partial' + \partial''$

Example 13.0.10. $C_* \otimes D_*$ is the total complex of double complex $C_{p,q} := C_p \otimes D_q$, $\partial' := \partial^C \otimes 1$, $\partial'' := (-1)^p 1 \otimes \partial^D$

Definition 13.0.11. A **filtered chain complex** is a filtered object in $Ch\mathscr{A}$

$$\cdots \to F_{p+1}C_\bullet \to F_pC_\bullet \to \cdots \to C_\bullet$$

Snake lemma

Lemma 13.0.12 (Snake lemma)**.** Given the following commutative diagram with exact rows, then we have an exact sequence

$$\begin{tikzcd}
& \ker a \arrow[r, "u_*"] \arrow[d] & \ker b \arrow[r, "v_*"] \arrow[d] & \ker c \arrow[d] \arrow[dddll, "\delta"] & \\
D \arrow[ur, "w_*"] \arrow[r] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r] \arrow[d] & D' \\
& \operatorname{coker} a \arrow[r, "u'_*"] & \operatorname{coker} b \arrow[r, "v'_*"] & \operatorname{coker} c \arrow[ur] &
\end{tikzcd}$$

Five lemma

Lemma 13.0.13 (Five lemma)**.** If b and d are monic and a is an epi, then c is monic. Dually, if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are iso, then c is also an iso

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d, "a"', "\cong"] & B' \arrow[r, "v'"] \arrow[d, "b"', "\cong"] & C' \arrow[r, "w'"] \arrow[d, "c"'] & D' \arrow[r, "x'"] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & D \arrow[r, "x"] & E
\end{tikzcd}$$

Horseshoe lemma

Lemma 13.0.14 (Horseshoe lemma)**.** Suppose $P_\bullet \xrightarrow{\varepsilon} M$, $Q_\bullet \xrightarrow{\eta} N$ are projective resolutions, then any exact sequence $0 \to M \xrightarrow{f} A \xrightarrow{g} N \to 0$ can be extended into commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

With $(P \oplus Q)_\bullet$ being a projective resolution, every row and column are exact

Proof. Since $A \xrightarrow{g} N$ is epi and Q_0 is projective, we get $Q_0 \xrightarrow{s_0} A$ such that $gs_0 = \partial_0$ which gives us $P_0 \oplus Q_0 \xrightarrow{\begin{pmatrix} f\partial_0 & s_0 \end{pmatrix}} A$, by Lemma 13.0.12, this is epi, and we get an exact sequence $0 \to Z_0P \to \ker i_0 \to Z_0Q \to 0$, similarly, we can construct $Q_1 \xrightarrow{s_1} \ker i_0$, then $P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} \iota_0\partial_0 & s_1 \end{pmatrix}} \ker i_0$ is again epi by Lemma 13.0.12, inductively, we can construct the commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\partial_1"'] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d, "\partial_1"] \arrow[dl, "s_1"] & 0 \\
0 \arrow[r] & Z_0P \arrow[r, "\iota_0"] \arrow[d] & \ker i_0 \arrow[r] \arrow[d] & Z_0Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\partial_0"'] & P_0 \oplus Q_0 \arrow[r] \arrow[d, "i_0"] & Q_0 \arrow[r] \arrow[d, "\partial_0"] \arrow[dl, "s_0"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\square$

Lemma for universal coefficient theorem for cohomology

Lemma 13.0.15. If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence and there is a homomorphism(retraction) $C \xrightarrow{r} B$ such that $rg = 1_B$, then there is an exact sequence $0 \to \mathrm{coker} f \to \mathrm{coker}(gf) \to \mathrm{coker} g \to 0$

Proof. First observe that we have $0 \to \mathrm{im} g/\mathrm{im}(gf) \to C/\mathrm{im}(gf) \to C/\mathrm{im} g \to 0$, $B \to \mathrm{im} g$, $\mathrm{im} f \to \mathrm{im}(gf)$, thus $B/\mathrm{im} f \to \mathrm{im} g/\mathrm{im}(gf)$, since $rg = 1_B$, $B/\mathrm{im} f \cong \mathrm{im} g/\mathrm{im}(gf)$, therefore, $0 \to B/\mathrm{im} f \to C/\mathrm{im}(gf) \to C/\mathrm{im} g \to 0$ $\square$

Lemma 13.0.16. Suppose $\mathscr{A}$ is abelian category, then $\mathrm{im} f = \ker \mathrm{coker} f = \mathrm{coker} \ker f$

Definition 13.0.17. Pick $p \in \mathbb{Z}$, define the **translation** of X by p is $X_\bullet[p]$ where $(X_\bullet[p])_n = X_{p+n}$, differential $X_\bullet[p]_n \to X_\bullet[p]_{n-1}$ is given by $(-1)^p\partial$ The **translation functor** $T : Ch(\mathscr{A}) \to Ch(\mathscr{A})$, $X \mapsto X_\bullet[1]$ is an auto morphism of $Ch(\mathscr{A})$

Acyclic model theorem

Theorem 13.0.18 (Acyclic model theorem)**.** [1] **Model** $\mathcal{M} = \{M_j\}$ is a subclass(possibly with repetition) of objects in $\mathscr{C}$, $F, G : \mathscr{C} \to Ch_{\geq 0}$ are functors, $H_n(G(M_j)) = 0$ for any $n \neq 0$, $M_j \in \mathcal{M}$. For any C, there exist $m_j \in F_kM_j$ such that $F_k(C)$ is free with basis $\left\{ F_k(\sigma)(m_j) \middle| M_j \xrightarrow{\sigma} C \right\}$

[1] Consult Theorem 9.12 of [1] or `https://amathew.wordpress.com/2010/09/11/the-method-of-acyclic-models/`

Theorem 13.0.19 (Universal coefficient theorem for cohomology)**.** There is an exact sequence

$$0 \to Ext^1(H_{n-1}, A) \to H^n(C; A) \to Hom(H_n, A) \to 0$$

Proof. Since C_n is a free group, so are subgroups B_n, Z_n, exact sequence

$$0 \to Z_n \hookrightarrow C_n \to B_{n-1} \to 0$$

Splits, i.e. we have a splitting homomorphism $B_{n-1} \xrightarrow{s} C_n$, $C_n \cong Z_n \oplus B_{n-1}$, thus exact sequence

$$0 \to H_n = Z_n/B_n \to C_n/B_n \to C_n/Z_n \cong B_{n-1} \to 0$$

Induces exact sequence

$$Hom(B_{n-1}, A) \to Hom(C_n/B_n, A) \to Hom(H_n, A) \to Ext^1(B_{n-1}, A) = 0$$

$Hom(H_n, A)$ is the cokernel of $Hom(B_{n-1}, A) \to Hom(C_n/B_n, A)$
Note that

$$H^n(C; A) = Z^n(C; A)/B^n(C; A) = \frac{\ker(Hom(C_n, A) \to Hom(C_{n+1}, A))}{\operatorname{im}(Hom(C_{n-1}, A) \to Hom(C_n, A))}$$

$C_n \xrightarrow{\phi} A \in Z^n(C; A) \Leftrightarrow \phi\partial = 0 \Leftrightarrow \phi \in Hom(C_n/B_n, A)$, thus $Hom(C_n/B_n, A) \cong Z^n(C; A)$

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial"] \arrow[dr] & C_n \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

$C_n \xrightarrow{\psi} A \in B^n(C; A) \Leftrightarrow \psi = \phi\partial$ for some $C_{n-1} \xrightarrow{\phi} A \Leftrightarrow \psi = \phi\partial$ for some $Z_{n-1} \xrightarrow{\phi} A$, and since $B^n(C; A) \subseteq Z^n(C; A) \cong Hom(C_n/B_n, A)$, we have $B^n(C; A) \cong \operatorname{im}(Hom(Z_{n-1}, A) \to Hom(C_n/B_n, A))$

$$\begin{tikzcd} C_n \arrow[r, "\partial"] \arrow[dr, "\psi"'] & C_{n-1} \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

Exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to B_{n-1}/Z_{n-1} = H_{n-1} \to 0$$

Induces exact sequence

$$Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A) \to Ext^1(H_{n-1}, A) \to Ext^1(Z_{n-1}, A) = 0$$

$Ext^1(H_{n-1}, A)$ is the cokernel of $Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A)$
Since composition $B_{n-1} \xrightarrow{s} C_n \to C_n/B_n \xrightarrow{\partial} B_{n-1}$ is identity, we have a homomorphism $r : Hom(C_n/B_n, A) \to Hom(B_{n-1}, A)$ induced by $B_{n-1} \to C_n/B_n$ such that composition $Hom(B_{n-1}, A) \to Hom(C_n/B_n, A) \xrightarrow{r} Hom(B_{n-1}, A)$ is identity
Apply Lemma 13.0.15 to $Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A) \to Hom(C_n/B_n, A)$, we get an exact sequence

$$0 \to Ext^1(H_{n-1}, A) \to H^n(C; A) \to Hom(H_n, A) \to 0$$

□

Remark 13.0.20. B_n is not necessarily a direct summand of C_n, a map $B_n \xrightarrow{\phi} A$ may not be possible to extended to $C_n \xrightarrow{\phi} A$, however a map $Z_n \xrightarrow{\phi} A$ can always be extended to $C_n \xrightarrow{\phi} A$

Theorem 13.0.21 (Algebraic Künneth formula)**.** C, D are free chain complexes, then

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor_1(H_p(C), H_q(D)) \to 0$$

Is exact

Proof. If D has trivial differentials, then $H_q(D) = D_q$ is free, hence

$$H_n(C \otimes D) = \bigoplus_{p+q=n} H_p(C \otimes D_q) = \bigoplus_{p+q=n} H_p(C) \otimes D_q = \bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$$

In general, consider exact sequence $0 \to Z \xrightarrow{i} D \xrightarrow{\partial} B[-1] \to 0$, $0 \to B \xrightarrow{i} Z \to H(D) \to 0$, then $0 \to C \otimes Z \to C \otimes D \to C \otimes B[-1] \to 0$ is exact since C_k are free, this gives us long exact sequence

$$\cdots \to H_n(C \otimes Z) \xrightarrow{1\otimes i} H_n(C \otimes D) \xrightarrow{1\otimes\partial} H_n(C \otimes B[-1]) \xrightarrow{1\otimes i} H_{n-1}(C \otimes Z) \to \cdots$$

$Z, B[-1]$ have trivial differentials, hence the connecting homomorphism is just

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(B[-1]) = \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(B) \xrightarrow{1\otimes i} \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(Z)$$

Then we have

$$0 \to \mathsf{coker}(1 \otimes i) \to H_n(C \otimes D) \to \mathsf{ker}(1 \otimes i) \to 0$$

We also have

$$0 \to Tor_1(H_p(C), H_q(D)) \to H_p(C) \otimes B_q \xrightarrow{1\otimes i} H_p(C) \otimes Z_q \to H_p(C) \otimes H_q(D) \to 0$$

Therefore, we have exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor_1(H_p(C), H_q(D)) \to 0$$

□

Definition 13.0.22. A **composition series** of A is a sequence of subobjects

$$A = A_n \supseteq \cdots \supseteq A_1 \supseteq A_0 = 0$$

With **composition factors** H_{i+1}/H_i simple and **composition length** $\ell(A) = n$

Lemma 13.0.23. $\ell(A)$ is indepent of the composition series

Chapter 14

Spectral sequence

Definition 14.0.1. Suppose $\mathscr{A}$ is an abelian category, a **spectral sequence** consists of objects $\{E_r\}_{r\geq r_0}(r_0$ is mostly $0)$, and morphisms $d_r : E_r \to E_r$ such that $d_r \circ d_r = 0$ and $E_{r+1} \cong H(E_r) = \ker d_r/\mathrm{im} d_r$

Definition 14.0.2. Suppose $\mathscr{A}$ is an abelian category, an **exact couple** is (D, E, i, j, k)

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Such that it is exact at each term, define differential $d = jk$, then $d^2 = jkjk = j(kj)k = 0$, we can define the **derived couple**

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Where $D' = i(D)$, $E' = \ker k/\mathrm{im} j$, $i'(a) = i(a)$, $j'(i(a)) = \overline{j(a)}$, $k'(b) = \overline{k(b)}$, then the derived couple is again an exact couple, thus we can carry this process indefinitely, giving the n-th derived couple $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$

Example 14.0.3. Suppose $\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots$ is a filteration of chain complex $C_\bullet$(or **filtered chain complex**), exact sequence $0 \to F_{p-1}C_\bullet \to F_pC_\bullet \to (grC_\bullet)_p \to 0$ give a long exact sequence

$$\cdots \to H_n(F_{p-1}C_\bullet) \xrightarrow{i_*} H_n(F_pC_\bullet) \xrightarrow{j_*} H_n(F_pC_\bullet/F_{p-1}C_\bullet) \xrightarrow{k_*} H_{n-1}(F_{p-1}C_\bullet) \to \cdots$$

If we write $D^1_{p,q} := H_{p+q}(F_pC_\bullet)$, $E^1_{pq} := H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet)$, then the long exact sequence become

$$\cdots \to D^1_{p,q} \to D^1_{p+1,q-1} \to E^1_{p,q} \to D^1_{p,q-1} \to \cdots$$

Consider $D^1 = \bigoplus D^1_{p,q}$, $E^1 = \bigoplus E^1_{p,q}$, then $(D^1, E^1, i_*, j_*, k_*)$ form an exact couple, note that $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

Remark 14.0.4. $grC_\bullet = \bigoplus_p F_pC_\bullet/F_{p-1}C_\bullet$ is called the **associated graded complex**
If X is a CW complex, we can take $F_pC_\bullet = C_\bullet(X^p)$, here X^p is the p-th skeleton of X

Definition 14.0.5. A **double cochain complex** $C^{\bullet,\bullet}$ is bigraded with anticommuting differetials d_h, d_v, i.e. $(d_h)^2 = 0$, $(d_v)^2 = 0$, $d_hd_v + d_vd_h = 0$

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & C^{0,1} \arrow[u] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "d_v^{1,0}"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Define the **total cochain complex** to be $C^n = \bigoplus_{p+q=n} C^{p,q}$, with total differential $d = d_h + d_v$, this is indeed a differential since $d^2 = (d_h + d_v)^2 = (d_h)^2 + d_h d_v + d_v d_h + (d_v)^2 = 0$
We can define the **horizontal filtration** of the total cochain complex $(F_p^h C)^n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C^{k,l}$ and the **vertical filtration** of the total cochain complex $(F_q^h C)^n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C^{k,l}$

A **double chain complex** $C_{\bullet,\bullet}$ is bigraded with anticommuting differetials ∂^h, ∂^v, i.e. $(\partial^h)^2 = 0$, $(\partial^h)^2 = 0$, $\partial^h \partial^v + \partial^v \partial^h = 0$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{1,1} \arrow[d, "\partial^v_{1,1}"] \arrow[r, "\partial^h_{1,1}"] & C_{0,1} \arrow[d, "\partial^v_{0,1}"] \arrow[r] & \cdots \\
\cdots \arrow[r] & C_{1,0} \arrow[d] \arrow[r, "\partial^h_{1,0}"] & C_{0,0} \arrow[d] \arrow[r] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

Define the **total chain complex** to be $C_n = \bigoplus_{p+q=n} C_{p,q}$, with total differential $\partial = \partial_h + \partial_v$
We can define the **horizontal filtration** of the total chain complex $(F_p^h C)_n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C_{k,l}$ and the **vertical filtration** of the total chain complex $(F_q^h C)_n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C_{k,l}$

Remark 14.0.6. If d_h, d_v commutes instead of anticommuting, then $C^{\bullet,\bullet}$ can be viewed as a cochain complex of cochain complexes, the total differential becomes $d^n(c) = d_h^p c + (-1)^p d_v^q c$ for any $c \in C^{p,q}$, this is indeed a differential since

$$\begin{aligned}
d^{n+1} d^n(c) &= d^{n+1}(d_r^p c + (-1)^p d_v^q c) \\
&= d^{n+1} d_h^p c + (-1)^p d^{n+1} d_v^q c \\
&= d_h^{p+1} d_h^p c + (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c + (-1)^{2p} d_v^{q+1} d_v^q c \\
&= (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c \\
&= (-1)^p (d_h^p d_v^q - d_v^q d_h^p) c \\
&= 0
\end{aligned}$$

However, these two types of definitions are equivalent

Proposition 14.0.7. Let $E^r_{p,q}$ be the spectral sequence corresponds to the horizontal filtration
(1) $E^0_{p,q} \cong C^{p,q}$
(2) $E^1_{p,q} \cong H_q(C_{p,\bullet})$
(3) $E^0_{p,q} \cong H_p(H^v_q(C))$
(4) If $C_{p,q}$ vanishes outside the first quadrant, i.e. $C_{p,q} = 0$ for any $p < 0$ or $q < 0$, then the spectral sequence converges to the homology of the total chain complex $E^r_{p,q} \Rightarrow H_{p+q}(C)$, i.e. $E^\infty_{p,q} \cong H_{p+q}(C)$

Proof. **(1)** By definition $E^0_{p,q} := (F^h_p C)_{p+q}/(F^h_{p-1}C)_{p+q} \cong C^{p,q}$
(2) $E^1_{p,q} = H_{p+q}(F^h_p C/F^h_{p-1}C) \cong H_{p+q}(C_{p,\bullet})$
(3) □

Part V

General topology

Chapter 15

General topology

15.1 General topology

Definition 15.1.1. A **topological space** X is a set with **topology** $\tau \subseteq \mathscr{P}(X)$, such that $\varnothing, X \in \tau$, $U_i \in \tau \Rightarrow \bigcup_i U_i \in \tau$, $U, V \in \tau \Rightarrow U \cap V \in \tau$, elements in τ are **open sets**, complements of open sets are **closed sets**
N is a **neighborhood** of $A \subseteq X$ if $A \subseteq U \subseteq N \subseteq X$ for some open set U
x is a **limit point** of A if any neighborhood of x intersects A. x is a **limit** of $\{x_n\}$ if for any neighborhood U of x, all but finitely many lies in U
A **subspace** is $A \subseteq X$ with **subspace topology** given by $\{U \cap A | U \in \tau\}$

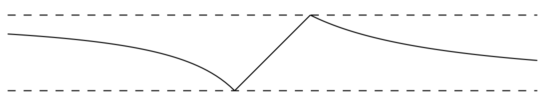
Definition 15.1.2. $X \xrightarrow{f} Y$ is **continuous** at x if for any neighborhood V of $y = f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$. Then f is continuous iff $f^{-1}(V)$ is open for any open set $V \subseteq Y$

Definition 15.1.3. A **base** for τ is $B \subseteq \tau$ such that B covers X and for any $U_1, U_2 \in B$ such that $U_1 \cap U_2 \neq \varnothing$, there exists $U_3 \in B$ such that $U_3 \subseteq U_1 \cap U_2$
A **local base** for τ at x is a collection of neighborhoods $B(x)$ of x such that any neighborhood of x contain an element of $B(x)$
A **subbase** for τ is $B \subseteq \tau$ such that B generates τ, i.e. by arbitrary union of finite intersections, equivalently, τ is the smallest topology containing B. Here empty union and empty intersection are $\varnothing$ and X

Definition 15.1.4. X is **first countable** if each point has a countable local base
X is **second countable** if it has a countable base

Definition 15.1.5. X is **regular** if any point and a disjoint closed set have disjoint neighborhoods. X is **normal** if disjoint closed sets have disjoint neighborhoods

Definition 15.1.6. $\{A_i\}$ can be **completely separated** if $\{A_i\}$ can be completely separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. Closed subsets $\{A_i\}$ can be **perfectly separated** if $\{A_i\}$ can be perfectly separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. $\mathbb{R}$ can be replaced with I considering $\mathbb{R} \to I$, $x \mapsto \begin{cases} \frac{x}{x-1} & x \le 0 \\ x & 0 \le x \le 1 \\ \frac{2}{x+1} & x \ge 1 \end{cases}$ and $I \hookrightarrow \mathbb{R}$



Definition 15.1.7 (Kolmogorov classification of topological spaces)**.** X is a T_0 **space** if for any two distinct points in X, at least one of them has a neighborhood which doesn't intersect the other piont, i.e. they are **topologically distinguishable**
X is a T_1 **space** if for any two distinct points in X, each of them has a neighborhood which doesn't intersect the other point. $T_1 \Leftrightarrow$ points are closed
X is a T_2 **space** or **Hausdorff space** if any two distinct points have disjoint neighborhoods. Then the limit of $\{x_n\}$ is unique, denotes the limit $x = \lim x_n$
X is a $T_{2\frac{1}{2}}$ **space** or **Urysohn space** if any two distinct points have disjoint closed neighborhoods
X is a T_3 **space** if X is regular Hausdorff
X is a $T_{3\frac{1}{2}}$ **space** if X is completely regular Hausdorff
X is a T_4 **space** if X is normal T_1 space $\Leftrightarrow$ normal Hausdorff
X is a T_5 **space** if X is completely normal Hausdorff
X is a T_6 **space** if X is perfectly normal $\Leftrightarrow$ perfectly normal Hausdorff

Definition 15.1.8. The **box topology** on $\prod\limits_{i\in I} X_i$ has base $\left\{\prod\limits_{i\in I} U_i \middle| U_i \subseteq X_i \text{ open}\right\}$

Lemma 15.1.9. X is Hausdorff iff the diagonal $\{(x,x)|x \in X\}$ is closed

Definition 15.1.10. $X \times I \xrightarrow{F} Y$ is a **homotopy** between $X \xrightarrow{f_0, f_1} Y$ if $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$, write $f_t = F(\cdot, t)$. $X \xrightarrow{f} Y$ is a **homotopy equivalence** if there is $Y \xrightarrow{g} X$ such that $gf \simeq 1_X$, $fg \simeq 1_Y$

Definition 15.1.11. $X \xrightarrow{f} Y$ is a **topological embedding** if f is injective and $f : X \to f(X)$ is a homeomorphism

Definition 15.1.12. $K \subseteq X$ is **compact** if any open cover has a finite subcover. Equivalently, K is disjoint from the intersection of a family of closed sets, then K is disjoint from the intersection of finitely many of them
X is **locally compact** if there is a compact neighborhood for each point
$Y \subseteq X$ is **precompact** if $\overline{Y}$ is compact

Definition 15.1.13. $A \subseteq X$ is **dense** if $\overline{A} = X$
X is **separable** if X has a countable dense subset

Definition 15.1.14. $X_\alpha \subseteq X$, $\{X_\alpha\}$ is **locally finite** if for any $x \in X$, there is a neighborhood of x intersecting only finitely many X_α's
$\mathcal{U} = \{U_\alpha\}$, $\mathcal{V} = \{V_\beta\}$ are covers of X, $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for any V_β, there exists U_α containing V_β
X is **paracompact** if every open cover has a locally finite open refinement

Lemma 15.1.15. Closed subsets of compact space are closed
The image of a compact set is compact
Compact subsets of a Hausdorff space are closed

X compact, Y Hausdorff, injective maps are embeddings

Lemma 15.1.16. X is compact, Y is Hausdorff, an injective map $X \xrightarrow{f} Y$ is a topological embedding

Proof. $f : X \to f(X)$ is a continuous bijection. If $K \subseteq X$ is closed, K is also compact since X is compact, thus $f(K)$ is compact, $f(K)$ is also closed since Y is Hausdorff $\square$

Definition 15.1.17. X is called **connected** if it can be written as the union of two open subsets
X is called **locally connected** if for any $x \in X$, there is a local basis that are connected

Proposition 15.1.18. Connected components are closed
Connectedness and local path connectedness implies path connectedness

Remark 15.1.19. Connected components may not be open

Definition 15.1.20. $E \xrightarrow{p} B$ has **lift extension property** for (X, A) if for any $X \xrightarrow{f} B$, a lift $A \xrightarrow{\tilde{f}} E$ can be extended to $\tilde{f} : X \to E$

$$\begin{tikzcd} A \arrow[r, "\tilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

$E \xrightarrow{p} B$ has **homotopy lifting property** for (X, A) if it has lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$

Proposition 15.1.21. If (X, A) satisfies homotopy extension property, and A is contractible, then the quotient map $X \xrightarrow{q} X/A$ is a homotopy equivalence

Proof. Consider $X \times \{0\} \cup A \times I \to A \hookrightarrow X$, where $(x, 0) \mapsto x$, $(a, 1) \mapsto *$ can be extended to $f : X \times I \to X$, $f_0 = 1_X$, $f_1(A) = \{*\}$, thus f_1 induces $r : X/A \to X$, $f_1 = rq$, $X \times I \xrightarrow{f} X \xrightarrow{q} X/A$ also induce $g : X/A \times I \to X/A$, where $qf_t = g_t q$, and $g_0 = 1_{X/A}$, $g_1 = qr$ thus $qr \simeq 1_{X/A}$ □

Definition 15.1.22. $U \subseteq X$ is open if $U \cap K$ is open for any compact subspace $K \subseteq X$ defines a topology. Equivalently, $F \subseteq X$ is closed if $F \cap K$ is closed for any compact subspace $K \subseteq X$. X is **compactly generated** if X has this topology

Definition 15.1.23. A map is **proper** if the preimage of a compact set is compact
A map is **discrete** if the preimage of a discrete set is discrete

Definition 15.1.24. X has **discrete topology** if $\tau = \mathcal{P}(X)$. X has **trivial topology** if $\tau = \{\varnothing, X\}$

Properties of discrete topology

Proposition 15.1.25. Suppose X has discrete topology
(a) Any map $f : Y \to X$ is continuous iff $f^{-1}(x)$ is open for all $x \in X$
(b) If continuous maps $f, g : X \to X$ are homotopic, then they are actually the same

Proof.
(a) For any subset $U \subseteq X$, $f^{-1}(U) = \bigcup_{x \in U} f^{-1}(x)$ is open
(b) If $F : X \times I \to X$ is a homotopy, then the restriction on $\{x\} \times I$ is gives a continuous map $I \to X$, the image has to be connected, thus the restriction is a constant, thus $f(x) = F(x, 0) = F(x, 1) = g(x)$ □

Pasting lemma

Lemma 15.1.26. $F_i \subseteq X$ are closed, $\bigcup_i F_i = X$, $f|_{F_i}$ are continuous, then f is continuous

X compact + Y Hausdorff => f:X->Y quotient map

Lemma 15.1.27. If X is compact, Y is Hausdorff, a surjective continuous map $f : X \to Y$ is a quotient map

Proof. Let's use the universal property of quotient space, consider a continuous map $g : X \to Z$ such that g maps fibers of f to points, thus we have a map $\tilde{g} : Y \to Z$, $\tilde{g}f = g$, for any closed set F in Z, so is $K = g^{-1}(F) = f^{-1}(\tilde{g}^{-1}(F))$, since X is compact, so is K, hence $f(K) = \tilde{g}^{-1}(F)$ is compact, and since Y is Hausdorff, $\tilde{g}^{-1}(F)$ is closed □

X locally compact+Hausdorff, F closed iff F intersects K is compact for any K compact

Lemma 15.1.28. X is locally compact, Hausdorff, $F \subseteq X$ is closed iff $F \cap K$ is compact for any compact subset $K \subseteq X$

Proof. F closed $\Rightarrow$ $F \cap K$ closed. Conversely, suppose $F \cap K$ is compact for any compact subsets $K \subseteq X$, for any $x \notin F$, there is a compact set K containing an open neighborhood U of x, $F \cap K$ is compact thus closed, hence $G = U - F \cap K$ is an open neighborhood of x which is disjoint of F, hence F is closed □

Lemma 15.1.29. X, Y are locally compact, Hausdorff, $p : X \to Y$ is continuous, proper, then p is closed

Proof. Suppose $F \subseteq X$ is closed, since $p\left(F \cap p^{-1}(K)\right) = p(F) \cap K$, by Lemma 15.1.28, we can take any $K \subseteq Y$ compact, hence F is closed □

Definition 15.1.30. X is noncompact, the **Alexandorff extension** of X is $X^* = X \cup \{\infty\}$ with open sets $\varnothing, X^*$, open sets in X and complements of closed compact sets of X
$X \hookrightarrow X^*$ is an open topological embedding
If X is also locally compact Hausdorff, X^* is the **one point compactification** of X which is Hausdorff

X,Y locally compact Hausdorff, f:X->Y proper, f send discrete sets to discrete sets

Lemma 15.1.31. X, Y are locally compact Hausdorff, $X \xrightarrow{f} Y$ is proper, then f sends discrete sets to discrete sets

Proof. Suppose $A \subseteq X$ is discrete, $x_0 \in A$, $y_0 = f(x_0) \in Y$, K is a compact neighborhood of y_0, then $f^{-1}(K)$ is a compact neighborhood of x_0, thus $f^{-1}(K) \cap A$ is finite, so is $K \cap f(A)$, since Y is Hausdorff, there is a neighborhood U of y_0 such that $U \cap f(A) = y_0$ □

Lemma 15.1.32. X, Y are locally compact, $X \xrightarrow{p} Y$ is proper and discrete, then $p^{-1}(y)$ is finite, and for any neighborhood V of $p^{-1}(y)$, there is a neighborhood U of y such that $p^{-1}(U) \subseteq V$

Lemma 15.1.33. X, Y are locally compact Hausdorff, $X \xrightarrow{p} Y$ is a proper local homeomorphism, then p is a finite sheeted covering

Definition 15.1.34. The **compact-open topology** on Y^X is given by a subbase $V(K, U) := \left\{f \in Y^X \middle| f(K) \subseteq U\right\}$, with $K \subseteq X$ compact and $U \subseteq Y$ open
A **normal family** $\{f_i\}$ is a precompact subset of Y^X

Lemma 15.1.35. $\{f_n\}$ converges pointwise on X iff $\{f_n\}$ converges in Y^X with the product topology $\prod\limits_{x \in X} Y$. Hence we call the product topology the **topology of pointwise convergence**

Proof. If f_n converges pointwise on X to f, then for any neighborhood V_i of $f(x_i)$, $i = 1, \cdots, k$, V_k contains all but finitely many $f_n(x_i)$, thus for n big enough, $f_n \in V_1 \cap \cdots \cap V_k \cap \prod\limits_{x \neq x_0} Y$, i.e. $\{f_n\}$ converges to f in Y^X □

Theorem 15.1.36. X is compact, Y is a complete metric space, then the topology induced by metric $d(f, g) = \sup\limits_{x \in X} d(f(x), g(x))$ is the same as the compact-open topology on Y^X

Theorem 15.1.37. $Y^* \cong Y$

Theorem 15.1.38. The composition $Z^Y \times Y^X \to Z^X, (g, f) \mapsto g \circ f$ is continuous, in particular, if $X = *$, then this becomes the evaluation map $\mathsf{eval} : Z^Y \times Y, (f, y) \mapsto f(y)$

Theorem 15.1.39. $Z^{X \times Y} \cong \left(Z^Y\right)^X$

Definition 15.1.40. A topological space X is reducible if $X = X_1 \cup X_2$, X_1, X_2 are proper nonempty closed subsets, $X_1 \nsubseteq X_2$, $X_2 \nsubseteq X_1$, X is **irreducible** if not reducible

Definition 15.1.41. A topological space X is **Noetherian** if $X \supsetneqq X_1 \supsetneqq X_2 \supsetneqq \cdots$ terminates, $\dim V = \sup\limits_d (X_0 \supsetneqq X_1 \supsetneqq \cdots \supsetneqq X_d)$, V_i's are closed and irreducible

Tychonoff's theorem

Theorem 15.1.42 (Tychonoff's theorem)**.** $\{K_i\}_{i \in I}$ are compact, so is $\prod\limits_{i \in I} K_i$

Proposition 15.1.43. Connected sets of $\mathbb{R}$ are intervals (a, b), $[a, b)$, $(a, b]$ or $[a, b]$

Jordan curve theorem

Theorem 15.1.44 (Jordan curve theorem)**.** $S^n \xrightarrow{i} \mathbb{R}^{n+1}$ is injective thus an open embedding by Lemma 15.1.16, denote $X = i(S^n)$, then $Y = \mathbb{R}^{n+1} \setminus X$ consists of exactly two connected components, the interior U which is bounded, and the exterior V which is not. When $n = 1$, U and V are homeomorphic to D and $\mathbb{R}^2 \setminus D$

Definition 15.1.45. A **locally closed set** X is the intersection of an open subset and a closed subset. Equivalently, X is relatively open in $\overline{X}$. A **constructible set** X if it is a finite union of locally closed sets

Lefschetz fixed point theorem

Theorem 15.1.46 (Lefschetz fixed point theorem)**.** X is a compact triangulable space of dimension n, the **Lefschetz number** of f is $\sum_{k=0}^{n} \mathrm{tr}(f_*|_{H_k(X;\mathbb{Q})})$. If the Lefschetz number of f is nonzero, then f has fixed points. The converse is not true, i.e. even if the Lefschetz number is zero, then could be fixed points

If $f = \mathrm{id}_X$, then the Lefschetz number is the Euler characteristic χ

Definition 15.1.47. The **join** of X, Y is

$$X * Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

We can also interpret it as all possible paths from X to Y. In general, $\underset{i}{*} X_i$ can be thought of as finite sum $\sum_i t_i x_i$, $t_i \in I$, $x_i \in X_i$

15.2 Retract

Definition 15.2.1. $A \xrightarrow{i} X$ is inclusion. A **deformation** of A into $B \subseteq X$ in X is a homotopy $A \xrightarrow{f_t} X$ such that $f_0 = i$ and $f_1(A) \subseteq B$, onto if equality holds. $X \xrightarrow{r} A$ is a **retraction** if $ri = 1_A$. r is a **weak retraction** if inclusion $A \xrightarrow{i} X$ has a left homotopy inverse, i.e. $ri \simeq 1_A$. A **deformation retraction** is a deformation $X \xrightarrow{f_t} X$ such that $f_1 = ri$ for some retraction $X \xrightarrow{r} A$. Deformation retraction f_t is **strong** if $f_t|_A = 1_A$
X is **contractible** if X deformation retracts onto a point
(X, A) is a **good pair** if A is a strong neighborhood deformation retract of X

Some rudimentary lemma about retract and deformation

Lemma 15.2.2. $A \xrightarrow{i} X$ is inclusion
(1) X is deformable into A iff i is a **weak section**, namely i has a right homotopy inverse, i.e. $ir \simeq 1_X$
(2) i is a homotopy equivalence iff A is a weak retract of X and X is deformable into A
(3) If X is deformable into a retract A, then A is a deformation retract of X
(4) If (X, A) is cofibered, then A is a weak retract of X iff A is a retract of X

Proof.
(1) If $X \times I \xrightarrow{H} X$ is a homotopy from 1_X to ir, then H is a deformation of X into A since $H_0 = 1_X$, $H_1(X) \subseteq A$. If H is a deformation of X into A, since $H_1(X) \subseteq A$, define $X \xrightarrow{r} A$ such that $ir = H_1$, then r is a right homotopy inverse of i
(2) i is a homotopy equivalence $\Leftrightarrow$ there exists $X \xrightarrow{r} A$ such that $ri \simeq 1_A$, $1_X \overset{H}{\simeq} ir \Leftrightarrow r$ is a weak retract, H is a deformation of X into A
(3) $X \xrightarrow{r} A$ is a retraction, $X \times I \xrightarrow{H} X$ is a deformation of X, then $1_X \simeq ir'$ for some $X \xrightarrow{r'} A$, hence $r \simeq rir' = r' \Rightarrow 1_X \simeq ir \simeq ir$ giving a deformation retract
(4) $A \times I \xrightarrow{H} A$ is a homotopy from ri to 1_A, since $r(a) = H_0(a)$ and (X, A) is cofibered, we have $X \times I \xrightarrow{F} A$, then $F_0 = r$, $F_1 i = 1_A$, i.e. r is homotopic to retraction F_1 $\square$

Definition 15.2.3. $\mathcal{C}$ is a class of topological spaces closed under homeomorphism and closed subsets. X is an **absolute retract** for $\mathcal{C}$ if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow X$ is a retract of Y. X is an **absolute neighborhood retract** for $\mathcal{C}$ if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow X$ is a neighborhood retract of Y

Part VI

Algebraic Topology

Chapter 16

Cell structure

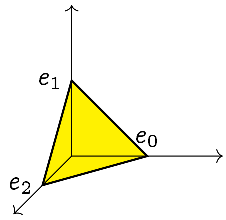
16.1 CW complexes

Standard simplex

Definition 16.1.1. With the standard basis $\{e_i\}$ for $\mathbb{R}^\infty$ as vertices, the **standard n-simplice** is

$$\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \subseteq \mathbb{R}^\infty \middle| \sum t_i = 1, 0 \leq t_i \leq 1\right\}$$

The **i-th face** of Δ^n is the face oppose to the i-th vertex, i.e. $\{t_i = 0\} \cap \Delta^n$
The **boundary** of Δ^n to be $\partial\Delta^n$ is the union of faces. $\partial\Delta^0 = \varnothing$



$\Delta^{n-1} \xrightarrow{d_{n,i}} \Delta^n$, $e_j \mapsto \begin{cases} e_j & j < i \\ e_{j+1} & j \geq i \end{cases}$ is i-th **face map** attaching Δ^{n-1} to the i-th face of Δ^n

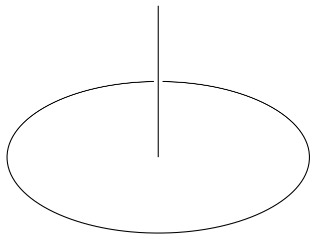
$\Delta^{n+1} \xrightarrow{s_{n,i}} \Delta^n$, $e_j \mapsto \begin{cases} e_j & j \leq i \\ e_{j-1} & j > i \end{cases}$ is the i-th **degeneracy map** which is a projection

Definition 16.1.2. X has a **cell decomposition** if X can be written as the disjoint union of open n cells, i.e. $X = \bigcup\limits_{n,\alpha} e^n_\alpha$, where cells e^n_α with subspace topology are homeomorphic to open n disks or open n simplices and disjoint, $X^n = \bigsqcup\limits_{k \leq n, \alpha} e^k_\alpha$ is called the **n-skeleton**, define $X^{-1} = \emptyset$
Suppose X, Y have cell decomposition $X = \bigcup\limits_{n,\alpha} e^n_\alpha$, $Y = \bigcup\limits_{m,\beta} e^m_\beta$, then $X \times Y$ also has a cell decomposition $X \times Y = \bigcup\limits_{k} \bigcup\limits_{\substack{n+m=k \\ \alpha,\beta}} e^n_\alpha \times e^m_\beta$, note that $e^n_\alpha \times e^m_\beta \cong e^{n+m}$
Every topological space has a cell decomposition into points

Definition 16.1.3. A **cellular map** is a map $f : X \to Y$ between topological spaces with cell decompositions such that $f(X^n) \subseteq Y^n$

Definition 16.1.4. X is called a **cell complex** if X is a Hausdorff space with cell decomposition $X = \bigcup\limits_{n,\alpha} e^n_\alpha$ and a cell complex structure: a family of characteristic maps $\Phi^n_\alpha : \Delta^n \to X$ such that Φ^n_α restricted on $\Delta^n \setminus \partial\Delta^n$ is a homeomorphism onto e^n_α and $\Phi^n_\alpha(\partial\Delta^n) \subseteq X^{n-1}$
Note that in the definition, we could also replace Δ^n with D^n

Remark 16.1.5. Since Δ^n is compact Hausdorff and X is Hausdorff, $\overline{e^n_\alpha} \subseteq \Phi^n_\alpha(\Delta^n)$, $\partial e^n_\alpha \subseteq \Phi^n_\alpha(\partial\Delta^n)$ for $n > 0$, on the other hand, if $\partial e^n_\alpha \subsetneqq \Phi^n_\alpha(\partial\Delta^n)$, then there exists $x \in \partial\Delta^n$ such that $y = \Phi^n_\alpha(x) \notin \overline{e^n_\alpha}$, this means there is an open neighborhood U of y disjoint from $\overline{e^n_\alpha}$, but then the preimage of U under Φ^n_α would be a nonempty open subset of Δ^n which intersects $\Delta^n \setminus \partial\Delta^n$ which is impossible, hence $\Phi^n_\alpha(\Delta^n) = \overline{e^n_\alpha}$, $\Phi^n_\alpha(\partial\Delta^n) = \partial e^n_\alpha$ for $n > 0$
A Hausdorff space X with a cell decomposition doesn't immediately give a cell complex structure, for example, consider an open disk union with an open segment right in the middle



Cell complex X can be seen as Δ^n is compact Hausdorff and X is Hausdorff and Lemma 15.1.27

Definition 16.1.6. A cell complex X is called **regular** if all characteristic maps are embeddings

Definition 16.1.7. Let X be a cell complex, it is **closure finite** if $\overline{e^n_\alpha}$ is contained in the union of finitely many cells, and we say X has the **weak topology**, meaning $F \subseteq X$ is closed iff $F \cap \overline{e^n_\alpha}$ is closed in $\overline{e^n_\alpha}$ for any cell, if a cell complex is both closure finite and has the weak topology, we say it is a **CW complex**
Closure finiteness is equivalent of saying $\partial e^n_\alpha \subseteq \bigcup\limits_{k<n,\alpha} e^k_\alpha$ a finite union of cells

Example 16.1.8. Consider $X = D^2$ with a cell complex structure $D^2 \to D^2$ and $* \to x$ for each $x \in \partial D^2$, this doesn't satisfy closure finiteness since $\overline{e^2} = X$, but the weak topology is the same as the original one, since if $F \subseteq D^2$ is closed in the weak topology, then $F \cap \overline{e^2} = F$ is closed
Consider $X = S^1$ with a cell complex structure $* \to x$ for each $x \in S^1$, the weak topology is the discrete topology on S^1 which doesn't match with the original topology on S^1, but it does satisfy closure finiteness

Remark 16.1.9. Suppose X is a CW complex
X^n is obviously closed due to the weak topology
Since $\overline{e^n_\alpha}$ is contained in the union of finitely many cells, $\overline{e^n_\alpha}$ contains at most finitely many 0 cells, thus any union of 0 cells F is closed because $F \cap \overline{e^n_\alpha}$ is finitely many points which is closed given that X is Hausdorff, therefore X^0 is discrete
Suppose $K \subseteq X$ is a compact subset, then $K \subseteq X = \bigcup e^n_\alpha \subseteq \bigcup \overline{e^n_\alpha} \setminus \partial e^n_\alpha$
contained in finitely many cells, since $K \subseteq \bigcup \overline{e^n_\alpha} \setminus \partial e^n_\alpha$
if $K \cap e^n_\alpha \neq \emptyset$,
, otherwise K intersects infinitely many cells,

Theorem 16.1.10. Another description of CW complexes is as follows:
These two definitions coincides

Proposition 16.1.11. Any compact set of a CW complex is contained in finitely many cells

Proposition 16.1.12. CW complexes are locally contractible, thus they are locally path connected, hence connectedness and path connectedness are equivalent for CW complexes

Theorem 16.1.13. CW complexes are normal, satisfies T_4 axiom

Proposition 16.1.14. If $A \subseteq X$ is a CW subcomplex, then (X, A) is a good pair

Theorem 16.1.15. CW complexes have partitions of unity

Proposition 16.1.16. Covering space of CW complexes are CW complexes

Proposition 16.1.17. The product of two countable CW complexes is again a CW complex

16.2 Graphs

Theorem 16.2.1. For every group G, there is a connected two dimensional CW complex X with $\pi_1(X) = G$

Proof. We can always find a surjection from a free group F to G, suppose F is generated by g_α's, and the kernel K is generated by r_β's, i.e. F has a group presentation $\langle g_\alpha | r_\beta \rangle$, then define X to be $\bigvee_\alpha S^1_\alpha$ attached with cells e^2_β's along each word r_β □

Definition 16.2.2. Cayley graphs, Cayley complexes

Definition 16.2.3. A **graph** G is a one dimensional CW complex, a **tree** T is a contractible graph, $T \subseteq G$ is maximal if T contains all vertices, note that in a tree there is a unique path between two vertices

Proposition 16.2.4. Let X be a connected graph, any tree in X is contained in a maximal tree, in particular, X has a maximal tree

Proof. Let's prove more generally any subgraph X_0 is the deformation retraction of subgraph Y which contains all the vertices
Construct $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$ as follows, X_{i+1} is obtained by adding the closures of all the edges that connected to X_i, $X = \bigcup_i X_i$, since X is path connected, let $Y_0 = X_0$, and construct $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \cdots$ as follows, for any vertex in $X_{i+1} - X_i$, choose one edge that connects to Y_i, and add the closure, so we have Y_{i+1} from Y_i, it is easy to see the Y_{i+1} deformation retracts onto Y_i, so $Y = \bigcup_i Y_i$ deformation retracts onto $Y_0 = X_0$
If $X_0 = T$ is a tree, so is Y since Y deformation retracts onto T which is contractible □

Free basis for connected graphs

Proposition 16.2.5 (Free basis for connected graphs)**.** For a connected graph X with maximal tree T, for any edge $e_\alpha \in X - T$, there is a corresponding loop f_α goes from x_0 to one endpoint of e_α, across e_α to the other, and go back to x_0, $\pi_1(X, x_0)$ is a free group generated by f_α

Proof. Consider $X \to X/T$ which is a homotopy equivalence □

Theorem 16.2.6. Any subgroup of a free group is also free

Proof. Let F be a free group, there exists a graph X such that $\pi_1(X) = F$ by just taking the wedge sum of circles at x_0, let $G \leq F$ be a subgroup, then there exists a covering $Y \xrightarrow{p} X$ such that $p_*(\pi_1(Y, y_0)) = G$, thus $\pi_1(Y, y_0) \cong G$, and since Y is a covering of X, Y is also a graph, by Proposition 16.2.5, $G \cong \pi_1(Y)$ is free □

16.3 Simplex category

Definition 16.3.1. The **simplex category** Simp has $[n] := \{0, 1, \cdots, n\}$ as objects, and order preserving functions as morphisms, there are two special types of morphisms: **Face maps** $d_{n,i} : [n-1] \to [n]$, $d_{n,i}(j) = \begin{cases} j & , j < i \\ j+1 & , j \geq i \end{cases}$ and the **degeneracy maps** $s_{n,i} : [n+1] \to [n]$, $s_{n,i}(j) = \begin{cases} j & , j \leq i \\ j-1 & , j > i \end{cases}$, they subject to **simplicial identities**:

$$d_j \circ d_i = d_i \circ d_{j-1}, i < j \Leftrightarrow i \leq j-1$$

$$s_j \circ s_i = s_i \circ s_{j+1}, i \leq j \Leftrightarrow i < j+1$$

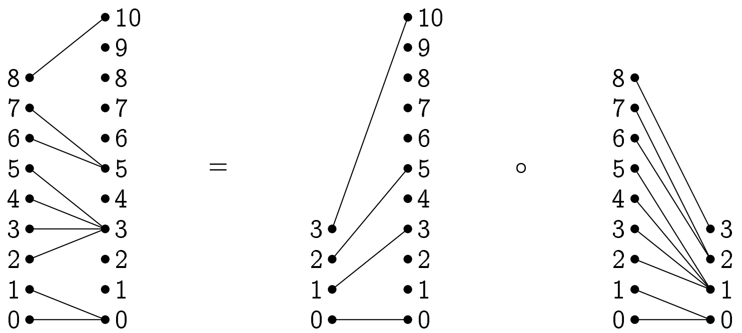
$$s_j \circ d_i = \begin{cases} d_{i-1} \circ s_j & , j \leq i-2 \Leftrightarrow j < i-1 \\ 1 & , j = i, i-1 \\ d_i \circ s_{j-1} & , j > i \Leftrightarrow j-1 \geq i \end{cases}$$

I find it easier to just think about $d_i, s_i : [\infty] \to [\infty]$
The **semisimple simplex category** is when discarding degeneracy maps, i.e. morphisms are strictly order preserving. The **augmented simplex category** is $\mathit{Simp} \cup \varnothing$. The **unordered simplex category** with the same objects as Simp and all functions as morphisms. The **unordered semisimplex category** with the same objects as Simp and all injections as morphisms

Unique decomposition of morphisms in simplex category

Lemma 16.3.2. Thanks to simplicial identities. Any morphism can be uniquely decomposed into a surjection compose with an injection. Any injection can be uniquely decomposed into composition of face maps with index strictly increasing. Any surjection can be uniquely decomposed into composition of degeneracy maps with index nondecreasing

Proof. For example



The right hand side can be written as $d_9 d_8 d_7 d_6 d_4 d_2 d_1 \circ s_2 s_2 s_1 s_1 s_1 s_0$ □

Definition 16.3.3. A **simplicial object** in $\mathscr{C}$ is a functor $\mathit{Simp}^{op} \to \mathscr{C}$, and a cosimplicial object is a functor $\mathit{Simp} \to \mathscr{C}$ If $\mathscr{C}$ is the category of sets, then the simplicial object is called a **simplicial set** $X : \mathit{Simp}^{op} \to \mathit{Set}$, $X([n]) = X_n$ is a family of sets, the face map $X(d_{n,i})$ sends elements of X_n to its i-th face
Similarly, we have semisimplicial object, augmented simplicial object, unordered simplicial object and unordered semisimplicial object

Example 16.3.4. The standard simplices $\{\Delta^n\}$ in Definition 16.1.1 with face and degeneracy maps is a cosimplicial object Δ in the category of topological spaces

$$\begin{tikzcd} {[n]} \arrow[r, mapsto] \arrow[d, "d_i"'] & \Delta^n \arrow[d, "d_i"] & {[n]} \arrow[r, mapsto] & \Delta^n \\ {[n+1]} \arrow[r, mapsto] & \Delta^{n+1} & {[n+1]} \arrow[r, mapsto] \arrow[u, "s_i"] & \Delta^{n+1} \arrow[u, "s_i"'] \end{tikzcd}$$

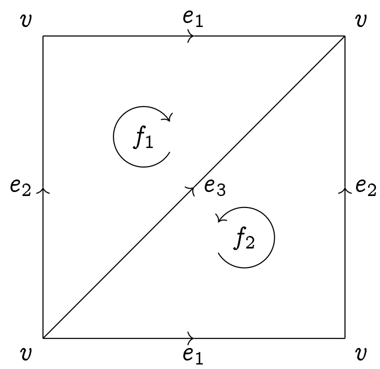
This functor is faithful and injective on objects, hence we may also just think of standard simplices as the simplex category
Due to Lemma 16.3.2, any morphism $\Delta^n \to \Delta^m$ can be uniquely written as an ordered degeneration compose with an ordered inclusion $\Delta^n \twoheadrightarrow \Delta^k \hookrightarrow \Delta^m$

Definition 16.3.5. A **Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi^n_\alpha : \Delta^n \to \overline{e^n_\alpha}$ to its i-th face is such that $\Phi^{n-1}_\beta = \Phi^n_\alpha \circ d_{n,i}$ for some Φ^{n-1}_β

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, hook, "d_{n,i}"] \arrow[d, "\Phi^{n-1}_\beta"'] & \Delta^n \arrow[d, "\Phi^n_\alpha"] \\
\overline{e^{n-1}_\beta} \arrow[r, hook] & \overline{e^n_\alpha}
\end{tikzcd}$$

Remark 16.3.6. A Δ complex X is also called semisimplicial complex because it can be regarded as a semisimple set $X : \mathit{Simp} \to \mathit{Set}$, with $X([n]) = X_n$ being all the n faces, $X(d_{n,i}) : X_n \to X_{n-1}$ being face maps that map each cell to its i-th face

Example 16.3.7. Consider a Δ complex structure on torus



Definition 16.3.8. An **unordered Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi^n_\alpha : \Delta^n \to \overline{e^n_\alpha}$ to any face is such that $\Phi^{n-1}_\beta = \Phi^n_\alpha \circ i$ for some Φ^{n-1}_β, i is an inclusion to that face regardless of order

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, hook, "i"] \arrow[d, "\Phi^{n-1}_\beta"'] & \Delta^n \arrow[d, "\Phi^n_\alpha"] \\
\overline{e^{n-1}_\beta} \arrow[r, hook] & \overline{e^n_\alpha}
\end{tikzcd}$$

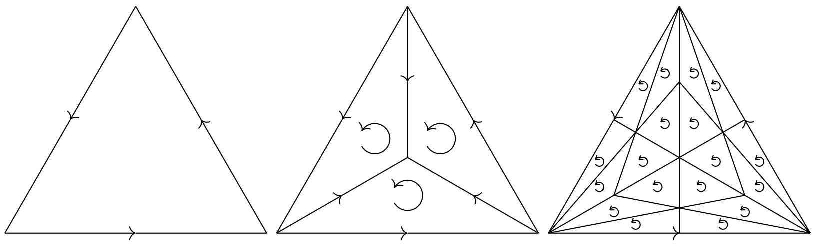
Remark 16.3.9. An unordered Δ complex X can be regarded as an unordered semisimple set $X : \mathit{Simp} \to \mathit{Set}$, with $X([n]) = X_n$ being all the n faces, $X(i) : X_n \to X_{n-1}$ being face maps that map each cell to the corresponding face

Definition 16.3.10. A regular unordered Δ complex is called a **multicomplex**, prefix multi- means different simplices can have the same faces
A regular unordered Δ complex in which each simplex is uniquely determined by its faces is called a simplicial complex

Definition 16.3.11. A **simplicial map** $f : K \to L$ is a map such that maps vertices of a simplex of K to the vertices of a simplex of L and linear on each simplex, two simplicial maps f, g are **contiguous** if for any simplex s in K, $f(s), g(s)$ are faces of the same simplex, in particular, f, g are homotopic, just consider $(1-t)f + tg$

Lemma 16.3.12. Any unordered Δ complex can be subdivided once to become a Δ complex, and any Δ complex can be subdivided to be a simplicial complex, therefore, every unordered Δ complex is homeomorphic to a Δ complex and is homeomorphic to a simplicial complex

Example 16.3.13. The one on the left with three edges identified is not a Δ complex, but an unordered Δ complex, the one in the middle is a Δ complex, but not a simplicial complex, the one on the right is a simplicial complex



Definition 16.3.14. A **singular Δ-complex** structure on a cell complex X is a CW complex structure where the restriction of a characteristic map $\Phi_\alpha^n : \Delta^n \to \overline{e_\alpha^n}$ to its i-th face is such that $\Phi_\beta^k \circ q = \Phi_\alpha^n \circ d_{n,i}$ for some $\Phi_\beta^k, k \leq n-1$, $q : \Delta^{n-1} \to \Delta^k$ is a degeneration

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, two heads, "q"] \arrow[d, hook, "d_{n,i}"'] & \Delta^k \arrow[d, "\Phi_\beta^k"] \\
\Delta^n \arrow[r, "\Phi_\alpha^n"] & X
\end{tikzcd}$$

Remark 16.3.15. A singular Δ complex is defined quite like a CW complex, where the characteristic maps are simplicial, instead of cellular
The vertices of a singular Δ complex can be given a partial order which is a total order on each simplex, just start at any point and use Zorn's lemma, in fact, it can be totally ordered

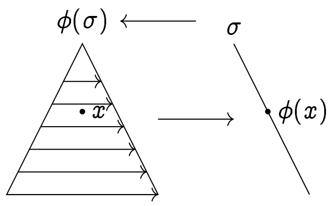
Definition 16.3.16. Suppose $X : \mathit{Simp} \to \mathit{Set}$ is a simplicial set, we can use this combinatorial information to construct its **geometric realization** $|X|$ with $X([n]) = X_n$ represents its n faces and morphisms $X_n \to X_{n-1}$ represents face maps and morphisms $X_n \to X_{n+1}$ represents degeneracy maps
The concrete construction is

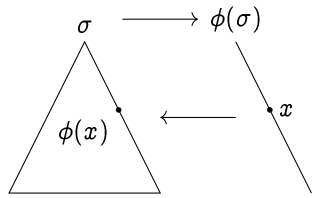
$$|X| := \frac{\bigsqcup_{n\geq 0} \Delta([n]) \times X([n])}{(\Delta\phi(x), \sigma) \sim (x, X\phi(\sigma))} = \frac{\bigsqcup_{n\geq 0} \Delta^n \times X_n}{(\phi(x), \sigma) \sim (x, \phi(\sigma))}$$

Where X_n is given the discrete topology, $x \in \Delta^n$, $\sigma \in X_n$, ϕ a morphism ranging over Simp which is the same as ranging over all face maps and degeneracy maps since every morphism can be decomposed uniquely as a degeneration compose with an inclusion, the difference being taking the transitive closure which is the definition of a quotient space
This basically means that give each n face an n simplex and if there is a face map or a degeneracy map, glue the corresponding simplices according to the map

Proposition 16.3.17. The geometric realization $|X|$ of a simplicial set X is a singular Δ complex, $|-|$ is a functor from the cateogory of simplicial sets to the category of singular Δ complexes, Moreover, if X is a semisimplicial set, then $|X|$ is a Δ complex

Proof. Let's first deal with the degeneration





Definition 16.3.18. Given a singular Δ complex(equivalently a simplicial set), its one skeleton with a partial order can be seen as a diagram, consider such a diagram in the category of spaces, we call this a complex of spaces

16.4 abstract simplicial complex

Definition 16.4.1. An **abstract simplicial complex** is $K \subseteq \mathscr{P}(S) \setminus \varnothing$ such that $X \in K \Rightarrow \mathscr{P}(X) \setminus \varnothing \subseteq K$. Finite elements of K are called **faces**. The **dimension** of a face X is $\dim X = |X| - 1$. The d skeleton K^d is the union of faces of dimension no more than d. $\dim K = \sup_{X \in K} \dim X$. K^0 are **vertices**. Maximal elements are **facets**. K is **pure** if all facets have dimension $\dim K$. A **simplex** is a subcomplex which contains all its nonempty subsets, for $X \in K$, $\overline{X}$ is the corresponding simplicial complex

Definition 16.4.2. The **closure** $\overline{L}$ of $L \subseteq K$ is smallest subcomplex of K containing L. The **star** of $Y \in K$ is $\operatorname{st} L = \{X | X \cap Y \neq 0\}$, the star of $L \subseteq K$ is $\operatorname{st} L = \bigcup_{Y \in L} \operatorname{st} Y$. The **link** of a face $Y \in K$ is or $\operatorname{lk} Y = \{X | X \cap Y = \varnothing, X \cup Y \in X\}$. Equivalently, $\operatorname{lk} Y = \overline{\operatorname{st} Y} - \operatorname{st} \overline{Y}$

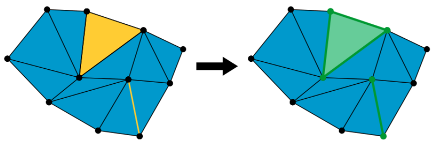


Figure 16.4.1: Two simplices and their closure

Closure of a complex

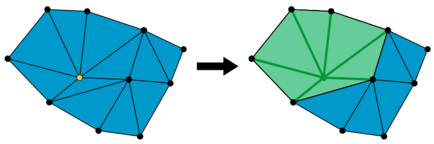


Figure 16.4.2: A vertex and its star

Star of a complex

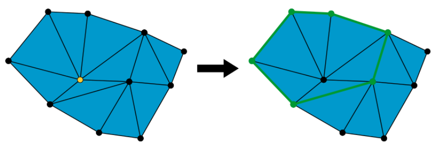


Figure 16.4.3: A vertex and its link

Link of a complex

Note. $\operatorname{lk} \varnothing = K$

Definition 16.4.3. If K, L has disjoint sets of vertices, then $K * L = \{X \sqcup Y | X \in K, Y \in L\}$

Definition 16.4.4. $L \subseteq K$, the **deletion** $K \setminus L$ consists of those sets which don't contain sets in L as subsets. The stellar subdivision of $X \in K$ is by introducing a new vertex x, and form $K \setminus X \cup (\overline{x} * \partial \overline{X} * \operatorname{lk} X)$

Definition 16.4.5. A simplicial map $f : K \to L$ is such that $f(K^d) \subseteq L^d$

16.5 CW approximation

CW approximation

Theorem 16.5.1 (CW approximation)**.** For any topological space X, there is a CW complex Z and a weak homotopy equivalence $f : Z \to X$, this is called a CW approximation

Whitehead's theorem

Theorem 16.5.2 (Whitehead's theorem)**.** Suppose $f : X \to Y$ is weak homotopy equivalence between CW complexes, then it is a homotopy equivalence

16.6 Triangulation

Definition 16.6.1. A **pseudomanifold** M is a pure triangulated space of dimension n such that M is not branching, i.e. any two n simplices have precisely one $(n-1)$ common face, M is strongly connected, i.e. any two n simplices can be linked with a sequence of simplices having common $(n-1)$ face pairwise

Note. The dual graph Γ of M is connected and n-regular

Example 16.6.2. The 0 dimensional pseudomanifold is the disjoint union of two points, since the empty set has to be the common face two point. The dual graph is a two points joined by an edge, this example is weird
A 1 dimensional pseudomanifold is a infinite chain or a loop, its dual graph is the same

Definition 16.6.3. D is a nonmaximal simplex, then $\operatorname{lk} D$ is a $n-|D|$ pure dimensional simplicial complex. A pseudomanifold such that $\operatorname{lk} D$ is also pseudomanifolds for any nonmaximal simplex is an **abstract polytope**

Chapter 17

Homology theory

17.1 Singular homology

Definition 17.1.1 (Eilenberg-Steenrod axioms)**.** Top is the category of topological spaces, Ab is the category of abelian groups , $\mathcal{T}$ is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces (X, A) such that $A \subseteq X$, $\mathcal{T}_A$ is the fully faithful subcategory of $\mathcal{T}$ with objects (X, A), $R : \mathcal{T} \to Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor
Relative homology are functors $H_n : \mathcal{T} \to Ab$, then $H_n(-, A)$ define functors $\mathcal{T}_A \to Ab$, **absolute homology** are functors $H_n(-, \varnothing) : Top \to Ab$, **reduced homology** are $\widetilde{H}_n = H_n(-, *)$.
$\partial_n : H_n \to H_{n-1}R$ are natural transformations

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \Big\downarrow{\scriptstyle \partial_n} & & \Big\downarrow{\scriptstyle \partial_n} \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) \end{array}$$

(H, ∂) is a **homology theory** if it satisfies axioms
Homotopy invariance: $f \simeq g : (X, A) \to (Y, B)$, then $H_n(f) = H_n(g)$
Additivity: $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$, then $\bigoplus_\alpha H_n(X_\alpha, A_\alpha) \xrightarrow{\bigoplus_\alpha H_n(i_\alpha)} H_n(X, A)$ is an isomorphism
Exactness:

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial_n} \cdots$$

Exicision: $\bar{Z} \subseteq \overset{\circ}{U}$, then $H_n(X - Z, U - Z) \xrightarrow{H_n(i)} H_n(X, U)$ is an isomorphism
Dimension: $H_n(*) = 0, \forall n \neq 0$, $H_0(*)$ is the **coefficient group**
(H, ∂) is an **extraordinary homology theory** without dimension axiom

Definition 17.1.2. A **singular n-simplex** in X is just a continuous map $\Delta \xrightarrow{\sigma} X$, the free abelian group $C_n(X)$ with singular n-simplices in X as basis consists of n-chains(singular chain) which are finite sums $\sum n_i \sigma_i, n_i \in \mathbb{Z}$, we can tensor $C_n(X)$ with a ring R, $C_n(X; R) := C_n(X) \otimes_{\mathbb{Z}} R$ to be chains with R coefficients, here R could be an abelian group(group ring) or a field
Also, if we only consider characteristic maps(for simplicial, Δ, cell complexes), we would get $C_n(X)$ to be simplicial, cellular chains

Remark 17.1.3. Given a topological space, we can form a huge Δ complex $S(X)$
Let $S(X)^0$ be X with discrete topology which can be identified with all the maps $\Delta^0 = * \to X$, then build on it inductively as a CW complex, suppose $S(X)^n$ is constructed, for each map $\Delta^{n+1} \to X$, we add an $n + 1$ cell by gluing its faces to its restrictions, preserving the order
Similarly, suppose X is a singular Δ complex, we can also construct a Δ complex $\Delta(X)$ by replacing continuous maps with simplicial maps above
The simplicial homology of $S(X), \Delta(X)$ is the same as the singular homology of X

Definition 17.1.4. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|[e_0, \cdots, \widehat{e_i}, \cdots, e_n]$$

Where $\sigma : \Delta^n \to X$ is a singular simplex, we can easily show that $\partial_n \partial_{n+1} = 0$, define cycles $Z_n(X) = \ker \partial_n$ and boundaries $B_n(X) = \mathrm{im}\partial_{n+1}$, and the (singular)homology group $H_n(X) = Z_n(X)/B_n(X)$
Similarly, we can define simplicial cycles, boundaries and homology groups correspondingly For cell complexes, if $\partial_n \sigma \subseteq X^{n-1}$, σ is called a cellular cycle, and cellular boundary is defined to be the image of some cellular chain, we can therefore define cellular homology

Definition 17.1.5. Define $C_n(X, A)$ to be $C_n(X)/C_n(A)$, $C_\bullet(X, A)$ form a chain complex, $Z_n(X, A)$ can be represented by n-chains with its boundary in A
The cellular homology could also be defined as the homology groups of $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \to \cdots$, where d_{n+1} is induced by $H_{n+1}(X^{n+1}, X^n) \to H_n(X^n) \to H_n(X^n, X^{n-1})$

Definition 17.1.6. Suppose $\mathcal{U} = \{U_j\}$ are a family of subspaces of X and interiors of U_j form an open cover of X, define $C_n^{\mathcal{U}}(X)$ to be n-chains $\sum n_i \sigma_i$ such that the image of each σ_i is contained in some U_j, $C_n^{\mathcal{U}}(X, A) := C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$

Theorem 17.1.7. The inclusion $C_n^{\mathcal{U}}(X, A) \to C_n(X, A)$ is a chain homotopy equivalence

Excision theorem for singular homology

Theorem 17.1.8 (Excision theorem for singular homology)**.** Singular homology satisfies excision theorem

Proof. Suppose $\bar{Z} \subseteq \mathring{U}$, let $A = U, B = X - Z, \mathcal{U} = \{A, B\}$, only need to show $H_n^{\mathcal{U}}(A \cup B, A) \cong H_n(A \cup B, A) \cong H_n(X, U) \cong H_n(X - Z, U - Z) \cong H_n(B, A \cap B) \cong H_n^{\mathcal{U}}(B, A \cap B)$
Consider $C_n^{\mathcal{U}}(B) \hookrightarrow C_n^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$ has kernel $C_n^{\mathcal{U}}(A \cap B)$, thus $C_n^{\mathcal{U}}(B)/C_n^{\mathcal{U}}(A \cap B) \cong C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$ □

Definition 17.1.9. (X, A) is called a good pair if A has a neighborhood U deformation retracts onto A

Definition 17.1.10. The reduced singular homology $\tilde{H}_n(X)$ is defined to be the homology group of the chain complex

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

Lemma 17.1.11. $\tilde{H}_n(X) \to H_n(X, *)$ is an isomorphism induced from $C_n(X) \to C_n(X, *)$

Proof. For $\sum n_i \sigma_i \in C_1(X)$, if $\sum n_i \partial \sigma_i \in C_0(*)$, then $\sum n_i \partial \sigma_i = 0$, thus $H_1(X, *) \cong \tilde{H}_1(X)$
For any $\sum n_i P_i \in C_0(X, *)$ where P_i are points, then $\sum n_i P_i - \sum n_i *$ is a preimage in $Z_0(X)$, a boundary in $Z_0(X)$ certainly maps to a boundary in $C_0(X, *)$, suppose $\sum n_i P_i \in C_0(X, *)$ is a boundary, $\sum n_i P_i - \sum n_i *$ has to be a boundary in $C_0(X)$, thus $H_0(X, *) \cong \tilde{H}_0(X)$ □

Theorem 17.1.12. If (X, A) is called a good pair, $H_n(X, A) \xrightarrow{q_*} \tilde{H}_n(X/A)$ is an isomorphism

Proof. Consider the quotient map $q : X \to X/A$ induces $H_n(X, A) \to H_n(X/A, *) \to \tilde{H}_n(X/A)$, we show that q_* is an isomorphism, suppose U is a neighborhood of A that deformation retracts onto it, consider the following diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(X, U) \arrow[d, "q_*"] & H_n(X - A, U - A) \arrow[l, "i_*"'] \arrow[d, "q_*"] \\
H_n(X/A, *) \arrow[r, "i_*"] & H_n(X/A, U/A) & H_n(X - A/A, U - A/A) \arrow[l, "i_*"']
\end{tikzcd}$$

$H_n(X,A) \xrightarrow{i_*} H_n(X,U)$, $H_n(X/A,*) \xrightarrow{i_*} H_n(X/A,U/A)$ are isomorphisms because of the deformation retraction, $H_n(X-A,U-A) \xrightarrow{i_*} H_n(X,U)$, $H_n(X-A/A,U-A/A) \xrightarrow{i_*} H_n(X/A,U/A)$ are isomorphisms because of the Theorem 17.1.8, $H_n(X-A,U-A) \xrightarrow{q_*} H_n(X-A/A,U-A/A)$is an isomorphism since $(X-A,U-A) \xrightarrow{q} (X-A/A,U-A/A)$ is a homeomorphism ☐

Theorem 17.1.13 (Mayer Vietoris sequence)**.** Suppose A,B are subspaces of X that the interior of A,B covers X, then we have an exact sequence of homology groups $\cdots \to H_n(A\cap B) \to H_n(A)\oplus H_n(B) \to H_n(A\cup B) \to \cdots$

Proof. It is not hard to see there is a short exact sequence $0 \to C_n^{\mathcal{U}}(A\cap B) \to C_n^{\mathcal{U}}(A)\oplus C_n^{\mathcal{U}}(B) \to C_n^{\mathcal{U}}(A\cup B) \to 0$, $x \mapsto (x,x)$ and $(x,y) \mapsto x-y$ ☐

Theorem 17.1.14. Suppose X has a Δ complex structure, $H_n^{\Delta}(X) \to H_n(X), \Phi_\alpha^n \mapsto \Phi_\alpha^n$ is an isomorphism

Definition 17.1.15. $S^n \xrightarrow{f} S^n$ induces $\mathbb{Z} \cong H_n S^n \xrightarrow{f_*} H_n S^n \cong \mathbb{Z}$, $f_*(1)$ is the **degree** of f

Proposition 17.1.16 (Properties of degrees)**.**

1. $\deg 1 = 1$

2. $\deg(fg) = \deg f \deg g$

3. If f is not surjective, $\deg f = 0$

4. If f is a reflection, $\deg f = -1$

5. Let a be the antipodal map, then $\deg a = (-1)^{n+1}$

6. If f has no fixed points on S^n, then f is homotopic to the antipodal map

Proof.

1. Let Δ_1^n, Δ_2^n maps to the upper and lower hemisphere be a Δ complex structure on S^n, then $\Delta_1^n - \Delta_2^n$ would be a generator, and f maps them to $\Delta_2^n - \Delta_1^n$, thus $\deg f = -1$

2. a is the composition of $n+1$ reflections

3. Since $f(x) \neq -x$, $\dfrac{(1-t)f(x)-tx}{|(1-t)f(x)-tx|}$ homotopy f to a

☐

Definition 17.1.17. View Δ^n as $\{0 \le x_1 \le \cdots \le x_n \le 1\}$, we can cut $\Delta^n \times \Delta^m = \{0 \le x_1 \le \cdots \le x_n \le 1\} \times \{0 \le x_{n+1} \le \cdots \le x_{n+m} \le 1\}$ into $\binom{n+m}{m}$ simplices

$$\Delta^n \times \Delta^m = \bigcup_\sigma \Delta_\sigma, \quad \Delta_\sigma = \{0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(n+m)} \le 1\}$$

σ runs over (n,m)-shuffles. Each σ can be viewed as a path through a grid as in Definition 4.2.2. Associate a linear map $\ell_\sigma : \Delta^{n+m} \to \Delta_\sigma \subseteq \Delta^n \times \Delta^m$, sending the k-th vertex to vertex in the grid. The **cross product**

$$\begin{gathered} C_n(X)\otimes C_m(Y) \to C_{n+m}(X\times Y) \\ f\otimes g \mapsto f\times g \end{gathered}$$

Where

$$f\times g = \sum_\sigma (-1)^{|\sigma|}(f\times g)\ell_\sigma$$

Here on the right hand side $f\times g : \Delta^n \times \Delta^m \to X\times Y$, $(a,b) \mapsto (f(a),g(b))$ is different from the left hand side. We have $\partial(f\times g) = \partial f \times g + (-1)^n f\times \partial g$

Eilenberg-Zilber theorem

Theorem 17.1.18 (Eilenberg-Zilber theorem)**.** $C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ is a natural equivalence

Proof. Consider $Top \times Top$ with model $\mathcal{M} = \{(\Delta^n, \Delta^m)\}$, $F, G : Top \times Top \to Ch_{\geq 0}$, $F(X,Y) = C_*(X \times Y)$, $G(X,Y) = C_*(X) \otimes C_*(Y)$, $H_i(\Delta^n \times \Delta^m) = 0$ for $i \neq 0$, $F_k(X,Y) = \left\{\Delta^k \xrightarrow{(\mathrm{id},\mathrm{id})} \Delta^k \times \Delta^k \xrightarrow{\sigma} X \times Y\right\}$. By Exercise 47.0.7, $H_i(C_*(X) \otimes C_*(Y)) = 0$ for $i \neq 0$, $C_k(X) = \{\Delta^k \xrightarrow{\mathrm{id}} \Delta^k \xrightarrow{\sigma} X\}$, $G_k(X,Y) = \left\{(\sigma \otimes \tau)(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})\middle| \Delta^p \xrightarrow{\sigma} X, \Delta^q \xrightarrow{\tau} Y\right\}$
There is a natural equivalence $\phi_0 : H_0F \to H_0G$ induced by $\varphi : C_0(X \times Y) = F_0(X,Y) \to G_0(X,Y) = C_0(X) \otimes C_0(Y)$, $(\sigma, \tau) \mapsto \sigma \otimes \tau$, since $H_0(X \times Y) = C_0(X \times Y)/(x_0, y_0) \sim (x_1, y_1)$, $(x_0, y_0), (x_1, y_1)$ are connected by a path, $H_0(C_*(X) \times C_*(Y)) = C_0(X) \otimes C_0(Y)/(x_0, y_0) \sim (x_1, y_0) \sim (x_1, y_1)$ □

Cross product and its dual for homology

Remark 17.1.19. We define the **cross product** $C_*(X) \otimes C_*(Y) \xrightarrow{\times} C_*(X \times Y)$ and its dual φ Define $T : C_*(X \times Y) \to C_*(Y \times X)$, $(x, y) \mapsto (y, x)$, $\tau : C_*(X) \otimes C_*(Y) \to C_*(Y) \otimes C_*(X)$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, $T^2 = 1$, $\tau^2 = 1$, $\tau\partial = \partial\tau$

$$\begin{array}{ccc} C*(X) \otimes C_*(Y) & \xrightarrow{\times} & C_*(X \times Y) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle T} \\ C_*(Y) \otimes C_*(X) & \xrightarrow{\times} & C_*(Y \times X) \end{array}$$

Is not commutative, but $\times$ and $T \circ \times \circ \tau$ are chain homotopic

$$\begin{array}{ccc} C_*(X \times Y) & \xrightarrow{\theta} & C*(X) \otimes C_*(Y) \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \tau} \\ C_*(Y \times X) & \xrightarrow{\theta} & C_*(Y) \otimes C_*(X) \end{array}$$

Is not commutative, but θ and $\tau \circ \theta \circ T$ are chain homotopic

Topological Kunneth formula

Theorem 17.1.20 (Topological Kunneth formula)**.**

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} Tor_1(H_p(X), H_q(Y)) \to 0$$

Is exact

Proof. Apply Theorem 17.1.18 and Theorem 13.0.21 □

17.2 Cellular homology

Chapter 18

Cohomology theory

18.1 Singular cohomology

Definition 18.1.1 (Eilenberg-Steenrod axioms)**.** Top is the category of topological spaces, Ab is the category of abelian groups , $\mathcal{T}$ is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces (X, A) such that $A \subseteq X$, $\mathcal{T}_A$ is the fully faithful subcategory of $\mathcal{T}$ with objects (X, A), $R : \mathcal{T} \to Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor
Relative cohomology are contravariant functors $H^n : \mathcal{T} \to Ab$, then $H^n(-, A)$ define contravariant functors $\mathcal{T}_A \to Ab$, **absolute cohomology** are contravariant functors $H^n(-, \varnothing)$: $Top \to Ab$, **reduced cohomology** are $\widetilde{H}^n = H^n(-, *)$. $\partial^n : H^n \to H^{n+1}R$ are natural transformations

$$\begin{array}{ccc} H^n(X,A) & \xrightarrow{H_n(f)} & H^n(Y,B) \\ \downarrow \partial^n & & \downarrow \partial^n \\ H^{n+1}(A) & \xrightarrow{H_{n-1}(f)} & H^{n+1}(B) \end{array}$$

(H, δ) is a **cohomology theory** if it satisfies axioms
Homotopy invariance: $f \simeq g : (X, A) \to (Y, B)$, then $H^n(f) = H^n(g)$
Additivity: $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$, then $\bigoplus_\alpha H^n(X_\alpha, A_\alpha) \xrightarrow{\bigoplus_\alpha H^n(i_\alpha)} H^n(X, A)$ is an isomorphism
Exactness:

$$\cdots \xrightarrow{\partial^{n-1}} H^n(X,A) \xrightarrow{H^n(j)} H^n(X) \xrightarrow{H^n(i)} H^n(A) \xrightarrow{\partial^n} \cdots$$

Exicision: $\bar{Z} \subseteq \mathring{U}$, then $H^n(X - Z, U - Z) \xrightarrow{H^n(i)} H^n(X, U)$ is an isomorphism
Dimension: $H^n(*) = 0, \forall n \neq 0$, $H^0(*)$ is the **coefficient group**
(H, δ) is an **extraordinary cohomology theory** without dimension axiom

Definition 18.1.2. Define singular n-cochains to be $C^n(X) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z})$, if R is a ring, then we can also define cohomology with R coefficients $C^n(X; R) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X), R)$, here R can be abelian groups(group ring) or fields
We can also define simplicial, cellular cochains correspondingly

Remark 18.1.3. Note that $\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$, $\mathrm{Hom}(C_n(X; R), \mathbb{Z}) = \mathrm{Hom}(C_n(X) \otimes R, \mathbb{Z}) \cong \mathrm{Hom}(C_n(X), \mathrm{Hom}(R, \mathbb{Z})) \ncong \mathrm{Hom}(C_n(X), R) = C^n(X; R)$

Definition 18.1.4. $\partial_{n+1} : C_{n+1}(X) \to C_n(X)$ induce the coboundary map $\delta^n : C^n(X) \to C^{n+1}(X)$, we can define cocycles $Z^n(X) = \ker \delta^n$, coboundaries $B^n = \mathrm{im}\delta^{n-1}$ and cohomology $H^n(X) = Z^n(X)/B^n(X)$

Definition 18.1.5. θ as in Remark 17.1.19, the cross product is composition $\times : C^*(X; R) \otimes C^*(Y; R) \xrightarrow{\theta^*} C^*(X \times Y; R \otimes R) \to C^*(X \times Y; R)$, here $R \otimes R \to R$ is the ring multiplication.

$\delta(f\times g)=\delta f\times g+(-1)^{|f|}f\times\delta g$, $\times$ is well defined on cohomology since θ is unique up to natural chain equivalence. If R is commutative, then $f\times g=(-1)^{|f||g|}g\times f$
For $[f]\in H^p(X;R)$, $[g]\in H^q(Y;R)$, $[a]\in H_p(X)$, $[b]\in H_q(Y)$, then $([f]\times[g])([a]\times[b])=f(a)g(b)\in R$

Lemma 18.1.6. If $a\in H^p(Y;R)$, then $1\times a=p_Y^*(a)\in H^p(X\times Y;R)$

Proof. $C_*(X\times Y)\to C_*(X)\otimes C_*(Y)\to C_*(x_0)\otimes C_*(Y)\xrightarrow{\epsilon\otimes 1}\mathbb{Z}\otimes C_*(Y)\cong C_*(Y)\xrightarrow{a}R$ and $C_*(X\times Y)\xrightarrow{p_Y}C_*(Y)\xrightarrow{a}R$ are chain homotopic □

Definition 18.1.7. $\Delta:X\to X\times X$ is the diagonal, for $a\in H^p(X;R)$, $b\in H^q(X;R)$, the **cup product** is $a\smile b=\Delta^*(a\times b)\in H^{p+q}(X;R)$, $f^*(a\smile b)=f^*(a)\smile f^*(b)$, if R is commutative, $a\smile b=(-1)^{|a||b|}b\smile a$, $1\smile a=\Delta^*(1\times a)=\Delta^*(p_X^*(a))=(p_X\Delta)^*(a)=1^*(a)=a$

Proposition 18.1.8. Cross product and cup product determine each other, $a\smile b=\Delta^*(a\times b)$, $a\times b=p_X^*(a)\smile p_Y^*(b)$

Proof. $p_X^*(a)\smile p_Y^*(b)=\Delta^*(p_X^*(a)\times p_Y^*(b))=\Delta^*(a\times 1\times 1\times b)=\Delta^*(1\times 1\times a\times b)=(1\times 1)\smile(a\times b)=1\smile(a\times b)=a\times b$ □

18.2 Cech cohomology

Definition 18.2.1. Given any open cover $\mathcal{U}$ of X, we can define a (abstract) simplicial complex, the nerve $N(\mathcal{U})$, with each U_α a vertex and an n-face if $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}} \neq \emptyset$, and we call $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ the carrier of this face, a cover is called a good cover if each $U_{\alpha_1} \cap \cdots \cap U_{\alpha_{n+1}}$ is contractible, in that case, $N(\mathcal{U})$ is homotopic to X

Definition 18.2.2. Suppose $\mathcal{V}$ is a refinement of $\mathcal{U}$, i.e. every V_β is contained in some U_α, refinement defines a preorder, then inclusion induce a simplicial map $N(\mathcal{V}) \to N(\mathcal{U})$, different choice of inclusions induce contiguous simplicial maps, thus this is well defined up to homotopy, we can define the direct limit $\varinjlim H^i(N(\mathcal{U}); G)$ to be the Čech cohomology group $H^i(X; G)$

18.3 Poincare duality

Chapter 19

Homotopy theory

19.1 Homotopy

Definition 19.1.1. $\pi_n(X) := [S^n, X]$ are the homotopy groups, the relative homotopy groups $\pi_n(X, A)$ are defined to be all homotopy classes of maps $(I^n, \partial I^n) \to (X, A)$ or equivalently all homotopy classes of maps $(S^n, s_0) \to (X, A)$, in particular, if $A = \{x_0\}$, we have $\pi_n(X, x_0) := \langle S^n, X \rangle$ with basepoints x_0, s_0, furthermore, we also define $\pi_n(X, A, x_0)$ to be all homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ or equivalently all homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$, here $J^{n-1} = \partial I^n - I^{n-1}$
$\pi_1(X, x_0)$ is called the fundamental group, note that $\pi_n(X, x, x) = \pi_n(X, x)$

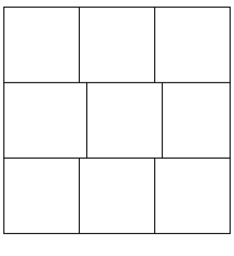
Definition 19.1.2. All homotopy classes of paths form the **fundamental groupoid** $\Pi_1(X)$ of X, suppose $A \subseteq X$, we can also define $\Pi_1(X, A)$ to be the subcategory with objects $x \in A$ and morphisms $Hom(x, y), x, y \in A$, $\Pi_1(X, x) = \pi_1(X, x)$, $\Pi_1(X)$ is a connected category if X is pathconnected connected since there is a morphism connecting any two objects, thus $\pi_1(X, x)$ is a skeleton of $\Pi_1(X)$, $\pi_1(X, x) \hookrightarrow \Pi_1(X)$ is an equivalence of categories

Proposition 19.1.3. $\pi_n(X, x_0)$ are groups, abelian if $n > 1$

Van Kampen's theorem

Theorem 19.1.4 (Van Kampen's theorem)**.** Suppose $X = \bigcup_\alpha A_\alpha$, interiors of A_α cover X, where $X, A_\alpha, A_\alpha \cap A_\beta$ are path connected and $x_0 \in A_\alpha$, then the map induced by inclusion $\mathop{*}_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ is surjective. Moreover, if $A_\alpha \cap A_\beta \cap A_\gamma$ are path connected, then then kernel is generated by $i_\alpha(w) i_\beta(w)^{-1}, w \in \pi_1(A_\alpha \cap A_\beta, x_0)$, where $i_\alpha : A_\alpha \to X$ are the inclusions

Proof. Since $A_\alpha \cap A_\beta$ are path connected, we can cut a loop in X into pieces such that each intermediate point is in $A_\alpha \cap A_\beta$ for some α, β, thus the map is surjective
Suppose $f_1 \cdots f_n$, $g_1 \cdots g_m$ are homotopic as loops, suppose F is the homotopy, consider the following diagram, each rectangle is so small that it is inside some A_α, then homotopy the path across a cube one at a time, and since $A_\alpha \cap A_\beta \cap A_\gamma$ are path connected, each vertex lies in $A_\alpha \cap A_\beta \cap A_\gamma$ for some α, β, γ, then we can connected it to x_0 through a path in $A_\alpha \cap A_\beta \cap A_\gamma$



□

Definition 19.1.5. $X \xrightarrow{f} Y$ is a **weak homotopy equivalence** if $\pi_0(X, x_0) \xrightarrow{f_*} \pi_0(Y, f(x_0))$ is bijective, and on each path connected component, $\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0))$ are isomorphisms

19.2 Model category

Model category

Definition 19.2.1. A **model structure** on $\mathscr{C}$ is three classes of morphisms $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ satisfying

1. $\mathbf{W}$ satisfies 2 out of 3, $\mathbf{F}, \mathbf{C}$ are closed under composition

2. $i \in \mathbf{C}$ has LLP for $p \in \mathbf{F} \cap \mathbf{W}$ and $p \in \mathbf{F}$ has RLP for $i \in \mathbf{C} \cap \mathbf{W}$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "g"'] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

3. For any morphism f, $f = pi$ with $i \in \mathbf{C} \cap \mathbf{W}$, $p \in \mathbf{F}$. $f = pi$ with $i \in \mathbf{C}$, $p \in \mathbf{F} \cap \mathbf{W}$

4. $\mathbf{F}, \mathbf{C}$ are closed under base change and cobase change. base change of $p \in \mathbf{F} \cap \mathbf{W}$ and cobase change of $i \in \mathbf{C} \cap \mathbf{W}$ are in $\mathbf{W}$. Isomorphisms $\mathbf{I} \subseteq \mathbf{F} \cap \mathbf{C}$

$(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a **closed model structure** if it satisfying 1,2,3 and

5. $\mathbf{W}, \mathbf{F}, \mathbf{C}$ are closed under retraction, i.e. if f is a retract of g in the arrow category, then f, g belong to the same class

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "j"] & X' \end{tikzcd}$$

By Theorem 19.2.3, a closed model structure is a model structure. Moreover, $\mathbf{F} \cap \mathbf{W}$ is closed under base change, $\mathbf{C} \cap \mathbf{W}$ is closed under cobase change
$\mathbf{W}$ are **weak equivalences**, $\mathbf{F}$ are **fibrations**, $\mathbf{C}$ are **cofibrations**, $\mathbf{F} \cap \mathbf{W}$ are **acyclic fibrations**, $\mathbf{C} \cap \mathbf{W}$ are **acyclic cofibrations**
A **model category** $\mathscr{C}$ is a complete and cocomplete category with a model structure

Remark 19.2.2. $\mathscr{C}^{op}$ is a also a model category where fibrations and cofibrations are switched, hence dual of true statements in $\mathscr{C}$ are also true

Equivalence of closed model structure and weak factorization system

Theorem 19.2.3. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure $\Leftrightarrow$ $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ are both weak factorization systems

Proof. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure. If $X \xrightarrow{i} Y$ has LLP for all $p \in \mathbf{F} \cap \mathbf{W}$, i can be decomposed as $X \xrightarrow{i'} Y' \xrightarrow{p} Y$ with $i' \in \mathbf{C}, p \in \mathbf{F} \cap \mathbf{W}$, then we have a lift $Y \xrightarrow{f} Y'$ by

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] & Y' \arrow[d, "p"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "f"] & Y \end{tikzcd}$$

Hence $i \in \mathbf{C}$ by

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "i"] & X \arrow[r, equal] \arrow[d, "i'"] & X \arrow[d, "i"] \\ Y \arrow[r, "f"] & Y' \arrow[r, "p"] & Y \end{tikzcd}$$

Suppose $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ and $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ are both weak factorization systems, and $\mathbf{F} \cap \mathbf{W}$ is closed under base change, $\mathbf{C} \cap \mathbf{W}$ is closed under cobase change
For any base cobase change j of $i \in \mathbf{C}$, and any $p \in \mathbf{F} \cap \mathbf{W}$, we get $Y \xrightarrow{h} E$ and then $Y \sqcup Z \xrightarrow{c} e$, hence $j \in \mathbf{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Z \arrow[r, "a"] \arrow[d, "j"] & E \arrow[d, "p"] \\ Y \arrow[r, "g"'] \arrow[urr, dashed, "h"] & Y \sqcup Z \arrow[r, "b"'] \arrow[ur, dashed, "c"] & B \end{tikzcd}$$

The class of isomorphisms $\mathbf{I} \subseteq \mathbf{W} \cap \mathbf{F} \cap \mathbf{C}$ since $1_X \in \mathbf{W} \cap \mathbf{F} \cap \mathbf{C}$ and

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ Y \arrow[r, "f^{-1}"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

□

Definition 19.2.4. Since $\mathscr{C}$ is complete and cocomplete, $\mathscr{C}$ initial object $\varnothing$ and final object $*$. X is **cofibrant** if $\varnothing \to X \in \mathbf{C}$, X is **fibrant** if $X \to * \in \mathbf{F}$

Definition 19.2.5. $A \times I$ is a **cylinder object** for A if the following diagram commutes for some $h \in \mathbf{W}$

$$\begin{tikzcd} A \coprod A \arrow[r, "i"] \arrow[dr, "1_A + 1_A"'] & A \times I \arrow[d, "h"] \\ & A \end{tikzcd}$$

$A \times I$ is **good** if $i \in \mathbf{C}$. $A \times I$ is **very good** if $i \in \mathbf{C}$, $h \in \mathbf{F} (\Rightarrow h \in \mathbf{F} \cap \mathbf{W})$. Very good cylinder object always exists by axiom 3 in Definition 19.2.1
Denote $i_0, i_1 : A \to A \coprod A \to A \times I$ by going through the first and second factor
A^I is a **path object** for A if the following diagram commutes for some $h \in \mathbf{W}$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "(1_A, 1_A)"'] & A^I \arrow[d, "p"] \\ & A \times A \end{tikzcd}$$

A^I is **good** if $p \in \mathbf{F}$. A^I is **very good** if $p \in \mathbf{F}$, $h \in \mathbf{C} (\Rightarrow h \in \mathbf{C} \cap \mathbf{W})$. Very good path object always exists by axiom 3 in Definition 19.2.1
Denote $p_0, p_1 : A^I \to A \times A \to A$ by going to the first and second factor

Lemma 19.2.6. If A is cofibrant and $A \times I$ is good, then $i_0, i_1 \in \mathbf{F} \cap \mathbf{W}$. If A is fibrant and A^I is good, then $p_0, p_1 \in \mathbf{C} \cap \mathbf{W}$

Definition 19.2.7. $f, g : A \to X$ is **left homotopic**, denoted $f \overset{l}{\sim} g$ if there exists a cylinder object $A \times I$ and a left homotopy $A \times I \xrightarrow{H} X$ such that

$$\begin{tikzcd} A \coprod A \arrow[r, "i"] \arrow[dr, "f+g"'] & A \times I \arrow[d, "H"] \\ & X \end{tikzcd}$$

$f, g : A \to X$ is **right homotopic**, denoted $f \overset{r}{\sim} g$ if there exists a path object A^I and a right homotopy $A \xrightarrow{H} X^I$ such that

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[dr, "(f,g)"'] & X^I \arrow[d, "p"] \\ & X \times X \end{tikzcd}$$

Example 19.2.8. $\mathscr{C} = Ch_{>-\infty}\mathscr{A}$ is the category of chain complexes bounded below. $\mathbf{W}$ are maps inducing isomorphisms on homologies. $\mathbf{F}$ are epimorphisms in $\mathscr{C}$. $\mathbf{C}$ are maps that are injective entrywise, and the cokernel is a chain complex of projectives of $\mathscr{A}$. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure on $\mathscr{C}$
The cofibrant objects are those with entries projective, then homotopy category is equivalent to the category with cofibrant objects with chain homootopy classes of maps

Example 19.2.9. $\mathscr{C}$ is the category of semisimplicial sets. $\mathbf{W}$ are morphisms that become homotopies after geometric realization. $\mathbf{F}$ are Kan fibrations. $\mathbf{C}$ are injective morphisms. $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ is a closed model structure on $\mathscr{C}$

19.3 Hurewicz fibration

Definition 19.3.1. $E \xrightarrow{p} B$ is a **Hurewicz fibration** if it has homotopy lifting property for any space X

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

$p^{-1}(A) \xrightarrow{p} A$ is also a fibration for any $A \subseteq B$

Definition 19.3.2. $A \xrightarrow{i} X$ is a **Hurewicz cofibration** if for any $X \xrightarrow{f_0} Y$, $A \times I \xrightarrow{g} Y$ such that $g_0 = f_0 i$, there there exists $X \times I \xrightarrow{f} Y$ extends f_0 such that $g = fi$

$$\begin{tikzcd} Y & X \arrow[l, "f_0"'] \arrow[dl, dashed, "f"'] \\ Y^I \arrow[u] & A \arrow[l, "g"] \arrow[u, "i"'] \end{tikzcd}$$

By Lemma 19.3.4, $A \hookrightarrow X$ is an embedding, (X, A) is a **cofibered pair** satisfying the **homotopy extension property**

Remark 19.3.3. The fiber of a fibration is the kernel in Top. The cofiber X/A of a cofibration is the cokernel in Top
$A \times I$ can be thought of as the "continuous" coproduct $\coprod_i A$, and A^I can be thought of as the "continuous" product $\prod_i A$

Cofibration is an embedding

Lemma 19.3.4. Cofibration $A \xrightarrow{i} X$ is a topological embedding

Proof. M is the mapping cylinder of $A \xrightarrow{i} X$, $X \times \{0\} \sqcup A \times I \xrightarrow{[]} M$ denote the quotient map, $f_0(x) = [(x, 0)]$, $g_t(a) = [(a, t)]$, then $g_0(a) = [(a, 0)] = [(i(a), 0)] = f_0 i(a)$

$$\begin{tikzcd} M & X \arrow[l, "f_0"'] \arrow[dl, dashed, "f"'] \\ M^I \arrow[u] & A \arrow[l, "g"] \arrow[u, "i"'] \end{tikzcd}$$

$M|_{A \times \{1\}} \xrightarrow{k} A$, $[(a, 1)] \mapsto a$ is a homeomorphism, $kf_1|_{i(A)}$ is the inverse of i because $kf_1 i(a) = kg_1(a) = k([(a, 1)]) = a$, $ikf_1(i(a)) = i(a)$ ☐

Lemma 19.3.5. A surjective fibration $E \xrightarrow{p} B$ with B locally path connected is a quotient map

Mapping cylinder of inclusion has subspace topology if it is a retraction

Lemma 19.3.6. If $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, then the topology of the mapping cylinder of the inclusion $A \hookrightarrow X$ is the the same as the subspace topology on $X \times \{0\} \cup A \times I$ induced from $X \times I$. In particular, f is continuous on $X \times \{0\} \cup A \times I$ iff f is continuous on both $X \times \{0\}$ and $A \times I$

Proof. This is trivial if A is closed due to Lemma 15.1.26
Write $Y = X \times \{0\} \cup A \times I$. If $O \subseteq Y$ is open in Y, then obviously $O \cap A \times I$ is open in $A \times I$, $O \cap X$ is open in X
Suppose $O \subseteq Y$ is such that $O \cap A \times I$ is open in $A \times I$, $O \cap X$ is open in X. Define $U_n = \bigcup_V \left\{ V \overset{\text{open}}{\subseteq} X \middle| (V \cap A) \times [0, \frac{1}{n}) \subseteq O \right\}$, i.e. U_n is the largest such open subset of X, $U = \bigcup_{n=1}^{\infty} U_n$, then $O \cap A \subseteq U$ since for any $(x, 0) \in O \cap A \subseteq O \cap A \times I$, because $O \cap A \times I$ is open $A \times I$,

there exists open subset $V \subseteq X$ containing x such that $(V \cap A) \times [0, \frac{1}{n}) \subseteq O \cap A \times I$ for some n, hence $x \in V \subseteq U_n \subseteq U$

$O \cap A \times (0,1]$ is open in Y, $O \cap (X \setminus \overline{A})$ is open in Y, we only need to show that for any $x \in \overline{A}$, there exists an open neighborhood of $(x, 0)$ contained in O, and it suffices to show that $x \in U$, then $x \in U_n$ for some n, $(U_n \cap O) \times [0, \frac{1}{n}) \cap Y$ is open in Y. Now fix $x \in \overline{A}$

Write the retraction r as (r_1, r_2), for $t > 0$, $r(x,t) = (r_1(x,t), r_2(x,t))$, since $x \in \overline{A}$, $r(a,t) = (a,t)$, we know $r_1(x,t) \in A$, $r_2(x,t) = t$. We claim: if $r_1(x,t) \in U_n$, then $x \in U_n$. Since U_n is open, there exists open neighborhood V of x such that $r_1(V \times (t-\varepsilon, t+\varepsilon)) \subseteq U_n$ for some $\varepsilon > 0$, in particular $r_1((V \cap A) \times \{t\}) \subseteq U_n$, thus $V \cap A \subseteq U_n \cap A$, by maximality of U_n, $V \subseteq U_n$

Suppose $x \notin U$, then by the claim, $r_1(x,t) \in A \setminus U$ for $t > 0$, then $r_1(x,t) \in A \setminus O$ since $A \cap O \subseteq U$, thus $x = r_1(x, 0) \in \overline{A} \setminus O$ which contradicts the fact $(x, 0) \in A$ □

A->X is a cofibration iff retraction exists

Proposition 19.3.7. (X, A) is cofibered iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$ iff $X \times \{0\} \cup A \times I$ is a strong deformation retract of $X \times I$

Proof. If $A \xrightarrow{i} X$ is a cofibration, then $X \times \{0\} \cup A \times I \xrightarrow{1} X \times \{0\} \cup A \times I$ induces a retraction. Conversely, by Lemma 19.3.6, $A \times I \xrightarrow{g} Y$, $X \xrightarrow{f_0} Y$ with $g_0 = f_0|_A$ gives a map $X \times \{0\} \cup A \times I \to Y$, composing with retraction gives $X \times I \to Y$

A strong deformation retraction is given by $H((x,t), s) = (\mathrm{Pr}_X\, r(x, st), s\, \mathrm{Pr}_I\, r(x,t) + (1-s)t)$ □

Lemma 19.3.8. If (X, A) is cofibered, so is $(X, \overline{A})$

Proof. Define $\phi(x) = \inf\limits_{t \in I}\{\mathrm{Pr}_I\, r(x,t) \neq 0\}$, then there is a retraction $X \times I \xrightarrow{r'} X \times \{0\} \cup \overline{A} \times I$

$$r'(x,t) = \begin{cases} (\mathrm{Pr}_X\, r(x,t), 0) & t \leq \phi(x) \\ (\mathrm{Pr}_X\, r(x, \phi(x)), t - \phi(x)) & t \geq \phi(x) \end{cases}$$

□

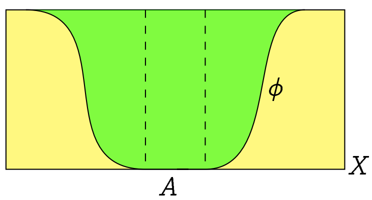
p:E->B fibration, i:A->X strong deformation retract, A can be perfectly separated => i has LLP

Lemma 19.3.9. $E \xrightarrow{p} B$ is a fibration, A is a strong deformation retract of X and A can be perfectly separated, then

$$\begin{tikzcd} A \arrow[r, "f''"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "f'"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

f is unique up to homotopy rel$A$

Proof. $X \xrightarrow{\phi} \mathbb{R}$ is a function such that $\phi^{-1}(0) = A$. $X \xrightarrow{r} A$ is the retract. $X \times I \xrightarrow{D} X$ is a homotopy from ir to 1_X. Define $H(x,t) = \begin{cases} D(x, t/\phi(x)) & t \leq \phi(x) \\ D(x, 1) & t \geq \phi(x) \end{cases}$



Since $E \xrightarrow{p} B$ is a fibration, we have a lift $X \times I \xrightarrow{F} E$ such that $pF = f'H$ and $F_0 = f''r$ because $pF_0 = pf''r = f'ir = f'H_0$. Define f to be the composition $X \xrightarrow{1 \times \phi} X \times I \xrightarrow{F} E$, then we have $fi = F(1 \times \phi) = F_0 i = f''$ and $pf = pF(1 \times \phi) = f'H(1 \times \phi) = f'D_1 = f'$

$$
\begin{tikzcd}
X \arrow[r, shift left, "r"] \arrow[d, hook] & A \arrow[l, shift left, "i"] \arrow[r, "f''"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[urr, dashed, "F"] & X \arrow[r, "f'"'] \arrow[l, bend left, "1\times\phi"] & B
\end{tikzcd}
$$

□

Fibrations have homotopy lifting property for closed cofibrations

Proposition 19.3.10. Fibrations have homotopy lifting property for closed cofibrations

$$
\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
$$

Proposition 19.3.11. If (X, A), (Y, B) are cofibered, $A \subseteq X$ is closed, then $(X \times Y, X \times B \cup A \times Y)$ is also cofibered. If in addition A or B is a strong deformation retract of X or Y, then $X \times B \cup A \times Y$ is a strong deformation retract of $X \times Y$

Fibers of fibration are homotopy equivalent

Proposition 19.3.12. $E \xrightarrow{p} B$ is a fibration, then the fibers over connected components of B are homotopic. More over, for any path $\gamma : I \to B$, we can get a lifting $g_t : F_{\gamma(0)} \to F_{\gamma(t)}$ of $F_{\gamma(0)} \hookrightarrow E$, define $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ to be g_1, if $\gamma \simeq \eta$ rel $\partial I : I \to B$, then $L_\gamma \simeq L_\eta$, and for any $\gamma, \eta : I \to B$, $\eta(0) = \gamma(1)$, $L_{\gamma\eta} \simeq L_\gamma L_\eta$

Proof. According to homotopy lifting property, lifting up $A \times F_{\gamma(0)} \to E$ is homeomorphic to $B \times F_{\gamma(0)} \to E$ □

Remark 19.3.13. We can think of this as an action of $\pi_1(B)$ on $H_*(F)$

Definition 19.3.14. $E_1 \xrightarrow{p_1} B_1$, $E_2 \xrightarrow{p_2} B_2$ are fibrations, $p_1 \xrightarrow{f_0, f_1} p_2$ are **fiber homotopic** if there exists $p_1 \xrightarrow{f_t} P_2$ varying from f_0 to f_1. p_1, p_2 are **fiber homotopy equivalent** if there are fiber homotopies $p_0 \xrightarrow{f} p_1$ and $p_1 \xrightarrow{g} p_0$ such that fg, gf fiber homotopic to 1

i:A->B cofibration is homotopy equivalence iff A strong deformation retract

Lemma 19.3.15. Cofibration $A \xrightarrow{i} X$ is a homotopy equivalence iff A is a strong deformation retract of X. Fibration $E \xrightarrow{p} B$ is a homotopy equivalence iff there exists a section $B \xrightarrow{s} E$ such that sp is fiber homotopic to 1

Proof. If i is a homotopy equivalence, then there exists $X \xrightarrow{r'} A$ such that $ir' \simeq 1_X$, $r'i \simeq 1_A$, by Lemma 15.2.2, since (X, A) is cofibered, $r' \simeq r$ is a retract and then A is a deformation retract of X. Suppose $X \times I \xrightarrow{F} X$ is a homotopy from 1_X to ir
$\Gamma = X \times \{0\} \cup A \times I \cup X \times \{1\} = X \times \{0\} \cup A \times [0, \frac{1}{2}] \cup A \times [\frac{1}{2}, 1] \cup X \times \{1\}$ is a retract of $X \times I$. Construct $\Gamma \times I \xrightarrow{G} X$

$$
G((x,t),s) = \begin{cases} F(x, (1-s)t) & (x,t) \in X \times \{0\} \cup A \times I \\ F(r(x), 1-s) & (x,t) \in X \times \{1\} \end{cases}
$$

G_1 can be extends to $X \times I \xrightarrow{H} X$, then $H_0(x) = G_1(x, 0) = F(x, 0) = x$, $H_1(x) = G_1(x, 1) = F(r(x), 0) = r(x)$, $H_t(a) = G_1(a, t) = F(a, 0) = a$, i.e. H is a strong deformation retraction □

fibration map is a homotopy equivalence iff it is a fiber homotopy equivalence

Lemma 19.3.16. $E \xrightarrow{p} B$ is a fibration, $A \xrightarrow{f} B$ is a map, $A \times_B E = f^*(E)$ is the **pullback fibration**, suppose $f_t : A \to B$ is a homotopy, then pullback fibrations $f_0^*(E) \to A$, $f_1^*(E) \to A$ are fiber homotopy equivalent. In particular, a morphism $p \xrightarrow{f} q$ between two fibrations is a homotopy equivalence iff f is a fiber homotopy equivalence

Proof. $A \times I \xrightarrow{F} B$ is a homotopy, we have the pullback fibration $F^*(E)$, it suffices to show that for any fibration $E \xrightarrow{p} B \times I$, $E_0 := p^{-1}(B \times \{0\}) \simeq p^{-1}(B \times \{1\}) =: E_1$ are fiber homotopy equivalent
Consider the following diagrams

$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_0 \times I \arrow[r, "(p,t)"'] \arrow[ur, dashed] & B \times I \end{tikzcd}$$

$$\begin{tikzcd} E_1 \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ E_1 \times I \arrow[r, "(p,1-t)"'] \arrow[ur, dashed] & B \times I \end{tikzcd}$$

Then we get fiber preserving maps $f : E_0 \to E_1$ and $g : E_1 \to E_0$, and restricts them to each fiber

$$\begin{tikzcd} p^{-1}(b,0) \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ p^{-1}(b,0) \times I \arrow[r, "(p,t)"'] \arrow[ur, dashed] & \{b\} \times I \end{tikzcd}$$

$$\begin{tikzcd} p^{-1}(b,1) \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\ p^{-1}(b,1) \times I \arrow[r, "(p,1-t)"'] \arrow[ur, dashed] & \{b\} \times I \end{tikzcd}$$

We get maps $f|_{p^{-1}(b,0)} : p^{-1}(b,0) \to p^{-1}(b,1)$, $g|_{p^{-1}(b,1)} : p^{-1}(b,1) \to p^{-1}(b,0)$, according to Proposition 19.3.12 they are homotopy equivalence and inverses to each other, hence f, g are fiber homotopy equivalences and inverses to each other ☐

Corollary 19.3.17. $E \xrightarrow{p} B$ is a fibration, B contractible, then p is fiber homotopy equivalent to $B \times F \to B$. If B is locally contractible, the fibration is locally homotopy equivalent to a product

Proof. Since B is contractible, identity map is homotopic to a constant map, and the pullback of E under the identity map is E itself, the pullback of E under a constant map is fiber bundle $B \times F$ ☐

Definition 19.3.18. (X, x_0) is a pointed space. The **loop space** ΩX consists of all the loops on X starting and ending at x_0, the constant loop being the basepoint. The **path space** PX consists of all the paths starting at x_0. $\Omega X \subseteq PX \subseteq X^I$ endowed with the subspace topology

Proposition 19.3.19. $\langle \Sigma X, Y \rangle = \langle X, \Omega Y \rangle$ is an adjunction

Definition 19.3.20. The **mapping path space** P_f is the pullback

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

P_f deformation retracts onto X by shrinking paths. The **homotopy fiber** F_f of f over y is the pullback

$$\begin{tikzcd} F_f \arrow[r] \arrow[d] & Y^I \arrow[d] \\ X \arrow[r, "f"] & Y \times Y \end{tikzcd}$$

The **mapping cylinder** M_f is the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r] & M_f \end{tikzcd}$$

M_f deformation retracts onto Y by sliding the cylinder The mapping cone $M_f/A \times \{0\} \cong C_f$ is the **homotopy cofiber** of f

Proposition 19.3.21. $X \xrightarrow{f} Y$ can be factorized as $X \hookrightarrow P_f \to Y$ or $X \hookrightarrow M_f \to Y$. $P_f \to Y$, $(x, \gamma) \mapsto \gamma(1)$, $M_f \to Y$ are Hurewicz fibrations. $X \hookrightarrow P_f$, $X \hookrightarrow M_f$ are closed Hurewicz cofibrations

Proof.

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & M_f \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

□

Example 19.3.22. PX is the mapping path space of $* \to X$, $\Omega X \to PX \to X$ is a Hurewicz fibration

Proposition 19.3.23. Suppose $E \xrightarrow{p} X$ is a fibration, then $E \hookrightarrow E_p$ is a fiber homotopy equivalence, and the restriction on each fiber to the homotopy fiber of p is a homotopy equivalence

Proposition 19.3.24. If $F \to E \to B$ is a fibration, and E is contractible, then F is weakly homotopic to ΩB

Theorem 19.3.25. $F \to E \to B$ is a fibration, the **fibration sequence** is

$$\to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

$A \to X \to X/A$ is a cofibration, the **cofibration(Puppe) sequence** is

$$A \to X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \to$$

Theorem 19.3.26. $E \xrightarrow{p} B$ is Serre fibration, fix $x_0 \in p^{-1}(b_0) = F$, $\pi_n(E, F, x_0) \xrightarrow{p^*} \pi_n(B, b_0)$ is an isomorphism, and we have long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p^*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Theorem 19.3.27. $\mathbf{W}$ consists of all homotopy equivalences, $\mathbf{F}$ consists of all Hurewicz fibrations, $\mathbf{C}$ consists of all closed Hurewicz cofibrations, $(\mathbf{W}, \mathbf{F}, \mathbf{C})$ defines a closed model structure on Top

Proof. 2 out of 3 is obvious
By Lemma 19.3.9 and Lemma 19.3.15, $i \in \mathbf{C} \cap \mathbf{W} \Rightarrow i$ has LLP for any $p \in \mathbf{F}$. Suppose i has LLP for any $p \in \mathbf{F}$, since $Y^I \to Y \in \mathbf{F}$, $i \in \mathbf{C}$, $A \to * \in \mathbf{F}$, we get a retraction $X \xrightarrow{r} A$ by

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook, "i"'] & A \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

$X^I \to X \times X \in F$, $\gamma \mapsto (\gamma(0), \gamma(1))$, then we get a strong deformation retraction

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & X^I \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "F"] & X \times X \end{tikzcd}$$

Hence $i \in \mathbf{C} \cap \mathbf{W}$ $\square$

19.4 Serre fibration

Definition 19.4.1. $E \xrightarrow{p} B$ is a **Serre fibration** if it has homotopy lifting property for all $(D^n, \partial D^n)$

Chapter 20

Isotopy

Definition 20.0.1. $f, g : X \to Y$ are topological embeddings, an **isotopy** from f to g is a homotopy H such that H_t are embeddings

Definition 20.0.2. g, h are embeddings of N in M, an **ambient isotopy** from g to h is $M \times I \xrightarrow{F} M$ such that F_t are homeomorphisms, and $F_0 = 1_M$, $F_1 \circ g = h$

Theorem 20.0.3 (Alexander's trick)**.** $D \subseteq \mathbb{R}^n$ is the unit ball, homeomorphisms of D that are isotopic on ∂D are also isotopic on D

Proof. Suppose $f, g : D \to D$ are homeomorphisms with $f|_{\partial D}$, $g|_{\partial D}$ isotopic $\square$

Chapter 21

Bundle

21.1 Bundles

Definition 21.1.1. A **bundle** is $E \xrightarrow{p} B$, where E is the **total space**, B is the **base space**, and p is the projection, $p^{-1}(b)$ is the **fiber** over b. A **cross section** is $s : X \to E$, such that $ps = 1_X$. The restriction $p^{-1}(A) \xrightarrow{\pi} A$, $A \subseteq B$ is also a bundle

Definition 21.1.2. Suppose $E \xrightarrow{p} B$ is a bundle, $f : A \to B$ is a map, then the pullback $f^*(E) = A \times_p E \to A$ is the **pullback bundle**, the pullback of a section $s : B \to E$ is defined as $f^*s := s \circ f$, notice $p(f^*s(y)) = p(s(f(y))) = f(y)$

Definition 21.1.3. A **fiber bundle** is a bundle $E \xrightarrow{p} B$ such that there exists an open neighborhood U of b and a homeomorphism ϕ making the following diagram commute

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

Definition 21.1.4. G is a topological group, a G **fiber bundle** $E \xrightarrow{p} B$ is a fiber bundle and also a morphism of G spaces

Lemma 21.1.5. A fiber bundle is a Serre fibration

Definition 21.1.6. $\mathbb{F}$ is a topological field, a **vector bundle** is a fiber bundle $E \xrightarrow{p} X$ with fiber being $\mathbb{F}^n$ and ϕ restricts on each fiber is an $\mathbb{F}$ isomorphism

Definition 21.1.7. G is a topological group, a **principal** G **bundle** $p : P \to B$ is a morphism of G spaces, B with the trivial G action, and for each $b \in B$, there is a local trivialization

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"] & U \times G \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

ϕ is an isomorphism

Remark 21.1.8. G action on P preserves fibers, and the action on fiber is free and transitive, each fiber is a G torsor. A morphism of principal G bundles is always an isomorphism. A principal G bundle is trivial iff it has a global section

Proposition 21.1.9. Suppose $P \to B$ is a principal G bundle, $G \to G/H$ is a principal H bundle, then $P \to P/H$ is a principal H bundle

Proof. $P \cong P \times_G G \to P \times_G (G/H) \cong P/H$ $\square$

Proposition 21.1.10. Suppose $P \to B$ is a principal G bundle, F is a left G space, $P \times_G F \to P \times_G * \cong B$ is a G fiber bundle. $X \xrightarrow{f} Y$ is a map, $f^*(P \times_G F) \to f^*(P) \times_G F$ is a natural homeomorphism

Proposition 21.1.11. $P \xrightarrow{p} B$ is a principal G bundle, X is a right G space, a morphism $P \xrightarrow{f} X$ induce $P \xrightarrow{\begin{pmatrix}1\\ f\end{pmatrix}} P \times X$, $B \cong P/G \to P \times X/G \cong P \times_G X$ which is a section s_f of $P \times_G X \to B$, this is a natural bijection

Proposition 21.1.12. $P \to B \times I$ is principal G bundle, then P and $P_0 \times I$ is an isomorphism, here P_0 is the restriction of P over $B \times \{0\}$

Proof.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, hook] & P \times_G (P_0 \times I) \arrow[d] \\
B \times I \arrow[r] \arrow[ur, dashed] & B \times I
\end{tikzcd}$$

□

21.2 Vector bundles

Proposition 21.2.1. $E \xrightarrow{p} X$ is trivial iff there exist global sections $s_1, \cdots, s_n$ that they are linearly independent on each fiber

Definition 21.2.2. Let $E \xrightarrow{p} X$ be a vector bundle, consider two trivializations $\varphi_U : E_U \to U \times \mathbb{R}^n$ and $\varphi_V : E_V \to V \times \mathbb{R}^n$ around $x \in X$, then $\varphi_V \circ \varphi_U^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$ is a local isomorphism with inverse $\varphi_U \circ \varphi_V^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$, it is also called a transition function and it can also be regard as a continuous map $g_{VU} : U \cap V \to GL(n, \mathbb{R})$ or $g_{VU} \in GL(n, C(U \cap V))$, such that $\varphi_V \circ \varphi_U(x, v) = (x, g_{VU}(x)v)$, notice then $g_{UV} = g_{VU}^{-1}$, and g_{VU}'s satisfy the cocycle relation $g_{WV}g_{VU} = g_{WU}$ on $U \cap V \cap W$
Conversely, given $\displaystyle\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{R}^n \times A$ transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ that satisfying cocycle relation $g_{\gamma\beta}g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, mod equivalence relation $(x, v, \alpha) \sim (x, g_{\beta\alpha}(v), \beta), x \in U_\alpha \cap U_\beta$, you will get back the vector bundle
Suppose $s : X \to E$, is a section, denote $\varphi_i \circ s|_{U_i}(x) = (x, f_i(x))$ over U_i, then $(x, f_j(x)) = \varphi_j \circ s|_{U_j}(x) = \varphi_j \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1} \circ \varphi_i \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1}(x, f_i(x)) = (x, g_{ji}(x)f_i(x)), \forall x \in U_i \cap U_j$, thus $f_j = g_{ji}f_i$, conversely, this relation also defines a section

Definition 21.2.3. The pullback of a transition function is defined to be $f^*g_{ij} := g_{ij} \circ f$

Definition 21.2.4. A morphism between vector bundles $\varphi : E \to F$ is map such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "p"] & F \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and $\varphi_x : E_x \to F_{f(x)}$ is a homomorphism between vector spaces

Definition 21.2.5. Let $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ be vector bundles, then direct sum $E \times F \xrightarrow{p \times q} X \times Y$ is also a vector bundle, suppose $\varphi_U : U \to U \times \mathbb{R}^n$, $\psi_V : V \to V \times \mathbb{R}^m$ are trivializations, then $\varphi_U \times \psi_V : U \times V \to U \times \mathbb{R}^n \times V \times \mathbb{R}^m \cong U \times V \times \mathbb{R}^{n+m}$ is also a trivialization

Proposition 21.2.6. Let $E \xrightarrow{p} X$ is a vector bundle, and $f : X \to Y$ is a homeomorphism, then $E \xrightarrow{f \circ p} Y$ is a vector bundle, suppose $\varphi_U : E_U \to U \times \mathbb{F}^n$ is a trivialization, then $(f \times 1) \circ \varphi_U =: \psi_{f(U)} : E_U \to U \times \mathbb{F}^n \to f(U) \times \mathbb{F}^n$ is a trivialization

Domain is homeomorphic to its graph

Proposition 21.2.7. $p : \Gamma_f \to X, (x, f(x)) \mapsto x$ is homeomorphism

Proof. p as a restriction on Γ_f of $X \times Y$ projecting to X is continuous, and define $q : X \to \Gamma_f, x \to (x, f(x))$, since the composition $X \xrightarrow{q} \Gamma_f \hookrightarrow X \times Y$ which is continuous because $X \xrightarrow{f} Y$, $X \xrightarrow{id} X$ are continuous, q is continuous, and p, q are inverses to each other □

Definition 21.2.8. $E \xrightarrow{\pi} X$ ia vector bundle, $f : Y \to X$ is a continuous map, then we can construct the pullback bundle $f^*E \xrightarrow{p} Y$

$$\begin{tikzcd} f^*E \arrow[r, "g"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

satisfying universal property

$$\begin{tikzcd}
F \arrow[dr, dashed, "\exists_1 h"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& Y \underset{X}{\times} E \arrow[r, "g"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Concrete construction: let $f^*E = Y \underset{X}{\times} E \subseteq Y \times X$ with subspace topology, where $Y \underset{X}{\times} E = \{(y, v) \in Y \times E | f(y) = \pi(v)\}$, let's check it is a vector bundle over Y, notice that $Y \underset{X}{\times} E \to Y$ factor through $Y \underset{X}{\times} E \to \Gamma_f \to Y$, $(y, v) \mapsto (y, \pi(v)) = (y, f(y)) \mapsto y$, where Γ_f is the graph of f which is homeomorphic to Y due to Proposition 21.2.7, notice that $Y \underset{X}{\times} E \to \Gamma_f$ is the restriction of vector bundle $Y \times E \xrightarrow{1\times\pi} Y \times X$ over Γ_f, thus $Y \underset{X}{\times} E \to Y$ is a vector bundle, suppose F as in the commutative diagram, then h is simply defined as $h(w) := (\beta(w), \alpha(w))$

Remark 21.2.9. In general, this is a pullback, but it has a vector bundle structure such that it induces an isomorphism on each fiber, now suppose $F \xrightarrow{q} Y$ is a another vector bundle such that not only the diagram commutes but also induce isomorphism on each fiber, then $F \cong f^*E$ Use this we have $(fg)^*E \cong g^*(f^*E)$, $f^*(E \oplus F) \cong f^*E \oplus f^*F$, $f^*(E \otimes F) \cong f^*E \otimes f^*F$, $1^*E \cong E$

Definition 21.2.10. Suppose E, F are vector bundles both trivialized over $\{U_\alpha\}$(this can easily be achieved, just take intersections), suppose the transition functions are $g_{\alpha\beta}, h_{\alpha\beta}$, then define the tensor product of vector bundles $E \otimes F$ by letting its transition functions be $g_{\alpha\beta} \otimes h_{\alpha\beta}$
Similarly, we can define symmetric power and exterior power of vector bundles by specifying its transition function
Does it have universal property also?

Definition 21.2.11. Let $E \xrightarrow{p} X$, $E \xrightarrow{p} X$ be vector bundles, then the direct sum $E \oplus F \xrightarrow{p} X$ is defined by transition functions $g_{\alpha\beta} \oplus h_{\alpha\beta}$, where $g_{\alpha\beta}, h_{\alpha\beta}$ are transition functions of E, F

Definition 21.2.12. Let $E \to X$ be a vector bundle, define its dual bundle as follows, if $g_{\alpha\beta}$ is a transition function, the transition function for E^* would be $\left(g_{\alpha\beta}^{-1}\right)^T$

Definition 21.2.13. quotient bundle, exterior and symmetric power of vector bundle

Proposition 21.2.14. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product

Definition 21.2.15. $F \subseteq E$ is called a vector subbundle if F is a subspace of E and $F \xrightarrow{p} X$ is also a vector space

Proposition 21.2.16. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space and $F \subseteq E$ is a vector subbundle, then there exists a vector subbundle $F^\perp \subseteq E$ such that $F_x \oplus F^\perp|_x = E|_x$ and $(F|_x)^\perp = F^\perp|_x$

Proof. □

X compact Hausdorff => E has complement

Theorem 21.2.17. If $E \xrightarrow{p} X$ is vector bundle over a compact Hausdorff space X, then there exists a vector bundle $E' \xrightarrow{p'} X$ such $E \oplus E'$ is a trivial bundle

Proposition 21.2.18. Every Lie group G is parallelizable

Proof. Pick an arbitrary basis $e_1, \cdots, e_n$ of T_1G, then $L_g^*(e_i)$ will be a basis of $T_{g^{-1}}G$ since L_g^* is an isomorphism, they form independent global sections of the tangent bundle □

Definition 21.2.19. Tautological bundle

Definition 21.2.20. Let X be a smooth manifold of dimension n(depending on the field), Ω denote the cotangent bundle, then $\omega := \bigwedge^n \Omega$ is called the canonical bundle

Definition 21.2.21. Universal bundle

Theorem 21.2.22. Let X be a paracompact Hausdoff space, there is a bijection $\left[X, \varinjlim Gr_{\mathbb{C}}(n,N)\right] \to \mathsf{Vect}^n_{\mathbb{C}}(X), [f] \mapsto [f^*(E)]$

Definition 21.2.23. If G is a topological group, then a principal G-bundle P is a fiber bundle with a continuous right G action $P \times G \to P$, and the action is free and transitive(thus regular), which imply each fiber is a G-torsor, also, $g \mapsto yg$ is a homeomorphism

Definition 21.2.24. Let $E \xrightarrow{p} X$ is a vector bundle, an inner product is a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product on E_x

Proposition 21.2.25. Let $E \xrightarrow{p} X$ is a vector bundle with an inner product $\langle , \rangle$, then we can local trivialization to be isometry on each fiber, i.e. $\langle v, w \rangle = (\varphi_U(v), \varphi_U(w)), v, w \in E_x$, where $(,)$ is the standard inner product on $U \times \mathbb{R}^n$

Proposition 21.2.26. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product

Definition 21.2.27. let G be a topological group, E, X be G-spaces, then $E \xrightarrow{p} X$ is a G-vector bundle if it is a vector bundle, p is a G map, and for any $x \in X$, $g : E_x \to E_{gx}$ is a linear map

Definition 21.2.28. Let G be a topological group, H be a closed subgroup, a G vector bundle $\pi : E \to G/H$ is called a homogeneous vector bundle

Lemma 21.2.29. Let $Y \xrightarrow{f} X, Z \xrightarrow{g} X$ be open surjective continuous maps, then the projection $p_Y : Y \times_X Z \to Y$ is open surjective

Proof. For surjectivity, if $y \in Y$, since g is surjective, $\exists z \in Z$ such that $g(z) = f(y)$, then $(z, y) \in Y \times_X Z$

To prove p_Y is open, suppose $(z_0, y_0) \in Y \times_X Z$ is in some open set, then $(z_0, y_0) \in U \times V \cap Y \times_X Z$ for some $y_0 \in U, z_0 \in V$ open, since f, g are open, $U' := f(U) \cap g(V)$ is open, let $V' := V \cap f^{-1}(U')$, then we can show V' is in the image of $U \times V \cap Y \times_X Z$, since $\forall y \in V'$, $f(y) \in U' \subseteq g(V)$, thus $f(y) = g(z)$ for some $z \in V$, hence $(y, z) \in U \times V \cap Y \times_X Z$ $\square$

Proposition 21.2.30. Let $\pi : E \to G/H$ be a homogeneous vector bundle, E_H be the fiber over the coset H, action $G \times E_H \to E$ can be regard as $\alpha : G \times_H E_H \to E$ which is an isomorphism of G vector bundles. Moreover, if H is locally compact, then for a given $\mathbb{R}H$ module E_H, $G \times_H E_H \to G/H$ is indeed a G vector bundle, hence G vector bundle E is in one to one correspondence with representations of H on E_H, so $K_G(G/H) \cong R(H)$

Proof. E_H is an $\mathbb{R}H$ module, let $G \times_H E_H$ denote the space of orbits of $G \times E_H$ under H by $h \cdot (g, \xi) = (gh^{-1}, h\xi)$, $G \times_H E_H$ is a G space with G action $g \cdot (g', \xi) \mapsto (gg', \xi)$, then the group action can be regarded as $\alpha : G \times_H E_H \to E, (g, \xi) \mapsto g\xi$, we can find its inverse $\beta : E \to G \times_H E_H, E_{gH} \ni \xi \mapsto (g, g^{-1}\xi)$, to show that this is continuous, consider $\gamma : G \times E \to G \times E, (g, \xi) \mapsto (g, g^{-1}\xi)$, then the preimage of $G \times E_H$ will be the pullback $G \times_{G/H} E := \{(g, \xi) \in G \times E | gH = \pi\xi\}$, then $G \times_{G/H} E \to G \times E_H \to G \times_H E_H, (g, \xi) \mapsto (g, g^{-1}\xi)$ factors as $G \times_{G/H} E \to E \xrightarrow{\beta} G \times_H E, (g, \xi) \mapsto \xi \mapsto (g, g^{-1}\xi)$ which open surjective, therefore β is continuous due to the previous Lemma $\square$

Definition 21.2.31. A clutching function for S^k is $f : S^{k-1} \to GL(n, \mathbb{C})$, then we can define vector bundle E_f with f being the transition function, conversely, if E is a vector bundle over S^k, since its upper and lower hemispheres are both contractible, $E = E_f$, where f is the transition function, denoting the corresonding matrix T_f

Theorem 21.2.32. $[S^{k-1}, GL(n,\mathbb{C})] \to \mathsf{Vect}^n_{\mathbb{C}}(S^k)$, $f \mapsto E_f$ is a bijection

Lemma 21.2.33. Suppose $f, g : S^{k-1} \to GL(n,\mathbb{C})$, then $(E_f \otimes E_g) \oplus \varepsilon^n \cong E_{fg} \oplus \varepsilon^n \cong E_f \oplus E_g$

Proof. Since $GL(n,\mathbb{C})$ is path connected, there is a path $A_t \in GL(2n,\mathbb{C})$ that $A_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $\begin{pmatrix} T_f & \\ & I \end{pmatrix} A_t \begin{pmatrix} I & \\ & T_g \end{pmatrix} A_t$ is $\begin{pmatrix} T_f & \\ & T_g \end{pmatrix}$ when $t = 0$ and $\begin{pmatrix} T_f T_g & \\ & I \end{pmatrix} = \begin{pmatrix} T_{fg} & \\ & I \end{pmatrix}$ when $t = 1$ □

Definition 21.2.34. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the sphere bundle $S(E)$ associated to E to be $S(E) = \bigcup_{x \in X} S(E_x)$ with the subspace topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{tikzcd}
S(E)_U \arrow[r, "\varphi_U"] \arrow[d, hook] & U \times S(\mathbb{R}^n) \arrow[d, hook] \\
E_U \arrow[r, "\varphi_U"] \arrow[dr, "p"'] & U \times \mathbb{R}^n \arrow[d] \\
& U
\end{tikzcd}$$

Definition 21.2.35. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the projective bundle $P(E)$ associated to E to be $P(E) = \bigcup_{x \in X} P(E_x)$ with the quotient topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{tikzcd}
S(E)_U \arrow[rr, "\varphi_U"] \arrow[dd, "q"] \arrow[dr] & & U \times S(\mathbb{R}^n) \arrow[dd, "q"] \arrow[dl] \\
& U & \\
P(E)_U \arrow[rr, "\varphi_U"] \arrow[ur] & & U \times P(\mathbb{R}^n) \arrow[ul]
\end{tikzcd}$$

Definition 21.2.36. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the flag bundle $F(E)$ associated to E to be $F(E) = \bigcup_{x \in X} F(E_x)$ with the subspace topology in $P(E) \times \cdots \times P(E)$

Remark 21.2.37. Consider the pullback of $\pi : F(E) \to X$, $\pi^*(E) \subseteq F(E) \times E$, consider its subbundles $L_1, \cdots, L_n$, where L_i is the subbundle that over a point in $F(E)$, it is the i-th factor, then $\pi^*(E) \cong L_1 \oplus \cdots \oplus L_n$

Definition 21.2.38. Let X be a paracompact and Hausdorff space, there exist unique functions $w_1, w_2, \cdots, w_i : \mathsf{Vect}_{\mathbb{R}}(\mathrm{X}) \to H^i(X, \mathbb{Z}_2)$, $E \to w_i(E)$, and they only depend on the isomorphism classes of E, satisfying

1. $w_i(f^*(E)) = f^*(w_i(E))$, for pullback bundle $f^*(E)$
2. $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ where $w = 1 + w_1 + w_2 + \cdots \in H^*(X, \mathbb{Z}_2)$
3. $w_i(E) = 0, \forall i > \dim E$
4. If $E \to \mathbb{R}P^\infty$ is the canonical line bundle, then $w_1(E)$ is the generator of $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$

$w_i(E)$ are called the Stiefel-Whitney classes of E

Definition 21.2.39. Let X be a paracompact and Hausdorff space, there exist unique functions $c_1, c_2, \cdots, c_i : \mathrm{Vect}_{\mathbb{C}}(\mathrm{X}) \to H^{2i}(X;\mathbb{Z})$, $E \to c_i(E)$, and they only depend on the isomorphism classes of E, satisfying
1. $c_i(f^*(E)) = f^*(c_i(E))$, for pullback bundle $f^*(E)$
2. $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ where $c = 1 + c_1 + c_2 + \cdots \in H^*(X;\mathbb{Z})$
3. $c_i(E) = 0, \forall i > \dim E$
4. If $E \to \mathbb{C}P^\infty$ is the canonical line bundle, then $c_1(E)$ is a generator of $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$, specify a generator in advance
$c_i(E)$ are called the Chern classes of E, also we define the Chern polynomial to be $c_t = 1 + c_1 t + c_2 t^2 + \cdots$ where t is just a formal variable used to keep tracking of the degree

Lemma 21.2.40. Let L_1, L_2 be line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$

Definition 21.2.41. Suppose L is a line bundle, define the Chern character $ch(L) = e^{c_1(L)} = 1 + c_1(L) + \dfrac{c_1(L)^2}{2!} + \cdots \in H^*(X;\mathbb{Q})$, then we have $ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = e^{c_1(L_1)}e^{c_1(L_2)} = ch(L_1)ch(L_2)$, If we assume $ch(L_1 \oplus L_2) = ch(L_1) + ch(L_2)$, then for $E = L_1 \oplus \cdots \oplus L_n$, $ch(E) = ch(L_1) + \cdots + ch(L_n) = n + (c_1(L_1) + \cdots + c_1(L_n)) + \left(c_1(L_1)^2 + \cdots + c_1(L_n)^2\right)/2! + \cdots$, on the other hand, we have $c(E) = c(L_1) \smile \cdots \smile c(L_n) = (1 + c_1(L_1)) \smile \cdots \smile (1 + c_1(L_n)) = 1 + c_1(E) + \cdots + c_n(E)$, where $c_i(E)$ would just be the i-th elementary symmetric polynomial of $c_1(L_1), \cdots, c_1(L_n)$, i.e. $c_i(E) = \sigma_i(c_1(L_1), \cdots, c_1(L_n))$, so we can express $c_1(L_1)^k + \cdots + c_1(L_n)^k$ in terms of $c_i(E)$, i.e. $c_1(L_1)^k + \cdots + c_1(L_n)^k = s_k(c_1(E), \cdots, c_n(E))$, thus we have an abstract definition of Chern character, $ch(E) = \dim E + s_1(c_1(E), \cdots, c_n(E)) + s_2(c_1(E), \cdots, c_n(E))/2! + \cdots$

Proposition 21.2.42. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$, $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$

21.3 Principal bundle

21.4 Topological K-theory

Definition 21.4.1. Two vector bundles $E \to X$, $F \to X$ are stably isomorphic if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^n$, denoted as $E \approx F$, we also denote $E \sim F$ if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^m$ for some n, m

Remark 21.4.2. Here stably isomorphic does not imply isomorphic, for example, $TS^2 \approx_s \varepsilon^2$, since $\varepsilon^3 \approx T^2 \oplus NS^2 \approx T^2 \oplus \varepsilon^1$ whereas TS^2 is not trivial by the hairy ball theorem, and $NS^2 \approx \varepsilon^1$ is trivial because it is very easy to find a nonvanishing global section

Definition 21.4.3. Define the reduced K group to be $\tilde{K}(X)$ which consists of $\sim$-equivalent classes, and define K group to be the formal difference of isomorphic classes $E - F$, and $E - F = E' - F'$ if $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ for some vector bundle G

Remark 21.4.4. When X is compact Hausdorff, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ is equivalent to $E \oplus F' \oplus \varepsilon^m \cong E' \oplus F \oplus \varepsilon^m$, since we can find G' such that $G \oplus G' \cong \varepsilon^m$ due to Theorem 21.2.17 $K(*) = \{\varepsilon^m - \varepsilon^n\} \cong \mathbb{Z}$, $\tilde{K}(*) = 0$, and when X compact Hausdorff we have an exact sequence $0 \to K(*) \to K(X) \to \tilde{K}(X) \to 0$, where $K(*) \to K(X)$ is simply given by $\varepsilon^m - \varepsilon^n \mapsto \varepsilon^m - \varepsilon^n$, $K(X) \to \tilde{K}(X)$ is defined as follows, given $E - F \in K(X)$, $E - F = E \oplus F' - F \oplus F' = E' - \varepsilon^m$ is mapped to E', this exact sequence splits since we have map $K(X) \to K(*)$ given by restriction

Conjecture 21.4.5. Let M be the Möbius line bundle over S^1, since $M \oplus M \cong \varepsilon^2$, and $M \otimes M \cong \varepsilon^1$, thus real K-theory of S^1 is isomorphic to $\mathbb{Z}[M]/(M^2 - 1, 2M - 2)$

Example 21.4.6. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, TS^n, NS^n be the tangent bundle and normal bundle, then $TS^n \oplus NS^n$ can be seen as the restriction of the trivial bundle $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ on S^n, thus $TS^n \oplus NS^n$ is trivial

Definition 21.4.7. Define external product $K(X) \otimes K(Y) \to K(X \times Y), a \otimes b \mapsto p_1^*(a) p_2^*(b) =: a \times b$, this is a ring homomorphism

21.5 Classifying space

Definition 21.5.1. Suppose G is a topological group, P_G is the contravariant functor from the category of CW complexes to the category of sets, mapping X to all the principal G bundles over X, a **classifying space** BG is a topological space such that $[-, BG] \to P_G(-)$ is a natural isomorphism

Lemma 21.5.2. BG is unique up to weak homotopy equivalence

Proof. Suppose $B'G$ is also a classifying space, then $[-, BG] \cong P_G(-) \cong [-, B'G]$ are natural isomorphic, by Theorem 16.5.1, we may assume BG, $B'G$ are both CW complexes, and by Lemma 3.2.1, $X \to Hom(-, X)$ is fully faithful functor, thus $BG, B'G$ are homotopic □

Theorem 21.5.3 (Milnor's contruction for classifying space)**.** Define E^nG to be $\overbrace{G * \cdots * G}^{n+1}$ are formal sums $t_0g_0 + t_1g_1 + \cdots + t_ng_n$, with $\sum t_i = 1$. $EG := \varinjlim E^nG$ are finite formal sums $\sum t_ig_i$ with $\sum t_i = 1$. $E^nG \to E^nG/G$, $EG \to EG/G =: BG$ are principal G bundles, any principal G bundle over X is a pullback bundle of $EG \xrightarrow{p} BG$

Proof. Define G right action on E^nG, EG

$$E^nG \times G \to E^nG, \left(\sum t_ig_i, g\right) \mapsto \sum t_ig_ig$$

$$EG \times G \to EG, \left(\sum t_ig_i, g\right) \mapsto \sum t_ig_ig$$

Let $U_i = \{p\left(\sum t_ig_i\right) | t_i \neq 0\}$, then we would have a equivariant homeomorphism $p^{-1}(U_i) \to U_i \times G, \sum t_ig_i \mapsto \left(p\left(\sum t_ig_i\right), g_i\right)$ with inverse $U_i \times G \to p^{-1}(U_i), \left(p\left(\sum t_ig_i\right), g\right) \mapsto \sum t_jg_jg_i^{-1}g$, this is well defined since $\left(p\left(\sum t_ig_ih\right), g\right) \mapsto \sum t_jg_jhh^{-1}g_i^{-1}g = \sum t_jg_jg_i^{-1}g$ □

Definition 21.5.4. A **topological category** $\mathscr{C}$ is a small category where $ob\mathscr{C}$, $mor\mathscr{C}$ are topological spaces and $i : ob\mathscr{C} \to mor\mathscr{C}, c \mapsto 1_c$, $s : mor\mathscr{C} \to ob\mathscr{C}, c \xrightarrow{f} d \mapsto c$, $t : mor\mathscr{C} \to ob\mathscr{C}, c \xrightarrow{f} d \mapsto d$, $\circ : mor\mathscr{C} \times mor\mathscr{C} \to mor\mathscr{C}$ are continuous. A **continuous functor** between topological categories is a functor that are continuous on both objects and morphisms

Nerve of a category

Definition 21.5.5. Define **nerve** $N\mathscr{C}$ on category $\mathscr{C}$ which is also a simplicial set, $N\mathscr{C}([n]) := Hom([n], \mathscr{C})$, the set of all functors from $[n]$ to $\mathscr{C}$, viewing $[n] = 0 \to 1 \to \cdots \to n$ as a category

Definition 21.5.6 (Segal's contruction for classifying space)**.** Define the classifying space of $\mathscr{C}$ to be $B\mathscr{C} := |N\mathscr{C}|$ as in Definition 21.5.5

Part VII

Differential topology

Chapter 22

Smooth manifold

Definition 22.0.1. An n dimensional **manifold** M is a topological space such that for any $p \in M$, there is a neighborhood $U \ni p$ such that U is homeomorphic to $\mathbb{R}^n$

Definition 22.0.2. A **chart** of $U \subseteq M$ is a homeomorphism $\varphi_U : U \to \varphi_U(U) \subseteq \mathbb{R}^n$, by abuse of notation, let $x_i : U \to \mathbb{R}$ be the composition $U \xrightarrow{\varphi_U} \mathbb{R}^n \xrightarrow{x_i} \mathbb{R}$, called **local coordinates**, suppose $\varphi_V : V \to \varphi_V(V) \subseteq \mathbb{R}^n$ is another chart, and $U \cap V \neq \varnothing$, the **transition map** is $\tau_{VU} = \varphi_V \circ \varphi_U^{-1} : \varphi_U(U \cap V) \to \varphi_V(U \cap V)$
M is a $\mathbf{C^k}$ **manifold** if transition maps $\{\tau_{VU}\} \subseteq C^k$, if $k = 0$, M is just a topological manifold, if $k = \infty$, M is a **smooth manifold**, if $k = \omega$, M is an **analytic manifold**
If n is even and transition maps are holomorphic, M is a **complex manifold**

Remark 22.0.3. A smooth manifold is locally ringed space locally ringed space with structure sheaf the sheaf of differentiable functions. An analytic manifold is locally ringed space locally ringed space with structure sheaf the sheaf of analytic functions. A complex manifold is locally ringed space with structure sheaf the sheaf of holomorphic functions

Definition 22.0.4. A **smooth map** $f : M \to N$ is map such that $\psi_V \circ f \circ \varphi_U^{-1}$ is smooth. $C^\infty(M)$ are smooth functions on M. $C_p^\infty(M)$ is the germ at p

Definition 22.0.5. A **submanifold** N is a inclusion and an immersion $i : N \hookrightarrow M$

Definition 22.0.6. The kernel of $C_p^\infty(M) \to \mathbb{R}, f \mapsto f(p)$ is a maximal ideal m_p, define the **cotangent space** $T_p^*M := \dfrac{m_p}{m_p^2}$, for $f \in C^\infty(M)$, define $(df)_p = f - f(p) \bmod m_p$, $(dx_1)_p, \cdots, (dx_n)_p$ form a basis of T_p^*M locally, $(df)_p = \dfrac{\partial f}{\partial x_1}(p)(dx_1)_p + \cdots + \dfrac{\partial f}{\partial x_n}(p)(dx_n)_p$

Definition 22.0.7. The **tangent space** T_pM at p are the derivations $Der(C_p^\infty(M))$
$\left(\dfrac{\partial}{\partial x_1}\right)_p, \cdots, \left(\dfrac{\partial}{\partial x_n}\right)_p$ form a basis of T_pM locally

Definition 22.0.8. The **pushforward(differential)** of smooth map $\phi : M \to N$ is $\phi_p : T_pM \to T_pN$, $\phi_p(X_p)(f) = X_p(f \circ \phi)$

Definition 22.0.9. The **pullback** of smooth map $\phi : M \to N$ is $\phi^* : C^\infty(N) \to C^\infty(N)$, $\phi^*(f) = f \circ \phi$

Definition 22.0.10. $(\varphi^*\alpha)_x(X) = \alpha_{\varphi(x)}(d\varphi_x(X)) = \alpha_{\varphi(x)}((\varphi_*)_x(X))$, or in short $\varphi^*\alpha(X) = \alpha(\varphi_*X)$, similarly, for k forms, $\varphi^*\alpha(X_1, \cdots, X_k) = \alpha(\varphi_*X_1, \cdots, \varphi_*X_k)$
In particular, $\varphi^*(dx) = d(x \circ \varphi)$, pullback is compatible with wedge product, $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$, and pullback is compatible with exterior derivative, $\varphi^*(d\alpha) = d(\varphi^*\alpha)$ The exterior multiplication by α is $\beta \mapsto \alpha \wedge \beta$ The interior multiplication by $v \in TM$ is $v \lrcorner : \omega(-) \mapsto \omega(v, -)$

Definition 22.0.11. $\pi : T^*M \to M$ is the cotangent bundle, ω is a one form, the **tautological one form** is $\pi^*\omega$

Definition 22.0.12. Let M be a smooth manifold, $X, Y \in C^\infty(M, TM)$ are vector fields, define Lie bracket $[X, Y] \in C^\infty(M, TM)$, $[X, Y](f) := (XY - YX)(f) = X(Y(f)) - Y(X(f))$

Remark 22.0.13. Check from local coordinates, $X(Y(f))$ is not well defined

Definition 22.0.14. Let M, N be smooth manifolds, $f : M \to N$ is a smooth map, it is called an immersion if df is injective at any point, it is called submersion if df is surjective at any point

Constant rank mapping theorem

Theorem 22.0.15. Suppose M, N are smooth manifolds with dimension m, n, $f : M \to N$ is a smooth map with constant rank r, then for any $p \in M$, denote $f(p) = q$, there are local charts (p, U), (q, V) such that $\chi_V \circ f \circ \chi_U(x_1, \cdots, x_m) = (x_1, \cdots, x_r, 0, \cdots, 0)$. Moreover, suppose M is second countable, if f is injective, then f is a immersion, if f is surjective, then f is a submersion, if f is bijective, then f is a diffeomorphism

Proof. If f is surjective but not a submersion, then $r < n$, but then by Theorem 34.3.2, f can't be surjective which is a contradiction □

Constant rank level set theorem

Theorem 22.0.16. Suppose M, N are smooth manifolds with dimension m, n, $f : M \to N$ is a smooth map with constant rank r, then a level set $S = f^{-1}(c)$ is an embedded submanifold in M of codimension r with $f|_S$ being a proper map

Proposition 22.0.17. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M, for any equivariant map $f : M \to N$, f has constant rank

Proof. For any $x \in M$, denote $y = f(x)$, it suffices to show $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ since G acts transitively on M, note that $f(gx) = gf(x)$, thus $fL_g = L_g f$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ □

Part VIII

Differential geometry

Chapter 23

Riemannian manifold

23.1 Differential geometry of surfaces

Definition 23.1.1. A **differentiable surface** is an embedding $S \hookrightarrow \mathbb{R}^3$

Lemma 23.1.2. $\gamma(t)$ is a geodesic iff $\ddot{\gamma}$ is parallel to the normal $\vec{n}$, meaning no acceleration in S
A geodesic γ on S has constant speed
The geodesic curvature of a curve γ is the curvature of the projection onto tangent plane, γ is a geodesic iff the geodesic curvature of γ is zero

Proof. $\frac{d}{dt}|\dot{\gamma}|^2 = 2\ddot{\gamma} \cdot \dot{\gamma} = 0$ □

23.2 Curvature

Definition 23.2.1. A **Riemannian manifold** is (M, g) where M is a smooth manifold and **Riemannian metric** $g_p : S^2(T_pM) \to \mathbb{R}$ is a positive definite

Definition 23.2.2. An **affine connection** is

$$\begin{aligned} \nabla : \Gamma(TM) \otimes \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

satisfying

- $\nabla_{fX} Y = f \nabla_X Y$, i.e. ∇ is $C^\infty(M, \mathbb{R})$ linear in the first variable
- $\nabla_X(fY) = XfY + f\nabla_X Y$, i.e. ∇ satisfies Leibniz rule in the second variable

From this we can define covariant derivative ∇, $\nabla_X f = X_f$, $\nabla_X(\alpha)(Y) = \nabla_X(\alpha(Y)) - \alpha(\nabla_X Y)$, here α is a covector, similarly for any tensor, Write contraction $(\nabla T)(\alpha_1, \cdots, \alpha_m, X_1, \cdots, X_n, X) = (\nabla_X T)(\alpha_1, \cdots, \alpha_m, X_1, \cdots, X_n)$, T is a tensor

Note. $\nabla_X(\alpha(Y)) = \nabla_X(\alpha)(Y) + \alpha(\nabla_X Y)$

Definition 23.2.3. ∇ is an affine connection, the **torsion tensor** is

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Definition 23.2.4. The Levi-Civita connection ∇ is the one satisfying

- $\nabla_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$, i.e. $\nabla g = 0$
- $\nabla_X Y - \nabla_Y X = [X, Y]$, i.e. ∇ is torsion free

Definition 23.2.5. ∇ is the Levi-Civita connection, the Riemannian curvature tensor is $R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$

Remark 23.2.6. X, Y are commuting vector fields around x_0, then $\frac{d}{ds}\frac{d}{dt}\tau_{sX}^{-1}\tau_{tY}^{-1}\tau_{sX}\tau_{tY} Z = R_{XY} Z$, τ is the parallel transport

$$\begin{tikzcd} \cdot \arrow[r, "\tau_{sX}"] & \cdot \arrow[d, "\tau_{sY}^{-1}"] \\ \cdot \arrow[u, "\tau_{sY}"] & \cdot \arrow[l, "\tau_{sX}^{-1}"] \end{tikzcd}$$

Proposition 23.2.7.

1. $R_{YX} = -R_{XY}$

2. $(R_{XY} Z, W) = -(R_{XY} W, Z)$

3. $R_{XY} Z + R_{YZ} X + R_{ZX} Y = 0$

4.

The **second Bianchi identity** follows

$$\nabla_X R_{YZ} + \nabla_Y R_{ZX} + \nabla_Z R_{XY} = 0$$

Remark 23.2.8. If write $(R_{XY} Z, W) = R(X, Y, Z, W)$, then R is antisymmetric about the first two variables and the last two variables, R satisfies Jacobi identity, the first two and the last two variables can switch place

Proof.

1.

2.

3.

4. Follow from above

□

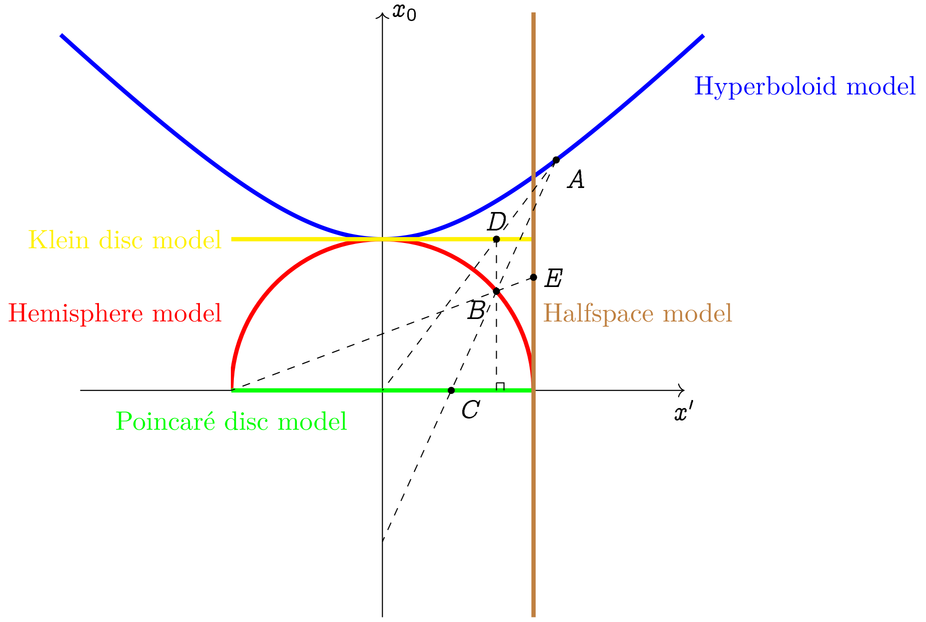
Definition 23.2.9. $\{e_i\}$ is an orthonomal basis, the **Ricci curvature** is $\mathrm{Ric}(X) = \sum R_{X,e_i} e_i$. The **scalar curvature** is $S = \mathrm{Tr}\,\mathrm{Ric} = \sum(\mathrm{Ric}(e_j), e_j) = \sum(R_{e_j,e_i} e_i, e_j)$. The **Einstein curvature** is $G = R - \frac{1}{2} g S$

23.3 Hyperbolic geometry

Definition 23.3.1. $\mathbb{R}^{n+1}$ with metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$ is the **Minkowski space**

The **hyperboloid model** is $\mathbb{H} = \{x_1^2 + \cdots + x_n^2 - x_0^2 = -1, x_0 > 0\}$. The Riemannian metric is the pullback metric $ds^2 = dx_1^2 + \cdots + dx_n^2 - dx_0^2$

The geodesics are intersections of $\mathbb{H}$ and two dimensional subspaces of $\mathbb{R}^{n+1}$ $(d\sinh s)^2 + (d\cosh s)^2 = \cosh^2 s ds^2 - \sinh^2 s ds^2 = ds^2$, thus $\mathbb{H}^1$ is isomorphic to $\mathbb{E}^1$



$(x', x_n) \mapsto \left(\dfrac{2x'}{1+x_n}, 1\right)$, $x' = (x_0, \cdots, x_{n-1})$ is the isometry from the hemisphere to the halfspace

$(x', 1) \mapsto \left(\dfrac{4x'}{4+|x'|^2}, \dfrac{4-|x'|^2}{4+|x'|^2}\right)$, $x' = (x_0, \cdots, x_{n-1})$ is the isometry from the halfspace to the hemisphere

$x \mapsto \left(\dfrac{x'}{1+x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to Poincaré disc

$x \mapsto \left(\dfrac{2x}{1-|x|^2}, \dfrac{1+|x|^2}{1-|x|^2}\right)$ is the isometry from Poincaré disc to the hyperboloid

$x \mapsto (1, x')$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to Klein disc

$x \mapsto \left(\dfrac{x'}{x_0}, \dfrac{1}{x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hyperboloid to the hemisphere

$x \mapsto \left(\dfrac{x'}{x_0}, \dfrac{1}{x_0}\right)$, $x' = (x_1, \cdots, x_n)$ is the isometry from the hemisphere to the hyperboloid

The **hemisphere model** is $\mathbb{H} = \{x_0 > 0\} \cap S^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
\sum_{i=0}^{n-1}\left[d\left(\frac{x_i}{x_n}\right)\right]^2 - \left[d\left(\frac{1}{x_n}\right)\right]^2 &= \sum_{i=0}^{n-1}\left(\frac{x_0dx_i - x_idx_0}{x_0^2}\right)^2 - \left(-\frac{dx_0}{x_0^2}\right)^2 \\
&= \sum_{i=0}^{n-1}\frac{x_0^2dx_i^2 - 2x_ix_0dx_idx_0 + x_i^2dx_0^2}{x_0^4} - \frac{dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(|x'|^2)d(x_0^2)}{2x_0^4} + \frac{|x'|^2dx_0^2 - dx_0^2}{x_0^4} \\
&= \frac{dx'^2}{x_0^2} - \frac{d(1-x_0^2)d(x_0^2)}{2x_0^4} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2}{x_0^2} + \frac{2dx_0^2}{x_0^2} - \frac{dx_0^2}{x_0^2} \\
&= \frac{dx'^2 + dx_0^2}{x_0^2}
\end{aligned}$$

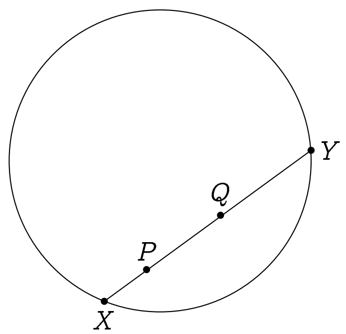
The **half space model** is $\mathbb{H} = \{x_0 > 0\} \cap \{x_n = 1\}$. The Riemannian metric is pullback metric

$$\begin{aligned}
\frac{\sum_{i=0}^{n-1} d\left(\frac{4x_i}{4+|x'|^2}\right)^2 + d\left(\frac{4-|x'|^2}{4+|x'|^2}\right)^2}{\left(\frac{4x_0}{4+|x'|^2}\right)^2} &\underset{X=4+|x'|^2}{=\!=} \frac{\sum_{i=0}^{n-1} d\left(\frac{4x_i}{X}\right)^2 + d\left(\frac{8}{X}-1\right)^2}{\left(\frac{4x_0}{X}\right)^2} \\
&= \frac{X^2}{x_0^2}\left(\sum_{i=0}^{n-1} d\left(\frac{x_i}{X}\right)^2 + 4d\left(\frac{1}{X}\right)^2\right) \\
&= \frac{X^2}{x_0^2}\left(\sum_{i=0}^{n-1}\left(\frac{X\,dx_i - x_i dX}{X^2}\right)^2 + 4\frac{dX^2}{X^4}\right) \\
&= \frac{1}{x_0^2}\left(\sum_{i=0}^{n-1}\frac{X^2dx_i^2 + x_i^2dX^2 - 2Xx_idX\,dx_i}{X^2} + 4\frac{dX^2}{X^2}\right) \\
&= \frac{1}{x_0^2}\left(dx'^2 + \frac{|x'|^2dX^2}{X^2} + \frac{4dX^2}{X^2} - \frac{dX\,d(|x'|^2)}{X}\right) \\
&= \frac{1}{x_0^2}\left(dx'^2 + \frac{X\,dX^2}{X^2} - \frac{dX\,d(X-4)}{X}\right) \\
&= \frac{dx'^2}{x_0^2}
\end{aligned}$$

The **Poincaré disc model** is $\mathbb{H} = D^n$. The Riemannian metric is pullback metric

$$\begin{aligned}
\sum_{i=1}^{n} d\left(\frac{2x_i}{1-|x|^2}\right)^2 - d\left(\frac{1+|x|^2}{1-|x|^2}\right)^2 &\overset{X=1-|x'|^2}{=\!=\!=} \sum_{i=1}^{n} d\left(\frac{2x_i}{X}\right)^2 - d\left(\frac{2}{X}-1\right)^2 \\
&= 4\sum_{i=1}^{n}\left(\frac{X dx_i + x_i dX}{X^2}\right)^2 - 4\left(-\frac{dX}{X^2}\right)^2 \\
&= 4\sum_{i=1}^{n}\frac{X^2 dx_i^2 + x_i^2 dX^2 - X x_i dX dx_i}{X^4} - 4\frac{dX^2}{X^4} \\
&= 4\left(\frac{dx^2}{X^2} + \frac{|x|^2 dX^2}{X^4} - \frac{dX^2}{X^4} - \frac{d(|x|^2)dX}{X^3}\right) \\
&= 4\left(\frac{dx^2}{X^2} - \frac{X dX^2}{X^4} - \frac{d(1-X)dX}{X^3}\right) \\
&= \frac{4dx^2}{X^2} = \frac{4dx^2}{(1-|x|^2)^2}
\end{aligned}$$

The **Klein disc model** is $\mathbb{H} = D^n$. The distance between P, Q is $\frac{1}{2}\ln\left(\frac{|XQ||PY|}{|XY||PQ|}\right) = \frac{1}{2}\ln(X, P; Q, Y)$, $(X, P; Q, Y)$ is the cross ratio



Theorem 23.3.2. $\mathrm{Isom}(\mathbb{H}^2) = PSL(2, \mathbb{R})$

Proof. An isometry sends half circles and orthogonal lines to half circles or orthogonal lines, by Schwarz reflection principle 33.2.3, it is can be regard as an isometry on $\mathbb{C}P^1$ sending $\mathbb{R}P^1$ to $\mathbb{R}P^1$, then it necessarily has to be in $PSL(2, \mathbb{R})$ □

Theorem 23.3.3. $\mathrm{Isom}(\mathbb{H}^3) = PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$

Proof. Since $\partial\mathbb{H}^3$ is the Riemann sphere, every isometry on $\mathbb{H}^3$ restricts to a conformal map on $\partial\mathbb{H}^3$ because it sends hemispheres and orthogonal planes to hemispheres or orthogonal planes, hence it is a Möbius transformation. On the other hand, Möbius transformations which can all be extended to an isometry on $\mathbb{H}^3$, translations $z \mapsto z + \lambda$ can be extended to $(z, x_3) \mapsto (z + \lambda, x_3)$, dilations $z \mapsto \lambda z$ can be extended to $(z, x_3) \mapsto (\lambda z, |\lambda| x_3)$, inversions $z \mapsto -\frac{1}{z}$ can be extended to $(z, x_3) \mapsto \left(\frac{-\bar{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2}\right)$. Therefore the isometry group for $\mathbb{H}^3$ is $PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z} \cong SL(2, \mathbb{C})$ □

Part IX

Complex geometry

Chapter 24

Riemann surface

Definition 24.0.1. A **Riemann surface** is a one dimensional complex manifold

Theorem 24.0.2 (Riemann's removable singularity theorem)**.** f is holomorphic on $X \setminus \{a\}$ and bounded near a, then f is holomophic on X

Theorem 24.0.3 (Principle of analytic continuation)**.** X is connected, $X \xrightarrow{f} Y$ is holomorphic and $f \equiv c$ on some nondiscrete subset of X, then $f \equiv c$ on X

Remark 24.0.4. This does not apply to higher dimensions, for example, $f(z, w) = z$, but in higher dimensions, we have Theorem 25.0.1

Theorem 24.0.5 (Local behaviour of holomorphic maps)**.** $X \xrightarrow{f} Y$ is a nonconstant holomorphic map, $a \in X$, $f(a) = b \in Y$. There are local charts $U \xrightarrow{\phi} \mathbb{C}$, $V \xrightarrow{\psi} \mathbb{C}$ of a, b such that $\psi f \phi^{-1} = z^k$ for some $k \geq 1$

Remark 24.0.6. If the **multiplicity** $k > 1$, a is a **branch point**

Theorem 24.0.7. $X \xrightarrow{f} Y$ is a proper nonconstant holomorphic map between Riemann surfaces, there exists some n such that f take every value $c \in Y$, counting multiplicities, n times

Theorem 24.0.8 (Radó's theorem)**.** A connected Riemann surface is second countable

Uniformization theorem

Theorem 24.0.9 (Uniformization theorem)**.** A simply connected Riemann surface is either $\mathbb{C}$, $\mathbb{P}$ or $\mathbb{H}^2$

Chapter 25

Complex manifold

Identity principle

Theorem 25.0.1 (Identity principle). X is connected, $X \xrightarrow{f} Y$ is holomorphic and $f \equiv c$ on some nonempty open subset of X, then $f \equiv c$ on X

Definition 25.0.2. M is a smooth manifold, an **almost complex structure** is $J : TM \to TM$ such that $J^2 = -1_{TM}$

Example 25.0.3. S^4 cannot be given an almost complex structure. S^6 can be given an almost complex structure but not a complex structure

A complex manifold always give an almost complex structure by $J\frac{\partial}{\partial z_i} = i\frac{\partial}{\partial z_i}$, $J\frac{\partial}{\partial \bar{z}_i} = -i\frac{\partial}{\partial \bar{z}_i}$

Definition 25.0.4. A is a $(1,1)$ form, the Nijenhuis tensor is

$$N_A(X,Y) = -A^2[X,Y] + A([AX,Y] + [X,AY]) - [AX,AY]$$

Theorem 25.0.5 (Newslander-Nirenberg theorem). J is **integrable** iff $N_J = 0$. Meaning there is a unique complex structure which will give J

Proposition 25.0.6. Given an almost complex structure, we can find coordinate charts $(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$ such that $\mathtt{Span}\left\{\frac{\partial}{\partial z_i}\right\}$, $\mathtt{Span}\left\{\frac{\partial}{\partial \bar{z}_i}\right\}$ to be the i and $-i$ eigenspaces of J

Definition 25.0.7. A **Hermitian manifold** M is a complex manifold with a **Hermitian metric** $h = \sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta$, where $h_{\alpha\bar{\beta}}$ is a positive definite Hermitian matrix. This gives a Riemannian metric

$$g = \frac{1}{2}(h + \bar{h}) = \frac{1}{2}\left(\sum h_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_\beta + \sum h_{\beta\bar{\alpha}} d\bar{z}_\alpha \otimes dz_\beta\right) = \sum h_{\alpha\bar{\beta}}(dz_\alpha \otimes d\bar{z}_\beta + d\bar{z}_\beta \otimes dz_\alpha)$$

Also gives **associate** $(1,1)$ **form**

$$\omega = -\frac{h - \bar{h}}{2i} = \frac{i}{2}(h - \bar{h}) = \frac{i}{2}\sum h_{\alpha\bar{\beta}}(dz_\alpha \otimes d\bar{z}_\beta - d\bar{z}_\beta \otimes dz_\alpha) = \frac{i}{2}\sum h_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

Definition 25.0.8. A **Kähler manifold** M is a complex and symplectic manifold with an integrable almost complex structure J with a Riemannian metric $g(u,v) = \omega(u, Jv)$

Chapter 26

Symplectic manifold

Definition 26.0.1. M is a smooth manifold, a **symplectic structure** on M is a 2 form ω that is nondegenerate and anti-symmetric on T_pM

Part X

Lie group

Chapter 27

Topological group

Definition 27.0.1. G is a **topological group** if it is a group and a topological space so that the group multiplication $G \times G \to G$ and the inverse map $G \to G$ are continuous maps

Definition 27.0.2. $f : G \to \mathbb{R}/\mathbb{C}$ is a continuous function, $L_y f(x) = f(y^{-1}x)$, $R_y f(x) = f(xy)$, $L_{yz} = L_y L_z$, $R_{yz} = R_y R_z$, f is called left/right uniformly continuous if $\forall \varepsilon > 0$, $\exists V \ni e$ such that $\|L_y f - f\| < \varepsilon / \|R_y f - f\| < \varepsilon$, $\forall y \in V$, $\|\cdot\|$ is the supremum norm

Proposition 27.0.3. If $f \in C_c(G)$, then f is both left and right uniformly continuous

Proof. Easy proof by a very standard analysis argument $\square$

Definition 27.0.4. If f is a Borel measurable function on G, then f factor through G/H, otherwise suppose $f(y) \neq f(z)$, $y, z \in xH$, $f^{-1}(f(y)) \cap xH \subsetneq xH$ is a Borel set which is impossible, because then $x^{-1}f^{-1}(f(y)) \cap H \subsetneq H$ will also be a Borel set, consider $\Gamma = \{S \in \mathscr{P} | H \subseteq H \text{ or } H \cap S = \emptyset\}$, then Γ is a sigma algebra containing all open sets hence Borel algebra, we reached a contradiction
Thus for most purposes one may as well work with G/H which is Hausdorff(L^p spaces for instance, mod almost everywhere vanishing function)
For a locally compact Hausdorff group, A Borel measure μ on G is called left/right invariant if $\mu(xE) = \mu(E)/\mu(Ex) = \mu(E)$, $x \in G, E \in \mathscr{B}(G)$
A linear functional I is left/right invariant if $I(L_x f) = I(f)/I(R_x f) = I(f)$
A left/right Haar measure on G is a left/right invariant Radon measure μ on G, for example, Lebesgue measure on $\mathbb{R}^n$, counting measure on G with discrete topology

Example 27.0.5. Continuous bijective group homomorphism doesn't imply homeomorphism, which is really obvious, by taking the identity map and a discrete topology on the topological group G

Definition 27.0.6. Let G be a topological group, then a 1-parameter subgroup means a continuous group homomorphism $\varphi : \mathbb{R} \to G$, $\varphi(s + t) = \varphi(s)\varphi(t)$, in the case of a Lie group, φ is required to be smooth

Definition 27.0.7. Suppose G is a connected, locally pathconnected and (semi-)locally simply connected topological space, then it has a universal cover $\tilde{G}$ which is unique up to an isomorphism, a connected Lie group certain satisfies this

Proposition 27.0.8. Denote $\pi : \tilde{G} \to G$ as the covering map, let $\bar{G}$ be the set of maps $T : \tilde{G} \to \tilde{G}$, such that $\pi(Tx) = g(\pi x)$ for some $g \in G$, i.e. the following diagram commutes

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{T} & \tilde{G} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G & \xrightarrow{g} & G \end{array}$$

Then $\bar{G}$ which is a group acts transitively and freely on $\tilde{G}$, thus we can think of the universal cover $\tilde{G}$ also as a topological group

Proof. Given $x, y \in \tilde{G}$, there is a unique $g \in G$ such that $g(\pi x) = \pi y$, since $\tilde{G}$ is the universal cover, there is a unique lift such that $T(x) = y$, thus the action is free and transitive $\square$

Chapter 28

Algebraic group

Definition 28.0.1. An **algebraic group** G is a group and an algebraic variety such that multiplication and inverse are morphisms, if G is an affine algebraic variety, then it is called a **linear algebraic group** or **affine algebraic group**. $\mathbb{F}$ group means linear algebraic group over $\mathbb{F}$

Chapter 29

Lie group

29.1 Lie groups

Definition 29.1.1. A **real Lie group** G is a group and a smooth manifold such that multiplication $G \times G \to G$ and inverse $G \to G$ are smooth
A **complex Lie group** is a group and complex manifold such that multiplication and inverse are holomorphic
a Lie subgroup H is a subgroup and an immersed submanifold

Definition 29.1.2. Left multiplication L_g by g is an isomorphism, a vector field X on G is called **left invariant** if $(L_g)_*X = X$, by Exercise 50.0.5, $[X, Y]$ is also left invariant since $(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$
Define **Lie algebra of** G to be left invariant vector fields. Equivalently, T_1G
If $\phi : G \to H$ is a homomorphism of Lie groups, then $d\phi : \mathsf{Lie}(G) \to \mathsf{Lie}(H)$ or $(d\phi)_1 : T_1G \to T_1H$ is an homomorphism of Lie algebras
Suppose $H \leq G$ is a Lie subgroup, then $\mathsf{Lie}(H) = T_1H \leq T_1G$

Proposition 29.1.3. Lie groups are parallelizable

Proof. For any $0 \neq X_1 \in T_1G$, we can define a vector field $X_g = (L_g)_1X_1$, this is a nonvanishing global section of the tangent bundle, G is parallelizable $\square$

Definition 29.1.4. A Lie group representation (ρ, V) is a Lie group homomorphism $\rho : G \to GL(V)$

Proposition 29.1.5. Let V be a complex vector space, (π, V) be a Lie group representation of a compact Lie group G, then there exists a positive definite Hermitian form such that (π, V) is unitary

Proof. Choose any positive definite Hermitian form $\langle, \rangle$, define Hermitian form

$$(v, w) := \int_G \langle \pi(g)v, \pi(g)w \rangle d\mu$$

Where μ is the Haar measure with $\int_G d\mu = 1$, integrals make sense since G is compact, then $(,)$ is G left invariant $\square$

Definition 29.1.6. Lie group G acts on smooth manifold M, G_p is the stablizer of p. The **isotropy representation** is $G_p \to GL(T_pM)$, $g \mapsto d_pg$

29.2 Exponential map

Lemma 29.2.1. The exponential map $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is defined on $M_n(\mathbb{C})$ and logarithmic map $\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(A-I)^k}{k}$ are defined on $|A - I| < 1$ and there inverses to each other locally, moreover, the exponential map is surjective onto $GL(n, \mathbb{C})$

Remark 29.2.2. Note that this also holds for a Banach algebra A

Proof. Just compare the coefficients of multiplication of series □

AV<=V <=> e^tAV<=V

Lemma 29.2.3. Let e^{tA} be a one parameter subgroup, then $V \leq \mathbb{R}^n$ is invariant under A iff invariant under $e^{tA}, \forall t$, in particular, $Av = 0$ iff $e^{tA}v = 0, \forall t$

Proof. If $AV \subseteq V$, then $e^{tA}V = \sum_{k=0}^{\infty} t^k \frac{A^k}{k!} V \subseteq V$

If $e^{tA}V \subseteq V, \forall t$, since V is closed, $\left.\frac{d}{dt}\right|_{t=0} e^{tA}V = AV \subseteq V$ □

Proposition 29.2.4. Observe that $v'(t) = Av(t)$ with $v(0) = v_0$ has the solution $v(t) = e^{tA}v_0$ Consider V_m to the vector space of homogeneous polynomials in n variables of degree m, define group action of $GL(n, \mathbb{C})$ on V_m, $g \cdot f(x) := f(g^{-1}x)$, consider $v(t) = e^{tA} \cdot f := f(e^{-tA}x)$, then $v'(t) = \left.\frac{d}{dt}\right|_{t=0} f(e^{-tA}x) =: D_A f$, where D_A is a linear differential operator $V_m \to V_m$ by Lemma 29.2.3, then we should have $f(e^{-tA}x) = v(t) = e^{tD_A} f$, therefore we would get $D_A = -A^T$, and it will be easy to check that $D_{[A,B]} = [D_A, D_B]$

Proof. If we denote $g = (g_{ij}) \in GL(n, \mathbb{C})$, $f(x) = \sum_{i_1, \cdots, i_n} C_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$, then $f(g^{-1}x) = \sum_{i_1, \cdots, i_n} C_{i_1, \cdots, i_n} (g_{11}x_1 + \cdots + g_{1n}x_n)^{i_1} \cdots (g_{n1}x_1 + \cdots + g_{nn}x_n)^{i_n}$ is still a homogeneous polynomial in n variables of degree m
Denote $A = (a_{ij})$,

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} f(e^{-tA}x) &= \nabla f(x) \cdot \left.\frac{d}{dt}\right|_{t=0} e^{-tA}x \\
&= -\nabla f(x) \cdot Ax \\
&= -\sum_{i,j} a_{ij} x_j \frac{\partial f}{\partial x_i} \\
&= \left(-\sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i} \right) f \\
&= (-\nabla^T A x) f \\
&= D_A f
\end{aligned}
$$

In particular, $D_A x_i = -\sum_{j=1}^{n} a_{ij} x_j$, thus D_A has matrix $-A^T$ with respect to $x_1, \cdots, x_n$, basis of V_1 □

Example 29.2.5. Consider Lie group $SL(2, \mathbb{C})$ whose Lie algebra is $\mathfrak{sl}(2, \mathbb{C})$, which is generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, thus $D_H = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, $D_X = -x_2 \frac{\partial}{\partial x_1}$, $D_Y = -x_1 \frac{\partial}{\partial x_2}$

Definition 29.2.6. Let G be a (Lie) group, then a 1-parameter subgroup means a (smooth) group homomorphism $\phi : \mathbb{R} \to G$, $\phi(s+t) = \phi(s)\phi(t)$

Lie group homomorphism induce Lie algebra homomorphism

Proposition 29.2.7. Let $\phi : G \to H$ be a homomorphism of Lie groups, then $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ or $(d\phi)_1 : T_1G \to T_1H$ is an homomorphism of Lie algebras

Proof. Suppose X is a left invariant vector field on G, then $(d\phi)_g X_g = (d\phi)_g (dL_g)_1 X_1(f) = X_1(f \circ \phi \circ L_g) = X_1(f \circ \phi \circ L_g) = (dL_{\phi(g)})_1 (d\phi)_1 X_1(f)$ which gives a left invariant vector field, thus using Lemma 50.0.4

$$\begin{aligned}
(d\phi)[X,Y](f) &= [X,Y](f \circ \phi) \\
&= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
&= X(((d\phi Y)f) \circ \phi) - Y(((d\phi X)f) \circ \phi) \\
&= ((d\phi X)(d\phi Y)f) \circ \phi - ((d\phi Y)(d\phi X)f) \circ \phi \\
&= ([d\phi X, d\phi Y]f) \circ \phi
\end{aligned}$$

Therefore $(d\phi)[X,Y] = [(d\phi X), (d\phi Y)]$, $d\phi$ is a Lie algebra homomorphism □

Proposition 29.2.8. One parameter subgroups are precisely the maximal integral curves of the left invariant vector fields starting at 1

Remark 29.2.9. There is a one to one correspondence, $\{\text{One parameter subgroups of } G\} \leftrightarrow \mathrm{Lie}(G) \leftrightarrow T_1G$

Proof. Suppose $\phi : \mathbb{R} \to G$ is a one parameter subgroup, let $X_1 = \phi'(0)$, then we have a left invariant vector field X on G, think of $\dfrac{\partial}{\partial t}$ as a left invariant vector field on $\mathbb{R}$, thus ϕ as Lie group homomorphism induces $(d\phi)\dfrac{\partial}{\partial t}$ which is also a left invariant vector field and $\phi'(s) = (d\phi)_s \left.\dfrac{\partial}{\partial t}\right|_s = X_{\phi(s)}$ as in Proposition 29.2.7

Conversely, if $\phi : \mathbb{R} \to G$ is the maximal integral curve of some left invariant vector field X, suppose the global flow generated by X is $\varphi : G \times \mathbb{R} \to G$, then $\varphi(1,t) = \phi(t)$, $\phi(t+s) = \varphi(1, t+s) = \varphi(\varphi(1,t), s) = \varphi(\phi(t), s)$, since $L_{\phi(t)}$ is an isomorphism, thus $L_{\phi(t)} \circ \phi$ is the maximal integral curve starting at $\phi(t)$, thus $\varphi(\phi(t), s) = \phi(t)\phi(s)$ □

Definition 29.2.10. For any $A \in T_1G$, define the exponential map $\exp A := \phi_A(1)$ where $\phi_A : \mathbb{R} \to G$ is the one parameter subgroup corresponding to A, also it is easy to see that $\exp tA := \phi_{tA}(1) = \phi_A(t)$ which is a scaling of the integral curve, and $\exp(t+s)A = \exp tA \exp sA$ since $\exp tA$ is a one parameter subgroup, and thus $(\exp A)^{-1} = \exp(-A)$

Properties of exponential map

Proposition 29.2.11. (**Properties of exponential map**)

Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$

(a) The exponential map is a smooth map

(b) $(d\exp)_0 : \mathfrak{g} \cong T_0\mathfrak{g} \to T_1G \cong \mathfrak{g}$ is the identity map, which implies that the exponential map is a local diffeomorphism around 0

(c) Suppose $\phi : G \to H$ is a Lie group homomorphism, then the following diagram commutes

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "(d\phi)_1"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"] \\
G \arrow[r, "\phi"] & H
\end{tikzcd}$$

Proof.

(a)

(b) For any $A \in \mathfrak{g}$, consider $\gamma : \mathbb{R} \to \mathfrak{g}, t \mapsto tA$ which is a one parameter subgroup of $\mathfrak{g}$, thus $A = \gamma'(0) \in T_0\mathfrak{g}$, and $\exp A = \gamma(1) = A$

(c) Define $\gamma(t) = \phi(\exp tA)$ which is a one parameter subgroup of H since $\gamma(t+s) = \phi(\exp(t+s)A) = \phi(\exp tA \exp sA) = \phi(\exp tA)\phi(\exp sA) = \gamma(t)\gamma(s)$, then $\gamma'(0) = \left.\frac{\partial}{\partial t}\right|_{t=0} \phi(\exp tA) = (d\phi)_1 \left.\frac{\partial}{\partial t}\right|_{t=0} \exp tA = (d\phi)_1 A$, on the other hand, $\exp(t(d\phi)_1 A)$ is one parameter subgroup of H corresponds to $(d\phi)_1 A = \gamma'(0)$, thus $\exp(t(d\phi)_1 A) = \gamma(t) = \phi(\exp tA)$ $\square$

Proposition 29.2.12. Let G be a Lie group and $H \leq G$ a Lie subgroup, then $\mathrm{Lie}(H) = \{A \in \mathrm{Lie}(G) | \exp tA \in H, \forall t \in \mathbb{R}\}$

Part XI

Algebraic geometry

Chapter 30

Variety

30.1 Affine Varieties

Definition 30.1.1. $V \subseteq \mathbb{A}^n$ is an algebraic set, $f \in k[V]$

$$D(f) = \{(x_1, \cdots, x_n) \in V | f(x_1, \cdots, x_n) \neq 0\} = V(f)^c$$

form a basis for the Zariski topology on V
$D(f)$ can also be thought of as an algebraic set

$$\{(x_1, \cdots, x_n, z) | zf(x_1, \cdots, x_n) = 0\}$$

The coordinate ring can be written as $k[V][\frac{1}{f}] = k[V]_f$, where z is just replaced by $\frac{1}{f}$

Theorem 30.1.2. $\sqrt{I} = \bigcap\limits_{P \supseteq I \text{ prime}} P$

Hilbert Nullstellensatz weak form

Theorem 30.1.3 (Hilbert Nullstellensatz weak form)**.** k is algebraically closed, $m < k[x_1, \cdots, x_n]$ is a maximal ideal, then $k[x]/m \cong k$

Theorem 30.1.4 (Hilbert Nullstellensatz strong form)**.** k is algebraically closed, $I(V(J)) = \sqrt{J}$

Proof. Since $\sqrt{J} = \bigcap\limits_{P \supseteq J \text{ prime}} P$, suppose $f \notin P$ for some $P \supseteq J$, consider $\varphi : k[x] \to k[x]/P \to A_{\overline{f}} \to A_{\overline{f}}/m$ which is a field, hence $\ker \varphi$ is a maximal ideal, by Theorem 30.1.3, $B/m \cong k[x]/\ker\varphi \cong k$, then $(\varphi(x_1), \cdots, \varphi(x_n)) \in V(P) \subseteq V(J)$ but $f(\varphi(x_1), \cdots, \varphi(x_n)) = \varphi(f) \neq 0 \Rightarrow f \notin I(V(J))$ $\square$

Proposition 30.1.5. Morphism $V \xrightarrow{\varphi} W$ induce a ring homomorphism $k[W] \xrightarrow{\varphi^*} k[V], f \mapsto f \circ \varphi$, and if $f(p) = q$, then $(\varphi^*)^{-1}(m_q) = m_p$, thus conversely, if $\alpha : k[W] \to k[V]$ is a ring homomorphism, then $\alpha^{-1} : Spmk[V] \to Spm[W]$ is a morphism which can be identified with $\varphi : V \to W$, and $\varphi^* = \alpha$

Proposition 30.1.6. A finite morphism $V \xrightarrow{\varphi} W$ between affine varieties is quasifinite

Proof. $\varphi(p) = q \Leftrightarrow (\varphi^*)^{-1}(m_p) = q$, $m_p \supseteq \varphi^*(\varphi^*)^{-1}(m_p) = \varphi^*(m_q)$

$$\varphi^{-1}(q) \leftrightarrow \left\{\text{maximal ideals of } B = \frac{k[W]}{\langle\varphi^*(m_q)\rangle}\right\}$$

Since $k[W]$ is a finite $k[V]$ algebra, so B is finite dimensional over $\dfrac{k[V]}{m_p} \cong k$ By Chinese Remainder theorem 5.2.11, $B \to B/m_1 \times \cdots \times B/m_s$ is surjective, $\dim B \geq s$, since $\dim B < \infty$, hence $s < \infty$, thus B has only finitely many maximal ideals $\square$

W->V dominant => k[V]->k[W] injective

Proposition 30.1.7. $W \xrightarrow{\varphi} V$ is dominant iff $k[V] \xrightarrow{\varphi} k[W]$ is injective

Proof. $f \in \ker \varphi^* \Leftrightarrow f \circ \varphi = 0$, $\mathrm{im}\varphi$ dense $\Rightarrow f = 0$. Conversely, $\overline{\mathrm{im}\varphi} \subsetneq V \Rightarrow 0 \neq f \in I(\overline{\mathrm{im}\varphi})$ □

Proposition 30.1.8. If $W \xrightarrow{\varphi} V$ is dominant and finite, then φ is surjective

Proof. By Proposition 30.1.7, $k[W]$ is integral over $k[V]$, by Theorem 5.2.21, for any $m_q < k[V]$, there exists maximal ideal $n < k[W]$ such that $n \cap k[V] = m_q$ □

Corollary 30.1.9. V is an algebraic set, $\dim V = \dim k[V]$. If V is irreducible, then $\dim V = \operatorname{trdeg} k(V)$

Example 30.1.10. $\dim \mathbb{A}^n = \dim k[x_1, \cdots, k_n] = \operatorname{trdeg}(k(x_1, \cdots, x_n)/k) = n$

Definition 30.1.11. V is an algebraic set, a **regular function** on $U \subseteq V$ is $\dfrac{f}{g}$, $f, g \in k[V]$ such that g doesn't vanish on U, i.e. a rational function that is regular on U

30.2 Varieties

Definition 30.2.1. A **prevariety** is a locally ringed space $(X, \mathcal{O})$ such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine variety $(V, \mathcal{O}_{SpmV})$

Definition 30.2.2. A morphism $W \xrightarrow{\varphi} V$ is **dominant** if $\varphi(W)$ is dense

Definition 30.2.3. A morphism $W \xrightarrow{\varphi} V$ is **quasifinite** if $\varphi^{-1}(p)$ is finite for any $p \in V$

Definition 30.2.4. A morphism $W \xrightarrow{\varphi} V$ is **finite** if $k[W]$ is finite $k[V]$ algebra

Proposition 30.2.5. A finite morphism is quasifinite

Proposition 30.2.6. A variety is an integral scheme X over k such that $X \to \mathsf{Spec}\, k$ is separated and of finite type

Chapter 31

Scheme

31.1 Affine schemes

Definition 31.1.1. An **affine scheme** is a ringed space $(\mathsf{Spec}\, R, \mathcal{O}_{\mathsf{Spec}\, R})$

Lemma 31.1.2. The inclusion of $\mathsf{Spec}\, k(p) \to \mathsf{Spec}\, A$ is given by $A \to A_p \to k(p)$

31.2 Schemes

Definition 31.2.1. A **scheme** is a ringed space $(X, \mathcal{O})$ such that for each $p \in X$, there is a open neighborhood $U \ni p$ such that $(U, \mathcal{O}|_U)$ is isomorphic to some affine scheme $(SpecR, \mathcal{O}_{SpecR})$

Definition 31.2.2. We say X is a scheme over Y if there is a morphism $X \to Y$, X is a scheme over R if there is a morphism $X \to \operatorname{Spec} R$
An R point is a morphism $\operatorname{Spec} R \to X$, we also write the set of R points as $X(R)$. If S is a commutative R algebra, then the set of S points $X(S)$ consists of morphisms $\operatorname{Spec} S \to X$ over $\operatorname{Spec} R$
$X(S)$ can also be constructed as the base change X_S

$$\begin{tikzcd} X_S \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} S \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Definition 31.2.3. A scheme X is **reduced/integral** if $\mathcal{O}(U)$ is reduced/integral for any open subset U

Definition 31.2.4. A morphism $f : X \to Y$ is **separated** if $\Delta(X)$ is closed, $\Delta : X \to X \times_Y X$ is the diagonal

Definition 31.2.5. A morphism $f : X \to Y$ is of **finite type** if Y has an affine open cover Y_i such that there is an affine open cover X_{ij} of $f^{-1}(Y_i)$ such that $f|_{X_{ij}} : X_{ij} \to Y_i$ are of finite type

Part XII

Analysis

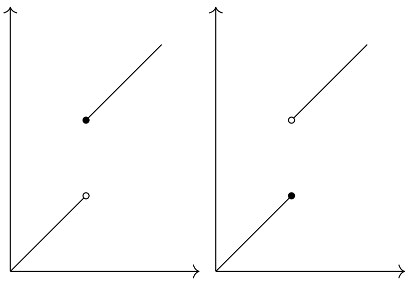
Chapter 32

Real analysis

Definition 32.0.1 (Hyperbolic functions)**.** $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\sinh z = -i\sin(iz) = \dfrac{e^{z} - e^{-z}}{2}$, $\cosh z = \cos(iz) = \dfrac{e^{z} + e^{-z}}{2}$

Definition 32.0.2. X is a convex set, $X \xrightarrow{f} \mathbb{R}$ is **convex** if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for $0 \leq t \leq 1$ and $x, y \in X$, f is **strictly convex** if $f(tx+(1-t)y) < tf(x)+(1-t)f(y)$ for $0 < t < 1$ and $x \neq y \in X$. f is **concave** if $-f$ is convex

Definition 32.0.3. $X \xrightarrow{f} [-\infty, \infty]$ is **upper semicontinuous** at x if for any $y > f(x)$, there exists a neighborhood U of x such that $f(U) < y$, i.e. f can only jump down at x. Thus $X \xrightarrow{f} [-\infty, \infty]$ is upper semicontinuous if $\{f < a\}$ are open. $X \xrightarrow{f} [-\infty, \infty]$ is **lower semicontinuous** if $-f$ is upper semicontinuous, i.e. f can only jump up



Lemma 32.0.4. $\{f_\alpha\}_{\alpha \in A}$ is a family of upper semicontinuous functions, $f = \inf\limits_{\alpha \in A} f_\alpha$ is also upper semicontinuous

Proof.

$$\{f < a\} = \bigcup_{\alpha \in A} \{f_\alpha < a\}$$

$\square$

Lemma 32.0.5. f is upper semicontinuous, K is compact, then f attains maximum over K

Definition 32.0.6. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **harmonic** at $x \in \Omega$ if u is continuous at x and for any ball $B(x, r)$, $u(x) = \dfrac{1}{|B(x, r)|}\displaystyle\int_{B(x,r)} u(y)dy$. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **subharmonic** at $x \in \Omega$ if u is upper semicontinuous at x and for any ball $B(x, r)$, any continuous v harmonic on $B(x, r)$, $u \leq v$ on $\partial B(x, r) \Rightarrow u \leq v$ on $\overline{B(x, r)}$. $\Omega \xrightarrow{u} [-\infty, \infty)$ is **superharmonic** if $-u$ is subharmonic Harmonic $\Leftrightarrow$ subharmonic and superharmonic

Lemma 32.0.7. $\Omega \xrightarrow{u} \mathbb{R}$ is subharmonic, $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is convex, then $f \circ u$ is also subharmonic

f holomorphic => log|f| subharmonic

Example 32.0.8. If f is holomorphic, then $\log|f|$ is subharmonic

Chapter 33

Complex analysis

33.1 Complex analysis

Definition 33.1.1. A **polydisc** $D(z,r) \subseteq \mathbb{C}^n$ is $D(z_1,r_1) \times \cdots \times D(z_n,r_n)$

Definition 33.1.2 (Wirtinger derivatives)**.**

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Note.

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}, \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$dz \wedge d\bar{z} = -2i dx \wedge dy$$

Definition 33.1.3. $f : \Omega \to \mathbb{C}$ is **holomorphic** at $z_0 \in \Omega$ if $f'(z)$ exists around z_0. f is **univalent** if f is injective

Theorem 33.1.4 (Cauchy-Riemann equations)**.** If we write $z = x + iy$, $f(z) = u(x,y) + iv(x,y)$, then the existence of $f'(z)$ implies that $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ which give the **Cauchy-Riemann equations**

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

If f satisfies Cauchy-Riemann equations around z_0, then f is holomorphic at z_0

Lemma 33.1.5. A univalent map is a biholomorphism to its image

Theorem 33.1.6 (Goursat)**.** If f is holomorphic on $\Omega \subseteq \mathbb{C}$, $\overline{T} \subseteq \Omega$ is a triangle, then $\oint_T f(z)dz = 0$

Theorem 33.1.7 (Cauchy's integral theorem)**.** If f is holomorphic on $\Omega \subseteq \mathbb{C}$, $\gamma \subseteq \Omega$ is a piecewise C^1 curve, then $\oint_\gamma f(z)dz = 0$

Theorem 33.1.8 (Morera's theorem)**.** $U \subseteq \mathbb{C}$ is open, if $\oint_T f(z)dz = 0$ for any triangle $T \subseteq U$, then f is holomorphic on D

Cauchy-Pompeiu formula

Theorem 33.1.9 (Cauchy-Pompeiu formula)**.** f is a complex valued C^1 function on a disc $D \subseteq \mathbb{C}$, then

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)dz}{z-\zeta} - \frac{1}{\pi}\iint_D \frac{\partial f(z)}{\partial \bar{z}}\frac{dx \wedge dy}{z-\zeta}$$

In particular, if f is holomorphic, then

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-\zeta}dz$$

Proof. Denote $D_\epsilon = D - B(0,\epsilon)$, consider

$$\eta = \frac{f(w)dw}{w-z}, d\eta = \frac{\partial f(w)}{\partial \bar{w}}\frac{d\bar{w}\wedge dw}{w-z}$$

By Stokes' theorem

$$\frac{1}{2\pi i}\int_{\partial D_\epsilon}\eta = \frac{1}{2\pi i}\int_{D_\epsilon} d\eta$$

As $\epsilon \searrow 0$

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)dw}{w-z} + \frac{1}{2\pi i}\iint_D \frac{\partial f(w)}{\partial \bar{w}}\frac{dw\wedge d\bar{w}}{w-z}$$

□

Osgood's lemma

Lemma 33.1.10 (Osgood's lemma)**.** f is continuous on an open subset $\Omega \subseteq \mathbb{C}^n$ and holomorphic on each variable, then f is holomorphic

Proof. For each $a \in \Omega$, pick $P = D(a,r) \subseteq \Omega$, since $\dfrac{\partial f}{\partial \bar{z}_j} \equiv 0$ on Ω, fix $z_2, \cdots, z_n$, then

$$f(w_1, z_2, \cdots, z_n) = \frac{1}{2\pi i}\int_{|z_1-a_1|=r_1}\frac{f(z_1,\cdots,z_n)}{z_1-w_1}dz_1$$

For $w_1 \in D(a_1, r_1)$, iterate and we get

$$f(w_1,\cdots,w_n) = \frac{1}{(2\pi i)^n}\int_{|z_1-a_1|=r_1}\cdots\int_{|z_n-a_n|=r_n}\frac{f(z_1,\cdots,z_n)}{\prod(z_j-w_j)}dz_1\cdots dz_n$$

For $w \in P$. Since f is continuous, it is bounded on $\overline{P}$, $\dfrac{1}{z_j-w_j} = \displaystyle\sum_{m=0}^{\infty}\frac{(w_j-a_j)^m}{(z_j-a_j)^{m+1}}$ converges uniformly on compact subsets of $D(a_j, r_j)$. Hence $f(w) = \sum c_\alpha (w-a)^\alpha$, where

$$c_\alpha = \frac{1}{(2\pi i)^n}\int_{|z_1-a_1|=r_1}\cdots\int_{|z_n-a_n|=r_n}\frac{f(z)}{\prod(z_j-a_j)^{\alpha_j+1}}dz_1\cdots dz_n$$

□

Corollary 33.1.11 (Cauchy inequality)**.**

Maximum principle

Theorem 33.1.12 (Maximum principle)**.**

Theorem 33.1.13. $\{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$, f_n are uniformly convergent on each compact subset, then f_n converges to a holomorphic function f, and $D^\alpha f_n \to D^\alpha f$ on each compact subset

Montel's theorem

Theorem 33.1.14 (Montel's theorem)**.** $\mathcal{F} = \{f_n\}$ are holomorphic on $\Omega \subseteq \mathbb{C}^n$ and locally uniformly bounded, i.e. for any $z_0 \in \Omega$, there exists a neighborhood U and M such that $\sup\limits_{z\in K}|f_n| \leq M$, then $\mathcal{F}$ is normal

Schwarz lemma

Lemma 33.1.15 (Schwarz lemma)**.** f is holomorphic on the unit disc $D \subseteq \mathbb{C}$, $f(0) = 0$ and $|f| \leq 1$ on D, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then $f(z) = az$, $a = f'(0)$

Proof. Define $g(z) = \dfrac{f(z)}{z}$, since $f(0) = 0$, 0 is a removable singularity, since $|f(z)| \leq 1$, $|g(z)| \leq 1$ on ∂D, by maximum principle 33.1.12, $|g(z)| \leq 1$ on D, thus $|f(z)| \leq |z|$ on D and $|f'(0)| = |g(0)| \leq 1$, if $|f(z)| = |z|$ for some nonzero z or $|f'(0)| = 1$, then g attains maximum within D, then $g \equiv a$ for some $|a| = 1$, thus $f(z) = az$ □

Corollary 33.1.16. $D \xrightarrow{f} D$ is a biholomorphic, then $f = e^{i\phi}\dfrac{z-a}{1-\bar{a}z}$ for some ϕ and $a \in D$

Proof. Denote $\psi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, ψ_{-a} is the inverse of ψ_a
Assume $f(a) = 0$, consider $g(z) = f \circ \psi_{-a}$, then $g(0) = 0$, by Schwarz lemma 33.1.15, $g = e^{i\phi}$, $f = g \circ \phi_a = e^{i\psi}\dfrac{z-a}{1-\bar{a}z}$ □

Lemma for Riemann mapping theorem

Lemma 33.1.17. Suppose $0 \in U \subsetneq D$ is a simply connected open set, there exists $U \xrightarrow{f} D$ univalent such that $f(0) = 0$, $|f'(0)| > 1$. Note that this is impossible if $U = D$ due to Schwarz lemma 33.1.15

Proof. Denote $\psi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $\psi_a'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$. Consider $f = \psi_{g(a)} \circ g \circ \psi_{-a}$ with some $\psi_{-a}(U) \xrightarrow{g} D$ univalent, then $f(0) = 0$

$$f'(0) = \frac{1-|g(a)|^2}{(1-|g(a)|^2)^2}g'(a)(1-|a|^2) = \frac{1-|a|^2}{1-|g(a)|^2}g'(a)$$

Since U is simply connected, so is $\psi_{-a}(U)$ given $-a \in D \setminus U$, we can take $g(z) = \sqrt{z}$ to be one branch, since $|a| < 1$, we get

$$|f'(0)| = \frac{1-|a|^2}{1-|a|}\frac{1}{2\sqrt{|a|}} = \frac{1+|a|}{2\sqrt{|a|}} > 1$$

□

Lemma for finding zeros

Lemma 33.1.18. φ is holomorphic on D, f is meromorphic on D and $f \neq 0$ on ∂D, $a_1, \cdots, a_m$ and $b_1, \cdots, b_n$ are the zeros and poles of order $k_1, \cdots, k_m$ and $l_1, \cdots, l_n$ of f in D, then

$$\frac{1}{2\pi i}\int_{\partial D}\varphi(z)\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{m}k_i\varphi(a_i) - \sum_{i=1}^{n}l_i\varphi(b_i)$$

Proof. $f(z) = g(z)\prod_{i=1}^{m}(z-z_i)^{q_i}$ with $g \neq 0$ on $\overline{D}$, z_i, q_i could be a_i, k_i or $b_i, -l_i$ depending on whether it is a zero or a pole, hence

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\partial D}\varphi(z)\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{\partial D}\varphi(z)\frac{g'(z)\prod_{i=1}^{m}(z-z_i) + g(z)\sum_{i=1}^{m}\prod_{j\neq i}(z-z_j)}{g(z)\prod_{i=1}^{m}(z-z_i)}dz \\
&= \frac{1}{2\pi i}\int_{\partial D}\left[\frac{\varphi(z)g'(z)}{g(z)} + \sum_{i=1}^{m}\frac{\varphi(z)}{z-z_i}\right]dz \\
&= \sum_{i=1}^{m}k_i\varphi(a_i) - \sum_{i=1}^{n}l_i\varphi(b_i)
\end{aligned}$$

□

Rouche's theorem

Theorem 33.1.19 (Rouché's theorem)**.**

Hurwitz's theorem

Theorem 33.1.20 (Hurwitz's theorem)**.** $U \subseteq \mathbb{C}$ is open connected, holomorphic functions $\{f_n\}$ converges uniformly to f on compact subsets of U and $f \not\equiv 0$, f has order m at z_0, for r small enough, there exists K such that for any $k \geq K$, f_k has precisely m zeros in $B(z_0, r)$, counting multiplicities, and these zeros converge to z_0 as $k \to \infty$

Remark 33.1.21. $B(z_0, r)$ can't be arbitrarily large. For example, $f_n(z) = z - 1 + \frac{1}{n}$ converges uniformly to $f(z) = z - 1$ on compact subsets, f has no zeros in the unit disc D, but f_n all have zeros in D

Proof. For r small enough, f doesn't vanish on $\partial B(z_0, r)$ on which $|f|$ attains minimum, then apply Rouché's theorem 33.1.19 □

Corollary 33.1.22. U is open connected, univalent maps $\{f_n\}$ converges to f on compact subsets, then f is either univalent or constant

Proof. If f is not a constant and $f(z_0) = f(w_0) = \zeta$, then $f(z) - \xi$ has z_0, w_0 as zeros, by Hurwitz's theorem 33.1.20, there exist $\{z_k\}, \{w_k\}$ converging to z_0, w_0 such that $f_{n_k}(z_k) = f_{n_k}(w_k) = \xi$, but f_n's are univalent, hence $z_k = w_k \Rightarrow z_0 = w_0$, i.e. f is univalent □

Riemann mapping theorem

Theorem 33.1.23 (Riemann mapping theorem)**.** $U \subsetneq \mathbb{C}$ is a nonempty simply connected open subset, $z_0 \in U$,then there is a unique biholomorphism f from U to the unit disc such that $f(z_0) = 0$, $f'(z_0) > 0$

Proof of uniqueness. Suppose $U \xrightarrow{f_1, f_2} D$ are biholomorphisms such that $f_i(z_0) = 0$, $f_i'(z_0) > 0$, consider $g = f_2 f_1^{-1}$, $g(0) = 0$, $|g| \leq 1$ on D and $g'(0) = \frac{f_2'(z_0)}{f_1'(z_0)} > 0$, by Schwarz lemma 33.1.15, $g(z) = z$, i.e. $f_1 = f_2$ □

Proof of existence. Fix $a \notin U$, $z_0 \in U$. Define

$$\mathcal{F} = \{f \text{ univalent on } U | |f| \leq 1, f(z_0) = 0\}$$

Since U is simply connected, we can pick one branch $h(z) = \sqrt{z - a}$, then $h(U) \cap -h(U) = \varnothing$, $\frac{h(z) - h(z_0)}{h(z) + h(z_0)}$ is univalent and bounded, scale to get some $f_0 \in \mathcal{F} \Rightarrow \mathcal{F}$ is nonempty

Let $A = \sup_{f \in \mathcal{F}} |f'(z_0)| > 0$, $f_n'(z_0) \to A$ for some $\{f_n\} \subseteq \mathcal{F}$, by Montel's theorem 33.1.14, f_{n_k} converges to g uniformly on compact subsets, then $|g| \leq 1$, $g(z_0) = 0$ and $0 < A = |g'(z_0)| < \infty$, according to Hurwitz's theorem 33.1.20, g is also univalent, i.e. $g \in \mathcal{F}$ attains maximal derivative at z_0

Suppose $0 \in g(U) \subsetneq D$, if not, by Lemma 33.1.17, there exists univalent map $g(U) \xrightarrow{f} D$ such that $f(0) = 0$, $|f'(0)| > 1$, then $f \circ g \in \mathcal{F}$, but $|(f \circ g)'(z_0)| = |f'(0) g'(z_0)| > |g'(z_0)|$ which is a contradiction □

Remark 33.1.24. Suppose $f_1, f_2 \in \mathcal{F}$ and f_1 is biholomorphic, then $g = f_2 f_1^{-1}$ is a map $D \to D$, with $g(0) = 0$, according to Schwarz lemma 33.1.15, $\frac{|f_2'(z_0)|}{|f_1'(z_0)|} = |g'(0)| \leq 1$, and if $|f_2'(z)| = |f_1'(z)|$, $g = e^{i\phi}$, f_2 is also biholomorphic

Example 33.1.25. $U = \mathbb{C} - \{z \geq 0\}$, then $h(z) = \sqrt{z}$ maps U to the upper half plane

Theorem 33.1.26 (Runge's theorem)**.** $K \subseteq \mathbb{C}$ is compact, then $\mathbb{C} \setminus K$ is the union of its connected components whereas the components are either bounded or not, denote

Hartogs's extension theorem

Theorem 33.1.27 (Hartogs's extension theorem)**.** An isolated singularity is always a removable singularity when $n \geq 2$

Proof. It suffices to consider the case $P = \{|z_1| \leq 1, |z_2| \leq 1\}$ is a polydisc, f is holomorphic on ∂P, then f is holomorphic on P □

Lemma for Remmert-Stein theorem

Lemma 33.1.28. $\Omega \subseteq \mathbb{C}^n$ is connected, $\Omega \xrightarrow{f} \partial B^n$ is holomorphic, then $f \equiv \mathsf{const}$

Proof. If h is holomorphic, then $\dfrac{\partial^2}{\partial z \partial \bar{z}}|h|^2 = |h'|^2$, hence

$$0 = \frac{\partial^2}{\partial z \partial \bar{z}}|f|^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial z \partial \bar{z}}|f_i|^2 = \sum_{i=1}^{n} |f_i'(z)|^2 \Rightarrow f_i'(z) = 0 \Rightarrow f \equiv \text{const}$$

□

Theorem 33.1.29 (Remmert-Stein)**.** $U_1 \subseteq \mathbb{C}^{n_1}, U_2 \subseteq \mathbb{C}^{n_2}$ are nonempty connected open subsets, $B = \{|z| < 1\} \subseteq \mathbb{C}^n$, then there is no proper holomorphic map $U_1 \times U_2 \to B$

Proof. Suppose $f : U_1 \times U_2 \to B$ is a proper holomorphic map. For any $(x, y) \in U_1 \times \partial U_2$, there is a discrete sequence $\{y_\nu\} \subseteq U_2$ converging to y as in Exercise 46.2.1, apply Lemma 15.1.31 to $f(x, y) : \{x\} \times U_2 \to B$, $\{f(x, y_\nu)\}$ is discrete, thus there exists a subsequence $\{y_\mu\} \subseteq \{y_\nu\}$ such that $f(x, y_\mu)$ such that $f(x, y) = \lim f(x, y_\mu) \in \partial B$. Then $f(x, y) : U_1 \times \{y\} \to \partial B$ is a holomorphic, by Lemma 33.1.28, $f(x, y)$ is constant on $U_1 \times \{y\}$, hence $U_1 \times \{y\} \subseteq f^{-1}(f(x, y))$ which is noncompact since it has noncompact image under projection to U_1. This contradicts the fact that f is proper □

Corollary 33.1.30 (Poincaré)**.** The 2 polydisc $P = \{|z_1| < 1, |z_2| < 1\}$ and the 2 ball $B = \{|z_1|^2 + |z_2|^2 < 1\}$ are not biholomorphic

Theorem 33.1.31 (Weierstrass preparation theorem)**.** f is analytic near 0, $f(0) = 0$, $f(z)$ written as power series around 0 has terms only involve z_1 which can always be achieved by a change of variables as in Exercise 46.2.2, then $f = wh$, where $w(z) = z_1^k + g_{k-1}z^{k-1} + \cdots + g_0$ is a **Weierstrass polynomial**, i.e. $g_i(z)$ are analytic around 0 and $g_i(0) = 0$, $h(z)$ is analytic around 0 and $h(0) \neq 0$

Theorem 33.1.32 (Weierstrass division theorem)**.** Suppose f, g are analytic near 0, g is a Weierstrass polynomial of degree k, then there exist unique h, r such that $f = gh + r$, where r is a polynomial of degree less than k

33.2 Conformal mapping

Definition 33.2.1. A conformal mapping is a map preserves angles and orientation

Note. Antiholomorphic map preserves angles but changes orientation

Definition 33.2.2. **Möbius transformations** are $f(z) = \dfrac{az+b}{cz+d}$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$, **Möbius group** acts regularly on $\mathbb{C}P^1$ and preserves cross ratio $(z_0, z_1; z_2, z_3) = \dfrac{(z_2 - z_0)(z_3 - z_1)}{(z_3 - z_0)(z_2 - z_1)}$

Schwarz reflection principle

Lemma 33.2.3 (Schwarz reflection principle)**.** If f is holomorphic on $\{\mathrm{Im} z > 0\}$ and continuous on $\{\mathrm{Im} z \geq 0\}$ with real values on $\mathrm{Im} z = 0$, then it can be extended to $\mathbb{C}$ with $f(\bar{z}) = \overline{f(z)}$ for $\mathtt{z} < 0$

33.3 Weierstrass functions

Definition 33.3.1. $\Lambda \subseteq \mathbb{C}$ is a lattice
The **Weierstrass σ-function** associated to lattice Λ is

$$\sigma(z) = z \prod_{\omega \in \Lambda^*} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

The **Weierstrass $\wp$-function** associated to lattice Λ is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right)$$

Hence

$$\wp'(z) = -\sum_{\omega \in \Lambda} \frac{2}{(z+\omega)^3}$$

33.4 Zeta function

Theorem 33.4.1 (Euler's reflection formula). $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$, $z \notin \mathbb{Z}$

Chapter 34

Functional analysis

34.1 Topological vector space

Definition 34.1.1. A **topological vector space** V over a topological field $\mathbb{F}$ is a topological abelian group such that scalar multiplication $\mathbb{F} \times V \to V$ is continuous

Definition 34.1.2. A **norm** on a group G is $G \xrightarrow{\|\|} \mathbb{R}_{\geq 0}$ such that $\|g\| = 0 \Leftrightarrow g = \mathrm{id}$, $\|g^{-1}\| = \|g\|$, $\|gh\| \leq \|g\|\|h\|$
A **norm** on a rng R is a normed abelian group such that $\|rs\| \leq \|r\|\|s\|$
A **norm** on a vector space V over a normed field is a normed abelian group such that $\|kv\| \leq |k|\|v\|$

Definition 34.1.3. A **Banach space** is a complete normed vector space

Definition 34.1.4. Y is a topological vector space, T is a set, $\mathcal{G} \subseteq \mathcal{P}(T)$ is a directed set by inclusion, $\mathcal{N}$ is a local base around $0 \in Y$. The **topology of uniform convergence** on sets in $\mathcal{G}$ or $\mathcal{G}$ **topology** is the unique translation invariant topology given by basis

$$U(G, N) = \left\{ f \in Y^T \middle| G \in \mathcal{G}, N \in \mathcal{N}, f(G) \subseteq N \right\}$$

Example 34.1.5. $\mathcal{G}$ is the set of compact subspaces, Y is a metric space

34.2 Arzela-Ascoli theorem

Definition 34.2.1. Let X, Y be a topological spaces, a family of continuous functions $A \subseteq Y^X$ is equicontinuous at $x \in X$, if for any open neighborhood V of $y = f(x)$, there is an open neighborhood U of x such that $f(U) \subseteq V, \forall f \in A$

Definition 34.2.2. A topological space X is called separable if X has a countable dense subset

Arzela-Ascoli theorem

Theorem 34.2.3. Let X be a topological space and Y be a complete metric space, $A \subseteq Y^X$ be a family of equicontinuous functions(meaning pointwise equicontinuous). If X is compact, and $A_x := \{f(x) | f \in A\} \subseteq Y$ is relatively compact for any $x \in A$, then A is relatively compact in Y^X. If X is separable with S being a countable dense subset, and A_x is relatively compact for any $x \in S$, then any sequence $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ converges uniformly on any compact subset of X

34.3 Baire category theorem

Definition 34.3.1. A topological space X is a **Baire space** if for any countable open dense subsets $\{U_i\}$, $\bigcap_{i=1}^{\infty} U_i$ is also dense

Baire category theorem

Theorem 34.3.2 (Baire category theorem)**.** Every complete metric space X is a Baire space

Proof. Let $\{U_i\}$ be a countable open dense subsets, suppose $\bigcap_{i=1}^{\infty} U_i$ is not dense, then the complement of its closure is open nonempty, suppose $B(x,r)$ is in the complement of the closure, since U_1 is dense, $U_1 \cap B(x,r) \neq \emptyset$, then there exists $\overline{B(x_1,r_1)} \subseteq U_1 \cap B(x,r)$, similarly, we can find $\overline{B(x_n,r_n)} \subseteq U_n \cap B(x_{n-1},r_{n-1})$, and we can also assume $r_i \to 0$, thus $x_i \to y \in X$ since X is complete, but $y \in B(x,r) \bigcap \bigcap_{i=1}^{\infty} U_i = \emptyset$ which is a contradiction $\square$

34.4 Distribution

Definition 34.4.1. $U \subseteq \mathbb{R}^n$ open, $\mathscr{D}(U) = C_c^\infty(U)$ is the **test function space**, $\{\phi_i\} \subseteq \mathscr{D}(U)$ converges if there exists $K \subseteq U$ compact such that $\mathrm{supp}\phi_i \subseteq K$ and $\partial^\alpha \phi_i$ converges uniformly

34.5 Banach algebra

Definition 34.5.1. A **Banach algebra** is an associative algebra A which is a complete normed rng such that $\|rs\| \leq \|r\|\|s\|$. A is **unital** if A is a ring with identity element having norm 1

Definition 34.5.2. A ***-algebra** is a Banach algebra over $\mathbb{C}$ such that there is an antilinear involution $* : A \to A$, such that $(xy)^* = y^* x^*$. A is a C^***-algebra** if $\|x^* x\| = \|x^*\|\|x\|$

Example 34.5.3. X is locally compact, $C_0(X)$ are the continuous functions vanishes at infinity, then $C_0(X)$ is a Banach algebra with the supremum norm, $C_0(X)$ is unital if X is compact with 1 being the identity. $C_0(X)$ is a C^*-algebra with complex conjugation as the involution

Definition 34.5.4. A is a unital Banach algebra over $\mathbb{R}, \mathbb{C}$, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ defines the **exponential**

$$\|e^x\| = \left\| \sum_{k=0}^{\infty} \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \left\| \frac{x^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|x\|^k}{k!} = e^{\|x\|}$$

The **logarithm** $\log x = \sum_{k=1}^{\infty} \frac{(-1)^{x+1}(x-1)^k}{k}$ is defined on $\|x-1\| < 1$

Lemma 34.5.5. e^x and $\log x$ are inverses to each other locally

Proposition 34.5.6. A is a Banach algebra, linear map $D : A \to A$ is a derivation iff e^{tD} is a group of automorphisms

Lie product formula

Theorem 34.5.7 (Lie product formula)**.** $e^{A+B} = \lim_{n \to \infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}} \right)^n$

Lie commutator formula

Theorem 34.5.8 (Lie commutator formula)**.** $e^{[A,B]} = \lim_{n \to \infty} \left[e^{\frac{A}{n}}, e^{\frac{B}{n}} \right]^{n^2}$, the left and right $[,]$ are Lie bracket and commutator

Lemma 34.5.9. If $[X, [X, Y]] = 0$, then $e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$

Proof. Let $A(t) = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]}$, $B(t) = e^{t(X+Y)}$, then $A(0) = B(0)$, $B'(t) = B(t)(X+Y)$ and

$$A'(t) = e^{tX} X e^{tY} e^{-\frac{t^2}{2}[X,Y]} + e^{tX} e^{tY} Y e^{-\frac{t^2}{2}[X,Y]} - e^{tX} e^{tY} t[X,Y] e^{-\frac{t^2}{2}[X,Y]}$$

Since $[X, [X, Y]] = 0$, $[Y, [X, Y]] = -[Y, [Y, X]] = 0$

$$e^{-tY} X e^{tY} = Ad_{e^{-tY}}(X) = e^{ad_{-tY}}(X) = X + t[X, Y]$$

$$A'(t) = e^{tX} e^{tY} (X+Y) e^{-\frac{t^2}{2}[X,Y]} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} (X+Y) = A(t)(X+Y)$$

Thus $A(t), B(t)$ satisfies the same ODE and initial condition, $A(t) = B(t) \Rightarrow e^X e^Y = A(1) = B(1) = e^{X+Y+\frac{1}{2}[X,Y]}$ □

Theorem 34.5.10 (Backer-Campbell-Hausdorff formula)**.** $e^X e^Y = e^Z$ around 0, where $Z =$

$X+\int_0^1 \psi(e^{ad_X}e^{tad_Y})dt(Y)$ and

$$\begin{aligned}\psi(x) &= \frac{x\log x}{x-1}\\ &\xlongequal{y=1-x} \frac{(1-y)\log(1-y)}{-y}\\ &= (1-y)\sum_{n=1}^{\infty}\frac{y^{n-1}}{n}\\ &= \sum_{n=1}^{\infty}\frac{y^{n-1}}{n}-\sum_{n=1}^{\infty}\frac{y^n}{n}\\ &= 1+\sum_{n=1}^{\infty}\left(\frac{y^n}{n+1}-\frac{y^n}{n}\right)\\ &= 1-\sum_{n=1}^{\infty}\frac{(1-x)^n}{n(n+1)}\end{aligned}$$

The first few terms are

$$e^Xe^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\cdots}$$

Proof. The Riemann sum $\sum_{k=0}^{m-1}\frac{1}{m}e^{-\frac{kx}{m}}$ converges to $\int_0^1 e^{-tx}dt = \frac{1-e^{-x}}{x}$, thus $\lim_{m\to\infty}\sum_{k=0}^{m-1}e^{-\frac{kX}{m}} = \frac{1-e^{-X}}{X}$, we have

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} e^{X+tY} &= \left.\frac{d}{dt}\right|_{t=0}\left(e^{\frac{X}{m}}e^{\frac{tY}{m}}\right)^m\\ &= \lim_{m\to\infty}\left.\frac{d}{dt}\right|_{t=0}\left(e^{\frac{X}{m}}e^{\frac{tY}{m}}\right)^m\\ &= \lim_{m\to\infty}\sum_{k=0}^{m-1}e^{\frac{kX}{m}}\frac{Y}{m}e^{\frac{(m-k)X}{m}}\\ &= \lim_{m\to\infty}\sum_{k=0}^{m-1}\frac{1}{m}e^{\frac{kX}{m}}Ye^{-\frac{kX}{m}}e^X\\ &= \left(\lim_{m\to\infty}\sum_{k=0}^{m-1}\frac{1}{m}e^{\frac{kad_X}{m}}\right)(Y)e^X\\ &= \frac{e^{ad_X}-1}{ad_X}(Y)e^X\end{aligned}$$

Let $e^{Z(t)} = e^Xe^{tY}$, $\frac{d}{dt}e^{Z(t)} = \frac{d}{dt}(e^Xe^{tY}) = e^Xe^{tY}Y = e^{Z(t)}Y$, but $\frac{d}{dt}e^{Z(t)} = \left.\frac{d}{ds}\right|_{s=t}e^{Z(s)} = \left.\frac{d}{ds}\right|_{s=t}e^{Z(t)+Z'(t)(s-t)} = \frac{e^{ad_{Z(t)}}-1}{ad_{Z(t)}}(Z'(t))e^{Z(t)}$, hence $\frac{e^{ad_{Z(t)}}-1}{ad_{Z(t)}}(Z'(t)) = e^{Z(t)}Ye^{-Z(t)} = Ad_{e^{Z(t)}}(Y) = e^{ad_{Z(t)}}(Y)$, $Z'(t) = \frac{ad_{Z(t)}e^{ad_{Z(t)}}}{e^{ad_{Z(t)}}-1}(Y)$, since $e^{ad_{Z(t)}} = Ad_{e^{Z(t)}} = Ad_{e^Xe^{tY}} = e^{ad_X}e^{tad_Y}$

$$\begin{aligned}Z &= Z(1)\\ &= Z(0)+\int_0^1\frac{ad_{Z(t)}e^{ad_{Z(t)}}}{1-e^{-ad_{Z(t)}}}(Y)dt\\ &= X+\int_0^1\frac{e^{ad_X}e^{tad_Y}\log(e^{ad_X}e^{tad_Y})}{e^{ad_X}e^{tad_Y}-1}dt(Y)\end{aligned}$$

□

34.6 Stone-Weierstrass theorem

Definition 34.6.1. $\mathcal{F} = \{f_i\}$ is a family of functions on X, $\mathcal{F}$ **separates points** in X if for any $x \neq y \in X$, some f_i separates x, y

Theorem 34.6.2. X is compact Hausdorff, $A \subseteq C(X, \mathbb{R})$ is a unital subalgebra. A is dense in $C(X, \mathbb{R})$ with the topology of uniform convergence iff A separates points
$S \subseteq C(X, \mathbb{C})$ is a unital *-algebra that separating points, then S is dense in $C(X, \mathbb{C})$

Part XIII

Differential equations

Chapter 35

Ordinary differential equations

Theorem 35.0.1. Linear differential equations $y'(t) = A(t)y(t) + b(t)$ with initial condition $y(0) = y_0$, where A, b, y are smooth, then there exists unique local solution

Proof. Define $Ty(t) = \int_0^t A(s)y(s) + b(s)ds + y_0$, note that $\|Ty(t) - Tz(t)\| = \left\| \int_0^t A(s)(y(s) - z(s))ds \right\| \leq |t|\|A\|\|y - z\|$, then there exists $\delta > 0$ such that $|t|\|A\| < 1, \forall |t| \leq \delta$, here we use $\|\cdot\|$ to denote the supremum norm in $|t| \leq \delta$, by Banach fixed point theorem, we have a unique local solution $\square$

Example 35.0.2. $v'(t) = Av(t), v(0) = v_0$, $A \in M_n(\mathbb{C})$, the solution is $v(t) = e^{tA}v_0$ since $\frac{d}{dt}A^{tA} = Ae^{tA}$

Picard-Lindelof theorem

Theorem 35.0.3. (**Picard-Lindelöf theorem**) Suppose $f(y, t)$ is uniformly Lipschitz continuous in y and continuous in t, then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a unique solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Remark 35.0.4. $f(y, t)$ is Lipschitz continuous in y and continuous in t would implie loacl uniformly Lipschitz in y and f uniformly continuous
When you have a local solution, you can try to extend it to a maximal length, i.e. $y(t)$ is defined on $(a, b) \supset [-\varepsilon, \varepsilon]$, it is open precisely because of the theorem

Proof. Define $Ty(t) = \int_0^t f(y(s), s)ds$, then
$\|Ty - Tz\| = \left\| \int_0^t f(y(s), s) - f(z(s), s)ds \right\| \leq \left\| C \int_0^t |y(s) - z(s)|ds \right\| \leq C|t|\|y - z\|$, then there exists $\varepsilon > 0$ such that $C|t| < 1$, $\forall t \in [-\varepsilon, \varepsilon]$, then by Banach fixed point theorem, we have a unique local solution $\square$

Peano existence theorem

Theorem 35.0.5. (**Peano existence theorem**) Let $f(y, t)$ be a continuous function around $(y_0, 0)$, then the ODE

$$\begin{cases} y'(t) = f(y(t), t) \\ y(0) = y_0 \end{cases}$$

Has a local solution $y(t)$ on $[-\varepsilon, \varepsilon]$

Proof. Say $|f| \leq M$ around $(y_0, 0)$, Define $\phi_n(t) = \begin{cases} y_0 & , x \leq 0 \\ y_0 + \displaystyle\int_0^t \phi_n\left(s - \frac{\varepsilon}{n}\right) ds & , 0 \leq x \leq \varepsilon \end{cases}$ for $n \geq 1$

By Arzelà-Ascoli theorem, we know that there is a subsequence ϕ_{n_k} converges on $[-\varepsilon, \varepsilon]$, and the limit $\phi(t)$ satisfies $\phi(t) = y_0 + \displaystyle\int_0^t \phi_n(s)\, ds$ which is a local solution to the problem $\square$

Remark 35.0.6. The uniqueness may fail without the Lipschitz condition in y, for example, consider $\dfrac{dy}{dt} = y^{\frac{1}{3}}$, $y(0) = 0$ has solutions $y(t) = 0$ or $y(t) = \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}}$

Chapter 36

Classical partial differential equations

36.1 Laplace's equation

36.2 Heat equation

Definition 36.2.1. The fundamental solution to solution to **heat equation** $u_t - \Delta u = 0$ is

$$E(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \le 0 \end{cases}$$

Theorem 36.2.2. $U \subseteq \mathbb{R}^n$ is open and bounded, $f \in C_c^1(U \times (0,T])$, then

$$u(x,t) = \int_{\mathbb{R}^{n+1}} E(x-y,t-s)f(s,y)dsdy$$

Satisfies

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = f(x,t)$$

Where u is C^1 in t and C^2 in x

Proof. $E(x,t)$ is supported in $t \ge 0$ and $\int_{\mathbb{R}^n} |\nabla_x E(x,t)|dx \le \frac{C}{\sqrt{t}}$ if $t > 0$, so $\nabla_x E(x,t)$ is integrable near $(0,0)$

$$\begin{aligned}
\nabla_x \int_{\mathbb{R}^{n+1}} E(y,s)f(x-y,t-s)dsdy &= \int_{\mathbb{R}^{n+1}} E(y,s)\nabla_x f(x-y,t-s)dsdy \\
&= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} E(y,s)\nabla_x f(x-y,t-s)dsdy \\
&= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\nabla E)(y,s)f(x-y,t-s)dsdy \\
&= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{n+1}} (\nabla E)(y,s)f(x-y,t-s)dsdy
\end{aligned}$$

And

$$\begin{aligned}
\Delta \int_{\mathbb{R}^{n+1}} E(y,s)f(x-y,t-s)dsdy &= \int_{\mathbb{R}^{n+1}} (\nabla E)(y,s)\cdot(\nabla f)(x-y,t-s)dsdy \\
&= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty (\nabla E)(y,s)\cdot(\nabla f)(x-y,t-s)dsdy
\end{aligned}$$

And

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right) \int_{\mathbb{R}^{n+1}} E(y,s)f(x-y,t-s)dsdy &= -\lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\nabla E)(y,s)\cdot(\nabla f)(x-y,t-s)dsdy \\
&\quad + \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} E(y,s)\frac{\partial f}{\partial x}(x-y,t-s)dsdy \\
&= \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial s} - \Delta_y\right) E(y,s)f(x-y,t-s)dsdy \\
&\quad + \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} E(y,\varepsilon)f(x-y,t-\varepsilon)dsdy \\
&= f(x,t)
\end{aligned}$$

Next, let $u \in C^2(U \times (0,T])$ and $u_t - \Delta u = 0$, $\chi \in C^\infty$, $\chi(x,t) = 1$ if $d((x,t),\Gamma_U) \ge 2$, $\chi(x,t) = 0$ if $d((x,t),\Gamma_U) \le \varepsilon$ and $(x,t) \in U \times (0,T]$, apply the previous argument to $f(x,t) = \left(\frac{\partial}{\partial t} - \Delta\right)(\chi(x,t)u(x,t)) = \left(\left(\frac{\partial}{\partial t} - \Delta\right)\chi(x,t)\right)u - 2\nabla\chi\cdot\nabla u \in C_c^1(U \times (0,T])$, we get

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\chi(x,t)u(x,t) - \int_{-\infty}^t \int_{\mathbb{R}^n} E(x-y,t-s)f(y,s)\right) = 0$$

And

$$u(x,t)\chi(x,t) - \int_{-\infty}^{t} E(x-y,t-s)f(y,s)dsdy = 0$$

if $t = 0$, so if $0 \leq t \leq T$

$$\chi(x,t)u(x,t) = \int_{-\infty}^{t} \int_{\mathbb{R}^n} E(x-y,t-s) \left(\frac{\partial}{\partial t} - \Delta \right) (\chi(y,s)u(y,s))dsdy$$

$\square$

36.3 Wave equation

Definition 36.3.1. The fundamental solution to **wave equation** $\square u = \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$ is

$$E(x,t) = \begin{cases} \frac{1}{2\pi^{\frac{n-1}{2}}} \chi_+^{\frac{1-n}{2}}(t^2 - |x|^2) & t > 0 \\ 0 & t < 0 \end{cases}$$

Theorem 36.3.2. $f \in C^2(\mathbb{R}^3)$, $u(x,t) = \frac{1}{4\pi t} \int_{\partial B(x,t)} f(y) dS_y = \frac{t}{4\pi} \int_{S^2} f(x + tw) dS_w$, then $u \in C^2\left(\mathbb{R}^3 \times [0,\infty)\right)$, $u(x,0) = 0$, $\left.\frac{\partial}{\partial t}\right|_{t=0} u(x,t) = f(x)$ and $\square u = 0$ for $t > 0$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} u(x,t) &= \frac{1}{4\pi} \int_{S^2} f(x+tw) dS_w + \frac{t}{4\pi} \int_{S^2} (w \cdot \nabla) f(x+tw) dS_w \\ &= \frac{1}{4\pi} \int_{S^2} f(x+tw) dS_w + \frac{1}{4\pi t} \int_{\partial B(x,t)} n \cdot \nabla f(y) dS_y \\ &= \frac{1}{4\pi} \int_{S^2} f(x+tw) dS_w + \frac{1}{4\pi t} \int_{B(x,t)} \Delta f(y) dy \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x,t) &= \frac{1}{4\pi} \int_{S^2} (w \cdot \nabla) f(x+tw) dS_w - \frac{1}{4\pi t^2} \int_{B(x,t)} \Delta f(y) dy \\ &\quad + \frac{1}{4\pi t} \frac{d}{dt} \int_0^t \int_{S^2} \lambda^2 \Delta f(x + \lambda w) dS_w d\lambda \\ &= \frac{1}{4\pi t^2} \int_{B(x,t)} \Delta f(y) dy - \frac{1}{4\pi t^2} \int_{B(x,t)} \Delta f(y) dy \\ &\quad + \frac{t}{4\pi} \int_{S^2} \Delta f(x + \lambda w) dS_w \\ &= \frac{1}{4\pi t} \int_{\partial B(x,t)} \Delta f(y) dS_y \\ &= \Delta u(x,t) \end{aligned}$$

$\square$

Theorem 36.3.3. $f \in C^2(\mathbb{R}^2)$, then $u(x,t) = \frac{1}{2\pi} \int_{|y|<t} \frac{1}{\sqrt{t^2 - |y|^2}} f(x-y) dy$ solves $\square u = 0$ for $t > 0$, $u(x,0) = 0$, $u_t(x,0) = f$

Proof. Consider $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x_1, x_2, x_3) = f(x_1, x_2)$ is independent of x_3, then $u(x,t) = \frac{1}{4\pi t} \int_{\partial B(x,t)} f(y) dy = \frac{1}{4\pi t} \int_{\partial B(0,t)} f(x-y) dS_y$

$y_3 = \pm\sqrt{t^2 - y_1^2 - y_2^2} = \gamma(y), ds = \sqrt{1 + |\nabla\gamma(y)|^2} dy_1 dy_2 = \frac{t}{t^2 - y_1^2 - y_2^2}$, upper + lower hemisphere

$= \frac{2}{4\pi t} \int_{|(y_1,y_2)|<t} f(x-y) \frac{t dy_1 dy_2}{\sqrt{t^2 - |(y_1,y_2)|^2}} = \frac{1}{2\pi} \int_{|y|<t} \frac{1}{\sqrt{t^2 - |y|^2}} f(x-y) dy$ $\square$

Theorem 36.3.4. $f \in C^\infty(\mathbb{R}^n \times [0,\infty))$, $u(x,t) = \int_0^t E(\cdot, t-s) * f(\cdot, s) ds$, then $\square u = f$, $u(x,0) = u_t(x,0) = 0$

Proof. Define $u(x,t,s) = E(\cdot, t-s) * f(\cdot, s) \in C^\infty$ for $t > s$

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= u(x,t,t) + \int_0^t \frac{\partial}{\partial t}u(x,t,s)ds \\ &= \int_0^t \frac{\partial}{\partial t}u(x,t,s)ds\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \int_0^t \frac{\partial^2}{\partial t^2}u(x,t,s)dx + \left.\frac{\partial}{\partial t}\right|_{t=s} u(x,t,s) \\ &= f(x,t) + \int_0^t \frac{\partial^2}{\partial t^2}u(x,t,s)dx\end{aligned}$$

Thus $\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = f(x,t) + \int_0^t \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t,s)dx$, the second term is zero for $s < t$

By the same argument, $\Box \int_{-\infty}^t E(\cdot, t-s) * f(\cdot, s)ds = f(\cdot, t)$, thus $\Delta E = \delta_{(x,t)}$ is the fundamental solution $\square$

1 dim wave equation reflection

Lemma 36.3.5. The solution to $\Box u = 0$ in $t > 0, x > 0$ with $u(0,t)$ for all $t > 0$, $u(x,0) = 0$, $u_t(x,0) = f(x)$, $f \in C^1([0,\infty))$, $f(0) = 0$ is

$$u(x,t) = \frac{1}{2}\int_{|t-x|}^{t+x} f(\lambda)d\lambda$$

Proof. Define $\tilde{f} : \mathbb{R} \to \mathbb{R}$, $\tilde{f}(x) = \begin{cases} f(x) & x \geq 0 \\ -f(-x) & x < 0 \end{cases}$ which solves $\Box \tilde{u} = 0$ for $t > 0, x \in \mathbb{R}$, $\tilde{u}(x,0) = 0$, $\tilde{u}_t(x,0) = \tilde{f}$, hence

$$\tilde{u}(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \tilde{f}(\lambda)d\lambda = \frac{1}{2}\int_{|x-t|}^{x+t} f(\lambda)d\lambda$$

$\square$

Laplacian of a spherical symmetric function

Lemma 36.3.6. $f(x) = f(|x|)$ is spherical symmetric in $\mathbb{R}^n$, then $(\Delta f)(x) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) f$

Proof. Δu is characterized by

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx = -\int_{\mathbb{R}^n} v\Delta u, \forall v \in C_c^\infty(\mathbb{R}^n)$$

If $u(x) = u(|x|)$, $v(x) = v(|x|)$

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx &= \int_{S^{n-1}} \int_0^\infty \frac{\partial u}{\partial r}\frac{\partial v}{\partial r} dr dS_w \\ &= -\int_{S^{n-1}} \int_0^\infty \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) v(r) r^{n-1} dr dS_w \\ &= -\int_{\mathbb{R}^n} \frac{1}{r^{n-}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) v(r)dx \\ &= -\int_{\mathbb{R}^n} \left(\frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) uv(r)dx\end{aligned}$$

Note that $\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right) = \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}$ $\square$

Theorem 36.3.7. The solution to $\Box u = 0$ in $\mathbb{R}^{3+1}$ with $u(x,0) = 0$, $u_t(x,0) = f(x) = f(|x|)$, $f \in C^\infty(\mathbb{R}^3)$ is

$$u(x,t) = \frac{1}{2|x|}\int_{t-|x|}^{t+|x|} \lambda f(\lambda) d\lambda$$

Proof. By Lemma 36.3.6, when $n = 3$, $\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) u = \frac{1}{\partial r}\frac{\partial^2}{\partial r^2}(ru)$, thus if $\Box u = 0$ in $\mathbb{R}^{3+1}$, $u(x,t) = u(|x|,t)$, then $\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)(ru(r,t)) = 0$ and $ru(r,t) = 0$ if $r = 0$, $\left.\frac{\partial}{\partial t}\right|_{t=0}(ru(r,t)) = rf(r)$, by Lemma 36.3.5, $ru(r,t) = \frac{1}{2}\int_{|t-r|}^{t+r} \lambda f(\lambda) d\lambda$. We can check $u \in C^1$ $\Box$

Theorem 36.3.8 (Energy estimate version 1)**.** $\Box u = 0$ for $t > 0$, then the energy $\frac{1}{2}\int_{\mathbb{R}^n} |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 36.3.9 (Energy estimate version 2)**.** $\Box u = 0$ in $U_T = U \times (0,T]$, $u = 0$ on Γ_U, $u_t(x,0) = 0$, implicitly, $u_t = 0$ on $\partial U \times [0,T]$, then $\frac{1}{2}\int_{\mathbb{U}} |u_t|^2 + |\nabla u|^2 dx$ is a constant

Theorem 36.3.10 (Energy estimate version 3)**.** $C = \left\{(x,t) \in \mathbb{R}^{n+1} \big| |x - x_0| \leq |t - t_0|\right\}$ is the cone, $D_t = \{x \in \mathbb{R}^n | |x - x_0| \leq |t - t_0|\}$ is the section at time t, considet the case $t < t_0$, then $\frac{1}{2}\int_{\mathbb{D}_t} |u_t|^2 + |\nabla u|^2 dx$ is decreasing on $0 \leq t \leq t_0$

36.4 Euler-Lagrange equation

36.5 Energy momentum tensor

Definition 36.5.1. ∇ is the gradient, write $\nabla^T\nabla = \nabla \cdot \nabla = \Delta$ is the laplacian, $\nabla \cdot 1 = \mathtt{div}$ is the divergence, $\nabla\nabla^T = D^2$ is the Hessian

Definition 36.5.2. $L(z, q) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is C^∞, u satisfies Euler-Langrange equation, then

$$\begin{aligned}\nabla_x L(u, \nabla u) &= \frac{\partial L}{\partial z}\nabla u + (\nabla\nabla^T u)(\nabla_q L)\\ &= (\nabla_x \cdot \nabla_q L)(\nabla u) + (\nabla\nabla^T u)(\nabla_q L)\\ &= (\nabla^T u \nabla_q L)\nabla_x\end{aligned}$$

Energy-momentum tensor $T_{\alpha\beta} = \dfrac{\partial u}{\partial x^\alpha}\dfrac{\partial L}{\partial q_\beta} - \delta_{\alpha\beta}L$, $T = \nabla^T u \nabla_q L - L1$, then $T\nabla_x = (\nabla^T \nabla_q L)\nabla_x - \nabla_x L = 0$

Example 36.5.3. $u_{tt} - \Delta u + u^3 = 0$, $L(u, \nabla_{x,t} u) = \dfrac{1}{2}(u_t^2 - |\nabla_x u|^2) - \dfrac{1}{4}u^4$, $T_{00} = u_t^2 - \left[\dfrac{1}{2}(u_t^2 - |\nabla u|^2) - \dfrac{1}{4}u^4\right] = \dfrac{1}{2}(u_t^2 + |\nabla u|^2) + \dfrac{1}{4}u^4$, $T_{0i} = -u_t\dfrac{\partial u}{\partial x^i}$, thus $0 = (T_{00}, \cdots, T_{0n})\nabla_x = \mathtt{div}(T_{00}, \cdots, T_{0n})$

Part XIV

Mathematical physics

Chapter 37

Special relativity

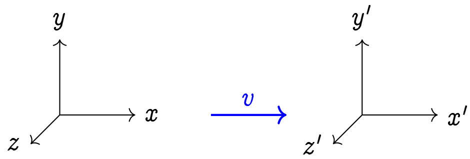
Definition 37.0.1 (Galilean group)**.** The **Galilean group** is the group of **Galilean transformations** generated by rotations in $\mathbb{R}^n$, translations in $\mathbb{R}^{n+1}$ and **Galilean boosts** $(x, t) \mapsto (x + tv, t)$

$$\begin{pmatrix} R & v & w \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} Rx + tv + w \\ t + s \\ 1 \end{pmatrix}$$

Definition 37.0.2 (Lorentz group)**.** The **Lorentz group** is the group of **Lorentz transformations** generated by rotations in $\mathbb{R}^n$ and **Lorentz boosts** $(x, t) \mapsto (\sinh sx - \cosh st, \sinh st - \cosh sx)$

Definition 37.0.3 (Poincaré group)**.** The **Galilean group** is the isometry group of the Minkowski space $\mathbb{R}^{n+1}$

Definition 37.0.4. $(x, ct) \mapsto \left(\gamma(x - vt), \gamma\left(t - \frac{vx}{c^2}\right)\right)$ $\beta = \frac{v}{c}$, $\alpha = \sqrt{1 - \beta^2}$. $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ is the **Lorentz factor** $\begin{cases} t' = \gamma\left(t - \frac{vx}{c^2}\right) \\ x' = \gamma(x - vt) \end{cases}$, where (x, t) and (x', t') are the coordinates of two frames, and frame (x', t') is moving towards the positive direction of the x axis with velocity v, and c is the speed of light, we can find the inverse transformation $\begin{cases} t = \gamma\left(t' + \frac{vx'}{c^2}\right) \\ x = \gamma(x' + vt') \end{cases}$ which makes perfect sense since relatively speaking, frame (x, t) is moving towards the negative direction of the x' axis with velocity v or rather moving towards the positive direction of the x' axis with velocity $-v$



More generally, if we consider $(\vec{x}, t), (\vec{x}', t')$ are the coordinates of two frames, with frame $(\vec{x}', t')$ moving with velocity $\vec{v}$, then the Lorentz transformation will be

Deduction 37.0.5 (Time dilation)**.** A frame moving (x', t') is at a constant speed v, then $\Delta t = \gamma \Delta t'$. Suppose you are on the train with constant speed v and height h, and let light bouncing up and down perpendicularly, then we have

$$2\sqrt{h^2 + \left(\frac{\Delta t}{2} v\right)^2} = c\Delta t, v\Delta t' = h$$

$$\Rightarrow \Delta t = \gamma \Delta t'$$

Things happen simultaneously in one frame may not be simultaneous in another frame

Deduction 37.0.6 (Length contraction)**.** Suppose a train is moving with speed v, shed a beam light from one end to get to the other end
A in frame (x,t) send a signal when the left end of train passes, B in frame (x',t') on the right end of the train receives and return the signal, suppose the length of the train is l', and the length appears to be l in frame (x,t), then it takes time $\frac{l'}{c}$ for B to receive the signal in (x',t'), which takes time $\frac{l'\gamma}{c}$ in (x,t), when B should be in distance $l+\frac{vl'\gamma}{c}$ from A in (x,t) but distance $l'+\frac{vl'}{c}$ in (x',t') which take time $\frac{l+\frac{vl'\gamma}{c}}{c}$ and $\frac{l'+\frac{vl'}{c}}{c}$ to get back to A in (x,t) and (x',t'), hence we should have $\frac{l+\frac{vl'\gamma}{c}}{c}=\frac{l'+\frac{vl'}{c}}{c}\gamma \Rightarrow l=\gamma l'$

Chapter 38

Maxwell's equations

Theorem 38.0.1 (Maxwell's equations)**.**

Part XV

Examples

Chapter 39

Examples in categories

Example 39.0.1. The image of a functor is not necessarily a category
Consider the following categories $\mathscr{C}$ and $\mathscr{D}$

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] & B \arrow[loop above, "1_B"] & C \arrow[loop above, "1_C"] \arrow[r, "g"] & D \arrow[loop above, "1_D"]
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[loop above, "1_E"] \arrow[r, "h"] \arrow[rr, bend right, "ih"'] & F \arrow[loop above, "1_F"] \arrow[r, "i"] & G \arrow[loop above, "1_G"]
\end{tikzcd}$$

Consider functor $F : \mathscr{C} \to \mathscr{D}$, $F(A) = E$, $F(B) = F$, $F(C) = F$, $F(D) = G$, $F(f) = h$, $F(g) = i$

Chapter 40

Examples in algebra

Example 40.0.1. Suppose $1 \mapsto k$ is an element in $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, then $m \mid kn \Rightarrow \frac{m}{(n,m)}$ divides $k\frac{n}{(n,m)}$, thus $\frac{m}{(n,m)}$ divides k, thus $k = \frac{im}{(n,m)}, i = 0, \cdots, (n,m)-1$, thus $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n,m)\mathbb{Z}$

Consider $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$, then $n(1 \otimes 1) = n \otimes 1 = 0$, $m(1 \otimes 1) = 1 \otimes m = 0$, thus $(n,m)(1 \otimes 1) = (rn + sm)(1 \otimes 1) = 0$

Apply functor $Hom(-, \mathbb{Z}/m\mathbb{Z})$ to short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, we get a left exact sequence $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \to 0$

Apply functor $- \otimes \mathbb{Z}/m\mathbb{Z}$ to short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, we get a left exact sequence $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to 0$

And the kernel and cokernel of $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z}$ are both $\mathbb{Z}/(n,m)\mathbb{Z}$

Example 40.0.2. $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}\left[\frac{1}{p}\right]$

Example 40.0.3. $O(1,1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}$

R^2=R

Example 40.0.4. F is a field, $R = \mathrm{End}(F^\infty) = \{\text{infinite dimensional matrices}\}$, Consider $R \hookrightarrow R$ by embedding into odd rows and even rows, we have $R^2 \cong R$ as right R modules

Example 40.0.5. $GL(2, \mathbb{F}_2) = SL(2, \mathbb{F}_2) \cong S_3$

Chapter 41

Examples in algebraic topology

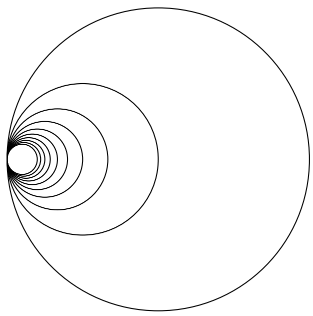
Example 41.0.1 (A surjective local homeomorphism may not be a covering)**.** $p : \mathbb{R} \setminus \{0\} \to S^1$, or n sheeted cover with a point missing, p is discrete but not proper

Example 41.0.2 (Bundle with fiber isomorphic to vector space but not a vector bundle)**.** $E := \bigsqcup_{x \in X} \mathbb{R}^n$

Example 41.0.3. $H'_n = H_{k+n}$ also defines a homology theory where the dimension axiom fails

Hawaiian earring

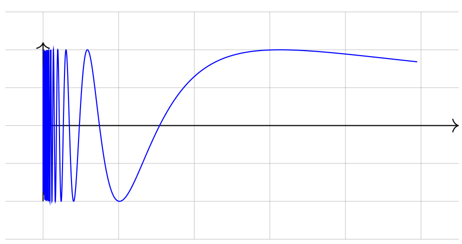
Example 41.0.4 (Hawaiian earring)**.** The **Hawaiian earring** H is the union of circles with radius $\frac{1}{n}$ and centered at $(\frac{1}{n}, 0)$ with subspace topology in $\mathbb{R}^2$



Proposition 41.0.5. Hawaiian earring is not a CW complex since it is not locally contractible

Example 41.0.6 (Topologist's sine curve)**.** The **topologist's sine curve** is

$$T = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \middle| x \in (0, 1] \right\} \cup \{(0, 0)\}$$



Proposition 41.0.7. The topologist's sine curve T is connected but not path connected

Example 41.0.8 (Warsaw circle)**.** The **Warsaw circle** W is the topologist's sine curve enclosed Bijective map $W \to [0, 1)$ is not a homeomorphism, thus not a quotient map. W is weakly homotopic to a point but not homotopic

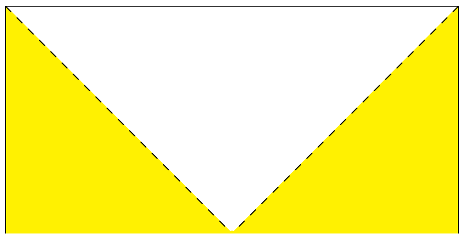
Example 41.0.9. $X = \mathbb{N}$ with discrete topology, $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ with subspace topology of $\mathbb{R}$, then $f : X \to Y, n \mapsto \frac{1}{n}$ is a weak homotopy equivalence, however, X, Y are not homotopy equivalent, otherwise suppose $g : X \to Y, h : Y \to X$ such that $hg \simeq 1_X$, $gh \simeq 1_Y$, suppose $F : Y \times I \to Y$ is a homotopy, then the restriction of F on $\{y\} \times I$ must be a constant map since the connected components of Y are just points, thus $F(y, 0) = F(y, 1)$, i.e. homotopic maps are in fact the same, for a similar argument on X, we have $hg = 1_X$, $gh = 1_Y$, thus h is injective which is impossible since $h^{-1}(h(0))$ consists of more then one point

Cofibration counterexample

Example 41.0.10. $D^2 = S^2 \setminus \{N\} \subseteq S^2$ is not a cofibration. $D^2 \setminus \{0\} \subseteq D^2$ is not a cofibration

Mapping cylinder of inclusion may have different topology then induced subspace topology

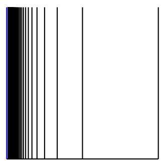
Example 41.0.11. $A = [-1, 0) \cup (0, 1]$, $X = [-1, 1]$, then the mapping cylinder of the inclusion $A \overset{i}{\hookrightarrow} X$ has different topology from the subspace topology $X \times \{0\} \cup A \times I$ induced from $X \times I$



Nonclosed cofibration

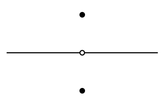
Example 41.0.12. $\{a, b\}$ with trivial topology, $\{a\} \subseteq \{a, b\}$ is a nonclosed cofibration since there is a retraction $I \sqcup I \to I \sqcup \{0\}$, $(s, t) \mapsto (s, 0)$

Example 41.0.13. The **comb space** is $[(0, 0), (1, 0)] \bigcup \bigcup_{n=1}^{\infty} [(\frac{1}{n}, 0), (\frac{1}{n}, 1)]$



A line with two origins

Example 41.0.14. A topological space with a cell decomposition may not be Hausdorff, consider $(-1, 1)$ with two origins, which has $(-1, 0), (0, 1)$ as 1 cells and two origins as 0 cells



Chapter 42

Examples in geometry

Definition 42.0.1 ($\mathcal{O}(n)$ bundle over Riemann sphere $S^2 \cong \mathbb{C}P^1$)**.** Suppose $(\mathbb{C}, z \mapsto z)$, $\left(S^2 \setminus 0, z \mapsto \dfrac{1}{z}\right)$ are the charts coordinate of S^2 with transition map $z \mapsto \dfrac{1}{z}$ both ways on $\mathbb{C} \setminus 0$, or equivalently

$(U_0, [1, z] \mapsto z)$, $(U_1, [z, 1] \mapsto z)$ are the corresponding charts of $\mathbb{C}P^1$ with transition map $z \mapsto \dfrac{1}{z}$ both ways on $U_0 \cap U_1$, namely, $S^2 \to \mathbb{C}P^1$, $z \mapsto [1, z]$, $\infty \mapsto [0, 1]$ is and isomorphism because of isomorphisms on charts

Now define $\mathcal{O}(n)$ line bundle on S^2 by specify transition functions $g_{10}(z) = z^{-n}$, $g_{01}(z) = z^n$, $\forall z \in \mathbb{C} \setminus 0 \cong U_0 \cap U_1$

Definition 42.0.2 (Tautological line bundle over Riemann sphere)**.** The tautological bundle is $\mathcal{O}(-1)$, tautological bundle is defined as a subspace E of $\mathbb{C}P^1 \times \mathbb{C}^2$ consists of (l, v) with $v \in l$ projects to the first factor, let's figure out the trivializations!

$\varphi_0 : U_0 \times \mathbb{C}^2 \cap E \to \mathbb{C} \times \mathbb{C}$, $([1, z], t(1, z)) \mapsto (z, t)$, and $\varphi_1 : U_1 \times \mathbb{C}^2 \cap E \to \mathbb{C} \times \mathbb{C}$, $([z, 1], t(z, 1)) \mapsto (z, t)$, since $\varphi_1 \circ \varphi_0^{-1} : (U_0 \cap U_1) \times \mathbb{C}^2 \cap E \to (U_0 \cap U_1) \times \mathbb{C}^2 \cap E$, $(z, t) \mapsto \left(\dfrac{1}{z}, zt\right)$, the transition function $g_{10}(z) = z$

Remark 42.0.3. $\mathcal{O}(-1)$ doesn't nonzero global section, suppose s is a global section of $\mathcal{O}(-1)$, then $s(x) = (x, f(x)) \in E \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}^2$ is holomorphic, but then image of σ has to be a point, and this point must be zero

Example 42.0.4. We still use U_0, U_1 to denote coordinate charts, φ_0, φ_1 to denote corresponding trivializations

Global sections of $\mathcal{O} = \mathcal{O}(0)$ are exactly holomorphic functions which are just constants, suppose $s : S^2 \to \mathcal{O}$ is a section, and $\varphi_0 \circ s|_{U_0}(z) = (z, f_0(z))$, $\varphi_1 \circ s|_{U_1}\left(\dfrac{1}{z}\right) = \left(\dfrac{1}{z}, f_1\left(\dfrac{1}{z}\right)\right)$, then we have $(z, f_1(z)) = \varphi_1 \circ s|_{U_1}(z) = \varphi_1 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1} \circ \varphi_0 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1}(z, f_0(z)) = (z, g_{10}(z) f_0(z)), \forall z \in U_0 \cap U_1$, thus $f_1(z) = g_{10}(z) f_0(z) = f_0(z)$ which precisely means s correspond to holomorphic function f over X, $f|_{U_0} = f_0, f|_{U_1} = f_1$

Let's show that the canonical bundle(which in the case of a Riemann surface is the same as the cotangent bundle) is $\mathcal{O}(-2)$, since $d\left(\dfrac{1}{z}\right) = -\dfrac{1}{z^2} dz$, the transition function would be $g_{10}(z) = -z^2$, but using dz or $-dz$ as the a basis element would be isomorphic

Proposition 42.0.5. $H^0\left(CP^1, \mathcal{O}(n)\right)$, the vector space of global sections of $\mathcal{O}(n) \to \mathbb{C}P^1, n \geq 0$ generated by homogeneous polynomials $z_0^n, z_0^{n-1} z_1, \cdots, z_0 z_1^{n-1}, z_1^n$

Proof. $z_0^k z_1^{n-k}$ have the forms z_1^{n-k} and z_0^k in U_0 and U_1 □

Example 42.0.6 (Line bundles on the projective space $\mathbb{C}P^n$)**.** Suppose $(U_0, [1, z_1, \cdots, z_n] \mapsto (z_1, \cdots, z_n))$, $(U_n, [z_0, z_1, \cdots, z_{n-1}, 1] \mapsto (z_0, \cdots, z_{n-1}))$ be coordi-

nate charts of $\mathbb{C}P^n$, with transition map $U_i \cap U_j \to U_i \cap U_j$, $\left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i}\right) \mapsto \left(\frac{z_0}{z_j}, \cdots, \widehat{\frac{z_j}{z_j}}, \cdots, \frac{z_n}{z_j}\right)$, which is kind of like multiply by $\frac{z_i}{z_j}$, then the line bundle $\mathcal{O}(m)$ is defined by transition function $g_{ji} = \frac{z_j}{z_i}$ which satisfies the cocycle condition
Similarly, we can check that the tautological bundle $E = \{(l, v) | v \in l\} \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$ projects to $\mathbb{C}P^n$ is $\mathcal{O}(1)$
It is obvious that any degree n polynomial are global section of $\mathcal{O}(n)$

Chapter 43

Examples in Lie groups and Lie algebras

Example 43.0.1. X is topological space, $End(X)$ is a unital nonassociative $\mathbb{R}$ algebra which is not symmetric, antisymmetric, nor does it satisfy Jacobi identity

Example 43.0.2. Consider $C^\infty(M)$ where M is a smooth manifold, then $\mathcal{L}(M) = Der(C^\infty(M))$ consists of vector fields, it is a Lie algebra, hence we can think of derivations as linear differential operator of order 1, then we know that the commutator of two such operators is again a linear differential operator of order 1

Example 43.0.3. Let $\mathfrak{g}$ be a Lie algebra, then ideals of $\mathfrak{g}$ precisely the Lie algebra subrepresentations of the adjoint representation $(ad, \mathfrak{g})$

Example 43.0.4 (Lie algebra of $M_n(\mathbb{R})$)**.** Suppose $X = \sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}}$ is a left invariant

$$\begin{aligned}
X_{kl}(A) &= \sum_{i,j} X_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\
&= X_A(x_{kl}) = (L_A)_0 X_0(x_{kl}) \\
&= X_0(x_{kl} \circ L_A) \\
&= \sum_{i,j} X_{ij}(0) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(0) \\
&= X_{kl}(0)
\end{aligned}$$

Thus X_{ij} are constants

$$\begin{aligned}
[X, Y] &= \left[\sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} Y_{kl} \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j,k,l} X_{ij} Y_{kl} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j} X_{ij} Y_{ij} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\
&= 0
\end{aligned}$$

Therefore $\mathsf{Lie}(M_n(\mathbb{R})) = 0$

Example 43.0.5 (Lie algebra of $GL(n,\mathbb{R})$)**.** Suppose $X = \sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}$ is a left invariant field

$$\begin{aligned}
c_{kl}(A) &= \sum_{i,j} c_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\
&= X_A(x_{kl}) = (L_A)_I X_I(x_{kl}) \\
&= X_I(x_{kl} \circ L_A) \\
&= \sum_{i,j} c_{ij}(I) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(I) \\
&= \sum_i a_{ki} c_{il}(I)
\end{aligned}$$

Hence $C(A) = AC(I)$, $\frac{\partial c_{kl}}{\partial x_{ij}} = \delta_{ki} c_{jl}(I)$

$$\begin{aligned}
[X,Y] &= \left[\sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} d_{kl} \frac{\partial}{\partial x_{kl}}\right] \\
&= \sum_{i,j,k,l} \left[c_{ij} \frac{\partial}{\partial x_{ij}}, d_{kl} \frac{\partial}{\partial x_{kl}}\right] \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial}{\partial x_{ij}} \left(d_{kl} \frac{\partial}{\partial x_{kl}}\right) - d_{kl} \frac{\partial}{\partial x_{kl}} \left(c_{ij} \frac{\partial}{\partial x_{ij}}\right) \\
&= \sum_{i,j,k,l} c_{ij} \left(\frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} + d_{kl} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}}\right) - d_{kl} \left(\frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} + c_{ij} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}}\right) \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} \\
&= \sum_{i,j,k,l} \left(c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} - d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}}\right) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{j,k,l} (c_{kj} d_{jl} - d_{kj} c_{jl}) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{k,l} \left(\sum_j c_{kj} d_{jl} - d_{kj} c_{jl}\right) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{k,l} b_{kl} \frac{\partial}{\partial x_{kl}}
\end{aligned}$$

Here $B = [C,D]$. Therefore $\mathsf{Lie}(GL(n,\mathbb{R})) = \mathfrak{gl}(n,\mathbb{R})$

Example 43.0.6. Consider the $\Phi : GL(n,\mathbb{R}) \to M_n(\mathbb{R}), A \mapsto A^T A$ which is a smooth map, and level set $\Phi^{-1}(I) = O(n,\mathbb{R})$ is the orthogonal group, to show this is a Lie subgroup, thanks to Theorem 22.0.16, it suffices to show Φ is of constant rank, but Φ is equivariant assuming $GL(n,\mathbb{R})$ acts on itself by right multiplication and acts on $M_n(\mathbb{R})$ by $X \cdot A = A^T X A, X \in M_n(\mathbb{R}), A \in GL(n,\mathbb{R})$, since $\Phi(A) \cdot B = B^T A^T A B = \Phi(AB)$
$(d\Phi)_I(B) = B^T + B$, and $T_I(O(n,\mathbb{R})) = \ker(d\Phi)_I = \{B \in M(n,\mathbb{R}) | B^T + B = 0\}$

Chapter 44

Examples in algebraic geometry

Example 44.0.1. Suppose $V \subseteq \mathbb{A}^n$ is an affine variety, $m_P \in \mathtt{Spm} k[V]$, $k[V]_{m_P}$ is the stalk of the sheaf of regular functions. Tow representatives $\frac{f}{u}, \frac{g}{v}$ are of the same germ $\Leftrightarrow$ $\frac{f}{u} = \frac{g}{v}$ on $D(wuv)$ for some $w(P) \neq 0 \Leftrightarrow w(fv - gu) = 0$

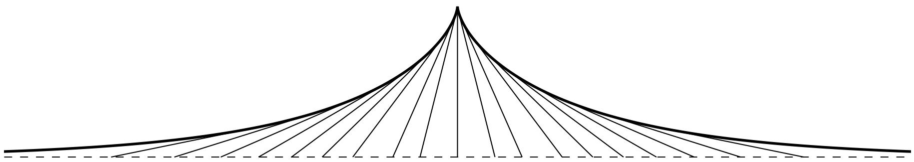
Example 44.0.2.

Chapter 45

Examples in analysis

Example 45.0.1. $D \subseteq \mathbb{C}$ is the unit disc, $f(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{n^2}$ is continuous on $\overline{D}$ and holomorphic on D but not on any point on ∂D

Example 45.0.2 (Tractrix)**.** An interval I with one end point pushed or dragged along the x axis gives a **Tractrix**. The velocity has the same direction as I, i.e. $\frac{dx}{dy} = \pm\frac{\sqrt{a^2 - y^2}}{y}$, which gives solution $x = \pm\left(\ln\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}\right)$



Part XVI

Exercises

Chapter 46

Exercises in combinatorics

Exercise 46.0.1. $[n] = \{1, \cdots, n\}$, what is the cardinality of $\{f \in Aut([n]) | f(i) \neq i, \forall i \in [n]\}$

Solution. Consider $A_k = \{f \in Aut([n]) | f(k) = k\}$, by Inclusion-exclusion principle 1.0.4, we have

$$\begin{aligned}
n! = \left| \bigcup_{i=1}^{n} A_n \right| &= \sum_{k=1}^{n} (-1)^k \sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}| \\
&= \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)! \\
&= \sum_{k=1}^{n} (-1)^k \frac{n!}{k!}
\end{aligned}$$

Thus the probability of picking such an auto morphism is $\sum_{i=1}^{n} \frac{(-1)^k}{k!}$ which approaches e^{-1} as n approaches infinity □

46.1 Exercises in abstract algebra

Exercise 46.1.1. If R is a domain, so is $R[x]$

Solution. Suppose $f = ax^n + \cdots$, $g = bx^m + \cdots$ for some $a, b \neq 0$, then $fg = abx^{n+m} + \cdots \neq 0$ □

Exercise 46.1.2. If E/F is a Galois extension, then $Tr_{E/F}(\alpha)$ is the sum of all conjugates of α, $N_{E/F}(\alpha)$ is the product of all conjugates of α

Solution. Suppose the minimal polynomial of α is $m(x) = x^n + a_1x^{n-1} + \cdots + a_n$ □

Exercise 46.1.3. If $F \subseteq E \subseteq L$ are field extensions, then $Tr_{L/F} = Tr_{E/F} \circ Tr_{L/E}$

Solution. Suppose $x_1, \cdots, x_n$ is a basis for L/E, $y_1, \cdots, y_m$ is a basis for E/F □

T:V->W<=W => Tr(T)=Tr(T|W)

Exercise 46.1.4. Suppose $T \in \mathrm{Hom}_{\mathbb{F}}(V, V)$ is a linear operator with $T(V) \leq W$, then $Tr(T) = Tr(T|_W)$

Mundane properties of rings

Exercise 46.1.5. R is a ring

1. $0x = 0$, $(-1)x = -x$

Solution.

1.

$$0x = (0+0)x = 0x + 0x \Rightarrow 0x = 0$$

$$0x = (1 + (-1))x = 1x + (-1)x = x + (-1)x \Rightarrow (-1)x = -x$$

□

Exercise 46.1.6. Let R be a commutative ring, and $I_1, \cdots, I_n \leq R$ be pairwise coprime ideals, then $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$

Solution. By induction □

Exercise 46.1.7. Every group G is naturally isomorphic to its opposite G^{op}

Solution. Consider $\phi : G \to G^{op}$, $g \mapsto g^{-1}$ □

Exercise 46.1.8. A morphism of G torsors is always an isomorphism

Exercise 46.1.9. X has a left G action and a right H action such that $(gx)h = g(xh)$

1. $X \times_G * \cong X/G$

2. $X \times_G G \cong X$

3. $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$

4. If $H \leq G$, then $X \times_G G \times_Y \cong X \times_H Y$

5. If $H \trianglelefteq G$, $X \times_G (G/H) \cong X/H$

Exercise 46.1.10. $SL(n, F)$ is a perfect group for $n \geq 3$. $SL(2, F)$ is a perfect group if $|k| \geq 4$

Solution. Denote $G_n = SL(n, F)$. Elementary matrices generate G_n and are in $[G_n, G_n]$ □

46.2 Exercises in analysis

U<R^n open, boundary point is the limit of some discrete sequence

Exercise 46.2.1. $U \subsetneq \mathbb{R}^n$ is a nonempty open set, $x \in \partial U$, then there exists a discrete sequence $\{x_i\} \subseteq U$ converges to x

Solution. x is necessarily an accumulation point since $\partial U \cap U = \varnothing$. Pick $x_0 \in U$, then we can find $\epsilon > 0$ such that $x_0 \notin B(x, \epsilon)$, then pick $x_1 \in B(x, \epsilon/2) \cap U$, and so on □

f analytic near 0, after change of variables, f has terms only involve one variable

Exercise 46.2.2. f is analytic near 0, by rotation of coordinates, we can always make f has terms only involve one variable

Exercise 46.2.3. Evaluate $\displaystyle\int_0^\infty e^{-s^2-\frac{1}{s^2}} ds$

Solution. $\left(s - \dfrac{1}{s}\right)^2 = s^2 + \dfrac{1}{s^2} - 2$, let $x = s - \dfrac{1}{s}$ which is increasing on $(0, \infty)$ since $0 < s < \infty$, $-\infty < x < \infty$, then $s = \dfrac{x + \sqrt{x^2+4}}{2}$ and

$$\int_0^\infty e^{-s^2-\frac{1}{s^2}} ds = e^{-2} \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{1}{2} + \frac{x}{2\sqrt{x^2+4}}\right) dx = e^{-2} \int_0^\infty e^{-x^2} dx = \frac{e^{-2}\sqrt{\pi}}{2}$$

□

Exercise 46.2.4. f is holomorphic on the punctured unit disc, $p > 0$, $\displaystyle\int_D |f(z)|^p dz < \infty$. What can we say about the singularity?

Solution. $|f(z)|^p = e^{p \log |f(z)|}$ is subharmonic by Example 32.0.8, thus essential singularity is impossible

$$|f(z)|^p \leq \frac{4}{\pi |z|^2} \int_{|w-z|<|z|/2} |f(w)|^p dw \leq \frac{C}{|z|^2}$$

Thus $|z|^{\frac{2}{p}} |f(z)| < \infty$ □

Exercise 46.2.5. $U \subseteq \Omega \subseteq \mathbb{C}$ are open, f is holomorphic on U, $\widehat{U}_\Omega$ be the union of U and compact connected components of $\Omega \setminus U$. There exist $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U iff there exists g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$

Solution. Assume $\widehat{U}_\Omega = U \cup K_1 \cup \cdots$, where K_i's are compact

Suppose $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U, by maximum principle, $\{f_n\}$ would be uniformly bounded around K_i, by Montel's theorem 33.1.14, there exists a subsequence of $\{f_n\}$ converges uniformly on K_i, thus converging to g holomorphic on $H(\widehat{U}_\Omega)$, hence $g|_U = f$

Conversely, suppose g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$, $\widehat{U}_\Omega$ is simply connected, by Riemann mapping theorem 33.1.23, we can think of $\widehat{U}_\Omega$ as the unit disc or $\mathbb{C}$, by Runge's theorem, there exist $\{f_n\}$ holomorphic on Ω uniformly converging to g on each disc. Thus there exist a subsequence of $\{f_n\}$ converging uniformly to g on compact subsets of $\widehat{U}_\Omega$ □

Exercise 46.2.6. Let Ω be an open subset of $\mathbb{C}$, $\mathscr{D} = \{D_i\}$ be an open cover of Ω with disks. Given meromorphic functions h_i on D_i, not identically zero. Assume $g_{ij} = \dfrac{h_i}{h_j}$ are holomorphic on $D_i \cap D_j$, then there exist holomorphic function f_i with no zeros on D_i such that $f_i = g_{ij} f_j$

Solution. It suffices to prove $H^1(\Omega, \mathcal{O}^*) = 0$, since then $H^1(\mathscr{D}, \mathcal{O}^*) = 0$, $(g_{ij}) \in Z^1(\mathscr{D}, \mathcal{O}^*) = B^1(\mathscr{D}, \mathcal{O}^*)$, i.e. there exists $(f_i) \in C^0(\mathscr{D}, \mathcal{O}^*)$ such that $f_i = g_{ij} f_j$

Consider exact sequence of sheaves $0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$, then we get a long exact sequence $\cdots \to H^1(\Omega, \mathcal{O}) \to H^1(\Omega, \mathcal{O}^*) \to H^2(\Omega, \mathbb{Z}) \to \cdots$, $H^1(\Omega, \mathcal{O}) = 0$ by Mittag-Leffler theorem □

Exercise 46.2.7. For each real r such that $0 < |r| < 1$, prove that there exists at most one real s with $0 < s < 1$ for which $\Omega := D \setminus \{0, r, s\}$ admits an analytic automorphism different from the identity

Solution. Suppose $\Omega \xrightarrow{\phi} \Omega$ is an analytic automorphism, then $0, r, s$ are all removable singularities, by continuity, ϕ can be extended to $D \xrightarrow{\phi} D$, so is ϕ^{-1}, by continuity, we know ϕ is an automorphism of D, sending $\{0, r, s\}$ to itself bijectively
By Schwarz lemma, we know that an automorphism ϕ of D with $\phi(\alpha) = 0$ iff $\phi = e^{i\theta}\dfrac{z-\alpha}{1-\overline{\alpha}z}$
Now suppose ϕ is an automorphism different from the identity, if $0 < r = s < 1$, then $\phi = -\dfrac{z-r}{1-rz}$ is a choice, now we assumen $r \neq s$

Case I: $\phi(0) = 0$

$\phi = e^{i\theta}z$, $\phi(r) = s$, but $0 < s < 1$, thus $s = |r|$

Case II: $\phi(r) = 0$

$\phi = e^{i\theta}\dfrac{z-r}{1-\overline{r}z}$, $\phi(0) = -re^{i\theta}$

Case i: $\theta = \pi$, $\phi(0) = r$, then $s = \phi(s) \Rightarrow \overline{r}s^2 - 2s + r = 0 \Rightarrow s = \dfrac{1+\sqrt{1-|r|^2}}{\overline{r}}$ or $\dfrac{1-\sqrt{1-|r|^2}}{\overline{r}}$, r has to be a positive real number and $s = \dfrac{1-\sqrt{1-r^2}}{r}$

Case ii: $s = \phi(0) = |r|$

Case III: $\phi(s) = 0$

$\phi = e^{i\theta}\dfrac{z-s}{1-sz}$, $\phi(0) = -se^{i\theta}$

Case i: $\theta = \pi$, $\phi(0) = s$, then $r = \phi(r) \Rightarrow sr^2 - 2r + s = 0 \Rightarrow s = \dfrac{2r}{1+r^2}$.

Case ii: $r = \phi(0) = -se^{i\theta}$, $s = \phi(r) \Rightarrow s^2 = 1$ which is impossible

$\square$

Exercise 46.2.8. $F \subseteq \mathbb{C}$ is closed, connected and noncompact, $\Omega = \mathbb{C} \setminus F$, then every $f \in \mathcal{O}(\Omega)$ has a primitive

Solution. It suffices to show that every connected component U of Ω is simply connected
Suppose U is not simply connected, then $\pi_1(U, z_0) \neq 0$, i.e. there is a simple(self non-intersecting) loop $\gamma \subseteq U$ with $\gamma(0) = \gamma(1)$ cannot be deform to z_0, by Jordan curve theorem 15.1.44, γ divides $\mathbb{C}$ into the exterior and the interior which is homeomorphic to the unit disc D, suppose $F \cap D$ is empty, then $\overline{D} \subseteq U$, γ can be deformed to z_0, giving a contradiction, hence $F \cap \overline{D}$ is a compact connected component of F which is also a contradiction $\square$

Exercise 46.2.9. Consider an open set $\Omega \subseteq \mathbb{C}^2$ such that

$$\{(z, w) \in \mathbb{C}^2 | |z| \leq R_1, |w| \leq R_2\} \subseteq \Omega$$

for some positive reals R_1 and R_2. Let $f \in \mathcal{H}(\Omega)$ be such that $f(z, w) \neq 0$ for every z and w for which $|z| \leq R_1$, $|w| = R_2$

1. Prove that the number (counted with multiplicities) of zeros of $w \mapsto f(z, w)$ in $D(0, R_2)$ is the same for every $|z| \leq R_1$

2. Let $w_1(z), \cdots, w_m(z)$ denote the zeros of $w \mapsto f(z, w)$ (counted with multiplicities). Prove that for each $n \in \mathbb{N}$ the function

$$z \mapsto w_1(z)^n + \cdots + w_m(z)^n$$

is holomorphic for $z \in D(0, R_1)$

3. Deduce that nth elementary symmetric function σ_n of $w_1(z), \cdots, w_m(z)$ is holomorphic.

4. Prove that there exists a function h that is holomorphic and without any zeros on $\{(z, w) \in \mathbb{C}^2 | |z| < R_1, |w| < R_2\}$ such that

$$f(z, w) = h(z, w)[w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)]$$

for every z and w such that $|z| < R_1$ and $|w| < R_2$

Solution.

1. By Lemma 33.1.18, $\dfrac{1}{2\pi i}\displaystyle\int_{\partial D(0,R_2)} \frac{f_w(z, w)}{f(z, w)} dw$ is the number of zeros in $D(0, R_2)$ which is continuous, hence the same for every $|z| \leq R_1$

2. By Lemma 33.1.18, $\dfrac{1}{2\pi i}\displaystyle\int_{\partial D(0,R_2)} w^n \frac{f_w(z, w)}{f(z, w)} dw = w_1(z)^n + \cdots + w_m(z)^n$ is holomorphic

3. Directly follows from (2) thanks to Newton's identities

4. Since $\displaystyle\prod_{i=1}^{m}(w - w_i(z)) = w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)$ is holomorphic

$$\frac{f(z, w)}{w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)}$$

has no zeros on D and holomorphic on $\{R_2 - \varepsilon < |w| < R_2\}$, hence by Hartogs's extension theorem 33.1.27, can be extended to a holomorphic function $h(z, w)$, then $f(z, w) = h(z, w)[w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)]$ on $\{R_2 - \varepsilon < |w| < R_2\}$, by identity theorem, this holds for all $|z| < R_1$ and $|w| < R_2$

□

Exercise 46.2.10. Suppose $p_1, \cdots, p_n$ are points on the compact Riemann surface X and $X' = X \setminus \{p_1, \cdots, p_n\}$. Suppose $f : X' \to \mathbb{C}$ is a non-constant holomorphic function. Show that the image of f comes arbitrarily close to every $c \in \mathbb{C}$

Solution. Suppose there exists $c \in \mathbb{C}$ such that $|f - c| \geq \varepsilon$ for some $\varepsilon > 0$, then $\dfrac{1}{f - c}$ would be a bounded holomorphic function on X', by Riemann's Removable singularity theorem, $\dfrac{1}{f - c}$ can be extended to a holomorphic function on X, but since X is compact, $\dfrac{1}{f - c}$ is a constant which is impossible □

Exercise 46.2.11. Let X be a compact Riemann surface and let $X \xrightarrow{\sigma} X$ be a biholomorphic map of X onto itself, different from the identity. Let $a \in X$ be a point with $\sigma(a) \neq a$, and suppose that there is a non-constant meromorphic function f on X, holomorphic on $X \setminus \{a\}$, with a pole of order k at a. Prove that σ can have at most $2k$ fixed points on X

Solution. Suppose there are more than $2k$ fixed points of σ, then consider $f - f \circ \sigma^{-1} : X \to \mathbb{P}^1$ is holomorphic on $X \setminus \{a, \sigma^{-1}(a)\}$ with at least $2k+1$ zeros and with poles of order k at $a, \sigma^{-1}(a)$, but it should have as many poles as zeros which is a contradiction □

Exercise 46.2.12. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are lattices in $\mathbb{C}$. Show that $\Lambda = \Lambda'$ iff there exists a matrix $A \in GL(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Solution. First not that

$$\Lambda \subseteq \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \text{ for some } A \in \mathrm{M}(2, \mathbb{Z})$$

Hence we have

$$\Lambda = \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = B \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some } A, B \in \mathrm{M}(2, \mathbb{Z})$$

Which is equivalent to $A \in \mathrm{GL}(2, \mathbb{Z})$ □

Exercise 46.2.13. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are lattices in $\mathbb{C}$ and $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$ are the corresponding complex tori

1. Prove that any holomorphic map $X \xrightarrow{f} X'$ is induced by a linear map $\mathbb{C} \xrightarrow{g} \mathbb{C}$ of the form $g(z) = \alpha z + \beta$, where $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda \subseteq \Lambda'$. f is biholomorphic if and only if $\alpha\Lambda = \Lambda'$
2. Show that every torus $X = \mathbb{C}/\Lambda$ is isomorphic to a torus of the form $X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$
3. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{Im}(\tau) > 0$. Let $\tau' := \dfrac{a\tau + b}{c\tau + d}$. Show that the tori $X(\tau)$ and $X(\tau')$ are biholomorphic

Solution.

1. Since $\mathbb{C}$ is the universal cover of $\mathbb{C}/\Lambda'$, $f \circ \pi : \mathbb{C} \to \mathbb{C}/\Lambda'$ has a lift $F : \mathbb{C} \to \mathbb{C}$, and locally we have $F = \pi'|_V^{-1} \circ f \circ \pi|_U$, thus F is holomorphic

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

Fix $\omega \in \Lambda$, since $\pi(z + \omega) = \pi(z)$ for any $z \in \mathbb{C}$, we have $F(z + \omega) - F(z) \in \Lambda'$, hence $F(z+\omega) - F(z)$ is a continuous function of z but Λ' is discrete, thus $F(z+\omega) - F(z) \equiv C_\omega$, where $C_\omega \in \Lambda'$ is a constant. Then $F'(z + \omega) = F'(z)$ which shows $F' : \mathbb{C} \to \mathbb{C}$ is doubly periodic function, thus induces $G : \mathbb{C}/\Lambda \to \mathbb{C}$ with $F = G \circ \pi$. Thus G must be a constant, so is F', therefore F has the form $F(z) = \alpha z + \beta$. Then for any $\omega \in \Lambda$, we have $F(\omega) - F(0) = \alpha\omega \in \Lambda'$, thus $\alpha\Lambda \subset \Lambda'$. If f is biholomorphic, then $\pi' \circ F = f \circ \pi \Rightarrow \pi \circ F^{-1} = f^{-1} \circ \pi'$, which implies $\begin{cases} \alpha\Lambda \subset \Lambda' \\ \alpha^{-1}\Lambda' \subset \Lambda \end{cases} \Rightarrow \alpha\Lambda = \Lambda'$

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"] & \mathbb{C} \arrow[l, "F^{-1}"'] \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda & \mathbb{C}/\Lambda' \arrow[l, "f^{-1}"'] \end{tikzcd}$$

Conversely, if $\alpha\Lambda = \Lambda'$, $\pi \circ F^{-1}$ is doubly periodic and induce f^{-1}, hence f is biholomorphic

2. Suppose $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$, define $\Lambda' = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \frac{\omega_2}{\omega_1}$, we have $\omega_1\Lambda' = \Lambda$, thus X and $X(\tau)$ are biholomorphic

3. $X(\tau)$ and $X(\tau')$ are biholomorphic iff $\begin{pmatrix}\tau'\\1\end{pmatrix} = \alpha A\begin{pmatrix}\tau\\1\end{pmatrix}, \alpha \in \mathbb{C} - \{0\}, A \in \mathrm{SL}(2,\mathbb{Z})$ If $X(\tau)$ and $X(\tau')$ are biholomorphic, then $\mathbb{Z} + \mathbb{Z}\tau' = \Lambda' = \alpha\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\alpha\tau$ for some $\alpha \in \mathbb{C} - \{0\}$, thus $\begin{pmatrix}\tau'\\1\end{pmatrix} = A\begin{pmatrix}\alpha\tau\\\alpha\end{pmatrix} = \alpha A\begin{pmatrix}\tau\\1\end{pmatrix}$, for some $A \in \mathrm{SL}(2,\mathbb{Z})$, the other direction is easy

□

Exercise 46.2.14. Determine the branch points(or ramification points) of the map $f : \mathbb{C} \to \mathbb{P}^1$ with

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Solution. $f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$ when $z \neq 0$, thus $1, -1$ are branch points.

Consider the chart $(P^1 - \{0\}, \varphi)$ with $\varphi(z) = \frac{1}{z}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[dr] & \mathbb{P}^1 - \{0\} \arrow[d, "\varphi"] \\ & \mathbb{C} \end{tikzcd}$$

Thus $g(z) = \varphi \circ f(z) = \frac{z}{2(z^2+1)}$, $g'(z) = \frac{1-z^2}{2(z^2+1)}$, hence 0 is not a branch point □

Exercise 46.2.15. If f and g are two elliptic functions with respect to the same lattice $\Omega \subseteq \mathbb{C}$, prove that there exists an irreducible polynomial $P(x,y) \in \mathbb{C}[x,y]$ such that $P(f,g) = 0$

Solution. If $f \equiv c$ is a constant, then $P(x,y) = x - c$ is an irreducible polynomial such that $P(f,g) = 0$, so we can assume f, g are not constants, Since $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$, there exists rational functions $R_0, \cdots, R_n$ such that $R_0(f) + R_1(f)g + \cdots + R_n(f)g^n = 0$, then after multiplying denominators, we get a polynomial $P(x,y) \in \mathbb{C}[x,y]$ such that $P(f,g) = 0$, since $\mathbb{C}[x,y]$ is a UFD, $P = P_1 \cdots P_k$, where P_i are prime hence irreducible, then $0 = P_1(f,g)\cdots P_k(f,g) \in \mathcal{M}(X)$ which is a field, thus $P_j(f,g) = 0$ for some irreducible polynomial $P_j \in \mathbb{C}[x,y]$ □

Exercise 46.2.16. f is an elliptic function of order $n > 0$, then f' is an elliptic function of order m such that $n + 1 \leq m \leq 2n$. Both bounds can be attained

Solution. f' is elliptic since $f(z+\omega) = f(z) \Rightarrow f'(z+\omega) = f'(z)$ for all $\omega \in \Omega$. Suppose f has poles $[P_1], \cdots, [P_k]$ with multiplicities $r_1, \cdots, r_k$, $\sum r_i = n$, then f' also has poles $[P_1], \cdots, [P_k]$ with multiplicities $r_1 + 1, \cdots, r_k + 1$, $\sum r_i = n + k = m$, since $1 \leq k \leq n$, $n + 1 \leq m \leq 2n$
We can find an elliptic function f of order n which has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $1, \cdots, 1, 2n+1-m$, then we get f' is another elliptic function which also has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $2, \cdots, 2, 2n+2-m$, thus f' is of order m □

Exercise 46.2.17. Prove that

$$\wp'(z) = \frac{2\,\sigma(z - \frac{\omega_1}{2})\,\sigma(z - \frac{\omega_2}{2})\,\sigma(z - \frac{\omega_3}{2})}{\sigma(\frac{\omega_1}{2})\,\sigma(\frac{\omega_2}{2})\,\sigma(\frac{\omega_3}{2})\,\sigma(z)^3}.$$

Solution. $\wp'(z)$ has a pole at $z = 0$ of order 3 and $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ as simple roots, thus

$$\wp'(z) = \lambda \frac{\sigma\left(z - \frac{\omega_1}{2}\right)\sigma\left(z - \frac{\omega_2}{2}\right)\sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma(z)^3}$$

for some $\lambda \in \mathbb{C}$, multiply by z^3 on both sides, and let $z \to 0$, since $\lim_{z\to 0} \frac{z}{\sigma(z)} = 1$, $\lim_{z\to 0} z^3\wp'(z) = -2$, we have

$$-2 = -\lambda\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right) \Rightarrow \lambda = \frac{2}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)}$$

Hence

$$\wp'(z) = \frac{2\sigma\left(z - \frac{\omega_1}{2}\right)\sigma\left(z - \frac{\omega_2}{2}\right)\sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)\sigma(z)^3}$$

□

Let $\Omega \subseteq \mathbb{C}$ be a lattice and $\wp(z)$ the associated Weierstrass $\wp$-function. We have seen that $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = p(\wp(z))$, where $p(x) = 4x^3 - g_2x - g_3$. The following three problems examine the conditions under which the coefficients g_2 and g_3 of $p(x)$ are real numbers

Exercise 46.2.18. Prove that the following conditions are equivalent

(i) $g_2, g_3 \in \mathbb{R}$

(ii) $G_k \in \mathbb{R}$ for all $k \geq 3$

(iii) $\wp(\bar{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$

(iv) $\overline{\Omega} = \Omega$ (the last condition says that Ω is a *real lattice*)

Solution. $(i) \Rightarrow (ii)$
$g_2 = 60G_4, g_3 = 140G_6 \in \mathbb{R} \Rightarrow G_4, G_6 \in \mathbb{R}$
Since

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + 7G_8z^6 + 9G_{10}z^8 + \cdots\end{aligned}$$

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^3} + \sum_{n=2}^{\infty}(2n-1)(2n-2)G_{2n}z^{2n-3} \\ &= -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + 42G_8z^5 + 72G_{10}z^7 + \cdots\end{aligned}$$

$$\begin{aligned}\wp''(z) &= \frac{6}{z^4} + \sum_{n=2}^{\infty}(2n-1)(2n-2)(2n-3)G_{2n}z^{2n-4} \\ &= \frac{6}{z^4} + 6G_4 + 60G_6z^2 + 210G_8z^4 + 504G_{10}z^6 + \cdots\end{aligned}$$

So we can conclude $\wp''(z) - 6\wp(z)^2 + 30G_4 = z\varphi(z)$, where $\varphi(z)$ is a holomorphic elliptic function, hence $\wp''(z) - 6\wp(z)^2 + 30G_4 = 0$, then the coefficients of $z^{2n}(n \geq 1)$ would be $(2n+1)(2n+2)(2n+3)(2n+4)G_{2n+4} - 6(2n+3)G_{2n+4}$ minus terms only involving $G_4, G_6, \cdots, G_{2n+2}$ and real numbers, thus by induction, we know $G_{2n+4} \in \mathbb{R}(n \geq 1)$
$(ii) \Rightarrow (iii)$
Since $\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$, if $G_k \in \mathbb{R}(k \geq 3)$, then $\wp(\bar{z}) = \overline{\wp(z)}$

$(iii) \Rightarrow (iv)$
The poles of $\overline{\wp(\bar{z})} = \wp(z)$ are exactly $\overline{\Omega}$, thus $\overline{\Omega} = \Omega$
$(iv) \Rightarrow (i)$
$g_2 = 60G_4 = 60 \sum\limits_{\omega \in \Omega^*} \dfrac{1}{\omega^4} = 60 \sum\limits_{\omega \in \overline{\Omega}^*} \dfrac{1}{\omega^4} = \overline{g_2} \Rightarrow g_2 \in \mathbb{R}$, similarly, $g_6 \in \mathbb{R}$ ☐

Exercise 46.2.19. We say that Ω is *real rectangular* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, and that Ω is *real rhombic* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_2 = \overline{\omega}_1$. Prove that a lattice Ω is real if and only if it is real rectangular or real rhombic

Solution. If Ω is real rectangular or real rhombic, Ω is obviously a real lattice
Conversely, if Ω is a real lattice, suppose $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then there exists $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, otherwise, $\omega_1 \in \mathbb{R}^*, \omega_2 \in i\mathbb{R}^*$ or $\omega_2 \in \mathbb{R}^*, \omega_1 \in i\mathbb{R}^*$, since ω_1, ω_2 are linear independent, but then $\omega = \omega_1 + \omega_2 \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$ which is a contradiction
Since $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \overline{\omega} \in \mathbb{R}^*, \omega - \overline{\omega} \in i\mathbb{R}^*$, thus $\Omega \cap \mathbb{R}^* \neq \varnothing, \Omega \cap i\mathbb{R}^* \neq \varnothing$, let $\eta_1 = \min\limits_{\eta \in \Omega \cap (0,\infty)} \eta$, then $\Omega \cap \mathbb{R} = \mathbb{Z}\eta_1$, otherwise $\exists \eta \in \mathbb{R} \setminus \mathbb{Z}\eta_1$, then $\eta - \left\lfloor \frac{\eta}{\eta_1} \right\rfloor \eta_1 \in \Omega \cap (0, \infty)$ which is a contradiction
Similarly, $\Omega \cap i\mathbb{R} = \mathbb{Z}\eta_2$ for some $\eta_2 \in i(0, \infty)$. If $\Omega = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$, then Ω is real rectangular, if not, $\exists \gamma \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, such that $|\gamma| = \min\limits_{\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})} |\omega|$, then $\gamma + \overline{\gamma} = \eta_1$ or $-\eta_1$, otherwise $\gamma + \overline{\gamma} = k\eta_1$ for some $|k| \geq 2$
If $k = 2$, then $\gamma - \eta_1 = \eta_1 - \overline{\gamma} = -\overline{(\gamma - \eta_1)} \Rightarrow \gamma - \eta_1 \in i\mathbb{R} \Rightarrow \gamma \in \mathbb{Z}\eta_1 + \mathbb{Z}(\gamma - \eta_1) \subseteq \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$
If $k > 2$, then $\gamma - \eta_1 \notin \mathbb{R} \cup i\mathbb{R}$ and $|\gamma - \eta_1| < |\gamma|$, similarly for $k \leq -2$, these are all contradictions
Similarly, we know that $\gamma - \overline{\gamma} = \eta_2$ or $-\eta_2$
Now, for any $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \overline{\omega} = k\eta_1 = k(\gamma + \overline{\gamma})$ for some $k \neq 0$, then $\omega - k\gamma = k\overline{\gamma} - \overline{\omega} = -\overline{(\omega - k\gamma)} \Rightarrow \omega - k\gamma \in i\mathbb{R}$, if $\omega \neq k\gamma$, then $\omega - k\gamma = l\eta_2 = l(\gamma - \overline{\gamma}) \Rightarrow \omega \in \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, therefore, we have $\Omega = \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, Ω is real rhombic ☐

Exercise 46.2.20. Let Ω be a real lattice. Define the real elliptic curve $E_{\mathbb{R}}$ to be the set $\{(x, y) \in \mathbb{R}^2 \mid y^2 = p(x)\}$. Prove that $E_{\mathbb{R}}$ has one or two connected components as Ω is real rhombic or real rectangular, respectively

Solution. The number of connected components of $E_{\mathbb{R}}$ is one or two if $p(x) = 0$ has one real root and two nonreal conjugate complex roots or three distinct real roots correspondingly
Since $\dfrac{\omega_1}{2}, \dfrac{\omega_2}{2}, \dfrac{\omega_3}{2}$ are simple roots of $\wp'(z)$, the three simple roots of $p(x)$ are $\wp\left(\dfrac{\omega_1}{2}\right), \wp\left(\dfrac{\omega_2}{2}\right), \wp\left(\dfrac{\omega_3}{2}\right)$, since Ω is a real lattice, $G_k \in \mathbb{R}$ and $\wp(z) = \dfrac{1}{z^2} + \sum_{n=2}^{\infty}(2n - 1)G_{2n}z^{2n-2}$
If Ω is real rectangular, then $\wp\left(\dfrac{\omega_1}{2}\right), \wp\left(\dfrac{\omega_2}{2}\right)$ are both real, thus $E_{\mathbb{R}}$ has two connected components
If Ω is real rhombic, then $\wp\left(\dfrac{\omega_3}{2}\right)$ is real $\wp\left(\dfrac{\omega_1}{2}\right) \neq \wp\left(\dfrac{\omega_2}{2}\right)$ are nonreal conjugate, thus $E_{\mathbb{R}}$ has only one connected component ☐

Complex structures on an open annulus

Exercise 46.2.21. $A(r, R) = \{r < |z| < R\}$ is biholomorphic to $\{s < |z| < S\}$ iff $R/r = S/s$, r can be 0, R can be ∞, but not at the same time

Solution. By scaling or inversion we can assume $r = s = 1$ and $|f(z)| \to 1$ as $|z| \to 1$. Suppose $f : A(r, R) \to A(s, S)$ is a biholomorphism, then consider the Laurent series $f = \sum\limits_{k=-\infty}^{\infty} c_k z^k$, for $1 < t < R$, by Stokes theorem we have

$$A(t) = \frac{1}{2i} \int_{f(\{|z|=t\})} \bar{z} dz = \frac{1}{2i} \int_{|z|=t} \overline{f(z)} df(z) = \frac{1}{2i} \int_{|z|=t} \overline{f(z)} f'(z) dz = \pi \sum_{k \in \mathbb{Z}} k |c_k|^2 t^{2k}$$

As $t \to 1$, we have $A(t) \to \pi \Rightarrow \sum k|c_k|^2 = 1$, thus

$$A(t) - \pi t^2 = \pi t^2 \sum_{k \in \mathbb{Z}} k|c_k|^2 \left(t^{2k-2} - 1\right) \geq 0$$

Thus $A(t) \geq \pi t^2$, as $t \to R$, $A(t) \to \pi S^2 \geq \pi R^2 \Rightarrow S \geq R$. Therefore we have $S = R$ $\square$

Chapter 47

Exercises in category

X1,X2 iso and Y1,Y2 iso implies Hom(X1,Y1),Hom(X2,Y2) iso

Exercise 47.0.1. In category $\mathscr{C}$, if $X \xrightarrow{\phi_X} X'$, $Y \xrightarrow{\phi_Y} Y'$ are isomorphisms, then $Hom(X, Y)$, $Hom(X', Y')$ are in bijective correspondence

Solution. Consider $Hom(X, Y) \to Hom(X', Y')$, $f \mapsto \phi_Y f \phi_X^{-1}$ and $Hom(X', Y') \to Hom(X, Y)$, $f' \mapsto \phi_Y^{-1} f' \phi_X$ which are inverses to each other

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

□

Exercise 47.0.2. Suppose the bottom row of the following commutative diagram is exact, $gf = 0$, then there exists a such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, dashed, "\exists a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}$$

Solution. Since $0 = cgf = g'bf$ and bottom row is exact, we have

$$\begin{tikzcd}
& A \arrow[dl, dashed, "\exists a"'] \arrow[d, "bf"] \\
A' \arrow[r, two heads, "f'"'] & \ker g'
\end{tikzcd}$$

□

Exercise 47.0.3. $F, G : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ are functors, $F \xrightarrow{\eta} G$ is a natural transformation iff η is natural on each factor

Solution. We have commutative diagram

$$\begin{tikzcd}
F(A, B) \arrow[r, "F(f,1)"] \arrow[d, "\eta_{A,B}"] & F(A', B) \arrow[r, "F(1,g)"] \arrow[d, "\eta_{A',B}"] & F(A', B') \arrow[d, "\eta_{A',B'}"] \\
G(A, B) \arrow[r, "G(f,1)"] & G(A', B) \arrow[r, "G(1,g)"] & G(A', B')
\end{tikzcd}$$

□

Fully faithfull functor is injective on objects up to isomorphism

Exercise 47.0.4. A fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ is injective on objects up to isomorphism

Solution. Suppose $F(X) = F(Y) = T$, let $f : X \to Y$ be the map corresponds to 1_T in $Hom(F(X), F(Y))$, then $f : X \to Y$ is an isomorphism because we can also let $g : Y \to X$ be the map corresponds to 1_T as in $Hom(F(Y), F(X))$, then $F(g \circ f) = F(g) \circ F(f) = 1_T \circ 1_T = 1_T$, thus $g \circ f$ corresponds to 1_T in $Hom(F(X), F(X))$, but $F(1_X) = 1_{F(X)} = 1_T$, thus $g \circ f = 1_X$, similarly, $f \circ g = 1_Y$ $\square$

Exercise 47.0.5. Suppose $\mathscr{A}$ is an abelian category, show $\mathscr{A}$ is balanced. For any $A \xrightarrow{f} B$, $\ker f \xrightarrow{i} A$ is a monomorphism, $B \xrightarrow{\pi} \mathrm{coker} f$ is an epimorphism, and $\mathrm{im} f := \ker \mathrm{coker} f$, $\mathrm{coim} f := \mathrm{coker} \ker f$ are isomorphic

Solution. Suppose $A \xrightarrow{f} B$ is a bimorphism, it is the equaliser of $B \underset{0}{\overset{\pi}{\rightrightarrows}} \mathrm{coker} f$, then $\pi = 0$, $\mathrm{coker} f = 0$, but $A \xrightarrow{1_A} A$ is the kernel of $A \to 0$, hence A, B are isomorphic
$\ker f \xrightarrow{i} A$ is a monomorphism due to the following diagram

$$\begin{tikzcd}
C \arrow[d, "g=0"'] \arrow[dr, "0"] \arrow[drr, "0"] & & \\
\ker f \arrow[r, "i"] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

$B \xrightarrow{\pi} \mathrm{coker} f$ is a monomorphism due to the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[drr, "0"'] & B \arrow[r, "\pi"] \arrow[dr, "0"] & \mathrm{coker} f \arrow[d, "g=0"] \\
& & C
\end{tikzcd}$$

Now let's show coimage and image are isomorphic. $fi = 0$ induces $\mathrm{coker} i \xrightarrow{g} B$, claim that g is monic
Suppose $X \xrightarrow{x} \mathrm{coker} i$ is morphism such that $gx = 0$, it induces $\mathrm{coker} x \xrightarrow{j} B$, since $qpk = 0$, $fk = jqpk = 0$ induces $\ker qp \xrightarrow{l} \ker f$, since qp is epi, $pk = pil = 0$ induces $\mathrm{coker} x \xrightarrow{r} \mathrm{coker} i$, since p is epi, $p = rqp \Rightarrow rq = 1_{\mathrm{coker} i}$, hence q is monic, $qx = 0 \Rightarrow x = 0$

$$\begin{tikzcd}
& \ker qp \arrow[dl, "l"'] \arrow[d, tail, "k"] & & \\
\ker f \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[d, two heads, "p"] & B \arrow[r, two heads, "\pi"] & \mathrm{coker} f \\
X \arrow[r, "x"] & \mathrm{coker} i \arrow[ur, "g"] \arrow[r, two heads, shift left, "q"] & \mathrm{coker} x \arrow[u, "j"'] \arrow[l, dashed, shift left, "r"] &
\end{tikzcd}$$

$\pi f = 0$ induces $A \xrightarrow{h} \ker \pi$, claim that h is epi
Suppose $\ker \pi \xrightarrow{y} Y$ is morphism such that $yh = 0$, it induces $A \xrightarrow{p} \ker y$, since $qjk = 0$, $qf = qjkp = 0$ induces $\mathrm{coker} f \xrightarrow{m} \mathrm{coker} jk$, since jk is monic, $qj = m\pi j = 0$ induces $\ker \pi \xrightarrow{s} \ker y$, since j is monic, $j = jks \Rightarrow ks = 1_{\ker \pi}$, hence k is epi, $yk = 0 \Rightarrow y = 0$

$$\begin{tikzcd}
& & \mathrm{coker} jk & \\
\ker f \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr, "h"] & B \arrow[r, two heads, "\pi"] \arrow[u, two heads, "q"] & \mathrm{coker} f \arrow[ul, "m"'] \\
& \ker y \arrow[r, tail, shift left, "k"] & \ker \pi \arrow[u, tail, "j"'] \arrow[l, dashed, shift left, "s"] \arrow[r, "y"] & Y
\end{tikzcd}$$

Since $\mathrm{im} f \to B$ is monic, $A \to \mathrm{coim} f$ is epi, g, h both induce $\mathrm{coim} f \xrightarrow{\phi} \mathrm{im} f$, then ϕ is monic and epi hence iso

$$\begin{tikzcd}
\ker f \arrow[r, tail] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads] & \mathrm{coker} f \\
& \mathrm{coim} f \arrow[r, dashed, "\phi"] & \mathrm{im} f \arrow[u, tail] &
\end{tikzcd}$$

□

bounded double complex with exact rows or exact columns has exact total complex

Exercise 47.0.6. C is a bounded double complex with exact rows or exact columns, then $Tot(C)$ is exact

Solution. Without loss of generality, we may assume C is bounded in the first quadrant and has exact rows, use d', d'', d to denote row, column and total differentials

$Tot(C)$ is exact for all $n < 0$ since $Tot(C)_n = 0$ for all $n < 0$. now suppose $n \geq 0$, $d\left(\sum_{k=0}^{n} x_{k,n-k}\right) = 0$, i.e. $d'x_{k+1,n-k-1} + d''x_{k,n-k} = 0$ for $0 \leq k < n$. Let $x_{0,n+1} = 0$, we can construct $x_{k,n+1-k}$ for $k > 0$ inductively such that $d''x_{k,n-k+1} + d'x_{k+1,n-k} = x_{k,n-k}$ for $0 \leq k \leq n$ as follow:

For $k \geq -1$

$$\begin{aligned}
d'(x_{k+1,n-k-1} - d''x_{k+1,n-k}) &= d'x_{k+1,n-k-1} - d'd''x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''d'x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''(d''x_{k,n-k+1} + d'x_{k+1,n-k}) \\
&= d'x_{k+1,n-k-1} + d''x_{k,n-k} \\
&= 0
\end{aligned}$$

By exactness of rows, there exists $x_{k+2,n-k-1}$ such that

$$d'x_{k+2,n-k-1} = x_{k+1,n-k-1} - d''x_{k+1,n-k} \Leftrightarrow d''x_{k+1,n-k} + d'x_{k+2,n-k-1} = x_{k+1,n-k-1}$$

Therefore

$$\begin{aligned}
d\left(\sum_{k=0}^{n+1} x_{k,n+1-k}\right) &= \sum_{k=1}^{n+1}(d'x_{k,n+1-k} + d''x_{k,n+1-k}) \\
&= \sum_{k=1}^{n+1}(x_{k-1,n-k+1} - d''x_{k-1,n-k+2} + d''x_{k,n+1-k}) \\
&= \sum_{k=0}^{n}(x_{k,n-k} - d''x_{k,n-k+1}) + \sum_{k=1}^{n+1} d''x_{k,n+1-k} \\
&= \sum_{k=0}^{n} x_{k,n-k}
\end{aligned}$$

□

C,D acyclic => C tensor D acyclic

Exercise 47.0.7. C, D are chain complexes with negative degree terms zeros, $H_n(C) = H_n(D) = 0$ for $n \neq 0$, then so is $C \otimes D$

Solution. Apply Exercise 47.0.6 □

Exercise 47.0.8. f is a retract of g in the arrow category, if g is an isomorphism, so is f

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\
X' \arrow[r, "i'"] & Y' \arrow[r, "r'"] & X'
\end{tikzcd}$$

Proof. $i'g^{-1}r$ is the inverse to f □

Chapter 48

Exercises in partial differential equations

Exercise 48.0.1. Consider the heat equation with Neumann's boundary condition:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma \times \mathbb{R}^+ \\ u(x,0) = v(x), & \text{in } \Omega \end{cases}$$

(a) Show that $\overline{u(t)} = \overline{v}$ for $t \geq 0$, where $\overline{v} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega v(x)dx$ denotes the average of v

(b) Show that $\|u(t) - \overline{v}\| \to 0$ as $t \to \infty$

Solution. **(a)** By divergence theorem, we have

$$0 = \int_\Omega u_t - \Delta u = \int_\Omega u_t + \nabla 1 \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega u_t = \left(\int_\Omega u\right)_t$$

Hence $\displaystyle\int_\Omega u = \int_\Omega v \Rightarrow \overline{u} = \overline{v}$

(b) By divergence theorem, we have

$$0 = \int_\Omega uu_t - u\Delta u = \frac{1}{2}\left(\int_\Omega u^2\right)_t + \int_\Omega |\nabla u|^2 \Rightarrow \frac{1}{2}\left(\int_\Omega u^2\right)_t = -\int_\Omega |\nabla u|^2 \leq 0$$

Hence $\displaystyle\int_\Omega u^2 \leq \int_\Omega v^2$. On the other hand, we have

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u)^2 \\ &= \int_\Omega u_t^2 - 2u_t\Delta u + (\Delta u)^2 \\ &= \int_\Omega u_t^2 - 2\nabla u_t \cdot \nabla u + (\Delta u)^2 \\ &= \int_\Omega 2(\Delta u)^2 + \left(\int_\Omega |\nabla u|^2\right)_t \end{aligned}$$

Which implies $\displaystyle\int_\Omega (\Delta u)^2 = -\frac{1}{2}\left(\int_\Omega |\nabla u|^2\right)_t$, thus

$$\left(\int_\Omega |\nabla u|^2\right)^2 = \left(\int_\Omega u\Delta u\right)^2 \leq \int_\Omega u^2 \cdot \int_\Omega (\Delta u)^2 \leq \int_\Omega v^2 \cdot \int_\Omega (\Delta u)^2 = -\frac{1}{2}\int_\Omega v^2 \cdot \left(\int_\Omega |\nabla u|^2\right)_t$$

Denote $\phi := \int_\Omega |\nabla u|^2$ which is a function of t, $C := \frac{1}{2}\int_\Omega v^2$, then the above equation becomes

$$\phi^2 \leq -C\phi' \Rightarrow 0 \geq \phi^2 + C\phi' \Rightarrow 0 \geq 1 + C\frac{\phi'}{\phi^2} = \left(t - \frac{C}{\phi}\right)'$$

Which implies

$$t - \frac{C}{\phi(t)} \leq -\frac{C}{\phi(0)} \Rightarrow \frac{C}{\phi(t)} \geq t + \frac{C}{\phi(0)} \geq t \Rightarrow \phi(t) \leq \frac{C}{t}$$

Thus $\phi(t) \to 0$ as $t \to \infty$

Now apply Poincaré's lemma, we get

$$\|u - \overline{v}\|_{L^2} = \|u - \overline{u}\|_{L^2} \leq C\|\nabla u\|_{L^2} \to 0, t \to \infty$$

$\square$

Exercise 48.0.2.

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)dS\right) &= \frac{d}{dr}\left(\frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+rz)dS\right) \\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} \frac{d}{dr}u(x+rz)dS \\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} z \cdot \nabla u(x+rz)dS \\
&= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \nu \cdot \nabla u(y)dS \\
&= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta u(y)dy \\
&= \frac{r}{n}\frac{1}{|B(x,r)|}\int_{B(x,r)} \Delta u(y)dy
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dr}\left(\int_{B(x,r)} u(y)dy\right) &= \frac{d}{dr}\int_0^r \left(\int_{\partial B(x,s)} u(y)dS\right)ds \\
&= \int_{\partial B(x,r)} u(y)dS
\end{aligned}$$

Exercise 48.0.3. $\square u = 0$ in $\mathbb{R}^{3+1}$, $u(x,0) = 0$, $u_t(x,0) = f(x) \in C^2(\mathbb{R}^3)$, show that $\int_0^\infty |u(0,t)|^2 dt \leq C\|f\|_{L^2(\mathbb{R}^3)}$

Proof. Hint: $u(x,t) = \frac{t}{4\pi}\int_{S^2} f(x+tw)dS_w = -\frac{1}{4\pi}\int_{S^2}\int_t^\infty \frac{d}{d\lambda}f(x+\lambda t)d\lambda$

$u(0,t) = \frac{t}{4\pi}\int_{S^2} f(tw)dS_w$, $|u(x,t)| \leq \frac{C}{t}\int_{\mathbb{R}^3} |\nabla f|dx$ $\square$

Chapter 49

Exercises in algebraic topology

Exercise 49.0.1. M is a locally Euclidean, Hausdorff and connected manifold, then paracompactness imples second countable

Proof. An open cover by precompact coordinate charts has a locally finite open refinement $\{U_i\}$, each U_i is precompact and second countable
Define $S_0 = \{U_0\}$ for some U_0, since $\{U_i\}$ is locally finite, define S_1 to be the union of S_0 and those intersects U_0, repeating this process, we get $S_2, \cdots, S_n, \cdots$, define $S = \bigcup_{n=0}^{\infty} S_n$
M is connected thus path connected, pick any $x_0 \in U_0$, for any $x \in M$, there is a path γ connecting x_0 and x, since γ is compact, it can be covered by S. Hence S is an open cover of M, thus M is second countable $\square$

Exercise 49.0.2. If G is a discrete group, P is connected, $P \xrightarrow{p} X$ is a principal G bundle iff it is a regular cover with $\mathtt{Aut}(p) = G$

Solution. $P \xrightarrow{p} X$ is a fiber bundle thus a cover, G acts regularly on fibers and $G \leq \mathtt{Aut}(p)$ $\square$

Exercise 49.0.3. Use Theorem 13.0.18 to prove homotopy invariance of maps on homology

Solution. Suppose $F : X \times I \to Y$ is a homotopy between f and g, we only need to prove i_0, i_1 are naturally chain homotopic since $Fi_0 = f, Fi_1 = g$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & C_n(X) \arrow[r] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & C_{n-1}(X) \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \\
C_{n+1}(X \times I) \arrow[r] \arrow[d, "F"] & C_n(X \times I) \arrow[r] \arrow[d, "F"] & C_{n-1}(X \times I) \arrow[d, "F"] \\
C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y)
\end{tikzcd}$$

Consider Top with model $\mathcal{M} = \{\Delta^n\}$, $F, G : Top \to Ch_{\geq 0}$, $F(X) = C_*(X)$, $G(X) = C_*(X \times I)$, $H_i(\Delta^n \times I) = 0$ for $i \neq 0$, $F_k(X) = \left\{\Delta^k \xrightarrow{\text{id}} \Delta^k \xrightarrow{\sigma} X\right\}$, there is an obvious natural equivalence $\phi_0 : H_0F \to H_0G$, then lifts i_0, i_1 are naturally chain homotopic $\square$

Exercise 49.0.4. K is a CW complex, $X \xrightarrow{f} Y$ is a weak equivalence, then $[K, X] \to [K, Y]$ is a bijection

Exercise 49.0.5. Quotient map $X \xrightarrow{q} Y$ is a homeomorphim iff q is bijective

Solution. If q is bijective, then for any open subset $U \subseteq X$, $U = q^{-1}(q(U))$, by definition, $q(U)$ is open, i.e. q^{-1} is continuous $\square$

Cofibration in a Hausdorff space is closed

Exercise 49.0.6. If X is Hausdorff, then cofibration $A \xrightarrow{i} X$ is closed. This is not true if X is not Hausdorff as showed in Example 41.0.12

Solution. Suppose $A \xrightarrow{i} X$ is a not closed, $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I$ is the retraction, pick any $x \in \overline{A} \setminus A$ with x_n converging to x, then $A \times \{1\} \ni r(x,1) = r(\lim x_n, 1) = \lim r(x_n, 1) = \lim(x_n, 1) = (x, 1)$ which is a contradiction ☐

Exercise 49.0.7. $\mathbb{R} \times \mathbb{R} \xrightarrow{\wedge} \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \xrightarrow{\vee} \mathbb{R}$ are continuous

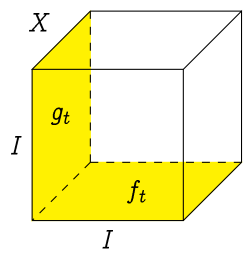
Solution. $x \wedge y = \dfrac{x + y - |x - y|}{2}$, $x \vee y = \dfrac{x + y + |x - y|}{2}$ ☐

Exercise 49.0.8. $Y^I \to Y$, $\gamma \mapsto \gamma(0)$ and $Y^I \to Y \times Y$, $\gamma \mapsto (\gamma(0), \gamma(1))$ are Hurewicz fibrations

Solution. Need $g(x,s) = H(x,0,s)$, $f(x,t) = H(x,t,0)$ so that $g(x,0) = f(x,0)$

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, hook] & Y^I \arrow[d] \\
X \times I \arrow[r, "f_t"'] \arrow[ur, dashed, "H"] & Y
\end{tikzcd}$$

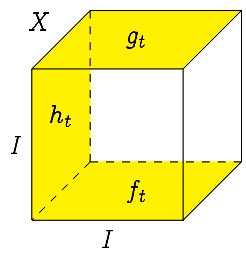
$X \times I^2$ can be deformed onto $X \times I \cup X \times I = X \times (I \cup I)$



Need $h(x,s) = H(x,0,s)$, $f(x,t) = H(x,t,0)$, $g(x,t) = H(x,t,1)$ so that $h(x,0) = f(x,0)$, $h(x,1) = g(x,0)$

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, hook] & Y^I \arrow[d] \\
X \times I \arrow[r, "(f_t, g_t)"'] \arrow[ur, dashed, "H"] & Y \times Y
\end{tikzcd}$$

$X \times I^2$ can be deformed onto $X \times I \cup X \times I \cup X \times I = X \times (I \cup I \cup I)$



☐

Chapter 50

Exercises in differential topology

Hom(V,W)=V^* tensor W

Exercise 50.0.1. $Hom(V,W) \to V^* \otimes W$, $A \mapsto \sum_{i,j} a_{ji} v_i^* \otimes w_j$ is an isomorphism where $A = (a_{ij})$ is the matrix with respect to basis $\{v_1^*, \cdots, v_m^*\}$, $\{w_1, \cdots, w_n\}$

Solution. $A(v_i) = \sum_j a_{ji} w_j$ □

Exercise 50.0.2. Suppose M, N are smooth manifolds of dimension m, n, $f : M \to N$ is a smooth map, $(x^1, \cdots, x^m)$, $(y^1, \cdots, y^n)$ are local coordinates around $p \in M$, $q = f(p) \in N$, then the corresponding matrix of df with respect to basis $\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\right)$, $\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right)$ is $\left(\frac{\partial y^i}{\partial x^j}\right)$. In particular, this gives the change of coordinates formula

Solution.

$$df\left(\frac{\partial}{\partial x^i}\right)(g) = \frac{\partial (g \circ f)}{\partial x^i} = \sum_j \frac{\partial g}{\partial y^j}\frac{\partial y^j}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}(g)$$

According to Exercise 50.0.1, $df = \sum_{i,j} \frac{\partial y^j}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^j}$, we can define higher differential $d^k f = \sum_{i_1, \cdots, i_k, j} \frac{\partial y^j}{\partial x^{i_1} \cdots \partial x^{i_k}} dx^{i_1} \cdots dx^{i_k} \otimes \frac{\partial}{\partial y^j}$ □

Exterior derivative of one form

Exercise 50.0.3. Suppose $\omega \in \Omega^1(M)$, $X, Y \in TM$, then $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$

Solution. By linearity, we can assume $\omega = udv$, then

$$\begin{aligned} d\omega(X,Y) &= d(udv)(X,Y) \\ &= du \wedge dv(X,Y) \\ &= du(X)dv(Y) - du(Y)dv(X) \\ &= XuYv - YuXv \end{aligned}$$

And

$$\begin{aligned} & X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ =\ & X(udv(Y)) - Y(udv(X)) - udv([X,Y]) \\ =\ & Xudv(Y) + uX(dv(Y)) - Yudv(X) - uY(dv(X)) - u[X,Y]v \\ =\ & XuYv + uXYv - YuXv - uYXv - uXYv + uYXv \\ =\ & XuYv - YuXv \end{aligned}$$

□

Pushforward of vector field

Exercise 50.0.4. Suppose $\phi : M \to N$ is a map of smooth manifolds, then $X(f \circ \phi) = ((\phi_* X) f) \circ \phi$

Solution. $X(f \circ \phi)(p) = X_p(f \circ \phi) = \phi_p X_p(f) = (\phi_* X)_{\phi(p)}(f) = ((\phi_* X)f)(\phi(p))$ □

Naturality of Lie bracket

Exercise 50.0.5. Suppose X, Y are vector fields on M, $\phi : M \to N$ is a smooth map, then $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$

Solution. Apply Exercise 50.0.4

$$\begin{aligned}
\phi_*[X,Y](f) &= [X,Y](f \circ \phi) \\
&= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
&= X(((\phi_* Y)f) \circ \phi) - Y(((\phi_* X)f) \circ \phi) \\
&= ((\phi_* X)(\phi_* Y)f) \circ \phi - ((\phi_* Y)(\phi_* X)f) \circ \phi \\
&= ([\phi_* X, \phi_* Y]f) \circ \phi \\
&= [\phi_* X, \phi_* Y]f
\end{aligned}$$

□

Chapter 51

Exercises in bundles

Exercise 51.0.1. $E \xrightarrow{p} B$ is a Serre fibration, $A \overset{i}{\hookrightarrow} X$ is a subcomplex, if either p or i is a weak equivalence, then we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & B \end{tikzcd}$$

Solution. If p is a weak equivalence, then fibers are weak contractible
If i is a weak equivalence, then X deformation retracts onto A $\square$

Chapter 52

Exercises in complex geometry

Chapter 53

Exercises in Lie groups and Lie algebras

Exercise 53.0.1. Suppose $\mathfrak{g}$ is a real semisimple Lie algebra with a negative definite Killing form, then $\mathfrak{g}$ is the Lie algebra of some compact Lie group G

A is the direct sum of ideals => A is the product of these ideals

Exercise 53.0.2. Suppose A is a nonassociative algebra, $A = I_1 \oplus \cdots \oplus I_n$ is a direct sum of ideals, then $I_i I_j \subseteq I_i \cap I_j = 0$, hence $A = I_1 \times \cdots \times I_n$ can be viewed as product of ideals

Remark 53.0.3. If $A = I_1 \oplus \cdots \oplus I_n$ is just a direct sum of subalgebras, Exercise 53.0.2 may not hold true

Pairwise commuting matrices can be diagonalized simultaneously

Exercise 53.0.4. Let $S \subseteq M(n, \mathbb{F}), (\overline{\mathbb{F}} = \mathbb{F})$ be a set such that $[X, Y] = 0, \forall X, Y \in S$, then elements in S can be diagonalized simultaneously

Solution. Suppose $0 \neq V_\lambda$ is the λ-eigenspace of $X \in S$, then for any $Y \in S$, $XYv = YXv = \lambda Yv$ for $v \in V_\lambda$, thus $YV_\lambda \subseteq V_\lambda$, thus V_λ is an invariant subspace for all $Y \in \mathfrak{g}$, since Y are all semisimple, by induction we can write V as a direct sum of V_λ's, and they are all invariant under any $Y \in \mathfrak{g}$, we only need to show that all elements of $\mathfrak{g}$ can be diagonalized simultaneously on each V_λ, if $X|_{V_\lambda} = 1_{V_\lambda}$, then we are done, otherwise we can decompose it into smaller eigenspaces ☐

Finite dimensional toral Lie algebra is abelian and its elements can be diagonalized simultaneously

Exercise 53.0.5. Let V be a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, and $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a toral Lie algebra, then $\mathfrak{g}$ is abelian. Moreover, $X \in \mathfrak{g}$ can be diagonalized simultaneously

Solution. Suppose $ad(X)|_{\mathfrak{g}} \neq 0$ for some $X \in \mathfrak{g}$, since $\overline{\mathbb{F}} = \mathbb{F}$, there exists $0 \neq Y \in \mathfrak{g}$ and $\lambda \neq 0$ such that $ad(X)(Y) = \lambda Y$, by Proposition 9.6.3, $ad(Y)$ is semisimple, suppose λ_j, X_j are the eigenvalues and linearly independent eigenvectors, then we have $X = \sum c_j X_j$ with $c_j \neq 0$, and $0 = ad(Y)(\lambda Y) = ad(Y)ad(X)(Y) = -ad(Y)^2(X) = -ad(Y)^2\left(\sum c_j X_j\right) = -\sum c_j \lambda_j^2 X_j$, thus $c_j \lambda_j^2 = 0 \Rightarrow \lambda_j = 0$, but $0 \neq \lambda Y = ad(X)(Y) = -ad(Y)(X) = -ad(Y)\left(\sum c_j X_j\right) = -\sum c_j \lambda_j X_j = 0$ which is a contradiction

Now that we know $[\mathfrak{g}, \mathfrak{g}] = 0$, use Lemma 53.0.4, we know all elements of $\mathfrak{g}$ are diagonalizable simultaneously ☐

Lie group homomorphism has constant rank

Exercise 53.0.6. Lie group homomorphism has constant rank

Solution. Let $\phi : G \to G'$ be a Lie group homomorphism, for any $g \in G$, it suffices to show $\mathtt{rank}(d\phi)_g = \mathtt{rank}(d\phi)_1$, since $\phi(gh) = \phi(g)\phi(h)$, thus $\phi \circ L_g = L_{\phi(g)} \circ \phi$, $(d\phi)_g(dL_g)_1 = d(L_{\phi(g)})_1(d\phi)_1$, and left multiplications are isomorphisms, we have $\mathtt{rank}(d\phi)_g = \mathtt{rank}(d\phi)_1$ ☐

Exercise 53.0.7. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M, for any equivariant map $f : M \to N$, f has constant rank

Solution. For any $x \in M$, denote $y = f(x)$, it suffices to show $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ since G acts transitively on M, note that $f(gx) = gf(x)$, thus $f \circ L_g = L_g \circ f$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ □

Exercise 53.0.8. If $\phi : G \to H$ be a bijective Lie group homomorphism, then it is an isomorphism

Solution. Apply Exercise 53.0.6 and Theorem 22.0.15 □

Exercise 53.0.9. Compact semisimple Lie group G has finite center

Solution. Since $\mathfrak{g} = \mathtt{Lie}(G)$ is semisimple, $\mathtt{Lie}(Z(G)) \leq Z(\mathfrak{g}) = 0$, thus $Z(G)$ is discrete, but G is compact, so $Z(G)$ is finite □

rudimentary facts about topological groups

Exercise 53.0.10. G is a topological group, A is called **symmetric** if $A = A^{-1}$

1. Topology of G is translation invariant, U is open $\Rightarrow$ xU, Ux are open

2. $e \in U$ is a neighborhood, then $e \in V \subseteq U$ a symmetric neighborhood

3. $e \in U$ is a neighborhood, then $e \in V \subseteq VV \subseteq U$ with V being a symmetric neighborhood

4. $H \leq G$ is a subgroup, then so is $\bar{H}$

5. Open subgroups of G are also closed(closed groups are not necessarily open, consider $\{e\}$)

6. $K_1, K_2 \subseteq G$ are compact sets, so is K_1K_2

7. Suppose G is a connected, U is a neighborhood of 1, then $G = \langle U \rangle$

Solution.

1. Multiplication by x is an isomorphism with x^{-1} being its inverse

2. Take $U \cap U^{-1}$

3. Since the multiplication $G \times G \to G$ is continuous, consider the preimage of U which contains $V_1 \times V_2$, take $V \subseteq V_1 \cap V_2$ symmetric

4. If $x_\alpha \to x$, $y_\beta \to y$, then $x_\alpha^{-1} \to x^{-1}$, $x_\alpha y_\beta \to xy$, since these maps are continuous. From this we know that $\bar{H} = \bigcap F$ wehre F runs over all closed subgroup containing H

5. Suppose $H \leq G$ is open, then $H = G \setminus \bigcup\limits_{x \neq e} xH$ is closed, thus if G is connected, then $H = G$

6. K_1K_2 is the image of $K_1 \times K_2$ under multiplication

7. By b, we there is a symmetric neighborhood $1 \in V \subseteq U$, let V_k be the subset of elements can be written in the product of no more the k elements in V, then $V_1 = V$, $V_k = V_1V_{k-1}$ is open by induction, $\langle V \rangle = \bigcup\limits_{k=1}^{\infty} V_k$ is also open, by e, G is generated by V hence by U, and if G is not connected, $G_0 = \langle V \rangle$ is called the identity component of G

□

Exercise 53.0.11. G is a topological group, if G is T_1, then G is Hausdorff, if G is not T_1, then $H := \overline{\{e\}}$ is normal subgroup, G/H is a Hausdorff topological group

Solution. If G is T_1, according to Exercise 53.0.10, for $x \neq y$, $\exists e \in VV \subseteq U$ with V a symmetric neighborhood of e disjoint from $y^{-1}x$, then $xV \cap yV = \emptyset$, suppose $z = xv_1 = yv_2$, then $y^{-1}x = v_2^{-1}v_1 \in VV$ thus reaches a contradiction
According to Proposiontion 53.0.10, $H = \bigcap H_i$, H_i runs over closed subgroups of G, thus H is the smallest closed subgroup, if H is normal, otherwise $xHx^{-1} \cap H$ is a smaller closed subgroup for some x
In G/H, identity is closed, by invariance of topology under translation, every point is closed, meaning G/H is T_1 thus Hausdorff
Checking G/H is still a topological group: $g \in \bigcup_x xH$ open in G, then $g^{-1} \in (\bigcup_x xH)^{-1} = \bigcup_x H^{-1}x^{-1} = \bigcup_x Hx^{-1} = \bigcup_x x^{-1}H$
If $V \times W \to VW \subseteq \bigcup_x xH$, then $vw \in \bigcup_x xH, \forall v \in V, w \in W$, then $\forall h \in H$, $vhw = vww^{-1}hw \in \bigcup_x xH$, therefore, $VH \times WH \to VHWH \subseteq \bigcup_x xH$, notice that VH is open as long as V is open since $VH = \bigcup_{h \in H} Vh$ □

Exercise 53.0.12. $(,)_B$ is the bilinear form given by matrix B, $O(B) = \{X \in GL_n(\mathbb{C}) | X^T BX = 1\}$, the Lie algebra is $\mathfrak{o}(B) = \{X \in M_n(\mathbb{C}) | X^T B + BX = 0\}$

Solution. $\left.\frac{d}{dX}\right|_{X=0} (e^{X^T} Be^X) = X^T B + BX = 0$ □

Exercise 53.0.13. $T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^\times \right\} \subseteq SL(2, \mathbb{C}) = G$ is the torus, the Weyl group $W(T) = N_G(T)/Z(T) = N/T \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

Solution. Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,$ad - bc = 1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} adx - bcx^{-1} & ab(x^{-1} - x) \\ cd(x - x^{-1}) & adx^{-1} - bcx \end{pmatrix} \in T$$

For any $x \in \mathbb{C}^\times$, which implies that $ab = cd = 0 \Rightarrow a = d = 0$ or $b = c = 0$ and

$$\begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} & -b^{-1} \\ b & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

□

Chapter 54

Exercises in algebraic geometry

Exercise 54.0.1. $H = V(f)$ is a hypersurface, $f(t_1, \cdots, t_n, x) = a_0(t_1, \cdots, t_n)x^m + \cdots + a_m(t_1, \cdots, t_n)$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{A}^{n+1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

φ is finite iff $a_0 \neq 0$ is a constant. φ is quasifinite $\Rightarrow$ $a_0, \cdots, a_m$ don't have common zeros

Chapter 55

Exercise in functional analysis

Part XVII

Miscellaneous

Chapter 56

Hodge structure

Theorem 56.0.1 (Stokes theorem). $\langle \partial\Omega, \omega\rangle = \langle \Omega, d\omega\rangle$, here $\langle \Omega, \omega\rangle = \int_\Omega \omega$

Theorem 56.0.2 (de Rham's theorem). M is a smooth manifold. $H^p_{\mathrm{dR}}(X;\mathbb{R}) \xrightarrow{\cong} H^p(X;\mathbb{R})$ is an isomorphism

Proof. Since $\mathbb{R}$ is a divisible abelian group, thus an injective $\mathbb{Z}$ module, hence $\mathtt{Ext}^1_{\mathbb{Z}}(A,\mathbb{R})$, thus universal coefficient theorem gives exact sequence

$$0 = \mathtt{Ext}^1_{\mathbb{Z}}(H_{p-1}(X;\mathbb{Z}),\mathbb{R}) \to H^p(M;\mathbb{R}) \to \mathtt{Hom}(H_p(X;\mathbb{Z}),\mathbb{R}) \to 0$$

The isomorphism is given by $H^p_{\mathrm{dR}}(X;\mathbb{R}) \to \mathtt{Hom}(H_p(X),\mathbb{R})$, $\omega \mapsto \int_{-} \omega$ $\square$

Chapter 57

Plucker embedding

Definition 57.0.1. Consider the Grassmannian $\mathtt{Span}\{v_1, \cdots, v_k\} \in \mathrm{Gr}_k(n)$

$$\begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{bmatrix}$$

Denote $W_{i_1,\cdots,i_k}$ as the minor of $i_1, \cdots, i_k$-th columns
Suppose $i_1 < \cdots < i_{k-1}$, $j_1 < \cdots < j_{k+1}$, the **Plücker relations** are

$$\sum_{l=0}^{k+1} W_{i_1,\cdots,i_{k-1},j_l} W_{j_1,\cdots,\widehat{j_l},\cdots,j_{k+1}}$$

Example 57.0.2. Consider $\mathrm{Gr}_2(4)$, the only Plücker relation is

$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0$$

Chapter 58

Graph theory

Definition 58.0.1. A **graph** is

Chapter 59

Moduli space

Consider a parametrized curve $C = \{(t, \mathbf{x}(t))\}_{t\in I}$, $\mathbf{x}(t) \in \mathbb{R}^n$, now we change I to some space X, $\mathbf{x}(t)$ to some algebro-geometric objects, then we have a parametrization of these objects by X

Definition 59.0.1. U is a family of some algebro-geometric objects. A parametrization of U by space X is a map $X \to U$, attaching some object U_x for each $x \in X$, we can also think of this map as a section of $X \times U \to X$
We say X is the parametrization space, U is parametrized over X
A moduli functor F is a contravariant functor $\mathit{Space} \to \mathit{Set}$ that takes a space X to the set of families of objects over X, and take a morphism f to the pullback f^* that taking section s to pullback section $f^*s(y) = (y, \mathrm{Pr}_U\, sf(y))$

$$\begin{tikzcd} Y \times U \arrow[r] \arrow[d] & X \times U \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

The category of spaces can be the category of schemes, manifolds, topological spaces, etc.
M is a fine moduli space if F is corepresentable by M, i.e. there is a natural isomorphism $\tau : F \to \mathtt{Hom}(-, M)$. There is a universal family over M corresponds to $1_M \in \mathtt{Hom}(M, M)$. Then any family over X is the pullback along some $X \xrightarrow{f} M$ of the universal family. The universal family is essentially unique and "tautological"
M is a coarse moduli space if there is exists a natural transformation $\tau : F \to \mathtt{Hom}(-, M)$ and universal among these natural transformations, i.e. for any natural transformation $\tau' : F \to \mathtt{Hom}(-, M')$, there is a morphism $M' \xrightarrow{\phi} M$ such that the following diagram commutes

$$\begin{tikzcd} & F \arrow[dl, "\tau'"'] \arrow[d, "\tau"] \\ \mathtt{Hom}(-, M') \arrow[r, "\exists_1 \phi"] & \mathtt{Hom}(-, M) \end{tikzcd}$$

Chapter 60

Teichmüller space

Definition 60.0.1. S is a orientable surface. A marked Riemann surface is a pair (X, f) where X is a Riemann surface and $S \xrightarrow{f} X$ is a isomorphism, i.e. giving X a complex structure. (X, f), (Y, g) are equivalent if $gf^{-1} : X \to Y$ is isotopic to an isomorphism, i.e. X, Y has isotopic complex structures. The Teichmüller space $T(S)$ of S is the the equivalence classes of marked Riemann surfaces. The mapping class group acts on $T(S)$ by $h \cdot (X, f) = (X, fh^{-1})$, then $T(S)$ mod the action is just S

Example 60.0.2. By Uniformization theorem 24.0.9, $T(\mathbb{S}^2)$ is a point corresponds to the Riemann sphere, $T(\mathbb{R}^2)$ is two points corresponds the complex plane and the unit disc. $T(A) = [0, 1)$, where A is the open annulus, and $\lambda \in [0, 1)$ corresponds to $\{\lambda < |z| < \lambda^{-1}\}$ according to Exercise 46.2.21

Chapter 61

Mapping class group

Definition 61.0.1. Suppose $\mathtt{Aut}(X)$ has a natural topology, the mapping class group is $\mathtt{Aut}(X)/\mathtt{Aut}_0(X)$, where $\mathtt{Aut}_0(X)$ is the path connected component of the identity, hence we have exact sequence

$$0 \to \mathtt{Aut}_0(X) \to \mathtt{Aut}(X) \to \mathrm{MCG}(X) \to 0$$

If X is a space, then a path connecting $f, g \in \mathtt{Aut}(X)$ is an isotopy

Example 61.0.2. $\mathrm{MCG}(S^2) = \mathbb{Z}/2\mathbb{Z}$

Bibliography

[1] *An Introduction to Algebraic Topology* - Joseph J. Rotman

[2] *Algebraic Topology* - Allen Hatcher

Index

C^*-algebra, 223
G set, 33
G space, 33
G-vector bundle, 163
R algebra, 41
$\mathcal{G}$ topology, 219
$\mathcal{O}$ modules, 29
$\mathfrak{g}$ module, 54
*-algebra, 223
étalé space, 28
étale map, 28
2 out of 3, 20

Abelian category, 105
Absolute neighborhood retract, 122
Absolute retract, 122
abstract simplicial complex, 132
Acyclic cofibration, 147
Acyclic fibration, 147
Additive category, 105
Additive funtor, 105
adjoint endomorphism, 54
Adjoint functors, 24
Affine algebraic group, 193
Affine connection, 176
Affine scheme, 205
Affine space, 49
Alexandorff extension, 120
Algebraic group, 193
Algebraic number field, 48
Almost complex structure, 185
Ambient isotopy, 157
Analytic manifold, 171
Arrow category, 27
Associate $(1, 1)$ form, 185
Associated graded complex, 111
Associated graded module, 40
Associated graded ring, 40
Augmented simplex category, 128

Baire space, 221
Balanced category, 17
Banach algebra, 223
Banach space, 219
Base change, 25
Base of a topology, 117
Basis of a module, 43
Bimorphism, 17
Biproduct, 23
Bloch-Wigner function, 94
Boundary maps, 106
Box topology, 118
Branch point, 183
Bundle, 159

Canonical bundle, 163
Cartan integers, 68
Cartan's criterion for semisimplicity, 60
Cartan's criterion for solvability, 60
Category of chain complexes, 106
Categroy, 17
Cauchy-Riemann equations, 211
Cell complex, 125
Cell decomposition, 125
Cellular map, 125
Center, 33
Chain complex, 105
Chart, 171
Chinese remainder theorem, 40
Closed set, 117
Cobase change, 25
Cocomplete, 22
Cocone, 22
Coequaliser, 22
Cofibered pair, 150
Cofibrant, 148
Cofibration sequence, 154
Cohomology theory, 141
Colimit, 22
Comb space, 252
Comma category, 27
Compact, 118
Compact-open topology, 120

Compactly generated, 119
Complete, 22
Complete bipartite, 13
Complete graph, 13
Completely reducible representation, 34
Completely separating function, 11
Complex manifold, 171
Composition factors, 109
composition length, 109
Composition series, 109
Concave function, 209
Concrete category, 19
Cone, 22
Connected, 118
Connected category, 20
Constant morphism, 19
Constant presheaf, 28
Constructible set, 121
Continuous, 117
Continuous functor, 168
Contractible, 122
Contravariant functor, 17
Convex function, 209
Convex set, 49
coroot, 68
Coslice category, 27
Cotangent space, 171
Coxeter diagram, 36
Coxeter element, 36
Coxeter group, 36
Coxeter matrix, 36
Coxeter number, 36
Cross product, 137, 138
Cross section, 159
Crystallographic root system, 68
Cup product, 142
Cylinder object, 148

Dedekind domain, 40
Deformation, 122
Deformation retraction, 122
Degeneracy map, 128
Dense, 118
Derivation, 53
Derived couple, 111
Derived series, 54
determinant, 40
Diagram, 22
Dilogarithm, 93
Direct limit, 22
Direct sum of chain complexes, 106
direct sum of vector bundles, 162
Direct system, 22
Directed set, 22
Discrete map, 119
Discrete topology, 119
Discrete valuation ring(DVR), 41
Division ring, 47
Dominant morphism, 203
Double cochain complex, 111
Double complex, 106
Dual category, 17

Einstein curvature, 177
Empty category, 19
Energy-momentum tensor, 238
Epimorphism, 17
Equaliser, 22
Equicontinuity, 220
Equivalent category, 18
Equivariant map, 33
Essentially surjective, 18
Exact couple, 111
Exponential object, 30

Face map, 128
Factorization system, 31
Faithful action, 33
Faithful functor, 18
Fiber bundle, 159
Fiber homotopy, 152
Fiber homotopy equivalence, 152
Fibrant, 148
Fibration sequence, 154
Field, 47
Field norm, 47
Field trace, 47
Filtered category, 26
Filtered chain complex, 106, 111
Filtered colimit, 26
Filtered object, 20
Filtration, 20
Final object, 23
Finite ring homomorphism, 41
Finite type ring homomorphism, 41
First countable, 117
Flag, 49
Fractional ideal, 40
Free action, 33
Free category, 20
Full functor, 18
Functor, 17
Functor category, 18
Fundamental groupoid, 145

Galilean group, 241

Generalized Cartan matrix, 72
Generalized eigenspace, 52
generalized eigenvector, 52
Geometric realization, 130
Germ, 28
Good pair of topological spaces, 122
Graded ring, 40
Graph, 127, 301
Grothedieck completion, 37
Grothendieck group, 37
Group, 33
Group action, 33
Group character, 47
Group representation, 34
Groupoid, 33

Half space model, 179
Harmonic, 209
Hasse diagram, 11
Hausdorff space, 118
Hawaiian earring, 251
Heat equation, 232
Hemisphere model, 179
Hermitian manifold, 185
Hermitian metric, 185
Holomorphic, 211
Homogeneous ideal, 40
Homogeneous space, 33
Homogeneous vector bundle, 163
Homology theory, 135
Homotopy, 118
Homotopy category of chain complexes, 106
Homotopy equivalence, 118
Homotopy extension property, 150
Homotopy lifting property, 119
Hurewicz cobfibration, 150
Hurewicz fibration, 150
Hurwitz's theorem, 213
Hyperboloid model, 178

Image of a morphism, 20
Inclusion-exclusion principle, 11
Indexed category, 22
Induced representation, 34
Inert, 48
Initial object, 23
Injective object, 19
Inner automorphism, 33
Inner derivations, 54
Integral domain, 40
Integral element, 41
Invariant bilinear form on Lie algebra, 57
Invariant subspace, 52
Inverse limit, 23
Inverse system, 22
Involution, 20
Irreducible topological spaces, 120
Isomorphic category, 18
Isomorphism, 17
Isotopy, 157
Isotropy representation, 195
Iterated integral, 93

Jacobi identity, 53
Join, 121
Jordan curve theorem, 121
Jordan-Chevalley decomposition, 52

Kähler manifold, 185
Killing form, 57
Klein disc model, 180
Kolmogorov classification of topological spaces, 118
Kronecker product, 51
Krull dimension, 41

Lattice, 11
Lefschetz fixed point theorem, 121
Lefschetz number, 121
Left R module, 43
Left homotopy, 148
Left lifting property, 20
Lie algebra, 54
Lie algebra of Lie group, 195
Lie algebra representation, 54
Lie bracket of vector fields, 172
Lie group, 195
Lift extension property, 119
Limit, 22
Limit point, 117
Linear algebraic group, 193
Linearly independent, 43
Local base, 117
Local ring, 40
Localization of a category, 20
Locally closed set, 121
Locally compact, 118
Locally connected, 118
Locally constant sheaf, 28
Locally finite, 118
Locally ringed space, 28
Locally small category, 18
Loop space, 153
Lorentz boost, 241
Lorentz factor, 241
Lorentz group, 241

Lower central series, 54

Möbius transformation, 216
manifold, 171
Minkowski space, 178
Model category, 147
Monoid, 33
Monoidal category, 32
Monomorphism, 17
Montel's theorem, 212
Morphism, 17
Multiple polylogarithm, 94
Multiplicatively closed, 44

Natural isomorphism, 18
Natural transformation, 17
Neighborhood, 117
Nerve, 168
Nilpotent Lie algebra, 54
Noetherian topological space, 120
Nonassociative $\mathbb{F}$ algebra, 53
Nonassociative algebra, 53
Normal family, 120
Normal topological space, 117
Normed group, 219
Normed rng, 219
Normed vector space, 219

Object, 17
One point compactification, 120
Open set, 117
Outer automorphism, 33

Paracompact, 118
Path object, 148
path space, 153
Perfect field, 47
Perfect group, 33
perfectly separated, 117
Perfectly separating function, 11
Permutation, 35
Plücker relations, 299
Poincaré disc model, 180
Pointed category, 23
Polydisc, 211
Polylogarithm, 93
preabelian category, 105
Preadditive category, 105
Precompact, 118
Presheaf, 28
Prevariety, 203
Prime element, 40
Principal G-bundle, 163
Product category, 18
Projective object, 19
Proper map, 119
pseudomanifold, 134
Pullback, 25, 171
Pullback bundle, 159
pullback bundle, 161
Pullback of sections, 159
Pushforward presheaf, 28
Pushforward(differential), 171
Pushout, 25

Quadratic Lie algebra, 57
Quasi-isomorphism, 106
quasifinite morphism, 203
Quiver, 20
quotient category, 19
Quotient object, 19

Radical ideal of Lie algebra, 54
Ramified, 48
Real form, 55
Refinement, 118
Regular function, 202
Regular topological space, 117
Residue field, 40
Restricted presheaf, 28
Restriction map, 28
Retraction, 27, 122
Ricci curvature, 177
Riemann mapping theorem, 214
Riemann surface, 183
Riemannian manifold, 176
Riemannian metric, 176
Right homotopy, 148
Right lifting property, 20
Ring, 39
Ringed space, 28
Root, 62
Root space, 62
Root system, 68
Rouche's theorem, 213

Scalar curvature, 177
Scheme, 206
Schur's lemma, 44
Schwarz lemma, 212
Second Bianchi identity, 176
Second countable, 117
Section, 27
semicategory, 17
Semicontinuity, 209
Semidirect product, 33

Semifunctor, 17
Semigroup, 33
Semilocal ring, 40
Semisimple linear operator, 52
Semisimple simplex category, 128
Separable space, 118
Separating function, 11
Serre fibration, 156
Sheaf, 28
Sheaf cohomology, 28
Sheafification, 28
Shuffle, 35
Simplex category, 128
Simplicial identities, 128
Simplicial map, 129
Simplicial object, 128
Skeleton, 125
Skeleton of a category, 20
Slice category, 27
Small category, 18
Smooth manifold, 171
Smooth map, 171
Snake lemma, 106
Solvable Lie algebra, 54
span, 43
Spectral sequence, 111
Stably isomorphic, 167
Stalk, 28
Strictly convex set, 49
Strong deformation retraction, 122
Structure sheaf, 28
Subbase, 117
Subharmonic, 209
Submanifold, 171
Subobject, 19
Subspace topology, 117
Superharmonic, 209
Sylow's theorem, 34
Symplectic structure, 187

Tangent space, 171
Tautological bundle, 253
Tautological one form, 172
tautological representation of Lie algebra, 55
tensor product of vector bundles, 162
Test function space, 222
Topological category, 168
Topological groups, 191
Topological space, 117
Topological vector space, 219
topologist's sine curve, 251
Topology, 117
Topology of pointwise convergence, 120
Topology of uniform convergence, 219
Toral Lie algebra, 54
Torsion tensor, 176
Torsor, 33
Total chain complex, 106
Total cochain complex, 112
Totally split, 48
Tractrix, 259
Transition map, 171
Transitive action, 33
Translation of a chain complex, 107
Tree, 127
Trivial action, 33
Trivial topology, 119
tropical semifield, 87

Uniformization theorem, 183
Unit, 39
Univalent, 211
Urysohn space, 118

Vector bundle, 159
Vector space, 49
vector subbundle, 162

Warsaw circle, 251
Wave equation, 234
Weak equivalence, 20
Weak factorization system, 31
Weak homotopy equivalence, 146
Weak retraction, 122
Weierstrass σ-function, 217
Weierstrass $\wp$-function, 217
Weierstrass polynomial, 215
Weyl chamber, 69
Weyl group, 69
Weyl's theorem, 55
Whitehead group, 39

Yoneda lemma, 21

Zero morphism, 19
Zero object, 23