Chapter 1

Examples in categories

Example 1.0.1. The image of a functor is not necessarily a category
Consider the following categories $\mathscr{C}$ and $\mathscr{D}$

$$\begin{tikzcd}
A \arrow["1_A"', loop above] \arrow[r, "f"] & B \arrow["1_B"', loop above] & C \arrow["1_C"', loop above] \arrow[r, "g"] & D \arrow["1_D"', loop above]
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow["1_E"', loop above] \arrow[r, "h"] \arrow[rr, bend right, "ih"'] & F \arrow["1_F"', loop above] \arrow[r, "i"] & G \arrow["1_G"', loop above]
\end{tikzcd}$$

Consider functor $F : \mathscr{C} \to \mathscr{D}$, $F(A) = E$, $F(B) = F$, $F(C) = F$, $F(D) = G$, $F(f) = h$, $F(g) = i$

Chapter 2

Examples in algebra

Example 2.0.1. Suppose $1 \mapsto k$ is an element in $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, then $m \mid kn \Rightarrow \frac{m}{(n,m)}$ divides $k\frac{n}{(n,m)}$, thus $\frac{m}{(n,m)}$ divides k, thus $k = \frac{im}{(n,m)}, i = 0, \cdots, (n,m)-1$, thus $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n,m)\mathbb{Z}$

Consider $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$, then $n(1 \otimes 1) = n \otimes 1 = 0$, $m(1 \otimes 1) = 1 \otimes m = 0$, thus $(n,m)(1 \otimes 1) = (rn + sm)(1 \otimes 1) = 0$

Apply functor $Hom(-, \mathbb{Z}/m\mathbb{Z})$ to short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, we get a left exact sequence $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \to 0$

Apply functor $- \otimes \mathbb{Z}/m\mathbb{Z}$ to short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times m} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$, we get a left exact sequence $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} \to 0$

And the kernel and cokernel of $\mathbb{Z}/m\mathbb{Z} \xrightarrow{\times n} \mathbb{Z}/m\mathbb{Z}$ are both $\mathbb{Z}/(n,m)\mathbb{Z}$

Example 2.0.2. $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}\left[\frac{1}{p}\right]$

Example 2.0.3. $O(1,1) = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}$

R^2=R

Example 2.0.4. F is a field, $R = \mathrm{End}(F^\infty) = \{$infinite dimensional matrices$\}$, Consider $R \hookrightarrow R$ by embedding into odd rows and even rows, we have $R^2 \cong R$ as right R modules

Example 2.0.5. $GL(2, \mathbb{F}_2) = SL(2, \mathbb{F}_2) \cong S_3$

Chapter 3

Examples in algebraic topology

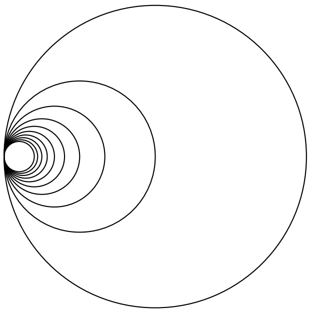
Example 3.0.1 (A surjective local homeomorphism may not be a covering)**.** $p : \mathbb{R} \setminus \{0\} \to S^1$, or n sheeted cover with a point missing, p is discrete but not proper

Example 3.0.2 (Bundle with fiber isomorphic to vector space but not a vector bundle)**.** $E := \bigsqcup_{x \in X} \mathbb{R}^n$

Example 3.0.3. $H'_n = H_{k+n}$ also defines a homology theory where the dimension axiom fails

Hawaiian earring

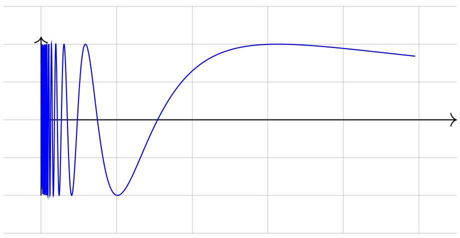
Example 3.0.4 (Hawaiian earring)**.** The **Hawaiian earring** H is the union of circles with radius $\frac{1}{n}$ and centered at $(\frac{1}{n}, 0)$ with subspace topology in $\mathbb{R}^2$



Proposition 3.0.5. Hawaiian earring is not a CW complex since it is not locally contractible

Example 3.0.6 (Topologist's sine curve)**.** The **topologist's sine curve** is

$$T = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \middle| x \in (0,1] \right\} \cup \{(0,0)\}$$



Proposition 3.0.7. The topologist's sine curve T is connected but not path connected

Example 3.0.8 (Warsaw circle)**.** The **Warsaw circle** W is the topologist's sine curve enclosed Bijective map $W \to [0,1)$ is not a homeomorphism, thus not a quotient map. W is weakly homotopic to a point but not homotopic

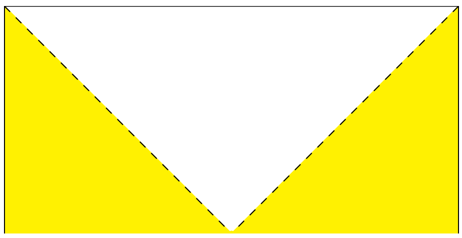
Example 3.0.9. $X = \mathbb{N}$ with discrete topology, $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ with subspace topology of $\mathbb{R}$, then $f : X \to Y, n \mapsto \frac{1}{n}$ is a weak homotopy equivalence, however, X, Y are not homotopy equivalent, otherwise suppose $g : X \to Y, h : Y \to X$ such that $hg \simeq 1_X$, $gh \simeq 1_Y$, suppose $F : Y \times I \to Y$ is a homotopy, then the restriction of F on $\{y\} \times I$ must be a constant map since the connected components of Y are just points, thus $F(y, 0) = F(y, 1)$, i.e. homotopic maps are in fact the same, for a similar argument on X, we have $hg = 1_X$, $gh = 1_Y$, thus h is injective which is impossible since $h^{-1}(h(0))$ consists of more then one point

Cofibration counterexample

Example 3.0.10. $D^2 = S^2 \setminus \{N\} \subseteq S^2$ is not a cofibration. $D^2 \setminus \{0\} \subseteq D^2$ is not a cofibration

Mapping cylinder of inclusion may have different topology then induced subspace topology

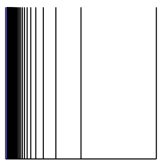
Example 3.0.11. $A = [-1, 0) \cup (0, 1]$, $X = [-1, 1]$, then the mapping cylinder of the inclusion $A \overset{i}{\hookrightarrow} X$ has different topology from the subspace topology $X \times \{0\} \cup A \times I$ induced from $X \times I$



Nonclosed cofibration

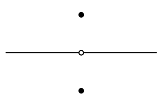
Example 3.0.12. $\{a, b\}$ with trivial topology, $\{a\} \subseteq \{a, b\}$ is a nonclosed cofibration since there is a retraction $I \sqcup I \to I \sqcup \{0\}$, $(s, t) \mapsto (s, 0)$

Example 3.0.13. The **comb space** is $[(0, 0), (1, 0)] \bigcup \bigcup_{n=1}^{\infty} [(\frac{1}{n}, 0), (\frac{1}{n}, 1)]$



A line with two origins

Example 3.0.14. A topological space with a cell decomposition may not be Hausdorff, consider $(-1, 1)$ with two origins, which has $(-1, 0), (0, 1)$ as 1 cells and two origins as 0 cells



Chapter 4

Examples in geometry

Definition 4.0.1 ($\mathcal{O}(n)$ bundle over Riemann sphere $S^2 \cong \mathbb{C}P^1$)**.** Suppose $(\mathbb{C}, z \mapsto z)$, $\left(S^2 \setminus 0, z \mapsto \frac{1}{z}\right)$ are the charts coordinate of S^2 with transition map $z \mapsto \frac{1}{z}$ both ways on $\mathbb{C} \setminus 0$, or equivalently $(U_0, [1, z] \mapsto z)$, $(U_1, [z, 1] \mapsto z)$ are the corresponding charts of $\mathbb{C}P^1$ with transition map $z \mapsto \frac{1}{z}$ both ways on $U_0 \cap U_1$, namely, $S^2 \to \mathbb{C}P^1$, $z \mapsto [1, z]$, $\infty \mapsto [0, 1]$ is and isomorphism because of isomorphisms on charts
Now define $\mathcal{O}(n)$ line bundle on S^2 by specify transition functions $g_{10}(z) = z^{-n}$, $g_{01}(z) = z^n$, $\forall z \in \mathbb{C} \setminus 0 \cong U_0 \cap U_1$

Definition 4.0.2 (Tautological line bundle over Riemann sphere)**.** The tautological bundle is $\mathcal{O}(-1)$, tautological bundle is defined as a subspace E of $\mathbb{C}P^1 \times \mathbb{C}^2$ consists of (l, v) with $v \in l$ projects to the first factor, let's figure out the trivializations!
$\varphi_0 : U_0 \times \mathbb{C}^2 \cap E \to \mathbb{C} \times \mathbb{C}$, $([1, z], t(1, z)) \mapsto (z, t)$, and $\varphi_1 : U_1 \times \mathbb{C}^2 \cap E \to \mathbb{C} \times \mathbb{C}$, $([z, 1], t(z, 1)) \mapsto (z, t)$, since $\varphi_1 \circ \varphi_0^{-1} : (U_0 \cap U_1) \times \mathbb{C}^2 \cap E \to (U_0 \cap U_1) \times \mathbb{C}^2 \cap E$, $(z, t) \mapsto \left(\frac{1}{z}, zt\right)$, the transition function $g_{10}(z) = z$

Remark 4.0.3. $\mathcal{O}(-1)$ doesn't nonzero global section, suppose s is a global section of $\mathcal{O}(-1)$, then $s(x) = (x, f(x)) \in E \hookrightarrow \mathbb{C}P^1 \times \mathbb{C}^2$ is holomorphic, but then image of σ has to be a point, and this point must be zero

Example 4.0.4. We still use U_0, U_1 to denote coordinate charts, φ_0, φ_1 to denote corresponding trivializations
Global sections of $\mathcal{O} = \mathcal{O}(0)$ are exactly holomorphic functions which are just constants, suppose $s : S^2 \to \mathcal{O}$ is a section, and $\varphi_0 \circ s|_{U_0}(z) = (z, f_0(z))$, $\varphi_1 \circ s|_{U_1}\left(\frac{1}{z}\right) = \left(\frac{1}{z}, f_1\left(\frac{1}{z}\right)\right)$, then we have $(z, f_1(z)) = \varphi_1 \circ s|_{U_1}(z) = \varphi_1 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1} \circ \varphi_0 \circ s|_{U_0}(z) = \varphi_1 \circ \varphi_0^{-1}(z, f_0(z)) = (z, g_{10}(z) f_0(z)), \forall z \in U_0 \cap U_1$, thus $f_1(z) = g_{10}(z) f_0(z) = f_0(z)$ which precisely means s correspond to holomorphic function f over X, $f|_{U_0} = f_0, f|_{U_1} = f_1$
Let's show that the canonical bundle(which in the case of a Riemann surface is the same as the cotangent bundle) is $\mathcal{O}(-2)$, since $d\left(\frac{1}{z}\right) = -\frac{1}{z^2} dz$, the transition function would be $g_{10}(z) = -z^2$, but using dz or $-dz$ as the a basis element would be isomorphic

Proposition 4.0.5. *$H^0\left(CP^1, \mathcal{O}(n)\right)$, the vector space of global sections of $\mathcal{O}(n) \to \mathbb{C}P^1, n \geq 0$ generated by homogeneous polynomials $z_0^n, z_0^{n-1} z_1, \cdots, z_0 z_1^{n-1}, z_1^n$*

Proof. $z_0^k z_1^{n-k}$ have the forms z_1^{n-k} and z_0^k in U_0 and U_1 □

Example 4.0.6 (Line bundles on the projective space $\mathbb{C}P^n$)**.** Suppose $(U_0, [1, z_1, \cdots, z_n] \mapsto (z_1, \cdots, z_n))$, $(U_n, [z_0, z_1, \cdots, z_{n-1}, 1] \mapsto (z_0, \cdots, z_{n-1}))$ be coordinate charts of $\mathbb{C}P^n$, with transition map

$U_i \cap U_j \to U_i \cap U_j$, $\left(\frac{z_0}{z_i}, \cdots, \widehat{\frac{z_i}{z_i}}, \cdots, \frac{z_n}{z_i}\right) \mapsto \left(\frac{z_0}{z_j}, \cdots, \widehat{\frac{z_j}{z_j}}, \cdots, \frac{z_n}{z_j}\right)$, which is kind of like multiply by $\frac{z_i}{z_j}$, then the line bundle $\mathcal{O}(m)$ is defined by transition function $g_{ji} = \frac{z_j}{z_i}$ which satisfies the cocycle condition

Similarly, we can check that the tautological bundle $E = \{(l, v) | v \in l\} \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$ projects to $\mathbb{C}P^n$ is $\mathcal{O}(1)$

It is obvious that any degree n polynomial are global section of $\mathcal{O}(n)$

Chapter 5

Examples in Lie groups and Lie algebras

Example 5.0.1. X is topological space, $End(X)$ is a unital nonassociative $\mathbb{R}$ algebra which is not symmetric, antisymmetric, nor does it satisfy Jacobi identity

Example 5.0.2. Consider $C^\infty(M)$ where M is a smooth manifold, then $\mathcal{L}(M) = Der(C^\infty(M))$ consists of vector fields, it is a Lie algebra, hence we can think of derivations as linear differential operator of order 1, then we know that the commutator of two such operators is again a linear differential operator of order 1

Example 5.0.3. Let $\mathfrak{g}$ be a Lie algebra, then ideals of $\mathfrak{g}$ precisely the Lie algebra subrepresentations of the adjoint representation $(ad, \mathfrak{g})$

Example 5.0.4 (Lie algebra of $M_n(\mathbb{R})$)**.** Suppose $X = \sum_{i,j} X_{ij} \dfrac{\partial}{\partial x_{ij}}$ is a left invariant

$$\begin{aligned}
X_{kl}(A) &= \sum_{i,j} X_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\
&= X_A(x_{kl}) = (L_A)_0 X_0(x_{kl}) \\
&= X_0(x_{kl} \circ L_A) \\
&= \sum_{i,j} X_{ij}(0) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(0) \\
&= X_{kl}(0)
\end{aligned}$$

Thus X_{ij} are constants

$$\begin{aligned}
[X, Y] &= \left[\sum_{i,j} X_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} Y_{kl} \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j,k,l} X_{ij} Y_{kl} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j} X_{ij} Y_{ij} \left[\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial x_{kl}} \right] \\
&= 0
\end{aligned}$$

Therefore $\mathsf{Lie}(M_n(\mathbb{R})) = 0$

Example 5.0.5 (Lie algebra of $GL(n, \mathbb{R})$)**.** Suppose $X = \sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}$ is a left invariant field

$$\begin{aligned}
c_{kl}(A) &= \sum_{i,j} c_{ij}(A) \frac{\partial x_{kl}}{\partial x_{ij}}(A) \\
&= X_A(x_{kl}) = (L_A)_I X_I(x_{kl}) \\
&= X_I(x_{kl} \circ L_A) \\
&= \sum_{i,j} c_{ij}(I) \frac{\partial (x_{kl} \circ L_A)}{\partial x_{ij}}(I) \\
&= \sum_i a_{ki} c_{il}(I)
\end{aligned}$$

Hence $C(A) = AC(I)$, $\frac{\partial c_{kl}}{\partial x_{ij}} = \delta_{ki} c_{jl}(I)$

$$\begin{aligned}
[X, Y] &= \left[\sum_{i,j} c_{ij} \frac{\partial}{\partial x_{ij}}, \sum_{k,l} d_{kl} \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j,k,l} \left[c_{ij} \frac{\partial}{\partial x_{ij}}, d_{kl} \frac{\partial}{\partial x_{kl}} \right] \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial}{\partial x_{ij}} \left(d_{kl} \frac{\partial}{\partial x_{kl}} \right) - d_{kl} \frac{\partial}{\partial x_{kl}} \left(c_{ij} \frac{\partial}{\partial x_{ij}} \right) \\
&= \sum_{i,j,k,l} c_{ij} \left(\frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} + d_{kl} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}} \right) - d_{kl} \left(\frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} + c_{ij} \frac{\partial^2}{\partial x_{ij} \partial x_{kl}} \right) \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{kl} \frac{\partial c_{ij}}{\partial x_{kl}} \frac{\partial}{\partial x_{ij}} \\
&= \sum_{i,j,k,l} c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} - \sum_{i,j,k,l} d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}} \frac{\partial}{\partial x_{kl}} \\
&= \sum_{i,j,k,l} \left(c_{ij} \frac{\partial d_{kl}}{\partial x_{ij}} - d_{ij} \frac{\partial c_{kl}}{\partial x_{ij}} \right) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{j,k,l} (c_{kj} d_{jl} - d_{kj} c_{jl}) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{k,l} \left(\sum_j c_{kj} d_{jl} - d_{kj} c_{jl} \right) \frac{\partial}{\partial x_{kl}} \\
&= \sum_{k,l} b_{kl} \frac{\partial}{\partial x_{kl}}
\end{aligned}$$

Here $B = [C, D]$. Therefore $\mathsf{Lie}(GL(n, \mathbb{R})) = \mathfrak{gl}(n, \mathbb{R})$

Example 5.0.6. Consider the $\Phi : GL(n, \mathbb{R}) \to M_n(\mathbb{R}), A \mapsto A^T A$ which is a smooth map, and level set $\Phi^{-1}(I) = O(n, \mathbb{R})$ is the orthogonal group, to show this is a Lie subgroup, thanks to Theorem **??**, it suffices to show Φ is of constant rank, but Φ is equivariant assuming $GL(n, \mathbb{R})$ acts on itself by right multiplication and acts on $M_n(\mathbb{R})$ by $X \cdot A = A^T X A, X \in M_n(\mathbb{R}), A \in GL(n, \mathbb{R})$, since $\Phi(A) \cdot B = B^T A^T A B = \Phi(AB)$
$(d\Phi)_I(B) = B^T + B$, and $T_I(O(n, \mathbb{R})) = \ker(d\Phi)_I = \{B \in M(n, \mathbb{R}) | B^T + B = 0\}$

Chapter 6

Examples in algebraic geometry

Example 6.0.1. Suppose $V \subseteq \mathbb{A}^n$ is an affine variety, $m_P \in \mathtt{Spm}k[V]$, $k[V]_{m_P}$ is the stalk of the sheaf of regular functions. Tow representatives $\frac{f}{u}, \frac{g}{v}$ are of the same germ $\Leftrightarrow$ $\frac{f}{u} = \frac{g}{v}$ on $D(wuv)$ for some $w(P) \neq 0 \Leftrightarrow w(fv - gu) = 0$

Example 6.0.2.

Chapter 7

Examples in analysis

Example 7.0.1. $D \subseteq \mathbb{C}$ is the unit disc, $f(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{n^2}$ is continuous on $\overline{D}$ and holomorphic on D but not on any point on ∂D

Example 7.0.2 (Tractrix)**.** An interval I with one end point pushed or dragged along the x axis gives a **Tractrix**. The velocity has the same direction as I, i.e. $\frac{dx}{dy} = \pm\frac{\sqrt{a^2 - y^2}}{y}$, which gives solution $x = \pm\left(\ln\frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}\right)$