MATH848I - Exterior differential systems

Haoran Li

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Contents

1 Introduction - 1/28/2020

Webpage: www.math.umd.edu/ kmelnick/eds20.html
Book recommendation:
1. T.A. Ivey and J.M. Landsberg: Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems (2nd ed.), AMS Graduate Studies in Mathematics 175, Providence, RI (2016)
2. R. Bryant, P. Griffiths, D. Grossman: Exterior Differential Systems and Euler-Lagrange Partial Differential Equations, Chicago Lectures in Mathematics, Chicago (2003)
3. R. Bryant, S.-S. Chern, R.B. Gardiner, H. Goldschmidt, P. Griffiths: Exterior Differential Systems, Springer (1990)
Note: Einstein summation convention is regularly used

Definition 1.1. We say a PDE is **overdetermined** if there are more equations than unknowns

Example 1.2. Suppose α is a 1 form on $U \subseteq \mathbb{R}^n$
Can we find f on U such that $df = \alpha$
In corrdinaties, $\alpha = a_i dx^i$, $\dfrac{\partial f}{\partial x^i} = a^i$
In general, there is no solution, a necessary condition is $d\alpha = d^2 f = 0$, i.e. $\dfrac{\partial a_i}{\partial x^j} = \dfrac{\partial a_j}{\partial x^i}$

Lemma 1.3 (Poincaré lemma)**.** *If $U \subseteq \mathbb{R}^n$ is contractible, $d\alpha = 0$ is also a sufficient condition, f is determined up to constants $c_0 = f(x_0), x_0 \in U$*

Example 1.4. Suppose $D \subseteq \mathbb{R}^2$ is the disk, $g, A : D \to 2 \times 2$ symmetric matrices with $g(x, y)$ positive definite
Can we find $\sigma : D \to \mathbb{R}^3$ such that g is the induced metric on D, and A is the second fundamental form, i.e. $g = d\sigma \cdot d\sigma$, $A = -dn \cdot d\sigma$
In coordinates, $\sigma = (\sigma^1, \sigma^2, \sigma^3)$, $g(x, y) = \begin{pmatrix} g_{11}(x, y) & g_{12}(x, y) \\ g_{21}(x, y) & g_{22}(x, y) \end{pmatrix}$, $A(x, y) = \begin{pmatrix} A_{11}(x, y) & A_{12}(x, y) \\ A_{21}(x, y) & A_{22}(x, y) \end{pmatrix}$
Write $\dfrac{\partial \sigma^i}{\partial x} = \sigma^i_1$, $\dfrac{\partial \sigma^i}{\partial y} = \sigma^i_2$, $n = \dfrac{\sigma_1 \times \sigma_2}{\|\sigma_1 \times \sigma_2\|}$, $g_{11} = (\sigma^i_1)^2$, $g_{12} = \sigma^i_1 \sigma^i_2 = g_{21}$, $g_{22} = (\sigma^i_2)^2$, $A_{11} = n^i \sigma^i_{11}$, $A_{12} = n^i \sigma^i_{12} = A_{21}$, $A_{22} = n^i \sigma^i_{22}$, there are 6 equations in total
There exists a solution iff satisfying Gauss-Codazzi equations:

$$\frac{\partial A_{11}}{\partial y} - \frac{\partial A_{12}}{\partial x} = A_{11}\Gamma^1_{12} + A_{12}(\Gamma^2_{12} - \Gamma^1_{11}) - A_{22}\Gamma^2_{11}$$

$$\frac{\partial A_{12}}{\partial y} - \frac{\partial A_{22}}{\partial x} = A_{11}\Gamma^1_{22} + A_{12}(\Gamma^2_{22} - \Gamma^1_{12}) - A_{22}\Gamma^2_{12}$$

Example 1.5. Given $\alpha = (\alpha^1(x, y, u), \alpha^2(x, y, u))$, $(x, y) \in U \subseteq \mathbb{R}^2$
Can we find $u : U \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial x} = \alpha^1(x, y, u) \frac{\partial u}{\partial y} = \alpha^2(x, y, u) \tag{1}$$

Introduce variables p, q and $J^1(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$(1-Jet), define $\theta = du - pdx - qdy$, $\Omega = dx \wedge dy$
Suppose $\Sigma \subseteq J^1(\mathbb{R}^2, \mathbb{R})$ is a surface such that $\Omega|_{T\Sigma}$ never vanishes and $\theta|_{T\Sigma}$ vanishes identically, then locally Σ is a graph($\Omega|_{T\Sigma} \neq 0$ is for nondegeneracy) $u = u(x, y)$, $p = p(x, y)$, $q = q(x, y)$ with $du = pdx + qdy$ on $T\Sigma$, but $du = u_x dx + u_y dy$ on $T\Sigma$, thus $p = u_x, q = u_y$ on Σ

Now consider $M \subseteq J^1(\mathbb{R}^2, \mathbb{R})$ be the solution to $p = \alpha^1(x, y, u), q = \alpha^2(x, y, u)$ which is a 3 manifold. Solution of (1) correspondes to surfaces $\Sigma \subseteq M$ on which $\Omega \neq 0, \theta = 0$
A necessary condition for existence of such a surface Σ is $d\theta = -dp \wedge dx - dq \wedge dy$ in $J^1(\mathbb{R}^2, \mathbb{R})$, suppose $j : M \hookrightarrow J^1(\mathbb{R}^2, \mathbb{R})$ is the inclusion, then

$$\begin{aligned} j^* d\theta &= -(\alpha^1_x dx + \alpha^1_y dy + \alpha^1_u du) \wedge dx - (\alpha^2_x dx + \alpha^2_y dy + \alpha^2_u du) \wedge dy \\ &= (\alpha^1_y - \alpha^2_x) dx \wedge dy - \alpha^1_u du \wedge dx - \alpha^2_u du \wedge dy \end{aligned}$$

On Σ
Suppose $i : \Sigma \hookrightarrow J^1(\mathbb{R}^2, \mathbb{R})$ is the inclusion, then

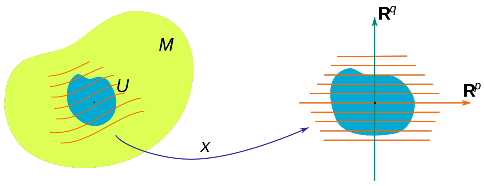
$$\begin{aligned} i^* d\theta &= (\alpha^1_y - \alpha^2_x) i^* d\Omega - \alpha^1_u (\alpha^1 dx + \alpha^2 dy) \wedge dx - \alpha^2_u (\alpha^1 dx + \alpha^2 dy) \wedge dy \\ &= (\alpha^1_y - \alpha^2_x + \alpha^1_u \alpha^2 - \alpha^2_u \alpha^1) i^* d\Omega \end{aligned}$$

Since $\Omega \neq 0$, $\alpha^1_y - \alpha^2_x + \alpha^1_u \alpha^2 - \alpha^2_u \alpha^1 = 0$ on Σ
Consider the following possible cases:
Case I: $\alpha^1_y - \alpha^2_x + \alpha^1_u \alpha^2 - \alpha^2_u \alpha^1 = 0$ on M
Case II: $\alpha^1_y - \alpha^2_x + \alpha^1_u \alpha^2 - \alpha^2_u \alpha^1 = 0$ on M
For case I, Apply Theorem 2.10, we know it is a sufficient condition since

$$\begin{aligned} d\theta &= (\alpha^1_y - \alpha^2_x + \alpha^1_u \alpha^2 - \alpha^2_u \alpha^1) dx \wedge dy + \alpha^1_u \theta \wedge dx - \alpha^2_u \theta \wedge dy \\ &= (\alpha^2_u dy - \alpha^1_u dx) \wedge \theta \end{aligned}$$

2 Frobenius theorem - 1/30/2020

Definition 2.1. A p dimension **foliation** of an n dimensional manifold M is decomposition of M into disjoint connected submanifolds $M = \bigsqcup_{\alpha\in A} N_\alpha$ such that for each point $p \in M$, there is a neighborhood of p and a local chart $(x^1, \cdots, x^n)$ such that each $N_\alpha \cap M$ is given by $x^{p+1} = \text{const}, \cdots, x^n = \text{const}$



Definition 2.2. An **integral submanifold** $N \subseteq M$ is a submanifold such that locally $TN = \text{Span}(X_1, \cdots, X_n)$ where X_i is a local basis, 1-dimensional integral submanifolds are just **integral curves**

Definition 2.3. Suppose M is a smooth manifold of dimension m, an n-dimensional **distribution** over M is

$$\Delta = \bigsqcup_p \Delta_p \subseteq TM, \Delta_p \leq T_pM, \dim \Delta_p = n$$

Which is locally spanned by a local basis $X_1, \cdots, X_n$

Remark. We can also define distributions on vector bundles

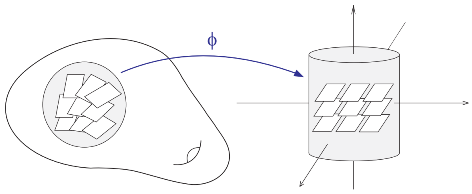
Definition 2.4. Δ is **involutive** if $[\Delta, \Delta] \subseteq \Delta$, Δ is **integrable** if for any point $p \in M$, there exists a integral submanifold $N \ni p$ such that $T_pN = \Delta_p$

Lemma 2.5. *If distribution Δ is integrable, then it is involutive*

Proof. Since Δ is integrable, for any $p \in M$, there is a integral submanifold $N \ni p$ such that $i_* : T_pN \hookrightarrow T_pM$ is injective with $i_*(T_pN) = \Delta_p$. Suppose $X, Y \in \Delta_p$, by the naturality of Lie bracket, $[X, Y] = i_*[i_*^{-1}X, i_*^{-1}Y] \in \Delta_p$ $\square$

Example 2.6. Consider $D = \langle V, W\rangle$ is a two dimensional distribution over $\mathbb{R}^3$, where $V = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$, $W = \frac{\partial}{\partial y}$, but $[X, Y] = -\frac{\partial}{\partial z} \notin D$, thus D is not involutive, by Lemma 2.5, D is not integrable

Definition 2.7. An n-dimensional distribution D over a m-dimensional smooth manifold M is **completely integrable** if for each point $p \in M$, there is a local coordinate chart (U, ϕ), such that $\phi : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ with $\phi(D) \subseteq \mathbb{R}^n$



Lemma 2.8. *Suppose M is an m dimensional manifold, D is an n-dimensional distribution around $p \in M$, (U, x) with $x(p) = 0$ is a local coordinate chart, then D has a local basis $X_1, \cdots, X_n$ around p such that such that*

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=n+1}^{m} a_i^j \frac{\partial}{\partial x^j}$$

Or in matrix form

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 & a_1^{n+1} & \cdots & a_1^m \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & a_n^{n+1} & \cdots & a_n^m \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^m} \end{pmatrix}$$

Proof. First pick a local basis $Y_1, \cdots, Y_n$ around p, then we have $Y_i = \sum_{j=1}^{m} b_i^j \frac{\partial}{\partial x^j}$, or in matrix form

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} b_1^1 & \cdots & b_1^m \\ \vdots & \ddots & \vdots \\ b_n^1 & \cdots & b_n^m \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^m} \end{pmatrix}$$

Since Y_i's are linearly independent, B is of full rank, reorder if needed, we can assume

$$\widetilde{B} = \begin{pmatrix} b_1^1 & \cdots & b_1^n \\ \vdots & \ddots & \vdots \\ b_n^1 & \cdots & b_n^n \end{pmatrix}$$

Is invertible, we can thus define $\begin{pmatrix} I & A \end{pmatrix} = \widetilde{B}^{-1} B$, $X = \widetilde{B}^{-1} Y$ □

Corollary 2.9. *Suppose M is an m dimensional manifold, D is an n-dimensional involutive distribution around $p \in M$, then D has a local basis $X_1, \cdots, X_n$ around p such that such that $[X_i, X_j] = 0$. In other words, we can choose a local commuting basis*

Proof. Suppose (U, x) with $x(p) = 0$ is a local coordinate chart, by Lemma 2.8, D has a local basis $X_1, \cdots, X_n$ around p such that such that

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=n+1}^{m} a_i^j \frac{\partial}{\partial x^j}$$

Then

$$\begin{aligned}
[X_i, X_j] &= \left[\frac{\partial}{\partial x^i} + \sum_{k=n+1}^{m} a_i^k \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} + \sum_{l=n+1}^{m} a_j^l \frac{\partial}{\partial x^l} \right] \\
&= \sum_{k=n+1}^{m} \left[a_i^k \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j} \right] + \sum_{l=n+1}^{m} \left[\frac{\partial}{\partial x^i}, a_j^l \frac{\partial}{\partial x^l} \right] + \sum_{k,l=n+1}^{m} \left[a_i^k \frac{\partial}{\partial x^k}, a_j^l \frac{\partial}{\partial x^l} \right] \\
&= \sum_{k=n+1}^{m} \frac{\partial a_i^k}{\partial x^j} \frac{\partial}{\partial x^k} + \sum_{l=n+1}^{m} \frac{\partial a_j^l}{\partial x^i} \frac{\partial}{\partial x^l} + \sum_{k,l=n+1}^{m} \left(a_i^k \frac{\partial a_j^l}{\partial x^k} \frac{\partial}{\partial x^l} - a_j^l \frac{\partial a_i^k}{\partial x^l} \frac{\partial}{\partial x^k} \right)
\end{aligned}$$

Is in the span of $\left\{ \frac{\partial}{\partial x^{n+1}}, \cdots, \frac{\partial}{\partial x^m} \right\}$, on the other hand, since D is involutive, $[X_i, X_j]$ is also in the span of $\{X_1, \cdots, X_n\}$, thus $[X_i, X_j] = 0$ □

Theorem 2.10 (Frobenius theorem)**.** *If distribution D is involutive, then it is completely integrable, alternatively, we could say that maximal integrable submanifolds form a foliation of M*

Remark. Frobenius theorem can be thought of as a generalization of the existence theorem in ODE

Proof. It suffices to show that for any $p \in M$, there is a local coordinate chart $x : U \to \mathbb{R}^m$ such that locally D is spanned by $\left\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right\}$, then integrable submanifolds are just $\{x^1, \cdots, x^{n-1}$ are constants$\}$, we prove this by induction on n

Base case: If $n = 1$, D is just a nonvanishing vector field X_m, for each $p \in M$, let $X_1, \cdots, X_m$ be a local basis for TM, define $\gamma_i : (-\varepsilon, \varepsilon)^i \to M$ by $\gamma_i(x^1, \cdots, x^i) = \phi_{X_i}^{x^i} \circ \cdots \circ \phi_{X_1}^{x^1}(p)$, where ϕ_X^t is the flow along X. Then $\gamma_m(0, \cdots, x^i, \cdots, 0) = \phi_{X_i}^{x^i}(p)$, $(\gamma_m)_* \left.\frac{\partial}{\partial x^i}\right|_{(0,\cdots,0)} = X_i(p)$ which are linearly independent, thus γ_m is invertible around origin, giving $x = \gamma_m^{-1}$ with $(\gamma_m)_* \left.\frac{\partial}{\partial x^m}\right|_{(x^1,\cdots,x^m)} = X_m(\gamma_m(x^1, \cdots, x^m))$, i.e. $\frac{\partial}{\partial x^m} = X_m$

Induction step: By Corollary 2.9, there exists local basis $X_1, \cdots, X_n$ for D such that $[X_i, X_j] = 0$, by induction hypothesis, there is a local chart y such that $\mathrm{Span}\,(X_1, \cdots, X_{n-1}) = \mathrm{Span}\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{n-1}}\right)$, write $X_n = \sum_{i=1}^m a^i \frac{\partial}{\partial y^i}$, then

$$\left[\frac{\partial}{\partial y^i}, X_n\right] = \sum_{j=1}^m \left[\frac{\partial}{\partial y^i}, a^j \frac{\partial}{\partial y^j}\right] = \sum_{j=1}^m \frac{\partial a^j}{\partial y^i} \frac{\partial}{\partial y^j}$$

Since D is involutive, $\left[\frac{\partial}{\partial y^i}, X_n\right] \in D$, which implies $\frac{\partial a^j}{\partial y^i} = 0, \forall n+1 \le j \le m$, let $Y := X_n - \sum_{i=1}^{n-1} a^i \frac{\partial}{\partial y^i} = \sum_{i=n}^m a^i \frac{\partial}{\partial y^i}$, then $\mathrm{Span}\,(X_1, \cdots, X_{n-1}, X_n) = \mathrm{Span}\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{n-1}}, Y\right)$

Now we restrict on the integral submanifold $N = \{y^1, \cdots, y^{n-1}$ are constants$\}$, $(y^n, \cdots, y^m)$ is a local coordinate chart on N, thus $Y \in TN$ is a nonvanishing distribution, this is again the base case, there exists coordinates $(x^n, \cdots, x^m)$ such that $\frac{\partial}{\partial x^n} = Y$, let $x^i = y^i, i < n$, then $x = (x_1, \cdots, x_m)$ becomes a local coordinate chart such that $\mathrm{Span}\,(X_1, \cdots, X_{n-1}, X_n) = \mathrm{Span}\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{n-1}}, Y\right) = \mathrm{Span}\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}, \frac{\partial}{\partial x^n}\right)$ □

Definition 2.11. A **differential ring** is a ring R with one or more derivations, a **derivation** d is a ring endomorphism satisfying Leibniz rule: $d(rs) = (dr)s + r(ds)$
A **differential ideal** is an ideal I closed under d, i.e. $dI \subseteq I$

Example 2.12. Given differential forms $v^1, \cdots, v^p \in \Omega^*(M)$, we can define the algebraic ideal

$$\langle v^1, \cdots, v^p \rangle_{\mathrm{alg}} = \left\{\sum_{i=1}^p v^i \wedge \alpha_i, \alpha_i \in \Omega^*(M)\right\}$$

Which is closed under wedge product $\wedge$, and the differential ideal

$$\langle v^1, \cdots, v^p \rangle_{\mathrm{diff}} = \left\{\sum_{i=1}^p (v^i \wedge \alpha_i + dv^i \wedge \beta_i), \alpha_i, \beta_i \in \Omega^*(M)\right\}$$

Which is closed under wedge product $\wedge$ and differential d

Lemma 2.13. *Suppose V is an m dimensional vector space, $v^1, \cdots, v^n \in V^*$ are linearly independent iff $v^1 \wedge \cdots \wedge v^n \neq 0$*
Suppose $v^1, \cdots, v^m$ is a basis of V^*, then $W := \bigcap_{i=1}^{m-n} \ker v^i$ is an n dimensional subspace, if 2-form $\omega \in \bigwedge^2 V$ vanishes on $W \times W$, then $\omega = \sum_{i=1}^{m-n} \alpha_j^i \wedge v^i$

Proof. Remember $v^1 \wedge \cdots \wedge v^n$ is a linear functional on $\overbrace{V \times \cdots \times V}^{n \text{ times}}$ given by

$$v^1 \wedge \cdots \wedge v^n(x_1, \cdots, x_n) = \sum_{\sigma} (-1)^{sgn\sigma} v^1(x_{\sigma 1}) \cdots v^n(x_{\sigma n}) = \det(v^i(x_j))$$

Assume $\omega = \sum_{i<j} c_{ij} v^i \wedge v^j$, denote $\nu = \sum_{m-n<i<j} c_{ij} v^i \wedge v^j$ □

Theorem 2.14. *Given a smooth manifold M of dimesion m, and $\theta^1, \cdots, \theta^{n-m} \in \Omega^1(M)$ such that*
(1) *$\theta^1, \cdots, \theta^{n-m} \in \Omega^1(M)$ are pointwise linearly independent*
(2) *$d\theta^j = \sum \alpha_i^j \wedge \theta^i$ for some $\alpha_i^j \in \Omega^1(M)$*
Then $\forall p \in M$, there exists a connected n dimensional submanifold N with $p \in N$, such that $\theta^i|_{TN} \equiv 0, \forall 1 \leq i \leq n-m$
According to Lemma 2.13, (1) $\Leftrightarrow \ker \theta^j \subseteq TM$ is an n-dimension distribution $\mathscr{D}$(subbundle of TM), locally $\mathscr{D} = span\{x_1, \cdots, x_n\}, x_i \in \mathfrak{X}(M)$. Since $d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$, (2) $\Leftrightarrow \mathscr{D}$ is involutive, i.e. $[x_i, x_j] \in \mathscr{D}$, denote $I_{\text{alg}} = \langle \theta^1, \cdots, \theta^{m-n} \rangle_{\text{alg}}$, $I_{\text{diff}} = \langle \theta^1, \cdots, \theta^{n-m} \rangle_{\text{diff}}$, then (2) $\Leftrightarrow I_{\text{alg}} = I_{\text{diff}}$
The result is actually stronger, $\forall p \in M$, there exists coordinates $(y^1, \cdots, y^m)$ on a neighborhood of p such that $\langle \theta^1, \cdots, \theta^{n-m} \rangle = \langle dy^1, \cdots, dy^{m-n} \rangle$, then the integral submanifolds are $\{y^1, \cdots, y^{n-m}$ are constants$\}$, giving a foliation of M

Proof. □

3 Maurer-Cartan formula - 2/4/2020

Example 3.1. $GL(n,\mathbb{C}) < GL(2n,\mathbb{R})$ is a real Lie group, $GL(n,\mathbb{C}) = \{g \in GL(2n,\mathbb{R}) | gJ = Jg$, where $J = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{pmatrix}$

Example 3.2. Given an inner product matrix B, $O(B) = \{g^T B g = B\}$ is a real Lie group, $O(2) = \{g^T g = I\}$ with $B = I$, $O(2) = SO(2) \rtimes \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$

Example 3.3. $\mathrm{Isom}^+(\mathbb{E}^n) = \{(r,t)\}$, $(r,t)x = rx + t$, $(I,0)$ is the identity, $(r,t)(r',t') = (rr', rt' + t)$, $(r,t)^{-1} = (r^{-1}, -rt)$ is a real Lie group, $\mathrm{Isom}^+(\mathbb{E}^n) = \left\{ \begin{pmatrix} 1 & 0 \\ t & r \end{pmatrix} \right\} \subseteq GL(n+1,\mathbb{R})$

Definition 3.4. Let G be a Lie group, left multiplication by g, denoted by L_g is an isomorphism, L_g acts on $\mathfrak{X}(G)$ by pushforward, $(L_{g*}X)_h = (dL_g)_{g^{-1}h}(X_{g^{-1}h})$, a vector field $X \in \mathfrak{X}(G)$ is left invariant if $L_{g*}X = X$, let $\mathfrak{X}^G(G)$ denote all the left invariant vector fields, $\mathfrak{X}^G(G) \cong T_e G$ is the Lie algebra, $T_e(G) \to \mathfrak{X}^G(G)$, $v \mapsto X$ with $X_g = (dL_g)_e(v)$ is a Lie algebra isomorphism

Example 3.5. For $O(B)$, a curve $\gamma(s)$ through I should satisfy $\gamma(s)^T B \gamma(s) = B \Rightarrow \gamma'(0)^T B + B\gamma'(0) = 0$, $\mathfrak{o}(B) = \{X \in M_n(\mathbb{R}) | X^T = -X\}$, $\mathfrak{o}(2) = \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \right\}$

Example 3.6. For $G = \mathrm{Isom}^+(\mathbb{E}^2)$, $\mathfrak{g} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ t_1 & 0 & -\theta \\ t_2 & \theta & 0 \end{pmatrix} \right\}$

Definition 3.7. $\Omega^n(M,V) = \Gamma(T^n M \otimes V)$ are **V-valued differential forms**
If $V = \mathfrak{g}$ is a Lie algebra, we can also define the "wedge" product, for any $w, v \in \Omega^1(M,\mathfrak{g})$, $[w,v](X,Y) = [w(X), v(Y)] - [w(Y), v(X)]$, this is kind of like wedge product, with product replaced by $[,]$

Definition 3.8. Define **Maurer-Cartan form** $\omega^G \in \Omega^1(G,\mathfrak{g})$ to be $\omega^G_g : T_e G \to \mathfrak{g}$ such that $\omega^G_g(v) = X$ where $X \in \mathfrak{X}^G(G)$ with $X_g = v \in T_g G$

Proposition 3.9. *ω^G is left invariant, i.e. $L_g^* \omega^G = \omega^G$*

Proof.

$$\begin{aligned}(L_g^*\omega^G)_h(v) &= \omega^G_{gh}((dL_g)_h(v)) \\ &= \omega^G_{gh}((dL_g)_h(X_h)) \\ &= \omega^G_{gh}((L_{g*}X)_{gh}) \\ &= \omega^G_{gh}(X_{gh}) \\ &= X\end{aligned}$$

Here $X \in \mathfrak{g}$ such that $X_h = v$, i.e. $\omega^G_h(v) = X$ □

Proposition 3.10. $d\omega^G + \frac{1}{2}[\omega^G, \omega^G] = 0$

Proof. First suppose $X, Y \in \mathfrak{g}$, then $\omega^G_g([X,Y]) = Z \in \mathfrak{g}$ with $Z_g = [X,Y]_g$, by definition, $Z = [X,Y]$
In general, let $X = f^i Z_i$, $Y = g^j Z_j$ with $Z_i \in \mathfrak{g}$ being a basis, then

$$\begin{aligned}
\omega^G([X,Y]) &= \omega^G(f^i Z_i(g^j)Z_j - g^j Z_j(f^i)Z_i + f^i g^j [Z_i, Z_j]) \\
&= (f^i Z_i(g^j) - g^i Z_i(f^j))\omega^G(Z_j) + f^i g^j \omega^G([Z_i, Z_j]) \\
&= (f^i Z_i(g^j) - g^i Z_i(f^j))Z_j + f^i g^j [Z_i, Z_j] \\
&= X(\omega^G(Y)) - Y(\omega^G(X)) + [\omega^G(X), \omega^G(Y)]
\end{aligned}$$

□

Theorem 3.11. *Given a smooth manifold M and $\omega \in \Omega^1(M, \mathfrak{g})$, if $d\omega + \dfrac{1}{2}[\omega, \omega] = 0$, then for any $p \in M$, there exists a neighborhood U and $f : U \to G$ such that $f^*\omega^G|_U = \omega|_U$, and f is unique up to a composition with L_g for some g*

4 Fundamental theorem of Maurer-Cartan form - 2/6/2020

Reference: Section 1.6 of I+L

Lemma 4.1. *If G is a matrix group, $g = (g^i_j) : U \to G$ is a local parametrization, then $\omega^G = g^{-1}dg = (g^i_j)^{-1}dg^k_j$ (matrix multiplication)*

Example 4.2. Suppose $G = \mathrm{Isom}^+(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes SO(2)$, $g = \begin{pmatrix} 1 & 0 & 0 \\ t_1 & \cos\theta & -\sin\theta \\ t_2 & \sin\theta & \cos\theta \end{pmatrix}$, $g^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & \cos\theta & \sin\theta \\ * & -\sin\theta & \cos\theta \end{pmatrix}$, $dg = \begin{pmatrix} 0 & 0 & 0 \\ dt_1 & -\sin\theta d\theta & -\cos\theta d\theta \\ dt_2 & \cos\theta d\theta & -\sin\theta d\theta \end{pmatrix}$, $g^{-1}dg = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & -d\theta \\ * & d\theta & 0 \end{pmatrix} \in \mathbb{R}^2 \rtimes \mathfrak{so}(2) = \mathfrak{g}$

Theorem 4.3. *Let M be a smooth manifold of dimension m, G be a Lie group, $\omega \in \Omega^1(M, \mathfrak{g})$, then*
(1) *For any $p \in M$, there exists a neighborhood U of p such that $\omega = f^*\omega^G \Leftrightarrow d\omega + \frac{1}{2}[\omega, \omega] = 0$*
(2) *Suppose $f, h : U \to G$ satisfying $f^*\omega^G = h^*\omega^G$, then there exists $g \in G$, such that $h = L_g \circ f$*
(3) *If M is simply connected, then f extends to M*

Proof. Given $\omega \in \Omega^1(M, \mathfrak{g})$, $d\omega + \frac{1}{2}[\omega, \omega] = 0$
(1) Define $\theta \in \Omega^1(M \times G, \mathfrak{g})$ by $\theta = \pi^*_M\omega - \pi^*_G\omega^G$, $\theta = \theta^i X_i$, $\{X_i\}$ being a basis of $\mathfrak{g}$, $\ker\theta \le T(M \times G)$, given $u \in T_pM$, $p \in M$, given $g \in G$, $\exists_1 v \in T_gG$ such that $\omega_p(u) = \omega^G_g(v)$ $\Rightarrow \forall (p, g) \in M \times G$, $T_pM \to (\ker\theta)_{(p,g)}$ is an isomorphism with inverse $(d\pi_M)_{(p,g)}$

$$\begin{aligned}
d\theta &= d(\pi^*_M\omega) - d(\pi^*_G\omega^G) \\
&= \pi^*_M d\omega - \pi^*_G d\omega^G \\
&= \frac{1}{2}(\pi^*_M[\omega, \omega] - \pi^*_G[\omega^G, \omega^G]) \\
&= \frac{1}{2}([\pi^*_M\omega, \pi^*_M\omega] - [\pi^*_G\omega^G, \pi^*_G\omega^G]) \\
&= \frac{1}{2}([\pi^*_M\omega, \pi^*_M\omega] - [\pi^*_G\omega^G, \pi^*_G\omega^G] - [\pi^*_G\omega^G, \pi^*_M\omega] + [\pi^*_G\omega^G, \pi^*_M\omega]) \\
&= \frac{1}{2}([\theta, \pi^*_M\omega] + [\pi^*_G\omega^G, \theta]) \\
&= \frac{1}{2}[\theta, \pi^*_M\omega - \pi^*_G\omega^G] \\
&= \frac{1}{2}[\theta, \theta]
\end{aligned}$$

$\frac{1}{2}[\theta^i X_i, \theta^j X_j](\xi, \eta) = \frac{1}{2}(\theta^i(\xi)\theta^j(\eta)[X_i, X_j] - \theta^i(\eta)\theta^j(\xi)[X_i, X_j]) = \frac{1}{2}[\theta^i, \theta^j]c^k_{ij}X_k$ where c^k_{ij} are structure constants of the Lie algebra $\mathfrak{g}$, i.e. $[X_i, X_j] = c^k_{ij}X_k$
Apply Frobenius Theorem2.10, $\forall (p, q)$, there exists a submanifol of dimension $\dim M$ everywhere tangent to $\ker\theta$, $(d\pi_M)_{(p,g)}) : T_{(p,g)} = (\ker\theta)_{(p,g)} \to T_pM$ is surjective, by inverse function theorem, there exists a neighborhood U of p and $f : U \to M \times G$, $f(U) \subseteq \Gamma$, $f|_U = \pi^{-1}_M \Rightarrow \Gamma$ is the graph of f and $f^*(\omega^G) = \omega$
(2) Let $f(p) = g$, $h(p) = g'$, $\exists_1 k \in G$ such that $g' = kg$, thus $(L_k \circ f)(p) = kg = g'$, thus $(L_k \circ f)^*\omega^G = f^*L^*_k\omega^G = f^*\omega^G = \omega$, thus the graph of $L_k \circ f$ coincides the graph of h on a neighborhood of p, because both are integral submanifolds of θ at (p, g)
(3) $\pi_M|_\Gamma : \Gamma \to M$ for Γ a maximal integral submanifold for $\ker\theta$ is a covering map ∎

Example 4.4. $M = I \subset \mathbb{R}$, $G = \mathrm{Isom}^+(\mathbb{R}^2)$, $\omega^G = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & -d\theta \\ * & d\theta & 0 \end{pmatrix}$, consider $\alpha, \beta : I \to \mathbb{R}^2$ are paths parametrized by arc length, $\widetilde{\alpha} : I \to G$, $\widetilde{\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ \alpha^1(t) & {\alpha^1}'(t) & -{\alpha^2}'(t) \\ \alpha^2(t) & {\alpha^2}'(t) & {\alpha^1}'(t) \end{pmatrix}$,

$\widetilde{\alpha}'(t) = \begin{pmatrix} 0 & 0 & 0 \\ {\alpha^1}'(t) & {\alpha^1}''(t) & -{\alpha^2}'(t) \\ {\alpha^2}'(t) & {\alpha^2}''(t) & {\alpha^1}''(t) \end{pmatrix}$, $\widetilde{\alpha}^* d\tau = \begin{pmatrix} {\alpha^1}' dt \\ {\alpha^2}' dt \end{pmatrix}$, $r_0^{-1} \circ \widetilde{\alpha} = \begin{pmatrix} {\alpha^1}' & {\alpha^2}' \\ -{\alpha^2}' & {\alpha^1}' \end{pmatrix}$

Thus $(r_0^{-1} \circ \widetilde{\alpha})(\widetilde{\alpha}^* d\tau) = \begin{pmatrix} ({\alpha^1}')^2 + ({\alpha^2}')^2 \\ 0 \end{pmatrix} dt = \begin{pmatrix} dt \\ 0 \end{pmatrix}$

$\theta = \arctan \left(\dfrac{{\alpha^2}'}{{\alpha^1}'} \right) \Rightarrow d\theta = \dfrac{1}{1 + \left(\dfrac{{\alpha^2}'}{{\alpha^1}'} \right)^2} \dfrac{{\alpha^2}''{\alpha^1}' - {\alpha^1}''{\alpha^2}'}{({\alpha^1}')^2} dt = ({\alpha^2}''{\alpha^1}' - {\alpha^1}''{\alpha^2}') dt$ Note that $\kappa(t) = -{\alpha^1}^{''}(t){\alpha^2}^{'}(t) + {\alpha^2}^{''}(t){\alpha^1}^{'}(t) = \begin{pmatrix} {\alpha^1}'' \\ {\alpha^2}'' \end{pmatrix} \cdot \begin{pmatrix} -{\alpha^2}' \\ {\alpha^1}' \end{pmatrix}$ is the curvature, $\widetilde{\alpha}^* \omega^G(t) =$
$\begin{pmatrix} 0 & 0 & 0 \\ dt & 0 & -\kappa(t)dt \\ 0 & \kappa(t)dt & 0 \end{pmatrix}$

Therefore, $\widetilde{\alpha}^* \omega^G = \widetilde{\beta}^* \omega^G \Leftrightarrow \widetilde{\alpha} = L_g \circ \widetilde{\beta} \Leftrightarrow \alpha = g\beta \Leftrightarrow \kappa_\alpha = \kappa_\beta$

5 Two identities about Maurer-Cartan form - 2/11/2020

Remark (Uniqueness of ω^G). ω^G is the unique left invariant $\mathfrak{g}$ valued 1-form on G given an isomorphism $\omega_e^G : T_eG \to \mathfrak{g}$, $\omega_g^G = L_{g^{-1}}^* \omega_e^G$

Proposition 5.1. *Due to the left invariance of ω^G and the fact that R_g, L_h commutes, we have $L_{h*}R_{g*}X = R_{g*}L_{h*}X = R_{g*}X$, for any $X \in \mathfrak{X}^G(G)$, thus pushforward of conjugation $C_{g^{-1}} = L_h R_g$ also preserves $\mathfrak{X}^G(G)$, giving an automorphism of $\mathfrak{g}$*
Similarly, it is easy to see

$$R_g^* \omega^G = L_{g^{-1}}^* R_g^* \omega^G = Ad(g)^{-1} \omega^G$$

Proposition 5.2. *Given $\alpha : U \to G$, $\alpha^* \omega^G \in \Omega^1(U, \mathfrak{g})$, $p : U \to G$, let $\beta(x) = \alpha(x)p(x)$, then $d\beta = R_{p*} \circ d\alpha + L_{\alpha *} \circ dp$, $\beta^* \omega^G = Ad(p)^{-1} \alpha^* \omega^G + p^* \omega^G$*

6 Schwarzian - 2/13/2020

Example 6.1. Consider a map $\alpha : U \subseteq \mathbb{C} \to \mathbb{C}P^1$
Let $G = \left\{ z \mapsto \dfrac{az+b}{cz+d} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \right\} / \pm \mathrm{id}$ be the group of Möbius transformations
The projection is defined by $G \to \mathbb{C}P^1$, $g \mapsto g[1:0] = [g_{11} : g_{21}]$, it is clear that this map is onto, thus G acts on $\mathbb{C}P^1$ transitively, $\mathbb{C}P^1$ is a homogeneous space, the stabilizer of $[1:0]$ is $\left\{ \begin{pmatrix} a & b^{-1} \\ 0 & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^\times, b \in \mathbb{C} \right\} =: P$, for any other $y = g[1:0] \in \mathbb{C}P^1$, the stabilizer would be gPg^{-1}
Pick a lift $\widehat{\alpha} : U \to G$, $z \mapsto \begin{pmatrix} \alpha(z) & -1 \\ 1 & 0 \end{pmatrix}$, $\widehat{\alpha}^{-1} d\widehat{\alpha} = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} \alpha' dz & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\alpha' dz & 0 \end{pmatrix}$, let $\widetilde{\alpha}(z) = \widehat{\alpha}(z) p(z)$ for some $p : U \to P < G$, $p(z) = \begin{pmatrix} a(z) & b(z) \\ 0 & a(z)^{-1} \end{pmatrix}$, apply Proposition 5.2, we have

$$\begin{aligned}
\widetilde{\alpha}^{-1} d\widetilde{\alpha} &= p^{-1} (\widehat{\alpha}^{-1} d\widehat{\alpha}) p + p^{-1} dp \\
&= \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\alpha' dz & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} + \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & -\dfrac{a'}{a^2} \end{pmatrix} dz \\
&= \begin{pmatrix} ab\alpha' + a^{-1} a' & b^2 \alpha' + a^{-1} b' + b a' a^{-2} \\ -a^2 \alpha' & -ab\alpha' - a^{-1} a' \end{pmatrix} dz
\end{aligned}$$

Set $a = (\alpha')^{-\frac{1}{2}}$, $b = \frac{1}{2} \alpha'' (\alpha')^{-\frac{3}{2}}$, $\widetilde{\alpha}^{-1} d\widetilde{\alpha}$ becomes $\begin{pmatrix} 0 & \frac{1}{2} S_\alpha(z) \\ 1 & 0 \end{pmatrix} dz$, here $S_\alpha(z) = \dfrac{\alpha'''}{\alpha'} - \dfrac{3}{2} \left(\dfrac{\alpha''}{\alpha'} \right)^2$ is called the **Schwarzian**

Remark. $\left\{ z \mapsto \dfrac{az+b}{cz+d} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\} = \mathrm{Isom}^+(\mathbb{H}^2)$, where $\mathbb{H}^2$ is the half space model for hyperbolic space, $\mathbb{H}^2 = \{\mathrm{Im} z > 0\}$ with metric $\dfrac{dx^2 + dy^2}{y^2}$

Example 6.2. Let $\beta : U \subseteq \mathbb{C}P^1 \to \mathbb{C}P^1$ be the identity map, $\widehat{\beta}(z) = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}$ is a lift of $\beta(z)$, $\widehat{\beta}^{-1} d\widehat{\beta} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, then we know $\alpha = g|_U$ for some $g \in SL(2,\mathbb{C}) \Leftrightarrow \alpha = g \circ \beta$ for some $g \in SL(2,\mathbb{C}) \Leftrightarrow \widehat{\beta}^{-1} d\widehat{\beta} = \widetilde{\alpha}^{-1} d\widetilde{\alpha}$ on $U \Leftrightarrow S_\alpha \equiv 0$ on U

Lemma 6.3. *If u, v are both solutions to the differential equation $X'' + qX = 0$, then $S_{u/v} = 2q$*

Proof. □

Lemma 6.4. *$Hom(V, W) \to V^* \otimes W$, $A = (a_{ij}) \mapsto \sum_{i,j} a_{ji} v_i^* \otimes w_j$ is an isomorphism*

Definition 6.5. A **tableau** is a linear subspace $A \leq Hom(V, W) \cong V^* \otimes W$ where V, W are linear vector spaces of dimension n and s, consider a smooth map $f : V \to W$, $D_x f : V \cong T_x V \to T_{f(x)} W \cong W \in Hom(V, W)$, $D_x f \in A, \forall x \in V$ if it satisfies a linear, constant coefficient PDE
Let $\{v^1, \cdots, v^n\}$ be a basis of V^*, $\{w_1, \cdots, w_s\}$ be a basis of W

$$A = \mathrm{Span} \left\{ A_i^{ta} \otimes w_a \middle| t = 1, \cdots, T \right\} = \bigcap_r \ker \left\{ B_a^{ri} v_i \otimes w^a \middle| r = 1, \cdots, R \right\}$$

where $R = \dim V^* \otimes W - \dim T$, $\{w^1, \cdots, w^s\}$, $\{v_1, \cdots, v_n\}$ are the dual basis, then

$$D_x f \in A, \forall x \in V \Leftrightarrow B_a^{ri} df^a(v^i) = 0, \forall r \Leftrightarrow B_a^{ri} \frac{\partial f^a}{\partial x^i} = 0, \forall r$$

$f(x) = f_0 + A_0 x,\ f_0 \in W,\ A_0 \in A$ is always a solution. Also

$D_x f \in A, \forall x \Rightarrow D_x^2 f(y, \cdot) \in A, \forall x, y \in V \Rightarrow \cdots \Rightarrow D_x^k f(y_1, \cdots, y_{k-1}, \cdot) \in A, \forall x, y_1, \cdots, y_{k-1}$

We define the l-th **prolongation** of A as

$$A^{(l)} = S^{l+1} V^* \otimes W \cap V^{* \otimes l} \otimes A = S^{l+1} V^* \otimes W \cap V^* \otimes A^{(l-1)}$$

Example 6.6. Consider Cauchy-Riemann equations, $(u(x,y), v(x,y)) : \mathbb{R}^2 \to \mathbb{R}^2$, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, $A \subseteq End(\mathbb{R}^2) = \left\{ \begin{pmatrix} A^1 & -A^2 \\ A^2 & A^1 \end{pmatrix} \middle| A^1, A^2 \in \mathbb{R} \right\} \cong \mathfrak{co}(2) \cong \mathbb{R} \otimes \mathfrak{so}(2) \cong \mathfrak{gl}_1(\mathbb{C}) \cong \mathbb{C}$

Example 6.7. $A = \mathfrak{so}(n) \subseteq End(\mathbb{R}^2) = \{X^T = -X\} = \left\{ \begin{pmatrix} 0 & & -A_i^j \\ & \ddots & \\ A_j^i & & 0 \end{pmatrix} \middle| i > j \right\}$, corresponds to $\dfrac{\partial f^j}{\partial x^i} = -\dfrac{\partial f^i}{\partial x^j}$
Let $\alpha \in S^2 \mathbb{R}^{n*} \otimes \mathbb{R}^n \cap \mathbb{R}^{n*} \otimes \mathfrak{so}(n)$, $X \in \mathfrak{so}(n) \Rightarrow \langle Xu, v \rangle = -\langle u, Xv \rangle$
$\langle \alpha(u,v), w \rangle = -\langle \alpha(u,w), v \rangle = \langle \alpha(v,w), u \rangle = -\langle \alpha(v,u), w \rangle = -\langle \alpha(u,v), w \rangle \Rightarrow \alpha = 0$. Thus the only solutions to $\dfrac{\partial u^j}{\partial x^i} = -\dfrac{\partial u^i}{\partial x^j}$ are $u = u_0 + X$

Proposition 6.8. $A^{(l)} = \{(p^1(x), \cdots, p^s(x))\}$ *where* $p^i(x)$ *are* $l+1$*-homogeneous symmetric polynomials such that* $D_x p^i \in A, \forall x \in V$

Proof. ∎

Index