11.1 Taylor Polynomials

$n! = 1\times 2\times 3\times \cdots \times (n-1)\times n$ is called the n-th factorial, $0! = 1$, $1! = 1$, $2! = 1\times 2 = 2$

$f^{(n)}(x)$ is the n-th derivative of f, $f^{(0)}(x) = f(x)$, $f^{(1)}(x) = f'(x)$, $f^{(2)}(x) = f''(x)$

Definition: the n-th Taylor polynomial of $f(x)$ at $x=a$ is the polynomial $P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

$P_n(a) = f(a)$, $P_n'(a) = f'(a)$, $P_n''(a) = f''(a)$, $\cdots$, $P_n^{(n)}(a) = f^{(n)}(a)$

In particular, the n-th Taylor polynomial of $f(x)$ at $x=0$ is the polynomial

$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$

$P_n(0) = f(0)$, $P_n'(0) = f'(0)$, $P_n''(0) = f''(0)$, $\cdots$, $P_n^{(n)}(0) = f^{(n)}(0)$

Example: The 2nd TP (Taylor polynomial) of $f(x) = \ln(1+5x^2)$ at $x=0$ which is

$P_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$, hence we need to calculate $f(0), f'(0), f''(0)$

$f(0) = \ln(1+5\cdot 0^2) = \ln(1) = 0$

$f'(x) = [\ln(1+5x^2)]' = \frac{1}{1+5x^2}\cdot 10x = \frac{10x}{1+5x^2}$, $f'(0) = 0$

$f''(x) = \left[\frac{10x}{1+5x^2}\right]' = 10\left[\frac{x}{1+5x^2}\right]' = 10\,\frac{x'(1+5x^2) - x(1+5x^2)'}{(1+5x^2)^2} = 10\,\frac{(1+5x^2) - x\cdot 10x}{(1+5x^2)^2}$

$f''(0) = 10$

hence $P_2(x) = 5x^2$

How about $P_4(x)$? There is a quicker way

Definition: Taylor expension (TE) of f at $x=0$ is $f(x) \sim f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$

without ending, if we truncate the first $n+1$ terms, we will get $P_n(x)$

consider the TE of $g(x) = \ln(1+X)$ at $X=0$, $g(0) = 0$, $g'(x) = \frac{1}{1+X}$, $g''(x) = -\frac{1}{(1+X)^2}$

thus $g'(0) = 1$, $g''(0) = -1$, thus $\ln(1+X) \sim g(0) + \frac{g'(0)}{1!}X + \frac{g''(0)}{2!}X^2 + \cdots$

$= X - \frac{1}{2}X^2 + \cdots$, replace X with $5x^2$, you have

$\ln(1+5x^2) \sim (5x^2) - \frac{1}{2}(5x^2)^2 + \cdots = 5x^2 - \frac{25}{2}x^4 + \cdots$, Notice

$P_2(x) = 5x^2$, $P_4(x) = 5x^2 - \frac{25}{2}x^4$