10.4a

A person planning for her retirement arranges to make continuous deposits into a savings account at the rate of \$3700 per year. The savings account earns 5% interest compounded continuously.

(a) Set up a differential equation that is satisfied by $f(t)$, the amount of money in the account at time t.

(b) Solve the differential equation in part (a), assuming that $f(0)=0$, and determine how much money will be in the account at the end of 20 years

Solution:

(a) $\qquad y' \qquad = \qquad 0.05y \qquad + \qquad 3700$

$$\begin{bmatrix}\text{rate of change} \\ \text{of } y\end{bmatrix} = \begin{bmatrix}\text{rate at which is} \\ \text{added}\end{bmatrix} + \begin{bmatrix}\text{rate at which money} \\ \text{is deposited}\end{bmatrix}$$

Here $\quad y(t) = y = f(t)$

(b) rewrite as $y' - 0.05y = 3700$, $a(t) = -0.05$, $b(t) = 3700$

$A(t) = \int a(t)\,dt = -0.05t$, thus $[ye^{-0.05t}]' = 3700e^{-0.05t}$, integrate

$ye^{-0.05t} = \int [ye^{-0.05t}]'dt = \int 3700e^{-0.05t} = -74000e^{-0.05t} + C$

Plug in $f(0)=0$, $0 = y(0)e^{-0.05\cdot 0} = -74000e^{-0.05\cdot 0} + C = -74000 + C$

$\Rightarrow C = 74000$, plug back, get $\quad ye^{-0.05t} = -74000e^{-0.05t} + 74000$

$\times e^{0.05t}$ on both sides $\quad y = e^{0.05t}(-74000e^{-0.05t} + 74000) = -74000 + 74000e^{0.05t}$

The money in the account at the end of 20 years is by definition

$y(20) = -74000 + 74000 \times e^{-0.05\cdot 20} = -74000 + 74000 \times e^{-1}$

A person purchased a new car for \$23100 and financed the entire amount. Suppose that the person can only afford to pay \$100 per month. Assume that the payments are made at a continuous annual rate and that interest is compounded continuously at the rate of 5%.

(a) Set up a differential equation that is satisfied by the amount $f(t)$ of money owed on the car loan at time t.

(b) How long will it take to pay off the car loan

Solution:

(a) $y' = 0.05y - 12\times 100$ ← 1 year = 12 month

$$\begin{bmatrix}\text{rate of change}\\ \text{of } y\end{bmatrix} = \begin{bmatrix}\text{rate at which}\\ \text{interest is added}\end{bmatrix} - \begin{bmatrix}\text{rate at which money}\\ \text{is paided annually}\end{bmatrix}$$

$y(0) = 23100$ since this is how much owed in the begining

(b) Rewrite as $y' - 0.05y = -1200$, $a(t) = -0.05$, $b(t) = -1200$

$A(t) = \int -0.05\,dt = -0.05t$, thus $[ye^{-0.05t}]' = -1200e^{-0.05t}$, integrate

$ye^{-0.05t} = \int [ye^{-0.05t}]'\,dt = \int -1200e^{-0.05t}\,dt = 24000e^{-0.05t} + C$

Plug in $y(0) = 23100$, $23100 = y(0)e^{-0.05\cdot 0} = 24000e^{-0.05\cdot 0} + C = 24000 + C$

$\Rightarrow C = 23100 - 24000 = -900$, plug back, $ye^{-0.05t} = 24000e^{-0.05t} - 900$,

$\times e^{0.05t}$ on both sides $y = e^{0.05t}(24000e^{-0.05t} - 900) = 24000 - 900e^{0.05t}$

It pays off in t years meaning solving $y(t) = 0 \Rightarrow$

$0 = 24000 - 900e^{0.05} \Rightarrow 900e^{0.05t} = 24000 \Rightarrow e^{0.05t} = \frac{24000}{900} = \frac{240}{9}$

$\Rightarrow 0.05t = \ln e^{0.05t} = \ln\left(\frac{240}{9}\right) \Rightarrow t = 20\ln\left(\frac{240}{9}\right) \approx 65.67$ years