0.1 Lie groups

Definition 0.1.1. A **real Lie group** G is a group and a smooth manifold such that multiplication $G \times G \to G$ and inverse $G \to G$ are smooth
A **complex Lie group** is a group and complex manifold such that multiplication and inverse are holomorphic
a Lie subgroup H is a subgroup and an immersed submanifold

Definition 0.1.2. Left multiplication L_g by g is an isomorphism, a vector field X on G is called **left invariant** if $(L_g)_*X = X$, by Exercise **??**, $[X, Y]$ is also left invariant since $(L_g)_*[X, Y] = [(L_g)_*X, (L_g)_*Y] = [X, Y]$
Define **Lie algebra of** G to be left invariant vector fields. Equivalently, T_1G
If $\phi : G \to H$ is a homomorphism of Lie groups, then $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ or $(d\phi)_1 : T_1G \to T_1H$ is an homomorphism of Lie algebras
Suppose $H \leq G$ is a Lie subgroup, then $\mathrm{Lie}(H) = T_1H \leq T_1G$

Proposition 0.1.3. Lie groups are parallelizable

Proof. For any $0 \neq X_1 \in T_1G$, we can define a vector field $X_g = (L_g)_1X_1$, this is a nonvanishing global section of the tangent bundle, G is parallelizable □

Definition 0.1.4. A Lie group representation (ρ, V) is a Lie group homomorphism $\rho : G \to GL(V)$

Proposition 0.1.5. Let V be a complex vector space, (π, V) be a Lie group representation of a compact Lie group G, then there exists a positive definite Hermitian form such that (π, V) is unitary

Proof. Choose any positive definite Hermitian form $\langle , \rangle$, define Hermitian form

$$(v, w) := \int_G \langle \pi(g)v, \pi(g)w \rangle d\mu$$

Where μ is the Haar measure with $\int_G d\mu = 1$, integrals make sense since G is compact, then $(,)$ is G left invariant □

Definition 0.1.6. Lie group G acts on smooth manifold M, G_p is the stablizer of p. The **isotropy representation** is $G_p \to GL(T_pM)$, $g \mapsto d_pg$

0.2 Exponential map

Lemma 0.2.1. The exponential map $e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is defined on $M_n(\mathbb{C})$ and logarithmic map $\log A = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(A-I)^k}{k}$ are defined on $|A-I| < 1$ and there inverses to each other locally, moreover, the exponential map is surjective onto $GL(n, \mathbb{C})$

Remark 0.2.2. Note that this also holds for a Banach algebra A

Proof. Just compare the coefficients of multiplication of series $\square$

AV<=V <=> e^tAV<=V

Lemma 0.2.3. Let e^{tA} be a one parameter subgroup, then $V \leq \mathbb{R}^n$ is invariant under A iff invariant under $e^{tA}, \forall t$, in particular, $Av = 0$ iff $e^{tA}v = 0, \forall t$

Proof. If $AV \subseteq V$, then $e^{tA}V = \sum_{k=0}^{\infty} t^k \frac{A^k}{k!} V \subseteq V$
If $e^{tA}V \subseteq V, \forall t$, since V is closed, $\left.\frac{d}{dt}\right|_{t=0} e^{tA}V = AV \subseteq V$ $\square$

Proposition 0.2.4. Observe that $v'(t) = Av(t)$ with $v(0) = v_0$ has the solution $v(t) = e^{tA}v_0$
Consider V_m to the vector space of homogeneous polynomials in n variables of degree m, define group action of $GL(n, \mathbb{C})$ on V_m, $g \cdot f(x) := f(g^{-1}x)$, consider $v(t) = e^{tA} \cdot f := f(e^{-tA}x)$, then $v'(t) = \left.\frac{d}{dt}\right|_{t=0} f(e^{-tA}x) =: D_A f$, where D_A is a linear differential operator $V_m \to V_m$ by Lemma 0.2.3, then we should have $f(e^{-tA}x) = v(t) = e^{tD_A}f$, therefore we would get $D_A = -A^T$, and it will be easy to check that $D_{[A,B]} = [D_A, D_B]$

Proof. If we denote $g = (g_{ij}) \in GL(n, \mathbb{C})$, $f(x) = \sum_{i_1, \cdots, i_n} C_{i_1, \cdots, i_n} x_1^{i_1} \cdots x_n^{i_n}$, then $f(g^{-1}x) = \sum_{i_1, \cdots, i_n} C_{i_1, \cdots, i_n} (g_{11}x_1 + \cdots + g_{1n}x_n)^{i_1} \cdots (g_{n1}x_1 + \cdots + g_{nn}x_n)^{i_n}$ is still a homogeneous polynomial in n variables of degree m
Denote $A = (a_{ij})$,

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} f(e^{-tA}x) &= \nabla f(x) \cdot \left.\frac{d}{dt}\right|_{t=0} e^{-tA}x \\
&= -\nabla f(x) \cdot Ax \\
&= -\sum_{i,j} a_{ij} x_j \frac{\partial f}{\partial x_i} \\
&= \left(-\sum_{i,j} a_{ij} x_j \frac{\partial}{\partial x_i} \right) f \\
&= (-\nabla^T Ax) f \\
&= D_A f
\end{aligned}
$$

In particular, $D_A x_i = -\sum_{j=1}^{n} a_{ij} x_j$, thus D_A has matrix $-A^T$ with respect to $x_1, \cdots, x_n$, basis of V_1 $\square$

Example 0.2.5. Consider Lie group $SL(2, \mathbb{C})$ whose Lie algebra is $\mathfrak{sl}(2, \mathbb{C})$, which is generated by $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, thus $D_H = -x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$, $D_X = -x_2 \frac{\partial}{\partial x_1}$, $D_Y = -x_1 \frac{\partial}{\partial x_2}$

Definition 0.2.6. Let G be a (Lie) group, then a 1-parameter subgroup means a (smooth) group homomorphism $\phi : \mathbb{R} \to G$, $\phi(s+t) = \phi(s)\phi(t)$

Lie group homomorphism induce Lie algebra homomorphism

Proposition 0.2.7. Let $\phi : G \to H$ be a homomorphism of Lie groups, then $d\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ or $(d\phi)_1 : T_1 G \to T_1 H$ is an homomorphism of Lie algebras

Proof. Suppose X is a left invariant vector field on G, then $(d\phi)_g X_g = (d\phi)_g (dL_g)_1 X_1(f) = X_1(f \circ \phi \circ L_g) = X_1(f \circ \phi \circ L_g) = (dL_{\phi(g)})_1 (d\phi)_1 X_1(f)$ which gives a left invariant vector field, thus using Lemma **??**

$$\begin{aligned}(d\phi)[X,Y](f) &= [X,Y](f \circ \phi)\\ &= X(Y(f \circ \phi)) - Y(X(f \circ \phi))\\ &= X(((d\phi Y)f) \circ \phi) - Y(((d\phi X)f) \circ \phi)\\ &= ((d\phi X)(d\phi Y)f) \circ \phi - ((d\phi Y)(d\phi X)f) \circ \phi\\ &= ([d\phi X, d\phi Y]f) \circ \phi\end{aligned}$$

Therefore $(d\phi)[X,Y] = [(d\phi X), (d\phi Y)]$, $d\phi$ is a Lie algebra homomorphism ☐

Proposition 0.2.8. One parameter subgroups are precisely the maximal integral curves of the left invariant vector fields starting at 1

Remark 0.2.9. There is a one to one correspondence, $\{\text{One parameter subgroups of } G\} \leftrightarrow \mathrm{Lie}(G) \leftrightarrow T_1 G$

Proof. Suppose $\phi : \mathbb{R} \to G$ is a one parameter subgroup, let $X_1 = \phi'(0)$, then we have a left invariant vector field X on G, think of $\dfrac{\partial}{\partial t}$ as a left invariant vector field on $\mathbb{R}$, thus ϕ as Lie group homomorphism induces $(d\phi)\dfrac{\partial}{\partial t}$ which is also a left invariant vector field and $\phi'(s) = (d\phi)_s \left.\dfrac{\partial}{\partial t}\right|_s = X_{\phi(s)}$ as in Proposition 0.2.7
Conversely, if $\phi : \mathbb{R} \to G$ is the maximal integral curve of some left invariant vector field X, suppose the global flow generated by X is $\varphi : G \times \mathbb{R} \to G$, then $\varphi(1,t) = \phi(t)$, $\phi(t+s) = \varphi(1, t+s) = \varphi(\varphi(1,t), s) = \varphi(\phi(t), s)$, since $L_{\phi(t)}$ is an isomorphism, thus $L_{\phi(t)} \circ \phi$ is the maximal integral curve starting at $\phi(t)$, thus $\varphi(\phi(t), s) = \phi(t)\phi(s)$ ☐

Definition 0.2.10. For any $A \in T_1 G$, define the exponential map $\exp A := \phi_A(1)$ where $\phi_A : \mathbb{R} \to G$ is the one parameter subgroup corresponding to A, also it is easy to see that $\exp tA := \phi_{tA}(1) = \phi_A(t)$ which is a scaling of the integral curve, and $\exp(t+s)A = \exp tA \exp sA$ since $\exp tA$ is a one parameter subgroup, and thus $(\exp A)^{-1} = \exp(-A)$

Properties of exponential map

Proposition 0.2.11. (**Properties of exponential map**)
Let G, H be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$
(a) The exponential map is a smooth map
(b) $(d\exp)_0 : \mathfrak{g} \cong T_0\mathfrak{g} \to T_1 G \cong \mathfrak{g}$ is the identity map, which implies that the exponential map is a local diffeomorphism around 0
(c) Suppose $\phi : G \to H$ is a Lie group homomorphism, then the following diagram commutes

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "(d\phi)_1"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & H \end{tikzcd}$$

Proof.
(a)
(b) For any $A \in \mathfrak{g}$, consider $\gamma : \mathbb{R} \to \mathfrak{g}, t \mapsto tA$ which is a one parameter subgroup of $\mathfrak{g}$, thus $A = \gamma'(0) \in T_0\mathfrak{g}$, and $\exp A = \gamma(1) = A$

(c) Define $\gamma(t) = \phi(\exp tA)$ which is a one parameter subgroup of H since $\gamma(t+s) = \phi(\exp(t+s)A) = \phi(\exp tA \exp sA) = \phi(\exp tA)\phi(\exp sA) = \gamma(t)\gamma(s)$, then $\gamma'(0) = \left.\frac{\partial}{\partial t}\right|_{t=0} \phi(\exp tA) = (d\phi)_1 \left.\frac{\partial}{\partial t}\right|_{t=0} \exp tA = (d\phi)_1 A$, on the other hand, $\exp(t(d\phi)_1 A)$ is one parameter subgroup of H corresponds to $(d\phi)_1 A = \gamma'(0)$, thus $\exp(t(d\phi)_1 A) = \gamma(t) = \phi(\exp tA)$ $\square$

Proposition 0.2.12. Let G be a Lie group and $H \leq G$ a Lie subgroup, then $\mathrm{Lie}(H) = \{A \in \mathrm{Lie}(G) | \exp tA \in H, \forall t \in \mathbb{R}\}$