More examples: $f(x)=e^x$, then $f'(x)=e^x$, $f''(x)=e^x$, $f'''(x)=e^x$, $\cdots$, $f^{(n)}(x)=e^x$, thus $f^{(n)}(0)=e^0=1$, hence the Taylor expasion of f at $x=0$ is $e^x=f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^2+\cdots=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$

notice: $\left(\frac{x^n}{n!}\right)'=\left(\frac{1}{n!}x^n\right)'=\frac{1}{n!}(x^n)'=\frac{1}{n!}nx^{n-1}=\frac{1}{n\cdot(n-1)\cdots1}\cdot n\cdot x^{n-1}=\frac{x^{n-1}}{(n-1)\cdots1}=\frac{x^{n-1}}{(n-1)!}$

Lower the index by 1. thus $(e^x)'=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)'$

$=1'+x'+\left(\frac{x^2}{2!}\right)'+\left(\frac{x^3}{3!}\right)'+\left(\frac{x^4}{4!}\right)'+\cdots=0+1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots=e^x$

$e^x-1=\int_0^x e^t dt=\int_0^x\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)dt=\int_0^x 1\,dt+\int_0^x t\,dt+\int_0^x\frac{t^2}{2!}dt+\int_0^x\frac{t^3}{3!}dt+\int_0^x\frac{t^4}{4!}dt+\cdots=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots=e^x-1$ notice $\int_0^x\frac{t^n}{n!}dt=\frac{1}{n!}\int_0^x t^n dt=\frac{1}{n!}\frac{t^{n+1}}{n+1}\Big|_0^x=\frac{x^{n+1}}{(n+1)!}$

Suppose $f(x)=\frac{1}{(1-x)^2}$, $f^{(100)}(0)=?$

notice $f(x)=\left(\frac{1}{1-x}\right)'=\left(1+x+x^2+\cdots+x^{101}+\cdots\right)'=1'+x'+(x^2)'+\cdots+(x^{101})'+\cdots$

$=0+1+2x+\cdots+101x^{100}+\cdots$

On the other hand, $f(x)=f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(100)}(0)}{100!}x^{100}+\cdots$

compare the coefficients, get $101=\frac{f^{(100)}(0)}{100!}\Rightarrow f^{(100)}(0)=101\cdot100!$

Taylor series for $e^{x^{11}}$ at $x=0$

$e^{x^{11}}=1+(x^{11})+\frac{(x^{11})^2}{2!}+\frac{(x^{11})^3}{3!}+\cdots=1+x^{11}+\frac{x^{22}}{2!}+\frac{x^{33}}{3!}+\cdots$