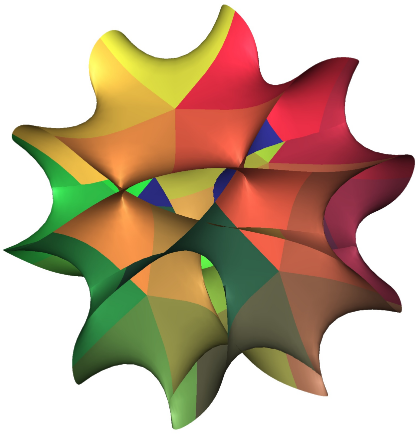
MATH742 - Geometric analysis



Taught by Tamás Darvas
Notes taken by Haoran Li
2020 Fall

Department of Mathematics
University of Maryland

Contents

1 Introduction

Theorem 1.1. $D \subsetneq \mathbb{R}^n$ is open bounded with smooth boundary, $f \in C^\infty(\partial D)$, then Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{on } \partial D \end{cases}$$

has a unique solution $u \in C^\infty(\overline{D})$. In the case $D = B(0, r)$, the solution is given by Poisson kernel

$$P[f](x) = \int_{\partial B(0,r)} f(\xi) \frac{r^2 - |x|^2}{r\omega_{n-1}|x - \xi|^2} d\sigma(\xi)$$

The uniqueness is guaranteed by integration by parts

Remark 1.2. Note that this always work as long as $\partial D \in C^\infty$

Theorem 1.3. M is a compact Riemannian manifold without boundary, $f \in C^\infty(M)$, if $\Delta u = f$ on M, then integration by parts demands $0 = \int_M \Delta u dx = \int_M f dx$, then $\Delta u = f$ on M has unique solution up to addting constants. Here $\Delta = \operatorname{Tr} \nabla^2_{X,Y}$ is the trace of Hessian, where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_Y X$

Theorem 1.4. M is a smooth manifold, the *de Rham complex* is

$$0 \to \mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots$$

Define the cohomology to be *de Rham cohomology* $H^k_{\mathrm{dR}}(\mathrm{X}, \mathbb{R})$, then

$$H^k_{\mathrm{dR}}(\mathrm{X}, \mathbb{R}) \cong H^k_{\mathrm{sing}}(\mathrm{X}, \mathbb{R}) = H^k(\mathrm{X}, \mathbb{R})$$

Where $H^k(\mathrm{X}, \mathbb{R})$ is the sheaf cohomology which is not so surprising by sheaf theory

Theorem 1.5. $s \in \Omega^k(\mathrm{X})$, the solution set S of $\Delta s = 0$ has $\dim S = \dim H^k(\mathrm{X}, \mathbb{R})$. Actually $S \hookrightarrow H^k(\mathrm{X}, \mathbb{R})$ with an explicit map

References

[1] *Differential Analysis On Complex Manifolds* - Raymond O. Wells

Index