MATH121-Lesson1

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1 Introduction

My name is Haoran Li, or Harrison Li, my office is at MATH4423, and my office hours are 2-3pm on Mondays and Wednesdays.

Computations on tests will involve only fairly easy numbers, and an exact answer will be required rather than a decimal approximation. For example

Exact	Not exact
$\sqrt{2}$	1.414
π	3.14159265358979323846
$\frac{2}{3}$	0.67
$\ln 6$	1.79
e^2	7.389

No calculators will be allowed on any of the tests!

2 Review

The following statements are mathematically equivalent:
a) Find the slope of the line tangent to the graph of f at a point (x, y)
b) Find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$
c) Find the first derivative of $f(x)$
d) Find $f'(x)$
e) Find $\dfrac{dy}{dx}$
Recall, however, that the first derivative is itself a function, which has its own domain and graph. Since it is a function, it has its own derivative. Given a function f, we can calculate the first derivative f' or $\dfrac{dy}{dx}$. We can then calculate the derivative of f', also called the second derivative of f, symbolically f'' or $\dfrac{d^2y}{dx^2}$

Important note: Just like $\dfrac{dy}{dx}$ is not a fraction, but is a notation for the first derivative, $\dfrac{d^2y}{dx^2}$ is also not a fraction but a notation. There is no multiplication involved! Rather, you need to interpret it this way: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ which means the derivative of $\dfrac{dy}{dx}$, the derivative of a derivative.

Example A:
Given $f(x) = x^3 - 8x + 2$, find $f'(x), f''(x), f(-1), f'(-1)$ and $f''(-1)$
Answers:
$3x^2 - 8, 6x, 9, -5, -6$

Example B:

Given $f(x) = \left(5x^4 - 1\right)^2$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, then find y when $x = -1$, $\left.\dfrac{dy}{dx}\right|_{x=-1}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=-1}$

Answers:

$200x^6 - 40x^3, 1400x^6 - 120x^2, 16, -160, 1280$

$$\frac{dy}{dx} = 2(5x^4-1)(20x^3) = 40x^3(5x^4-1) = 200x^7 - 40x^3$$

$$\frac{d^2y}{dx^2} = 40\left[(3x^2)(5x^4-1) + x^3(20x^3)\right]$$

$$= 40\left[15x^6 - 3x^2 + 20x^6\right]$$

$$= 40(35x^6 - 3x^2)$$

$$= 40x^2(35x^4 - 3) = 1400x^6 - 120x^2$$

Example C:
Public health officials use rates of change to quantify the spread of an epidemic into an equation, which they then use to determine the most effective measures to counter it. A recent measles epidemic followed the equation $45t^2 - t^3$ where $y =$ the number of people infected and $t =$ time in days. a) What is the domain of this function? b) How many people are infected after 5 days? c) What is the rate of spread after 5 days? d) After how many days does the number of cases reach its maximum? e) Use the above to sketch the graph of y
Answers:
$0 \le x \le 45$, 1000 people, 375 cases per day, 30 days

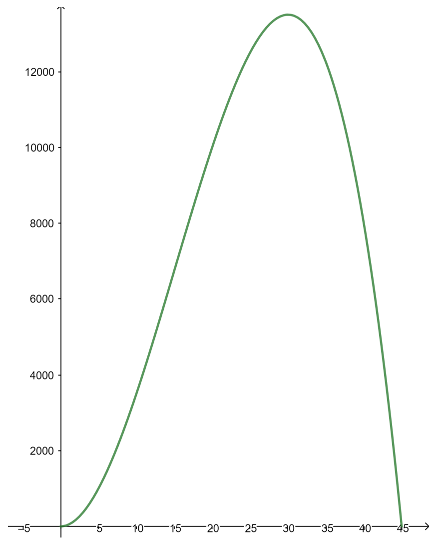

Figure 1: Example C, e)

(a) $0 \leq y = 45t^2 - t^3 = t^2(45-t) \Rightarrow 0 \leq t \leq 45$

(b) take $t=5$, $y = 5^2(45-5) = 25 \times 40 = 1000$

(c) $\frac{dy}{dt} = y' = (45t^2 - t^3)' = 90t - 3t^2 = 3t(30-t)$

take $t=5$, $y' = 90 \times 5 - 3 \times 5^2 = 450 - 75 = 375$

(d) Amounts to find the maximum, first find the extrema which is given by the zeros of equation

$y' = 0 \Leftrightarrow 3t(30-t) = 0 \Rightarrow t=0$ or $t=30$

and if $t<0$, $y'>0$

if $0<t<30$, $y'>0$

if $t>30$, $y'<0$

hence when $t=30$, y reaches its maximum

Example D:
Optimization does not always involve a maximum. The fuel, maintenance and labor costs (in dollars per mile) of operating a truck on an interstate highway are described as a function of the trucks velocity (miles per hour) by the algebraic rule $C(v) = 78 + 1.2v + 5880v^{-1}$. What speed should the driver maintain on a 600 mile haul to minimize costs?
Answers:
70mph

$$C'(v) = 1.2 - \frac{5880}{v^2}$$

$$C'(v) = 0 \Rightarrow v^2 = \frac{5880}{1.2} = \frac{5880}{\frac{6}{5}} = \frac{\overset{980}{\cancel{5880}} \times 5}{\cancel{6}} = 4900$$

$$\Rightarrow v = 70$$

[$v = -70$ is eliminated since speed v should be a non-negtive number]

Example E:

Given $f(x) = \sqrt{2x+1}(\sqrt{x}-1)$, find $\dfrac{dy}{dx}$

Answers:

$\dfrac{4x+1-2\sqrt{x}}{2\sqrt{x(2x+1)}}$

$$\frac{dy}{dx} = \left(\frac{2}{2\sqrt{2x+1}}\right)(\sqrt{x}-1) + (\sqrt{2x+1})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\sqrt{x}-1}{\sqrt{2x+1}} + \frac{\sqrt{2x+1}}{2\sqrt{x}}$$

$$= \frac{(\sqrt{x}-1)\,2\sqrt{x}}{2\sqrt{x(2x+1)}} + \frac{2x+1}{2\sqrt{x(2x+1)}}$$

$$= \frac{2x-2\sqrt{x}+2x+1}{2\sqrt{x(2x+1)}}$$

$$= \frac{4x+1-2\sqrt{x}}{2\sqrt{x(2x+1)}}$$

Example F:
Given $h(x) = \dfrac{3x+1}{x-2}$, find h'
Answers:
$\dfrac{-7}{(x-2)^2}$

You can use the quotient rule, or

$$\frac{3x+1}{x-2} = \frac{3x - 3\times 2 + 3\times 2 + 1}{x-2}$$

$$= \frac{3(x-2) + 6 + 1}{x-2}$$

$$= 3 + \frac{7}{x-2}$$

$$h' = -\frac{7}{(x-2)^2}$$

Example G:
Determine whether $\sqrt[3]{x+\sqrt{2x}}$ has any extrema, either relative or absolute
Answers:
Absolute minimum at $(0, 0)$, no maximum

First notice the domain should be $x \geq 0$

Let $h = \sqrt[3]{x+\sqrt{2x}} = \left(x+\sqrt{2x}\right)^{\frac{1}{3}}$

$$h' = \frac{1}{3}\left(x+\sqrt{2x}\right)^{-\frac{2}{3}}\left(1+\frac{2}{2\sqrt{2x}}\right)$$

$$= \frac{1}{3}\left(x+\sqrt{2x}\right)^{-\frac{2}{3}}\left(1+\frac{1}{\sqrt{2x}}\right)$$

$h' = 0 \Leftrightarrow x = 0$

and $h' > 0$ when $x > 0$

hence h has absolute minimum at $(0, 0)$
and no maximum

Example H:
Given $h(x) = e^{x^2-x}$, find the first derivative and determine the location of any relative extrema
Answers:
$x = \frac{1}{2}$

$$h' = (2x-1)e^{x^2-x}$$

$h' = 0 \Leftrightarrow x = \frac{1}{2}$ since e^x is always greater than zero

Example I:
Given $f(x)=\ln\left(x^2e^x\right)$, find the first and second derivatives.
Answers:
$\frac{2}{x}+1, -\frac{2}{x^2}$
Note that domain is not an issue. For f and both derivatives, x can be any real number except 0

$$f'(x)=\frac{2\cancel{x}\cancel{e^x}+x^{\cancel{2}}\cancel{e^x}}{x^{\cancel{2}}\cancel{e^x}}=\frac{2+x}{x}=\frac{2}{x}+1$$

$$f''(x)=-\frac{2}{x^2}$$

Example J:
The number of units a new worker can produce on an assembly line after t days on the job is given by the formula $N(t) = 40 - 40e^{-0.35t}$. This function is called a learning curve. a) How many units can the worker make when she or he first begins? b) What is the workers rate of production? c) What is the maximum number he or she can be expected to make?
Answers:
0 units, $14e^{-0.35t}$ units per day, 40 units

(a) $N(0) = 40 - 40 \times 1 = 0$

(b) $N'(t) = -40 \cdot (-0.35)\, e^{-0.35t}$

$= 14e^{-0.35t}$

(c) as $t \to +\infty$, $e^{-0.35t} \to 0$

hence the maximum expected is 40
but never quite

Example K:

Find $\int \left(3x^{-6} - 2e^{5x} + 4x^{-1} - 7\right) dx$

Answers:

$-\frac{3}{5}x^{-5} - \frac{2}{5}e^{5x} + 4\ln|x| - 7x + C$

Note that domain is an issue. For f and its integral, x can be any real number except 0.

$$\int \left(3x^{-6} - 2e^{5x} + 4x^{-1} - 7\right) dx$$

$$= \frac{3}{-5}x^{-5} + 4\ln|x| - 7x - 2\int e^{5x}\,dx$$

$$= -\frac{3}{5}x^{5} + 4\ln|x| - 7x - \frac{2}{5}\int e^{5x}\,d(5x)$$

$$= -\frac{3}{5}x^{5} + 4\ln|x| - 7x - \frac{2}{5}e^{5x} + C$$

where C is an arbitrary constant

Example L:
Find the area under the curve $y = e^x + e^{-x}$, on the interval $0 \leq x \leq \ln(8)$
Answers:
$\frac{63}{8}$

Fun fact: the graph of this function is called a catenary, matching the shape of a hanging chain

$$\int_0^{\ln 8} (e^x + e^{-x})\,dx = \int_0^{\ln 8} e^x\,dx + \int_0^{\ln 8} e^{-x}\,dx$$

$$= e^x\Big|_0^{\ln 8} + \left[-e^{-x}\right]\Big|_0^{\ln 8}$$

$$= (8-1) + \left[\left(-\frac{1}{8}\right) - (-1)\right]$$

$$= 7 + \frac{7}{8}$$

$$= \frac{63}{8}$$

$$e^{-\ln 8} = \left\{ \begin{array}{c} \frac{1}{e^{\ln 8}} \\ e^{\ln\left(\frac{1}{8}\right)} \end{array} \right\} = \frac{1}{8}$$