(DE)

Definition: a differential equation is an equation involving $t, y, y', y'', \cdots$

for example: $y''-4y'-5y=0$ (1) $\quad y' = t^4 y + 2t^4$ (2)

Definition: a solution is a function $y=f(t)$ satisfying the equation, for example,
$y=f(t)=Ce^{5t}$ (C being arbitsary constant) is a solution of equation
$y''-4y'-5y=0$ since $(Ce^{5t})''-4(Ce^{5t})'-5(Ce^{5t}) = 25Ce^{5t}-20Ce^{5t}-5Ce^{5t}=0$

Constant solutions are solutions that are constants, for example

Find constant solutions of $y' = t^4 y + 2t^4$

suppose $y=C$ is a solution, then $0 = y' = Ct^4 + 2t^4 = (C+2)t^4 \Rightarrow C+2=0$
$\Rightarrow C=-2$, hence $y=-2$ is the only constant solution

DE with initial conditions (IC)

Example: $\begin{cases} y''-4y'-5y=0 \\ y(0)=3 \end{cases}$, $\quad y=3e^{5t}$ is a solution since $y(0)=3$

however, $y=4e^{5t}$ is not even though it solves the DE because it doesn't satisfy IC since $y(0)=4$ now

Problem:

If $f(t)$ is a solution of the initial value problem $y'=3y-1$, $y(0)=3$, find $f(0)$ and $f'(0)$

Solution: Since $f(t)$ is a solution, $f'(t)=3f(t)-1$ and $f(0)=3$, take $t=0$,
$f'(0)=3f(0)-1=8$

Let $f(t)$ be the balance in a savings account at the end of t years and suppose $y=f(t)$ satisfies the differential equation $y'=-0.04y+8000$

(a) Suppose that after one year the balance is \$220000, Is the balance increasing or decreasing at that time?

Answer: we are given $f(1)=220000$, hence $f'(1)=-0.04f(1)+8000=-800<0$
hence the balance is decreasing

Remark: $f'(t)=-0.04f(t)+8000=-0.04(f(t)-200000)$

If pocess $>$ \$200000 at t years, i.e. $f(t)>200000$, then $f'(t)<0$, the balance is decreasing, if $f(t)<200000$, then $f'(t)>0$, the balance is increasing, if $f(t)=200000$, $f'(t)=0$, $f(t)$ is "stable" $\Leftrightarrow$ constant solutions

(Direction field)

Slope field: $y' = \frac{dy}{dt} = 3t - y$, suppose you know $y(t)$ is a solution, and it passes point $(0,-3)$ [meaning $(0,-3)$ is on the solution curve $y(t)$, i.e. $y(0) = -3$ equivalent to giving an IC], then the slope of the tangent to $y(t)$ at $(0,-3)$ is $y'(0) = 3\times 0 - y(0) = 3$, thus the tangent line has the form $y = 3x + b$, since it passes $(0,-3)$, $-3 = 3\times 0 + b \Rightarrow b = -3$, hence the equation of the tangent line is $y = 3x - 3$, take a small piece of it at $(0,-3)$, similarly, for each point on the yt-plane, it gives an IC, and you can compute the slope of the tangent to the solution curve at that point, for example: $(0,1)$ gives $\begin{cases} y' = 3t - y \\ y(0) = 1 \end{cases} \Rightarrow y'(0) = -1$, $(2,5)$ gives

$\begin{cases} y' = 3t - y \\ y(2) = 5 \end{cases} \Rightarrow y'(0) = 1$ and all these small pieces form the slope field

as showed in the graph below: each point (can be considered as an IC) is on a solution curve, the slope (direction) of each small piece tells you the slope (tangent) of the solution curve

the general solution is $y = 3t - 3 + Ce^{-t}$