Chapter 1

Hodge structure

Use $H_{\mathbb{F}}$ or $H(\mathbb{F})$ to indicate coefficients in $\mathbb{F}$

Definition 1.0.1. A *pure Hodge structure* of weight n on $H_{\mathbb{Z}}$ is a decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$. Equivalently, $H_{\mathbb{C}} = F^p \oplus \overline{F^{n+1-p}}$ by introducing the decreasing *Hodge filtration* $F^p = \bigoplus_{i\geq p} H^{i,n-i}$, then $\overline{F^q} = \bigoplus_{j\leq p} H^{j,n-j}$, $H^{p,q} = F^p \cap \overline{F^q}$, $F^p \cap \overline{F^{n+1-p}} = 0$

Example 1.0.2. X is a complex manifold, $H_{\mathbb{Z}} = H^n(X;\mathbb{Z})$, then

$$H^n(X;\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q} = \bigoplus_{p+q=n} H^p(X;\mathbb{C}) \wedge \overline{H^p(X;\mathbb{C})}$$

Example 1.0.3. *Tate structure* $\mathbb{Z}(-k)$ is of weight $2k$ given by $H_{\mathbb{Z}} = \mathbb{Z}$ with filtration $F^k = \begin{cases} H_{\mathbb{C}} = \mathbb{C} & k \leq p \\ 0 & k > p \end{cases}$

Definition 1.0.4. A *polarization* over $\mathbb{Q}$ of a Hodge structure over $\mathbb{Q}$ of weight k is a $(-1)^k$ symmetric nondegenerate flat bilinear map $\beta : \mathbb{V}_{\mathbb{Q}} \times \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}$ such that the Hermitian form $\beta_x(C_x v, \bar{w})$ on each fiber $\mathcal{V}_x$ is positive definite, here C_x is the *Weil operator*, given as the direct sum of multiplication i^{p-q} on $H_x^{p,q}$

Definition 1.0.5. A *mixed Hodge structure* on $H_{\mathbb{Z}}$ consists of an increasing *weight filtration* $W_\bullet$ on $H_{\mathbb{Q}}$ and a decreasing filtration $F^\bullet$ that are compatible, i.e.

$$F^p(\operatorname{gr}_k W)_{\mathbb{C}} = F^p\left(\frac{W_{k+1}}{W_k}\right)_{\mathbb{C}} = \frac{F^p \cap W_{k+1}(\mathbb{C})}{W_k(\mathbb{C})} = \frac{F^p \cap W_{k+1}(\mathbb{C}) + W_k(\mathbb{C})}{W_k(\mathbb{C})}$$

is a pure Hodge structure of weight k of $\operatorname{gr}_k W$

Definition 1.0.6. A *variation* of Hodge structure of weight k over $\mathbb{Q}$ and a complex manifold X is $(\mathbb{V}_{\mathbb{Q}}, \mathcal{F}^\bullet)$, $\mathbb{V}_{\mathbb{Q}}$ is a locally constant sheaf of $\mathbb{Q}$ vector spaces, $\mathcal{F}^\bullet$ is a decreasing filtration of holomorphic subbundles of the locally free sheaf $\mathcal{V} = \mathcal{O}_X \otimes \mathbb{V}_{\mathbb{Q}}$ such that

- $(\mathcal{V}_x, \mathcal{F}_x^\bullet)$ has a pure Hodge structure of weight k, i.e. $\mathcal{V}_x = \mathcal{F}^p \oplus \overline{\mathcal{F}^{k+1-p}}$
- (Griffiths transversality) $\nabla \mathcal{F}^p \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$

Definition 1.0.7. A variation of mixed Hodge structure over $\mathbb{Q}$ and a complex manifold X is $(\mathbb{V}_{\mathbb{Q}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$, $\mathcal{W}_\bullet$ is an increasing filtration of $\mathbb{V}_{\mathbb{Q}}$ by locally constant subsheaves such that

- $(\mathcal{V}_x, (\mathcal{W}_\bullet)_x, \mathcal{F}_x^\bullet)$ has a mixed Hodge structure, i.e. $()$ is a pure Hodge structure of weight k
- $\nabla \mathcal{F}^p \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{F}^{p-1}$

Remark 1.0.8. Given a locally constant sheaf is equivalent to given a monodromy representation $\rho_{\mathbf{x}} : \pi_1(X, \mathbf{x}) \to \mathtt{Aut}_{\mathbb{Q}}(\mathcal{V}_x)$. A variation is *unipotent* if the the monodromy representation is unipotent

Deligne's theorem on unipotent VMHS

Theorem 1.0.9 (Deligne). $\tilde{X}$ is a normalization of X, $(\mathbb{V}_{\mathbb{Q}}, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$ is a unipotent variation of mixed Hodge structure of weight k, then there is a unique extension $\tilde{\mathcal{V}}$ over $\tilde{X}$ such that

- Inside every section of $\tilde{\mathcal{V}}$, flat sections increase at most at the rate of $O(\log(\|x\|^k))$ on each compact set of $\tilde{X} - X$
- Every flat section of $\mathcal{V}^{\vee}$ increases at most at the rate of $O(\log(\|x\|^k))$

These conditions are equivalent to

- In a local basis of $\tilde{\mathcal{V}}$, the connection matrix $\boldsymbol{\omega}$ of $\mathcal{V}$ has at most logarithmic singularities along $\tilde{X} - X$
- The residue of $\boldsymbol{\omega}$ along any irreducible component of $\tilde{X} - X$ is nilpotent

Chapter 2

Plucker embedding

Definition 2.0.1. Consider Grassmannian $W \in Gr_k(n)$, the *Plücker coordinates* $W_{i_1,\cdots,i_k}$ to be the minor of $i_1, \cdots, i_k$-th columns. For $1 \le i_1 < \cdots < i_{k-1} \le n$, $1 \le j_1 < \cdots < j_k \le n$, $r \le k$, the **Plücker relations** is

$$W_{i_1,\cdots,i_k} W_{j_1,\cdots,j_k} = \sum W_{i'_1,\cdots,i'_k} W_{j'_1,\cdots,j'_k}$$

The summation is over all swaps of a size r order set of $\{i_1, \cdots, i_k\}$ with $w_1, \cdots, w_r$, respectively

Proof. If $r = k$, it is trivial. So we may assume $r < k$. For $v_1, \cdots, v_k, w_1, \cdots, w_k \in \mathbb{C}^k$, consider multilinear function

$$f(v_1, \cdots, v_k, w_1, \cdots, w_k) = |v_1 \cdots v_k||w_1 \cdots w_k| - \sum |v'_1 \cdots v'_k||w'_1 \cdots w'_k| = \text{LHS} - \text{RHS}$$

Let's first show that f is skew-symmetric, it is suffices to prove if $v_i = v_{i+1}$ or $v_k = w_k$, then $f = 0$

(i) If $v_i = v_{i+1}$, LHS $= 0$, RHS consists of terms $|\cdots v_i \cdots||\cdots v_{i+1} \cdots|$ or $|\cdots v_{i+1} \cdots||\cdots v_i \cdots|$, and each pair will cancel out in summation

(ii) If $v_k = w_k$, through a linear transformation, $v_k = w_k$ can be taken to be $(0, \cdots, 0, 1)^T$, and then it reduces to a lower case

Since w_k, v_k can be move to any column up to a sign, we know f is indeed skew-symmetric $\square$

Example 2.0.2. Consider $\text{Gr}_2(4)$, the only Plücker relation is

$$W_{12}W_{34} - W_{13}W_{24} + W_{14}W_{23} = 0$$

Theorem 2.0.3. The *Plücker embedding* is

$$\begin{aligned} \text{Gr}_k(n) &\to \mathbb{P}(\textstyle\bigwedge^k \mathbb{C}) \\ \text{Span}(v_1, \cdots, v_k) &\mapsto [v_1 \wedge \cdots \wedge v_k] \end{aligned}$$

The image is an irreducible projective algebraic variety defined exactly by Plücker relations on Plücker coordinates

Chapter 3

Graph theory

Definition 3.0.1. A **graph** is

Chapter 4

Moduli space

Consider a parametrized curve $C = \{(t, \mathbf{x}(t))\}_{t\in I}$, $\mathbf{x}(t) \in \mathbb{R}^n$, now we change I to some space X, $\mathbf{x}(t)$ to some algebro-geometric objects, then we have a parametrization of these objects by X

Definition 4.0.1. U is a family of some algebro-geometric objects. A parametrization of U by space X is a map $X \to U$, attaching some object U_x for each $x \in X$, we can also think of this map as a section of $X \times U \to X$
We say X is the parametrization space, U is parametrized over X
A moduli functor F is a contravariant functor $\mathit{Space} \to \mathit{Set}$ that takes a space X to the set of families of objects over X, and take a morphism f to the pullback f^* that taking section s to pullback section $f^*s(y) = (y, \mathrm{Pr}_U\, sf(y))$

$$\begin{tikzcd} Y \times U \arrow[r] \arrow[d] & X \times U \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

The category of spaces can be the category of schemes, manifolds, topological spaces, etc.
M is a fine moduli space if F is corepresentable by M, i.e. there is a natural isomorphism $\tau : F \to \mathtt{Hom}(-, M)$. There is a universal family over M corresponds to $1_M \in \mathtt{Hom}(M, M)$. Then any family over X is the pullback along some $X \xrightarrow{f} M$ of the universal family. The universal family is essentially unique and "tautological"
M is a coarse moduli space if there is exists a natural transformation $\tau : F \to \mathtt{Hom}(-, M)$ and universal among these natural transformations, i.e. for any natural transformation $\tau' : F \to \mathtt{Hom}(-, M')$, there is a morphism $M' \xrightarrow{\phi} M$ such that the following diagram commutes

$$\begin{tikzcd} & F \arrow[dl, "\tau'"'] \arrow[d, "\tau"] \\ \mathtt{Hom}(-, M') \arrow[r, "\exists_1 \phi"] & \mathtt{Hom}(-, M) \end{tikzcd}$$

Chapter 5

Teichmüller space

Definition 5.0.1. S is a orientable surface. A marked Riemann surface is a pair (X, f) where X is a Riemann surface and $S \xrightarrow{f} X$ is a isomorphism, i.e. giving X a complex structure. (X, f), (Y, g) are equivalent if $gf^{-1} : X \to Y$ is isotopic to an isomorphism, i.e. X, Y has isotopic complex structures. The Teichmüller space $T(S)$ of S is the the equivalence classes of marked Riemann surfaces. The mapping class group acts on $T(S)$ by $h \cdot (X, f) = (X, fh^{-1})$, then $T(S)$ mod the action is just S

Example 5.0.2. By Uniformization theorem **??**, $T(\mathbb{S}^2)$ is a point corresponds to the Riemann sphere, $T(\mathbb{R}^2)$ is two points corresponds the complex plane and the unit disc. $T(A) = [0, 1)$, where A is the open annulus, and $\lambda \in [0, 1)$ corresponds to $\{\lambda < |z| < \lambda^{-1}\}$ according to Exercise **??**

Chapter 6

Mapping class group

Definition 6.0.1. Suppose $\mathtt{Aut}(X)$ has a natural topology, the mapping class group is $\mathtt{Aut}(X)/\mathtt{Aut}_0(X)$, where $\mathtt{Aut}_0(X)$ is the path connected component of the identity, hence we have exact sequence

$$0 \to \mathtt{Aut}_0(X) \to \mathtt{Aut}(X) \to \mathrm{MCG}(X) \to 0$$

If X is a space, then a path connecting $f, g \in \mathtt{Aut}(X)$ is an isotopy

Example 6.0.2. $\mathrm{MCG}(S^2) = \mathbb{Z}/2\mathbb{Z}$

Chapter 7

Weil conjecture

Definition 7.0.1. X is a non-singular n dimensional projective algebraic variety over F_q, the *zeta function* is

$$\zeta(X, s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-ms}\right)$$

Where N_m are the number of points of X over F_{q_m}. The *Weil conjectures* are

1. Let $T = q^{-s}$

$$\zeta(X, s) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)} = \prod_{i=0}^{2n} P_i(T)^{(-1)^{i+1}}$$

Where $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^n T$, $P_i(T)$ can be split into $\prod_j (1 - \alpha_{ij} T)$ over $\mathbb{C}$. In particular, $\zeta(X, s)$ is a rational function of T

2.

$$\zeta(X, n - s) = \pm q^{\frac{nE}{2} - Es} \zeta(X, s)$$

Or equivalently

$$\zeta(X, q^{-n}T^{-1}) = \pm q^{\frac{nE}{2}} T^E \zeta(X, T)$$

E is the Euler characteristic. $\{\alpha_{2n-i,1}, \alpha_{2n-i,2}, \cdots\}$ coincide with $\left\{\frac{q^n}{\alpha_{i,1}}, \frac{q^n}{\alpha_{i,2}}, \cdots\right\}$ in some order

3. $|\alpha_{i,j}| = q^{i/2}$

4.

Example 7.0.2. If X is the n dimensional projective space, $N_m = 1 + q^m + \cdots + q^{nm}$, $\zeta(\mathbb{P}^n, s) = \frac{1}{(1 - q^{-s}) \cdots (1 - q^{n-s})}$

Chapter 8

Elliptic curves

Consider ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the circumference is

$$4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}dx = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2 \theta}d\theta$$

Definition 8.0.1. The elliptic integral of the *first* kind is

$$\int_0^{\varphi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Let $t = \sin\theta$, $x = \sin\varphi$, we have

$$\int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}$$

The elliptic integral of the *second* kind is

$$\int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \theta}d\theta$$

Let $t = \sin\theta$, $x = \sin\varphi$, we have

$$\int_0^x \frac{\sqrt{1 - k^2t^2}}{\sqrt{1 - t^2}}dt$$

The elliptic integral of the *third* kind is

$$\int_0^{\varphi} \frac{d\theta}{(1 - n\sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

Let $t = \sin\theta$, $x = \sin\varphi$, we have

$$\int_0^x \frac{dt}{(1 - nt^2)\sqrt{(1 - t^2)(1 - k^2t^2)}}$$

These elliptic integrals are called *incomplete*, they are *complete* if $\varphi = \frac{\pi}{2}$

Legendre's relation

Theorem 8.0.2 (Legendre's relation)**.** For $k^2 + k'^2 = 1$, E, E' are corresponding complete elliptic integrals of the second kind, K, K' are corresponding complete elliptic integrals of the first kind, then they satisfy the *Legendre's relation*

$$KE' + K'E - KK' = \frac{\pi}{2}$$

Equivalently

$$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$$

ω_1, ω_2 are the periods of Weierstrass $\wp$ function, η_1, η_2 are the quasiperiods of Weierstrass zeta function

Definition 8.0.3. An *elliptic integral* is of the form

$$\int_c^x R\left(x, \sqrt{P(x)}\right) dx$$

Here $R(x, w)$ is a rational function of x, w and $P(x)$ is a polynomial of degree 3 or 4. Every elliptic integral can be reduced into elliptic integrals of the first, second and third kinds

Definition 8.0.4. An *abelian integral* is of the form

$$\int_{z_0}^z R(x, w)dx$$

R is a rational function of x, w, and $F(x, w) = 0$ for some

$$\varphi_n(x)w^n + \cdots + \varphi_0(x) = 0$$

$\varphi_i(x)$ are rational functions of x. It is called a *hyperelliptic integral* if $F(x, w) = w^2 - P(x)$ for some polynomial P, note that if degree of P is 3 or 4 than it is an elliptic integral

Definition 8.0.5. C is a compact algebraic curve of genus g, $H^0(X, K) = \mathbb{C}^g$ is generated by one forms $\omega_1, \cdots, \omega_g$, K is a canonical bundle, the *Abel-Jacobi map* is

$$\begin{aligned} J : C &\to J(C) = \mathbb{C}^g/\Lambda \\ P &\mapsto \left(\int_{P_0}^P \omega_1, \cdots, \int_{P_0}^P \omega_g\right) \bmod \Lambda \end{aligned}$$

Theorem 8.0.6 (Abel-Jacobi theorem)**.** Abel-Jacobi map J is an isomorphism

Chapter 9

Logarithmic form

Definition 9.0.1. D is a simple normal crossing divisor of X, $Y = X - D$, $Y \overset{j}{\hookrightarrow} X$ is the embedding, the *log de Rham commplex* of X along D is $\Omega_X^*(\log D)$, which is the smallest chain complex of $j_*\Omega_Y^*$ closed under wedge product such that for any $f \in j_*\mathcal{O}_X^*(U)$ meromorphic along D, $\frac{df}{f} \in \Omega_X^*(\log D)(U)$. A section of $j_*\Omega_Y^*$ has *logarithmic poles* if it is a section of $\Omega_X^*(\log D)$

Proposition 9.0.2.

1. Section ω of $j_*\Omega_Y^*$ has logarithmic poles along D iff both $\omega, d\omega$ have at most simple poles along D
2. $\Omega_X^1(\log D)$ is locally free and $\Omega_X^p(\log D) = \bigwedge^p \Omega_X^1(\log D)$
3. For $(X, D) = (X_1, D_1) \times (X_2, D_2) = (X_1 \times X_2, X_1 \times D_2 \cup X_2 \times D_1)$, isomorphism $\Omega_{Y_1}^* \boxtimes \Omega_{Y_2}^* \to \operatorname{pr}_{X_1}^* \Omega_{X_1}^* \otimes \operatorname{pr}_{X_2}^* \Omega_{X_2}^*$ induces isomorphism $\Omega_{X_1}^*(\log D_1) \boxtimes \Omega_{X_2}^*(\log D_2) \to \Omega_X^*(\log D)$
4. For $f : X_1 \to X_2$, $f^{-1}(D_2) = D_1$, $f^* : j_{2*}\Omega_{Y_2}^* \to j_{1*}\Omega_{Y_1}^*$ induces $f^* : \Omega_{X_2}^*(\log D_2) \to \Omega_{X_1}^*(\log D_1)$

Lemma 9.0.3. $X = D^n$, $D = \bigcup_{1 \le i \le k} D_i$ with $D_i = \operatorname{pr}_i^{-1}(0)$, $Y = D^{*k} \cup D^{n-k}$. Then $\Omega_X^1(\log D)$ is a free sheaf with base $\left\{ \frac{dz_i}{z_i} \right\}_{1 \le i \le k}$ and $\{dz_i\}_{k \le i \le n}$. In fact, any section of $j_*\mathcal{O}_Y^*$ meromorphic along D can be written locally as $f = g \prod_{i=1}^{k} z_i^{n_i}$, then

$$\frac{df}{f} = \frac{dg}{g} + \sum_{i=1}^{k} \frac{n_i}{z_i} dz_i$$

Chapter 10

Jet

Definition 10.0.1. A *jet*

Chapter 11

Algebraic K theory

Definition 11.0.1. The Grothendieck group of R is $K_0(R)$, the Grothendieck group of monoid of finitely generated projective modules over R

Swan's theorem

Theorem 11.0.2 (Swan's theorem)**.** X is a compact Hausdorff space, $K(X) = K_0(C(X, \mathbb{R}))$

Proof. If $E \to X$ is a vector bundle, then it is the direct summand of some trivial vector bundle. Conversely, if P is a finitely generated module over $R = C(X, \mathbb{R})$, then P is image of some idempotent endomorphism of R^n which is a vector bundle $\square$

Definition 11.0.3 (Whitehead group)**.** The **Whitehead group** of ring R is an abelian group $K_1(R)$ satisfying universal property

$$\begin{tikzcd}
GL(R) \arrow[d, "\pi"'] \arrow[dr] & \\
K_1(R) \arrow[r, dashed, "\exists_1"'] & A
\end{tikzcd}$$

For any abelian group A

Construction 11.0.4. Thanks to Whitehead's lemma **??**, $K_1(R) = GL(R)/[GL(R), GL(R)] = GL(R)/E(R)$

Definition 11.0.5. If R is commutative, $SL(R)$ is the kernel of $GL(R) \xrightarrow{\det} R^\times$, the special Whitehead group $SK_1(R) = SL(R)/E(R)$ is the kernel of $K_1(R) \xrightarrow{\det} R^\times$, $GL(R) \cong SL(R) \rtimes R^\times$, $K_1(R) \cong SK_1(R) \oplus R^\times$. $K_1(F) = F^\times$

Lemma 11.0.6. Since $GL(R_1 \times R_2) = GL(R_1) \times GL(R_2)$, $K_1(R_1 \times R_2) = K_1(R_1) \oplus K_1(R_2)$

Chapter 12

Thinking shortcut

Remark 12.0.1.

$$\begin{aligned}
a^k + \cdots + a^l &= (a^k + \cdots) - (a^{l+1} + \cdots) \\
&= \frac{a^k}{1-a} - \frac{a^{l+1}}{1-a} \\
&= \frac{a^k - a^{l+1}}{1-a}
\end{aligned}$$