MATH868C - Several Complex Variables

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1 Subharmonic functions

Definition 1.1. $\Omega \subseteq \mathbb{C}$ is an open set, $h \in C^2(\Omega)$ is *harmonic* if $\Delta h = \dfrac{4\partial^2}{\partial z \partial \bar{z}} h = 0$, denote the set of harmonic functions $H(\Omega)$

Definition 1.2. $u : \Omega \to [-\infty, +\infty)$ is subharmonic, denoted $u \in SH(\Omega)$ if

- u is upper semi-continuous, i.e. $\{u < r\}$ is open
- For any compact $K \subseteq \Omega$, and $h \in H(\operatorname{Int} K) \cap C(K)$ such that $u \leq h$ on ∂K, then $u \leq h$ on K

Theorem 1.3. $\{u_j\} \subseteq SH(\Omega)$, $v = \sup\limits_j u_j$. If v is upper semi-continuous, then $v \in SH(\Omega)$, $u = \inf\limits_j u_j$ is upper semi-continuous generally doesn't imply $u \in SH(\Omega)$, but if $\{u_j\}$ is decreasing, then $u \in SH(\Omega)$

Theorem 1.4. $u : \Omega \to [-\infty, +\infty)$ is upper semi-continuous. The following are equivalent

1. $u \in SH(\Omega)$
2. For any $\overline{D} \subseteq \Omega$, and any polynomial $f(z)$, if $u \leq \operatorname{Re} f$ on ∂D, then $u \leq \operatorname{Re} f$
3. $\Omega_\delta = \{z \in \Omega | \operatorname{dist}(z, \partial\Omega) > \delta\} \subseteq \Omega$, for $z \in \Omega_\delta$
$$2\pi u(z) \int_0^\delta d\mu(r) \leq \int_0^\delta \int_0^{2\pi} u(z + re^{i\theta}) d\theta d\mu(r)$$
here $d\mu$ is any measure on $[0, \delta]$, take $d\mu(r) = rdr$, the average of the disk, take $d\mu(r)$ to be Dirac measure, the average of the circle

Proof. 1. $\Rightarrow$ 2. is by definition. 2. $\Rightarrow$ 3.

- If $p(z) = \sum_{j=0}^k a_j z^j$, then $2\pi \operatorname{Re} p(z) \displaystyle\int_0^\delta d\mu(r) = \int_0^\delta \int_0^{2\pi} \operatorname{Re} p(z + re^{i\theta} d\theta d\mu(r))$
- $\varphi \in C(\partial D(z, r))$, $r \in [0, \delta]$ such that $u \leq \varphi$ on $\partial D(z, r)$. Fourier: $\exists p_k = \sum_{j=0}^l a_j^k z^j$ such that $\varphi \leq \operatorname{Re} p_k \leq \varphi + \frac{1}{k}$ (Rudin). $u \leq \operatorname{Re} p_k$ on $\partial D(z, r)$, by 2. $u(z) \leq \operatorname{Re} p_k(z)$, then $2\pi u(z) \leq 2\pi \operatorname{Re} p_k(z) = \int_0^{2\pi} \operatorname{Re} p_k(z + re^{i\theta}) d\theta \to \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta$ as $k \to \infty$
- $u : X \to [-\infty, \infty)$ is upper semi-continuous and bounded above, $\{f_j\} \subseteq C(X)$ such that $f_j \searrow u$, then there exists $\{\varphi_j\} \subseteq C(\partial D(z, r))$ such that $\varphi_j \searrow u$ on $\partial D(z, r)$, then $2\pi u(z) \leq \int_0^{2\pi} \varphi_j(z + re^{i\theta}) d\theta \to \int_0^{2\pi} u(z + re^{i\theta}) d\theta$, integrate this over $[0, \delta]$ of $d\mu$

3. $\Rightarrow$ 1. Assume 1. doesn't hold, $\exists K \subseteq \Omega$ compact, $h \in C(K) \cap H(\operatorname{Int} K)$ such that $u \leq h$ on ∂K but $u(z) > h(z)$ for some $z \in K$, define $F = \{z \in K | u(z) = \max_K (u - h)\} \neq \varnothing$ and closed, compact, thus $\exists x \in F$ such that $\operatorname{dist}(x, \partial K)$ is a minimizer. For some r, an open part of $\partial D(z, r)$ lies outside F, $\int_0^{2\pi} (u - h)(x + re^{i\theta}) d\theta < (u - h)(x)$ which is a contradiction ☐

Corollary 1.5. $f \in \mathcal{O}(\Omega) \Rightarrow \log|f| \in SH(\Omega)$, if $f = 0$, $\log|f| = -\infty$

Proof. $\overline{D} \subseteq \Omega$, $p = \sum_{j=0}^k a_j z^j$, if $\log|f| \leq \operatorname{Re} p$ on ∂D, then $|f| \leq e^{\operatorname{Re} p} \Leftrightarrow |f| \leq |e^p|$ on $\partial D \Rightarrow |\frac{f}{e^p}| \leq 1$ on $\partial D \Rightarrow \cdots$ ☐

Corollary 1.6. $\varphi : \mathbb{R} \to \mathbb{R}$ is convex and increasing, $u \in SH(\Omega)$, then $\varphi \circ u \in SH(\Omega)$

Proof. Sub-mean value inequality + Jensen inequality ☐

Theorem 1.7. $u \in SH(\Omega)$ $u \not\equiv -\infty$ on a component of Ω, then $u \in L^1_{\text{loc}}(\Omega)$. $\Delta u \geq 0$ as a distribution, i.e. $\int_\Omega u \Delta v \geq 0$ $\forall v \in C_0^2(\Omega), v \geq 0$

Proof. $r = \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$, $x \in \Omega_r$, $2\pi u(x) \leq \int_0^{2\pi} u(x + \delta e^{i\theta}) d\theta$, $\delta \in [0, r]$. $2\pi \int_\Omega u(x)v(x)d\lambda \leq \int_\Omega \int_0^{2\pi} u(x + \delta e^{i\theta})v(x)d\theta d\lambda(x) \Rightarrow 0 \leq \int_\Omega u(x) \int_0^{2\pi} u(x + \delta e^{i\theta}) \frac{v(x-\delta e i\theta)-v(x)}{\delta^2} d\theta d\lambda$ as $\delta \to 0$, $0 \leq \int_\Omega u(x) 2\pi \Delta v(x) d\lambda$ □

Theorem 1.8 (Implicit function theorem)**.** $(w, z) = (w_1, \cdots, w_m, z_1, \cdots, z_n)$, $f_j(w, z)$ are analytic in a neighborhood of $(w^0, z^0) \in \mathbb{C}^{m+n}$, suppose $f_j(w^0, z^0) = 0$, $\det(\frac{\partial f_j}{\partial w_k}) \neq 0$ at (w^0, z^0), then $\exists w(z)$ analytic in a neighborhood of z^0 with $w(z^0) = w^0$, $F(w(z), z) = 0$

2 Cauchy's formula

Definition 2.1. $D = D(x_1, r_1) \times \cdots \times D(x_n, r_n)$ is called a *polydisc*. $\partial_0 D = \partial D_1 \times \cdots \times \partial D_n \subsetneq \partial D$ is the *distinguished boundary* of D

Theorem 2.2. $u \in C(\overline{D}) \cap \mathcal{O}(D)$, then

$$u(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\xi_1, \cdots, \xi_n)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} d\xi_1 \cdots d\xi_n, \forall z \in D$$

Proof.

$$u(z_1, \cdots, z_{n-1}, z) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{u(z_1, \cdots, z_{n-1}, \xi)}{\xi_n - z_n} d\xi$$

$z_{n-1} \mapsto u(z_1, \cdots, z_{n-2}, z_{n-1}, z_n^k) \Rightarrow z_{n-1} \mapsto u(z_1, \cdots, z_{n-2}, z_{n-1}, \xi_n)$ is uniform convergence $\square$

Theorem 2.3. $K \subseteq K$ is compact, $\exists C_{K,\alpha} > 0$ such that

$$\sup_K |\partial^\alpha u| \leq C_{K,\alpha} \sup_K |u|$$

Proof. If $K = D_1 \times \cdots \times D_n$, in general, cover K with a finite number of polydisks $\square$

Corollary 2.4. $\{u_k\} \subseteq \mathcal{O}(\Omega)$

1. (Montel) $\{u_k\}$ uniformly bounded on every compact $K \subseteq \Omega$, then $\exists u \in \mathcal{O}(\Omega)$, $k_j \in \mathbb{N}$ such that $u_{k_j} \Rightarrow u$ uniformly on compact subsets
2. If $u_j \Rightarrow u$, then $u \in \mathcal{O}(\Omega)$

Proof.

1. $\{\partial^\alpha u_j\}$ are equicontinuous(Arzela-Ascoli), $\{\partial^\alpha u_j\}$ is relatively compact w.r.t. uniform convergence. To finish, exhaust Ω by compact subsets, and take a diagonal process to assure relative compactness for all partial derivatives, Cauchy-Riemann conditions is satisfied for the limit

$\square$

Theorem 2.5 (Cauchy's estimates)**.** If $|u(z)| \leq M$ on D, $|\partial^j u(0)| \leq M j_1! \cdots j_m! \frac{1}{r_1^{j_1}} \cdots \frac{1}{r_m^{j_m}}$

Theorem 2.6 (Hartogs' theorem)**.** $f : \Omega \to \mathbb{C}^n$, f is holomorphic in every variable separately, then $f \in \mathcal{O}(\Omega)$

Example 2.7. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} \\ 0 \end{cases}$, this can't be a counterexample in complex variables since $z_1^2 + z_2^2 = 0$ at points other than $(0, 0)$

Theorem 2.8 (Cauchy-Pompeiu formula)**.** $f \in C^1(\overline{U})$, $\int_{\partial U} f dz = \int_U d(f dz) = \int_U \bar{\partial} f \wedge dz = \int_U \frac{\partial f}{\partial \bar{z}}$

Theorem 2.9. $f \in C_0^\infty(\mathbb{C})$, $\frac{\partial u}{\partial \bar{z}} = f$ always has a solution $u \in C^\infty(\mathbb{C})$

Proof.

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}$$

$\square$

Theorem 2.10. $f = \sum f_j d\bar{z}_j$, $f_j \in C_0^\infty(\mathbb{C}^n)$, $\bar{\partial} f = 0$, then there exists unique $u \in C_0^\infty(\mathbb{C}^n)$ such that $\bar{\partial} u = f$

Proof.

$$u(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\tau - z_1} f_1(\tau, z-2, \cdots, z_n) d\tau \wedge d\bar{\tau}$$

$$\bar{\partial} f = 0 \Leftrightarrow \frac{\partial f_j}{\partial \bar{z}_k} = \frac{\partial f_k}{\partial \bar{z}_j}$$

Need to show $\frac{\partial u}{\partial \bar{z}_k} = f_k$. $k = 1$, Cauchy-Pompeiu implies $\frac{\partial u}{\partial \bar{z}_1} f_1(z_1, \cdots, z_n)$, $k > 1$, $\frac{\partial u}{\partial \bar{z}_k} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\tau} \frac{\partial f_k}{\partial \bar{z}_1}(z_1 - \tau, z_2, \cdots, z_n) d\tau \wedge d\bar{\tau} = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{\tau - z_1} \frac{\partial f_k}{\partial \bar{z}_1}(z_1 - \tau, z_2, \cdots, z_n) d\tau \wedge d\bar{\tau}$ Why is u compactly supported $\square$

3 Homeworks

3.1 Homework1

Problem 3.1. Let (X, d) be a metric space

1. Let $f_j : X \to [-\infty, \infty)$ be a decreasing sequence of upper semi-continuous functions. Show that $f_j := \lim f_j$ is also upper semi-continuous
2. Let $f : X \to [-\infty, \infty)$ be an upper semi-continuous function such that $f(x) \le M \in \mathbb{R}$ for all $x \in X$. Show that there exist a decreasing sequence of continuous functions such that $f_j \searrow f$ pointwise everywhere on X. [Hint: Show that the functions $f_j(x) := \sup_{y \in X}(f(y) - jd(y, x))$ satisfy the requirements]

Solution.

1. The infinimum of a family of upper semicontinuous functions is again upper semicontinuous
2. Consider $f_n(x) = \sup_{y \in X}(f(y) - nd(y, x))$ which surely is monotone decreasing, for any fixed x, it is obvious $f(x) \le f_n(x)$, suppose $\lim_{n\to\infty} f_n(x) > f(x)$, then $\exists y_n, f(y_n) - nd(y_n, x) - f(x) > \eta$ for some $\eta > 0$, hence $d(y_n, x) < \dfrac{f(y_n) - f(x) - \eta}{n} \le \dfrac{M - f(x) - \eta}{n}$, thus $\lim_{n\to\infty} y_n = x$, since f is semicontinuous, $\exists \delta > 0$, such that$f(y) < f(x) + \eta, \forall y \in B(x, \delta)$, thus $\exists N$, such that $f(y_n) < f(x) + \eta, \forall n > N$, but then $\eta > f(y_n) - f(x) \ge f(y_n) - nd(y_n, x) - f(x) > \eta$ which is a contradiction. Therefore, $\lim_{n\to\infty} f_n(x) = f(x)$. Next we will prove that f_n is indeed continuous, since $f_n(x)$ could be seen as the supremum of a family of continuous functions in x over the family $\{f(y) - nd(y, x)\}_{y\in X}$, it is lower semicontinuous. To show that f_n is also upper semicontinuous, we only need to show that, $\forall x \in \{f_n < a\}, \exists \delta > 0$, such that $B(x, \delta) \in \{f_n < a\}$. we have $f(z) - nd(z, y) \le f(z) - nd(x, z) + nd(y, x) \le f_n(x) + nd(y, x) < a \Rightarrow f_n(y) < a$, as long as δ is small enough

$\square$

Problem 3.2. Let $\Omega \subset \mathbb{R}^n$ and $f \in C^2(\Omega)$ a real valued. If $x \in \Omega$ show that

$$\lim_{r\to 0} \frac{\int_{\mathbb{S}^{n-1}} f(x + r\xi)d\xi - f(x)}{r^2\mu(\mathbb{S}^{n-1})} = \frac{1}{n}\Delta f(x) := \frac{1}{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_i} f(x),$$

where $d\xi$ is the surface measure of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, and $\mu(\mathbb{S}^{n-1})$ is the surface area of the unit sphere. [Hint: Use Taylor's formula and some linear algebra wisdom. Also, it was pointed out to me that the constant $\frac{1}{n}$ may need to adjusted in front of $\Delta f(x)$ on the right hand side. I leave it up to you to find the correct constant, which your precise calculations should naturally yield]

Solution.

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} f(x + r\xi)\mathrm{d}\xi - \mu(\mathbb{S}^{n-1})f(x) &= \int_{\mathbb{S}^{n-1}} (f(x + r\xi) - f(x)) \\
&= \int_{\mathbb{S}^{n-1}} \left(r\xi^T Df(x) + \frac{r^2}{2}\xi^T D^2 f(\eta)\xi\right) \\
&= \int_{\mathbb{S}^{n-1}} \frac{r^2}{2}\xi^T D^2 f(\eta)\xi
\end{aligned}$$

Where $\eta = x + \theta r\xi, 0 < \theta < 1$ depends on $r\xi$. Then we have

$$
\begin{aligned}
\lim_{r\to 0}\frac{\int_{\mathbb{S}^{n-1}} f(x+r\xi)\mathrm{d}\xi-\mu(\mathbb{S}^{n-1})f(x)}{r^2\mu(\mathbb{S}^{n-1})} &= \frac{1}{2\mu(\mathbb{S}^{n-1})}\lim_{r\to 0}\int_{\mathbb{S}^{n-1}}\xi^T D^2 f(\eta)\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\xi^T D^2 f(x)\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\xi^T P^T \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n^2}(x)\end{pmatrix} P\xi \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\zeta^T \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & & \\ & \ddots & \\ & & \frac{\partial^2 f}{\partial x_n^2}(x)\end{pmatrix} \zeta \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}}\frac{\partial^2 f}{\partial x_1^2}(x)\zeta_1^2+\cdots+\frac{\partial^2 f}{\partial x_n^2}(x)\zeta_n^2 \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} V\cdot\zeta \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{B}^{n}} \mathrm{div}V \\
&= \frac{1}{2\mu(\mathbb{S}^{n-1})}\int_{\mathbb{B}^{n}} \Delta f(x) \\
&= \frac{1}{2n}\Delta f(x)
\end{aligned}
$$

Where $P\in O(n)$, $\zeta := P\xi$, $V=\left(\frac{\partial^2 f}{\partial x_1^2}(x)\zeta_1,\cdots,\frac{\partial^2 f}{\partial x_n^2}(x)\zeta_n\right)^T$ □

3.2 Homework2

Problem 3.3 (Unique analytic continuation). Let $\Omega \subset \mathbb{C}^n$ open and connected, and $f, g \in A(\Omega)$. If $f = g$ on an open subset of Ω show that $f = g$ everywhere on Ω

Problem 3.4. Show that the function $u \in C_0^k(\mathbb{C}^n)$ constructed in Theorem 2.3.1 is unique

3.3 Homework3

Problem 3.5. Let $\Omega \subset \mathbb{C}^n$ be open and let P be a polydisk, whose closure is contained in Ω. Show that $\widehat{\partial_0 P}_\Omega = \overline{P}$, where $\partial_0 P$ is the distinguished boundary of P

Problem 3.6. Argue precisely why the function f constructed in the proof of Theorem 2.5.5 can not be identically zero!

Problem 3.7. $\mathbb{C}^n$ can be viewed as a $2n$-dimensional real vector space and an n-dimensional complex vector space. Show that any $\mathbb{R}$-linear functional on $\mathbb{C}^n$ is the real part of a $\mathbb{C}$-linear functional on $\mathbb{C}^n$

3.4 Homework4

Problem 3.8. Let $\lambda := (\lambda_1, \ldots, \lambda_j)$, $z := (z_1, \ldots, z_j)$ and $\xi := (\xi_1, \ldots, \xi_j)$ be as in the proof of Corollary 2.5.8. Show that

$$\sum_{i=1}^{j} \lambda_i \log|z_i| \leq \sup_{\xi \in k} \sum_{i=1}^{j} \lambda_j \log|\xi_i|, \ \forall \lambda \in \mathbb{R}_+^n \quad \text{with} \ \ \lambda_1 + \ldots + \lambda_j = 1$$

is equivalent with $(\log|z_1|, \ldots, \log|z_j|)$ being in the convex hull of the set of all points $(\eta_1, \ldots, \eta_j)$ such that $\eta_i \leq \log|\xi_i|$ for $1 \leq i \leq j$. [Hint: One direction is easy. For show that being in the convex hull implies the inequality use the fact that a closed convex set is always the intersection of half spaces]

Problem 3.9. Let δ as defined by Hörmander on page 37. Show that $z \to \delta(z, \Omega^c)$ is a continuous on $\mathbb{C}^n$, where Ω is an open subset of $\mathbb{C}^n$

3.5 Homework5

Problem 3.10. In Theorem 1.1 of Chapter VIII.1 (Demailly's textbook): argue carefully that $T^{**} = T$ and $Ker\ T^{\perp} = \overline{Im\ T^{*}}$

3.6 Homework6

Problem 3.11. Given a Hermitian metric $h := \sum_{j,k} h_{j,\bar{k}} dz_j \wedge \overline{dz_k}$ on a complex manifold Ω, show that it is possible to define a Hermitian metric on the vector bundle of $(p, \bar{q})$-forms on Ω

References

[1] *An Introduction to Complex Analysis in Several Variables* - Lars Hörmander

[2] *Complex Analytic and Differential Geometry* - Jean-Pierre Demailly

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