# 11.4 Series with positive terms

November 4, 2019

**Remark:**

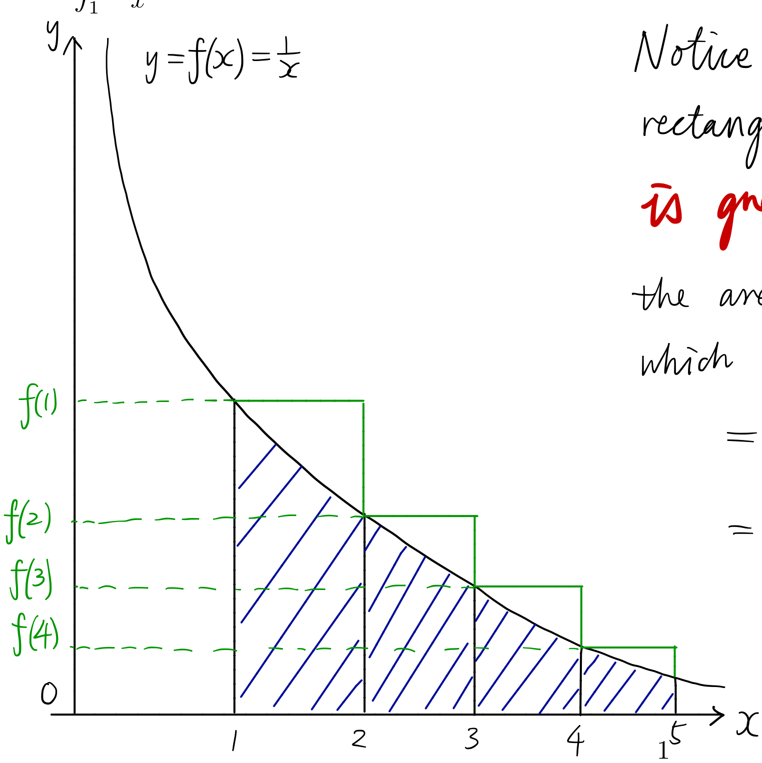
$$\sum_{k=1}^{\infty} a_k \text{ is convergent/ divergent} \Leftrightarrow \sum_{k=N}^{\infty} a_k \text{ is convergent/divergent}$$

$$\int_1^{\infty} f(x)dx \text{ is convergent/divergent} \Leftrightarrow \int_A^{\infty} f(x)dx \text{ is convergent/divergent}$$

**Integral test:** Let $f(x)$ be a continuous, nonincreasing, and nonnegative function for $x \geq 1$, Then the infinite series $\sum_{k=1}^{\infty} f(k)$ is convergent or divergent if the improper integral $\int_1^{\infty} f(x)dx$ is convergent or divergent

There are nice geometric explanations!

**Example:** Consider the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}$, we can find the corresponding function $f(x) = \frac{1}{x}$(check it is indeed continuous, nonincreasing and nonnegative), then notice that $\int_1^{\infty} \frac{1}{x}dx$ is divergent, hence the series is also divergent



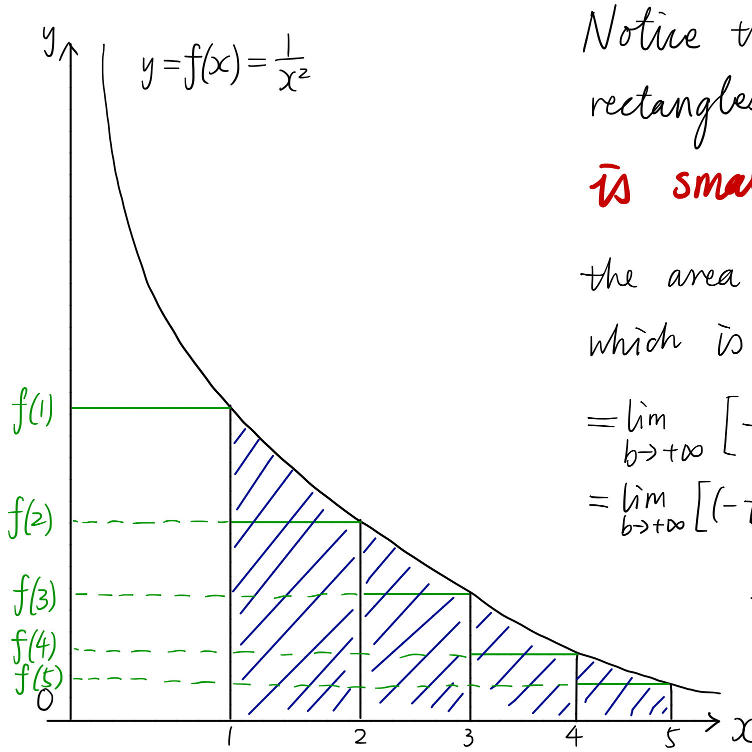

Notice the sum of the areas of rectangles is $\sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{\infty} \frac{1}{k}$ **is greater (because containing)** the area under the graph of $y=f(x)=\frac{1}{x}$ which is $\int_1^{+\infty} f(x)dx = \int_1^{+\infty} \frac{1}{x}dx$

$$= \lim_{b\to+\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to+\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to+\infty} [\ln b - \ln 1] \to +\infty \text{ is divergent}$$

Therefore $\sum_{k=1}^{\infty} \frac{1}{k}$ which is greater is divergent

**Example:** Consider the series in Basel problem $1+\frac{1}{4}+\frac{1}{9}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k^2}$, we can find the corresponding function $f(x)=\frac{1}{x^2}$(check it is indeed continuous, nonincreasing and nonnegative), then notice that $\int_1^{\infty}\frac{1}{x^2}dx$ is convergent, hence the series is also convergent




Notice the sum of the areas of rectangles is $\sum_{k=2}^{\infty} f(k)=\sum_{k=2}^{\infty}\frac{1}{k^2}$

is smaller (because being contained)

the area under the graph of $y=f(x)=\frac{1}{x^2}$

which is $\int_1^{+\infty} f(x)dx=\int_1^{+\infty}\frac{1}{x^2}dx=\lim_{b\to+\infty}\int_1^b x^{-2}dx$

$=\lim_{b\to+\infty}\left[-x^{-1}\right]\Big|_1^b=\lim_{b\to+\infty}\left[(-b^{-1})-(-1^{-1})\right]$

$=\lim_{b\to+\infty}\left[(-\frac{1}{b})-(-1)\right]=\left[(-0)-(-1)\right]=1$ is convergent

Therefore $\sum_{k=2}^{\infty}\frac{1}{k^2}$ which is smaller is convergent, so is $\sum_{k=1}^{\infty}\frac{1}{k^2}$

**Problems:** Use the integral test to determine if the following series or improper integrals are convergent

$$\sum_{k=3}^{\infty}\frac{2}{7k\sqrt{\ln k}}$$

First notice $f(x)=\frac{2}{7x\sqrt{\ln x}}$ is continuous, nonincreasing and nonnegative

$$\int_3^{\infty}\frac{2}{7x\sqrt{\ln x}}dx=\lim_{b\to+\infty}\frac{2}{7}\int_3^b\frac{1}{x\sqrt{\ln x}}dx\ \frac{u=\ln x}{du=\frac{1}{x}dx}\ \frac{2}{7}\lim_{b\to+\infty}\int_{\ln 3}^{\ln b}\frac{1}{\sqrt{u}}du$$

$$=\frac{2}{7}\lim_{b\to+\infty}\int_{\ln 3}^{\ln b}u^{-\frac{1}{2}}du=\frac{2}{7}\lim_{b\to+\infty}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]\Bigg|_{\ln 3}^{\ln b}=\frac{2}{7}\lim_{b\to+\infty}\left[2u^{\frac{1}{2}}\right]\Big|_{\ln 3}^{\ln b}$$

$$=\frac{4}{7}\lim_{b\to+\infty}u^{\frac{1}{2}}\Big|_{\ln 3}^{\ln b}=\frac{4}{7}\left[\sqrt{\ln b}-\sqrt{\ln 3}\right]\to+\infty \text{ divergent}$$

Therefore $\sum_{k=3}^{\infty}\frac{2}{7k\sqrt{\ln k}}$ is divergent

$$\sum_{k=2}^{\infty} \frac{k}{(2k^2+9)^{\frac{4}{3}}}$$

$f(x) = \dfrac{x}{(2x^2+9)^{\frac{4}{3}}}$ is continuous, nonincreasing and nonnegative

$$\int_2^{\infty} \frac{x}{(2x^2+9)^{\frac{4}{3}}}\,dx = \lim_{b\to+\infty}\int_2^b \frac{x}{(2x^2+9)^{\frac{4}{3}}}\,dx \;\overset{\underset{du=2xdx}{u=x^2}}{=}\; \lim_{b\to+\infty}\int_{2^2}^{b^2} \frac{1}{(2u+9)^{\frac{4}{3}}}\,\frac{1}{2}du$$

$$= \frac{1}{2}\lim_{b\to+\infty}\int_4^b \frac{1}{(2u+9)^{\frac{4}{3}}}\,du \;\overset{\underset{dv=2du}{v=2u+9}}{=}\; \frac{1}{2}\lim_{b\to+\infty}\int_{2\times4+9}^{2b+9} \frac{1}{v^{\frac{4}{3}}}\,\frac{1}{2}dv$$

$$= \frac{1}{4}\lim_{b\to+\infty}\int_{17}^{2b+9} v^{-\frac{4}{3}}\,dv = \frac{1}{4}\lim_{b\to+\infty}\left[\frac{v^{\frac{1}{3}}}{\frac{1}{3}}\right]\Bigg|_{17}^{2b+9} = \frac{3}{4}\lim_{b\to+\infty} v^{\frac{1}{3}}\Big|_{17}^{2b+9}$$

$$= \frac{3}{4}\lim_{b\to+\infty}\left[\sqrt[3]{2b+9} - \sqrt[3]{17}\right] = +\infty$$

$$\int_1^{\infty} \frac{3^x}{10^x}\,dx$$

$f(x) = \dfrac{3^x}{10^x} = \left(\dfrac{3}{10}\right)^x = (0.3)^x$ is continuous, nonincreasing and nonnegative

Notice $\sum_{k=1}^{\infty} 0.3^k = 0.3 + 0.3^2 + 0.3^3 + 0.3^4 + \cdots$ is a GS with

$a = 0.3,\ r = 0.3,\quad S = \dfrac{0.3}{1-0.3} = \dfrac{0.3}{0.7} = \dfrac{0.3\times10}{0.7\times10} = \dfrac{3}{7}$ convergent

Therefore $\int_1^{\infty} \frac{3^x}{10^x}dx$ is convergent