Review of chapters 10 and 11

December 8, 2019

Problems:
Review the definition of Taylor polynomials

(1) What is the Taylor series(expension) of $\frac{x^2+2}{e^{x^3}}$ at $x=0$, at least four terms

Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

$$\frac{x^2+2}{e^{x^3}} = (x^2+2)e^{-x^3} = (x^2+2)\left(1+(-x^3)+\frac{(-x^3)^2}{2!}+\frac{(-x^3)^3}{3!}+\cdots\right)$$
$$= (x^2+2)\left(1-x^3+\frac{x^6}{2}-\frac{x^9}{6}+\cdots\right)$$
$$= 2+x^2-2x^3-x^5+\cdots$$

Review the definition of geometric series

(2) A patient is receiving medication, and he will take one pill per day, and each pill consists of 5 mg, and and 10% of the drug will be absorbed into the body, determine the equilibrium amount of drug in the body of that patient. That is, after a lenthy period of time, how much medication would we expect to be in patient's body

The 1st day: 0 The 2nd day: $5\cdot 0.1$

The 3rd day: $(5+5\cdot 0.1)\cdot 0.1 = 5\cdot 0.1+5\cdot 0.1^2$ The 4th day: $(5+5\cdot 0.1+5\cdot 0.1^2)\cdot 0.1 = 5\cdot 0.1+5\cdot 0.1^2+5\cdot 0.1^3$

The 5th day: $(5+5\cdot 0.1+5\cdot 0.1^2+5\cdot 0.1^3)\cdot 0.1 = 5\cdot 0.1+5\cdot 0.1^2+5\cdot 0.1^3+5\cdot 0.1^4$

After a long period it reaches an equilibrium which is a geometric series

$$5\cdot 0.1+5\cdot 0.1^2+5\cdot 0.1^3+5\cdot 0.1^4+\cdots = \frac{5\cdot 0.1}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}\,mg$$ (Only the last line is need)

(3) Find the constant solutions of the differential equation $y' - y^2 = 4y - 5$

Suppose y is a constant solution, then $y'=0$, we get

$0-y^2 = y'-y^2 = 4y-5 \Rightarrow y^2+4y-5=0$ which is a quadratic equation

solve it to get $y=1$ or $y=-5$

(4) $yy' = \frac{t}{y}$ $\Leftrightarrow$ $y\frac{dy}{dt} = \frac{t}{y}$ $\Rightarrow$ $y^2dy = tdt$ $\Rightarrow$ $\int y^2dy = \int tdt$

$\Rightarrow$ $\frac{1}{3}y^3 = \frac{1}{2}t^2 + C$ $\Rightarrow$ $y^3 = \frac{3}{2}t^2 + C$ (there is no difference between using C or 3C)

$\Rightarrow$ $y = \sqrt[3]{\frac{3}{2}t^2 + C}$

(5) Solve the differential equation $y' - e^{\ln x}y = x$

Equivalent to $y' - xy = x$

$a(x) = -x$, $b(x) = x$

$A(x) = -\frac{x^2}{2}$

$\left(ye^{-\frac{x^2}{2}}\right)' = xe^{-\frac{x^2}{2}}$

$\Rightarrow ye^{-\frac{x^2}{2}} = \int xe^{-\frac{x^2}{2}}dx \overset{u=-\frac{x^2}{2}}{=\!=\!=} -\int e^u du = -e^u + C$

$= -e^{-\frac{x^2}{2}} + C$

$\Rightarrow y = 1 + Ce^{\frac{x^2}{2}}$

$y' + a(t)y = b(t)$

$A'(t) = a(t)$

$\left[ye^{A(t)}\right]' = b(t)e^{A(t)}$

$\Rightarrow ye^{A(t)} = \int b(t)e^{A(t)}dt$

$\Rightarrow y = e^{-A(t)}\int b(t)e^{A(t)}dt$

(6) Use the Trapezoidal Rule with $n = 3$ partitions to approximate the area under the curve y^4 on the interval $1 \le x \le 3$

$\Delta x = \frac{3-1}{3} = \frac{2}{3}$, $x_0 = 1$, $x_1 = x_0 + \Delta x = \frac{5}{3}$, $x_2 = x_1 + \Delta x = \frac{7}{3}$, $x_3 = x_2 + \Delta x = 3$

Thus $\int_1^3 x^4 dx = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]$

$= \frac{\frac{2}{3}}{2}\left[1^4 + 2\cdot\left(\frac{5}{3}\right)^4 + 2\cdot\left(\frac{7}{3}\right)^4 + 3^4\right]$

$= \frac{1}{3}\left(1 + \frac{1250}{81} + \frac{4802}{81} + 81\right)$

$= \frac{12694}{243}$

$= 52.239$