## 0.1 Singular homology

**Definition 0.1.1** (Eilenberg-Steenrod axioms)**.** $Top$ is the category of topological spaces, $Ab$ is the category of abelian groups , $\mathcal{T}$ is the fully faithful subcategory of $Top \times Top$ with objects pairs of topological spaces $(X, A)$ such that $A \subseteq X$, $\mathcal{T}_A$ is the fully faithful subcategory of $\mathcal{T}$ with objects $(X, A)$, $R : \mathcal{T} \to Top$, $(X, A) \mapsto A$, $f \mapsto f|_A$ is a functor
**Relative homology** are functors $H_n : \mathcal{T} \to Ab$, then $H_n(-, A)$ define functors $\mathcal{T}_A \to Ab$, **absolute homology** are functors $H_n(-, \varnothing) : Top \to Ab$, **reduced homology** are $\widetilde{H}_n = H_n(-, *)$. $\partial_n : H_n \to H_{n-1}R$ are natural transformations

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{H_n(f)} & H_n(Y,B) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) \end{array}$$

$(H, \partial)$ is a **homology theory** if it satisfies axioms
Homotopy invariance: $f \simeq g : (X, A) \to (Y, B)$, then $H_n(f) = H_n(g)$
Additivity: $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$, then $\bigoplus_\alpha H_n(X_\alpha, A_\alpha) \xrightarrow{\bigoplus_\alpha H_n(i_\alpha)} H_n(X, A)$ is an isomorphism
Exactness:

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial_n} \cdots$$

Exicision: $\bar{Z} \subseteq \overset{\circ}{U}$, then $H_n(X - Z, U - Z) \xrightarrow{H_n(i)} H_n(X, U)$ is an isomorphism
Dimension: $H_n(*) = 0, \forall n \neq 0$, $H_0(*)$ is the **coefficient group**
$(H, \partial)$ is an **extraordinary homology theory** without dimension axiom

**Definition 0.1.2.** A **singular $n$-simplex** in $X$ is just a continuous map $\Delta \xrightarrow{\sigma} X$, the free abelian group $C_n(X)$ with singular $n$-simplices in $X$ as basis consists of $n$-chains(singular chain) which are finite sums $\sum n_i\sigma_i, n_i \in \mathbb{Z}$, we can tensor $C_n(X)$ with a ring $R$, $C_n(X; R) := C_n(X) \otimes_{\mathbb{Z}} R$ to be chains with $R$ coefficients, here $R$ could be an abelian group(group ring) or a field
Also, if we only consider characteristic maps(for simplicial, $\Delta$, cell complexes), we would get $C_n(X)$ to be simplicial, cellular chains

**Remark 0.1.3.** Given a topological space, we can form a huge $\Delta$ complex $S(X)$
Let $S(X)^0$ be $X$ with discrete topology which can be identified with all the maps $\Delta^0 = * \to X$, then build on it inductively as a CW complex, suppose $S(X)^n$ is constructed, for each map $\Delta^{n+1} \to X$, we add an $n + 1$ cell by gluing its faces to its restrictions, preserving the order
Similarly, suppose $X$ is a singular $\Delta$ complex, we can also construct a $\Delta$ complex $\Delta(X)$ by replacing continuous maps with simplicial maps above
The simplicial homology of $S(X), \Delta(X)$ is the same as the singular homology of $X$

**Definition 0.1.4.** The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ given by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma|[e_0, \cdots, \widehat{e_i}, \cdots, e_n]$$

Where $\sigma : \Delta^n \to X$ is a singular simplex, we can easily show that $\partial_n\partial_{n+1} = 0$, define cycles $Z_n(X) = \ker \partial_n$ and boundaries $B_n(X) = \mathrm{im}\partial_{n+1}$, and the (singular)homology group $H_n(X) = Z_n(X)/B_n(X)$
Similarly, we can define simplicial cycles, boundaries and homology groups correspondingly For cell complexes, if $\partial_n\sigma \subseteq X^{n-1}$, $\sigma$ is called a cellular cycle, and cellular boundary is defined to be the image of some cellular chain, we can therefore define cellular homology

**Definition 0.1.5.** Define $C_n(X, A)$ to be $C_n(X)/C_n(A)$, $C_\bullet(X, A)$ form a chain complex, $Z_n(X, A)$ can be represented by $n$-chains with its boundary in $A$
The cellular homology could also be defined as the homology groups of $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \to \cdots$, where $d_{n+1}$ is induced by $H_{n+1}(X^{n+1}, X^n) \to H_n(X^n) \to H_n(X^n, X^{n-1})$

**Definition 0.1.6.** Suppose $\mathcal{U} = \{U_j\}$ are a family of subspaces of $X$ and interiors of $U_j$ form an open cover of $X$, define $C_n^{\mathcal{U}}(X)$ to be $n$-chains $\sum n_i \sigma_i$ such that the image of each $\sigma_i$ is contained in some $U_j$, $C_n^{\mathcal{U}}(X, A) := C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$

**Theorem 0.1.7.** The inclusion $C_n^{\mathcal{U}}(X, A) \to C_n(X, A)$ is a chain homotopy equivalence

Excision theorem for singular homology

**Theorem 0.1.8** (Excision theorem for singular homology)**.** Singular homology satisfies excision theorem

*Proof.* Suppose $\bar{Z} \subseteq \overset{\circ}{U}$, let $A = U, B = X - Z, \mathcal{U} = \{A, B\}$, only need to show $H_n^{\mathcal{U}}(A \cup B, A) \cong H_n(A \cup B, A) \cong H_n(X, U) \cong H_n(X - Z, U - Z) \cong H_n(B, A \cap B) \cong H_n^{\mathcal{U}}(B, A \cap B)$
Consider $C_n^{\mathcal{U}}(B) \hookrightarrow C_n^{\mathcal{U}}(X) \to C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$ has kernel $C_n^{\mathcal{U}}(A \cap B)$, thus $C_n^{\mathcal{U}}(B)/C_n^{\mathcal{U}}(A \cap B) \cong C_n^{\mathcal{U}}(X)/C_n^{\mathcal{U}}(A)$ $\square$

**Definition 0.1.9.** $(X, A)$ is called a good pair if $A$ has a neighborhood $U$ deformation retracts onto $A$

**Definition 0.1.10.** The reduced singular homology $\tilde{H}_n(X)$ is defined to be the homology group of the chain complex

$$\cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\varepsilon} \mathbb{Z}$$

**Lemma 0.1.11.** $\tilde{H}_n(X) \to H_n(X, *)$ is an isomorphism induced from $C_n(X) \to C_n(X, *)$

*Proof.* For $\sum n_i \sigma_i \in C_1(X)$, if $\sum n_i \partial \sigma_i \in C_0(*)$, then $\sum n_i \partial \sigma_i = 0$, thus $H_1(X, *) \cong \tilde{H}_1(X)$
For any $\sum n_i P_i \in C_0(X, *)$ where $P_i$ are points, then $\sum n_i P_i - \sum n_i *$ is a preimage in $Z_0(X)$, a boundary in $Z_0(X)$ certainly maps to a boundary in $C_0(X, *)$, suppose $\sum n_i P_i \in C_0(X, *)$ is a boundary, $\sum n_i P_i - \sum n_i *$ has to be a boundary in $C_0(X)$, thus $H_0(X, *) \cong \tilde{H}_0(X)$ $\square$

**Theorem 0.1.12.** If $(X, A)$ is called a good pair, $H_n(X, A) \xrightarrow{q_*} \tilde{H}_n(X/A)$ is an isomorphism

*Proof.* Consider the quotient map $q : X \to X/A$ induces $H_n(X, A) \to H_n(X/A, *) \to \tilde{H}_n(X/A)$, we show that $q_*$ is an isomorphism, suppose $U$ is a neighborhood of $A$ that deformation retracts onto it, consider the following diagram

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(X, U) \arrow[d, "q_*"] & H_n(X - A, U - A) \arrow[l, "i_*"'] \arrow[d, "q_*"] \\
H_n(X/A, *) \arrow[r, "i_*"] & H_n(X/A, U/A) & H_n(X - A/A, U - A/A) \arrow[l, "i_*"']
\end{tikzcd}$$

$H_n(X, A) \xrightarrow{i_*} H_n(X, U)$, $H_n(X/A, *) \xrightarrow{i_*} H_n(X/A, U/A)$ are isomorphisms because of the deformation retraction, $H_n(X - A, U - A) \xrightarrow{i_*} H_n(X, U)$, $H_n(X - A/A, U - A/A) \xrightarrow{i_*} H_n(X/A, U/A)$ are isomorphisms because of the Theorem 0.1.8, $H_n(X - A, U - A) \xrightarrow{q_*} H_n(X - A/A, U - A/A)$is an isomorphism since $(X - A, U - A) \xrightarrow{q} (X - A/A, U - A/A)$ is a homeomorphism $\square$

**Theorem 0.1.13** (Mayer Vietoris sequence)**.** Suppose $A, B$ are subspaces of $X$ that the interior of $A, B$ covers $X$, then we have an exact sequence of homology groups $\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(A \cup B) \to \cdots$

*Proof.* It is not hard to see there is a short exact sequence $0 \to C_n^{\mathcal{U}}(A \cap B) \to C_n^{\mathcal{U}}(A) \oplus C_n^{\mathcal{U}}(B) \to C_n^{\mathcal{U}}(A \cup B) \to 0$, $x \mapsto (x, x)$ and $(x, y) \mapsto x - y$ $\square$

**Theorem 0.1.14.** Suppose $X$ has a $\Delta$ complex structure, $H_n^{\Delta}(X) \to H_n(X), \Phi_\alpha^n \mapsto \Phi_\alpha^n$ is an isomorphism

**Definition 0.1.15.** $S^n \xrightarrow{f} S^n$ induces $\mathbb{Z} \cong H_n S^n \xrightarrow{f_*} H_n S^n \cong \mathbb{Z}$, $f_*(1)$ is the **degree** of $f$

**Proposition 0.1.16** (Properties of degrees)**.**

**1.** $\deg 1 = 1$

**2.** $\deg(fg) = \deg f \deg g$

**3.** If $f$ is not surjective, $\deg f = 0$

**4.** If $f$ is a reflection, $\deg f = -1$

**5.** Let $a$ be the antipodal map, then $\deg a = (-1)^{n+1}$

**6.** If $f$ has no fixed points on $S^n$, then $f$ is homotopic to the antipodal map

*Proof.*

**1.** Let $\Delta_1^n, \Delta_2^n$ maps to the upper and lower hemisphere be a $\Delta$ complex structure on $S^n$, then $\Delta_1^n - \Delta_2^n$ would be a generator, and $f$ maps them to $\Delta_2^n - \Delta_1^n$, thus $\deg f = -1$

**2.** $a$ is the composition of $n+1$ reflections

**3.** Since $f(x) \neq -x$, $\dfrac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$ homotopy $f$ to $a$

□

**Definition 0.1.17.** View $\Delta^n$ as $\{0 \leq x_1 \leq \cdots \leq x_n \leq 1\}$, we can cut $\Delta^n \times \Delta^m = \{0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \times \{0 \leq x_{n+1} \leq \cdots \leq x_{n+m} \leq 1\}$ into $\binom{n+m}{m}$ simplices

$$\Delta^n \times \Delta^m = \bigcup_\sigma \Delta_\sigma, \quad \Delta_\sigma = \{0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n+m)} \leq 1\}$$

$\sigma$ runs over $(n, m)$-shuffles. Each $\sigma$ can be viewed as a path through a grid as in Definition **??**. Associate a linear map $\ell_\sigma : \Delta^{n+m} \to \Delta_\sigma \subseteq \Delta^n \times \Delta^m$, sending the $k$-th vertex to vertex in the grid. The **cross product**

$$C_n(X) \otimes C_m(Y) \to C_{n+m}(X \times Y)$$
$$f \otimes g \mapsto f \times g$$

Where

$$f \times g = \sum_\sigma (-1)^{|\sigma|}(f \times g)\ell_\sigma$$

Here on the right hand side $f \times g : \Delta^n \times \Delta^m \to X \times Y$, $(a, b) \mapsto (f(a), g(b))$ is different from the left hand side. We have $\partial(f \times g) = \partial f \times g + (-1)^n f \times \partial g$

Eilenberg-Zilber theorem

**Theorem 0.1.18** (Eilenberg-Zilber theorem)**.** $C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$ is a natural equivalence

*Proof.* Consider $Top \times Top$ with model $\mathcal{M} = \{(\Delta^n, \Delta^m)\}$, $F, G : Top \times Top \to Ch_{\geq 0}$, $F(X, Y) = C_*(X \times Y)$, $G(X, Y) = C_*(X) \otimes C_*(Y)$, $H_i(\Delta^n \times \Delta^m) = 0$ for $i \neq 0$, $F_k(X, Y) = \left\{\Delta^k \xrightarrow{(\mathrm{id},\mathrm{id})} \Delta^k \times \Delta^k \xrightarrow{\sigma} X \times Y\right\}$. By Exercise **??**, $H_i(C_*(X) \otimes C_*(Y)) = 0$ for $i \neq 0$, $C_k(X) = \{\Delta^k \xrightarrow{\mathrm{id}} \Delta^k \xrightarrow{\sigma} X\}$, $G_k(X, Y) = \left\{(\sigma \otimes \tau)(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \middle| \Delta^p \xrightarrow{\sigma} X, \Delta^q \xrightarrow{\tau} Y\right\}$ There is a natural equivalence $\phi_0 : H_0F \to H_0G$ induced by $\varphi : C_0(X \times Y) = F_0(X, Y) \to G_0(X, Y) = C_0(X) \otimes C_0(Y)$, $(\sigma, \tau) \mapsto \sigma \otimes \tau$, since $H_0(X \times Y) = C_0(X \times Y)/(x_0, y_0) \sim (x_1, y_1)$, $(x_0, y_0), (x_1, y_1)$ are connected by a path, $H_0(C_*(X) \times C_*(Y)) = C_0(X) \otimes C_0(Y)/(x_0, y_0) \sim (x_1, y_0) \sim (x_1, y_1)$ □

Cross product and its dual for homology

**Remark 0.1.19.** We define the **cross product** $C_*(X) \otimes C_*(Y) \xrightarrow{\times} C_*(X \times Y)$ and its dual $\varphi$ Define $T : C_*(X \times Y) \to C_*(Y \times X)$, $(x, y) \mapsto (y, x)$, $\tau : C_*(X) \otimes C_*(Y) \to C_*(Y) \otimes C_*(X)$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$, $T^2 = 1$, $\tau^2 = 1$, $\tau\partial = \partial\tau$

$$\begin{array}{ccc} C*(X)\otimes C_*(Y) & \xrightarrow{\times} & C_*(X\times Y) \\ \big\downarrow\tau & & \big\downarrow T \\ C_*(Y)\otimes C_*(X) & \xrightarrow{\times} & C_*(Y\times X) \end{array}$$

Is not commutative, but $\times$ and $T\circ\times\circ\tau$ are chain homotopic

$$\begin{array}{ccc} C_*(X\times Y) & \xrightarrow{\theta} & C*(X)\otimes C_*(Y) \\ \big\downarrow T & & \big\downarrow\tau \\ C_*(Y\times X) & \xrightarrow{\theta} & C_*(Y)\otimes C_*(X) \end{array}$$

Is not commutative, but $\theta$ and $\tau\circ\theta\circ T$ are chain homotopic

Topological Kunneth formula

**Theorem 0.1.20** (Topological Kunneth formula)**.**

$$0\to\bigoplus_{p+q=n}H_p(X)\otimes H_q(Y)\to H_n(X\times Y)\to\bigoplus_{p+q=n-1}Tor_1(H_p(X),H_q(Y))\to 0$$

Is exact

*Proof.* Apply Theorem 0.1.18 and Theorem **??** □

## 0.2 Cellular homology