MATH730 - Fundamental Concepts of Topology

Taught by `Sarah Yeakel`
Notes taken by `Haoran Li`
2018 Fall

Department of Mathematics
University of Maryland

Contents

1 General topology **2**

2 Homeworks **5**

2.1 Homework1 5
2.2 Homework2 7
2.3 Homework3 9
2.4 Homework4 10
2.5 Homework5 12
2.6 Homework6 14
2.7 Homework7 15
2.8 Homework8 18
2.9 Homework9 20
2.10 Homework10 23
2.11 Homework11 25
2.12 Homework12 28

Index **31**

1 General topology

Example 1.1. $\mathbb{R}/\mathbb{Q}$ is not Hausdorff

Proposition 1.2. If Y is discrete, then X is connected if every continuous function $f : X \to Y$ is a constant

Proposition 1.3. X is compact, Y is Hausdorff, any continuous function $f : X \to Y$ is closed. In particular, if f is bijective, then f is a homeomorphism

Fact 1.4. X is Hausdorff and locally compact iff X is homeomorhphic to an open subset of a compact Hausdorff space Y through one point compactification

$$\mathtt{Hom}(X \times A, Y) \cong \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$$

as a set. However as topological spaces $\mathtt{Hom}(X \times A, Y) \to \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$ is not surjective. Consider $X = \mathtt{Hom}(A, Y)$

Note. Here $\mathtt{Hom}(A, Y)$ is endowed with compact-open topology

Theorem 1.5. A is locally compact and Hausdorff, then $f : X \times A \to Y$ is continuous iff $f : X \to \mathtt{Hom}(A, Y)$ is continuous. Furthermore, if X is also locally compact and Hausdorff, then

$$\mathtt{Hom}(X \times A, Y) \cong \mathtt{Hom}(X, \mathtt{Hom}(A, Y))$$

as topological spaces

Proposition 1.6. If $g : A \to Y$ is injective, then $\iota_X : X \to X \cup_A Y$ is also injective. If $f : A \to Y$ is surjective, then $\iota_Y : X \to Y \cup_A Y$ is also surjective, moreover, if f is a homeomorphism, so is ι_Y

Proof. Proof of homeomorphism: Show that Y satisfies the universal property of the pushout

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, equal] \arrow[ddr, bend left, "\varphi_2"] & \\
X \arrow[r, "gf"] \arrow[drr, bend right, "\varphi_1"'] & Y \arrow[dr, dashed, "\Phi"] & \\
 & & T
\end{tikzcd}$$

$\forall \varphi_1, \varphi_2$ such that $\varphi_1 f = \varphi_2 g$, $\Phi g f^{-1} = \varphi_1$, thus $\Phi = \varphi_2$ ☐

Definition 1.7 (CW complexes)**.** For $x, y \in X$, define $\varphi : S^0 \to X$ with $\varphi(-1) = x$, $\varphi(1) = y$, Write $X \cup_\varphi D^1$ for pushout

$$\begin{tikzcd}
S^0 \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\
D^1 \arrow[r, "\iota"] & X \cup_\varphi D^1
\end{tikzcd}$$

The image $\iota(\mathtt{Int}(D^1))$ is called a 1-cell, denoted e^1
In general, we have

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X \arrow[d] \\
D^n \arrow[r, "\iota"] & X \cup_\varphi D^n
\end{tikzcd}$$

The image $\iota(\mathtt{Int}(D^n))$ is called an n-cell, denoted e^n. Attaching cells does not disturb the interiors of the cells
A CW complex is built up in the following way

1. Starting with a discrete set X_0, the set of 0-cells or the 0-skeleton
2. Given $(n-1)$-skeleton X_{n-1}, then n-skeleton X_n is obtained by attaching n-cells to X_{n-1}, that is

$$\begin{tikzcd} \coprod_{\alpha \in A_n} S_\alpha^{n-1} \arrow[r, "\phi_\alpha"] \arrow[d, hook] & X_{n-1} \arrow[d, hook] \\ \coprod_{\alpha \in A_n} D_\alpha^n \arrow[r, "\Phi_\alpha"] & X_n \end{tikzcd}$$

3. The space X is the union of X_n's, topologized by weak topology

The third condition ensures $X_n \hookrightarrow X$ is continuous. As a set, X is the disjoint union of $\Phi_\alpha(\mathrm{Int}(D^n))$, $\Phi_\alpha : D^n \to X_n \to X$ are called characteristic maps, CW complexes are determined by characteristic maps

Definition 1.8. G is a discrete group. $Y \times G \to Y$ is a continuous action, then $q : Y \to Y/G$ is continuous, the action is *properly discontinuous* if $\forall y \in Y$, $\exists U$ neighborhood such that $U \cap Ug = \varnothing, \forall g \neq 1$, this implies that the action is free, then q is a covering map, furthermore, $p : Y/H \to Y/G$ is a covering map for $H \leq G$

Fact 1.9. A finite group which acts freely on a Hausdorff space Y is properly discontinuous

Theorem 1.10 (Monodromy)**.** Let $\tilde{\gamma}_x$ be the lift of γ against p starting at x

$$\begin{aligned} \pi_1(B, b_0) \times F &\to F \\ ([\gamma], x) &\mapsto \tilde{\gamma}_x(1) \end{aligned}$$

specifies a transitive left action of $\pi_1(B, b_0)$ on F, called the monodromy action

Proof. Let c_{b_0} be the constant loop, which is the identity element, by transitivity, path-connectedness and orbit-stabilizer theorem, $F \cong G/G_{x_0}$ ☐

Proposition 1.11. The stabilizer of $x \in F$ under the monodromy action isthe subgroup $p_*(\pi_1(E, x)) \leq \pi_1(B, b_0)$

Corollary 1.12. $\pi_1(B, b_0)/p_*(\pi_1(E, x)) \to F$ is an isomorphism

Proposition 1.13. $\varphi : E_1 \to E_2$ induce map on fibers $F_1 \to F_2$ is $\pi_1(B, b_0)$ equivariant, i.e. $[\gamma] \cdot \varphi(x) = \tilde{\gamma}_{\varphi(x)}(1) = \varphi(\tilde{\gamma}_x)(1) = \varphi(\tilde{\gamma}_x(1)) = \varphi([\gamma] \cdot x)$

Proposition 1.14. $H, K \leq G$, every G equivariant map $\varphi : G/H \to G/K$ is of the form $gH \mapsto g\gamma K$ for some $\gamma \in G$ such that $\gamma H \gamma^{-1} \leq K$, in short, H is subconjugate to K

Proof. An equivariant map is determined by the value at one element, suppose $eH \mapsto \gamma K$, for some $\gamma \in G$. then $gH \mapsto g\gamma K$ which is well-defined should have $ghH = gH$, $gh\gamma H = h\gamma H$, so we need $\gamma^{-1} h \gamma \in K \Rightarrow \gamma^{-1} H \gamma \leq K$ ☐

Corollary 1.15. An equivariant map $\varphi : G/H \to G/K$ exists iff H is subconjugate ot K. The two orbits are isomorphic as G-sets iff H is conjugate to K

Theorem 1.16. There is a bijection of sets

$$\mathrm{Hom}_B(E_1, E_2) \cong \mathrm{Hom}_{\pi_1(B, b_0)}(F_1, F_2)$$

Corollary 1.17.

$$\mathrm{Aut}_B(E) \cong \mathrm{Hom}_G(G/H, G/H) \cong N_G(H)/H = W_G(H)$$

here $H = p_*(\pi_1(E))$

Proof. There exists a surjective homomorhpism $N_G(H) \to \mathrm{Hom}_G(G/H, G/H)$, $\gamma \mapsto gh \mapsto g\gamma H$, thus $eH \mapsto \gamma H \Rightarrow \gamma \in H$, thus $\mathrm{Hom}_G(G/H, G/H) \cong N_G(H)/H$ $\square$

Proposition 1.18. *X is the universal cover of B, $\mathrm{Aut}_B(X) \to F$, $\varphi \mapsto \varphi(x)$, $x \in q^{-1}(b)$ is a bijection as sets*

2 Homeworks

2.1 Homework1

1.
Suppse τ is a topology on X with exactly 7 elements. Since $\mathscr{P}(X)$ has exactly 8 elements, and $\varnothing, X \in \tau$. Thus, without loss of generality, we could assume either $\tau = \mathscr{P}(X) - \{3\}$ or $\tau = \mathscr{P}(X) - \{2,3\}$
Case I: $\tau = \mathscr{P}(X) - \{3\}$
$\{1,3\}, \{2,3\} \in \tau$ which implies $\{3\} = \{1,3\} \cap \{2,3\} \in \tau$, that is a contradiction.
Case II: $\tau = \mathscr{P}(X) - \{2,3\}$
$\{2\}, \{3\} \in \tau$ which implies $\{2,3\} = \{2\} \cup \{3\} \in \tau$, that is also a contradiction.
Therefore, it isn't a topology on X with exactly 7 elements.
2.
Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of $f(X)$, then $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X since f is continuous, X has a finite subcover $\{f^{-1}(U_{\alpha_1}), \cdots, f^{-1}(U_{\alpha_n})\}$ since X is compact, hence $f(X)$ also has a finite subcover $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$, thus $f(X)$ is also compact.
3.
Consider the following commutative diagram

$$\begin{tikzcd} D^n \arrow[r, "h"] \arrow[d, "q"] \arrow[dr, "g"] & \mathbb{R}^{n+1} \\ D^n/S^{n-1} \arrow[r, "f"] & S^n \arrow[u, hook, "j"] \end{tikzcd}$$

Where h is given explicitly by

$$h(x) = \left(\cos(\pi|x|), \frac{x}{|x|}\sin(\pi|x|)\right)$$

h is continuous, so is g since $h = jg$, so is f since $g = fq$. Since D^n is compact, so is $D^n/S^{n-1} = q(D^n)$, since S^n is Hausdorff and f is bijective, thus f is a homeomorphism. Therefore, $D^n/S^{n-1} \cong S^n$
4.
The pushout of

$$\begin{tikzcd} \mathbb{R} - \{0\} \arrow[r, hook] \arrow[d, hook] & \mathbb{R} \arrow[d] \\ \mathbb{R} \arrow[r] & X \end{tikzcd}$$

is not Hausdorff.
Consider the two origins O_1 and O_2 of X, then any two neighborhoods of them would intersect, thus X cannot be Hausdorff.
5.
Pushout $X \cup_A Y$ is constructed by $X \sqcup Y/\sim$, where $\sim$ is the equivalence relationship generated by binary relationship $f(a) \sim g(a)$, hence $\forall x \neq x' \in X$, suppose $\iota_X(x) = \iota_X(x')$, then $x = f(a_1) \sim g(a_1) = g(a_2) \sim \cdots \sim f(a_n) = x'$, however, since g is injective, $a_1 = a_2$, thus $x = x'$, which is a contradiction. Therefore, ι_X is injective.
6.
Consider the trivial topology on $\mathbb{R}$, then a neighborhood of any point can only be $\mathbb{R}$ itself, thus it is not Hausdorff.
7.
(a)

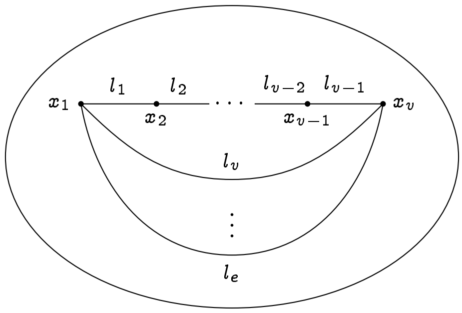
$\forall \epsilon > 0$, let $\delta = \min\left(1, \dfrac{\epsilon}{1+|x_0|+|y_0|}\right)$, then $\forall (x,y) \in (x_0-\delta, x_0+\delta) \times (y_0-\delta, y_0+\delta)$, $|xy - x_0y_0| \le |x||(y-y_0)| + |(x-x_0)||y_0| \le (|x_0|+\delta)\delta + \delta|y_0| = \delta(\delta+|x_0|+|y_0|) \le \delta(1+|x_0|+|y_0|) < \epsilon$, hence f is continuous.

(b)

Consider $x_0 = y_0 = -1$, then $(0,\infty)$ is a neighborhood of x_0y_0, however, any neighborhood $(a,\infty)\times(b,\infty)$ of (x_0,y_0) with $a<-1, b<-1$, we have $f(0,0) = 0 \notin (0,\infty)$, therefore, f isn't continuous.

2.2 Homework2

Hatcher 0.10.
Suppose X is contractible, $\{*\}$ is the one point space, $\exists\varphi : X \to \{*\}, \psi : \{*\} \to X$ such that $\psi \circ \varphi \simeq \mathrm{id}_X$, thus for any map $f : X \to Y$ and arbitrary space Y we have $f = f \circ \mathrm{id}_X \simeq f \circ \psi\varphi$ where $\psi\varphi$ is constant map, hence f is nullhomotopic. Conversely, suppose for any map $f : X \to Y$ and arbitrary space Y, f is nullhomotopic. Take Y to be X, and f to be id_X, then $\mathrm{id}_X \simeq c_{x_0}$ where $x_0 \in X$ and c_{x_0} is the constant map, define $\varphi : X \to \{*\}$ by $\varphi(x) = *$ for any $x \in X$, and define $\psi : \{*\} \to X$ by $\psi(*) = x_0$, then $\psi \circ \varphi = c_{x_0} \simeq \mathrm{id}_X$ and $\psi \circ \varphi = \mathrm{id}_{\{*\}}$, hence X is contractible. Similarly, suppose X is contractible, $\{*\}$ is the one point space, $\exists\varphi : Y \to \{*\}, \psi : \{*\} \to Y$ such that $\psi \circ \varphi \simeq \mathrm{id}_Y$, thus for any map $f : Y \to X$ and arbitrary space Y we have $f = f \circ \mathrm{id}_Y \simeq f \circ \psi\varphi$ where $\psi\varphi$ is constant map, hence f is nullhomotopic. Conversely, suppose for any map $f : Y \to X$ and arbitrary space Y, f is nullhomotopic. Take Y to be X, and f to be id_X, then $\mathrm{id}_X \simeq c_{x_0}$ where $x_0 \in X$ and c_{x_0} is the constant map, define $\varphi : X \to \{*\}$ by $\varphi(x) = *$ for any $x \in X$, and define $\psi : \{*\} \to X$ by $\psi(*) = x_0$, then $\psi \circ \varphi = c_{x_0} \simeq \mathrm{id}_X$ and $\psi \circ \varphi = \mathrm{id}_{\{*\}}$, hence X is contractible.
Hatcher 0.14.
Since $f \geq 1$, $1 \geq 2 - f = v - e \Rightarrow e \geq v - 1$, hence we can construct as follows, where $x_1, \cdots, x_v$ are 0-cells, and $l_1, \cdots, l_e$ are 1-cells, which divide $\mathbb{S}^2$ into f 2-cells, and this give $\mathbb{S}^2$ a CW complex structure as we wanted.



Hatcher 1.1.1
Since $g_0 \simeq g_1$, there exists continuous map $H : I \times I \to X$, such that $H(s, 0) = g_0(s), H(s, 1) = g_1(s)$, then $\overline{H}(s, t) := H(1 - s, t)$ is so a continuous map from $I \times I$ to X with $\overline{H}(s, 0) = \overline{g_0}(s), \overline{H}(s, 1) = \overline{g_1}(s)$, thus $\overline{g_0} \simeq \overline{g_1}$, hence $f_0 = f_0 \cdot g_0 \cdot \overline{g_0} \simeq f_1 \cdot g_1 \cdot \overline{g_1} = f_1$
1.

$$\mathbb{S}^{n-1} \xhookrightarrow{i} \mathbb{R}^n \setminus \{0\} \xrightarrow{\pi} \mathbb{S}^{n-1}$$

Where i is inclusion map and π is given by $\pi(x) = \dfrac{x}{|x|}$, then $\pi \circ i = \mathrm{id}_{\mathbb{S}^{n-1}}$, and $i \circ \pi \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ by homotopy $H : \mathbb{R}^n \setminus \{0\} \times I \to \mathbb{R}^n \setminus \{0\}, (x, t) \mapsto tx + (1-t)\dfrac{x}{|x|}$, which demonstrate $\mathbb{S}^{n-1} \simeq \mathbb{R}^n \setminus \{0\}$
2.
$H : \mathbb{S}^n \setminus N \times I \to \mathbb{S}^n \setminus N$, given by

$$H(x, t) = \left(\frac{x}{|x|} \sqrt{1 - (tx_{n+1} - (1-t))^2}, tx_{n+1} - (1-t) \right)$$

Where N is north pole. H is a homotopy between $\mathrm{id}_{\mathbb{S}^n \setminus N}$ and c_N, where c_N is the constant map that maps $\mathbb{S}^n \setminus N$ to N, thus $\mathrm{id}_{\mathbb{S}^n \setminus N} \simeq c_N$. Since ϕ is continuous but not surjective, we can assume $N \notin \phi(\mathbb{S}^n)$, then $\phi \simeq \mathrm{id}_{\mathbb{S}^n \setminus N} \circ \phi \simeq c_N \circ \phi$, hence ϕ is nullhomotopic.
3.
If n is odd, then $n + 1$ is even, then we could define $H : \mathbb{S}^n \times I \to \mathbb{S}^n$ by rotating the first two coordinates while fix the rest, written explictly

$$H(x_1, x_2, \cdots, x_{n+1}) = (x_1 \cos \pi t - x_2 \sin \pi t, x_1 \sin \pi t + x_2 \cos \pi t, \cdots, x_{n+1})$$

The remaining coordinates are of even number, thus we could could repeat this process $\dfrac{n+1}{2}$ times, and compose all these homotopy, thus shows that $a \simeq \mathrm{id}_{\mathbb{S}^n}$

4.

Reflexivity: Suppose $f : X \to Y$, then $f \overset{H}{\simeq} f, \mathrm{rel}\, Z$, where $H(x,t) = f(x)$

Symmetry: Suppose $f, g : X \to Y$, and $f \overset{H}{\simeq} g, \mathrm{rel}\, Z$, then $g \overset{H'}{\simeq} f, \mathrm{rel}\, Z$ where $H'(x,t) = H(x, 1-t)$

Transitivity: Suppose $f, g, h : X \to Y$, and $f \overset{F}{\simeq} g, \mathrm{rel}\, Z$, $g \overset{G}{\simeq} h, \mathrm{rel}\, Z$, then $f \overset{H}{\simeq} h, \mathrm{rel}\, Z$, where

$$H(x,t) = \begin{cases} F\left(x, 2t\right), t \in \left[0, \frac{1}{2}\right] \\ G\left(x, 2t-1\right), t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Therefore, homotopy relative a given subspace $Z \subset X$ is an equivalence relation on the set of maps from X to Y

2.3 Homework3

Hatcher 1.1.6.

If X is path-connected, for any $[u] \in [S^1, X]$, u is a representative of $[u]$, denote $x_1 := u(0)$, then there is a path γ connecting x_0 and x_1 with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, then $[\gamma \cdot u \cdot \overline{\gamma}] \in \pi_1(X, x_0)$, define $H : S^1 \times I \to X$ where

$$H(s,t) = \begin{cases} \gamma\left(t + 3s(1-t)\right), \; s \in [0, \frac{1}{3}] \\ u(3s-1), \; s \in [\frac{1}{3}, \frac{2}{3}] \\ \gamma\left(t + 3(1-s)(1-t)\right), \; s \in [\frac{2}{3}, 1] \end{cases}$$

Then H is a homotopy between $\gamma \cdot u \cdot \overline{\gamma} \simeq H(s,0) \simeq H(s,1) \simeq u$, thus $\gamma \cdot u \cdot \overline{\gamma}$ is also a representative of $[u]$, thus Φ is surjective.

If $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$, assume $\gamma \cdot g \cdot \overline{\gamma} \simeq f$ for some $\gamma \in \pi_1(X, x_0)$ with some path homotopy H, then $\Phi([f]) = \Phi([\gamma \cdot g \cdot \overline{\gamma}]) = \Phi([g])$, the second equality holds using the same construction above.

Conversely, suppose $\Phi([f]) = \Phi([g])$, then there is a homotopy H between f and g, let $\gamma(t) = H(0,t)$, then $F : I \times I \to X$

$$F(s,t) = \begin{cases} \gamma\left(3st\right), \; s \in [0, \frac{1}{3}] \\ H\left(3s-1, t\right), \; s \in [\frac{1}{3}, \frac{2}{3}] \\ \gamma\left(3(1-s)t\right), \; s \in [\frac{2}{3}, 1] \end{cases}$$

shows that $\gamma \cdot g \cdot \overline{\gamma} \simeq f$, thus f and g are conjugate

Hatcher 1.1.12.

Suppose $f \in \pi_1(S^1)$ is a generator, and a homomorphism $\alpha : \pi_1(S^1) \to \pi_1(S^1)$ with $\alpha(f) = f^n$, then define $\varphi : S^1 \to S^1$ as $\phi(e^{2\pi i\theta}) = e^{2\pi i n\theta}$, then we would have $\varphi_* = \alpha$

1.

Define $\Psi : \pi_1(X, x_0) \times \pi_1(Y, y_0) \to \pi_1\left(X \times Y, (x_0, y_0)\right)$, $\Psi([f],[g]) = [(f,g)]$, to check Ψ is well-defined, suppose $f^{'} \simeq f, g^{'} \simeq g$, and F, G are homotopies between $f^{'}, f$ and $g^{'}, g$ respectively, then (F, G) is a homotopy between $(f^{'}, g^{'})$ and (f, g), also, $\Psi\left(([f_1],[g_1])([f_2],[g_2])\right) = \Psi\left([f_1 f_2, g_1 g_2]\right) = [(f_1 f_2), (g_1 g_2)] = [(f_1, g_1)][(f_2, g_2)]$, thus Ψ is a homomorphism, similarly, define $\Phi : \pi_1\left(X \times Y, (x_0, y_0)\right) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$, $\Phi[(h^1, h^2)] = ([h^1],[h^2])$ for $h = (h^1, h^2)$, to check Ψ is well-defined, suppose $h^{'} \simeq h$ by H, then $h^{1^{'}} \simeq h^1, h^{2^{'}} \simeq h^2$by H^1, H^2, also, $\Phi(h_1 h_2) = \Phi\left([h_1^1, h_1^2][h_2^1, h_2^2]\right) = \Phi\left([h_1^1 h_1^2, h_2^1 h_2^2]\right) = [h_1^1 h_1^2, h_2^1 h_2^2] = [h_1^1, h_1^2][h_2^1, h_2^2] = \Phi(h_1)\Phi(h_2)$, thus Φ is a homomorphism, $\Psi \circ \Phi = 1, \Phi \circ \Psi = \mathbb{1}$, hence Ψ is bijective thus an isomorphism.

2.

Since $r \circ i = \mathrm{id}_A$, $r_* i_* = \mathbb{1}_*$, thus i_* is injectivce.

3.

By problem 1 we have $\pi_1(D^2 \times S^1) \cong \mathbb{Z}$ and $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$, there is no injective map from $\pi_1(S^1 \times S^1)$ to $\pi_1(D^2 \times S^1)$, since if there is such a injective homomorphism, we would have $(1,0) \mapsto n, (0,1) \mapsto m$ with $n \neq m$, but then $(-m, n) \mapsto 0$ which has the same image as $(0,0)$, but $(-m,n) \neq (0,0)$, thus we have reached a contradiction, then by problem 2 we know that there is no such retraction.

2.4 Homework4

Hatcher 1.3.1.
$\forall x \in A \subset X, \exists U \ni x$ is a neighborhood of x such that U is evenly covered by $p^{-1}(U) \bigsqcup U_\alpha$, $\phi_\alpha := p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism with ψ_α as its inverse. Then $U \cap A \ni x$ is a neighborhood of x in A such that $p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A) = \left(\bigsqcup_\alpha U_\alpha\right) \cap \widetilde{A} = \bigsqcup_\alpha \left(U_\alpha \cap \widetilde{A}\right)$, and $\phi_\alpha|_{U_\alpha \cap \widetilde{A}} = p|_{U_\alpha \cap \widetilde{A}} : U_\alpha \cap \widetilde{A} \to U \cap A$ is a homeomorphism with $\psi_\alpha|_{U\cap A}$ as its inverse. Hence, $p : \widetilde{A} \to A$ is a covering space
1.
a)
Lemma: A finite CW complex is Hausdorff and compact.
We prove by induction, assume X is a finite CW complex and it is Hausdorff and compact, using $\phi : S^{n-1} \to X$ to attach D^n to X, the resulting adjunction space $X \cup_\phi D^n = X \bigsqcup D^n / \sim$, since $X \bigsqcup D^n$ is compact, so is $X \cup_\phi D^n$, for $x \neq y \in X \cup_\phi D^n$, ι_X is injective by homework 1, thus $\iota_X^{-1}(x) \neq \iota_X^{-1}(y)$ thus have disjoint neighborhoods in X, also, $\iota_{D^n}^{-1}(x), \iota_{D^n}^{-1}(y)$ are disjoint compact set in D^n,but D^n is a metric space, thus they have disjoint neighborhoods in D^n, hence x, y has disjoint neighborhood in $X \cup_\phi D^n$, therefore $X \cup_\phi D^n$ is Hausdorff

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "\phi"] & D^n \arrow[d, "\iota_{D^n}"] \\ X \arrow[r, hook, "\iota_X"] & X \cup_\phi D^n \end{tikzcd}$$

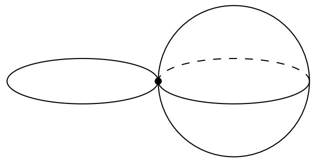
Proposition: Let $X = X^n$ be a finite CW complex, x_0 is a 0-cell, then the map $\pi_1(X^{n-1}, x_0) \to \pi_1(X^n, x_0)$ induced by inclusion is surjective where $n \geq 2$
There are finitely many n-cells, say $D_1^n, \cdots, D_m^n$, denote their origin as $O_1, \cdots, O_m$, let $U_1 := D_1^n \setminus \partial D_1^n, \cdots, U_m := D_m^n \setminus \partial D_m^n, U_{m+1} := X \setminus \{O_1, \cdots, O_m\}$ is an open cover of X, X^{n-1} would be a deformation retraction of U_{m+1}, since $X^{n-1}, D_1^n \setminus \{O_1\}, \cdots, D_m^n \setminus \{O_m\}$ is a close cover of U_{m+1}, X^{n-1} is closed because it is compact and X^n is Hausdorff, using pasting lemma, define the homotopy $F : U_{m+1} \times I \to U_{m+1}$ by

$$F(x,t) = x, \, \forall x \in X^{n-1}$$
$$F(x,t) = (1-t)x + t\frac{x}{|x|}, \, \forall x \in D_i^n \setminus \{O_i\}$$

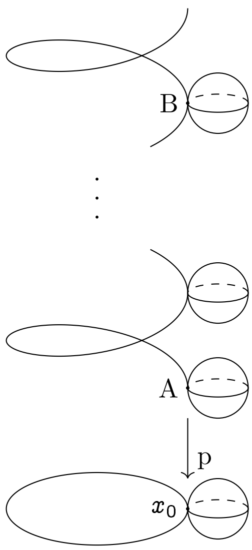
Hence for any loop $\gamma \in \pi_1(X^n, x_0)$, using Lebesgue's number lemma, we can divide the time into intervals $[s_j, s_{j+1}]$ so that each interval live inside one of U_i's, then the ones that are not contained in U_{m+1} must be contained in some $U_i, i \leq m$ and went through O_i, thus by wiggle it a little we can make γ contained in U_{m+1}, then compose with the deformation retract, we know that γ is homotopic to an element in $\pi_1(X^{n-1}, x_0)$
This would imply $\pi_1(X^1, x_0) \to \cdots \to \pi_1(X^n, x_0)$ is surjective
Remark: As it came to me afterwards, I could just define $F : X^{n-1} \bigsqcup_{i=1}^m D_i^n \times I \to X^{n-1} \bigsqcup_{i=1}^m D_i^n$ by

$$F(x,t) = x, \, \forall x \in X^{n-1}$$
$$F(x,t) = (1-t)x + t\frac{x}{|x|}, \, \forall x \in D_i^n \setminus \{O_i\}$$

Then F could be factor through the quotient space $X \times I$, thus induce $G : X \times I \to X^{n-1} \bigsqcup_{i=1}^m D_i^n \to X \times I$ is a homotopy for the deformation retract
b)
$S^1 \vee S^n$ could be given a CW complex structure as below with one 0-cell, one 1-cell and one n-cell



Thus by a), $\mathbb{Z} \cong \pi_1(S^1) \xrightarrow{\varphi} \pi_1(S^1 \vee S^n)$ is surjective, then $\pi_1(S^1 \vee S^n) \cong \pi_1(S^1)/\ker\varphi$, then $p : X \to S^1 \vee S^n$ is a covering map, let $H \subset X$ be the helix which is $p^{-1}(S^1)$, thus by Hatcher 1.3.1. we know $p|_H : H \to S^1$ is a covering map, suppose $\ker\varphi \neq 0$, then $\exists\gamma$ such that $0 \neq [\gamma] \in \pi_1(S^1, x_0)$ but $0 = [\gamma] \in \pi_1(S^1 \vee S^n, x_0)$, suppose $\widetilde{\gamma}$ is a path from A to B,such that $p(\widetilde{\gamma}) = \gamma$, then $\widetilde{\gamma}$ is a lift of γ under either $p|_H$ or p, by the uniqueness of homotopy lifting, we know that $A \neq B$, but for the same reason, $[\gamma] \neq 0$ in $\pi_1(S^1 \vee S^n)$ which is a contradiction, thus $\ker\varphi = 0$ and hence $\pi_1(S^1 \vee S^n) \cong \mathbb{Z}$



2.

Let $p : \mathbb{R} \to S^1, t \mapsto \mathrm{e}^{2\pi i t}$ be a covering map, and $e_1 : I \to S^1, t \mapsto \mathrm{e}^{2\pi i t}$ be the generator of $\pi_1(S^1, 1)$, by homotopy lift we can find F is a lift of $f(e_1)$, and $1 \neq n := \deg(f) = F(1) - F(0) \in \mathbb{Z}$. Define $G(t) := F(t) - t$, then $G(0) = F(0), G(1) = F(1) - 1$, thus $|G(0) - G(1)| = |F(1) - F(0) - 1| \geq ||F(1) - F(0)| - 1| \geq 1$, since G is continuous, $\exists t \in I$ such that $F(t) - t = G(t) \in \mathbb{Z}$, thus $f(e_1(t)) = p \circ F(t) = p(t) = \mathrm{e}^{2\pi i t} = e_1(t)$, thus f has a fixed point

3.

It is easy to show that $g : Y \to Y, y \mapsto yg$ is a homeomorphism, denote the quotient map as q, then for any $q(y) \in Y/G$, since G is a properly discontinuous action on Y, there exists a open neighborhood U of y such that $U \cap Ug = \varnothing$, thus $q^{-1}(q(U)) = \bigsqcup_{g \in G} Ug$ which is open, so is $q(U)$, and also $q|_{Ug}$ is continuous, injective and surjective, for any open set in Ug which would have the form $Ug \cap Vg = (U \cap V)g$ where V is open in Y, $q|_{Ug}((U \cap V)g) = U \cap V$ is open, thus $q|_{Ug}$ is also open, hence a homeomorphism. Therefore q is indeed a covering map

2.5 Homework5

Hatcher 1.3.2.
$\forall (x_1, x_2) \in X_1 \times X_2$, $\exists U_1, U_2$ are open neighborhoods of x_1, x_2, such that $p_1^{-1}(U_1) = \bigsqcup_\alpha U_{1\alpha}$, $p_1|_{U_{1\alpha}}$ is a homeomorphism and $p_2^{-1}(U_2) = \bigsqcup_\beta U_{2\beta}$, $p_1|_{U_{2\beta}}$ is a homeomorphism, then $U_1 \times U_2$ is an open neighborhood of (x_1, x_2), such that $(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2) = \bigsqcup_{\alpha,\beta} U_{1\alpha} \times U_{2\beta}$ and $(p_1 \times p_2)|_{U_{1\alpha} \times U_{2\beta}} = p_1|_{U_{1\alpha}} \times p_2|_{U_{2\beta}}$ is a homeomorphism
Hatcher 0.11.

$$\mathbb{1} \simeq hf \simeq hfgf \simeq gf$$

Hatcher 1.3.8.
Assume $p : X \to Y$ and $q : Y \to X$ are the covering maps, and $\varphi : X \to Y$ and $\psi : Y \to X$ satisfying $\psi\varphi \simeq \mathrm{id}_X, \varphi\psi \simeq \mathrm{id}_Y$, Since $\widetilde{X}$ and $\widetilde{Y}$ is simply connected, thus $\exists \widetilde{\varphi} : \widetilde{X} \to \widetilde{Y}$ and $\exists \widetilde{\psi} : \widetilde{Y} \to \widetilde{X}$ are lifts of φp and ψq, then $p\widetilde{\psi}\widetilde{\varphi} = \psi q\widetilde{\varphi} = \psi\varphi p$, thus $\widetilde{\psi}\widetilde{\varphi}$ is a lift of $\psi\varphi p$, suppose $\psi\varphi \overset{H}{\simeq} \mathrm{id}_X$, then by homotopy lifting property, we have $\widetilde{H}$, where $\widetilde{H}(x, 0) = \widetilde{\psi}\widetilde{\varphi}$, $\widetilde{H}(x, 1)$ is the lift of $\mathrm{id}_X p = p$, pick $x_0 \in \widetilde{X}$, $\widetilde{H}(x_0, 1) \in p^{-1}(x_0)$ might not be x_0, but $\exists \Phi : \widetilde{X} \to \widetilde{X}$ sending $\widetilde{H}(x_0, 1)$ to x_0 which is a lift of p, then $\Phi\widetilde{H}(x, 1)$ is a lift of p sending x_0 to x_0, by uniqueness, we have $\Phi\widetilde{H}(x, 1) = \mathrm{id}_{\widetilde{X}}$, also, $\widetilde{H}(x, 1)\Phi = \mathrm{id}_{\widetilde{X}}$, thus $\widetilde{\psi}\widetilde{\varphi} \simeq \widetilde{H}(x, 1)$ is a homotopy equivalence, $\mathrm{id}_{\widetilde{X}} = \Phi\widetilde{H}(x, 1) \simeq \left(\Phi\widetilde{\psi}\right)\widetilde{\varphi}$, similarly, $\exists \Psi : \widetilde{Y} \to \widetilde{Y}$, such that $\mathrm{id}_{\widetilde{Y}} \simeq \widetilde{\varphi}\left(\widetilde{\psi}\Psi\right)$, by Hatcher 0.11, we have $\widetilde{X} \overset{\widetilde{\varphi}}{\simeq} \widetilde{Y}$
1.
There exists an open neighborhood U of x such that $p_1^{-1}(U) = \bigsqcup_\alpha U_\alpha$, $p_1|_{U_\alpha}$ is a homeomorphism, and $p_2^{-1}(U) = \bigsqcup_\beta V_\beta$, $p_2|_{U_\beta}$ is a homeomorphism, assume there is a lift $\widetilde{p}_1$, fix U_α then $\widetilde{p}_1(U_\alpha) \subset p_2^{-1}(U)$ is connected, thus $\widetilde{p}_1(U_\alpha) \subset V_\beta$ for some β, but since $p_2(\widetilde{p}_1(U_\alpha)) = U$, $\widetilde{p}_1(U_\alpha) = V_\beta$, since $p_1|_{U_\alpha}, p_2|_{V_\beta}$ are homeomorphisms, so is $\widetilde{p}_1|_{U_\alpha} = \left(p_2|_{V_\beta}\right)^{-1} \circ p_1|_{U_\alpha}$, thus $\widetilde{p}_1$ is a covering map
2.
Case I: $n = 1$
Since $\mathbb{RP} \cong S^1$, $\pi_1(\mathbb{RP}, [x_0]) \cong \mathbb{Z}$
Case II: $n \geq 2$
For any $\gamma \in \pi_1(\mathbb{RP}^n, [x_0])$, it can be lifts up to $\widetilde{\gamma}$ which starts at x_0 and ends at x_0 or $-x_0$, since $\pi_1(S^n, x_0) = 0$, thus $\widetilde{\gamma}$ has only two different choice up to homotopy, ending at x_0 or $-x_0$, thus γ also has only to different choice, thus $|\pi_1(\mathbb{RP}^n, [x_0])| = 1 \Rightarrow \pi_1(\mathbb{RP}^n, [x_0]) = \mathbb{Z}/2\mathbb{Z}$
3.
Case I: $n = 1$
The statement is false. Consider $\mathbb{RP} \cong S^1 \xrightarrow{\mathrm{id}} S^1$
Case II: $n \geq 2$
The statement is true. since $\mathbb{RP} \xrightarrow{f} S^1$ would induce a injective group homomorphism $\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^n) \xrightarrow{f_*} \pi_1(S^1) \cong \mathbb{Z}$, hence $f_*(\pi_1(\mathbb{RP})) = \{0\}$, thus there exists a lift $\mathbb{RP} \xrightarrow{\widetilde{f}} \mathbb{R}$, then $\widetilde{f} \simeq c$ by a linear homotopy, so $f \simeq c$
4.
Assume $\pi : X \to T$ is the covering map, $T = I^2/\sim$ is the torus, where $(x, 0) \sim (x, 1), (0, y) \sim (1, y)$
Case I: $X \simeq \{*\}$
$X = \mathbb{R}^2$, $\pi(x, y) = [(x, y) - (\lfloor x \rfloor, \lfloor y \rfloor)]$
Case II: $X \simeq S^1$

$X = I \times \mathbb{R}/\sim$, where $(0, y) \sim (1, y)$, $\pi[(x, y)] = [(x, y) - (\lfloor x \rfloor, \lfloor y \rfloor)]$
Case III: $X \simeq S^1 \times S^1$
$X = 2I \times 2I/\sim$, where $(x, 0) \sim (x, 2), (0, y) \sim (2, y)$, $\pi[(x, y)] = [(x, y) - (\lfloor x \rfloor, \lfloor y \rfloor)]$

2.6 Homework6

Hatcher 1.3.22.
Since G_1, G_2 acts proper discontinuously on X_1, X_2, thus, $\forall x_1 \in X_1, x_2 \in X_2$, there exist neighborhoods $U_1 \ni x_1, U_2 \ni x_2$, such that $g_1U_1 \cap U_1 = \varnothing, g_2U_2 \cap U_2 = \varnothing, \forall g_1 \in G_1, g_2 \in G_2$, thus $(g_1, g_2)(U_1, U_2) \cap (U_1, U_2) = (g_1U_1, g_2U_2) \cap (U_1, U_2) = (g_1U_1 \cap U_1, g_2U_2 \cap U_2) = \varnothing, \forall (g_1, g_2) \in G_1 \times G_2$, hence $G_1 \times G_2$ acts proper discontinuously on $X_1 \times X_2$
Define $\varphi : X_1/G_1 \times X_2/G_2 \to (X_1 \times X_2)/(G_1 \times G_2)$, $([x_1], [x_2]) \mapsto [(x_1, x_2)]$ which is well-defined, since $[x_1] = [y_1], [x_2] = [y_2] \Rightarrow y_1 = g_1 \cdot x_1, y_2 = g_2 \cdot x_2 \Rightarrow (y_1, y_2) = (g_1, g_2) \cdot (x_1, x_2) \Rightarrow [(y_1, y_2)] = [(x_1, x_2)]$
Similarly, there is a well-defined map $\psi : (X_1 \times X_2)/(G_1 \times G_2) \to X_1/G_1 \times X_2/G_2$, $[(x_1, x_2)] \mapsto ([x_1], [x_2])$
As defined above, since $\varphi([U_1] \times [U_2]) = [U_1 \times U_2] \ni [(x_1, x_2)]$, we know φ is continuous, similarly, ψ is also continuous, and since $\varphi \circ \psi = \mathbb{1}, \psi \circ \varphi = \mathbb{1}$, $X_1/G_1 \times X_2/G_2$ and $(X_1 \times X_2)/(G_1 \times G_2)$ are homeomorphic
1.
$\Gamma(Ev(\cdot)) : Aut(X) \to \pi_1(B, b)$ is an isomorphism
(a)
The main reason is that $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ is abelian
Let $\alpha, \beta \in \pi_1(S^1 \times S^1)$ be two generators and $\phi_1, \phi_2 \in Aut(X)$ such that $\Gamma(Ev(\phi_1)) = \alpha, \Gamma(Ev(\phi_2)) = \beta$
Since any element in $\pi_1(B, b)$ can be written uniquely as $\gamma = \alpha^n \star \beta^m$, then $\Gamma(Ev(\phi_1^n \cdot \phi_2^m)) = \Gamma(Ev(\phi_1))^n \star \Gamma(Ev(\phi_2))^m = \gamma$, $\forall x \in F$, we have $x \cdot \gamma = (\alpha^n \star \beta^m) \cdot x = (\beta^m \star \alpha^n) \cdot x$, and $x \cdot \gamma = x \cdot (\phi_1^n \cdot \phi_2^m) = \phi_2^m(\phi_1^n(x))$, they coincide
(b)
The main reason is that $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$ is not abelian
Let $\alpha, \beta \in \pi_1(S^1 \vee S^1)$ be two generators and $\phi_1, \phi_2 \in Aut(X)$ such that $\Gamma(Ev(\phi_1)) = \alpha, \Gamma(Ev(\phi_2)) = \beta$
Then we have $\alpha \star \beta \neq \beta \star \alpha$, consider $(\alpha \star \beta) \cdot x \neq (\beta \star \alpha) \cdot x = \phi_2(\phi_1(x)) = x \cdot (\phi_1 \cdot \phi_2)$, hence these two actions don't coincide
2.
Suppose $\phi : E \to E$ is a covering homomorphism, $\forall x \in E$, let $b = p(x)$ and U be a evenly covered path connected neighborhood of b, let $p^{-1}(U) = \bigsqcup_{i=1}^{n} V_i$, $x \in V_j$, then $\phi(V_j) \subseteq p^{-1}(U)$, since $V_j \cong U$ is connected, so is $\phi(V_j)$, thus $\phi(V_j) \subseteq V_k$ for some k, but then $\phi|_{V_j} = p|_{V_k}^{-1} \circ p|_{V_k}$, thus $\phi(V_j) = V_k$ and ϕ is a local homeomorphism, For any $y \in F := p^{-1}(b)$, there is a path α from $\phi(x)$ to y, let β be the lift of $p \circ \alpha$ starting at x against p, then $\phi \circ \beta$ will be the lift of $p \circ \alpha$ starting at $\phi(x)$ which is just α, thus $\phi(\beta(1)) = y$, ϕ is surjective, since p is finitely sheeted, ϕ is bijective, hence ϕ is a homeomorphism
3.
Lemma: $H \leq G$ is a subgroup satisfies $H \leq g^{-1}Hg, \forall g \in G$, then H is a normal subgroup
Proof: $H \leq g^{-1}H(g^{-1})^{-1} = g^{-1}Hg$, thus $gHg^{-1} \leq H$, hence $H = g^{-1}Hg, \forall g \in G$
Suppose $p_*(\pi_1(E, x)) \leq \pi_1(B, b)$ is a normal subgroup, $\forall x' \in F$, assume $\tilde{\gamma}$ is a path from x to x', $\gamma = p(\tilde{\gamma})$, then $p_*(\pi_1(E, x)) = [\gamma]p_*(\pi_1(E, x))[\gamma]^{-1} = p_*(\pi_1(E, x'))$, hence there is a lift φ of p against p, with $\varphi(x) = x'$, thus φ is a deck transformation
Conversely, $\forall [\gamma] \in \pi_1(B, b)$, there is a lift $\tilde{\gamma}$ starting at x, let $\tilde{\gamma}(1) = x'$, since there is a deck transformation φ with $\varphi(x) = x'$
Thus $p_*(\pi_1(E, x)) \leq p_*(\pi_1(E, x')) = [\gamma]p_*(\pi_1(E, x))[\gamma]^{-1}$, by lemma we know $p_*(\pi_1(E, x)) \leq \pi_1(B, b)$ is a normal subgroup

2.7 Homework7

Hatcher 1.3.10.
Using the homotopy lifting property and its uniqueness we get all connected 2-sheeted coverings up to isomorphisms are

And all connected 3-sheeted coverings up to isomorphisms are

Hatcher 1.3.12.
Consider normal covering

Then $p_*(\pi_1(E,e)) \leq \pi_1(B,b)$ is a normal subgroup, $a^2, b^2, (ab)^4 \in p_*(\pi_1(E,e))$, and any element in $\pi_1(E,e)$ could be written as a product of $a^2, b^2, (ab)^4$, thus $p_*(\pi_1(E,e))$ is contained in the normal subgroup generated by $a^2, b^2, (ab)^4$, hence they are the same
Hatcher 1.3.29.
$p_1 : Y \to Y/G_1$, $y \mapsto G_1 y$, $p_2 : Y \to Y/G_2$, $y \mapsto G_2 y$ are both quotient and covering maps
If $G_2 = gG_1g^{-1}$ are conjugate subgroups, for some $g \in \texttt{Homeo}(Y)$, define $\overline{g} : Y/G_1 \to Y/G_2$, $G_1 y \mapsto G_2 gy$, it is well-defined since $\overline{g}(G_1 g y) = G_2 g g_1 y = G_2 g_2 g y = G_2 g y$ for some $g_2 \in G_2$ such that $g g_1 g^{-1} = g_2$
Since, thus the following diagram commutes

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "p_1"] \arrow[dr, "p_2 g"] & Y \arrow[d, "p_2"] \\ Y/G_1 \arrow[r, "\overline{g}"] & Y/G_2 \end{tikzcd}$$

Also, $p_2 g$ is continuous and p_1 is a quotient map, hence $\overline{g}$ is continunous
Similarly, define $\overline{g}^{-1} : Y/G_2 \to Y/G_1$, $G_2 y \mapsto G_1 g^{-1} y$ is also continuous, then $\overline{g}, \overline{g}^{-1}$ are inverses of each other, hence $\overline{g} : Y/G_1 \to Y/G_2$ is a homeomorphism
Conversely, suppose $\varphi : Y/G_1 \to Y/G_2$ is a homeomorphism, let $\psi = \varphi^{-1}$, Since Y is path connected, locally path connected, simply connected, choose some $y_1, y_2 \in Y$, such that $\varphi(G_1 y) = G_2 y$, then there exists a lift of φp_1 against p_2, $\Phi : Y \to Y$ such that the following diagram commutes with $\Phi(y_1) = y_2$

$$\begin{tikzcd} Y \arrow[r, "\Phi"] \arrow[d, "p_1"] & Y \arrow[d, "p_2"] \\ Y/G_1 \arrow[r, "\varphi"] & Y/G_2 \end{tikzcd}$$

Similarly, $\exists \Psi : Y \to Y$ such that the following diagram commutes with $\Psi(y_2) = y_1$

$$\begin{tikzcd}
Y \arrow[d, "p_1"] & Y \arrow[l, "\Psi"'] \arrow[d, "p_2"] \\
Y/G_1 & Y/G_2 \arrow[l, "\psi"]
\end{tikzcd}$$

Then $\Psi\Phi : Y \to Y$ is a deck transformation sending y_1 to y_1, thus $\Psi\Phi = \mathbb{1}$, similarly, $\Phi\Psi = \mathbb{1}$, hence $\Phi \in \text{Homeo}(Y), \Psi = \Phi^{-1}$, we need to show $\Phi G_1 \Phi^{-1} = G_2$
$\forall y \in Y$, $G_2\Phi g_1\Phi^{-1}y = p_2\Phi g_1\Phi^{-1}y = \varphi p_1 g_1\Phi^{-1}y = \varphi G_1 g_1\Phi^{-1}y = \varphi G_1\Phi^{-1}y = \varphi p_1\Phi^{-1}y = p_2\Phi\Phi^{-1}y = p_2 y = G_2 y$ which implies $\Phi g_1\Phi^{-1}$ is a deck transformation of $p_2 : Y \to Y/G_2$, hence $\Phi g_1\Phi^{-1} \in G_2 \Rightarrow \Phi G_1\Phi^{-1} \leq G_2$, similarly, $\Psi G_2\Psi^{-1} \leq G_1 \Rightarrow G_2 \leq \Phi G_1\Phi^{-1}$, thus $G_2 = \Phi G_1 \Phi^{-1}$

1.

Suppose $E_1 \overset{p_1}{\to} B$, $E_2 \overset{p_2}{\to} B$ are any two simply connected covers, $b = p_1(e_1) = p_2(e_2)$ for some $e_1 \in E_1, e_2 \in E_2$, then $\exists\Phi : E_1 \to E_2$ with $\Phi(e_1) = e_2$ and $\exists\Psi : E_2 \to E_1$ with $\Phi(e_2) = e_1$ such that the following diagram commutes

$$\begin{tikzcd}
E_1 \arrow[r, "\Phi"] \arrow[d, "p_1"] & E_2 \arrow[r, "\Psi"] \arrow[d, "p_2"] & E_1 \arrow[d, "p_1"] \\
B & B & B
\end{tikzcd}$$

Thus $\Psi\Phi : E_1 \to E_1$ sends e_1 to e_1, by the uniqueness of liftings, $\Psi\Phi = \mathbb{1}$, similarly, $\Phi\Psi = \mathbb{1}$, hence $E_1 \overset{\Phi}{\to} E_2$ is a homeomorphism

2.

Denote the set of G-equivariant maps $G/H \to Y$ as Y
Define $\varphi : Y \to X^H$, by $\varphi(f) = f(H)$, since f is G-equivariant, $hf(H) = f(hH) = f(H)$, thus $f(H) \in X^H$ and also define $\psi : X^H \to Y$ by $\psi(x) = f$ where $f(gH) = gx$ is a G-equivariant map, it is obvious that $\varphi\psi = \psi\varphi = \mathbb{1}$, we only need to show both φ and ψ are continuous
Let $X^{G/H}$ be the set of continuous functions from G/H to X, $\{U \cup X^H \,|U \text{ open in } X\}$ are the open sets of X^H, and $\bigcap_{i=1}^{n} \left\{ f \in X^{G/H} \,| f(g_i(H)) \in U_{g_i H} \right\}$ form a basis for $X^{G/H}$, where $U_{g_i H}$ is just some open set in X with $g_i H$ as its index
Thus

$$\begin{aligned}
\bigcap_{i=1}^{n} \left\{ f \in X^{G/H} \,| f(g_i(H)) \in U_{g_i H} \right\} \bigcap Y &= \bigcap_{i=1}^{n} \{ f \in Y \,| g_i f(H) \in U_{g_i H} \} \\
&= \bigcap_{i=1}^{n} \left\{ f \in Y \,\middle| f(H) \in g_i^{-1} U_{g_i H} \cap X^H \right\}
\end{aligned}$$

Form a basis for Y, Since

$$\varphi\left(\left\{ f \in Y \,\middle| f(H) \in U \cap X^H \right\}\right) \subseteq U \cap X^H$$

And

$$\psi\left(\bigcap_{i=1}^{n} g_i^{-1} U_{g_i H} \bigcap X^H\right) \subseteq \bigcap_{i=1}^{n} \left\{ f \in Y \,\middle| f(H) \in g_i^{-1} U_{g_i H} \cap X^H \right\}$$

φ and ψ are both continuous

3.

Let $X = \mathbb{R} \times I$ which is simply connected, define group action $\mathbb{Z} \times X \to X$ by $(n, (x, t)) \mapsto f^n(x, t)$, then $p : X \to X/\mathbb{Z}$ is a universal cover

(a)

Since isomorphism classes of covering spaces are in one to one correspondence to conjugacy classes of subgroups, all path connected covering spaces of the Mobius band up to a covering isomorphism are $p_n : X/n\mathbb{Z} \to X/\mathbb{Z}, n\mathbb{Z}(x,t) \mapsto \mathbb{Z}(x,t)$, where $n = 0, 1, 2\cdots$, thus $p_{n*}(X/n\mathbb{Z}) = n\mathbb{Z}$

(b)

Since $m|n \Leftrightarrow p_{m*}(X/m\mathbb{Z}) = m\mathbb{Z} \leq n\mathbb{Z} = p_{n*}(X/n\mathbb{Z})$, hence all homomorphisms between these covering spaces are $\varphi_{n,m} : X/n\mathbb{Z} \to X/m\mathbb{Z}, n\mathbb{Z}(x,t) \mapsto m\mathbb{Z}(x,t)$

2.8 Homework8

Hatcher 1.2.3.
Assume this finite set of points are $S = \{p_1, \cdots, p_n\}$, let $U_i = \mathbb{R}^n \setminus \{p_i\}$, then $U_i \cong S^n$ is simply connected, $U_i \cap U_j$ is path connected and $\{U_i\}$ forms an open cover for $\mathbb{R}^n \setminus S$, by Van-Kampen's theorem, $*\pi_1(U_i) \to \pi_1(\mathbb{R}^n \setminus S)$ is surjective, hence $\pi_1(\mathbb{R}^n \setminus S) = 0$
Hatcher 1.2.8.
Could imagine put a torus T_1 exactly on top of the other T_2, then they overlap on a circle C, C has a neighborhood W which is open and can be deformation retract to C, let $U_1 = T_1 \cup W, U_2 = T_2 \cup W$, then U_1 is homotopic to T_1, U_2 is homotopic to T_2, $U_1 \cap U_2 = W$ is homotopic to C, by Van-Kampen's theorem, the resulting space X has fundamental group

$$\begin{aligned}\pi_1(X) &= \frac{F(a,b)/\langle aba^{-1}b^{-1}\rangle * F(c,d)/\langle cdc^{-1}d^{-1}\rangle}{\langle ac^{-1}\rangle}\\ &= F(a,b,c,d)/\langle aba^{-1}b^{-1}, cdc^{-1}d^{-1}, ac^{-1}\rangle\\ &= F(a,b,d)/\langle aba^{-1}b^{-1}, ada^{-1}d^{-1}\rangle\\ &= F(b,d) \times \langle a\rangle\\ &\cong F_2 \times \mathbb{Z}\end{aligned}$$

Which isn't surprising since $X = (S^1 \vee S^1) \times S^1$, $\pi_1(X) = F_2 \times \mathbb{Z}$
Hatcher 1.2.21.
Let $q : X \times Y \times I \to X * Y$ be the quotient map
First, assume Y is path connected, consider $U = q\left(X \times Y \times [0, \frac{2}{3})\right), V = q\left(X \times Y \times (\frac{1}{3}, 1]\right)$ which form an open cover for $X * Y$, $X \times Y \times [0, \frac{2}{3})$ deformation retracts to $q(X \times Y \times \{0\}) \cong X$, $X \times Y \times (\frac{1}{3}, 1]$ deformation retracts to $q(X \times Y \times \{1\}) \cong Y$, $U \cap V = X \times Y \times (\frac{1}{3}, \frac{2}{3})$ is path connected and deformation retracts to $q(X \times Y \times \{\frac{1}{2}\}) \cong X \times Y$, thus we have $\pi_1(U) = \pi_1(X), \pi_1(V) = \pi_1(Y)$ and $i_{U*}(\pi_1(U \cap V)) = \pi_1(X), i_{V*}(\pi_1(U \cap V)) = \pi_1(Y)$, where $i_U : U \cap V \to U, i_V : U \cap V \to V$ are inclusion maps, by Van-Kampen's theorem, we get $\pi_1(X * Y) = 0$
In general, consider $\{Y_i\}$ to be the path connected components of Y, let $Z = q\left(X \times Y \times [0, \frac{1}{2})\right), U_i = (X \times Y_i \times I) \cup Z$, $\{U_i\}$ forms an open cover of $X * Y$, and $U_i \cap U_j = Z$ deformation retracts to $q(X \times Y \times \{0\}) \cong X$, which is path connected, by Van-Kampen's theorem, $*\pi_1(x \times Y_i \times I) \cong *\pi_1(U_i) \to \pi_1(X * Y)$ is surjective, hence $\pi_1(X * Y) = 0$
1.
X could be given the following CW structure: 2 0-cells, 3 1-cells and 2 2-cells

Let X^1 be the 1-skeleton of X, then by attaching these two cells on X^1, we have $\pi_1(X) = \pi_1(X^1)/\langle ca^{-1}\rangle \cong \mathbb{Z}$
2.
Suppose M has α_n n-cells, and N has β_n n-cells, then $M \times N$ also has a CW structure with $\gamma_n = \sum_{k+l=n} \alpha_k \beta_l$ n-cells, thus

$$\begin{aligned}\chi(M \times N) &= \sum_{n \geq 0} (-1)^n \gamma_n\\ &= \sum_{n \geq 0} \sum_{k+l=n} (-1)^{k+l} \alpha_k \beta_l\\ &= \left(\sum_{k \geq 0} (-1)^k \alpha_k\right)\left(\sum_{l \geq 0} (-1)^l \beta_l\right)\\ &= \chi(M)\chi(N)\end{aligned}$$

3.

For a connected graph X, assuming it has v 0-cells and e 1-cells, find a maximal tree T of X, then we have $\chi(X) + n = v - (e - n) = \chi(T) = 1 \Rightarrow n = 1 - \chi(X)$ where n is the number of generators of free group $\pi_1(X)$

Let X be n copies of S^1 wedge at a point, then $G = \pi_1(X)$ is a free group on n generators, for a subgroup $H \leq G$ of index k, we could find a covering $p : E \to X$ with $p_*(\pi_1(E)) = H$, then E is also a connected graph with kv 0-cells and ke 1-cells, thus we have the number of generators of free group H is $1 - \chi(E) = 1 - (kv - ke) = 1 - k\chi(X) = 1 - k(1 - n) = 1 - k + nk$

2.9 Homework9

1.

Thus the standard form is $eje^{-1}j^{-1}ihi^{-1}h^{-1}$ which is M_2

2.

(a)

$F : G \to \mathsf{Set}$ is a functor, then $F(*) = S$ is a set, $F(g) : S \to S$ is a map, also, $F(e) = \mathrm{id}_S, F(gh) = F(g)F(h)$, thus F is precisely a group action and vice versa

(b)

$F : G \to \mathsf{Vec}$ is a functor, then $F(*) = V$ is a K-vector space, $F(g) : V \to V$ is a linear transformation, also, $F(e) = \mathrm{id}_V, F(gh) = F(g)F(h)$, thus F is precisely a group representation and vice versa

(c)

Define $F : \mathsf{Set} \to \mathsf{Group}$ by sending set S to the free group $F(S)$ generated by S, sending map $\sigma : S \to T$ to $F(\sigma) : F(S) \to F(T)$ by sending the corresponding generators, $F(\mathrm{id}_S) = \mathrm{id}_{F(S)}, F(\mu\sigma) = F(\mu)F(\sigma)$, thus F is a functor

(d)

If $F : \mathbb{P}(\{0,1\}) \to \mathsf{Top}$ is a functor, denote $F(\varnothing) = A, F(\{0\}) = X, F(\{1\}) = Y, F(\{0,1\}) = Z$ and $F(\varnothing \to \{0\}) = f, F(\varnothing \to \{1\}) = g, F(\{0\} \to \{0,1\}) = \alpha, F(\{1\} \to \{0,1\}) = \beta$,$F(\varnothing \to \{0,1\}) = \alpha \cdot f = \beta \cdot g$, then the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"] & Z \end{tikzcd}$$

This induce a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "\iota_X"] \arrow[ddr, bend left, "\alpha"] & \\
Y \arrow[r, "\iota_Y"] \arrow[drr, bend right, "\beta"] & X \cap_A Y \arrow[dr, dotted, "h"] & \\
 & & Z
\end{tikzcd}$$

3.

(a)

Define composition as $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ then it is associative

$$\begin{aligned}
&(f'', g'') \circ [(f', g') \circ (f, g)] = (f'', g'') \circ (f' \circ f, g' \circ g) = (f'' \circ (f' \circ f), g' \circ (g' \circ g)) \\
&= ((f'' \circ f') \circ f, (g' \circ') \circ g) = (f'' \circ f', g'' \circ g) \circ (f, g) = [(f'', g'') \circ (f', g')] \circ (f, g)
\end{aligned}$$

For any object (C, D) in $\mathscr{C} \times \mathscr{D}$, we have identity $(1_C, 1_D) : (C, D) \to (C, D)$, then $\forall (f, g) : (C, D) \to (C', D')$, $(f', g') : (C', D') \to (C, D)$, we have

$$(f, g) \circ (1_c, 1_D) = (f \circ 1_C, g \circ 1_D) = (f, g), (1_C, 1_D) \circ (f', g') = (1_C \circ f', 1_D \circ g') = (f', g')$$

Therefore, $\mathscr{C} \times \mathscr{D}$ is indeed a category

(b)

The group operation $F : G \times G \to G, F(g_1, g_2) = g_1 g_2$ won't be a functor, if so, then $g_1 g_2 g_1' g_2' = F(g_1 g_2) F(g_1' g_2') = F[(g_1, g_2)(g_1', g_2')] = F(g_1 g_1', g_2 g_2') = g_1 g_1' g_2 g_2' \Rightarrow g_2 g_1' = g_1' g_2$ so that g_1' and g_2 are commutative which is not true in general

4.

h induces $[h(x, \cdot)] = [h_x] : h_0(x) \to h_1(x)$, such that the following diagram commutes

$$\begin{tikzcd}
h_0(x) \arrow[r, "{[h_x]}"] \arrow[d, "{[h_0 \circ \gamma]}"] & h_1(x) \arrow[d, "{[h_1 \circ \gamma]}"] \\
h_0(x') \arrow[r, "{[h_{x'}]}"] & h_1(x')
\end{tikzcd}$$

Where $[\gamma] \in \pi_1(X, x, x')$, thus h induces a natural transformation between the induced functors $(h_0)_*$ and $(h_1)_*$ **5.**

(a)

Reflexivity: $h_n = 0$ is a chain homotopy between f and f since $h_{n-1}\partial_n^C + \partial_{n+1}^D h_n = 0 = f_n - f_n$

Symmetry: If h_n is a homotopy between f and g, then $-h_n$ is a chain homotopy between g and f since $-h_{n-1}\partial_n^C - \partial_{n+1}^D h_n = 0 = -(g_n - f_n) = f_n - g_n$

Transitivity: If α_n is a chain homotopy between f and g, β_n is a chain homotopy between g and h, then

$$\begin{aligned}
h_n - f_n &= (h_n - g_n) + (g_n - f_n) \\
&= (\beta_{n-1}\partial_n^C + \partial_{n+1}^D \beta_n) + (\alpha_{n-1}\partial_n^C + \partial_{n+1}^D \alpha_n) \\
&= (\beta_{n-1} + \alpha_{n-1})\partial_n^C + \partial_{n+1}^D(\beta_n + \alpha_n)
\end{aligned}$$

Thus $\beta_n + \alpha_n$ is a chain homotopy between f and h

(b)

$$
\begin{aligned}
\partial^2 \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} &= \partial \begin{pmatrix} \partial(c_0) + (-1)^n c_2 \\ \partial(c_1) + (-1)^{n+1} c_2 \\ \partial(c_2) \end{pmatrix} \\
&= \begin{pmatrix} \partial\left(\partial(c_0) + (-1)^n c_2\right) + (-1)^{n-1}\partial(c_2) \\ \partial\left(\partial(c_1) + (-1)^{n+1} c_2\right) + (-1)^n\partial(c_2) \\ \partial^2(c_2) \end{pmatrix} \\
&= \begin{pmatrix} \partial^2(c_0) + (-1)^n\partial(c_2) + (-1)^{n-1}\partial(c_2) \\ \partial^2(c_1) + (-1)^{n+1}\partial(c_2) + (-1)^n\partial(c_2) \\ \partial^2(c_2) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

(c)

Define $\iota_0(c) = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}$, $\iota_1(c) = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$, then

$$
\partial\iota_0(c) = \partial \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial(c) \\ 0 \\ 0 \end{pmatrix} = \iota_0\partial(c)
$$

$$
\partial\iota_1(c) = \partial \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial(c) \\ 0 \end{pmatrix} = \iota_1\partial(c)
$$

(d)

Define $\xi_n : C_n \to (C_* \otimes D_*)_{n+1}$ by $\xi_n(c) = \begin{pmatrix} 0 \\ 0 \\ (-1)^n c \end{pmatrix}$, then

$$
\begin{aligned}
(\partial\xi_n + \xi_{n-1}\partial)(c) &= \partial \begin{pmatrix} 0 \\ 0 \\ (-1)^n c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ (-1)^{n-1}\partial(c) \end{pmatrix} \\
&= \partial \begin{pmatrix} (-1)^{n+1}(-1)^n c \\ (-1)^{n+2}(-1)^n c \\ (-1)^n\partial(c) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ (-1)^{n-1}\partial(c) \end{pmatrix} \\
&= \begin{pmatrix} -c \\ c \\ 0 \end{pmatrix}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\left(\partial(h\xi_n) + (h\xi_{n-1})\partial\right)(c) &= h(\partial\xi_n + \xi_{n-1}\partial)(c) \\
&= h \begin{pmatrix} -c \\ c \\ 0 \end{pmatrix} = h \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix} - h \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \\
&= h \circ \iota_1(c) - h \circ \iota_0(c)
\end{aligned}
$$

Thus $h\xi$ is the corresponding chain homotopy between $h \circ \iota_0$ and $h \circ \iota_1$

2.10 Homework10

Hatcher 2.1.11.
Suppose $r : X \to A$ is a retraction, $i : A \to X$ is the inclusion, then $r \circ i = \mathrm{id}_A$, $i_*[\sum n_\alpha \sigma_\alpha^n] = [\sum n_\alpha i \circ \sigma_\alpha^n]$, if $[\sum n_\alpha i \circ \sigma_\alpha^n] = 0$, $\sum n_\alpha i \circ \sigma_\alpha^n = \partial \left(\sum n_\beta \tau_\beta^{n+1}\right)$ where $\tau_\beta^{n+1} : \Delta^{n+1} \to X$ par
Then we have $\sum n_\alpha \sigma_\alpha^n = r\left(\sum n_\alpha i \circ \sigma_\alpha^n\right) = r\left(\partial\left(\sum n_\beta \tau_\beta^{n+1}\right)\right) = \partial\left(\sum n_\beta r \circ \tau_\beta^{n+1}\right)$, $r \circ \tau_\beta^{n+1} : \Delta^{n+1} \to A$
Hence $[\sum n_\alpha \sigma_\alpha^n] = 0$, i_* is injective
Hatcher 2.1.30.
If $\alpha : A \to B$ is an isomorphism, so is $\alpha^{-1} : B \to A$, then we can write the remaining map as a composition of the others, so it is also an isomorphism
1.
Suppose h_n is a chain homotopy between f_n and g_n, then $g_n - f_n = \partial_{n+1}^D h_n + h_{n-1}\partial_n^C$
$\forall [f_n(c)] \in H_n(f)$, where $c \in \ker \partial_n^C$, $g_n(c) = f_n(c) + \partial_{n+1}^D h_n(c) + h_{n-1}\partial_n^C(c) = f_n(c) + \partial_{n+1}^D h_n(c)$, since $\partial_{n+1}^D h_n(c) \in \partial_{n+1}^D(D_{n+1})$, $[f_n(c)] = [g_n(c)] \in H_n(g)$, $H_n(f) \subseteq H_n(g)$, similarly, $H_n(g) \subseteq H_n(f)$, thus $H_n(f) = H_n(g)$
2.
$(a) \Rightarrow (b)$: Since the sequence is split exact, the following diagram commutes

$$\begin{tikzcd}
 & & B \arrow[dd, "\varphi"] \arrow[dr, "p"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "\iota_A"'] & & C \arrow[r] & 0 \\
 & & A \oplus C \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

Where π is the projection, ι_A is the inclusion
Define $s = \varphi^{-1}\iota_C$, where $\iota_C : C \to A \oplus C$ is the inclusion, then $p \circ s = p \circ \varphi^{-1} \circ \iota_C = \pi \circ \varphi^{-1} \circ \varphi \circ \iota_C = \pi \circ \iota_C = \mathrm{id}_C$
$(b) \Rightarrow (c)$: $\forall b \in B$, let $b_1 = b - s \circ p(c)$, then $p(b_1) = p(b) - p \circ s \circ p(b) = p(b) - p(b) = 0$, $\Rightarrow b \in \ker p = \mathrm{im} i$, since i is injective, $\exists_1 a \in A$, such that $i(a) = b$, define $a := r(b)$, then we have $r \circ i(a) = a, \forall a \in A$, hence $r \circ i = \mathrm{id}_A$
$(c) \Rightarrow (a)$: $\forall b \in B$, define $\varphi : B \to A \oplus C, b \mapsto (r(b), p(b))$, $\forall a \in A, a = r \circ i(a) \in r(B)$, also $C = p(B)$, thus $A \oplus C \subseteq \varphi(B)$, φ is surjective
On the other hand, if $\varphi(b) = 0$, then $r(b) = p(b) = 0$, $\exists a \in A$ such that $i(a) = b$, but then $a = r \circ i(a) = r(b) = 0 \Rightarrow b = i(a) = 0$, hence φ is also injective, φ is an isomorphism
$\forall b \in B, \pi \circ \varphi(b) = \pi(r(b), p(b)) = p(b)$, $\forall a \in A, \varphi \circ i(a) = (r \circ i(a), p \circ i(a)) = (a, 0) = \iota_A(a)$, thus the diagram above commutes
3.
If $C = 0$, then $A \to B$ is obviously surjective and $D \to E$ is obviously injective
Conversely, since $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D \xrightarrow{\xi} E$ is exact, if α is surjective, ξ is injective, $\forall c \in C$, we have

$$\begin{aligned}
\xi \circ \gamma(c) = 0 &\Rightarrow \gamma(c) = 0 \\
&\Rightarrow \exists b \in B, \text{such that } \beta(b) = c \\
&\Rightarrow \exists a \in A, \text{such that } \alpha(a) = b \\
&\Rightarrow c = \beta \circ \alpha(a) = 0
\end{aligned}$$

Hence $C = 0$
4.
Define $\varepsilon : C_0(X) \to \mathbb{Z}$ by sending $\sum n_\alpha \sigma_\alpha^0$ to $\sum n_\alpha$
$\forall \sigma : \Delta^1 \to X$, $\varepsilon(\partial\sigma) = \varepsilon\left(\sigma|[v_1] - \sigma|[v_0]\right) = 1 - 1 = 0$, thus $\varepsilon \circ \partial = 0$

Also, $\forall \sum n_\alpha \sigma_\alpha^0 \in C_0(X)$, if $\varepsilon(\sum n_\alpha \sigma_\alpha^0) = \sum n_\alpha = 0$, then there are same number of σ_α^0's with coefficients 1 and coefficients -1, and for each pair, there is a path: $\Delta^1 \to X$ from the one with coefficient -1 to the one with coefficient 1, then the sum of these paths $\xi \in C_1(X)$, and $\partial\xi = \sum n_\alpha \sigma_\alpha^0$, hence $C_1(X) \xrightarrow{\partial} C_0X \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is exact, $H_0(X) = C_0(X)/\mathrm{im}\partial = C_0(X)/\ker\varepsilon \cong \mathbb{Z}$

2.11 Homework11

Hatcher 2.1.20.
Let $q : X \times I \to SX \cong X \times I/X \times \{0\} \cup X \times \{1\}$ is the quotient map, define $U = q\left(X \times [0, \frac{2}{3})\right)$, $V = q\left(X \times (\frac{1}{3}, 0]\right)$ are both contractible, and $U \cap V = q\left(X \times (\frac{1}{3}, \frac{2}{3})\right) \cong X$
Apply Mayer-Vietoris theorem we have

$$H_{n+1}(U) \oplus H_{n+1}(V) \to H_{n+1}(SX) \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \quad (n \geq 1)$$

is exact, thus

$$0 \to H_{n+1}(SX) \to H_n(X) \to 0 \quad (n \geq 1)$$

is exact, hence $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X) \quad (n \geq 1)$
Also

$$H_1(U) \oplus H_1(V) \to H_1(SX) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(SX) \to 0$$

is exact and SX is path connected, thus

$$0 \to H_1(SX) \to H_0(X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

is exact
Since $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is surjective and $\mathbb{Z} \oplus \mathbb{Z}$ is free, $\operatorname{Im}(H_0(X) \to \mathbb{Z} \oplus \mathbb{Z}) = \operatorname{Ker}(\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}) \cong \mathbb{Z}$, thus

$$0 \longrightarrow H_1(SX) \xrightarrow{\alpha} H_0(X) \underset{\gamma}{\overset{\beta}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

is exact, since $\mathbb{Z}$ is free, $\exists \gamma : \mathbb{Z} \to H_0(X)$ such that $\beta \circ \gamma = \mathrm{id}_{\mathbb{Z}}$, hence by splitting lemma, $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$, on the other hand, $H_0(X) \cong \widetilde{H}_0 \oplus \mathbb{Z}$, thus $\widetilde{H}_1(SX) \cong \widetilde{H}_0(X)$, $\widetilde{H}_0(SX) = 0 = \widetilde{H}_{-1}(X)$ and $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X) \quad (n < -1)$ is trivial
More generally, let $q : X \times I \to CX \cong X \times I/X \times \{1\}$ be the quotient map, $Y = \bigsqcup_{i=1}^{m} CX \Big/ \sim$ with their bases identified, define $U = \bigsqcup_{i=1}^{m-1} q\left(X \times [0, 1)\right) \Big/ \sim \simeq \{*\}$, $V = CX \bigsqcup \bigsqcup_{i=1}^{m-1} q\left(X \times (\dfrac{1}{2}, 1]\right) \Big/ \sim \simeq \bigsqcup_{i=1}^{m-1} \{*\}$
Apply Mayer-Vietoris theorem we have

$$H_{n+1}(U) \oplus H_{n+1}(V) \to H_{n+1}(Y) \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \quad (n \geq 1)$$

is exact, thus

$$0 \to H_{n+1}(Y) \to \bigoplus_{i=1}^{m-1} H_n(X) \to 0 \quad (n \geq 1)$$

is exact, hence $H_{n+1}(Y) \cong \bigoplus_{i=1}^{m-1} H_n(X) \quad (n \geq 1)$
Also

$$H_1(U) \oplus H_1(V) \to H_1(Y) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(Y) \to 0$$

is exact and Y is path connected, thus

$$0 \to H_1(Y) \to \bigoplus_{i=1}^{m-1} H_0(X) \to \mathbb{Z}^{m-1} \oplus \mathbb{Z} = \mathbb{Z}^m \to \mathbb{Z} \to 0$$

is exact, since $\mathbb{Z}^m$ is free

$$0 \to H_1(Y) \to \bigoplus_{i=1}^{m-1} H_0(X) \to \mathbb{Z}^{m-1} \to 0$$

is exact

Since $\mathbb{Z}^{m-1}$ is free, by splitting lemma, $\bigoplus_{i=1}^{m-1} H_0(X) \cong H_0(Y) \oplus \mathbb{Z}^{m-1}$, on the other hand, $\bigoplus_{i=1}^{m-1} H_0(X) \cong \bigoplus_{i=1}^{m-1} \left(\widetilde{H}_0(X) \oplus \mathbb{Z}\right) \cong \left(\bigoplus_{i=1}^{m-1} \widetilde{H}_0(X)\right) \bigoplus \mathbb{Z}^{m-1}$, thus $\widetilde{H}_1(Y) \cong \bigoplus_{i=1}^{m-1} \widetilde{H}_0(X)$

$\widetilde{H}_0(Y) = 0 = \bigoplus_{i=1}^{m-1} \widetilde{H}_{-1}(X)$, and $\widetilde{H}_{n+1}(Y) \cong \bigoplus_{i=1}^{m-1} \widetilde{H}_n(X) \quad (n < -1)$ is trivial

Hatcher 2.1.21.

Let $q : X \times I \to SX$ still be the quotient map, define $U = q\left(X \times [0, \frac{2}{3})\right)$, $V = q\left(X \times (\frac{1}{3}, 0]\right)$, $v = q(X \times \{0\})$, $v = q(X \times \{1\})$, then define

$$s(\sigma) = [v_0, \cdots, v_n, v] - [v_0, \cdots, v_n, w]$$

Where $\sigma = [v_0, \cdots, v_n]$, and $[v_i, v]$ be the segment from v_i to v along $v_i \times \left[0, \frac{1}{2}\right]$, $[v_i, w]$ be the segment from v_i to w along $v_i \times \left[\frac{1}{2}, 1\right]$

$$\begin{aligned}
\partial s[v_0, \cdots, v_{n+1}] &= \partial\left([v_0, \cdots, v] - [v_0, \cdots, w]\right) \\
&= \sum_{k=0}^{n+1} (-1)^k \left([v_0, \cdots, \hat{v}_k, \cdots, v] - [v_0, \cdots, \hat{v}_k, \cdots, w]\right) \\
s\partial[v_0, \cdots, v_{n+1}] &= s\left(\sum_{k=0}^{n+1} (-1)^k [v_0, \cdots, \hat{v}_k \cdots, v_{n+1}]\right) \\
&= \sum_{k=0}^{n+1} (-1)^k \left([v_0, \cdots, \hat{v}_k, \cdots, v] - [v_0, \cdots, \hat{v}_k, \cdots, w]\right)
\end{aligned}$$

Thus $\partial s = s\partial$, hence s is a chain map which could induce a homomorphism from $\tilde{H}_n(X)$ to $\tilde{H}_{n+1}(SX)$

Suppose $\sum n_i \sigma_i \in Z_n(X)$, namely

$$0 = \partial\left(\sum n_i \sigma_i\right) = \partial\left(\sum n_i [v_0^i, \cdots, v_n^i]\right) = \sum n_i \sum_{k=0}^{n} (-1)^k [v_0^i, \cdots, \hat{v}_k^i, \cdots, v_n^i]$$

Then

$$s\left(\sum n_i \sigma_i\right) = \sum n_i \left([v_0^i, \cdots, v] - [v_0^i, \cdots, w]\right) = \left(\sum n_i [v_0^i, \cdots, v] - \sum n_i [v_0^i, \cdots, w]\right)$$

But $(j_U - j_V)\left(\sum n_i [v_0^i, \cdots, v]\right) \bigoplus \left(\sum n_i [v_0^i, \cdots, v]\right) = s\left(\sum n_i \sigma_i\right)$, also

$$\begin{aligned}
(i_U \oplus i_V)\left((-1)^{n+1} \sum n_i \sigma_i\right) &= \left((-1)^{n+1} \sum n_i \sigma_i\right) \bigoplus \left((-1)^{n+1} \sum n_i \sigma_i\right) \\
&= \partial\left(\sum n_i \sum_{k=0}^{n} (-1)^k [v_0^i, \cdots, \hat{v}_k^i, \cdots, v] + \sum n_i [v_0^i, \cdots, v]\right) \bigoplus \\
&\quad \partial\left(\sum n_i \sum_{k=0}^{n} (-1)^k [v_0^i, \cdots, \hat{v}_k^i, \cdots, w] + \sum n_i [v_0^i, \cdots, w]\right) \\
&= \partial\left(\sum n_i [v_0^i, \cdots, v]\right) \bigoplus \partial\left(\sum n_i [v_0^i, \cdots, w]\right)
\end{aligned}$$

Therefore $\delta s = (-1)^{n+1}\mathrm{id}$, since δ is an isomorphism, s also induce an isomorphism

1.

Let U be a small neighborhood of the left S^1, V be a small neighborhood of the right S^1, $U, V \simeq S^1$, $U \cap V \simeq \{*\}$

Apply Mayer-Vietoris theorem, we have

$$H_{n+1}(U) \oplus H_{n+1}(V) \to H_{n+1}(S^1 \vee S^1) \to H_n(U \cap V) \quad (n \geq 1)$$

is exact, thus $0 \to H_{n+1}(S^1 \vee S^1) \to 0 \quad (n \geq 1)$ is exact, hence $H_k(S^1 \vee S^1) = 0 \quad (k \geq 2)$ Since $S^1 \vee S^1$ is path connected, $H_0(S^1 \vee S^1) = \mathbb{Z}$ and

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(S^1 \vee S^1) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(S^1 \vee S^1) \to 0$$

is exact, thus

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(S^1 \vee S^1) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

hence $0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_1(S^1 \vee S^1) \to 0$ is exact, $H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

2.

Let U be a small neighborhood of A, V be the interior of the n-cell Δ^n, then V is contractible, $U \simeq A$, $U \cap V \simeq S^{n-1}$

Apply Mayer-Vietoris theorem, we have

$$H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(U \cap V) \to H_{n-1}(U) \oplus H_{n-1}(V) \to H_{n-1}(X) \to H_{n-2}(U \cap V)$$

is exact, thus

$$H_n(S^{n-1}) \to H_n(A) \oplus H_n(*) \to H_n(X) \to H_{n-1}(S^{n-1}) \to H_{n-1}(A) \oplus H_{n-1}(*) \to H_{n-1}(X) \to H_{n-2}(S^{n-1})$$

is exact

(a)

If $n > 2$

$$0 \to H_n(A) \to H_n(X) \to \mathbb{Z} \to H_{n-1}(A) \to H_{n-1}(X) \to 0$$

is exact

(b)

If $k \geq 1, k \neq n, n-1$, then $0 \to H_k(A) \to H_k(X) \to 0$ is exact, thus $H_k(A) \cong H_k(X)$

3.

If f is not surjective, without loss of generality, assume $\mathrm{Im} f \subset S^n \setminus \{N\}$, where N is the north pole, then $f \simeq c$, $H_n f = H_n c = 0$ which is a contradiction, hence f is surjective

2.12 Homework12

Hatcher 2.1.17.

(a)

Assume $A = \{a_1, \cdots, a_n\}$, $\forall[\xi_0] \in C_0(S^2, A)$, $\xi_0 = \sum n_k v_0^k$ is a sum of points on $S^2 \setminus A$, then we can find paths from a_1 to v_0^k contained in $S^2 \setminus A$, then define $\xi_1 = \sum n_k[a_1, v_0^k] \in C_1(S^2)$, $\partial[\xi_1] = [\xi_0]$, hence $\partial C_1(S^2, A) = C_0(S^2, A)$, $H_0(S^2, A) = 0$, similarly we have $H_0(S^1 \vee S^1, A) = 0$

$$H_1(S^2) \to H_1(S^2, A) \to H_0(A) \to H_0(S^2) \to H_0(S^2, A)$$

$$H_1(A) \to H_1(S^1 \vee S^1) \to H_1(S^1 \vee S^1, A) \to H_0(A) \to H_0(S^1 \vee S^1) \to H_0(S^1 \vee S^1, A)$$

are exact, thus

$$0 \to H_1(S^2, A) \to \mathbb{Z}^n \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}^2 \to H_1(S^1 \vee S^1, A) \to \mathbb{Z}^n \to \mathbb{Z} \to 0$$

are exact, hence $H_1(S^2, A) \cong \mathbb{Z}^{n-1}$, $H_1(S^1 \vee S^1, A) \cong \mathbb{Z}^{n+1}$

$$0 = H_2(A) \to H_2(S^2) \to H_2(S^2, A) \to H_1(A) = 0$$

$$0 = H_2(A) \to H_2(S^1 \vee S^1) \to H_2(S^1 \vee S^1, A) \to H_1(A) = 0$$

are exact, hence $H_2(S^2, A) \cong H_2(S^2) \cong \mathbb{Z}$, $H_2(S^1 \vee S^1, A) \cong H_2(S^1 \vee S^1) \cong \mathbb{Z}$

$$0 = H_k(S^2) \to H_k(S^2, A) \to H_{k-1}(A) = 0$$

$$0 = H_k(S^1 \vee S^1) \to H_k(S^1 \vee S^1, A) \to H_{k-1}(A) = 0$$

are exact, hence $H_k(S^2, A) = 0$, $H_k(S^1 \vee S^1, A) = 0$, where $k \geq 3$

(b)

Both (X, A) and (X, B) are good pairs, it is easy to give X/A and X/B CW complex structures, X/A consists of one 0-cell, four 1-cells and two 2-cells, X/B consists of two 0-cells, five 1-cells and two 2-cells, then we can use cellular homology to get singular homology, the exact sequence for cellular homology are the following

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z}^4 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}} \mathbb{Z}^5 \xrightarrow{\begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0$$

Which give us $H_n(X, A) \cong \tilde{H}_n(X/A) = \begin{cases} \mathbb{Z}^4, & n = 1 \\ \mathbb{Z}, & n = 2 \\ 0, & \text{else} \end{cases}$ and $H_n(X, B) \cong \tilde{H}_n(X/B) = \begin{cases} \mathbb{Z}^3, & n = 1 \\ \mathbb{Z}, & n = 2 \\ 0, & \text{else} \end{cases}$

Hatcher 2.1.22.

(a)

$H_i(X) = H_i(X^n) = 0, \forall i > n$

$0 = H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1})$ is exact and $H_n(X^n, X^{n-1})$ is free, so is $H_n(X^n)$

(b)

$X^{n-1} = X^{n-2}$, thus $0 = H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}) = H_{n-1}(X^{n-2}) = 0$ is exact implies $H_n(X^n) \cong H_n(X^n, X^{n-1}) \cong \bigoplus \mathbb{Z}\{e^n\}$

(c)

Similarly as in (a), $H_{n-1}(X^{n-1})$ is also free, Then exact sequence $H_n(X^{n-1}) \to H_n(X^n) \to H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ implies $0 \to H_n(X^n) \to \mathbb{Z}^k \to \mathbb{Z}^m \to 0$ is exact for some $m \le k$, hence $H_n(X^n) \cong \mathbb{Z}^{k-m}$

Hatcher 2.1.27.

(a)

$\exists g : Y \to X$, $h : B \to A$ such that $fg \simeq \mathbb{1} \simeq gf$, $fh \simeq \mathbb{1} \simeq hf$, then we have $f_* : H_n(X) \to H_n(Y)$, $f_* : H_n(A) \to H_n(B)$ are isomorphisms, hence f_* is an isomorphism by five lemma

$$\begin{tikzcd}
H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(X) \arrow[d, "f_*"] \\
H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(Y)
\end{tikzcd}$$

(b)

Suppose $g : (Y, B) \to (X, A)$ such that $fg \overset{\varphi}{\simeq} \mathbb{1}_Y$, $gf \overset{\psi}{\simeq} \mathbb{1}_X$, $f|_A g|_B \overset{\varphi|_{B\times I}}{\simeq} \mathbb{1}_B$, $g|_A f|_B \overset{\psi|_{A\times I}}{\simeq} \mathbb{1}_A$
Thus we have $f|_{\bar{A}} : \bar{A} \to \bar{B}$, $g|_{\bar{B}} : \bar{B} \to \bar{A}$, $\varphi|_{\bar{B}\times I} : \bar{B} \times I \to \bar{B}$, $\psi|_{\bar{A}\times I} : \bar{A} \times I \to \bar{A}$ such that $f|_{\bar{A}} g|_{\bar{B}} \overset{\varphi|_{\bar{B}\times I}}{\simeq} \mathbb{1}_{\bar{B}}$, $g|_{\bar{A}} f|_{\bar{B}} \overset{\psi|_{\bar{A}\times I}}{\simeq} \mathbb{1}_{\bar{A}}$
Hence $f_* : H_n(X, \bar{A}) \to H_n(Y, \bar{B})$ is also an isomorphism, in this case it is impossible since $H_n(X, \bar{A}) = H_n(D^n, S^[n-1]) \cong \tilde{H}_n(D^n/S^{n-1}) \cong \tilde{H}_n(S^n) = \mathbb{Z}^n$, but $H_n(X, \bar{B}) = H_n(D^n, D^n) = \tilde{H}_n(D^n/D^n) = 0$ which is a contradiction

Hatcher 2.1.29.

$S^1 \vee S^1 \vee S^2$ has a CW complex structure, one 0-cell, two 1-cells, 1 2-cell, then we get an exact sequence

$$H_3(X^3, X^2) \to H_2(X^2, X^1) \to H_1(X^1, X^0) \to H_0(X^0) \to 0$$

Which gives $0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$
On the other hand, $\mathbb{R}^2$ is a universal cover of $S^1 \times S^1$, and attach to each vertex of the caylay graph which is a universal cover of $S^1 \vee S^1$ a S^2, call this Γ, then Γ is a universal cover of $S^1 \vee S^1 \vee S^2$, but $H_2(\mathbb{R}^2) = 0 \neq H_2(\Gamma)$

2.

(a)

Simply use Hatcher 2.1.27, we know that j induces an isomorphism

(b)

Let $r : V \to A$ be the retraction and $i : A \to V$ be inclusion, then $\exists h : V \times I \to V$ such that $ri = \mathbb{1}_A$, $ir \overset{h}{\simeq} \mathbb{1}_V$, then $\tilde{r} : V/A \to A/A$ is also a retraction, $\tilde{i} : A/A \to V/A$ is also an inclusion, h also induces $\tilde{h} : V/A \times I \to V/A$, $\tilde{r}\tilde{i} = \mathbb{1}_{A/A}$, $\tilde{i}\tilde{r} \overset{\tilde{h}}{\simeq} \mathbb{1}_{V/A}$, thus $\tilde{r}$ is also a deformation retraction

(c)

Directly check that the following diagram is commutative

$$\begin{tikzcd}
(X, A) \arrow[r, "j"] \arrow[d, "p"] & (X, V) \arrow[d, "q"] \\
(X/A, A/A) \arrow[r, "g"] & (X/A, V/A)
\end{tikzcd}$$

Similarly as in (a), g_* and j_* are isomorphisms, so is q_* by 1(c), hence p_* is also an isomorphism

References

[1] *Algebraic Topology* - Allen Hatcher

[2] *A Concise Course in Algebraic Topology* - Peter May

Index

Properly discontinuous group action, 3