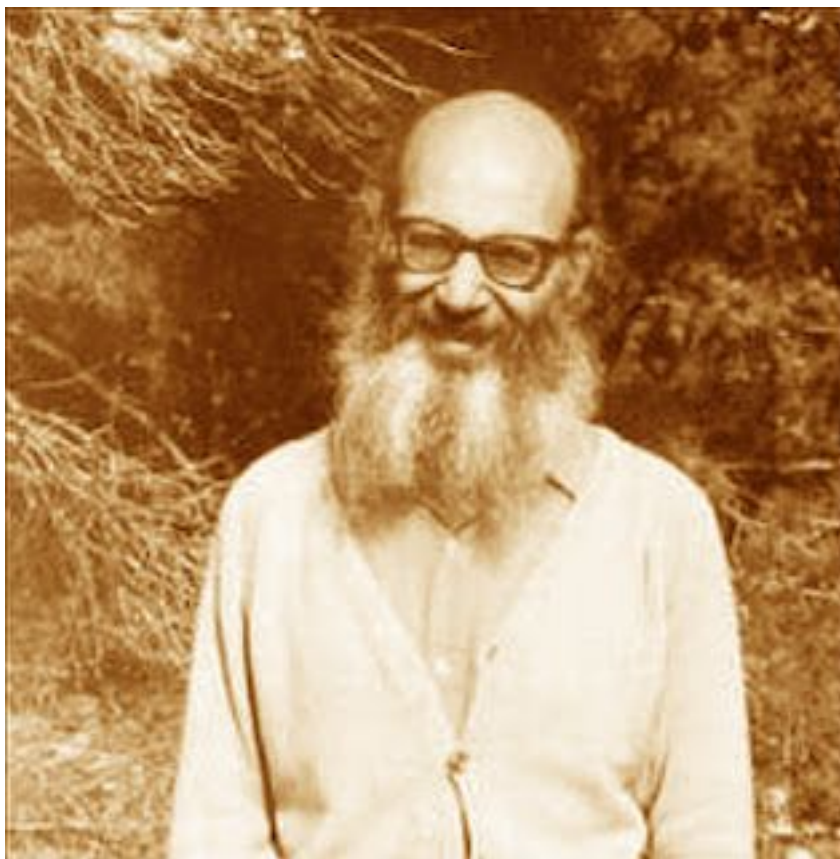


MATH808K - Algebraic K-Theory



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1 Projective modules

K-theory is the study of categories of vector bundles or similar objects. A vector bundle is a parametrized family of vector spaces: $p : E \rightarrow X$ is a vector bundle, X is a topological space. For each $x \in X$, $E_x = p^{-1}(x)$ is a vector space depending "continuously" on $x \in X$. K-theory deals with parametrized linear algebra. Often we don't deal directly with geometry, but with rings

Swan-Serre Theorem

Theorem 1.1 (Swan-Serre). There is an equivalence of categories between vector bundles over X and finitely generated projective modules over an associated ring of functions on X . Here are 3 categories in which this works

1. X compact Hausdorff, $R = C(X)$ the continuous function on X
2. X affine variety over a field k , $R = \mathcal{O}(X)$ is the ring of regular functions. If X is projective, it is more complicated
3. X stein manifold(holomorphic submanifold of \mathbb{C}^n), $R = \mathcal{O}(X)$ holomorphic functions on X , category of vector bundles is holomorphic category

Review of projective modules

In this course, a ring almost always have units but not necessarily commutative

Definition 1.2. R is a ring with unit. A free R -module is one isomorphic to R^I , I is some index set. A finitely generated free R -module is one isomorphic to R^n . R is said to have the invariant basis property if $R^n \cong R^m \Rightarrow n = m$. Note that this is always true if R is commutative(reason: true for fields, and if R is commutative, $k = R/m$ is field, $R \otimes k$ is a vector space over k)

Example 1.3 (Counter-example). k is a field, $R = \text{End}_k(k^\infty)$ doesn't have the invariant basis property, as $R \cong R^2$. Idea: $k^\infty \oplus k^\infty \cong k^\infty$

Solution. R corresponds to a matrix with rows and columns ranging in \mathbb{N} , and each column has all but finitely many non-zero elements, and we can decompose any matrix to the direct sum of two matrices with even columns or odd columns

$$\begin{bmatrix} \circ & \times & \circ & \times & \circ & \times & \dots \\ \circ & \times & \circ & \times & \circ & \times & \dots \\ \circ & \times & \circ & \times & \circ & \times & \dots \\ \circ & \times & \circ & \times & \circ & \times & \dots \\ \circ & \times & \circ & \times & \circ & \times & \dots \\ \circ & \times & \circ & \times & \circ & \times & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

□

Theorem 1.4. R is ring, P is an R -module. The following are equivalent

1. P is a direct summand in a free R -module, i.e. $F \cong P \oplus Q$ for some free R -module F
2. $\text{Hom}_R(P, -)$ is an exact functor
3. P has the property that if $\phi : M \rightarrow N$ is a surjective R -module map and we are given $\alpha : P \rightarrow N$, there exists $\beta : P \rightarrow M$ such that $\alpha = \phi \circ \beta$

An R -module with these 3 equivalent conditions is called *projective*

Proof. $2 \Rightarrow 3$ is due to the fact that $\text{Hom}_R(P, -)$ can only fail to be exact on the right, i.e. given $0 \rightarrow M' \rightarrow M \rightarrow N \rightarrow 0$

□

The first invariant of K-theory is $K_0(R)$, then Grothendieck group of finitely generated projective modules over R . If P and Q are finitely generated projective R modules, we can "add" by taking direct sum, but not subtract. $K_0(R)$ is the group with generators $[P]$, P finitely generated R -module with relations $[P] = [Q]$ if $P \cong Q$. We build in the relation $[P] + [Q] = [P \oplus Q]$. Note that every element of $K_0(R)$ is of the form $[P] - [Q]$ for some P, Q , $[P] - [Q] = [P'] - [Q'] \iff P \oplus Q' \oplus S \cong R' \oplus Q \oplus S$ for some S . In general, "addition" of projective modules does not have the cancellation property, just as addition of vector bundles does not

Example 1.5. TS^2 is not free since $\chi(S^2) = 2 \neq 0$

Fact 1.6 (reference: Hatcher's K-theory book). 1. Any vector bundle (by definition) is locally trivial, then $\text{rank } X \rightarrow \mathbb{N}$ is continuous, hence locally constant

2. Any vector bundle can be equipped with a metric, i.e. a family of inner products varying continuously with $x \in X$. (Construction: use local triviality and patch with partition of unity)
3. Any vector bundle can be embedded into a trivial vector bundle $X \times \mathbb{F}^n$ for n large enough. (Use local triviality and partition of unity)
4. 2+3 \Rightarrow Any vector bundle is a direct summand in a trivial bundle

proof of Theorem 1.1. Send $p : E \rightarrow X$ to the set of sections $\Gamma(E)$, then $\Gamma(E)$ is a $\mathcal{O}(X)$ -module, from above, $\Gamma(E)$ is finitely generated and projective. The rest is formal \square

Example 1.7. Observation: Any vector bundle over $S^n, n \geq 1$ is obtained by gluing ("clutching"): two trivial vector bundles over the upper and lower hemispheres via a map $S^{n-1} \rightarrow \text{GL}(k, \mathbb{F})$. This is because any vector bundle over a contractible space is trivial, so

$$\text{Vect}_{\mathbb{F}}^k(S^n) \cong [S^{n-1}, \text{GL}(k, \mathbb{F})] \cong \begin{cases} \pi_{n-1}(O(k)) & \mathbb{F} = \mathbb{R} \\ \pi_{n-1}(U(k)) & \mathbb{F} = \mathbb{C} \\ \pi_{n-1}(Sp(k)) & \mathbb{F} = \mathbb{H} \end{cases}$$

$X = S^2, \mathbb{F} = \mathbb{R}$, what is the classification of rank n vector bundles over X ? We see that rank k vector bundles over S^2 are classified by $\pi_1(O(k))$, since S^1 is connected, any map $S^1 \rightarrow O(k)$ lies in a single component, both isomorphic to $SO(k)$, for $k \geq 3$, $SO(k)$ is a simple Lie group and $\pi_1(SO(k)) \cong \mathbb{Z}/2$ for $k \geq 3$ (except $SO(4)$ is only semi-simple with two cover?)

Note. $SU(2)$ as quaternions acts on S^3 , gives a double cover of $SO(3)$

Implication for K-theory: The stable isomorphic classes of vector bundles E over S^2 is characterized by

$$\begin{cases} \text{rank} \in \mathbb{N} \\ \text{Stiefel-Whitney number} = \langle w_2(E), [S^2] \rangle \in \mathbb{Z}/2 \end{cases}$$

Similar analysis holds for S^n

$$\begin{cases} \text{rank} \in \mathbb{N} \\ \text{something in } \pi_{n-1}(\text{SO}) \end{cases}$$

Here $\pi_{n-1}(\text{SO}) = \pi_{n-1}(SO(\infty)) = \varinjlim_k \pi_{n-1}(SO(k))$, $\pi_{n-1}(SO(k))$ stabilizes as $k \rightarrow \infty$

Theorem 1.8 (Bott periodicity theorem).

$$\pi_{n-1}(\text{SO}) = \begin{cases} \mathbb{Z}, & n \text{ is a multiple of } 4 \\ \mathbb{Z}/2, & n \equiv 1, 2 \pmod{8} \\ 0, & \text{otherwise} \end{cases}$$

Lessons from this example: stable classification is much easier than the unstable classification. A stably trivial bundle need not to be trivial. These lessons carry over to the purely algebraic setting of projective modules over a ring. To get a corresponding example with projective modules over a Noetherian commutative ring, take $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$, $\text{Spec } R$ is an "algebraic model" for S^2 . Our non-trivial but stably trivial vector bundle can be constructed as $\{(x, y, z, u, v, w) \mid x^2 + y^2 + z^2 = 1, xu + yv + zw = 0\}$

2 Homotopy invariance

theorem1 - 1/29/2021

Theorem 2.1. The classification of the topological vector bundles over a compact Hausdorff space X is homotopy invariant. In other words, if $f, g : X \rightarrow Y$ are maps of compact spaces and E is an \mathbb{F} bundle over Y , then $f \simeq g \Rightarrow f^*(E) \cong g^*(E)$

Corollary 2.2. Every vector bundle over a contractible space is trivial

theorem2 - 1/29/2021

Theorem 2.3. A is a unital Banach algebra (For application, $A = C(X, M_n(\mathbb{F}))$). Let $\text{Idem } A$ be the set of idempotents in A ($x^2 = x$). If $e, f \in \text{Idem } A$ lies in the same component, then they are conjugate under $\text{GL}_1(A)$

Proof. It's enough to show that if $e, f \in \text{Idem } A$ are sufficiently close in norm, then e, f are conjugate. Suppose e, f are close and let $a = e + f - 1 \in A$, then a^2 is close to $(2e - 1)^2 = 1$, so a^2 is invertible, thus a is invertible. $ae = fa$ since $ae = (e + f - 1)e = fe$, $fa = f(e + f - 1) = fe$, thus $aea^{-1} = f$ \square

Proof Theorem 2.1. Embed E as a direct summand in a trivial bundle of rank n , then $f^*(E), g^*(E)$ are obtained by projecting down from $X \times \mathbb{F}^n$ via homotopic idempotents in $C(X, M_n(\mathbb{F}))$ \square

Projective modules over a local ring

Definition 2.4. R is a ring with unit, R is called local if the non-invertible elements in R constitute a 2-sided ideal \mathfrak{m} . Obviously \mathfrak{m} is the unique maximal 2-sided ideal

Caution: In the non-commutative case, having a unique maximal 2-sided ideal is not good enough! Since $M_n(\mathbb{F})$ has this property for \mathbb{F} a field, and this ring is not local

Note: If R is local and $x \in R$ has a left inverse, then it also has a right inverse. Suppose $ax = 1$, then $ax \notin \mathfrak{m}$, so $x \notin \mathfrak{m}$, so x is invertible

Fact 2.5. 1. If R is local with maximal ideal \mathfrak{m} and $x \in \mathfrak{m}$, then $1 + x$ is invertible. If not, then $1 + x \in \mathfrak{m} \Rightarrow 1 \in \mathfrak{m}$ which is a contradiction

2. (Nakayama's lemma) R is a local ring with maximal ideal \mathfrak{m} , and let M be a finitely generated R -module, if $\mathfrak{m}M = M$, then $M = 0$ Proof: Let $M = Rx_1 + \cdots + Rx_n$ such that n is minimal, then since $\mathfrak{m}M = M$, $x_n = r_1x_1 + \cdots + r_nx_n$ with $r_j \in \mathfrak{m}$, now $(1 - r_n)x_n = r_1x_1 + \cdots + r_{n-1}x_{n-1}$, but $1 - r_n$ is invertible, we can divide to get $x_n = \cdots$, contradicting the minimality unless $n = 0$, i.e. $M = 0$

Theorem 2.6. Let R be a local ring, M a finitely generated projective R -module, then M is free

Proof. $M \oplus N \cong R^n$. $\mathfrak{m}(M \oplus N) = \mathfrak{m}^n$, so $(R/\mathfrak{m})M$ is a direct summand in $(R/\mathfrak{m})^n$, but R/\mathfrak{m} is a division ring, so $(R/\mathfrak{m})M \cong (R/\mathfrak{m})^k$ for some $0 \leq k \leq n$, and $(R/\mathfrak{m})N = (R/\mathfrak{m})^{n-k}$. Let $\hat{x}_1, \dots, \hat{x}_k$ be a free basis for $(R/\mathfrak{m})M \cong M/\mathfrak{m}M$ and extend it to a free basis by adding $\hat{x}_{k+1}, \dots, \hat{x}_n$ for $(R/\mathfrak{m})N$, pull these back to $x_1, \dots, x_k \in M$ and $x_{k+1}, \dots, x_n \in N$. $M = Rx_1 + \cdots + Rx_k$ by Nakayama's lemma x_1, \dots, x_k is another generating set for R^n with n elements, writing the the matrix of x_i 's and e_i 's gives the linear independence \square

Corollary 2.7. For R is a local ring, $K_0(R) = \mathbb{Z}$, with the class of a projective module given by its rank (this is only stable case, note that the theorem actually prove the non-stable case, which is more general)

theorem 1, 2021-2-1

Theorem 2.8 (Lemma 2.4, [1]). R is a commutative ring, M is a finitely generated R -module. The following are equivalent

1. M is projective

2. M is locally free, i.e. for each prime ideal \mathfrak{p} , $\exists s \in R \setminus \mathfrak{p}$ such that $M[1/s]$ is free of finite rank over $R[1/s]$
3. For each prime ideal \mathfrak{p} of M , $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module, and M is finitely presented, i.e. there is an exact sequence $R^m \rightarrow R^n \rightarrow M \rightarrow 0$ (If R is Noetherian, then finitely generated implies finitely presented)

Proof. $1 \Rightarrow 2, 3$: Assume M is projective, $R^n \cong M \oplus P$, then M is finitely presented, i.e. $R^n \rightarrow R^n \rightarrow M \rightarrow 0$. Also $R_{\mathfrak{p}}^n \cong M_{\mathfrak{p}} \oplus P_{\mathfrak{p}}$ over the local ring $R_{\mathfrak{p}}$, by Theorem 2.8, $M_{\mathfrak{p}}$ is finitely generated and free over $R_{\mathfrak{p}}$. So after inverting things in $R \setminus \mathfrak{p}$, we get a free module. But we really only have to invert finitely many elements since M is finitely presented, which is the same as inverting their product, which is a single $s \in R \setminus \mathfrak{p}$, so $M[1/s]$ is free of finite rank over $R[1/s]$

$3 \Rightarrow 1$: Since M is finitely presented, we have an exact sequence $R^m \rightarrow R^n \xrightarrow{\epsilon} M \rightarrow 0$. To show M is projective, it suffices to show that $\epsilon^* : \text{Hom}(M, R^n) \rightarrow \text{Hom}(M, M)$ is surjective, since id_M then has a preimage. Given a prime ideal \mathfrak{p} , $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$, so $\epsilon_{\mathfrak{p}}^*$ is split surjective, so is ϵ^* , since $\text{Hom}(M, M) / \text{Hom}(M, R^n)$ is 0 after localizing at any prime ideal, hence 0 \square

So at least over a Noetherian commutative ring, M finitely generated projective $\iff M_{\mathfrak{p}}$ finitely generated free for each $\mathfrak{p} \in \text{Spec } R$. This suggests (and we used this to study vector bundles over spheres) that to construct interesting projective modules, we get them by patching (clutching) finitely generated free modules over different open sets

Example 2.9 (Milnor square, his book on algebraic K-theory). Suppose $f : R \rightarrow S$ is a map of rings, $I \cong f(I)$, then R is the pushout of the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \downarrow & & \downarrow \\ R/I & \xrightarrow{f} & S/I \end{array}$$

Think of S as R/J , so we are patching over I and J

Example 2.10 (Topological example). $R = C(S^2)$, I is the ideal of functions which vanish on the open upper hemisphere and tend to zero along the equator. Then $R/I \cong C(D_-^2)$, continuous functions on the closed lower hemisphere. Let S be the continuous functions on the closed upper hemisphere $C(D_+^2)$, $R \rightarrow S$ is just restriction S and R/I are both of the form $C(D^2)$, and since D^2 is contractible, every vector bundle on D^2 is trivial, and so every finitely generated projective S -module or R/I -module is free. But S^2 is contractible, we want to describe finitely generated projective R -modules in terms of patching of trivial bundles along the equator

Theorem 2.11 (Milnor patching theorem). In a Milnor square

1. An R -module P obtained by patching a finitely generated projective S -module P_1 and a finitely generated projective R/I -module P_2 is finitely generated and projective
2. In the situation of 1, $S \otimes_R P \cong P_1$ and $P/IP \cong P \otimes_R R/I \cong P_2$
3. (important) Every finitely generated projective R -module arises this way
4. Suppose P is obtained by patching S^n and $(R/I)^n$ via $g \in \text{GL}_n(S/I)$, and Q is obtained similarly via $g^{-1} \in \text{GL}_n(S/I)$, Then $P \oplus Q \cong R^{2n}$
5. More generally, if P is obtained by patching S^n and $(R/I)^n$ via $g \in \text{GL}_n(S/I)$, and Q is obtained by patching S^n and $(R/I)^n$ via h , then $P \oplus Q \cong$ patching of S^n and $(R/I)^n$ via $gh \oplus R^n$

Proof. Translate to algebra the idea of gluing vector bundles when $X = X_1 \cup_Y X_2$, here X_1, X_2 are closed subsets of a compact Hausdorff space X_1 and $Y = X_1 \cap X_2$. Alternatively, the Milnor patching theorem is a kind of "algebraic Mayer-Vietoris" proof of 3: Recall R is embedded in $S \oplus R/I$. Let P be a finitely generated projective R -module. Let $P_1 = S \otimes_R P$ which is finitely generated projective S -module, let $P_2 = P/IP$ is a finitely generated projective R/I -module. We want to show that P is obtained by patching P_1 and P_2 . We have an exact sequence of R -modules

$$0 \rightarrow R \rightarrow S \oplus R/I \rightarrow S/I \rightarrow 0$$

$\text{Hom}_R(P, -)$ and $- \otimes_R P$ is exact since P is projective, so we have exact sequence

$$0 \rightarrow P \rightarrow P_1 \oplus P_2 \rightarrow S/I \otimes_R P$$

which says that P is patched from P_1 and P_2 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ through homotopy $\begin{bmatrix} 1-t & -t \\ t & 1-t \end{bmatrix}$ which has determinant $(1-t)^2 + t^2 > 0$, hence $\begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} = \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -g^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix} = \begin{bmatrix} g & 0 \\ 0 & g^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & gh \end{bmatrix}$, here $\begin{bmatrix} 1 & g \\ 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ through homotopy $\begin{bmatrix} 1 & tg \\ 0 & 1 \end{bmatrix}$ □

3 Line bundles and Picard groups

We assume in this section R is commutative. We showed that a finitely generated R -module P is projective exactly when it's locally free. We call it (by analogy with topology) a line bundle if P is locally free of rank 1. Commutativity of R enables you to tensor two R -modules together to get a new one. This preserves the property of being a line bundle. So we can define the Picard group $\text{Pic}(R)$ of R , to be the abelian group of line bundles over R up to isomorphism with tensor product as the group operation and $[R]$ as the identity element. The dual line bundle $P^\vee = \text{Hom}_R(P, R)$, and $P \otimes_R P^\vee \rightarrow R$ is an isomorphism, so $[P^\vee] = [P]^{-1}$. For topological line bundles over \mathbb{C} , the dual line bundle can also be identified with the complex conjugate bundle. The dual of vector bundle $E \rightarrow X$ is the bundle whose fibers are the dual vector spaces. Over \mathbb{C} , $\mathbb{C}^* \cong \overline{\mathbb{C}}$ via the usual inner product

Example 3.1.

1. $R = C^\mathbb{R}(X)$, X compact Hausdorff. $\text{Pic}(R) \cong H^1(X, \mathbb{Z}/2)$
2. $R = C^\mathbb{C}(X)$, X compact Hausdorff. $\text{Pic}(R) \cong H^1(X, \mathbb{Z})$

Theorem 3.2. If M is a manifold of dimension n and E is a vector bundle of rank k which is stably trivial, then E is trivial provided $\begin{cases} k > n & \mathbb{F} = \mathbb{R} \\ k > (n+1)/2 & \mathbb{F} = \mathbb{C} \end{cases}$

Example 3.3. The tangent bundle of S^2 is stably trivial but not trivial, in this case, $n = 2, k = 2, \mathbb{F} = \mathbb{R}$. But every stably trivial 3 plane bundle over S^2 is trivial

Proof. There is a fibration $O(n-1) \rightarrow O(n) \rightarrow S^{n-1}$, we have $\pi_{j+1}(S^{n-1}) \rightarrow \pi_j(O(n)) \rightarrow \pi_j(O(n-1)) \rightarrow \pi_j(S^{n-1})$. The map $\pi_j(O(n-1)) \rightarrow \pi_j(O(n))$ is surjective if $j < n-1$. an iso if $j < n-2$. and we can decompose M as $(n-1)$ -skeleton $\cup D^n$ via gluing along the $(n-1)$ -skeleton. Similarly with $U(n-1) \rightarrow U(n) \rightarrow S^{2n-1}$, which gives $\pi_j(U(n-1)) \rightarrow \pi_j(U(n))$ surjective if $j < 2n-1$, an iso if $j < 2n-2$ \square

The rank function: R is a commutative ring, we get a notion of local rank function of a finitely generated projective R -mod P , $\text{Spec}(R) \rightarrow N, p \mapsto \text{rank}(P_p)$. Claim: The rank function is continuous in Zariski topology

Proof of claim. p is contained in a maximal ideal m , $\text{rank Spec } R[1/s] \text{rank}(P_p) = \text{rank}(P_m)$. If $\text{rank}(P_p) \neq \text{rank}(P_q) = \text{rank Spec } R[1/s']$ \square

Theorem 3.4 (Stable rank theorem, Bass-Serre, Bass's book on algebraic K theory, not easy). R commutative and Noetherian of dimension d . If R has a finitely projective module P of constant rank k . then if $k > d$

1. $P = P_0 \oplus R^{k-d}$ for some projective module P_0 of rank d
2. P is stably isomorphic to P' of same rank $\Rightarrow P \cong P'$
3. P stably free $\Rightarrow P$ is free

Corollary 3.5. If R is a commutative ring with no idempotents other than 0 and 1, then every finitely generated projective R -module has constant rank

Proof. $\text{Spec } R$ is connected iff R has no non-trivial idempotent. And an integral domain is connected (if x is an idempotent, so is $1-x$, and $x(1-x) = 0$) \square

Back to line bundles

Let R be a commutative ring, P finitely generated projective R -module. We can form new modules $\bigwedge^j P$. If P has constant rank k , then $\bigwedge^j P$ is projective of rank $\binom{k}{j}$. So we can define the determinant of P , $\det(P) = \bigwedge^k P, k = \text{rank } P$. If R has no non-trivial idempotents, the determinant module is defined for all finitely generated projective modules

1. $\det P$ is a line bundle
2. $\det(P \oplus Q) = \det P \otimes \det Q$

From these facts we deduce

Theorem 3.6. R is a commutative ring, if two line bundles are stably isomorphic iff isomorphic, i.e. define the same class in $\text{Pic}(R)$

Proof. Assume $P \oplus R^n \cong P' \oplus R^n$, then $P \cong \det P \otimes R = \det(P \oplus R^n) \cong \det(P' \oplus R^n) = \det P' \otimes R \cong P'$ \square

Theorem 3.7. R commutative Noetherian of dimension 1 (in particular, Dedekind domain: Noetherian integral domain which is integrally closed in the field of fractions). Then every finitely generated projective R -module of constant rank is of the form $P \oplus R^{k-1}$ for some k and for some line bundle P , namely the determinant bundle of the module. Thus finitely generated projective R -module are completely classified by rank and determinant

Proof. Just use stable rank theorem \square

Corollary 3.8. If R is a commutative Noetherian integral domain of dimension 1, then $K_0(R) \cong \mathbb{Z} \oplus \text{Pic}(R)$, $[P] \mapsto (\text{rank } P, \det P)$

Note $\det(P \oplus Q) = \det P \otimes \det Q$ says it is a homomorphism in Pic , rank adding says it is a homomorphism in \mathbb{Z}

4 Cartier divisor

R commutative integral domain, F its field of fractions. A fractional ideal of R is a non-zero R -submodule I of F , is such that $xI \subseteq R$ for some $x \in F^\times$ (i.e. we can clear denominators). I is invertible if $IJ = R$ for some other fraction ideal J . Principal fractional ideals are Rx for $x \in F^\times$. Principal fractional ideals are invertible. The invertible fractional ideals form the Cartier group $\text{Cart}(R)$ of Cartier divisors under multiplication

Proposition 4.1. The line bundles over R are the same as invertible ideals up to isomorphism. If I, J are fractional ideals and I is invertible, then $I \otimes_R J \cong IJ$. There is an exact sequence

$$1 \rightarrow R^\times \rightarrow F^\times \xrightarrow{\text{div}} \text{Cart}(R) \rightarrow \text{Pic}(R) \rightarrow 0$$

Proof. Suppose I is an invertible fractional ideal, $IJ = R$, then there exist $x_i \in I, y_i \in J$, $\sum_{i=1}^n x_i y_i = 1$, $\{x_i\}$ define homomorphisms $R^n \rightarrow I$, $\{y_i\}$'s define homomorphisms $I \rightarrow R^n$. $I \rightarrow R^n \rightarrow I$ is the identity, so I is projective, and has rank one because $I \otimes_R F = F$, so I is a line bundle. Conversely, suppose L is a line bundle, we get an embedding $L = R \otimes_R L \subseteq F \otimes_R L = F$, L is isomorphic to a nonzero R -submodule of F , i.e. a fractional ideal. Since I is finitely generated and projective, we have $R \cong L \otimes_R L^\vee$, hence L is invertible \square

Equivalent definition for Dedekind domain: Integral domain for which every fractional ideal is invertible

Theorem 4.2. R Dedekind domain, every finitely generated torsion-free R -module M is projective and isomorphic to $I \otimes R^{n-1}$ for some fractional ideal I

Proof. Induction on $\text{rank } M = \dim_F(F \otimes_R M)$. M torsion-free means M embeds into $F \otimes_R M$, so if $\text{rank } M = 0$, $M = 0$. Suppose $\text{rank } M = n$, M embeds into F^n and then projects to the last factor F , image can't be zero, so we have $0 \rightarrow M_0 \rightarrow M \rightarrow I \rightarrow 0$ with I fractional ideal projective, so $M = M_0 \oplus I$, by induction, $M = I \oplus J \oplus R^{n-2}$. $I \oplus J = IJ \oplus R$ since they have the same determinant \square

Applications: R is a PID: $1 \rightarrow R^\times \rightarrow F^\times \rightarrow \text{Cart}(R) \rightarrow 1$ since $\text{Pic}(R) = 0$

Example 4.3. $R = \mathbb{Z}, F = \mathbb{Q}$, get

$$1 \rightarrow \mathbb{Z}^\times \rightarrow \mathbb{Q}^\times \rightarrow \mathbb{Q}^\times / \{\pm 1\} \rightarrow 1$$

$$R = \mathbb{Z}[i], F = \mathbb{Q}[i], R^\times = \{\pm 1, \pm i\}$$

$$R = k[t], F = k(t), R^\times = k^\times$$

F be a number field, $R = \mathcal{O}_F$. Then R is a Dedekind domain, the $\text{Pic}(R)$ is finite, and R^\times is finitely generated, in fact, $\text{rank}(R^\times) = r_1 + r_2 - 1$. Most of the time, R is not a PID

k is a field, C is a smooth affine curve over k , R is the ring of regular functions on C , then R is a Dedekind domain. In the case $k = \mathbb{C}$, $\text{Pic}(R)$ is the group of algebraic line bundles. $\text{Cart}(R)$ indeed corresponds to the group of divisors (finite linear combinations of points)

5 Homeworks

References

- [1] *The K-Book* - Charles Weibel

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