Section 9.5: Some Applications of the integral

Present value of an income stream

If you have 100 dollars, its value will decrease in 10 years, namely, you can't buy as much as you can now, another way to put it is that A dollars you make in t years only has present value

$P = Ae^{-rt}$ (this expression is coming from Section 5.2)

Now suppose the annual rate of income is $K(t)$

hence in a infinitesimally small amount of time dt at time t, you can make $K(t)dt$, and its present value is $e^{-rt}K(t)dt$

hence the present value of the stream of income between $t=0$ to $t=T$, is to integrate ("sum") all these infinitesimally amount of present value which is $\int_0^T K(t)e^{-rt}dt$

Example: If r (interest rate) is $50\% = 0.5$, and $K(t) = 1+t$ (you can make money faster and faster), $T = 10$, then

$$\int_0^{10}(1+t)e^{-0.5t}dt = \frac{1}{-0.5}\int_0^{10}(1+t)e^{-0.5t}(-0.5dt)$$

$$= -2\int_0^{10}(1+t)e^{-0.5t}d(-0.5t) \overset{u=-0.5t}{=\!=\!=} -2\int_{-0.5\times 0}^{-0.5\times 10}(1+(-2u))e^u du$$

$$= -2\int_0^{-5}(1-2u)e^u du = -2\int_0^{-5}(1-2u)de^u$$

$$= -2\left[\left((1-2u)e^u\right)\Big|_0^{-5} - \int_0^{-5}e^u d(1-2u)\right]$$

$$= -2\left[\left((1-2\times(-5))e^{-5}\right) - \left((1-2\times 0)e^0\right) - \int_0^{-5}e^u\cdot(-2)du\right]$$

$$= -2\left[11e^{-5} - 1 + 2\int_0^{-5}e^u du\right]$$

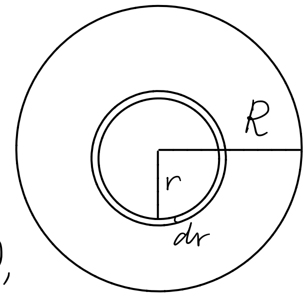
$$= -2\left[11e^{-5} - 1 + 2e^u\Big|_0^{-5}\right]$$

$$= -2\left[11e^{-5} - 1 + 2(e^{-5} - e^0)\right]$$

$$= -2\left[11e^{-5} - 1 + 2e^{-5} - 2\right]$$

$$= -2(13e^{-5} - 3) = 6 - 26e^{-5}$$

Demographic Model



In a disk shaped city, on each infinesimally narrow (width dr) circular band with radius r has population density $D(r)$, the whole disk is of radius R, then the infinitesimally amount of population on this circular band is

$$D(r)\cdot \overbrace{\boxed{\underbrace{2\pi r}_{\text{circumference}} \cdot \underbrace{dr}_{\text{width}}}}^{\text{area}} = 2\pi r D(r)\,dr$$

then the total population would be to integrate ("sum") all these infinitesimally amount of people

$$\int_0^R 2\pi r D(r)\,dr = 2\pi\int_0^R rD(r)\,dr$$

Example: $D(r) = \frac{1}{1+r}$ (less people away from the city center), $R = 2$ (say miles), then total population is

$$2\pi\int_0^2 \frac{r}{1+r}\,dr = 2\pi\int_0^2 \frac{r}{r+1}\,d(r+1) \xlongequal{u=r+1} 2\pi\int_{0+1}^{2+1} \frac{u-1}{u}\,du$$

$$= 2\pi\int_1^3 \left(1-\frac{1}{u}\right)du = 2\pi\left[u - \ln u\right]\Big|_1^3$$

$$= 2\pi\left[(3-\ln 3) - (1-\ln 1)\right] = 2\pi(2-\ln 3)$$

Example with Simpon's rule: say $n=4$, want to approximate $\int_4^6 x^2\,dx$, now check the previous definition you will see $f(x) = x^2$,

$a_0 = 4$, $a_n = a_4 = 6$, $\Delta x = \frac{a_n - a_0}{n} = \frac{6-4}{4} = \frac{2}{4} = 0.5$

$a_1 = a_0 + \Delta x = 4 + 0.5 = 4.5$, $a_2 = a_1 + \Delta x = 5$, $a_3 = a_2 + \Delta x = 5.5$

$x_1 = \frac{a_0 + a_1}{2} = \frac{4+4.5}{2} = 4.25$, $x_2 = \frac{a_1 + a_2}{2} = 4.75$, $x_3 = \frac{a_2 + a_3}{2} = 5.25$, $x_4 = \frac{a_3 + a_4}{2} = 5.75$

generally $x_i = \frac{a_{i-1} + a_i}{2}$ is the midpoint between points a_{i-1} and a_i

then $$\int_4^6 x^2\,dx \approx \frac{\Delta x}{6}\Big[f(a_0) + 4f(x_1) + 2f(a_1) + 4f(x_2) + 2f(a_2) + 4f(x_3) + 2f(a_3) + 4f(x_4) + f(a_4)\Big]$$

$$= \frac{0.5}{6}\left[4^2 + 4\times 4.25^2 + 2\times 4.5^2 + 4\times 4.75^2 + 2\times 5^2 + 4\times 5.25^2 + 2\times 5.5^2 + 4\times 5.75^2 + 6^2\right]$$

$$= \cdots$$