Chapter 1

Exercises in combinatorics

Exercise 1.0.1. $[n] = \{1, \cdots, n\}$, what is the cardinality of $\{f \in Aut([n]) | f(i) \neq i, \forall i \in [n]\}$

Solution. Consider $A_k = \{f \in Aut([n]) | f(k) = k\}$, by Inclusion-exclusion principle **??**, we have

$$\begin{aligned} n! = \left| \bigcup_{i=1}^{n} A_n \right| &= \sum_{k=1}^{n} (-1)^k \sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}| \\ &= \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)! \\ &= \sum_{k=1}^{n} (-1)^k \frac{n!}{k!} \end{aligned}$$

Thus the probability of picking such an auto morphism is $\sum_{i=1}^{n} \frac{(-1)^k}{k!}$ which approaches e^{-1} as n approaches infinity $\square$

Chapter 2

Exercises in abstract algebra

Exercise 2.0.1. If R is a domain, so is $R[x]$

Solution. Suppose $f = ax^n + \cdots$, $g = bx^m + \cdots$ for some $a, b \neq 0$, then $fg = abx^{n+m} + \cdots \neq 0$ □

Exercise 2.0.2. If E/F is a Galois extension, then $Tr_{E/F}(\alpha)$ is the sum of all conjugates of α, $N_{E/F}(\alpha)$ is the product of all conjugates of α

Solution. Suppose the minimal polynomial of α is $m(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ □

Exercise 2.0.3. If $F \subseteq E \subseteq L$ are field extensions, then $Tr_{L/F} = Tr_{E/F} \circ Tr_{L/E}$

Solution. Suppose $x_1, \cdots, x_n$ is a basis for L/E, $y_1, \cdots, y_m$ is a basis for E/F □

T:V->W<=W => Tr(T)=Tr(T|W)

Exercise 2.0.4. Suppose $T \in \mathtt{Hom}_{\mathbb{F}}(V, V)$ is a linear operator with $T(V) \leq W$, then $Tr(T) = Tr(T|_W)$

Mundane properties of rings

Exercise 2.0.5. R is a ring

1. $0x = 0$, $(-1)x = -x$

Solution.

1.

$$0x = (0+0)x = 0x + 0x \Rightarrow 0x = 0$$

$$0x = (1 + (-1))x = 1x + (-1)x = x + (-1)x \Rightarrow (-1)x = -x$$

□

Exercise 2.0.6. Let R be a commutative ring, and $I_1, \cdots, I_n \leq R$ be pairwise coprime ideals, then $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$

Solution. By induction □

Exercise 2.0.7. Every group G is naturally isomorphic to its opposite G^{op}

Solution. Consider $\phi : G \to G^{op}$, $g \mapsto g^{-1}$ □

Exercise 2.0.8. A morphism of G torsors is always an isomorphism

Exercise 2.0.9. X has a left G action and a right H action such that $(gx)h = g(xh)$

1. $X \times_G * \cong X/G$

2. $X \times_G G \cong X$

3. $(X \times_G Y) \times_H Z \cong X \times_G (Y \times_H Z)$

4. If $H \leq G$, then $X \times_G G{\times}_Y \cong X \times_H Y$

5. If $H \trianglelefteq G$, $X \times_G (G/H) \cong X/H$

Exercise 2.0.10. $SL(n, F)$ is a perfect group for $n \geq 3$. $SL(2, F)$ is a perfect group if $|k| \geq 4$

Solution. Denote $G_n = SL(n, F)$. Elementary matrices generate G_n and are in $[G_n, G_n]$ □

Exercise 2.0.11. M is a finitely presented, then $N^* \otimes M \cong \mathtt{Hom}_R(M, N)^*$
R is a local ring, then flat, projective, free modules are equivalent notions

Solution. Finite presented and flat always imply projective
M has minimal generating set $m_1, \cdots, m_n$, $0 \to K \to R^n \to M \to 0$ is a split exact sequence, tensor with $k = R/m$, we have $0 \to k \otimes K \to k^n \to k \otimes M \to 0$, but $\dim k^n = \dim k \otimes M = n$, $K/mK = k \otimes K = 0$, by Nakayama's lemma **??**, $K = 0$, hence $M = R^n$ □

Exercise 2.0.12. Let K be a field, and let n be a positive integer. Let $K(x_1, \cdots, x_n)$ be the field of rational functions over K with n variables, and let $L = K[x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}]$ be the subring of $K(x_1, \cdots, x_n)$. Let $R = K[x_1, \cdots, x_n, y_1, \cdots, y_n]$

1. For an element $p \in R$, let $\varphi(p)$ denote the element of L obtained by substituting x_i^{-1} into each variable y_i in p. This map $\varphi : R \to L$ is a ring homomorphism. Show that for an ideal J of L, $\varphi^{-1}(J)$ is an ideal of R

2. For $1 \leq i \leq n$, let $g_i = x_i y_i - 1$. Let

$$R' = \{r \in R | \text{for } 1 \leq i \leq n, \text{ every monomial in } r \text{ does not involve } x_i \text{ and } y_i \text{ simultaneously}\}$$

Show that for an arbitrary element $p \in R$, there exist $h_1, \cdots, h_n \in R$ and $r \in R'$ such that $p = h_1 g_1 + \cdots + g_n h_n + r$

3. Let I denote the ideal of R generated by $g_1, \cdots, g_n$. Show that $\ker \varphi = I$ and that L is isomorphic to the quotient ring R/I

Solution.

1. By definition

2. Suppose monomial q containing factor $(x_1 y_1)^k$ but not $(x_1 y_1)^{k+1}$, since $(x_1 y_1)^k = (g_1 + 1)^k$, q can be written as $u g_1 + v$, where every monomial in v does not involve x_1 and y_1 simultaneously. Repeat for $g_2, \cdots, g_n$, then we are done

3. $\varphi(g_i) = 0 \Rightarrow I \leq \ker \varphi$, conversely, if $p \in \ker \varphi$, then $0 = \varphi(r) \Rightarrow r = 0$, thus $\ker \varphi = I$. Since φ is surjective, by first isomorphism theorem, we have $L \cong R/I$

□

Chapter 3

Exercises in analysis

U<R^n open, boundary point is the limit of some discrete sequence

Exercise 3.0.1. $U \subsetneqq \mathbb{R}^n$ is a nonempty open set, $x \in \partial U$, then there exists a discrete sequence $\{x_i\} \subseteq U$ converges to x

Solution. x is necessarily an accumulation point since $\partial U \cap U = \varnothing$. Pick $x_0 \in U$, then we can find $\epsilon > 0$ such that $x_0 \notin B(x, \epsilon)$, then pick $x_1 \in B(x, \epsilon/2) \cap U$, and so on □

f analytic near 0, after change of variables, f has terms only involve one variable

Exercise 3.0.2. f is analytic near 0, by rotation of coordinates, we can always make f has terms only involve one variable

Exercise 3.0.3. Evaluate $\displaystyle\int_0^\infty e^{-s^2-\frac{1}{s^2}} ds$

Solution. $\left(s - \dfrac{1}{s}\right)^2 = s^2 + \dfrac{1}{s^2} - 2$, let $x = s - \dfrac{1}{s}$ which is increasing on $(0, \infty)$ since $0 < s < \infty$, $-\infty < x < \infty$, then $s = \dfrac{x + \sqrt{x^2+4}}{2}$ and

$$\int_0^\infty e^{-s^2-\frac{1}{s^2}} ds = e^{-2} \int_{-\infty}^{+\infty} e^{-x^2} \left(\frac{1}{2} + \frac{x}{2\sqrt{x^2+4}}\right) dx = e^{-2} \int_0^\infty e^{-x^2} dx = \frac{e^{-2}\sqrt{\pi}}{2}$$

□

Exercise 3.0.4. f is holomorphic on the punctured unit disc, $p > 0$, $\displaystyle\int_D |f(z)|^p dz < \infty$. What can we say about the singularity?

Solution. $|f(z)|^p = e^{p \log |f(z)|}$ is subharmonic by Example **??**, thus essential singularity is impossible

$$|f(z)|^p \leq \frac{4}{\pi |z|^2} \int_{|w-z|<|z|/2} |f(w)|^p dw \leq \frac{C}{|z|^2}$$

Thus $|z|^{\frac{2}{p}} |f(z)| < \infty$ □

Exercise 3.0.5. $U \subseteq \Omega \subseteq \mathbb{C}$ are open, f is holomorphic on U, $\widehat{U}_\Omega$ be the union of U and compact connected components of $\Omega \setminus U$. There exist $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U iff there exists g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$

Solution. Assume $\widehat{U}_\Omega = U \cup K_1 \cup \cdots$, where K_i's are compact
Suppose $\{f_n\}$ holomorphic on Ω converging uniformly to f on compact subsets of U, by maximum principle, $\{f_n\}$ would be uniformly bounded around K_i, by Montel's theorem **??**, there exists a subsequence of $\{f_n\}$ converges uniformly on K_i, thus converging to g holomorphic on $H(\widehat{U}_\Omega)$, hence $g|_U = f$
Conversely, suppose g holomorphic on $H(\widehat{U}_\Omega)$ such that $g|_U = f$, $\widehat{U}_\Omega$ is simply connected, by Riemann mapping theorem **??**, we can think of $\widehat{U}_\Omega$ as the unit disc or $\mathbb{C}$, by Runge's theorem,

there exist $\{f_n\}$ holomorphic on Ω uniformly converging to g on each disc. Thus there exist a subsequence of $\{f_n\}$ converging uniformly to g on compact subsets of $\widehat{U}_\Omega$ $\square$

Exercise 3.0.6. Let Ω be an open subset of $\mathbb{C}$, $\mathscr{D} = \{D_i\}$ be an open cover of Ω with disks. Given meromorphic functions h_i on D_i, not identically zero. Assume $g_{ij} = \dfrac{h_i}{h_j}$ are holomorphic on $D_i \cap D_j$, then there exist holomorphic function f_i with no zeros on D_i such that $f_i = g_{ij} f_j$

Solution. It suffices to prove $H^1(\Omega, \mathcal{O}^*) = 0$, since then $H^1(\mathscr{D}, \mathcal{O}^*) = 0$, $(g_{ij}) \in Z^1(\mathscr{D}, \mathcal{O}^*) = B^1(\mathscr{D}, \mathcal{O}^*)$, i.e. there exists $(f_i) \in C^0(\mathscr{D}, \mathcal{O}^*)$ such that $f_i = g_{ij} f_j$
Consider exact sequence of sheaves $0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$, then we get a long exact sequence $\cdots \to H^1(\Omega, \mathcal{O}) \to H^1(\Omega, \mathcal{O}^*) \to H^2(\Omega, \mathbb{Z}) \to \cdots$, $H^1(\Omega, \mathcal{O}) = 0$ by Mittag-Leffler theorem $\square$

Exercise 3.0.7. For each real r such that $0 < |r| < 1$, prove that there exists at most one real s with $0 < s < 1$ for which $\Omega := D \setminus \{0, r, s\}$ admits an analytic automorphism different from the identity

Solution. Suppose $\Omega \xrightarrow{\phi} \Omega$ is an analytic automorphism, then $0, r, s$ are all removable singularities, by continuity, ϕ can be extended to $D \xrightarrow{\phi} D$, so is ϕ^{-1}, by continuity, we know ϕ is an automorphism of D, sending $\{0, r, s\}$ to itself bijectively
By Schwarz lemma, we know that an automorphism ϕ of D with $\phi(\alpha) = 0$ iff $\phi = e^{i\theta}\dfrac{z - \alpha}{1 - \overline{\alpha} z}$
Now suppose ϕ is an automorphism different from the identity, if $0 < r = s < 1$, then $\phi = -\dfrac{z - r}{1 - rz}$ is a choice, now we assumen $r \neq s$

Case I: $\phi(0) = 0$

$\phi = e^{i\theta} z$, $\phi(r) = s$, but $0 < s < 1$, thus $s = |r|$

Case II: $\phi(r) = 0$

$\phi = e^{i\theta}\dfrac{z - r}{1 - \overline{r} z}$, $\phi(0) = -re^{i\theta}$

Case i: $\theta = \pi$, $\phi(0) = r$, then $s = \phi(s) \Rightarrow \overline{r} s^2 - 2s + r = 0 \Rightarrow s = \dfrac{1 + \sqrt{1 - |r|^2}}{\overline{r}}$ or $\dfrac{1 - \sqrt{1 - |r|^2}}{\overline{r}}$, r has to be a positive real number and $s = \dfrac{1 - \sqrt{1 - r^2}}{r}$

Case ii: $s = \phi(0) = |r|$

Case III: $\phi(s) = 0$

$\phi = e^{i\theta}\dfrac{z - s}{1 - sz}$, $\phi(0) = -se^{i\theta}$

Case i: $\theta = \pi$, $\phi(0) = s$, then $r = \phi(r) \Rightarrow sr^2 - 2r + s = 0 \Rightarrow s = \dfrac{2r}{1 + r^2}$.

Case ii: $r = \phi(0) = -se^{i\theta}$, $s = \phi(r) \Rightarrow s^2 = 1$ which is impossible

$\square$

Exercise 3.0.8. $F \subseteq \mathbb{C}$ is closed, connected and noncompact, $\Omega = \mathbb{C} \setminus F$, then every $f \in \mathcal{O}(\Omega)$ has a primitive

Solution. It suffices to show that every connected component U of Ω is simply connected
Suppose U is not simply connected, then $\pi_1(U, z_0) \neq 0$, i.e. there is a simple(self non-intersecting) loop $\gamma \subseteq U$ with $\gamma(0) = \gamma(1)$ cannot be deform to z_0, by Jordan curve theorem **??**, γ divides $\mathbb{C}$ into the exterior and the interior which is homeomorphic to the unit disc D, suppose $F \cap D$ is empty, then $\overline{D} \subseteq U$, γ can be deformed to z_0, giving a contradiction, hence $F \cap \overline{D}$ is a compact connected component of F which is also a contradiction $\square$

Exercise 3.0.9. Consider an open set $\Omega \subseteq \mathbb{C}^2$ such that

$$\{(z,w) \in \mathbb{C}^2 || z| \leq R_1, |w| \leq R_2\} \subseteq \Omega$$

for some positive reals R_1 and R_2. Let $f \in \mathcal{H}(\Omega)$ be such that $f(z,w) \neq 0$ for every z and w for which $|z| \leq R_1$, $|w| = R_2$

1. Prove that the number (counted with multiplicities) of zeros of $w \mapsto f(z,w)$ in $D(0,R_2)$ is the same for every $|z| \leq R_1$

2. Let $w_1(z), \cdots, w_m(z)$ denote the zeros of $w \mapsto f(z,w)$ (counted with multiplicities). Prove that for each $n \in \mathbb{N}$ the function

$$z \mapsto w_1(z)^n + \cdots + w_m(z)^n$$

is holomorphic for $z \in D(0,R_1)$

3. Deduce that nth elementary symmetric function σ_n of $w_1(z), \cdots, w_m(z)$ is holomorphic.

4. Prove that there exists a function h that is holomorphic and without any zeros on $\{(z,w) \in \mathbb{C}^2 || z| < R_1, |w| < R_2\}$ such that

$$f(z,w) = h(z,w)[w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)]$$

for every z and w such that $|z| < R_1$ and $|w| < R_2$

Solution.

1. By Lemma **??**, $\frac{1}{2\pi i}\int_{\partial D(0,R_2)} \frac{f_w(z,w)}{f(z,w)} dw$ is the number of zeros in $D(0,R_2)$ which is continuous, hence the same for every $|z| \leq R_1$

2. By Lemma **??**, $\frac{1}{2\pi i}\int_{\partial D(0,R_2)} w^n \frac{f_w(z,w)}{f(z,w)} dw = w_1(z)^n + \cdots + w_m(z)^n$ is holomorphic

3. Directly follows from (2) thanks to Newton's identities

4. Since $\prod_{i=1}^{m}(w - w_i(z)) = w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)$ is holomorphic

$$\frac{f(z,w)}{w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)}$$

has no zeros on D and holomorphic on $\{R_2 - \varepsilon < |w| < R_2\}$, hence by Hartogs's extension theorem **??**, can be extended to a holomorphic function $h(z,w)$, then $f(z,w) = h(z,w)[w^m + \sigma_1(z)w^{m-1} + \cdots + \sigma_{m-1}(z)w + \sigma_m(z)]$ on $\{R_2 - \varepsilon < |w| < R_2\}$, by identity theorem, this holds for all $|z| < R_1$ and $|w| < R_2$

□

Exercise 3.0.10. Suppose $p_1, \cdots, p_n$ are points on the compact Riemann surface X and $X' = X \setminus \{p_1, \cdots, p_n\}$. Suppose $f : X' \to \mathbb{C}$ is a non-constant holomorphic function. Show that the image of f comes arbitrarily close to every $c \in \mathbb{C}$

Solution. Suppose there exists $c \in \mathbb{C}$ such that $|f - c| \geq \varepsilon$ for some $\varepsilon > 0$, then $\frac{1}{f-c}$ would be a bounded holomorphic function on X', by Riemann's Removable singularity theorem, $\frac{1}{f-c}$ can be extended to a holomorphic function on X, but since X is compact, $\frac{1}{f-c}$ is a constant which is impossible □

Exercise 3.0.11. Let X be a compact Riemann surface and let $X \xrightarrow{\sigma} X$ be a biholomorphic map of X onto itself, different from the identity. Let $a \in X$ be a point with $\sigma(a) \neq a$, and suppose that there is a non-constant meromorphic function f on X, holomorphic on $X \setminus \{a\}$, with a pole of order k at a. Prove that σ can have at most $2k$ fixed points on X

Solution. Suppose there are more than $2k$ fixed points of σ, then consider $f - f \circ \sigma^{-1} : X \to \mathbb{P}^1$ is holomorphic on $X \setminus \{a, \sigma^{-1}(a)\}$ with at least $2k+1$ zeros and with poles of order k at $a, \sigma^{-1}(a)$, but it should have as many poles as zeros which is a contradiction ☐

Exercise 3.0.12. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are lattices in $\mathbb{C}$. Show that $\Lambda = \Lambda'$ iff there exists a matrix $A \in GL(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Solution. First not that

$$\Lambda \subseteq \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \text{ for some } A \in \mathrm{M}(2, \mathbb{Z})$$

Hence we have

$$\Lambda = \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = B \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some } A, B \in \mathrm{M}(2, \mathbb{Z})$$

Which is equivalent to $A \in \mathrm{GL}(2, \mathbb{Z})$ ☐

Exercise 3.0.13. $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ are lattices in $\mathbb{C}$ and $X = \mathbb{C}/\Lambda$, $X' = \mathbb{C}/\Lambda'$ are the corresponding complex tori

1. Prove that any holomorphic map $X \xrightarrow{f} X'$ is induced by a linear map $\mathbb{C} \xrightarrow{g} \mathbb{C}$ of the form $g(z) = \alpha z + \beta$, where $\alpha \in \mathbb{C}$ is such that $\alpha\Lambda \subseteq \Lambda'$. f is biholomorphic if and only if $\alpha\Lambda = \Lambda'$
2. Show that every torus $X = \mathbb{C}/\Lambda$ is isomorphic to a torus of the form $X(\tau) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, where $\tau \in \mathbb{C}$ satisfies $\mathrm{Im}(\tau) > 0$
3. Assume that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ and $\mathrm{Im}(\tau) > 0$. Let $\tau' := \dfrac{a\tau + b}{c\tau + d}$. Show that the tori $X(\tau)$ and $X(\tau')$ are biholomorphic

Solution.

1. Since $\mathbb{C}$ is the universal cover of $\mathbb{C}/\Lambda'$, $f \circ \pi : \mathbb{C} \to \mathbb{C}/\Lambda'$ has a lift $F : \mathbb{C} \to \mathbb{C}$, and locally we have $F = \pi'|_V^{-1} \circ f \circ \pi|_U$, thus F is holomorphic

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

Fix $\omega \in \Lambda$, since $\pi(z + \omega) = \pi(z)$ for any $z \in \mathbb{C}$, we have $F(z + \omega) - F(z) \in \Lambda'$, hence $F(z + \omega) - F(z)$ is a continuous function of z but Λ' is discrete, thus $F(z + \omega) - F(z) \equiv C_\omega$, where $C_\omega \in \Lambda'$ is a constant. Then $F'(z + \omega) = F'(z)$ which shows $F' : \mathbb{C} \to \mathbb{C}$ is doubly periodic function, thus induces $G : \mathbb{C}/\Lambda \to \mathbb{C}$ with $F = G \circ \pi$. Thus G must be a constant, so is F', therefore F has the form $F(z) = \alpha z + \beta$. Then for any $\omega \in \Lambda$, we have $F(\omega) - F(0) = \alpha\omega \in \Lambda'$, thus $\alpha\Lambda \subset \Lambda'$. If f is biholomorphic, then $\pi' \circ F = f \circ \pi \Rightarrow \pi \circ F^{-1} = f^{-1} \circ \pi'$, which implies $\left\{ \begin{array}{c} \alpha\Lambda \subset \Lambda' \\ \alpha^{-1}\Lambda' \subset \Lambda \end{array} \right. \Rightarrow \alpha\Lambda = \Lambda'$

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"] & \mathbb{C} \arrow[l, "F^{-1}"'] \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda & \mathbb{C}/\Lambda' \arrow[l, "f^{-1}"] \end{tikzcd}$$

Conversely, if $\alpha\Lambda = \Lambda'$, $\pi \circ F^{-1}$ is doubly periodic and induce f^{-1}, hence f is biholomorphic

2. Suppose $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$, define $\Lambda' = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \frac{\omega_2}{\omega_1}$, we have $\omega_1\Lambda' = \Lambda$, thus X and $X(\tau)$ are biholomorphic

3. $X(\tau)$ and $X(\tau')$ are biholomorphic iff $\begin{pmatrix}\tau' \\ 1\end{pmatrix} = \alpha A \begin{pmatrix}\tau \\ 1\end{pmatrix}$, $\alpha \in \mathbb{C} - \{0\}, A \in \mathrm{SL}(2, \mathbb{Z})$ If $X(\tau)$ and $X(\tau')$ are biholomorphic, then $\mathbb{Z} + \mathbb{Z}\tau' = \Lambda' = \alpha\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\alpha\tau$ for some $\alpha \in \mathbb{C} - \{0\}$, thus $\begin{pmatrix}\tau' \\ 1\end{pmatrix} = A\begin{pmatrix}\alpha\tau \\ \alpha\end{pmatrix} = \alpha A \begin{pmatrix}\tau \\ 1\end{pmatrix}$, for some $A \in \mathrm{SL}(2, \mathbb{Z})$, the other direction is easy

$\square$

Exercise 3.0.14. Determine the branch points(or ramification points) of the map $f : \mathbb{C} \to \mathbb{P}^1$ with

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

Solution. $f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$ when $z \neq 0$, thus $1, -1$ are branch points.

Consider the chart $(P^1 - \{0\}, \varphi)$ with $\varphi(z) = \frac{1}{z}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[dr] & \mathbb{P}^1 - \{0\} \arrow[d, "\varphi"] \\ & \mathbb{C} \end{tikzcd}$$

Thus $g(z) = \varphi \circ f(z) = \frac{z}{2(z^2+1)}$, $g'(z) = \frac{1 - z^2}{2(z^2+1)}$, hence 0 is not a branch point $\square$

Exercise 3.0.15. If f and g are two elliptic functions with respect to the same lattice $\Omega \subseteq \mathbb{C}$, prove that there exists an irreducible polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$

Solution. If $f \equiv c$ is a constant, then $P(x, y) = x - c$ is an irreducible polynomial such that $P(f, g) = 0$, so we can assume f, g are not constants, Since $\mathcal{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$, there exists rational functions $R_0, \cdots, R_n$ such that $R_0(f) + R_1(f)g + \cdots + R_n(f)g^n = 0$, then after multiplying denominators, we get a polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$, since $\mathbb{C}[x, y]$ is a UFD, $P = P_1 \cdots P_k$, where P_i are prime hence irreducible, then $0 = P_1(f, g) \cdots P_k(f, g) \in \mathcal{M}(X)$ which is a field, thus $P_j(f, g) = 0$ for some irreducible polynomial $P_j \in \mathbb{C}[x, y]$ $\square$

Exercise 3.0.16. f is an elliptic function of order $n > 0$, then f' is an elliptic function of order m such that $n + 1 \leq m \leq 2n$. Both bounds can be attained

Solution. f' is elliptic since $f(z + \omega) = f(z) \Rightarrow f'(z + \omega) = f'(z)$ for all $\omega \in \Omega$. Suppose f has poles $[P_1], \cdots, [P_k]$ with multiplicities $r_1, \cdots, r_k$, $\sum r_i = n$, then f' also has poles $[P_1], \cdots, [P_k]$ with multiplicities $r_1 + 1, \cdots, r_k + 1$, $\sum r_i = n + k = m$, since $1 \leq k \leq n$, $n + 1 \leq m \leq 2n$
We can find an elliptic function f of order n which has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $1, \cdots, 1, 2n+1-m$, then we get f' is another elliptic function which also has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $2, \cdots, 2, 2n + 2 - m$, thus f' is of order m $\square$

Exercise 3.0.17. Prove that

$$\wp'(z) = \frac{2\,\sigma(z-\frac{\omega_1}{2})\,\sigma(z-\frac{\omega_2}{2})\,\sigma(z-\frac{\omega_3}{2})}{\sigma(\frac{\omega_1}{2})\,\sigma(\frac{\omega_2}{2})\,\sigma(\frac{\omega_3}{2})\,\sigma(z)^3}\,.$$

Solution. $\wp'(z)$ has a pole at $z=0$ of order 3 and $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ as simple roots, thus

$$\wp'(z) = \lambda \frac{\sigma\left(z-\frac{\omega_1}{2}\right)\sigma\left(z-\frac{\omega_2}{2}\right)\sigma\left(z-\frac{\omega_3}{2}\right)}{\sigma(z)^3}$$

for some $\lambda \in \mathbb{C}$, multiply by z^3 on both sides, and let $z \to 0$, since $\lim_{z\to 0}\frac{z}{\sigma(z)} = 1$, $\lim_{z\to 0} z^3\wp'(z) = -2$, we have

$$-2 = -\lambda\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right) \Rightarrow \lambda = \frac{2}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)}$$

Hence

$$\wp'(z) = \frac{2\sigma\left(z-\frac{\omega_1}{2}\right)\sigma\left(z-\frac{\omega_2}{2}\right)\sigma\left(z-\frac{\omega_3}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)\sigma(z)^3}$$

□

Let $\Omega \subseteq \mathbb{C}$ be a lattice and $\wp(z)$ the associated Weierstrass $\wp$-function. We have seen that $\wp(z)$ satisfies the differential equation $(\wp'(z))^2 = p(\wp(z))$, where $p(x) = 4x^3 - g_2x - g_3$. The following three problems examine the conditions under which the coefficients g_2 and g_3 of $p(x)$ are real numbers

Exercise 3.0.18. Prove that the following conditions are equivalent

(i) $g_2, g_3 \in \mathbb{R}$

(ii) $G_k \in \mathbb{R}$ for all $k \geq 3$

(iii) $\wp(\bar{z}) = \overline{\wp(z)}$ for all $z \in \mathbb{C}$

(iv) $\overline{\Omega} = \Omega$ (the last condition says that Ω is a *real lattice*)

Solution. $(i) \Rightarrow (ii)$
$g_2 = 60G_4, g_3 = 140G_6 \in \mathbb{R} \Rightarrow G_4, G_6 \in \mathbb{R}$
Since

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2} \\ &= \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + 7G_8z^6 + 9G_{10}z^8 + \cdots\end{aligned}$$

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^3} + \sum_{n=2}^{\infty}(2n-1)(2n-2)G_{2n}z^{2n-3} \\ &= -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + 42G_8z^5 + 72G_{10}z^7 + \cdots\end{aligned}$$

$$\begin{aligned}\wp''(z) &= \frac{6}{z^4} + \sum_{n=2}^{\infty}(2n-1)(2n-2)(2n-3)G_{2n}z^{2n-4} \\ &= \frac{6}{z^4} + 6G_4 + 60G_6z^2 + 210G_8z^4 + 504G_{10}z^6 + \cdots\end{aligned}$$

So we can conclude $\wp''(z) - 6\wp(z)^2 + 30G_4 = z\varphi(z)$, where $\varphi(z)$ is a holomorphic elliptic function, hence $\wp''(z) - 6\wp(z)^2 + 30G_4 = 0$, then the coefficients of $z^{2n}(n \geq 1)$ would be $(2n+1)(2n+$

$2)(2n+3)(2n+4)G_{2n+4} - 6(2n+3)G_{2n+4}$ minus terms only involving $G_4, G_6, \cdots, G_{2n+2}$ and real numbers, thus by induction, we know $G_{2n+4} \in \mathbb{R}(n \geq 1)$
$(ii) \Rightarrow (iii)$
Since $\wp(z) = \dfrac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$, if $G_k \in \mathbb{R}(k \geq 3)$, then $\wp(\bar{z}) = \overline{\wp(z)}$
$(iii) \Rightarrow (iv)$
The poles of $\overline{\wp(\bar{z})} = \wp(z)$ are exactly $\overline{\Omega}$, thus $\overline{\Omega} = \Omega$
$(iv) \Rightarrow (i)$
$g_2 = 60G_4 = 60 \sum_{\omega \in \Omega^*} \dfrac{1}{\omega^4} = 60 \sum_{\omega \in \overline{\Omega}^*} \dfrac{1}{\omega^4} = \overline{g_2} \Rightarrow g_2 \in \mathbb{R}$, similarly, $g_6 \in \mathbb{R}$ □

Exercise 3.0.19. We say that Ω is *real rectangular* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1 \in \mathbb{R}$ and $\omega_2 \in i\mathbb{R}$, and that Ω is *real rhombic* if $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_2 = \overline{\omega}_1$. Prove that a lattice Ω is real if and only if it is real rectangular or real rhombic

Solution. If Ω is real rectangular or real rhombic, Ω is obviously a real lattice
Conversely, if Ω is a real lattice, suppose $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then there exists $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, otherwise, $\omega_1 \in \mathbb{R}^*, \omega_2 \in i\mathbb{R}^*$ or $\omega_2 \in \mathbb{R}^*, \omega_1 \in i\mathbb{R}^*$, since ω_1, ω_2 are linear independent, but then $\omega = \omega_1 + \omega_2 \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$ which is a contradiction
Since $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \overline{\omega} \in \mathbb{R}^*$, $\omega - \overline{\omega} \in i\mathbb{R}^*$, thus $\Omega \cap \mathbb{R}^* \neq \varnothing$, $\Omega \cap i\mathbb{R}^* \neq \varnothing$, let $\eta_1 = \min_{\eta \in \Omega \cap (0,\infty)} \eta$, then $\Omega \cap \mathbb{R} = \mathbb{Z}\eta_1$, otherwise $\exists \eta \in \mathbb{R} \setminus \mathbb{Z}\eta_1$, then $\eta - \left\lfloor \frac{\eta}{\eta_1} \right\rfloor \eta_1 \in \Omega \cap (0, \infty)$ which is a contradiction
Similarly, $\Omega \cap i\mathbb{R} = \mathbb{Z}\eta_2$ for some $\eta_2 \in i(0,\infty)$. If $\Omega = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$, then Ω is real rectangular, if not, $\exists \gamma \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, such that $|\gamma| = \min_{\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})} |\omega|$, then $\gamma + \overline{\gamma} = \eta_1$ or $-\eta_1$, otherwise $\gamma + \overline{\gamma} = k\eta_1$ for some $|k| \geq 2$
If $k = 2$, then $\gamma - \eta_1 = \eta_1 - \overline{\gamma} = -\overline{(\gamma - \eta_1)} \Rightarrow \gamma - \eta_1 \in i\mathbb{R} \Rightarrow \gamma \in \mathbb{Z}\eta_1 + \mathbb{Z}(\gamma - \eta_1) \subseteq \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$
If $k > 2$, then $\gamma - \eta_1 \notin \mathbb{R} \cup i\mathbb{R}$ and $|\gamma - \eta_1| < |\gamma|$, similarly for $k \leq -2$, these are all contradictions
Similarly, we know that $\gamma - \overline{\gamma} = \eta_2$ or $-\eta_2$
Now, for any $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, $\omega + \overline{\omega} = k\eta_1 = k(\gamma + \overline{\gamma})$ for some $k \neq 0$, then $\omega - k\gamma = k\overline{\gamma} - \overline{\omega} = -\overline{(\omega - k\gamma)} \Rightarrow \omega - k\gamma \in i\mathbb{R}$, if $\omega \neq k\gamma$, then $\omega - k\gamma = l\eta_2 = l(\gamma - \overline{\gamma}) \Rightarrow \omega \in \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, therefore, we have $\Omega = \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, Ω is real rhombic □

Exercise 3.0.20. Let Ω be a real lattice. Define the real elliptic curve $E_{\mathbb{R}}$ to be the set $\{(x, y) \in \mathbb{R}^2 \mid y^2 = p(x)\}$. Prove that $E_{\mathbb{R}}$ has one or two connected components as Ω is real rhombic or real rectangular, respectively

Solution. The number of connected components of $E_{\mathbb{R}}$ is one or two if $p(x) = 0$ has one real root and two nonreal conjugate complex roots or three distinct real roots correspondingly
Since $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ are simple roots of $\wp'(z)$, the three simple roots of $p(x)$ are $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_3}{2}\right)$, since Ω is a real lattice, $G_k \in \mathbb{R}$ and $\wp(z) = \dfrac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$
If Ω is real rectangular, then $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right)$ are both real, thus $E_{\mathbb{R}}$ has two connected components
If Ω is real rhombic, then $\wp\left(\frac{\omega_3}{2}\right)$ is real $\wp\left(\frac{\omega_1}{2}\right) \neq \wp\left(\frac{\omega_2}{2}\right)$ are nonreal conjugate, thus $E_{\mathbb{R}}$ has only one connected component □

Complex structures on an open annulus

Exercise 3.0.21. $A(r, R) = \{r < |z| < R\}$ is biholomorphic to $\{s < |z| < S\}$ iff $R/r = S/s$, r can be 0, R can be ∞, but not at the same time

Solution. By scaling or inversion we can assume $r = s = 1$ and $|f(z)| \to 1$ as $|z| \to 1$. Suppose $f : A(r, R) \to A(s, S)$ is a biholomorphism, then consider the Laurent series $f = \sum_{k=-\infty}^{\infty} c_k z^k$, for

$1 < t < R$, by Stokes theorem we have

$$A(t) = \frac{1}{2i}\int_{f(\{|z|=t\})} \bar{z}dz = \frac{1}{2i}\int_{|z|=t} \overline{f(z)}df(z) = \frac{1}{2i}\int_{|z|=t} \overline{f(z)}f'(z)dz = \pi\sum_{k\in\mathbb{Z}} k|c_k|^2 t^{2k}$$

As $t \to 1$, we have $A(t) \to \pi \Rightarrow \sum k|c_k|^2 = 1$, thus

$$A(t) - \pi t^2 = \pi t^2 \sum_{k\in\mathbb{Z}} k|c_k|^2 \left(t^{2k-2} - 1\right) \geq 0$$

Thus $A(t) \geq \pi t^2$, as $t \to R$, $A(t) \to \pi S^2 \geq \pi R^2 \Rightarrow S \geq R$. Therefore we have $S = R$ $\square$

Exercise 3.0.22. Let $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{R}^+$ be a σ-additive set function defined on the Borel σ-algebra on $\mathbb{R}$ and let $D \subseteq \mathbb{R}$ be a discrete set with the property $x \in D$ if and only if there exists an open set U such that $x \in U$ and $\mu(U) > 0$. Show that μ can be expressed as a countable linear combination of measures of the form

$$\delta_x(A) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Where $x \in D$ and $A \in \mathcal{B}(\mathbb{R})$

Proof. For each $x \in D$, denote $\mu(\{x\}) = c_x$. Since D is discrete, it is countable and any subset of D is closed. For any open set V in $\mathbb{R}$, by definition, $\mu(V - D) = 0$ since $V - D$ is an open set which doesn't intersect D. Hence

$$\mu(V) = \mu(V \cap D) + \mu(V - D) = \mu(V \cap D) = \sum_{x\in V\cap D} c_x = \sum_{x\in D} c_x\delta_x(V)$$

i.e. μ can be expressed as a countable linear combination of Dirac measures $\square$

Exercise 3.0.23. Let $c : [0,1] \to [0,1]$ denote the Cantor (ternary) function (known also as the Devil's staircase, or the Cantor-Lebesgue function). For each continuous real-valued function $f : [0,1] \to \mathbb{R}$, let $l(f)$ denote the Riemann integral

$$l(f) = \int_0^1 f(c(x))dx$$

Find explicitly a Borel measure $\mu : \mathcal{B}([0,1]) \to \mathbb{R}$ so that

$$l(f) = \int_{[0,1]} f d\mu$$

where $\mathcal{B}([0,1])$ denotes the Borel σ-algebra over $[0,1]$

Proof. Write m as the Lebesgue measure. Since c is a continuous function, the pushforward measure $\mu(E) = m(c^{-1}(E))$ is a Borel measure on $\mathcal{B}([0,1])$. Note that

$$\int_{[0,1]} \chi_E d\mu = \mu(E) = m(c^{-1}(E)) = \int_0^1 \chi_E(c(x))dx$$

and continuous functions are approximated by simple functions, thus

$$\int_{[0,1]} f d\mu = \int_0^1 f(c(x))dx$$

$\square$

Chapter 4

Exercises in category

X1,X2 iso and Y1,Y2 iso implies Hom(X1,Y1),Hom(X2,Y2) iso

Exercise 4.0.1. In category $\mathscr{C}$, if $X \xrightarrow{\phi_X} X'$, $Y \xrightarrow{\phi_Y} Y'$ are isomorphisms, then $Hom(X,Y)$, $Hom(X',Y')$ are in bijective correspondence

Solution. Consider $Hom(X,Y) \to Hom(X',Y')$, $f \mapsto \phi_Y f \phi_X^{-1}$ and $Hom(X',Y') \to Hom(X,Y)$, $f' \mapsto \phi_Y^{-1} f' \phi_X$ which are inverses to each other

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\
X' \arrow[r, "f'"] & Y'
\end{tikzcd}$$

□

Exercise 4.0.2. Suppose the bottom row of the following commutative diagram is exact, $gf = 0$, then there exists a such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, dashed, "\exists a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'
\end{tikzcd}$$

Solution. Since $0 = cgf = g'bf$ and bottom row is exact, we have

$$\begin{tikzcd}
& A \arrow[dl, dashed, "\exists a"'] \arrow[d, "bf"] \\
A' \arrow[r, two heads, "f'"'] & \ker g'
\end{tikzcd}$$

□

Exercise 4.0.3. $F, G : \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ are functors, $F \xrightarrow{\eta} G$ is a natural transformation iff η is natural on each factor

Solution. We have commutative diagram

$$\begin{tikzcd}
F(A,B) \arrow[r, "F(f,1)"] \arrow[d, "\eta_{A,B}"] & F(A',B) \arrow[r, "F(1,g)"] \arrow[d, "\eta_{A',B}"] & F(A',B') \arrow[d, "\eta_{A',B'}"] \\
G(A,B) \arrow[r, "G(f,1)"] & G(A',B) \arrow[r, "G(1,g)"] & G(A',B')
\end{tikzcd}$$

□

Fully faithfull functor is injective on objects up to isomorphism

Exercise 4.0.4. A fully faithful functor $F : \mathscr{C} \to \mathscr{D}$ is injective on objects up to isomorphism

Solution. Suppose $F(X) = F(Y) = T$, let $f : X \to Y$ be the map corresponds to 1_T in $Hom(F(X), F(Y))$, then $f : X \to Y$ is an isomorphism because we can also let $g : Y \to X$ be the map corresponds to 1_T as in $Hom(F(Y), F(X))$, then $F(g \circ f) = F(g) \circ F(f) = 1_T \circ 1_T = 1_T$, thus $g \circ f$ corresponds to 1_T in $Hom(F(X), F(X))$, but $F(1_X) = 1_{F(X)} = 1_T$, thus $g \circ f = 1_X$, similarly, $f \circ g = 1_Y$ $\square$

Exercise 4.0.5. Suppose $\mathscr{A}$ is an abelian category, show $\mathscr{A}$ is balanced. For any $A \xrightarrow{f} B$, $\ker f \xrightarrow{i} A$ is a monomorphism, $B \xrightarrow{\pi} \operatorname{coker} f$ is an epimorphism, and $\operatorname{im} f := \ker \operatorname{coker} f$, $\operatorname{coim} f := \operatorname{coker} \ker f$ are isomorphic

Solution. Suppose $A \xrightarrow{f} B$ is a bimorphism, it is the equaliser of $B \overset{\pi}{\underset{0}{\rightrightarrows}} \operatorname{coker} f$, then $\pi = 0$, $\operatorname{coker} f = 0$, but $A \xrightarrow{1_A} A$ is the kernel of $A \to 0$, hence A, B are isomorphic

$\ker f \xrightarrow{i} A$ is a monomorphism due to the following diagram

$$\begin{tikzcd}
C \arrow[d, "g=0"'] \arrow[dr, "0"] \arrow[drr, "0"] & & \\
\ker f \arrow[r, "i"] & A \arrow[r, "f"'] & B
\end{tikzcd}$$

$B \xrightarrow{\pi} \operatorname{coker} f$ is a monomorphism due to the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[drr, "0"'] & B \arrow[r, "\pi"] \arrow[dr, "0"] & \operatorname{coker} f \arrow[d, "g=0"] \\
& & C
\end{tikzcd}$$

Now let's show coimage and image are isomorphic. $fi = 0$ induces $\operatorname{coker} i \xrightarrow{g} B$, claim that g is monic

Suppose $X \xrightarrow{x} \operatorname{coker} i$ is morphism such that $gx = 0$, it induces $\operatorname{coker} x \xrightarrow{j} B$, since $qpk = 0$, $fk = jqpk = 0$ induces $\ker qp \xrightarrow{l} \ker f$, since qp is epi, $pk = pil = 0$ induces $\operatorname{coker} x \xrightarrow{r} \operatorname{coker} i$, since p is epi, $p = rqp \Rightarrow rq = 1_{\operatorname{coker} i}$, hence q is monic, $qx = 0 \Rightarrow x = 0$

$$\begin{tikzcd}
& \ker qp \arrow[dl, "l"'] \arrow[d, tail, "k"] & & \\
\ker f \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[d, two heads, "p"] & B \arrow[r, two heads, "\pi"] & \operatorname{coker} f \\
X \arrow[r, "x"] & \operatorname{coker} i \arrow[ur, "g"] \arrow[r, two heads, shift left, "q"] & \operatorname{coker} x \arrow[u, "j"'] \arrow[l, dashed, shift left, "r"] &
\end{tikzcd}$$

$\pi f = 0$ induces $A \xrightarrow{h} \ker \pi$, claim that h is epi

Suppose $\ker \pi \xrightarrow{y} Y$ is morphism such that $yh = 0$, it induces $A \xrightarrow{p} \ker y$, since $qjk = 0$, $qf = qjkp = 0$ induces $\operatorname{coker} f \xrightarrow{m} \operatorname{coker} jk$, since jk is monic, $qj = m\pi j = 0$ induces $\ker \pi \xrightarrow{s} \ker y$, since j is monic, $j = jks \Rightarrow ks = 1_{\ker \pi}$, hence k is epi, $yk = 0 \Rightarrow y = 0$

$$\begin{tikzcd}
& & \operatorname{coker} jk & \\
\ker f \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr, "h"] & B \arrow[r, two heads, "\pi"] \arrow[u, two heads, "q"] & \operatorname{coker} f \arrow[ul, "m"'] \\
& \ker y \arrow[r, tail, shift left, "k"] & \ker \pi \arrow[u, tail, "j"'] \arrow[l, dashed, shift left, "s"] \arrow[r, "y"] & Y
\end{tikzcd}$$

Since $\operatorname{im} f \to B$ is monic, $A \to \operatorname{coim} f$ is epi, g, h both induce $\operatorname{coim} f \xrightarrow{\phi} \operatorname{im} f$, then ϕ is monic and epi hence iso

$$\begin{tikzcd}
\ker f \arrow[r, tail] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads] & \mathrm{coker} f \\
& \mathrm{coim} f \arrow[r, dashed, "\phi"] & \mathrm{im} f \arrow[u, hook]
\end{tikzcd}$$

□

bounded double complex with exact rows or exact columns has exact total complex

Exercise 4.0.6. C is a bounded double complex with exact rows or exact columns, then $Tot(C)$ is exact

Solution. Without loss of generality, we may assume C is bounded in the first quadrant and has exact rows, use d', d'', d to denote row, column and total differentials
$Tot(C)$ is exact for all $n < 0$ since $Tot(C)_n = 0$ for all $n < 0$. now suppose $n \geq 0$, $d\left(\sum_{k=0}^{n} x_{k,n-k}\right) = 0$, i.e. $d'x_{k+1,n-k-1} + d''x_{k,n-k} = 0$ for $0 \leq k < n$. Let $x_{0,n+1} = 0$, we can construct $x_{k,n+1-k}$ for $k > 0$ inductively such that $d''x_{k,n-k+1} + d'x_{k+1,n-k} = x_{k,n-k}$ for $0 \leq k \leq n$ as follow:
For $k \geq -1$

$$\begin{aligned}
d'(x_{k+1,n-k-1} - d''x_{k+1,n-k}) &= d'x_{k+1,n-k-1} - d'd''x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''d'x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''(d''x_{k,n-k+1} + d'x_{k+1,n-k}) \\
&= d'x_{k+1,n-k-1} + d''x_{k,n-k} \\
&= 0
\end{aligned}$$

By exactness of rows, there exists $x_{k+2,n-k-1}$ such that

$$d'x_{k+2,n-k-1} = x_{k+1,n-k-1} - d''x_{k+1,n-k} \Leftrightarrow d''x_{k+1,n-k} + d'x_{k+2,n-k-1} = x_{k+1,n-k-1}$$

Therefore

$$\begin{aligned}
d\left(\sum_{k=0}^{n+1} x_{k,n+1-k}\right) &= \sum_{k=1}^{n+1}(d'x_{k,n+1-k} + d''x_{k,n+1-k}) \\
&= \sum_{k=1}^{n+1}(x_{k-1,n-k+1} - d''x_{k-1,n-k+2} + d''x_{k,n+1-k}) \\
&= \sum_{k=0}^{n}(x_{k,n-k} - d''x_{k,n-k+1}) + \sum_{k=1}^{n+1} d''x_{k,n+1-k} \\
&= \sum_{k=0}^{n} x_{k,n-k}
\end{aligned}$$

□

C,D acyclic => C tensor D acyclic

Exercise 4.0.7. C, D are chain complexes with negative degree terms zeros, $H_n(C) = H_n(D) = 0$ for $n \neq 0$, then so is $C \otimes D$

Solution. Apply Exercise 4.0.6 □

Exercise 4.0.8. f is a retract of g in the arrow category, if g is an isomorphism, so is f

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\
X' \arrow[r, "i'"] & Y' \arrow[r, "r'"] & X'
\end{tikzcd}$$

Proof. $i'g^{-1}r$ is the inverse to f □

Chapter 5

Exercises in partial differential equations

Exercise 5.0.1. Consider the heat equation with Neumann's boundary condition:

$$\begin{cases} u_t - \Delta u = 0, & \text{in } \Omega \times \mathbb{R}^+ \\ \dfrac{\partial u}{\partial n} = 0, & \text{on } \Gamma \times \mathbb{R}^+ \\ u(x,0) = v(x), & \text{in } \Omega \end{cases}$$

(a) Show that $\overline{u(t)} = \overline{v}$ for $t \geq 0$, where $\overline{v} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega v(x)dx$ denotes the average of v
(b) Show that $\|u(t) - \overline{v}\| \to 0$ as $t \to \infty$

Solution. **(a)** By divergence theorem, we have

$$0 = \int_\Omega u_t - \Delta u = \int_\Omega u_t + \nabla 1 \cdot \nabla u - \int_{\partial\Omega} \frac{\partial u}{\partial n} = \int_\Omega u_t = \left(\int_\Omega u\right)_t$$

Hence $\displaystyle\int_\Omega u = \int_\Omega v \Rightarrow \overline{u} = \overline{v}$
(b) By divergence theorem, we have

$$0 = \int_\Omega u u_t - u\Delta u = \frac{1}{2}\left(\int_\Omega u^2\right)_t + \int_\Omega |\nabla u|^2 \Rightarrow \frac{1}{2}\left(\int_\Omega u^2\right)_t = -\int_\Omega |\nabla u|^2 \leq 0$$

Hence $\displaystyle\int_\Omega u^2 \leq \int_\Omega v^2$. On the other hand, we have

$$\begin{aligned} 0 &= \int_\Omega (u_t - \Delta u)^2 \\ &= \int_\Omega u_t^2 - 2u_t\Delta u + (\Delta u)^2 \\ &= \int_\Omega u_t^2 - 2\nabla u_t \cdot \nabla u + (\Delta u)^2 \\ &= \int_\Omega 2(\Delta u)^2 + \left(\int_\Omega |\nabla u|^2\right)_t \end{aligned}$$

Which implies $\displaystyle\int_\Omega (\Delta u)^2 = -\frac{1}{2}\left(\int_\Omega |\nabla u|^2\right)_t$, thus

$$\left(\int_\Omega |\nabla u|^2\right)^2 = \left(\int_\Omega u\Delta u\right)^2 \leq \int_\Omega u^2 \cdot \int_\Omega (\Delta u)^2 \leq \int_\Omega v^2 \cdot \int_\Omega (\Delta u)^2 = -\frac{1}{2}\int_\Omega v^2 \cdot \left(\int_\Omega |\nabla u|^2\right)_t$$

Denote $\phi := \int_\Omega |\nabla u|^2$ which is a function of t, $C := \frac{1}{2}\int_\Omega v^2$, then the above equation becomes

$$\phi^2 \le -C\phi' \Rightarrow 0 \ge \phi^2 + C\phi' \Rightarrow 0 \ge 1 + C\frac{\phi'}{\phi^2} = \left(t - \frac{C}{\phi}\right)'$$

Which implies

$$t - \frac{C}{\phi(t)} \le -\frac{C}{\phi(0)} \Rightarrow \frac{C}{\phi(t)} \ge t + \frac{C}{\phi(0)} \ge t \Rightarrow \phi(t) \le \frac{C}{t}$$

Thus $\phi(t) \to 0$ as $t \to \infty$

Now apply Poincaré's lemma, we get

$$\|u - \overline{v}\|_{L^2} = \|u - \overline{u}\|_{L^2} \le C\|\nabla u\|_{L^2} \to 0, t \to \infty$$

□

Exercise 5.0.2.

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)dS\right) &= \frac{d}{dr}\left(\frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} u(x+rz)dS\right)\\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} \frac{d}{dr}u(x+rz)dS\\
&= \frac{1}{|\partial B(0,1)|}\int_{\partial B(0,1)} z\cdot\nabla u(x+rz)dS\\
&= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} \nu\cdot\nabla u(y)dS\\
&= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta u(y)dy\\
&= \frac{r}{n}\frac{1}{|B(x,r)|}\int_{B(x,r)} \Delta u(y)dy
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dr}\left(\int_{B(x,r)} u(y)dy\right) &= \frac{d}{dr}\int_0^r\left(\int_{\partial B(x,s)} u(y)dS\right)ds\\
&= \int_{\partial B(x,r)} u(y)dS
\end{aligned}$$

Exercise 5.0.3. $\Box u = 0$ in $\mathbb{R}^{3+1}$, $u(x,0) = 0$, $u_t(x,0) = f(x) \in C^2(\mathbb{R}^3)$, show that $\int_0^\infty |u(0,t)|^2 dt \le C\|f\|_{L^2(\mathbb{R}^3)}$

Proof. Hint: $u(x,t) = \frac{t}{4\pi}\int_{S^2} f(x+tw)dS_w = -\frac{1}{4\pi}\int_{S^2}\int_t^\infty \frac{d}{d\lambda}f(x+\lambda t)d\lambda$

$u(0,t) = \frac{t}{4\pi}\int_{S^2} f(tw)dS_w$, $|u(x,t)| \le \frac{C}{t}\int_{\mathbb{R}^3}|\nabla f|dx$ □

Exercise 5.0.4. Consider the differential equation

$$\frac{\partial}{\partial t}x(n,t) = x(n-1,t) + x(n+1,t) - 2x(n,t) \tag{5.0.1}$$

for a function $x(n,t)$ of an integer n and real number $t \ge 0$. Assume that the function $x(n,t)$ satisfies

$$x(n+N,t) = x(n,t) \tag{5.0.2}$$

for any integer n, where N is an integer larger than or equal to 3. Furthermore, let $e(m,n) = \exp\left(i\frac{2\pi mn}{N}\right)$ for integers m and n

1. Let $f_m(t)$ be a function of a real number $t \geq 0$ for an integer m with $f_m(0) = c_m$, where c_m is a complex number. Assume that the function of the form $x(n,t) = e(m,n)f_m(t)$ satisfies differential equation (5.0.1) and (5.0.2). Find $f_m(t)$

2. Let $(g_0, \cdots, g_{N-1})$ be an N-dimensional complex vector. Under the initial condition $x(n,0) = g_n$, $n = 0, \cdots, N-1$, find the solution of the differential equation (5.0.1) with condition (5.0.2)

3. Find $\lim_{t\to\infty} x(n,t)$ for the solution $x(n,t)$ found in 2.

Solution.

1. Note that $e(m, n+N) = e(m,n)$, hence (5.0.2) is justified. Plug $x(n,t) = e(m,n)f_m(t)$ in (5.0.1), we have

$$e^{i\frac{2\pi mn}{N}} f_m'(t) = \left(e^{i\frac{2\pi m(n-1)}{N}} + e^{i\frac{2\pi m(n+1)}{N}} - 2e^{i\frac{2\pi mn}{N}}\right) f_m(t)$$

Which can be simplified as

$$f_m'(t) = \left(e^{-i\frac{2\pi m}{N}} + e^{i\frac{2\pi m}{N}} - 2\right) f_m(t) = -4\sin^2\left(\frac{\pi m}{N}\right) f_m(t) = K_m f_m(t)$$

Solve this with initial condition $f_m(0) = c_m$ we get $f_m(t) = c_m e^{K_m t}$

2. Consider $x(n,t) = \sum_{m=0}^{N-1} e(m,n)f_m(t)$, then $x(n,0) = \sum_{m=0}^{N-1} e(m,n)f_m(0) = g_n$ which can be uniquely solved since $E = \{e(m,n)\}_{0\leq m,n<N}$ is a Vandermonde matrix, suppose the solutions are $f_m(0) = c_m$, then use 1. to solve $f_m(t)$

3. Note that $K_m \leq 0$ for $0 \leq m < N$ and $K_m = 0$ iff $m = 0$. Let $\omega = e^{\frac{2\pi i}{N}}$, then $\omega^N = 1$ and the determinant of

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \omega & \omega^2 & \cdots & \omega^{N-1} \\ \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{N-1} & \omega^{2(N-1)} & \cdots & \omega^{(N-1)^2} \end{bmatrix}$$

without the $j+1$-th row would be

$$\begin{aligned} &\omega \cdots \widehat{\omega^j} \cdots \omega^{N-1} \prod_{\substack{0\leq p<q\leq N-1 \\ p,q\neq j}} (\omega^q - \omega^p) \\ &= (-1)^j \omega^{\frac{N(N-1)}{2} - j} \frac{\prod_{0\leq p<q\leq N-1} (\omega^q - \omega^p)}{\prod_{k\neq j} (\omega^k - \omega^j)} \\ &= (-1)^j \omega^{\frac{N(N-1)}{2} - j} \frac{\det E}{\omega^{j(N-1)} \prod_{k\neq 0} (\omega^k - 1)} \\ &= \frac{(-1)^j \omega^{\frac{N(N-1)}{2}}}{N} \det E \end{aligned}$$

thus by Cramer's rule we have

$$
\begin{aligned}
\lim_{t\to\infty} x(n,t) = c_0 &= \frac{\begin{vmatrix} g_0 & 1 & \cdots & 1 \\ g_1 & \omega & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1} & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \cdots & \omega^{(N-1)^2} \end{vmatrix}} \\
&= \frac{\sum_{j=0}^{N-1} (-1)^j g_j \frac{(-1)^j \omega^{\frac{N(N-1)}{2}}}{N} \det E}{\sum_{j=0}^{N-1} (-1)^j \frac{(-1)^j \omega^{\frac{N(N-1)}{2}}}{N} \det E} \\
&= \frac{g_0 + \cdots + g_{N-1}}{N}
\end{aligned}
$$

□

Chapter 6

Exercises in algebraic topology

Exercise 6.0.1. M is a locally Euclidean, Hausdorff and connected manifold, then paracompactness imples second countable

Proof. An open cover by precompact coordinate charts has a locally finite open refinement $\{U_i\}$, each U_i is precompact and second countable
Define $S_0 = \{U_0\}$ for some U_0, since $\{U_i\}$ is locally finite, define S_1 to be the union of S_0 and those intersects U_0, repeating this process, we get $S_2, \cdots, S_n, \cdots$, define $S = \bigcup_{n=0}^{\infty} S_n$
M is connected thus path connected, pick any $x_0 \in U_0$, for any $x \in M$, there is a path γ connecting x_0 and x, since γ is compact, it can be covered by S. Hence S is an open cover of M, thus M is second countable □

Exercise 6.0.2. If G is a discrete group, P is connected, $P \xrightarrow{p} X$ is a principal G bundle iff it is a regular cover with $\mathtt{Aut}(p) = G$

Solution. $P \xrightarrow{p} X$ is a fiber bundle thus a cover, G acts regularly on fibers and $G \leq \mathtt{Aut}(p)$ □

Exercise 6.0.3. Use Theorem **??** to prove homotopy invariance of maps on homology

Solution. Suppose $F : X \times I \to Y$ is a homotopy between f and g, we only need to prove i_0, i_1 are naturally chain homotopic since $Fi_0 = f, Fi_1 = g$

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & C_n(X) \arrow[r] \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] & C_{n-1}(X) \arrow[d, shift right, "i_0"'] \arrow[d, shift left, "i_1"] \\
C_{n+1}(X \times I) \arrow[r] \arrow[d, "F"] & C_n(X \times I) \arrow[r] \arrow[d, "F"] & C_{n-1}(X \times I) \arrow[d, "F"] \\
C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y)
\end{tikzcd}$$

Consider Top with model $\mathcal{M} = \{\Delta^n\}$, $F, G : Top \to Ch_{\geq 0}$, $F(X) = C_*(X)$, $G(X) = C_*(X \times I)$, $H_i(\Delta^n \times I) = 0$ for $i \neq 0$, $F_k(X) = \left\{\Delta^k \xrightarrow{\text{id}} \Delta^k \xrightarrow{\sigma} X\right\}$, there is an obvious natural equivalence $\phi_0 : H_0F \to H_0G$, then lifts i_0, i_1 are naturally chain homotopic □

Exercise 6.0.4. K is a CW complex, $X \xrightarrow{f} Y$ is a weak equivalence, then $[K, X] \to [K, Y]$ is a bijection

Exercise 6.0.5. Quotient map $X \xrightarrow{q} Y$ is a homeomorphim iff q is bijective

Solution. If q is bijective, then for any open subset $U \subseteq X$, $U = q^{-1}(q(U))$, by definition, $q(U)$ is open, i.e. q^{-1} is continuous □

Cofibration in a Hausdorff space is closed

Exercise 6.0.6. If X is Hausdorff, then cofibration $A \xrightarrow{i} X$ is closed. This is not true if X is not Hausdorff as showed in Example **??**

Solution. Suppose $A \xrightarrow{i} X$ is a not closed, $X \times I \xrightarrow{r} X \times \{0\} \cup A \times I$ is the retraction, pick any $x \in \overline{A} \setminus A$ with x_n converging to x, then $A \times \{1\} \ni r(x,1) = r(\lim x_n, 1) = \lim r(x_n, 1) = \lim(x_n, 1) = (x, 1)$ which is a contradiction $\square$

Exercise 6.0.7. $\mathbb{R} \times \mathbb{R} \xrightarrow{\wedge} \mathbb{R}$, $\mathbb{R} \times \mathbb{R} \xrightarrow{\vee} \mathbb{R}$ are continuous

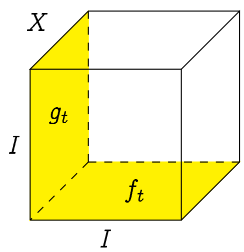
Solution. $x \wedge y = \dfrac{x+y-|x-y|}{2}$, $x \vee y = \dfrac{x+y+|x-y|}{2}$ $\square$

Exercise 6.0.8. $Y^I \to Y$, $\gamma \mapsto \gamma(0)$ and $Y^I \to Y \times Y$, $\gamma \mapsto (\gamma(0), \gamma(1))$ are Hurewicz fibrations

Solution. Need $g(x,s) = H(x,0,s)$, $f(x,t) = H(x,t,0)$ so that $g(x,0) = f(x,0)$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, hook] & Y^I \arrow[d] \\ X \times I \arrow[r, "f_t"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

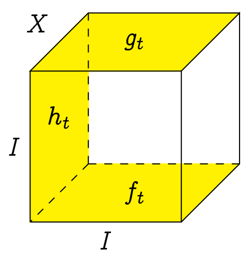
$X \times I^2$ can be deformed onto $X \times I \cup X \times I = X \times (I \cup I)$



Need $h(x,s) = H(x,0,s)$, $f(x,t) = H(x,t,0)$, $g(x,t) = H(x,t,1)$ so that $h(x,0) = f(x,0)$, $h(x,1) = g(x,0)$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, hook] & Y^I \arrow[d] \\ X \times I \arrow[r, "(f_t, g_t)"'] \arrow[ur, dashed, "H"] & Y \times Y \end{tikzcd}$$

$X \times I^2$ can be deformed onto $X \times I \cup X \times I \cup X \times I = X \times (I \cup I \cup I)$



$\square$

Chapter 7

Exercises in differential topology

Hom(V,W)=V^* tensor W

Exercise 7.0.1. $Hom(V,W) \to V^* \otimes W$, $A \mapsto \sum_{i,j} a_{ji} v_i^* \otimes w_j$ is an isomorphism where $A = (a_{ij})$ is the matrix with respect to basis $\{v_1^*, \cdots, v_m^*\}$, $\{w_1, \cdots, w_n\}$

Solution. $A(v_i) = \sum_j a_{ji} w_j$ □

Exercise 7.0.2. Suppose M, N are smooth manifolds of dimension m, n, $f : M \to N$ is a smooth map, $(x^1, \cdots, x^m)$, $(y^1, \cdots, y^n)$ are local coordinates around $p \in M$, $q = f(p) \in N$, then the corresponding matrix of df with respect to basis $\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}\right)$, $\left(\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\right)$ is $\left(\frac{\partial y^i}{\partial x^j}\right)$. In particular, this gives the change of coordinates formula

Solution.

$$df\left(\frac{\partial}{\partial x^i}\right)(g) = \frac{\partial(g \circ f)}{\partial x^i} = \sum_j \frac{\partial g}{\partial y^j}\frac{\partial y^j}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}(g)$$

According to Exercise 7.0.1, $df = \sum_{i,j} \frac{\partial y^j}{\partial x^i} dx^i \otimes \frac{\partial}{\partial y^j}$, we can define higher differential $d^k f = \sum_{i_1, \cdots, i_k, j} \frac{\partial y^j}{\partial x^{i_1} \cdots \partial x^{i_k}} dx^{i_1} \cdots dx^{i_k} \otimes \frac{\partial}{\partial y^j}$ □

Exterior derivative of one form

Exercise 7.0.3. Suppose $\omega \in \Omega^1(M)$, $X, Y \in TM$, then $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$

Solution. By linearity, we can assume $\omega = udv$, then

$$\begin{aligned} d\omega(X,Y) &= d(udv)(X,Y) \\ &= du \wedge dv(X,Y) \\ &= du(X)dv(Y) - du(Y)dv(X) \\ &= XuYv - YuXv \end{aligned}$$

And

$$\begin{aligned} &X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \\ =\ &X(udv(Y)) - Y(udv(X)) - udv([X,Y]) \\ =\ &Xudv(Y) + uX(dv(Y)) - Yudv(X) - uY(dv(X)) - u[X,Y]v \\ =\ &XuYv + uXYv - YuXv - uYXv - uXYv + uYXv \\ =\ &XuYv - YuXv \end{aligned}$$

□

Pushforward of vector field

Exercise 7.0.4. Suppose $\phi : M \to N$ is a map of smooth manifolds, then $X(f \circ \phi) = ((\phi_* X)f) \circ \phi$

Solution. $X(f \circ \phi)(p) = X_p(f \circ \phi) = \phi_p X_p(f) = (\phi_* X)_{\phi(p)}(f) = ((\phi_* X)f)(\phi(p))$ □

Naturality of Lie bracket

Exercise 7.0.5. Suppose X, Y are vector fields on M, $\phi : M \to N$ is a smooth map, then $\phi_*[X, Y] = [\phi_* X, \phi_* Y]$

Solution. Apply Exercise 7.0.4

$$\begin{aligned}
\phi_*[X, Y](f) &= [X, Y](f \circ \phi) \\
&= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
&= X(((\phi_* Y)f) \circ \phi) - Y(((\phi_* X)f) \circ \phi) \\
&= ((\phi_* X)(\phi_* Y)f) \circ \phi - ((\phi_* Y)(\phi_* X)f) \circ \phi \\
&= ([\phi_* X, \phi_* Y]f) \circ \phi \\
&= [\phi_* X, \phi_* Y]f
\end{aligned}$$

□

Chapter 8

Exercises in bundles

Exercise 8.0.1. $E \xrightarrow{p} B$ is a Serre fibration, $A \overset{i}{\hookrightarrow} X$ is a subcomplex, if either p or i is a weak equivalence, then we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] \arrow[ur, dashed, "\exists h"] & B \end{tikzcd}$$

Solution. If p is a weak equivalence, then fibers are weak contractible
If i is a weak equivalence, then X deformation retracts onto A □

Chapter 9

Exercises in complex geometry

Chapter 10

Exercises in Lie groups and Lie algebras

Exercise 10.0.1. Suppose $\mathfrak{g}$ is a real semisimple Lie algebra with a negative definite Killing form, then $\mathfrak{g}$ is the Lie algebra of some compact Lie group G

A is the direct sum of ideals => A is the product of these ideals

Exercise 10.0.2. Suppose A is a nonassociative algebra, $A = I_1 \oplus \cdots \oplus I_n$ is a direct sum of ideals, then $I_i I_j \subseteq I_i \cap I_j = 0$, hence $A = I_1 \times \cdots \times I_n$ can be viewed as product of ideals

Remark 10.0.3. If $A = I_1 \oplus \cdots \oplus I_n$ is just a direct sum of subalgebras, Exercise 10.0.2 may not hold true

Pairwise commuting matrices can be diagonalized simultaneously

Exercise 10.0.4. Let $S \subseteq M(n, \mathbb{F}), (\overline{\mathbb{F}} = \mathbb{F})$ be a set such that $[X, Y] = 0, \forall X, Y \in S$, then elements in S can be diagonalized simultaneously

Solution. Suppose $0 \neq V_\lambda$ is the λ-eigenspace of $X \in S$, then for any $Y \in S$, $XYv = YXv = \lambda Yv$ for $v \in V_\lambda$, thus $YV_\lambda \subseteq V_\lambda$, thus V_λ is an invariant subspace for all $Y \in \mathfrak{g}$, since Y are all semisimple, by induction we can write V as a direct sum of V_λ's, and they are all invariant under any $Y \in \mathfrak{g}$, we only need to show that all elements of $\mathfrak{g}$ can be diagonalized simultaneously on each V_λ, if $X|_{V_\lambda} = 1_{V_\lambda}$, then we are done, otherwise we can decompose it into smaller eigenspaces $\square$

Finite dimensional toral Lie algebra is abelian and its elements can be diagonalized simultaneously

Exercise 10.0.5. Let V be a finite dimensional $\mathbb{F}$ vector space with $\overline{\mathbb{F}} = \mathbb{F}$, and $\mathfrak{g} \leq \mathfrak{gl}(V)$ be a toral Lie algebra, then $\mathfrak{g}$ is abelian. Moreover, $X \in \mathfrak{g}$ can be diagonalized simultaneously

Solution. Suppose $ad(X)|_{\mathfrak{g}} \neq 0$ for some $X \in \mathfrak{g}$, since $\overline{\mathbb{F}} = \mathbb{F}$, there exists $0 \neq Y \in \mathfrak{g}$ and $\lambda \neq 0$ such that $ad(X)(Y) = \lambda Y$, by Proposition **??**, $ad(Y)$ is semisimple, suppose λ_j, X_j are the eigenvalues and linearly independent eigenvectors, then we have $X = \sum c_j X_j$ with $c_j \neq 0$, and $0 = ad(Y)(\lambda Y) = ad(Y)ad(X)(Y) = -ad(Y)^2(X) = -ad(Y)^2\left(\sum c_j X_j\right) = -\sum c_j \lambda_j^2 X_j$, thus $c_j \lambda_j^2 = 0 \Rightarrow \lambda_j = 0$, but $0 \neq \lambda Y = ad(X)(Y) = -ad(Y)(X) = -ad(Y)\left(\sum c_j X_j\right) = -\sum c_j \lambda_j X_j = 0$ which is a contradiction
Now that we know $[\mathfrak{g}, \mathfrak{g}] = 0$, use Lemma 10.0.4, we know all elements of $\mathfrak{g}$ are diagonalizable simultaneously $\square$

Lie group homomorphism has constant rank

Exercise 10.0.6. Lie group homomorphism has constant rank

Solution. Let $\phi : G \to G'$ be a Lie group homomorphism, for any $g \in G$, it suffices to show $\mathtt{rank}(d\phi)_g = \mathtt{rank}(d\phi)_1$, since $\phi(gh) = \phi(g)\phi(h)$, thus $\phi \circ L_g = L_{\phi(g)} \circ \phi$, $(d\phi)_g(dL_g)_1 = d(L_{\phi(g)})_1(d\phi)_1$, and left multiplications are isomorphisms, we have $\mathtt{rank}(d\phi)_g = \mathtt{rank}(d\phi)_1$ $\square$

Exercise 10.0.7. Let G be a Lie group, M, N be smooth manifolds with a G action, and G acts transitively on M, for any equivariant map $f : M \to N$, f has constant rank

Solution. For any $x \in M$, denote $y = f(x)$, it suffices to show $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ since G acts transitively on M, note that $f(gx) = gf(x)$, thus $f \circ L_g = L_g \circ f$, $(df)_{gx}(dL_g)_x = d(L_g)_y(df)_x$, and group actions are isomorphisms, we have $\mathtt{rank}(df)_x = \mathtt{rank}(df)_{gx}$ □

Exercise 10.0.8. If $\phi : G \to H$ be a bijective Lie group homomorphism, then it is an isomorphism

Solution. Apply Exercise 10.0.6 and Theorem **??** □

Exercise 10.0.9. Compact semisimple Lie group G has finite center

Solution. Since $\mathfrak{g} = \mathtt{Lie}(G)$ is semisimple, $\mathtt{Lie}(Z(G)) \leq Z(\mathfrak{g}) = 0$, thus $Z(G)$ is discrete, but G is compact, so $Z(G)$ is finite □

rudimentary facts about topological groups

Exercise 10.0.10. G is a topological group, A is called **symmetric** if $A = A^{-1}$

1. Topology of G is translation invariant, U is open $\Rightarrow xU, Ux$ are open

2. $e \in U$ is a neighborhood, then $e \in V \subseteq U$ a symmetric neighborhood

3. $e \in U$ is a neighborhood, then $e \in V \subseteq VV \subseteq U$ with V being a symmetric neighborhood

4. $H \leq G$ is a subgroup, then so is $\bar{H}$

5. Open subgroups of G are also closed(closed groups are not necessarily open, consider $\{e\}$)

6. $K_1, K_2 \subseteq G$ are compact sets, so is K_1K_2

7. Suppose G is a connected, U is a neighborhood of 1, then $G = \langle U \rangle$

Solution.

1. Multiplication by x is an isomorphism with x^{-1} being its inverse

2. Take $U \cap U^{-1}$

3. Since the multiplication $G \times G \to G$ is continuous, consider the preimage of U which contains $V_1 \times V_2$, take $V \subseteq V_1 \cap V_2$ symmetric

4. If $x_\alpha \to x$, $y_\beta \to y$, then $x_\alpha^{-1} \to x^{-1}$, $x_\alpha y_\beta \to xy$, since these maps are continuous. From this we know that $\bar{H} = \bigcap F$ wehre F runs over all closed subgroup containing H

5. Suppose $H \leq G$ is open, then $H = G \setminus \bigcup_{x \neq e} xH$ is closed, thus if G is connected, then $H = G$

6. K_1K_2 is the image of $K_1 \times K_2$ under multiplication

7. By b, we there is a symmetric neighborhood $1 \in V \subseteq U$, let V_k be the subset of elements can be written in the product of no more the k elements in V, then $V_1 = V$, $V_k = V_1V_{k-1}$ is open by induction, $\langle V \rangle = \bigcup_{k=1}^{\infty} V_k$ is also open, by e, G is generated by V hence by U, and if G is not connected, $G_0 = \langle V \rangle$ is called the identity component of G

□

Exercise 10.0.11. G is a topological group, if G is T_1, then G is Hausdorff, if G is not T_1, then $H := \overline{\{e\}}$ is normal subgroup, G/H is a Hausdorff topological group

Solution. If G is T_1, according to Exercise 10.0.10, for $x \neq y$, $\exists e \in VV \subseteq U$ with V a symmetric neighborhood of e disjoint from $y^{-1}x$, then $xV \cap yV = \emptyset$, suppose $z = xv_1 = yv_2$, then $y^{-1}x = v_2^{-1}v_1 \in VV$ thus reaches a contradiction
According to Proposiontion 10.0.10, $H = \bigcap H_i$, H_i runs over closed subgroups of G, thus H is the smallest closed subgroup, if H is normal, otherwise $xHx^{-1} \cap H$ is a smaller closed subgroup for some x
In G/H, identity is closed, by invariance of topology under translation, every point is closed, meaning G/H is T_1 thus Hausdorff
Checking G/H is still a topological group: $g \in \bigcup_x xH$ open in G, then $g^{-1} \in (\bigcup_x xH)^{-1} = \bigcup_x H^{-1}x^{-1} = \bigcup_x Hx^{-1} = \bigcup_x x^{-1}H$
If $V \times W \to VW \subseteq \bigcup_x xH$, then $vw \in \bigcup_x xH, \forall v \in V, w \in W$, then $\forall h \in H$, $vhw = vww^{-1}hw \in \bigcup_x xH$, therefore, $VH \times WH \to VHWH \subseteq \bigcup_x xH$, notice that VH is open as long as V is open since $VH = \bigcup_{h\in H} Vh$ □

Exercise 10.0.12. $(,)_B$ is the bilinear form given by matrix B, $O(B) = \{X \in GL_n(\mathbb{C}) | X^T BX = 1\}$, the Lie algebra is $\mathfrak{o}(B) = \{X \in M_n(\mathbb{C}) | X^T B + BX = 0\}$

Solution. $\left.\dfrac{d}{dX}\right|_{X=0} (e^{X^T} Be^X) = X^T B + BX = 0$ □

Exercise 10.0.13. $T = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \middle| a \in \mathbb{C}^\times \right\} \subseteq SL(2, \mathbb{C}) = G$ is the torus, the Weyl group $W(T) = N_G(T)/Z(T) = N/T \cong \mathbb{Z}/2\mathbb{Z}$ is generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$

Solution. Consider $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$,$ad - bc = 1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} adx - bcx^{-1} & ab(x^{-1} - x) \\ cd(x - x^{-1}) & adx^{-1} - bcx \end{pmatrix} \in T$$

For any $x \in \mathbb{C}^\times$, which implies that $ab = cd = 0 \Rightarrow a = d = 0$ or $b = c = 0$ and

$$\begin{pmatrix} b & \\ & b^{-1} \end{pmatrix} \begin{pmatrix} & -b^{-1} \\ b & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

□

Chapter 11

Exercises in algebraic geometry

Exercise 11.0.1. $H = V(f)$ is a hypersurface, $f(t_1, \cdots, t_n, x) = a_0(t_1, \cdots, t_n)x^m + \cdots + a_m(t_1, \cdots, t_n)$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr, "\varphi"'] & \mathbb{A}^{n+1} \arrow[d] \\ & \mathbb{A}^n \end{tikzcd}$$

φ is finite iff $a_0 \neq 0$ is a constant. φ is quasifinite $\Rightarrow$ $a_0, \cdots, a_m$ don't have common zeros

Chapter 12

Exercise in functional analysis

Chapter 13

Exercises in linear algebra

Exercise 13.0.1. Let n be a positive integer. Let A be a real square matrix of size n, and let B be a real symmetric positive-definite matrix of size n

1. Show that there exists a unique real square matrix C of size n satisfying

$$BC + CB = A \tag{13.0.1}$$

In the following, this matrix C is denoted by $C_{A,B}$

2. Show that $BC_{A,B} = C_{A,B}B$ iff $AB = BA$

Solution.

1. $B = PDP^T$ for some orthogonal matrix P and digonal matrix $D = \mathtt{diag}(d_1, \cdots, d_n)$ with $d_i > 0$, then (13.0.1) becomes

$$DP^TCP + P^TCPD = P^TAP$$

Denote as

$$DC' + C'D = A'$$

Then we have

$$(d_i + d_j)c'_{ij} = a'_{ij} \Rightarrow c'_{ij} = \frac{a'_{ij}}{d_i + d_j}$$

Therefore C' is uniquely determined, so is $C = PC'P^T$

2. We only need to prove $DC' = CD'$ iff $A'D = DA'$. But $A'D = DA'$ is equivalent to $D^2C' = C'D^2$. Therefore it suffices to prove $d_ic'_{ij} = d_jc'_{ij}$ iff $d_i^2c'_{ij} = d_j^2c'_{ij}$, i.e. $(d_i - d_j)c'_{ij} = (d_i + d_j)(d_i - d_j)c'_{ij}$ which is obviously true since $d_i + d_j > 0$

□

Exercise 13.0.2. Let n be a positive integer, and all matrices are supposed to be over the real numbers

1. Let A be a symmetric positive definite matrix of order n. Show that there exists a unique symmetric positive definite matrix R such that $R^2 = A$. We denote such R by $\sqrt{A}$

2. Let B be a nonsingular matrix of order n. Show that an orthogonal matrix Q that maximizes $f(Q) = \mathtt{tr}(QB)$ satisfies

$$Q = \sqrt{B^TB}^{-1}B^T = B^T\sqrt{BB^T}^{-1}$$

3. Let G and H be symmetric positive definite matrices of order n. Find a square matrix L that minimizes

$$g(L) = \mathrm{tr}\{(I-L)G(I-L)^T\}$$

subject to $LGL^T = H$

Solution.

1. $A = PDP^T$, here P is an orthogonal matrix, $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$, $\lambda_i > 0$, take $R = P\sqrt{D}P^T$ where $\sqrt{D} = \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$, and the uniqueness of R is obvious

2. Consider polar decomposition $B = \sqrt{BB^T}(\sqrt{BB^T}^{-1}B) = (B\sqrt{B^TB}^{-1})\sqrt{B^TB}$, $\sqrt{BB^T}, \sqrt{B^TB}$ are symmetric and positive definite, $\sqrt{BB^T}^{-1}B, B\sqrt{B^TB}^{-1}$ are orthogonal, then $\mathrm{tr}(QB) = \mathrm{tr}((QB\sqrt{B^TB}^{-1})\sqrt{B^TB}) = \mathrm{tr}((\sqrt{BB^T}^{-1}BQ)\sqrt{BB^T})$. Hence the problem can be rewritten as given a symmetric positive definite matrix B, the orthogonal maximizer of $f(Q) = \mathrm{tr}(QB)$ is $Q = I$, and by writing $B = PDP^T$, $\mathrm{tr}(QB) = \mathrm{tr}(P^TQPD)$, we may even assume $B = D$ is diagonal, but then $\mathrm{tr}(QD) = \sum q_{kk}d_k$ which obtains maximum iff $q_{kk} = 1$ since $|q_{kk}| \leq 1$, this justifies our simplified version

3. $LGL^T = H$ can be rewritten as $Q^TQ = I$ with orthogonal matrix $Q = \sqrt{H}^{-1}L\sqrt{G}$, note that

$$\mathrm{tr}((I-L)G(I-L)^T) = \mathrm{tr}(G+H) - \mathrm{tr}(LG) - \mathrm{tr}(GL^T) = \mathrm{tr}(G+H) - 2\,\mathrm{tr}(LG)$$

Hence we only need to maximize $\mathrm{tr}(LG) = \mathrm{tr}(\sqrt{H}Q\sqrt{G}) = \mathrm{tr}(Q\sqrt{G}\sqrt{H})$, by 2. we know

$$Q = \sqrt{H}\sqrt{G}\sqrt{\sqrt{G}H\sqrt{G}}^{-1} \Rightarrow L = H\sqrt{G}\sqrt{\sqrt{G}H\sqrt{G}}^{-1}\sqrt{G}^{-1}$$

$\square$

Chapter 14

Exercises in polylogarithm

Exercise 14.0.1 (Derivatives of polylogarithms)**.**

If $m_i > 1$ for $1 \leq i \leq d$, then

$$\begin{aligned}\frac{\partial}{\partial z_i}\operatorname{Li}_{m_1,\cdots,m_d}(z_1,\cdots,z_d) &= \sum_{k_1>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots z_i^{k_i-1}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_i^{m_i-1}\cdots k_d^{m_d}}\\ &= \frac{1}{z_i}\operatorname{Li}_{m_1,\cdots,m_i-1,\cdots,m_d}(z_1,\cdots,z_d)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial z_1}\operatorname{Li}_{1,m_2,\cdots,m_d}(z_1,\cdots,z_d) &= \sum_{k_1>\cdots>k_d\geq 1}\frac{z_1^{k_1-1}z_2^{k_2}\cdots z_d^{k_d}}{k_2^{m_2}\cdots k_d^{m_d}}\\ &= \sum_{k_2>\cdots>k_d\geq 1}\frac{z_2^{k_2}\cdots z_d^{k_d}}{k_2^{m_2}\cdots k_d^{m_d}}\sum_{k_1=k_2+1}^{\infty} z_1^{k_1-1}\\ &= \sum_{k_2>\cdots>k_d\geq 1}\frac{z_2^{k_2}\cdots z_d^{k_d}}{k_2^{m_2}\cdots k_d^{m_d}}\frac{z_1^{k_2}}{1-z_1}\\ &= \frac{1}{1-z_1}\sum_{k_2>\cdots>k_d\geq 1}\frac{(z_1z_2)^{k_2}\cdots z_d^{k_d}}{k_2^{m_2}\cdots k_d^{m_d}}\\ &= \frac{1}{1-z_1}\operatorname{Li}_{m_2,\cdots,m_d}(z_1z_2,\cdots,z_d)\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial z_d}\operatorname{Li}_{m_1,\cdots,m_{d-1},1}(z_1,\cdots,z_d) &= \sum_{k_1>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots z_{d-1}^{k_{d-1}}z_d^{k_d-1}}{k_2^{m_2}\cdots k_{d-1}^{m_{d-1}}}\\
&= \sum_{k_1>\cdots>k_{d-1}\geq 2}\frac{z_1^{k_1}\cdots z_{d-1}^{k_{d-1}}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\sum_{k_d=1}^{k_{d-1}-1}z_d^{k_d-1}\\
&= \sum_{k_1>\cdots>k_{d-1}\geq 2}\frac{z_1^{k_1}\cdots z_{d-1}^{k_{d-1}}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\frac{1-z_d^{k_{d-1}-1}}{1-z_d}\\
&= \sum_{k_1>\cdots>k_{d-1}\geq 1}\frac{z_1^{k_1}\cdots z_{d-1}^{k_{d-1}}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\frac{1-z_d^{k_{d-1}-1}}{1-z_d}\\
&= \frac{1}{1-z_d}\sum_{k_1>\cdots>k_{d-1}\geq 1}\frac{z_1^{k_1}\cdots z_{d-1}^{k_{d-1}}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\\
&\quad -\frac{1}{z_d(1-z_d)}\sum_{k_1>\cdots>k_{d-1}\geq 1}\frac{z_1^{k_1}\cdots(z_{d-1}z_d)^{k_{d-1}}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\\
&= \frac{1}{1-z_d}\operatorname{Li}_{m_1,\cdots,m_{d-1}}(z_1,\cdots,z_{d-1})\\
&\quad -\frac{1}{z_d(1-z_d)}\operatorname{Li}_{m_1,\cdots,m_{d-1}}(z_1,\cdots,z_{d-1}z_d)
\end{aligned}$$

If $1<i<d$, then

$$\begin{aligned}
&\frac{\partial}{\partial z_i}\operatorname{Li}_{m_1,\cdots,1,\cdots,m_d}(z_1,\cdots,z_d)\\
&= \sum_{k_1>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots z_i^{m_i-1}\cdots z_d^{k_d}}{k_2^{m_2}\cdots k_d^{m_d}}\\
&= \sum_{\substack{k_1>\cdots>k_{i-1}>k_{i+1}>\cdots>k_d\geq 1\\ k_{i-1}>k_{i+1}+1}}\frac{z_1^{k_1}\cdots z_{i-1}^{k_{i-1}}z_{i+1}^{k_{i+1}}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\sum_{k_i=k_{i+1}+1}^{k_{i-1}-1}z_i^{k_i-1}\\
&= \sum_{\substack{k_1>\cdots>k_{i-1}>k_{i+1}>\cdots>k_d\geq 1\\ k_{i-1}>k_{i+1}+1}}\frac{z_1^{k_1}\cdots z_{i-1}^{k_{i-1}}z_{i+1}^{k_{i+1}}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\frac{z_i^{k_{i+1}}-z_i^{k_{i-1}-1}}{1-z_i}\\
&= \sum_{k_1>\cdots>k_{i-1}>k_{i+1}>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots z_{i-1}^{k_{i-1}}z_{i+1}^{k_{i+1}}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\frac{z_i^{k_{i+1}}-z_i^{k_{i-1}-1}}{1-z_i}\\
&= \frac{1}{1-z_i}\sum_{k_1>\cdots>k_{i-1}>k_{i+1}>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots z_{i-1}^{k_{i-1}}(z_iz_{i+1})^{k_{i+1}}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\\
&\quad -\frac{1}{z_i(1-z_i)}\sum_{k_1>\cdots>k_{i-1}>k_{i+1}>\cdots>k_d\geq 1}\frac{z_1^{k_1}\cdots(z_{i-1}z_i)^{k_{i-1}}z_{i+1}^{k_{i+1}}\cdots z_d^{k_d}}{k_1^{m_1}\cdots k_{d-1}^{m_{d-1}}}\\
&= \frac{1}{1-z_i}\operatorname{Li}_{m_1,\cdots,m_{i-1},m_{i+1},\cdots,m_d}(z_1,\cdots,z_{i-1},z_iz_{i+1},\cdots,z_d)\\
&\quad -\frac{1}{z_i(1-z_i)}\operatorname{Li}_{m_1,\cdots,m_{i-1},m_{i+1},\cdots,m_{d-1}}(z_1,\cdots,z_{i-1}z_i,z_{i+1},\cdots,z_d)
\end{aligned}$$