11.5a Taylor Series

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Definition: Series of the form $a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots = \sum_{k=0}^{\infty} a_k(x-a)^k$, notice that if $a = 0$, then it has the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{k=0}^{\infty} a_k x^k$, and geometric series $a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k$ is a special case of power series

Definition: The Taylor series(Taylor expansion) of $f(x)$ at $x = a$ is a power series $f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, if $a = 0$, then it has the form $f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$

Remark: Not very rigorously, you can think of $f(x)$ equals to its Taylor series, i.e. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Crucial examples:

1: What is the Taylor series for $\frac{1}{1-x}$ at $x = 0$

The direct(or brutal way): if $f(x) = \frac{1}{1-x} = (1-x)^{-1}$, then $f^{(n)}(x) = n!(1-x)^{n+1}$, $f^{(n)}(0) = n!$, $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} \frac{k!}{k!}x^k = \sum_{k=0}^{\infty} x^k$

A smarter way: consider geometric series $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ which of course coincides beautifully with previous formula

Note: the equality only holds if $|x| < 1$

2: What is the Taylor series for $\ln(1-x)$ at $x = 0$

The direct(or brutal way): if $f(x) = \ln(1-x)$, $f'(x) = -\frac{1}{1-x}$, then $f^{(n)}(x) = -(n-1)!(1-x)^n$ for $n \geq 1$ by the previous example, $f^{(n)}(0) = -(n-1)!$ for $n \geq 1$ and $f(0) = 0$, $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{\infty} \frac{-(k-1)!}{k!}x^k = -\sum_{k=1}^{\infty} \frac{x^k}{k}$

A smarter way: $\ln(1-x) = -\int \frac{1}{1-x}dx = -\int \left(\sum_{k=0}^{\infty} x^k\right) dx = \sum_{k=0}^{\infty} \int x^k dx = -\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)} =$

$-\sum_{k=1}^{\infty}\frac{x^k}{k}$ which coinsides with the formula above nicely!

Note: the equality only holds if $|x|<1$(actually $x=-1$ is also fine)

Also $\ln(1+x)=\ln(1-(-x))=-\sum_{k=1}^{\infty}\frac{(-x)^k}{k}=-\sum_{k=1}^{\infty}\frac{(-1)^k x^k}{k}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$

Note: the equality only holds if $|x|<1$(but actually $x=1$ is also fine)

Remark: You can integrate the series term by term! **3:** What is the Taylor series for e^x at $x=0$

If we let $f(x)=e^x$, because $(e^x)'=e^x$, $f^{(n)}(x)=e^x$, $f^{(n)}(0)=e^0=1$, then $\frac{f^{(n)}(0)}{n!}=\frac{1}{n!}$, thus the Taylor series is $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}\frac{x^k}{k!}$

Notice this makes perfect sense, since $(e^x)'=\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}x^k\right)'=\left(1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\right)'=\sum_{k=1}^{\infty}\left(\frac{x^k}{k!}\right)'=$ $\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{x^k}{k!}=e^x$

Remark: You can differentiate the series term by term! **4:** What is the Taylor series for $\sin x$ at $x=0$

If we let $f(x)=\sin x$, then $f'(x)=\cos x$, $f''(x)=-\sin x$, $f^{(3)}(x)=-\cos x$, $f^{(4)}(x)=\sin x$, then it repeats periodically with period 4, actually $f^{(n)}(x)=\sin\left(x+\frac{n\pi}{2}\right)$, $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f^{(3)}(0)=-1$, $f^{(4)}(0)=0$, and repeating periodically with period 4, we get the Taylor series $=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

5: What is the Taylor series for $\cos x$ at $x=0$

The direct(or brutal way): Mimic what we have done above

A smarter way: we can differentiate the series for $\sin x$ term by term, $\cos x=(\sin x)'=\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)'=$ $\sum_{k=0}^{\infty}\left(\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)'=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$, how magical!

Problems: Find the Taylor series(First few terms, that also is what will be needed in real life because Taylor polynomials(truncates of Taylor series) is an approximation of the original function, then more terms(the higher the degree of the Taylor polynomial), the more accurate, normally, a few terms is accurate enough) of the following functions at $x=0$

1: $5\left(e^{-2x}-3\right)$

Notice $e^{-2x}=\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}x^k=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots$

$e^{-2x}-3=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots-3=-2-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots$

$5\left(e^{-2x}-3\right)=5\left(-2-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots\right)=-10-10x+10x^2-\frac{20}{3}x^3+\frac{10}{3}x^4+\cdots$ **2;** $\frac{6}{(1+2x)^2}$

Notice $\frac{1}{1+2x}=\frac{1}{1-(-2x)}=\sum_{k=0}^{\infty}(-2x)^k=\sum_{k=0}^{\infty}(-2)^k x^k=1-2x+4x^2-8x^3+\cdots$, thus

$$
\begin{aligned}
\frac{1}{(1+2x)^2} &= \left(\frac{1}{1-(-2x)}\right)^2 = \left(1-2x+4x^2-8x^3+\cdots\right)^2 \\
&= \left(1-2x+4x^2-8x^3+\cdots\right)\left(1-2x+4x^2-8x^3+\cdots\right) \\
&= 1\cdot\left(1-2x+4x^2-8x^3+\cdots\right) \\
&- 2x\cdot\left(1-2x+4x^2-8x^3+\cdots\right) \\
&+ 4x^2\cdot\left(1-2x+4x^2-8x^3+\cdots\right) \\
&- 8x^3\cdot\left(1-2x+4x^2-8x^3+\cdots\right) \\
&+ \cdots \\
&= 1-2x+4x^2-8x^3+\cdots \\
&- 2x+4x^2-8x^3+16x^4+\cdots \\
&+ 4x^2-8x^3+16x^4-32x^5+\cdots \\
&- 8x^3+16x^4+-32x^5+64x^6+\cdots \\
&+ \cdots \\
&= 1-4x+12x^2-32x^3+\cdots
\end{aligned}
$$

Hence $\frac{6}{(1+2x)^2} = 6\left(1-4x+12x^2-32x^3+\cdots\right) = 6-24x+72x^2-192x^3+\cdots$

This may not seem much, **BUT** try this! $\frac{6}{(1+2x^9)^2}$

Just replace x with x^9, thus the Taylor series is $6-24x^9+72\left(x^9\right)^2-192\left(x^9\right)^3+\cdots = 6-24x^9+72x^{18}-192x^{27}+\cdots$