STAT400

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2020 Fall

Contents

1 Practice problems

1.1 Practice 1 - 9/3/2020

Exercise 1.1.1 ([1] Section 2.1, Exercise 1)**.**
Four universities (1, 2, 3, and 4) are participating in a holiday basketball tournament. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will play for the championship, and the two losers will also play. One possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4)

a. List all outcomes in $\mathcal{S}$

b. Let A denote the event that 1 wins the tournament. List outcomes in A

c. Let B denote the event that 2 gets into the championship game. List outcomes in B

d. What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A'?

Solution. The first two digits represent the two winners of the first round, the last two digits represent the two losers of the first round, the order of the first two digits determine the first and second place from the second round, the order of the last two digits determine the third and fourth place from the second round. Note that what the first two digits are automatically determine what the last two digits are and vice versa

a. The first two digits must consist one from $\{1, 2\}$ and the other from $\{3, 4\}$

$$\mathcal{S} = \begin{Bmatrix} 1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, \\ 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231 \end{Bmatrix}$$

b. The first digit must be 1

$$A = \{1324, 1342, 1423, 1432\}$$

c. The first two digits must contain 2

$$B = \{2314, 2341, 3214, 3241, 2413, 2431, 4213, 4231\}$$

d. Note that if 1 wons the first place is equivalent of saying 2 doesn't make into the championship game, hence A, B are disjoint, therefore $A \cap B = \varnothing$

$$A \cup B = \{1324, 1342, 1423, 1432, 2314, 2341, 3214, 3241, 2413, 2431, 4213, 4231\}$$

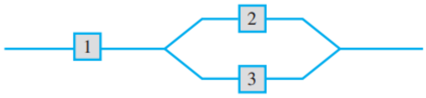
and

$$A' = \{2314, 2341, 2413, 2431, 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231\}$$

□

Exercise 1.1.2 ([1] Section 2.1, Exercise 3)**.**
Three components are connected to form a system as shown in the accompanying diagram. Because the components in the 2-3 subsystem are connected in parallel, that subsystem will function if at least one of the two individual components functions. For the entire system to function, component 1 must function and so must the 2-3 subsystem. The experiment consists of determining the condition of each component [S (success) for a functioning component and F (failure) for a nonfunctioning component]

a. Which outcomes are contained in the event A that exactly two out of the three components function?



b. Which outcomes are contained in the event B that at least two of the components function?

c. Which outcomes are contained in the event C that the system functions?

d. List outcomes in C', $A \cup C$, $A \cap C$, $B \cup C$, and $B \cap C$

Solution. Write $T_1T_2T_3$ for the outcome, where $T_i \in \{S, F\}$ stands for whether component i is functioning successfully or not, thus the sample space is

$$\mathcal{S} = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

a.

$$A = \{SSF, SFS, FSS\}$$

b.

$$B = \{SSF, SFS, FSS, SSS\}$$

c. Since component 1 and subsystem 2-3 are connected in series, the whole system functions if both component 1 and subsystem function, and since component 2,3 are connected in parallel, subsystem 2-3 functions if at least one of components 2,3 works

$$C = \{SSF, SFS, SSS\}$$

d.

$$C' = \{FSS, SFF, FFS, FSF, FFF\}$$
$$A \cup C = \{SSSF, SFS, FSS, SSS\}$$
$$A \cap C = \{SSF, SFS\}$$
$$B \cup C = \{SSF, SFS, FSS, SSS\}$$
$$B \cap C = \{SSF, SFS, SSS\}$$

□

1.2 Practice 2 - 9/10/2020

Exercise 1.2.1. Suppose that vehicles taking a particular highway exit can either turn right (denoted as R), turn left (denoted as L), or go straight (denoted as S). Consider observing three successive cars who takes this exit

(a) Let A denote the event that only one car turns right. List all the outcomes in the event A. (An event is represented as a union of outcomes in sample space occurrence of which makes this event occur)

(b) List all outcomes for the event that all three cars take different directions. Call this event as B

(c) Describe the elements in the sample space and and compute the size of the sample space

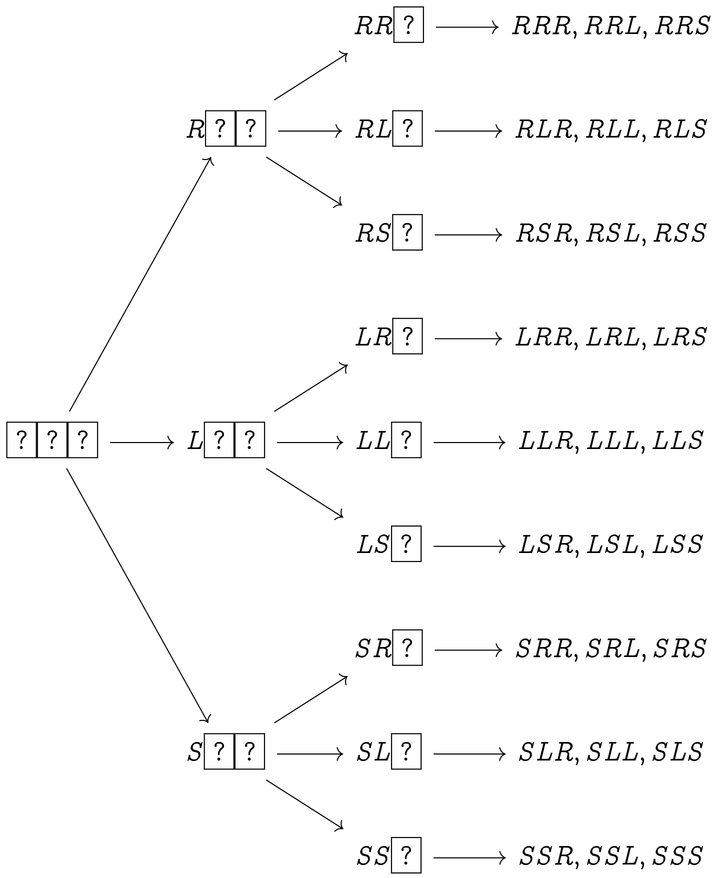
(d) List the outcomes in $A \cap B$, $A \cap B'$

Solution. Represent the outcomes as three letter sequences, where the first letter indicates the first car's choice, the second indicates the second car's choice and the third letter indicates the third car's choice. For example, RLS means the first car turns right, the second car turns left and the third car goes straight

(a)
$$A = \{RLS, RSL, RSS, RLL, SRL, LRS, SRS, LRL, SLR, LSR, SSR, LLR\}$$

(b)
$$B = \{RSL, RLS, SRL, SLR, LRS, LSR\}$$

(c) Suppose the outcome is covered with cards, and you try to reveal what they are one by one



You can see there are in total $3 \times 3 \times 3 = 27$ different outcomes

(d) $A \cap B = B$ as B is contained in A. $A \cap B' = A \setminus B = \{RLL, LRL, LLR, RSS, SRS, SSR\}$

□

Exercise 1.2.2. Let P denote the probability function which is defined as a function that satisfies Axiom1, Axiom 2, Axiom 3 in Section 2.2 of Devore's book (these are the axioms we talked about in class). Let A, B, C denote three events defined on a sample space, justify that the following two expressions are correct. Note: Using Venn diagrams to display the sets is fine, please make sure to provide enough explanations and refer to the right axioms

(a) $P(A \cup B) = P(A \cap B') + P(B)$

(b) $P(A) = P(A - (B \cup C)) + P(A \cap C) + P(A \cap B) - P(A \cap B \cap C)$

Solution.

(a) $A \cup B = (A \setminus B) \cup B$, since $A \setminus B$ and B are disjoint, thus by Axiom 3

$$P(A \cup B) = P(A \setminus B) + P(B) = P(A \cap B') + P(B)$$

(b)

$$P(A) = P(1 \cup 2 \cup 3 \cup 4) = P(1) + P(2) + P(3) + P(4) \tag{1.1}$$

where

$$P(1) = P(A \setminus (B \cup C))$$
$$P(2) = P(A \cap B) - P(A \cap B \cap C)$$
$$P(3) = P(A \cap C) - P(A \cap B \cap C)$$
$$P(4) = P(A \cap B \cap C)$$

Then equation (1.1) gives the result

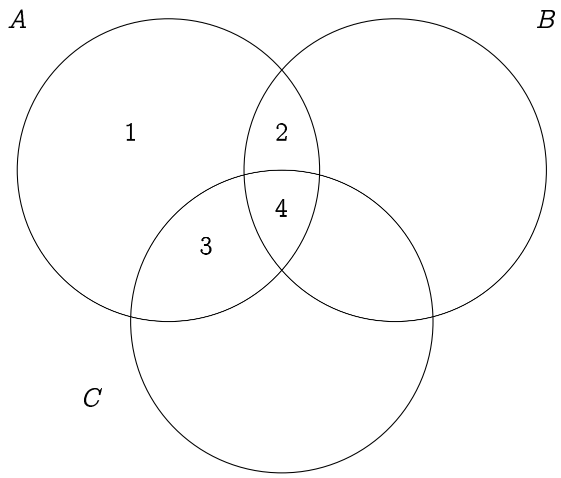


Figure 1.1: The Venn diagram

□

Exercise 1.2.3. A purse contains five 10\$, four 5\$, and eight 1\$ bills. The bills are selected one by one -without replacement- in random order, Let A denote the event that at least two bills are selected to obtain the first 10\$ bill. Describe this event as the complement of another event which is much a smaller set and easier to comprehend. Next week you will compute the probability of the event A

Solution. The complement event would be the event that a 10$ bill is selected at the first drawal □

Exercise 1.2.4 ([1] Section 2.2, Exercise 26)**.**
A certain system can experience three different types of defects. Let $A_i (i = 1, 2, 3)$ denote the event that the system has a defect of type i. Suppose that

$$P(A_1) = .12 \quad P(A_2) = .07 \quad P(A_3) = .05$$
$$P(A_1 \cup A_2) = .13 \quad P(A_1 \cup A_3) = .14$$
$$P(A_2 \cup A_3) = .10 \quad P(A_1 \cap A_2 \cap A_3) = .01$$

a. What is the probability that the system does not have a type 1 defect?

b. What is the probability that the system has both type 1 and type 2 defects?

c. What is the probability that the system has both type 1 and type 2 defects but not a type 3 defect?

d. What is the probability that the system has at most two of these defects?

Solution.

a. $P(A_1') = 1 - P(A_1) = 1 - .12 = .88$

b.

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = .12 + .07 - .13 = .06$$

c. According to a.

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3') &= P(A_1 \cap A_2 - A_3) \\ &= P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) \\ &= .06 - .01 = .05 \end{aligned}$$

d. Denote the event that the system has at most two of these defects as E, then E' will be the event that all three types of defects occurs at the same time, i.e. $A_1 \cap A_2 \cap A_3$, hence

$$P(E) = 1 - P(E') = 1 - P(A_1 \cap A_2 \cap A_3) = 1 - .01 = .99$$

□

1.3 Practice 3 - 9/17/2020

Exercise 1.3.1. A purse contains five 10\$, four 5\$, and eight 1\$ bills. If the bills are selected one by one in random order, what is the probability that at least two bills must be selected to obtain the first 10\$ bill. Hint: Describe this event as the complement of another event whose probability is relatively easier to compute

Solution. Let A be the event that at least two bills to get the first 10\$ bill, then A' would be the event of getting the 10\$ bill at the first drawal. Let the sample space $\mathcal{S}$ be all the possible outcomes from the first drawal, then

$$\mathcal{S} = \{10_1, 10_2, 10_3, 10_4, 10_5, 5_1, 5_2, 5_3, 5_4, 1_1, 1_2, 1_3, 1_4, 1_5, 1_6, 1_7, 1_8\}$$

$$A' = \{10_1, 10_2, 10_3, 10_4, 10_5\}$$

The subscripts indicates different dollar bills with same value. Hence $P(A') = \dfrac{5}{17}$ and $P(A) = 1 - P(A') = \dfrac{12}{17}$ ☐

Exercise 1.3.2. Solve the following counting problems

(a) Given a group consisting of five couples, we would like to choose two couples and an additional person for a study for which people will be asked about their opinion on the same matter. In how many ways this selection can be done

(b) Given 260 seats in our classroom, 250 students can be seated in these seats in $P_{250,260}$ different ways. Explain why this is true. Note that: $P_{250,260}$ denotes permutations of size 250 that can be formed from 260 items

(c) Using the counting techniques you have mastered in our class, show that total number of subsets of a set of size n is 2^n

Solution.

(a) We have $\binom{5}{2}$ ways to choose two couples out of five couples, and then there are still three couples(6 persons) remaining, so we have $\binom{6}{1} = 6$ ways to choose an additional person. Thus in total there are $\binom{5}{2}\binom{6}{1} = 10 \cdot 6 = 60$ different choices

(b) Recall $P_{250,260}$ means the number of ways of ordering a subset of size 250 from the whole set of size 260. Let's write $1, \cdots, 260$ on 260 cards, each representing a seat, giving 250 of these cards to the 250 students constitute as a seating plan, which is the same as ordering a subset of size 250 from $\{1, \cdots, 260\}$, thus justifying the answer
Or you can reason like this: For student 1 there are 260 seats to choose from, after making a selection for student 1, there are 259 seats to choose from for student 2, and after that there are 258 choices for seating the third student, etc. there are only 11 choices for the last student, so in total there are

$$260 \times 259 \times 258 \times \cdots \times 11 = \frac{260!}{10!} = P_{250,260}$$

different ways to assign seats for 250 students

(c) Assume the set is $\mathcal{S} = \{1, \cdots, n\}$, every subset can be represented as a sequence of n digits of 0's and 1's, put 1 on the k-th digit if element k is in the subset, 0 if not, then is a well defined one-to-one correspondence between the subsets of $\mathcal{S}$ and these sequences, so we just need to figure out how many such sequences are there. For the first digit there are 2 choices(put either 0 or 1), there are 2 choices for the second digit, etc. The total number of such sequences is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ of them}} = 2^n$$

□

Exercise 1.3.3. The composer Beethoven wrote 9 symphonies, 5 piano concertos and 32 piano sonatas

(a) A music director would like to choose one symphony and 2 piano sonatas from Beethoven, in how many ways this selection can be done. Justify the steps

(b) If every year at the birthday of Beethoven a concert featuring Beethoven's music is organized and three pieces of music is going to be presented, how many different programs are possible. order in which music pieces are played is not significant and the same piece of music can not be played more than once in a program

(c) When a program consisting of three (distinct) pieces of Beethoven music is formed, compute the probability that the program consists of two piano sonatas and one symphony. Justify to yourselves that any three pieces of music are equally likely to come together

Solution.

(a) The number of ways of choosing the symphony is $\binom{9}{1} = 9$, for each choice of symphony there are $\binom{32}{2}$ choices of piano sonatas, so in total $\binom{9}{1} \cdot \binom{32}{2} = 9 \times \frac{32 \cdot 31}{2} = 9 \times 16 \times 31 = 4464$

(b) 3 pieces from the total 46 pieces of music which is

$$\binom{46}{3} = \frac{46 \times 45 \times 44}{3 \times 2 \times 1} = 46 \times 15 \times 22 = 15180$$

(c) The probability of a 3-piece program consists of exactly one symphony and 2 sonatas is

$$P = \frac{\binom{9}{1}\binom{32}{2}}{\binom{46}{3}} = \frac{4464}{15180} \approx 0.294$$

□

1.4 Practice 4 - 9/24/2020

Exercise 1.4.1. Using the multiplication rule $P(A \cap B) = P(A|B) \times P(B)$, given events A, B, C

(a) First, show that $P(A \cap B \cap C) = P(A|B \cap C) \times P(B \cap C)$

(b) Then, prove that $P(A \cap B \cap C) = P(A|B \cap C) \times P(B|C) \times P(C)$

(c) Derive a similar rule for the probability of intersection of k events, i.e. $P(A_1 \cap A_2 \cdots \cap A_k) = P(A_1|\cdots) \times \cdots \times P(A_k)$

Solution.

(a) $P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A|B \cap C) \times P(B \cap C)$

(b) Since $P(B \cap C) = P(B|C) \times P(C)$, $P(A \cap B \cap C) = P(A|B \cap C) \times P(B|C) \times P(C)$

(c)

$$
\begin{aligned}
& P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_k) \\
&= P(A_1 \cap (A_2 \cap A_3 \cap \cdots \cap A_k)) \\
&= P(A_1|A_2 \cap A_3 \cap \cdots \cap A_k) \times P(A_2 \cap A_3 \cap \cdots \cap A_k) \\
&= P(A_1|A_2 \cap A_3 \cap \cdots \cap A_k) \times P(A_2|A_3 \cap \cdots \cap A_k) \times P(A_3 \cap \cdots \cap A_k) \\
&= \vdots \\
&= P(A_1|A_2 \cap A_3 \cap \cdots \cap A_k) \times P(A_2|A_3 \cap \cdots \cap A_k) \times \cdots \times P(A_k)
\end{aligned}
$$

□

Exercise 1.4.2. Suppose an individual is going to be randomly selected from the population of all adult males in the US. Let A be the event that the selected individual is taller than 6ft in height, and B the event that the selected individual is a professional Basketball player. Assume additionally that all basketball players are taller than 6 feet

(a) Compare $P(A|B)$ to $P(B|A)$ in magnitude, which one do you think is larger? explain why you think so

(b) Also compare $P(B)$ to $P(B|A)$ in magnitude?

(c) Assume that 35% of adult men living in the US are taller than 6 feet, and 0.001% of these men are professional basketball players, represent these (two) numbers as probabilities (or conditional probabilities) about the events A and B

(d) If we would like to compute $P(B)$, can we derive it from the given information? If not, what is missing?

Solution.

(a) By the assumption, we know that $B \subseteq A$, thus $A \cap B = B$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} \leq 1$$

Since not every person taller than 6 feet is a basketball player, it may well be $P(A) > P(B) \Rightarrow \frac{P(B)}{P(A)} < 1$, hence in general $P(A|B) > P(B|A)$

(b) In general there could be people no taller than 6 feet, thus $P(A) < 1 \Rightarrow P(B|A) = \dfrac{P(B)}{P(A)} > P(B)$

(c) $P(A) = 0.35$, $P(B|A) = 0.00001$

(d) $P(B|A) = \dfrac{P(B)}{P(A)} \Rightarrow P(B) = P(A) \times P(B|A) = 0.35 \times 0.00001 = 35 \times 10^{-7}$

$\square$

Exercise 1.4.3. At a certain gas station, 25% of customers use regular gas, 45% use plus gas, and 30% use premium gas. Among those who use regular gas, only 40% fill their tanks. Of those who use plus, 60% fill their tanks, whereas of those using premium, 90% fill their tanks

(a) What is the probability that next customer will request premium gas and not fill his tank? While solving this problem, think about how the above percentages translate into probabilities of some events

(b) What is the probability that next customer fills his tank?

(c) Given that the next customer fills his tank, what is the probability that premium gas is requested?

Solution. Let A, B, C deonte the events that the next customer use regular, plus and premium gas respectively, F denote the event that the next customer fill the tank. The given informations translates to $P(A) = 0.25$, $P(B) = 0.45$, $P(C) = 0.3$, $P(F|A) = 0.4$, $P(F|B) = 0.6$, $P(F|C) = 0.9$

(a) $P(C \cap F') = P(F'|C)P(C) = (1 - P(F|C))P(C) = (1 - 0.9) \times 0.3 = 0.03$

(b) $P(F) = P(F \cap A) + P(F \cap B) + P(F \cap C) = P(F|A)P(A) + P(F|B)P(B) + P(F|C)P(C) = 0.64$. Note that this is the law of total probability

(c) $P(C|F) = \dfrac{P(F \cap C)}{P(F)} = \dfrac{P(F|C)P(C)}{P(F)} = \dfrac{0.27}{0.64}$. Note that this is Baye's theorem

$\square$

1.5 Practice 5 - 10/1/2020

Exercise 1.5.1. Show that for any two independent events A, B; the events A and B are also independent

Solution. A', B are independent since $P(A' \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = P(B)[1 - P(A)] = P(B)P(A')$ □

Exercise 1.5.2. Let A and B be two disjoint events defined on a sample space, and assume that $P(A) = 1/3$, $P(B) = 1/3$. Are A and B independent events or not? You should provide a proof to your answer

Solution. Since $0 = P(\varnothing) = P(A \cap B) \neq P(A)P(B) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$, A, B are not independent □

Exercise 1.5.3. An urn contains four slips of paper, each having the same dimensions, which are associated with some prices. The paper with number one on it wins Price 1; paper with number 2 wins Price 2; paper with number 3 wins Price 3; paper with number four on it wins all prices 1,2,3. In an experiment only one slip of paper is going to be drawn from the urn. Let A_1 denote the event of winning Price 1. and let A_2 denote the event of winning Price 2, and let A_3 denote the event of winning price 3 associated to this experiment

(a) Show that A_1 and A_2 are independent events, similarly show that A_1 and A_3 are independent events; A_2 and A_3 are independent events. This is called pairwise independence

(b) However show that $P(A_1 \cap A_2 \cap A_3) \neq P(A_1) \times P(A_2) \times P(A_3)$, therefore the three events are not mutually independent

(c) Think of an analogous problem where in the urn there are FIVE cards of equal dimensions, where the first four cards are associated to single prices Price 1, Price 2, Price 3, Price 4 respectively and the fifth card wins all four prices. Let A_1 denote the event of winning the first price, and A_2 be the event of winning Price 2, are A_1 and A_2 independent or not?

Solution. $P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$, $P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = P(A_1 \cap A_2 \cap A_3) = \frac{1}{4}$

(a) $\frac{1}{4} = P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, similarly for A_1, A_3 and A_2, A_3, hence A_1, A_2, A_3 are pairwise independent

(b) $\frac{1}{4} = P(A_1 \cap A_2 \cap A_3) \neq P(A_1)P(A_2)P(A_3) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

(c) $\frac{1}{5} = P(A_1 \cap A_2) = P(A_1)P(A_2) = \frac{2}{5} \times \frac{2}{5} = \frac{4}{25}$

□

Exercise 1.5.4. Answer if each X defined as below is a random variable or not. If X is not a random variable, try to create a random variable X' based on X

(a) X is defined as the genders of next two consecutive births in a particular hospital

(b) X is defined as the outcome of a coin toss experiment in which a coin is tossed three times

(c) X is defined as 0 when a randomly chosen adult man living in the US has a college degree,and is defined as 5 if he does not have a college degree

(d) X denotes the trials until a success occurs in a gambling machine.

Solution.

(a) X is not a random variable. Let X' be the number of boys of the next two consecutive births in a particular hospital

(b) X is not a random variable. Let X' be the number of heads of the three tosses

(c) X is a random variable

(d) X is not a random variable. Let X' be the number failures until a success occurs in a gambling machine

$\square$

Exercise 1.5.5. Let X be the number of boy babies in next 5 consecutive births a midwife attends. Assume that this midwife believes that probability of a boy baby birth is 0.45 while probability of a girl baby birth is 0.55. Compute all of the following probabilities: $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, $P(X = 3)$, $P(X = 4)$, $P(X = 5)$, $P(X = 6)$ according to her belief. Note that outcomes of the births are independent from each other. (You can leave the results as products of numbers)

Solution. Let $A_i = B$ or G, $i = 1, \cdots, 5$ to denote the i-th baby's gender, then A_i's are mutually independent. Since these events are independent, $P(BGBGB) = P(B)P(G)P(B)P(G)P(B) = P(B)^3P(G)^2$, and $P(BBBGG) = P(B)^3P(G)^2$, so the order of the gender doesn't matter, what really matters is the number of each gender

1. $P(X = 6) = 0$ since it's just impossible to have six kids!
2. $P(X = 5) = P(BBBBB) = P(B)^5 = (0.45)^5$
3. $P(X = 4) = \binom{5}{4}P(B)^4P(G) = 5 \times (0.45)^4 \times 0.55$
4. $P(X = 3) = \binom{5}{3}P(B)^3P(G)^2 = 10 \times (0.45)^3 \times (0.55)^2$
5. $P(X = 2) = \binom{5}{2}P(B)^2P(G)^3 = 10 \times (0.45)^2 \times (0.55)^3$
6. $P(X = 1) = \binom{5}{1}P(B)P(G)^4 = 5 \times 0.45 \times (0.55)^4$
7. $P(X = 0) = P(GGGGG) = P(G)^5 = (0.55)^5$

This example as you will see in the future is an example of a *binomial distribution* $\square$

1.6 Practice 6 - 10/9/2020

Exercise 1.6.1. Let X be the number of students who show up for a professor's office hours on a day. Pmf function of X is denoted as $p(x)$ and its values are as follows

$$p(0) = 0.1, \quad p(1) = 0.2, \quad p(2) = 0.3, \quad p(3) = 0.2, \quad p(4) = 0.2, \quad p(x) = 0, \text{ for } x \geq 5$$

(a) Compute the probability that at least one student shows up during the office hours

(b) Let $F(x)$ denote the cdf function of X, compute $F(x)$ at all integer x's

(c) Compute the expected value for X

(d) Let Y denote the amount of time the professor spends to answer questions during her office hours. Assume that Y is equal to X^2, derive the pmf function of Y and compute the expected value of Y using its own pmf function. Additionally compute the expected value of Y using its relation to X and the pmf function of X

Solution.

(a) $P(X \geq 1) = 1 - P(X \leq 0) = 1 - p(0) = 1 - 0.1 = 0.9$

(b)

X	0	1	2	3	4
$p(x)$	0.1	0.2	0.3	0.2	0.2
$F(x)$	0.1	0.3	0.6	0.8	1
$Y = X^2$	0	1	4	9	16
$p_Y(y) = p(x^2)$	0.1	0.2	0.3	0.2	0.2

(c)

$$EX = \sum_x xp(x) = 0 \times 0.1 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.2 = 2.2$$

(d) Since $Y = X^2$, the possible values of Y are $\{0, 1, 4, 9, 16\}$, the pmf function p_Y of Y is given as follows

$$p_Y(0) = 0.1, \quad p_Y(1) = 0.2, \quad p_Y(4) = 0.3, \quad p_Y(9) = 0.2, \quad p_Y(16) = 0.2$$

$$EY = \sum_y yp_Y(y) = 0 \times 0.1 + 1 \times 0.2 + 4 \times 0.3 + 9 \times 0.2 + 16 \times 0.2 = 6.4$$

Or rather

$$EY = EX^2 = \sum_x x^2 p(x) = 0^2 \times 0.1 + 1^2 \times 0.2 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.2 = 6.4$$

□

Exercise 1.6.2. The variance of a discrete random variable X with pmf function $p(x)$ is defined as $\sum_x (x - \mu_X)^2 p(x)$, which is the expected value of the squared distance from the mean, μ_X. Show that this sum is equal to $\sum_x x^2 p(x) - \mu_X^2$, which is also equal to $E(X^2) - \mu_X^2$

Solution.

$$\begin{aligned}
\mathrm{Var}(x) &= \sum_x (x-\mu_X)^2 p(x) \\
&= \sum_x (x^2 - 2\mu_X x + \mu_X^2) p(x) \\
&= \sum_x x^2 p(x) - 2\mu_X \sum_X x p(x) + \mu_X^2 \sum_x p(x) \\
&= \sum_x x^2 p(x) - 2\mu_X^2 + \mu_X^2 \\
&= \sum_x x^2 p(x) - \mu_X^2 \\
&= E(X^2) - \mu_X^2
\end{aligned}$$

□

Exercise 1.6.3. Let X denote the random variable which counts number of births observed until the first boy is born, in an experiment. Probability of a boy baby birth is p, and outcomes of births are independent. You computed the pdf function for X, the cdf function and the expected value of it. Compute the variance of X. Hint: You will need to apply the trick you used for the expected value computation

Solution. The pmf of X is $p(x) = p(1-p)^{x-1}$, $x = 0, 1, 2, \cdots$. Denote the largest integer less than or equal to x by $\lfloor x \rfloor$. For example $\lfloor 4.2 \rfloor = 4$. And denote the cdf of X by $F(x)$. (Be aware that $F(x)$ is defined on all real line). For $x < 1$, $F(x) = P(X \leq x) = 0$. For $x \geq 1$

$$F(x) = P(X \leq x) = \sum_{n=1}^{\lfloor x \rfloor} p(1-p)^{n-1} = 1 - (1-p)^{\lfloor x \rfloor}$$

In summary

$$F(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ 1 - (1-p)^{\lfloor x \rfloor} & x \geq 1 \end{cases}$$

The expectation

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x p(x) = \sum_{x=1}^{\infty} x p(1-p)^{x-1} \\
&= -p \sum_{x=1}^{\infty} \frac{d}{dp}(1-p)^x \\
&= p \frac{d}{dp} \left(\sum_{x=1}^{\infty} (1-p)^x \right) \\
&= -p \frac{d}{dp} \frac{1}{p} = \frac{1}{p}
\end{aligned}$$

The variance can be computed by first computing

$$\begin{aligned}
E(X^2) &= \sum_{x=1}^{\infty} x^2 p(x) \\
&= \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1} \\
&= \sum_{x=1}^{\infty} (x^2 - x + x) p(1-p)^{x-1} \\
&= \sum_{x=1}^{\infty} (x^2 - x) p(1-p)^{x-1} + \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1}
\end{aligned}$$

The second term is $E(X) = \dfrac{1}{p}$. For the first term, we have

$$\begin{aligned}
\sum_{x^2-x} p(1-p)^{x-1} &= \sum_{x=2}^{\infty} (x^2 - x) p(1-p)^{x-2}(1-p) \\
&= p(1-p) \sum_{x=2}^{\infty} x(x-1)(1-p)^{x-2} \\
&= p(1-p) \sum_{x=2}^{\infty} \frac{d^2}{dp^2}(1-p)^x \\
&= p(1-p) \frac{d^2}{dp^2} \left(\sum_{x=2}^{\infty} (1-p)^x \right) \\
&= p(1-p) \frac{d^2}{dp^2} \left(\frac{(1-p)^2}{1-(1-p)} \right) \\
&= p(1-p) \frac{d^2}{dp^2} \left(\frac{1}{p} - 2 + p \right) \\
&= p(1-p) \frac{2}{p^3} = \frac{2(1-p)}{p^2}
\end{aligned}$$

Therefore

$$E(X^2) = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2-p}{p^2}$$

Finally by Exercise 1.6.1(d)

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

□

Exercise 1.6.4. Derive the expected value for a binomial random variable $B(n, p)$

Solution. Let $X \sim B(n, p)$, the pmf of X is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

Then

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} xP(X=x) = \sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x} \\
&= \sum_{x=0}^{n} x\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!}p^x(1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!((n-1)-(x-1))!}pp^{x-1}(1-p)^{(n-1)-(x-1)} \\
&\overset{y=x-1,m=n-1}{=\!=\!=\!=\!=} np\sum_{y=0}^{m} \frac{m!}{y!(m-y)!}p^y(1-p)^{m-y} \\
&= np\binom{m}{y}p^y(1-p)^{m-y} \\
&= np\sum_{y} P(Y=y) \\
&= np
\end{aligned}
$$

Note that here $Y \sim B(n-1,p)$, thus $\sum_y P(Y=y) = 1$ □

Exercise 1.6.5. Show that $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ for any scalar a and b

Solution. First recall $\mathrm{Var}(X) = E[(X-EX)^2]$, $E(aX+b) = aE(X)+b$

$$
\begin{aligned}
\mathrm{Var}(X+b) &= E[((X+b)-E(X+b))^2] \\
&= E[((X+b)-(EX+b))^2] \\
&= E[(X-EX)^2] = \mathrm{Var}(X)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(aX) &= E[(aX-E(aX))^2] \\
&= E[(aX-aEX)^2] \\
&= E[a^2(X-EX)^2] = a^2\,\mathrm{Var}(X)
\end{aligned}
$$

$$
\mathrm{Var}(aX+b) = \mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)
$$

□

1.7 Practice 7 - 10/22/2020

Exercise 1.7.1. A personnel director is going to interview 13 engineers for four job openings. Six of the interviews will be scheduled for the first day and the rest for the second day of the interviews. Assume that the assignment of applicants to the interview slots is completely random

(a) Derive the function for the probability that x of the top four candidates are interviewed in the first day

(b) Compute the expected value for the number of candidates that are interviewed in the first day who are among the top four, you can use a formula

Solution.

(a) Let X denote the number of the top four candidates are interviewed in the first day. X follows a hypergeometric distribution with parameters $n = 4, M = 6, N = 13$. The pmf function of X is:
$$P(X = x) = \frac{\binom{6}{x}\binom{9}{4-x}}{\binom{13}{4}}, x = 0, 1, 2, 3, 4$$

(b) $E(X) = \dfrac{nM}{N} = \dfrac{24}{13}$

□

Exercise 1.7.2. The probability that a random box of a certain type of cereal has a particular prize is 0.3. Suppose that you will purchase box after box until you have obtained four of these prizes

(a) Let X denote the number of boxes purchased, compute the pmf function for X

(b) Compute probability of purchasing not more than 5 boxes in the described experiment

(c) What is the expected value for the number of boxes without the prize in the above experiment

Solution. Let Y be the number of boxes without the prize, which can be understood as the number of failures before 4 successes are obtained, then $Y \sim NB(r = 4, p = 0.3)$

(a) A sequence with x boxes purchased, 4 of which have the prize and the other $x - 4$ do not have the prize, has probability $0.7^{x-4}0.3^4$. And there are $\binom{x-1}{3}$ such sequences (last spot has to be success and $\binom{x-1}{3}$ ways to choose the other the 3 success spots from $x - 1$ spots). Therefore,
$$P(X = x) = P(Y = x-r) = \binom{(x-r)+r-1}{r-1}p^r(1-p)^{x-r} = \binom{x-1}{3}0.7^{x-4}0.3^4, x = 4, 5, 6$$

(b)
$$P(X = 4 \text{ or } 5) = P(X = 4) + P(X = 5) = \binom{3}{3}0.7^0 0.3^4 + \binom{4}{3}0.7^1 0.3^4 = 0.03078$$

(c)
$$E(Y) = r \cdot \frac{1-p}{p} = 4 \cdot \frac{0.7}{0.3} = \frac{28}{3}$$

□

Exercise 1.7.3. An article in the Los Angeles Times (Dec 3, 1993) reports that 1 in 200 people carry the defective gene that causes inherited colon cancer. In a random sample of 2000 people

(a) What is the exact distribution for the number of people who carry this gene?

(b) What is the approximating distribution to the distribution of the number of people who carry this gene in a random sample of size 2000?

(c) Use the approximating distribution to estimate the probability that the number of people who have this defective gene is between 10 and 14 (including 10 and 14).

Solution.

(a) Let X denote the number of people who carry the defective gene in a random sample of 2000 people. $X \sim B(n = 2000, p = 1/200 = 0.005)$

(b) X can be approximated by a Poisson distribution $P(\mu = 2000 \times 0.0005 = 10)$

(c) Let F be the cdf of a Poisson distribution with parameter 10

$$P(10 \leq X \leq 14) \approx F(14) - F(9) = 0.917 - 0.458 = 0.459$$

□

Exercise 1.7.4. In some applications the distribution of a discrete random variable X resembles the Poisson distribution, except that zero is not a possible value of X. For example let X be the number of tattoos an individual wants removed at a tattoo removal facility. Suppose the pmf function of X is

$$k\frac{e^{-\theta}\theta^x}{x!}$$

for $x = 1, 2, \cdots$. Additionally θ is an integer, which you will eventually have to compute

(a) Determine the value of k, so that total sum of probabilities is equal to 1

(b) If the mean value for X is 2.313035, what is the probability that an individual wants at most two tattoos removed

(c) Given the mean as above, compute the variance and standard deviation of X

Solution. Suppose $Y \sim P(\theta)$ is a Poisson distribution, then $p(y;\theta) = \dfrac{e^{-\theta}\theta^y}{y!}$, $EY = \displaystyle\sum_{y=0}^{\infty} yp(y;\theta) = \theta$, $E(Y^2) = \displaystyle\sum_{y=0}^{\infty} y^2 p(y;\theta) = \text{Var}(Y) + (EY)^2 = \theta + \theta^2$. Note that $p(x) = k\dfrac{e^{-\theta}\theta^x}{x!} = kp(x;\theta)$

(a)

$$\begin{aligned} 1 = \sum_{x=1}^{\infty} p(x) &= \sum_{x=1}^{\infty} kp(x;\theta) = k\sum_{x=1}^{\infty} p(x;\theta) \\ &= k\left[\sum_{x=1}^{\infty} p(x;\theta) - p(0;\theta)\right] \\ &= k(1 - e^{-\theta}) = 1 \Rightarrow k = \frac{1}{1 - e^{-\theta}} \end{aligned}$$

(b)

$$\begin{aligned} 2.313.35 = E(X) &= \sum_{x=1}^{\infty} xp(x) = k\sum_{x=1}^{\infty} xp(x;\theta) \\ &= k\left[\sum_{x=0}^{\infty} xp(x;\theta) - 0 \cdot p(0;\theta)\right] \\ &= k\theta = \frac{\theta}{1 - e^{-\theta}} \end{aligned}$$

Since θ is an integer

$$\frac{\theta}{1-e^{-\theta}} = 2.313035 \Rightarrow \theta = 2$$

Thus

$$P(X \leq 2) = P(X=1) + P(X=2) = k\frac{e^{-\theta}\theta}{1!} + k\frac{e^{-\theta}\theta^2}{2!} = \frac{4e^{-2}}{1-e^{-2}} \approx 0.626$$

(c)

$$\begin{aligned}
E(X^2) &= \sum_{x=1}^{\infty} x^2 p(x) = k\sum_{x=1}^{\infty} x^2 p(x;\theta) \\
&= k\left[\sum_{x=0}^{\infty} x^2 p(x;\theta) - 0^2 \cdot p(0;\theta)\right] \\
&= k(\theta + \theta^2)
\end{aligned}$$

Then

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = k(\theta+\theta^2) - (k\theta)^2 \approx 1.649$$

and $\sigma(X) = \sqrt{\mathrm{Var}(X)} \approx 1.284$

□

1.8 Practice 8 - 10/29/2020

Exercise 1.8.1. The time to failure (in years) for an electronic device follows a continuous distribution with the pdf function

$$f(x) = \begin{cases} 0.2 \times e^{-0.2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(a) Write the integral that gives expected value for the time to failure (which is also known as life-time) for this device and compute it

(b) Compute the probability that the life-time of this electronic device will be less than a number "t". Compute this probability for $t = 5$, i.e. compute the probability that the electronic device will function for less than 5 years

Solution.

(a)

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} 0.2 x e^{-0.2x} dx \\ &\xlongequal{y=0.2x} \frac{1}{0.2} \int_{0}^{\infty} y e^{-y} dy \\ &= 5 \int_{0}^{\infty} y(-e^{-y})' dy \\ &\xlongequal{\text{Integration by parts}} 5 \left((-y e^{-y})|_0^{\infty} + \int_{0}^{\infty} e^{-y} y' dy \right) \\ &= 5 \int_{0}^{\infty} e^{-y} dy \\ &= 5(-e^{-y})|_0^{\infty} = 5 \end{aligned}$$

(b)

$$P(X \leq t) = \int_{0}^{t} f(x) dx = \int_{0}^{\infty} 0.2 e^{-0.2x} dx \xlongequal{y=0.2x} \int_{0}^{0.2t} e^{-y} dy = 1 - e^{-0.2t}$$

Thus $P(X \leq 5) = 1 - e^{-1}$

□

Exercise 1.8.2. Let X be a continuous random variable with pdf function $f(x)$. Show that $\text{Var}(X)$ -which is by definition equal to $\int_{-\infty}^{\infty} (x - E(X))^2 f(x) dx$ is equal to $\int_{-\infty}^{\infty} x^2 f(x) dx - E(X)^2$

Solution. Let's denote $\mu = E(X)$

$$\begin{aligned} \text{Var}(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2\mu + \mu^2) f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2\mu \int_{-\infty}^{\infty} x f(x) dx + \mu^2 \int_{-\infty}^{\infty} f(x) dx \\ &= E(X^2) - 2\mu^2 + \mu^2 \\ &= E(X^2) - (EX)^2 \end{aligned}$$

□

Exercise 1.8.3. Let X be a continuous random variable with pdf function $f(x)$. Also assume that X has mean = 5 and variance = 16. Compute the following:

(a) $E(2X) = \int_{-\infty}^{\infty} 2xf(x)dx$

(b) $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$

(c) $\int_{-\infty}^{\infty} (x^2 + x + 1)f(x)dx$

Solution. We know that $E(X) = 5$, $\text{Var}(X) = 16$

(a) $E(2X) = \int_{-\infty}^{\infty} 2xf(x)dx = 2\int_{-\infty}^{\infty} xf(x)dx = 2E(X) = 10$

(b) $E(X^2) = \text{Var}(X) + (EX)^2 = 16 + 5^2 = 41$

(c)

$$\begin{aligned}\int_{-\infty}^{\infty} (x^2 + x + 1)f(x)dx &= \int_{-\infty}^{\infty} x^2 f(x)dx + \int_{-\infty}^{\infty} xf(x)dx + \int_{-\infty}^{\infty} f(x)dx \\ &= E(X^2) + EX + 1 = 41 + 5 + 1 = 47\end{aligned}$$

□

Exercise 1.8.4. Assume that masses (in kg's) of stone pieces on a beach follow a normal distribution with pdf function

$$f(x) = \frac{1}{\sqrt{2\pi}0.3} e^{-\frac{1}{2}\left(\frac{x-1.5}{0.3}\right)^2}$$

(a) What is average mass of stones on this beach

(b) If a stone piece is randomly selected compute the probability of it weighing more than 1.2 kg's

Solution. $X \sim N(\mu, \sigma^2)$ is a normal random variable, we can see from the pdf that $\mu = 1.5$ and $\sigma = 0.3$

(a) The average mass of stones on this beach is $\mu = 1.5$

(b)

$$P(X > 1.2) = 1 - P(X \leq 1.2) = 1 - \Phi\left(\frac{1.2 - 1.5}{0.3}\right) = 1 - \Phi(-1) = 1 - 0.1587 = 0.8413$$

□

Exercise 1.8.5. By central limit theorem we know that distribution of sum of identically distributed independent random variables can be approximated by a normal distribution. Therefore a binomial random variable, which is a sum of results of Bernoulli trials, can be approximated by a normal distribution. Assume that at a music school the probability of a student becoming a professional musician is 0.03 (and also assume that all assumptions about the binomial process hold). Given that currently there are 500 students enrolled at this school and use normal approximation to estimate the probability that 12 or more among 500 will become professional musicians.If needed you can relax the criteria for normal approximation

Solution. In this context, central limit theorem asserts that if $X \sim B(n,p)$, as n gets large, we can approximate X by a normal random variable such that

$$P(X \leq x) = \Phi\left(\frac{x - np + 0.5}{\sqrt{np(1-p)}}\right)$$

In this problem, $n = 500$, $p = 0.03$, so $np = 15$, $np(1-p) = 14.55$, hence

$$\begin{aligned}
P(X \geq 12) &= 1 - P(X \leq 11) \\
&= 1 - \Phi\left(\frac{11 - 15 + 0.5}{\sqrt{14.55}}\right) \\
&= 1 - \Phi(-0.92) \\
&= 1 - 0.1788 = 0.8212
\end{aligned}$$

$\square$

1.9 Practice 9 - 11/5/2020

Exercise 1.9.1. Suppose that the number of events occurring in a time interval of length t follows a Poisson distribution with parameter λt. This means that the events occur at the rate equal to λ per 1 unit of time, according to a Poisson process. It is true that the time between two consecutive events follows an exponential distribution with parameter λ. For this problem you are asked to show that the time until the occurrence of the first event in the given process, follows an exponential distribution with parameter λ

Solution. Let X be the number of events occuring in a time interval of length t, then $X \sim \text{Poisson}(\lambda t)$. Let T be the time until the first occurrence of the first event. Then $\{T \leq t\}$ and $\{X \geq 1\}$ describe the same event. hence the cdf of T is

$$P(T \leq t) = P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda t}$$

Thus the pdf of T is the derivative of the cdf which is $\lambda e^{-\lambda t}$, $t \geq 0$. hence T follows an exponential distribution with parameter λ ☐

Exercise 1.9.2. Let X be a continuous r.v. and show that $E(aX + b) = aE(X) + b$, where a, b are scalar numbers

Solution.

$$\begin{aligned} E(aX + b) &= \int_0^\infty (ax + b)p(x)dx \\ &= a\int_0^\infty xp(x)dx + b\int_0^\infty p(x)dx \\ &= aE(X) + b \end{aligned}$$

☐

Exercise 1.9.3. Let X be a continuous r.v show that $\text{Var}(aX + b) = a^2 \text{Var}(X)$ where a, b are scalar numbers

Solution.

$$\begin{aligned} \text{Var}(aX + b) &= E[((aX + b) - E(aX + b))^2] \\ &= E[(aX + b - aEX - b)^2] \\ &= E[(a(X - EX))^2] \\ &= a^2 E[(X - EX)^2] \\ &= a^2 \text{Var}(X) \end{aligned}$$

☐

Exercise 1.9.4. Let $X \sim N(\mu, \sigma^2)$, show that $aX + b$ has also a normal distribution

Solution. Let $Y = aX + b$, then the cdf of Y is

$$F_Y(y) = P(Y \leq y) = P\left(X \leq \frac{y-b}{a}\right) = \int_0^{\frac{y-b}{a}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, G(y) = \int_0^y g(x)dx$$

Then $G'(y) = g(y)$ and $F_Y(y) = G\left(\frac{y-b}{a}\right)$. Thus the pdf of Y is by chain rule

$$\begin{aligned}
F'_Y(y) = G'\left(\frac{y-b}{a}\right)\left(\frac{y-b}{a}\right)' \\
&= g\left(\frac{y-b}{a}\right)\frac{1}{a} \\
&= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right)^2} \\
&= \frac{1}{(a\sigma)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(a\mu+b)}{a\sigma}\right)^2}
\end{aligned}$$

Hence $Y \sim N(a\mu+b, a^2\sigma^2)$, which totally make sense since $EY = E(aX+b) = aEX + b = a\mu + b$ and $\operatorname{Var} Y = \operatorname{var}(aX+b) = a^2 \operatorname{var} X = a^2\sigma^2$ □

Exercise 1.9.5. The gamma function is defined as follows:
Definition 1: For $\alpha > 0$ the gamma function, $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Properties of Gamma Function: For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha 1)\Gamma(\alpha 1)$ and $\Gamma(1/2) = \sqrt{\pi}$

(a) Using the fact that $\Gamma(1/2) = \sqrt{\pi}$, show that the pdf function of a normal distribution is a legitimate pdf, i.e. its integral from $-\infty$ to ∞ is equal to 1

(b) Show that the function defined as below is a legitimate pdf , i.e the are under the curve defined by this function is equal to 1

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & , x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

Where $\alpha > 0, \beta > 0$

Definition 2: The random variable that has the pdf function

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & , x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

For $\alpha > 0, \beta > 0$, is said to have a gamma distribution with parameters α, β

Solution.

□

Exercise 1.9.6.
Definition 3: The pdf function for chi-squared distribution with parameter ν is as follows:

$$f(x;\nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{(\nu/2)-1} e^{-x/2} & , x \geq 0 \\ 0 & \text{, otherwise} \end{cases}$$

Where $\nu > 0$. As you can notice this pdf function is a special case of the pdf for gamma distribution. I.e chi-squared is a gamma distribution where the parameter $\alpha =, \beta =$. Fill in the blanks.

$\alpha = \nu/2, \beta = 2$

Solution.

□

Exercise 1.9.7. Show that the σ term that appears in the pdf function of a normally distributed random variable corresponds to the standard deviation (square root of the variance) of the random variable. (In other words, you are expected to compute the variance of a normally distributed random variable using its pdf function and show that the computed variance is equal to σ^2 from the pdf function)

Solution. Suppose $X \sim N(\mu, \sigma^2)$, need to show that $\operatorname{var} X = \sigma^2$. Recall that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, so we have

$$\begin{aligned}
\operatorname{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\, dx \\
&\overset{y=\frac{x-\mu}{\sqrt{2}\sigma}}{=\!=\!=} \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2}\, dy \\
&= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} y^2 e^{-y^2}\, dy \\
&\overset{z=y^2}{=\!=\!=} \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^{1/2} e^{-z}\, dz \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^{(3/2)-1} e^{-z}\, dz \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\
&= \sigma^2
\end{aligned}$$

□

1.10 Practice 10 - 11/12/2020

Exercise 1.10.1. Assume that a device consists of 2 different parts where for each of them the time to failure (in years) follows a continuous distribution with the pdf function defined by

$$f(x) = \begin{cases} 0.2e^{-0.2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Also assume that the two parts have life-times independent form each other and for the device to work both parts should work

(a) Compute the joint pdf function for two parts

(b) Compute the probability that the life-time of the device will be more than t years

(c) Compute the cdf for the life-time of the given device

(d) Compue the pdf function for the life time of the device above

(e) Set the integral that gives the expected value for the life time of this device

Solution. Let T be the random variable denoting the life time of the device

(a) The joint pdf function is

$$f(x,y) = \begin{cases} 0.04e^{-0.2(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} P(T \geq t) &= \int_t^\infty \int_t^\infty f(x,y)dxdy \\ &= \int_t^\infty \int_t^\infty 0.04e^{-0.2(x+y)}dxdy \\ &= \int_t^\infty 0.2e^{-0.2x}dx \int_t^\infty 0.2e^{-0.2y}dy \\ &= e^{-0.2t}e^{-0.2t} = e^{-0.4t} \end{aligned}$$

For $t \geq 0$

(c) The cdf of T is

$$F(t) = P(T \leq t) = 1 - P(T \geq t) = 1 - e^{-0.4t}, t \geq 0$$

(d) The pdf of T is

$$f(t) = \frac{d}{dt}F(t) = 0.4e^{-0.4t}, t \geq 0$$

(e)

$$\begin{aligned} E(T) &= \int_0^\infty tf(t)dt \\ &= \int_0^\infty 0.4te^{-0.4t}dt \\ &= 2.5 \end{aligned}$$

□

Exercise 1.10.2. Let X and Y be two random variables with joint pdf function $f(x, y)$ that is defined as below

$$f(x,y) = \begin{cases} \frac{1}{16}xy & 0 \le x \le 2, 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases}$$

show that X and Y are independent random variables

Solution.

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \int_0^4 \frac{1}{16}xydy = \frac{x}{2}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \int_0^2 \frac{1}{16}xydx = \frac{y}{8}$$

Then

$$f_X(x)f_Y(y) = \begin{cases} \frac{x}{2}\frac{y}{8} = \frac{xy}{16} & 0 \le x \le 2, 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases} = f(x,y)$$

Therefore, X, Y are independent random variables ☐

Exercise 1.10.3. Let X and Y be two independent random variables where each has the normal distribution $N(\mu = 10, \text{variance} = 18)$. It is known that a linear combination of independent normal random variables is also normal, therefore $X + Y$ has a normal distribution

(a) Compute the probability that $X + Y \ge 14$

(b) Compute the expected value and variance for $2X - 5Y - 10$

(c) Compute the probability that $2X - 5Y - 10 < -5$

Solution.

(a) Denote $U = X+Y$, then $\mu_U = EU = EX+EY = 2\mu = 20$, $\sigma_U^2 = \operatorname{Var} U = \operatorname{Var} X + \operatorname{Var} Y = 2\sigma^2 = 36$, hence

$$P(U \ge 14) = P\left(\frac{U-\mu_U}{\sigma_U} \ge \frac{14-20}{6} = -1\right) = 1 - \Phi(-1) = 10.1587 = 0.8413$$

(b) Denote $V = 2X - 5Y - 10$, then $\mu_V = EV = 2EX - 5EY - 10 = -3\mu - 10 = -40$

$$\sigma_V^2 = \operatorname{Var} V = \operatorname{Var}(2X+(-5)Y) = \operatorname{Var}(2X)+\operatorname{Var}((-5)Y) = 2^2 \operatorname{Var} X+(-5)^2 \operatorname{Var} Y = 29\sigma^2 = 29{\cdot}18$$

(c)

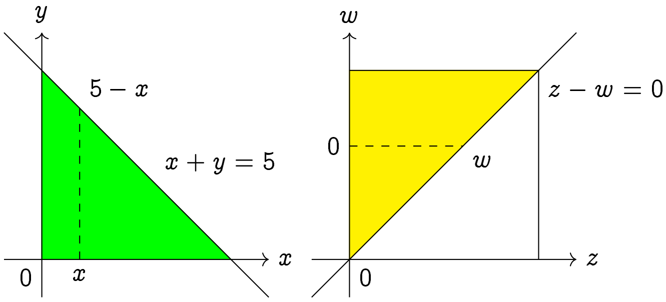
$$P(V < -5) = P\left(\frac{V-\mu_V}{\sigma_V} < \frac{-5-(-40)}{\sqrt{29 \cdot 18}}\right) = \Phi(1.53) = 0.9370$$

☐

Exercise 1.10.4. Let $f(x, y)$ defined as below be the joint pdf function of random variables X, Y. Compute the coefficient k that makes the function $f(x, y)$ a legitimate joint pdf function

$$f(x,y) = \begin{cases} k(x+y) & x \ge 0, y \ge 0, x+y \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

Solution.



$$
\begin{aligned}
\iint_{\text{green region}} f(x,y)dxdy &= k\int_0^5\left(\int_0^{5-x}(x+y)dy\right)dx \\
&= k\int_0^5 x(5-x)+\frac{(5-x)^2}{2}dx \\
&= k\int_0^5 \frac{25}{2}-\frac{x^2}{2}dx \\
&= k\frac{125}{3}
\end{aligned}
$$

Or equivalently, consider change of variables $\begin{cases} z = x \\ w = x+y \end{cases}$ with Jacobian 1, thus

$$
\begin{aligned}
k\iint_{\text{green region}}(x+y)dxdy &= \iint_{\text{yellow region}} wdzdw \\
&= k\int_0^5\left(\int_0^w wdz\right)dw \\
&= k\int_0^5 w^2dw \\
&= k\frac{125}{3}
\end{aligned}
$$

Therefore

$$1 = k\frac{125}{3} \Rightarrow k = \frac{3}{125}$$

$\square$

1.11 Practice 11 - 12/3/2020

Exercise 1.11.1. Assume that a device consists of 3 different parts where for each of them the time to failure (in years) follows a continuous distribution with the pdf function defined by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Also assume that all the three parts have life-times independent form each other and the device works until all parts die. The parts of devices are observed until all parts die. Counting from day 0, the first part dies after 3 years, the second part dies after 2.5 years and the third part dies after 4 years

(a) Write the likelihood function for the observed values

(b) Compute the value of λ that maximizes the above likelihood function. This value is the mle of λ

Solution.

(a) The the likelihood function is $L(\lambda) = \lambda^3 e^{-\lambda(3+2.5+4)} = \lambda^3 e^{-9.5\lambda}$

(b) Consider $\log L(\lambda) = 3\log\lambda - 9.5\lambda$, solve $\frac{d}{d\lambda}[\log L(\lambda)] = \frac{3}{\lambda} - 9.5 = 0$, $L(\lambda)$ has a critical point $\lambda = \frac{3}{9.5}$ which is a maximizer, since $\log$ is strictly increasing, the mle of λ is also the maximizer $\hat{\lambda} = \frac{3}{9.5}$

□

Exercise 1.11.2. Let X and Y denote two unbiased estimator of a parameter θ. Show that the statistic $\frac{X+Y}{2}$ is also an unbiased estimator of θ

Solution. Since X, Y are both unbiased estimator of θ, $E(X) = E(Y) = \theta$, hence

$$E\left(\frac{X+Y}{2}\right) = \frac{EX + EY}{2} = \theta$$

□

Exercise 1.11.3. Let X denote a biased estimators of parameter θ and Y denote an unbiased estimator of the same parameter. Show that the statistic $\frac{X+Y}{2}$ is a biased estimator of θ

Solution. Since X is while Y is not an unbiased unbiased estimator of θ, $E(X) \neq \theta$, $E(Y) = \theta$, hence

$$E\left(\frac{X+Y}{2}\right) = \frac{EX + EY}{2} \neq \theta$$

□

Exercise 1.11.4. Given a random sample X_1, X_2, X_3, X_4, X_5 Show that sample variance defined as $S^2 = \frac{1}{5-1}\sum_{i=1}^{5}(X_i - \bar{X})^2$, where $\bar{X}$ denotes the average of the five observation in the sample, is an unbiased estimator of the population variance

Solution. Suppose $X_1, \cdots, X_n$ is a random sample, i.e. X_i's are i.i.d(independent and identically distributed), say, with expected value μ and variance σ^2, then

$$\begin{aligned}
E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] &= E\left[\sum_{i=1}^{n}(X_i^2 - 2\bar{X}X_i + \bar{X}^2)\right] \\
&= E\left[\sum_{i=1}^{n} X_i^2 - 2\bar{X}\sum_{i=1}^{n} X_i + n\bar{X}^2\right] \\
&= E\left[\sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2\right] \\
&= E\left[\sum_{i=1}^{n} X_i^2 - n\bar{X}^2\right] \\
&= \sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2) \\
&= \sum_{i=1}^{n}(\mathrm{Var}(X_i) + (EX_i)^2) - n(\mathrm{Var}(\bar{X}) + (E\bar{X})^2) \\
&= \sum_{i=1}^{n} \mathrm{Var}(X_i) + n(EX_i)^2] - n\,\mathrm{Var}(\bar{X}) - n(E\bar{X})^2 \\
&= \sum_{i=1}^{n}(\sigma^2 + \mu^2) - n\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) - n\left(\frac{\sum_{i=1}^{n} EX_i}{n}\right)^2 \\
&= \sigma^2 + n\mu^2 - \frac{1}{n}\sum_{i=1}^{n}\sigma^2 - n\left(\frac{\sum_{i=1}^{n}\mu}{n}\right)^2 \\
&= \sigma^2 + n\mu^2 - \sigma^2 - n\mu^2 \\
&= (n-1)\sigma^2
\end{aligned}$$

Therefore $E(S^2) = \sigma^2$ is an unbiased estimator of population variance □

Exercise 1.11.5. Did you need to use the normality assumption in your computations in the previous question

Solution. No, we don't need the assumption on normality □

Exercise 1.11.6. Given a set of data points 2, 1, 4, 3.5, 7.2 from a population, using sample mean as the estimator, compute an estimate for the population mean

Solution.

$$\hat{\mu} = \bar{X} = \frac{2+1+4+3.5+7.2}{5} = \frac{17.5}{5} = 3.5$$

□

2 Exams

2.1 Exam 1 - 10/14/2020

Question 2.1.1. A town has two fire engines operating independently. The probability that a specific engine is available at a random time is 0.8

(a) What is the probability that none of the two are available at a random time

(b) What is the probability that only one is available at a random time

Solution.

(a) $P(\text{none of the two are available}) = (1 - 0.8)^2 = 0.04$

(b) $P(\text{only one is available}) = (1 - 0.8) \cdot 0.8 + 0.8 \cdot (1 - 0.8) = 0.32$

□

Question 2.1.2. Consider 3 independents events A_1, A_2, A_3 where $P(A_i) = p_i$ for $i = 1, 2, 3$. Express the probability that at most one of these events occur in terms of p_i's. Provide enough explanations

Solution. At most one of these 3 independent events happens is equivalent to none of these events happen, or exactly one event happens

$$\begin{aligned}
&P(\text{at most one of } A_i\text{'s occur}) \\
&= P(A_1^c \cap A_2^c \cap A_3^c) + P(A_1 \cap A_2^c \cap A_3^c) + P(A_1^c \cap A_2 \cap A_3^c) + P(A_1^c \cap A_2^c \cap A_3) \\
&= (1 - p_1)(1 - p_2)(1 - p_3) + p_1(1 - p_2)(1 - p_3) + (1 - p_1)p_2(1 - p_3) + (1 - p_1)(1 - p_2)p_3
\end{aligned}$$

□

Question 2.1.3. Assume that an urn contains 4 blue chips and 4 red chips where chips are of equal dimensions and are equally likely to be selected when a chip is drawn from the urn. Let X denote the total number of blue chips obtained when 3 chips are consecutively drawn from the urn without replacement

(a) Compute the probability that $X = 2$

(b) Compute the probability that first two chips drawn are blue and the third one is red?

(c) Write the expression (with the most detail possible) that gives the expected value of X. You are not asked to compute it

Solution.

(a) $X = 2$ means get exactly 2 blue chips and 1 red chip, use counting techique

$$P(X = 2) = \frac{\binom{4}{2}\binom{4}{1}}{\binom{8}{3}} = \frac{6 \times 4}{56} = \frac{3}{7}$$

(b) Denote A_i to be the event that the i-th chip being drawn is blue

$$\begin{aligned}
P(\text{first two blue and the third red}) &= P(A_1 \cap A_2 \cap A_3') \\
&= P(A_3' | A_1 \cap A_2) P(A_1 \cap A_2) \\
&= P(A_3' | A_1 \cap A_2) P(A_1 | A_2) P(A_1) \\
&= \frac{4}{8} \times \frac{3}{7} \times \frac{4}{6} \\
&= \frac{1}{7}
\end{aligned}$$

(c)

$$E(X) = \sum_{x=0}^{3} xp(x) = \sum_{x=0}^{3} x \frac{\binom{4}{x}\binom{4}{3-x}}{\binom{8}{3}}$$

□

Question 2.1.4. For a discrete random variable X, with pmf function $p(x)$, $E(X) = 4$ and $E(X^2) = 29$. If possible

(a) Compute $\mathtt{Var}(X)$

(b) Compute $E(X^2 + 5X9)$, include the justifications of your steps

Solution.

(a) $\mathtt{Var}(X) = E(X^2) - [E(X)]^2 = 29 - 4^2 = 13$

(b) $E(X^2 + 5X9) = E(X^2) + 5E(X) - 9 = 29 + 5 \times 4 - 9 = 40$

□

2.2 Exam 2 - 11/19/2020

Question 2.2.1. Assume that with prob 0.008 a professor accidentally types in a wrong grade when she enters the grade of each student. Additionally, assume that errors occur independently of one another. The professor has 250 students and therefore 250 grades to submit

(a) Write the numerical expression that gives the probability that at least 3 of the grades are entered incorrectly

(b) Use a Poisson approximation to estimate the probability in part (a)

Solution.

(a) Denote the number of grades entered incorrectly X, then $X \sim B(0.008, 250)$

$$\begin{aligned} P(X \geq 3) &= 1 - P(X \leq 2) \\ &= 1 - \binom{250}{0} 0.992^{250} - \binom{250}{1} 0.008 \cdot 0.992^{249} - \binom{250}{2} 0.008^2 \cdot 0.992^{248} \end{aligned}$$

(b) The parameter for Poisson distribution should be $\mu = np = 250 \cdot 0.008 = 2$, hence

$$\begin{aligned} P(X \geq 3) &\approx 1 - \frac{e^{-2}2^0}{0!} - \frac{e^{-2}2}{1!} - \frac{e^{-2}2^2}{2!} \\ &= 1 - 5e^{-2} = 0.3233 \end{aligned}$$

□

Question 2.2.2. Assume that X_1 denotes a random variable which has a normal distribution with mean 4 and standard deviation 0.5, similarly X_2 is a normal random variable with mean -4 and std deviation 1.2. Additionally, assume that X_1 and X_2 are independent

(a) Compute the mean and variance for $\frac{X_1+X_2}{2}$

(b) What kind of distribution does the random variable $\frac{X_1+X_2}{2}$ follow. Explain the reason

(c) If possible, compute $P(\frac{X_1+X_2}{2} \leq 1.3)$, if not, explain the reason

Solution.

(a)

$$E\left(\frac{X_1 + X_2}{2}\right) = \frac{E(X_1) + E(X_2)}{2} = 0$$

$$\text{Var}\left(\frac{X_1 + X_2}{2}\right) = \frac{\text{Var}(X_1) + \text{Var}(X_2)}{4} = \frac{0.5^2 + 1.2^2}{4} = 0.4225$$

(b) The linear combination of independent normal random variables is again a normal random variable

(c)

$$\begin{aligned} P\left(\frac{X_1 + X_2}{2} \leq 1.3\right) &= P\left(\frac{\frac{X_1+X_2}{2} - 0}{\sqrt{0.4225}} \leq \frac{1.3 - 0}{\sqrt{0.4225}}\right) \\ &= P(Z \leq 2) = \Phi(2) = 0.9772 \end{aligned}$$

□

Question 2.2.3. X denotes a positive random variable with probability density function $f(x) = 10e^{-10x}$ where $x \geq 0$. Y is a positive random variable with pdf given as $g(y) = 5e^{-5y}$ for $y \geq 0$. Also, assume that X and Y are independent

(a) Write the joint probability density function of X and Y, please specify the regions as well

(b) Set the integral that computes the probability that $X + Y < 1$. DO NOT COMPUTE IT

(c) Set the integral that computes the expected value for XY and show that $E(XY) = E(X) \times E(Y)$. Note that this relationship is true for any two independent random variables, and I want you to derive it for this particular case of X and Y

(d) Compute the expected value for $X + Y$

Note. In this problem you can use the result $\int_0^\infty x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}$, without showing it

Solution.

(a) Since X and Y are independent, the joint pdf of X and Y is

$$f_{X,Y}(x, y) = f(x)g(y) = \begin{cases} 50e^{-10x-5y} & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$P(X + Y < 1) = \int_0^1 \int_0^{1-x} 50e^{-10x-5y} dy dx$$

Or

$$P(X + Y < 1) = \int_0^1 \int_0^{1-y} 50e^{-10x-5y} dx dy$$

(c)

$$\begin{aligned} E(XY) &= \int_0^\infty \int_0^\infty xy f_{X,Y}(x, y) dx dy \\ &= \int_0^\infty \int_0^\infty (xf(x))(yg(y)) dx dy \\ &= \left(\int_0^1 xf(x) dx \right) \left(\int_0^1 yg(y) dy \right) \\ &= E(X)E(Y) \end{aligned}$$

(d)

$$E(X + Y) = E(X) + E(Y) = \frac{1}{10} + \frac{1}{5} = \frac{3}{10}$$

□

2.3 Final - 12/17/2020

Question 2.3.1. Assume that there are 7 distinct items, named a, b, c, d, e, f, g respectively

(a) Using these 7 items how many different ordered triplets (an ordered subset of size 3) can be formed? Explain your answer

(b) Assume that all ordered triplets are equally likely to be formed and compute the probability that the first object of a randomly generated ordered triplet is g? Explain your solution

(c) Assume that two ordered triplets are going to be formed INDEPENDENTLY one after another, using the available set of 7 items. Compute the probability that the first triplet contains a or b (or both) and the second triplet includes at most one of the a or d (neither or one of the given two). Explain your steps briefly, without any unnecessary details

Solution.

(a) For the first object, there are 7 choices, for the second object, there are 6 choices, for the third object, there are 5 choices, hence in total $7 \times 6 \times 5 = 210$ choices for the triplet

(b) Since the first object has to be g, there are 6 choices for the second object and 5 choices for the third object, hence in total $6 \times 5 = 30$ choices for the triplet, and the probability is $\frac{30}{210} = \frac{1}{7}$

(c) Let A denote the event that the first triplet contains a or b or both, B denote the event that the second triplet includes at most one of a or d Since the A, B are independent

$$
\begin{aligned}
P(A \cap B) &= P(A)P(B) \\
&= \frac{3! \cdot \binom{5}{2} + 3! \cdot \binom{5}{2} + 3! \cdot \binom{5}{1}}{210} \cdot \frac{3! \cdot \binom{5}{2} + 3! \cdot \binom{5}{2} + 3! \cdot \binom{5}{3}}{210} \\
&= \frac{150}{210} \cdot \frac{180}{210} = \frac{30}{49}
\end{aligned}
$$

□

Question 2.3.2. Let X denote the wait time (for an individual) to be seated at a restaurant and Y denote the wait time to be served food after sitting at a table. Assume that X and Y are independent random variables with identical distributions, which is $N(\mu = 15, \sigma^2 = 36)$. Additionally let $Z = X + Y$ denote the total wait time of an individual in a visit

(a) Compute joint probability density function of X and Y, explain how you derived it

(b) Using the joint pdf of X and Y , compute the expected value and variance for Z

(c) Let T_7 denote the total wait time at this restaurant at the end of 7 visits. What kind of distribution does T_7 have, what is the mean and variance for it? Explain the reasons for your answers. Note: You should assume that wait times at different visits are independent

(d) For T_7 defined as above, compute the probability that T_7 takes a value between 210 and 280

Solution.

(a) Since X, Y are independent and have identical distributions $N(\mu = 15, \sigma^2 = 36)$, hence the pdf for X, Y are

$$
f_X(x) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-15}{6}\right)^2}, f_Y(y) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-15}{6}\right)^2}
$$

And the joint pdf of X, Y is

$$
f_{X,Y}(x, y) = \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-15}{6}\right)^2} \cdot \frac{1}{6\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-15}{6}\right)^2}
$$

(b) $EZ = E(X+Y) = EX + EY = 30$, $\operatorname{Var} Z = \operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) = 72$

(c) $T_7 = Z_1 + \cdots + Z_7 \sim N(7 \times 30 = 210, 7 \times 72 = 504)$

(d)

$$\begin{aligned} P(210 < T_7 < 280) &= P\left(\frac{210-210}{\sqrt{504}} < \frac{T_7 - 210}{\sqrt{504}} < \frac{280-210}{\sqrt{504}}\right) \\ &= \Phi\left(\frac{70}{\sqrt{504}}\right) - \Phi(0) \\ &= \Phi(3.12) - 0.5 = 0.9991 - 0.5 = 0.4991 \end{aligned}$$

□

Question 2.3.3. A test for syphilis gives a positive result with probability 0.98 when tested on a person who has the disease. The test gives a negative result with probability 0.9 when tested on a person who does not have the disease. Assume that 3% of a society has the disease, and compute the following probabilities in the order they are asked. Note that for each individual there are two disease status(disease, no disease) and two types of test results(positive, negative)

(a) Compute the probability that an individual has the disease and has a positive test result. Explain your computation

(b) Compute the probability that an individual does not have the disease and has a positive test result

(c) What does the sum of two probabilities given above correspond to? Explain your answer

(d) If possible, using the three probabilities computed above compute the probability that an individual with a positive test result has the disease. Give reasons to your computation

Solution. Let A denote the event that the individual has the disease, B denote the event that this individual has a positive test result. Then from the information we have $P(B|A) = 0.98$, $P(B^c|A^c) = 0.9$, $P(A) = 0.03$

(a) $P(A \cap B) = P(B|A)P(A) = 0.98 \cdot 0.03 = 0.0294$

(b) $P(A^c \cap B) = P(B|A^c)P(A^c) = [1 - P(B^c|A^c)][1 - P(A)] = 0.1 \cdot 0.97 = 0.097$

(c) The probabililty that an individual has a positive test result, i.e. $P(B)$. Since $(A \cap B) \cup (A^c \cap B) = (A \cup A^c) \cap B = B$

(d) It's the Bayes formula

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{0.98 \cdot 0.03}{0.0294 + 0.097} = 0.2326$$

□

Question 2.3.4. Assume that body masses of Goldfinch birds follow a normal distribution, with standard deviation equal to 0.04 oz. An ornithologist who would like to make some inference about the average body mass of Goldfinch birds. In particular, at significance level 0.01, she would like to test the null hypothesis H_0: Average body mass of the Goldfinch bird is 0.5 oz, against the alternative which claims that average body size is less than 0.5 oz

(a) Assume that a random sample of 4 Goldfinch birds are going to be used to test the above hypothesis. Compute the largest value of the sample mean, that will allow the experimenter reject the null and prove the alternative at significance level 0.01. For this problem, make sure that your solution includes all steps of your computation. You will not earn points if you just use a formula

(b) If the true average weight of Goldfinch bird is 0.49, compute the probability of committing a type II error in the hypothesis testing procedure described so far. Explain what are the main factors effecting the probability of type II error in a hypothesis testing procedure and suggest ways(all you can think of) to reduce the probability of committing a type II error

Solution.

(a) Suppose L is the largest value of the sample mean that will allow the experimenter reject the null and prove the alternative at significance level 0.01, then assuming H_0 is true

$$P(\bar{X} < L) = P\left(\frac{\bar{X} - 0.5}{0.04/\sqrt{4}} < \frac{L - 0.5}{0.04/\sqrt{4}}\right) = 0.01$$

Thus $\dfrac{L-0.5}{0.02} = -2.325 \Rightarrow L = 0.4535$

(b)

$$P(\bar{X} > 0.4535) = P\left(\frac{\bar{X} - 0.49}{0.04/\sqrt{4}} > \frac{0.4535 - 0.49}{0.04/\sqrt{4}}\right) = 1 - \Phi(-1.825) = 0.9656$$

To reduce this probability, we can increase the significance level α

□

Question 2.3.5. Create a biased estimator for a population mean, show why it is biased and compute its bias

Solution. Let X_i, satisfy uniform distribution on $(0, \theta)$, where θ is an unknown parameter, then $EX_i = \theta/2$, let $\hat{\theta}/2 = \max_i X_i$ be an estimator for the population mean(see the textbook [1] Example 6.4), then $E(\hat{\theta}/2) = \dfrac{n\theta}{2(n+1)}$, and the bias would be $E(\hat{\theta}/2) - \theta/2 = -\dfrac{\theta}{2(n+1)}$ □

Question 2.3.6. Assume that the number of house fires in a neighborhood over the period of a year follows a Poisson distribution with parameter λ
Given that in the most recent four consecutive years, the number of fires in each year was x_1, x_2, x_3, x_4 respectively, write the likelihood function for the given data and compute the value of λ that maximizes the likelihood function. Note that this value is going to be as a function of data values x_1, x_2, x_3, x_4

Solution. The likelihood function would be

$$\begin{aligned} L(\lambda) &= \frac{e^{-\lambda}\lambda^{x_1}}{x_1!} \cdot \frac{e^{-\lambda}\lambda^{x_2}}{x_2!} \cdot \frac{e^{-\lambda}\lambda^{x_3}}{x_3!} \cdot \frac{e^{-\lambda}\lambda^{x_4}}{x_4!} \\ &= \frac{e^{-4\lambda}\lambda^{(x_1+x_2+x_3+x_4)}}{x_1!x_2!x_3!x_4!} \end{aligned}$$

Then

$$\frac{d}{d\lambda}\log L(\lambda) = -4 + \frac{x_1 + x_2 + x_3 + x_4}{\lambda}$$

Solve $\dfrac{d}{d\lambda}\log L(\lambda) = 0$ we have the mle $\dfrac{x_1 + x_2 + x_3 + x_4}{4}$ □

References

[1] *Probability and Statistics for Engineering and the Sciences (9th edition)* - Jay Devore