Definition 0.0.1. *Iterated integral* is defined inductively by

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b f_1(\tau)d\tau \cdots f_{r-1}(\tau)\left(\int_a^\tau f_r(t)dt\right)d\tau$$

If $\alpha : I \to M$ is a curve, $\alpha^*\omega_i = f_i(t)dt$, then

$$\int_\alpha \omega_1 \cdots \omega_r = \int_0^1 f_1(t)dt \cdots f_r(t)dt$$

is well defined, independent of the parametrization. Set the integral to be 1 if $r = 0$

Proposition 0.0.2.

1. $\displaystyle\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_j \int_\beta \omega_1 \cdots \omega_j \int_\alpha \omega_{j+1} \cdots \omega_r$

2. $\displaystyle\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_\alpha \omega_r \cdots \omega_1$

3. $\displaystyle\int_\alpha \omega_1 \cdots \omega_r \int_\alpha \omega_{r+1} \cdots \omega_{r+s} = \sum_\sigma \int_\alpha \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}$, here σ runs over (r, s) shuffles

Lemma 0.0.3. $\omega_i^{(j)}$, $1 \le i \le r$, $1 \le j \le n$ are closed one forms such that $\displaystyle\sum_j \omega_{i-1}^{(j)} \wedge \omega_i^{(j)} = 0$ for $2 \le i \le r$, then $\displaystyle\int_\alpha \sum_j \omega_1^{(j)} \cdots \omega_r^{(j)}$ only depends on the homotopy class of α

Definition 0.0.4. The *Polylogarithms* are

$$\mathrm{Li}_n(z) = \sum_{k=1}^\infty \frac{z^k}{k^n}$$

Note that

$$\mathrm{Li}_{n+1}(z) = \int_0^z \frac{\mathrm{Li}_n(t)}{t}dt, \quad \mathrm{Li}_1(z) = -\ln(1-z)$$

Hence

$$\mathrm{Li}_n(z) = \int_0^z \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{1-t} = \int_0^1 \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{z^{-1}-t}$$

Dilogarithm $\displaystyle\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u}du$ is the analytic continuation on $\mathbb{C} \setminus \{0, 1\}$, avoiding the the cut $[1, \infty]$

Definition 0.0.5. The *Bloch-Wigner function* is $D_2(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\ln|z|$, $z \in \mathbb{C} \setminus \{0, 1\}$

Definition 0.0.6. The *multiple polylogarithms* are

$$\mathrm{Li}_{\mathbf{n}}(\mathbf{z}) = \sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \int_0^1 \left(\frac{dt}{t}\right)^{n_1-1} \frac{dt}{a_1-t} \cdots \left(\frac{dt}{t}\right)^{n_d-1} \frac{dt}{a_d-t}$$

Here $\mathbf{k}$ runs over $k_1 > \cdots > k_d \ge 1$, $a_j = a_j(\mathbf{z}) = (z_1 \cdots z_j)^{-1}$, $a_0 = 1$, $a_{n+1} = 0$

Note. For $\mathbf{k}$ runs over $(k_1, \cdots, k_d) \in \mathbb{Z}_{\ge 1}^d$

$$\sum_{\mathbf{k}} \frac{\mathbf{z}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \left(\sum_{k_1} \frac{z_1^{k_1}}{k_1^{n_1}}\right) \cdots \left(\sum_{k_d} \frac{z_d^{k_d}}{k_d^{n_d}}\right) = \mathrm{Li}_{n_1}(z_1) \cdots \mathrm{Li}_{n_d}(z_d)$$

Total differential on L__n

Lemma 0.0.7. Write $\mathfrak{S}_n = \{0,1\}^n$, $\mathbf{0} = (0,\cdots,0)$, $\mathbf{1} = (1,\cdots,1)$, $\mathbf{u}_s = (0,\cdots,\underset{\substack{\uparrow\\ s\text{-th}}}{1},\cdots,0)$, define *multiple logarithm* $\mathcal{L}_n = \mathrm{Li}_{\mathbf{1}}$, $\mathcal{L}_0 = 1$, then

$$\begin{aligned} d_j\mathcal{L}_n(\mathbf{x}) &= \sum_{k_1>\cdots\widehat{k_j}>\cdots k_n} \frac{x_1^{k_1}\cdots\widehat{x_j^{k_j}}\cdots x_n^{k_n}}{k_1\cdots\widehat{k_j}\cdots k_n} \sum_{k_j=k_{j+1}+1}^{k_{j-1}-1} x_j^{k_j-1}dx_j \\ &= \sum_{k_1>\cdots\widehat{k_j}>\cdots k_n} \frac{x_1^{k_1}\cdots\widehat{x_j^{k_j}}\cdots x_n^{k_n}}{k_1\cdots\widehat{k_j}\cdots k_n} \frac{x_j^{k_{j-1}-1}-x_j^{k_{j+1}}}{x_j-1}dx_j \end{aligned}$$

Denote $\mathbf{x}_j = (x_1,\cdots,x_jx_{j+1},x_{j+2},\cdots,x_n)$, $\mathbf{x}_n = (x_1,\cdots,x_{n-1})$, we have

$$d_j\mathcal{L}_n(\mathbf{x}) = \mathcal{L}_{n-1}(\mathbf{x}_{j-1})\frac{dx_j}{x_j(x_j-1)} - \mathcal{L}_{n-1}(\mathbf{x}_j)\frac{dx_j}{x_j-1}$$

For $2\le j\le n$, and

$$d_1\mathcal{L}_n(\mathbf{x}) = -\mathcal{L}_{n-1}(\mathbf{x}_1)\frac{dx_1}{x_1-1}$$

Therefore

$$\begin{aligned} d\mathcal{L}_n(\mathbf{x}) &= \sum_{j=1}^{n} d_j\mathcal{L}_n(\mathbf{x}) \\ &= \sum_{j=1}^{n-1}\left(\mathcal{L}_{n-1}(\mathbf{x}_j)\frac{dx_j}{1-x_j} + \mathcal{L}_{n-1}(\mathbf{x}_j)\frac{dx_{j+1}}{x_{j+1}(x_{j+1}-1)}\right) + \mathcal{L}_{n-1}(\mathbf{x}_n)\frac{dx_n}{x_n-1} \\ &= \sum_{j=1}^{n-1}\mathcal{L}_{n-1}(\mathbf{x}_j)\left(-d\ln(1-x_j) + d\ln\left(\frac{x_{j+1}-1}{x_{j+1}}\right)\right) + \mathcal{L}_{n-1}(\mathbf{x}_n)\frac{dx_n}{x_n-1} \\ &= \sum_{j=1}^{n}\mathcal{L}_{n-1}(\mathbf{x}_j)d\ln\left(\frac{1-x_{j+1}^{-1}}{1-x_j}\right) \end{aligned}$$

Here $x_{n+1} = \infty$, $\mathcal{L}_0 = 1$

Suppose $\mathbf{i}\in\mathfrak{S}_n$, $|\mathbf{i}| = k$ and $i_{\tau_1} = \cdots = i_{\tau_k} = 1$ for some $1\le\tau_1\le\cdots\le\tau_k\le n$, set

$$\mathbf{x}(\mathbf{i}) = \mathbf{y}, \quad y_m = \prod_{j=\tau_{m-1}+1}^{\tau_m} x_j = \frac{a_{\tau_{m-1}}}{a_{\tau_m}}$$

$$w_j(\mathbf{x}) = d\ln\left(\frac{1-x_{j+1}^{-1}}{1-x_j}\right)$$

With $\tau_0 = 0$, $w_0(\mathbf{x}) = 1$. A partial order $\preceq$ on $\mathfrak{S}_n$ is given by $\mathbf{i}\preceq\mathbf{j}$ if $i_k\le j_k$
Define

$$X_n = \left\{\prod_{j\le k}(1-x_j\cdots x_k) = 0\right\}, \quad S_n = \mathbb{C}^n\setminus X_n$$

$$X_n' = \left\{\prod_i x_i\prod_{j\le k}(1-x_j\cdots x_k) = 0\right\}, \quad S_n' = \mathbb{C}^n\setminus X_n'$$

$$D_n = \bigcap_j\left\{\left|x_j-\frac{1}{2}\right| < \frac{1}{2}\right\}$$

Theorem 0.0.8. Multiple logarithm $\mathcal{L}_n(\mathbf{x})$ is a multi-valued holomorphic function on S_n

$$\mathcal{L}_n(\mathbf{x}) = \sum_{0 \neq \mathbf{j}_1 \prec \cdots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2 - \mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1))$$

Here $\mathbf{j} - \mathbf{i} = \begin{cases} s' & j_t = i_t + \delta_{st} \\ 0 & \text{otherwise} \end{cases}$ for $\mathbf{i} \prec \mathbf{j}$, j_s is the s'-th nonzero element in $\mathbf{j}$, and the integration is taken over $\alpha : I \to \mathbb{C}^n$

Proof. Use induction and Lemma 0.0.7

$$\begin{aligned}
\mathcal{L}_n(\mathbf{x}) &= \int_{\mathbf{0}}^{\mathbf{x}} d\mathcal{L}_n(\mathbf{x}) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{k=1}^{n-1} \mathcal{L}_{n-1}(\mathbf{x}_k) d\ln\left(\frac{1 - x_{k+1}^{-1}}{1 - x_k}\right) + \mathcal{L}_{n-1}(\mathbf{x}_n) \frac{dx_n}{1 - x_n} \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{j=1}^{n} w_{\mathbf{1} - \mathbf{f}_k}(\mathbf{x}) \sum_{0 \neq \mathbf{p}_1 \prec \cdots \prec \mathbf{p}_{n-1}} w_{\mathbf{p}_{n-1} - \mathbf{p}_{n-2}}(\mathbf{x}_k(\mathbf{p}_{n-1})) \cdots w_{\mathbf{p}_2 - \mathbf{p}_1}(\mathbf{x}_k(\mathbf{p}_2)) w_1(\mathbf{x}_k(\mathbf{p}_1)) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \sum_{j=1}^{n} w_{\mathbf{1} - \mathbf{f}_k}(\mathbf{x}) \sum_{0 \neq \mathbf{q}_1 \prec \cdots \prec \mathbf{q}_{n-1}} w_{\mathbf{q}_{n-1} - \mathbf{q}_{n-2}}(\mathbf{x}(\mathbf{q}_{n-1})) \cdots w_{\mathbf{q}_2 - \mathbf{q}_1}(\mathbf{x}(\mathbf{q}_2)) w_1(\mathbf{x}(\mathbf{q}_1)) \\
&= \sum_{0 \neq \mathbf{j}_1 \prec \cdots \prec \mathbf{j}_n} \int_0^{\mathbf{x}} w_{\mathbf{j}_n - \mathbf{j}_{n-1}}(\mathbf{x}(\mathbf{j}_n)) \cdots w_{\mathbf{j}_2 - \mathbf{j}_1}(\mathbf{x}(\mathbf{j}_2)) w_1(\mathbf{x}(\mathbf{j}_1))
\end{aligned}$$

Here $\mathbf{f}_k = (1, \cdots, \underset{\substack{\uparrow \\ k\text{-th}}}{0}, \cdots, 1)$, $\mathbf{q}_i$ is $\mathbf{p}_i$ with 0 inserted in as the k-th entry. Note that S_n is given so that $w_i(\mathbf{x}(\mathbf{j}))$ are defined □

Example 0.0.9. When $n = 1$

$$\begin{aligned}
\mathcal{L}_1(x_1) &= \int_0^{x_1} w_1(\mathbf{x}(1)) \\
&= \int_0^{x_1} d\ln\left(\frac{1}{1 - x_1}\right) \\
&= \int_0^{x_1} \frac{dx_1}{1 - x_1}
\end{aligned}$$

When $n = 2$

$$\begin{aligned}
\mathcal{L}_2(\mathbf{x}) &= \int_{\mathbf{0}}^{\mathbf{x}} w_{(1,1)-(1,0)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(1, 0)) + w_{(1,1)-(0,1)}(\mathbf{x}(\mathbf{1})) w_1(\mathbf{x}(0, 1)) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} w_2(\mathbf{x}) w_1(x_1) + w_1(\mathbf{x}) w_1(x_1 x_2) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} d\ln\left(\frac{1}{1 - x_2}\right) d\ln\left(\frac{1}{1 - x_1}\right) + d\ln\left(\frac{1 - x_2^{-1}}{1 - x_1}\right) d\ln\left(\frac{1}{1 - x_1 x_2}\right) \\
&= \int_{\mathbf{0}}^{\mathbf{x}} \frac{dx_2}{1 - x_2} \frac{dx_1}{1 - x_1} + \left(\frac{dx_2}{x_2(x_2 - 1)} + \frac{dx_1}{1 - x_1}\right) \frac{d(x_1 x_2)}{1 - x_1 x_2}
\end{aligned}$$

When $n = 3$

$$
\begin{aligned}
\mathcal{L}_3(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} & w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(1,0,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(1,0,0))+ \\
& w_{(1,1,1)-(1,1,0)}(\mathbf{x}(\mathbf{1}))w_{(1,1,0)-(0,1,0)}(\mathbf{x}(1,1,0))w_1(\mathbf{x}(0,1,0))+ \\
& w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(1,0,0)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(1,0,0))+ \\
& w_{(1,1,1)-(1,0,1)}(\mathbf{x}(\mathbf{1}))w_{(1,0,1)-(0,0,1)}(\mathbf{x}(1,0,1))w_1(\mathbf{x}(0,0,1))+ \\
& w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,1,0)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,1,0))+ \\
& w_{(1,1,1)-(0,1,1)}(\mathbf{x}(\mathbf{1}))w_{(0,1,1)-(0,0,1)}(\mathbf{x}(0,1,1))w_1(\mathbf{x}(0,0,1)) \\
= \int_{\mathbf{0}}^{\mathbf{x}} & w_3(\mathbf{x})w_2(x_1,x_2)w_1(x_1) + w_3(\mathbf{x})w_1(x_1,x_2)w_1(x_1x_2)+ \\
& w_2(\mathbf{x})w_2(x_1,x_2x_3)w_1(x_1) + w_2(\mathbf{x})w_1(x_1,x_2x_3)w_1(x_1x_2x_3)+ \\
& w_1(\mathbf{x})w_2(x_1x_2,x_3)w_1(x_1x_2) + w_1(\mathbf{x})w_1(x_1x_2,x_3)w_1(x_1x_2x_3) \\
= \int_{\mathbf{0}}^{\mathbf{x}} & \frac{dx_3}{1-x_3}\frac{dx_2}{1-x_2}\frac{dx_1}{1-x_1} + \frac{dx_3}{1-x_3}\left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\frac{d(x_1x_2)}{1-x_1x_2}+ \\
& \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2}\right)\frac{d(x_2x_3)}{1-x_2x_3}\frac{dx_1}{1-x_1}+ \\
& \left(\frac{dx_3}{x_3(x_3-1)} + \frac{dx_2}{1-x_2}\right)\left(\frac{d(x_2x_3)}{x_2x_3(x_2x_3-1)} + \frac{dx_1}{1-x_1}\right)\frac{d(x_1x_2x_3)}{1-x_1x_2x_3}+ \\
& \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\frac{dx_3}{1-x_3}\frac{d(x_1x_2)}{1-x_1x_2}+ \\
& \left(\frac{dx_2}{x_2(x_2-1)} + \frac{dx_1}{1-x_1}\right)\left(\frac{dx_3}{x_3(x_3-1)} + \frac{d(x_1x_2)}{1-x_1x_2}\right)\frac{d(x_1x_2x_3)}{1-x_1x_2x_3}
\end{aligned}
$$

Definition 0.0.10. $\mathbf{i},\mathbf{j} \in \mathfrak{S}_n$, $|\mathbf{i}| = k$, $|\mathbf{j}| = l$, the $\mathbf{i}$-th *retraction* map $\rho_{\mathbf{i}} : \mathfrak{S}_n \to \mathfrak{S}_k$ is defined by

- If $\mathbf{i} \not\succeq \mathbf{j}$, $\rho_{\mathbf{i}}(\mathbf{j}) = \mathbf{0}$
- If $\mathbf{i} \succeq \mathbf{j}$, assume $\tau_1, \cdots, \tau_k$ and $t_1, \cdots, t_l$ are the nonzero entries in $\mathbf{i}$ and $\mathbf{j}$, suppose $\tau_{\alpha_r} = t_r$, then $\alpha_1, \cdots, \alpha_l$ are the nonzero entries of $\rho_{\mathbf{i}}(\mathbf{j})$

Write $\theta_s = \theta_s(\mathbf{x}) = \dfrac{dt}{t - a_s}$, the $2^n \times 2^n$ *variation matrix* $\mathcal{M}_{\mathbf{1}}(\mathbf{x}) = (2\pi i)^l E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ associated with $\mathcal{L}_n(\mathbf{x})$ is defined by

$$
\begin{aligned}
E_{\mathbf{i},\mathbf{j}} = \gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{y}) &= (-1)^{k-l} \prod_{r=0}^{l} \int_{a_{\alpha_{r+1}}(\mathbf{y})}^{a_{\alpha_r}(\mathbf{y})} \theta_{\alpha_r+1}(\mathbf{y}) \cdots \theta_{\alpha_{r+1}-1}(\mathbf{y}) \\
&= (-1)^{k-l} \prod_{r=0}^{l} \int_{a_{t_{r+1}}}^{a_{t_r}} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x}) \\
&= (-1)^{k-l} \prod_{r=0}^{l} \int_{p_r} \theta_{\tau_{\alpha_r+1}}(\mathbf{x}) \cdots \theta_{\tau_{\alpha_{r+1}-1}}(\mathbf{x})
\end{aligned}
$$

$\tau_{k+1} = t_{l+1} = n+1$, $\alpha_{l+1} = k+1$. p_r are independent from $\mathbf{i}$ These thetas are very weird

Proposition 0.0.11.

$$
\begin{aligned}
E_{\mathbf{i},\mathbf{j}} &= \prod_{r=0}^{l} \mathcal{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{a_{t_r}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_r+1}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})},\cdots,\frac{a_{\tau_{\alpha_{r+1}-2}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}{a_{\tau_{\alpha_{r+1}-1}}(\mathbf{x})-a_{t_{r+1}}(\mathbf{x})}\right)\\
&= \mathcal{L}_{k-\alpha_l}(x_{1+t_l}\cdots x_{\tau_{\alpha_l+1}},\cdots,x_{1+\tau_{k-1}}\cdots x_{\tau_k})\cdot\\
&\quad\prod_{r=0}^{l-1} \mathcal{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{1+\tau_{\alpha_r+1}}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{1+\tau_{\alpha_{r+1}-2}}\cdots x_{t_{r+1}}}{1-x_{1+\tau_{\alpha_{r+1}-1}}\cdots x_{t_{r+1}}}\right)
\end{aligned}
$$

Proof.

□

Example 0.0.12.

$$
\begin{aligned}
E_{\mathbf{1},\mathbf{j}}(\mathbf{x}) = \gamma^n_{\mathbf{j}}(\mathbf{x}) &= \prod_{r=0}^{l} \mathcal{L}_{t_{r+1}-t_r-1}\left(\frac{a_{t_r}-a_{t_{r+1}}}{a_{t_r+1}-a_{t_{r+1}}},\cdots,\frac{a_{t_{r+1}-2}-a_{t_{r+1}}}{a_{t_{r+1}-1}-a_{t_{r+1}}}\right)\\
&= \prod_{r=0}^{l} \mathcal{L}_{t_{r+1}-t_r-1}\left(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{2+t_r}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{t_{r+1}-1}x_{t_{r+1}}}{1-x_{t_{r+1}}}\right)\\
&= \mathcal{L}_{k-\alpha_l}(x_{1+t_l}\cdots x_{t_l+1},\cdots,x_n)\cdot\\
&\quad\prod_{r=0}^{l-1} \mathcal{L}_{t_{r+1}-t_r-1}\left(\frac{1-x_{1+t_r}\cdots x_{t_{r+1}}}{1-x_{2+t_r}\cdots x_{t_{r+1}}},\cdots,\frac{1-x_{t_{r+1}-1}x_{t_{r+1}}}{1-x_{t_{r+1}}}\right)
\end{aligned}
$$

In particular we have

$$
E_{\mathbf{1},\mathbf{0}}(\mathbf{x}) = \gamma^n_{\mathbf{0}}(\mathbf{x}) = \mathcal{L}_n(\mathbf{x}),\quad E_{\mathbf{1},\mathbf{1}}(\mathbf{x}) = \gamma^n_{\mathbf{1}}(\mathbf{x}) = \prod_{r=0}^{n}\mathcal{L}_0 = 1
$$

$E = E(\mathbf{x})$ has columns

$$
C_{\mathbf{j}} = \sum_{\mathbf{i}\succeq\mathbf{j}} \gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i}))e_{\mathbf{i}}
$$

$e_{\mathbf{i}}$ is the standard unit column vector, using the complete order $<$ on $\mathfrak{S}_n$: if $|\mathbf{i}|<|\mathbf{j}|$, then $\mathbf{i}<\mathbf{j}$, if $|\mathbf{i}|=|\mathbf{j}|$, then compare the lexicographic order from left to right with $1<0$. By definition, the $\mathbf{i}$-th row of $\mathcal{M}_{\mathbf{1}}$ is

$$
R_{\mathbf{i}} = \sum_{\mathbf{i}\succeq\mathbf{j}} (2\pi i)^{|\mathbf{j}|}\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i}))e^T_{\mathbf{j}}
$$

And the $\mathbf{j}$-th column of $\mathcal{M}_{\mathbf{1}}$ is $(2\pi i)^{|\mathbf{j}|}C_{\mathbf{j}}$ Note that $\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{i})}(\mathbf{x}(\mathbf{i}))=1$, and the first entry of $R_{\mathbf{i}}$ is $\mathcal{L}_{|\mathbf{i}|}(\mathbf{x}(\mathbf{i}))$

Example 0.0.13. The variation matrix associated with double logarithm is

$$
\begin{aligned}
\mathcal{M}_{1,1} &= \begin{bmatrix}
\gamma^0_{\rho(0,0)(0,0)} & (2\pi i)\gamma^0_{\rho(0,0)(1,0)} & (2\pi i)\gamma^0_{\rho(0,0)(0,1)} & (2\pi i)^2\gamma^0_{\rho(0,0)(1,1)}\\
\gamma^1_{\rho(1,0)(0,0)}(x_1) & (2\pi i)\gamma^1_{\rho(1,0)(1,0)}(x_1) & (2\pi i)\gamma^1_{\rho(1,0)(0,1)}(x_1) & (2\pi i)^2\gamma^1_{\rho(1,0)(1,1)}(x_1)\\
\gamma^1_{\rho(0,1)(0,0)}(x_1x_2) & (2\pi i)\gamma^1_{\rho(0,1)(1,0)}(x_1x_2) & (2\pi i)\gamma^1_{\rho(0,1)(0,1)}(x_1x_2) & (2\pi i)^2\gamma^1_{\rho(0,1)(1,1)}(x_1x_2)\\
\gamma^2_{\rho(1,1)(0,0)}(x_1,x_2) & (2\pi i)\gamma^2_{\rho(1,1)(1,0)}(x_1,x_2) & (2\pi i)\gamma^2_{\rho(1,1)(0,1)}(x_1,x_2) & (2\pi i)^2\gamma^2_{\rho(1,1)(1,1)}(x_1,x_2)
\end{bmatrix}\\
&= \begin{bmatrix}
1 & & & \\
\mathcal{L}_1(x_1) & 2\pi i & & \\
\mathcal{L}_1(x_1x_2) & & 2\pi i & \\
\mathcal{L}_2(x_1,x_2) & (2\pi i)\mathcal{L}_1(x_2) & (2\pi i)\mathcal{L}_1\left(\frac{1-x_1x_2}{1-x_2}\right) & (2\pi i)^2
\end{bmatrix}
\end{aligned}
$$

Variation matrix of multiple logarithm is lower triangular

Lemma 0.0.14. The variation matrix is lower triangular. The principal submatrix of $\mathcal{M}_{\mathbf{1}}$ with $|\mathbf{i}|=|\mathbf{j}|=k$ is $(2\pi i)^k$ times the $\binom{n}{k}\times\binom{n}{k}$ identity matrix

Proof. By definition □

Proposition 0.0.15. $\mathcal{M}_{\mathbf{1}}(\mathbf{x})$ is the fundamental matrix of linear differential equations

$$\begin{cases} dX_{\mathbf{i}} &= \sum_{|\mathbf{k}|=|\mathbf{i}|-1, \mathbf{k}\prec \mathbf{i}} X_k d\rho_{\mathbf{i}}^{|\mathbf{i}|}(\mathbf{k})(\mathbf{x}(\mathbf{i})) \\ dX_{\mathbf{0}} &= \mathbf{0} \end{cases}$$

Theorem 0.0.16.

Corollary 0.0.17. The monodromy representation $\rho_{\mathbf{x}} : \pi_1(S_n, \mathbf{x}) \to \mathrm{GL}_{2^n}(\mathbb{Z})$ is unipotent

Definition 0.0.18. Let $\mathcal{D}_n = X_n' \cup (\mathbb{CP}^n \setminus \mathbb{C}^n)$ and

$$\boldsymbol{\omega} = (c_{\mathbf{i},\mathbf{j}}) \in H^0(\mathbb{CP}^n, \Omega^1_{\mathbb{CP}^n}(\log(\mathcal{D}_n))) \otimes M_{2^n}(\mathbb{C})$$

Here

$$c_{\mathbf{i},\mathbf{j}} = \begin{cases} d\gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i})) & |\mathbf{j}| = |\mathbf{i}| - 1, \mathbf{j} \prec \mathbf{i} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

All one forms in $\boldsymbol{\omega}$ has logarithmic singularity. $\mathcal{M}_{\mathbf{1}}(\mathbf{x})$ is invertible, $d\boldsymbol{\omega} = 0$, $\boldsymbol{\omega} \wedge \omega = 0$, thus $\boldsymbol{\omega}$ is integrable. Define a meromorphic connection on trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{2^n} \to \mathbb{CP}^n$

$$\nabla f = df - \boldsymbol{\omega} f$$

Here $f : S_n \to \mathbb{C}^{2^n}$ is a section

Definition 0.0.19. Let $V_{\mathbf{1}}(\mathbf{x})$ be the locally constant sheaf of $\mathbb{Q}$ vector space generated by the column vectors in $\mathcal{M}_{\mathbf{1}}(\mathbf{x})$, define a weight filtration $W_\bullet$ by letting $W_{2k+1} = W_{2k}$ and W_{-2k} is generated by $(2\pi i)^{|\mathbf{j}|} C_{\mathbf{j}}(\mathbf{x})$, $|\mathbf{j}| \geq k$, and $W_{2k} = 0$, note that $W_{-2k} = V_{\mathbf{1}}(\mathbf{x})$ for $k \geq n$. define filtration $\mathcal{F}^\bullet$ by $\mathcal{F}^{-k} V_{\mathbf{1}}(\mathbb{C}) = \langle e_{\mathbf{i}}, |\mathbf{i}| \leq k \rangle_{\mathbb{C}}$, $\mathcal{F}^k = 0$, note that $\mathcal{F}^{-k} = V_{\mathbf{1}}(\mathbb{C})$ for $k \geq n$