0.1 Bundles

Definition 0.1.1. A **bundle** is $E \xrightarrow{p} B$, where E is the **total space**, B is the **base space**, and p is the projection, $p^{-1}(b)$ is the **fiber** over b. A **cross section** is $s : X \to E$, such that $ps = 1_X$. The restriction $p^{-1}(A) \xrightarrow{\pi} A$, $A \subseteq B$ is also a bundle

Definition 0.1.2. Suppose $E \xrightarrow{p} B$ is a bundle, $f : A \to B$ is a map, then the pullback $f^*(E) = A \times_p E \to A$ is the **pullback bundle**, the pullback of a section $s : B \to E$ is defined as $f^*s := s \circ f$, notice $p(f^*s(y)) = p(s(f(y))) = f(y)$

Definition 0.1.3. A **fiber bundle** is a bundle $E \xrightarrow{p} B$ such that there exists an open neighborhood U of b and a homeomorphism ϕ making the following diagram commute

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

Definition 0.1.4. G is a topological group, a G **fiber bundle** $E \xrightarrow{p} B$ is a fiber bundle and also a morphism of G spaces

Lemma 0.1.5. A fiber bundle is a Serre fibration

Definition 0.1.6. $\mathbb{F}$ is a topological field, a **vector bundle** is a fiber bundle $E \xrightarrow{p} X$ with fiber being $\mathbb{F}^n$ and ϕ restricts on each fiber is an $\mathbb{F}$ isomorphism

Definition 0.1.7. G is a topological group, a **principal** G **bundle** $p : P \to B$ is a morphism of G spaces, B with the trivial G action, and for each $b \in B$, there is a local trivialization

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi"] \arrow[d, "p"] & U \times G \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

ϕ is an isomorphism

Remark 0.1.8. G action on P preserves fibers, and the action on fiber is free and transitive, each fiber is a G torsor. A morphism of principal G bundles is always an isomorphism. A principal G bundle is trivial iff it has a global section

Proposition 0.1.9. Suppose $P \to B$ is a principal G bundle, $G \to G/H$ is a principal H bundle, then $P \to P/H$ is a principal H bundle

Proof. $P \cong P \times_G G \to P \times_G (G/H) \cong P/H$ $\square$

Proposition 0.1.10. Suppose $P \to B$ is a principal G bundle, F is a left G space, $P \times_G F \to P \times_G * \cong B$ is a G fiber bundle. $X \xrightarrow{f} Y$ is a map, $f^*(P \times_G F) \to f^*(P) \times_G F$ is a natural homeomorphism

Proposition 0.1.11. $P \xrightarrow{p} B$ is a principal G bundle, X is a right G space, a morphism $P \xrightarrow{f} X$ induce $P \xrightarrow{\begin{pmatrix} 1 \\ f \end{pmatrix}} P \times X$, $B \cong P/G \to P \times X/G \cong P \times_G X$ which is a section s_f of $P \times_G X \to B$, this is a natural bijection

Proposition 0.1.12. $P \to B \times I$ is principal G bundle, then P and $P_0 \times I$ is an isomorphism, here P_0 is the restriction of P over $B \times \{0\}$

Proof.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, hook] & P \times_G (P_0 \times I) \arrow[d] \\
B \times I \arrow[r] \arrow[ur, dashed] & B \times I
\end{tikzcd}$$

□

0.2 Vector bundles

Proposition 0.2.1. $E \xrightarrow{p} X$ is trivial iff there exist global sections $s_1, \cdots, s_n$ that they are linearly independent on each fiber

Definition 0.2.2. Let $E \xrightarrow{p} X$ be a vector bundle, consider two trivializations $\varphi_U : E_U \to U \times \mathbb{R}^n$ and $\varphi_V : E_V \to V \times \mathbb{R}^n$ around $x \in X$, then $\varphi_V \circ \varphi_U^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$ is a local isomorphism with inverse $\varphi_U \circ \varphi_V^{-1}$ restricted on $U \cap V \times \mathbb{R}^n$, it is also called a transition function and it can also be regard as a continuous map $g_{VU} : U \cap V \to GL(n, \mathbb{R})$ or $g_{VU} \in GL(n, C(U \cap V))$, such that $\varphi_V \circ \varphi_U(x, v) = (x, g_{VU}(x)v)$, notice then $g_{UV} = g_{VU}^{-1}$, and g_{VU}'s satisfy the cocycle relation $g_{WV} g_{VU} = g_{WU}$ on $U \cap V \cap W$
Conversely, given $\bigsqcup_{\alpha \in A} U_\alpha \times \mathbb{R}^n \times A$ transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ that satisfying cocycle relation $g_{\gamma\beta} g_{\beta\alpha} = g_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, mod equivalence relation $(x, v, \alpha) \sim (x, g_{\beta\alpha}(v), \beta), x \in U_\alpha \cap U_\beta$, you will get back the vector bundle
Suppose $s : X \to E$, is a section, denote $\varphi_i \circ s|_{U_i}(x) = (x, f_i(x))$ over U_i, then $(x, f_j(x)) = \varphi_j \circ s|_{U_j}(x) = \varphi_j \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1} \circ \varphi_i \circ s|_{U_i}(x) = \varphi_j \circ \varphi_i^{-1}(x, f_i(x)) = (x, g_{ji}(x) f_i(x)), \forall x \in U_i \cap U_j$, thus $f_j = g_{ji} f_i$, conversely, this relation also defines a section

Definition 0.2.3. The pullback of a transition function is defined to be $f^* g_{ij} := g_{ij} \circ f$

Definition 0.2.4. A morphism between vector bundles $\varphi : E \to F$ is map such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "p"] & F \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and $\varphi_x : E_x \to F_{f(x)}$ is a homomorphism between vector spaces

Definition 0.2.5. Let $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ be vector bundles, then direct sum $E \times F \xrightarrow{p \times q} X \times Y$ is also a vector bundle, suppose $\varphi_U : U \to U \times \mathbb{R}^n$, $\psi_V : V \to V \times \mathbb{R}^m$ are trivializations, then $\varphi_U \times \psi_V : U \times V \to U \times \mathbb{R}^n \times V \times \mathbb{R}^m \cong U \times V \times \mathbb{R}^{n+m}$ is also a trivialization

Proposition 0.2.6. Let $E \xrightarrow{p} X$ is a vector bundle, and $f : X \to Y$ is a homeomorphism, then $E \xrightarrow{f \circ p} Y$ is a vector bundle, suppose $\varphi_U : E_U \to U \times \mathbb{F}^n$ is a trivialization, then $(f \times 1) \circ \varphi_U =: \psi_{f(U)} : E_U \to U \times \mathbb{F}^n \to f(U) \times \mathbb{F}^n$ is a trivialization

Domain is homeomorphic to its graph

Proposition 0.2.7. $p : \Gamma_f \to X, (x, f(x)) \mapsto x$ is homeomorphism

Proof. p as a restriction on Γ_f of $X \times Y$ projecting to X is continuous, and define $q : X \to \Gamma_f, x \to (x, f(x))$, since the composition $X \xrightarrow{q} \Gamma_f \hookrightarrow X \times Y$ which is continuous because $X \xrightarrow{f} Y$, $X \xrightarrow{id} X$ are continuous, q is continuous, and p, q are inverses to each other □

Definition 0.2.8. $E \xrightarrow{\pi} X$ ia vector bundle, $f : Y \to X$ is a continuous map, then we can construct the pullback bundle $f^*E \xrightarrow{p} Y$

$$\begin{tikzcd} f^*E \arrow[r, "g"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

satisfying universal property

$$\begin{tikzcd}
F \arrow[dr, dashed, "\exists_1 h"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& Y \underset{X}{\times} E \arrow[r, "g"] \arrow[d, "p"] & E \arrow[d, "\pi"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd}$$

Concrete construction: let $f^*E = Y \underset{X}{\times} E \subseteq Y \times X$ with subspace topology, where $Y \underset{X}{\times} E = \{(y, v) \in Y \times E | f(y) = \pi(v)\}$, let's check it is a vector bundle over Y, notice that $Y \underset{X}{\times} E \to Y$ factor through $Y \underset{X}{\times} E \to \Gamma_f \to Y$, $(y, v) \mapsto (y, \pi(v)) = (y, f(y)) \mapsto y$, where Γ_f is the graph of f which is homeomorphic to Y due to Proposition 0.2.7, notice that $Y \underset{X}{\times} E \to \Gamma_f$ is the restriction of vector bundle $Y \times E \xrightarrow{1\times\pi} Y \times X$ over Γ_f, thus $Y \underset{X}{\times} E \to Y$ is a vector bundle, suppose F as in the commutative diagram, then h is simply defined as $h(w) := (\beta(w), \alpha(w))$

Remark 0.2.9. In general, this is a pullback, but it has a vector bundle structure such that it induces an isomorphism on each fiber, now suppose $F \xrightarrow{q} Y$ is a another vector bundle such that not only the diagram commutes but also induce isomorphism on each fiber, then $F \cong f^*E$
Use this we have $(fg)^*E \cong g^*(f^*E)$, $f^*(E \oplus F) \cong f^*E \oplus f^*F$, $f^*(E \otimes F) \cong f^*E \otimes f^*F$, $1^*E \cong E$

Definition 0.2.10. Suppose E, F are vector bundles both trivialized over $\{U_\alpha\}$(this can easily be achieved, just take intersections), suppose the transition functions are $g_{\alpha\beta}, h_{\alpha\beta}$, then define the tensor product of vector bundles $E \otimes F$ by letting its transition functions be $g_{\alpha\beta} \otimes h_{\alpha\beta}$
Similarly, we can define symmetric power and exterior power of vector bundles by specifying its transition function
Does it have universal property also?

Definition 0.2.11. Let $E \xrightarrow{p} X$, $E \xrightarrow{p} X$ be vector bundles, then the direct sum $E \oplus F \xrightarrow{p} X$ is defined by transition functions $g_{\alpha\beta} \oplus h_{\alpha\beta}$, where $g_{\alpha\beta}, h_{\alpha\beta}$ are transition functions of E, F

Definition 0.2.12. Let $E \to X$ be a vector bundle, define its dual bundle as follows, if $g_{\alpha\beta}$ is a transition function, the transition function for E^* would be $\left(g_{\alpha\beta}^{-1}\right)^T$

Definition 0.2.13. quotient bundle, exterior and symmetric power of vector bundle

Proposition 0.2.14. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product

Definition 0.2.15. $F \subseteq E$ is called a vector subbundle if F is a subspace of E and $F \xrightarrow{p} X$ is also a vector space

Proposition 0.2.16. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space and $F \subseteq E$ is a vector subbundle, then there exists a vector subbundle $F^\perp \subseteq E$ such that $F_x \oplus F^\perp|_x = E|_x$ and $(F|_x)^\perp = F^\perp|_x$

Proof. □

X compact Hausdorff => E has complement

Theorem 0.2.17. If $E \xrightarrow{p} X$ is vector bundle over a compact Hausdorff space X, then there exists a vector bundle $E' \xrightarrow{p'} X$ such $E \oplus E'$ is a trivial bundle

Proposition 0.2.18. Every Lie group G is parallelizable

Proof. Pick an arbitrary basis $e_1, \cdots, e_n$ of T_1G, then $L_g^*(e_i)$ will be a basis of $T_{g^{-1}}G$ since L_g^* is an isomorphism, they form independent global sections of the tangent bundle □

Definition 0.2.19. Tautological bundle

Definition 0.2.20. Let X be a smooth manifold of dimension n(depending on the field), Ω denote the cotangent bundle, then $\omega := \bigwedge^n \Omega$ is called the canonical bundle

Definition 0.2.21. Universal bundle

Theorem 0.2.22. Let X be a paracompact Hausdoff space, there is a bijection $\left[X, \varinjlim Gr_{\mathbb{C}}(n,N)\right] \to \mathsf{Vect}^n_{\mathbb{C}}(X), [f] \mapsto [f^*(E)]$

Definition 0.2.23. If G is a topological group, then a principal G-bundle P is a fiber bundle with a continuous right G action $P \times G \to P$, and the action is free and transitive(thus regular), which imply each fiber is a G-torsor, also, $g \mapsto yg$ is a homeomorphism

Definition 0.2.24. Let $E \xrightarrow{p} X$ is a vector bundle, an inner product is a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product on E_x

Proposition 0.2.25. Let $E \xrightarrow{p} X$ is a vector bundle with an inner product $\langle , \rangle$, then we can local trivialization to be isometry on each fiber, i.e. $\langle v, w \rangle = (\varphi_U(v), \varphi_U(w)), v, w \in E_x$, where $(,)$ is the standard inner product on $U \times \mathbb{R}^n$

Proposition 0.2.26. $E \xrightarrow{p} X$ is a vector bundle with X being a paracompact space, then there exists a continuous map $\langle , \rangle : E \oplus E \to \mathbb{R}$ with $\langle , \rangle|_{E_x}$ defines an inner product

Definition 0.2.27. let G be a topological group, E, X be G-spaces, then $E \xrightarrow{p} X$ is a G-vector bundle if it is a vector bundle, p is a G map, and for any $x \in X$, $g : E_x \to E_{gx}$ is a linear map

Definition 0.2.28. Let G be a topological group, H be a closed subgroup, a G vector bundle $\pi : E \to G/H$ is called a homogeneous vector bundle

Lemma 0.2.29. Let $Y \xrightarrow{f} X, Z \xrightarrow{g} X$ be open surjective continuous maps, then the projection $p_Y : Y \times_X Z \to Y$ is open surjective

Proof. For surjectivity, if $y \in Y$, since g is surjective, $\exists z \in Z$ such that $g(z) = f(y)$, then $(z, y) \in Y \times_X Z$
To prove p_Y is open, suppose $(z_0, y_0) \in Y \times_X Z$ is in some open set, then $(z_0, y_0) \in U \times V \cap Y \times_X Z$ for some $y_0 \in U, z_0 \in V$ open, since f, g are open, $U' := f(U) \cap g(V)$ is open, let $V' := V \cap f^{-1}(U')$, then we can show V' is in the image of $U \times V \cap Y \times_X Z$, since $\forall y \in V'$, $f(y) \in U' \subseteq g(V)$, thus $f(y) = g(z)$ for some $z \in V$, hence $(y, z) \in U \times V \cap Y \times_X Z$ $\square$

Proposition 0.2.30. Let $\pi : E \to G/H$ be a homogeneous vector bundle, E_H be the fiber over the coset H, action $G \times E_H \to E$ can be regard as $\alpha : G \times_H E_H \to E$ which is an isomorphism of G vector bundles. Moreover, if H is locally compact, then for a given $\mathbb{R}H$ module E_H, $G \times_H E_H \to G/H$ is indeed a G vector bundle, hence G vector bundle E is in one to one correspondence with representations of H on E_H, so $K_G(G/H) \cong R(H)$

Proof. E_H is an $\mathbb{R}H$ module, let $G \times_H E_H$ denote the space of orbits of $G \times E_H$ under H by $h \cdot (g, \xi) = (gh^{-1}, h\xi)$, $G \times_H E_H$ is a G space with G action $g \cdot (g', \xi) \mapsto (gg', \xi)$, then the group action can be regarded as $\alpha : G \times_H E_H \to E, (g, \xi) \mapsto g\xi$, we can find its inverse $\beta : E \to G \times_H E_H, E_{gH} \ni \xi \mapsto (g, g^{-1}\xi)$, to show that this is continuous, consider $\gamma : G \times E \to G \times E, (g, \xi) \mapsto (g, g^{-1}\xi)$, then the preimage of $G \times E_H$ will be the pullback $G \times_{G/H} E := \{(g, \xi) \in G \times E | gH = \pi\xi\}$, then $G \times_{G/H} E \to G \times E_H \to G \times_H E_H, (g, \xi) \mapsto (g, g^{-1}\xi)$ factors as $G \times_{G/H} E \to E \xrightarrow{\beta} G \times_H E, (g, \xi) \mapsto \xi \mapsto (g, g^{-1}\xi)$ which open surjective, therefore β is continuous due to the previous Lemma $\square$

Definition 0.2.31. A clutching function for S^k is $f : S^{k-1} \to GL(n, \mathbb{C})$, then we can define vector bundle E_f with f being the transition function, conversely, if E is a vector bundle over S^k, since its upper and lower hemispheres are both contractible, $E = E_f$, where f is the transition function, denoting the corresonding matrix T_f

Theorem 0.2.32. $[S^{k-1}, GL(n,\mathbb{C})] \to \mathrm{Vect}^n_{\mathbb{C}}(S^k), f \mapsto E_f$ is a bijection

Lemma 0.2.33. Suppose $f, g : S^{k-1} \to GL(n,\mathbb{C})$, then $(E_f \otimes E_g) \oplus \varepsilon^n \cong E_{fg} \oplus \varepsilon^n \cong E_f \oplus E_g$

Proof. Since $GL(n,\mathbb{C})$ is path connected, there is a path $A_t \in GL(2n,\mathbb{C})$ that $A_0 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $\begin{pmatrix} T_f & \\ & I \end{pmatrix} A_t \begin{pmatrix} I & \\ & T_g \end{pmatrix} A_t$ is $\begin{pmatrix} T_f & \\ & T_g \end{pmatrix}$ when $t = 0$ and $\begin{pmatrix} T_f T_g & \\ & I \end{pmatrix} = \begin{pmatrix} T_{fg} & \\ & I \end{pmatrix}$ when $t = 1$ □

Definition 0.2.34. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the sphere bundle $S(E)$ associated to E to be $S(E) = \bigcup_{x\in X} S(E_x)$ with the subspace topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{tikzcd}
S(E)_U \arrow[r, "\varphi_U"] \arrow[d, hook] & U \times S(\mathbb{R}^n) \arrow[d, hook] \\
E_U \arrow[r, "\varphi_U"] \arrow[dr, "p"] & U \times \mathbb{R}^n \arrow[d] \\
 & U
\end{tikzcd}$$

Definition 0.2.35. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the projective bundle $P(E)$ associated to E to be $P(E) = \bigcup_{x\in X} P(E_x)$ with the quotient topology, this is a fiber bundle, suppose φ_U is a local trivialization, since we can choose φ_U to be isometry over each fiber, thus the following diagram commutes

$$\begin{tikzcd}
S(E)_U \arrow[rr, "\varphi_U"] \arrow[dd, "q"] \arrow[dr] & & U \times S(\mathbb{R}^n) \arrow[dd, "q"] \arrow[dl] \\
 & U & \\
P(E)_U \arrow[rr, "\varphi_U"] \arrow[ur] & & U \times P(\mathbb{R}^n) \arrow[ul]
\end{tikzcd}$$

Definition 0.2.36. Let $E \xrightarrow{p} X$ be vector bundle of rank n, and there is a inner product over E, we can define the flag bundle $F(E)$ associated to E to be $F(E) = \bigcup_{x\in X} F(E_x)$ with the subspace topology in $P(E) \times \cdots \times P(E)$

Remark 0.2.37. Consider the pullback of $\pi : F(E) \to X$, $\pi^*(E) \subseteq F(E) \times E$, consider its subbundles $L_1, \cdots, L_n$, where L_i is the subbundle that over a point in $F(E)$, it is the i-th factor, then $\pi^*(E) \cong L_1 \oplus \cdots \oplus L_n$

Definition 0.2.38. Let X be a paracompact and Hausdorff space, there exist unique functions $w_1, w_2, \cdots, w_i : \mathrm{Vect}_{\mathbb{R}}(\mathrm{X}) \to H^i(X, \mathbb{Z}_2)$, $E \to w_i(E)$, and they only depend on the isomorphism classes of E, satisfying

1. $w_i(f^*(E)) = f^*(w_i(E))$, for pullback bundle $f^*(E)$
2. $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ where $w = 1 + w_1 + w_2 + \cdots \in H^*(X, \mathbb{Z}_2)$
3. $w_i(E) = 0, \forall i > \dim E$
4. If $E \to \mathbb{R}P^\infty$ is the canonical line bundle, then $w_1(E)$ is the generator of $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$

$w_i(E)$ are called the Stiefel-Whitney classes of E

Definition 0.2.39. Let X be a paracompact and Hausdorff space, there exist unique functions $c_1, c_2, \cdots, c_i : \mathsf{Vect}_{\mathbb{C}}(\mathrm{X}) \to H^{2i}(X;\mathbb{Z})$, $E \to c_i(E)$, and they only depend on the isomorphism classes of E, satisfying
1. $c_i(f^*(E)) = f^*(c_i(E))$, for pullback bundle $f^*(E)$
2. $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ where $c = 1 + c_1 + c_2 + \cdots \in H^*(X;\mathbb{Z})$
3. $c_i(E) = 0, \forall i > \dim E$
4. If $E \to \mathbb{C}P^\infty$ is the canonical line bundle, then $c_1(E)$ is a generator of $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$, specify a generator in advance
$c_i(E)$ are called the Chern classes of E, also we define the Chern polynomial to be $c_t = 1 + c_1 t + c_2 t^2 + \cdots$ where t is just a formal variable used to keep tracking of the degree

Lemma 0.2.40. Let L_1, L_2 be line bundles, then $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$

Definition 0.2.41. Suppose L is a line bundle, define the Chern character $ch(L) = e^{c_1(L)} = 1 + c_1(L) + \dfrac{c_1(L)^2}{2!} + \cdots \in H^*(X;\mathbb{Q})$, then we have $ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = e^{c_1(L_1)}e^{c_1(L_2)} = ch(L_1)ch(L_2)$, If we assume $ch(L_1 \oplus L_2) = ch(L_1) + ch(L_2)$, then for $E = L_1 \oplus \cdots \oplus L_n$, $ch(E) = ch(L_1) + \cdots + ch(L_n) = n + (c_1(L_1) + \cdots + c_1(L_n)) + \left(c_1(L_1)^2 + \cdots + c_1(L_n)^2\right)/2! + \cdots$, on the other hand, we have $c(E) = c(L_1) \smile \cdots \smile c(L_n) = (1 + c_1(L_1)) \smile \cdots \smile (1 + c_1(L_n)) = 1 + c_1(E) + \cdots + c_n(E)$, where $c_i(E)$ would just be the i-th elementary symmetric polynomial of $c_1(L_1), \cdots, c_1(L_n)$, i.e. $c_i(E) = \sigma_i(c_1(L_1), \cdots, c_1(L_n))$, so we can express $c_1(L_1)^k + \cdots + c_1(L_n)^k$ in terms of $c_i(E)$, i.e. $c_1(L_1)^k + \cdots + c_1(L_n)^k = s_k(c_1(E), \cdots, c_n(E))$, thus we have an abstract definition of Chern character, $ch(E) = \dim E + s_1(c_1(E), \cdots, c_n(E)) + s_2(c_1(E), \cdots, c_n(E))/2! + \cdots$

Proposition 0.2.42. $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$, $ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$

0.3 Principal bundle

0.4 Topological K-theory

Definition 0.4.1. Two vector bundles $E \to X$, $F \to X$ are stably isomorphic if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^n$, denoted as $E \approx F$, we also denote $E \sim F$ if $E \oplus \varepsilon^n \cong F \oplus \varepsilon^m$ for some n, m

Remark 0.4.2. Here stably isomorphic does not imply isomorphic, for example, $TS^2 \approx_s \varepsilon^2$, since $\varepsilon^3 \approx T^2 \oplus NS^2 \approx T^2 \oplus \varepsilon^1$ whereas TS^2 is not trivial by the hairy ball theorem, and $NS^2 \approx \varepsilon^1$ is trivial because it is very easy to find a nonvanishing global section

Definition 0.4.3. Define the reduced K group to be $\tilde{K}(X)$ which consists of $\sim$-equivalent classes, and define K group to be the formal difference of isomorphic classes $E - F$, and $E - F = E' - F'$ if $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ for some vector bundle G

Remark 0.4.4. When X is compact Hausdorff, $E \oplus F' \oplus G \cong E' \oplus F \oplus G$ is equivalent to $E \oplus F' \oplus \varepsilon^m \cong E' \oplus F \oplus \varepsilon^m$, since we can find G' such that $G \oplus G' \cong \varepsilon^m$ due to Theorem 0.2.17 $K(*) = \{\varepsilon^m - \varepsilon^n\} \cong \mathbb{Z}$, $\tilde{K}(*) = 0$, and when X compact Hausdorff we have an exact sequence $0 \to K(*) \to K(X) \to \tilde{K}(X) \to 0$, where $K(*) \to K(X)$ is simply given by $\varepsilon^m - \varepsilon^n \mapsto \varepsilon^m - \varepsilon^n$, $K(X) \to \tilde{K}(X)$ is defined as follows, given $E - F \in K(X)$, $E - F = E \oplus F' - F \oplus F' = E' - \varepsilon^m$ is mapped to E', this exact sequence splits since we have map $K(X) \to K(*)$ given by restriction

Conjecture 0.4.5. Let M be the Möbius line bundle over S^1, since $M \oplus M \cong \varepsilon^2$, and $M \otimes M \cong \varepsilon^1$, thus real K-theory of S^1 is isomorphic to $\mathbb{Z}[M]/(M^2 - 1, 2M - 2)$

Example 0.4.6. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, TS^n, NS^n be the tangent bundle and normal bundle, then $TS^n \oplus NS^n$ can be seen as the restriction of the trivial bundle $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ on S^n, thus $TS^n \oplus NS^n$ is trivial

Definition 0.4.7. Define external product $K(X) \otimes K(Y) \to K(X \times Y), a \otimes b \mapsto p_1^*(a) p_2^*(b) =: a \times b$, this is a ring homomorphism

0.5 Classifying space

Definition 0.5.1. Suppose G is a topological group, P_G is the contravariant functor from the category of CW complexes to the category of sets, mapping X to all the principal G bundles over X, a **classifying space** BG is a topological space such that $[-, BG] \to P_G(-)$ is a natural isomorphism

Lemma 0.5.2. BG is unique up to weak homotopy equivalence

Proof. Suppose $B'G$ is also a classifying space, then $[-, BG] \cong P_G(-) \cong [-, B'G]$ are natural isomorphic, by Theorem **??**, we may assume BG, $B'G$ are both CW complexes, and by Lemma **??**, $X \to Hom(-, X)$ is fully faithful functor, thus $BG, B'G$ are homotopic ☐

Theorem 0.5.3 (Milnor's contruction for classifying space)**.** Define E^nG to be $\overbrace{G * \cdots * G}^{n+1}$ are formal sums $t_0g_0 + t_1g_1 + \cdots + t_ng_n$, with $\sum t_i = 1$. $EG := \varinjlim E^nG$ are finite formal sums $\sum t_ig_i$ with $\sum t_i = 1$. $E^nG \to E^nG/G$, $EG \to EG/G =: BG$ are principal G bundles, any principal G bundle over X is a pullback bundle of $EG \xrightarrow{p} BG$

Proof. Define G right action on E^nG, EG

$$E^nG \times G \to E^nG, \left(\sum t_ig_i, g\right) \mapsto \sum t_ig_ig$$

$$EG \times G \to EG, \left(\sum t_ig_i, g\right) \mapsto \sum t_ig_ig$$

Let $U_i = \{p(\sum t_ig_i) | t_i \neq 0\}$, then we would have a equivariant homeomorphism $p^{-1}(U_i) \to U_i \times G, \sum t_ig_i \mapsto (p(\sum t_ig_i), g_i)$ with inverse $U_i \times G \to p^{-1}(U_i), (p(\sum t_ig_i), g) \mapsto \sum t_jg_jg_i^{-1}g$, this is well defined since $(p(\sum t_ig_ih), g) \mapsto \sum t_jg_jhh^{-1}g_i^{-1}g = \sum t_jg_jg_i^{-1}g$ ☐

Definition 0.5.4. A **topological category** $\mathscr{C}$ is a small category where $ob\mathscr{C}$, $mor\mathscr{C}$ are topological spaces and $i : ob\mathscr{C} \to mor\mathscr{C}, c \mapsto 1_c$, $s : mor\mathscr{C} \to ob\mathscr{C}, c \xrightarrow{f} d \mapsto c$, $t : mor\mathscr{C} \to ob\mathscr{C}, c \xrightarrow{f} d \mapsto d$, $\circ : mor\mathscr{C} \times mor\mathscr{C} \to mor\mathscr{C}$ are continuous. A **continuous functor** between topological categories is a functor that are continuous on both objects and morphisms

Nerve of a category

Definition 0.5.5. Define **nerve** $N\mathscr{C}$ on category $\mathscr{C}$ which is also a simplicial set, $N\mathscr{C}([n]) := Hom([n], \mathscr{C})$, the set of all functors from $[n]$ to $\mathscr{C}$, viewing $[n] = 0 \to 1 \to \cdots \to n$ as a category

Definition 0.5.6 (Segal's contruction for classifying space)**.** Define the classifying space of $\mathscr{C}$ to be $B\mathscr{C} := |N\mathscr{C}|$ as in Definition 0.5.5