MATH848T - Scissors congruence, spectral sequences, Hilbert's third problem

Haoran Li

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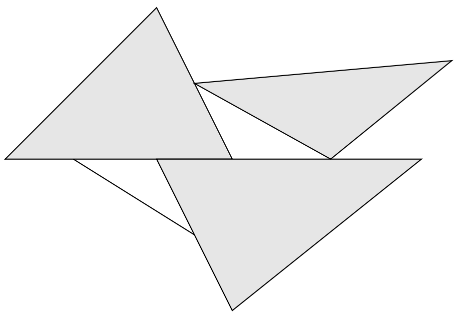
Contents

1 Hilbert's third problem - 1/28/2020

Definition 1.1. A **n-simplex** Δ^n is the convex hull of $n+1$ general positioned points $v_0, \cdots, v_{n+1}$, called its **vertices**, a **face** is the convex hull of some vertices

Definition 1.2. A **polytope** P is such that $P = \Delta_1 \cup \cdots \cup \Delta_m$, where Δ_i's are simplices and the interiors of Δ_i are disjoint, and $\Delta_i \cap \Delta_j$ is precisely a common face
We say P is a **generalized polytope** if without the last condition



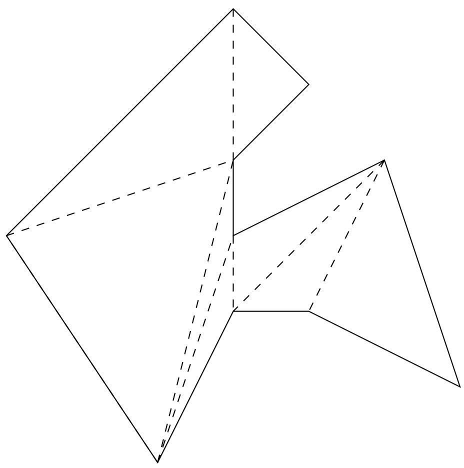
Suppose P_1, P_2 are polyhedra, we write $P = P_1 \sqcup P_2$ if $P = P_1 \cup P_2$ and the interiors of P_1, P_2 are disjoint. Therefore, any polyhedron P must have a finite decomposition into polyhedra $P = P_1 \sqcup \cdots \sqcup P_m$
we say P is **scissors congruent**(s.c.) to Q, denote $P \sim Q$, if there are decompositions $P = P_1 \sqcup \cdots \sqcup P_m$, $Q = Q_1 \sqcup \cdots \sqcup Q_m$ such that $Q_i = g_i P_i$, where $g_i \in \mathrm{Isom}(\mathbb{R}^n)$ is an isometry, we can also define more generally G-scissors congruence $\underset{G}{\sim}$, meaning $g_i \in G \leq \mathrm{Isom}(\mathbb{R}^n)$

Remark. Two dimensional polytopes are called **polygons**, and three dimensional polytopes are called **polyhedrons**

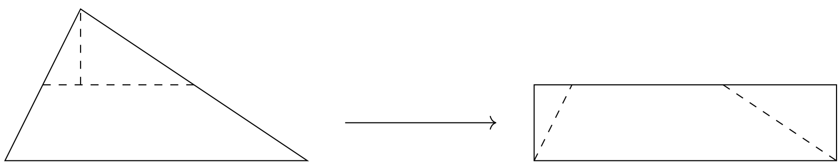
Example 1.3.

Theorem 1.4. *Suppose P, Q are polygons in $\mathbb{R}^2$, $P \sim Q$ iff* $\mathrm{Area}(P) = \mathrm{Area}(Q)$

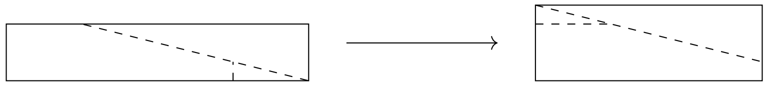
Proof.
Step I: Each polygon is triangulizable



Step II: Each triangle is scissors congruent to a rectangle



Step III: A rectangle with the shorter side between $\frac{1}{2}$ and 1
Step IV:



$\square$

Hilbert's third problem. Is there any two polyhedra of the same volume which is not scissors congruent

Answer. YES!

Definition 1.5. Given a polyhedron P, we can define **Dehn invariant**

$$D(P) = \sum_{e} l(e) \otimes \frac{\theta(e)}{\pi} \in \mathbb{R} \underset{\mathbb{Z}}{\otimes} \mathbb{R}/\mathbb{Z}$$

Where e runs over all the edges of P, $l(e)$ is the length of e, $\theta(e)$ is the dehedral angle

Theorem 1.6. *D is an invariant of scissors congruence*

Proof. $\square$

Lemma 1.7. If $b \notin \mathbb{Q}$, then $a \otimes b \in \mathbb{R} \underset{\mathbb{Z}}{\otimes} \mathbb{R}/\mathbb{Z}$ is not zero

Proof. We can define a $\mathbb{Z}$-bilinear map $\langle a \rangle \times \langle b \rangle \to \mathbb{R}/\mathbb{Z}$, $(na, mb) \mapsto (nmb)$, which induces a homomorphism $\langle a \otimes b \rangle = \langle a \rangle \otimes \langle b \rangle \to \mathbb{R}/\mathbb{Z}$, $nm(a \otimes b) = na \otimes mb \mapsto (nmb)$, this is not a zero map, thus $a \otimes b$ is not zero $\square$

Example 1.8. The Dehn invariant of a cube of side length l is

$$6l \otimes \frac{1}{2} = 0$$

The Dehn invariant of a tetrahedron of side length l is

$$4l \otimes \frac{\theta}{\pi}$$

Where $\cos\theta = \frac{1}{3}$
Let's show that $\frac{\theta}{\pi} \notin \mathbb{Q}$, suppose otherwise, then there is a positive integer k such that $\cos k\theta = 0$, let's show $\cos k\theta = \frac{a_k}{3^k}$, $3 \nmid a_k$ by induction, when $k = 1$, $\cos\theta = \frac{1}{3}, 3 \nmid 1$

$$\begin{aligned}
\cos(k+1)\theta &= 2\cos k\theta \cos\theta - \cos(k-1)\theta \\
&= 2\frac{a_k}{3^k}\frac{1}{3} - \frac{a_{k-1}}{3^{k-1}} \\
&= \frac{2a_k - 9a_{k-1}}{3^{k+1}}
\end{aligned}$$

$3 \mid 9a_{k-1}$ but $3 \nmid 2a_k$
According to Lemma 1.7, we know the Dehn invariant of a tetrahedron isn't zero, thus a cube and a tetrahedron can never be scissors congruent due to Theorem 1.6

Theorem 1.9 (Sydler)**.** *Suppose P, Q are polyhedra in $\mathbb{R}^3$, $P \sim Q$ iff* $\mathrm{Volume}(P) = \mathrm{Volume}(Q)$ *and* $D(P) = D(Q)$

Proof. □

Definition 1.10. Suppose X is a metric space with the notion of a polytope, for example $\mathbb{R}^n$, S^n, $\mathbb{H}^n$

The **scissors congruence group** $\mathcal{P}(X)$ is defined to the free abelian group generated by polytopes $[P]$, modulo relations:

i: $[P] = [P_1] + [P_2]$, for $P = P_1 \sqcup P_2$

ii: $[gP] = [P]$, for $g \in \mathrm{Isom}(\mathbb{R}^n)$

$\mathrm{Isom}(\mathbb{R}^n)$ is the Isometry group of $\mathbb{R}^n$

We can also define more generally $\mathcal{P}(X, G)$, meaning $g \in G \leq \mathrm{Isom}(\mathbb{R}^n)$

P and Q is **stably scissors congruent** if $P \sqcup R \sim Q \sqcup R'$

Proposition 1.11. If P, Q are scissors congruent, $[P] = [Q]$

Proof. By Definition 1.2, $P = P_1 \sqcup \cdots \sqcup P_m$, $Q = Q_1 \sqcup \cdots \sqcup Q_m$ and $Q_i = g_i P_i$, $g_i \in G$, thus $[P] = [P_1] + \cdots + [P_m] = [Q_1] + \cdots + [Q_m] = [Q]$ □

Remark. We can give isometric classes of polytopes a commutative monoid stucture, then $\mathcal{P}(X)$ is the Grothendieck K-group, $[P] = [Q]$ iff $P \sqcup R \sim Q \sqcup R'$, where $R' = gR$

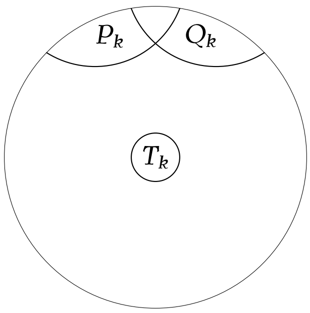
Lemma 1.12. Suppose P, Q are generalized polytopes, and $\mathrm{Volume}(P) > \mathrm{Volume}(Q)$, then there exists a generalized polytope $R \subseteq P$ such that $Q \sim R$

Proof. Consider dividing into small cubes

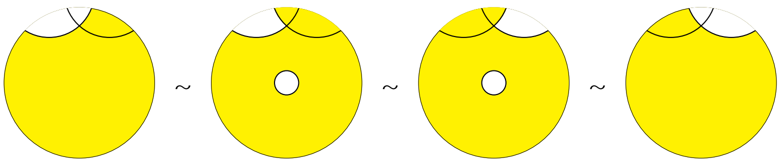
On S^2, we can consider barycentric subdivision of the tetrahedron over and over again □

Theorem 1.13 (Zylev). *Suppose G acts on X transitively, stable scissors congruence implies scissors congruence*

Proof. Let's only consider the special case $X = \mathbb{R}^n$, $G = \mathrm{Isom}(\mathbb{R}^n)$, if $[P] = [Q]$, then $P \sqcup R \sim Q \sqcup R'$, where $R' = gR$, it suffices to prove: if $Y = P \bigcup \bigcup_{i=1}^{k} P_i = Q \bigcup \bigcup_{i=1}^{k} Q_i$ and $P_i \sim Q_i$, then $P \sim Q$, we also assume that $\mathrm{Volume}(Y) > 3\mathrm{Volume}(P_i)$, by then Lemma 1.12 there exists a generalized polytope $T_k \subseteq Y - P_k \cup Q_k$ such that $T_k \sim P_k \cap Q_k$, schematically shown as follow



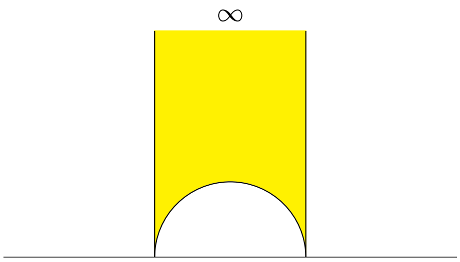
Then we would have



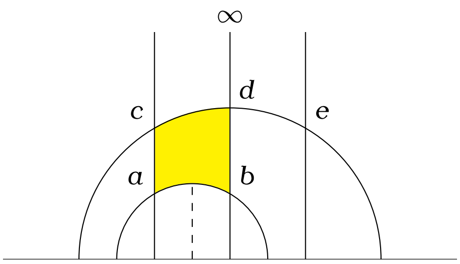
Thus $P \bigcup \bigcup_{i=1}^{k-1} P_i \sim Q \bigcup \bigcup_{i=1}^{k-1} Q_i$, by induction, we get $P \sim Q$ □

2 Scissors congruence group - 1/30/2020

Example 2.1. This is an example where Theorem 1.13 fails
In $\mathbb{H}^2$, we can polygons with ideal vertex(vertex at ∞)



Consider the following situation



Note here angle ∞cd, ∞ed, bac and abd are all $60°$
Denote $P_1 = cd\infty$, $P_2 = de\infty$, $P_3 = abdc$, then $P_1 \sqcup P_2 \sim P_1 \sqcup P_3 \Rightarrow [P_2] = [P_3]$, but $P_2 \not\sim P_3$ since P_3 doesn't have ideal vertex

Lemma 2.2.
(a) If X is a G set, then $\mathbb{Z}[X]$ is a $\mathbb{Z}[G]$-module(G-module)
(b) If M is an right R-module, N is a left R-module, then $M \otimes_R N$ is an abelian group

Proposition 2.3. Treating $\mathbb{Z}$ as a trivial G-module, i.e. $g \cdot 1 = 1$
(a) If M is a G-module, $M \otimes_{\mathbb{Z}[G]} \mathbb{Z} = M/\{gm \sim m\}$
(b) If X is a G-set, $\mathbb{Z}[X] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}[X/G]$
(c) If $H \trianglelefteq G$, $\mathbb{Z}[X] \otimes_{\mathbb{Z}[G]} \mathbb{Z} = \mathbb{Z}[X/H] \otimes_{\mathbb{Z}[G/H]} \mathbb{Z} = \mathbb{Z}[X/G]$
(d) If $H \trianglelefteq G \leq \mathrm{Isom}(X)$, $\mathcal{P}(X, G) = \mathcal{P}(X, H) \otimes_{\mathbb{Z}[G/H]} \mathbb{Z}$

Proof.
(a) Consider $M \to M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, $m \mapsto m \otimes 1$, since $(gm) \otimes 1 = g(m \otimes 1) = m \otimes (g \cdot 1) = m \otimes 1$, this induce $M/\{gm \sim m\} \to M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, on the other hand, $M/\{gm \sim m\}$ satisfies the universal property
(b) $\mathbb{Z}[X]/\{gx \sim x\} \cong \mathbb{Z}[X/G]$
(c) $\mathbb{Z}[X/H]/\{\overline{g}x \sim x\} \cong \mathbb{Z}[X/G]$
(d) Let S be the set of simplices in X, then $\mathcal{P}(X, G) = \mathbb{Z}[S/G]$ is a G-module $\square$

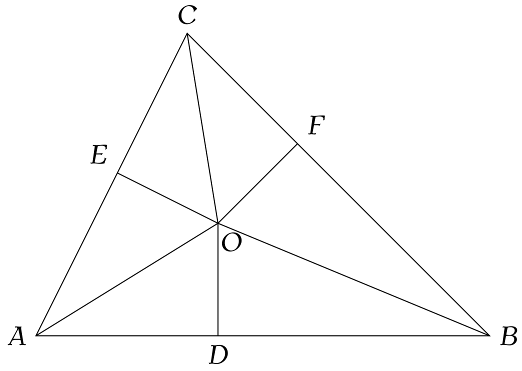
Example 2.4. $H = T \trianglelefteq G = \mathrm{Isom}^+(\mathbb{R}^n)$ is the translation group, $G/H = SO(n)$, $\mathcal{P}(\mathbb{R}^n, SO(n)) = \mathcal{P}(\mathbb{R}^n, T) \otimes_{\mathbb{Z}[SO(n)]} \mathbb{Z}$, $\mathcal{P}(X) = \mathcal{P}(X, \{1\}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$

Theorem 2.5.
(a) $\mathcal{P}(\mathbb{R}^1) \cong \mathbb{R}$
(b) $\mathcal{P}(S^1) \cong \mathbb{R}$
(c) $\mathcal{P}(\mathbb{H}^1) \cong \mathbb{R}$
(d) $\mathcal{P}(\mathbb{R}^2) \cong \mathbb{R}$

Proof.
(a) Consider homomorphism $\phi : \mathscr{P}(\mathbb{R}^1) \to \mathbb{R}$, $[I] \mapsto |I|$, sending an interval to its length, this is obviously surjective and injective, thus an isomorphism
(b) Consider homomorphism $\phi : \mathscr{P}(S^1) \to \mathbb{R}$, $[\theta] \mapsto |\theta|$, sending an arc(angle) to its length, this is obviously surjective and injective, thus an isomorphism
(c) $\mathbb{R}^1 \xrightarrow{\exp} \mathbb{H}^1 = \{y|y > 0\}$ is an isomorphism
(d) Consider homomorphism $\phi : \mathscr{P}(\mathbb{R}^2) \to \mathbb{R}$, $[P] \mapsto |P|$, sending a polygon to its area, according to Theorem 1.4, this is injective, this is clearly surjective, thus an isomorphism $\square$

Lemma 2.6. $\mathscr{P}(X)$ is 2 divisible

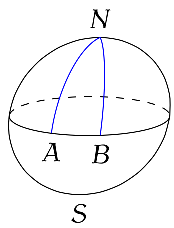
Proof. Consider the following decomposition about "inscribed sphere center"



Denote $P = ABC$, $P_1 = AOD$, $Q_1 = AOE$, $P_2 = BOD$, $Q_2 = BOF$, $P_3 = COE$, $Q_3 = COF$, then we have $[P] = [P_1] + [P_2] + [P_3] + [Q_1] + [Q_2] + [Q_3] = 2([P_1] + [P_2] + [P_3])$, hence $\mathscr{P}(S^2)$ is 2 divisible, i.e. $\mathscr{P}(S^2) = 2\mathscr{P}(S^2)$ $\square$

Theorem 2.7. $\mathscr{P}(S^2) \cong \mathbb{R}$

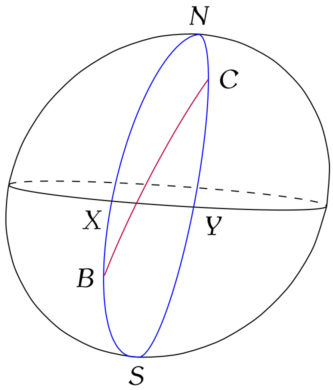
Proof. Consider homomorphism $\phi : \mathscr{P}(S^1) \to \mathbb{R}$, $[\theta] \mapsto |\theta|$, $\psi : \mathscr{P}(S^2) \to \mathbb{R}$, $[P] \mapsto |P|$ and $\Sigma : \mathscr{P}(S^1) \to \mathscr{P}(S^2)$ defined as follow: suppose $S^1 \hookrightarrow S^2$ as equator, N, S are the north and south poles, Σ maps arc $AB \subseteq S^1$ to ABN which is clearly injective



Then the following diagram commutes

$$\begin{tikzcd}
\mathscr{P}(S^1) \arrow[r, "\Sigma"] \arrow[dr, "\phi"'] & \mathscr{P}(S^2) \arrow[d, "\psi"] \\
& \mathbb{R}
\end{tikzcd}$$

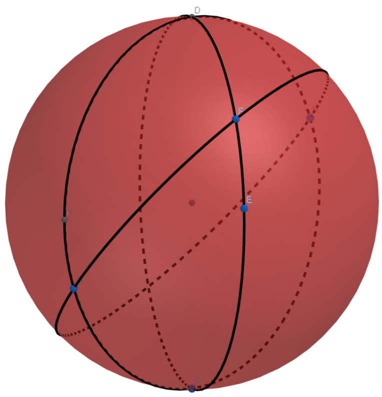
According to the following picture, $[NBC] + [SBC] = [XYN] + [XYS] = 2[XYN] \in \Sigma\mathscr{P}(S^1)$



Denote $\overline{A}$ to be the antipodal point of A and $G = \mathscr{P}(S^2)/\Sigma\mathscr{P}(S^1)$, then $[ABC] = -[\overline{A}BC]$ in G, Then

$$\begin{aligned}
0 &= ([ABC] + [\overline{A}BC]) + ([\overline{AB}C] + [\overline{ABC}]) \\
&= [ABC] + ([\overline{A}BC] + [\overline{AB}C]) + [\overline{ABC}] \\
&= [ABC] + [\overline{ABC}] \\
&= 2[ABC]
\end{aligned}$$

In G, thus every element in G is of 2 torsion, i.e. $2G = 0$, by Lemma 2.6, $G = 2G = 0$, Σ is an isomorphism



$\square$

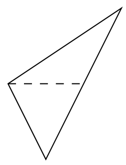
3 More elementary calculations on scissors congruence groups - 2/4/2020

Theorem 3.1. *$\mathcal{P}(\mathbb{H}^2) \to \mathbb{R}$, $[P] \mapsto \mathrm{Area}(P)$ is an isomorphism*

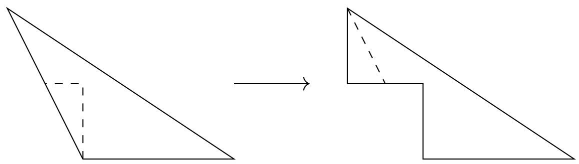
Proof. $\square$

Theorem 3.2. *Suppose $\varphi \in \mathrm{Isom}(\mathbb{R}^2)$ is rotation by $180°$, $T < \mathrm{Isom}(\mathbb{R}^2)$ is the translation group, $\mathcal{P}(\mathbb{R}^2, \langle T, \varphi\rangle) \cong \mathbb{R}$*

Proof. In Step II of Theorem 1.4, first we can divide any triangle into triangles with one horizontal side



The following triangle can be turn into the case in Step II of Theorem 1.4



Also, any two congruent rectangle on $\mathbb{R}^2$ are scissors congruent just by cutting and translating $\square$

Remark. There is a complete set of invariants needed for $\mathcal{P}(\mathbb{R}^2, T)$ called Hadwiger invariants

Theorem 3.3. *$\mathcal{P}(X) = \mathcal{P}(X, \mathrm{Isom}^+(X))$, i.e. only orientation preserving isometries are needed*

Proof. By Proposition 2.3, $\mathcal{P}(X) = \mathcal{P}(X, \mathrm{Isom}^+(X)) \otimes_{\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{Z}$ since $\mathrm{Isom}(X)/\mathrm{Isom}^+(X) \cong \mathbb{Z}/2\mathbb{Z}$ which is generated by some reflection r, we just need to prove $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ acts trivially on $\mathcal{P}(X, \mathrm{Isom}^+(X))$ also due to Proposition 2.3
As in Lemma 2.6, suppose r_1, r_2, r_3 are the reflections over AO, BO, CO, and there are $s_i, i = 1, 2, 3$ such that $r_i = s_i g$, then

$$\begin{aligned}
[gP] &= [gP_1] + [gQ_1] + [gP_2] + [gQ_2] + [gP_3] + [gQ_3] \\
&= [s_1 g P_1] + [s_1 g Q_1] + [s_2 g P_2] + [s_2 g Q_2] + [s_3 g P_3] + [s_3 g Q_3] \\
&= [r_1 P_1] + [r_1 Q_1] + [r_2 P_2] + [r_2 Q_2] + [r_3 P_3] + [r_3 Q_3] \\
&= [Q_1] + [P_1] + [Q_2] + [P_2] + [Q_3] + [P_3] \\
&= [P]
\end{aligned}$$

$\square$

Definition 3.4. Suppose X is a set, we can define the **tuple chain complex** $C_*(X)$, where $C_n(X)$ to be the free abelian group generated by $n+1$ tuples $(x_0, \cdots, x_n)$, define the boundary map $\partial : C_n(X) \to C_{n-1}(X)$, $(x_0, \cdots, x_n) \mapsto (-1)^n (x_0, \cdots, \widehat{x}_i, \cdots, x_n)$

Lemma 3.5. $H_n(C_*(X)) = 0$ for $n > 0$, i.e. tuple chain complex is acyclic

Proof. Fix $b \in X$, consider $P : C_n(X) \to C_{n+1}(X)$, $(x_0, \cdots, x_n) \mapsto (b, x_0, \cdots, x_n)$, then $\partial P + P\partial = 1$ $\square$

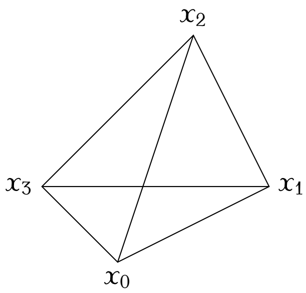
Definition 3.6. If $X = \mathbb{R}^n, S^n$ or $\mathbb{H}^n$, we can defined the convex hull $|(x_0, \cdots, x_n)|$ to be the **carrier** of $(x_0, \cdots, x_n)$, let $C^p_*(X)$ be those tuples such that their carriers lie in a dimension $\le p$ subspace, note that $C^k_n(X) = C_n(X)$ for $k \ge n$

Remark. Note that a 0-dimensional hyperplane in S^n are antipodal points

Theorem 3.7. *Suppose* $X = \mathbb{R}^n, \mathbb{H}^n$, *then* $C_*(X) = C^n_*(X)$, *we have an isomorphism* $H_n(C_*(X)/C^{n-1}_*(X)) \cong \mathscr{P}(X, \{1\})$

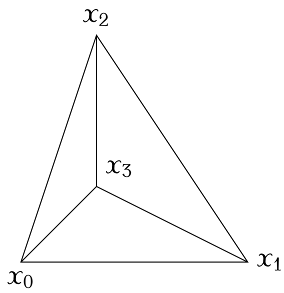
Proof. Define the homomorphism $\varphi : C_n(X) \to \mathscr{P}(X)$, $(x_0, \cdots, x_n) \mapsto \varepsilon[|(x_0, \cdots, x_n)|]$, $\varepsilon = 0$ if $|(x_0, \cdots, x_n)|$ is degenerate, otherwise $\varepsilon = 1$ or $\varepsilon = -1$ depending on if the orientation of the carrier matches with $n+1$ tuple $(x_0, \cdots, x_n)$, so by definition, φ is really a map $C_n(X)/C^{n-1}_n(X) \to \mathscr{P}(X)$, $C_n(X)/C^{n-1}_n(X) = Z_n(C_*(X)/C^{n-1}_*(X))$ since $\partial C_n(X) \subsetneq C_{n-1}(X) = C^{n-1}_{n-1}(X)$, also $\varphi(B_n(C_*(X)/C^{n-1}_*(X))) = \varphi(\partial(C_{n+1}(X))) = \varphi(\partial(C^n_{n+1}(X))) = 0$, such as

$$\begin{aligned}
\varphi(\partial(x_0, x_1, x_2, x_3)) &= \varphi((x_1, x_2, x_3)) - \varphi((x_0, x_2, x_3)) + \varphi((x_0, x_1, x_3)) - \varphi((x_0, x_1, x_2)) \\
&= [|(x_1, x_2, x_3)|] - [|(x_0, x_2, x_3)|] + [|(x_0, x_1, x_3)|] - [|(x_0, x_1, x_2)|] \\
&= ([|(x_1, x_2, x_3)|] + [|(x_0, x_1, x_3)|]) - ([|(x_0, x_1, x_2)|] - [|(x_0, x_2, x_3)|]) \\
&= [|(x_0, x_1, x_2, x_3)|] - [|(x_0, x_1, x_2, x_3)|] \\
&= 0
\end{aligned}$$



Or

$$\begin{aligned}
\varphi(\partial(x_0, x_1, x_2, x_3)) &= \varphi((x_1, x_2, x_3)) - \varphi((x_0, x_2, x_3)) + \varphi((x_0, x_1, x_3)) - \varphi((x_0, x_1, x_2)) \\
&= [|(x_1, x_2, x_3)|] + [|(x_0, x_2, x_3)|] + [|(x_0, x_1, x_3)|] - [|(x_0, x_1, x_2)|] \\
&= ([|(x_1, x_2, x_3)|] + [|(x_0, x_2, x_3)|] + [|(x_0, x_1, x_3)|]) - [|(x_0, x_1, x_2)|] \\
&= [|(x_0, x_1, x_2)|] - [|(x_0, x_1, x_2)|] \\
&= 0
\end{aligned}$$



Therefore we get a well-defined map $\varphi : H_n(C_*(X)/C^{n-1}_*(X)) \to \mathscr{P}(X, \{1\})$
We can also define map $\psi : \mathscr{P}(X, \{1\}) \to H_n(C_*(X)/C^{n-1}_*(X))$, $[P] \mapsto (x_0, \cdots, x_n)$, where x_i are the vertices of P that matches up to the orientation

$$\partial(x_0, x_1, x_0, x_2) = (x_1, x_0, x_2) - (x_0, x_0, x_2) + (x_0, x_1, x_2) - (x_0, x_1, x_0)$$

Is equivalent to

$$(x_0, x_1, x_2) = (x_1, x_0, x_2) + (x_0, x_0, x_2) + (x_0, x_1, x_0) + \partial(x_0, x_1, x_0, x_2)$$

Where $(x_0, x_0, x_2) + (x_0, x_1, x_0) + \partial(x_0, x_1, x_0, x_2) = 0$ in $H_n(C_*(X)/C_*^{n-1}(X))$
Thus ψ is well-defined, clearly φ, ψ are inverses to each other, hence $H_n(C_*(X)/C_*^{n-1}(X)) \cong \mathcal{P}(X, \{1\})$ are isomorphic $\square$

4 Spectral sequence - 2/6/2020

Definition 4.1. A filtered chain complex C_* is a chain complex with a filtration

$$\cdots \to F_{p-1}C_* \to F_pC_* \to F_{p+1}C_* \to \cdots \to C_*$$

Where F_pC_* are chain complexes, and the maps are chain maps, more concretely

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \vdots \\
\cdots \arrow[r] & F_{p-1}C_{i+1} \arrow[r] \arrow[u] & F_pC_{i+1} \arrow[r] \arrow[u] & F_{p+1}C_{i+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_{i+1} \arrow[u] \\
\cdots \arrow[r] & F_{p-1}C_i \arrow[r] \arrow[u] & F_pC_i \arrow[r] \arrow[u] & F_{p+1}C_i \arrow[r] \arrow[u] & \cdots \arrow[r] & C_i \arrow[u] \\
\cdots \arrow[r] & F_{p-1}C_{i-1} \arrow[r] \arrow[u] & F_pC_{i-1} \arrow[r] \arrow[u] & F_{p+1}C_{i-1} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_{i-1} \arrow[u] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u]
\end{tikzcd}$$

Example 4.2. Tuple chain complex $C_*(X)$ in Definition 3.4 is a filtered chain complex with filtration

$$\cdots \to C_*^{p-1}(X) \to C_*^p(X) \to C_*^{p+1}(X) \to \cdots \to C_*(X)$$

If X is a topological space with a filtrations of subspaces(CW complex with skeletons is a special case)

$$\cdots \subseteq X^{p-1} \subseteq X^p \subseteq X^{p+1} \subseteq X$$

Then the singular chain complex $C_*^{\mathrm{sing}}(X)$ is a filtered chain complex with filtration

$$\cdots \to C_*(X^{p-1}) \to C_*(X^p) \to C_*(X^{p+1}) \to \cdots \to C_*(X)$$

Definition 4.3. Suppose R is a commutative ring with identity
A **graded module** is a module with grading $A = \bigoplus_n A_n$, $n \in I$ is a totally ordered set, mostly we just consider $\mathbb{Z}$
A **filtered module** is a module with a filtration

$$\cdots \hookrightarrow F_{p-1}A \hookrightarrow F_pA \hookrightarrow F_{p+1}A \hookrightarrow \cdots \hookrightarrow A$$

We can define the graded module associated to the filtration $grA = \bigoplus_p F_{p+1}A/F_pA$
A **filtered graded module** is a graded module $A = \bigoplus_n A_n$ with a filtration of graded modules

$$\cdots \hookrightarrow F_{p-1}A \hookrightarrow F_pA \hookrightarrow F_{p+1}A \hookrightarrow \cdots \hookrightarrow A$$

Such that filtration preserving grading, i.e. $F_pA \subseteq \bigoplus_{n \le p} A_n$, then we have

$$F_pA = F_pA \bigcap \bigoplus_{n\le p} A_n = \bigoplus_{n \le p} F_pA \cap A_n$$

Define $(F_pA)_n$ or $F_pA_n := F_pA \cap A_n$, then

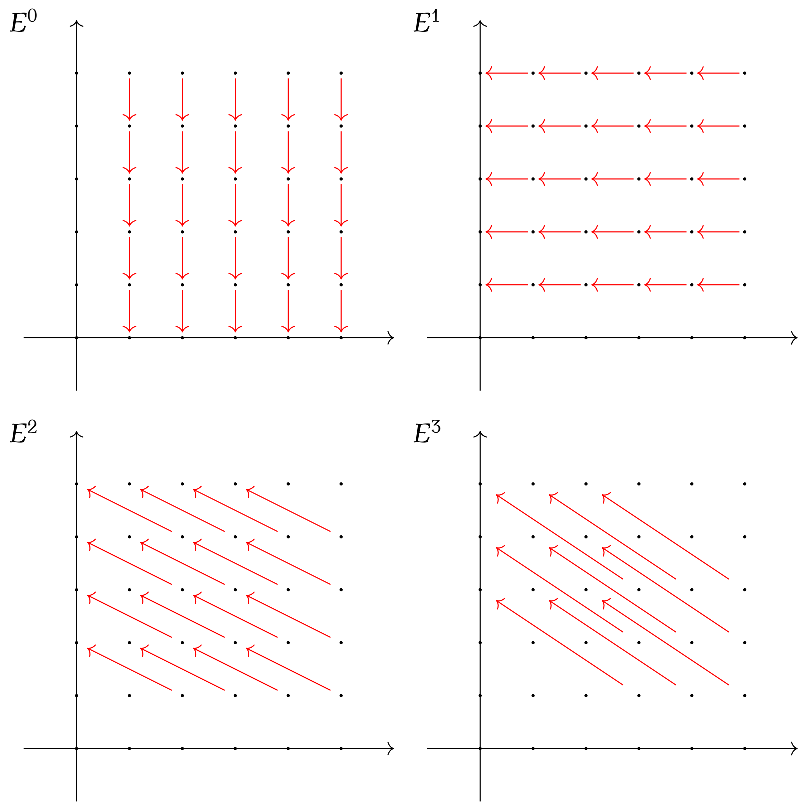
$$\cdots \hookrightarrow F_{p-1}A_n \hookrightarrow F_pA_n \hookrightarrow F_{p+1}A_n \hookrightarrow \cdots \hookrightarrow A_n$$

Is a filtration of A_n, and $(F_pA)_n$ are the direct summands of F_pA, we also have a grid

$$\begin{tikzcd}
\cdots \arrow[r, hook] & F_{p-1}A \arrow[r, hook] & F_pA \arrow[r, hook] & F_{p+1}A \arrow[r, hook] & \cdots \arrow[r, hook] & A \\
& \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & & \vdots \arrow[u, hook] \\
\cdots \arrow[r, hook] & F_{p-1}A_{i+1} \arrow[r, hook] \arrow[u, hook] & F_pA_{i+1} \arrow[r, hook] \arrow[u, hook] & F_{p+1}A_{i+1} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & A_{i+1} \arrow[u, hook] \\
\cdots \arrow[r, hook] & F_{p-1}A_i \arrow[r, hook] \arrow[u, hook] & F_pA_i \arrow[r, hook] \arrow[u, hook] & F_{p+1}A_i \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & A_i \arrow[u, hook] \\
\cdots \arrow[r, hook] & F_{p-1}A_{i-1} \arrow[r, hook] \arrow[u, hook] & F_pA_{i-1} \arrow[r, hook] \arrow[u, hook] & F_{p+1}A_{i-1} \arrow[r, hook] \arrow[u, hook] & \cdots \arrow[r, hook] & A_{i-1} \arrow[u, hook] \\
& \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & \vdots \arrow[u, hook] & & \vdots \arrow[u, hook]
\end{tikzcd}$$

Example 4.4. Let C_* be a filtered chain complex, the homology $H_*C = \bigoplus_p H_pC$ is a graded graded module with filtration with $F_pH_nC = \mathrm{im}(H_n(F_pC) \to H_*C)$

Definition 4.5. A **spectral sequence** is a sequence of bigraded module $\{E^r_{*,*}\}$ together with differentials $\partial^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ such that $\partial^r \circ \partial^r = 0$ and $E^{r+1} \cong \ker \partial^r / \mathrm{im}\partial^r =: Z^r/B^r$, E^r are called the r-th page

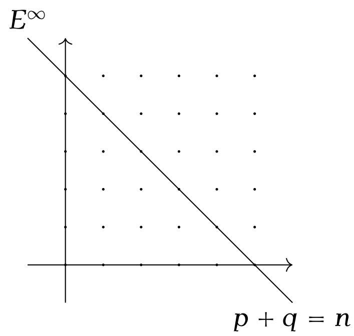


Since Z^{i+1}, B^{i+1} are submodules of Z^i/B^i, we have

$$\cdots \subseteq B^i \subseteq B^{i+1} \subseteq \cdots \subseteq Z^{i+1} \subseteq Z^i \subseteq \cdots$$

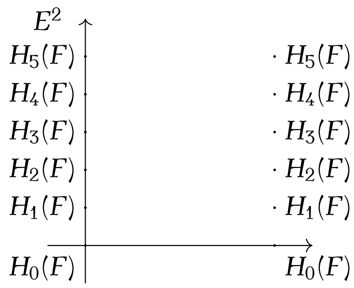
Define $B^\infty = \bigcup_r B^r$, $Z^\infty = \bigcap_r Z^r$, $E^\infty = Z^\infty/B^\infty$

Definition 4.6. A spectral sequence converges to a filtered graded module A if $E^{\infty}_{p,q} = F_pA_{p+q}/F_{p-1}A_{p+q}$, or equivalently $\bigoplus\limits_{p+q=n} E^{n}_{p,q} = grA_n$, we write as $E^1_{p,q} \Rightarrow A_{p+q}$

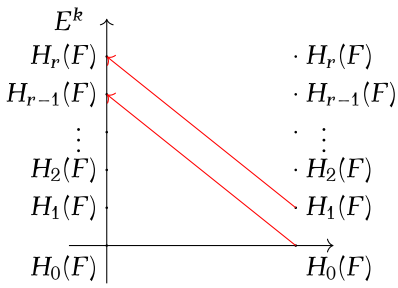


Theorem 4.7 (Serre spectral sequence)**.** *Let $F \to E \to B$ be a filtration, suppose $\pi_1(B)$ acts on $H_*(F)$ trivially, then there is a spectral sequence with $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E)$, meaning converges to $H_*(E)$*

Example 4.8 (Wang sequence)**.** Suppose $F \to E \to B$ be a filtration, with $B = S^k$, then $E^2_{p,q} = H_p(B; H_q(F)) = H_p(B) \otimes H_q(F)$ by universal coefficient theorem, since $H_p(B) = \begin{cases} \mathbb{Z}, & p = 0, k \\ 0, & \text{Otherwise} \end{cases}$, we have the second page



The only non trivial differential appears on page k, thus we have $E^2 = E^3 = \cdots = E^k$, $E^{k+1} = E^{k+2} = \cdots = E^{\infty}$, on page k, we have $\partial^k : E^k_{k,r} \to E^k_{0,r+k-1}$, thus $\ker \partial^k = E^{k+1}_{k,r} = E^{\infty}_{k,r}$, $\text{coker}\partial^k = E^{k+1}_{0,r+k-1} = E^{\infty}_{k0,r+k-1}$



Therefore we get an exact sequence

$$0 \to E^{\infty}_{k,r} \to H_r(F) \to H_{r+k-1}(F) \to E^{\infty}_{0,r+k-1} \to 0$$

Replace r with $n-k$, we get

$$0 \to E^{\infty}_{k,n-k} \to H_{n-k}(F) \to H_{n-1}(F) \to E^{\infty}_{0,n-1} \to 0$$

Since $H_n(E) = E^\infty_{k,n-k} \oplus E^\infty_{0,n}$, there is also an exact sequence

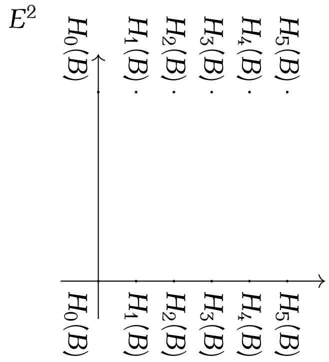
$$0 \to E^\infty_{k,n-k} \to H_n(E) \to E^\infty_{0,n} \to 0$$

Put them together, we get the **Wang sequence**

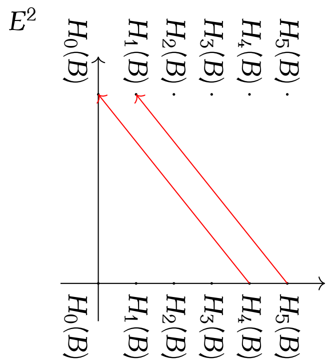
$$\cdots \to H_n(E) \to H_{n-k}(F) \to H_{n-1}(F) \to H_{n-1}(E) \to \cdots$$

5 Double chain complex - 2/11/2020

Example 5.1 (Gysin sequence). Suppose $F \to E \to B$ be a filtration, with $F = S^k$, then $E^2_{p,q} = H_p(B; H_q(F)) = H_p(B) \otimes H_q(F)$ by universal coefficient theorem, since $H_q(F) = \begin{cases} \mathbb{Z}, & q = 0, k \\ 0, & \text{Otherwise} \end{cases}$, we have the second page



The only non trivial differential appears on page $k+1$, thus we have $E^2 = E^3 = \cdots = E^{k+1}$, $E^{k+2} = E^{k+3} = \cdots = E^\infty$, on page $k+1$, we have $\partial^{k+1} : E^{k+1}_{r,0} \to E^{k+1}_{r-k-1,k}$, thus $\ker \partial^{k+1} = E^{k+2}_{r,0} = E^\infty_{r,0}$, $\operatorname{coker} \partial^{k+1} = E^{k+2}_{r-k-1,k} = E^\infty_{r-k-1,k}$



Therefore we get an exact sequence

$$0 \to E^\infty_{r,0} \to H_r(B) \to H_{r-k-1}(B) \to E^\infty_{r-k-1,k} \to 0$$

Since $H_n(E) = E^\infty_{n-k,k} \oplus E^\infty_{n,0}$, there is also an exact sequence

$$0 \to E^\infty_{n-k,k} \to H_n(E) \to E^\infty_{n,0} \to 0$$

Put them together, we get the **Gysin sequence**

$$\cdots \to H_n(E) \to H_n(B) \to H_{n-k-1}(B) \to H_{n-1}(E) \to \cdots$$

Theorem 5.2. *Suppose C_* is filtered chain complex, then there is a spectral sequence $E^1_{p,q} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}(C)$ with differential $\partial^1 : H_{p+q}(F_pC/F_{p-1}C) \to H_{p+q-1}(F_{p-1}C/F_{p-2}C)$ induced by the composition of boundary map and quotient map $H_{p+q}(F_pC/F_{p-1}C) \to H_{p+q-1}(F_{p-1}C) \to H_{p+q-1}(F_{p-1}C/F_{p-2}C)$*

Remark. Suppose the filtration is in the first quadrant, we can view $Z^r_{p,q}$ as

$$\{c \in F_pC_{p+q} | \partial c \in F_{p-r}C_{p+q-1}\} + F_{p-1}C_{p+q}$$

$B_{p,q}^r$ as
$$\partial F_{p+r-1}C_{p+q+1} + F_{p-1}C_{p+q}$$
$E_{p,q}^r$ as
$$\frac{\{c \in F_p C_{p+q} | \partial c \in F_{p-r}C_{p+q-1}\} + F_{p-1}C_{p+q}}{\partial F_{p+r-1}C_{p+q+1} + F_{p-1}C_{p+q}}$$
Then $Z_{p,q}^0 = F_p C_{p,q}$, $B_{p,q}^0 = F_{p-1}C_{p+q}$, $E_{p,q}^0 = F_p C_{p+q}/F_{p-1}C_{p+q}$, $E_{p,q}^1 = H_{p+q}(F_p C_*/F_{p-1}C_*)$
$Z_{p,q}^\infty = \{c \in F_p C_{p+q} | \partial c = 0\} + F_{p-1}C_{p+q}$, $B_{p,q}^\infty = \partial C_{p+q+1} + F_{p-1}C_{p+q}$, $E_{p,q}^\infty = \dfrac{Z_{p+q}(F_p C_*) + F_{p-1}C_{p+q}}{B_{p+q}(C_*) + F_{p-1}C_{p+q}} = \dfrac{F_p H_{p+q}(C)}{F_{p-1}H_{p+q}(C)} = \dfrac{\mathrm{im}(H_{p+q}(F_p C) \to H_{p+q}(C))}{\mathrm{im}(H_{p+q}(F_{p-1}C) \to H_{p+q}(C))}$

Example 5.3. Let X be a CW complex, consider $C_* = C_*^{\text{sing}}(X)$, $F_p C_* = C_*^{\text{sing}}(X^p)$, $E_{p,q}^1 = H_{p+q}(X^p, X^{p-1}) = \begin{cases} C_p^{\text{cell}}(X), & q = 0 \\ 0, & \text{Otherwise} \end{cases}$, ∂^1 is just the cellular boundary map, thus $E_{p,0}^2 = E_{p,0}^\infty = H_p^{\text{cell}}(X) \cong H_p^{\text{sing}}(X)$

Example 5.4. Suppose $A \subseteq X$ is a subspace, consider $0 \subseteq C_*(A) \subseteq C_*(X)$, $E_{1,q}^1 = H_{q+1}(X, A)$, $E_{0,q}^1 = H_q(A)$ with $\partial^1 : H_{q+1}(X, A) \to H_q(A)$ induced by the boundary map, then we have exact sequences
$$0 \to E_{1,q}^\infty \to H_{q+1}(X, A) \to H_q(A) \to E_{0,q}^\infty \to 0$$
$$0 \to E_{0,n}^\infty \to H_n(X) \to E_{1,n-1}^\infty \to 0$$
Which give us the long exact sequence for (X, A)

Definition 5.5. A **double complex** $C_{*,*}$ is $\mathbb{Z}$-bigraded with two differentials $\partial' : C_{p,q} \to C_{p-1,q}$, $\partial' : C_{p,q} \to C_{p,q-1}$ such that $(\partial')^2 = (\partial'')^2 = 0$ and $\partial'\partial'' + \partial''\partial' = 0(\partial', \partial''$ anticommutes)
Define the **total chain complex** $Tot_n := \bigoplus_{p+q=n} C_{p,q}$, $\partial = \partial' + \partial''$

Example 5.6. Suppose C_*, D_* are chain complexes, $C_* \otimes D_*$ is defined to be the total complex of the double complex $C_{p,q} := C_p \otimes D_q$, $\partial' := \partial^C \otimes 1$, $\partial'' := (-1)^p 1 \otimes \partial^D$

6 Total chain complex - 2/13/2020

Theorem 6.1 (Algebraic Künneth formula). *If C_*, D_* are chain complexes, then we have an exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor(H_p(C), H_q(D) \to 0$$

Theorem 6.2 (Topological Künneth formula). *Since $C_*(X \times Y) \cong C_*(X) \otimes C_*(Y)$, apply Theorem 6.1, we get an exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} Tor(H_p(X), H_q(Y) \to 0$$

Example 6.3. Consider $C_{p,q} := C_q^{\text{sing}}(\overbrace{X \times \cdots \times X}^{p})$, $\partial' := \sum(-1)^i \varepsilon_i$, where $\varepsilon_i : C_q^{\text{sing}}(X^p) \to C_q^{\text{sing}}(X^{p-1})$ by removing the i-th coordinate, $\partial'' := (-1)^p \partial$

Definition 6.4. Suppose $C_{*,*}$ is a double complex, there are two natural filtrations on Tot_*, $'F_p Tot_n = \bigoplus_{r \le p} C_{r,n-r}$, $''F_p Tot_n = \bigoplus_{r \ge p} C_{n-r,r}$, hence we have two corresponding spectral sequences
For $'F_p Tot_*$, $'E^1_{p,q} = H_{p+q}('F_p Tot_* / 'F_{p-1} Tot_*) = H_q(C_{p,*})$, $\partial^1 = \partial'$, $'E^2_{p,q} = H_p(H_q(C_{p,*}))$
For $''F_p Tot_*$, $''E^1_{p,q} = H_{p+q}('F_p Tot_* / ''F_{p-1} Tot_*) = H_p(C_{*,q})$, $\partial^1 = \partial''$, $''E^2_{p,q} = H_p(H_q(C_{p,*}))$
However, unlike ∂' has bidegree $(-1, 0)$ which matches up with that ∂^1 normally has bidegree $(-1, 0)$, ∂'' has bidegree $(0, -1)$, there are two ways to work around this
(1) Flip the double complex $C'_{p,q} = C_{q,p}$
(2) Flip the differential

7 Applications of double complex - 2/18/2020

Example 7.1. Define $A_{p,q} = C_{p+q}(V^q) \otimes_{\mathbb{Z}[GL(q,V)]} \mathbb{Z}$, V is a vector space, and C_* is the tuple complex, $\partial' : A_{p,q} \to A_{p-1,q}$ is $\partial \otimes 1$, $\partial'' : A_{p,q} \to A_{p,q-1}$, $(v_0, \cdots, v_{p+q}) \mapsto \sum (-1)^i (\overline{v_0}, \cdots, \widehat{\overline{v_i}}, \cdots, \overline{v_{p+q}})$ where $\overline{v}$ is the projection of v in $V/\langle v \rangle$

Theorem 7.2. *Recall* $Tor_i(M,N) = H_i(F_* \otimes N)$ *where* F_* *is a free resolution of* M. $Tor_i(M,N) = Tor_i(N,M)$

Proof. Let F_*, G_* be free resolutions of M, N, define $C_{*,*} = F_* \otimes G_*$, then

$$'E^1_{p,q} = H_q(F_p \otimes G_*) \cong F_p \otimes H_q(G_*) = \begin{cases} F_p \otimes N, & q = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$''E^1_{p,q} = H_p(F_* \otimes G_q) \cong H_p(F_*) \otimes G_q = \begin{cases} M \otimes G_q, & p = 0 \\ 0, & \text{otherwise} \end{cases}$$

$'E^2_{p,0} = Tor_p(M,N)$, $''E^2_{0,q} = Tor_q(N,M)$, since both spectral sequences converge to $H_*(Tot)$, $Tor_i(M,N) \cong grH_i(Tot) \cong Tor_i(N,M)$ □

Theorem 7.3 (Comparison theorem). *Suppose* C_*, D_* *are filtered chain complexes,* $f_* : C_* \to D_*$ *is a chain map preserving the filtration, i.e.* $f_* : F_pC_* \to F_pD_*$*, then it induces a map on the first page* $E^1_{p,q}(C_*) = H_{p+q}(F_pC_*/F_{p-1}C_*) \to H_{p+q}(F_pD_*/F_{p-1}D_*) \cong E^1_{p,q}(D_*)$*, which then induce maps on every page, including the infinity page*
If the induced map $f_* : E^r(C_*) \to E^r(D_*)$ *is an isomorphism on some page, then* $f : H_*(C) \to H_*(D)$ *is also an isomorphism*

Corollary 7.4. Suppose $f : C_{*,*} \to D_{*,*}$ is a map between double complexes, and induce isomorphism $f_* :' E^1(C_*) \to' E^1(D_*)$ or $f_* :'' E^1(C_*) \to'' E^1(D_*)$, then $f : H_*(Tot(C_*)) \to H_*(Tot(D_*))$ is also an isomorphism

Definition 7.5. Suppose M is a Riemannian manifold, a δ neighborhood of p is the image of $B_\delta(0) \subseteq T_pM \xrightarrow{\exp} M$, denoted by $U_p = \exp(B_\delta(0))$ where any two points in U_p can be connected by a unique geodesic

Remark. A prototypical example is the sphere

Theorem 7.6. *Let* $C^\delta_*(M)$ *denote the subcomplex of the tuple complex where each tuple lie in some* δ *neighborhood, the inclusion* $C^\delta_*(M) \hookrightarrow C^{\text{sing}}_*(M)$ *induces an isomorphism* $H^\delta_*(M) \to H^{\text{sing}}_*(M)$

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of M consists of δ neighborhoods, then $U_{i_0} \cap \cdots \cap U_{i_p}$ are contractible(pick a point x and connect with other points by the unique geodesics, then look at the exponential map at x), define double complexes

$$C^\delta_{p,q} = \bigoplus_{(i_0,\cdots,i_p)} C^\delta_q(U_{i_0} \cap \cdots \cap U_{i_p}), \quad C^{\text{sing}}_{p,q} = \bigoplus_{(i_0,\cdots,i_p)} C^{\text{sing}}_q(U_{i_0} \cap \cdots \cap U_{i_p})$$

With $\partial' = \sum_{j=0}^p (-1)^j \varepsilon_{i_j}$, where $\varepsilon_i : C_n(U_{i_0} \cap \cdots \cap U_{i_p}) \hookrightarrow C_n(U_{i_0} \cap \cdots \cap \widehat{U_{i_j}} \cap \cdots \cap U_{i_p})$ is inclusion, and $\partial'' = (-1)^p \partial$. Then the inclusion $C^\delta_*(M) \hookrightarrow C^{\text{sing}}_*(M)$ induces a map on the double complex
It is easy to show that $\bigoplus C^\delta_q(U_{i_0} \cap U_{i_1}) \to \bigoplus C^\delta_q(U_i) \to C^\delta_q(M) \to 0$ is exact, hence $H_0(C^\delta_{*,q}) = C^\delta_q(M)$. Define $P_p : C^\delta_{p,q} \to C^\delta_{p+1,q}$, for any q geodesic simplex σ, fix U_j containing σ, sending σ in $C^\delta_q(U_{i_0} \cap \cdots \cap U_{i_p})$ to σ in $C^\delta_q(U_j \cap U_{i_0} \cap \cdots \cap U_{i_p})$, then $\partial P + P\partial = 1$, hence $H_p(C^\delta_{*,q}) = 0, p > 0$
Similar for $H_p(C^{\text{sing}}_{*,q})$, then apply Theorem 7.3 □

Example 7.7. We are mostly interested in tuple chain complex of S^n with each tuple lie in a hemisphere, due to Theorem 7.6, $H^{\frac{\pi}{2}}_k(S^n) = H^{\text{sing}}_k(S^n)$ is $\mathbb{Z}$ when $k = 0, n$ and 0 otherwise

8 Cohomological spectral sequence - 2/20/2020

Definition 8.1. A **cohomology spectral sequence** is a bigraded module $\{E_r^{p,q}\}$ with differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r \circ d_r = 0$
Suppose A^* is a filtered graded module with a descending filtration

$$A \supseteq \cdots \supseteq F^{p+1}A^* \supseteq F^p A^* \supseteq F^{p-1}A^* \supseteq \cdots \supseteq \cdots$$

We say $E_1^{p,q} \Rightarrow A^*$ if $E_\infty^{p,q} = \dfrac{F^p A^{p+q}}{F^{p+1}A^{p+q}}$

Remark. The convergence also preserves cup product

Theorem 8.2 (Serre's cohomology spectral sequence)**.** *Suppose $F \to E \to B$ is a fibration, $\pi_1(B)$ acts on $H_*(F)$ trivially, then we have a cohomology spectral sequence with $E_r^{p,q} = H^p(B; H^q(F)) \Rightarrow H^{p+q}(E)$*

Example 8.3. $H^*(SU(n)) = \bigwedge_{\mathbb{Z}}[x_3, x_5, \cdots, x_{2n-1}]$, where $x_i \in H^i(SU(n))$, $\bigwedge_R[a_1, \cdots, a_n]$ is the exterior algebra generated by $a_1, \cdots, a_n$

Proof. Use induction, we already knew $H^*(SU(1)) = H^*(pt) = \mathbb{Z}$ with trivial multiplication, $H^*(SU(2)) = H^*(S^3) = \bigwedge_{\mathbb{Z}}[x_3]$, now suppose we know $H^*(SU(n-1)) = \bigwedge_{\mathbb{Z}}[x_3, x_5, \cdots, x_{2n-3}]$ Consider fibration $SU(n-1) \to SU(n) \to S^{2n-1}$, $U \mapsto Ue_1$, apply Theorem 8.2, $E_2^{p,q} = H^p(S^{2n-1}, H^q(SU(n-1))) \cong Hom_{\mathbb{Z}}(H^p(S^{2n-1}), H^q(SU(n-1)))$, we can show $E_2 = E_\infty$, thus $H^q(SU(n)) = E_2^{0,q} \oplus E_2^{2n-1,q-2n+1} = H^q(SU(n-1)) \oplus H^{q-2n+1}(SU(n-1))$ ∎

9 Exact couple - 2/25/2020

Definition 9.1. Suppose $\mathscr{A}$ is an abelian category, an **exact couple** is (D, E, i, j, k)

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Such that it is exact at each term, define differential $d = jk$, then $d^2 = jkjk = j(kj)k = 0$, we can define the **derived couple**

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Where $D' = i(D)$, $E' = \ker k/\mathrm{im} j$, $i'(a) = i(a)$, $j'(i(a)) = \overline{j(a)}$, $k'(b) = \overline{k(b)}$, then the derived couple is again an exact couple, thus we can carry this process indefinitely, giving the n-th derived couple $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$

Example 9.2. Suppose $\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots$ is a filteration of chain complex $C_\bullet$, exact sequence $0 \to F_{p-1}C_\bullet \to F_pC_\bullet \to (grC_\bullet)_p \to 0$ give a long exact sequence

$$\cdots \to H_n(F_{p-1}C_\bullet) \xrightarrow{i_*} H_n(F_pC_\bullet) \xrightarrow{j_*} H_n(F_pC_\bullet/F_{p-1}C_\bullet) \xrightarrow{k_*} H_{n-1}(F_{p-1}C_\bullet) \to \cdots$$

If we write $D^1_{p,q} := H_{p+q}(F_pC_\bullet)$, $E^1_{pq} := H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet)$, then the long exact sequence become

$$\cdots \to D^1_{p,q} \to D^1_{p+1,q-1} \to E^1_{p,q} \to D^1_{p,q-1} \to \cdots$$

Consider $D^1 = \bigoplus D^1_{p,q}$, $E^1 = \bigoplus E^1_{p,q}$, then $(D^1, E^1, i_*, j_*, k_*)$ form an exact couple, note that $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

Lemma 9.3. Suppose $\dim X = n$, we have

$$F_pC_k(X)/F_{p-1}C_k(X) = \begin{cases} 0, & p < k \text{ or } p > n \\ \bigoplus\limits_U F_p(C_k(U)/F_{p-1}C_k(U), & \text{otherwise} \end{cases}$$

Where $U \leq X$ runs over p dimensional planes

Lemma 9.4. Suppose $\dim X = p$, $H_{p+q}(C_*(X)/F_{p-1}C_*(X)) = 0$ for any $q > 0$

Theorem 9.5. *Consider spectral sequence*

$$E^1_{p,q} = \bigoplus_U H_{p+q}(C_*(U)/F_{p-1}C_*(U)) = \begin{cases} \bigoplus\limits_U \mathscr{P}(U,1), & q = 0 \\ 0, & q > 0 \end{cases}$$

Since E^r converges to $H_*(X)$, if $X = \mathbb{R}^n, \mathbb{H}^n$, we have exact sequence

$$0 \to \bigoplus_{\dim U = n} \mathscr{P}(U,1) \to \cdots \to \bigoplus_{\dim U = 0} \mathscr{P}(U,1) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

If $X = S^n$, we have exact sequence

$$0 \to \mathbb{Z} \to \bigoplus_{\dim U = n} \mathscr{P}(U,1) \to \cdots \to \bigoplus_{\dim U = 0} \mathscr{P}(U,1) \to \mathbb{Z} \to 0$$

Here the maps can be thought of as take k dimensional polygons on the k dimensional faces

10 Simplicial set - 2/27/2020

Definition 10.1. G is group, define **group homology** $H_i(G) = Tor_i^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$

Example 10.2. Suppose $G = T$, then $C_*(X)$ is a free $\mathbb{Z}[T]$ module, then

$$C_n(X) \to \cdots \to C_0(X) \to \mathbb{Z}$$

is a free $\mathbb{Z}[T]$ resolution of $\mathbb{Z}$, hence $H_i(C_*(X) \otimes_{\mathbb{Z}[T]} \mathbb{Z}) = Tor_i^{\mathbb{Z}[T]}(\mathbb{Z}, \mathbb{Z}) = H_i(T)$

Definition 10.3. Since

$$C_{n+1}(X)/F_{n-1}C_{n+1}(X) \to C_n(X)/F_{n-1}C_n(X) \to C_{n-1}(X)/F_{n-1}C_{n-1}(X) = 0$$

is exact and $- \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is a right exact functor

$$\begin{aligned}
\mathcal{P}(X, G) &= \mathcal{P}(X, \{1\}) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \\
&= H_n(C_*(X)/F_{n-1}C_*(X)) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \\
&\cong H_n\left(C_*(X)/F_{n-1}C_*(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}\right) \\
&\cong H_n\left(\frac{C_*(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}}{F_{n-1}C_*(X) \otimes_{\mathbb{Z}[G]} \mathbb{Z}}\right)
\end{aligned}$$

Definition 10.4. The **simplex category** $Simp$ has $[n] := \{0, 1, \cdots, n\}$ as objects, and order preserving functions as morphisms, there are two special types of morphisms: **Face maps** $\varepsilon_{n,i} : [n-1] \to [n]$, $\varepsilon_{n,i}(j) = \begin{cases} j & , j < i \\ j+1 & , j \geq i \end{cases}$ and the **degeneracy maps** $\eta_{n,i} : [n+1] \to [n]$, $\eta_{n,i}(j) = \begin{cases} j & , j \leq i \\ j-1 & , j > i \end{cases}$, and they subject to the **simplicial identities**:

$$\varepsilon_j \circ \varepsilon_i = \varepsilon_i \circ \varepsilon_{j-1}, i < j \Leftrightarrow i \leq j-1$$

$$\eta_j \circ \eta_i = \eta_i \circ \eta_{j+1}, i \leq j \Leftrightarrow i < j+1$$

$$\eta_j \circ \varepsilon_i = \begin{cases} \varepsilon_{i-1} \circ \eta_j & , j \leq i-2 \Leftrightarrow j < i-1 \\ 1 & , j = i, i-1 \\ \varepsilon_i \circ \eta_{j-1} & , j > i \Leftrightarrow j-1 \geq i \end{cases}$$

Definition 10.5. A **simplicial set** is a functor $X : Simp^{op} \to Set$

Definition 10.6. An element in S_p is called **degenerate** if it is the image of some element in S_{p-1} under some η_j

Definition 10.7. Given a simplicial set S, we can form a chain complex

$$\cdots \to C_p(S) \xrightarrow{\partial_p} C_{p-1}(S) \to \cdots \to C_0(S) \to 0$$

Where $C_p(S) = \mathbb{Z}[S_p]$, $\partial_p = \sum_{i=0}^{p}(-1)^i \varepsilon_i$

Define homology $H_*(S) = H_*(C_*(S))$

Example 10.8. **(a)** S_p is the set of $(p+1)$ tuples, $\varepsilon_i : S_p \to S_{p-1}$, $(x_0, \cdots, x_p) \mapsto (x_0, \cdots, \widehat{x_i}, \cdots, x_p)$, $\eta_i : S_p \to S_{p+1}$, $(x_0, \cdots, x_p) \mapsto (x_0, \cdots, x_i, x_i, \cdots, x_p)$, $C_*(S)$ is the tuple complex

(b) X is a topological space, $S_p = [\Delta^p, X]$, $C_*(S)$ is the singular chain complex

(c) X is topological space with an open cover $\mathcal{U} = \{U_j\}$, $S_p = \{(p+1) \text{ folded intersections}\}$, $C_*(S)$ is Čech chain complex

Definition 10.9. Realization

Lemma 10.10. $C_*(S)$ the simplicial chain complex of the realization of S

Lemma 10.11. Degenerate simplices generate a subcomplex $DC_*(S)$

Example 10.12. $\partial(x_0, x_1, x_1, x_2) = (x_1, x_1, x_2) - (x_0, x_1, x_2) + (x_0, x_1, x_2) - (x_0, x_1, x_1) = (x_1, x_1, x_2) - (x_0, x_1, x_1)$

Theorem 10.13. *$C_*(S)$ and $C_*(S)/DC_*(S)$ are chain homotopy equivalent*

Definition 10.14. A p **flag** in X is $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_p$, $\varnothing \subsetneqq U_i \subsetneqq X$, let S_p be the set of p flags, then S form a simplicial set, $\varepsilon_i : S_p \to S_{p-1}$, $U_0 \supseteq \cdots \supseteq U_p \mapsto U_0 \supseteq \cdots \supseteq \widehat{U_i} \supseteq \cdots \supseteq U_p$, $\eta_i : S_p \to S_{p+1}$, $U_0 \supseteq \cdots \supseteq U_p \mapsto U_0 \supseteq \cdots \supseteq U_i \supseteq U_i \supseteq \cdots \supseteq U_p$. Clearly, a flag is degenrate iff it has to consecutive U_i's, thus $H_k(S) = 0$ for $k > n$

Lemma 10.15. Let S be the simplicial set of flags, $H_k(S) \cong H_k(F_{n-1}C_*(X)) \overset{\text{LES}}{\cong} H_{k-1}(C_*(X)/F_{n-1}C_*(X))$

Proof. Define double chain complex $A_{p,q} = \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} C_q(U_p)$, $\partial' = \sum (-1)^i \varepsilon_i$, $\partial'' = (-1)^p \partial$, then we have

$$'E^1_{p,q} = H_q(A_{p,*}) = \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} H_q(C_*(U_p)) = \begin{cases} \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} \mathbb{Z} = C_p(S) & , q = 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$'E^\infty_{p,q} =' E^2_{p,q} = \begin{cases} H_p(S) & , q = 0 \\ 0 & , \text{otherwise} \end{cases}$$

We can construct $P : A_{p,q} \to A_{p+1,q}$ as follows, for each $\sigma \in C_*(X)$, let U_σ be the intersection of all subspaces containing σ which is again a subspace, for $\sigma \in C_q(U_p)$ with $U_0 \supseteq \cdots \supseteq U_p$, define $P(\sigma) = \sigma \in C_q(U_\sigma)$ with $U_0 \supseteq \cdots \supseteq U_p \supseteq U_\sigma$, then $\partial P + P\partial = 1$

$$\begin{tikzcd} A_{p+1,q} \arrow[r] \arrow[d] & A_{p,q} \arrow[r] \arrow[d] \arrow[dl, "P"'] & A_{p-1,q} \arrow[d] \arrow[dl, "P"] \\ A_{p+1,q} \arrow[r] & A_{p,q} \arrow[r] & A_{p-1,q} \end{tikzcd}$$

Also, $\bigoplus_{U_0 \supseteq U_1} C_q(U_1) \to \bigoplus_{U_0} C_q(U_0) \to F_{n-1}(X) \to 0$ is exact, hence

$$''E^1_{p,q} = H_p(A_{*,q}) = \begin{cases} F_{n-1}C_*(X) & , q = 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$''E^\infty_{p,q} ='' E^2_{p,q} = \begin{cases} H_q(F_{n-1}C_*(X)) & , p = 0 \\ 0 & , \text{otherwise} \end{cases}$$

Since $'E,'' E$ both converge to the homology of total complex, $H_k(S) = H_k(F_{n-1}C_*(X))$ $\square$

Remark. The map $H_k(S) \to H_k(F_{n-1}C_*(X))$ is actually given by

$$(x_0, \cdots, x_n) \mapsto \sum_\sigma \left\{ (x_{\sigma(0)}, \cdots, x_{\sigma(1)}) \supseteq \cdots \supseteq (x_{\sigma(0)}) \right\}$$

11 Group homology - 3/5/2020

Theorem 11.1. *$K(-,n)$ is a functor from of the category of groups to the category of topological spaces or to the category of CW complexes. $K(-,n)$ respects product, i.e. $K(G\times H,n)=K(G,n)\times K(H,n)$ since $\pi_n(\prod X_i)=\prod_i \pi_n(X_i)$*

Lemma 11.2. G is a group, there exists a contractible space X on which G acts freely

Proof. □

Theorem 11.3. *G is a group, $H_i(G)\cong H_i(K(G,1))$*

Proof. By Lemma 11.2, $X\xrightarrow{p} X/G$ is a covering with deck transformation group G, thus $\pi_1(X/G)=G$, also, $\pi_i(X/G)=0,\forall i>1$ since

$$\begin{tikzcd} & X \arrow[d, "p"] \\ S^n \arrow[r] \arrow[ur, dashed] & X/G \end{tikzcd}$$

By CW approximation theorem and cellular approximation theorem, we may assume $G\times X\to X$ is cellular, X/G is a CW complex, $X/G=K(G,1)$, We have free resolution $\cdots\to C_1^{\text{cell}}(X)\to C_0^{\text{cell}}(X)\to\mathbb{Z}\to 0$, since $C_i^{\text{cell}}(X/G)\cong C_i^{\text{cell}}(X)\otimes_{\mathbb{Z}[G]}\mathbb{Z}$

$$H_i(K(G,1))=H_i(X/G)\cong H_i^{\text{cell}}(X/G)=H_i(G)$$

Since X/G is connected, $H_0(G)=\mathbb{Z}$ □

Remark. Since G acts on $C_*(G)$ freely, tuple complex $C_*(G)$ on G give a free resolution of $\mathbb{Z}$, $C_k(G)$ consists of $(k+1)$-tuples $(g_0,\cdots,g_k)$, denote $a_{ij}=g_i^{-1}g_j$, then $a_{ij}=a_{ji}^{-1}$, $a_{ij}a_{jk}=a_{ik}$ satisfies **cocycle relation**, $B_k(G)=C_k(G)\otimes_{\mathbb{Z}[G]}\mathbb{Z}$ consists of $[a_{01}|\cdots|a_{k-1,k}]$, then boundary map on $B_*(G)$ would be

$$\partial[a_1|\cdots|a_n]=[a_2|\cdots|a_n]+\sum_{k=1}^{n-1}(-1)^k[a_1|\cdots|a_ka_{k+1}|\cdots|a_n]+(-1)^n[a_1|\cdots|a_{n-1}]$$

In particular, $\partial[a]=[]-[]=0$, $\partial[a_1|a_2]=[a_2]-[a_1a_2]+[a_1]$, $\partial[a_1|a_2|a_3]=[a_2|a_3]-[a_1a_2|a_3]+[a_1|a_2a_3]-[a_1|a_2]$

Example 11.4. **(a)** Since $K(\mathbb{Z},1)=S^1$, $K(\mathbb{Z}^n,1)=\overbrace{S^1\times\cdots\times S^1}^{n}=\mathbb{T}^n$, $H_i(\mathbb{Z}^n)=H_i(\mathbb{T}^n)=\mathbb{Z}^{\binom{n}{i}}$ by Kunneth theorem

(b) $K(\mathbb{Z}/2\mathbb{Z},1)=\mathbb{R}P^\infty$, $H_i(\mathbb{Z}/2\mathbb{Z},1)=H_i(\mathbb{R}P^\infty)=\begin{cases}\mathbb{Z}, & i=0\\ \mathbb{Z}/2\mathbb{Z}, & i>0\text{ even}\\ 0, & i\text{ odd}\end{cases}$

Lemma 11.5. $H_1(G)=G^{\text{ab}}=G/[G,G]$

Proof. □

Definition 11.6. Given a group homomorphism $\phi: G_1\to G_2$. ϕ induce $\phi: K(G_1,1)\to K(G_2,1)$ which then induce $\phi_*: H_i(G_1)=H_i(K(G_1,1))\to H_i(K(G_2,1))=H_i(G_2)$. Equivalently, if $F_\bullet\xrightarrow{\varepsilon}\mathbb{Z}\to 0$, $F'_\bullet\xrightarrow{\varepsilon}\mathbb{Z}\to 0$ are free resolutions, $F'_\bullet$ can be viewed as $\mathbb{Z}[G_1]$ modules via $\phi:\mathbb{Z}[G_1]\to\mathbb{Z}[G_2]$, i.e. $g\cdot x=\phi(g)x$, since F_i's are free we can find $f_\bullet: F_\bullet\to F'_\bullet$, $f(gx)=g\cdot f(x)=\phi(g)f(x)$, which induce a chain map between free resolutions and then a morphism on homology. In particular, ϕ induce $B_n(G_1)\to B_n(G_2)$, $[g_1|\cdots|g_n]\mapsto[\phi(g_1)|\cdots|\phi(g_n)]$

Example 11.7. $C_a : G \to G$, $g \mapsto aga^{-1}$ is the conjugation, $F_\bullet \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}[G]$ modules, $f(x) = ax$, $f(gx) = agx = aga^{-1}ax = C_a(g)f(x)$ give a homomorphism $f_\bullet : F_\bullet \to F_\bullet$ which becomes the identity when tensoring with $\mathbb{Z}[G_1]$ and $\mathbb{Z}[G_2]$, thus C_a induce identity on group homology

Example 11.8. $G = \mathbb{Z}/n\mathbb{Z} = \langle t \rangle$, $N = 1 + t + \cdots + t^{n-1}$, then we have free resolution

$$\cdots \to \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Tensor with $\mathbb{Z}$, we get

$$\cdots \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Hence $H_i(G) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}/n\mathbb{Z}, & i \text{ odd} \\ 0, & i > 0 \text{ even} \end{cases}$

Corollary 11.9. $A = \mathbb{Z}^n \times \mathbb{Z}/n_1\mathbb{Z} \times \cdots \times \mathbb{Z}/n_k\mathbb{Z}$ is a finitely generated abelian group, use Kunneth formula, we can determine the group homology

12 Homology of abelian groups - 3/10/2020

Definition 12.1. View Δ^n as $\{0 \le x_1 \le \cdots \le x_n \le 1\}$, we can decompose $\Delta^n \times \Delta^m = \{0 \le x_1 \le \cdots \le x_n \le 1\} \times \{0 \le x_{n+1} \le \cdots \le x_{n+m} \le 1\}$ into $\binom{n+m}{m}$ simplices

$$\Delta^n \times \Delta^m = \sum_\sigma (-1)^{|\sigma|}\Delta_\sigma, \quad \Delta_\sigma = \{0 \le x_{\sigma(1)} \le \cdots \le x_{\sigma(n+m)} \le 1\}$$

σ runs over (n, m)-shuffles. This gives $C_k(X) \times C_l(Y) \to C_{k+l}(X \times Y)$

Definition 12.2. We have $H_k(G) \times H_l(H) \to H_{k+l}(G \times H)$. If A is an abelian group, we have $H_k(A) \times H_l(A) \to H_{k+l}(A \times A) \xrightarrow{+_*} H_{k+l}(A)$, $[a_1|\cdots|a_k] \otimes [a_{k+1}|\cdots|a_{k+l}] \mapsto \sum(-1)^{|\sigma|}[a_{\sigma(1)}|\cdots|a_{\sigma(k+l)}]$, σ runs over (k, l)-shuffles. Making $H_*(A)$ a graded $\mathbb{Z}$ algebra

Example 12.3. $[a_1] \otimes [a_2] = [a_1|a_2] - [a_2|a_1]$
$[a_1|a_2] \otimes [a_3] = [a_1|a_2|a_3] - [a_1|a_3|a_2] + [a_2|a_3|a_1]$
$[a_1] \otimes [a_2|a_3] = [a_1|a_2|a_3] - [a_2|a_1|a_3] + [a_3|a_1|a_2]$

Theorem 12.4. *If A is a free abelian group, $\bigwedge^*_{\mathbb{Z}} A \xrightarrow{\cong} \bigwedge^*_{\mathbb{Z}} H_1(A) \to H_*(A)$, $a_1 \wedge \cdots \wedge a_n \mapsto \sum(-1)^{|\sigma|}[a_{\sigma(1)}|\cdots|a_{\sigma(n)}]$ is an isomorphism, σ runs over S_n*

Proof. $H_k(\mathbb{Z}^n) = \mathbb{Z}^{\binom{n}{k}} = \bigwedge^k_{\mathbb{Z}} \mathbb{Z}^n$. Every abelian group is the direct limit of its finitely generated subgroups and functors H_*, $\bigwedge^*_{\mathbb{Z}}$ commute with direct limit □

Example 12.5. Geometrically, the isomorphism $P : \bigwedge^n_{\mathbb{Z}} \mathbb{Z} \to H_n(\mathbb{Z}^n) \cong H_n(K(\mathbb{Z}^n, 1)) = H_n(\mathbb{T}^n)$ maps $e_1 \wedge \cdots \wedge e_k$ to the triangulation of $\mathbb{T}^n$

13 Translational scissors congruence - 3/12/2020

Definition 13.1 (Homology with local coefficients)**.** S is a simplicial set, G_s is a family of abelian groups for all $s \in S$ with homomorphisms $\varepsilon_i : G_s \to G_{\varepsilon_i s}$ satisfying simplicial identities, we can form a complex $C_p(S, G) = \bigoplus_{s \in S_p} G_s$, write $H_p(S, G)$ as the homology

Example 13.2. F is the simplicial set of flags, $\bigwedge^q_{U_0 \supseteq \cdots \supseteq U_p} = \bigwedge^q_{\mathbb{Z}} U_p$

Definition 13.3. Denote $M_G = M \otimes_{\mathbb{Z}[G]} \mathbb{Z}$, then $\mathcal{P}(X, T) = H_n(C_*(X)_T / F_{n-1} C_*(X)_T)$

Theorem 13.4. *$\mathcal{P}(\mathbb{R}^n, T)$ is a $\mathbb{Q}$ vector space*

Remark. $\mathcal{P}(\mathbb{R}^n, T)$ is in fact an $\mathbb{R}$ vector space

Proof. Define double complex

$$A_{p,q} = \bigoplus_{\mathbb{R}^n \supsetneqq U_0 \supseteq \cdots \supseteq U_p} \widetilde{C}_q(U_p)_{T(U_p)} = \begin{cases} \mathbb{Z} & , p = q = -1 \\ C_q(\mathbb{R}^n)_T & , p = -1, q \geq 0 \\ \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} \mathbb{Z} & , p \geq 0, q = -1 \\ \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} C_q(U_p)_{T(U_p)} & , p, q \geq 0 \end{cases}$$

$\partial' = \sum (-1)^i \varepsilon_i$, $\partial'' = (-1)^p \partial$. As additive groups $U_p \cong T(U_p)$, hence

$$'E^1_{p,q} = \begin{cases} \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} H_q(\widetilde{C}_*(U_p)_{U_p}) = \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} H_q(U_p) = \bigoplus_{U_0 \supseteq \cdots \supseteq U_p} \bigwedge^*_{\mathbb{Z}} U_p & , p \geq 0 \\ H_q(\widetilde{C}_*(\mathbb{R}^n)_T) = H_q(\mathbb{R}^n) = \bigwedge^*_{\mathbb{Z}} \mathbb{R}^n & , p = -1 \end{cases}$$

This is complex $C_*(F, \bigwedge^q)$
Claim: higher differentials of $'E$ is zero
For any $a \in \mathbb{Z}$, consider $\mu_a : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto ax$ which induces multiplication by a^q on $\bigwedge^q_{\mathbb{Z}} U \to \bigwedge^q_{\mathbb{Z}} U$, $A_{p,q} \to A_{p,q}$, since μ_a commutes with differentials $\partial^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, we have

$$a^q \partial^r x = \partial^r \mu_a x = \mu_a \partial^r x = a^{q+r-1} \partial^r x \Rightarrow r = 1 \text{ or } \partial^r x = 0$$

Hence $'E^2_{p,q} =' E^\infty_{p,q} = H_p(F, \bigwedge^q)$. Since $U \leq \mathbb{R}^n$ is a $\mathbb{Q}$ vector space, $\bigwedge^*_{\mathbb{Z}} U = \bigwedge^*_{\mathbb{Q}} U$ is also a $\mathbb{Q}$ vector space, $H_k(Tot(A))$ has a $\mathbb{Q}$ vector space filtration $0 \subseteq F_{-1} \subseteq \cdots \subseteq F_{k-1} = H_k(Tot(A))$, $H_k(Tot(A)) = E^\infty_{-1,k} \oplus \cdots \oplus E^\infty_{k-1,2}$ is the eigenspace decomposition

$$''E^1_{p,q} = \begin{cases} 0 & , q \geq 0 \\ C_*(\mathbb{R}^n)/F_{n-1} C_*(\mathbb{R}^n) & , p = -1 \end{cases}$$

$$''E^\infty_{-1,q} ='' E^2_{-1,q} = H_q(C_*(\mathbb{R}^n)/F_{n-1} C_*(\mathbb{R}^n))$$

Hence $\mathcal{P}(\mathbb{R}^n, T) ='' E^2_{-1,n} = H_{n-1}(Tot(A))$ is a $\mathbb{Q}$ vector space □

Definition 13.5. Define $\Gamma : B_n(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n, T)$, $[v_1 | \cdots | v_n] \mapsto (0, v_1, v_1 + v_2, \cdots, v_1 + \cdots + v_n)$

14 Hyperbolic scissors congruence

Definition 14.1. R is a commutative ring, M is right $R[G]$ module, $C_*(G)$ is the tuple complex, equivalently, $B_*(G)$ is the bar complex, $\overline{B}_*(G) = B_*(G) \otimes_{R[G]} R$, thus

$$M \otimes_{R[G]} B_*(G) \cong M \otimes_R R \otimes_{R[G]} B_*(G) \cong M \otimes_R \overline{B}_*(G)$$

Group homology with coefficients in M is

$$\begin{aligned} H_k(G; M) &= H_k(M \otimes_{R[G]} C_*(G)) \\ &= H_k(M \otimes_{R[G]} B_*(G)) \\ &= H_k(M \otimes_R \overline{B}_*(G)) \\ &= Tor_k^{R[G]}(M, R) \end{aligned}$$

The differential of $M \otimes_{\mathbb{Z}[G]} C_*(G)$ is given by

$$\begin{aligned} \partial(m \otimes [g_1|\cdots|g_n]) &= \partial(m \otimes (1, g_1, g_1g_2, \cdots, g_1\cdots g_n)) \\ &= mg_1 \otimes [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i m \otimes [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &\quad + (-1)^n m \otimes [g_1|\cdots|g_{n-1}] \end{aligned}$$

$H_0(G; M) = M \otimes_{R[G]} R = M_G$. Write $H_k(G)$ for $H_k(G; \mathbb{Z})$

Remark. A right $R[G]$ module M can be viewed as a left $R[G]$ module and vice versa via $g^{-1}m = mg$. Therefore if M is a left $\mathbb{Z}G$ module, then

$$\begin{aligned} \partial(m \otimes [g_1|\cdots|g_n]) &= g_1^{-1}m \otimes [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i m \otimes [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &\quad + (-1)^n m \otimes [g_1|\cdots|g_{n-1}] \end{aligned}$$

Definition 14.2. $\mathcal{A}$ is an abelian category with enough projectives, a **Cartan-Eilenberg resolution** P_{**} of chain complex A_* is an upper half plane double complex consist of projectives and a augmentation $P_{*0} \xrightarrow{\varepsilon} A_*$ such that

1. If $A_p = 0$, then $P_{p*} = 0$
2. $B_p^h(P) \xrightarrow{B_p(\varepsilon)} B_p(A_*)$, $H_p^h(P) \xrightarrow{H_p(\varepsilon)} H_p(A_*)$ are projective resolutions

Lemma 14.3. Every chain complex A_* has a Cartan-Eilenberg resolution, and $Z_p^h(P) \xrightarrow{Z_p(\varepsilon)} Z_p(A_*)$, $P_{p*} \xrightarrow{\varepsilon_p} A_p$ are projective resolutions

Lemma 14.4. $f, g : A_* \to B_*$ are chain homotopic, $P \to A_*$, $Q \to B_*$ are Cartan-Eilenberg resolutions, $\widetilde{f}, \widetilde{g} : P \to Q$ are over f, g, then $\widetilde{f}, \widetilde{g}$ are chain homotopic
Any two Cartan-Eilenberg resolutions of $P \to A_*$, $Q \to A_*$ are chain homotopic. If F is an additive functor, then $Tot^{\oplus}(F(P))$, $Tot^{\oplus}(F(Q))$ are chain homotopic

Definition 14.5. $\mathcal{A}, \mathcal{B}$ are abelian categories, $\mathcal{A}$ has enough projectives, $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, $f : A_* \to B_*$ is a map of chain complexes. The **left hyper-derived functor** of F is $\mathbb{L}_i F : \mathbf{Ch}\mathcal{A} \to \mathcal{B}$ given by $\mathbb{L}_i F(A_*) = H_i(Tot^{\oplus}(F(P)))$ which we just write as $H_i(F(P))$, is independent of the choice of Cartan-Eilenberg resolution P thanks to Lemma 14.4

Definition 14.6. A_* is a chain of RG modules. Since $A_* \otimes_{R[G]} B_*(G) = A_* \otimes_R \overline{B}_*(G)$ is a Cartan-Eilenberg resolution of $F(A_*)$ with $F = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}$. Hence we define the **hyperhomology** of a chain complex of RG modules A_* to be

$$\mathbb{H}_i(G, A_*) = \mathbb{L}_i F(A_*) = H_i(A_* \otimes_{\mathbb{Z}G} B_*(G))$$

Lemma 14.7. A_* is an acyclic chain complex with $H_0(A_*) = M$, then $\mathbb{L}_i F(A_*) = L_i F(M)$. In particular, $\mathbb{H}_i(G; A_*) \cong H_i(G; M)$

Proof. By Lemma 14.4, it suffices to consider the case where A_* is the chain complex with only one nonzero term M at degree zero. Suppose $P \to M$ is a projective resolution of M, it can be regared as a Cartan-Eilenberg resolution of A_*, thus $\mathbb{L}_i F(A_*) = H_i(F(P)) = L_i F(M)$ □

Theorem 14.8 (Hyperhomology spectral sequence)**.** $L_p F(H_q(A_*)) \Rightarrow \mathbb{L}_{p+q} F(A_*)$. *If* A_* *is bounded below, then* $H_p(L_q F(A_*)) \Rightarrow \mathbb{L}_{p+q} F(A_*)$

Proof. Consider the double complex P of a Cartan-Eilenberg resolution $P \to A_*$. Since $H_p^h(P) \to H_p A_*$, $P_{p*} \to A_p$ are projective resolutions, we have

$$L_p F(H_q(A_*)) = H_p^v F(H_q^h(P)) = H_p^v(H_q^h F(P)) ='' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A_*)$$

If A_* is bounded below, then $'E_{pq}^1 = L_q F(A_p) = H_q^v(F(P_{p*}))$ and

$$H_p(L_q F(A_*)) = H_p^h H_q^v(F(P)) =' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A_*)$$

□

Theorem 14.9 (Grothendieck spectral sequence)**.** *$\mathcal{A}, \mathcal{B}$ have enough projectives, $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{A} \to \mathcal{B}$ are right exact functors and G sends projectives to F-acyclic objects, then*

$$(L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A)$$

Proof. Suppose $P \to A$ is a projective resolution, then by Theorem 14.8, we have

$$(L_p F)(L_q G)(A) \cong L_p F(H_q G(P)) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

$$H_p(L_q F(G(P))) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

Since $G(A)$ is F-acyclic, $'E_2^{pq} = 0$ for $q \neq 0$ and

$$E_2^{p0} = H_p(FG(P)) = L_p(FG)(A) \cong \mathbb{L}_p(FG)(A)$$

□

Corollary 14.10 (Hochschild-Serre spectral sequence)**.** $N \trianglelefteq G$ is a normal subgroup, A is a $\mathbb{Z}G$ module, then

$$H_p(G/N; H_q(N; A)) \Rightarrow H_{p+q}(G; A)$$

Proof. Consider right exact functors

$$F = - \otimes_{\mathbb{Z}[G/N]} \mathbb{Z} : \mathbb{Z}[G/N]\text{-mod} \to \mathbb{Z}\text{-mod}$$

$$G = - \otimes_{\mathbb{Z}[N]} \mathbb{Z} = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N] : \mathbb{Z}[G]\text{-mod} \to \mathbb{Z}[G/N]\text{-mod}$$

The left derived functors of $FG = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is $L_*(FG)(A) = Tor_*^{\mathbb{Z}[G]}(A, \mathbb{Z}) = H_*(G; A)$. For any $\mathbb{Z}[G]$ module A and $\mathbb{Z}[G/N]$ module B, we have natural isomorphism

$$Hom_{\mathbb{Z}[G/N]}(A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N], B) \cong Hom_{\mathbb{Z}[G]}(A, B) = Hom_{\mathbb{Z}[G]}(A, U(B))$$

Hence G is left adjoint to forgetful functor U which is exact, this implies that G preserves projectives which are exactly F-acyclic objects. Apply Theorem 14.9 we have

$$H_p(G/N; H_q(N; A)) = (L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A) = H_{p+q}(G; A)$$

□

Lemma 14.11 (Center kills lemma). M is a right $R[G]$ module, $\gamma \in Z(G)$ such that $x\gamma = rx, \forall x \in M$ for some $r \in R$, then $(r-1)$ annihilates $H_*(G;M)$

Proof. Any $\gamma \in G$ induce endomorphism $M \otimes_R \overline{B}_*(G)$

$$\gamma_*(x \otimes [g_1|\cdots|g_q]) = x\gamma \otimes [\gamma g_1 \gamma^{-1}|\cdots|\gamma g_q \gamma^{-1}]$$

Which is chain homotopic to identity through

$$s(x \otimes [g_1|\cdots|g_q]) = \sum_{i=0}^{q} x \otimes [g_1|\cdots|g_i|\gamma|\gamma g_{i+1}\gamma^{-1}|\cdots|\gamma g_1 \gamma^{-1}]$$

If $\gamma \in Z(G)$ such that $x\gamma = rx, \forall x \in M$ for some $r \in R$, then $\gamma_*(x \otimes [g_1|\cdots|g_q]) = rx \otimes [g_1|\cdots|g_q]$ which equals $x \otimes [g_1|\cdots|g_q]$ in $H_*(G;M)$ □

Lemma 14.12 (Shapiro's lemma). $H \leq G$, M is a left $R[G]$ module, then there is a natural isomorphism $H_*(H;M) \cong H_*(G; M \otimes_{R[H]} R[G])$

Proof. $B_*(H), B_*(G)$ are both free resolutions of $R[H]$ modules of R, thus inclusion $M \otimes_{RH} B_*(H) \hookrightarrow M \otimes_{RH} B_*(G)$ induces an isomorphism on homology

$$\begin{aligned} H_k(G; M \otimes_{R[H]} R[G]) &= H_k(M \otimes_{R[H]} R[G] \otimes_{R[G]} B_*(G)) \\ &\cong H_k(M \otimes_{R[H]} B_*(G)) \\ &\cong Tor_k^{R[H]}(M, R) \\ &\cong H_k(M \otimes_{R[H]} B_*(H)) \\ &\cong H_k(H; M) \end{aligned}$$

□

Lemma 14.13. Torsion free divisible abelian groups are exactly $\mathbb{Q}$ vector spaces

Proposition 14.14. A is an abelian group

1. $H_0(A) = \mathbb{Z}$ and $A \xrightarrow{\cong} H_1(A)$, $a \mapsto [a]$ is a natural isomorphism
2. If A is torsion free, then $\bigwedge_{\mathbb{Z}}^k(A) \xrightarrow{\cong} \bigwedge_{\mathbb{Z}}^k(H_1(A)) \xrightarrow{\wedge^k} H_k(A)$ is a natural isomorphism
3. If A is a divisible group, then $A \cong A/T \oplus T$, T is the torsion subgroup of A, and $H_k(A) \cong \bigwedge_{\mathbb{Q}}^k(A/T) \bigoplus H_k(T)$

Proof.

1. $H_0(A) = \mathbb{Z}$ is clear
2.
3.

□

Example 14.15. If F is a field, $H_k(F) = \bigwedge_{\mathbb{Q}}^k(F)$

Theorem 14.16. $H_n\left(\frac{C_*(X)}{F_{n-1}C_*(X)}\right)^t \to \mathscr{P}(X,\{1\})$ *is an isomorphism given* $H_n\left(\frac{C_*(X)}{F_{n-1}C_*(X)}\right)^t$ *with group action* $g(a_0,\cdots,a_n) = (\det g)(ga_0,\cdots,ga_n)$

$$\begin{aligned}
\mathscr{P}(X,G) &= \mathbb{Z}\otimes_{\mathbb{Z}[G]}\mathscr{P}(X,\{1\})^t \\
&= \mathbb{Z}\otimes_{\mathbb{Z}[G]} H_n\left(\frac{C_*(X)}{F_{n-1}C_*(X)}\right)^t = H_0\left(G; H_n\left(\frac{C_*(X)}{F_{n-1}C_*(X)}\right)^t\right) \\
&= H_n\left(\mathbb{Z}\otimes_{\mathbb{Z}[G]}\frac{C_*(X)}{F_{n-1}C_*(X)}\right) = H_n\left(H_0\left(G;\frac{C_*(X)}{F_{n-1}C_*(X)}\right)\right) \\
&= H_n\left(\frac{\mathbb{Z}\otimes_{\mathbb{Z}[G]} C_*(X)}{\mathbb{Z}\otimes_{\mathbb{Z}[G]} F_{n-1}C_*(X)}\right) = H_n\left(\frac{H_0(G;C_*(X))}{H_0(G;F_{n-1}C_*(X))}\right)
\end{aligned}$$

Definition 14.17. On $FP^1 = F\cup\{\infty\}$ the group of **Möbius transformations** $f(z) = \frac{az+b}{cz+d}$ can be identified with the group of projective transformations $PGL(2,F) = PSL(2,F)$ since denoting $z = \frac{z_1}{z_2}$ we have

$$\begin{aligned}
[z,1] = [z_1,z_2] \mapsto \begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot[z_1,z_2] \\
&= [az_1+bz_2, cz_1+dz_2] \\
&= \left[\frac{az_1+bz_2}{cz_1+dz_2},1\right] \\
&= \left[\frac{az+b}{cz+d},1\right]
\end{aligned}$$

The **Cross ratio** of $z_0,z_1,z_2,z_3\in FP^1$ is

$$(z_0,z_1;z_2,z_3) = \frac{(z_2-z_0)(z_3-z_1)}{(z_3-z_0)(z_2-z_1)}$$

Lemma 14.18.

1. Cross ratio is a projective invariant
2. There is a unique $g \in PSGL(2,F) = PSL(2,F)$ such that $(z_0,z_1,z_2,z_3) = g(\infty,0,1,(z_0,z_1;z_2,z_3))$

Proof.

1. Since

$$\frac{az+b}{cz+d} - \frac{aw+b}{cw+d} = \frac{(ad-bc)(z-w)}{(cz+d)(cw+d)}$$

We have

$$\begin{aligned}
\left(\frac{az_0+b}{cz_0+d},\frac{az_1+b}{cz_1+d};\frac{az_2+b}{cz_2+d},\frac{az_3+b}{cz_3+d}\right) &= \frac{\frac{(ad-bc)^4(z_2-z_0)(z_3-z_1)}{(cz_0+d)(cz_1+d)(cz_2+d)(cz_3+d)}}{\frac{(ad-bc)^4(z_3-z_0)(z_2-z_1)}{(cz_0+d)(cz_1+d)(cz_2+d)(cz_3+d)}} \\
&= \frac{(z_2-z_0)(z_3-z_1)}{(z_3-z_0)(z_2-z_1)} \\
&= (z_0,z_1;z_2,z_3)
\end{aligned}$$

2. $g = \begin{pmatrix} a_2 - a_0 & a_1(a_0 - a_2) \\ a_2 - a_1 & a_0(a_1 - a_2) \end{pmatrix}$

$\square$

Definition 14.19. The **ideal points** $\partial\mathbb{H}^n$ of $\mathbb{H}^n$ in the Klein and Poincaré disc models are the boundary circles, in the half space model is $\mathbb{R}^n \cup \{\infty\}$, $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$
$\mathcal{P}(\partial\mathbb{H}^n)$ is the K_0-group generated by simplices in $\overline{\mathbb{H}^n}$ with vertices on $\partial\mathbb{H}^n$. Explicitly, $\mathcal{P}(\partial\mathbb{H}^n) = H_0(G; H_n(C_*(X)/F_{n-1}C_*(X))^t)$ are $(n+1)$-tuples modulo relations $(a_0, \cdots, a_n) = 0$ if lies in a subspace, $(ga_0, \cdots, ga_n) = (\det g)(a_0, \cdots, a_n)$, and $\sum_{i=0}^{n+1}(a_0, \cdots, \widehat{a_i}, \cdots, a_{n+1}) = 0$ for $a_i \in \partial\mathbb{H}^n$

Example 14.20. $\partial\mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, every isometry on $\mathbb{H}^3$ restricts to a conformal map on $\partial\mathbb{H}^3$ since isometry sends hemispheres to hemispheres or orthogonal planes. Isometries on $\partial\mathbb{H}^3$ are möbius transformations, and translations $z \mapsto z+\lambda$ can be extended to $(z, x_3) \mapsto (z + \lambda, x_3)$, dilations $z \mapsto \lambda z$ can be extended to $(z, x_3) \mapsto (\lambda z, |\lambda| x_3)$, inversions $z \mapsto -\frac{1}{z}$ can be extended to $(z, x_3) \mapsto \left(\frac{-\overline{z}}{|z|^2 + x_3^2}, \frac{x_3}{|z|^2 + x_3^2}\right)$. Thus the isometry group for $\mathbb{H}^3$ is $PSL(2, \mathbb{C}) \ltimes \mathbb{Z}/2\mathbb{Z}$

Definition 14.21. F is a field, define $\mathcal{P}_F$ to be the abelian group generated by $z \in F \setminus \{0, 1\}$ modulo relation

$$z_1 - z_2 + \frac{z_2}{z_1} - \frac{1 - z_2}{1 - z_1} + \frac{1 - z_2^{-1}}{1 - z_1^{-1}} = 0 \tag{14.1}$$

For $z_1 \neq z_2$

Lemma 14.22. If $\overline{F} = F$, then $z + z^{-1} = 0$, $z + \{1 - z\} = 0$

Remark. By adding $0 = 1 = \infty = 0$, these relations are true for all $z \in F \cup \{\infty\}$

Theorem 14.23 (K-groups of fields)**.** *F is a field. The K_0, K_1, K_2 groups of F are*

$$K_0(F) = \mathbb{Z},\ K_1(F) = F^\times$$
$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / \langle a \otimes (1 - a) \rangle,\ a \neq 0, 1$$

Theorem 14.24 (Bloch-Wigner)**.** *F is algebraically closed field with characteristic 0, write $G = SL(2, F)$, we have an exact sequence*

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(G; \mathbb{Z}) \xrightarrow{\sigma} \mathcal{P}_F \xrightarrow{\lambda} \textstyle\bigwedge_{\mathbb{Z}}^2(F^\times/\mu_F) \xrightarrow{\text{sym}} H_2(G; \mathbb{Z}) \to 0$$

$\mathbb{Q}/\mathbb{Z} \cong H_3(\mu_F; \mathbb{Z}) \to H_3(G; \mathbb{Z})$ is induced by

$$\mu_F \to G,\ z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$$

For any $(g_0, g_1, g_2, g_3) \in B_3(G)$

$$\sigma(g_0, g_1, g_2, g_3) = (g_0\infty, g_1\infty, g_2\infty, g_3\infty)$$

Or for any $[g_1|g_2|g_3] \in \overline{B}_3(G)$

$$\sigma([g_1|g_2|g_3]) = \sigma(1, g_1, g_1g_2, g_1g_2g_3) = (\infty : g_1\infty : g_1g_2\infty : g_1g_2g_3\infty)$$

This doesn't depend on the choice of ∞

$$\lambda(z) = z \wedge (1 - z)$$

$$\text{sym}(u \wedge v) = u \otimes v$$

$K_2(F) \cong H_2(G; \mathbb{Z})$

Proof. Let C_* be the tuple complex of FP^1. The stabilizer of $\infty = [1, 0]$ is the Borel subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a \in F^\times, b \in F \right\}$$

Hence $FP^1 \cong G/B$. The stabilizer of ∞ and $0 = [0, 1]$ is the split torus

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a \in F^\times \right\} \cong F^\times$$

Hence $FP^1 \times FP^1 \cong G/T$. The stabilizer of $\infty, 0$ and $1 = [1, 1]$ is

$$\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} \cong \mathbb{Z}/2\mathbb{Z}$$

Hence $FP^1 \times FP^1 \times FP^1 \cong G$. Therefore we have

$$\begin{aligned} C_0 &= \mathbb{Z}[G/B] = \mathbb{Z}G \otimes_{\mathbb{Z}B} \mathbb{Z} \\ C_1 &= \mathbb{Z}[G/T] = \mathbb{Z}G \otimes_{\mathbb{Z}T} \mathbb{Z} \\ C_2 &= \mathbb{Z}[G/(\mathbb{Z}/2\mathbb{Z})] = \mathbb{Z}G \otimes_{\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{Z} \end{aligned}$$

By Shapiro's lemma 14.12, we get

$$\begin{aligned} H_*(G; C_0) &= H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}B} \mathbb{Z}) = H_*(B; \mathbb{Z}) \\ H_*(G; C_1) &= H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}T} \mathbb{Z}) = H_*(T; \mathbb{Z}) \\ H_*(G; C_2) &= H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{Z}) = H_*(\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2\mathbb{Z} & * \text{ odd} \\ 0 & * > 0 \text{ even} \end{cases} \end{aligned}$$

$$C_0 \otimes_{\mathbb{Z}T} B_1(T) \hookrightarrow C_0 \otimes_{\mathbb{Z}T} B_1(G) \hookrightarrow C_0 \otimes_{\mathbb{Z}G} B_1(G)$$

induce isomorphisms on homology Consider split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, hook] & B \arrow[r] & T \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in F \right\} \cong F$ is the unipotent subgroup. Use Hochshild-Serre spectral sequence 14.10 we have

$$H_p(F^\times; \textstyle\bigwedge_{\mathbb{Q}}^q(F)) = H_p(F^\times; H_q(F; \mathbb{Z})) = H_p(B/U; H_q(U; \mathbb{Z})) \Rightarrow H_{p+q}(B; \mathbb{Z})$$

For any $r \in F$, consider

$$\mu_r : \textstyle\bigwedge_{\mathbb{Q}}^q(F) \to \bigwedge_{\mathbb{Q}}^q(F),\ x_1 \wedge \cdots \wedge x_q \mapsto (rx_1) \wedge \cdots \wedge (rx_q) = r^q(x_1 \wedge \cdots \wedge x_q)$$

Apply Center kills lemma 14.11, $(r^q - 1)$ annihilates $H_q(F^\times; \bigwedge_{\mathbb{Q}}^q(F))$, hence $H_q(F^\times; \bigwedge_{\mathbb{Q}}^q(F)) = 0$ for $q > 0$, therefore inclusion $T \hookrightarrow B$ and projection $B \to T$ induce isomorphisms

$$H_*(F^\times; \mathbb{Z}) \cong H_*(B; \mathbb{Z})$$

Note that $\mu_F \cong \mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $F^\times$, by Proposition 14.14, we get

$$H_*(G; C_0) = H_*(G; C_1) = H_*(F^\times; \mathbb{Z}) = \textstyle\bigwedge_{\mathbb{Q}}^*(F^\times/\mu_F) \bigoplus H_*(\mu_F; \mathbb{Z})$$

Since Tor preserves direct sums and filtered colimits, or through some Bockstein, we have

$$\begin{aligned}
H_*(\mu_F;\mathbb{Z}) &= Tor_*^{\mathbb{Z}[\mu_F]}(\mathbb{Z},\mathbb{Z}) \\
&= Tor_*^{\mathbb{Z}[\mu_F]}\left(\bigoplus_p \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z};\mathbb{Z}\right) \\
&= Tor_*^{\mathbb{Z}[\mu_F]}\left(\bigoplus_p \varinjlim_n \mathbb{Z}/p^n\mathbb{Z};\mathbb{Z}\right) \\
&= \bigoplus_p Tor_*^{\mathbb{Z}[\mu_F]}\left(\varinjlim_n \mathbb{Z}/p^n\mathbb{Z};\mathbb{Z}\right) \\
&= \bigoplus_p \varinjlim_n Tor_*^{\mathbb{Z}[\mu_F]}(\mathbb{Z}/p^n\mathbb{Z};\mathbb{Z}) \\
&= \begin{cases} \bigoplus_p \varinjlim_n \mathbb{Z} = \bigoplus_p \mathbb{Z} & i = 0 \\ \bigoplus_p \varinjlim_n \mathbb{Z}/p^n\mathbb{Z} = \mu_F & i \text{ odd} \\ 0 & i > 0 \text{ even} \end{cases}
\end{aligned}$$

Here $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} = \varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$ is the Prüfer group

By Lemma 14.7 and Theorem 14.8 we know

$$E^1_{pq} = H_q(G;C_p) = H_q(C_p \otimes_{\mathbb{Z}[G]} B_*(G)) \Rightarrow \mathbb{H}_{p+q}(G;C_*) = H_{p+q}(G;\mathbb{Z})$$

Hence the E^1 page looks like

$$\begin{array}{c|ccccc}
3 & \bigwedge^3_{\mathbb{Q}}(F^\times/\mu_F)\bigoplus\mathbb{Q}/\mathbb{Z} & \bigwedge^3_{\mathbb{Q}}(F^\times/\mu_F)\bigoplus\mathbb{Q}/\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & & \\
2 & \bigwedge^2_{\mathbb{Q}}(F^\times/\mu_F) & \bigwedge^2_{\mathbb{Q}}(F^\times/\mu_F) & 0 & & \\
1 & F^\times & F^\times & \mathbb{Z}/2\mathbb{Z} & & \\
0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & H_0(G;C_3) & H_0(G;C_4) \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}$$

By Definition 14.21 of $\mathcal{P}_F$ we know

$$\begin{aligned}
E^2_{30} &= \frac{\ker(H_0(G;C_3) \to H_0(G;C_2))}{\operatorname{im}(H_0(G;C_4) \to H_0(G;C_3))} \\
&= \frac{\ker(C_3 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_2 \otimes_{\mathbb{Z}G} \mathbb{Z})}{\operatorname{im}(C_4 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_3 \otimes_{\mathbb{Z}G} \mathbb{Z})} \\
&= \frac{C_3 \otimes_{\mathbb{Z}G} \mathbb{Z}}{\operatorname{im}(C_4 \otimes_{\mathbb{Z}G} \mathbb{Z} \to C_3 \otimes_{\mathbb{Z}G} \mathbb{Z})} \\
&= \mathcal{P}_F
\end{aligned}$$

Since

$$\begin{aligned}
\cdots \to C_4 \to C_3 \to C_2 \to C_1 \to Z_0 \to 0 \\
\cdots \to C_4 \to C_3 \to C_2 \to Z_1 \to 0 \\
\cdots \to C_4 \to C_3 \to Z_2 \to 0
\end{aligned}$$

are free $\mathbb{Z}G$ resolutions of Z_0, Z_1, Z_2, we have

$$H_2(G;Z_0) \cong H_1(G;Z_1) \cong H_0(G;Z_2) \cong \mathcal{P}_F$$

$$\begin{aligned}
H_0(G;C_2) \cong C_2 \otimes_{\mathbb{Z}G} \mathbb{Z} \cong \mathbb{Z} &\to \mathbb{Z} \cong C_1 \otimes_{\mathbb{Z}G} \mathbb{Z} \cong H_0(G;C_1) \\
(\infty,0,1)\otimes 1 &\mapsto ((0,1)-(\infty,1)+(\infty,0))\otimes 1 \\
&= (\infty,0)\otimes 1
\end{aligned}$$

is an isomorphism. Similarly, $H_0(G; C_1) \to H_0(G; C_0)$ is zero map, thus

$$E^2_{00} = \mathbb{Z},\ E^2_{10} = E^2_{20} = 0$$

$w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ is a generator of the Weyl group $W(T) = N_G(T)/T$ of T, w switches $0, \infty$ and $w^2 = -I$

$$\begin{aligned} C_1 \otimes_{\mathbb{Z}G} B_1(G) &\to C_0 \otimes_{\mathbb{Z}G} B_1(G) \\ (\infty, 0) \otimes (g_0, g_1) &\mapsto ((\infty) - (0)) \otimes (g_0, g_1) \\ &= (\infty) \otimes (g_0, g_1) + (0w) \otimes (wg_0, wg_1) \\ &= 2(\infty)\otimes \end{aligned}$$

is an isomorphism. Similarly
We get the E^2 page

3	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Q}/\mathbb{Z}$		
2	$\bigwedge^2_{\mathbb{Q}}(F^\times/\mu_F)$	$\bigwedge^2_{\mathbb{Q}}(F^\times/\mu_F)$	0	
1	0	0	0	
0	$\mathbb{Z}$	0	0	$\mathcal{P}_F$
	0	1	2	3

□

Proof.

4	$H_0(G;C_4)$			
3	$H_0(G;C_3)$	$H_1(G;C_3)$	$H_2(G;C_3)$	$H_3(G;C_3)$
2	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
1	$\mathbb{Z}$	$F^\times$	$\bigwedge_{\mathbb{Q}}^2(F^\times/\mu_F)$	$\bigwedge_{\mathbb{Q}}^3(F^\times/\mu_F)\bigoplus\mathbb{Q}/\mathbb{Z}$
0	$\mathbb{Z}$	$F^\times$	$\bigwedge_{\mathbb{Q}}^2(F^\times/\mu_F)$	$\bigwedge_{\mathbb{Q}}^3(F^\times/\mu_F)\bigoplus\mathbb{Q}/\mathbb{Z}$
	0	1	2	3

$H_1(G;C_3)$ are of 2 torsion We get the E^2 page

3	$\mathscr{P}_F$			
2	0	0	0	
1	0	0	$\bigwedge_{\mathbb{Q}}^2(F^\times/\mu_F)$	$\mathbb{Q}/\mathbb{Z}$
0	$\mathbb{Z}$	0	$\bigwedge_{\mathbb{Q}}^2(F^\times/\mu_F)$	$\mathbb{Q}/\mathbb{Z}$
	0	1	2	3

$\square$

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