Section 12.4b

December 1, 2019

Remark: When we say random variable X satisfy some distribution, we mean its cdf(cumulative distribution function is of that form)

Motivating example: Galton Board $\to$ binomial distribution $\xrightarrow[\text{De Moivre-Laplace theorem}]{\text{in large scale}}$ Normal distribution

Definition: $X \sim N(\mu, \sigma^2)$, we say X is continuous random variable satisfying a **normal distribution** with mean μ, and variance σ^2(which means the standard deviation is $\sigma > 0$), where the pdf(probability density function) is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $\sigma > 0$, notice here $A = -\infty, B = +\infty$, $E(X) = \mu, Var(X) = \sigma^2, \sigma(X) = \sigma$, the graph of f is called a **normal curve(sometimes bell curve)**, with symmetric axis $x = \mu$, if σ is greater, variance is bigger, the normal curve is shorter and wider, if σ is smaller, the variance is smaller, the normal curve is taller and narrower(centered around the mean)

The most special case is when $\mu = 0, \sigma = 1$, namely(for some psychological reasons, people often use Z,z,ϕ,Φ), $Z \sim N(0, 1^2) = N(0, 1)$(meaning Z is normally distributed with mean 0 and variance 1, sometimes also called the **standard normal distribution**), the correspponding pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ still with $A = -\infty, B = +\infty$, and $E(Z) = 0$, $Var(Z) = 1$, $\sigma(Z) = 1$, the cdf(cumulative distribution function) of $Z \sim N(0, 1)$ is not possible expressed in finite terms and compositions of those elementary functions(which has a mathematically rigorous proof!), instead we just invent a name to denote this function, define $\Phi(z) := Pr(Z \leq z) = \int_{-\infty}^{z} f(t)dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$, corresponding we would have $Pr(a \leq Z \leq b) = \int_a^b f(z)dz = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dx = \Phi(b) - \Phi(a)$, $Pr(Z > a) = 1 - Pr(Z \leq a) = 1 - \Phi(a)$, and informally $\Phi(-\infty) = 0, \Phi(+\infty) = 1$, note that $\phi(z)$ is symmetric over the y axis, thus $\Phi(0) = Pr(Z \leq 0) = Pr(Z \geq 0) = 1 - \Phi(0) \Rightarrow Pr(Z \leq 0) = Pr(Z \geq 0) = \Phi(0) = \frac{1}{2}$, use this symmetry, people created the **table for areas under the standard normal curve(z table)** to help with calculations(especially in the past when all calculations are done purely by hand!), from the table, $A(z)$ is the area between 0 and z(here z can be negative), we have $A(z)$ is symmetric over y axis, meaning $A(z) = A(-z)$, and $A(0) = 0$, thus for $z \geq 0$, $\Phi(z) = \frac{1}{2} + A(z)$, $z < 0$, $\Phi(z) = \frac{1}{2} - A(z)$

Standard normal random variable is of particular interest because every normal random variable can be normalized(translate and scale) into a standard normal random variable as follow, suppose $X \sim N(\mu, \sigma^2)$, then $Z := \frac{X - \mu}{\sigma} \sim N(0, 1)$

Problems:

1. Find the expected value and the standard deviation(by inspection) of the normal random variable with the density function $\dfrac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-6}{4}\right)^2}$

$X \sim N(6, 4^2)$, thus $E(X) = 6$, $\sigma(X) = 4$

2. Let Z be a standard normal random variable. Calculate the following probabilities using the table for areas under the standard normal curve(z table)
$Pr(-1.5 \leq Z \leq 0) = 0.4332$, $Pr(0.83 \leq Z) = 0.2033$, $Pr(-1 \leq Z \leq 2.5) = 0.8351$, $Pr(Z \leq 1) = 0.8413$

Use the symmetry of the graph of $A(z)$ and the z-table

$Pr(-1.5 \leq z \leq 0) = Pr(0 \leq z \leq 1.5) = A(1.5) = 0.4332$

$Pr(0.83 \leq z) = Pr(0 \leq z) - Pr(0 \leq z \leq 0.83) = \frac{1}{2} - A(0.83) = 0.2033$

$Pr(-1 \leq z \leq 2.5) = Pr(-1 \leq z \leq 0) + Pr(0 \leq z \leq 2.5) = Pr(0 \leq z \leq 1) + Pr(0 \leq z \leq 2.5)$

$= A(1) + A(2.5) = 0.8351$

$Pr(z \leq 1) = Pr(z \leq 0) + Pr(0 \leq z \leq 1) = \frac{1}{2} + A(1) = 0.8413$

3. Suppose that the life span of a certain automobile tire is normally distributed with $\mu = 28,000$ miles, $\sigma = 2500$ miles

(a) Find the probability that a tire will last between $33,000$ and $35,500$ miles

(b) Find the probability that a tire will last more than $33,000$ miles

Let X be the life span, then $X \sim N(\mu, \sigma) = N(28000, 2500)$

recall $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X - 28000}{2500} \sim N(0,1)$

(a) $Pr(33000 \leq X \leq 35000) = Pr\left(\dfrac{33000-28000}{2500} \leq Z \leq \dfrac{35000-28000}{2500}\right)$

$= Pr(2 \leq Z \leq 3) = Pr(0 \leq Z \leq 3) - Pr(0 \leq Z \leq 2) = A(3) - A(2)$

(b) $Pr(X \geq 33000) = Pr\left(Z \geq \dfrac{33000-28000}{2500}\right) = Pr(Z \geq 2)$

$= Pr(0 \leq Z) - Pr(0 \leq Z \leq 2)$

$= \frac{1}{2} - A(2)$

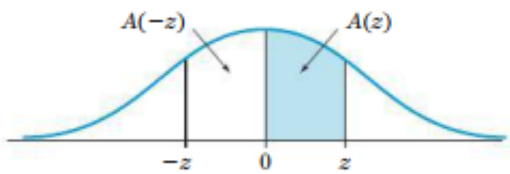

TABLE 1 Areas under the Standard Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3820
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998