0.1 General topology

Definition 0.1.1. A **topological space** X is a set with **topology** $\tau \subseteq \mathscr{P}(X)$, such that $\varnothing, X \in \tau$, $U_i \in \tau \Rightarrow \bigcup_i U_i \in \tau$, $U, V \in \tau \Rightarrow U \cap V \in \tau$, elements in τ are **open sets**, complements of open sets are **closed sets**
N is a **neighborhood** of $A \subseteq X$ if $A \subseteq U \subseteq N \subseteq X$ for some open set U
x is a **limit point** of A if any neighborhood of x intersects A. x is a **limit** of $\{x_n\}$ if for any neighborhood U of x, all but finitely many lies in U
A **subspace** is $A \subseteq X$ with **subspace topology** given by $\{U \cap A | U \in \tau\}$

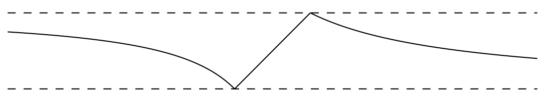
Definition 0.1.2. $X \xrightarrow{f} Y$ is **continuous** at x if for any neighborhood V of $y = f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$. Then f is continuous iff $f^{-1}(V)$ is open for any open set $V \subseteq Y$

Definition 0.1.3. A **base** for τ is $B \subseteq \tau$ such that B covers X and for any $U_1, U_2 \in B$ such that $U_1 \cap U_2 \neq \varnothing$, there exists $U_3 \in B$ such that $U_3 \subseteq U_1 \cap U_2$
A **local base** for τ at x is a collection of neighborhoods $B(x)$ of x such that any neighborhood of x contain an element of $B(x)$
A **subbase** for τ is $B \subseteq \tau$ such that B generates τ, i.e. by arbitrary union of finite intersections, equivalently, τ is the smallest topology containing B. Here empty union and empty intersection are $\varnothing$ and X

Definition 0.1.4. X is **first countable** if each point has a countable local base
X is **second countable** if it has a countable base

Definition 0.1.5. X is **regular** if any point and a disjoint closed set have disjoint neighborhoods. X is **normal** if disjoint closed sets have disjoint neighborhoods

Definition 0.1.6. $\{A_i\}$ can be **completely separated** if $\{A_i\}$ can be completely separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. Closed subsets $\{A_i\}$ can be **perfectly separated** if $\{A_i\}$ can be perfectly separated by a continuous function $X \xrightarrow{f} \mathbb{R}$. $\mathbb{R}$ can be replaced with I considering $\mathbb{R} \to I$, $x \mapsto \begin{cases} \frac{x}{x-1} & x \leq 0 \\ x & 0 \leq x \leq 1 \\ \frac{2}{x+1} & x \geq 1 \end{cases}$ and $I \hookrightarrow \mathbb{R}$



Definition 0.1.7 (Kolmogorov classification of topological spaces)**.** X is a T_0 **space** if for any two distinct points in X, at least one of them has a neighborhood which doesn't intersect the other piont, i.e. they are **topologically distinguishable**
X is a T_1 **space** if for any two distinct points in X, each of them has a neighborhood which doesn't intersect the other point. $T_1 \Leftrightarrow$ points are closed
X is a T_2 **space** or **Hausdorff space** if any two distinct points have disjoint neighborhoods. Then the limit of $\{x_n\}$ is unique, denotes the limit $x = \lim x_n$
X is a $T_{2½}$ **space** or **Urysohn space** if any two distinct points have disjoint closed neighborhoods
X is a T_3 **space** if X is regular Hausdorff
X is a $T_{3½}$ **space** if X is completely regular Hausdorff
X is a T_4 **space** if X is normal T_1 space $\Leftrightarrow$ normal Hausdorff
X is a T_5 **space** if X is completely normal Hausdorff
X is a T_6 **space** if X is perfectly normal $\Leftrightarrow$ perfectly normal Hausdorff

Definition 0.1.8. The **box topology** on $\prod_{i\in I} X_i$ has base $\left\{\prod_{i\in I} U_i \middle| U_i \subseteq X_i \text{ open}\right\}$

Lemma 0.1.9. X is Hausdorff iff the diagonal $\{(x,x)|x \in X\}$ is closed

Definition 0.1.10. $X \times I \xrightarrow{F} Y$ is a **homotopy** between $X \xrightarrow{f_0,f_1} Y$ if $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$, write $f_t = F(\cdot,t)$. $X \xrightarrow{f} Y$ is a **homotopy equivalence** if there is $Y \xrightarrow{g} X$ such that $gf \simeq 1_X$, $fg \simeq 1_Y$

Definition 0.1.11. $X \xrightarrow{f} Y$ is a **topological embedding** if f is injective and $f : X \to f(X)$ is a homeomorphism

Definition 0.1.12. $K \subseteq X$ is **compact** if any open cover has a finite subcover. Equivalently, K is disjoint from the intersection of a family of closed sets, then K is disjoint from the intersection of finitely many of them
X is **locally compact** if there is a compact neighborhood for each point
$Y \subseteq X$ is **precompact** if $\overline{Y}$ is compact

Definition 0.1.13. $A \subseteq X$ is **dense** if $\overline{A} = X$
X is **separable** if X has a countable dense subset

Definition 0.1.14. $X_\alpha \subseteq X$, $\{X_\alpha\}$ is **locally finite** if for any $x \in X$, there is a neighborhood of x intersecting only finitely many X_α's
$\mathcal{U} = \{U_\alpha\}$, $\mathcal{V} = \{V_\beta\}$ are covers of X, $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for any V_β, there exists U_α containing V_β
X is **paracompact** if every open cover has a locally finite open refinement

Lemma 0.1.15. Closed subsets of compact space are closed
The image of a compact set is compact
Compact subsets of a Hausdorff space are closed

X compact, Y Hausdorff, injective maps are embeddings

Lemma 0.1.16. X is compact, Y is Hausdorff, an injective map $X \xrightarrow{f} Y$ is a topological embedding

Proof. $f : X \to f(X)$ is a continuous bijection. If $K \subseteq X$ is closed, K is also compact since X is compact, thus $f(K)$ is compact, $f(K)$ is also closed since Y is Hausdorff ☐

Definition 0.1.17. X is called **connected** if it can be written as the union of two open subsets
X is called **locally connected** if for any $x \in X$, there is a local basis that are connected

Proposition 0.1.18. Connected components are closed
Connectedness and local path connectedness implies path connectedness

Remark 0.1.19. Connected components may not be open

Definition 0.1.20. $E \xrightarrow{p} B$ has **lift extension property** for (X,A) if for any $X \xrightarrow{f} B$, a lift $A \xrightarrow{\tilde{f}} E$ can be extended to $\tilde{f} : X \to E$

$$\begin{tikzcd} A \arrow[r, "\tilde{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$

$E \xrightarrow{p} B$ has **homotopy lifting property** for (X,A) if it has lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$

Proposition 0.1.21. If (X,A) satisfies homotopy extension property, and A is contractible, then the quotient map $X \xrightarrow{q} X/A$ is a homotopy equivalence

Proof. Consider $X \times \{0\} \cup A \times I \to A \hookrightarrow X$, where $(x, 0) \mapsto x$, $(a, 1) \mapsto *$ can be extended to $f : X \times I \to X$, $f_0 = 1_X$, $f_1(A) = \{*\}$, thus f_1 induces $r : X/A \to X$, $f_1 = rq$, $X \times I \xrightarrow{f} X \xrightarrow{q} X/A$ also induce $g : X/A \times I \to X/A$, where $qf_t = g_t q$, and $g_0 = 1_{X/A}$, $g_1 = qr$ thus $qr \simeq 1_{X/A}$ ☐

Definition 0.1.22. $U \subseteq X$ is open if $U \cap K$ is open for any compact subspace $K \subseteq X$ defines a topology. Equivalently, $F \subseteq X$ is closed if $F \cap K$ is closed for any compact subspace $K \subseteq X$. X is **compactly generated** if X has this topology

Definition 0.1.23. A map is **proper** if the preimage of a compact set is compact
A map is **discrete** if the preimage of a discrete set is discrete

Definition 0.1.24. X has **discrete topology** if $\tau = \mathcal{P}(X)$. X has **trivial topology** if $\tau = \{\varnothing, X\}$

Properties of discrete topology

Proposition 0.1.25. Suppose X has discrete topology
(a) Any map $f : Y \to X$ is continuous iff $f^{-1}(x)$ is open for all $x \in X$
(b) If continuous maps $f, g : X \to X$ are homotopic, then they are actually the same

Proof.
(a) For any subset $U \subseteq X$, $f^{-1}(U) = \bigcup_{x \in U} f^{-1}(x)$ is open
(b) If $F : X \times I \to X$ is a homotopy, then the restriction on $\{x\} \times I$ is gives a continuous map $I \to X$, the image has to be connected, thus the restriction is a constant, thus $f(x) = F(x, 0) = F(x, 1) = g(x)$ ☐

Pasting lemma

Lemma 0.1.26. $F_i \subseteq X$ are closed, $\bigcup_i F_i = X$, $f|_{F_i}$ are continuous, then f is continuous

X compact + Y Hausdorff => f:X->Y quotient map

Lemma 0.1.27. If X is compact, Y is Hausdorff, a surjective continuous map $f : X \to Y$ is a quotient map

Proof. Let's use the universal property of quotient space, consider a continuous map $g : X \to Z$ such that g maps fibers of f to points, thus we have a map $\tilde{g} : Y \to Z$, $\tilde{g}f = g$, for any closed set F in Z, so is $K = g^{-1}(F) = f^{-1}(\tilde{g}^{-1}(F))$, since X is compact, so is K, hence $f(K) = \tilde{g}^{-1}(F)$ is compact, and since Y is Hausdorff, $\tilde{g}^{-1}(F)$ is closed ☐

X locally compact+Hausdorff, F closed iff F intersects K is compact for any K compact

Lemma 0.1.28. X is locally compact, Hausdorff, $F \subseteq X$ is closed iff $F \cap K$ is compact for any compact subset $K \subseteq X$

Proof. F closed $\Rightarrow$ $F \cap K$ closed. Conversely, suppose $F \cap K$ is compact for any compact subsets $K \subseteq X$, for any $x \notin F$, there is a compact set K containing an open neighborhood U of x, $F \cap K$ is compact thus closed, hence $G = U - F \cap K$ is an open neighborhood of x which is disjoint of F, hence F is closed ☐

Lemma 0.1.29. X, Y are locally compact, Hausdorff, $p : X \to Y$ is continuous, proper, then p is closed

Proof. Suppose $F \subseteq X$ is closed, since $p\left(F \cap p^{-1}(K)\right) = p(F) \cap K$, by Lemma 0.1.28, we can take any $K \subseteq Y$ compact, hence F is closed ☐

Definition 0.1.30. X is noncompact, the **Alexandorff extension** of X is $X^* = X \cup \{\infty\}$ with open sets $\varnothing, X^*$, open sets in X and complements of closed compact sets of X
$X \hookrightarrow X^*$ is an open topological embedding
If X is also locally compact Hausdorff, X^* is the **one point compactification** of X which is Hausdorff

X,Y locally compact Hausdorff, f:X->Y proper, f send discrete sets to discrete sets

Lemma 0.1.31. X, Y are locally compact Hausdorff, $X \xrightarrow{f} Y$ is proper, then f sends discrete sets to discrete sets

Proof. Suppose $A \subseteq X$ is discrete, $x_0 \in A$, $y_0 = f(x_0) \in Y$, K is a compact neighborhood of y_0, then $f^{-1}(K)$ is a compact neighborhood of x_0, thus $f^{-1}(K) \cap A$ is finite, so is $K \cap f(A)$, since Y is Hausdorff, there is a neighborhood U of y_0 such that $U \cap f(A) = y_0$ ☐

Lemma 0.1.32. X, Y are locally compact, $X \xrightarrow{p} Y$ is proper and discrete, then $p^{-1}(y)$ is finite, and for any neighborhood V of $p^{-1}(y)$, there is a neighborhood U of y such that $p^{-1}(U) \subseteq V$

Lemma 0.1.33. X, Y are locally compact Hausdorff, $X \xrightarrow{p} Y$ is a proper local homeomorphism, then p is a finite sheeted covering

Definition 0.1.34. The **compact-open topology** on Y^X is given by a subbase $V(K, U) := \left\{f \in Y^X \middle| f(K) \subseteq U\right\}$, with $K \subseteq X$ compact and $U \subseteq Y$ open
A **normal family** $\{f_i\}$ is a precompact subset of Y^X

Lemma 0.1.35. $\{f_n\}$ converges pointwise on X iff $\{f_n\}$ converges in Y^X with the product topology $\prod_{x \in X} Y$. Hence we call the product topology the **topology of pointwise convergence**

Proof. If f_n converges pointwise on X to f, then for any neighborhood V_i of $f(x_i)$, $i = 1, \cdots, k$, V_k contains all but finitely many $f_n(x_i)$, thus for n big enough, $f_n \in V_1 \cap \cdots \cap V_k \cap \prod_{x \neq x_0} Y$, i.e. $\{f_n\}$ converges to f in Y^X ☐

Theorem 0.1.36. X is compact, Y is a complete metric space, then the topology induced by metric $d(f, g) = \sup_{x \in X} d(f(x), g(x))$ is the same as the compact-open topology on Y^X

Theorem 0.1.37. $Y^* \cong Y$

Theorem 0.1.38. The composition $Z^Y \times Y^X \to Z^X, (g, f) \mapsto g \circ f$ is continuous, in particular, if $X = *$, then this becomes the evaluation map $\mathsf{eval} : Z^Y \times Y, (f, y) \mapsto f(y)$

Theorem 0.1.39. $Z^{X \times Y} \cong \left(Z^Y\right)^X$

Definition 0.1.40. A topological space X is reducible if $X = X_1 \cup X_2$, X_1, X_2 are proper nonempty closed subsets, $X_1 \nsubseteq X_2$, $X_2 \nsubseteq X_1$, X is **irreducible** if not reducible

Definition 0.1.41. A topological space X is **Noetherian** if $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$ terminates, $\dim V = \sup_d (X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_d)$, V_i's are closed and irreducible

Tychonoff's theorem

Theorem 0.1.42 (Tychonoff's theorem)**.** $\{K_i\}_{i \in I}$ are compact, so is $\prod_{i \in I} K_i$

Proposition 0.1.43. Connected sets of $\mathbb{R}$ are intervals (a, b), $[a, b)$, $(a, b]$ or $[a, b]$

Jordan curve theorem

Theorem 0.1.44 (Jordan curve theorem)**.** $S^n \xrightarrow{i} \mathbb{R}^{n+1}$ is injective thus an open embedding by Lemma 0.1.16, denote $X = i(S^n)$, then $Y = \mathbb{R}^{n+1} \setminus X$ consists of exactly two connected components, the interior U which is bounded, and the exterior V which is not. When $n = 1$, U and V are homeomorphic to D and $\mathbb{R}^2 \setminus D$

Definition 0.1.45. A **locally closed set** X is the intersection of an open subset and a closed subset. Equivalently, X is relatively open in $\overline{X}$. A **constructible set** X if it is a finite union of locally closed sets

Lefschetz fixed point theorem

Theorem 0.1.46 (Lefschetz fixed point theorem)**.** X is a compact triangulable space of dimension n, the **Lefschetz number** of f is $\sum_{k=0}^{n} \mathrm{tr}(f_*|_{H_k(X;\mathbb{Q})})$. If the Lefschetz number of f is nonzero, then f has fixed points. The converse is not true, i.e. even if the Lefschetz number is zero, then could be fixed points
If $f = \mathrm{id}_X$, then the Lefschetz number is the Euler characteristic χ

Definition 0.1.47. The **join** of X, Y is

$$X * Y = \frac{X \times Y \times I}{(x, y_1, 0) \sim (x, y_2, 0), (x_1, y, 1) \sim (x_2, y, 1)}$$

We can also interpret it as all possible paths from X to Y. In general, $\underset{i}{*} X_i$ can be thought of as finite sum $\sum_i t_i x_i$, $t_i \in I$, $x_i \in X_i$

0.2 Retract

Definition 0.2.1. $A \xrightarrow{i} X$ is inclusion. A **deformation** of A into $B \subseteq X$ in X is a homotopy $A \xrightarrow{f_t} X$ such that $f_0 = i$ and $f_1(A) \subseteq B$, onto if equality holds. $X \xrightarrow{r} A$ is a **retraction** if $ri = 1_A$. r is a **weak retraction** if inclusion $A \xrightarrow{i} X$ has a left homotopy inverse, i.e. $ri \simeq 1_A$. A **deformation retraction** is a deformation $X \xrightarrow{f_t} X$ such that $f_1 = ri$ for some retraction $X \xrightarrow{r} A$. Deformation retraction f_t is **strong** if $f_t|_A = 1_A$
X is **contractible** if X deformation retracts onto a point
(X, A) is a **good pair** if A is a strong neighborhood deformation retract of X

Some rudimentary lemma about retract and deformation

Lemma 0.2.2. $A \xrightarrow{i} X$ is inclusion
(1) X is deformable into A iff i is a **weak section**, namely i has a right homotopy inverse, i.e. $ir \simeq 1_X$
(2) i is a homotopy equivalence iff A is a weak retract of X and X is deformable into A
(3) If X is deformable into a retract A, then A is a deformation retract of X
(4) If (X, A) is cofibered, then A is a weak retract of X iff A is a retract of X

Proof.
(1) If $X \times I \xrightarrow{H} X$ is a homotopy from 1_X to ir, then H is a deformation of X into A since $H_0 = 1_X$, $H_1(X) \subseteq A$. If H is a deformation of X into A, since $H_1(X) \subseteq A$, define $X \xrightarrow{r} A$ such that $ir = H_1$, then r is a right homotopy inverse of i
(2) i is a homotopy equivalence $\Leftrightarrow$ there exists $X \xrightarrow{r} A$ such that $ri \simeq 1_A$, $1_X \overset{H}{\simeq} ir \Leftrightarrow r$ is a weak retract, H is a deformation of X into A
(3) $X \xrightarrow{r} A$ is a retraction, $X \times I \xrightarrow{H} X$ is a deformation of X, then $1_X \simeq ir'$ for some $X \xrightarrow{r'} A$, hence $r \simeq rir' = r' \Rightarrow 1_X \simeq ir \simeq ir$ giving a deformation retract
(4) $A \times I \xrightarrow{H} A$ is a homotopy from ri to 1_A, since $r(a) = H_0(a)$ and (X, A) is cofibered, we have $X \times I \xrightarrow{F} A$, then $F_0 = r$, $F_1 i = 1_A$, i.e. r is homotopic to retraction F_1 ☐

Definition 0.2.3. $\mathcal{C}$ is a class of topological spaces closed under homeomorphism and closed subsets. X is an **absolute retract** for $\mathcal{C}$ if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow$ X is a retract of Y. X is an **absolute neighborhood retract** for $\mathcal{C}$ if for $Y \in \mathcal{C}$, embedding $X \hookrightarrow Y$ is closed $\Rightarrow$ X is a neighborhood retract of Y

0.3 Covering space

Definition 0.3.1. A covering space is a fiber bundle with discrete fibers

Unique lifting iff fundamental group is a subgroup

Proposition 0.3.2. $Z \xrightarrow{p} X$ is a covering, $f(y_0) = p(z_0)$. f lifts $\tilde{f} : Y \to Z$ with $f(y_0) = z_0$ iff $f_*\pi_1(Y, y_0) \leq p_*\pi_1(Z, z_0)$

$$\begin{tikzcd} & Z \arrow[d, "p"] \\ Y \arrow[ur, "\exists_1 \tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proposition 0.3.3. Covering $Y \xrightarrow{p} X$ is regular if $\mathtt{Aut}(Y/X)$ is a normal subgroup of $\pi_1(X, x_0)$

Proof. Assume $p(y_1) = p(y_2) = x_0$, by Proposition 0.3.2, $p_*\pi_1(Y, y_1) = p_*\pi_1(Y, y_2)$ are conjugate, hence normal $\square$