MATH606 - Algebraic Geometry I

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Contents

1 Homeworks

1.1 Homework1

Problem 1.1. First not that

$$\Lambda \subseteq \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, A \in \mathrm{M}(2, \mathbb{Z})$$

Hence we have

$$\Lambda = \Lambda' \quad \Leftrightarrow \quad \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = A \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}, \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = B \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, A, B \in \mathrm{M}(2, \mathbb{Z})$$

Which is equivalent to $A \in \mathrm{GL}(2, \mathbb{Z})$

Problem 1.2. Since $\mathbb{C}$ is the universal cover of $\mathbb{C}/\Lambda'$, $f \circ \pi : \mathbb{C} \to \mathbb{C}/\Lambda'$ has a lift $F : \mathbb{C} \to \mathbb{C}$, and locally we have $F = \pi'|_V^{-1} \circ f \circ \pi|_U$, thus F is holomorphic

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d, "\pi"] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda \arrow[r, "f"] & \mathbb{C}/\Lambda' \end{tikzcd}$$

Fix $\omega \in \Lambda$, since $\pi(z+\omega) = \pi(z)$ for any $z \in \mathbb{C}$, we have $F(z+\omega) - F(z) \in \Lambda'$, hence $F(z+\omega) - F(z)$ is a continuous function of z but Λ' is discrete, thus $F(z + \omega) - F(z) \equiv C_\omega$, where $C_\omega \in \Lambda'$ is a constant. Then $F'(z + \omega) = F'(z)$ which shows $F' : \mathbb{C} \to \mathbb{C}$ is doubly periodic function, thus induces $G : \mathbb{C}/\Lambda \to \mathbb{C}$ with $F = G \circ \pi$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F'"] \arrow[d, "\pi"] & \mathbb{C} \\ \mathbb{C}/\Lambda \arrow[ur, "G"] & \end{tikzcd}$$

Thus G must be a constant, so is F', therefore F has the form $F(z) = \alpha z + \beta$. Then for any $\omega \in \Lambda$, we have $F(\omega) - F(0) = \alpha\omega \in \Lambda'$, thus $\alpha\Lambda \subset \Lambda'$ If f is biholomorphic, then $\pi' \circ F = f \circ \pi \Rightarrow \pi \circ F^{-1} = f^{-1} \circ \pi'$, which implies $\left\{ \begin{array}{c} \alpha\Lambda \subset \Lambda' \\ \alpha^{-1}\Lambda' \subset \Lambda \end{array} \right. \Rightarrow \alpha\Lambda = \Lambda'$

$$\begin{tikzcd} \mathbb{C} \arrow[d, "\pi"] & \mathbb{C} \arrow[l, "F^{-1}"'] \arrow[d, "\pi'"] \\ \mathbb{C}/\Lambda & \mathbb{C}/\Lambda' \arrow[l, "f^{-1}"'] \end{tikzcd}$$

Conversely, if $\alpha\Lambda = \Lambda'$, $\pi \circ F^{-1}$ is doubly periodic and induce f^{-1}, hence f is biholomorphic

Problem 1.3. Suppose $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \mathrm{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$, define $\Lambda' = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \frac{\omega_2}{\omega_1}$, we have $\omega_1\Lambda' = \Lambda$, thus X and $X(\tau)$ are biholomorphic.
$X(\tau)$ and $X(\tau')$ are biholomorphic if and only if $\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \alpha A \begin{pmatrix} \tau \\ 1 \end{pmatrix}, \alpha \in \mathbb{C} - \{0\}, A \in \mathrm{SL}(2, \mathbb{Z})$ If $X(\tau)$ and $X(\tau')$ are biholomorphic, then $\mathbb{Z} + \mathbb{Z}\tau' = \Lambda' = \alpha\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\alpha\tau$ for some $\alpha \in \mathbb{C} - \{0\}$, thus $\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = A \begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \alpha A \begin{pmatrix} \tau \\ 1 \end{pmatrix}$, for some $A \in \mathrm{SL}(2, \mathbb{Z})$, the other direction is easy

Problem 1.4. Assume it is not the case, then $\exists \delta > 0$ and $c \in \mathbb{C}$ such that $B(c, \delta) \cap f(\mathrm{X}') = \varnothing$, then consider non-constant holomorphic function $\frac{1}{f - c} : \mathrm{X}' \to \mathbb{C}$, then it is bounded since $\left|\frac{1}{f - c}\right| \leq \delta^{-1}$, by Riemann's removable singularity theorem, we can extend $\frac{1}{f - c}$ to a non-constant holomorphic function $g : \mathrm{X} \to \mathbb{C}$, but since X is compact, g should be a constant, that is a contradiction

Problem 1.5. $f'(z) = \dfrac{1}{2}\left(1 - \dfrac{1}{z^2}\right)$ when $z \neq 0$, thus $1, -1$ are branch points.

Consider the chart $(P^1 - \{0\}, \varphi)$ with $\varphi(z) = \dfrac{1}{z}$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "f"] \arrow[dr] & \mathbb{P}^1 - \{0\} \arrow[d, "\varphi"] \\ & \mathbb{C} \end{tikzcd}$$

Thus $g(z) = \varphi \circ f(z) = \dfrac{z}{2(z^2+1)}$, $g'(z) = \dfrac{1 - z^2}{2(z^2+1)}$, hence 0 is not a branch point

1.2 Homework2

Problem 1.6. Using the exactness of

$$0 \to \mathbb{C} \hookrightarrow \mathcal{O} \xrightarrow{d} \mathcal{O} \to 0$$

we have exact sequence

$$0 \to H^0(\Omega, \mathbb{C}) \to H^0(\Omega, \mathcal{O}) \xrightarrow{d} H^0(\Omega, \mathcal{O}) \to H^1(\Omega, \mathbb{C}) \to H^1(\Omega, \mathcal{O})$$

but according to Mittag-Leffler's theorem, we know that $H^1(\Omega, \mathcal{O}) = 0$, thus we have exact sequence

$$0 \to \mathbb{C}(\Omega) \to \mathcal{O}(\Omega) \xrightarrow{d} \mathcal{O}(\Omega) \to H^1(\Omega, \mathbb{C}) \to 0$$

Thus $\dim \operatorname{coker} d = \dim H^1(\Omega, \mathbb{C})$, also from this we know that $H^1(\Omega, \mathbb{C}) = 0$ if Ω is simply connected

(i)

For any $k \geq 0$, Take Ω to be $\mathbb{C} - \{1, \cdots, k\}$, Consider covering

$$U_1 := \mathbb{C} - [1, \infty), U_i := \mathbb{C} - (-\infty, i-1] - [i, \infty), \cdots, U_{k+1} := \mathbb{C} - (-\infty, k], (\forall 2 \leq i \leq k)$$

Then $\{U_i\}_{i=1}^{k+1}$ forms a Leray covering since U_i is simply connected and $H^1(U_i, \mathbb{C}) = 0$, hence $H^1(\Omega, \mathbb{C}) = H^1(\mathcal{U}, \mathbb{C})$

Notice that $U_i \cap U_j = \{\mathrm{Im} z > 0\} \cup \{\mathrm{Im} z < 0\}, i \neq j$

For any $(c_{ij}) \in Z^1(\mathcal{U}, \mathbb{C})$, according to cocycle relation, we only need to know $\{c_{12}, c_{23}, \cdots, c_{k,k+1}\}$, and each $c_{i,i+1}$ has two components, let's denote as $c^+_{i,i+1}, c^-_{i,i+1} \in \mathbb{C}$, thus $Z^1(\mathcal{U}, \mathbb{C}) \cong \mathbb{C}^{2k}$, on the other hand, for any $(c_i) \in C^0(\mathcal{U}, \mathbb{C})$, $\delta((c_i)) = (c_i - c_j)$, thus $B^1(\mathcal{U}, \mathbb{C})$ is the subgroup of $Z^1(\mathcal{U}, \mathbb{C})$ consists of exactly elements with $c^+_{i,i+1} = c^-_{i,i+1}$, thus $Z^1(\mathcal{U}, \mathbb{C}) \cong \mathbb{C}^k$ and $H^1(\mathcal{U}, \mathbb{C}) = Z^1(\mathcal{U}, \mathbb{C})/B^1(\mathcal{U}, \mathbb{C}) \cong \mathbb{C}^k$, thus $\dim H^1(\mathcal{U}, \mathbb{C}) = k$

(ii)

Take Ω to be $\mathbb{C} - \mathbb{Z}_{>0}$, and consider covering

$$U_1 := \mathbb{C} - [1, \infty), \cdots, U_i := \mathbb{C} - (-\infty, i-1] - [i, \infty), \cdots, (i \geq 2)$$

Then $\{U_i\}_{i=1}^{\infty}$ forms a Leray covering since U_i is simply connected and $H^1(U_i, \mathcal{C}) = 0$, hence $H^1(\Omega, \mathcal{C}) = H^1(\mathcal{U}, \mathcal{C})$

Notice that $U_i \cap U_j = \{\mathrm{Im} z > 0\} \cup \{\mathrm{Im} z < 0\}, i \neq j$

For any $(c_{ij}) \in Z^1(\mathcal{U}, \mathbb{C})$, according to cocycle relation, we only need to know $\{c_{12}, \cdots, c_{k,k+1}, \cdots\}$, and each $c_{i,i+1}$ has two components, let's denote as $c^+_{i,i+1}, c^-_{i,i+1} \in \mathbb{C}$, on the other hand, for any $(c_i) \in C^0(\mathcal{U}, \mathbb{C})$, $\delta((c_i)) = (c_i - c_j)$, thus $B^1(\mathcal{U}, \mathbb{C})$ is the subgroup of $Z^1(\mathcal{U}, \mathbb{C})$ consists of exactly elements with $c^+_{i,i+1} = c^-_{i,i+1}$, thus $H^1(\mathcal{U}, \mathbb{C}) = Z^1(\mathcal{U}, \mathbb{C})/B^1(\mathcal{U}, \mathbb{C})$ is of infinite dimension

Problem 1.7. **(a)**

Suppose $\alpha_X(f) = 0, f \in H^0(X, \mathcal{F}) = \mathcal{F}(X)$, then $\alpha_x(f|_x) = 0$, $f|_x$ being the germ of f at x, by the injectivity of α, we know that $f|_x = 0 = 0|_x$, thus $\exists U_x \ni x$ such that $f|_{U_x} = 0$, and also $X = \bigcup_{x \in X} U_x$, thus by the sheaf axiom, we know that $f = 0$, hence α_X is injective

(b)

We know that $\beta\alpha = 0$, thus $(\beta\alpha)_x(f|_x) = 0, \forall f \in \mathcal{F}(X)$, as argued above, we know that $f = 0$, thus $\beta_X \alpha_X = (\beta\alpha)_X = 0$

On the other hand, $\forall g \in \ker \beta_X$, $\beta_X(g) = 0$, thus $\beta_x(g|_x) = 0$, $\exists V_x \ni x$ such that $\beta_{V_x}(g) = 0$, also, $\exists f_x \in \mathcal{F}(U_x)$ such that $\alpha_{U_x}(f_x) = g|_{U_x}$ where $x \in U_x \subset V_x$. Thus, for any $U_x \cap U_y \neq \varnothing$, then $\alpha_{U_x \cap U_y}(f_x - f_y) = 0$, but for the same reason we know that $\alpha_{U_x \cap U_y}$ is injective, thus $f_x = f_y$ on $U_x \cap U_y$, by sheaf axiom, there exists $f \in \mathcal{F}(X)$ such that $f|_{U_x} = f_x$, hence $g = \alpha(f)$, therefore $\mathrm{im}\alpha_X = \ker \beta_X$

(c)

$\forall g \in H^0(X, \mathcal{G}), \forall \mathcal{U}, \beta_{U_i}(g|_{U_i}) = \beta_X(g)|_{U_i}, g - g|_{U_i \cap U_j} = 0 \in \ker \beta_{U_i \cap U_j} = \mathrm{im}\alpha_{U_i \cap U_j}$, and since $\alpha_{U_i \cap U_j}$ is injective, thus $\delta \circ \beta_X(g) = 0$

On the other hand, $\forall h \in \ker\delta, \exists\mathcal{U}, \exists g_i \in \mathcal{G}(U_i)$, such that $\beta_{U_i}(g_i) = h|_{U_i}$, and $g_i - g_j|U_i \cap U_j \in \ker\beta_{U_i\cap U_j} = \mathrm{im}\alpha_{U_i\cap U_j}$, $\exists_1 f_{ij} \in \mathcal{F}(U_i \cap U_j)$ with $\alpha_{U_i\cap U_j}(f_{ij}) = g_i - g_j|U_i \cap U_j$, then $\exists\{\rangle \in \mathcal{F}(U_i)$ such that $f_{ij} = f_i - f_j|U_i \cap U_j$, observe that on $U_i \cap U_j$

$$\begin{aligned}(\alpha_{U_i}(f_i) - g_i) - (\alpha_{U_j}(f_j) - g_j) &= (\alpha_{U_i}(f_i) - \alpha_{U_j}(f_j)) - (g_i - g_j)\\ &= \alpha_{U_i\cap U_j}(f_i - f_j) - (g_i - g_j)\\ &= 0\end{aligned}$$

Thus $\exists g \in \mathcal{G}(X)$ such that $g|_{U_i} = \alpha_{U_i}(f_i) - g_i$, then we have $\beta_{U_i}(g|_{U_i}) = \beta_{U_i}(\alpha_{U_i}(f_i) - g_i) = \beta_{U_i}(g_i) = h|_{U_i}$, hence $\beta_X(g) = h$, therefore $\mathrm{im}\beta_X = \ker\delta$

Problem 1.8. (a)
Suppose s_1, s_2 are nonzero meromorphic sections of L over X, then there exists nonzero meromorphic function f on X such that $s_2 = fs_1$, if f is constant, surely $\mathrm{div}(s_1) = \mathrm{div}(s_2)$, otherwise, $f : X \to \mathbb{P}^1$ would be a non-constant proper holomorphic mapping which has as many zeros as poles, thus $\mathrm{div}(f) = 0$, therefore $\mathrm{div}(s_1) = \mathrm{div}(f) + \mathrm{div}(s_2) = \mathrm{div}(s_2)$, thus $\deg(L)$ is well-defined
(b)
Define $f_1(z) = z^k$ on U_1 and $f_2(z) = 1$ on U_2, then $f_1 = g_{12}f_2$, thus f_i defines a section s of L_k over $\mathbb{P}^1$, hence $\deg L_k = \mathrm{div}(s) = k$
(c)
Define $f_1(z) = g_{12}(z)$ on U_1 and $f_2(z) = 1$ on U_2, then $f_1 = g_{12}f_2$, thus f_i defines a section s of L_k over $\mathbb{P}^1$, since $g_{12}(z) \neq 0$ is holomorphic on $U_1 \cap U_2$, thus f_1 can only have zeros or poles at 0, hence $\deg L_k = \mathrm{div}(s) = \dfrac{1}{2\pi i}\displaystyle\int_{|z|=1} \frac{g_{12}'(z)}{g_{12}(z)}dz$ by argument principle

Problem 1.9. Suppose L and L^* both have non-trivial global holomorphic sections s_1 and s_2, let $D_1 := \mathrm{div}(S_1)$, $D_2 := \mathrm{div}(S_2)$, then $L \cong L(D_1)$, $L^* \cong L(D_2) \cong L(-D_1)$, which is equivalent to $D_1, D_2 \geq 0$ and $D_1 + D_2 = \mathrm{div}(f)$ for some meromorphic function f on X. Since $0 \leq \deg(D_1 + D_2) = \deg\mathrm{div}(f) = 0$, $D_1 + D_2 = 0$, hence $D_1 = -D_2 \leq 0 \leq D_1$, thus $D_1 = 0$, $L \cong \mathcal{O}$ is trivial
Conversely, if L is trivial, constant could be a global holomorphic section on both L and L^*

Problem 1.10. There exists such a sequence.
Consider $K_n := \{0\} \cup \{re^{i\theta} \in \mathbb{C} | r \in [\frac{1}{n}, n], \theta \in [\frac{1}{n}, 2\pi]\} \subset \mathbb{C}$ is a compact set, we can define a holomorphic function f_n on a open neighborhood of K_n such that $f(0) = 1$ and $f|_{K_n\setminus\{0\}} = 0$, then using Runge's approximation theorem, we can find a polynomial Q_n such that $|Q_n - f_n| < \dfrac{1}{n}$ on K_n, then define $P_n = \dfrac{Q_n}{Q_n(0)}$, we can easily verify that P_n could be a desired sequence

1.3 Homework3

Problem 1.11. In the proof of finiteness theorem, we have $\|s\|_U \leq A\|s\|_V$
where $A := \max_{i,j} \sup_{x \in U_i \cap U_j} \|g_{ij}(x)\|$, $\forall s \in H^0(X, L)$
$\forall s \in H^0(X, L), \operatorname{ord}_{a_i} s_i \geq l$, $\|s\|_U \leq A\|s\|_V \leq 2^{-l} A \|s\|_U$, thus

$$H^0(X, L) \to \bigoplus_{i=1}^{N} \mathbb{C} \otimes \left(\mathcal{O}_{a_i}/m_i^l\right),\ s \mapsto \bigoplus_{i=1}^{N} \left(s_i \bmod z_i^l\right)$$

is injective if $2^l > A$, thus $\dim H^0(X, L) \leq Nl =: C$
But now we have $\|s\|_U \leq A^k \|s\|_V$, $\forall s \in H^0(X, L)$
since $|s_i(x)| = |g_{ij}(x)^k s_j(x)| \leq \|g_{ij}(x)\|^k |s_j(x)|$
$\forall s \in H^0(X, L^k), \operatorname{ord}_{a_i} s_i \geq kl$, $\|s\|_U \leq A\|s\|_V \leq 2^{-kl} A \|s\|_U$, thus

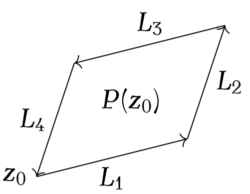
$$H^0(X, L^k) \to \bigoplus_{i=1}^{N} \mathbb{C} \otimes \left(\mathcal{O}_{a_i}/m_i^{kl}\right),\ s \mapsto \bigoplus_{i=1}^{N} \left(s_i \bmod z_i^{kl}\right)$$

is injective if $2^l > A \Rightarrow 2^{kl} > A^k$, thus $\dim H^0(X, L^k) \leq Nkl =: Ck$

Problem 1.12. Suppose there are more than $2k$ fixed points of σ, then consider $f - f \circ \sigma^{-1} : X \to \mathbb{P}^1$ is holomorphic on $X \backslash \{a, \sigma^{-1}(a)\}$ with at least $2k+1$ zeros and with poles of order k at $a, \sigma^{-1}(a)$, but it should have as many poles as zeros which is a contradiction

Problem 1.13. The genus g of $\mathbb{P}^1$ is 0
If $\deg D < 0$, then $\forall f \in H^0(\mathbb{P}^1, \mathcal{O}_D) \subseteq \mathcal{M}(\mathbb{P}^1)$, then $0 = \deg \operatorname{div}(f) \geq \deg(-D) = -\deg D > 0$ which is impossible, hence $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = 0$ and $\dim H^1(\mathbb{P}^1, \mathcal{O}_D) = -1 - \deg D$
If $\deg D \geq 0$, since from the long exact sequence we already knew that $H^1(\mathbb{P}^1, \mathcal{O}'_D) \to H^0(\mathbb{P}^1, \mathcal{O}_D) \to 0$ is exact if $D' \leq D$, and we can always find a D' such that $D' \leq D$ and $\deg D' < 0$, then $0 = \dim H^1(\mathbb{P}^1, \mathcal{O}'_D) \geq \dim H^1(\mathbb{P}^1, \mathcal{O}_D) \geq 0$, thus $\dim H^1(\mathbb{P}^1, \mathcal{O}_D) = 0$ and $\dim H^0(\mathbb{P}^1, \mathcal{O}_D) = 1 + \deg D$
Therefore, we have

$$\begin{aligned} \dim H^0(\mathbb{P}^1, \mathcal{O}_D) &= \max(0, 1 + \deg D) \\ \dim H^1(\mathbb{P}^1, \mathcal{O}_D) &= \max(0, -1 - \deg D) \end{aligned}$$



Problem 1.14.

a.
Since f is elliptic, $\int_{L_1} f(z)dz = -\int_{L_3} f(z)dz$, $\int_{L_2} f(z)dz = -\int_{L_4} f(z)dz$, thus $\int_{\partial P(z_0)} f(z)dz = \int_{L_1} f(z)dz + \int_{L_2} f(z)dz + \int_{L_3} f(z)dz + \int_{L_4} f(z)dz = 0$

b.
As the same reason in (a), we have $\frac{1}{2\pi i}\int_{\partial P(z_0)} \frac{f'(z)}{f(z)}dz = 0$, by argument principle, we know in the interior of $P(z_0)$, there are as many zeros as poles, counted multiplicities

Problem 1.15. Consider $\frac{1}{2\pi i}\int_{\partial P(z_0)} \frac{zf'(z)}{f(z)}dz = \sum_{j=1}^{n}(z_j - w_j)$

On the other hand,

$$\begin{aligned} -\frac{1}{2\pi i}\int_{L_3} \frac{zf'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{L_1} \frac{(z+\omega_2)f'(z)}{f(z)}dz \\ \frac{1}{2\pi i}\int_{L_2} \frac{zf'(z)}{f(z)}dz &= -\frac{1}{2\pi i}\int_{L_4} \frac{(z+\omega_1)f'(z)}{f(z)}dz \end{aligned}$$

Thus

$$\frac{1}{2\pi i}\int_{\partial P(z_0)}\frac{zf'(z)}{f(z)}dz=\frac{1}{2\pi i}\left(\int_{L_1}+\int_{L_2}+\int_{L_3}+\int_{L_4}\right)\frac{zf'(z)}{f(z)}dz=-\frac{\omega_1}{2\pi i}\int_{L_4}\frac{f'(z)}{f(z)}dz-\frac{\omega_2}{2\pi i}\int_{L_1}\frac{f'(z)}{f(z)}dz$$

Hence we only need to show $\frac{1}{2\pi i}\int_{L_1}\frac{f'(z)}{f(z)}dz, \frac{1}{2\pi i}\int_{L_4}\frac{f'(z)}{f(z)}dz \in \mathbb{Z}$, but

$$\frac{1}{2\pi i}\int_{L_1}\frac{f'(z)}{f(z)}dz=\frac{1}{2\pi i}\int_{L_1}\frac{d(f(z))}{f(z)}=\frac{1}{2\pi i}\int_{f(L_1)}\frac{dz}{z}=k$$

For some $k \in \mathbb{Z}$

1.4 Homework4

Problem 1.16. If $f \equiv c$ is a constant, then $P(x, y) = x - c$ is an irreducible polynomial such that $P(f, g) = 0$, so we can assume f, g are not constants, Since $\mathscr{M}(X)$ is a finite algebraic extension of $\mathbb{C}(f)$, there exists rational functions $R_0, \cdots, R_n$ such that $R_0(f) + R_1(f)g + \cdots + R_n(f)g^n = 0$, then after multiplying denominators, we get a polynomial $P(x, y) \in \mathbb{C}[x, y]$ such that $P(f, g) = 0$, since $\mathbb{C}[x, y]$ is a UFD, $P = P_1 \cdots P_k$, where P_i are prime hence irreducible, then $0 = P_1(f, g) \cdots P_k(f, g) \in \mathscr{M}(X)$ which is a field, thus $P_j(f, g) = 0$ for some irreducible polynomial $P_j \in \mathbb{C}[x, y]$

Problem 1.17. If f is an elliptic function of order n, then $f(z + \omega) = f(z), \forall \omega \in \Omega$, which implies $f'(z + \omega) = f'(z), \forall \omega \in \Omega$, thus f' is also elliptic, suppose f has $[P_1], \cdots, [P_k]$ as its poles with multiplicities $r_1, \cdots, r_k$, $\sum r_i = n$, then f' also has $[P_1], \cdots, [P_k]$ as its poles with multiplicities $r_1 + 1, \cdots, r_k + 1$, $\sum r_i = n + k = m$, since $1 \leq k \leq n$, $n + 1 \leq m \leq 2n$
We can find an elliptic function f of order n which has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $1, \cdots, 1, 2n+1-m$, then we get f' is another elliptic function which also has $[P_1], \cdots, [P_{n-m}]$ as its poles with multiplicities $2, \cdots, 2, 2n + 2 - m$, thus f' is of order m

Problem 1.18. $\wp'(z)$ has a pole at $z = 0$ of order 3 and $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ as simple roots, thus

$$\wp'(z) = \lambda \frac{\sigma\left(z - \frac{\omega_1}{2}\right)\sigma\left(z - \frac{\omega_2}{2}\right)\sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma(z)^3}$$

for some $\lambda \in \mathbb{C}$, multiply by z^3 on both sides, and let $z \to 0$, since $\lim\limits_{z\to 0} \frac{z}{\sigma(z)} = 1, \lim\limits_{z\to 0} z^3\wp'(z) = -2$, we have $-2 = -\lambda\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right) \Rightarrow \lambda \frac{2}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)}$
Hence $\wp'(z) = \dfrac{2\sigma\left(z - \frac{\omega_1}{2}\right)\sigma\left(z - \frac{\omega_2}{2}\right)\sigma\left(z - \frac{\omega_3}{2}\right)}{\sigma\left(\frac{\omega_1}{2}\right)\sigma\left(\frac{\omega_2}{2}\right)\sigma\left(\frac{\omega_3}{2}\right)\sigma(z)^3}$

Problem 1.19. $(i) \Rightarrow (ii)$
$g_2 = 60G_4, g_3 = 140G_6 \in \mathbb{R} \Rightarrow G_4, G_6 \in \mathbb{R}$
Since

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2} = \frac{1}{z^2} + 3G_4z^2 + 5G_6z^4 + 7G_8z^6 + 9G_{10}z^8 + \cdots$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=2}^{\infty}(2n-1)(2n-2)G_{2n}z^{2n-3} = -\frac{2}{z^3} + 6G_4z + 20G_6z^3 + 42G_8z^5 + 72G_{10}z^7 + \cdots$$

$$\wp''(z) = \frac{6}{z^4} + \sum_{n=2}^{\infty}(2n-1)(2n-2)(2n-3)G_{2n}z^{2n-4} = \frac{6}{z^4} + 6G_4 + 60G_6z^2 + 210G_8z^4 + 504G_{10}z^6 + \cdots$$

So we can conclude $\wp''(z) - 6\wp(z)^2 + 30G_4 = z\varphi(z)$, where $\varphi(z)$ is a holomorphic elliptic function, hence $\wp''(z) - 6\wp(z)^2 + 30G_4 = 0$, then the coefficients of $z^{2n}(n \geq 1)$ would be $(2n + 1)(2n + 2)(2n + 3)(2n + 4)G_{2n+4} - 6(2n + 3)G_{2n+4}$ minus terms only involving $G_4, G_6, \cdots, G_{2n+2}$ and real numbers, thus by induction, we know $G_{2n+4} \in \mathbb{R}(n \geq 1)$
$(ii) \Rightarrow (iii)$
Since $\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$, if $G_k \in \mathbb{R}(k \geq 3)$, then $\wp(\bar{z}) = \overline{\wp(z)}$
$(iii) \Rightarrow (iv)$
The poles of $\overline{\wp(\bar{z})} = \wp(z)$ are exactly $\overline{\Omega}$, thus $\overline{\Omega} = \Omega$
$(iv) \Rightarrow (i)$
$g_2 = 60G_4 = 60\sum\limits_{\omega\in\Omega^*}\frac{1}{\omega^4} = 60\sum\limits_{\omega\in\overline{\Omega}^*}\frac{1}{\omega^4} = \overline{g_2} \Rightarrow g_2 \in \mathbb{R}$, similarly, $g_6 \in \mathbb{R}$

Problem 1.20. If Ω is real rectangular or real rhombic, Ω is obviously a real lattice
Conversely, if Ω is a real lattice, suppose $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then there exists $\omega \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, otherwise, $\omega_1 \in \mathbb{R}^*, \omega_2 \in i\mathbb{R}^*$ or $\omega_2 \in \mathbb{R}^*, \omega_1 \in i\mathbb{R}^*$, since ω_1, ω_2 are linear independent, but then $\omega = \omega_1 + \omega_2 \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$ which is a contradiction

Since $\omega \in \Omega\backslash(\mathbb{R}\cup i\mathbb{R})$, $\omega+\overline{\omega} \in \mathbb{R}^*, \omega-\overline{\omega} \in i\mathbb{R}^*$, thus $\Omega\cap\mathbb{R}^* \neq \varnothing, \Omega\cap i\mathbb{R}^* \neq \varnothing$, let $\eta_1 = \min_{\eta\in\Omega\cap(0,\infty)} \eta$, then $\Omega\cap\mathbb{R} = \mathbb{Z}\eta_1$, otherwise $\exists\eta \in \mathbb{R}\backslash\mathbb{Z}\eta_1$, then $\eta - \left\lfloor \frac{\eta}{\eta_1} \right\rfloor \eta_1 \in \Omega\cap(0,\infty)$ which is a contradiction
Similarly, $\Omega \cap i\mathbb{R} = \mathbb{Z}\eta_2$ for some $\eta_2 \in i(0,\infty)$. If $\Omega = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$, then Ω is real rectangular, if not, $\exists\gamma \in \Omega \setminus (\mathbb{R} \cup i\mathbb{R})$, such that $|\gamma| = \min_{\omega\in\Omega\backslash(\mathbb{R}\cup i\mathbb{R})} |\omega|$, then $\gamma + \overline{\gamma} = \eta_1$ or $-\eta_1$, otherwise $\gamma + \overline{\gamma} = k\eta_1$ for some $|k| \geq 2$
If $k = 2$, then $\gamma - \eta_1 = \eta_1 - \overline{\gamma} = -\overline{(\gamma - \eta_1)} \Rightarrow \gamma - \eta_1 \in i\mathbb{R} \Rightarrow \gamma \in \mathbb{Z}\eta_1 + \mathbb{Z}(\gamma - \eta_1) \subseteq \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2$
If $k > 2$, then $\gamma-\eta_1 \notin \mathbb{R}\cup i\mathbb{R}$ and $|\gamma-\eta_1| < |\gamma|$, similarly for $k \leq -2$, these are all contradictions
Similarly, we know that $\gamma - \overline{\gamma} = \eta_2$ or $-\eta_2$
Now, for any $\omega \in \Omega\backslash(\mathbb{R}\cup i\mathbb{R})$, $\omega + \overline{\omega} = k\eta_1 = k(\gamma + \overline{\gamma})$ for some $k \neq 0$, then $\omega - k\gamma = k\overline{\gamma} - \overline{\omega} = -\overline{(\omega - k\gamma)} \Rightarrow \omega - k\gamma \in i\mathbb{R}$, if $\omega \neq k\gamma$, then $\omega - k\gamma = l\eta_2 = l(\gamma - \overline{\gamma}) \Rightarrow \omega \in \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, therefore, we have $\Omega = \mathbb{Z}\gamma + \mathbb{Z}\overline{\gamma}$, Ω is real rhombic

Problem 1.21. The number of connected components of $E_{\mathbb{R}}$ is one or two if $p(x) = 0$ has one real root and two nonreal conjugate complex roots or three distinct real roots correspondingly
Since $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ are simple roots of $\wp'(z)$, the three simple roots of $p(x)$ are $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_3}{2}\right)$, since Ω is a real lattice, $G_k \in \mathbb{R}$ and $\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}z^{2n-2}$
If Ω is real rectangular, then $\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right)$ are both real, thus $E_{\mathbb{R}}$ has two connected components
If Ω is real rhombic, then $\wp\left(\frac{\omega_3}{2}\right)$ is real $\wp\left(\frac{\omega_1}{2}\right) \neq \wp\left(\frac{\omega_2}{2}\right)$ are nonreal conjugate, thus $E_{\mathbb{R}}$ has only one connected component

1.5 Homework5

Problem 1.22. $\left(e^{\frac{2\pi ki}{n}}, 0\right)$, $(0 \le k \le n-1)$ are branch points with degree n, apply Riemann-Hurwitz formula, we have $2g - 2 = n(2 \cdot 0 - 2) + n(n-1)$, thus $g = \dfrac{(n-1)(n-2)}{2}$

Problem 1.23. Need to check $0 \to \mathbb{C}_x \to \mathscr{M}_x \xrightarrow{\mathrm{d}} \mathscr{N}_x \to 0$ is exact

$\mathbb{C}_x \to \mathscr{M}_x$ is injective is easy, if we write $\sum_{\nu=-k}^{\infty} a_\nu (z-x)^\nu \in \mathscr{M}_x$, $\mathrm{d}a_0 = 0$ and $\mathrm{res}_x \left(\sum_{\nu=-k}^{\infty} a_\nu (z-x)^\nu\right) = 0 \Rightarrow a_0 = 0$, thus $\mathbb{C}_x \to \mathscr{M}_x \xrightarrow{\mathrm{d}} \mathscr{N}_x$ is exact, also, since $\mathrm{d}(z-x)^\nu = \nu(z-x)^{\nu-1}$, $\mathscr{M}_x \xrightarrow{\mathrm{d}} \mathscr{N}_x$ is surjective. Therefore, we have long exact sequence

$$0 \to \mathbb{C}(X) \to \mathscr{M}(X) \xrightarrow{\mathrm{d}} \mathscr{N}(X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathscr{M}) = 0$$

Thus $H^1(X, \mathbb{C}) \cong \mathscr{N}(X)/\mathrm{d}\mathscr{M}(X)$

Problem 1.24.

Problem 1.25. Define $D = (2g+1)P$, $L = L(D)$ is line bundle satisfies $\deg L > 2g$, using embedding theorem, choose a basis $\{s_0, \cdots, s_N\}$ of $H^0(X, L)$, such that $s_0 = s_D$, $\varphi_L : X \to \mathbb{P}^N$, by sending x to $[s_0, \cdots, s_N]$, but since $s_D(x) \neq 0$, $\forall x \neq P$, hence $\varphi_L : X \setminus \{P\} \to \mathbb{C}^N$ by sending x to $\left(\frac{s_1}{s_0}, \cdots, \frac{s_N}{s_0}\right)$ is also an embedding

Problem 1.26. Since $k^{n+1} = \bigsqcup_{\alpha \in k} \alpha + k^n$, $|k^{n+1}| = |k||k^n|$, by induction and $|k| = q = p^r$, $|k^n| = q^n = p^{nr}$

On the other hand, since $P^n(k) = k^{n+1} - \{0\}/\sim$,

$$|P^n(k)| = \frac{|k^{n+1}| - 1}{|k^*|} = \frac{q^n - 1}{q - 1} = \frac{p^{nr} - 1}{p^r - 1}$$

Problem 1.27. **(a)**

Consider all the picks of two different checkers in the same row, which essentially give picks of two different columns, the two columns given by these picks are all distinct, otherwise, four of the checkers will be centered on the vertices of a rectangle with sides parallel to the sides of the board, hence the number of picks is no more than the number of picks of two different columns. Therefore, we have

$$\sum_{i=1}^{k} \frac{x_i(x_i - 1)}{2} \le \frac{k(k-1)}{2}$$

(b)

First let's state a fact: If $x_i \ge 0$, then

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i,j=1}^{n} x_i x_j \le \sum_{i,j=1}^{n} \left(\frac{x_i^2}{2} + \frac{x_j^2}{2}\right) = n\left(\sum_{i=1}^{n} x_i^2\right)$$

Equality holds if and only if x_i are the same

For any k such that $k - 1 = q(q-1)$ where $q \ge 1$ is some integer, if we let $q = p + 1$, then $k = 1 + p + p^2$

Use the fact above, we know that

$$\frac{k(k-1)}{2} \ge \sum_{i=1}^{k} \frac{x_i(x_i-1)}{2} = \frac{\sum_{i=1}^{k} x_i^2}{2} - \frac{\sum_{i=1}^{k} x_i}{2} \ge \frac{\left(\sum_{i=1}^{k} x_i\right)^2}{2k} - \frac{\sum_{i=1}^{k} x_i}{2} = \frac{n^2}{2k} - \frac{n}{2}$$

But, when $x_i = q = p + 1$, both of inequalities become equalities, thus $n = k(p+1)$ is the maximum possible value for n

(i)
$k = 7 = 1 + 2 + 2^2 \Rightarrow p = 2 \Rightarrow n = 21$
(ii)
$k = 13 = 1 + 3 + 3^2 \Rightarrow p = 3 \Rightarrow n = 52$

References

[1] *Compact Riemann Surfaces* - R. Narasimhan

[2] *Lectures on Riemann Surfaces* - Otto Forster