

## Chapter 1

# Polylogarithms



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## 1.1 Iterated integral

**Definition 1.1.1.** Chen's *Iterated integral* is defined inductively by

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b \left( \int_a^t f_1(\tau)d\tau \cdots f_{r-1}(\tau)d\tau \right) f_r(t)dt$$

It can also be written as

$$\int_{a \leq t_1 \leq \cdots \leq t_n \leq b} f_1(t_1)dt_1 \wedge \cdots \wedge f_n(t_n)dt_n$$

If  $\alpha : [a, b] \rightarrow M$  is a curve,  $\alpha^*\omega_i = f_i(t)dt$ , then

$$\int_{\alpha} \omega_1 \cdots \omega_r = \int_a^b f_1(t)dt \cdots f_r(t)dt$$

Set the integral to be 1 if  $r = 0$

**Proposition 1.1.2.**

1. The iterated integral is independent of the parametrization
2.  $\int_{\alpha\beta} \omega_1 \cdots \omega_r = \sum_{j=0}^r \int_{\alpha} \omega_1 \cdots \omega_j \int_{\beta} \omega_{j+1} \cdots \omega_r$ , here  $\beta(0) = \alpha(1)$
3.  $\int_{\alpha^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_{\alpha} \omega_r \cdots \omega_1$
4.  $\int_{\alpha} \omega_1 \cdots \omega_r \int_{\alpha} \omega_{r+1} \cdots \omega_{r+s} = \sum_{\sigma} \int_{\alpha} \omega_{\sigma^{-1}(1)} \cdots \omega_{\sigma^{-1}(r+s)}$ , here  $\sigma$  runs over  $(r, s)$ -shuffles
5. If  $\omega_i^{(j)}$ ,  $1 \leq i \leq r$ ,  $1 \leq j \leq n$  are closed one forms such that  $\sum_j \omega_{i-1}^{(j)} \wedge \omega_i^{(j)} = 0$  for  $2 \leq i \leq r$ ,  
then  $\int_{\alpha} \sum_j \omega_1^{(j)} \cdots \omega_r^{(j)}$  only depends on the homotopy class of  $\alpha$

*Proof.*

1. Suppose  $\beta : [c, d] \rightarrow M$  is another parametrization of the same curve as  $\alpha$  with  $\beta^*\omega_i = g_i(s)ds$
- 2.
- 3.
- 4.
- 5.

□

## 1.2 Polylogarithm

**Definition 1.2.1.** The *Polylogarithms* are

$$\text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

Note that

$$\text{Li}_{n+1}(z) = \int_0^z \frac{\text{Li}_n(t)}{t} dt, \quad \text{Li}_1(z) = -\ln(1-z)$$

Hence

$$\text{Li}_n(z) = \int_0^z \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{1-t} = \int_0^1 \left(\frac{dt}{t}\right)^{n-1} \frac{dt}{z^{-1}-t}$$

*Dilogarithm*  $\text{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du$  is the analytic continuation on  $\mathbb{C} \setminus \{0, 1\}$ , avoiding the cut  $[1, \infty]$

**Lemma 1.2.2.**  $\text{Li}_k(z)$  satisfies differential equation

$$\left[ (1-z) \frac{d}{dz} \right] \left( z \frac{d}{dz} \right)^{k-1} y = 1$$

Other solutions are  $\frac{\ln^j z}{j!}$ ,  $1 \leq j \leq k-1$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $x = \epsilon e^{it}$ , we get 0.

To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $x = (1-t)\epsilon + t(1-\epsilon)$ ,  $x = 1 - \epsilon e^{it}$  and  $x = (1-t)(1-\epsilon) + t\epsilon$ , we get  $-\frac{2\pi i}{(n-1)!} \log^{n-1} x$

The variation matrix is

$$\Lambda = \begin{bmatrix} 1 & & & & & \\ \text{Li}_1(x) & 1 & & & & \\ \text{Li}_2(x) & \log x & 1 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ \text{Li}_{n-1}(x) & \frac{\log^{n-2} x}{(n-2)!} & \cdots & \log x & 1 & \\ \text{Li}_n(x) & \frac{\log^{n-1} x}{(n-1)!} & \cdots & \frac{\log^2 x}{2!} & \log x & 1 \end{bmatrix} \tau_n(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & & & \\ -dv & 0 & & & & \\ 0 & du & 0 & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ 0 & 0 & \cdots & du & 0 & \\ 0 & 0 & \cdots & 0 & du & 0 \end{bmatrix}$$

The monodromy representation  $\rho$  is as follows

For monodromy around  $x = 0$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & 1 & 1 & & \\ & \vdots & \vdots & \ddots & \\ & \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & 1 \end{bmatrix}$$

For monodromy around  $x = 1$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\widehat{\mathbb{C}} = \{(u, v) \in \mathbb{C}^2 | e^u + e^v = 1\}, u = \log x, v = \log(1 - x)$$

$$\widehat{\mathcal{L}}_1 : \widehat{\mathbb{C}} \rightarrow \mathbb{C}/(2\pi i)\mathbb{Z} \text{ is just } \text{Li}_1$$

$$\widehat{\mathcal{L}}_2 : \widehat{\mathbb{C}} \rightarrow \mathbb{C}/(2\pi i)^2\mathbb{Z} \text{ is } \text{Li}_2(x) - \frac{1}{2} \log x \text{Li}_1(x), [M_1, M_0]\widehat{\mathcal{L}}_2 = (2\pi i)^2, d\widehat{\mathcal{L}}_2 = \frac{1}{2}(udv - vdu)$$

## 1.3 Multiple polylogarithm

**Definition 1.3.1.** The *multiple polylogarithms* are

$$\text{Li}_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \int_0^1 \frac{dt}{a_1 - t} \left( \frac{dt}{t} \right)^{n_1-1} \cdots \frac{dt}{a_d - t} \left( \frac{dt}{t} \right)^{n_d-1}$$

Here  $\mathbf{k}$  runs over  $0 < k_1 < \cdots < k_d$ ,  $a_j = a_j(\mathbf{x}) = (x_j \cdots x_d)^{-1}$

$$\text{Define } \text{Li}_0(x) = \frac{x}{1-x}$$

*Note.* For  $\mathbf{k}$  runs over  $(k_1, \dots, k_d) \in \mathbb{Z}_{\geq 1}^d$

$$\sum_{\mathbf{k}} \frac{\mathbf{x}^{\mathbf{k}}}{\mathbf{k}^{\mathbf{n}}} = \left( \sum_{k_1} \frac{x_1^{k_1}}{k_1^{n_1}} \right) \cdots \left( \sum_{k_d} \frac{x_d^{k_d}}{k_d^{n_d}} \right) = \text{Li}_{n_1}(x_1) \cdots \text{Li}_{n_d}(x_d)$$

Can be written in terms of multiple polylogarithms

*Note.*

$$\begin{aligned} \text{Li}_{n_1, \dots, n_{i-1}, 0, n_{i+1}, \dots, n_d}(x_1, \dots, x_d) &= \sum_{0 < k_1 < \dots < k_d} \frac{x_1^{k_1-1} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \\ &= \sum_{0 < k_1 < \dots < k_d} \frac{x_1^{k_1-1} \cdots x_{i-1}^{k_{i-1}-1} x_{i+1}^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{x_i^{k_{i-1}+1} - x_i^{k_{i+1}}}{1 - x_i} \\ &= \sum_{0 < k_1 < \dots < k_d} \frac{x_1^{k_1-1} (\cdots x_{i-1} x_i)^{k_{i-1}-1} x_{i+1}^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{x_i}{1 - x_i} \\ &\quad - \sum_{0 < k_1 < \dots < k_d} \frac{x_1^{k_1-1} \cdots x_{i-1}^{k_{i-1}-1} (x_i x_{i+1})^{k_{i+1}} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_{i-1}^{n_{i-1}} k_{i+1}^{n_{i+1}} \cdots k_d^{n_d}} \frac{1}{1 - x_i} \\ &= \text{Li}_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d}(x_1, \dots, x_{i-1} x_i, x_{i+1}, \dots, x_d) \frac{x_i}{1 - x_i} \\ &\quad - \text{Li}_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d}(x_1, \dots, x_{i-1}, x_i x_{i+1}, \dots, x_d) \frac{1}{1 - x_i} \end{aligned}$$

$$\text{Li}_{n_1, \dots, n_{d-1}, 0}(x_1, \dots, x_d) = \text{Li}_{n_1, \dots, n_{d-1}}(x_1, \dots, x_{d-1} x_d) \frac{x_d}{1 - x_d}$$

$$\text{Li}_{0, n_2, \dots, n_d}(x_1, \dots, x_d) = \text{Li}_{n_2, \dots, n_{d-1}}(x_2, \dots, x_d) \frac{x_1}{1 - x_1} - \text{Li}_{n_2, \dots, n_{d-1}}(x_1 x_2, \dots, x_d) \frac{1}{1 - x_1}$$

**Exercise 1.3.2** (Derivatives of polylogarithms). Observe the following

$$\begin{aligned} \frac{\partial}{\partial x_i} \left( \sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}-1} x_i^{k_i} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}-1} k_i^{n_i} k_{i+1}^{n_{i+1}} \cdots} \right) &= \sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}-1} x_i^{k_i-1} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}-1} k_i^{n_i-1} k_{i+1}^{n_{i+1}} \cdots} \\ &= \sum_{\mathbf{k}} \frac{\cdots x_{i-1}^{k_{i-1}-1} x_i^{k_i} x_{i+1}^{k_{i+1}} \cdots}{\cdots k_{i-1}^{n_{i-1}-1} k_i^{n_i-1} k_{i+1}^{n_{i+1}} \cdots} \frac{1}{x_i} \end{aligned}$$

Write  $u_i = \log(x_i)$ ,  $v_i = \log(1 - x_i)$ ,  $u_{ij} = \log(x_i \cdots x_j)$ ,  $v_{ij} = \log(1 - x_i \cdots x_j)$ . If  $m_i > 1$ , then

$$\begin{aligned} d_i \text{Li}_{n_1, \dots, n_d}(z_1, \dots, z_d) &= \text{Li}_{n_1, \dots, n_{i-1}, \dots, n_d}(z_1, \dots, z_d) \frac{dz_i}{x_i} \\ &= \text{Li}_{n_1, \dots, n_{i-1}, \dots, n_d}(z_1, \dots, z_d) du_i \end{aligned}$$

$$\begin{aligned} d_d \operatorname{Li}_{n_1, \dots, n_{d-1}, 1}(z_1, \dots, z_d) &= \operatorname{Li}_{n_1, \dots, n_{d-1}}(z_1, \dots, z_{d-1} z_d) \frac{dx_d}{1 - x_d} \\ &= -\operatorname{Li}_{n_1, \dots, n_{d-1}}(z_1, \dots, z_{d-1} z_d) dv_d \end{aligned}$$

$$\begin{aligned} d_1 \operatorname{Li}_{1, n_2, \dots, n_d}(z_1, \dots, z_d) &= \operatorname{Li}_{n_2, \dots, n_d}(z_2, \dots, z_d) \frac{dx_1}{1 - x_1} \\ &\quad - \operatorname{Li}_{n_2, \dots, n_d}(z_1 z_2, \dots, z_d) \frac{dx_1}{x_1(1 - x_1)} \end{aligned}$$

$$\begin{aligned} d_i \operatorname{Li}_{n_1, \dots, n_{i-1}, 1, n_{i+1}, \dots, n_d}(z_1, \dots, z_d) &= \operatorname{Li}_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d}(z_1, \dots, z_{i-1} z_i, z_{i+1}, \dots, z_d) \frac{dx_i}{1 - x_i} \\ &\quad - \operatorname{Li}_{n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d}(z_1, \dots, z_{i-1}, z_i z_{i+1}, \dots, z_d) \frac{dx_i}{x_i(1 - x_i)} \end{aligned}$$

**Remark 1.3.3.**

**Theorem 1.3.4.**  $\operatorname{Li}_n(x) + (-1)^n \operatorname{Li}_n(x^{-1}) =$

*Proof.*  $\operatorname{Li}_0(x) + \operatorname{Li}_0(x^{-1}) = -1$ ,  $\operatorname{Li}_1(x) - \operatorname{Li}_1(x^{-1}) = \pi i - \log x$ ,  $\operatorname{Li}_2(x) + \operatorname{Li}_2(x^{-1}) = -\frac{\pi^2}{6} - \frac{\log^2(-x)}{2}$

$$d(\operatorname{Li}_n(x) + (-1)^n \operatorname{Li}_n(x^{-1})) = (\operatorname{Li}_{n-1}(x) + (-1)^{n-1} \operatorname{Li}_{n-1}(x^{-1})) \frac{dx}{x}$$

□



1.4  $\text{Li}_{1,1}$ 

$$\begin{aligned}
\text{Li}_{1,1}(x, y) &= \int \frac{dy}{1-y} \frac{dx}{1-x} + \frac{d(xy)}{1-xy} \left( \frac{dy}{1-y} - \frac{dx}{x(1-x)} \right) \\
&= \int d \log(1-y) d \log(1-x) + d \log(1-xy) d \log \frac{x(1-y)}{1-x} \\
&= \int dv_2 dv_1 + dv_{12} dw_1
\end{aligned}$$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = \epsilon)$ , we get 0.  
To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon, y = \epsilon e^{it})$ , we get 0.  
To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$ , we get 0.  
To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$ , we get  $-2\pi i \text{Li}_1(x)$ .  
To compute the monodromy around  $xy = 1$ , take  $q$  to be the loop  $(x = x^0, y \text{ such that } \int_q d \log(1-xy) = 2\pi i)$ , we get  $-2\pi i \text{Li}_1(\frac{1-xy}{1-x})$

The variation matrix is

$$\Lambda = \begin{bmatrix} 1 & & & \\ \text{Li}_1(y) & 1 & & \\ \text{Li}_1(xy) & & 1 & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \text{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 \end{bmatrix} \tau_{1,1}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & \\ -dv_2 & 0 & & \\ -dv_{12} & 0 & 0 & \\ 0 & -dv_1 & -dw_1 & 0 \end{bmatrix}$$

Note that  $\text{Li}_1\left(\frac{1-xy}{1-x}\right) = -\log\left(\frac{x(y-1)}{1-x}\right) = \text{Li}_1(y) - \text{Li}_1(x^{-1}) = \text{Li}_1(y) - \text{Li}_1(x) - \log x - i\pi$

The monodromy representation  $\rho$  is as follows

For monodromy around  $x = 0$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}$$

For monodromy around  $y = 0$ , identity.

For monodromy around  $x = 1$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 1 & 1 \end{bmatrix}$$

For monodromy around  $y = 1$

$$\begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & & 1 & \\ & & -1 & 1 \end{bmatrix}$$

For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ -1 & & 1 & \\ & & & 1 \end{bmatrix}$$

## 1.5 $\text{Li}_{1,2}$

$$\text{Li}_{1,2}(x, y) = dv_{12}du_{12}(du_1 - dv_1) + dv_2du_2dv_1 - (dv_2dv_1 + dv_{12}dw_{1,2})du_2$$

$$\Lambda = \begin{bmatrix} 1 & & & & & & \\ \text{Li}_1(y) & 1 & & & & & \\ \text{Li}_1(xy) & 0 & 1 & & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \text{Li}_1(y) - \text{Li}_1(x^{-1}) & 1 & & & \\ \text{Li}_2(y) & \log y & & & 1 & & \\ \text{Li}_2(xy) & 0 & \log(xy) & & & 1 & \\ \text{Li}_{1,2}(x, y) & \text{Li}_1(x) \log y & g(x, y) & \log y \text{Li}_1(x) - \text{Li}_1(x^{-1}) & 1 & & \end{bmatrix} \tau_{1,2}(2\pi i)$$

Where  $g(x, y) = \text{Li}_2(y) - \log y \text{Li}_1(x^{-1}) - \frac{1}{2} \log^2 x - \log x \text{Li}_1(x) + \text{Li}_2(x)$   $g(x, y) = -I((xy)^{-1}; y^{-1}, 0; 1)$

$$\omega = \begin{bmatrix} 0 & & & & & & \\ -dv_2 & 0 & & & & & \\ -dv_{12} & 0 & 0 & & & & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 & & & \\ 0 & du_2 & 0 & 0 & 0 & & \\ 0 & 0 & d(u_1 + u_2) & 0 & 0 & 0 & \\ 0 & 0 & 0 & du_2 & -dv_1 & d(v_1 - u_1) & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & & & \\ -I(0; y^{-1}; 1) & 1 & & & & & \\ -I(0; x^{-1}y^{-1}; 1) & 0 & 1 & & & & \\ -I(0; y^{-1}; 0; 1) & I(y^{-1}; 0; 1) & 0 & 1 & & & \\ -I(0; x^{-1}y^{-1}; 0; 1) & 0 & -I(x^{-1}y^{-1}; 0; 1) & 0 & 1 & & \\ I(0; x^{-1}y^{-1}, y^{-1}; 1) & -I(0; x^{-1}y^{-1}; y^{-1}) & -I(x^{-1}y^{-1}; y^{-1}; 1) & 0 & 0 & 1 & \\ I(0; x^{-1}y^{-1}, y^{-1}, 0; 1) & -I(0; x^{-1}y^{-1}; y^{-1})I(y^{-1}; 0; 1) & -I(x^{-1}y^{-1}; y^{-1}, 0; 1) & -I(0; x^{-1}y^{-1}; y^{-1}) & -I(x^{-1}y^{-1}; y^{-1}, 0) & I(y^{-1}; 0; 1) & 1 \end{bmatrix}$$

For  $x = 0$

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For  $y = 0$

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For  $x = 1$

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For  $y = 1$

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For  $xy = 1$

$$\begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

## 1.6 $\text{Li}_{2,1}$

$$\begin{aligned}\text{Li}_{2,1}(x, y) &= \int \frac{dy}{1-y} \frac{dx}{1-x} \frac{dx}{x} + \frac{d(xy)}{1-xy} \left( \frac{dy}{1-y} - \frac{dx}{x(1-x)} \right) \frac{dx}{x} + \frac{d(xy)}{1-xy} \frac{d(xy)}{xy} \frac{dy}{1-y} \\ &= \int dv_2 dv_1 du_1 + dv_{12} dw_{1,2} du_1 + dv_{12} du_{12} dv_2\end{aligned}$$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = \epsilon)$ , we get 0  
 To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon, y = \epsilon e^{it})$ , we get 0  
 To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$ , we get  $-2\pi i \text{Li}_2(x)$   
 To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$ , we get 0  
 To compute the monodromy around  $xy = 1$ , take  $q$  to be the loop  $(x = x^0, y \text{ such that } \int_q d \log(1-xy) = 2\pi i)$

The variation matrix is

$$\Lambda = \begin{bmatrix} \text{Li}_1(y) & 1 & & & \\ \text{Li}_1(xy) & & 1 & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \text{Li}_1(y) - \text{Li}_1(x^{-1}) & 1 & \\ \text{Li}_2(xy) & & \log(xy) & & 1 \\ \text{Li}_{2,1}(x, y) & \text{Li}_2(x) & \log(xy) & \text{Li}_1(y) - \text{Li}_2(y) + \text{Li}_2(x^{-1}) & \log x & \text{Li}_1(y) & 1 \end{bmatrix} \tau_{2,1}(2\pi i)$$

$$- \text{Li}_{1,1}(1-x, \frac{1-xy}{1-x}) = \log(xy) \text{Li}_1(y) - \text{Li}_2(y) + \text{Li}_2(x^{-1})$$

$$\omega = \begin{bmatrix} 0 & & & & & & \\ -dv_2 & 0 & & & & & \\ -dv_{12} & 0 & 0 & & & & \\ 0 & -dv_1 & -w_1 & 0 & & & \\ 0 & 0 & du_{12} & 0 & 0 & & \\ 0 & 0 & 0 & du_1 & -dv_2 & 0 & \end{bmatrix}$$

The monodromy representation  $\rho$  is as follows

For monodromy around  $x = 0$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -1 & 1 & & \\ & & 1 & & 1 & \\ & & & 1 & & 1 \end{bmatrix}$$

For monodromy around  $y = 0$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & 1 & & 1 & \\ & & & & & 1 \end{bmatrix}$$

For monodromy around  $x = 1$

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & -1 & 1 & 1 & \\ & & & & & 1 \\ & & & & & & 1 \end{bmatrix}$$

For monodromy around  $y = 1$

$$\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & 1 & & \\ & & -1 & 1 & \\ & & & & 1 \\ & & & & -1 & 1 \end{bmatrix}$$

For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ -1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & & 1 \end{bmatrix}$$

$\text{Li}_{2,1}(x, y) - \text{Li}_2(xy) \text{Li}_1(y) - \text{Li}_{1,1}(x, y) \log x$  is well defined on the universal abelian cover, and

$$d\widehat{L}_{2,1} = -u_1 v_{12} du_1 + u_1 v_{12} dv_1 - u_1 v_{12} dv_2 - u_1 v_2 dv_1 - v_2 v_{12} du_{12}$$

## 1.7 $\text{Li}_{1,1,1}$

$$\begin{aligned}\text{Li}_{1,1,1}(x, y, z) = & \int -(dv_3 dv_2 + dv_{23} dw_{2,3}) dv_1 \\ & - (dv_3 dv_{12} + dv_{123} dw_{12,3}) dw_{1,2} \\ & - (dv_{23} dv_1 + dv_{123} dw_{1,23}) dw_{2,3}\end{aligned}$$

$$\Lambda = \begin{bmatrix} \text{Li}_1(z) & 1 & & & & \\ \text{Li}_1(yz) & & 1 & & & \\ \text{Li}_1(xyz) & & & 1 & & \\ \text{Li}_{1,1}(y, z) & \text{Li}_1(y) & \text{Li}_1(\frac{1-yz}{1-y}) & & 1 & \\ \text{Li}_{1,1}(xy, z) & \text{Li}_1(xy) & & \text{Li}_1(\frac{1-xyz}{1-xy}) & & 1 \\ \text{Li}_{1,1}(x, yz) & & \text{Li}_1(x) & \text{Li}_1(\frac{1-xyz}{1-x}) & & 1 \\ \text{Li}_{1,1,1}(x, y, z) & \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \text{Li}_1(\frac{1-yz}{1-y}) & \text{Li}_{1,1}(\frac{1-xy}{1-x}, \frac{1-xyz}{1-xy}) & \text{Li}_1(x) & \text{Li}_1(\frac{1-xy}{1-x}) & \text{Li}_1(\frac{1-yz}{1-y}) & 1 \end{bmatrix} \tau_{1,1,1}(2\pi i)$$

Where

$$\text{Li}_{1,1}\left(\frac{1-xy}{1-x}, \frac{1-xyz}{1-xy}\right) = \text{Li}_{1,1}(y, 1) - \text{Li}_{1,1}(y, x^{-1}y^{-1}) + \text{Li}_1(x^{-1}y^{-1}) \text{Li}_1(x^{-1})$$

$$\omega = \begin{bmatrix} 0 & & & & & & & & \\ -dv_3 & 0 & & & & & & & \\ -dv_{23} & 0 & 0 & & & & & & \\ -dv_{123} & 0 & 0 & 0 & & & & & \\ 0 & -dv_2 & -dw_{2,3} & 0 & 0 & & & & \\ 0 & -dv_{12} & 0 & -dw_{12,3} & 0 & 0 & & & \\ 0 & 0 & -dv_1 & -dw_{1,23} & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & -dv_1 & -dw_{1,2} & -dw_{2,3} & 0 & \end{bmatrix}$$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = z = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = z = \epsilon, y = \epsilon e^{it})$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $z = 0$ , take  $q(\epsilon)$  to be the loop  $(x = y = \epsilon, z = \epsilon e^{it})$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = - \int_q dv_3 \int_p (dv_2 dv_1 + dv_{12} dw_{1,2}) = 0$$

To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = z = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = z = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = z = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = z = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = z = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = z = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $z = 1$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} &= - \int_q dv_3 \int_p (dv_2 dv_1 + dv_{12} dw_{1,2}) \\ &= -2\pi i \text{Li}_{1,1}(x, y) \end{aligned}$$

To compute the monodromy around  $xy = 1$ , take  $z = 0$ ,  $x$  to be constant

$$\int_{qp} - \int_p = 0$$

To compute the monodromy around  $yz = 1$ , take  $x, y$  to be constants

$$\begin{aligned} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} &= - \int_{q(\epsilon)} dv_{23} dv_3 \int_{p(\epsilon)} dv_1 - \int_{q(\epsilon)} dv_{123} dv_{23} \int_{p(\epsilon)} dw_{2,3} \\ &\quad - \int_{q(\epsilon)} dv_{23} \int_{p(\epsilon)} dw_{2,3} dv_1 - \int_{q(\epsilon)} dv_{23} \int_{p(\epsilon)} dv_1 dw_{2,3} \\ &= - \int_{q(\epsilon)^{-1}} dv_3 dv_{23} \int_{p(\epsilon)} dv_1 - \int_{q(\epsilon)} dv_{123} dv_{23} \int_{p(\epsilon)} dw_{2,3} \\ &\quad - \int_{q(\epsilon)} dv_{23} \int_{p(\epsilon)} dw_{2,3} \int_{p(\epsilon)} dv_1 \\ &= 2\pi i \log \frac{1 - \epsilon^{-1}}{1 - \epsilon} \log \frac{1 - x}{1 - \epsilon} \\ &\quad - 2\pi i \left( \log \frac{1 - \epsilon}{1 - \epsilon^3} + \log \frac{1 - x}{1 - \epsilon} \right) \left( \log \frac{y}{\epsilon} + \log \frac{1 - z}{1 - y} \right) \end{aligned}$$

Let  $\epsilon \rightarrow 0$ , we get

$$2\pi i \log(1 - x) \log \frac{y - 1}{y(1 - z)} = -2\pi i \text{Li}_1(x) \text{Li}_1\left(\frac{1 - yz}{1 - y}\right)$$

To compute the monodromy around  $xyz = 1$

$$\begin{aligned} \int_{pq} - \int_p &= - \int_q dv_{123} dv_{23} dv_3 = \int_{q^{-1}} dv_3 dv_{23} dv_{123} \\ &= -2\pi i \int_z^{x^{-1}y^{-1}} \frac{dt}{1 - t} \frac{dyt}{1 - yt} \\ &= -2\pi i \int_{xyz}^1 \frac{dt}{xy - t} \frac{dt}{x - t} \\ &= -2\pi i \int_1^{xyz} \frac{dt}{x - t} \frac{dt}{xy - t} \\ &= -2\pi i \int_0^{xyz-1} \frac{dt}{(x - 1) - t} \frac{dt}{(xy - 1) - t} \\ &= -2\pi i \int_0^1 \frac{dt}{\frac{1-x}{1-xyz} - t} \frac{dt}{\frac{1-xy}{1-xyz} - t} \\ &= -2\pi i \text{Li}_{1,1} \left( \frac{1 - xy}{1 - x}, \frac{1 - xyz}{1 - xy} \right) \end{aligned}$$

The monodromy representation  $\rho$  is as follows

For monodromy around  $x = 0$

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$





For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

For monodromy around  $yz = 1$

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

For monodromy around  $xyz = 1$

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & 1 & \\ & & & & & & & 1 \end{bmatrix}$$

## 1.8 $\text{Li}_{3,1}$

$$\begin{aligned}\text{Li}_{3,1}(x, y) &= \int \frac{dy}{1-y} \frac{dx}{1-x} \left( \frac{dx}{x} \right)^2 + \frac{d(xy)}{1-xy} \left( \frac{dy}{1-y} - \frac{dx}{x(1-x)} \right) \left( \frac{dx}{x} \right)^2 \\ &\quad + \frac{d(xy)}{1-xy} \frac{d(xy)}{xy} \frac{dy}{1-y} \frac{dx}{x} + \frac{d(xy)}{1-xy} \left( \frac{d(xy)}{xy} \right)^2 \frac{dy}{1-y} \\ &= \int dv_2 dv_1 (du_1)^2 + dv_{12} dw_1 (du_1)^2 + dv_{12} du_{12} dv_2 du_1 + dv_{12} (du_{12})^2 dv_2\end{aligned}$$

Here we write  $w_1 = u_1 + v_2 - v_1 = \log \frac{x(1-y)}{1-x}$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = \epsilon)$ , we get 0.

To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon, y = \epsilon e^{it})$ , we get 0.

To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$ , we get 0.

To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$ , we get  $-2\pi i \text{Li}_3(x)$ .

To compute the monodromy around  $xy = 1$ , take  $q$  to be the loop  $(x = x^0, y \text{ such that } \int_q d \log(1 - xy) = 2\pi i)$ , we get  $-2\pi i \text{Li}_1(\frac{1-xy}{1-x})$

The variation matrix is

$$\Lambda = \begin{bmatrix} \text{Li}_1(y) & 1 & & & & & & & & \\ \text{Li}_1(xy) & & 1 & & & & & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & & \text{Li}_1(y) - \text{Li}_1(x^{-1}) & & 1 & & & & \\ \text{Li}_2(xy) & & \log(xy) & & & & 1 & & & \\ \text{Li}_{2,1}(x, y) & \text{Li}_2(x) & & \log(xy) \text{Li}_1(y) - \text{Li}_2(y) + \text{Li}_2(x^{-1}) & & \log x & & \text{Li}_1(y) & & 1 \\ \text{Li}_3(xy) & & \log^2(xy)/2 & & & & \log(xy) & & & \\ \text{Li}_{3,1}(x, y) & \text{Li}_3(x) & \frac{1}{2} \log^2(xy) & \text{Li}_1(y) - \log(xy) \text{Li}_2(y) + \text{Li}_3(y) - \text{Li}_3(x^{-1}) & \log^2 x/2 & \log(xy) \text{Li}_1(y) - \text{Li}_2(y) & \log x & \text{Li}_1(y) & 1 & \end{bmatrix} \tau_{3,1}(2\pi i)$$

$$\begin{bmatrix} -I(0; y^{-1}; 1) & 1 & & & & & & & & \\ -I(0; (xy)^{-1}; 1) & 0 & & 1 & & & & & & \\ I(0; (xy)^{-1}, y^{-1}; 1) & -I(0; (xy)^{-1}, y^{-1}) & & -I((xy)^{-1}, y^{-1}; 1) & & 1 & & & & \\ -I(0; (xy)^{-1}, 0; 1) & 0 & & -I((xy)^{-1}, 0; y^{-1}) & & 0 & & 1 & & \\ I(0; (xy)^{-1}, 0, y^{-1}; 1) & -I(0; (xy)^{-1}, 0; y^{-1}) & & -I((xy)^{-1}, 0; y^{-1}; 1) & & I((xy)^{-1}, 0; y^{-1}) & & -I(0; y^{-1}; 1) & & 1 \\ -I(0; (xy)^{-1}, 0, 0; 1) & 0 & & I((xy)^{-1}, 0, 0; 1) & & 0 & & I((xy)^{-1}, 0; 0) & & 0 \\ I(0; (xy)^{-1}, 0, 0, y^{-1}; 1) & -I(0; (xy)^{-1}, 0, 0; y^{-1}) & & -I((xy)^{-1}, 0, 0, y^{-1}; 1) & & I((xy)^{-1}, 0, 0; y^{-1}) & & -I((xy)^{-1}, 0; 0) I(0; y^{-1}; 1) - I(0; 0, y^{-1}; 1) & & I((xy)^{-1}, 0; 0) + I(0; 0, y^{-1}; 1) \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 & & & & & & & & & \\ -dv_2 & 0 & & & & & & & & \\ -dv_{12} & 0 & 0 & & & & & & & \\ & -dv_1 & -dw_1 & 0 & & & & & & \\ & & du_{12} & 0 & 0 & & & & & \\ & & & du_1 & -dv_2 & 0 & & & & \\ & & & & du_{12} & 0 & 0 & & & \\ & & & & & du_1 & -dv_2 & 0 & & \end{bmatrix}$$

The monodromy representation  $\rho$  is as follows

For monodromy around  $x = 0$

$$\begin{bmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & -1 & 1 & & & & & & \\ & & 1 & & 1 & & & & & \\ & & & 1 & & 1 & & & & \\ & & & & \frac{1}{2} & & 1 & & & \\ & & & & & 1 & & 1 & & \end{bmatrix}$$

For monodromy around  $y = 0$

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

For monodromy around  $x = 1$

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

For monodromy around  $y = 1$

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \\ & & & & & & & 1 \end{bmatrix}$$

## 1.9 $\text{Li}_{2,2}$

$$\begin{aligned} \text{Li}_{2,2} = & (dv_{12}du_{12}(du_1 - dv_1) + dv_2du_2dv_1 - (dv_2dv_1 + dv_{12}dw_{1,2})dv_2)du_1 \\ & + (dv_2dv_1du_1 + dv_{12}dw_{1,2}du_1 + dv_{12}du_{12}dv_2)du_2 \end{aligned}$$

Monodromy around  $x = 0$  Monodromy around  $y = 0$  Monodromy around  $x = 1$  Monodromy around  $y = 1$  Monodromy around  $xy = 1$

**1.10**  $\text{Li}_{4,1}$ 

$$\text{Li}_{n,1}(x, y) = \int (dv_2 dv_1 + dv_{12} dw_1)(du_1)^{n-1} + dv_{12} \left( \sum_{k=1}^{n-1} du_{12}^k dv_2 (du_1)^{n-1-k} \right)$$

Here  $w_1 = u_1 + v_2 - v_1$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon, y = \epsilon e^{it})$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 \cdots (du_1)^{n-1} = 0$$

To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 (du_1)^{n-1} = -2\pi i \text{Li}_n(x)$$

To compute the monodromy around  $xy = 1$ , take  $q$  to be the loop  $(x = x^0, y \text{ such that } \int_q d \log(1-xy) = 2\pi i)$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{pq} - \int_p &= \sum_{k=0}^{n-1} \int_p dv_{12} (du_{12})^k \int_q (du_{12})^{n-1-k} dv_2 + \int_q dv_{12} du_{12}^{n-1} dv_2 \\ &= (-1)^{n+1} \int_{q^{-1}} dv_2 du_2^{n-1} dv_{12} \\ &= (-1)^{n+1} \int_{q^{-1}} \frac{\text{Li}_n(y) - \text{Li}_n(y^0)}{y - 1/x^0} \\ &= (-1)^n 2\pi i (\text{Li}_n(y^0) - \text{Li}_n(1/x^0)) \end{aligned}$$

The variation matrix is

$$\omega = \begin{bmatrix} 0 & & & & & & & & & \\ -dv_2 & 0 & & & & & & & & \\ -dv_{12} & & 0 & & & & & & & \\ & -dv_1 & -dw_1 & 0 & & & & & & \\ & & du_{12} & & 0 & & & & & \\ & & & du_1 & -dv_2 & 0 & & & & \\ & & & & du_{12} & & 0 & & & \\ & & & & & du_1 & -dv_2 & 0 & & \\ & & & & & & du_{12} & & 0 & \\ & & & & & & & du_1 & -dv_2 & 0 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & & & & & & & & & \\ \text{Li}_1(y) & 1 & & & & & & & & \\ \text{Li}_1(xy) & & 1 & & & & & & & \\ \text{Li}_{1,1}(x, y) \text{Li}_1(x) & & & \text{Li}_1(y) - \text{Li}_1(x^{-1}) & & & 1 & & & \\ \text{Li}_2(xy) & & & \log(xy) & & & & 1 & & \\ \text{Li}_{2,1}(x, y) \text{Li}_2(x) & & & \log(xy) \text{Li}_1(y) - \text{Li}_2(y) + \text{Li}_2(x^{-1}) & & & \log x & & \text{Li}_1(y) \\ \text{Li}_3(xy) & & & \log^2(xy)/2 & & & & \log(xy) & & \\ \text{Li}_{3,1}(x, y) \text{Li}_3(x) & & \frac{1}{2} \log^2(xy) \text{Li}_1(y) - \log(xy) \text{Li}_2(y) + \text{Li}_3(y) - \text{Li}_3(x^{-1}) & & & \log^2 x/2 & & \log(xy) \text{Li}_1(y) - \text{Li}_2(y) \\ \text{Li}_4(xy) & & & \log^3(xy)/3! & & & & \log^2(xy)/2 & & \\ \text{Li}_{4,1}(x, y) \text{Li}_4(x) & \frac{1}{3!} \log^3(xy) \text{Li}_1(y) - \frac{1}{2} \log^2(xy) \text{Li}_2(y) + \log(xy) \text{Li}_3(y) - \text{Li}_4(y) + \text{Li}_4(x^{-1}) & \log^3 x/3! & \frac{1}{2} \log^2(xy) \text{Li}_1(y) - \log(xy) \text{Li}_2(y) + \text{Li}_3(y) & & & & & \end{bmatrix}$$



For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ -1 & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{bmatrix}$$

### 1.11 $\text{Li}_{n,1}$

$$\text{Li}_{1,1}(x, y) = \int_{(0,0)}^{(x,y)} dv_2 dv_1 + dv_{12} dw_{1,2}$$

Here  $w_{1,2} = u_1 + v_2 - v_1$ , then

$$\text{Li}_{n,1}(x, y) = \int_{(0,0)}^{(x,y)} (dv_2 dv_1 + dv_{12} dw_{1,2}) (du_1)^{n-1} + dv_{12} \left( \sum_{k=1}^{n-1} du_{12}^k dv_2 (du_1)^{n-1-k} \right)$$

since inductively

$$\begin{aligned} \text{Li}_{n,1}(x, y) &= \int_{(0,0)}^{(x,y)} \text{Li}_{n-1,1}(x, y) du_1 - \text{Li}_n(xy) dv_2 \\ &= \int_{(0,0)}^{(x,y)} \left[ (dv_2 dv_1 + dv_{12} dw_{1,2}) (du_1)^{n-2} + dv_{12} \left( \sum_{k=1}^{n-2} du_{12}^k dv_2 (du_1)^{n-2-k} \right) \right] du_1 \\ &\quad + dv_{12} (du_{12})^{n-1} dv_2 \end{aligned}$$

Suppose  $p$  is the path from  $(0, 0)$  to  $(x, y)$  that give the value of  $\text{Li}_{n,1}(x, y)$ ,  $q$  is a loop based at  $(x, y)$  that induces monodromy, we can take  $p$  to  $p(\epsilon)$  to start at  $(\epsilon, \epsilon)$  and then take limit  $\epsilon \rightarrow 0$ , then  $pq$  can be homotoped to some  $q(\epsilon)p(\epsilon)$

To compute the monodromy around  $x = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon e^{it}, y = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 0$ , take  $q(\epsilon)$  to be the loop  $(x = \epsilon, y = \epsilon e^{it})$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 (du_1)^{n-1} = 0$$

To compute the monodromy around  $x = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = (1-t)\epsilon + t(1-\epsilon), y = \epsilon)$ ,  $(x = 1 - \epsilon e^{it}, y = \epsilon)$  and  $(x = (1-t)(1-\epsilon) + t\epsilon, y = \epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = 0$$

To compute the monodromy around  $y = 1$ , take  $q(\epsilon)$  to be the composition of  $(x = \epsilon, y = (1-t)\epsilon + t(1-\epsilon))$ ,  $(x = \epsilon, y = 1 - \epsilon e^{it})$  and  $(x = \epsilon, y = (1-t)(1-\epsilon) + t\epsilon)$

$$\lim_{\epsilon \rightarrow 0} \int_{q(\epsilon)p(\epsilon)} - \int_{p(\epsilon)} = \int_q dv_2 \int_p dv_1 (du_1)^{n-1} = -2\pi i \text{Li}_n(x)$$

To compute the monodromy around  $xy = 1$ , take  $q$  to be the loop  $(x = x^0, y \text{ such that } \int_q d \log(1-xy) = 2\pi i)$  First it's easy to show inductively

$$\int_{(x^0, y^0)}^{(x, y)} dv_2 du_2^{n-1} = \text{Li}_n(y) - \sum_{k=1}^n \frac{(\log y - \log y^0)^{n-k}}{(n-k)!} \text{Li}_k(y^0)$$

Thus

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{pq} - \int_p &= \sum_{k=0}^{n-1} \int_p dv_{12} (du_{12})^k \int_q (du_{12})^{n-1-k} dv_2 + \int_q dv_{12} du_{12}^{n-1} dv_2 \\ &= (-1)^{n+1} \int_{q^{-1}} dv_2 du_2^{n-1} dv_{12} \\ &= (-1)^{n+1} \int_{q^{-1}} \frac{\text{Li}_n(y) - \sum_{k=1}^n \frac{(\log y - \log y^0)^{n-k}}{(n-k)!} \text{Li}_k(y^0)}{y - 1/x^0} dy \\ &= (-1)^n 2\pi i (-g_n(x^0, y^0) - \text{Li}_n(1/x^0)) \end{aligned}$$



Where

$$g_m(x, y) = \sum_{k=1}^m (-1)^{k+1} \frac{\log^{m-k}(xy)}{(m-k)!} \text{Li}_k(y)$$

The variation matrix is

$$\Lambda = \begin{bmatrix} \text{Li}_1(y) & 1 & & & & & & \\ \text{Li}_1(xy) & 0 & & & & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & g_1(x, y) - \text{Li}_1(x^{-1}) & 1 & & & & \\ \text{Li}_2(xy) & 0 & \log x & 0 & 1 & & & \\ \text{Li}_{2,1}(x, y) & \text{Li}_2(x) & g_2(x, y) + \text{Li}_2(x^{-1}) & \log x & g_1(x, y) & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ \text{Li}_n(xy) & 0 & \frac{\log^{n-1}(xy)}{(n-1)!} & 0 & \frac{\log^{n-2}(xy)}{(n-2)!} & 0 & \dots & 1 \\ \text{Li}_{n,1}(x, y) & \text{Li}_n(x) & g_n(x, y) + (-1)^n \text{Li}_n(x^{-1}) & \frac{\log^{n-1} x}{(n-1)!} & g_{n-1}(x, y) & \frac{\log^{n-2} x}{(n-2)!} & \dots & g_1(x, y) & 1 \end{bmatrix} \tau_{n,1}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & & & & & & \\ -dv_2 & 0 & & & & & & & \\ -dv_{12} & 0 & 0 & & & & & & \\ & -dv_2 & -dw_1 & 0 & & & & & \\ & & du_{12} & 0 & 0 & & & & \\ & & & du_1 & -dv_2 & 0 & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & du_{12} & 0 & 0 & \\ & & & & & & du_1 & -dv_2 & 0 \end{bmatrix}$$

$$dg_m(x, y) = g_{m-1}(x, y) du_1 - \frac{\log^{m-1}(xy)}{(m-1)!} dv_2$$

Justifies the differential equation  $d\Lambda = \omega\Lambda$

Monodromy of  $\log^m x/m!$  around  $x = 0$  is

$$\sum_{k=0}^{m-1} \frac{\log^k x}{k!} \frac{(2\pi i)^{m-k}}{(m-k)!}$$

Monodromy of  $\log^m(xy)/m!$  around  $x = 0$  is

$$\sum_{k=0}^{m-1} \frac{\log^k(xy)}{k!} \frac{(2\pi i)^{m-k}}{(m-k)!}$$

Monodromy of  $\text{Li}_n(x^{-1})$  around  $x = 0$  is  $2\pi i \log^{n-1}(x^{-1})/(n-1)! = (-1)^{n-1} 2\pi i \log^{n-1}(x)/(n-1)!$

Monodromy of  $g_n$  around  $x = 0$  or  $y = 0$  is

$$\begin{aligned} \sum_{k=1}^m \sum_{l=1}^{m-k} (-1)^{k+1} \text{Li}_k(y) \frac{\log^{m-k-l}(xy)}{(m-k-l)!} \frac{(2\pi i)^l}{l!} &= \sum_{l=1}^m \sum_{k=1}^{m-l} (-1)^{k+1} \text{Li}_k(y) \frac{\log^{m-k-l}(xy)}{(m-k-l)!} \frac{(2\pi i)^l}{l!} \\ &= \sum_{l=1}^{m-1} \frac{(2\pi i)}{l!} g_{m-l}(x, y) \end{aligned}$$

Monodromy of  $g_n$  around  $y = 1$  is

$$2\pi i \sum_{k=1}^m (-1)^k \frac{\log^{m-k}(xy)}{(m-k)!}$$



For monodromy around  $xy = 1$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ -1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & \ddots \\ & & & & & \ddots \\ & & & & & & 1 \end{bmatrix}$$

## 1.12 Variation matrix

**Theorem 1.12.1.**  $\Lambda$  is the fundamental solution of the system of linear differential equations

$$d\Lambda = \omega\Lambda$$

**Example 1.12.2.** For

$$\begin{aligned} \text{Li}_{1,1}(x, y) &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 + dv_{12} d(u_1 - v_1 + v_2) \\ &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 + dv_{12} du_1 - dv_{12} dv_1 + dv_{12} dv_2 \end{aligned}$$

$(0, 0) < (0, 1) < (1, 0) < (1, 1)$  in  $\mathfrak{S}(1, 1)$

$$\Lambda = \begin{bmatrix} 1 & & & \\ \text{Li}_1(y) & 1 & & \\ \text{Li}_1(xy) & & 1 & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \text{Li}_1\left(\frac{1-xy}{1-x}\right) & 1 \end{bmatrix} \tau_{1,1}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & \\ -dv_2 & 0 & & \\ -dv_{12} & 0 & 0 & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 \end{bmatrix}$$

**Example 1.12.3.** For

$$\begin{aligned} \text{Li}_{2,1}(x, y) &= \int_{(0,0)}^{(x,y)} (dv_1 dv_2 + dv_{12} d(u_1 - v_1 + v_2)) du_1 + dv_{12} d(u_1 + u_2) dv_2 \\ &= \int_{(0,0)}^{(x,y)} dv_1 dv_2 du_1 + dv_{12} du_1 du_1 - dv_{12} dv_1 du_1 \\ &\quad + dv_{12} dv_2 du_1 + dv_{12} du_1 dv_2 + dv_{12} u_2 dv_2 \end{aligned}$$

$(0, 0) < (0, 1) < (1, 0) < (1, 1) < (2, 0) < (2, 1)$  in  $\mathfrak{S}(2, 1)$

$$\Lambda = \begin{bmatrix} 1 & & & & & \\ \text{Li}_1(y) & 1 & & & & \\ \text{Li}_1(xy) & & 1 & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \log \frac{1-x}{(1-y)x} & 1 & & \\ \text{Li}_2(xy) & & \log(xy) & & 1 & \\ \text{Li}_{2,1}(x, y) & \text{Li}_2(x) & g(x, y) & \log x & \text{Li}_1(y) & 1 \end{bmatrix} \tau_{2,1}(2\pi i)$$

Where  $dg = \log \frac{1-x}{(1-y)x} \frac{dx}{x} + \log(xy) \frac{dy}{1-y}$

$$\omega = \begin{bmatrix} 0 & & & & & \\ -dv_2 & 0 & & & & \\ -dv_{12} & 0 & 0 & & & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 & & \\ 0 & 0 & d(u_1 + u_2) & 0 & 0 & \\ 0 & 0 & 0 & du_1 & -dv_2 & 0 \end{bmatrix}$$

**Example 1.12.4.** For

$$\begin{aligned} \text{Li}_{1,1,1}(x, y, z) &= \int_{(0,0,0)}^{(x,y,z)} \\ &= \int_{(0,0,0)}^{(x,y,z)} \end{aligned}$$

$(0, 0, 0) < (0, 0, 1) < (0, 1, 0) < (1, 0, 0) < (0, 1, 1) < (1, 0, 1) < (1, 1, 0) < (1, 1, 1)$  in  $\mathfrak{S}(1, 1, 1)$

$$\Lambda = \begin{bmatrix} 1 & & & & & & & & \\ \text{Li}_1(z) & 1 & & & & & & & \\ \text{Li}_1(yz) & & 1 & & & & & & \\ \text{Li}_1(xyz) & & & 1 & & & & & \\ \text{Li}_{1,1}(y, z) & \text{Li}_1(y) & \log \frac{1-y}{(1-z)y} & & 1 & & & & \\ \text{Li}_{1,1}(xy, z) & \text{Li}_1(xy) & & \log \frac{1-xy}{(1-z)xy} & & 1 & & & \\ \text{Li}_{1,1}(x, yz) & & \text{Li}_1(x) & \log \frac{1-x}{(1-yz)x} & & & 1 & & \\ \text{Li}_{1,1,1}(x, y, z) & g(x, y) & \text{Li}_1(x) \log \frac{1-y}{(1-z)y} & h(x, y) & \text{Li}_1(x) \log \frac{1-y}{(1-x)x} & \log \frac{1-z}{(1-y)y} & 1 & & \end{bmatrix} \tau_{1,1,1}(2\pi i)$$

Where

$$\omega = \begin{bmatrix} 0 & & & & & & & & \\ -dv_3 & 0 & & & & & & & \\ -dv_{23} & 0 & 0 & & & & & & \\ -dv_{13} & 0 & 0 & 0 & & & & & \\ 0 & -dv_2 & d(v_2 - u_2 - v_3) & 0 & 0 & & & & \\ 0 & -dv_{12} & 0 & d(v_{12} - u_1 - u_2 - v_3) & 0 & 0 & & & \\ 0 & 0 & -dv_1 & d(v_1 - u_1 - v_{23}) & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & -dv_1 & d(v_1 - u_1 - v_2) & d(v_2 - u_2 - v_3) & 0 & \end{bmatrix}$$

**Example 1.12.5.** For

$$\begin{aligned} \text{Li}_{1,2}(x, y) &= dv_{12}d(u_1 + u_2)d(u_1 - v_2) + (dv_1dv_2 + dv_{12}d(u_1 - v_1 + v_2))du_2 \\ &= dv_{12}du_1du_1 - dv_{12}du_1dv_2 + dv_{12}du_2du_1 - dv_{12}du_2dv_2 \\ &\quad + dv_1dv_2du_2 + dv_{12}du_1du_2 - dv_{12}dv_1du_2 + dv_{12}dv_2du_2 \end{aligned}$$

$(0, 0) < (0, 1) < (1, 0) < (1, 1) < (0, 2) < (1, 2)$  in  $\mathfrak{S}(1, 2)$

$$\Lambda = \begin{bmatrix} 1 & & & & & & & & \\ \text{Li}_1(y) & 1 & & & & & & & \\ \text{Li}_1(xy) & 0 & 1 & & & & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \log \frac{1-x}{(1-y)x} & 1 & & & & & \\ \text{Li}_2(y) & \log y & & & 1 & & & & \\ \text{Li}_2(xy) & 0 & \log(xy) & & & 1 & & & \\ \text{Li}_{1,2}(x, y) & \text{Li}_1(x) \log y & g(x, y) & \log y & \text{Li}_1(x) & -\text{Li}_1(x^{-1}) & 1 & & \end{bmatrix} \tau_{1,2}(2\pi i)$$

Where  $g(x, y) = -I((xy)^{-1}; y^{-1}, 0; 1)$

$$\omega = \begin{bmatrix} 0 & & & & & & & & \\ -dv_2 & 0 & & & & & & & \\ -dv_{12} & 0 & 0 & & & & & & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 & & & & & \\ 0 & du_2 & 0 & 0 & 0 & & & & \\ 0 & 0 & d(u_1 + u_2) & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & du_2 & -dv_1 & d(v_1 - u_1) & 0 & & \end{bmatrix}$$

**Example 1.12.6.** For

$$\begin{aligned} \text{Li}_{2,2}(x, y) &= (dv_{12}du(u_1 + u_2)d(u_1 - v_2) + (dv_1dv_2 + dv_{12}d(u_1 - v_1 + v_2))du_2)du_1 \\ &\quad + ((dv_1dv_2 + dv_{12}d(u_1 - v_1 + v_2))du_1 + dv_{12}d(u_1 + u_2)dv_2)du_2 \\ &= dv_{12}du_1du_1du_1 - dv_{12}du_1dv_2du_1 + dv_{12}du_2du_1du_1 - dv_{12}du_2dv_2du_1 \\ &\quad + dv_1dv_2du_2du_1 + dv_{12}du_1du_2du_1 - dv_{12}dv_1du_2du_1 + dv_{12}dv_2du_2du_1 \\ &\quad + dv_1dv_2du_1du_2 + dv_{12}du_1du_1du_2 - dv_{12}dv_1du_1du_2 \\ &\quad + dv_{12}dv_2du_1du_2 + dv_{12}du_1dv_2du_2 + dv_{12}u_2dv_2du_2 \end{aligned}$$

$(0, 0) < (0, 1) < (1, 0) < (1, 1) < (0, 2) < (2, 0) < (1, 2) < (2, 1) < (2, 2)$  in  $\mathfrak{S}(2, 2)$

$$\Lambda = \begin{bmatrix} \text{Li}_1(y) & 1 & & & & & & & & \\ \text{Li}_1(xy) & & 1 & & & & & & & \\ \text{Li}_{1,1}(x, y) & \text{Li}_1(x) & \log \frac{1-x}{(1-y)x} & 1 & & & & & & \\ \text{Li}_2(y) & \log y & & & 1 & & & & & \\ \text{Li}_2(xy) & & \log(xy) & & & 1 & & & & \\ \text{Li}_{1,2}(x, y) & \text{Li}_1(x) \log y & g(x, y) & \log y & \text{Li}_1(x) & \log \frac{1-x}{x} & 1 & & & \\ \text{Li}_{2,1}(x, y) & \text{Li}_2(x) & h(x, y) & \log x & & \text{Li}_1(y) & & 1 & & \\ \text{Li}_{2,2}(x, y) & \text{Li}_2(x) \log y & i(x, y) & \log x \log y & \text{Li}_2(x) & \text{Li}_2(y) - \text{Li}_2(x) - \frac{1}{2} \log^2 x \log x \log y & 1 & & & \end{bmatrix} \tau_{2,2}(2\pi i)$$

$$\omega = \begin{bmatrix} 0 & & & & & & & & & \\ -dv_2 & 0 & & & & & & & & \\ -dv_{12} & 0 & 0 & & & & & & & \\ 0 & -dv_1 & d(-u_1 + v_1 - v_2) & 0 & & & & & & \\ 0 & du_2 & 0 & 0 & 0 & & & & & \\ 0 & 0 & d(u_1 + u_2) & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & du_2 & -dv_1 & d(v_1 - u_1) & 0 & & & \\ 0 & 0 & 0 & du_1 & 0 & -dv_2 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & du_1 & du_2 & 0 & \end{bmatrix}$$

### 1.13 Bloch-Wigner polylogarithm

**Definition 1.13.1.** The *Bloch-Wigner polylogarithm* is defined as

$$\mathcal{L}_n(z) = \Re_n \left( \sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \operatorname{Li}_{n-r}(z) \log^r |z| \right)$$

Here  $\Re_n$  is  $\operatorname{Re}$  if  $n$  is odd and  $\operatorname{Im}$  if  $n$  is even.  $B_n$  are Bernoulli numbers. For instance,  $\mathcal{L}_1(z) = \log |1 - z|$ ,  $\mathcal{L}_2(z) = \operatorname{Im}(\operatorname{Li}_2(z)) + \operatorname{Im}(\log(1 - z)) \log |z|$

**Lemma 1.13.2.**  $\mathcal{L}_n(z) + (-1)^n \mathcal{L}_n(z^{-1}) = 0$ .  $\mathcal{L}_3(z) + \mathcal{L}_3\left(\frac{1}{1-z}\right) + \mathcal{L}_3(1 - z^{-1}) = \zeta(3)$ .  $\mathcal{L}_2(z) - \mathcal{L}_2\left(\frac{1}{1-z}\right) = 0$

## 1.14 Hopf algebra structure

**Definition 1.14.1.** Iterated integrals form a Hopf algebra  $H$  with coproduct

$$\Delta I(a_0; a_1, \dots; a_n; a_{n+1}) = \sum_{0=i_0 < i_1 < \dots < i_k < i_{k+1}=n+1} I(a_{i_0}; a_{i_1}, \dots, a_{i_k}; a_{i_{k+1}}) \otimes \prod_{p=1}^k I(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

The product is the just shuffle product,  $\Delta_{i_1, \dots, i_k}$  means those in grading  $(i_1, \dots, i_k)$ .  $\Delta'(x) = \Delta(x) - 1 \otimes x - x \otimes 1$  is the reduced coproduct. The space of indecomposables  $Q(H) = H/(H_{>0} \cdot H_{>0})$  is mod products. The projection  $\frac{1}{n}R = P : H \rightarrow Q(H)$ , where  $R$  is defined inductively as  $R(x) = nx - \mu(1 \otimes R)\Delta'(x)$ ,  $\mu$  is multiplication. The cobracket is defined as  $\delta(x) = (P \otimes P)(1 - \tau)\Delta(x)$ ,  $\tau(x \otimes y) = y \otimes x$

Symbol of a multiple polylogarithm is defined to be  $\Delta_{1, \dots, 1}(x)$ , and omit log sign