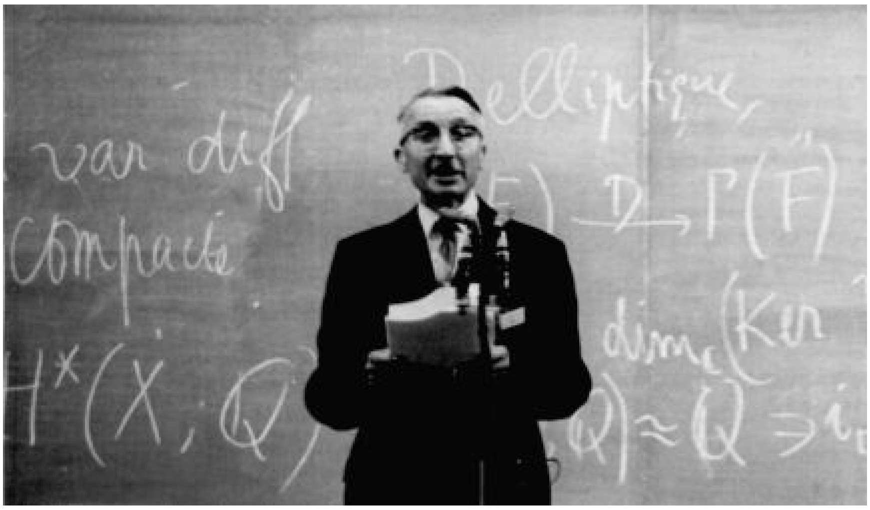
MATH602 - Homological algebra



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1 Review of category theory

1.1 Categories - 1/27/2020

Definition 1.1. A **category** $\mathscr{C}$ consists of $\mathrm{Ob}\mathscr{C}$ class of **objects** and $\mathrm{Hom}\mathscr{C}$ class of **morphisms**, for $f : A \to B$, $g : B \to C$, $\exists g \circ f : A \to C$, the composition is associative $(h \circ g) \circ f = h \circ (g \circ f)$, $\exists 1_A : A \to A$ such that $1_A f = f, \forall f : B \to A$ and $g 1_A = g, \forall g : A \to B$(thus 1_A is unique), we denote $Hom_{\mathscr{C}}(A, B)$ to be all the morphisms from A to B

Definition 1.2. Let $\mathscr{C}$ be a category, $f : A \to B$ is an **isomorphism** if there exists $g : B \to A$ such that $gf = 1_A$, $fg = 1_B$

Example 1.3. Let M be a monoid, we can view it as a category $\mathscr{C}_M$, where $Ob\mathscr{C}_M = \{*\}$, $Hom_{\mathscr{C}_M}(*, *) = M$

Remark 1.4. Book recommandation: Abelian category - Fregd, it defines a category use only morphisms

Lemma 1.5. An isomorphism $f : X \to Y$ has an unique inverse, denoted f^{-1}

Proof.

□

Definition 1.6. A category $\mathscr{C}$ is called a **small category** if $Ob\mathscr{C}$ is a set

Definition 1.7. A category $\mathscr{C}$ is called an **essentially small category** if $Ob\mathscr{C}/\sim$ is a set, here $Ob\mathscr{C}/\sim$ is the isomorphic classes of objects

Example 1.8. Let k be a field, then the category of finite dimensional k vector fields is not small but essentially small, two k vector spaces are isomorphic iff they have the same dimension

Example 1.9. Let R be a commutative ring, the category of R modules, $RMod$ is not essentially small

Definition 1.10. Let P be a poset, we can view it as a category $\mathscr{C}_P$, where $Ob\mathscr{C}_p = P$, $Hom_{\mathscr{C}_p}(x, y) = \begin{cases} \{*\}, & x \leq y \\ \varnothing, & \text{else} \end{cases}$

Exercise 1.11. Suppose small category $\mathscr{C}$ satisfies
$|Hom(x, y)| \leq 1$
$x \neq y \Rightarrow x \ncong y$
Then $\mathscr{C}$ is poset

Proof.

□

Definition 1.12. A category is a **groupoid** if every morphism is an isomorphism, thus a groupoid with only one object is a group

1.2 Functors - 1/29/2020

Definition 1.13. $\mathscr{C}, \mathscr{D}$ are categories, $F : \mathscr{C} \to \mathscr{D}$ is a **functor** if it is a mapping: $\mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D}$, $\mathscr{C}(A, B) \to \mathscr{D}(F(A), F(B))$, $F(1_A) = 1_{F(A)}$, given $f : A \to B, g : B \to C$, $F(g \circ f) = F(g) \circ F(f) : F(A) \to F(C)$, this kind of functor is called **covariant fucntor**, if $\mathrm{Ob}\mathscr{C} \to \mathrm{Ob}\mathscr{D}$, $\mathscr{C}(A, B) \to \mathscr{D}(F(B), F(A))$, $F(1_A) = 1_{F(A)}$, given $f : A \to B, g : B \to C$, $F(g \circ f) = F(f) \circ F(g) : F(C) \to F(A)$, then this is called a **contravariant functor**
The **dual category** of a category $\mathscr{C}$ is denoted as $\mathscr{C}^{op}$ with the same objects but morphisms reversed, a contravariant functor is just a functor in the dual

Example 1.14. **(1):** Let M, N be monoids, a functor $F : \mathscr{C}_M \to \mathscr{C}_N$ is just a homomorphism of monoids
(2): Let M, N be groups, a functor $F : \mathscr{C}_M \to \mathscr{C}_N$ is just a homomorphism of groups
(3): Let L/F be a field extension, $- \otimes L$ is a functor $Vect_F \to Vect_L, V \mapsto V \otimes_F L$, $\phi \mapsto \phi \otimes 1_L$
(4): Homology H_* is a functor $Top \to Abgp$, $X \mapsto H_*(X)$
(5): Cohomology H^* is a contravariant functor $Top \to Abgp$, $X \mapsto H^*(X)$
(6): Let $FinAbgp$ be the category of finite abelian groups, then $D : FinAbgp \to FinAbgp$, $X \mapsto Hom(X, \mathbb{Q}/\mathbb{Z})$ is a contravariant functor, or we could use $Hom(X, \mathbb{C}^\times)$, this is called **Pontrjagin duality**
(7): $D : Vect_K \to Vect$, $V \mapsto V^*$ is a contravariant functor

Notation. Suppose $f : X \to Y$ is a morphism in category $\mathscr{C}$, for $Z \in ob\mathscr{C}$, we define

$$f_* : Hom(Z, X) \to Hom(Z, Y), \quad g \mapsto fg$$

$$f^* : Hom(Y, Z) \to Hom(X, Z), \quad g \mapsto gf$$

Definition 1.15. A morphism $f : X \to Y$ is called a monomorphism if f_* is 1-1 for all $Z \in ob\mathscr{C}$
A morphism $f : X \to Y$ is called a epimorphism if f^* is 1-1 for all $Z \in ob\mathscr{C}$

Definition 1.16. In category $\mathscr{C}$, an object X is called an **initial object** if $Hom(X, Y)$ consists of exactly one element for all Y, X is called a **final object** if $Hom(Y, X)$ consists of exactly one element for all Y, X is called a **zero object** if it is both initial and final

Example 1.17. **(1):** In the category of sets, $\emptyset$ is an initial object, $\{1\}$ is a final object
(2): In the category of abelian groups, 0 is a zero object

Definition 1.18. $F, G : \mathscr{C} \to \mathscr{D}$ are covariant functors, $\eta_A : F(A) \to G(A)$ is a family of morphisms such that the following diagram commutes for any $f : A \to B$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

For contravariant functors, we have the following commutative diagram for any $f : A \to B$

$$\begin{tikzcd} F(B) \arrow[r, "F(f)"] \arrow[d, "\eta_B"] & F(A) \arrow[d, "\eta_A"] \\ G(B) \arrow[r, "G(f)"] & G(A) \end{tikzcd}$$

η is called a **natural transformation**
If η_A are isomorphisms, then η is called a **natural isomorphism**, denoted $F \cong G$

1.3 Presheaves and Yoneda lemma - 1/31/2020

Definition 1.19. Suppose $\mathscr{C}$, $\mathscr{D}$ are categories, we can define the **functor category** $\mathrm{Fun}(\mathscr{C}, \mathscr{D}) = \mathscr{D}^{\mathscr{C}}$ with objects functors from $\mathscr{C}$ to $\mathscr{D}$ and morphisms natural transformations

Remark 1.20. If I is a small category, then $Hom_{\mathscr{C}^I}(F, G)$ is a set

Definition 1.21. we say categories $\mathscr{C}, \mathscr{D}$ are **isomorphic** if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F = 1_{\mathscr{C}}$, $F \circ G = 1_{\mathscr{D}}$ and we say $\mathscr{C}, \mathscr{D}$ are **equivalent** if $G \circ F$ is naturally isomorphic to $1_{\mathscr{C}}$ and $F \circ G$ is naturally isomorphic to $1_{\mathscr{D}}$

Example 1.22. Let $\mathscr{C} = Vect_K$ be te category of K vector spaces, define functor $F : \mathscr{C} \to \mathscr{C}$, $V \mapsto V \otimes_K K$ is an equivalence with inverse $G = 1_{\mathscr{C}}$, but this is not an isomorphism, since not every vector space is in the form of a tensor product

Definition 1.23. Suppose $\mathscr{C}, \mathscr{D}$ are locally small categories, $F : \mathscr{C} \to \mathscr{D}$ is a functor
F is **faithful** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is injective for any X, Y
F is **full** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is surjective for any X, Y
F is **fully faithful** if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is bijective for any X, Y
F is **essentially surjective** if $\forall d \in ob\mathscr{D}$, $\exists c \in ob\mathscr{C}$ such that $Fc \cong d$

X1,X2 iso and Y1,Y2 iso implies Hom(X1,Y1),Hom(X2,Y2) iso

Lemma 1.24. In category $\mathscr{C}$, if $\phi_X : X \to X'$, $\phi_Y : Y \to Y'$ are isomorphisms, then $Hom(X, Y)$, $Hom(X', Y')$ are in bijective correspondence

Proof. We can define maps $Hom(X, Y) \to Hom(X', Y')$, $f \mapsto \phi_Y f \phi_X^{-1}$ and $Hom(X', Y') \to Hom(X, Y)$, $f' \mapsto \phi_Y^{-1} f' \phi_X$ which are inverses to each other

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

□

A functor F is an equivalence iff it is fully faithful and essentially surjective

Theorem 1.25. A functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence iff it is fully faithful and essentially surjective

Proof. If F is an equivalence, there exist functor $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : 1_{\mathscr{C}} \to GF$, $\xi : 1_{\mathscr{D}} \to FG$, $\forall d \in \mathscr{C}$, $\xi_d : d = 1_{\mathscr{D}}(d) \to FG(d) = F(Gd)$ is an isomorphism, i.e. F is essentially surjective, similarly, so is G
The composition of

$$Hom(c, c') \xrightarrow{F} Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc'), \quad f \mapsto Ff \mapsto GFf$$

Is the same as

$$Hom(c, c') \xrightarrow{\eta} Hom(GFc, GFc'), \quad f \mapsto \eta'_c f \eta_c^{-1}$$

By Lemma 1.24, this is bijective, thus $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is injective, i.e. F is faithful. Similarly, consider the composition

$$Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$$

We know $Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$ is surjective, but we also have the following diagram

$$\begin{tikzcd} Hom(c, c') \arrow[r, "F"] \arrow[d, "\eta"'] & Hom(Fc, Fc') \arrow[d, "\xi"] \\ Hom(GFc, GFc') \arrow[r, "F"] & Hom(FGFc, FGFc') \end{tikzcd}$$

Since η, ξ are bijective, $Hom(c,c') \xrightarrow{F} Hom(Fc,Fc')$ is surjective, i.e. F is full
Conversely, suppose F is fully faithful and essentially surjective, then for any $d \in \mathscr{D}$, there exists c and an isomorphism $d \xrightarrow{\xi_d} Fc$, denote this c as Gd, we can define a functor $G : \mathscr{D} \to \mathscr{C}$, $d \mapsto Gd$(Here we have used the axiom of choice), $d \xrightarrow{f} d' \mapsto c \xrightarrow{Gf} c'$ where $FGf = \xi_d^{-1} f \xi_{d'}$ since F is fully faithful

$$\begin{tikzcd}
d \arrow[r, "f"] \arrow[d, "\xi_d"'] & d' \arrow[d, "\xi_{d'}"] \\
FGd \arrow[r, "FGf"] & FGd' \\
Gd \arrow[r, "Gf"] \arrow[u, "F"] & Gd' \arrow[u, "F"']
\end{tikzcd}$$

$\xi : 1_{\mathscr{D}} \to FG$ is a natural isomorphism
Since F is fully faithful, there are unique $\eta_c : c \to GFc$, $F(\eta_c) = \xi_{Fc}$
If $f, g : c \to c'$ such that $\eta_{c'} f = \eta_{c'} g$, then $\xi_{Fc'} Ff = \xi_{Fc'} Fg \Rightarrow Ff = Fg \Rightarrow f = g$
If $f, g : c \to c'$ such that $f \eta_c = g \eta_c$, then $Ff \xi_{Fc} = Fg \xi_{Fc} \Rightarrow Ff = Fg \Rightarrow f = g$

$$\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dd, bend right=60, "\eta_c"'] & c' \arrow[d] \arrow[dd, bend left=60, "\eta_{c'}"] \\
Fc \arrow[r] \arrow[d, "G"'] \arrow[dd, bend right=60, "\xi_{Fc}"'] & Fc' \arrow[d, "G"] \arrow[dd, bend left=60, "\xi_{Fc'}"] \\
GFc \arrow[r] \arrow[d, "F"'] & GFc' \arrow[d, "F"] \\
FGFc \arrow[r] & FGFc'
\end{tikzcd}$$

$\eta : 1_{\mathscr{C}} \to GF$ is a natural isomorphism ☐

Definition 1.26. $\mathscr{D}^{\mathscr{C}^{op}}$ is the category of **presheaves**. Denote $\mathscr{C}^\vee := Sets^{\mathscr{C}^{op}}$. In particular, if X is a topological space, open subsets with inclusion form a category $\mathscr{C}$, $PreSh(X, \mathscr{D})$ is the category of presheaves on X with values in $\mathscr{D}$

Yoneda lemma

Lemma 1.27 (Yoneda lemma)**. Yoneda embedding** $h : \mathscr{C} \to \mathscr{C}^\vee$ defined as follows is a fully faithful functor
For $X \in ob\mathscr{C}$, $h(X) = Hom_{\mathscr{C}}(-, X)$ is a contravariant functor $\mathscr{C} \to Sets$:
For $Z \in ob\mathscr{C}$, $h(X)(Z) = Hom_{\mathscr{C}}(Z, X)$, for $\phi : Z \to W$, $h(X)(\phi) = \phi^* : h(X)(W) = Hom_{\mathscr{C}}(W, X) \to Hom_{\mathscr{C}}(Z, X) = h(X)(Z)$, hence $h(X)$ is an object in $\mathscr{C}^\vee$
For $\psi : X \to Y$, $h(\psi) = \psi_*$ is a natural transformation $h(X) \to h(Y)$:
For $\phi : Z \to W$, we have the commutative diagram

$$\begin{tikzcd}
Hom_{\mathscr{C}}(W,X) \arrow[r, "\psi_*"] \arrow[d, "\phi_*"'] & Hom_{\mathscr{C}}(W,Y) \arrow[d, "\phi_*"] \\
Hom_{\mathscr{C}}(Z,X) \arrow[r, "\psi_*"] & Hom_{\mathscr{C}}(Z,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
h(X)(W) \arrow[r, "h(\psi)_W"] \arrow[d, "h(Y)(\phi)"'] & h(Y)(W) \arrow[d, "h(Y)(\phi)"] \\
h(X)(Z) \arrow[r, "h(\psi)_Z"] & h(X)(Z)
\end{tikzcd}$$

Proof.

☐

Definition 1.28. A **subcategory** $\mathscr{D}$ of $\mathscr{C}$ is a category with objects a subclass of $ob\mathscr{C}$ and morphisms a subclass of $Hom\mathscr{C}$, with the original composition

Example 1.29. The image of a functor is not necessarily a category
Consider the following categories $\mathscr{C}$ and $\mathscr{D}$

$$\begin{tikzcd}
A \arrow[loop above, "1_A"] \arrow[r, "f"] & B \arrow[loop above, "1_B"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[loop above, "1_C"] \arrow[r, "g"] & D \arrow[loop above, "1_D"]
\end{tikzcd}$$

$$\begin{tikzcd}
E \arrow[loop above, "1_E"] \arrow[r, "h"] \arrow[rr, bend right, "ih"'] & F \arrow[loop above, "1_F"] \arrow[r, "i"] & G \arrow[loop above, "1_G"]
\end{tikzcd}$$

Consider functor $F : \mathscr{C} \to \mathscr{D}$, $F(A) = E$, $F(B) = F$, $F(C) = F$, $F(D) = G$, $F(f) = h$, $F(g) = i$

Theorem 1.30. A functor $F : \mathscr{C} \to \mathscr{D}$ is fully faithful iff it induces an equivalence of categories from $\mathscr{C}$ to a full subcategory of $\mathscr{D}$

Proof.

□

1.4 Limits - 2/3/2020

Definition 1.31. A functor F in $\mathscr{C}^\vee$ is called **representable** if there exists $X \in ob\mathscr{C}$ such that $h(X) \cong F$, here h is the Yoneda embedding. In other words, there exists a natural isomorphism $Hom_{\mathscr{C}}(Y, X) \to F(Y)$, since h is fully faithful, if $F \cong h(X) \cong h(X')$, the natural isomorphism $h(X) \cong h(X')$ comes from an isomorphism $\phi : X \to X'$, hence X is unique to isomorphism

Definition 1.32. Let I be a small category, $\mathscr{C}$ be a category, for any $X \in ob\mathscr{C}$, we can define the **constant functor** $K_X : I \to \mathscr{C}$, $i \mapsto X$, $i \xrightarrow{f} j \mapsto 1_X$, hence $K : \mathscr{C} \to \mathscr{C}^I$, $X \mapsto K_X$ is a functor, a natural transformation f between constant functors $K_X \to K_Y$ is just a morphism $f : X \to Y$

Definition 1.33. Suppose $F : I \to \mathscr{C}$ is a functor, we get a presheaf P, $P(X) = Hom_{\mathscr{C}^I}(K_X, F)$
If P is representable, i.e. $h(L) \cong P$, we write $L = \varprojlim F$ which is called the **limit** of F
We also have a functor $F^{op} : I^{op} \to \mathscr{C}^{op}$. The **colimit** is defined to be $\varprojlim F^{op}$

Remark 1.34. Unravel $Hom_{\mathscr{C}^I}(K_X, F) = P(X) \cong Hom_{\mathscr{C}}(X, L)$
If we take $X = L$, 1_X corresponds to a natural transformation $\phi : K_L \to F$, i.e. $\phi_i : L \to F(i)$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "\phi_i"] \arrow[dr, "\phi_j"'] & F(i) \arrow[d, "F(i\to j)"] \\
 & F(j)
\end{tikzcd}$$

Each natural transformation $\psi : K_X \to F$ corresponds to a unique morphism $\widehat{\psi} : X \to L$, due to naturality, the following diagram commutes

$$\begin{tikzcd}
X \arrow[d, "\widehat{\psi}"'] \arrow[dr, "\psi_i"] & \\
L \arrow[r, "\phi_i"] & F(i)
\end{tikzcd}$$

Definition 1.35. A category I is called **discrete** if all morphisms are just identities, it is clear that a discrete category is the same as a class of objects, and a functor $F : I \to \mathscr{C}$ is the same as giving $X_i = F(i)$

Example 1.36. Suppose I is a discrete category, $F : I \to \mathscr{C}$ is a functor, we also get functor $F^{op} : I^{op} \to \mathscr{C}^{op}$. The **product** is defined to be the limit $\prod_{i \in I} X_i := \varprojlim F$, and the **coproduct** is $\varprojlim F^{op}$

1.5 Equalizers and fiber product - 2/5/2020

Definition 1.37. A category $\mathscr{C}$ is **complete** if $\mathscr{C}$ contains all limits, $\mathscr{C}$ is **cocomplete** if $\mathscr{C}$ contains all colimits

Definition 1.38. Let I be the category $\bullet \rightrightarrows \bullet$, a functor $F : I \to \mathscr{C}$ is just $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, the limit is defined to be the **equalizer**, the dual notion is called a **coequalizer**

Theorem 1.39. If a category $\mathscr{C}$ contains all products and equalizers, then $\mathscr{C}$ is complete

Proof. The limit of $A \xrightarrow{f} B$ is the same as the equaliser of $A \times B \overset{fp_A}{\underset{p_B}{\rightrightarrows}} B$

$$\begin{tikzcd} & C \arrow[d, dashed] \arrow[ddl, "g"'] \arrow[ddr, "h"] & \\ & A \times B \arrow[dl, "p_A"] \arrow[dr, "p_B"'] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

Then by induction, we can find the limit of $A_i \to \varprojlim_{j \neq i} A_j$ which is $\varprojlim_i A_i$ $\square$

Definition 1.40. Let I be the category $\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$, a functor $F : I \to \mathscr{C}$ is just $\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$, the limit is defined to be the **fiber product(pullback)**, the dual notion is called a **pushforward(pushout)**

Definition 1.41. An **Ab-category** $\mathscr{C}$ is a category such that $Hom_{\mathscr{C}}(X, Y)$ are equipped with an abelian group structure, such that $f(g + h) = fg + fh$, $(f + g)h = fh + gh$

Remark 1.42. An Ab-category is also called a **preadditive category**
$End_{\mathscr{C}}(X)$ is a ring, $Aut_{\mathscr{C}}(X) = End_{\mathscr{C}}(X)^\times$ is a group

1.6 Abelian category - 2/7/2020

Definition 1.43. The **biproducts** $(A_1 \oplus \cdots \oplus A_n, p_1, \cdots, p_n, i_1, \cdots, i_n)$ of $A_1, \cdots, A_n$ is such that $(A_1 \oplus \cdots \oplus A_n, p_1, \cdots, p_n)$ is the product of $A_1, \cdots, A_n$ and $(A_1 \oplus \cdots \oplus A_n, i_1, \cdots, i_n)$ is the coproduct of $A_1, \cdots, A_n$

Lemma 1.44. Suppose $\mathscr{A}$ is an Ab category, then for any $A_1, \cdots, A_n$, if the product $\prod A_i$ exists, then it is a biproduct, similarly, if the coproduct $\coprod A_i$ exsits, then it is a biproduct

Proof. Suppose $(A \times B, p_A, p_B)$ is the product of A, B, then we can define morphisms $i_A = (1_A, 0) : A \to A \times B$, $i_B = (0, 1_B) : B \to A \times B$

$$\begin{tikzcd}
& A \arrow[dl, "1_A"'] \arrow[d, dashed, "i_A"] \arrow[dr, "0"] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& B \arrow[dl, "0"'] \arrow[d, dashed, "i_B"] \arrow[dr, "1_B"] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}$$

Thus $p_A i_A = 1_A$, $p_B i_A = 0$, $p_A i_B = 0$, $p_B i_B = 1_B$, also if we consider the following commutative diagram

$$\begin{tikzcd}
& A \times B \arrow[dl, bend right, "p_A"'] \arrow[d, dashed, "i_A p_A + i_B p_B"] \arrow[dr, bend left, "p_B"] & \\
A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B
\end{tikzcd}$$

By the uniqueness of the induced map, $i_A p_A + i_B p_B = 1_{A \times B}$, let's show that $(A \times B, i_A, i_B)$ is the coproduct of A, B, suppose $h : A \times B \to C$ is a morphism such that $h i_A = f$, $h i_B = g$, then $h = h(i_A p_A + i_B p_B) = h i_A p_A + h i_B p_B = f p_A + g p_B$

$$\begin{tikzcd}
& C & \\
A \arrow[ur, "f"] \arrow[r, shift left, "i_A"] & A \times B \arrow[u, dashed, "\exists_1 h"] \arrow[l, shift left, "p_A"] \arrow[r, shift left, "p_B"] & B \arrow[ul, "g"'] \arrow[l, shift left, "i_B"]
\end{tikzcd}$$

□

Definition 1.45. An **additive category** is an Ab category with all finite biproducts, including empty biproduct 0, the zero object

Definition 1.46. An **abelian category** $\mathscr{A}$ is an additive category satisfying
(AB1) Every map has a kernel and a cokernel
(AB2) Every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel

Example 1.47. The category of free $\mathbb{Z}$ modules(free abelian groups) is not an abelian category, $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ has 0 as its cokernel but this is not the kernel of $\mathbb{Z} \to 0$

Example 1.48 (A category with two different Ab structures)**.** Consider rings $\mathbb{Q}[x]$, $\mathbb{Q}[x, y]$ as abelian categories with a single object and morphisms being the elements, multiplication as composition, addition gives an abelian group structure
$\mathbb{Q}[x]$, $\mathbb{Q}[x, y]$ as categories are isomorphic to its underlying monoids, since $\mathbb{Q}[x]$, $\mathbb{Q}[x, y]$ are UFD's, and $\mathbb{Q}[x] = \{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_f \mathbb{N}$, $\mathbb{Q}[x, y] = \{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_g \mathbb{N}$, where f, g run over all irreducible polynomials of $\mathbb{Q}[x] \setminus \mathbb{Q}$ and $\mathbb{Q}[x, y] \setminus \mathbb{Q}$ which are both countably many, thus as monoids they are both isomorphic to $\{0\} \bigcup \mathbb{Q}^\times \times \bigoplus_{i \in \mathbb{N}} \mathbb{N}$, Where $0 \circ 0 = 0$, $0 \circ (q, (i_0, i_1, \cdots)) = (q, (i_0, i_1, \cdots)) \circ 0 = 0$, $(q, (i_0, i_1, \cdots)) \circ (q', (i'_0, i'_1, \cdots)) = (qq', (i_0 + i'_0, i_1 + i'_1, \cdots))$ with $(1, (0, 0, \cdots))$ as the identity
but $\mathbb{Q}[x]$, $\mathbb{Q}[x, y]$ are not isomorphic as rings

Remark 1.49. Being an abelian category is purely a property of a category
If all finite products and coproducts are biproducts, i.e. $X \sqcup Y = X \times Y$, with some other exactness properties, then the abelian group structure on $Hom(X, Y)$ comes from this
See Freyd - Abelian category

Definition 1.50. Define the **diagonal functor** $\Delta : \mathscr{C} \to \mathscr{C}^I$ mapping A to the constant functor K_A

1.7 Adjunction - 2/10/2020

Definition 1.51. Let $L : \mathscr{D} \to \mathscr{C}$, $R : \mathscr{C} \to \mathscr{D}$ be functors, and there is a natural isomorphism $\Phi_{X,Y}$, $X \in \mathscr{C}, Y \in \mathscr{D}$

$$\begin{tikzcd} Hom_{\mathscr{C}}(LX, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "(Lf,g)"'] & Hom_{\mathscr{D}}(X, RY) \arrow[d, "(g,Rf)"] \\ Hom_{\mathscr{C}}(LX', Y') \arrow[r, "\Phi_{X',Y'}"] & Hom_{\mathscr{D}}(X', RY') \end{tikzcd}$$

Here $f : X' \to X$, $g : Y \to Y'$, $Hom_{\mathscr{C}}(Lf, g)(h) = h \circ g \circ Lf$
We say L is the **left adjoint** of R and R is the **right adjoint** of L

Example 1.52. Let $G : Group \to Set$ be the forgetful functor, then the functor $F : Set \to Group$, sending S to $F(S)$ is the left adjoint of G
In the category of R-modules Mod, consider functor $F := - \otimes B$ and functor $G := Hom(B, -)$, then F, G are adjoint pairs, i.e. $Hom(A \otimes B, C) \cong Hom(A, Hom(B, C))$

Theorem 1.53. Suppose $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$ are a pair of adjoint functors, then there exist natural transformations $\eta : 1_{\mathscr{A}} \to RL$ and $\varepsilon : LR \to 1_{\mathscr{B}}$ such that the right adjoint of $LX \xrightarrow{f} Y$ is $X \xrightarrow{R(f)\eta_X} RY$ and left adjoint of $g : X \to RY$ is $LX \xrightarrow{\varepsilon_Y L(g)} Y$. Moreover, the following composites are identity, $LX \xrightarrow{L(\eta_X)} LRLX \xrightarrow{\varepsilon_{LX}} LX$, $RY \xrightarrow{\eta_{RY}} RLRY \xrightarrow{R(\varepsilon_Y)} RY$

Proof. □

Theorem 1.54. Suppose F, G is an adjunction pair, then F preserve colimits, G preserve limits

Proof. Suppose $\Phi : I \to \mathscr{D}$ is a functor, $L = \varprojlim_{i \in I} \Phi(i)$ exists, applying G to commutative diagram $L \xrightarrow{\varphi_i} \Phi(i)$, we get another commutative diagram $GL \xrightarrow{G\varphi_i} G\Phi(i)$. For any commutative diagram $X \xrightarrow{\psi_i} G\Phi(i)$, by adjunction, we have a commutative diagram $FX \to \Phi(i)$, which induce a map $FX \to L$, by adjunction again, we have $X \to GL$ □

Definition 1.55. A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **left exact** if F preserve all finite limits, and **right exact** if F preserve all finite colimits

Example 1.56. A left exact functor preserves all equalizers, and all kernels if the category is abelian, a right exact functor preserves all coequalizers, and all cokernels if the category is abelian, left adjoints are right exact, right adjoints are left exact, for example, $- \otimes B$ is right exact and $Hom(B, -)$ is left exact

2 Chain complexes

2.1 Chain complexes - 2/12/2020

Definition 2.1. Let $\mathscr{A}$ be an abelian category, a ($\mathbb{Z}$-graded) **chain complex** $C_\bullet$ is

$$\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0$, ∂_i are called **boundary maps(differentials)**
We can define chain maps, chain homotopy, boundaries, cycles, and homology groups, and we say the chain complex is exact if each homology groups is zero, the chain complexes form the **category of chain complexes** $Ch_\bullet\mathscr{A}$
Similarly, we can also define cochain complex $C^\bullet$

$$\cdots \to C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \to \cdots$$

Such that $d^{n+1} \circ d^n = 0$, d^i are called **coboundary maps**, cochain complexes form the **category of cochain complexes** $Ch^\bullet\mathscr{A}$

Lemma 2.2. $\phi : Ch^\bullet\mathscr{A} \to Ch_\bullet\mathscr{A}$, $(\phi C_\bullet)^n = C_{-n}$, $\phi(d^n) = \partial_n$

Proof. $\square$

Definition 2.3. Suppose $X_\bullet$ is a chain complex, we can define **cycles** $Z_n(X) := \ker(X_n \xrightarrow{\partial_n} X_{n-1})$, **boundaries** $B_n(X) := \mathrm{im}(X_{n+1} \xrightarrow{\partial_n} X_n)$ and **homology** $H_n(X) := \mathrm{coker}(B_n \to Z_n)$, actually, Z_n, B_n, H_n are functors $Ch\mathscr{A} \to \mathscr{A}$

Definition 2.4. $\phi : X_\bullet \to Y_\bullet$ is called a **quasi-isomorphism** if $H_n(\phi) : H_nX \to H_nY$ are isomorphisms

Example 2.5. Consider

$$\begin{tikzcd}
X_\bullet : & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 5"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{mod}\,5"] & 0 \\
Y_\bullet : & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/5\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Definition 2.6. Pick $p \in \mathbb{Z}$, define the **translation** of X by p is $X_\bullet[p]$ where $(X_\bullet[p])_n = X_{p+n}$, differential $X_\bullet[p]_n \to X_\bullet[p]_{n-1}$ is given by $(-1)^p\partial$ The **translation functor** $T : Ch(\mathscr{A}) \to Ch(\mathscr{A})$, $X \mapsto X_\bullet[1]$ is an auto morphism of $Ch(\mathscr{A})$

Example 2.7. Suppose X is a topological space, R is a ring, $C_*^{\mathrm{sing}}(X)$ is the singular chain complex, ΣX is the suspension of X, we have the Freudenthal theorem $H^k(\Sigma X) \cong H^{k-1}(X)$ for $k > 0$

Definition 2.8. Pick $p \in \mathbb{Z}$, define the **truncation** of X at p is $\tau_{\geq p}X$, where $(\tau_{\geq p}X)_k = \begin{cases} 0, & k < p \\ Z_pX, & k = p \\ X_k, & k > p \end{cases}$, and define the cokernel of $\tau_{\geq p}X \to X$ to be $\tau_{<p}X$. We get the **truncation functors** $\tau_{\geq p}X \to X$ and $X \to \tau_{<p}X$ Moreover, $H_* : \tau_{\geq p}X \to X$ induce isomorphisms for $k \geq p$ and zero maps for $k < p$, $H_* : X \to \tau_{<p}X$ induce isomorphisms for $k < p$ and zero maps for $k \geq p$

Example 2.9. Consider $p = 0$

$$\begin{tikzcd}
X_2 \arrow[r] \arrow[d, equal] & X_1 \arrow[r] \arrow[d, equal] & Z_0X \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
X_2 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d, two heads] & X_{-1} \arrow[r] \arrow[d, equal] & X_{-2} \arrow[d, equal] \\
0 \arrow[r] & 0 \arrow[r] & X_0/Z_0 \arrow[r] & X_{-1} \arrow[r] & X_{-2}
\end{tikzcd}$$

2.2 Arrow category - 2/14/2020

Definition 2.10. A chain complex of **amplitude** $[p, q]$ are chains of the following form

$$0 \longrightarrow X_q \xrightarrow{d} \cdots \xrightarrow{d} X_p \longrightarrow 0$$

Let $Ch_{[p,q]}\mathscr{C}$ denote the full subcategory of $Ch\mathscr{C}$ consist of chain complexes of amplitude $[p, q]$

Definition 2.11. Suppose $\mathscr{C}$ is a category, we can define the **arrow category** $Ar\mathscr{C}$, where the objects are morphisms in $\mathscr{C}$, and $Hom(X \xrightarrow{f} Y, Z \xrightarrow{g} W)$ consists of commutative diagrams

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\
Z \arrow[r, "g"] & W
\end{tikzcd}$$

Equivalently, $Ar\mathscr{C} = Ch_{[0,1]}\mathscr{C}$

Lemma 2.12. Suppose $\mathscr{A}$ is an abelian category, then $\ker, \operatorname{coker} : Ar\mathscr{A} \to \mathscr{A}$ are two functors given by the following diagram

$$\begin{tikzcd}
\ker f \arrow[r, hook] \arrow[d, "u_*"] & X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, two heads] \arrow[d, "v"] & \operatorname{coker} f \arrow[d, "v_*"] \\
\ker g \arrow[r, hook] & Z \arrow[r, "g"] & W \arrow[r, two heads] & \operatorname{coker} g
\end{tikzcd}$$

Let $F_1 : \mathscr{A} \to Ar\mathscr{A}$, $X \mapsto 0 \to X \to 0$, where X is of degree 1, $F_0 : \mathscr{A} \to Ar\mathscr{A}$, $X \mapsto 0 \to X \to 0$, where X is of degree 0. Then $\ker$ is the right adjoint to F_1 and coker is the left adjoint to F_0

Proof.

□

2.3 Chain homotopy - 2/17/2020

Snake lemma

Lemma 2.13 (Snake lemma)**.** Given the following commutative diagram with exact rows, then we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r, dashed] & \ker a \arrow[r, "u_*"] \arrow[d] & \ker b \arrow[r, "v_*"] \arrow[d] & \ker c \arrow[d] \arrow[dddll, dashed, "\delta"] & \\
0 \arrow[r, dashed] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & 0 \\
 & \operatorname{coker} a \arrow[r, "u'_*"] & \operatorname{coker} b \arrow[r, "v'_*"] & \operatorname{coker} c \arrow[r, dashed] & 0
\end{tikzcd}$$

Proof. □

Lemma 2.14. $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is exact iff $0 \to A_n \to B_n \to C_n \to 0$ are exact

Proof. □

Theorem 2.15. Suppose $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is exact, then we have $\partial : H_nC \to H_{n-1}A$ yielding a long exact sequence

$$\cdots \to H_nA \to H_nB \to H_nC \xrightarrow{\partial} H_{n-1}A \to H_{n-1}B \to H_{n-1}C \to \cdots$$

Proof. Fisrtly, by Lemma 2.13, we have

$$\begin{tikzcd}
0 \arrow[r] & Z_nA \arrow[r] \arrow[d] & Z_nB \arrow[r] \arrow[d] & Z_nC \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & A_n/\partial A_{n-1} \arrow[r] & B_n/\partial B_{n-1} \arrow[r] & C_n/\partial C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Then apply Lemma 2.13 again, we get

$$\begin{tikzcd}
 & H_nA \arrow[r] \arrow[d] & H_nB \arrow[r] \arrow[d] & H_nC \arrow[d] \arrow[dddll, dotted] & \\
 & A_n/\partial A_{n+1} \arrow[r] \arrow[d] & B_n/\partial B_{n+1} \arrow[r] \arrow[d] & C_n/\partial C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[d] & \\
 & H_{n-1}A \arrow[r] & H_{n-1}B \arrow[r] & H_{n-1}C &
\end{tikzcd}$$

□

Five lemma

Lemma 2.16 (Five lemma)**.** If b and d are monic and a is an epi, then c is monic. Dually, if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are iso, then c is also an iso

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d, "a"', "\cong"] & B' \arrow[r, "v'"] \arrow[d, "b"', "\cong"] & C' \arrow[r, "w'"] \arrow[d, "c"'] & D' \arrow[r, "x'"] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & D \arrow[r, "x"] & E
\end{tikzcd}$$

Definition 2.17. We can define a full subcategory $S(\mathscr{A})$ of short exact sequences, or equivalently just $Ch_{[0,2]}\mathscr{A}$, and define a full subcategory $L(\mathscr{A})$ of long exact sequences

Lemma 2.18. *H gives a functor $S(Ch\mathscr{A}) \to L(\mathscr{A})$, sending $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ to its long exact sequence*

Proof.

$\square$

Definition 2.19. Suppose $X_\bullet, Y_\bullet \in Ch\mathscr{A}$, define a complex $Hom_\bullet(X_\bullet, Y_\bullet) \in Ch\mathscr{A}b$ as follows: for each $k \in \mathbb{Z}$, $Hom_k(X_\bullet, Y_\bullet) = \prod_{n\in\mathbb{Z}} Hom(X_n, Y_{k+n})$, and $d(f_n : X_n \to Y_{n+k})_{n\in\mathbb{Z}} = (g_n : X_n \to Y_{n+k-1})_{n\in\mathbb{Z}}$ where $g_n = \partial f_n - (-1)^k f_{n-1}\partial$

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\partial"] \arrow[d, "f"] \arrow[dr, "g"] & X_n \arrow[r, "\partial"] \arrow[d, "f_n"] \arrow[dr, "g_n"] & X_{n-1} \arrow[r, "\partial"] \arrow[d, "f_{n-1}"] \arrow[dr, "g"] & X_{n-2} \arrow[d, "f"] \\
Y_{n+k+1} \arrow[r, "\partial"] & Y_{n+k} \arrow[r, "\partial"] & Y_{n+k-1} \arrow[r, "\partial"] & Y_{n+k-2}
\end{tikzcd}$$

$Hom_\bullet(X_\bullet, Y_\bullet)$ is a chain complex since for any $f \in Hom_k(X_\bullet, Y_\bullet)$

$$\begin{aligned}
d^2 f &= d(\partial f - (-1)^k f\partial) \\
&= \partial(\partial f - (-1)^k f\partial) + (-1)^{k-1}(\partial f - (-1)^k f\partial)\partial \\
&= \partial^2 f - (-1)^k \partial f\partial + (-1)^k \partial f\partial + (-1)^k f\partial^2 \\
&= 0
\end{aligned}$$

If $f \in Hom_0(X_\bullet, Y_\bullet)$, then $df = \partial f - f\partial = 0 \Leftrightarrow f \in Hom(X_\bullet, Y_\bullet)$, i.e. $Hom(X_\bullet, Y_\bullet) = Z_0(Hom_\bullet(X_\bullet, Y_\bullet))$, $f \in B_0 Hom_\bullet(X_\bullet, Y_\bullet) \Leftrightarrow f - 0 = f = ds = \partial s + s\partial$. i.e. f is chain homotopy equivalent to 0. Therefore we define the **chain homotopy** classes of morphisms from $X_\bullet \to Y_\bullet$ to be $H_0(Hom(X_\bullet, Y_\bullet))$

2.4 Chain homotopy category - 2/19/2020

Definition 2.20. Suppose $\mathscr{A}$ is an abelian category, define $K(\mathscr{A})$ to be the **homotopy category** with $obK(\mathscr{A}) = obCh(\mathscr{A})$, $Hom_{K(\mathscr{A})}(X_\bullet, Y_\bullet) = H_0(Hom_\bullet(X_\bullet, Y_\bullet))$

Definition 2.21. Suppose $f : X_* \to Y_*$ is chain map, then the **mapping cone** of f is defined to be the object $C(f)$ in $Ch(\mathscr{A})$ with $C(f)_n = X_{n-1} \oplus Y_n$, $d_{C(f)} = \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix}$, note that $d_{C(f)}^2 = \begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix}\begin{pmatrix} -d_X & 0 \\ -f & d_Y \end{pmatrix} = \begin{pmatrix} d_X^2 & 0 \\ fd_X - d_Yf & d_Y^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Lemma 2.22. We have a short exact sequence

$$(x, y) \longmapsto x$$

$$0 \longrightarrow Y \longrightarrow C(f) \longrightarrow X[-1] \longrightarrow 0$$

$$y \longmapsto (0, y)$$

Remark 2.23. If $f : X_* \to 0$ is the zero morphism, then $C(f) = X[-1]$

Proof. $\square$

Corollary 2.24. $f : X_* \to Y_*$ is a quasi-isomorphism $\Leftrightarrow$ $C(f)$ is exact

2.5 Freyd-Mitchell embedding - 2/21/2020

Theorem 2.25. We have a homology functor $H_* : K(\mathscr{A}) \to \mathscr{A}$ such that the following diagram commutes

$$\begin{tikzcd} Ch(\mathscr{A}) \arrow[r, "H_*"] \arrow[d] & \mathscr{A} \\ K(\mathscr{A}) \arrow[ur, "H_*"] & \end{tikzcd}$$

Theorem 2.26 (Freyd-Mitchell embedding theorem)**.** Suppose $\mathscr{A}$ is a small abelian category, then there exists a ring R and a fully faithful embedding $\mathscr{A} \to R\text{-}mod$, i.e. $\mathscr{A}$ embeds in $R\text{-}mod$ as a full subcategory. Moreover, the embedding is an exact functor

Lemma 2.27. Suppose $\mathscr{A}$ is an abelian category, $\mathscr{C}$ is a subcategory, then
(1) $\mathscr{C}$ is additive $\Leftrightarrow$ if $\mathscr{C}$ is closed under direct sum, including 0
(2) $\mathscr{C}$ is abelian and $\mathscr{C} \hookrightarrow \mathscr{A}$ is exact $\Leftrightarrow$ $\mathscr{C}$ is additive and contain kernels, cokernels

Proof.

□

Definition 2.28. Suppose $\mathscr{A}, \mathscr{B}$ are abelian categories, a covariant homological $\boldsymbol{\delta}$ **functor** is a family of functors $T_n : \mathscr{A} \to \mathscr{B}$ and for each short exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$, a family of morphisms $\delta_n : T_n C \to T_{n-1} A$ which induces a long exact sequence

$$\cdots \to T_n(A) \xrightarrow{u_n} T_n(B) \xrightarrow{v_n} T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \to \cdots$$

And any chain map

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

The following diagram commutes

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta_n"] \arrow[d] & T_{n-1}(A) \arrow[d] \\ T_n(C') \arrow[r, "\delta_n"] & T_{n-1}(A') \end{tikzcd}$$

Similarly, we can define covariant cohomological δ functors

Example 2.29. $H_n : Ch_\bullet(\mathscr{A}) \to \mathscr{A}$, $C_* \mapsto H_n(C)$, $\delta = \partial$ is a homological δ functor, $H^n : Ch^\bullet(\mathscr{A}) \to \mathscr{A}$, $C^* \mapsto H^n(C)$, $\delta = d$ is a cohomological δ functor

Example 2.30. Let $\mathscr{A}b$ be the category of abelian groups, define functors $T_1 : \mathscr{A}b \to \mathscr{A}b$, $A \mapsto A_p$, where A_p is $\ker(A \xrightarrow{\times p} A)$ is the p torsion of A, and $T_0 : \mathscr{A}b \to \mathscr{A}b$, $A \mapsto A/pA$, where $A/pA = \operatorname{coker}(A \xrightarrow{\times p} A)$. For a short exact sequence $0 \to A \to B \to C \to 0$, by Snake Lemma 2.13, we have long exact sequence

$$0 \to T_1 A \to T_1 B \to T_1 C \xrightarrow{\delta_1} T_0 A \to T_0 B \to T_0 C \to 0$$

Definition 2.31. A morphism between delta functors $\{S_i\}$, $\{T_i\}$ is a sequence of natural transformations $\eta_n : S_n \to T_n$ commuting with δ, i.e.

$$\begin{tikzcd} \cdots \arrow[r] & S_n A \arrow[r, "S_n u"] \arrow[d, "\eta"] & S_n B \arrow[r, "S_n v"] \arrow[d, "\eta"] & S_n C \arrow[r, "\delta_S"] \arrow[d, "\eta"] & S_{n-1} A \arrow[r] \arrow[d, "\eta"] & \cdots \\ \cdots \arrow[r] & T_n A \arrow[r, "T_n u"] & T_n B \arrow[r, "T_n v"] & T_n C \arrow[r, "\delta_T"] & T_{n-1} A \arrow[r] & \cdots \end{tikzcd}$$

Therefore we get a category of homological δ functors

Definition 2.32. A δ functor $\{T_n\}$ is called **universal** if for any δ functor $\{S_n\}$, given a natural transformation $\eta_0 : S_0 \to T_0$, this can be uniquely extended to $\eta_n : S_n \to T_n$ up to isomorphism, in other words, $\{T_n\}$ is a final object in the category of homological δ functors

2.6 Resolutions - 2/24/2020

Definition 2.33. Suppose $\mathscr{C}$ is an abelian category, P is **projective** if functor $Hom(P,-):\mathscr{C}\to Sets$ sends epi to epi, or equivalently

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] \\ X \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

I is **injective** if functor $Hom(-,Q):\mathscr{C}\to Sets$ sends mono to epi, or equivalently

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, dashed, "\exists h"] \\ Q & \end{tikzcd}$$

Coproduct of projetives is projective, product of injectives is injective

Lemma 2.34. Coproduct of projective objects is projective, product of injective objects is injective

Proof. Suppose I_α are injective, $A\hookrightarrow B$ is a monomorphism, we have

$$\begin{tikzcd} Hom(B,\prod I_\alpha) \arrow[r] \arrow[d, leftrightarrow] & Hom(A,\prod I_\alpha) \arrow[d, leftrightarrow] \\ \prod Hom(B,I_\alpha) \arrow[r] & \prod Hom(A,I_\alpha) \end{tikzcd}$$

□

Definition 2.35. $\mathscr{C}$ has **enough projectives** if for any X, there is an epi $P\to X$ from a projective object, $\mathscr{C}$ has **enough injectives** if for any X, there is a mono $X\to Q$ to an injective object

Lemma 2.36. Suppose $\mathscr{A}$ is an abelian category, $Hom(P,-)$, $Hom(-,I)$ are left exact. We have

$$P \text{ is projective} \Leftrightarrow Hom(P,-) \text{ is right exact} \Leftrightarrow Hom(P,-) \text{ is exact}$$

$$I \text{ is injective} \Leftrightarrow Hom(-,I) \text{ is right exact} \Leftrightarrow Hom(-,I) \text{ is exact}$$

Proof.

□

Remark 2.37. It is obvious that $A\to B\to C\to 0$ is exact iff $0\to Hom(C,D)\to Hom(B,D)\to Hom(A,D)$ is exact for all D, $0\to A\to B\to C$ is exact iff $0\to Hom(D,A)\to Hom(D,B)\to Hom(D,C)$ is exact for all D

Left adjoint to exact functor preserves projectives

Lemma 2.38. Functors between abelian categories $F:\mathcal{A}\to\mathcal{B}$, $G:\mathcal{B}\to\mathcal{A}$ is an adjoint pair. If F is left exact, then G preserves injectives. If G is right exact, then F preserves projectives

Proof.

$$Hom_{\mathcal{B}}(-,G(I)) \text{ is right exact} \Leftrightarrow Hom_{\mathcal{A}}(F(-),I) = Hom_{\mathcal{A}}(-,I)\circ F \text{ is right exact}$$

$$Hom_{\mathcal{B}}(F(P),-) \text{ is right exact} \Leftrightarrow Hom_{\mathcal{A}}(P,G(-)) = Hom_{\mathcal{A}}(P,-)\circ G \text{ is right exact}$$

□

Lemma 2.39. An R module M is projective iff M is a direct summand of a free module

Proof.

□

Definition 2.40. Exact sequence $C_\bullet$ **split** at if there are $s_n: C_n\to C_{n+1}$ such that $\partial_{n+1}s_n\partial_{n+1} = \partial_{n+1}$

Equivalent conditions for split and split exact

Lemma 2.41. Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. C splits if and only if there are R-module decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n \cong B_n \oplus H'_n$. C is split exact iff $H'_n = 0$

Proof. If $C_n \cong Z_n \oplus B'_n$ and $Z_n \cong B_n \oplus H'_n$, claim that any element in B_n has a unique preimage in B'_{n+1}: if $x, y \in B'_{n+1}$ are such that $\partial_{n+1}x = \partial_{n+1}y$, then $\partial_{n+1}(x-y) = 0 \Rightarrow (x-y) \in Z_{n+1} \cap B'_{n+1} = 0 \Rightarrow x = y$
Hence we can define a unique bijective homomorphism $s_n : B_n \to B'_{n+1}$ sending elements to its preimage, then extend s_n to $s_n : C_n \to C_{n+1}$ such that $s_n(C_n) = B'_{n+1}$, $s_n(H'_n \oplus B'_n) = 0$, then $\partial_{n+1}s_n\partial_{n+1} = \partial_{n+1}$, i.e. C split
If C split, denote $B'_n = s_{n-1}\partial_n(C_n)$, $H'_n = \ker \partial_{n+1}s_n \cap Z_n$, we claim $C_n = Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$: For any $s_{n-1}\partial_n(a_n) \in Z_n$, $0 = \partial_n(s_{n-1}\partial_n(a_n)) = \partial_n a_n \Rightarrow s_{n-1}\partial_n(a_n) = 0$. For $a_n \in C_n$, $\partial_n(a_n - s_{n-1}\partial_n(a_n)) = 0$. For any $\partial_{n+1}a_{n+1} \in \ker \partial_{n+1}s_n$, $\partial_{n+1}(a_{n+1}) = \partial_{n+1}s_n\partial_{n+1}(a_{n+1}) = 0$. For any $a_n \in Z_n$, $a_n - \partial_{n+1}s_n(a_n) \in \ker \partial_{n+1}s_n \cap Z_n$
It is obvious that C is exact $\Leftrightarrow Z_n \cong B_n \Leftrightarrow H'_n = 0$ ☐

Splic iff nullhomotopic

Lemma 2.42. $C_\bullet$ split iff $1_C \simeq 0$

Proof. Suppose the identity map on C is null homotopic, then there exists $s_n : C_n \to C_{n+1}$ such that $1_{C_n} = s_{n-1}\partial_n + \partial_{n+1}s_n$, then $\partial_n = \partial_n s_{n-1}\partial_n$, i.e. C split, for any $a_n \in Z_n$, $a_n = (s_{n-1}\partial_n + \partial_{n+1}s_n)a_n = \partial_{n+1}(s_n a_n) \in B_n$, i.e. C is exact
Suppose C split exact, according to exercise 1.4.2, $C_n \cong Z_n \oplus B'_n \cong B_n \oplus B'_n$ with $H'_n = 0$, then we can define $s_n : C_n \to C_{n+1}$ such that $s_n(B_n) = B'_{n+1}$ bijective, $s_n(H'_n \oplus B'_n) = 0$, $\partial_{n+1}s_n\partial_{n+1} = \partial_{n+1}$, thus $B'_n = s_{n-1}(B_n) = s_{n-1}\partial_n(C_n)$, $s_n s_{n-1}\partial_n(C_n) = s_n(B'_n) = 0$. Therefore for any element in C_n which can be written as $\partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n$, we have $(s_{n-1}\partial_n + \partial_{n+1}s_n)(\partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n) = \partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n$, i.e. $1_{C_n} = s_{n-1}\partial_n + \partial_{n+1}s_n$, the identity map on C is nullhomotopic ☐

Lemma 2.43. $P_\bullet$ is a projective in $Ch(\mathscr{A})$ iff $P_\bullet$ is a split exact sequence of projectives. [Hint: To see that P must be split exact, consider the surjection from $cone(\mathrm{id}_P)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \cong P_0 \to 0$]

Proof. Consider chain complex C with $C_n = P_n \oplus P_{n+1}$

$$\cdots \to P_n \oplus P_{n+1} \xrightarrow{\begin{pmatrix} \partial & 0 \\ 1 & -\partial \end{pmatrix}} P_{n-1} \oplus P_n \to \cdots$$

and first coordinate projection $C \twoheadrightarrow P$ which is a surjection, if P is projective, then there exists s such that

$$\begin{tikzcd} & P \arrow[dl, "s"'] \arrow[d, equal] \\ C \arrow[r, two heads] & P \end{tikzcd}$$

In order to make s a chain map and the diagram commute, we must have $s : P_n \to C_n$, $x \to \begin{pmatrix} x \\ s_n x \end{pmatrix}$ and

$$\begin{pmatrix} \partial_n x \\ s_{n-1}\partial_n x \end{pmatrix} = \begin{pmatrix} \partial_n & 0 \\ 1 & -\partial_{n+1} \end{pmatrix} \begin{pmatrix} x \\ s_n x \end{pmatrix} = \begin{pmatrix} \partial_n x \\ x - \partial_{n+1}s_n x \end{pmatrix}$$

Hence $s_{n-1}\partial_n + \partial_{n+1}s_n = 1$, by Lemma 2.42, P split exact
To prove P_n are projectives, given

$$\begin{tikzcd} & P_n \arrow[d, "g_n"] \\ B \arrow[r, two heads, "f"] & A \end{tikzcd}$$

consider the following commutative diagram with $g_{n+1} = g_n \partial_{n+1}$

$$\begin{tikzcd}
P_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[d, "g_n"'] \arrow[dd, dashed, bend left, "h"] & P_{n-1} \arrow[d] \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & B \arrow[r, equal] \arrow[u, two heads, "f"'] & B \arrow[r] \arrow[u, two heads, "f"] & 0 \arrow[u]
\end{tikzcd}$$

Since $P_\bullet$ is projective, there exists $h : P_n \to B$ such that $fh = g_n$
Conversely, suppose P is a split exact sequence of projectives, by Lemma 2.41, there exist bijection $s_n : Z_n = B_n \to B'_{n+1}$ and $P_n \cong B_n \oplus B'_n$, thus P is the direct sum of $0 \to B'_{n+1} \to B_n \to 0$, and we know the coproducts of projectives are projective, it suffices to consider $0 \to P_1 \xrightarrow{\cong} P_0 \to 0$. Since ψ_1 is epi and P_1 is projective, there exists $P_1 \xrightarrow{\phi_1} B_1$ such that $\xi_1 = \psi_1\phi_1$, let $\phi_0 = b_1\phi_1 p_1^{-1}$, then $\xi_0 = a_1\xi_1 p_1^{-1} = a_1\psi_1\phi_1 p_1^{-1} = \psi_0 b_1\phi_1 p_1^{-1} = \psi_0\phi_0$ and $b_0\phi_0 = b_0 b_1\phi_1 p_1^{-1} = 0$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_1 \arrow[r, "p_1", "\cong"'] \arrow[d, dashed, "\phi_1"] \arrow[dd, bend right, "\xi_1"'] & P_0 \arrow[r] \arrow[d, dashed, "\phi_0"'] \arrow[dd, bend left, "\xi_0"] & 0 \arrow[d] \\
B_2 \arrow[r] \arrow[d, two heads, "\psi_2"'] & B_1 \arrow[r, "b_1"] \arrow[d, two heads, "\psi_1"] & B_0 \arrow[r, "b_0"] \arrow[d, two heads, "\psi_0"'] & B_{-1} \arrow[d, two heads, "\psi_{-1}"] \\
A_2 \arrow[r] & A_1 \arrow[r, "a_1"] & A_0 \arrow[r] & A_{-1}
\end{tikzcd}$$

□

Definition 2.44. A **left resolution** is morphism $P_\bullet \xrightarrow{\varepsilon} M$ in $Ch_{\geq 0}(\mathscr{A})$, here M means $0 \to M \to 0$ with M at degree 0, then $\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$ is exact. A projective resolution is a left resolution with projectives, an injective resolution is a right resolution with injectives

Lemma 2.45. If $\mathscr{A}$ has enough projectives, then for any M, there is a projective resolution $P_\bullet \xrightarrow{\varepsilon} M$

Proof. First there exists exact sequence $0 \to \ker\varepsilon \xrightarrow{i_0} P_0 \xrightarrow{\varepsilon} M \to 0$ where P_0 is a projective, then there exists another exact sequence $0 \to \ker i_0 \to P_1 \to \ker\varepsilon \to 0$ where P_1 is a projective, then we can splice them to get exact sequence $P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$, inductively we get a projective resolution □

Comparison theorem

Theorem 2.46 (Comparison theorem)**.** Suppose $P_\bullet \xrightarrow{\varepsilon} M$ is a complex with P_n projectives, $Q_\bullet \xrightarrow{\eta} N$ is a left resolution, then for any $M \xrightarrow{f} N$, it can be extend to chain map $f_\bullet : P_\bullet \to Q_\bullet$, and $f_\bullet$ is unique up to homotopy

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

Proof. Since P_0 is projective, there exists $P_0 \xrightarrow{f_0} Q_0$ such that $f\varepsilon = \eta f_0$, then we have $\eta f_0 \partial_1 = f\varepsilon\partial_1 = 0$, $f_0\partial_1 : P_1 \to Z_0 Q$, and $Q_1 \xrightarrow{\partial_1} Z_0 Q$ is epi, we have $P_1 \xrightarrow{f_1} Q_1$, inductively, we can extend f to a chain map $f_\bullet : P_\bullet \to Q_\bullet$
Suppose $f = 0$, we need to show $f_\bullet \simeq 0$, Write $P_{-1} := M$, $Q_{-1} := N$, $P_n = Q_n = 0, \forall n < -1$, define $s_n : P_n \to Q_{n+1}, \forall n < 0$ to be zero. Since $\eta f_0 = 0$, thus $f_0 : P_0 \to Z_0 Q$, and $Q_1 \xrightarrow{\partial_1} Z_0 Q$ is epi, we get $P_0 \xrightarrow{s_0} Q_1$ such that $f_0 = \partial_1 s_0 = \partial_1 s_0 + s_{-1}\partial_0$, then since $\partial_1 f_1 = f_0\partial_1 = \partial_1 s_0 \partial_1 \Rightarrow f_1 - s_0\partial_1 : P_1 \to Z_1 Q$, and $Q_2 \xrightarrow{\partial_2} Z_1 Q$ is epi, we get $s_1 : P_1 \to Q_2$ such that $\partial_2 s_1 = f_1 - s_0\partial_1$, inductively we construct a null homotopy $s_\bullet$

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] \arrow[dl, "s_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] \arrow[dl, "s_0"] & M \arrow[r] \arrow[d, "0"] \arrow[dl, "0"] & 0 \arrow[dl, "0"] \\
\cdots \arrow[r] & Q_2 \arrow[r, "\partial_2"] & Q_1 \arrow[r, "\partial_1"] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}
$$

□

Horseshoe lemma

Lemma 2.47 (Horseshoe lemma). Suppose $P_\bullet \xrightarrow{\varepsilon} M$, $Q_\bullet \xrightarrow{\eta} N$ are projective resolutions, then any exact sequence $0 \to M \xrightarrow{f} A \xrightarrow{g} N \to 0$ can be extended into commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

With $(P \oplus Q)_\bullet$ being a projective resolution, every row and column are exact

Proof. Since $A \xrightarrow{g} N$ is epi and Q_0 is projective, we get $Q_0 \xrightarrow{s_0} A$ such that $gs_0 = \partial_0$ which gives us $P_0 \oplus Q_0 \xrightarrow{\begin{pmatrix} f\partial_0 & s_0 \end{pmatrix}} A$, by Lemma 2.13, this is epi, and we get an exact sequence $0 \to Z_0 P \to \ker i_0 \to Z_0 Q \to 0$, similarly, we can construct $Q_1 \xrightarrow{s_1} \ker i_0$, then $P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} \iota_0\partial_0 & s_1 \end{pmatrix}} \ker i_0$ is again epi by Lemma 2.13, inductively, we can construct the commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\partial_1"'] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d, "\partial_1"] \arrow[dl, "s_1"] & 0 \\
0 \arrow[r] & Z_0 P \arrow[r, "\iota_0"] \arrow[d] & \ker i_0 \arrow[r] \arrow[d] & Z_0 Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\partial_0"'] & P_0 \oplus Q_0 \arrow[r] \arrow[d, "i_0"] & Q_0 \arrow[r] \arrow[d, "\partial_0"] \arrow[dl, "s_0"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

□

2.7 Baer's criterion - 2/28/2020

Baer's criterion

Theorem 2.48 (Baer's criterion)**.** A left R module E is injective in the category of left R modules iff every left ideal homomorphism $\phi : I \to E$ can be extended to a homomorphism $R \to E$

Proof. If E is injective, we can certainly extend $\phi : I \to E$

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d, "\phi"'] & R \arrow[dl, dashed] \\ E & \end{tikzcd}$$

Now suppose extension is always possible, $A \hookrightarrow B$ is a submodule, $\alpha : A \to E$ is a homomorphism, consider poset $\Gamma := \{A \le C \le B, \alpha_C : C \to E\}$, $(C, \alpha_C) \le (D, \alpha_D)$ meaning $C \le D$ and $\alpha_D|_C = \alpha_C$, by Zorn's lemma, we can pick a maximal element (C, α), suppose $C \lneq B$, there exists $b \in B \setminus C$, consider $J = \{r \in R | rb \in C\}$, we have

$$\begin{tikzcd} J \arrow[r, "\times b"] \arrow[dr, hook] & C \arrow[r, "\alpha"] & E \\ & R \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

define $\beta : C + \langle b \rangle \to E$, $c + rb \mapsto \alpha(c) + f(r)$, contradicting the maximality $\square$

Definition 2.49. A left R module M is r **divisible** if $M \xrightarrow{\times r} M$ is surjetive, M is divisible if M is r divisible for any $0 \ne r \in R$

Injective <=> Divisible

Corollary 2.50. Suppose R is a PID, a left R module M is injective iff M is divisible

Proof. Suppose M is injective, for any $0 \ne r \in R$, $R \xrightarrow{\times r} rR$ is an isomorphism since R is a PID, by Theorem 2.48, we have

$$\begin{tikzcd} R \arrow[r, "\times r"] \arrow[dr, "m"'] & rR \arrow[r, hook] & R \arrow[dl, dashed, "m'"] \\ & M & \end{tikzcd}$$

Here $R \xrightarrow{m} M$, $1 \mapsto m$, thus $m = rm'$
Suppose M is divisible, since R is a PID, for any homomorphism $rR \to M$, $r \mapsto m$, we have $m = rm'$ for some m', giving the extension $R \to M$, $1 \mapsto m'$ $\square$

Corollary 2.51. The category of abelian groups $\mathscr{A}b$ has enough injectives

Proof. By corollary 2.50, $\mathbb{Q}/\mathbb{Z}$ is injective since $\mathbb{Q}/\mathbb{Z}$ is divisible
Suppose M is an abelian group, define $I = \prod_f \mathbb{Q}/\mathbb{Z}$ which is also injective due to Lemma 2.34, here f runs over $Hom(M, \mathbb{Q}/\mathbb{Z})$, thus we get $h : M \to I$. Suppose $0 \ne m \in \ker h$, then $f(m) = 0$ for any $f \in Hom(M, \mathbb{Q}/\mathbb{Z})$. Define $H = \mathbb{Z}m$ which is isomorphism to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, then we can define $\beta : H \to \mathbb{Q}/\mathbb{Z}$, $m \mapsto \dfrac{1}{n}$, since $\mathbb{Q}/\mathbb{Z}$ is a injective, we can extend to $\alpha : M \to \mathbb{Q}/\mathbb{Z}$, but then $\alpha(m) = \beta(m) \ne 0$ which is a contradiction $\square$

Theorem 2.52. Suppose $\mathscr{A}, \mathscr{B}$ are abelian categories, $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$ are adjoint functors, L is exact, $I \in \mathrm{ob}\mathscr{B}$ is injective, then $R(I)$ is also injective

Proof. For any monomorphism $A \hookrightarrow A'$, $L(A) \hookrightarrow L(A')$ is also monic, we have

$$\begin{tikzcd} Hom(L(A'), I) \arrow[r] \arrow[d, leftrightarrow] & Hom(L(A), I) \arrow[d, leftrightarrow] \\ Hom(A', R(I)) \arrow[r] & Hom(A, R(I)) \end{tikzcd}$$

$\square$

2.8 Enough injectives in R-Mod - 3/2/2020

Lemma 2.53. If M is a left R module, A is an abelian group, then $Hom(M, A)$ is a right R module. Similarly, if M is a ring R module, then $Hom(A, M)$ is a left R module

Proof. $(fr)(m) = f(rm)$, $(frs)(m) = f(rsm) = (fr)(sm) = ((fr)s)(m)$
$(rf)(m) = f(mr)$, $(rsf)(m) = f(mrs) = (sf)(rm) = (r(sf))(m)$ □

Proposition 2.54. If M is a left R module, A is an abelian group, viewing R as a right R module, then the natural map $Hom_{\mathscr{A}b}(M, A) \to Hom_{R\text{-}Mod}(M, Hom(R, A))$ is an isomorphism. In other words, $Hom(R, -)$ is the right adjoint to the forgetful functor $R\text{-}Mod \to \mathscr{A}b$, sending a right R module to its underlying abelian group, the forgetfull is clearly an exact functor, thus $Hom(R, -)$ maps injectives to injectives

Proof.

□

Corollary 2.55. $R\text{-}Mod$ has enough injectives

Proof.

□

Definition 2.56. Suppose $\mathscr{A}, \mathscr{B}$ are abelian category, $F : \mathscr{A} \to \mathscr{B}$ is an **additive funtor** if $F(A \oplus B)$ is naturally isomorphic to $F(A) \oplus F(B)$, and F is **additive**, i.e. $F(f + g) = F(f) + F(g)$, $f, g \in Hom(A, B)$, here $f + g$ is given by the composition $A \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} B \oplus B \xrightarrow{\nabla} Y$, here ∇ is the **codiagonal**

Example 2.57. $\bigwedge^2 : \mathrm{Vect}_F \to \mathrm{Vect}_F$ is exact but not additive

2.9 Universality of derived functors - 3/6/2020

Definition 2.58. The **left derived functor** of F is $L_iF(A) = H_iF(P)$, where $P \to A$ is a projective resolution

Definition 2.59. $Y \in ob\mathscr{A}$ is F-**acyclic** if $L_iF(Y) = 0$ for all $i \geq 1$. Projectives are acyclic

Theorem 2.60. $F : \mathscr{A} \to \mathscr{B}$ is right exact, $\mathscr{A}$ has enough projectives, the **left derived functor** L_iF is a universal homological δ functor

Proof. Suppose $0 \to X \to Y \to Z \to 0$ is exact, P_X, P_Z are projective resolutions of X, Z, then $(P_Y)_i = (P_X)_i \oplus (P_Z)_i$ is a projective resolution of Y by Lemma 2.47, since $F(P_Y)_i = F(P_Y)_i \oplus F(P_Z)_i$, $0 \to F(P_X)_i \to F(P_Y)_i \to F(P_Z)_i \to 0$ split, by Lemma 2.13, we have $\cdots \to L_iF(X) \to L_iF(Y) \to L_iF(Z) \xrightarrow{\delta} L_{i-1}F(X) \to \cdots$, i.e. L_iF is a homological δ functor. Suppose T_i is another homological δ functor, $\phi_0 : T_0 \to L_0F$ is a natural transformation, since $\mathscr{A}$ has enough projecives, there exists $P \twoheadrightarrow X$ with P projetive, let K be the kernel, we have a short exact sequence $0 \to K \to P \to X \to 0$

$$\begin{tikzcd}
 & T_1X \arrow[r, "\delta"] \arrow[d, dashed, "\exists_1 \phi_1"] & T_0K \arrow[r] \arrow[d, "\phi_0"] & T_0P \arrow[r] \arrow[d, "\phi_0"] & T_0X \arrow[d, "\phi_0"] & \\
0 \arrow[r] & L_1FX \arrow[r, "\delta"] & L_0FK \arrow[r] & L_0FP \arrow[r] & L_0FX \arrow[r] & 0
\end{tikzcd}$$

And then inductively for $i > 0$

$$\begin{tikzcd}
 & T_{i+1}X \arrow[r, "\delta"] \arrow[d, dashed, "\exists_1 \phi_{i+1}"] & T_iK \arrow[d, "\phi_i"] & \\
0 \arrow[r] & L_{i+1}FX \arrow[r, "\delta"] & L_iFK \arrow[r] & 0
\end{tikzcd}$$

□

Corollary 2.61. $F : \mathscr{A} \to \mathscr{B}$ is left exact, $\mathscr{A}$ has enough injectives, the **right derived functor** R^iF is a universal cohomological δ functor

Example 2.62. $F_M : R\text{-}mod \to Ab$, $N \mapsto Hom_R(M, N)$ is left exact, $R^iF_M(N) = Ext^i_R(M, N)$

2.10 Filtered category - 3/9/2020

Lemma 2.63. A left adjoint is a right exact functor, a right adjoint is a left exact functor

Proof. Suppose (L, R) are adjoint pair of functors of abelian categories, $L : \mathscr{A} \to \mathscr{B}$, $R : \mathscr{B} \to \mathscr{A}$, $A \to B \to C \to 0$ is exact, then $0 \to Hom(C, RD) \to Hom(B, RD) \to Hom(A, RD)$ is exact, $0 \to Hom(LC, D) \to Hom(LB, D) \to Hom(LA, D)$ is exact, thus $LA \to LB \to LC \to 0$ is exact ☐

Proposition 2.64. I is small, $\mathscr{C}$ is cocomplete, $K : \mathscr{C} \to \mathscr{C}^I$, $X \mapsto K_X$ is the right adjoint to $\mathrm{colim} : \mathscr{C}^I \to \mathscr{C}$

Proof.

☐

Corollary 2.65. I, J is small, $\mathscr{C}, \mathscr{C}^I, \mathscr{C}^J$ are cocomplete, $F : I \times J \to \mathscr{C}$ is a bifunctor, which give $F_I : I \to \mathscr{C}^J$, $F_J : J \to \mathscr{C}^I$, then $\mathrm{colim} F \cong \mathrm{colim}\mathrm{colim} F_I \cong \mathrm{colim}\mathrm{colim} F_J$

Proof.

☐

Definition 2.66. I is a **filtered category** if for any i, j, there exist $i \to k \leftarrow j$ and for any $i \underset{\psi}{\overset{\phi}{\rightrightarrows}} j$, there exists $j \xrightarrow{\xi} k$ coequalizes ϕ, ψ, i.e. $\xi\phi = \xi\psi$

Equivalently, for finitely many i_α, there exist $i_\alpha \xrightarrow{\phi_\alpha} j$ and for finitely many $i \xrightarrow{\phi_\alpha} j$, there exists $j \xrightarrow{\psi} k$ coequalizes ϕ_α

Lemma 2.67. $F : I \to R\text{-}mod$ is a functor, then $\mathrm{colim} F = \bigoplus_i F(i)/E$, E is generated by $F(\phi)(a_i) - a_i$, here $i \xrightarrow{\phi} j$

Proof. We have the following diagrams

$$\begin{tikzcd}
F(i) \arrow[r, "F(\phi)"] \arrow[dr] & F(j) \arrow[d] \\
 & \bigoplus_i F(i)/E
\end{tikzcd}
\qquad
\begin{tikzcd}
F(i) \arrow[rr, "F(\phi)"] \arrow[dr] \arrow[ddr] & & F(j) \arrow[dl] \arrow[ddl] \\
 & \bigoplus_i F(i)/E \arrow[d, dashed, "\exists_1"] & \\
 & M &
\end{tikzcd}$$

☐

Finitely many morphisms in a filtered category can goes to a common end

Lemma 2.68. Suppose I is a filtered category, for finitely many $i \to j$, there exist a k and $i \to k$, $j \to k$ such that $i \to k = i \to j \to k$

Proof. Use induction

(a) For a single morphism $i \xrightarrow{\phi} j$, there exist $i \xrightarrow{\lambda} k$, $j \xrightarrow{\mu} k$, then there exists $k \xrightarrow{\psi} j$ such that $\psi\lambda = \psi\mu\phi$

$$\begin{tikzcd}
i \arrow[r, "\phi"] \arrow[rr, bend right, "\lambda"'] & j \arrow[r, "\mu"] & k \arrow[r, "\psi"] & l
\end{tikzcd}$$

(b) For $\bullet \xrightarrow{\alpha_d} l$, $i \xrightarrow{\phi} j$, by (a), there exist $i \xrightarrow{\lambda} k$, $j \xrightarrow{\mu} k$ such $\lambda = \mu\phi$, then there exist $l \xrightarrow{\beta} m$, $k \xrightarrow{\gamma} m$

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha_d"] & l \arrow[r, "\beta"] & m \\
i \arrow[r, "\phi"] \arrow[rr, bend right, "\lambda"'] & j \arrow[r, "\mu"] & k \arrow[u, "\gamma"']
\end{tikzcd}$$

(c) For $\bullet \xrightarrow{\alpha_d} j$, $\bullet \xrightarrow{\beta} i$, there exist $j \xrightarrow{\lambda} k$, $i \xrightarrow{\mu} k$, then there exists $k \xrightarrow{\phi} l$ such that $\phi\lambda\alpha_d = \phi\mu\beta$

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha_d"] \arrow[dr, "\beta"'] & j \arrow[r, "\lambda"] & k \arrow[r, "\phi"] & l \\
& i \arrow[ur, "\mu"'] & &
\end{tikzcd}$$

(d) For $\bullet \xrightarrow{\alpha_d} j$, $i \xrightarrow{\beta} \bullet$, there exists $j \xrightarrow{\phi} k$ equalizes $\alpha_d\beta$

$$i \xrightarrow{\beta} \bullet \xrightarrow{\alpha_d} j \xrightarrow{\phi} k$$

(e) For $\bullet \xrightarrow{\alpha_d} i$, $\bullet \xrightarrow{\beta} \bullet$, there exists $i \xrightarrow{\phi} j$ equalizes $\alpha_d\beta$

$$\begin{tikzcd}
\bullet \arrow[r, "\alpha_d"] \arrow[loop below, "\beta"] & i \arrow[r, "\phi"] & j
\end{tikzcd}$$

□

Lemma 2.69. Suppose I is a filtered category, $F : I \to R\text{-}mod$ is a functor, then
(a) Every element of $\mathrm{colim}F$ lies in the image of some $F(i) \to \mathrm{colim}F$
(b) $\ker(F(i) \to \mathrm{colim}F) = \bigcup_{i \xrightarrow{\phi} j} \ker F(\phi)$

Proof. **(a)** Any element of $\mathrm{colim}F$ is a finite sum $\sum a_i$, since I is filtered, there exist k and $i \xrightarrow{\phi_i} k$
(b) $a_i \in \ker(F(i) \to \mathrm{colim}F)$ can be written as finite sum $\sum(F(\phi)(a_j) - a_j)$ with $j \xrightarrow{\phi} k$, by Lemma 2.68, there exist $j \xrightarrow{\psi_j} l$ such that for each $j \xrightarrow{\phi} k$ we have $\psi_j = \psi_k\phi$, then $F(\psi_k)F(\phi) = F(\psi_j)$, hence

$$\begin{aligned}
F(\psi_i)(a_i) &= \left(\sum F(\psi_j)\right)(a_i) \\
&= \left(\sum F(\psi_j)\right)\left(\sum(F(\phi)(a_j) - a_j)\right) \\
&= \sum \left(F(\psi_k)F(\phi)(a_j) - F(\psi_j)a_j\right) \\
&= 0
\end{aligned}$$

Therefore $a_i \in \ker F(\psi_i)$ □

Definition 2.70. A **sheaf** is a presheaf F such that

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Is an equaliser. $Sh(X)$ is the category of sheaves over X

Proposition 2.71. If $\mathscr{D}$ is complete, then $\mathscr{D}^{\mathscr{C}^{\mathrm{op}}}$ is complete, $(\lim F_i)(X) = \lim F_i(C)$. If $\mathscr{D}$ is cocomplete, then $\mathscr{D}^{\mathscr{C}^{\mathrm{op}}}$ is cocomplete, $(\mathrm{colim}F_i)(X) = \mathrm{colim}F_i(X)$

Proof.

□

Corollary 2.72. $PreSh(X, R\text{-}mod)$, $Sh(X, R\text{-}mod)$ are abelian categories

Theorem 2.73. Inclusion $Sh(X, R\text{-}mod) \to PreSh(X, R\text{-}mod)$ is the right adjoint to the sheafification $PreSh(X, R\text{-}mod) \to Sh(X, R\text{-}mod)$, hence inclusion is left exact, sheafification is actually exact

Proof.

□

Lemma 2.74. $0 \to F \to G \to H \to 0$ is exact iff $0 \to F_x \to G_x \to H_x \to 0$ is exact

Proof. $\square$

Example 2.75. $X = \mathbb{C} \setminus \{0\}$, $\mathbb{Z}$ is the sheaf of locally integer constant functions, $\mathcal{O}$ is the sheaf of holomorphic functions, $\mathcal{O}^\times$ is the sheaf of nonvanishing holomorphic functions, $0 \to \mathbb{Z} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$ is exact in $\mathit{Sh}(X)$, but not in $\mathit{PreSh}(X)$, since $z \in \mathcal{O}^\times$ is not in $\mathrm{im}(\mathcal{O}(X) \xrightarrow{\exp} \mathcal{O}^\times(X))$

2.11 Hom cochain complex

Definition 2.76. P is a chain complex with differentials ∂, I is a cochain complex with codifferentials d, we get a double complex $Hom(P, I) = \{Hom(P_p, I^q)\}$ with horizontal and vertical codifferentials d', d'' defined as $d''(f) = f\partial$, $d'(f) = (-1)^{p+q+1}df$. The **Hom cochain complex** is the product total complex $Tot^{\Pi}(Hom(P, I))$

2.12 Group homology

Definition 2.77. R is a commutative ring, M is right $R[G]$ module, $C_*(G)$ is the tuple complex, equivalently, $B_*(G)$ is the bar complex, $\bar{B}_*(G) = B_*(G) \otimes_{R[G]} R$, thus

$$M \otimes_{R[G]} B_*(G) \cong M \otimes_R R \otimes_{R[G]} B_*(G) \cong M \otimes_R \bar{B}_*(G)$$

Group homology with coefficients in M is

$$\begin{aligned} H_k(G; M) &= H_k(M \otimes_{R[G]} C_*(G)) \\ &= H_k(M \otimes_{R[G]} B_*(G)) \\ &= H_k(M \otimes_R \bar{B}_*(G)) \\ &= Tor_k^{R[G]}(M, R) \end{aligned}$$

The differential of $M \otimes_{\mathbb{Z}[G]} C_*(G)$ is given by

$$\begin{aligned} \partial(m \otimes [g_1|\cdots|g_n]) &= \partial(m \otimes (1, g_1, g_1g_2, \cdots, g_1\cdots g_n)) \\ &= mg_1 \otimes [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i m \otimes [g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &\quad + (-1)^n m \otimes [g_1|\cdots|g_{n-1}] \end{aligned}$$

$H_0(G; M) = M \otimes_{R[G]} R = M_G$

3 Spectral sequence

3.1 Spectral sequence

Lemma 3.1. $E \xrightarrow{f} E'$ is a morphism of spectral sequences, and $E^r_{pq} \xrightarrow{f^r_{pq}} E'^r_{pq}$ are isomorphisms for any p, q, then $E^s_{pq} \xrightarrow{f^s_{pq}} E'^s_{pq}$ are isomorphisms for any p, q, $s \geq r$

Proof. By five lemma 2.16 $\square$

Definition 3.2. A spectral sequence C is **bounded** if for all n, r, all but finitely many $E^r_{p,n-p}$ vanish

Definition 3.3. $H_* \in \mathscr{A}^{\mathbb{Z}}$, $\mathbb{Z}$ is the discrete category, $F_p H_*$ is a filtration of H_*. E **weakly converge** to H_* if $E^\infty_{pq} \cong F_p H_{p+q}/F_{p-1}H_{p+q}$. If $F_p H_n$ are Hausdorff and exhaustive, then E **approaches** or **abuts** H_*. If $F_p H_n$ are complete, then E **converges** to H_*
Bounded spectral sequence E converge to H_* if $F_p H_n$ are bounded, and $E^\infty_{pq} = F_p H_{p+q}/F_{p-1}H_{p+q}$, denote $E^r_{pq} \Rightarrow H_{p+q}$

3.2 Spectral sequence of a filtered chain complex

Definition 3.4. C is a chain complex, $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$ is a filtration of chain complexes. FC is **exhaustive** if $\bigcup F_pC = C$. FC is **Hausdorff** if $\bigcap F_pC = 0$. $\widehat{C} = \varprojlim C/F_pC$ is the **completion**. FC is **complete** if $\widehat{C} \cong C$, since $C \to \widehat{C}$ has kernel $\bigcap F_pC$, hence completeness implies Hausdorff. FC is **bounded below** if $\forall n$, $F_pC_n = 0$ for p small enough. FC is **bounded above** if $\forall n$, $F_pC_n = C_n$ for p big enough. FC is **bounded** if bounded below and above
$F_pH_n(C) = \mathrm{im}(H_n(F_pC) \to H_n(C))$

Definition 3.5. F_nC is a filtered chain complex, $E^0_{pq} = \dfrac{F_pC_{p+q}}{F_{p+1}C_{p+q-1}}$ defines a spectral sequence E^1_{pq} **converges** to H_*C if $E^1_{pq} = H_{p+q}(F_pC/F_{p-1}C) \Rightarrow H_{p+q}C$

Definition 3.6. $E^0_{pq} = F_pC_{p+q}$ defines a spectral sequence

Theorem 3.7. $A^r_p = \{x \in F_pC | dx \in F_{p-r}C\}$, $Z^r_p = A^r_p + F_{p-1}C$, $B^r_p = dA^{r-1}_{p+r-1} + F_{p-1}C$, $A^r_p \cap F_{p-1}C = A^{r-1}_{p-1}$

$$E^r_p = \frac{Z^r_p}{B^r_p} = \frac{A^r_p + F_{p-1}C}{dA^{r-1}_{p+r-1} + F_{p-1}C} = \frac{\dfrac{A^r_p + F_{p-1}C}{F_{p-1}C}}{\dfrac{dA^{r-1}_{p+r-1} + F_{p-1}C}{F_{p-1}C}} = \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1}}{dA^{r-1}_{p+r-1} \cap F_{p-1}C}}$$
$$= \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1}}{dA^{r-1}_{p+r-1} \cap A^{r-1}_{p-1}}} = \frac{\dfrac{A^r_p}{A^{r-1}_{p-1}}}{\dfrac{dA^{r-1}_{p+r-1} + A^{r-1}_{p-1}}{A^{r-1}_{p-1}}} = \frac{A^r_p}{dA^{r-1}_{p+r-1} + A^{r-1}_{p-1}}$$

Lemma 3.8. C and $\widehat{C}$ give the same spectral sequence

Theorem 3.9. If F_*C is bounded, then $E^1_{p,q}$ converges to H_*C
If F_*C is bounded below and exhaustive, then $E^1_{p,q}$ converges to H_*C, the convergence is natural

Theorem 3.10. C is a complete filtration, then

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim{}^1 H_{n+1}(C/F_pC) \arrow[r] \arrow[d, equal] & H_n(C) \arrow[r] & H_n(C/F_pC) \arrow[r] \arrow[d, equal] & 0 \\
& \bigcap F_pH_n(C) & & \varprojlim H_n(C)/F_pH_n(C) \arrow[d, equal] & \\
& & & H_*(C)/\bigcap F_pH_n(C) &
\end{tikzcd}$$

Lemma 3.11. F_*C is Hausdorff and exhaustive, then

1. $A^\infty_{pq} = \ker(F_pC_{p+q} \xrightarrow{d} F_pC_{p+q-1})$
2. $F_pH_{p+q}(C) \cong A^\infty / \bigcup dA^r_{p+r,q-r+1}$
3. The subgroup $e^\infty_{pq} = A^\infty_{pq} + B^\infty_{pq}$ is isomorphic to $F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C)$

3.3 Spectral sequence of a double complex

Definition 3.12. C_{pq} is a double complex. Filtration by columns $'F_n(Tot(C))$ is the total complex of a truncation of C with $C_{pq} = 0$ for $q > n$

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 & 0 \end{array}$$

Filtration by rows $''F_n(Tot(C))$ is the total complex of a truncation of C with $C_{pq} = 0$ for $p > n$

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{array}$$

We have

$$'E^0_{pq} = C_{pq},' E^1_{pq} = H_q(C_{p*}),' E^2_{pq} = H^h_p H^v_q C$$

$$''E^0_{pq} = C_{qp},'' E^1_{pq} = H_q(C_{*p}),' E^2_{pq} = H^v_q H^h_p C$$

3.4 Hyperhomology

Definition 3.13. $\mathcal{A}$ is an abelian category with enough projectives, a **Cartan-Eilenberg resolution** P_{**} of chain complex A_* is an upper half plane double complex consist of projectives and a augmentation $P_{*0} \xrightarrow{\varepsilon} A_*$ such that

1. If $A_p = 0$, then $P_{p,*} = 0$

2. $B_p^h(P) \xrightarrow{B_p(\varepsilon)} B_p(A_*)$, $H_p^h(P) \xrightarrow{H_p(\varepsilon)} H_p(A_*)$ are projective resolutions

Lemma 3.14. Every chain complex A_* has a Cartan-Eilenberg resolution, and $Z_p^h(P) \xrightarrow{Z_p(\varepsilon)} Z_p(A_*)$, $P_{p*} \xrightarrow{\varepsilon_p} A_p$ are projective resolutions

Lemma 3.15. $f : A \to B$ is a chain map, $P \to A$, $Q \to B$ are Cartan-Eilenberg resolutions, there exists a double complex map $\widetilde{f} : P \to Q$ over f

Definition 3.16. $f, g : D \to E$ are maps between double complexes, a chain homotopy from f to g consists of $s^h : D_{pq} \to E_{p+1,q}$ and $s^v : D_{pq} \to E_{p,q+1}$ satisfying

$$f - g = (s^h d^h + d^h s^h) = (s^v d^v + d^v s^v)$$

$$s^v d^h + d^h s^v = s^h d^v + d^v s^h = 0$$

So that $s^h + s^v : Tot(D)_n \to Tot(E)_{n+1}$ is a chain homotopy between $Tot(f), Tot(g) : Tot^{\oplus}(D) \to Tot^{\oplus}(E)$

Chain homotopy between Cartna-Eilenberg resolutions

Lemma 3.17. $f, g : A \to B$ are chain homotopic, $P \to A$, $Q \to B$ are Cartan-Eilenberg resolutions, $\tilde{f}, \tilde{g} : P \to Q$ are over f, g, then $\tilde{f}, \tilde{g}$ are chain homotopic
Any two Cartan-Eilenberg resolutions of $P \to A$, $Q \to A$ are chain homotopic. F is an additive functor, then $Tot^{\oplus}(F(P))$, $Tot^{\oplus}(F(Q))$ are chain homotopic

Definition 3.18. $\mathcal{A}, \mathcal{B}$ are abelian categories, $\mathcal{A}$ has enough projectives, $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, $f : A \to B$ is a chain map. Define $\mathbb{L}_i F(A) = H_i(Tot^{\oplus}(F(P)))$, by Lemma 3.17, $\mathbb{L}_i F(A)$ is independent of the choice of P, $\mathbb{L}_i F(f) = H_i(Tot(F(f)))$. $\mathbb{L}_i F : \mathbf{Ch}\mathcal{A} \to \mathcal{B}$ is the left hyper-derived functor of F

Lemma 3.19. $0 \to A \to B \to C \to 0$ is a short exact sequence of bounded below chain complexes, then we have a long exact sequence

$$\cdots \to \mathbb{L}_{i+1} F(C) \xrightarrow{\delta} \mathbb{L}_i F(A) \to \mathbb{L}_i F(B) \to \mathbb{L}_i(C) \xrightarrow{\delta} \cdots$$

Hyperhomology spectral sequence

Proposition 3.20 (Hyperhomology spectral sequence)**.** $L_p F(H_q(A)) \Rightarrow \mathbb{L}_{p+q} F(A)$. If A is bounded below, then $H_p(L_q F(A)) \Rightarrow \mathbb{L}_{p+q} F(A)$

Proof. Consider the double complex P of a Cartan-Eilenberg resolution $P \to A$. Since $H_p^h(P) \to H_p A$ is a projective resolution, we have

$$L_p F(H_q(A)) = H_p^v F(H_q^h(P)) = H_p^v H_q^h(F(P)) ='' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A)$$

If A is bounded belwo, then

$$H_p(L_q F(A)) = H_p^h H_q^v(F(P)) =' E_{pq}^2 \Rightarrow H_{p+q} F(P) = \mathbb{L}_{p+q} F(A)$$

□

Corollary 3.21.

1. If A is exact, then $\mathbb{L}_i F(A) = 0$

2. If $f : A \to B$ is a quasi-isomorphism, then $\mathbb{L}_* F(f) : \mathbb{L}_* F(A) \to \mathbb{L}_* F(B)$ are isomorphisms

3. If A is bounded below and A_p are F acyclic, then $\mathbb{L}_p F(A) = H_p F(A)$

Grothendieck spectral sequence

Theorem 3.22 (Grothendieck spectral sequence)**.** $\mathcal{A}, \mathcal{B}$ have enough projectives, $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{A} \to \mathcal{B}$ are right exact functors and G sends projectives to F-acyclic objects, then

$$(L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A)$$

Proof. Suppose $P \to A$ is a projective resolution, then by Proposition 3.20, we have

$$(L_p F)(L_q G)(A) \cong L_p F(H_q G(P)) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

$$H_p(L_q F(G(P))) \Rightarrow \mathbb{L}_{p+q}(FG)(A)$$

Since $G(A)$ is F-acyclic, $'E_2^{pq} = 0$ for $q \neq 0$ and

$$E_2^{p0} = H_p(FG(P)) = L_p(FG)(A) \cong \mathbb{L}_p(FG)(A)$$

□

Hochschild-Serre spectral sequence

Corollary 3.23 (Hochschild-Serre spectral sequence)**.** $N \trianglelefteq G$ is a normal subgroup, A is a $\mathbb{Z}G$ module, then

$$H_p(G/N; H_q(N; A)) \Rightarrow H_{p+q}(G; A)$$

Proof. Consider right exact functors

$$F = - \otimes_{\mathbb{Z}[G/N]} \mathbb{Z} : \mathbb{Z}[G/N]\text{-mod} \to \mathbb{Z}\text{-mod}$$

$$G = - \otimes_{\mathbb{Z}[N]} \mathbb{Z} = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N] : \mathbb{Z}[G]\text{-mod} \to \mathbb{Z}[G/N]\text{-mod}$$

The left derived functors of $FG = - \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is $L_*(FG)(A) = Tor_*^{\mathbb{Z}[G]}(A, \mathbb{Z}) = H_*(G; A)$. For any $\mathbb{Z}[G]$ module A and $\mathbb{Z}[G/N]$ module B, we have natural isomorphism

$$Hom_{\mathbb{Z}[G/N]}(A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/N], B) \cong Hom_{\mathbb{Z}[G]}(A, B) = Hom_{\mathbb{Z}[G]}(A, U(B))$$

Hence G is left adjoint to forgetful functor U which is exact, by Lemma 2.38, G preserves projectives which are exactly F-acyclic objects. Apply Theorem 3.22 we have

$$H_p(G/N; H_q(N; A)) = (L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A) = H_{p+q}(G; A)$$

□

4 Homeworks

4.1 Homework1

Exercise A1.1. Show that $\mathbb{Z} \subset \mathbb{Q}$ is epi in **Rings**. Show that $\mathbb{Q} \subset \mathbb{R}$ is epi in the category of Hausdorff topological spaces

Proof. Suppose we have a commutative diagram of rings

$$\mathbb{Z} \xhookrightarrow{i} \mathbb{Q} \overset{f}{\underset{g}{\rightrightarrows}} R$$

Then $f(n) = g(n), \forall n \in \mathbb{Z}$, hence

$$f\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right)g(1) = f\left(\frac{1}{n}\right)g(n)g\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right)f(n)g\left(\frac{1}{n}\right) = f(1)g\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$$

Thus $f\left(\frac{m}{n}\right) = f(m)f\left(\frac{1}{n}\right) = g(m)g\left(\frac{1}{n}\right) = g\left(\frac{m}{n}\right) \Rightarrow f = g$
Therefore, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism
Suppose we have a commutative diagram of Hausdorff spaces

$$\mathbb{Q} \xhookrightarrow{i} \mathbb{R} \overset{f}{\underset{g}{\rightrightarrows}} X$$

Suppose $f \neq g$, then there exists $r \in \mathbb{R}$ such that $x := f(r) \neq g(r) =: y$. Since X is Hausdorff, there are disjoint open neighborhoods $x \in U$, $y \in V$, then $r \in W := f^{-1}(U) \cap g^{-1}(V)$ is an open neighborhood of r. Since $\mathbb{Q} \hookrightarrow \mathbb{R}$ is dense, there exists $q \in \mathbb{Q} \cap W$, we have $f(q) \in U, g(q) \in V$, $f(q) = g(q)$ which is a contradiction, hence $f = g$. Therefore, $\mathbb{Q} \hookrightarrow \mathbb{R}$ is an epimorphism ☐

Exercise A1.2. In **Groups**, show that monics are just injective set maps, and kernels are monics whose image is a normal subgroup

Proof. Consider a commutative diagram of groups

$$K \overset{g}{\underset{h}{\rightrightarrows}} G \xrightarrow{f} H \tag{4.1}$$

Given f is an injective map, if $fg = fh$, then $f(g(k)) = f(h(k)) \Rightarrow g(k) = h(k)$, thus f is a monomorphism
Conversely, given f is a monomorphism. Suppose f is not an injective map, then $f(g_1) = f(g_2)$ for some $g_1 \neq g_2 \in G$, if we take K in (4.1) to be the infinite cyclic group $\langle x \rangle$ generated by x and $g(x) = g_1$, $h(x) = g_2$, then $g \neq h$ but $fg = fh$ which is a contradiction. Therefore f is an injective map
Given $i : K \hookrightarrow G$ is a monomorphism and $N := i(K)$ is a normal subgroup of G. $\pi i = 0$ is the zero morphism where $\pi : G \to G/N$ is the quotient homomorphism, suppose $i' : K' \to G$ is a homomorphism such that $\pi i' = 0$, then $i'(K') \subseteq N = i(K)$, we can define $\phi : K' \to K$, $k' \mapsto i^{-1}i'(k')$, ϕ is a homomorphism since

$$\begin{aligned} i(i^{-1}i'(k_1')i^{-1}i'(k_2')) = i(i^{-1}i'(k_1'))i(i^{-1}i'(k_2')) = i'(k_1')i'(k_2') = i'(k_1'k_2') \\ \Rightarrow \phi(k_1'k_2') = i^{-1}i'(k_1'k_2') = i^{-1}i'(k_1')i^{-1}i'(k_2') = \phi(k_1')\phi(k_2') \end{aligned}$$

$$\begin{tikzcd} K \arrow[r, hook, "i"] & G \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & G/N \\ K' \arrow[u, dashed, "\phi"] \arrow[ru, "i'"'] & & \end{tikzcd}$$

Conversely, given $i : K \hookrightarrow G$ is a kernel of $G \xrightarrow{\pi} H$. Suppose $f, g : M \to K$ such that $if = ig$, then $\pi if = \pi ig = 0$, by universal property, $f = g$

$$\begin{tikzcd}
M \arrow[d, "g"'] \arrow[dr, "ig"] & & \\
K \arrow[r, hook, "i"] & G \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & H \\
M \arrow[u, "f"] \arrow[ur, "if"'] & &
\end{tikzcd}$$

Let N be the kernel of π, since $\pi i = 0$, $i(K) \subseteq N$, on the other hand, by universal property, there is a homomorphism $\phi : N \to K$ such that $i\phi = \iota$, where $\iota : N \hookrightarrow G$ is inclusion, thus $N = \iota(N) = \phi i(K) \subseteq i(K)$. Therefore $i(K) = N$ is a normal subgroup

$$\begin{tikzcd}
K \arrow[r, hook, "i"] & G \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & H \\
N \arrow[u, dashed, "\phi"] \arrow[ur, "\iota"'] & &
\end{tikzcd}$$

□

Exercise A1.3. (Pontrjagin duality) Show that the category $\mathcal{G}$ of finite abelian groups is isomorphic to its opposite category $\mathcal{G}^{op}$, but that this fails for the category $\mathcal{T}$ of torsion abelian groups. We will see in exercise 6.11.4 that $\mathcal{T}^{op}$ is the category of profinite abelian groups

Proof. Make the use of the following facts

$$Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n,m)\mathbb{Z}$$

And

$$Hom\left(\bigoplus_{j=1}^{n}\mathbb{Z}/n_j\mathbb{Z}, \bigoplus_{i=1}^{m}\mathbb{Z}/m_i\mathbb{Z}\right) \cong \bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} Hom\left(\mathbb{Z}/n_j\mathbb{Z}, \mathbb{Z}/m_i\mathbb{Z}\right)$$

We could think of elements in $Hom\left(\bigoplus_{j=1}^{n}\mathbb{Z}/n_j\mathbb{Z}, \bigoplus_{i=1}^{m}\mathbb{Z}/m_i\mathbb{Z}\right)$ as $m \times n$ matrices

$$\begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{pmatrix}$$

With $k_{ij} \in \mathbb{Z}/(n_j, m_i)\mathbb{Z}$, and composition can be thought of as matrix multiplication
The opposite category $\mathcal{G}^{op}$ can be thought of as a category with finite abelian groups as objects and homomorphism direction reversed as morphisms
Define $F : \mathcal{G} \to \mathcal{G}^{op}$, sending any object to itself and sending morphisms as follows

$$Hom\left(\bigoplus_{j=1}^{n}\mathbb{Z}/n_j\mathbb{Z}, \bigoplus_{i=1}^{m}\mathbb{Z}/m_i\mathbb{Z}\right) \to Hom_{\mathcal{G}^{op}}\left(\bigoplus_{j=1}^{n}\mathbb{Z}/n_j\mathbb{Z}, \bigoplus_{i=1}^{m}\mathbb{Z}/m_i\mathbb{Z}\right)$$

$$\begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{m1} & \cdots & k_{mn} \end{pmatrix} \mapsto \begin{pmatrix} k_{11} & \cdots & k_{m1} \\ \vdots & \ddots & \vdots \\ k_{1n} & \cdots & k_{mn} \end{pmatrix}^{op}$$

This is a functor since $F(I) = I^{op}$, $F(AB) = \left((AB)^T\right)^{op} = \left(B^T A^T\right)^{op} = \left(A^T\right)^{op}\left(B^T\right)^{op} = F(A)F(B)$
Similarly, define $G : \mathcal{G}^{op} \to \mathcal{G}$, $G(X) = X, \forall X \in \mathcal{G}^{op}$, $G(A^{op}) = A^T$, this is also a functor since $G(I^{op}) = I$, $G(A^{op}B^{op}) = G\left((BA)^{op}\right) = (BA)^T = A^T B^T = G(A)G(B)$
Therefore $GF = 1_{\mathcal{G}}$, $FG = 1_{\mathcal{G}^{op}}$, $\mathcal{G}$ is isomorphic to $\mathcal{G}^{op}$
Now let's show $\mathcal{T}$ is not equivalent to $\mathcal{T}^{op}$, suppose there are functors $F : \mathcal{T} \to \mathcal{T}^{op}$, $G : \mathcal{T}^{op} \to \mathcal{T}$ such that $GF \cong 1_{\mathcal{T}}$, $FG \cong 1_{\mathcal{T}^{op}}$

Claim I: If $f : A \to B$ is a monomorphism in $\mathcal{T}$, then f is an injective map, otherwise $0 \neq \ker f \overset{i}{\hookrightarrow} A$ is a nonzero subgroup, but then $fi = f0$ which is a contradiction
Claim II: If $f : A \to B$ is a surjective map in $\mathcal{T}$, then f is an epimorphism
Claim III: Let Ab be the category of abelian groups, suppose $A_i \in ob\mathcal{T}$, the colimit of $A_1 \to A_2 \to \cdots$ in Ab is $\oplus A_i / \sim$ which is also a torsion abelian group, it's clear that $\oplus A_i / \sim$ also satisfies universal property for colimits in $\mathcal{T}$, thus $\oplus A_i / \sim$ is also the colimit in $\mathcal{T}$, moreover, if $A_i \neq 0$, $A_i \hookrightarrow A_j$ are monomorphisms thus injective, then $\oplus A_i / \sim$ is not zero
Claim IV: 0 is obviously the zero object in $\mathcal{T}$ thus unique up to isomorphism
Claim V: The limit of $\mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\bmod p^n} \mathbb{Z}/p^n\mathbb{Z}$ in $\mathcal{T}$ is 0 since for any torsion abelian group A, if diagram

$$\begin{tikzcd}
\cdots & \mathbb{Z}/p^n\mathbb{Z} \arrow[l] & \mathbb{Z}/p^{n+1}\mathbb{Z} \arrow[l, "\bmod p^n"'] & \cdots \arrow[l] \\
 & & A \arrow[ul, "f_n"] \arrow[u, "f_{n+1}"'] &
\end{tikzcd}$$

Commutes, suppose $0 \neq a \in A$ is of order k such that $f_n(a) \neq 0$, since $0 = f_m(ka) = kf_m(a)$ for any m, $k = p^l$ for some l, but $f_m(a) \equiv f(a) \bmod p^n$ for any $m \geq n$, thus $p^l f_{n+l}(a) = 0 \Rightarrow f_n(a) = 0$ which is a contradiction. Therefore $f_i = 0$, giving the unique zero morphism $A \xrightarrow{0} 0$ such that the following diagram commutes

$$\begin{tikzcd}
\cdots & \mathbb{Z}/p^n\mathbb{Z} \arrow[l] & \mathbb{Z}/p^{n+1}\mathbb{Z} \arrow[l, "\bmod p^n"'] & \cdots \arrow[l] \\
 & & 0 \arrow[ul, "0"] \arrow[u, "0"'] & \\
 & & A \arrow[u, dashed, "0"] \arrow[uul, bend left, "f_n"] \arrow[uu, bend right, "f_{n+1}"'] &
\end{tikzcd}$$

Claim VI: F is fully faithful, F maps initial objects in $\mathcal{T}$ to final objects in $\mathcal{T}$ and maps final objects in $\mathcal{T}$ to initial objects in $\mathcal{T}$, $F(0) = 0$, similarly, $G(0) = 0$, thus $F(A) \neq 0$ for any $A \neq 0$, otherwise $A = GF(A) = 0$, F maps epimorphisms in $\mathcal{T}$ to monomorphisms in $\mathcal{T}$, F maps limits in $\mathcal{T}$ to colimits in $\mathcal{T}$
$\mathbb{Z}/p^{n+1}\mathbb{Z} \xrightarrow{\bmod p^n} \mathbb{Z}/p^n\mathbb{Z}$ are surjective, F map these to monomorphisms between nonzero objects with colimit 0 which contradicts Claim III
Therefore, $\mathcal{T}$ is not equivalent to $\mathcal{T}^{op}$ ☐

Exercise A1.4. Show that

$$Hom_{\mathcal{C}}\left(A, \prod C_i\right) \cong \prod_{i \in I} Hom_{\mathcal{C}}(A, C_i)$$

And that

$$Hom_{\mathcal{C}}\left(\coprod C_i, A\right) \cong \prod_{i \in I} Hom_{\mathcal{C}}(C_i, A)$$

Proof. Given $\phi \in Hom_{\mathcal{C}}(A, \prod C_i)$, compose with $p_i : \prod C_i \to C_i$, we have $\phi_i := p_i\phi \in Hom_{\mathcal{C}}(A, C_i)$. Conversely, given $\phi_i \in Hom_{\mathcal{C}}(A, C_i), \forall i \in I$, by the universal property of product, there is a morphism $\phi \in Hom_{\mathcal{C}}(A, \prod C_i)$ such that $\phi_i = p_i\phi$
Given $\phi \in Hom_{\mathcal{C}}(\coprod C_i, A)$, compose with $\iota_i : C_i \to \coprod C_i$, we have $\phi_i := \phi\iota_i \in Hom_{\mathcal{C}}(C_i, A)$. Conversely, given $\phi_i \in Hom_{\mathcal{C}}(C_i, A), \forall i \in I$, by the universal property of coproduct, there is a morphism $\phi \in Hom_{\mathcal{C}}(\coprod C_i, A)$ such that $\phi_i = \phi\iota_i$ ☐

Exercise A4.1. Let $\mathcal{A}$ be an **Ab**-category and $f : B \to C$ a morphism. Show that:
1. f is monic $\Leftrightarrow$ for every nonzero $e : A \to B$, $fe \neq 0$
2. f is an epi $\Leftrightarrow$ for every nonzero $g : C \to D$, $gf \neq 0$

Proof. **1.** Given f is a monomorphism. Suppose there is a nonzero $e : A \to B$, such that $fe = 0$, since $f0 = 0$, $e = 0$ which is a contradiction. Therefore, for every nonzero $e : A \to B$, $fe \neq 0$. Conversely, given for every nonzero $e : A \to B$, $fe \neq 0$. Suppose $e, e' : A \to B$ are homomorphisms such that $fe = fe'$, then $f(e - e) = 0 \Rightarrow e - e' = 0 \Rightarrow e = e'$. Therefore, f is a monomorphism

2. Given f is an epimorphism. Suppose there is a nonzero $g : C \to D$, such that $gf = 0$, since $0f = 0$, $g = 0$ which is a contradiction. Therefore, for every nonzero $g : C \to D$, $gf \neq 0$. Conversely, given for every nonzero $g : C \to D$, $gf \neq 0$. Suppose $g, g' : C \to D$ are homomorphisms such that $gf = g'f$, then $(g - g')f = 0 \Rightarrow g - g' = 0 \Rightarrow g = g'$. Therefore, f is an epimorphism $\square$

4.2 Homework2

Exercise A4.2. Show that $\mathcal{A}^{op}$ is an abelian category if $\mathcal{A}$ is an abelian category

Proof. $Hom_{\mathcal{A}^{op}}(X, Y)$ has an obvious abelian group structure with addition $f^{op} + g^{op} = (f+g)^{op}$, 0^{op} being the zero morphism, the inverse of f^{op} being $(-f)^{op}$, $f^{op}(g^{op} + h^{op}) = f^{op}g^{op} + f^{op}h^{op}$ and $(g^{op} + h^{op})f^{op} = g^{op}g^{op} + h^{op}f^{op}$. Thus $\mathcal{A}^{op}$ has an Ab structure
Since $\mathcal{A}$ is an abelian category, 0 is the zero object in $\mathcal{A}^{op}$, for any $A_1, \cdots, A_n \in ob\mathcal{A}^{op}$, suppose $(\bigoplus A_i, \pi_i, \iota_i)$ is the biproduct in $\mathcal{A}$, then $(\bigoplus A_i, \iota_i^{op}, \pi_i^{op})$ is the biproduct in $\mathcal{A}^{op}$. Hence $\mathcal{A}^{op}$ is additive
Since $\mathcal{A}$ is an abelian category, for any morphism $f^{op} : A \to B$ in $Hom_{\mathcal{A}^{op}}(A, B)$, suppose $(\ker f, i)$ is the kernel and $(\text{coker} f, \pi)$ is the cokernel of $f : B \to A$, then $(\ker f, i^{op})$ is the cokernel and $(\text{coker} f, \pi^{op})$ is the kernel of f^{op}. For any monomorphism $e^{op} : A \to B$ and any epimorphism $m^{op} : A \to B$, suppose e is the cokernel of its kernel $i : C \to B$, m is the kernel of its cokernel $\pi : A \to D$, then e^{op} is the kernel of its cokernel $i^{op} : B \to C$, m^{op} is the cokernel of its kernel $\pi^{op} : D \to A$. Hence $\mathcal{A}^{op}$ is an abelian category ☐

Exercise A4.3. Given a category I and an abelian category $\mathcal{A}$, show that the functor category $\mathcal{A}^I$ is also an abelian category and that the kernel of $\eta : B \to C$ is the functor A, $A(i) = \ker(\eta_i)$

Proof. Suppose $\eta, \xi \in Hom_{\mathcal{A}^I}(B, C)$ are natural transformations, define addition $(\eta + \xi)_i = \eta_i + \xi_i$, then $0 = \{0_i\}$ is the zero morphism, $-\eta = \{-\eta_i\}$ is the inverse of η, $(\eta + \xi)\mu = \eta\mu + \xi\mu$ and $\eta(\xi + \mu) = \eta\xi + \eta\mu$. Thus $\mathcal{A}^I$ has an Ab structure
Constant functor $K_0 \in ob\mathcal{A}^I$, $K_0(B) = 0$, $K(f) = 0$ is the zero object in $\mathcal{A}^I$. For any functors $B, C \in ob\mathcal{A}^I$, $(B \times C, \pi_B, \pi_C)$ is the product, where $(B \times C)(i) = B(i) \times C(i)$, $(\pi_B)_i = \pi_{B(i)}$, $(\pi_C)_i = \pi_{C(i)}$. Hence $\mathcal{A}^I$ is additive
Now let's show A is the kernel of $\eta : B \to C$, note that morphism $A(f)$ is defined by the following commutative diagram because $\eta_j B(f)\iota_i = C(f)\eta_i\iota_i$

$$\begin{tikzcd}
\ker \eta_i \arrow[r, hook, "\iota_i"] \arrow[d, dashed, "\exists_1 A(f)"] & B_i \arrow[r, "\eta_i"] \arrow[d, "B(f)"] & C_i \arrow[d, "C(f)"] \\
\ker \eta_j \arrow[r, hook, "\iota_j"] & B_j \arrow[r, "\eta_j"] & C_j
\end{tikzcd}$$

Suppose $A' \in ob\mathcal{A}^I$ is another functor making the following diagram commute

$$\begin{tikzcd}
A' \arrow[d, "\xi"] \arrow[dr, "\eta\xi"] & \\
B \arrow[r, "\eta"] & C
\end{tikzcd}$$

There are unique morphisms μ_i's making the following diagram commute

$$\begin{tikzcd}
& A'_i \arrow[dl, dashed, "\exists_1 \mu_i"'] \arrow[d, "\xi_i"] \arrow[dr, "\eta_i\xi_i"] & \\
\ker \eta_i \arrow[r] & B_i \arrow[r, "\eta_i"] & C_i
\end{tikzcd}$$

Which gives the unique natural transformation $\mu : A' \to A$ making the diagram commute

$$\begin{tikzcd}
& A' \arrow[dl, dashed, "\exists_1 \mu"'] \arrow[d, "\xi"] \arrow[dr, "\eta\xi"] & \\
A \arrow[r] & B \arrow[r, "\eta"] & C
\end{tikzcd}$$

Hence A is indeed the kernel of $\eta : B \to C$
Similarly, the cokernel of $\eta : B \to C$ would be functor D, $D(i) = \text{coker}(\eta_i)$, $D(f)$ is defined by the following commutative diagram

$$\begin{tikzcd}
B_i \arrow[r, "\eta_i"] \arrow[d, "B(f)"] & C_i \arrow[r, two heads, "\pi_i"] \arrow[d, "C(f)"] & \mathrm{coker}\eta_i \arrow[d, dashed, "\exists_1 D(f)"] \\
B_j \arrow[r, "\eta_j"] & C_j \arrow[r, two heads, "\pi_j"] & \mathrm{coker}\eta_j
\end{tikzcd}$$

Similarly, it is easy to verify that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. Hence $\mathcal{A}^I$ is an abelian category $\square$

Exercise A5.1. Show that an abelian category is complete iff it has all products

Proof. Let $\mathcal{A}$ be an abelian category, I be a small category and $F : I \to \mathcal{A}$ be a functor and $\mathcal{A}$ contains all products

For $j \xrightarrow{f} k \in Hom(j, k)$, let ϕ_f be the composition $\prod_{i \in obI} F(i) \xrightarrow{\pi_j} F(j) \xrightarrow{F(f)} F(k)$, ψ_f be the composition $\prod_{i \in obI} F(i) \xrightarrow{\pi_k} F(k) \xrightarrow{1_{F(k)}} F(k)$, they induce two maps

$$\phi, \psi : \prod_{i \in obI} F(i) \to \prod_f F(k)$$

Write (E, ι) as the equalizer of ϕ, ψ, ι_j as the composition $E \xrightarrow{\iota} \prod_{i \in obI} F(i) \xrightarrow{\pi_j} F(j)$, then

$$F(f)\iota_j = F(f)\pi_j\iota = \phi_f\iota = \pi_f\phi\iota = \pi_f\psi\iota = \psi_f\iota = 1_{F(k)}\pi_k\iota = \pi_k\iota = \iota_k$$

$$\begin{tikzcd}
 & E \arrow[d, "\iota"] \arrow[ddl, bend right, "\iota_j"'] \arrow[ddr, bend left, "\iota_k"] & \\
 & \prod_i F(i) \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_k"] & \\
F(j) \arrow[rr, "F(f)"] & & F(k)
\end{tikzcd}$$

Let's show (E, ι_j) is the limit $\varprojlim F$

Given a commutative digram

$$\begin{tikzcd}
 & A \arrow[dl, "\alpha_j"'] \arrow[dr, "\alpha_k"] & \\
F(j) \arrow[rr, "F(f)"] & & F(k)
\end{tikzcd}$$

Which induce unique $\eta : A \to \prod_i F(i)$, $\xi : A \to \prod_f F(k)$ such that $\pi_j\eta = \alpha_j$, $\pi_f\xi = \alpha_k$, then we have a commutative diagram

$$\begin{tikzcd}
A \arrow[d, "\eta"'] \arrow[dr, "\xi"] & \\
\prod_i F(i) \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \prod_f F(k)
\end{tikzcd}$$

Since

$$\pi_f\phi\eta = \phi_f\eta = F(f)\pi_j\eta = F(f)\alpha_j = \alpha_k = \pi_f\xi \Rightarrow \phi\eta = \xi$$

And

$$\pi_f\psi\eta = \psi_f\eta = 1_{F(k)}\pi_k\eta = \pi_k\eta = \alpha_k = \pi_f\xi \Rightarrow \psi\eta = \xi$$

Which induces unique $\zeta : A \to E$ such that $\iota\zeta = \eta$, $\iota_j\zeta = \pi_j\iota\zeta = \pi_j\eta = \alpha_j$, suppose $\zeta' : A \to E$ is another map such that $\iota_j\zeta' = \alpha_j$, then $\iota_i\zeta' = \pi_i\iota\zeta' = \alpha_j = \pi_j\eta \Rightarrow \zeta' = \eta \Rightarrow \zeta' = \zeta$ $\square$

Exercise A6.1. Fix categories I and $\mathcal{A}$. When every functor $F : I \to \mathcal{A}$ has a limit, show that $\mathsf{lim} : \mathcal{A}^I \to \mathcal{A}$ is a functor. Show that the universal property of $\mathsf{lim}\, F_i$ is nothing more than the assertion that lim is right adjoint to Δ. Dually, show that the universal property of $\mathsf{colim} F_i$ is nothing more than the assertion that $\mathsf{colim} : \mathcal{A}^I \to \mathcal{A}$ is the left adjoint to Δ

Proof. Suppose $\eta : F \to G$ is a natural transformation, then we have morphisms $\varprojlim F \to F(i) \xrightarrow{\eta_i} G(i)$ which induces uniquely a morphism $\varprojlim \eta : \varprojlim F \to \varprojlim G$, it is obvious that $\varprojlim(1_F) = 1_{\varprojlim F}$, $\varprojlim(\eta\xi) = \varprojlim(\eta)\varprojlim(\xi)$. Hence $\varprojlim : \mathcal{A}^I \to \mathcal{A}$ is a functor
Suppose $\eta \in Hom_{\mathcal{A}^I}(\Delta A, F)$ is a natural transformation, we have morphisms $\eta_i : A \to F(i)$, which induces uniquely a morphism $\varprojlim \eta : A \to \varprojlim F$, conversely, if we have a morphism $\phi : A \to \varprojlim F$, then we have morphisms $\eta_i : A \to \varprojlim F \to F(i)$, this gives a natural transformation $\eta : \Delta A \to F$. Therefore $\eta \to \varprojlim \eta$ is a bijective correspondence, this correspondence is natural since for any $B \xrightarrow{f} A$ and $F \xrightarrow{\xi} G$ the following diagram commutes

$$\begin{tikzcd}
Hom_{\mathcal{A}^I}(\Delta A, F) \arrow[rrr] \arrow[ddd] & & & Hom_{\mathcal{A}}(A, \varprojlim F) \arrow[ddd] \\
& \eta \arrow[r, mapsto] \arrow[d, mapsto] & \varprojlim \eta \arrow[d, mapsto] & \\
& \xi\eta f \arrow[r, mapsto] & \varprojlim \xi \varprojlim \eta f & \\
Hom_{\mathcal{A}^I}(\Delta B, G) \arrow[rrr] & & & Hom_{\mathcal{A}}(B, \varprojlim G)
\end{tikzcd}$$

Hence $\varprojlim$ is the right adjoint of Δ, dually, it is easy to show $\varinjlim$ is the left adjoint of Δ $\square$

Exercise A6.2. Suppose given functor $L : \mathcal{A} \to \mathcal{B}$, $R : \mathcal{B} \to \mathcal{A}$ and natural transformations $\eta : \mathrm{id}_{\mathcal{A}} \Rightarrow RL$, $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathcal{B}}$ such that the composites $LX \xrightarrow{L(\eta_X)} LRLX \xrightarrow{\varepsilon_{LX}} LX$, $RY \xrightarrow{\eta_{RY}} RLRY \xrightarrow{R(\varepsilon_Y)} RY$ are the identities. Show that (L, R) is an adjoint pair of functions

Proof. We have a bijective correspondence between $Hom_{\mathscr{B}}(LX, Y)$ and $Hom_{\mathscr{A}}(X, RY)$ giving by commutative diagram

$$\begin{tikzcd}
& Hom_{\mathscr{A}}(RLX, RY) \arrow[dr, "\eta_X^*"] & \\
Hom_{\mathscr{B}}(LX, Y) \arrow[ur, "R"] & & Hom_{\mathscr{A}}(X, RY) \arrow[dl, "L"] \\
& Hom_{\mathscr{B}}(LX, LRY) \arrow[ul, "\varepsilon_{Y*}"] &
\end{tikzcd}$$

Since

$$\varepsilon_Y \circ L(R(f) \circ \eta_X) = \varepsilon_Y \circ LR(f) \circ L(\eta_X) = f \circ \varepsilon_{LX} \circ L(\eta_X) = f \circ 1_{LX} = f$$

$$R(\varepsilon_Y \circ L(g)) \circ \eta_X = R(\varepsilon_Y) \circ RL(g) \circ \eta_X = R(\varepsilon_Y) \circ \eta_{RY} \circ g = 1_{RY} \circ g = g$$

The bijective correspondence is natural by the following commutative diagram for any given $\alpha : X' \to X$, $\beta : Y \to Y'$

$$\begin{tikzcd}
Hom_{\mathscr{B}}(LX, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "\beta_*(L\alpha)^*"'] & Hom_{\mathscr{A}}(X, RY) \arrow[d, "(R\beta)_*\alpha^*"] \\
Hom_{\mathscr{B}}(LX', Y') \arrow[r, "\Phi_{X',Y'}"] & Hom_{\mathscr{A}}(X', RY')
\end{tikzcd}$$

Since

$$R(\beta \circ f \circ L(\alpha) \circ \eta_{X'}) = R(\beta) \circ R(f) \circ RL(\alpha\eta_{X'}) = R(\beta) \circ R(f) \circ \eta_X \circ \alpha$$

Therefore (L, R) is an adjoint pair of functors $\square$

Exercise 1.2.5. Given an elementary proof that $Tot(C)$ is acyclic whenever C is a bounded double complex with exact rows(or exact columns). We will see later that this result follows from the Acyclic Assembly Lemma 2.7.3. It also follows from a spectral sequence argument(see Definition 5.6.2 and exercise 5.6.4)

Proof. Without loss of generality, we may assume C is bounded in the first quadrant and has exact rows, use d', d'', d to denote row, column and total differentials
$Tot(C)$ is exact for all $n < 0$ since $Tot(C)_n = 0$ for all $n < 0$. now suppose $n \geq 0$, $d\left(\sum_{k=0}^{n} x_{k,n-k}\right) = 0$, i.e. $d'x_{k+1,n-k-1} + d''x_{k,n-k} = 0$ for $0 \leq k < n$. Let $x_{0,n+1} = 0$, we can construct $x_{k,n+1-k}$ for $k > 0$ inductively such that $d''x_{k,n-k+1} + d'x_{k+1,n-k} = x_{k,n-k}$ for $0 \leq k \leq n$ as follow:
For $k \geq -1$

$$\begin{aligned}
d'(x_{k+1,n-k-1} - d''x_{k+1,n-k}) &= d'x_{k+1,n-k-1} - d'd''x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''d'x_{k+1,n-k} \\
&= d'x_{k+1,n-k-1} + d''(d''x_{k,n-k+1} + d'x_{k+1,n-k}) \\
&= d'x_{k+1,n-k-1} + d''x_{k,n-k} \\
&= 0
\end{aligned}$$

By exactness of rows, there exists $x_{k+2,n-k-1}$ such that

$$d'x_{k+2,n-k-1} = x_{k+1,n-k-1} - d''x_{k+1,n-k} \Leftrightarrow d''x_{k+1,n-k} + d'x_{k+2,n-k-1} = x_{k+1,n-k-1}$$

Therefore

$$\begin{aligned}
d\left(\sum_{k=0}^{n+1} x_{k,n+1-k}\right) &= \sum_{k=1}^{n+1}(d'x_{k,n+1-k} + d''x_{k,n+1-k}) \\
&= \sum_{k=1}^{n+1}(x_{k-1,n-k+1} - d''x_{k-1,n-k+2} + d''x_{k,n+1-k}) \\
&= \sum_{k=0}^{n}(x_{k,n-k} - d''x_{k,n-k+1}) + \sum_{k=1}^{n+1} d''x_{k,n+1-k} \\
&= \sum_{k=0}^{n} x_{k,n-k}
\end{aligned}$$

□

Exercise 1.2.7. If C is a complex, show that there are exact sequences of complexes:

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0$$

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0$$

Proof. Let $Z_i \hookrightarrow C_i$ be the kernel of $C_i \xrightarrow{\partial} C_{i-1}$ and $C_i \to B_{i-1} = B[-1]_i$ be the image of $C_i \xrightarrow{\partial} C_{i-1}$, then $0 \to Z_i \to C_i \to B_{i-1} \to 0$ are exact sequences, and we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & Z_2 \arrow[r, "\partial"] \arrow[d, hook] & Z_1 \arrow[r, "\partial"] \arrow[d, hook] & Z_0 \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & C_2 \arrow[r, "\partial"] \arrow[d, two heads, "\partial"] & C_1 \arrow[r, "\partial"] \arrow[d, two heads, "\partial"] & C_0 \arrow[r] \arrow[d, two heads, "\partial"] & \cdots \\
\cdots \arrow[r] & B_1 \arrow[r, "-\partial"] \arrow[d] & B_0 \arrow[r, "-\partial"] \arrow[d] & B_{-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Hence we have an exact sequence of complexes

$$0 \to Z(C) \to C \xrightarrow{d} B(C)[-1] \to 0$$

Consider the following commutative diagram

$$\begin{tikzcd}
 & B_i \arrow[r, equal] \arrow[d, hook] & B_i \arrow[r, hook] \arrow[d, hook] & Z_i \arrow[r, "\partial"] \arrow[d, hook] & B_{i-1} \arrow[d, hook] & \\
 & Z_i \arrow[r, hook] \arrow[d, two heads] & C_i \arrow[r, equal] \arrow[d, two heads] & C_i \arrow[ur, "\partial"] \arrow[d, two heads] & Z_{i-1} \arrow[d, two heads] & \\
0 \arrow[r] & Z_i/B_i \arrow[r, dashed] & C_i/B_i \arrow[r, dashed] & C_i/Z_i \arrow[uur, dashed] & Z_{i-1}/B_{i-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & B_i \arrow[r, equal] \arrow[d, hook] & B_i \arrow[r, hook] \arrow[d, hook] & Z_i \arrow[r, "\partial"] \arrow[d, hook] & B_{i-1} \arrow[d, hook] & \\
 & Z_i \arrow[r, hook] \arrow[d, two heads] & C_i \arrow[r, equal] \arrow[d, two heads] & C_i \arrow[ur, "\partial"] \arrow[d, two heads] & Z_{i-1} \arrow[d, two heads] & \\
0 \arrow[r] & Z_i/B_i \arrow[r, dashed] & C_i/B_i \arrow[r, dashed] \arrow[urr, dashed, bend right, red, "d"'] & C_i/Z_i \arrow[uur, dashed] & Z_{i-1}/B_{i-1} \arrow[r] & 0
\end{tikzcd}$$

We get an exact sequence $0 \to H_i \to C_i/B_i \xrightarrow{d} Z_{i-1} \to H_{i-1} \to 0$, and thus an exact sequence of complexes

$$0 \to H(C) \to C/B(C) \xrightarrow{d} Z(C)[-1] \to H(C)[-1] \to 0$$

□

Exercise 1.3.1. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third

Proof. Since we have a long exact sequence

$$\cdots \to H_nA \to H_nB \to H_nC \xrightarrow{\partial} H_{n-1}A \to H_{n-1}B \to H_{n-1}C \to \cdots$$

If either two of complexes of A, B, C are exact, then their homologies vanish, thus the homologies of the third one also vanishes, i.e. the third complex is exact □

4.3 Homework3

Exercise 1.3.2. (3 × 3 lemma) Suppose given a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d, "a'"] & B' \arrow[r, "v'"] \arrow[d, "b'"] & C' \arrow[r] \arrow[d, "c'"] & 0 \\
0 \arrow[r] & A'' \arrow[r, "u''"] \arrow[d] & B'' \arrow[r, "v''"] \arrow[d] & C'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In an abelian category, such that every column is exact. Show the following:
1. If the bottom two rows are exact, so is the top row
2. If the top two rows are exact, so is the bottom row
3. If the top and bottom rows are exact, and the composite $A' \to C'$ is zero, the middle row is also exact
Hint: Show the remaining row is a complex, and apply exercise 1.3.1

Proof. 1. Since third column is exact, $C = \ker c$, $c(vu) = v'bu = (v'u')a = 0$, $A \xrightarrow{0} C' \to C''$ must induce the unique map $A \to C$ which is $0 = vu$, i.e. the first row is a complex
2. Since first column is exact, $A'' = \operatorname{coker} a$, $(v''u'')a' = v''b'u' = c'(v'u') = 0$, $A \to A' \xrightarrow{0} C''$ must induce the unique map $A'' \to C''$ which is $0 = v''u''$, i.e. the third row is a complex
3. If composite $A' \to C'$ is zero, then the second row is a complex
Finally, apply exercise 1.3.1, two of three exact implies the remaining one is also exact ☐

Exercise 1.3.3. (5-Lemma) In any commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d, "a", "\cong"'] & B' \arrow[r, "v'"] \arrow[d, "b", "\cong"'] & C' \arrow[r, "w'"] \arrow[d, "c"] & D' \arrow[r, "x'"] \arrow[d, "d", "\cong"'] & E' \arrow[d, "e", "\cong"'] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & D \arrow[r, "x"] & E
\end{tikzcd}$$

With exact rows in any abelian category, show that if a, b, d and e are isomorphisms, then c is also an isomorphism. More precisely, show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi

Proof. Since rows are exact, we can apply snake lemma on following commutative diagrams

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, two heads] & B' \arrow[r, two heads] \arrow[d, tail] & \operatorname{im} v' = \ker w' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C &
\end{tikzcd}$$

$$\begin{tikzcd}
 & C' \arrow[r] \arrow[d] & D' \arrow[r, two heads] \arrow[d, two heads] & E' \arrow[r] \arrow[d, tail] & 0 \\
0 \arrow[r] & \operatorname{coker} v \arrow[r] & D \arrow[r] & E &
\end{tikzcd}$$

Thus $\operatorname{im} v' = \ker w' \to C$ is monic, $C' \to \operatorname{im} v = \ker w$ is epi
Suppose $Y \xrightarrow{y} C'$ satisfies $cy = 0$, then $0 = wcy = d(w'y)$, since d is monic, $w'y = 0$ which induce $Y \xrightarrow{y'} \ker w' = \operatorname{im} v'$, then $Y \xrightarrow{y'} \ker w' \to C$ is zero implies $y' = 0$ which in turn implies $y = 0$, i.e. c is monic
Suppose $C \xrightarrow{z} Z$ satisfies $zc = 0$, then $0 = zcv' = (zv)b$, since b is epi, $zv = 0$ which induce $\operatorname{coker} v \xrightarrow{z'} Z$, then $C' \to \operatorname{coker} v \xrightarrow{z'} C$ is zero implies $z' = 0$ which in turn implies $z = 0$, i.e. c is epi ☐

Exercise 1.4.1. The previous example shows that even an acyclic chain complex of free R-modules need not to split exact
1. Show that acyclic bounded below chain complexes of free R modules are always split exact
2. Show that an acyclic chain complex of finitely generated free abelian groups is always split exact, even when it is not bounded below

Proof. 1. Suppose $\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0$ is an acyclic chain complex of free R-modules, since ∂_1 is surjective, we can define $s_0 : F_0 \to F_1$ sending each generator of F_0 to a preimage, then $\partial_1 s_0 \partial_1 = \partial_1$
Now let's construct $s_n : F_n \to F_{n+1}$ inductively that satisfying $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$
Suppose we have already constructed s_{n-1}, we claim $F_n = \partial_{n+1}(F_{n+1}) \oplus s_{n-1}\partial_n(F_n)$: if $\partial_{n+1}(a_{n+1}) = s_{n-1}\partial_n(a_n)$, then $\partial_n a_n = \partial_n s_{n-1}\partial_n a_n = \partial_n\partial_{n+1}(a_{n+1}) = 0 \Rightarrow s_{n-1}\partial_n(a_n) = 0$. For any $a_n \in F_n$, $\partial_n(a_n - s_{n-1}\partial_n(a_1)) = 0 \Rightarrow a_n - s_{n-1}\partial_n(a_n) \in \ker \partial_n = \mathrm{im}\partial_{n+1}$
Define $s_n : F_n \to F_{n+1}$, sending each generator of F_n to a preimage of its direct sum part in $\partial_{n+1}(F_{n+1})$, then $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$
2. In the case of finitely generated free abelian groups, subgroups $\partial_n(F_n)$ are also finitely generated free abelian, we can define $s_{n-1} : \partial_n(F_n) \to F_n$ by sending each generator of $\partial_n(F_n)$ to a preimage, then $\partial_n = \partial_n s_{n-1}\partial_n$, similarly we have $F_n = \partial_{n+1}(F_{n+1}) \oplus s_{n-1}\partial_n(F_n)$, thus $\partial_{n+1}(F_{n+1})$ is a direct summand of F_n, thus we can extend s_n such that $s_n : F_n \to F_{n+1}$ with $s_n(s_{n-1}\partial_n(F_n)) = 0$, we still have $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$ ☐

Exercise 1.4.2. Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. Show that C split if and only if there are R-module decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n \cong B_n \oplus H'_n$. Show that C is split exact iff $H'_n = 0$

Proof. If $C_n \cong Z_n \oplus B'_n$ and $Z_n \cong B_n \oplus H'_n$, claim that any element in B_n has a unique preimage in B'_{n+1}: if $x, y \in B'_{n+1}$ are such that $\partial_{n+1}x = \partial_{n+1}y$, then $\partial_{n+1}(x - y) = 0 \Rightarrow (x - y) \in Z_{n+1} \cap B'_{n+1} = 0 \Rightarrow x = y$
Hence we can define a unique bijective homomorphism $s_n : B_n \to B'_{n+1}$ sending elements to its preimage, then extend s_n to $s_n : C_n \to C_{n+1}$ such that $s_n(C_n) = B'_{n+1}$, $s_n(H'_n \oplus B'_n) = 0$, then $\partial_{n+1} s_n \partial_{n+1} = \partial_{n+1}$, i.e. C split
If C split, denote $B'_n = s_{n-1}\partial_n(C_n)$, $H'_n = \ker \partial_{n+1} s_n \cap Z_n$, we claim $C_n = Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$: For any $s_{n-1}\partial_n(a_n) \in Z_n$, $0 = \partial_n(s_{n-1}\partial_n(a_n)) = \partial_n a_n \Rightarrow s_{n-1}\partial_n(a_n) = 0$. For $a_n \in C_n$, $\partial_n(a_n - s_{n-1}\partial_n(a_n)) = 0$. For any $\partial_{n+1}a_{n+1} \in \ker \partial_{n+1} s_n$, $\partial_{n+1}(a_{n+1}) = \partial_{n+1} s_n \partial_{n+1}(a_{n+1}) = 0$. For any $a_n \in Z_n$, $a_n - \partial_{n+1} s_n(a_n) \in \ker \partial_{n+1} s_n \cap Z_n$
It is obvious that C is exact $\Leftrightarrow Z_n \cong B_n \Leftrightarrow H'_n = 0$ ☐

Exercise 1.4.3. Show that C is a split exact chain complex if and only if the identity map on C is null homotopic

Proof. Suppose the identity map on C is null homotopic, then there exists $s_n : C_n \to C_{n+1}$ such that $1_{C_n} = s_{n-1}\partial_n + \partial_{n+1} s_n$, then $\partial_n = \partial_n s_{n-1} \partial_n$, i.e. C split, for any $a_n \in Z_n$, $a_n = (s_{n-1}\partial_n + \partial_{n+1} s_n) a_n = \partial_{n+1}(s_n a_n) \in B_n$, i.e. C is exact
Suppose C split exact, according to exercise 1.4.2, $C_n \cong Z_n \oplus B'_n \cong B_n \oplus B'_n$ with $H'_n = 0$, then we can define $s_n : C_n \to C_{n+1}$ such that $s_n(B_n) = B'_{n+1}$ bijective, $s_n(H'_n \oplus B'_n) = 0$, $\partial_{n+1} s_n \partial_{n+1} = \partial_{n+1}$, thus $B'_n = s_{n-1}(B_n) = s_{n-1}\partial_n(C_n)$, $s_n s_{n-1}\partial_n(C_n) = s_n(B'_n) = 0$. Therefore for any element in C_n which can be written as $\partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n$, we have $(s_{n-1}\partial_n + \partial_{n+1} s_n)(\partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n) = \partial_{n+1}a_{n+1} + s_{n-1}\partial_n a_n$, i.e. $1_{C_n} = s_{n-1}\partial_n + \partial_{n+1} s_n$, the identity map on C is nullhomotopic ☐

Exercise 1.5.1. Let $cone(C)$ denote the mapping cone of the identity map id_C of C; it has $C_{n-1} \oplus C_n$ in degree n. Show that $cone(C)$ is split exact, with $s(b, c) = (-c, 0)$ defining the splitting map

Proof. Suppose $\begin{pmatrix} -\partial & \\ -1 & \partial \end{pmatrix}\begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -\partial b \\ -b + \partial c \end{pmatrix} = 0$ for some $(b, c) \in cone(C)_n = C_{n-1} \oplus C_n$, then $b = \partial c$, $\begin{pmatrix} -\partial & \\ -1 & -\partial \end{pmatrix}\begin{pmatrix} -c \\ 0 \end{pmatrix} = \begin{pmatrix} \partial c \\ c \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}$, i.e. $cone(C)$ is exact, also $\begin{pmatrix} -\partial & \\ -1 & -\partial \end{pmatrix} s_n(b, c) = (b, c)$, i.e. $cone(C)$ split ☐

Exercise 1.5.2. Let $f : C \to D$ be a map of complexes. Show that f is null homotopic if and only if f extends to a map $(-s, f) : cone(C) \to D$

Proof. Suppose $(-s_{n-1}, f_n) : cone(C)_n \to D_n$ are maps, then $(-s, f)$ is chain map iff

$$\begin{pmatrix} -s_{n-1} & f_n \end{pmatrix} \begin{pmatrix} -\partial_n & \\ -1_{C_n} & \partial_{n+1} \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \partial_{n+1} \begin{pmatrix} -s_n & f_{n+1} \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

Which equivalent to

$$(s_{n-1}\partial_n + \partial_{n+1}s_n - f_n)a_n = (\partial_{n+1}f_{n+1} - f_n\partial_{n+1})a_{n+1} = 0$$

Which equivalent to $s_{n-1}\partial_n + \partial_{n+1}s_n = f_n$, i.e. f is null homotopic □

Exercise 1.. Let X denote the chain complex

$$\cdots \to 0 \to 0 \to \mathbb{Q} \xrightarrow{\text{id}} \mathbb{Q} \to 0 \to 0 \to \cdots$$

With $X_1 = X_0 = \mathbb{Q}$ and all other $X_i = 0$. Consider it in the category $\mathscr{C}_*(\mathbf{Vect}_{\mathbb{Q}})$ of complexes of $\mathbb{Q}$-vector spaces
(a) Compute the complex $\mathrm{Hom}_\bullet(X, X)$
(b) Compute the homology $H_*(\mathrm{Hom}_\bullet(X, X))$ of the above complex
(c) Show that X is not isomorphic to 0 in $\mathscr{C}(\mathbf{Vect}_{\mathbb{Q}})$, but X is isomorphic to 0 in the homotopy category $\mathscr{K}(\mathbf{Vect}_{\mathbb{Q}})$

Proof. **(a)** It is obvious that $Hom_n(X, X) = 0$ for $|n| \geq 2$, $Hom_0(X, X) = \mathbb{Q} \oplus \mathbb{Q}$, $Hom_1(X, X) = Hom_{-1}(X, X) = \mathbb{Q}$, and differentials are

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{Q} \arrow[r, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & \mathbb{Q} \oplus \mathbb{Q} \arrow[r, "\begin{pmatrix} 1 & -1 \end{pmatrix}"] & \mathbb{Q} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

(b) According to (a), it is not hard to see this is an exact sequence, hence $H_n(\mathrm{Hom}_\bullet(X, X)) = 0$ for all n
(c) X is not isomorphic to 0 in $\mathscr{C}(\mathbf{Vect}_{\mathbb{Q}})$ since $\mathbb{Q} \to 0 \to \mathbb{Q}$ can never be $1_{\mathbb{Q}}$, but $X \to 0$ and $0 \to X$ are actually inverses to each other in $\mathscr{K}(\mathbf{Vect}_{\mathbb{Q}})$, we only need to show $X \xrightarrow{1_X} X$ is null homotopic which is true due to the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q} \arrow[r, "1_{\mathbb{Q}}"] \arrow[d, "1_{\mathbb{Q}}"'] \arrow[dl] & \mathbb{Q} \arrow[r] \arrow[d, "1_{\mathbb{Q}}"] \arrow[dl, "1_{\mathbb{Q}}"'] & 0 \arrow[dl] \\ 0 \arrow[r] & \mathbb{Q} \arrow[r, "1_{\mathbb{Q}}"'] & \mathbb{Q} \arrow[r] & 0 \end{tikzcd}$$

Note that $1_{\mathbb{Q}} = 1_{\mathbb{Q}} \circ 1_{\mathbb{Q}} + 0 \circ 0$, $1_{\mathbb{Q}} = 0 \circ 0 + 1_{\mathbb{Q}} \circ 1_{\mathbb{Q}}$ □

Exercise 2.. Suppose $\mathscr{A}$ is an abelian category. In class, I defined the category $\mathbf{Ar}(\mathscr{A})$ to be the full subcategory of $\mathscr{C}_*(\mathscr{A})$ consisting of complexes of amplitude $[0, 1]$. We can think of objects in $\mathbf{Ar}(\mathscr{A})$ as morphisms $f : M \to N$ in $\mathscr{A}$
I defined functors $F_0, F_1 : \mathscr{A} \to \mathbf{Ar}(\mathscr{A})$ where $F_i(M)$ is the complex X_* with $X_i = M$ and $X_j = 0$ for all $j \neq i$ (and $F_i(\phi) : F_i(M) \to F_i(N)$ is the obvious morphism induced by ϕ)
(a) In class, I claimed that $F_1 : \mathscr{A} \to \mathbf{Ar}(\mathscr{A})$ is left adjoint to the functor $\ker : \mathbf{Ar}(\mathscr{A}) \to \mathscr{A}$ taking a morphism $f : M \to N$ to $\ker f$. Prove this
(b) Prove similarly that F_0 is right adjoint to the functor $\mathrm{coker} : \mathbf{Ar}(\mathscr{A}) \to \mathscr{A}$
(c) Suppose

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Is a commutative diagram with exact rows. Using (a) and (b), show that

$$0 \to \ker a \to \ker b \to \ker c$$

And

$$\mathrm{coker} a \to \mathrm{coker} b \to \mathrm{coker} c \to 0$$

Are exact. (Obviously, don't use the proof of the Snake Lemma sketched in class)

Proof. **(a)** For any $M \xrightarrow{f} N \in \mathbf{Ar}(\mathscr{A})$, $W \in \mathscr{A}$, there is clearly bijective map $Hom(F_1(W), M \xrightarrow{f} N) \to Hom(W, \ker f)$ and Hom given by the following diagram

$$\begin{tikzcd}
 & W \arrow[r] \arrow[d] \arrow[dl, dashed] & 0 \arrow[d] \\
\ker f \arrow[r, tail] & M \arrow[r, "f"] & N
\end{tikzcd}$$

This bijective correspondence is natural due to the universal property of kernel and the following diagram

$$\begin{tikzcd}
 & & W \arrow[rr] \arrow[dd] \arrow[dll, dashed] & & 0 \arrow[dd] & \\
 & & & W' \arrow[ul, "g"] \arrow[rr] \arrow[d, bend left] \arrow[dll, dashed, bend right] & & 0 \arrow[ul] \arrow[dd] \\
\ker f \arrow[rr] \arrow[dr, dashed] & & M \arrow[rr, "f"] \arrow[dr] & & N \arrow[dr] & \\
 & \ker f' \arrow[rr] & & M' \arrow[rr, "f'"] & & N'
\end{tikzcd}$$

(b) For any $M \xrightarrow{f} N \in \mathbf{Ar}(\mathscr{A})$, $W \in \mathscr{A}$, there is clearly bijective map $Hom(\mathrm{coker} f, W) \to Hom(M \xrightarrow{f} N, F_0(W))$ and Hom given by the following diagram

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d] & N \arrow[r, two heads] \arrow[d] & \mathrm{coker} f \arrow[dl, dashed] \\
0 \arrow[r] & W &
\end{tikzcd}$$

This bijective correspondence is natural due to the universal property of cokernel and the following diagram

$$\begin{tikzcd}
M \arrow[rr, "f"] \arrow[dd] & & N \arrow[rr, two heads] \arrow[dd, bend right] & & \mathrm{coker} f \arrow[dll, dashed, bend left] & \\
 & M' \arrow[ul] \arrow[rr, "f'"] \arrow[dd] & & N' \arrow[ul] \arrow[rr, two heads] \arrow[dd] & & \mathrm{coker} f' \arrow[ul, dashed] \arrow[dll, dashed] \\
0 \arrow[rr] \arrow[dr] & & W \arrow[dr, "g"] & & & \\
 & 0 \arrow[rr] & & W' & &
\end{tikzcd}$$

(c) By (a), we have

$$\begin{tikzcd}
 & \ker a \arrow[r, "u_*"] \arrow[d, tail, "i_a"] & \ker b \arrow[d, tail, "i_b"] \\
M \arrow[ur, "f"] \arrow[r] \arrow[d] & A \arrow[r, tail, "u"] \arrow[d, "a"] & B \arrow[d, "b"] \\
0 \arrow[r] & A' \arrow[r, tail, "u'"] & B'
\end{tikzcd}$$

If $u_* f = 0$, then $0 = i_b u_* f = u i_a f$, since u, i_a are monic, $f = 0$, i.e. $\ker a \xrightarrow{u_*} \ker b$ is monic
By universal property of kernel, we know $\ker a \xrightarrow{u_*} \ker b \xrightarrow{v_*} \ker c$ is zero, consider $v_* f = 0$ which induce unique $M \to A$ and then induce unique $M \to \ker a$, by universal property of kernel, we know $\ker a$ is $\ker v_*$

$$
\begin{tikzcd}
& \ker a \arrow[r, "u_*"] \arrow[d, tail, "i_a"] & \ker b \arrow[r, "v_*"] \arrow[d, tail, "i_b"] & \ker c \arrow[d, tail, "i_c"] \\
M \arrow[r] \arrow[d] \arrow[ur, dashed] \arrow[urr, bend left, "f"] & A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads, "v"] \arrow[d, "a"] & C \arrow[d, "b"] \\
0 \arrow[r] & A' \arrow[r, tail] & B' \arrow[r, two heads, "v'"] & C'
\end{tikzcd}
$$

Hence $0 \to \ker a \to \ker b \to \ker c$ is exact
By (b), we have

$$
\begin{tikzcd}
B \arrow[r, two heads, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "b"] & 0 \arrow[d] \\
B' \arrow[r, two heads, "v'"] \arrow[d, two heads, "\pi_b"] & C' \arrow[r] \arrow[d, two heads, "\pi_c"] & M \\
\mathrm{coker} b \arrow[r, "v'^*"] & \mathrm{coker} c \arrow[ur, "f"'] &
\end{tikzcd}
$$

If $f v'^* = 0$, then $0 = f v'^* \pi_b = f \pi_c v'$, since v', π_c are epi, $f = 0$, i.e. $\mathrm{coker} b \xrightarrow{v'^*} \mathrm{coker} c$ is epi
By universal property cokernel, we know $\mathrm{coker} a \xrightarrow{u'^*} \mathrm{coker} b \xrightarrow{v'^*} \mathrm{coker} c$ is zero, consider $f v'^* = 0$ which induce unique $C' \to M$ and then induce unique $\mathrm{coker} c \to M$, by universal property of cokernel, we know $\mathrm{coker} c$ is $\mathrm{coker} u'^*$

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, two heads, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "b"] & 0 \arrow[d] \\
A' \arrow[r, "u'"] \arrow[d, two heads, "\pi_a"] & B' \arrow[r, two heads, "v'"] \arrow[d, two heads, "\pi_b"] & C' \arrow[r] \arrow[d, two heads, "\pi_c"] & M \\
\mathrm{coker} a \arrow[r, "u'^*"] & \mathrm{coker} b \arrow[r, "v'^*"] \arrow[urr, bend right, "f"'] & \mathrm{coker} c \arrow[ur, dashed] &
\end{tikzcd}
$$

Hence $\mathrm{coker} a \to \mathrm{coker} b \to \mathrm{coker} c \to 0$ is exact $\square$

4.4 Homework4

Exercise 2.1.2. If $F : \mathscr{A} \to \mathscr{B}$ is an exact functor, show that $T_0 = F$ and $T_n = 0$ for $n \neq 0$ defines a universal δ-functor(of both homological and cohomological type)

Proof. Since F is exact, for any short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, we get another short exact sequence $0 \to F(A) \to F(B) \to F(C) \to 0$, we can view this as a long exact sequence with $\delta = 0$, thus T is both a homological and cohomological δ functor

Suppose S_n is another homological or cohomological δ functor, $S_0 \xrightarrow{\phi} F$, $S^0 \xrightarrow{\psi} F$, then we have

$$\begin{tikzcd} S_1C \arrow[r, "\delta_1"] \arrow[d] & S_0A \arrow[r, "S_0f"] \arrow[d, "\phi"] & S_0B \arrow[r] \arrow[d, "\phi"] & S_0C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & FA \arrow[r, "Ff"] & FB \arrow[r] & FC \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & FA \arrow[r] \arrow[d] & FB \arrow[r, "Fg"] \arrow[d, "\psi"'] & FC \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[d] \\ 0 \arrow[r] & S^0A \arrow[r] & S^0B \arrow[r, "S^0g"] & S^0C \arrow[r, "\delta^0"] & S^1A \end{tikzcd}$$

$0 = \phi S_0 f \delta_1 = Ff\phi\delta_1$ and Ff is monic $\Rightarrow \phi\delta_1 = 0$. $0 = \delta^0 S^0 g\psi = \delta^0\psi Fg$ and Fg is epi $\Rightarrow$ $\delta^0\psi = 0$. Thus T is both a universal homological and cohomological δ functor $\square$

Exercise 2.2.1. Show that a chain complex P is a projective object in Ch if and only if it is a split exact complex of projectives. Hint: To see that P must be split exact, consider the surjection from $cone(\mathrm{id}_P)$ to $P[-1]$. To see that split exact complexes are projective objects, consider the special case $0 \to P_1 \overset{\cong}{\to} P_0 \to 0$

Proof. Consider chain complex C with $C_n = P_n \oplus P_{n+1}$

$$\cdots \to P_n \oplus P_{n+1} \xrightarrow{\begin{pmatrix} \partial & 0 \\ 1 & -\partial \end{pmatrix}} P_{n-1} \oplus P_n \to \cdots$$

and first coordinate projection $C \twoheadrightarrow P$ which is a surjection, if P is projective, then there exists s such that

$$\begin{tikzcd} & P \arrow[dl, "s"'] \arrow[d, equal] \\ C \arrow[r, two heads] & P \end{tikzcd}$$

In order to make s a chain map and the diagram commute, we must have $s : P_n \to C_n$, $x \to \begin{pmatrix} x \\ s_n x \end{pmatrix}$ and

$$\begin{pmatrix} \partial_n x \\ s_{n-1}\partial_n x \end{pmatrix} = \begin{pmatrix} \partial_n & 0 \\ 1 & -\partial_{n+1} \end{pmatrix} \begin{pmatrix} x \\ s_n x \end{pmatrix} = \begin{pmatrix} \partial_n x \\ x - \partial_{n+1} s_n x \end{pmatrix}$$

Hence $s_{n-1}\partial_n + \partial_{n+1}s_n = 1$, i.e. 1 is null homotopic, by Exercise 1.4.3, P split exact. To prove P_n are projectives, given

$$\begin{tikzcd} & P_n \arrow[d, "g_n"] \\ B \arrow[r, two heads, "f"] & A \end{tikzcd}$$

consider the following commutative diagram with $g_{n+1} = g_n \partial_{n+1}$

$$\begin{tikzcd}
P_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[d] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "g_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[d, "g_n"'] \arrow[dd, dashed, bend left, "h"] & P_{n-1} \arrow[d] \\
0 \arrow[r] & A \arrow[r, equal] & A \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & B \arrow[r, equal] \arrow[u, two heads, "f"] & B \arrow[r] \arrow[u, two heads, "f"'] & 0 \arrow[u]
\end{tikzcd}$$

Since $P_\bullet$ is projective, there exists $h : P_n \to B$ such that $fh = g_n$

Conversely, suppose P is a split exact sequence of projectives, by Exercise 1.4.2, there exist bijection $s_n : Z_n = B_n \to B'_{n+1}$ and $P_n \cong B_n \oplus B'_n$, thus P is the direct sum of $0 \to B'_{n+1} \to B_n \to 0$, and we know the coproducts of projectives are projective, it suffices to consider $0 \to P_1 \xrightarrow{\cong} P_0 \to 0$. Since ψ_1 is epi and P_1 is projective, there exists $P_1 \xrightarrow{\phi_1} B_1$ such that $\xi_1 = \psi_1\phi_1$, let $\phi_0 = b_1\phi_1 p_1^{-1}$, then $\xi_0 = a_1\xi_1 p_1^{-1} = a_1\psi_1\phi_1 p_1^{-1} = \psi_0 b_1\phi_1 p_1^{-1} = \psi_0\phi_0$ and $b_0\phi_0 = b_0 b_1\phi_1 p_1^{-1} = 0$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P_1 \arrow[r, "p_1", "\cong"'] \arrow[d, dashed, "\phi_1"] \arrow[dd, bend right, "\xi_1"'] & P_0 \arrow[r] \arrow[d, dashed, "\phi_0"'] \arrow[dd, bend left, "\xi_0"] & 0 \arrow[d] \\
B_2 \arrow[r] \arrow[d, two heads, "\psi_2"'] & B_1 \arrow[r, "b_1"] \arrow[d, two heads, "\psi_1"] & B_0 \arrow[r, "b_0"] \arrow[d, two heads, "\psi_0"'] & B_{-1} \arrow[d, two heads, "\psi_{-1}"] \\
A_2 \arrow[r] & A_1 \arrow[r, "a_1"] & A_0 \arrow[r] & A_{-1}
\end{tikzcd}$$

□

Exercise 2.3.1. Let $R = \mathbb{Z}/m$. Use Baer's criterion to show that R is an injective R-module. Then show that $\mathbb{Z}/d$ is not an injective R-module when $d \mid m$ and some prime p divides both d and m/d.(The hypothesis ensures that $\mathbb{Z}/m \neq \mathbb{Z}/d \oplus \mathbb{Z}/e$)

Proof. Any ideal of R is of the form $I = \langle d \rangle$, $m = de$, if $f : I \to R$ is a homomorphism, then $m \mid ef(d) \Rightarrow d \mid f(d)$, thus we can extend $f : R \to R$, $1 \mapsto \dfrac{f(d)}{d}$, R is an injective R-module

If $m = de$ but $\mathbb{Z}/m\mathbb{Z} \neq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}$, suppose $\mathbb{Z}/d\mathbb{Z}$ is injective, then we would have

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/d\mathbb{Z} \cong e\mathbb{Z}/m\mathbb{Z} \arrow[r, hook, "i"] \arrow[d, equal] & \mathbb{Z}/m\mathbb{Z} \arrow[r, two heads] \arrow[dl, dashed, "s"] & \mathbb{Z}/m\mathbb{Z}/\mathbb{Z}/d\mathbb{Z} \cong \mathbb{Z}/e\mathbb{Z} \arrow[r] & 0 \\
& \mathbb{Z}/d\mathbb{Z} & & &
\end{tikzcd}$$

Then the exact sequence split, $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}$ which is a contradiction □

Exercise 2.4.2. (Preserving derived functors) If $U : \mathscr{B} \to \mathscr{C}$ is an exact functor, show that $U(L_iF) \cong L_i(UF)$

Remark 4.1. Forgetful functors such as $\mathsf{mod}\text{-}R \to Ab$ are often exact, and it is often easier to compute the derived functor of UF due to the absence of cluttering restrictions

Proof. For any chain complex C and exact functor U, we have

$$\begin{tikzcd}
& UB_{n-1} \arrow[r, hook] & UZ_{n-1} \arrow[d, hook] \\
UC_{n+1} \arrow[r] \arrow[d, two heads] & UC_n \arrow[r] \arrow[u, two heads] & UC_{n-1} \\
UB_n \arrow[r, hook] & UZ_n \arrow[r, two heads] \arrow[u, hook] & UH_n
\end{tikzcd}$$

Then $U(H_nC) = \mathsf{coker}(UB_n \to FZ_n) = H_n(UC)$ are naturally isomorphic, hence $U(L_iF(A)) = U(H_i(F(P))) = H_i(UF(P)) = L_i(UF(P))$ are naturally isomorphic □

Exercise 2.4.3. (Dimension shifting) If $0 \to M \to P \to A \to 0$ is exact with P projective(or F-acyclic 2.4.3), show that $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and that $L_1F(A)$ is the kernel of $F(M) \to F(P)$. More generally, show that if

$$0 \to M_m \to P_m \to P_{m-1} \to \cdots \to P_0 \to A \to 0$$

is exact with the P_i projective(or F-acyclic), then $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$ and $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$. Conclude that if $P \to A$ is an F-acyclic resolution of A, then $L_iF(A) = H_i(F(P))$

Remark 4.2. The object M_m, which obviously depends on the choices made, is called m^{th} syzygy of A. The word "syzygy" comes from astronomy, where it was originally used to describe the alignment of the Sun, Earth, and Moon

Proof. We have long exact sequence

$$L_iF(P) \to L_iF(A) \to L_{i-1}F(M) \to L_{i-1}F(P) \to \cdots \to L_1F(P) \to L_1F(A) \to F(M) \to F(P)$$

Here $L_iF(P) = 0$ for $i \neq 0$ since P is F-acyclic, hence $L_iF(A) \cong L_{i-1}F(M)$ for $i \geq 2$ and $L_1F(A)$ is the kernel of $F(M) \to F(P)$

Suppose $M_m = \operatorname{im}(P_{m+1} \to P_m) = \ker(P_m \to P_{m-1})$, $M_{-1} = A$, we have

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 & & & 0 \arrow[d] & 0 \\
 & & & M_{m-1} \arrow[ur] \arrow[d] & & & & M_0 \arrow[ur] \arrow[d] & \\
\cdots \arrow[r] & P_{m+1} \arrow[r] \arrow[d] & P_m \arrow[r] \arrow[ur] & P_{m-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[ur] & P_0 \arrow[d] & \\
 & M_m \arrow[ur] & & M_{m-2} \arrow[d] & & M_1 \arrow[ur] & & A \arrow[d] & \\
0 \arrow[ur] & & & 0 & 0 \arrow[ur] & & & 0 &
\end{tikzcd}$$

Then we can split this up into short exact sequences

$$0 \to M_m \to P_m \to M_{m-1} \to 0$$

Hence $L_iF(M_{m-1}) \cong L_{i-1}F(M_m)$ for $i \geq 2$ and $L_1F(M_{m-1})$ is the kernel of $F(M_m) \to F(P_m)$. This implies $L_iF(A) \cong L_{i-m-1}F(M_m)$ for $i \geq m+2$ and $L_{m+1}F(A)$ is the kernel of $F(M_m) \to F(P_m)$

Suppose $P \to A$ is an F-acyclic resolution of A, $L_0F(A) = F(A) = H_0(F(P))$ is still true because F is right exact. For $m \geq 1$, $F(M_{m-1}) \cong \operatorname{coker}(F(M_m) \to F(P_m)) = \operatorname{coker}(F(P_{m+1}) \to F(P_m))$, hence $L_mF(A) \cong \ker(F(M_{m-1}) \to F(P_{m-1})) = \ker\operatorname{coker}(F(P_{m+1}) \to F(P_m)) = H_m(F(P))$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \\
 & & & F(M_{m-1}) \arrow[ur] \arrow[d] & \\
 & F(P_{m+1}) \arrow[r] \arrow[d] & F(P_m) \arrow[r] \arrow[ur] & F(P_{m-1}) & \\
 & F(M_m) \arrow[ur] \arrow[d] & & & \\
0 \arrow[ur] & 0 & & &
\end{tikzcd}$$

□

Exercise 2.4.4. Show that homology $H_* : Ch_{\geq 0}(\mathscr{A}) \to \mathscr{A}$ and cohomology $H^* : Ch^{\geq 0}(\mathscr{A}) \to \mathscr{A}$ are universal δ-functors. Hint: Copy the proof above, replacing P by the mapping cone $cone(A)$ of exercise 1.5.1

Proof. Suppose T is a homological δ functor, $\phi_0 : T_0 \to H_0$ is given. From a short exact sequence $0 \to A[1] \to cone(A[1]) \to A \to 0$ we get a long exact sequence

$$\cdots \to H_{n+1}(A) \to H_n(A[1]) \to H_n(cone(A[1])) \to H_n(A) \to H_{n-1}(A[1]) \to \cdots$$

Since $H_n(A) = H_{n-1}(A[1])$, we have $H_n(cone(A[1])) = 0$. T_n induce a unique T_{n+1} as follows

$$\begin{tikzcd}
& T_{n+1}(A) \arrow[r] \arrow[d, dashed, "\phi_{n+1}"] & T_n(A[1]) \arrow[r] \arrow[d, "\phi_n"] & T_n(cone(A[1])) \arrow[d, "\phi_n"] \\
0 \arrow[r] & H_{n+1}(A) \arrow[r] & H_n(A[1]) \arrow[r] & 0
\end{tikzcd}$$

Check ϕ_n's are natural transformations. For $f : A \to B$, we have

$$\begin{tikzcd}
0 \arrow[r] & A[1] \arrow[r] \arrow[d, "f[1]"] & cone(A[1]) \arrow[r] \arrow[d, "cone(f[1])"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & B[1] \arrow[r] & cone(B[1]) \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Suppose $\phi_{n-1}T_{n-1}(f[-1]) = H_{n-1}(f[-1])\phi_{n-1}$, $\delta_n T_n(f) = T_{n-1}(f[1])\delta_n$, $\delta_n H_n(f) = H_{n-1}(f[1])\delta_n$, $\delta_n\phi_n = \phi_{n-1}\delta_n$, then $\delta_n\phi_n T_n(f) = \phi_{n-1}\delta_n T_n(f) = \phi_{n-1}T_{n-1}(f[1])\delta_n = H_{n-1}(f[1])\phi_{n-1}\delta_n = H_{n-1}(f[1])\delta_n\phi_n = \delta_n H_n(f)\phi_n$, since δ_n is an isomorphism, $\phi_n T_n(f) = H_n(f)\phi_n$

$$\begin{tikzcd}
T_{n-1}(A[1]) \arrow[rrr, "T_{n-1}(f[1])"] \arrow[ddd, "\phi_{n-1}"] & & & T_{n-1}(B[1]) \arrow[ddd, "\phi_{n-1}"] \\
& T_n(A) \arrow[r, "T_n(f)"] \arrow[d, "\phi_n"] \arrow[ul, "\delta_n"] & T_n(B) \arrow[d, "\phi_n"] \arrow[ur, "\delta_n"] & \\
& H_n(A) \arrow[r, "H_n(f)"] \arrow[dl, "\delta_n"] & H_n(B) \arrow[dr, "\delta_n"] & \\
H_{n-1}(A[1]) \arrow[rrr, "H_{n-1}(f[1])"] & & & H_{n-1}(B[1])
\end{tikzcd}$$

Check ϕ_n's commute with δ_n's. For any $f : A[1] \to C$, we can construct commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "f_n"] & A_n \oplus A_{n+1} \arrow[r] \arrow[d] & A_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C_n \arrow[r] & C_n \oplus A_n \arrow[r] & A_n \arrow[r] & 0
\end{tikzcd}$$

Since

$$\begin{tikzcd}
T_n(A[1]) \arrow[r] \arrow[d, "T_n(f)"] & T_n(cone(A)) \arrow[r] \arrow[d] & T_n(A) \arrow[r, "\delta_n"] \arrow[d, equal] & T_{n-1}(A[1]) \arrow[d, "T_{n-1}(f)"] \\
T_n(C) \arrow[r] & T_n(C \oplus A) \arrow[r] & T_n(A) \arrow[r, "\delta_n"] & T_{n-1}(C)
\end{tikzcd}$$

We have $T_{n-1}(f)\delta_n = \delta_n$. Hence

$$\begin{tikzcd}
T_n(A) \arrow[r, "\delta_n"] \arrow[d, "\phi_n"] & T_{n-1}(A[1]) \arrow[r] \arrow[d, "\phi_{n-1}"] & T_{n-1}(C) \arrow[d, "\phi_{n-1}"] \\
H_n(A) \arrow[r, "\delta_n"] & H_{n-1}(A[1]) \arrow[r] & H_{n-1}(C)
\end{tikzcd}$$

Gives

$$\begin{tikzcd}
T_n(A) \arrow[r, "\delta_n"] \arrow[d, "\phi_n"] & T_{n-1}(C) \arrow[d, "\phi_{n-1}"] \\
H_n(A) \arrow[r, "\delta_n"] & H_{n-1}(C)
\end{tikzcd}$$

□

4.5 Homework5

Exercise 1.2.6. Gives examples of
(1) a second quadrant double complex C with exact columns such that $Tot^{\Pi}(C)$ is acyclic but $Tot^{\oplus}(C)$ is not
(2) a second quadrant double complex C with exact rows such that $Tot^{\oplus}(C)$ is acyclic but $Tot^{\Pi}(C)$ is not
(3) a double complex(in the entire plane) for which every row and every column is exact, yet neither $Tot^{\Pi}(C)$ nor $Tot^{\oplus}(C)$ is acyclic

Proof. (1) Consider the second quadrant double complex C with $C_{-p,p} = C_{-p,p+1} = \mathbb{Z}$ for $p \leq 0$ and identity maps for all $\mathbb{Z} \to \mathbb{Z}$, then $(\cdots, 0, 1) \in Tot_0^{\oplus}(C)$ doesn't have a preimage in $Tot_1^{\oplus}(C)$, hence $Tot^{\oplus}(C)$ is not acyclic. On the other hand, $Tot_1^{\Pi}(C) \to Tot_0^{\Pi}(C)$ is an isomorphism, hence $Tot^{\Pi}(C)$ is acyclic

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\cdots & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] \\
\cdots & 0 \arrow[l] & 0 \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

(2) Consider the second quadrant double complex C with $C_{-p,p} = C_{-p-1,p} = \mathbb{Z}$ for $p \leq 0$ and identity maps for all $\mathbb{Z} \to \mathbb{Z}$, then $(\cdots, -1, 1, -1, 1) \in Tot_1^{\Pi}(C)$ maps to $0 \in Tot_0^{\Pi}(C)$, hence $Tot^{\Pi}(C)$ is not acyclic. On the other hand, $Tot_1^{\oplus}(C) \to Tot_0^{\oplus}(C)$ is an isomorphism, hence $Tot^{\oplus}(C)$ is acyclic

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\cdots & 0 \arrow[l] & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l]
\end{tikzcd}$$

(3) Consider the double complex C with $C_{-p,p} = C_{-p,p+1} = \mathbb{Z}$ and identity maps for all $\mathbb{Z} \to \mathbb{Z}$, then $(\cdots, 0, 1, 0, \cdots) \in Tot_0^{\oplus}(C)$ doesn't have a preimage in $Tot_1^{\oplus}(C)$, hence $Tot^{\oplus}(C)$ is not acyclic. On the other hand, $(\cdots, -1, 1, -1, 1, \cdots) \in Tot_1^{\Pi}(C)$ maps to $0 \in Tot_0^{\Pi}(C)$ is an isomorphism, hence $Tot^{\Pi}(C)$ is not acyclic

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$\square$

Exercise 2.7.1. Let C be the periodic upper half-plane complex with $C_{p,q} = \mathbb{Z}/4$ for all p and $q \geq 0$, all differentials being multiplication by 2

$$\begin{tikzcd}
 & \arrow[d, "2"] & \arrow[d, "2"] & \arrow[d, "2"] & \\
\cdots & \mathbb{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \mathbb{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \mathbb{Z}/4 \arrow[l, "2"'] \arrow[d, "2"] & \cdots \arrow[l, "2"'] \\
\cdots & \mathbb{Z}/4 \arrow[l, "2"'] & \mathbb{Z}/4 \arrow[l, "2"'] & \mathbb{Z}/4 \arrow[l, "2"'] & \cdots \arrow[l, "2"']
\end{tikzcd}$$

1. Show that $H_0(Tot^{\Pi}(C)) \cong \mathbb{Z}/2$ on the cycle $(\cdots, 1, 1, 1) \in \prod C_{-p,p}$ even though the rows of C are exact. Hint: First show that the 0-boundaries are $\prod 2\mathbb{Z}/4$
2. Show that $Tot^{\oplus}(C)$ is acyclic
3. Now extend C downward to form a doubly periodic plane double complex D with $D_{pq} = \mathbb{Z}/4$ for all $p, q \in \mathbb{Z}$. Show that $H_0(Tot^{\Pi}(D))$ maps onto $H_0(Tot^{\Pi}(C)) \cong \mathbb{Z}/2$. Hence $Tot^{\Pi}(D)$ is not acyclic, even though every row and column of D is exact. Finally, show that $Tot^{\oplus}(D)$ is acyclic

Proof. 1. It is obvious that $B_0(Tot^{\Pi}(C)) \subseteq \prod 2\mathbb{Z}/4$, for any $(\cdots, x_{-2,2}, x_{-1,1}, x_{0,0}) \in \prod 2\mathbb{Z}/4$, we can find $(\cdots, x_{-2,3}, x_{-1,2}, x_{0,1}, 0)$ inductively such that $2x_{0,1} = x_{0,0}$, $2x_{0,1} + 2x_{-1,2} = x_{-1,1}$, $2x_{-1,2} + 2x_{-2,3} = x_{-2,2}$, $\cdots$, hence $B_0(Tot^{\Pi}(C)) = \prod 2\mathbb{Z}/4$. Similarly, we can show any element in $Tot_0^{\Pi}(C)$ that maps to 0 has entries of the same parity, hence $H_0(Tot^{\Pi}(C)) \cong \mathbb{Z}/2$ on the cycle $(\cdots, 1, 1, 1) \in Tot_0^{\Pi}(C)$
2. Apply acyclic assembly lemma. C is an upper half-plane complex with exact rows, thus $Tot^{\oplus}(C)$ is acyclic
As in 1. $B_0(Tot^{\Pi}(C)) = \bigoplus 2\mathbb{Z}/4$, any element in $Tot_0^{\oplus}(C)$ that maps to 0 has entries of the same parity, thus $Z_0(Tot^{\oplus}(C)) = \bigoplus 2\mathbb{Z}/4$, hence $Tot^{\oplus}(C)$ is acyclic
3. We have an obvious map $D \to C$. For any $(\cdots, x_{-2,2}, x_{-1,1}, x_{0,0}) \in Z_0(Tot^{\Pi}(C))$, it comes from $(\cdots, x_{-2,2}, x_{-1,1}, x_{0,0}, x_{0,0}, x_{0,0}, \cdots) \in Z_0(Tot^{\Pi}(D))$, also, $B_0(Tot^{\Pi}(D)) \subseteq \prod 2\mathbb{Z}/4$ maps into $B_0(Tot^{\Pi}(C)) = \prod 2\mathbb{Z}/4$, thus $H_0(Tot^{\Pi}(D))$ maps onto $H_0(Tot^{\Pi}(C)) \cong \mathbb{Z}/2$
As in 2. $B_0(Tot^{\oplus}(D)) = Z_0(Tot^{\oplus}(D)) = \bigoplus 2\mathbb{Z}/4$, hence $Tot^{\oplus}(D)$ is acyclic $\square$

Exercise 2.7.3. To see why $Tot^{\oplus}$ is used for the tensor product $P \otimes_R Q$ of right and left R module complexes, while Tot^{Π} is used for Hom, let I be a cochain complex of abelian groups. Show that there is a natural isomorphism of double complexes:

$$Hom_{Ab}(Tot^{\oplus}(P \otimes_R Q), I) \cong Hom_R(P, Tot^{\Pi}(Hom_{Ab}(Q, I)))$$

Proof. Since

$$\begin{aligned}
Hom\left(\bigoplus_{p+q=r} P_p \otimes Q_q, I^r\right) &\cong \prod_{p+q=r} Hom\left(P_p \otimes Q_q, I^r\right) \\
&\cong \prod_{p+q=r} Hom\left(P_p, Hom(Q_q, I^r)\right) \\
&\cong Hom\left(P_p, \prod_{p+q=r} Hom(Q_q, I^r)\right)
\end{aligned}$$

is natural isomorphic. $Hom_{Ab}(Tot^{\oplus}(P \otimes_R Q), I) \cong Hom_R(P, Tot^{\Pi}(Hom_{Ab}(Q, I)))$ is also natural isomorphic $\square$

Exercise 3.1.2. Suppose that T is a commutative domain with field of fractions F. Show that $Tor_1^R(F/R, B)$ is the torsion submodule $\{b \in B : (\exists r \neq 0) rb = 0\}$ of B for every R module B

Proof. Since localization is exact, F is a flat R module, tensoring the flat resolution $0 \to R \to F \to F/R \to 0$ with B, we get $0 \to B = R \otimes B \to F \otimes B \to F/R \otimes B \to 0$, and $Tor_1^R(F/R, B) = \ker(B \to F \otimes B) = T(B)$ is the torsion submodule of B $\square$

Exercise 3.1.3. Show that $Tor_1^R(R/I, R/J) \cong \dfrac{I \cap J}{IJ}$ for every right ideal I and left ideal J of R. In particular, $Tor_1(R/I, R/I) \cong I/I^2$ for every 2 sided ideal I. Hint: Apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I \otimes R/J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R \otimes R/J & \longrightarrow & 0 \end{array}$$

Proof. Since $0 \to I \to R \to R/I \to 0$ is exact, we have exact sequence

$$0 = Tor_1^R(R, R/J) \to T_1^R(R/I, R/J) \to I \otimes R/J \to R \otimes R/J$$

Apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & IJ & \longrightarrow & I & \longrightarrow & I \otimes R/J & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R \otimes R/J & \longrightarrow & 0 \end{array}$$

We get exact sequence

$$0 = \ker(I \to R) \to \ker(I \otimes R/J \to R \otimes R/J) \to \mathrm{coker}(IJ \to J) = J/IJ \to \mathrm{coker}(I \to R) = R/J$$

Hence

$$Tor_1^R(R/I, R/J) \cong \ker(I \otimes R/J \to R \otimes R/J) = \ker(J/IJ \to R/I) = \frac{I \cap J}{IJ}$$

□

Exercise 3.2.1. Show that the following are equivalent for every left R module B
1. B is flat
2. $Tor_n^R(A, B) = 0$ for all $n \neq 0$ and all A
3. $Tor_1^R(A, B) = 0$ for all A

Proof. $2 \Rightarrow 3$: By definition
$3 \Rightarrow 1$: For any short exact sequence $0 \to K \to F \to A \to 0$, we have $0 = Tor_1^R(A, B) \to K \otimes B \to F \otimes B \to A \otimes B \to 0$, hence B is flat
$1 \Rightarrow 2$: Since B is flat, $- \otimes B$ is exact, tensor any projective resolution of A with B, $Tor_n^R(A, B)$, $n \neq 0$ which are the homologies are 0 □

Exercise 3.2.3. We saw in the last section that if $R = \mathbb{Z}$(or more generally, if R is a principal ideal domain), a module B is flat iff B is torsion free. Here is an example of a torsion free ideal I that is not a flat R module. Let k be a field and set $R = k[x, y]$, $I = (x, y)R$. Show that $k = R/I$ has the projective resolution

$$0 \to R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to k \to 0$$

Then compute that $Tor_1^R(I, k) \cong Tor_2^R(k, k) \cong k$, showing that I is not flat

Proof. If $R = k[x, y]$ is a UFD, $I = (x, y)$ is a maximal ideal, then we have a projective resolution

$$0 \to R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to R/I \cong k \to 0$$

For any $h \in R$, $\begin{pmatrix} -yh \\ xh \end{pmatrix} = 0 \Rightarrow h = 0$, hence $R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2$ is injective. $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} = 0$ and if $\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = xf + yg = 0$, then $xf = -yg \Rightarrow g = xh \Rightarrow f = -yh \Rightarrow \begin{pmatrix} f \\ g \end{pmatrix} = h \begin{pmatrix} -y \\ x \end{pmatrix}$. any element of R is in I iff it can be written as $xf + yg = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$, hence $R^2 \to R \to k$ is exact. $R \to k$ is obviously surjective

Tensoring $0 \to I \to R \to k \to 0$ with k we get $0 = Tor_2^R(R, k) \to Tor_2^R(k, k) \to Tor_1^R(I, k) \to Tor_1^R(R, k) = 0$. Tensoring the projective resolution with k we get $0 \to k \xrightarrow{0} k^2 \to k \to 0$, hence $Tor_2^R(k, k) = \ker(k \xrightarrow{0} k^2) \cong k$ □

Exercise 3.2.4. Show that a sequence $A \to B \to C$ is exact iff its dual $C^* \to B^* \to A^*$ is exact

Proof. Combine the fact that $Hom(-, R)$ is a left exact contravariant functor and Lemma 3.2.5. we know $A \to B \to C$ is exact iff its dual $C^* \to B^* \to A^*$ is exact □

Exercise 3.3.1. Show that $Ext_{\mathbb{Z}}^1\left(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}\right) \cong \widehat{\mathbb{Z}}_p/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$. This shows that $Ext_{\mathbb{Z}}^1(-, \mathbb{Z})$ does not vanish on flat abelian groups

Proof. Consider short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}\left[\frac{1}{p}\right] \to \mathbb{Z}_{p^\infty} \to 0$, then we have exact sequence

$$\mathbb{Z} \cong Hom(\mathbb{Z}, \mathbb{Z}) \to Ext_{\mathbb{Z}}^1(\mathbb{Z}_{p^\infty}, \mathbb{Z}) \cong (\mathbb{Z}_{p^\infty})^* \cong \widehat{\mathbb{Z}}_p \to Ext_{\mathbb{Z}}^1\left(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}\right) \to Ext_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) = 0$$

Hence $Ext_{\mathbb{Z}}^1\left(\mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}\right) \cong \widehat{\mathbb{Z}}_p/\mathbb{Z} \cong \mathbb{Z}_{p^\infty}$ □

Exercise 3.3.2. When $R = \mathbb{Z}/m$ and $B = \mathbb{Z}/p$ with $p \mid m$, show that

$$0 \to \mathbb{Z}/p \overset{l}{\hookrightarrow} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \mathbb{Z}/m \xrightarrow{p} \mathbb{Z}/m \xrightarrow{m/p} \cdots$$

is an infinite periodic injective resolution of B. Then compute the groups $Ext_{\mathbb{Z}/m}^n(A, \mathbb{Z}/p)$ in terms of $A^* = Hom(A, \mathbb{Z}/m)$. In particular, show that if $p^2 \mid m$, then $Ext_{\mathbb{Z}/m}^n(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p$ for all n

Proof. For any ideal $k\mathbb{Z}/m$, $k \mid m$, any map $k\mathbb{Z}/m \to \mathbb{Z}/m$ must send k to a multiple of k, thus can be extended to a map $\mathbb{Z}/m \to \mathbb{Z}/m$, hence $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$ module. $\mathbb{Z}/p \to \mathbb{Z}/m$ is obviously injective. If $m \mid pk$, then $\frac{m}{p} \mid k$, hence this is an injective resolution of B. Then we have $0 \to A^* \xrightarrow{p} A^* \xrightarrow{\frac{m}{p}} \cdots$, hence $Ext_{\mathbb{Z}/m}^n(A, \mathbb{Z}/p) = \begin{cases} Hom(A, \mathbb{Z}/p) & n = 0 \\ \dfrac{Hom(A, \mathbb{Z}/\frac{m}{p})}{pA^*} & n \text{ odd} \\ \dfrac{Hom(A, \mathbb{Z}/p)}{\frac{m}{p}A^*} & n > 0 \text{ even} \end{cases}$

If $p^2 \mid m$, then we would have $0 \to (\mathbb{Z}/p)^* \xrightarrow{0} (\mathbb{Z}/p)^* \xrightarrow{0} \cdots$, hence $Ext_{\mathbb{Z}/m}^n(\mathbb{Z}/p, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^* = Hom(\mathbb{Z}/p, \mathbb{Z}/m) \cong \mathbb{Z}/p$ □

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Exercise 3.4.1. Show that if p is prime, there are exactly p equivalence classes of extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$ in **Ab**: the split extension and the extensions

$$0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{i} \mathbb{Z}/p \to 0 \quad (i = 1, 2, \cdots, p-1)$$

Proof. Suppose $0 \to \mathbb{Z}/p \to A \to \mathbb{Z}/p \to 0$ is exact, then

$$\frac{A}{\mathbb{Z}/p} \cong \mathbb{Z}/p \Rightarrow |A| = |\mathbb{Z}/p|^2 = p^2 \Rightarrow A = \mathbb{Z}/p^2 \text{ or } A = \mathbb{Z}/p \times \mathbb{Z}/p$$

If $A = \mathbb{Z}/p \times \mathbb{Z}/p$, and $0 \to \mathbb{Z}/p \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} \mathbb{Z}/p \xrightarrow{\begin{pmatrix} c & d \end{pmatrix}} \mathbb{Z}/p \to 0$ is exact, then $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} c \\ d \end{pmatrix}$ are perpendicular in $\mathbb{F}_p^2$, thus $\begin{pmatrix} c \\ d \end{pmatrix} = \lambda \begin{pmatrix} -b \\ a \end{pmatrix}$ for some $\lambda \in \mathbb{F}_p^\times$, and $\begin{pmatrix} a \\ b \end{pmatrix}$ can be any nonzero vector in $\mathbb{F}_p^2$. We have $\lambda(av - bu) = 1$ for some u, v such that

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] \arrow[d, equal] & \mathbb{Z}/p \times \mathbb{Z}/p \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] \arrow[d, "\cong", "\begin{pmatrix} a & u \\ b & v \end{pmatrix}"'] & \mathbb{Z}/p \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "\begin{pmatrix} a \\ b \end{pmatrix}"'] & \mathbb{Z}/p \times \mathbb{Z}/p \arrow[r, "\begin{pmatrix} -\lambda b & \lambda a \end{pmatrix}"'] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

If $A = \mathbb{Z}/p^2$, $0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{i} \mathbb{Z}/p \to 0$, $i = 1, 2, \cdots, p-1$ are all the possible exact sequences. Suppose the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "p"] \arrow[d] & \mathbb{Z}/p^2 \arrow[r, "i"] \arrow[d, "k"] & \mathbb{Z}/p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, "p"'] & \mathbb{Z}/p^2 \arrow[r, "j"'] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

$kp \equiv p \bmod p^2 \Leftrightarrow p(k-1) \equiv 0 \bmod p^2 \Rightarrow k \equiv 1 \bmod p \Rightarrow jk \equiv i \bmod p$, thus $0 \to \mathbb{Z}/p \xrightarrow{p} \mathbb{Z}/p^2 \xrightarrow{i} \mathbb{Z}/p \to 0$, $i = 1, 2, \cdots, p-1$ are nonequivalent □

Exercise 3.5.1. Let $\{A_i\}$ be a tower in which the maps $A_{i+1} \to A_i$ are inclusions. We may regard $A = A_0$ as a topological group in which the sets $a + A_i (a \in A, i \geq 0)$ are open sets. Show that $\varprojlim A_i = \bigcap A_i$ is zero iff A is Hausdorff. Then show that $\varprojlim^1 A_i = 0$ iff A is complete in the sense that every Cauchy sequence has a limit, not necessarily unique. Hint: Show that A is Hausdorff and complete iff $A \cong \varprojlim (A/A_i)$

Proof. Consider $\varprojlim(A/A_i) \cong \{(\cdots, a_i, \cdots) \in \prod A/A_i | a_j \equiv a_i \bmod A_i, \forall j \geq i\}$, $A \xrightarrow{\phi} \varprojlim(A/A_i)$, $a \mapsto (\cdots, a \bmod A_i, \cdots)$. Let's show ϕ is injective iff A is Hausdorff, ϕ is surjective iff A is complete

Note that ϕ is injective $\Leftrightarrow \varprojlim A_i = \bigcap A_i = 0$. If $\bigcap A_i = 0$, then for any $a \neq b \in A$, $a - b \in A_{i-1} \setminus A_i$ for some i, then $a + A_i \cap b + A_i = \varnothing$, otherwise $a + c = b + d$ for some $c, d \in A_i$, but then $a - b = d - c \in A_i$ which is a contradiction, hence A is Hausdorff. If A is Hausdorff, for any $a \neq 0$, $a \notin 0 + A_i = A_i$ for some i, thus $\bigcap A_i = 0$

If ϕ is surjective, suppose $\{a_n\}$ converges, then there is a subsequence $\{a_{n_i}\}$ such that $a_{n_j} - a_{n_i} \in A_i$ for any $j \geq i$, i.e. $(\cdots, a_{n_i}, \cdots) \in \varprojlim(A/A_i)$, then there exists $a \in A$ such that $a \equiv a_{n_i} \bmod A_i$.

If A is complete, for any $(\cdots, a_i, \cdots) \in \varprojlim(A/A_i)$, $\{a_i\}$ is a Cauchy sequence since $a_j - a_i \in A_i$ for any $j \geq i$, there exists a limit $a \in A$ such that $a - a_i \in A_i$, thus a is a preimage

Use snake lemma on the following commutative diagram

$$\begin{tikzcd}
 & & A \arrow[r] \arrow[d] & \varprojlim(A/A_i) \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & \prod A_i \arrow[r] \arrow[d] & \prod A \arrow[r] \arrow[d] & \prod A/A_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod A_i \arrow[r] \arrow[d] & \prod A \arrow[r] & \prod A/A_i \arrow[r] & 0 \\
 & \varprojlim{}^1 A_i & & &
\end{tikzcd}$$

Thus $\varprojlim{}^1 A_i = 0$ iff ϕ is surjective $\square$

Exercise 3.5.2. Show that $\varprojlim{}^1 A_i = 0$ if $\{A_i\}$ is a tower of finite abelian groups, or a tower of finite dimensional vector spaces over a field

Proof. If $\{A_i\}$ is a tower of finite abelian groups, or a tower of finite dimensional vector spaces over a field, then $A_i = 0$ for i big enough, hence $\{A_i\}$ satisfies trivial Mittag-Leffler condition, thus $\varprojlim{}^1 A_i = 0$ $\square$

Exercise 3.5.4. Let C be a second quadrant double complex with exact rows, and let B^h_{pq} be the image of $d^h : C_{pq} \to C_{p-1,q}$. Show that $H_{p+q}Tot(T_{-p}C) \cong H_q(B^h_{p*}, d^v)$. Then let $b = d^h(a)$ be an element of B^h_{pq} representing a cycle ξ in $H_{p+q}Tot(T_{-p}C)$ and show that the image of ξ in $H_{p+q}Tot(T_{-p-1}C)$ is represented by $d^v(a) \in B^h_{p+1,q-1}$. This provides an effective method for calculating $H_*Tot(C)$

Proof. Since $d^v x_{p,q} + d^h x_{p+1,q-1} = 0 \Rightarrow d^v d^h x_{p,q} = -d^h d^v x_{p,q} = (d^h)^2 x_{p+1,q-1} = 0$, $C_{p,q} \to B^h_{p,q}$ defines a map $Z_{p+q}Tot(T_{-p}C) \xrightarrow{\phi_q} Z_q(B^h_{p*}, d^v)$. For any $d^v d^h x_{p,q} = 0$, by diagram chasing, we can find element in $Z_{p+q}Tot(T_{-p}C)$ with (p,q) entry $x_{p,q}$, thus ϕ_q is surjective. Again by some diagram chasing, we can show that $\phi_q^{-1}(B_q(B^h_{p*}, d^v)) = B_{p+q}Tot(T_{-p}C)$. Hence $H_{p+q}Tot(T_{-p}C) \cong H_q(B^h_{p*}, d^v)$
Suppose an representative in of ξ in $Z_{p+q}Tot(T_pC)$ has a in (p,q) entry and c in $(p+1, q-1)$ entry, then the image of ξ has a representative in $Z_{p+q}Tot(T_{-p-1}C)$ having c in $(p+1, q-1)$ entry, thus the image of ξ in $H_{p+q}Tot(T_{-p-1}C)$ is represented by $d^v(a) = -d^h(c)$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 & B^h_{p,2} \arrow[l] \arrow[d] & C_{p,2} \arrow[l] \arrow[d] & C_{p+1,2} \arrow[l] \arrow[d] & C_{p+2,2} \arrow[l] \arrow[d] & {} \arrow[l] \\
0 & B^h_{p,1} \arrow[l] \arrow[d] & C_{p,1} \arrow[l] \arrow[d] & C_{p+1,1} \arrow[l] \arrow[d] & C_{p+2,1} \arrow[l] \arrow[d] & {} \arrow[l] \\
0 & B^h_{p,0} \arrow[l] \arrow[d] & C_{p,0} \arrow[l] \arrow[d] & C_{p+1,0} \arrow[l] \arrow[d] & C_{p+2,0} \arrow[l] \arrow[d] & {} \arrow[l] \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

$\square$

Exercise 3.5.5. (Pullback) Let $\to\leftarrow$ denote the poset $\{x, y, z\}$, $x < z$ and $y < z$, so that $\lim_{\to\leftarrow} A_i$ is the pullback of A_x and A_y over A_z. Show that $\lim^1_{\to\leftarrow} A_i$ is the cokernel of the difference map $A_x \times A_y \to A_z$ and that $\lim^n_{\to\leftarrow} = 0$ for $n \neq 0, 1$

Proof. Consider the construction in Vista 3.5.12

$$A_x \xrightarrow{f} A_z \xleftarrow{g} A_y$$

$C_0 = A_x \times A_y \times A_z$, $C_1 = A_z \times A_z$, $C_n = 0$ for $n \geq 2$. $d^0 = (p_x f, p_y g)$, $d^1 = (p_z, p_z)$, where p_x, p_y, p_z are projections of $A_x \times A_y \times A_z$ onto each factor, thus we have a cochain complex

$$0 \to A_x \times A_y \times A_z \xrightarrow{\begin{pmatrix} f & 0 & -1 \\ 0 & g & -1 \end{pmatrix}} A_z \times A_z \to 0$$

The image being $B_1 = \left\{ \begin{pmatrix} fa - c \\ gb - c \end{pmatrix} \middle| (a, b, c) \in A_x \times A_y \times A_z \right\}$

Consider surjection $A_z \times A_z \xrightarrow{\begin{pmatrix} 1 & -1 \end{pmatrix}} A_z$, the preimage of $\operatorname{im} d = \{fa - gb | (a, b) \in A_x \times A_y\}$ which is the image of the difference map $A_x \times A_y \xrightarrow{d} A_z$ is precisely B_1. Hence $\varprojlim^1 A_i = \operatorname{coker}(C_0 \to C_1) \cong \operatorname{coker} d$

It is clear that $\varprojlim^n = 0$ for $n \neq 0, 1$ $\square$

Exercise 5.1.1. Suppose that the double complex E consists solely of the two columns p and $p - 1$. Fix n and set $q = n - p$, so that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{pq}$. Show that we have calculated the homology of $T = Tot(E)$ up to extension in the sense that there is a short exact sequence

$$0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E^2_{pq} \to 0$$

Proof. Consider $E^1_{p-1,q+1} \to H_{p+q}T$ induced by $E_{p-1,q+1} \hookrightarrow T_{p+q}$, if $\bar{a} \mapsto 0$ with $a \in E_{p-1,q+1}$ being a representative, then $a = d''b + d'c$ for some $(b, c) \in E_{p-1,q+2} \times E_{p,q+1}$, thus $\bar{a} \in E^1_{p-1,q+1}$ is the image of $\bar{c} \in E^1_{p,q+1}$, therefore $E^2_{p-1,q+1} = \operatorname{coker}(E^1_{p-1,q+1} \to H_{p+q}T)$

Consider $H_{p+q}T \to E^1_{p,q}$ induced by $T_{p+q} \to E_{p,q}$, for any $(a, b) \in E_{p-1,q+1} \times E_{p,q}$, $d'b + d''a = 0$, thus $\bar{b}$ maps to zero in $E^1_{p-1,q}$. On the other hand, if $\bar{b} \in E^1_{p,q}$ mapsto zero in $E^1_{p-1,q}$, then $d'b = d''a$, then $(-a, b)$ is a preimage of $\bar{b}$ under $Z_{p+q}T \to E^1_{p,q}$, therefore $E^2_{p,q} \cong \ker(H_{p+q}T \to E^1_{p,q})$ $\square$

Exercise 5.2.1. (2 columns) Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $p = 0, 1$. Show that there are exact sequences

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0$$

Proof. Since $E^2_{p,q} = 0$ unless $p = 0, 1$, $E^2_{p,q} = E^\infty_{p,q}$, $F_{-1}H_n = 0$, $E^2_{0,n} = \dfrac{F_0 H_n}{F_{-1} H_n} = F_0 H_n$, $E^2_{1,n-1} = \dfrac{F_1 H_n}{F_0 H_n}$ and $F_1 H_n = H_n$, thus we have exact sequences

$$0 \to E^2_{0n} \to H_n \to E^2_{1,n-1} \to 0$$

$\square$

Exercise 5.2.2. (2 rows) Suppose that a spectral sequence converging to H_* has $E^2_{pq} = 0$ unless $q = 0, 1$. Show that there is a long exact sequence

$$\cdots H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \cdots$$

Remark 4.3. If a spectral sequence is not bounded, everything is more complicated, and there is no uniform terminology in the literature. For exacmple, a filtration in [CE] is "regular" if for each n there is an N such that $H_n(F_p C) = 0$ for $p < N$, and all filtrations are exhaustive. In [MacH] exhaustive filtrations are called "convergent above". In [EGA, $0_{\text{III}}(11.2)$] even the definition of spectral sequence is different, and "regular" spectral sequences are not only convergent but also bounded below. In what follows, we shall mostly follow the terminology of Bourbaki [BX, p175]

Proof. Since $E^2_{pq} = 0$ unless $q = 0, 1$, $E^3_{p,q} = E^\infty_{p,q}$, $E^3_{p,0} = \dfrac{F_p H_p}{F_{p-1}H_p} = \ker(E^2_{p,0} \to E^2_{p-2,1})$, $E^3_{p,1} = \dfrac{F_p H_{p+1}}{F_{p-1}H_{p+1}} = \operatorname{coker}(E^2_{p+2,0} \to E^2_{p,1})$, thus $F_{n-2}H_n = 0$, $F_n H_n = H_n$, we have exact sequences

$$0 \to \operatorname{coker}(E^2_{p+1,0} \to E^2_{p-1,1}) = \frac{F_{p-1}H_p}{F_{p-2}H_p} \to H_p \to \frac{F_p H_p}{F_{p-1}H_p} = \ker(E^2_{p,0} \to E^2_{p-2,1}) \to 0$$

Splice these together we get

$$\cdots H_{p+1} \to E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p0} \xrightarrow{d} E^2_{p-2,1} \to H_{p-1} \cdots$$

□

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Exercise 5.3.2. If $n \neq 0$, the complex projective n-space $\mathbb{C}P^n$ is a simply connected manifold of dimension $2n$. As such $H_p(\mathbb{C}P^n) = 0$ for $p > 2n$. Given that there is a fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, show that for $0 \leq p \leq 2n$

$$H_p(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

Proof. The E^2 page looks like

$$\begin{tikzcd}
H_0(\mathbb{C}P^n) & H_1(\mathbb{C}P^n) & H_2(\mathbb{C}P^n) & H_3(\mathbb{C}P^n) & H_4(\mathbb{C}P^n) & H_5(\mathbb{C}P^n) \\
H_0(\mathbb{C}P^n) & H_1(\mathbb{C}P^n) & H_2(\mathbb{C}P^n) \arrow[ull] & H_3(\mathbb{C}P^n) \arrow[ull] & H_4(\mathbb{C}P^n) \arrow[ull] & H_5(\mathbb{C}P^n) \arrow[ull]
\end{tikzcd}$$

Thus we have

$$E^\infty_{p1} = E^3_{p1} = \operatorname{coker}(H_{p+2}(\mathbb{C}P^n) \to H_p(\mathbb{C}P^n))$$

$$E^\infty_{p0} = E^3_{p0} = \begin{cases} \ker(H_p(\mathbb{C}P^n) \to H_{p-2}(\mathbb{C}P^n)) & p \geq 2 \\ 0 & p = 0, 1 \end{cases}$$

$$\bigoplus_{p=0}^{k} E^3_{p,k-p} = \bigoplus_{p=0}^{k} E^\infty_{p,k-p} = H_k(S^{2n+1}) = \begin{cases} \mathbb{Z} & k = 0, 2n+1 \\ 0 & \text{otherwise} \end{cases}$$

Since $H_0(\mathbb{C}P^n) = \mathbb{Z}$ and $H_k(S^{2n+1}) = 0$ for $k = 2, \cdots, 2n$, we know $H_1(\mathbb{C}P^n) = 0$ and $H_k(\mathbb{C}P^n) \to H_{k-2}(\mathbb{C}P^n)$ are isomorphisms for $k = 2, \cdots, 2n$, hence for $0 \leq p \leq 2n$

$$H_p(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & p \text{ even} \\ 0 & p \text{ odd} \end{cases}$$

□

Exercise 5.4.1. Recall that the completion $\widehat{C}$ is a filtered complex. Show that $C/F_{p-k}C$ and $\widehat{C}/F_{p-k}\widehat{C}$ are naturally isomorphic

Proof. Fix p, we have exact sequence

$$0 \to F_pC/F_{p-k}C \to C/F_{p-k}C \to C/F_pC \to 0$$

Since $F_pC/F_{p-k}C$ satisfies Mittag-Leffler condition, take limit we get exact sequence

$$0 \to F_p\widehat{C} \to \widehat{C} \to C/F_pC \to \varprojlim_k{}^1 F_pC/F_{p-k}C = 0$$

Thus C/F_pC is naturally isomorphic to the cokernel $\widehat{C}/F_p\widehat{C}$ □

Exercise 5.4.2. Show that the spectral sequences for C, $\bigcup F_pC$, and $C/\bigcap F_pC$ are all isomorphic

Proof. The spectral sequence of C and $\bigcup F_pC$ are isomorphic since they define the same $A^r_p = \{c \in F_pC | dc \in F_{p-r}C\}$ thus the same Z^r_p, B^r_p, E^r_p

The spectral sequence of C and $C/\bigcap F_pC$ are isomorphic since $\varprojlim_k \dfrac{C/\bigcap F_pC}{F_kC/\bigcap F_pC} \cong \varprojlim_k C/F_kC$, i.e. they have the same completion □

Exercise 5.4.3. (Shifting or Décalage) Given a filtration F on a chain complex C, define two new filtrations $\widetilde{F}$ and Dec$F$ on C by $\widetilde{F}_p C_n = F_{p-n}C_n$ and $(\mathrm{Dec}F)_p C_n = \{x \in F_{p+n}C_n | dx \in F_{p+n-1}C_{n-1}\}$. Show that the spectral sequences for these three filtrations are isomorphic after reindexing: $E^r_{pq}(F) \cong E^{r+1}_{2p+q,-p}(\widetilde{F})$ for $r \geq 0$, and $E^r_{pq}(F) \cong E^{r-1}_{-q,p+2q}(\mathrm{Dec}F)$ for $r \geq 2$

Proof. $E^r_{pq}(F) \cong E^{r+1}_{2p+q,-p}(\widetilde{F})$ for $r \geq 0$ since

$$\widetilde{A}^{r+1}_{2p+q,-p} = \left\{x \in \widetilde{F}_{2p+q}C_{p+q} \middle| dx \in \widetilde{F}_{2p+q-r-1}C_{p+q-1}\right\} = \left\{x \in F_p C_{p+q} \middle| dx \in \widetilde{F}_{p-r}C_{p+q-1}\right\} = A^r_{p,q}$$

$$\begin{aligned}
(\mathrm{Dec}A)^{r-1}_{-q,p+2q} &= \{x \in (\mathrm{Dec}F)_{-q}C_{p+q} | dx \in (\mathrm{Dec}F)_{-q-r+1}C_{p+q-1}\} \\
&= \left\{x \in F_p C_{p+q} \middle| dx \in F_{p-1}C_{p+q-1}, dx \in F_{p-r}C_{p+q-1}, 0 = d^2x \in F_{p-r-1}C_{p+q-2}\right\} \\
&= \{x \in F_p C_{p+q} | dx \in F_{p-r}C_{p+q-1}\} \\
&= A^r_{p,q}
\end{aligned}$$

□

4.8 Homework8

Exercise 5.5.1. Give an example of a complete Hausdorff filtered complex C such that the filtration on $H_0(C)$ is Hausdorff, that is, such that $\bigcap F_pH_0(C) \neq 0$

Proof. Consider $\mathbb{Z}_3 = \varprojlim_k \mathbb{Z}/3^k\mathbb{Z}$, $F_pC_n = \begin{cases} 3^{-p}\mathbb{Z}_3 & p \leq 0 \\ \mathbb{Z}_3 & p \geq 0 \end{cases}$ for $n = 0, 1$

$$0 \to \mathbb{Z}_3 \xrightarrow{\times 2} \mathbb{Z}_3 \to 0$$

is a complete Hausdorff filtered chain complex. However

$$\begin{aligned} F_p(H_0C) &= \mathrm{im}(H_0(F_pC) \to H_pC) \\ &= \mathrm{im}\left(\frac{F_pC_0}{B_p(F_pC_0)} \to \frac{C_0}{B_0C}\right) \\ &= F_pC_0 + B_0C \\ &= 3^{-p}\mathbb{Z}_3 + 2\mathbb{Z}_3 \\ &= \mathbb{Z}_3 + 2\mathbb{Z}_3 \end{aligned}$$

The last equality holds since 3^{-p} and 2 are coprime, hence $\bigcap F_pH_0(C) \neq 0$ □

Exercise 5.5.3. Suppose that the filtration on C is Hausdorff and exhaustive. If for any $p+q = n$ we have $E^r_{pq} = 0$, show that $F_pH_n(C) = F_{p-1}H_n(C)$. Conclude that $H_n(C) = \bigcap F_pH_n(C)$, provided that every E^r_{pq} with $p + q$ equalling n vanishes

Proof. Since $E^r_{pq} = 0$, $0 = E^\infty_{pq} \supseteq e^\infty_{pq} \cong F_pH_n(C)/F_{p-1}H_n(C) \Rightarrow F_pH_n(C) = F_{p-1}H_n(C)$, then

$$\cdots \subseteq F_{p-1}H_n(C) \subseteq F_pH_n(C) \subseteq \cdots \subseteq H_n(C) = \bigcup F_pH_n(C)$$

implies $H_n(C) = F_pH_n(C), \forall p$, hence $H_n(C) = \bigcap F_pH_n(C)$ □

Exercise 5.6.3. (Base-change for Ext) Let $f : R \to S$ be a ring map. Show that there is a first quadrant cohomology spectral sequence

$$E_2^{p,q} = Ext^p_S(A, Ext^q_R(S, B)) \Rightarrow Ext^{p+q}_R(A, B)$$

For every S-module A and every R-module B

Proof. Let $P_* \to A$ be an S-module projective resolution, $B \to I^*$ be an R-module injective resolution, consider the first quadrant double complex $Hom_R(P, I)$ and write $H_*(Hom_R(P, I)) = H_*(Tot^\Pi(Hom_R(P, I)))$. Since $Hom_R(P_p, -)$ is an exact functor, the p^{th} column of $Hom_R(P, I)$ is a resolution of $Hom(P_p, B)$. Therefore the first spectral sequence collapse at $'E^1 = H^v_q(Hom(P, I))$ to yield $H_*(Hom_R(P, I)) \cong H_*(Hom_R(P, B)) \cong Ext^*_R(A, B)$. Therefore the second spectral sequence converges to $Ext^*_R(A, B)$ and

$$\begin{aligned} ''E^1_{pq} &= H_q(Hom_R(P, I^p)) \\ &= H_q(Hom_S(P, Hom_R(S, I^p)) \\ &= Hom_S(P, H_q(Hom_R(S, I^p))) \\ &= Hom_S(P, Ext^q_R(S, B)) \end{aligned}$$

Hence $H_p(''E^1_{pq}) = Ext^p_S(A, Ext^q_R(S, B))$ □

Exercise 5.7.4.

1. If A is an object of $\mathcal{A}$, considered as a chain complex concentrated in degree zero, show that $\mathbb{L}_iF(A)$ is the ordinary derived functor $L_iF(A)$

2. Let $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ be a subcategory of complexes A with $A_p = 0$ for $p < 0$. Show that the functors $\mathbb{L}_iF$ restricted to $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ are the left derived functors of the right exact functor H_0F

3. (Dimension shifting) Show that $\mathbb{L}_i F(A[n]) = \mathbb{L}_{n+i} F(A)$ for all n. Here $A[n]$ is the translate of A with $A[n]_i = A_{n+i}$

Proof.

1. Suppose $P \to A$ is a projective resolution of A, then it can also be regared as a Cartan-Eilenberg resolution of A, then $\mathbb{L}_i F(A) = H_i(Tot^{\oplus}(F(P))) = H_i(F(P)) = L_i F(A)$

2. Use Grothendieck spectral sequence theorem 5.8.4, we have

$$(L_p H_0)(L_q F)(A) \Rightarrow L_{p+q}(H_0 F)(A)$$

On the other hand, by proposition 5.7.6, we have

$$(L_p H_0)(L_q F)(A) = H_p(L_q F)(A) \Rightarrow \mathbb{L}_{p+q} F(A)$$

Therefore, $\mathbb{L}_i F(A) \cong L_i(H_0 F)(A)$

3. Suppose $P \to A$ is a Cartan-Eilenberg resolution of A, then $\tilde{P} \to A[n]$ with $\tilde{P}_{ij} = P_{i+n,j}$ is also a Cartan-Eilenberg resolution, and

$$\mathbb{L}_i F(A[n]) = H_i(Tot^{\oplus}(F(\tilde{P}))) = H_{n+i}(Tot^{\oplus}(F(P))) = \mathbb{L}_{n+i} F(A)$$

□

Exercise 5.7.6. Let A be the mapping cone complex $0 \to A_1 \xrightarrow{f} A_0 \to 0$ with only two nonzero rows. Show that there is a long exact sequence

$$\cdots \to \mathbb{L}_{i+1} F(A) \to L_i F(A_1) \xrightarrow{f} L_i F(A_0) \to \mathbb{L}_i F(A) \to L_{i-1} F(A_1) \to \cdots$$

Proof. We have exact sequence

$$0 \to A_1 \to A \to A_0[-1] \to 0$$

Thus we have long exact sequence

$$\cdots \to \mathbb{L}_{i+1} F(A) \to \mathbb{L}_{i+1} F(A_0[-1]) = L_i F(A_0) \xrightarrow{f} \mathbb{L}_i F(A_1) = L_i F(A_1) \to \mathbb{L}_i F(A) \to \cdots$$

□

References

[1] *An introduction to homological algebra* - Charles Weibel

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