0.1 Rings

Definition 0.1.1 (**Rings**). R is an abelian group with addition $+$ and additive identity 0, a monoid with multiplication $\cdot$ and multiplicative identity 1, and distributive, $a \cdot (b+c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$

Definition 0.1.2. Ring R is **commutative** if $ab = ba$

Definition 0.1.3. u is a **unit** if there exists $v \in R$ such that $uv = vu = 1$. The set of units $R^\times$ is a multiplicative group

Definition 0.1.4. A **semiring** or **rig** is a ring without negatives

Definition 0.1.5. A **rng** is ring without identity

0.2 Commutative rings

Definition 0.2.1. The *determinant* of a matrix is

Definition 0.2.2. $I, J \leq R$ are ideals, the *ideal quotient* $(I : J) = \{r \in R | rJ \subseteq I\}$ is also an ideal

Definition 0.2.3. $S \subseteq R$ is *multiplictive closed*, the localization $S^{-1}R$ of R with respect to S is $R \times S/\sim$, $(r, s) \sim (r', s')$ iff there exists $t \in S$ such that $t(rs' - sr) = 0$. $S^{-1}R$ has the universal property that for any $f : R \to T$ such that maps S to units, then there exists a unique $g : S^{-1}R \to T$ such that $gi = f$

$$\begin{tikzcd} R \arrow[r, "i"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, dashed, "\exists_1 g"] \\ & T \end{tikzcd}$$

Definition 0.2.4. Given a ring R and a proper ideal I, we can define an *associated graded ring* $gr_I R := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$, if M is a left R-module, we can define *associated graded module* $gr_I M := \bigoplus_{n=0}^{\infty} I^n M/I^{n+1}M$

Definition 0.2.5. R is a *local ring* if it has a unique maximal ideal m. The *residue field* is $k = R/m$

Definition 0.2.6. R is a *semilocal ring* if it has only finitely many maximal ideal

Proposition 0.2.7. Let R be a UFD, f a prime element, then $ht(f) = 1$

Proof. Suppose there exists prime ideal P such that $0 \subsetneq P \subsetneq (f)$, then we can find a prime element in $g \in P$, thus we have $0 \subsetneq (g) \subseteq P \subsetneq (f)$, but then $g = fh$ for some h, but since f is prime, thus $(f) = (g)$ which is a contradiction, such a prime element exists since we can pick any element $0 \neq q = q_1 \cdots q_m \in P$ where q_i's are prime, but then at least one of them has to be in P $\square$

Theorem 0.2.8. Let $A \subseteq B$ be finitely generated k-algebras, and A, B are both domains, $0 \neq b \in B \Rightarrow \exists 0 \neq a \in A$ such that for any k-algebra homomorphism $\alpha : A \to k$ with $\alpha(a) \neq 0$ can be extended to k-algebra homomorphism $\beta : B \to k$ with $\beta(b) \neq 0$

Definition 0.2.9. Suppose R is a commutative ring with identity, a prime element $p \in R$ is an element which is nonzero nor a unit and $p|fg \Rightarrow p|f$ or $p|g$

Definition 0.2.10. A *graded ring* R is a ring such that $R = \bigoplus_i R_i$ is a direct sum of abelian groups and $R_i R_j \subseteq R_{i+j}$
An ideal is called a *homogeneous ideal* if it consists of only homogeneous elements

Chinese remainder theorem

Theorem 0.2.11 (Chinese remainder theorem)**.** Let R be a commutative ring, and $I_1, \cdots, I_n \leq R$ be pairwise coprime ideals, then $R \cong R/I_1 \times \cdots R/I_n, r \mapsto (r \bmod I_1, \cdots, r \bmod I_n)$

Definition 0.2.12. An *integral domain* is a commutative ring R such that (0) is a prime ideal. Equivalently, $rs \in R \Rightarrow r \in R$ or $s \in R$

Definition 0.2.13. Suppose R is a domain, K is the field of fractions, a *fractional ideal* is an R submodule $I \leq K$ such that $rI \subseteq R$ for some nonzero $r \in R$. I is invertible if $IJ = R$ for some other fractional ideal J

Definition 0.2.14. A *Dedekind domain* is an integral domain such that every proper ideal is a product of prime ideal

Definition 0.2.15. A *discrete valuation ring(DVR)* is a PID with a unique nonzero prime ideal

Definition 0.2.16. A local ring homomorphism $\phi : R \to S$ between local rings is such that $\phi(m_R) \subseteq m_S$

Definition 0.2.17. R is a commutative ring. An R-*linear category* $\mathscr{C}$ is a category enriched over R-modules, i.e. $\mathtt{Hom}(A, B)$ are R-modules, $\mathtt{Hom}(B, C) \otimes_R \mathtt{Hom}(A, B) \to \mathtt{Hom}(A, C)$ is R-bilinear

Definition 0.2.18. A unital associative R-algebra A is a monoid in the monoidal category of R-modules, coalgebras are comonoids

Definition 0.2.19. Commutative ring R is a preadditive category with a single object $\bullet$. An R-*algebra* is an additive functor $\phi \in R^{\text{RMod}}$, write $\phi(\bullet) = S$, $\phi(r)s = rs$. A ring A is an R *algebra* is a ring homomorphism $R \xrightarrow{\phi} A$, $ra = \phi(r)a$

Definition 0.2.20. A *coalgebra* is the categorical dual to a unital associative algebra

Definition 0.2.21. A is *finite* or ϕ is *finite* if A is a finitely generated R module
ϕ is of *finite type* if A is *finitely generated* R algebra

Definition 0.2.22. For $p \in \mathsf{Spec}\, A$, $q \in \mathsf{Spec}\, B$, $A \subseteq B$, p *lies under* q or q *lies over* p if $q \cap A = p$ $A \subseteq B$ satisfies *lying over property* if every $p \in \mathsf{Spec}\, A$ lies under some $q \in \mathsf{Spec}\, B$
$A \subseteq B$ satisfies the *incomparability property* if different prime ideals q, q' both lie over p are incomparable, i.e. they don't contain each other
$A \subseteq B$ satisfies *going up property* if for any chain of prime ideals $p_1 \subseteq \cdots \subseteq p_n$, $q_1 \subseteq \cdots \subseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \subseteq \cdots \subseteq q_n$ with q_i lies over p_i
$A \subseteq B$ satisfies *going down property* if for any chain of prime ideals $p_1 \supseteq \cdots \supseteq p_n$, $q_1 \supseteq \cdots \supseteq q_m$ with q_i lies over p_i and $m < n$ can be extended to a chain of prime ideals $q_1 \supseteq \cdots \supseteq q_n$ with q_i lies over p_i

Definition 0.2.23. $R \subseteq S$ are commutative rings, $a \in S$ is *integral* over R if it is a root of some monic polynomial in $R[x]$. The *integral closure* of R in S are the integral elements of S

Going up and Going down theorems

Theorem 0.2.24. B is integral over A, then $A \subseteq B$ satisfies going up property and incomparability property

Definition 0.2.25. The *height* of prime ideal p is $\operatorname{ht} p = \sup_d p_0 \subsetneqq \cdots \subsetneqq p_d = p$. The *Krull dimension* of a ring R is $\dim R = \sup_d p_0 \subsetneqq \cdots \subsetneqq p_d = \sup_p \operatorname{ht} p$, p_i are prime ideals

Theorem 0.2.26 (Krull's height theorem)**.** R is Noetherian, I is an ideal which can be generated by n elements, then the minimal prime over I is of height at most n

Proposition 0.2.27. A is a integral domain, finitely generated over some subfield k, then $\dim A = \mathtt{trdeg}(\mathrm{Frac} A/k)$

Definition 0.2.28. A *finitely presented algebra* over R is of the form $R[x_1, \cdots, x_n]/I$, I is a finitely generated ideal

Definition 0.2.29. $W(R) = 1 + tR[[t]]$ is the ring of *Witt vectors*, $1 - t$ is the multiplicative identity. Formally every element can be factored as

$$f(t) = \prod_{i=1}^{\infty}(1 - r_i t^i)$$

Definition 0.2.30. A commutative ring R is a λ ring if it has λ operations λ^k satisfying

$$\lambda^0(x) = 1, \lambda^1(x) = x, \lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$$

The last condition is equivalent to homomorphism

$$\begin{aligned} \lambda_t : R &\to W(R) = 1 + tR[[t]] \\ x &\mapsto \sum \lambda^k(x)t^k \end{aligned}$$

An λ ideal is an ideal $I \leq R$ such that $\lambda^k(I) \subseteq I$ for any k A special λ ring is a λ ring such that

$$\begin{aligned} \lambda^k(1) &= 0, k > 2 \\ \lambda^k(xy) &= P_k(\lambda^1(x), \cdots, \lambda^k(x), \lambda^1(y), \cdots, \lambda^k(y)) \\ \lambda^n(\lambda^k(x)) &= P_{n,k}(\lambda^1(x), \cdots, \lambda^{nk}(x)) \end{aligned}$$

$P_n, P_{n,k}$ are defined through

$$\sum P_{n,k}(s_1(X), \cdots, s_{nk}(X))t^n = \prod_{1 \leq X_{i_1} \leq \cdots \leq X_{i_n} \leq nk} (1 + tX_{i_1} \cdots X_{i_n})$$

$$\sum P_n(s_1(X), \cdots, s_n(X), s_1(Y), \cdots, s_n(Y))t^n = \prod_{i,j=1}^{n} (1 + tX_iX_j)$$

s_i's are elementary symmetric polynomials

Example 0.2.31. A binomial ring is a $\mathbb{Q}$ algebra R with $\lambda_t = (1+t)^k$, formally $\lambda^k(x) = \binom{x}{k}$

Example 0.2.32. $K_0(R)$ is a λ ring with $\lambda^k(P) = \bigwedge^k P$

Definition 0.2.33. $R \xrightarrow{\varepsilon} \mathbb{Z}$ is the augmentation. The Adams operation ψ is defined by

$$\psi_t(x) = \sum \psi^k(x)t^k = \varepsilon(x) - t\frac{d}{dt}\log \lambda_{-t}(x)$$

Proposition 0.2.34. If R satisfies splitting principle, then ψ^k's are endomorphisms of R and $\psi^j\psi^k = \psi^{jk}$

Definition 0.2.35. Let $s = \dfrac{t}{1-t}$, then $t = \dfrac{s}{1+s}$, $R[[t]] = R[[s]]$. The γ operation is defined by

$$\gamma_t(x) = \sum \gamma^k(x)t^k = \lambda_s(x)$$

Example 0.2.36. $\gamma^k(x) = \lambda^k(x+k-1) = \binom{x+k-1}{k} = (-1)^k\binom{-x}{k}$

Definition 0.2.37. The γ dimension $\dim_\gamma x$ is the greatest integer k such that $\gamma^k(x - \varepsilon(x)) \neq 0$, $\dim_\gamma R = \sup_x \dim_\gamma x$. The γ filtration is

$$R = F_\gamma^0 R \supseteq F_\gamma^1 R \supseteq \cdots$$

Here $F_\gamma^1 R = \ker \varepsilon$, $F_\gamma^k R$ is the ideal generated by products $\gamma^{i_1}(x)\cdots\gamma^{i_m}(x)$ whereas $\sum i_j \geq k$, $x_j \in F_\gamma^1 R$

0.3 Hopf algebra

Definition 0.3.1. Topological space X is an H-*space* if there is a continuous map $\mu : X \times X \to X$ and an identity element e such that $\mu(x, e) = \mu(e, x) = e$

Definition 0.3.2. A *Hopf algebra* H is a bialgebra with an *antipode* $S : H \to H$ such that the following diagram commutes

$$\begin{tikzcd}
H \otimes H \arrow[rr, "S\otimes 1"] & & H \otimes H \arrow[d, "\mu"] \\
H \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\epsilon"] & I \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "1\otimes S"] & & H \otimes H \arrow[u, "\mu"']
\end{tikzcd}$$