0.1 Cluster algebra

ZP is a UFD

Lemma 0.1.1. $\mathbb{P}$ is a torsion free abelian group written multiplicatively, then the group ring $\mathbb{Z}\mathbb{P}$ is a UFD

Proof. Finitely generated torsion free abelian groups are free □

Definition 0.1.2 (Exchange pattern)**.** $I = \{1, \cdots, n\}$, $\mathbb{T}_n$ is the regular n tree, the coefficient group $\mathbb{P}$ is a torsion free abelian group under multiplication, thus the group ring $\mathbb{Z}\mathbb{P}$ is a domain. Cluster variables are $\mathbf{x}(t) = \{x_i(t)\}_{i \in I}$ for $t \in \mathbb{T}_n$ such that for $\neq j$ and $t \overset{j}{\text{———}} t'$

$$x_i(t) = x_i(t')$$

$\mathcal{M} = \{M_j(t)\}$ are monomials such that

$$M_j(t)(\mathbf{x}) = p_j(t) \prod_i x_i^{b_i}, p_j(t) \in \mathbb{P}, b_i \geq 0$$

and for $t \overset{j}{\text{———}} t'$, b_i's depend on j and t

$$x_j(t) x_j(t') = M_j(t)(\mathbf{x}(t)) + M_j(t')(\mathbf{x}(t'))$$

satisfying **exchange pattern**

(E1) $x_j \nmid M_j(t)$

(E2) $x_i \mid M_j(t) \Rightarrow x_i \nmid M_j(t')$ for $t \overset{j}{\text{———}} t'$

(E3) $x_j \mid M_i(t) \Leftrightarrow x_i \mid M_j(t')$ for $t \overset{i}{\text{———}} t' \overset{j}{\text{———}} t_1$

(E4) $\dfrac{M_i(t_3)}{M_i(t_4)} = \left.\dfrac{M_i(t_2)}{M_i(t_1)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$ for $t_1 \overset{i}{\text{———}} t_2 \overset{j}{\text{———}} t_3 \overset{i}{\text{———}} t_4$, $M_0 = (M_j(t_2) + M_j(t_3))|_{x_i = 0}$

Remark 0.1.3. The substitution $x_j \leftarrow \dfrac{M_0}{x_j}$ is effectively a monomial. Since if $M_j(t_2)$ nor $M_j(t_3)$ contain x_i, then $M_i(t_2)$ nor $M_i(t_3)$ contain x_j which it substitute for nothing

$$\frac{M_i(t_2)}{M_i(t_1)} = \left.\left(\left.\frac{M_i(t_2)}{M_i(t_1)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}\right)\right|_{x_j \leftarrow \frac{M_0}{x_j}} = \left.\frac{M_i(t_3)}{M_i(t_4)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Definition 0.1.4. There is an involution between $(\mathbf{x}, \mathcal{M})$ and $(\mathbf{x}', \mathcal{M}')$ where $x'_j(t) = x_j(t')$, $M'_j(t) = M_j(t')$ for every $t \overset{j}{\text{———}} t'$

Definition 0.1.5. Suppose $J \subsetneq I$ is a subset of size m, delete sides labeled in $I - J$ in $\mathbb{T}_n$ and choose choose a connected component which would be $\mathbb{T}_m$, add to the coefficient group x_k's $k \in I - J$. This is called a restriction

Definition 0.1.6. Exchange pattern on exponents is a family of $B(t)$ such that for each $t \text{———} t'$

$$\frac{M_j(t)}{M_j(t')} = \frac{p_j(t)}{p_j(t')} \prod_i x_i^{b_{ij}(t)}$$

Thus

$$M_j(t) = p_j(t) \prod_i x_i^{[b_{ij}(t)]_+}, M_j(t') = p_j(t') \prod_i x_i^{[-b_{ij}(t)]_+}$$

Definition 0.1.7. An $n \times n$ matrix B is **sign-skew-symmetric** if $b_{ii} = 0$ and for $i \neq j$, b_{ij}, b_{ji} are both zeros or of opposite signs. B is **skew-symmetrizable** if there is a diagonal matrix D such that DB is skew symmetric, i.e. $d_i b_{ij} = -d_j b_{ji}$. Skew-symmetrizable matrices are obviously sign-skew-symmetric

Lemma on (|a|b+a|b|)/2

Lemma 0.1.8.

$$\frac{|a|b + a|b|}{2} = \begin{cases} ab & a, b > 0 \\ -ab & a, b < 0 \\ 0 & ab < 0 \end{cases} = \operatorname{sgn}(a)[ab]_+ = \operatorname{sgn}(b)[ab]_+$$

Note. $|a| = [a]_+ + [-a]_+$

Definition 0.1.9. A **mutation** on a $m \times n$ $(m > n)$ matrix B in direction k denoted by μ_k is given by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + \dfrac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} = b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

Here $\mu_k(B) = B'$. μ_k is involutive

Theorem 0.1.10. If $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \overset{k}{\text{———}} t'$, then it gives a exchange pattern

Proof. Suppose $B(t)$ is an exchange pattern, then $B(t)$ is obviously sign-skew-symmetric. For $t \overset{k}{\text{———}} t'$, we have

$$\frac{M_k(t)}{M_k(t')} = \frac{p_k(t)}{p_k(t')} \prod_i x_i^{b_{ik}}, \frac{M_k(t')}{M_k(t)} = \frac{p_k(t')}{p_k(t)} \prod_i x_i^{b'_{ik}}$$

Hence $b'_{ik} = -b_{ik}$. Consider $t_1 \overset{j}{\text{———}} t' \overset{k}{\text{———}} t \overset{j}{\text{———}} t_2$

$$\frac{M_j(t')}{M_j(t_1)} = \left.\frac{M_j(t)}{M_j(t_2)}\right|_{x_k \leftarrow \frac{M_0}{x_k}}$$

becomes

$$\frac{p_j(t')}{p_j(t_1)} \prod_i x_i^{b'_{ij}} = \left.\frac{p_j(t)}{p_j(t_2)} \prod_i x_i^{b_{ij}}\right|_{x_k \leftarrow \frac{M_0}{x_k}}$$

Where

$$M_0 = \left.\left(p_k(t) \prod_i x_i^{[b_{ik}]_+} + p_k(t') \prod_i x_i^{[-b_{ik}]_+}\right)\right|_{x_j = 0}$$

Case 1: $b_{jk} > 0 \Leftrightarrow b_{kj} < 0$, then $M_0 = p_k(t') \prod\limits_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j,k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j,k} x_i^{[-b_{ik}]_+}\right)^{b_{kj}} = \prod_{i \neq j,k} x_i^{b_{ij} + b_{kj}[-b_{ik}]_+} x_k^{-b_{kj}}$$

Case 2: $b_{jk} < 0 \Leftrightarrow b_{kj} > 0$, then $M_0 = p_k(t') \prod\limits_{i \neq j} x_i^{[-b_{ik}]_+}$, thus

$$\prod_{i \neq j} x_i^{b'_{ij}} = \prod_{i \neq j,k} x_i^{b_{ij}} \cdot \left(x_k^{-1} \prod_{i \neq j,k} x_i^{[b_{ik}]_+}\right)^{b_{kj}} = \prod_{i \neq j,k} x_i^{b_{ij} + b_{kj}[b_{ik}]_+} x_k^{-b_{kj}}$$

Case 3: $b_{jk} = 0 \Leftrightarrow b_{kj} = 0$, then
$$\prod_{i\neq j,k} x_i^{b'_{ij}} = \prod_{i\neq j,k} x_i^{b_{ij}}$$

Therefore $b'_{kj} = -b_{kj}$ and $b'_{ij} = b_{ij} + \mathtt{sgn}(b_{ik})[b_{ik}b_{kj}]_+$

Conversely, if $B(t)$ are sign-skew-symmetric and $\mu_k(B(t)) = B(t')$ for each $t \overset{k}{\text{———}} t'$, take
$$M_k(t) = \prod_i x_i^{[b_{ik}(t)]_+}, M_k(t') = \prod_i x_i^{[-b_{ik}(t)]_+}$$

for $t \overset{k}{\text{———}} t'$, then obviously $x_k \nmid M_k(t)$ since $b_{kk} = 0$ and
$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow -b_{jk} < 0 \Rightarrow x_j \nmid M_k(t')$$

For $t \overset{k}{\text{———}} t' \overset{j}{\text{———}} t_1$
$$x_j \mid M_k(t) \Leftrightarrow b_{jk} > 0 \Leftrightarrow b'_{kj} = -b_{kj} > 0 \Leftrightarrow x_k \mid M_j(t')$$

For $t_1 \overset{j}{\text{———}} t' \overset{k}{\text{———}} t \overset{j}{\text{———}} t_2$, it is the exact argument above by taking $p_j(t) \equiv 1$ □

Mutation of a skew-symmetrizable matrix preserves the skew-symmetrizing matrix

Proposition 0.1.11. Given a skew-symmetrizable matrix B, the all possible mutations $B(t)$ in $\mathbb{T}_n$ are skew-symmetrizable with the same skew-symmetrizing matrix D

Proof. True for each mutation μ_k □

Remark 0.1.12. For cluster algebra of rank $n \leq 2$, the exchange pattern is skew-symmetrizable. If $n = 1$, $B(t) \equiv 0$. If $n = 2$, $B(t_n) = (-1)^n \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$

Definition 0.1.13. Denote $2n$ tuple $\mathbf{p}(t)$ the coefficients $p_j(t), p_j(t')$ for $t \overset{j}{\text{———}} t'$. $\Sigma(t) = (\mathbf{x}(t), \mathbf{p}(t), B(t))$ is a **seed**, $\mathbf{x}(t)$ is the **cluster** of the seed. If we assume $\mathbf{x}(t_0)$ are algebraically independent ($\mathbf{x}(t_0)$ is a cluster of rank n), then so are $\mathbf{x}(t)$ since they are all mutationally equivalent. Denote the collection of all cluster variables $\mathcal{X}$, the collection of all coefficients $\mathcal{P}$, the collection of exchange matrices $\mathcal{B}$, the collection of $M_j(t)$'s $\mathcal{M}$, the collection of seeds $\mathcal{S}$
We can take $\mathcal{F} = \mathbb{ZP}(x_1, \cdots, x_n)$ to be the **ambient field**, $\mathbf{x}$ can be some cluster $\mathbf{x}(t_0)$. The **cluster algebra** is the subalgebra $\mathbb{Z}\mathcal{P}[\mathcal{X}]$

Proposition 0.1.14. Given $B(t)$ that give rise to exchange pattern, the coefficients must satisfy
$$p_i(t_1)p_i(t_3)p_i(t_3)^{[b_{ji}(t_3)]_+} = p_i(t_2)p_i(t_4)p_i(t_2)^{[b_{ji}(t_2)]_+} \tag{0.1.1}$$

Proof. For $t_1 \overset{i}{\text{———}} t_2 \overset{j}{\text{———}} t_3 \overset{i}{\text{———}} t_4$
$$\frac{p_i(t_3)}{p_i(t_4)}\prod_k x_k^{b_{ki}(t_3)} = \frac{M_i(t_3)}{M_i(t_4)} = \left.\frac{M_i(t_3)}{M_i(t_4)}\right|_{x_j \leftarrow \frac{M_0}{x_j}} = \left.\frac{p_i(t_2)}{p_i(t_1)}\prod_k x_k^{b_{ki}(t_2)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Here
$$M_0 = (M_j(t_2) + M_j(t_3))|_{x_i=0} = \left.\left(p_j(t_2)\prod_k x_k^{[b_{kj}(t_2)]_+} + p_j(t_3)\prod_k x_k^{[b_{kj}(t_3)]_+}\right)\right|_{x_i=0}$$

Take $x_k = 1$ for $k \neq j$, and use the fact that $B(t)$ are sign-skew-symmetric, we get
$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)M_0^{b_{ji}(t_2)}$$

With

Case 1: $b_{ij}(t_2) > 0$. Then $b_{ij}(t_3) < 0$, $M_0 = p_j(t_3)$ and

$$p_i(t_1)p_i(t_3)p_i(t_3)^{b_{ji}(t_3)} = p_i(t_2)p_i(t_4)$$

Case 2: $b_{ij}(t_2) < 0$. Then $b_{ij}(t_3) > 0$, $M_0 = p_j(t_2)$ and

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)p_i(t_2)^{b_{ji}(t_2)}$$

Case 3: $b_{ij}(t_2) = 0$. Then $b_{ij}(t_3) = 0$, $M_0 = p_j(t_2) + p_j(t_3)$, but $b_{ji}(t_2) = b_{ji}(t_3) = 0$, hence

$$p_i(t_1)p_i(t_3) = p_i(t_2)p_i(t_4)$$

$\square$

Note. A trivial solution of (0.1.1) is $p_j(t) = 1$

Proposition 0.1.15. The **universal coefficient group** $\mathcal{P}$ of $\mathbb{P}$ is the free abelian group generated by $p_i(t)$ modulo (0.1.1). $\mathcal{P}$ is torsion free, more precisely, it is the free abelian group generated by $p_i(t_0), p_i(t)$ for every $t_0 \overset{i}{\text{———}} t$ and exactly one of $p_i(t), p_i(t')$ for every $t \overset{i}{\text{———}} t'$ where $t, t' \neq t_0$

Definition 0.1.16. Take the field of rational functions of cluster variables $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}\mathcal{P}$ to be the ambient field $\mathcal{F}$, all other cluster variables $\mathbf{x}(t)$ are also in $\mathcal{F}$ by Theorem 0.2.3. The **universal cluster algebra** $\mathcal{A}$ is the subalgebra generated by all cluster variables with coefficients in $\mathbb{Z}\mathcal{P}$

M-equivalence

Definition 0.1.17. $t, t' \in \mathbb{T}_n$ are $\mathcal{M}$**-equivalent** if there is a permutation σ of I such that

- $x_{\sigma(i)}(t) = x_i(t')$
- $M_{\sigma(j)}(t)(\mathbf{x}(t)) = M_j(t')(\mathbf{x}(t'))$ and $M_{\sigma(j)}(t_1)(\mathbf{x}(t)) = M_j(t'_1)(\mathbf{x}(t'))$ for $t \overset{\sigma(j)}{\text{———}} t_1$ and $t' \overset{j}{\text{———}} t'_1$

0.2 Laurent phenomenon

Caterpillar lemma

Lemma 0.2.1 (Caterpillar lemma)**.** Define the caterpillar tree $\mathbb{T}_{n,m}$ consists of a spine of $m+2$ nodes, with an orientation from t_{tail} to t_{head} with t_{base} connected to t_{tail}, as illustrated in Figure 0.2.1

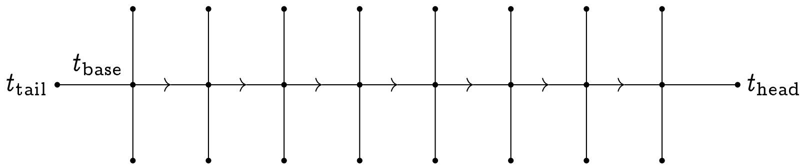

Figure 0.2.1: $\mathbb{T}_{4,8}$

T4,8

Let $\mathbb{A}$ be a UFD, exchange polynomial $P \in \mathbb{A}[x_1, \cdots, x_n]$ for each edge $t \overset{j}{\longrightarrow} t'$, denoted $x \underset{P}{\overset{j}{\longrightarrow}} t'$ satisfying the generalized exchange pattern

- P doesn't depend on x_j and x_i doesn't divide P
- For $t_0 \underset{P}{\overset{i}{\longrightarrow}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2$, P, Q_0 are coprime in $\mathbb{A}[x_1, \cdots, x_n]$, where $Q_0 = Q|_{x_i=0}$
- For $t_0 \underset{P}{\overset{i}{\longrightarrow}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2 \underset{R}{\overset{i}{\longrightarrow}} t_3$, $LQ_0^bP = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$ for some $b \geq 0$ and some Laurent monomial L with coefficients in $\mathbb{A}$ coprime with P

Cluster variables $\mathbf{x}(t) = \{x_i(t)\}$ for $t \in \mathbb{T}_{n,m}$ satisfying for each $t \underset{P}{\overset{i}{\longrightarrow}} t'$

- $x_i(t) = x_i(t')$ for any $i \neq j$
- $x_j(t)x_j(t') = P(t)(\mathbf{x}(t))$

Then $\mathbf{x}(t_{\mathsf{head}})$ are Laurent polynomials in $\mathbf{x}(t_0)$ with coefficients in $\mathbb{A}$

Proof. Write the subring of Laurent polynomials generated by $\mathbf{x}(t)$ as

$$\mathcal{L}(t) = \mathbb{A}[x_1(t)^{\pm 1}, \cdots, x_n(t)^{\pm 1}]$$

Make induction on m. If $m = 1$, consider $t_{\mathsf{tail}} = t \underset{P}{\overset{i}{\longrightarrow}} t_{\mathsf{base}} = t' \underset{Q}{\overset{j}{\longrightarrow}} t_{\mathsf{head}} = t_1$, we have for $k \neq i, j$

$$\begin{aligned} x_k(t_1) &= x_k(t') = x_k(t) \\ x_i(t_1) &= x_i(t') = \frac{P(\mathbf{x}(t))}{x_i(t)} \\ x_j(t_1) &= \frac{Q(\mathbf{x}(t'))}{x_j(t')} = \frac{Q(\mathbf{x}(t'))}{x_j(t)} \end{aligned}$$

Now suppose $m \geq 2$, let's show that $X = x_k(t_{\mathsf{head}}) \in \mathcal{L}(t_0)$, by induction, $X \in \mathcal{L}(t_1) \cap \mathcal{L}(t_3)$. Since $X, x_i(t_1) = \dfrac{P(\mathbf{x}(t_0))}{x_i(t_0)} \in \mathcal{L}(t_0)$, $X = \dfrac{f_0}{x_i(t_1)^a}$ for some $f_0 \in \mathcal{L}(t_0)$ and $a \geq 0$, similarly, $X = \dfrac{g_0}{x_j(t_2)^b x_i(t_3)^c}$ for some $g_0 \in \mathcal{L}(t_0)$ and $b, c \geq 0$, thanks to Lemma 0.2.2, $X \in \mathcal{L}(t_0)$ $\square$

Lemma for caterpillar lemma

Lemma 0.2.2. For $t_0 \underset{P}{\overset{i}{\longrightarrow}} t_1 \underset{Q}{\overset{j}{\longrightarrow}} t_2 \underset{R}{\overset{i}{\longrightarrow}} t_3$, $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3) \in \mathcal{L}(t_0)$, and

$$\mathsf{gcd}(x_i(t_1), x_i(t_3)) = \mathsf{gcd}(x_j(t_2), x_i(t_1)) = 1$$

in $\mathcal{L}(t_0)$

Note. $\mathcal{L}(t_0)$ is a UFD, $\mathcal{L}(t_0)^\times$ consists of Laurent monomials with coefficients $\mathbb{A}^\times$

Proof. Denote $x = x_i(t_0)$, $y = x_j(t_0) = x_j(t_1)$, $z = x_i(t_1) = x_i(t_2)$, $u = x_j(t_2) = x_j(t_3)$, $v = x_i(t_3)$, think of P, Q, R as functions of x_j, x_i, x_j respectively, then

$$\begin{aligned} z &= \frac{P(y)}{x} \\ u &= \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y} \\ v &= \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} + \frac{R\left(\frac{Q(0)}{y}\right)}{z} \end{aligned}$$

$$\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} = R'\left(\frac{Q_0}{y}\right)\frac{Q'(0)}{y} + \frac{1}{2}\left.R\left(\frac{Q(z)}{y}\right)''\right|_{z=0} z + \cdots \equiv R'\left(\frac{Q_0}{y}\right)\frac{Q'(0)}{y} \bmod z$$

$$\frac{R\left(\frac{Q_0}{y}\right)}{z} = \frac{L(y)Q_0(y)^b P(y)}{z} = L(y)Q_0(y)^b x$$

Thus $v \in \mathcal{L}(t_0)$
Since $\gcd(P, Q_0) = \gcd(P, L) = 1$

$$\gcd(z, v) = \gcd\left(\frac{P(y)}{x}, L(y)Q_0(y)^b x\right) = \gcd\left(P(y), L(y)Q_0(y)^b\right) = 1$$

Since $\frac{Q(z)}{y} \equiv \frac{Q_0}{y} \bmod z$

$$\gcd(z, u) = \gcd\left(z, \frac{Q_0}{y}\right) = \gcd\left(P(y), Q_0\right) = 1$$

□

Laurent phenonmenon

Theorem 0.2.3. Catepillar lemma 0.2.1 implies that in a cluster algebra, any cluster variable can be expressed as a Laurent polynomial in a given $\mathbf{x}(t_0)$ with coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$ since there is no subtraction involved

Proof. $\mathbb{T}_{n,m}$ can be embedded in $\mathbb{T}_n$. $M_j(t) + M_j(t')$ doesn't depend on x_j and not divisible by x_i for $t \overset{j}{\text{———}} t'$ and any $i \neq j$
For $t_0 \underset{P}{\overset{i}{\text{———}}} t_1 \xrightarrow[Q]{j} t_2 \underset{R}{\overset{i}{\text{———}}} t_3$, we have

$$\frac{P}{M_i(t_0)} = 1 + \frac{M_i(t_1)}{M_i(t_0)} = 1 + \left.\frac{M_i(t_2)}{M_i(t_3)}\right|_{x_j \leftarrow \frac{M_0}{x_j}} = \left.\frac{R}{M_i(t_3)}\right|_{x_j \leftarrow \frac{M_0}{x_j}}$$

Where $M_0 = \left.(M_j(t_1) + M_j(t_2))\right|_{x_i=0} = Q_0$, thus

$$\frac{R|_{x_j \leftarrow \frac{Q_0}{x_j}}}{P} = \frac{M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}}}{M_i(t_0)}$$

Note that $M_i(t_0) = p_i(t_0)\prod_k x_k^{[b_{ki}(t_0)]_+}$ and

$$\begin{aligned} M_i(t_3)|_{x_j \leftarrow \frac{Q_0}{x_j}} &= p_i(t_3)\prod_k x_k^{[b_{ki}(t_3)]_+}\Bigg|_{x_j \leftarrow \frac{Q_0}{x_j}} \\ &= p_i(t_3)\left(\frac{Q_0}{x_j}\right)^{[b_{ji}(t_3)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+} \\ &= p_i(t_3)Q_0^{[b_{ji}(t_3)]_+} x_j^{-[b_{ji}(t_3)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+} \end{aligned}$$

Hence

$$R|_{x_j \leftarrow \frac{Q_0}{x_j}} = \frac{p_i(t_3)}{p_i(t_0)} x_j^{-[b_{ji}(t_3)]_+ - [b_{ji}(t_0)]_+}\prod_{k\neq i,j} x_k^{[b_{ki}(t_3)]_+ - [b_{ki}(t_0)]_+} Q_0^{[b_{ji}(t_3)]_+} P = LQ_0^b P$$

Since the sum of two monomials P doesn't depend on x_i and is not divisible by any x_k for $k \neq i$, Q_0 is a monomial, L is a Laurent monomial, Q_0, P are coprime in $\mathbb{A}[\mathbf{x}]$ and L, P are coprime in $\mathcal{L}[\mathbf{x}]$ $\square$

0.3 Y-system

A is the Cartan matrix of root system Φ with simple system Π, denote $[\alpha : \alpha_i]$ as the coefficients of $\alpha \in \Phi$, write $\Phi_{\geq -1} = \Phi_+ \cup (-\Pi)$. Since the Coxeter graph is a tree, it is bipartite, up to renaming $I = \{1, \cdots, n\} = I_- \sqcup I_+$, $\varepsilon(i) = \varepsilon$ for $i \in I_\varepsilon$ be the indicator. Let $t_\varepsilon = \prod_{i \in I_\varepsilon} s_i$, $t = t_- t_+$ is a Coxeter element, h is the Coxeter number. $s_{\mathbf{i}_-}, s_{\mathbf{i}_+}$ are reduced words of t_-, t_+, then

$$w_\circ = \underbrace{s_{\mathbf{i}_-} s_{\mathbf{i}_+} \cdots s_{\mathbf{i}_\pm}}_{h \text{ times}} = s_{\mathbf{i}_\circ}$$

is the element of longest length

Definition 0.3.1. Suppose Φ is irreducible, then

$$[s_i(\alpha) : \alpha_k] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} [\alpha : \alpha_j] & k = i \\ [\alpha : \alpha_k] & k \neq i \end{cases}$$

Define a piecewise linear modification

$$[\sigma_i(\alpha) : \alpha_k] = \begin{cases} -[\alpha : \alpha_i] - \sum_{j \neq i} a_{ij} [\alpha : \alpha_j]_+ & k = i \\ [\alpha : \alpha_k] & k \neq i \end{cases}$$

Proposition 0.3.2.

1. σ_i are involutions

2. σ_i, σ_j commutes if i, j are not adjacent in the Coxeter graph

3. σ_i preserves $\Phi_{\geq -1}$

Proposition 0.3.3. Let $\tau_\varepsilon = \prod_{i \in I_\varepsilon} \sigma_i$

1. τ_ε are involutions that preserve $\Phi_{\geq -1}$

2. $\tau_\varepsilon \alpha = t_\varepsilon \alpha$ for $\alpha \in \mathbb{Z}_{\geq 0} \Pi$

3. $\Phi_{\geq -1} \to \Phi^\vee_{\geq -1}$, $\alpha \mapsto \alpha^\vee$ are τ_ε equivariant, i.e. $(\tau_\varepsilon \alpha)^\vee = \tau_\varepsilon \alpha^\vee$

Definition 0.3.4. A **Y-system** is a family of commuting variables $Y_i(t)$, $i \in I = \{1, \cdots, n\}$, $t \in \mathbb{Z}$ such that

$$Y_i(t+1) Y_i(t-1) = \prod_{j \neq i} (1 + Y_i(t))^{-a_{ij}} \tag{0.3.1}$$

Remark 0.3.5. (0.3.1) only really involve those $Y_j(k)$ with $\varepsilon(j) \cdot (-1)^k = \mathsf{const}$, assume $Y_j(k) = Y_j(k+1)$ for $\varepsilon(i) = (-1)^k$, then we have

$$Y_i(k+1) = \begin{cases} \dfrac{\prod_{j \neq i} (1 + Y_i(k))^{-a_{ij}}}{Y_i(k)} & \varepsilon(i) = (-1)^k + 1 \\ Y_i(k) & \varepsilon(i) = (-1)^k \end{cases}$$

Denote $\mathcal{Y}$ as the collection of all $Y_j(k)$'s, $u_i = Y_i(0)$, define

$$\tau_\varepsilon(u_i) = \begin{cases} \dfrac{\prod_{j \neq i} (1 + u_i)^{-a_{ij}}}{u_i} & \varepsilon(i) = \varepsilon \\ u_i & \text{otherwise} \end{cases}$$

Theorem 0.3.6 (Zamolodichikov)**.** $Y_i(t)$'s are $2(h+2)$ periodic, i.e. $Y_i(t+2(h+2)) = Y_i(t)$

Theorem 0.3.7. There is a unique family $\{F[\alpha]\}_{\alpha\in\Phi_{\geq -1}}$ of polynomials in u_i such that $F[-\alpha_i] = -1$ and

$$\tau_\varepsilon(F[\alpha]) = \frac{\prod\limits_{\varepsilon(i)=-\varepsilon}(u_i+1)^{[\alpha^\vee:\alpha_i^\vee]}}{\prod\limits_{\varepsilon(i)=\varepsilon} u_i^{[\alpha^\vee:\alpha_i^\vee]_+}} F[\tau_{-\varepsilon}(\alpha)]$$

Furthermore, $F[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ has constant term 1. Call $F[\alpha]$ Fibonacci polynomials
Any $\alpha \in \Phi_{\geq -1}$ can be written as $\alpha(k,i) = (\tau_-\tau_+)^k(-\alpha_i)$, denote $N[\alpha] = \prod\limits_{j\neq i} F[\alpha(-k,i)]^{-a_{ij}}$, note that $N[\alpha] \in \mathbb{Z}_{\geq 0}[\mathbf{u}]$ also has constant term 1

Theorem 0.3.8. There is a unique bijection $\Phi_{\geq} - 1 \to \mathcal{Y}$, $\alpha \mapsto Y[\alpha] = \dfrac{N[\alpha]}{\mathbf{u}^{\alpha^\vee}}$ such that $Y[-\alpha_i] = u_i$, $\tau_\varepsilon(Y[\alpha]) = Y[\tau_\varepsilon(\alpha)]$

Definition 0.3.9. A **tropical specialization** $r_{\mathtt{trop}}$ of a rational expression r is changing the addition $+$ and multiplication $\cdot$ into $\oplus$ and $\odot$ where $a \oplus b = \max(a,b)$, $a \odot b = a+b$
The **compatibility degree** for $\alpha, \beta \in \Phi_{\geq -1}$ is

$$(\alpha||\beta) = (Y[\alpha]+1)(\beta)_{\mathtt{trop}}$$

Here $(Y[\alpha]+1)(\beta)$ is evaluation at $\{u_i = [\beta : \alpha_i]\}$. α, β are **compatible** if $(\alpha||\beta) = 0$
$\Delta(\Phi)$ is a simplicial complex with $\Phi_{\geq -1}$ as vertices and mutually compatible subsets of $\Delta(\Phi)$ are simplices, the maximal simplices are **clusters**. The **exchange graph** $E(\Phi)$ is an unoriented graph with clusters as vertices and an edge between clusters which has intersection of cardinality $n-1$

Remark 0.3.10. $(||)$ is uniquely characterized by

$$(-\alpha_i||\beta) = (Y[-\alpha_i]+1)(\beta)_{\mathtt{trop}} = (u_i+1)(\beta)_{\mathtt{trop}} = [\beta : \alpha_i]_+$$
$$(\tau_\varepsilon(\alpha)||\tau_\varepsilon(\beta)) = (Y[\tau_\varepsilon(\alpha)]+1)(\tau_\varepsilon(\beta))_{\mathtt{trop}} = (\tau_\varepsilon(Y[\alpha])+1)(\tau_\varepsilon(\beta))_{\mathtt{trop}} = (\alpha||\beta)$$

Proposition 0.3.11. Consider perfect bilinear pairing

$$\mathbb{Z}\Pi^\vee \times \mathbb{Z}\Pi \to \mathbb{Z}$$
$$(\xi,\gamma) \mapsto \{\xi,\gamma\}$$

Where $\{\xi,\gamma\} = \sum \varepsilon(i)[\xi : \alpha_i^\vee][\gamma : \alpha_i]$. Then

$$(\alpha||\beta) = \max(\{\tau_+\alpha^\vee, \beta\}, \{\alpha^\vee, \tau_+\beta\}, 0) = \max(-\{\tau_-\alpha^\vee, \beta\}, -\{\alpha^\vee, \tau_-\beta\}, 0)$$

Note. $(||)$ doesn't depend on the choice of the indicator ε

Proposition 0.3.12.

1. $(\alpha||\beta) = (\beta^\vee||\alpha^\vee)$, in particular, if Φ is simply laced, then $(\alpha||\beta) = (\beta||\alpha)$

2. If $(\alpha||\beta) = 0$, then $(\beta||\alpha) = 0$

3. $J \subseteq I$, $\Phi(J) \subseteq \Phi$ is a root subsystem, $(||)$ on $\Phi(J)$ is the same as the restriction

Theorem 0.3.13. $\Delta(\Phi)$ is pure of dimension $n-1$, and each facet forms a $\mathbb{Z}$-basis for the root lattice

Theorem 0.3.14. The simplicial cones of all clusters form a complete simplicial fan

Corollary 0.3.15. The geometric realization of $\Delta(\Phi)$ is $\mathbb{S}^{n-1}$

Conjecture 0.3.16. The simplicial fan of $\Delta(\Phi)$ is the normal fan of some convex polytope $P(\Phi)$

Theorem 0.3.17. $E(\Phi)$ is a regular n tree

Example 0.3.18.

0.4 Associahedron

Definition 0.4.1. Any n regular polygon has $\binom{n}{2} - n = \frac{n(n-3)}{2}$ diagonals, with these as vertices, noncrossing subsets as simplexes, we have given it a abstract simplicial complex structure

0.5 Cluster algebra of geometric type

Semifield is multiplicative torison free

Lemma 0.5.1. Semifield $\mathbb{P}$ is multiplicative torison free

Proof. Suppose $p^m = 1$, then

$$p = \frac{p^m \oplus p^{m-1} \oplus \cdots + p}{p^{m-1} \oplus p^{m-2} \oplus \cdots + 1} = \frac{1 \oplus p^{m-1} \oplus \cdots + p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} = 1$$

□

Definition 0.5.2. Exchange pattern is **normalized** if $\mathbb{P}$ is a semifield and $p_j(t) \oplus p_j(t') = 1$ for any $t \overset{j}{\text{———}} t'$

Normalized exchange pattern determines the cluster algebra

Proposition 0.5.3. Given p_j, r_j in a semifield $\mathbb{P}$ such that $p_j \oplus r_j = 1$, and exchange matrix $B(t)$ on $\mathbb{T}_n$, define $p_j(t_0) = p_j$, $p_j(t) = r_j$ for each $t_0 \overset{i}{\text{———}} t$, this completely determines the cluster algebra

Proof. Define $u_j(t) = \dfrac{p_j(t)}{p_j(t')}$ for $t \overset{j}{\text{———}} t'$, then

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)}, p_j(t') = \frac{1}{1 \oplus u_j(t)}$$

Then (0.1.1) becomes

$$u_i(t_3)p_j(t_3)^{[b_{ji}(t_3)]_+} = u_i(t_2)p_j(t_2)^{[b_{ji}(t_2)]_+}$$

Case 1: $u_i(t_3)p_j(t_3)^{b_{ji}(t_3)} = u_i(t_2) \Rightarrow u_i(t_3) = u_i(t_2)(1 \oplus u_j(t_2))^{b_{ji}(t_2)}$

Case 2: $u_i(t_3) = u_i(t_2)p_j(t_2)^{b_{ji}(t_2)} = u_i(t_2)\left(\dfrac{u_j(t_2)}{1 \oplus u_j(t_2)}\right)^{b_{ji}(t_2)}$

Thus for $t \overset{j}{\text{———}} t'$, we have

$$u_i(t') = u_i(t)u_j(t)^{[b_{ji}(t)]_+}(1 \oplus u_j(t))^{-b_{ji}(t)}$$

□

Remark 0.5.4. $\mathbf{p}$ determines $\mathbf{u}$ which in turn determines $\mathbf{p}$
Fix semifield $\mathbb{P}$, B is skew-symmetrizable, then $(B, \mathbf{p})$ determines the cluster algebra $\mathcal{A} = \mathcal{A}(B, \mathbf{p})$ up to isomorphism

Corollary 0.5.5. The exchange graph of a normalized cluster algebra is n-regular

Definition 0.5.6. The **tropical semifield** $(\mathbb{R}, \oplus, \odot)$ is a semifield with multiplication as $\odot$, $\min$ or $\max$ as $\oplus$
The tropical semifield generated by p is the free abelian group generated multiplicatively by p with $p^a \oplus p^b = p^{\min(a,b)}$

Definition 0.5.7. A normalized cluster algebra is of geometric type if $\mathbb{P}$ is the tropical semifield generated by $\{p_i\}_{i \in I'}$ and each $p_j(t)$ is a monomial with nonnegative exponents

Remark 0.5.8. In this particular case, normality just means that for $t \overset{j}{\text{———}} t'$, $p_j(t), p_j(t')$ doesn't have a common variable, or the support doesn't intersect

Proposition 0.5.9. $\mathbb{P}$ is the tropical semifield generated by $p_i, i \in I'$, $B(t)$ is the exchange pattern of exponents, $p_j(t)$ give rise to a cluster algebra of geometric type iff $C(t)$ satisfies the exchange pattern of coefficients, i.e. $p_j(t) = \prod_i p_i^{[c_{ij}(t)]_+}$ and

$$c'_{ij} = \begin{cases} -c_{ij} & j = k \\ c_{ij} + \dfrac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2} & \text{otherwise} \end{cases}$$

Here the mutation is in direction k

Proof. Suppose $p_j(t)$ give rise to a cluster algebra of geometric type. Define $u_j(t) = \dfrac{p_j(t)}{p_j(t')} = \prod_{i \in I'} p_i^{c_{ij}(t)}$ for each $t \overset{j}{\longrightarrow} t'$, then according to Proposition 0.5.3

$$p_j(t) = \frac{u_j(t)}{1 \oplus u_j(t)} = \frac{\prod_i p_i^{c_{ij}}}{1 \oplus \prod_i p_i^{c_{ij}}} = \frac{\prod_i p_i^{c_{ij}}}{\prod_i p_i^{-[-c_{ij}]_+}} = \prod_i p_i^{[c_{ij}]_+}$$

$$1 = u_k(t)u_k(t') = \prod_i p_i^{c_{ik}+c'_{ik}} \Rightarrow c'_{ik} = -c_{ik}$$

And

$$\begin{aligned} \prod_i p_i^{c'_{ij}} &= \prod_i p_i^{c_{ij}} \left(\prod_i p_i^{c_{ik}} \right)^{[b_{kj}]_+} \left(1 \oplus \prod_i p_i^{c_{ik}} \right)^{-b_{kj}} \\ &= \prod_i p_i^{c_{ij}} \prod_i p_i^{c_{ik}[b_{kj}]_+} \prod_i p_i^{b_{kj}[-c_{ik}]_+} \\ &= \prod_i p_i^{c_{ij} + \frac{|c_{ij}|b_{jk} + c_{ij}|b_{jk}|}{2}} \end{aligned}$$

□

Remark 0.5.10. Note if we take $\tilde{B}(t) = (\tilde{b}_{ij})_{i \in I \cup I', j \in I}$ where $\tilde{b}_{ij} = b_{ij}$ for $i, j \in I$ is the principal part of $\tilde{B}$, $\tilde{b}_{ij} = c_{ij}$ for $i \in I', j \in I$

Corollary 0.5.11. Given $\tilde{B}_0$ with a skew-symmetrizable principal part B_0, then there exists a unique exchange pattern of geometric type such that $\tilde{B}(t_0) = \tilde{B}_0$ for $t_0 \in \mathbb{T}_n$

Proof. By Proposition 0.1.11 □

Remark 0.5.12. The class of exchange patterns of geometric type is stable under restriction and direct product

0.6 Rank two case

Cluster algebra of rank 2

Example 0.6.1. If $n = 2$, consider $\mathbb{T}_2$

$$\overset{1}{\text{———}}\ t_0\ \overset{2}{\text{———}}\ t_1\ \overset{1}{\text{———}}\ t_2\ \overset{2}{\text{———}}\ t_3\ \overset{1}{\text{———}}\ t_4\ \overset{2}{\text{———}}\ t_5\ \overset{1}{\text{———}}$$

The cluster variables are y_i, y_{i+1} for t_i

$$y_{2k+1} = x_1(t_{2k}) = x_1(t_{2k+1}), y_{2k} = x_2(t_{2k-1}) = x_2(t_{2k})$$

$M_2(t_0)$ and $M_2(t_1)$ don't have x_1 and can't both have x_2
If both of them don't have x_2, then $M_2(t_0), M_2(t_1) \in \mathbb{P}$, thus

$$\cdots x_2 \nmid M_1(t_{-1}) \Leftrightarrow x_1 \nmid M_2(t_0) \Leftrightarrow x_2 \nmid M_1(t_1) \Leftrightarrow x_1 \nmid M_2(t_2) \cdots$$

$$\cdots x_2 \nmid M_1(t_0) \Leftrightarrow x_1 \nmid M_2(t_1) \Leftrightarrow x_2 \nmid M_1(t_2) \Leftrightarrow x_1 \nmid M_2(t_3) \cdots$$

So is every $M_*(t_*) \in \mathbb{P}$, write q_m, r_m as the two monomials of t_{m-1} ——— t_m, then we have

$$y_{m-1}y_{m+1} = q_m + r_m$$

And for t_{m-2} ——— t_{m-1} ——— t_m ——— t_{m+1} we have

$$\frac{q_{m+1}}{r_{m+1}} = \frac{r_{m-1}}{q_{m-1}} \Leftrightarrow q_{m-1}q_{m+1} = r_{m-1}r_{m+1}$$

If $M_2(t_0) = q_1x_1^b$, $M_2(t_1) = r_1$(the other case corresponds to the involution) for some $b > 0$, then $M_1(t_1) = q_2x_2^c$, $M_1(t_2) = r_2$ for some $c > 0$, we have

$$\frac{M_2(t_2)}{M_2(t_3)} = \left.\frac{M_2(t_1)}{M_2(t_0)}\right|_{x_1 \leftarrow \frac{M_0}{x_1}} = \left.\frac{r_1}{q_1x_1^b}\right|_{x_1 \leftarrow \frac{r_2}{x_1}} = \frac{r_1x_1^b}{q_1r_2^b}$$

Since $x_1 \mid M_2(t_3) \Rightarrow x_2 \nmid M_2(t_2)$ gives a contradiction, $x_1 \nmid M_2(t_3) \Rightarrow M_2(t_3) = r_3$, thus $M_2(t_2) = q_3x_1^b$, periodically, we can conclude

$$\begin{aligned} M_2(t_{2k}) &= q_{2k+1}x_1^b \\ M_2(t_{2k+1}) &= r_{2k+1} \\ M_1(t_{2k-1}) &= q_{2k}x_2^c \\ M_1(t_{2k}) &= r_{2k} \end{aligned}$$

Therefore we have

$$\begin{aligned} y_{2k-1}y_{2k+1} &= q_{2k}y_{2k}^c + r_{2k} \\ y_{2k}y_{2k+2} &= q_{2k+1}y_{2k+1}^b + r_{2k+1} \end{aligned}$$

For $t_{2k-1}\ \overset{1}{\text{———}}\ t_{2k}\ \overset{2}{\text{———}}\ t_{2k+1}\ \overset{1}{\text{———}}\ t_{2k+2}$ we have

$$q_{2k}q_{2k+2}r_{2k+1}^c = r_{2k}r_{2k+2}$$

For $t_{2k-2}\ \overset{2}{\text{———}}\ t_{2k-1}\ \overset{1}{\text{———}}\ t_{2k}\ \overset{2}{\text{———}}\ t_{2k+1}$ we have

$$q_{2k-1}q_{2k+1}r_{2k}^b = r_{2k-1}r_{2k+1}$$

The exchange matrices are

$$B(t_m) = (-1)^m \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$$

Conversely, given such relation, we can always find a corresponding cluster algebra

In particular, consider the coordinate ring of $\mathrm{Gr}_2(5)$, let

$$\begin{aligned} y_m &= [\overline{2m-1}, \overline{2m+1}] \\ q_m &= [\overline{2m-2}, \overline{2m+2}] \\ r_m &= [\overline{2m-2}, \overline{2m-1}][\overline{2m+1}, \overline{2m+2}] \\ b &= c = 1 \end{aligned}$$

Note. If we denote $m \bmod 2$ as $\langle m \rangle$, then

$$\begin{aligned} p_{\langle m \rangle}(t_m) &= q_{m+1} \\ p_{\langle m+1 \rangle}(t_m) &= r_m \\ x_{\langle m \rangle}(t_m) &= q_m \\ x_{\langle m+1 \rangle}(t_m) &= y_{m+1} \end{aligned}$$

Example 0.6.2. As in Example 0.6.1, by Theorem 0.2.3, we know that

$$y_m = \frac{N_m(y_1, y_2)}{y_1^{d_1(m)} y_2^{d_2(m)}}$$

Where $N_m(y_1, y_2) \in \mathbb{ZP}[y_1, y_2]$ not divisible by y_1 or y_2

0.7 Cluster algebra of finite type

Definition 0.7.1. Seeds $\Sigma(t), \Sigma(t')$ are **equivalent** if t, t' are $\mathcal{M}$-equivalent, i.e. there is a permutation σ of I such that $x_{\sigma(i)(t)} = x_i(t')$, $b_{\sigma(i)\sigma(j)}(t') = b_{ij}(t)$, $p_{\sigma(j)}(t) = p_j(t')$. For geometric type, $c_{i\sigma(j)}(t) = c_{ij}(t')$, or rather, $\tilde{b}_{\sigma(i)\sigma(j)}(t') = \tilde{b}_{ij}(t)$ since σ as only a permutation of I fixes I'. By Proposition 0.5.3, if t, t' are equivalent, and $t \xrightarrow{\sigma(j)} t_1$ and $t' \xrightarrow{j} t'_1$, then t_1, t'_1 are equivalent
Cluster algebras $\mathcal{A}(\mathcal{S})$, $\mathcal{A}'(\mathcal{S}')$ are strongly isomorphic if there is a field isomorphism $\mathcal{F} \to \mathcal{F}'$ that sends seeds in $\mathcal{S}$ to seeds $\mathcal{S}'$, thus inducing bijection $\mathcal{S} \to \mathcal{S}'$ and an isomorphism $\mathcal{A} \to \mathcal{A}'$
$\mathcal{A}(B, -)$ are all the possible normalized cluster algebras. $\mathcal{A}(B), \mathcal{A}(B')$ are strongly isomorphic if there is a one-to-one correspondence between $\mathcal{A}(B, \mathbf{p})$ and $\mathcal{A}(B', \mathbf{p}')$, this is true iff B, B' are mutationally equivalent modulo relabelling rows and columns
$\mathcal{A}$ is of **finite type** if it has finitely many seeds up to equivalences

Definition 0.7.2. The **Cartan counterpart** of B is the **generalized Cartan matrix** $A(B) = (a_{ij})$, $a_{ii} = 2$, $a_{ij} = -|b_{ij}|$ for $i \neq j$, with the same symmetrizing matrix D, i.e. $d_i a_{ij} = d_j a_{ji}$

A Cartan matrix of finite type, A(B)=A, bijbik>0, p normalized, then cluster algebra is of finite type

Theorem 0.7.3. A is a Cartan matrix of finite type, there is a sign-skew symmetric $B_\circ$ such that $A(B_\circ) = A$ and $b_{ij}b_{ik} > 0$ for all i, j, k, and $\mathbf{p}_\circ$ is normalized, then $\mathcal{A}(B_\circ, \mathbf{p}_\circ)$ is of finite type. Any cluster algebra of finite type is strongly isomorphic to one such data

Remark 0.7.4. Since the Coxeter graph of A is a tree which is bipartite, thus we can certainly divide them into sinks and sources. Since $b_{ij} > 0$ would be there is a directed edge from i to j, thus we can always find such a $B_\circ$

Theorem 0.7.5. B, B' sign-skew symmetric, $\mathcal{A}(B), \mathcal{A}(B')$ iff $A(B), A(B')$ are of the same Cartan-Killing type

Theorem 0.7.6. $\mathcal{A}$ is a cluster algebra, the following are equivalent

(i) $\mathcal{A}$ is of finite type

(ii) $|\mathcal{X}| < \infty$

(iii) For every seed $(\mathbf{x}, \mathbf{p}, B)$, $|b_{xy}b_{yx}| < 3$ for $x, y \in \mathbf{x}$

(iv) $\mathcal{A} = \mathcal{A}(B_\circ, p_\circ)$ as in Theorem 0.7.3

Theorem 0.7.7. $\mathcal{A}(B)$ consists of cluster algebras all simultaneously of finite type or of infinite type. There is a bijective correspondence between generalized Cartan matrices of finite type and strong isomorphic classes of normalized cluster algebras, through $B \to A(B)$

Bijection between almost positive roots and X

Theorem 0.7.8. There is a unique bijection $\Phi_{\geq -1} \to \mathcal{X}$, $\alpha \mapsto x[\alpha] = \dfrac{P_\alpha(\mathbf{x}_\circ)}{\mathbf{x}^\alpha}$, $P_\alpha \in \mathbb{Z}_{\geq 0}\mathcal{P}$ with nonzero constant term such that $X[-\alpha_i] = x_i$

Theorem 0.7.9. Every seed $(\mathbf{x}, \mathbf{p}, B)$ in $\mathcal{A}$ is uniquely determined by the cluster $\mathbf{x}$, and for any $x \in \mathbf{x}$, there is a unique cluster $\mathbf{x}'$ such that $\mathbf{x} \cap \mathbf{x}' = \mathbf{x} - \{x\}$. The the cluster complex $\Delta(\mathcal{A})$ encodes the combinatorics of seed mutations

Theorem 0.7.10. The bijection in Theorem 0.7.8 identifies $\Delta(\mathcal{A})$ and $\Delta(\Phi)$, in particular, the cluster complex doesn't depend on $\mathbb{P}$ nor $\mathbf{p}_\circ$