Introduction to Linear Algebra

Haoran Li

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Contents

1. Lecture 1 - System of linear equations

1.1. Linear systems

Throughout this course, we adopt the following notations:

- Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, ...\}$
- Integers: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
- Rational numbers: $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\right\}$ is the set of fractions. Here $\in$ means belong to.
- Real numbers: $\mathbb{R}$ is the set of numbers on the whole real number line. It includes:
  - irrational numbers (like $\sqrt{2}$, $\sqrt[3]{3}$)
  - transcendental numbers (like π, e)
- Complex numbers: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$ is the imaginary number such that $i^2 = -1$.
- $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$
- $\mathbb{R}^n = \{(r_1, r_2, r_3, ..., r_n) \mid r_1, r_2, ..., r_n \in \mathbb{R}\}$ is the set of all n-tuples of real numbers. Geometrically:
  - $\mathbb{R}^1 = \mathbb{R}$ is a line.
  - $\mathbb{R}^2$ is a plane.
  - $\mathbb{R}^3$ is our usual physical space.

Definition 1.1.1: A linear equation in the variables $x_1, x_2, x_3, ..., x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n = b \tag{1.1}$$

where the coefficients $a_1, a_2, a_3, ..., a_n$ and b are real or complex numbers, usually known in advance.

Example 1.1.1:

- $x_1 + \frac{1}{2} x_2 = 2$, ✓
- $\pi(x_1 + x_2) - 9.9x_3 = e$, ✓. Because if we expand it, we got $\pi x_1 + \pi x_2 - 9.9x_3 = e$ in which case $a_1 = \pi, a_2 = \pi, a_3 = -9.9, b = e$ as in the form of (1.1)
- $|x_2| - 1 = 0$, ✘
- $x_1 + x_2^2 = 9$, ✘
- $\sqrt{x_1} + \sqrt{x_2} = 1$, ✘

Definition 1.1.2: A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables, say $x_1, x_2, x_3, ..., x_n$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{1.2}$$

Example 1.1.2: For $n = m = 2$, (1.2) is just

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{1.3}$$

Example 1.1.3: (The Nine Chapters on the Mathematical Art). In a cage full of chickens and rabbits. The total number of heads is 10 and the total number of legs is 26. Calculate the number of chickens and rabbits.

Solution: Let's assume the number of chickens and rabbits are x_1 and x_2, then we can write down a linear system

$$\begin{cases} x_1 + x_2 = 10 \\ 2x_1 + 4x_2 = 26 \end{cases} \tag{1.4}$$

Let's solve this linear system
step 1. Replace `Row2` by `Row2` − `2Row1`, we get

$$\begin{cases} x_1 + 2x_2 = 20 \\ 2x_2 = 6 \end{cases} \tag{1.5}$$

step 2. Divide `Row2` by 2, we get

$$\begin{cases} x_1 + 2x_2 = 20 \\ x_2 = 3 \end{cases} \tag{1.6}$$

step 3. Replace `Row1` by `Row1` − `2Row 2`, we have the solution

$$\begin{cases} x_1 = 14 \\ x_2 = 3 \end{cases} \tag{1.7}$$

Remark: This process is call the Gaussian elimination

Definition 1.1.3: A solution of the linear system (1.2) is

$$\begin{cases} x_1 = s_1 \\ x_2 = s_2 \\ x_3 = s_3 \\ \ldots \\ x_n = s_n \end{cases} \tag{1.8}$$

where $s_1, s_2, s_3, \ldots, s_n$ are numbers that make (1.2) true. The set of all possible solutions is called the solution set of the linear system. To solve a linear system is to find all its solutions.

1.2. Geometric interpretation

Example 1.2.1:

$$\begin{cases} x_1 + x_2 = 10 \\ 2x_1 + 4x_2 = 26 \end{cases} \tag{1.9}$$

describes two lines in $\mathbb{R}^2$, and the solution set is the intersection.

Question: How many solutions does the following linear system have?

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{1.10}$$

Answer: It may have
- a *unique* solution if these two lines *intersect*.
- (uncountably) *infinitely many* solutions if these two lines *overlap*.
- *no* solutions if these two lines are *parallel* but not overlapping.

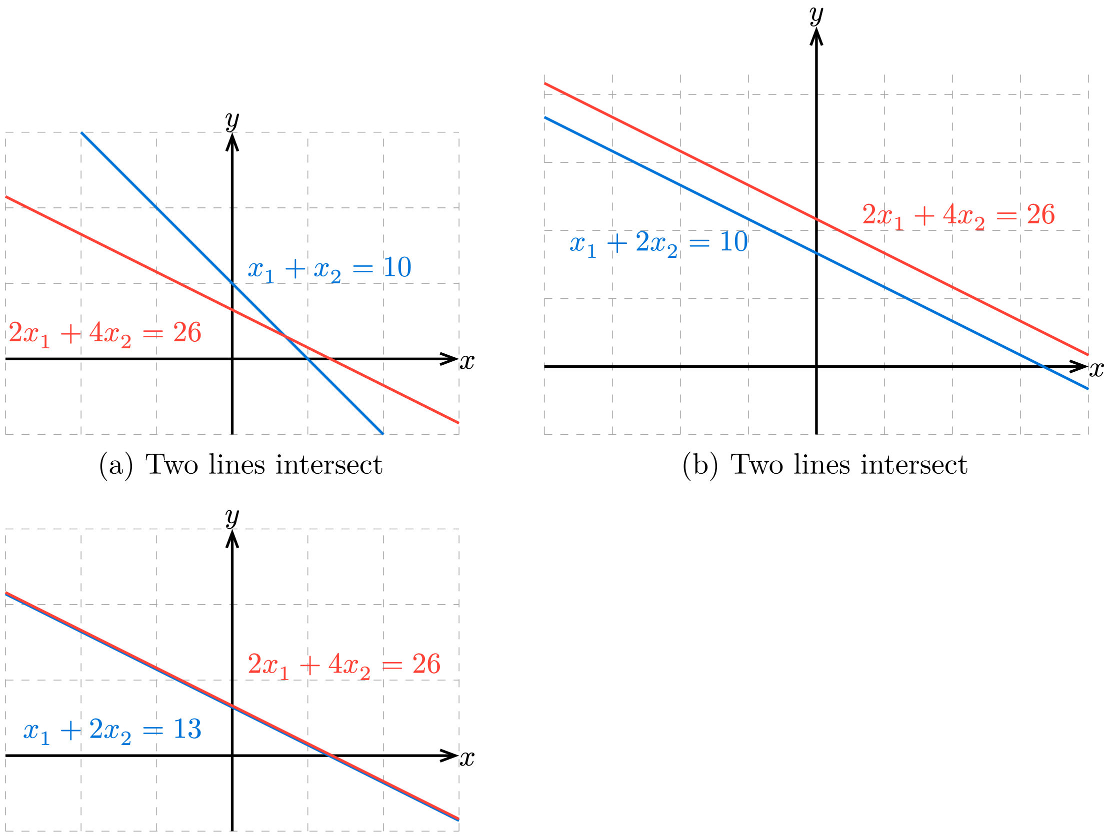
Example 1.2.2: Compare the following three linear systems

$$\begin{cases} x_1 + x_2 = 10 \\ 2x_1 + 4x_2 = 26 \end{cases} \quad (1.11) \qquad \begin{cases} x_1 + 2x_2 = 10 \\ 2x_1 + 4x_2 = 26 \end{cases} \quad (1.12) \qquad \begin{cases} x_1 + 2x_2 = 13 \\ 2x_1 + 4x_2 = 26 \end{cases} \quad (1.13)$$

- (1.11) has a unique solution

$$\begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases} \tag{1.14}$$

- (1.12) has no solutions since the 1st equation contradicts the 2nd.
- (1.13) has infinitely many solutions since the 2nd equation is twice of the 1st.



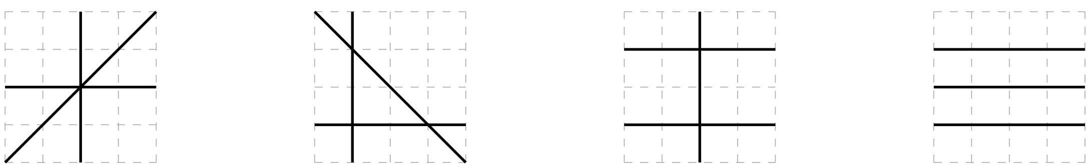
(a) Two lines intersect

(b) Two lines intersect

(c) Two lines intersect

Figure 1: Two lines in a plane

If we increase the number of equations, we get more lines, it might looks like



If we increase the number of variables, we get
- $a_1x_1 + a_2x_2 + a_3x_3 = b$ describes a plane in $\mathbb{R}^3$.
- $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$ describes a *hyperplane* in $\mathbb{R}^n$.
- Therefore the solution set of (1.2) is the intersection of m hyperplanes.

Example 1.2.3: Geometric interpretation of

$$\begin{cases} x_1 - 3x_2 + 2x_3 = 0 \\ -5x_1 + 12x_2 - x_3 = 0 \end{cases} \tag{1.15}$$

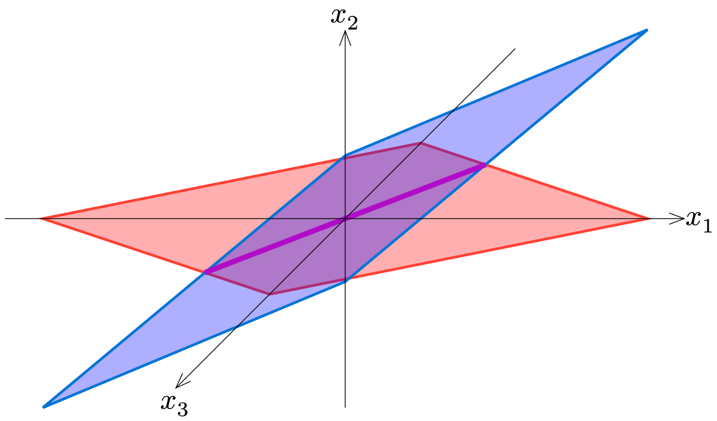


Figure 3: Two planes intersect

Remark: It is geometrically clear that for a system of 2 equations in 3 variables, there are either no solutions or infinitely many, since two planes either intersects at a line, or overlap, or simply parallel.

Definition 1.2.1: We say a linear system is consistent if it has solution(s), and inconsistent if it has none.

Exercise 1.2.1:

1. Try Gaussian elimination on the following linear systems
   - $$\begin{cases} x_1 + 5x_2 = 7 \\ -2x_1 - 7x_2 = -5 \end{cases} \tag{1.16}$$
   - $$\begin{cases} 2x_1 + 4x_2 = -4 \\ 5x_1 + 7x_2 = 11 \end{cases} \tag{1.17}$$
   - $$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \end{cases} \tag{1.18}$$
2. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.
3. For what values of h and k is the following system consistent?

$$\begin{cases} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{cases} \tag{1.19}$$

2. Lecture 2 - Matrices and row echelon form

2.1. Matrices

Definition 2.1.1: A m by n (or $m \times n$) matrix is a rectangular array of numbers with m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{2.20}$$

We use the (i, j)-th entry to mean the entry on the i-th row and j-column (i.e. a_{ij}).

Definition 2.1.2: A matrix is

- a zero matrix is a matrix with all entries zeros.

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.21}$$

- a square matrix is a matrix with the same number of rows and columns, i.e. $m = n$.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2.22}$$

- a vector if it only has one column, i.e. $n = 1$.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \tag{2.23}$$

- a row vector if it only has one row, i.e. $m = 1$.

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \tag{2.24}$$

- the identity matrix if it is a square matrix with diagonal elements $1'$s, and $0'$s otherwise. Here the diagonal are the (i, i)-th entries.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2.25}$$

Definition 2.1.3: Soon we will be getting tired of writing all these equations in the linear system (2.2), instead we write down its augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{bmatrix} \tag{2.26}$$

Which is obtained by omitting x_i's, pluses, and equal signs. If we delete the last column, we will get the coefficient matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{2.27}$$

Example 2.1.1:

- For (2.4), its augmented matrix and coefficient matrix are

$$\begin{bmatrix} 1 & 1 & 10 \\ 2 & 4 & 26 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \tag{2.28}$$

- For

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \end{cases} \tag{2.29}$$

  , its augmented matrix and coefficient matrix are

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \tag{2.30}$$

- In general, a linear system of m equations in n variables has a m by $(n+1)$ augmented matrix and a m by n coefficient matrix.

Definition 2.1.4: Inspired by Gaussian elimination, we define the following three elementary row operations

- Replacement: Replace one row by the sum of itself and a multiple of another row.
- Interchangement: Interchange two rows.
- Scaling: Multiply all entries in a row by a *nonzero* constant.

We say matrices A, B are row equivalent ($A \sim B$) if B can obtained by applying a sequence of elementary row operations to A (or vise versa).

Example 2.1.2: Let's rewrite the process in Example Example 1.1.3

$$\begin{bmatrix} 1 & 1 & 10 \\ 2 & 4 & 26 \end{bmatrix} \xrightarrow{\text{R2}\to\text{R2}-2\text{R1}} \begin{bmatrix} 1 & 1 & 10 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow{\text{R2}\to\frac{\text{R2}}{2}} \begin{bmatrix} 1 & 1 & 10 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{R1}\to\text{R1}-\text{R2}} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \end{bmatrix} \tag{2.31}$$

Example 2.1.3: Solve

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \end{cases} \tag{2.32}$$

with augmented matrix.

$$\begin{aligned} &\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{R2}\to\text{R2}-2\text{R1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \xrightarrow{\text{R3}\to\text{R3}-\text{R1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \\ &\xrightarrow{\text{R3}\to\text{R3}-\text{R2}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \xrightarrow{\text{R3}\to\frac{\text{R3}}{3}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{R2}\to\text{R2}+3\text{R3}]{\text{R1}\to\text{R1}-\text{R3}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &\xrightarrow{\text{R2}\to\frac{\text{R2}}{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{R1}\to\text{R1}+\text{R2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned} \tag{2.33}$$

This gives the unique solution

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{cases} \tag{2.34}$$

2.2. Row echelon form

Definition 2.2.1:
- A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row).
- A matrix is of row echelon form (REF) if it is of a "staircase shape".

$$\overset{\text{REF}}{\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}} \tag{2.35}$$

$\blacksquare$ are the leading entries, $*$ are some unknown numbers.
- The leading entries of an REF matrix are called pivots.
- The position of pivots are called pivot positions.
- The column pivots are in are called pivot columns.
- An REF of reduced row echelon form (RREF) if all its pivots are 1's and in each pivot column, every entry except the pivot are 0's.

$$\overset{\text{RREF}}{\begin{bmatrix} 1 & * & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}} \tag{2.36}$$

Example 2.2.1: In Example Example 2.1.3,

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \tag{2.37}$$

is not an REF.

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \tag{2.38}$$

is an REF, but not an RREF.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.39}$$

is an RREF.

Theorem 2.2.1: Every matrix is row equivalent to some REF matrix (which is not in general unique), but it is row equivalent to some unique RREF matrix.

Remark: This ensures that the pivot positions are well-defined, i.e. you won't get different pivot positions if you applied different row operations

Example 2.2.2: In Example Example 2.1.3,

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \tag{2.40}$$

is an REF that is row equivalent to the original matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \tag{2.41}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{2.42}$$

is its unique row equivalent RREF.

Remark: A linear system has a unique solution if and only if its RREF deleting the last column gives the identity matrix.

Example 2.2.3: Solve

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \\ x_1 + x_2 - 2x_3 = 1 \end{cases} \tag{2.43}$$

with augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \xrightarrow[\text{R3}\to\text{R3}-\text{R1}]{\text{R2}\to\text{R2}-2\text{R1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & -3 & 0 \end{bmatrix} \xrightarrow{\text{R3}\to\text{R3}-\text{R2}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.44}$$

You might notice that the last row represents $0x_1 + 0x_2 + 0x_3 = 1$, this is a contradiction, therefore the linear system is inconsistent.

Remark: This only happens if and only if the last pivot column is the last column

Example 2.2.4:

$$\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \end{cases} \tag{2.45}$$

we write down its augmented matrix

$$\begin{aligned} &\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{R2}\to\text{R2}-2\text{R1}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \end{bmatrix} \\ &\xrightarrow{\frac{\text{R2}}{2}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{\text{R1}\to\text{R1}+\text{R2}} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned} \tag{2.46}$$

This gives the solution set

$$\begin{cases} x_1 - \frac{1}{2}x_3 = \frac{1}{2} \\ x_2 - \frac{3}{2}x_3 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}x_3 + \frac{1}{2} \\ x_2 = \frac{3}{2}x_3 - \frac{1}{2} \end{cases} \tag{2.47}$$

Let's formalize these as row reduction algorithm

step 1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position should be at the top.

step 2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

step 3. Use row replacement operations to create zeros in all positions below the pivot.

step 4. Cover (or ignore) the rows containing the pivot positions. Apply Steps 1-3 to the rows that remains. Repeat the process until you are left with an REF.

step 5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Steps 1-4 are call forward phase, after which you get an REF. Step 5 is called backward phase, after which you get the RREF.

Example 2.2.5: Consider the augmented matrix

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \tag{2.48}$$

- Forward phase

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \xrightarrow[]{\substack{\text{R1}\leftrightarrow\text{R4} \\ \text{Step 1,2}}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \tag{2.49}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[]{\substack{\text{R2}\to\text{R2+R1} \\ \text{R3}\to\text{R3+2R1} \\ \text{Step 3}}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \tag{2.50}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \xrightarrow[]{\substack{\text{R3}\to\text{R3}-\frac{5}{2}\text{R2} \\ \text{R4}\to\text{R4}+\frac{3}{2}\text{R2} \\ \text{Step 4,1,2,3}}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \tag{2.51}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix} \xrightarrow[]{\substack{\text{R3}\leftrightarrow\text{R4} \\ \text{Step 4,1}}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.52}$$

- Backward phase

$$\begin{gathered} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[]{\substack{\text{R3}\to\frac{\text{R3}}{-5} \\ \text{Step 5}}} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[]{\substack{\text{R1}\to\text{R1+9R3} \\ \text{R2}\to\text{R2+6R3} \\ \text{Step 5}}} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 2 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \xrightarrow[]{\substack{\text{R2}\to\frac{\text{R2}}{2} \\ \text{Step 5}}} \begin{bmatrix} 1 & 4 & 5 & 0 & -7 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[]{\substack{\text{R1}\to\text{R1}-\text{4R2} \\ \text{Step 5}}} \begin{bmatrix} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{gathered} \tag{2.53}$$

Definition 2.2.2: The variables corresponding to pivot columns in a matrix are called basic variables, the other variables are called free variables. In a solution set, basic variables are expressed in terms of free variables, and a free variable can take any value.

Example 2.2.6: In Example Example 2.2.4, x_1, x_2 are basic variables and x_3 is a free variable. And we formally write our solution set as

$$\begin{cases} x_1 = \frac{1}{2}x_3 + \frac{1}{2} \\ x_2 = \frac{3}{2}x_3 - \frac{1}{2} \\ x_3 \text{ is free} \end{cases} \tag{2.54}$$

Exercise 2.2.1: Find the general solution of the system

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 4x_3 + 8x_4 = 2 \end{cases} \tag{2.55}$$

Solution:

$$\begin{gathered} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -4 & 8 & 2 \end{bmatrix} \xrightarrow[\text{R3}\to\text{R3}-3\text{R1}]{\text{R2}\to\text{R2}+2\text{R1}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{R3}\to(-1)\cdot\text{R3}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \\ \xrightarrow{\text{R2}\to\text{R2}-3\text{R3}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & -2 & 9 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{\text{R2}\leftrightarrow\text{R3}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 9 \end{bmatrix} \\ \xrightarrow{\text{R3}\to\frac{\text{R3}}{-}2} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xrightarrow[\text{R1}\to\text{R1}-3\text{R3}]{\text{R2}\to\text{R2}-\text{R3}} \begin{bmatrix} 1 & -2 & -1 & 0 & \frac{27}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xrightarrow{\text{R1}\to\text{R1}+\text{R2}} \begin{bmatrix} 1 & -2 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \end{gathered} \tag{2.56}$$

Write this as solution set, we get

$$\begin{cases} x_1 - 2x_2 = 16 \\ x_3 = \frac{5}{2} \\ x_4 = -\frac{9}{2} \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 + 16 \\ x_2 \ \text{ is free} \\ x_3 = \frac{5}{2} \\ x_4 = -\frac{9}{2} \end{cases} \tag{2.57}$$

Theorem 2.2.2: Suppose the augmented matrix of a linear system is $[A \ \ \boldsymbol{b}]$, and its RREF is $[U \ \ \boldsymbol{d}]$, then the linear system has

1. no solutions $\Leftrightarrow$ $\boldsymbol{d}$ is a pivot column, i.e. contains a pivot.
2. has solutions $\Leftrightarrow$ $\boldsymbol{d}$ is not a pivot column
   - a unique solution $\Leftrightarrow$ every column of U is a pivot column.
   - infinitely many solutions $\Leftrightarrow$ some columns of U is not a pivot column.

Example 2.2.7:

- In Example Example 2.2.3, the linear system has no solutions since

$$[A\ \ \boldsymbol{b}] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [U\ \ \boldsymbol{d}] \tag{2.58}$$

- In Example Example 2.1.3, the linear system has a unique solution since

$$[A\ \ \boldsymbol{b}] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [U\ \ \boldsymbol{d}] \tag{2.59}$$

- In Example Exercise 2.2.1, the linear system has infinitely many solutions since

$$[A\ \ \boldsymbol{b}] = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -4 & 8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} = [U\ \ \boldsymbol{d}] \tag{2.60}$$

Exercise 2.2.2: Find the general solutions of the system with given augmented matrix, name the pivot columns, pivot positions, basic and free variables.

-
$$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{bmatrix} \tag{2.61}$$

-
$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \tag{2.62}$$

Question: How does the size of the augmented matrix affect the solution set?

3. Lecture 3 - Matrix algebra

3.1. Matrix addition and scalar multiplication

Definition 3.1.1: Let's use $M_{m\times n}(\mathbb{R})$ to denote the set of all (real-valued) m by n matrices.

Definition 3.1.2: Suppose A, B are $m \times n$ matrices, c is a scalar (i.e. a number), then we can define

- Addition

$$\begin{aligned} &\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\ = &\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \end{aligned} \tag{3.63}$$

- Scalar multiplication

$$c\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix} \tag{3.64}$$

Example 3.1.1:

1.
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \tag{3.65}$$

2.
$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \tag{3.66}$$

3.2. Matrix multiplication

Definition 3.2.1: Suppose A is a $m \times n$ matrix, and B is a $n \times p$ matrix, we can define matrix multiplication AB to be the $m \times p$ matrix, computed via the row-column rule: The (i, j)-entry is to multiply the i-row and j-th column

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} & & \\ & \blacksquare & \\ & & \end{bmatrix} \tag{3.67}$$

Where the (i, j)-entry $\blacksquare = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

If A is a square matrix, then we could define matrix power A^k to be simply $\overbrace{A \cdots A}^{k \text{ times}}$

Example 3.2.1:

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{\{31\}} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{\{31\}} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{\{31\}} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix} \tag{3.68}$$

- $$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 + 2\cdot 1 + 2\cdot 2 & 1\cdot 1 + 2\cdot 2 + 2\cdot 3 \\ 2\cdot 3 + 1\cdot 1 + 1\cdot 2 & 2\cdot 1 + 1\cdot 2 + 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 9 & 7 \end{bmatrix} \tag{3.69}$$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_1 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_1 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \tag{3.70}$$

- $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 1 + 1\cdot 0 & 0\cdot 0 + 1\cdot 0 \\ 0\cdot 1 + 0\cdot 0 & 0\cdot 0 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.71}$$

- $$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 0\cdot 0 & 1\cdot 1 + 0\cdot 0 \\ 0\cdot 0 + 0\cdot 0 & 0\cdot 1 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.72}$$

item

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.73}$$

item

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.74}$$

Example 3.2.2:

1.
$$\begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \begin{bmatrix}a_{11}-3a_{31} & a_{12}-3a_{32} & a_{13}-3a_{33}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{3.75}$$

2.
$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix} = \begin{bmatrix}a_{11}+2a_{13} & a_{12} & a_{13}\\ a_{21}+2a_{23} & a_{22} & a_{23}\\ a_{31}+2a_{33} & a_{32} & a_{33}\end{bmatrix} \tag{3.76}$$

3.
$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21} & 2a_{22} & 2a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{3.77}$$

4.
$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}a_{11} & 3a_{12} & a_{13}\\ a_{21} & 3a_{22} & a_{23}\\ a_{31} & 3a_{32} & a_{33}\end{bmatrix} \tag{3.78}$$

5.
$$\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \begin{bmatrix}a_{31} & a_{32} & a_{33}\\ a_{21} & a_{22} & a_{23}\\ a_{11} & a_{12} & a_{13}\end{bmatrix} \tag{3.79}$$

6.
$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}a_{12} & a_{11} & a_{13}\\ a_{22} & a_{21} & a_{23}\\ a_{32} & a_{31} & a_{33}\end{bmatrix} \tag{3.80}$$

Exercise 3.2.1: Suppose $A = \begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}, B = \begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}$, computes matrix multiplication AB

Fact: label{fact:mat-alg} Suppose A, B, C, D are matrices, c is a scalar, 0 is the zero matrix, I is the identity matrix. we have the following facts

- Matrix multiplication is generally *NOT commutative*, i.e. $AB \neq BA$
- Matrix multiplication is *associative*, i.e. the order of multiplication doesn't matter, in other words $(AB)C = A(BC)$, so it makes sense to write successive multiplication $A_1A_2A_3 \cdots A_n$
- Scalar multiplication and matrix multiplication commutes, $A(cB) = c(AB) = (cA)B$. so it makes sense to write $cA_1A_2A_3 \cdots A_n$
- Matrix multiplication is *distributive* over addition, i.e. $A(B + C) = AB + AC$, $(A + B)C = AC + BC$
- Zero matrix and identity matrix acts as 0 and 1, i.e. $A + 0 = 0 + A = A$, $A0 = 0A = 0$, $IA = AI = A$
- Even if $A \neq 0$, $B \neq 0$, AB could still be 0, take eqref{ eq:equation1} for an example
- $AB = AC$ does NOT imply $B = C$

Remark: Some of the properties of matrices are really similar to that of numbers, so we dub this the name of *matrix algebra*

3.3. Partitioned matrix

Definition 3.3.1: A is a partitioned (or block) matrix if is divided into smaller submatrix by some horizontal and vertical lines. And the submatrices are the blocks

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right] = \left[\begin{array}{cc|cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \end{array}\right] \tag{3.81}$$

Here the blocks are $A_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_{12} = \begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix}$, $A_{13} = \begin{bmatrix} a_{17} \\ a_{27} \end{bmatrix}$, $A_{21} = \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}$,

$A_{22} = \begin{bmatrix} a_{33} & a_{34} & a_{35} & a_{36} \end{bmatrix}$, $A_{23} = \begin{bmatrix} a_{37} \end{bmatrix}$, $A_{31} = \begin{bmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix}$, $A_{32} = \begin{bmatrix} a_{43} & a_{44} & a_{45} & a_{46} \\ a_{53} & a_{54} & a_{55} & a_{56} \\ a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$,

$A_{33} = \begin{bmatrix} a_{47} \\ a_{57} \\ a_{67} \end{bmatrix}$.

Fact: Suppose $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & & \vdots \\ B_{q1} & B_{q2} & \cdots & B_{qr} \end{bmatrix}$ are partitioned matrices, and the number of columns of A_{1k} is equal to the number of rows of B_{k1} (so that all submatrices multiplications make sense). Then the usual row-column rule still WORKS!!! By treating submatrices as if they are numbers.

$$AB = C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1r} \\ C_{21} & C_{22} & \cdots & C_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ C_{p1} & C_{p2} & \cdots & C_{pr} \end{bmatrix}, C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{iq}B_{qj} \tag{3.82}$$

Example 3.3.1: Consider $\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 1 \\ \hline 2 & 1 & 1 \end{array}\right]$, $\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 2 & 2 & 1 \\ \hline 2 & 1 & 1 \end{array}\right]$, then

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [2] = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{3.83}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \;\; 1] = \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix} \tag{3.84}$$

$$A_{21}B_{11} + A_{22}B_{21} = [2 \;\; 1] \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [1][2] = [6] \tag{3.85}$$

$$A_{21}B_{12} + A_{22}B_{22} = [2 \;\; 1] \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} + [1][1 \;\; 1] = [5 \;\; 4] \tag{3.86}$$

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{cc|c} 5 & 4 & 3 \\ 8 & 7 & 5 \\ \hline 6 & 5 & 4 \end{array}\right] \tag{3.87}$$

Index