Introduction to Linear Algebra

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Contents

1. Lecture 1 - System of linear equations

1.1. Linear systems

Throughout this course, we adopt the following notations:

- Natural numbers: $\mathbb{N} = \{0, 1, 2, 3, ...\}$
- Integers: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
- Rational numbers: $\mathbb{Q} = \left\{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\right\}$ is the set of fractions. Here $\in$ means belong to.
- Real numbers: $\mathbb{R}$ is the set of numbers on the whole real number line. It includes:
  - irrational numbers (like $\sqrt{2}$, $\sqrt[3]{3}$)
  - transcendental numbers (like π, e)
- Complex numbers: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, $i = \sqrt{-1}$ is the imaginary number such that $i^2 = -1$.
- $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$
- $\mathbb{R}^n = \{(r_1, r_2, r_3, ..., r_n) \mid r_1, r_2, ..., r_n \in \mathbb{R}\}$ is the set of all n-tuples of real numbers. Geometrically:
  - $\mathbb{R}^1 = \mathbb{R}$ is a line.
  - $\mathbb{R}^2$ is a plane.
  - $\mathbb{R}^3$ is our usual physical space.

Definition 1.1.1: A linear equation in the variables $x_1, x_2, x_3, c..., x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \tag{1}$$

where the coefficients $a_1, a_2, a_3, ..., a_n$ and b are real or complex numbers, usually known in advance.

Example:

- $x_1 + \frac{1}{2}x_2 = 2$, ✓
- $\pi(x_1 + x_2) - 9.9x_3 = e$, ✓. Because if we expand it, we got $\pi x_1 + \pi x_2 - 9.9x_3 = e$ in which case $a_1 = \pi, a_2 = \pi, a_3 = -9.9, b = e$ as in the form of (1)
- $|x_2| - 1 = 0$, ✘
- $x_1 + x_2^2 = 9$, ✘
- $\sqrt{x_1} + \sqrt{x_2} = 1$, ✘

Definition 1.1.2: A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables, say $x_1, x_2, x_3, ..., x_n$.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{cases} \tag{2}$$

Example: For $n = m = 2$, (2) is just

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \tag{3}$$