MATH240 Summer 2023

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1 Worksheets

1.1 Worksheet for Section 6.1

1. A cylindrical tank standing upright, with a volume of 40π cubic meters, is to be built. Materials for the sides cost 10 dollars per square meter, and for the top and bottom the materials cost 25 dollar per square meter. Find the dimension that will minimize the cost.

2. Let $g(x) = \displaystyle\int_x^{x+\pi} \sin^{2/3} t dt$, for any given value of $x \geq 0$. Show that g is a constant function.

3. Consider the solid region whose base R is bounded by the negative x-axis, the positive y-axis, and the curve $y = 4 - x^2$ for $-2 \leq x \leq 0$.

   (a) Let D_1 be the solid with base R, and assume that the cross sections of D_1 perpendicular to the x-axis are squares. Draw a picture of the base, and then draw a representative cross section perpendicular to the x-axis at some arbitrary x in the interval $(-2, 0)$. Finally, find the cross-sectional area A_1 of the cross section.

   (b) Now let D_2 be the solid with base R, and assume that the cross sections of D_2 perpendicular to the x-axis are semi-circles. Draw a second picture of the base, and then draw a representative cross section perpendicular to the x-axis at some arbitrary x in the interval $(-2, 0)$. Finally, find the cross-sectional area A_2 of the cross section.

   (c) Suppose that the solid D_3 has the same base R, but has equilateral triangles for cross sections perpendicular to the x-axis. Without necessarily calculating the volumes, order the volumes V_1, V_2, V_3 of D_1, D_2, D_3, respectively, from smallest to largest.

4. Let f and g be continuous, and $0 \leq g(x) \leq f(x)$ on $[a, b]$. The washer method yields the volume formula $V = \displaystyle\int_a^b \pi[(f(x))^2 - (g(x))^2]dx$ when the region between the graphs of f and g are revolved about the x-axis. Now suppose that $g(x) < 0 < f(x)$ for x in $[a, b]$, and $|g(x)| \leq |f(x)|$ for x in $[a, b]$. What should be the resulting formula for V when the bounded region between the graphs of f and g on $[a, b]$ is revolved about the x-axis? Justify your answer.

5. (This problem is related to problem 44 in Section 6.1.) Consider the sphere of radius r that is centered at the origin. Take a vertical slice through the sphere at a distance h to the right of the center.

   (a) Draw a picture of the situation, and then show that the volume V of the smaller piece cut off from the ellipse (i.e., the piece to the right of $x = h$) satisfies the formula

   $$V = \frac{\pi(r-h)^2}{3}(2r + h)$$

   (b) Suppose that $r = 2$. Approximate the value of h such that $V =$ half the volume of the right hemisphere of the whole sphere, that is, so that $V = 8\pi/3$. (Hint: You likely will need to use the Newton-Raphson method to solve the equation.)

Solution.

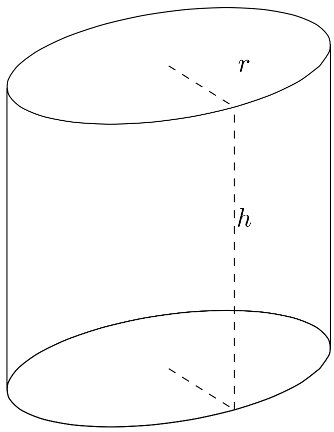
1. Denote the radius, height and cost as r, h, f, we have the volume is $\pi r^2 h = 40\pi$, so the cost

   $$f = 10 \cdot 2\pi\frac{40}{r} + 25 \cdot 2\pi r^2$$

   Solve

   $$\frac{df}{dr} = -\frac{800\pi}{r^2} + 100\pi r = 0$$

   we get $r = 2$, so $h = 10$. These minimizes the cost.

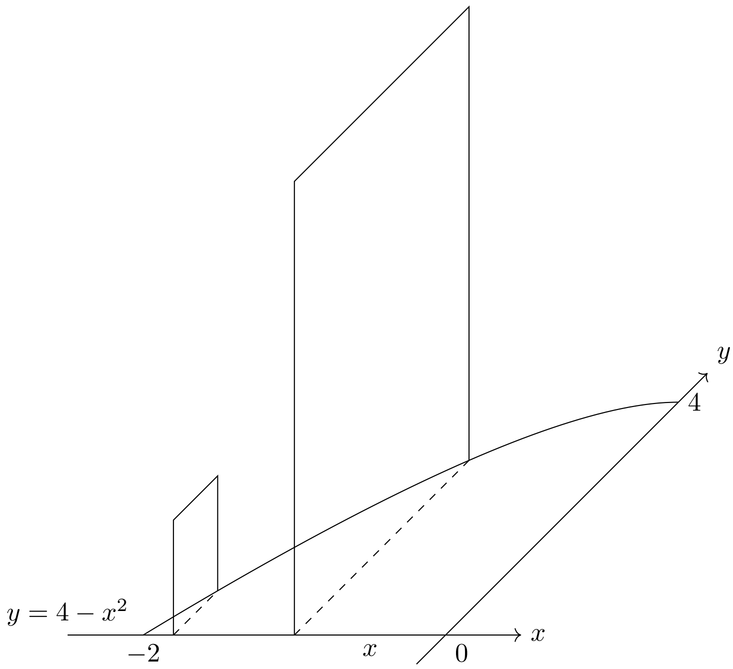


2.

$$g'(x) = \sin^{2/3}(x+\pi) - \sin^{2/3}(x) = [-\sin(x)]^{2/3} - [\sin(x)]^{2/3} = 0$$

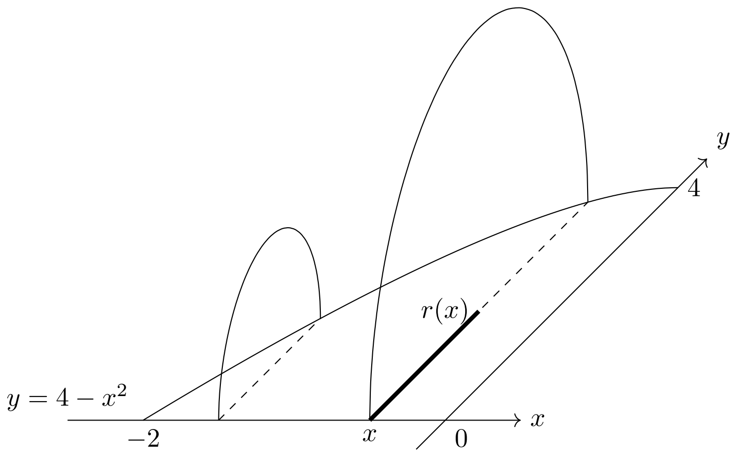
3. (a)

$$A_1(x) = (4-x^2)^2$$



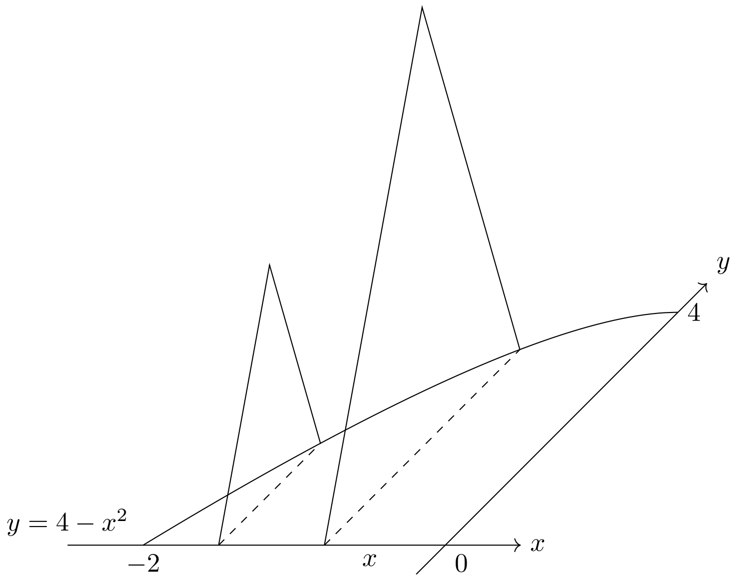
(b)

$$A_2(x) = \frac{\pi}{2}\left(\frac{(4-x^2)^2}{2}\right)$$



(c)

$$A_3(x) = \frac{\sqrt{3}(4 - x^2)^2}{4}$$



It is not hard to see that

$$A_1(x) > A_3(x) > A_2(x), \qquad V_1 > V_3 > V_2$$

The bigger sectional area, the bigger volume.

4. $V = \int_a^b \pi [f(x)]^2 dx$

5. (a)

$$V = \int_h^r \pi (r^2 - x^2) dx = \pi \left(\frac{2r^3}{3} - r^2 h + \frac{h^3}{3} \right)$$

(b)

□

1.2 Worksheet for Section 6.4

1. In calculating the work, W necessary to stretch a spring from its natural length of 8 centimeters to a length of 12 centimeters, we need some more information about the spring. There are two distinct kinds of information, each of which separately would provide the necessary assistance so we could find W. Describe what the two additional kinds of information are.

2. We wish to empty a hemispherical tank full of water out the top. Which should entail the larger amount of work: when the tank is like a cup full of water, or when the tank is upside down with a hole at the top? Explain your answer, and include a picture of the situation.

3. In Example 4 (page 393), a hemispherical tank with radius 10 feet and open at the top is filled with water (Figure 6.43). In determining the work W required to pump all the water to 6 feet above the top of the tank, we assumed that the origin was at the top of the tank, and we derived the formula
$$W = \int_{-10}^{0} 62.5(6-x)\pi(100-x^2)dx$$
Now let us assume that the origin is at the *bottom* of the tank. Find a formula for the radius $r(x)$ of the cross section x units above the origin, and use this formula to help write down the integral for the work W. (If you have time, you should check that the two formulas for W yield the same number of foot-pounds.)

4. A painter weighing 120 pounds is hoisted by means of a rope windlass from the ground to a window on the tenth story of a hotel, 80 feet above ground level.
   (a) If the weight of the entire rope is considered as negligible, then find the work W required to lift the person the 80 feet above ground level.
   (b) Assume that instead of a rope, a substantial chain weighing 2 pounds per foot is used in the windlass. Find the work W_1 required to lift the person the 80 feet from ground level. (Hint: Note that as the person is lifted, the length of the chain that must be lifted decreases.)

5. (a) At a party, the punch bowl is hemispherical, with 2 foot radius and open at the top. The fruit punch weighs 66 pounds per cubic foot. At the beginning of the party the bowl is full to the brim. The rules for serving punch require each person to fill the ladle, raise it exactly 1 foot above the top of the bowl (without spilling!), and then pour the punch into a cup at that level. After 10 minutes, 30 people have filled 30 cups, and the punch remaining in the bowl is 4 inches deep. Draw a picture of the situation, and then set up the integral for the work W involved in raising the punch until the punch remaining in the bowl has a depth of only 4 inches. Include all pertinent information in the integral, but do not evaluate the integral.
   (b) Suppose that each person is to raise the ladle exactly 1 foot (*not* 1 foot above the top of the bowl). Set up the integral for the work W_1 involved until the punch is 4 inches deep. Does this activity make any sense for the partygoers? Explain why or why not.

Solution.

1. We need either the spring constant k or the resultant force F.

2. The tank upside down requires more work since more water are concentrated at the bottom

3. The radius is $r(x) = \sqrt{100-(10-x)^2}$, and the work needed is
$$W = \int_{0}^{10} 62.5(16-x)(100-(10-x)^2)dx$$

4. (a) The work need is
$$W = 120 * 80 = 9600 \text{ pound-foot}$$
   (b) The work should be
$$W = 9600 + \int_{0}^{80} 2(80-x)dx = 9600 + 6400 = 16000$$

5. (a)

$$W = \int_{-5/3}^{0} 66(1-x)(4-x^2)dx$$

(b)

$$W_1 = \int_{-5/3}^{0} 66(4-x^2)dx$$

It doesn't make sense raise the ladle by exactly 1 foot, some punch will stay in the bowl forever.

□

1.3 Worksheet for Section 6.5

1. Let $f(x) = 3\sin x$ for $0 \le x \le \pi$, and consider the region R bounded by the x-axis and the graph of f.

   (a) Explain in a complete sentence why you don't need to evaluate any integrals in order to determine the x-coordinate $\bar{x}$ of the center of gravity.

   (b) Find the center of gravity $(\bar{x}, \bar{y})$ of R.

2. Let $r > 0$, and let R be the semicircular region bounded below by the x-axis and above by the circle $x^2 + y^2 = r^2$, that is, $x^2 + y^2 = r^2$ with $y \ge 0$.

   (a) Find the center of gravity $(\bar{x}, \bar{y})$ of R.

   (b) Find the radius r for which $(\bar{x}, \bar{y}) = (0, \pi)$.

   (c) Suppose S is the quarter circular region in the first quadrant, bounded by the x-axis, the y-axis, and the quarter circle $x^2 + y^2 = r^2$. Without evaluating any integrals or making further computations, use symmetry considerations and computations in (a) to determine $(\bar{x}, \bar{y})$.

3. Let $a > 0$ and $h > 0$. Consider the triangular region R whose vertices are $(0, 0), (a, 0)$, and $(0, h)$.

   (a) Find a formula for the line that represents the hypotenuse of the triangle.

   (b) Show that the center of gravity of R is $(a/3, h/3)$. (The center of gravity of R turns out to be precisely the centroid of the triangle, which is by definition the intersection of the 3 medians of the triangle, that is, the lines from the vertices to the midpoints of the opposite sides.)

4. Let f and g be continuous on $[a, b]$, with $a > 0$ and $0 \le g(x) \le f(x)$ for $a \le x \le b$. Also let R be the region between the graphs of f and g on $[a, b]$, and A the area of R.

   (a) Write down a formula for the volume V of the solid obtained by revolving the region R about the y-axis. (Note: If you are in doubt of the formula, please consult Section 6.1.)

   (b) Write down a formula for the moment M_y of R about the y-axis.

   (c) Use the formulas in (a) and (b) to derive the formula $V = 2\pi\bar{x}A$. (This is the result of Pappus and Guldin in Theorem 6.6, where b is substituted for $\bar{x}$.)

   (d) Suppose someone says that $\bar{x} = 0$. What hypothesis in the problem helps to prevent $\bar{x} = 0$?

5. Find the center of gravity $(\bar{x}, \bar{y})$ of the region R in Exercise 32(d) in Section 6.5, which region can be split into two rectangles. Then use the formulas at the beginning of Exercise 32 to help find $(\bar{x}, \bar{y})$.

Solution.

1. (a) $\bar{x} = \pi/2$ since R is symmetric over $x = \pi/2$.

(b)

$$\bar{y} = \frac{\int_0^\pi \frac{(3\sin x)^2}{2} dx}{\int_0^\pi 3\sin x dx} = \frac{3\pi}{16}$$

2. $\bar{x} = 0$ since R is symmetric over $x = 0$, $A = \pi r^2/2$

$$\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{r^2 - x^2}{2} dx = \frac{4r}{3\pi}$$

3. (a) $y = -\frac{h}{a}x + h$

   (b)

$$M_x = \int_0^a \frac{\left(-\frac{h}{a}x + h\right)^2}{2} dx \xlongequal{u=-\frac{h}{a}x+h} -\frac{a}{h}\int_h^0 \frac{u^2}{2} du = \frac{ah^2}{6}, \qquad \bar{y} = \frac{h}{3}$$

$$M_y = \int_0^a x\left(-\frac{h}{a}x + h\right) dx = \frac{ha^2}{6}, \qquad \bar{x} = \frac{a}{3}$$

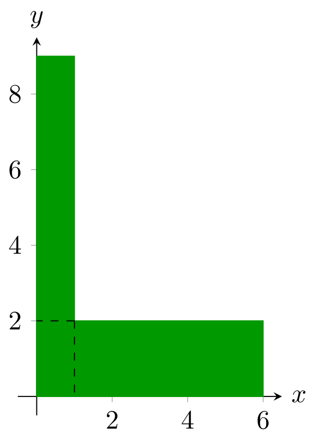
4. (a) $V = \int_a^b 2\pi x(f(x) - g(x))dx$

   (b) $M_y = \int_a^b x(f(x) - g(x))dx$

   (c) From (a) and (b) we know that $V = 2\pi M_y = 2\pi \bar{x} A$.

   (d) $a > 0$

5. $(\bar{x}, \bar{y}) = (39.5/19, 50.5/19)$



□

1.4 Worksheet for Sections 6.7, 6.2, and 6.8

1. Consider the curves C_1 and C_2 parametrized by $x = \cos t, y = \sin t$, and $x = \sin 2t, y = \cos 2t$, respectively, for $0 \leq t \leq 27$. Draw the graphs of C_1 and C_2, and indicate all values of t in $[0, 2\pi]$, any, for which $C_1(t) = C_2(t)$.

2. (a) Under what conditions is it possible to consider a parametrized curve as the graph of a function? (Hint: If $x = f(t)$ and $y = g(t)$ for $a < t < b$, what might one let $f(t)$ be?)

   (b) Write down parametric equations for the line l defined by $y = ax + b$, where a and b are arbitrary constants with $a \neq 0$ and b.

(c) We are given a differentiable function f on $[a, b]$, whose graph is C. Derive the formulas for the length L of C from the formula for the length of C when f is given parametrically.

3. Let $f(x) = \sin x$ for $0 \le x \le \pi/2$.

   (a) Write down a formula for the length L of the graph of f.

   (b) By using the Comparison Property for Integrals, show that $\pi/2 \le L \le \pi$.

4. (a) Let $f(t) = t - \sin t$ for all t. Show that f is a strictly increasing function of t, and determine the inflection points of the graph of f.

   (b) Consider the cycloid C parametrized by $x = t - \sin t$ and $y = 1 - \cos t$, for all real t. Find $C(0), C(\pi)$, and $C(2\pi)$, and show that the highest point on C occurs for $t = \pi$. Tell why this shows that one arch of the cycloid is not a semicircle.

   (c) Let $P(t) = (t - \sin t, 1 - \cos t)$ for all real t. Use (a) to show that if $t_1 \ne t_2$, then $x(t_1) \ne x(t_2)$. (This means that the cycloid C is the graph of a function!)

5. This problem is taken from Project 1 on page 383, which is related to Archimedes' approximation of the circumference of a circle by finding lengths of inscribed (regular) polygons such as those in Figure 6.26. Here we will assume that the point $(1, 0)$ is one vertex of each of the inscribed polygons.

   (a) Consider the inscribed regular polygon of n sides. Show that a vertex adjacent to $(1, 0)$ is $(\cos(2\pi/n), \sin(2\pi/n))$.

   (b) For the inscribed polygon of n sides in (a), show that the length of the side S determined by the 2 points in (a) is $\sqrt{2 - 2\cos(2\pi/n)}$, and show that this expression equals $2\sin(\pi/n)$.

   (c) Using the result of (b), find the length L_n of the inscribed regular polygon of n sides.

   (d) By letting n be large, and using a limit involving $(\sin t)/t$, show why the limit L of L_n as n increases without bound should be 2π.

6. Let $r > 0$. The equations $x = r\cos^3 t$ and $y = r\sin^3 t$ parametrize an astroid (Figure 6.84 in the book). Draw the graph, and find the length L of the astroid. (Hint: Square roots are nonnegative!)

Solution.

1. $C_1(t) = C_2(t)$ reads $\begin{cases} \cos t = \sin 2t \\ \sin t = \cos 2t = 1 - 2\sin^2 t \end{cases}$, solve the second equation we get $\sin t = -1$ or $\sin t = \frac{1}{2}$, so t could be $\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$, all three satisfies the first equation as well.

2. (a)

   (b)

   (c)

3. (a)

$$L = \int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x} dx$$

   (b) Note that $1 \le \sqrt{1 + \cos^2 x} \le 2$, so we have

$$\int_0^{\frac{\pi}{2}} 1 dx \le \int_0^{\frac{\pi}{2}} \sqrt{1 + \cos^2 x} dx \le \int_0^{\frac{\pi}{2}} 2 dx$$

   which is precisely $\pi/2 \le L \le \pi$

4. (a) $f'(x) = 1 - \cos t \ge 0$, so f is increasing. $f''(x) = \sin t = 0 \Rightarrow x = k\pi$ for all $k \in \mathbb{Z}$.

   (b) $C(0) = (0, 0), C(\pi) = (\pi, 2), C(2\pi) = (2\pi, 0)$, it cannot be a semicircle since the radius would not be defined!

   (c) Since $x(t) = f(t)$ is strictly increasing.

5. (a) Since it is a regular polygon, we just divide $2n$ into n equal angles, hence the vertex adjcent to $(1,0)$ will be $(\cos(2\pi/n), \sin(2\pi/n))$.

   (b)
$$\sqrt{(1-\cos(2\pi/n))^2+\sin^2(2\pi/n)} = \sqrt{2-2\cos(2\pi/n)} = 2\sin(\pi/n)$$

   (c)
$$L_n = 2n\sin(\pi/n)$$

   (d)
$$L = \lim_{n\to\infty} L_n = \lim_{n\to\infty} 2\pi\frac{\sin(\pi/n)}{\pi/n} = 2\pi$$

6.
$$L = \int_0^{\frac{\pi}{2}} \sqrt{(-3r\cos^2 t\sin t)^2 + (3r\sin^2 t\cos t)^2}dt = \int_0^{\frac{\pi}{2}} \sqrt{9r^2\sin^2 t\cos^2 t(\cos^2 t+\sin^2 t)}dt$$
$$= \int_0^{\frac{\pi}{2}} 3r\sin t\cos t dt \overset{u=\sin t}{=\!=\!=} \int_0^1 3rudu = \frac{3r}{2}$$

□

1.5 Sample Exam 1 MATH 141 - Calculus II

Instructions: Please put everything away and no calculators are permitted on this exam. Legibly print your full name on each answer sheet, and write and sign the honor pledge on the first page. Write your solutions neatly on the answer sheets, doing problem 1 on the first page, problem 2 on the second page, and so on. Continue your work onto the back of the page, if necessary. When you have finished, arrange the answer sheets in order and turn them in to your TA.

1. (13 points) Let R be the region between the graphs of $f(x) = 1+\sqrt{x}$ and $g(x) = e^{-x}$ over the interval $[0,1]$. Find the volume of the solid of revolution obtained by revolving the region R around the x-axis.

2. (15 points) Suppose the base of a solid figure S is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$. Cross-sections perpendicular to the x-axis are semicircles. Calculate the volume of S.

3. (a) (15 points) A hemispherical swimming pool has a radius of 9 feet. It is filled with water to a depth of 6 feet. Set up, but DO NOT EVALUATE, the integral for the work required to pump all but 3 feet of water to a platform 1 foot above the top of the pool. Assume the weight of water is 62.5 pounds per cubic foot.

   (b) (12 points) A spring is expanded 5 inches from its natural length and 20 pounds of force is required to hold it there. Find the work done in stretching the spring an additional 5 inches.

4. (15 points) Find the center of mass (centroid) of the region bounded by the curves $y = \sqrt{x}$ and $y = x$.

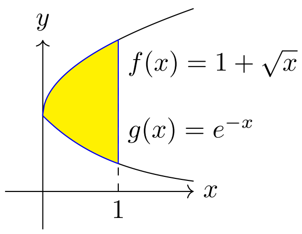
5. (a) (15 points) Find the length of the curve
$$y = \frac{x^3}{6} + \frac{1}{2x}$$
   for $1 \le x \le 2$.

   (b) (15 points) Calculate the length of the curve described parametrically by $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \le t \le \pi/2$.

Solution.

1. First we sketch the situation

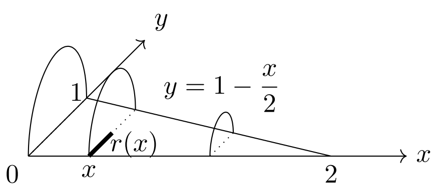
1



Note that $0 < g(x) \leq f(x)$. So

$$V = \int_0^1 \pi[f(x)^2 - g(x)^2]dx = \int_0^1 \pi\left((1+\sqrt{x})^2 - e^{-2x}\right)dx = \int_0^1 \pi\left(1 + 2\sqrt{x} + x - e^{-2x}\right)dx$$
$$= \left(x + \frac{4}{3}x^{3/2} + \frac{x^2}{2} + \frac{e^{-2x}}{2}\right)\Bigg|_0^1 = \frac{7}{3} + \frac{e^{-2}}{2}$$

2. We first sketch the solid



From this we can see that $r(x) = \dfrac{1-\frac{1}{2}x}{2}$, and $A(x) = \dfrac{\pi r(x)^2}{2} = \frac{1}{2}\left(\dfrac{-\frac{1}{2}x+1}{2}\right)^2$

$$V = \int_0^2 \frac{1}{2}\left(\frac{-\frac{1}{2}x+1}{2}\right)^2 dx = \int_0^2 A(x)dx = \frac{1}{12}$$

3. (a)

$$W = \int_{-6}^{-3} 62.5(1-x)\pi(9^2 - x^2)dx$$

(b) We know $20 = 5k \Rightarrow k = 4$, so

$$W = \int_5^{10} kxdx = \int_5^{10} 4xdx = 150$$

4.

$$M_x = \int_0^1 \frac{(\sqrt{x})^2 - x^2}{2} dx = \frac{1}{12}, \qquad M_y = \int_0^1 x(\sqrt{x} - x)dx = \frac{1}{15}, \qquad A = \int_0^1 (\sqrt{x} - x)dx = \frac{1}{6}$$

$$\bar{x} = \frac{M_y}{A} = \frac{2}{5}, \qquad \bar{y} = \frac{M_x}{A} = \frac{1}{2}$$

5. (a)

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2} dx = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2} dx = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \frac{17}{12}$$

(b)

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2 dt} = \int_0^{\frac{\pi}{2}} \sqrt{9\sin^2 t \cos^2 t(\cos^2 t + \sin^2 t)} dt$$
$$= \int_0^{\frac{\pi}{2}} 3\sin t \cos t dt \overset{u=\sin t}{=\!=\!=} \int_0^1 3udu = \frac{3}{2}$$

□

1.6 Worksheet for Section 7.1

1. Simplify the following:

   a) $(\ln 2 + \ln 3)/\ln 36$ b) $e^{-2\ln x}$ c) $\ln(e^{-x^2})$

2. Let $f(x) = mx + b$, where $m \neq 0$. Show that f has an inverse, and find a formula for f^{-1}. Finally, explain why f does *not* have an inverse if $m = 0$.

3. (a) Suppose that the formula $y = f(x)$ defines a function f that has an inverse on $[a, b]$. Describe the process you would use in order to find a formula for the inverse f^{-1}.

   (b) Let $f(x) = 3/x$. Draw the graph of f, then find a formula for f^{-1}. Does it appear that $f^{-1} = f$? Explain. Do the same for $g(x) = -3/x$.

   (c) What property of the graph of a function h implies that $h^{-1} = h$?

4. (a) In a complete sentence, tell why an increasing function f defined on an interval I must have an inverse. Must the inverse be an increasing function? Explain your answer.

   (b) Draw the graph of a function f defined on the interval $[0, 1]$ that has an inverse but is *neither* increasing on $[0, 1]$ *nor* decreasing on $[0, 1]$.

   (c) Suppose that g is defined and differentiable on $[0, 2]$, with $g(1) = 3$ and $g'(1) = 0$. Assume that g has an inverse. Can $(g^{-1})'(3)$ exist? Explain why or why not.

   (d) Suppose that $h(x) = x^5 + x^3 + 2x$. Show that h has an inverse h^{-1}, and that although h is a differentiable function, h^{-1} is *not* a differentiable function.

5. The function $f(x) = x^2$ has no inverse. However, if we restrict the domain to, say, $[0, \infty)$ or $[1, 100)$, then the new function f_1 *does* have an inverse. In the following, find the largest interval containing 0 for which the restricted function f_1 has an inverse:

   a) $f(x) = \sin x$ b) $f(x) = \tan x$ c) $f(x) = x^3 - x$ d) $f(x) = x + \sin x$

6. Suppose a ball is thrown vertically (either upward or downward) from a balcony h_0 meters above the ground, with an initial velocity v_0 and constant acceleration due to gravity (-9.8 meters per second squared). Assume further that the ball hits the ground after c seconds.

   (a) Write down a formula for the height h as a function of time t, for $0 \le t \le c$. (*Hint:* See p. 323.)

   (b) Under what physical conditions does h have an inverse h^{-1}?

   (c) In the event that h does have an inverse, what does h^{-1} tell us in physical terms?

Solution.

1. a) $\dfrac{\ln(2 \cdot 3)}{\ln(6^2)} = \dfrac{\ln 6}{2 \ln 6} = 1/2.$

   b) $e^{-2 \ln x} = e^{\ln(x^{-2})} = x^{-2}.$

   c) $\ln(e^{-x^2}) == -x^2.$

2. $f^{-1}(x) = \dfrac{x-b}{m}$, f has no inverse since m as the denominator cannot be zero, also because $f(x) = b$ is horizontal line, which evidently fails the horizontal line test.

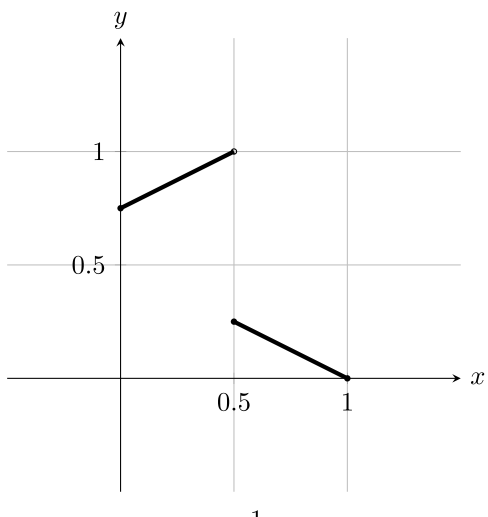
3. (a) First solve x in terms of y, and then change y into x, we get an expression of $f^{-1}(x)$

   (b) Yes.

   (c) The graph of h is symmetric over the line $y = x$.

4. (a) Because it is one-to-one, the inverse function is also increasing since suppose $b_1 < b_2$, we have unique $a_1 < a_2$ such that $f(a_1) = b_1, f(a_2) = b_2$, i.e. $f^{-1}(b_1) < f^{-1}(b_2)$.

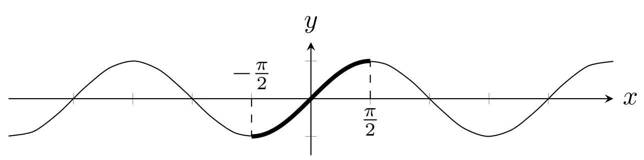
   (b) We can consider a discontinuous function which increases on $[0, 0.5)$, drops at $x = 0.5$ and decreases on $[0.5, 1]$.



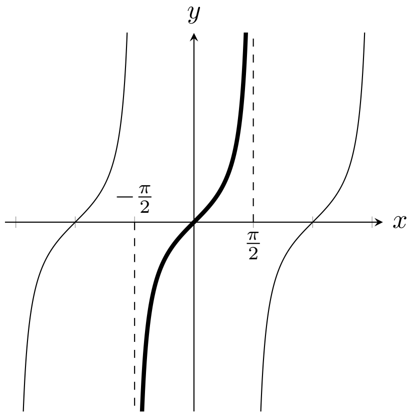
   (c) No. Since otherwise it will be $(g^{-1})'(3) = \dfrac{1}{g'(1)}$ which is undefined.

   (d) $h'(x) = 5x^4 + 3x^2 + 2 \ge 2$, hence h is a strictly increasing function.

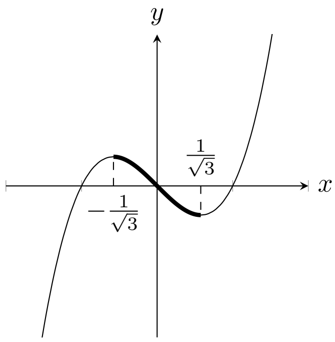
5. a) $[-\pi/2, \pi/2]$



b) $(-\pi/2, \pi/2)$



c) Solve $f'(x) = 3x^2 - 1 = 0$ we get $x = \pm 1/\sqrt{3}$, so the interval should be $[-1/\sqrt{3}, 1/\sqrt{3}]$



d) Since $f'(x) = 1 + \cos x \geq 0$, f is increasing, so the interval should be $\mathbb{R} = (-\infty, \infty)$.

6. (a) $h(t) = h_0 + v_0 t - 4.9t^2$
   (b) Thrown downwards.
   (c) The time needed to reach certain height.

□

1.7 Worksheet for Section 7.2

1. Let $f(x) = e^{(x^2)}$ and $g(x) = (e^x)^2$.
   (a) Find $f'(x)$ and $g'(x)$.
   (b) One of the two integrals $\int e^{(x^2)} dx$ and $\int (e^x)^2 dx$ can be evaluated by the methods of Chapter 5. Determine which integral it is, and evaluate the integral.

2. Iodine 131, which has been used for treating cancer of the thyroid gland, is also used in order to detect leaks in water pipes. It has a half-life of (approximately) 8.14 days. Suppose that at noon today you have a bottle with 5 grams of Iodine 131.
   (a) Find a formula for the amount $f(t)$ of Iodine 131 in the bottle t hours after noon today, that is, for each $t \geq 0$. (Hint: Section 4.4 might be helpful.)
   (b) Show that f has an inverse f^{-1}, and find a formula for f^{-1}.
   (c) In a complete sentence, indicate what the function f^{-1} tells us physically.

3. (a) Let b be any real number. Use the product rule for derivatives (and *not* the Law of Exponents) to show that $\frac{d}{dx}(e^{-x}e^{b+x}) = 0$. Consequently by Theorem 4.6, $e^{-x}e^{b+x}$ is a constant function.

(b) By using (a) and letting $x = 0$, show that the constant function in (a) is e^b, so that $e^{-x}e^{b+x} = e^b$ for all x.

(c) Use (b) with $b = 0$ to prove that $e^{-x} = 1/e^x$ for all real x.

(d) Use (b) and (c) to prove that $e^{b+c} = e^b e^c$ for all real numbers b and c. (You have just proved the Law of Exponents!)

4. Let $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where μ and σ are constants, and $\sigma > 0$.

(a) Show that $f'(x) = 0$ if $x = \mu$. Then, without taking the second derivative, show that $f(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}}$ is the maximum value of f.

(b) Show that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$.

(c) Let $\sigma = 1$ and $\mu = 0$. Use the left sum formula with $n = 50$ to get an approximate value of $\displaystyle\int_{-3}^{3} f(x)dx$. From your answer, guess the value of $\displaystyle\lim_{k\to\infty}\int_{-k}^{k} f(x)dx$.

Solution.

1. (a) $f'(x) = 2xe^{(x^2)}$, $g'(x) = 2e^{2x}$

(b) The second one.

2. (a) We know $f(t) = f(0)e^{kt} = 5e^{kt}$ and

$$2.5 = f(8.14) = 5e^{k8.14} \Rightarrow e^{k8.14} = \frac{1}{2} \Rightarrow k = -\frac{\ln 2}{8.14} \approx -0.085$$

Thus $f(t) = 5e^{-0.085t}$.

(b) $f'(t) > 0$, $f^{-1}(x) = \frac{1}{0.085}\ln(x/5)$

(c) The number of days to get decay to certain amount of Iodine.

3. (a)

$$\frac{d}{dx}(e^{-x}e^{b+x}) = -e^{-x}e^{b+x} + e^{-x}e^{b+x} = 0$$

(b) $e^{-0}e^{b+0} = e^b$

(c) Take $b = 0$ in (b) we have $e^{-x}e^x = e^{-x}e^{0+x} = e^0 = 1 \Rightarrow e^{-x} = 1/e^x$.

(d) Take $x = c$ in (b) we get $e^{-c}e^{b+c} = e^b$, use (c) we know $\frac{1}{e^c}e^{b+c} = e^b \Rightarrow e^{b+c} = e^b e^c$.

4. (a)

$$f'(x) = -\frac{x-\mu}{\sigma^3\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

So $f'(\mu) = 0$, and f attain maximum at $x = \mu$.

(b)

(c)

$$\int_{-3}^{3} f(x)dx \approx \sum_{k=0}^{49} \frac{1}{50} f\left(-3 + \frac{6k}{50}\right) = \sum_{k=0}^{49} \frac{1}{50} e^{-\frac{(-3+\frac{6k}{50})^2}{2}}$$

$\square$

1.8 Worksheet for Section 7.3 and 7.5

1. (a) Let $f(t) = a(b^t)$, with $f(-1) = 1/2$ and $f(2) = 108$. Find the values of a and b.

   (b) Let $x > 0$. Order the following functions from smallest to largest, and sketch their graphs:

$$3^{-x}, 3^x, 2^{-x}, 2^x, e^x$$

2. (a) Let a and b be positive constants, with $a \neq 1$ and $b \neq 1$. Using Theorem 7.8, prove the general change of base formula

$$\log_b x = \log_b a \log_a x, \qquad \text{for all } x > 0$$

   (b) We know that $\log_2 7 \approx 2.807355$, $\log_{15} 7 \approx 0.718565$ and $\log_7 15 \approx 1.391663$. Using (a) and whichever such approximations are relevant, approximate $\log_2 15$.

3. For certain numerical values of c, the answer when we evaluate the integral $\int \frac{1}{x^2 + 6x + c} dx$ involves an arctangent. By completing the square in the denominator of the integrand, determine those values of c, and evaluate the corresponding integral for an arbitrary such value of c.

4. Let g be a differentiable function defined on $[0, 1]$, with $|g(t)| < 3$ for $0 \leq t \leq 1$. Thus $16-(g(t))^2$ is strictly positive on $[0, 1]$.

   (a) Substitute $u = g(t)$, and then evaluate the integral $\int \frac{g'(t)}{\sqrt{16 - (g(t))^2}} dt$.

   (b) Suppose $g(0) = 0$. Find a value of $g(1)$ so that $\int_0^1 \frac{g'(t)}{\sqrt{16 - (g(t))^2}} dt = \frac{\pi}{3}$.

5. Jo drops a marble from her apartment window, 20 meters above the ground. At the same height but 12 meters away, Doe watches the marble fall. When the height of the marble above the ground is h, let θ be the angle between L and M, where L is the horizontal line joining Jo and Doe, and where M represents Doe's line of sight to the marble.

   (a) Draw a picture of the scene, including L, M, and θ Put the horizontal axis at ground level, and indicate the position of the marble $h(t)$ meters above ground level at time t.

   (b) Find θ in terms of h, and then calculate $d\theta/dh$.

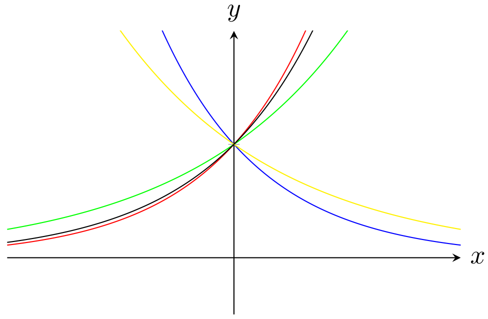
   (c) Assume that $h(t) = 20 - 4.9t^2$ until the marble hits the ground. Find $d\theta/dt$. Show geometrically why $d\theta/dt > 0$, and then tell why the formula for $d\theta/dt$ tells us that $d\theta/dt > 0$.

   Note: It turns out that there is a value t_0 to for which $d\theta/dt$ has a maximum value. For $\theta(t_0)$ the marble appears to be falling the fastest. Can you find the value of t_0?

Solution.

1. (a) From $ab^{-1} = f(-1) = 1/2$ we know $a = \frac{b}{2}$, plug it into $ab^2 = f(2) = 108$ we get $b^3 = 216 \Rightarrow b = 6$, so $a = 3$.

   (b) Let's use red to indicate 3^x, blue to indicate 3^{-x}, green to indicate 2^x, yellow to indicate 2^{-x} and black to indicate e^x.



2. (a) The left hand side is $\log_b x = \dfrac{\ln x}{\ln b}$ which coincide with the right hand side

$$\log_b a \log_a x = \frac{\ln a}{\ln b}\frac{\ln x}{\ln a} = \frac{\ln x}{\ln b}$$

(b) We can simply use (a)

$$\log_2 15 = \log_2 7 \log_7 15 = 2.807355 \cdot 1.391663 \approx 3.90$$

3. First we complete the squares

$$x^2 + 6x + c = (x+3)^2 + (c-9)$$

Therefore

$$\int \frac{1}{x^2+6x+c}dx = \int \frac{1}{(x+3)^2+(c-9)}dx = \frac{1}{\sqrt{c-9}} \arctan\left(\frac{x+3}{\sqrt{c-9}}\right) + C$$

So $c > 9$.

4. (a)

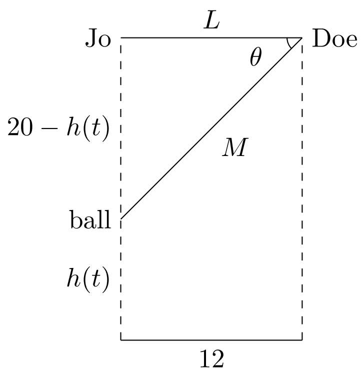
$$\int \frac{g'(t)}{\sqrt{16-(g(t))^2}}dt \overset{u=g(t)}{=\!=\!=} \int \frac{du}{\sqrt{16-u^2}} = \arcsin\left(\frac{g(t)}{4}\right) + C$$

(b) By Fundamental Theore of Calculus, we have

$$\begin{aligned}\int_0^1 \frac{g'(t)}{\sqrt{16-(g(t))^2}}dt &= \arcsin\left(\frac{g(t)}{4}\right)\Big|_0^1 = \arcsin\left(\frac{g(1)}{4}\right) - \arcsin\left(\frac{g(0)}{4}\right) \\ &= \arcsin\left(\frac{g(1)}{4}\right) - \arcsin\left(\frac{0}{4}\right) \\ &= \arcsin\left(\frac{g(1)}{4}\right) = \frac{\pi}{3}\end{aligned}$$

So $\dfrac{g(1)}{4} = \sin(\frac{\pi}{3}) = \dfrac{\sqrt{3}}{2} \Rightarrow g(1) = 2\sqrt{3}$.

5. The picture looks like



(a) $h(t) = 20 - 4.9t^2$

(b) By examining the picture we see that $\tan\theta = \dfrac{20-h}{12}$, so $\theta = \arctan\left(\dfrac{20-h}{12}\right)$, so

$$\frac{d\theta}{dh} = \frac{1}{\left(\frac{20-h}{12}\right)^2+1} \cdot \left(-\frac{1}{12}\right)$$

(c)

$$\frac{d\theta}{dt}=\frac{d\theta}{dh}\frac{dh}{dt}=\frac{1}{\left(\frac{20-h}{12}\right)^2+1}\cdot\left(-\frac{1}{12}\right)(-9.8t)=\frac{1}{\left(\frac{20-h}{12}\right)^2+1}\cdot\left(\frac{9.8t}{12}\right)>0$$

There for $\frac{\theta}{dt}>0$, and it achieves its maximum when it hits the ground, i.e. when $h(t_0)=0\Rightarrow t_0=\sqrt{\frac{20}{4.9}}$.

□

1.9 Worksheet for Section 7.6

1. (a) Show that 1^∞ is an indeterminate form by finding 2 values of c for which $\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x$ yields two different numbers.
   (b) Show that 0^∞ is a determinate form by looking at $\lim_{x\to\infty} f(x)^{g(x)}$, where $\lim_{x\to\infty} f(x)=0$ and $\lim_{x\to\infty} g(x)=\infty$, and telling the behavior of the values oft if $f(x)^{g(x)}$ as x approaches ∞.
2. Exercise 51 in the textbook asks us to evaluate one of the limits that appeared in l'Hôpital's original text on calculus: $\lim_{x\to a}\frac{a^2-ax}{a-\sqrt{ax}}$ with $a>0$.
   (a) Use l'Hôpital's Rule to evaluate the limit.
   (b) Rewrite the numerator as the product of two expressions, one of which is the denominator. Then simplify the resulting expression, and evaluate the limit *without* using l'Hôpital's Rule.
3. Consider $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$.
   (a) Apply l'Hôpital's Rule twice. What do you obseve?
   (b) Factor out x^2 from x^2+1 in the denominator, then cancel, and next evaluate the limit without l'Hôpital's Rule.
   (c) Corroborate your answer to part (b) by plotting. and examining the portion of the graph of $y=x/\sqrt{x^2+1}$ for large values of x.
4. Consider $\lim_{x\to 0}\frac{\sin x}{x+1}$.
   (a) Use the l'Hôpital's Rule process to obtain value of the limit.
   (b) Use the limit rules and *not* l'Hôpital's Rule in order to obtain the value of the Limit.
   (c) In a sentence, tell why the result of (a) is wrong.
5. Suppose a right circular cylinder with top and bottom is to be made in such a way that if the radis is r, then the height of the cylinder is 2^{-r}. Determine what happens to the volume V of the cylinder as r incresses without bound.
6. EXTRA: Solve Exercise 60 in Section 7.6, which is the following: Show that $\lim_{x\to 0^+} x^{(x^x)}=0$.

Solution.

1. (a)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e,\qquad \lim_{x\to\infty}\left(1+\frac{-1}{x}\right)^x=1/e,\qquad \lim_{x\to\infty}\left(1+\frac{0}{x}\right)^x=1=e^0$$

In general, first we know

$$\lim_{x\to\infty} x\ln\left(1+\frac{c}{x}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{c}{x}\right)}{1/x}\overset{\text{l'H }\frac{0}{0}}{=\!=}\lim_{x\to\infty}\frac{\frac{1}{1+\frac{c}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}=\lim_{x\to\infty}\frac{1}{1+\frac{c}{x}}=c$$

so we have

$$\lim_{x\to\infty}\left(1+\frac{c}{x}\right)^x=\lim_{x\to\infty} e^{\ln\left(1+\frac{c}{x}\right)^x}=\lim_{x\to\infty} e^{x\ln\left(1+\frac{c}{x}\right)}=e^{\lim_{x\to\infty} x\ln\left(1+\frac{c}{x}\right)}=e^c$$

(b)

2. (a)

$$\lim_{x\to a}\frac{a^2-ax}{a-\sqrt{ax}} \overset{\text{l'H } \frac{0}{0}}{=\!=\!=} \lim_{x\to a}\frac{-a}{-\frac{\sqrt{a}}{2\sqrt{x}}} = \frac{-a}{-\frac{1}{2}} = 2a$$

(b)

$$\lim_{x\to a}\frac{a^2-ax}{a-\sqrt{ax}} = \lim_{x\to a}\frac{(a-\sqrt{ax})(a+\sqrt{ax})}{a-\sqrt{ax}} = \lim_{x\to a}(a+\sqrt{ax}) = 2a$$

3. (a)

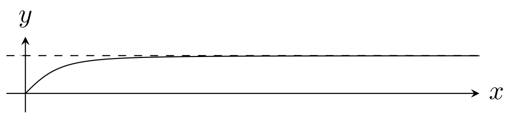
$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{1}{\frac{x}{\sqrt{x^2+1}}} = \lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty}\frac{\frac{x}{\sqrt{x^2+1}}}{1} = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$$

This has no progress.

(b)

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x}{x\sqrt{1+x^{-2}}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+x^{-2}}} = \frac{1}{1+0} = 1$$

(c)



4. (a)

$$\lim_{x\to 0}\frac{\sin x}{x+1} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

(b)

$$\lim_{x\to 0}\frac{\sin x}{x+1} = \frac{0}{0+1} = 0$$

(c) $\frac{0}{a}$ where $a \neq 0$ is not an indeterminate form.

5. First we know $V = \pi r^2 \cdot 2^{-r} = \dfrac{\pi r^2}{2^r}$, so

$$\lim_{r\infty} V = \lim_{r\to\infty}\frac{\pi r^2}{2^r} = \lim_{r\to\infty}\frac{2\pi r}{2^r \ln 2} = \lim_{r\to\infty}\frac{2\pi}{2^r(\ln 2)^2} = 0$$

6. First consider

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/x} = \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x = 0$$

So we know

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^x \ln x = e^0 = 1$$

And

$$\lim_{x\to 0^+} x^{(x^x)} = \lim_{x\to 0^+} e^{(x^x)\ln x} = e^{-\infty} = 0$$

$\square$

1.10 Worksheet for Section 7.6 and 8.1

1. Determine which of the following limits exist as a number, which as ∞, and which don't exist at all. For any limit that exists as a number, determine the number.

   (a) (i) $\lim_{x\to 0^+} x^x$ (ii) $\lim_{x\to 0^+} (1/x)^x$ (iii) $\lim_{x\to\infty} x^{\sin x}$

   (b) (i) $\lim_{x\to 0^+} (1-1/x)^x$ (ii) $\lim_{x\to 0^+} (1-1/x)^{1/x}$ (iii) $\lim_{x\to\infty} (1-1/x)^x$

2. In the following, one integral in each pair can be evaluated by substitution and the other integral can be evaluated by integration by parts. In each case, write down which method applies, and evaluate the integral.

   (a) (i) $\int x\cos(x^2)dx$ (ii) $\int x^2\cos x dx$

   (b) (i) $\int \frac{x^2}{e^x}dx$ (ii) $\int xe^{(x^2)}dx$

   (c) (i) $\int \frac{1}{x^4}\ln x dx$ (ii) $\int \frac{1}{x}(\ln x)^4 dx$

3. Consider the integral $\int e^{\pi x}\sin(2x)dx$.

   (a) In half of your group, solve the integral by performing integration by parts (twice!) with $u = \sin(2x)$ and $du = e^{\pi x}dx$. In the other half of the group, solve the integral by performing integration by parts (twice!) with $u = e^{\pi x}$ and $dv = \sin(2x)dx$. Are the answers equivalent to each other? Explain in a sentence.

   (b) Which of integration by parts processes is easier, or are they both about the same? Explain your answer. (Note: Sometimes it is not so simple to determine which part of the integrand is better as u, and which is better as dv.)

4. In each of the following, try to use the technique of integration by parts with the given u. In each, explain why that choice of u is not effective. Then write down a different function u and also write down dv that will make integration by parts effective, and tell why you chose your u and dv. (Don't perform the integration, however.)

   (i) $\int e^{-3x}x^{10}dx$; $u = e^{-3x}$ (ii) $\int x^4\sin x dx$; $u = \sin x$

Solution.

1. (a) (i)
$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

   (ii)
$$\lim_{x\to 0^+} (1/x)^x = \lim_{x\to 0^+} 1/x^x = 1$$

   (iii) It doesn't exist since $\sin x$ oscillates between -1 and 1, $x^{\sin x}$ oscillates between really small and really big, so it doesn't have a limit.

   (b) (i) $(1-1/x)^x = e^{x\ln(1-1/x)}$, but $\lim_{x\to 0^+}\ln(1-1/x)$ is not well defined.

   (ii) $(1-1/x)^{1/x} = e^{\ln(1-1/x)/x}$, but $\lim_{x\to 0^+}\ln(1-1/x)$ is not well defined.

   (iii) As in 1(a) of worksheet 7.6, we get e^{-1}.

2. (i)
$$\int x\cos(x^2)dx \overset{u=x^2}{=\!=\!=} \frac{1}{2}\int \cos u du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C$$

(ii)

$$\int x^2 \cos x dx \xlongequal{u=x^2, dv=\cos x dx = d\sin x} x^2 \sin x - \int 2x \sin x dx$$
$$\xlongequal{u=2x, dv=\sin x dx = d(-\cos x)} x^2 \sin x + 2x\cos x - \int 2\cos x dx$$
$$= x^2 \sin x + 2x \cos x - 2\sin x + C$$

(iii) (i)

$$\int \frac{x^2}{e^x} dx \xlongequal{u=x^2, dv=e^{-x}dx=d(-e^{-x})} -x^2 e^{-x} + \int 2xe^{-x} dx$$
$$\xlongequal{u=2x, dv=e^{-x}dx=d(-e^{-x})} -x^2e^{-x} - 2xe^{-x} + \int 2e^{-x}dx$$
$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

(ii)

$$\int xe^{(x^2)} dx \xlongequal{u=x^2} \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{(x^2)} + C$$

(iv) (i)

$$\int \frac{1}{x^4} \ln x dx \xlongequal{u=\ln x, dv=x^{-4}dx=d(-1/3x^{-3})} -\frac{1}{3x^3}\ln x + \int \frac{1}{3x^3}\frac{1}{x}dx$$
$$= -\frac{1}{3x^3}\ln x - \frac{1}{9x^3} + C$$

(ii)

$$\int \frac{1}{x}(\ln x)^4 dx \xlongequal{u=\ln x} \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}(\ln x)^5 + C$$

3. (a) Assume $I = \int e^{\pi x}\sin(2x)dx$.

$$\begin{aligned} I &\xlongequal{u=\sin(2x), dv=e^{\pi x}dx=d(e^{\pi x}/\pi)} \frac{e^{\pi x}}{\pi}\sin(2x) - \frac{2}{\pi}\int e^{\pi x}\cos(2x)dx \\ &\xlongequal{u=\cos(2x), dv=e^{\pi x}dx=d(e^{\pi x}/\pi)} \frac{e^{\pi x}}{\pi}\sin(2x) - \frac{2e^{\pi x}\cos(2x)}{\pi^2} - \frac{4}{\pi^2}I \end{aligned} \tag{1.10.1}$$

$$\begin{aligned} I &\xlongequal{u=e^{\pi x}, dv=\sin(2x)dx=d(-\cos(2x)/2)} -\frac{\cos(2x)}{2}e^{\pi x} + \frac{\pi}{2}\int e^{\pi x}\cos(2x)dx \\ &\xlongequal{u=e^{\pi x}, dv=\cos(2x)dx=d(\sin(2x)/2)} -\frac{\cos(2x)}{2}e^{\pi x} + \frac{\pi}{4}e^{\pi x}\sin(2x) - \frac{\pi^2}{4}I \end{aligned} \tag{1.10.2}$$

Both equations give us

$$I = \frac{e^{\pi x}(\pi\sin(2x) - 2\cos(2x))}{4+\pi^2}$$

You will get the same answer since they the same integral after all!

(b) It is about the same difficulty.

4. (a) It is ineffective since it raised the power of x. If we choose $u = x^{10}$, then $du = 10x^9 dx$ which reduces down the power on x.

(b) It is ineffective since it raised the power of x. If we choose $u = x^4$, then $du = 4x^3 dx$ which reduces down the power on x.

□

1.11 Sample Exam 2 MATH 141 - Calculus II

Instructions: Please put everything away and no calculators are permitted on this exam. Legibly print your full name on each answer sheet, and write and sign the honor pledge on the first page. Write your solutions neatly on the answer sheets, doing problem 1 on the first page, problem 2 on the second page, and so on. Continue your work onto the back of the page, if necessary. When you have finished, arrange the answer sheets in order and turn them in to your TA.

1. Let $f(x) = 3x + \sqrt{x+10}$.
   (a) (10 points) Find the largest interval on which f has an inverse function f^{-1}. What are the domain and range of f^{-1}?
   (b) (10 points) Calculate $(f^{-1})'(0)$.

2. Find the derivative y'. You do not have to simplify your answer.
   (a) (10 points)
   $$y = 3\log_{10}(2x) + 2^{x-1} + \sin^{-1}(x^3)$$
   (b) (10 points)
   $$y = (\sin^{-1}(x^2))^{\tan^{-1} x}$$

3. (a) (10 points) Let $f(x) = \sin\left(\cos^{-1}\left(\frac{5x}{7}\right)\right)$. Rewrite the function f so it does not contain a trigonometric or inverse trigonometric function.
   (b) (10 points) Evaluate the integral
   $$\int \frac{1}{x^2+3x+3}dx$$

4. Find the limit.
   (a) (10 points)
   $$\lim_{x\to 0} \frac{1-\cos x}{(1-e^x)^2}$$
   (b) (10 points)
   $$\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{2x}$$

5. Evaluate the integral.
   (a) (10 points)
   $$\int_0^{\pi/4} x^2 \sin 2x dx$$
   (b) (10 points)
   $$\int_0^{\pi/2} e^{2x} \cos x dx$$

Solution.

1. (a) Since $f'(x) = 3 + \frac{1}{2\sqrt{x+10}} \geq 3$, f is always strictly increases, so f has an inverse on $[-10,\infty)$. The domain of f^{-1} is the range of f which is $[-30,\infty)$, and range of f^{-1} is the domain of f which is $[-10,\infty)$.
   (b) First solve $0 = f(x) = 3x+\sqrt{x+10} = \iff 3x = -\sqrt{x+10}$, so we have $9x^2 = x+10 \iff (9x-10)(x+1) = 0 \iff x = -1$ or $x = 10/9$ which we discard since it is not a solution(we can also see this because $x = -\sqrt{x+10}/3$ should be non-positive)
   $$(f^{-1})'(0) = \frac{1}{f'(-1)} = \frac{1}{3+\frac{1}{2\cdot 3}} = \frac{6}{19}$$

2. (a) First we can change the base for log

$$y = \frac{3\ln(2x)}{\ln 10} + e^{(x-1)\ln 2} + \sin^{-1}(x^3)$$

Therefore

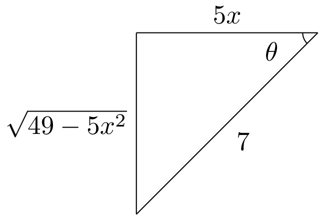
$$y' = \frac{3}{(\ln 10)x} + 2^{x-1}\ln 2 + 3x^2\sqrt{1+x^6}$$

(b) First we rewrite it as

$$y = e^{\tan^{-1}x \ln(\sin^{-1}(x^2))}$$

$$y' = (\sin^{-1}(x^2))^{\tan^{-1}x}\left[\frac{\ln(\sin^{-1}(x^2))}{1+x^2} + \tan^{-1}x\frac{1}{\sin^{-1}(x^2)}\frac{1}{1+x^4}2x\right]$$

3. (a) Suppose $\theta = \cos^{-1}\left(\frac{5x}{7}\right)$, then $\cos\theta = \frac{5x}{7}$, so $\sin\theta = \frac{\sqrt{49-25x^2}}{7}$



(b) First complete the squares $x^2+3x+3 = (x+\frac{3}{2})^2 + \frac{3}{4}$, then

$$\int \frac{dx}{x^2+3x+3} = \int \frac{dx}{(x+\frac{3}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+3}{\sqrt{3}}\right) + C$$

4. (a)

$$\lim_{x\to 0}\frac{1-\cos x}{(1-e^x)^2} \overset{\text{l'H } \frac{0}{0}}{=\!=} \lim_{x\to 0}\frac{\sin x}{-2e^x(1-e^x)} = \lim_{x\to 0}\frac{1}{2e^x}\lim_{x\to 0}\frac{\sin x}{-(1-e^x)} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{-(1-e^x)}$$

$$\overset{\text{l'H } \frac{0}{0}}{=\!=} \frac{1}{2}\lim_{x\to 0}\frac{\cos x}{e^x} = \frac{1}{2}$$

(b) We can covert this into an expression that we have dealt before

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{2x} = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{-2x} = \lim_{x\to\infty}\left(\left(x+\frac{1}{x}\right)^{x}\right)^{-2} = e^{-2}$$

5. (a)

$$\int_0^{\pi/4} x^2\sin 2x dx \overset{u=x^2, dv=\sin(2x)dx=d(-\cos(2x)/2)}{=\!=\!=\!=} -\frac{1}{2}x^2\cos(2x)\Big|_0^{\pi/4} + \int_0^{\pi/4} x\cos(2x)dx$$

$$\overset{u=x, dv=\cos(2x)dx=d(\sin(2x)/2)}{=\!=\!=\!=} \frac{1}{2}x\sin(2x)\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\sin(2x)dx$$

$$= \frac{\pi}{8} - \frac{1}{4}\cos(2x)\Big|_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}$$

(b) Suppose $I = \int_0^{\pi/2} e^{2x}\cos x dx$

$$I \overset{u=e^{2x}, dv=\cos x dx=d(\sin x)}{=\!=\!=\!=} e^{2x}\sin x|_0^{\pi/2} - 2\int_0^{\pi/2} e^{2x}\sin x dx$$

$$\overset{u=e^{2x}, dv=\sin x dx=d(-\cos x)}{=\!=\!=\!=} e^{\pi} + 2e^{2x}\cos x|_0^{\pi/2} - 4I$$

Solve the equal we get $I = \frac{e^{\pi}-2}{5}$

$\square$

1.12 Worksheet for Section 8.2

1. (Warmup): Let $f(x) = \sin x$ for $-\pi/2 \leq x \leq 0$, and R the region between the graph of f and the x-axis on $[-\pi/2, 0]$. Sketch R, and find the volume V of the solid obtained by revolving R about the x-axis.

2. (a) Consider the two integrals $\int \sin^8 x \cos^6 x dx$ and $\int \sin^7 x \cos^4 x dx$. Perform the integration for the integral that is easier to integrate.

   (b) Evaluate each of the following three integrals:

$$\int \tan^6 x \sec^8 x dx, \qquad \int \tan^3 x \sec^4 x dx, \qquad \int \tan^5 x \sec^3 x dx$$

3. Consider the integral $\int \sin^3 x dx$.

   (a) Evaluate the integral by an appropriate reduction formula in Section 8.1, and write the constant added at the end as C_1.

   (b) Evaluate the integral by the method of Section 8.2 (i.e., by factoring out $\sin x$ and by writing the rest of the integrand in terms of $\cos x$), and write the constant added at the end as C_2.

   (c) Show that the results found in (a) and (b) are compatible, by finding a relationship between C_1 and C_2. (*Hint:* Eliminate $\sin x$ from the answers in (a) and/or (b).)

4. Let r be any positive number (rational or irrational), and consider $\int_0^{\pi/2} \sin^r x \cos^3 x dx$.

   (a) Without evaluating the integral, show that $0 < \int_0^{\pi/2} \sin^r x \cos^3 x dx < 2$.

   (b) Evaluate the given integral, and find the smallest positive value of s such that the value of the integral is always less than s, irrespective of the positive value of r.

5. In a short paragraph, describe 3 different types of functions you have learned how to integrate in Section 8.2, and the method employed in integrating each type.

Solution.

1.

$$V = \int_{-\pi/2}^{0} \pi \sin^2 x dx = \int_{-\pi/2}^{0} \frac{\pi(1 - \cos(2x))}{2} dx = \pi \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Big|_{-\pi/2}^{0} = \frac{\pi^2}{4}$$

2. (a)

$$\int \sin^7 x \cos^4 x dx \overset{u=\cos x}{=\!=\!=} - \int (1 - u^2)^3 u^4 du = - \int (1 - u^2)^3 u^4 du$$
$$= - \int u^4 - 3u^6 + 3u^8 - u^{10} du = -\frac{\cos^5}{5} + \frac{3\cos^7 x}{7} - \frac{\cos^9}{3} + \frac{\cos^{11} x}{11} + C$$

   (b)

$$\int \tan^6 x \sec^8 x dx \overset{u=\tan x}{=\!=\!=} \int u^6(1 + u^2)^3 du = \int u^6 + 3u^8 + 3u^{10} + u^{12} du$$
$$= \frac{\tan^7 x}{7} + \frac{\tan^9}{3} + \frac{3\tan^{11} x}{11} + \frac{\tan^{13} x}{13} + C$$

$$\int \tan^3 x \sec^4 x dx \overset{u=\sec x}{=\!=\!=} \int (u^2 - 1)u^3 du = \int u^5 - u^3 du = \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C$$

$$\int \tan^5 x \sec^3 x dx \xlongequal{u=\sec x} \int (u^2-1)^2 u^2 du = \int u^6 - 2u^4 + u^2 du$$
$$= \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

3. (a) Recall the reduction formula

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx$$

Thus

$$\int \sin^3 x dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C_1$$

(b)

$$\int \sin^3 x dx \xlongequal{u=\cos x} - \int (1-u^2) du = -u + \frac{u^3}{3} + C_2 = \frac{\cos^3 x}{3} - \cos x + C_2$$

(c) If we convert $\sin x$ into $\cos x$ in (a), it becomes

$$\int \sin^3 x dx = -\frac{1}{3}(1-\cos^2 x)\cos x - \frac{2}{3}\cos x + C_1 = \frac{1}{3}\cos^3 x - \cos x + C_1$$

So we see that $C_1 = C_2$

4. (a) On $[0, \pi/2]$, $0 \le \sin x \le 1, 0 \le \cos x \le 1$, and therefore $0 \le \sin^r x \cos^3 x \le 1$, by comparison property, we have

$$0 = \int_0^{\pi/2} 0 dx < \int_0^{\pi/2} \sin^r x \cos^3 x dx \le \int_0^{\pi/2} 1 dx = \frac{\pi}{2} < 2$$

(b)

$$\int_0^{\pi/2} \sin^r x \cos^3 x dx \xlongequal{u=\sin x} \int_0^1 u^r (1-u^2) du = \left. \frac{u^{r+1}}{r+1} - \frac{u^{r+3}}{r+3} \right|_0^1 = \frac{2}{(r+1)(r+3)}$$

It is easy to see that $I(r) = \displaystyle\int_0^{\pi/2} \sin^r x \cos^3 x dx$ as a function of r is decreasing, so it attains its minimum $2/3$ at $r = 0$.

5.

□

1.13 Worksheet for Section 8.3

1. (Warmup):

(a) Let $g(t) = \dfrac{t^2}{\sqrt{4-t^2}}$, for $-2 < t < 2$. Find the largest interval containing $t = 1$ for which g^{-1} exists.

(b) Find $\lim\limits_{x\to\infty} (x^2 5^{-x})$, showing your work.

(c) Evaluate $\displaystyle\int \sin^5 t \cos^6 t dt$.

2. In a sentence, describe the difference between integrating by ordinary substitution (as described in Section 5.6), and integrating by trigonometric substitution.

3. A trigonometric substitution is appropriate for only one of the following integrals, and ordinary substitution (as in Section 5.6) is appropriate for the other two. Evaluate each integral by using an appropriate method, and specify which method you use.

(a) $\int x\sqrt{x^2-9}dx$ (b) $\int \frac{1}{x\sqrt{x^2-9}}dx$ (c) $\int \frac{x}{\sqrt{x^2-9}}dx$

4. (a) Use integration by parts and trigonometric substitution in order to find $\int x\sin^{-1} x dx$.

   (b) Find $\int x\tan^{-1} x dx$.

5. Find the area A of the region R between the graph of $f(x) = \frac{x^2}{\sqrt{4-x^2}}$ and the x-axis on the interval $[-1, 1]$.

6. (a) Use trigonometric substitution to show that if $a > 0$ and $b > 0$, then the region enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area $A = \pi ab$

   (b) If $a = b = 1$ for part (a), and you want to find the area of the circular region (which is evidently π) by integration, do you need to use trigonometric substitution? Explain why, or why not.

Solution.

1. (a) First realize that $g(t) = 1 - \frac{4}{\sqrt{4-t^2}}$

$$g'(t) = \frac{t(8-t^2)}{(\sqrt{4-t^2})^3}$$

   Which means that g decreases in $(-2, 0]$ and increases in $[0, 2)$, therefore, $[0, 2)$ is the largest interval containing $t = 1$ where g^{-1} exists.

   (b)

$$\lim_{x\to\infty}(x^2 5^{-x}) = \lim_{x\to\infty}\frac{x^2}{5^x} \overset{\text{l'H } \frac{\infty}{\infty}}{=\!=} \frac{2x}{(\ln 5)5^x} \overset{\text{l'H } \frac{\infty}{\infty}}{=\!=} \frac{2}{(\ln 5)^2 5^x} = 0$$

   (c)

$$\begin{aligned}\int \sin^5 t\cos^6 t dt &\overset{u=\cos t}{=\!=} -\int (1-u^2)^2 u^6 du = \int -u^6 + 2u^8 - u^{10} du \\ &= -\frac{\cos^7 t}{7} + \frac{2\cos^9 t}{9} - \frac{\cos^{11} t}{11} + C\end{aligned}$$

2. Ordinary u-substitution you substitute u for a function of x, for trigonometry substitution you substitute x for a trigonometric function.

3. (a)

$$\int x\sqrt{x^2-9}dx \overset{u=x^2-9}{=\!=} \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}(x^2-9)^{3/2} + C$$

   (b)

$$\int \frac{1}{x\sqrt{x^2-9}}dx \overset{x=3\sec t}{=\!=} \int \frac{1}{9\sec t\tan t}3\sec t\tan t dt = \int \frac{1}{3}dt = \frac{t}{3} + C = \frac{\text{arcsec}(x/3)}{3} + C$$

   (c)

$$\int \frac{x}{\sqrt{x^2-9}}dx \overset{u=x^2-9}{=\!=} \frac{1}{2}\int u^{-1/2}du = \sqrt{x^2-9} + C$$

4. (a)

$$\int x\sin^{-1}x dx \overset{u=\sin^{-1}x, dv=xdx}{=\!=\!=} \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx \overset{x=\sin t}{=\!=\!=}$$

$$\frac{1}{2}t\sin^2 t - \frac{1}{2}\int \frac{\sin^2 t}{\cos t}\cos t dt = \frac{1}{2}t\sin^2 t - \frac{t}{4} + \frac{\sin(2t)}{8} + C$$

$$= \frac{1}{2}t\sin^2 t - \frac{t}{4} + \frac{2\sin t\cos t}{8} + C$$

$$= \frac{x^2\sin^{-1}x}{2} - \frac{\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C$$

(b)

$$\int x\tan^{-1}x dx \overset{u=\tan^{-1}x, dv=xdx}{=\!=\!=} \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$$

We can do this via

$$\overset{x=\tan t}{=\!=\!=} \frac{x^2\arctan x}{2} - \frac{1}{2}\int \frac{\tan^2 t}{\sec^2 t}\sec^2 t dt = \frac{x^2\arctan x}{2} - \frac{1}{2}\int \tan^2 t dt$$

$$= \frac{x^2\arctan x}{2} - \frac{1}{2}\int \sec^2 t - 1 dt = \frac{x^2\arctan x}{2} - \frac{\tan t - t}{2} + C$$

$$= \frac{x^2\arctan x}{2} - \frac{x - \arctan x}{2} + C$$

Or

$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}dx = \frac{1}{2}x^2\tan^{-1}x - \frac{x-\arctan x}{2} + C$$

5.

$$\int \frac{x^2}{\sqrt{4-x^2}}dx \overset{x=2\sin t}{=\!=\!=} \int \frac{4\sin^2 t}{2\cos t}2\cos t dt = \int 4\sin^2 t dt$$

$$= 2\int 1 - \cos(2t)dt = 2t - \sin(2t) + C = 2t - 2\sin t\cos t + C$$

$$= 2\arcsin(x/2) + x\sqrt{1-\frac{x^2}{4}} + C$$

So

$$A = \int_{-1}^{1} \frac{x^2}{\sqrt{4-x^2}}dx = \left(2\arcsin(x/2) - x\sqrt{1-\frac{x^2}{4}}\right)\Bigg|_{-1}^{1} = \frac{2\pi}{3} - \sqrt{3}$$

6. (a)

$$A = 4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}dx \overset{x=a\sin t}{=\!=\!=} 4\int_0^{\pi/2} ab\cos^2 t dt = 4ab$$

(b) Yes.

□

1.14 Worksheet for Section 8.4 and 8.6

1. Where applicable in (a) - (c), carry out the necessary division (as in Example 1 on p. 530) in preparation for integration. If division is not needed, give a reason why it is not needed.

(a) $\dfrac{x^2}{x^2-x+1}$ (b) $\dfrac{x-1}{x^2-x+1}$ (c) $\dfrac{x^4+4}{x^2+1}$

2. In the following, write the rational function as a combination of factors like those appearing in (4) and/or (5) of Section 8.4, without finding the numerical values of the constants $A_1, \cdots, B_1, \cdots, C_1, \cdots$

(a) $\dfrac{6x^2-4x+1}{(x-2)^2}$ (b) $\dfrac{x^3+2x^2+3x+4}{x^4-16}$

3. (a) Show that $\dfrac{x^2}{(x^2-4)^2} = \dfrac{1}{4}\left(\dfrac{1}{x-2}+\dfrac{1}{x+2}\right)^2$.

(b) Use partial fractions to solve $\displaystyle\int \frac{x^2}{(x^2-4)^2}dx$.

(c) Could the integral in (b) be solved by trigonometric substitution? Explain your answer.

(d) Find $\displaystyle\int \frac{\sqrt{x+4}}{x^2}dx$. (Hint: Substitute $u=\sqrt{x+4}$, and then use the result in part (b)!)

4. (a) In a complete sentence, tell why there needs to be an even number of subintervals in Simpson's Rule.

(b) Does there need to be an even number of subintervals in the Trapezoidal Rule? Explain your answer in a complete sentence.

5. Suppose f is continuous and positive on the interval $[x_{k-1}, x_{k+1}]$, and let $x_{k-1} < x_k < x_{k+1}$.

(a) Draw the trapezoid above the interval $[x_{k-1}, x_k]$, with vertices $(x_{k-1}, 0)$, $(x_k, 0)$, $(x_k, f(x_k))$, $(x_{k-1}, f(x_{k-1}))$. Then find the area A of the trapezoid.

(b) Using the result of (a), find the sum A_1 of the area of two trapezoids, the one trapezoid above the interval $[x_{k-1}, x_k]$ and the other trapezoid above the interval $[x_k, x_{k+1}]$. Draw a picture for the two trapezoids.

(c) Using the formula derived in (b) as a guide, explain why the 2's appear inside the brackets of the Trapezoidal Rule ((2) in Section 8.6). Why is there no 2 accompanying $f(x_0)$ or $f(x_n)$?

Solution.

1. (a)

$$\frac{x^2}{x^2-x+1} = 1 + \frac{x-1}{x^2-x+1}$$

(b) Not needed, since the degree of numerator is less than that of the denominator.

(c)

$$\frac{x^4+1}{x^2+4} = x^2 - 1 + \frac{5}{x^2+1}$$

2. (a)

$$\frac{6x^2-4x+1}{(x-2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{A_3}{(x-2)^3}$$

(b)

$$\frac{x^3+2x^2+3x+4}{x^4-16} = \frac{A_1}{x-2} + \frac{B_1}{x+2} + \frac{C_1x+D_1}{x^2+2}$$

3. (a)

$$\frac{1}{4}\int\left(\frac{1}{x-2}+\frac{1}{x+2}\right)^2 = \frac{1}{4}\int\left(\frac{2x}{(x-2)(x+2)}\right)^2 = \left(\frac{x}{x^2-4}\right)^2 = \frac{x^2}{(x^2-4)^2}$$

(b)

$$\begin{aligned}\int \frac{x^2}{(x^2-4)^2}dx &= \frac{1}{4}\int\left(\frac{1}{x-2}+\frac{1}{x+2}\right)^2 dx\\ &= \frac{1}{4}\int \frac{1}{(x-2)^2}+\frac{1}{(x+2)^2}+\frac{2}{(x-2)(x+2)}dx\\ &= \frac{1}{4}\int \frac{1}{(x-2)^2}+\frac{1}{(x+2)^2}+\frac{1/2}{x-2}+\frac{-1/2}{x+2}dx\\ &= -\frac{1}{4(x-2)}-\frac{1}{4(x+2)}+\frac{1}{8}\ln|x-2|-\frac{1}{8}\ln|x+2|+C\end{aligned}$$

(c) We can substitute x for $2\sec u$.

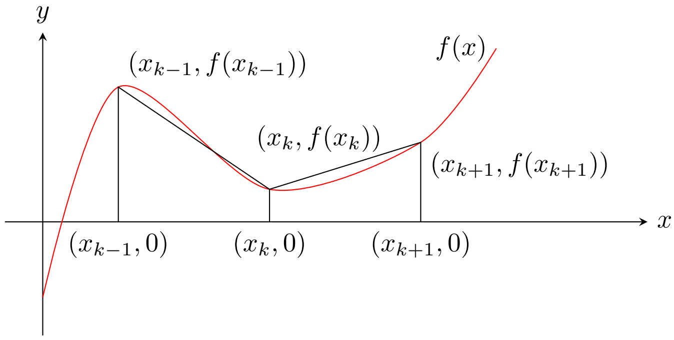
(d)

$$\int \frac{\sqrt{x+4}}{x^2}dx \overset{u=\sqrt{x+4}}{\underset{x=u^2-4}{=\!=\!=}} \int \frac{u}{(u^2-4)^2}2udu = 2\int \frac{u^2}{(u^2-4)^2}du$$

4. (a) For Simpson's Rule, you need to add up areas under parabolas which is determined by three points and two intervals, so there needs to be an even number of subintervals.

   (b) No need for Trapezoidal rules.

5.



(a)

$$A = \frac{x_k - x_{k-1}}{2}(f(x_{k-1}) + f(x_k))$$

(b)

$$A_1 = \frac{x_k - x_{k-1}}{2}(f(x_{k-1}) + f(x_k)) + \frac{x_{k+1} - x_k}{2}(f(x_{k+1}) + f(x_k))$$

(c) The middle sides get added twice whereas the end sides get only added once.

□

1.15 Worksheet for Section 8.6 and 8.7

1. Suppose f is continuous on $[a, b]$ and the graph of f is concave downward on (a, b). Does this mean that the Trapezoidal Rule yields a number that is larger than, or a number that is smaller than, the integral $\int_a^b f(x)dx$? Draw a picture with your answer and explanation.

2. (a) Find the exact value A of $\int_0^1 x^3 dx$ by performing the integration.

   (b) Approximate the integral in (a) by Simpson's Rule with $n = 4$, obtaining the value B. Tell what the relationship between A and B is, and confirm it by using (8) in Section 8.6.

(c) Tell why the Trapezoidal Rule for the integral in part (a) with any value of $n > 1$ is (should be) larger than the exact value of the integral. (*Hint:* Draw a graph of the function x^3 on the appropriate interval, and appropriate line segments corresponding to the Trapezoidal Rule.) Then corroborate the assertion by obtaining the Trapezoidal Rule value C with $n = 50$.

3. Consider $\displaystyle\int_{-1}^{1} x^4 dx$. Find the exact value D of the integral, and Simpson's Rule's value E, where $n = 2$. By comparing the graphs of x^2 and x^4, tell why D should indeed be less than E.

4. Find examples of functions f, g and h that are continuous on $[1, \infty)$, and such that (a) $\displaystyle\int_1^\infty f(x)dx = \infty$ (b) $\displaystyle\int_1^\infty g(x)dx = 2$ (c) $\displaystyle\int_1^\infty h(x)dx$ does not converge to equal ∞ or $-\infty$.

5. (a) Suppose f and g are continuous on $[a, \infty)$. Prove that if both $\displaystyle\int_a^\infty f(x)dx$ and $\displaystyle\int_a^\infty f(x)dx$ converge, then $\displaystyle\int_a^\infty (f(x) + g(x))dx$ converges.

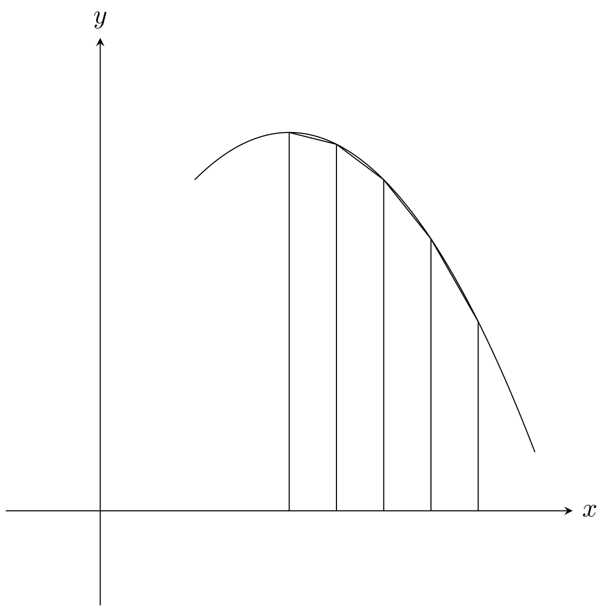
   (b) Give an example of f and g are continuous on $[a, \infty)$, for which $\displaystyle\int_a^\infty (f(x) + g(x))dx$ converges, but neither $\displaystyle\int_a^\infty f(x)dx$ nor $\displaystyle\int_a^\infty g(x)dx$ converges

6. For (a) and (b), determine whether the area A of the region of the region between the graph of f and the x-axis on the interval $[0, \infty)$ is finite, or is infinite. If it is finite, determine its numerical value.

   (a) $\displaystyle\int_0^\infty \frac{x^3}{\sqrt{2 + x^4}}dx$ (b) $\displaystyle\int_0^\infty \frac{x^3}{(2 + x^4)^2}dx$

Solution.

1. The approximation is smaller than the actuall integral since the trapezoids are below the graph of the funtion.



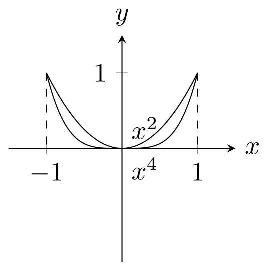
2. (a)

$$A = \int_0^1 x^3 dx = \frac{1}{4}$$

(b)
$$B = \frac{1/4}{3}(0^3 + 4\cdot(1/4)^3 + 2\cdot(1/2)^3 + 4\cdot(3/4)^3) = \frac{1}{4}$$
$A = B$ because $(x^3)^{(4)} = 0$ and the error
$$E \le \frac{\max|f^{(4)}(x)|}{180n^4}(b-a)^4 = 0$$
Which implies that $E = 0$.

(c) Similar to 1 but the function concave downwards.

3. Using Simpson's rule we have
$$E = \frac{1}{3}((-1)^4 + 4\cdot 0^4 + 1^4) = \frac{2}{3}$$
We also know that $D = \int_{-1}^{1} x^4 dx = \frac{2}{5}$. We see that x^2 is the parabolic approximation of x^4 passing $(-1,1),(0,0),(1,1)$, and $D < E$.



4. (a) $f(x)$ could be x
   (b) $g(x)$ could be $2/x^2$
   (c) $h(x)$ could be $\sin x$

5. (a)
$$\int_a^\infty (f(x)+g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$$
since the right hand side converges, so is the left hand side.

   (b) $f(x) = x$, $g(x) = -x$.

6. (a)
$$\int \frac{x^3}{\sqrt{2+x^4}}dx \overset{u=2+x^4}{=\!=\!=} \frac{du}{4\sqrt{u}} = \frac{1}{2}\sqrt{u} + C = \frac{1}{2}\sqrt{2+x^4} + C$$
So
$$\int_0^\infty \frac{x^3}{\sqrt{2+x^4}}dx = \lim_{b\to\infty}\int_0^b \frac{x^3}{\sqrt{2+x^4}}dx = \lim_{b\to\infty} \frac{1}{2}\sqrt{2+x^4}\Big|_0^b = \infty$$

   (b)
$$\int \frac{x^3}{(2+x^4)^2}dx \overset{u=2+x^4}{=\!=\!=} \frac{du}{4u^2} = -\frac{1}{4u} + C = -\frac{1}{4(2+x^4)} + C$$
So
$$\int_0^\infty \frac{x^3}{(2+x^4)^2}dx = \lim_{b\to\infty}\int_0^b \frac{x^3}{(2+x^4)^2}dx = \left(-\frac{1}{4(2+b^4)}\right) - \left(-\frac{1}{4(2+0)}\right) = \frac{1}{8}$$

□

1.16 Worksheet for Section 9.1

Definition: For any positive integer n, the definition of n *factorial*, which is written as $n!$, is given by $n! = n(n1)(n2)(2)(1)$. We also define $0! = 1$.
Definition: Let $f^{(k)}(0)$ exist for $k = 0, 1, 2, \cdots, n$. Then the nth Taylor polynomial of f about 0 is defined as the polynomial $p_n(x)$ of degree at most n, given by

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Thus $1! = 1$, $2! = (2)(1) = 2$, $3! = (3)(2)(1) = 6$, etc. We remark that $(n+1)! = (n+1)n!$ for $n \geq 0$.

1. Calculate $4!$ and $7!$.

2. Prove that $(n+1)! = (n+1)n!$ for every $n \geq 0$.

3. For any integer $n \geq 2$, find the value of $\dfrac{(n+1)!(n-1)!}{(n!)^2}$.

4. (a) Show that p_0 has the same value at 0 as f does.
   (b) Show that p_1 has the same value at 0 and the same slope at 0 as f does.
   (c) Show that p_2 has the same value at 0, the same slope at 0, and the same concavity near $(0, 0)$ as f does.

5. Let a, b, and c be nonzero constants, and let $g(x) = ax^2 + bx + c$. Write down formulas for the Taylor polynomials p_2 and p_4 of g, and explain how you arrived at them.

6. Let $f(x) = \sin x$. We want to find some Taylor polynomials for f.
   (a) Find $f(0), f'(0), f''(0), \cdots, f^{(8)}(0)$.
   (b) Using (a) and the formula above for $p_n(x)$, write down the Taylor polynomials $p_0(x), p_1(x), \cdots, p_8(x)$ for f.
   (c) What pattern do you see from your answers in (b)?
   (d) Using the information from (b) and (c), find the highest degree of x in each of the following:

      i. $p_{13}(x)$ ii. $p_{30}(x)$ iii. $p_{99}(x)$

Solution.

1. $4! = 24$, $7! = 5040$.

2. $(n+1)! = (n+1)(n)(n-1)\cdots 1 = (n+1)n!$, note that this definition only works when $n \geq 1$, for $n = 0$, we have the extra definition $0! = 1$, so we check that $(0+1)! = 1 = (0+1)0!$.

3. $$\frac{(n+1)!(n-1)!}{(n!)^2} = \frac{(n+1)(n-1)!}{n!} = \frac{(n+1)(n-1)!}{n(n-1)!} = \frac{(n+1)}{n}$$

4. (a) $p_0(x) = f(0)$
   (b) $p_1(x) = f(0) + f'(0)x$, $p_1'(x) = f'(0)$, so $p_1(0) = f(0)$, $p_1'(0) = f'(0)$.
   (c) $p_2(x) = f(0) + f'(0)x + f''(0)x^2/2$, $p_2'(x) = f'(0) + f''(0)x$, $p_2''(x) = f''(0)$, so $p_2(0) = f(0)$, $p_2'(0) = f'(0)$, $p_2''(0) = f''(0)$

5. $g'(x) = 2ax + b$, $g''(x) = 2a$, so

$$p_2(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 = c + bx + ax^2 = g(x)$$

$$p_4(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g^{(3)}(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 = c + bx + ax^2 + 0 + 0 = g(x)$$

6. It is not hard to see that $f^{(n)}(x) = \begin{cases} \sin x, & n \bmod 1 \equiv 0 \\ \cos x, & n \bmod 1 \equiv 1 \\ -\sin x, & n \bmod 1 \equiv 2 \\ -\cos x, & n \bmod 1 \equiv 3 \end{cases}$, so $f^{(n)}(0) = \begin{cases} 0, & n \bmod 1 \equiv 0 \\ 1, & n \bmod 1 \equiv 1 \\ 0, & n \bmod 1 \equiv 2 \\ -1, & n \bmod 1 \equiv 3 \end{cases}$

   (a) $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = 1$, $f^{(6)}(0) = 0$, $f^{(7)}(0) = -1$, $f^{(8)}(0) = 0$.

   (b) $p_0(x) = 0$, $p_1(x) = x$, $p_2(x) = x$, $p_3(x) = x - \frac{x^3}{3!}$, $p_4(x) = x - \frac{x^3}{3!}$, $p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$, $p_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$, $p_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$, $p_8(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

   (c)
$$p_{2k}(x) = p_{2k+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{(k-1)} \frac{x^{2k-1}}{(2k-1)!}$$
$$p_{2k+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

   (d) i. The highest degree of x in $p_{13}(x)$ is 13
   ii. The highest degree of x in $p_{30}(x)$ is 29
   iii. The highest degree of x in $p_{99}(x)$ is 99

$\square$

1.17 Sample Exam 3 MATH 141 - Calculus II

Instructions: Please put everything away and no calculators are permitted on this exam. Legibly print your full name on each answer sheet, and write and sign the honor pledge on the first page. Write your solutions neatly on the answer sheets, doing problem 1 on the first page, problem 2 on the second page, and so on. Continue your work onto the back of the page, if necessary. When you have finished, arrange the answer sheets in order and turn them in to your TA.

1. (10 points) Evaluate the integral
$$\int \sin^5 x \cos^4 x dx$$

2. (15 points) Evaluate the integral
$$\int \frac{1}{(1-9x^2)^{3/2}} dx$$

3. (15 points) Evaluate the integral
$$\int \frac{2x+3}{x^3+3x} dx$$

4. Determine whether the improper integral converges. If it does, determine the value of the integral.

   (a) (15 points)
$$\int_0^1 \frac{e^x}{\sqrt{e^x - 1}} dx$$

   (b) (15 points)
$$\int_2^\infty x e^{-2x+3} dx$$

5. (a) (15 points) Find an upper bound

$$E_n^T \leq \frac{K_T}{12n^2}(b-a)^3$$

for the error E_n^T in approximating the integral

$$\int_0^1 e^{x^2}\,dx$$

by the trapezoidal rule with $n = 10$ subintervals.

(b) (15 points) Let

$$f(x) = 1 + e^{-x}$$

Find the third Taylor polynomial $p_3(x)$ of f.

Solution.

1.

$$\begin{aligned}\int \sin^5 x \cos^4 x dx &\overset{u=\cos x}{=\!=\!=} -\int (1-u^2)^2 u^4 du = -\int (1-2u^2+u^4)u^4 du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C\end{aligned}$$

2.

$$\begin{aligned}\int \frac{1}{(1-9x^2)^{\frac{3}{2}}} dx &\overset{x=\sin u/3}{=\!=\!=} \int \frac{1}{(1-9x^2)^{\frac{3}{2}}} dx = \int \frac{1}{\cos^2 u}\frac{1}{3}\cos u du \\ &= \frac{1}{3}\int \sec^2 u du = \frac{\tan u}{3} + C = \frac{x}{\sqrt{1-9x^2}} + C\end{aligned}$$

3. Write it as partial fractions, we have

$$\frac{2x+3}{x^3+3x} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Clear the denominators we get

$$2x+3 = A(x^2+3) + (Bx+C)x$$

By taking $x = 0$ we have $3 = 3A \Rightarrow A = 1$ and that $2x = (B+1)x^2 + Cx$, by comparing the coefficients, we get $B = -1$, $C = 2$, hence

$$\begin{aligned}\int \frac{2x+3}{x^3+3x} dx &= \int \frac{1}{x} + \frac{-x+2}{x^2+3} dx \\ &= \int \frac{1}{x} - \frac{x}{x^2+3} + \frac{2}{x^2+3} dx \\ &= \ln x - \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\arctan(x/\sqrt{3}) + C\end{aligned}$$

4. (a)

$$\int_0^1 \frac{e^x}{\sqrt{e^x-1}} dx \overset{u=e^x-1}{=\!=\!=} \int_0^{e-1} u^{-\frac{1}{2}} du = 2u^{\frac{1}{2}}\Big|_0^{e-1} = 2\sqrt{e-1}$$

(b)

$$\int xe^{-2x+3} dx \overset{u=x, dv=e^{-2x+3}dx}{=\!=\!=\!=\!=} -\frac{1}{2}xe^{-2x+3} + \frac{1}{2}\int e^{-2x+3} dx = -\frac{1}{2}xe^{-2x+3} - \frac{1}{4}e^{-2x+3} + C$$

$$
\begin{aligned}
\int_2^\infty xe^{-2x+3}dx &= \lim_{b\to\infty}\int_2^b xe^{-2x+3}dx \\
&= \lim_{b\to\infty} -\frac{1}{2}xe^{-2x+3} - \frac{1}{4}e^{-2x+3}\Big|_2^b \\
&= \lim_{b\to\infty}\left(-\frac{1}{2}be^{-2b+3} - \frac{1}{4}e^{-2b+3}\right) - \left(-\frac{1}{2}2e^{-1} - \frac{1}{4}e^{-1}\right) \\
&= -\frac{1}{2}\lim_{b\to\infty}\frac{b}{e^{2b-3}} + \frac{5}{4}e^{-1} \\
&\overset{\text{l'H } \frac{\infty}{\infty}}{=\!=} -\frac{1}{2}\lim_{b\to\infty}\frac{1}{2e^{2b-3}} + \frac{5}{4}e^{-1} \\
&= \frac{5}{4e}
\end{aligned}
$$

5. (a) Suppose $f(x) = e^{x^2}$, then $f'(x) = 2xe^{x^2}$, $f''(x) = (2+4x^2)e^{x^2}$, Since both $w+4x^2$ and e^{x^2} increases on $[0,1]$, so is $f^{(2)}(x)$, thus $K_T = \max|f''(x)| = |f''(1)| = 6e$.

   (b) We can write a table

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$1+e^{-x}$	2
1	$-e^{-x}$	-1
2	e^{-x}	1
3	$-e^{-x}$	-1

   so the third Taylor polynomial is

$$p_3(x) = 2 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

□

1.18 Worksheet for Section 9.2

1. Find the following limits:

   a) $\lim\limits_{n\to\infty}\dfrac{(5/n)-3n^2}{(2/n)-4n^2}$ b) $\lim\limits_{k\to\infty}(2k)^{3/k}$ c) $\lim\limits_{n\to\infty}\left(n\tan\dfrac{\pi}{n}\right)$

2. (a) Let $a_0 = 1$, $a_1 = 1+\frac{1}{3}$, $a_2 = 1+\frac{1}{3}+\frac{1}{9}$, $a_3 = 1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}$, and so on. For an arbitrary positive integer n, write down the formula for a_n. Then find $\lim\limits_{n\to\infty} a_n$.

   (b) Let $r > 0$, and let $b_n = 1 + r + r^2 + \cdots + r^n$, for $n \geq 1$. Find $\lim\limits_{n\to\infty} b_n$. (Note that there are two distinct cases, depending on the value of r.)

3. You deposit \$100 in a savings account that pays 5 percent interest, compounded annually. Thus, after 1 year, there is the original \$100 plus the interest $(100)(0.05)$ dollars in the account, that is, there are $100 + (100)(0.05)$ dollars in the account.

   (a) Show that after 2 years there are $100(1+0.05)^2$ dollars in the account.

   (b) Let n be an arbitrary positive integer. Find a formula for the amount in the account after n years.

   (c) Determine how many years it would take for the amount in the account to reach \$200.

4. (a) Evaluate $\displaystyle\int_0^1 t^2 dt$.

   (b) Let $f(t) = t^2$ for $0 \leq t \leq 1$. Find the right sum for f, where the interval $[0,1]$ is subdivided into n subintervals of equal length.

(c) Let $a_n = \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{n^2}{n^3}$. Use (a) and (b) to find $\lim_{n\to\infty} a_n$.

(d) Suppose the integral in (a) were $\int_0^2 t^2\, dt$, and that for the right sum the interval $[0,2]$ is subdivided into n subintervals (as in (b)). Find a formula for a_n.

5. (Extra): Use the ideas of Problem 4 to find $\lim_{n\to\infty} a_n$, where $a_n = \frac{1^5}{n^6} + \frac{2^5}{n^6} + \cdots + \frac{n^5}{n^6}$

Solution.

1. (a)

$$\lim_{n\to\infty} \frac{(5/n) - 3n^2}{(2/n) - 4n^2} = \lim_{n\to\infty} \frac{((5/n) - 3n^2)\frac{1}{n^2}}{((2/n) - 4n^2)\frac{1}{n^2}} = \lim_{n\to\infty} \frac{(5/n^3) - 3}{(2/n^3) - 4} = \frac{0-3}{0-4} = \frac{3}{4}$$

(b) First we know

$$\lim_{k\to\infty} \frac{3\ln(2k)}{k} \overset{\text{l'H } \frac{\infty}{\infty}}{=\!=\!=} \lim_{k\to\infty} \frac{\frac{3}{2k}}{1} = 0$$
$$\lim_{k\to\infty} (2k)^{3/k} = \lim_{k\to\infty} e^{\frac{3\ln(2k)}{k}} = 1$$

(c)

$$\lim_{n\to\infty} \left(n \tan\frac{\pi}{n}\right) = \lim_{n\to\infty} \pi \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} \overset{x=\frac{\pi}{n}}{=\!=\!=} \lim_{x\to 0} \pi \frac{\tan x}{x} \overset{\text{l'H } \frac{0}{0}}{=\!=\!=} \pi \lim_{x\to 0} \frac{\sec^2 x}{1} = \pi$$

2. (a) By inspection we found out that

$$a_n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n-1}} + \frac{1}{3^n}$$

(b) First we note that

$$\begin{aligned} rb_n - b_n &= r(1 + r + r^2 + \cdots + r^{n-1} + r^n) - (1 + r + r^2 + \cdots + r^{n-1} + r^n) \\ &= (r + r^2 + r^3 + \cdots + r^n + r^{n+1}) - (1 + r + r^2 + \cdots + r^{n-1} + r^n) \\ &= r^{n+1} - 1 \end{aligned}$$

If $r \neq 1$ we have

$$(r-1)b_n = r^{n+1} - 1 \Rightarrow b_n = \frac{r^{n+1} - 1}{r - 1}$$

If $r = 1$ we have $b_n = 1 + n$, so it is not hard to see that $\lim_{n\to\infty} b_n \begin{cases} \infty, r \geq 1 \\ \frac{1}{1-r}, r < 1 \end{cases}$

3. (a)

$$100(1 + 0.05)^2 = 100 + 100(0.05) + (100 + 100(0.05))(0.05)$$

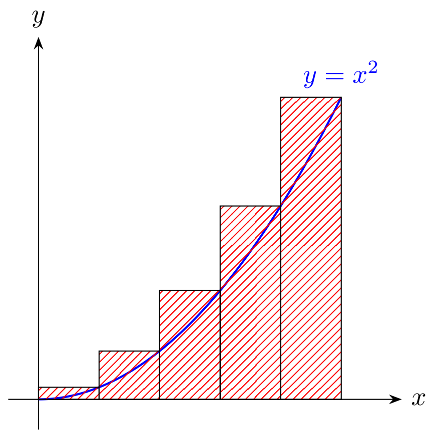
(b) The amount in the account after n years is $100(1 + 0.05)^n$

(c) We want solve the smallest integer n such that

$$100(1 + 0.05)^n \geq 200 \Rightarrow 1.05^n \geq 2 \Rightarrow n \geq \log_{1.05} 2 = \frac{\ln 2}{\ln 1.5} \approx 1.7$$

So $n = 2$.

4.



(a) $\int_0^1 t^2 dt = \frac{1}{3}$

(b) The right sum would be $\frac{1}{n}\left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right)$.

(c) Note that a_n is the right sum in (b), the limit of Riemann sum is the actual integral in (a), so $\lim_{n\to\infty} a_n = \int_0^1 t^2 dt = \frac{1}{3}$.

(d) The right sum of $\int_0^2 t^2 dt$ is $\frac{2}{n}\left(\left(\frac{2}{n}\right)^2 + \left(\frac{4}{n}\right)^2 + \cdots + \left(\frac{2n}{n}\right)^2\right)$.

5. a_n is the right sum of $\int_0^1 x^5 dx$ with n subintervals of equal length. Therefore

$$\lim_{n\to\infty} a_n = \int_0^1 x^5 dx = \frac{1}{6}$$

$\square$

1.19 Worksheet for Section 9.3

1. (a) Suppose that $\lim_{n\to\infty} a_n = L$. Tell in a complete sentence why $\lim_{n\to\infty} a_{n+1} = L$.

   (b) Suppose that $\lim_{n\to\infty} a_n = 3$. Is it necessarily true that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$? Explain why it is true, or why it is not necessarily true. (*Hint:* Does (4) in Section 9.3 apply?)

   (c) Suppose that $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \pi$. Prove that $\lim_{n\to\infty} a_n b_n = 0$.

2. When a superball is dropped onto a hardwood floor, it bounces up to approximately 80% of its original height. Suppose that the ball is dropped initially from a height of 5 feet above the floor, and let b_n = the maximum height of the nth bounce.

   a) Evaluate b_1, b_2, and b_3.

   b) Prove that $\lim_{n\to\infty} b_n = 0$.

3. Let the sequence $\{a_n\}_{n=1}^{\infty}$ be defined recursively by

$$a_1 = 1, \quad a_2 = \sin 1, \quad a_3 = \sin(\sin 1), \quad \text{and in general, } a_{n+1} = \sin a_n, \text{ for all } n \geq 1$$

   (a) By Exercise 4.3.60(b), we know that $0 < a_{n+1} < a_n < 1$ for $n > 1$. From these inequalities, we can use a theorem in Section 9.3 to conclude that $\{a_n\}_{n=1}^{\infty}$ has a limit. Write out the statement of the theorem.

(b) Let the limit in part (a) be denoted L. Convince yourself and your group that because $a_n \to L$ as $n \to \infty$, and because $\sin x$ is continuous,

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sin a_n = \sin L$$

(c) Find the numerical value of L.

4. (a) Show that $\ln(n+1) - \ln n = \int_n^{n+1} \frac{1}{t}\,dt - \int_n^{n+1} \frac{1}{t}\,dt = \int_n^{n+1} \frac{1}{t}\,dt > \frac{1}{n+1}$.

(b) Let $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$, for $n = 1, 2, \ldots$. Use (a) to show that $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence. (*Hint:* Show that $a_n - a_{n+1} > 0$.)

(c) Use the left sum of $\int_1^{n+1} \frac{1}{t}\,dt$ with partition $\{1, 2, \ldots, n+1\}$ to show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$.

(d) Use (c) and the definition of a_n to show that $a_n > 0$ for all $n \geq 1$.

(e) Use (b) and (d) to show that $\{a_n\}_{n=1}^{\infty}$ converges to a number r. ($r \approx 0.577216$, and is known as the Euler-Mascheroni constant.)

Solution.

1. (a) Since the sequence $\{a_n\}_{n=1}^{\infty}$ approaches L, so is $\{a_{n+1}\}_{n=1}^{\infty}$, and they have the same limit.

(b) Yes. Since $\lim_{n\to\infty} a_n = 3 \neq 0$, use (a), we know that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\lim_{n\to\infty} a_{n+1}}{\lim_{n\to\infty} a_n} = \frac{3}{3} = 1$.

(c) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = 0 \cdot \pi = 0$.

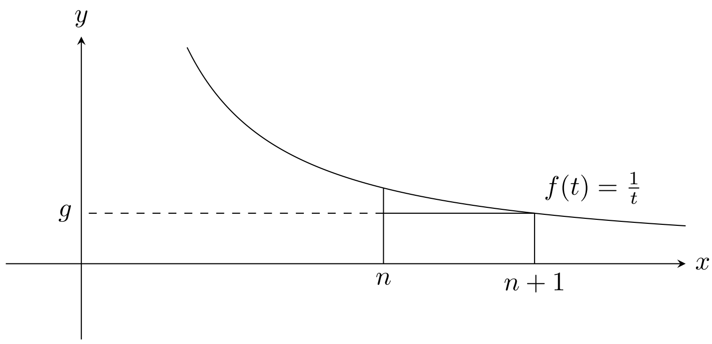
2. (a) We may suppose $b_0 = 5$, then $b_1 = b_0 \cdot 0.8 = 4$, $b_2 = b_1 \cdot 0.8 = 3.2$, $b_3 = b_2 \cdot 0.8 = 2.56$.

(b) From the fact that $b_n = b_{n-1} \cdot 0.8$ we may conclude $b_n = b_0(0.8)^n = 5(0.8)^n$, so $\lim_{n\to\infty} b_n = 0$.

3. (a) This uses the Bolzano-Weierstrass theorem.

(b)

(c) Consider $f(x) = x - \sin x$, $f'(x) = 1 - \cos x \geq 0$, so $f(x)$ is increasing, and we know that $f(0) = 0$, so L must be 0.



4. (a) We can use comparison property of integrals

$$\ln(n+1) - \ln n = \int_n^{n+1} \frac{1}{t}dt > \int_n^{n+1} \frac{1}{n+1}dt = \frac{1}{n+1}$$

This is geometrically clear because the area under the curve between $[n, n+1]$ is bigger than the area of the rectangle between $[n, n+1]$.

(b) Notice that

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)$$
$$a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} - \ln(n+1)$$

$a_n - a_{n+1} = \ln(n+1) - \ln n - \dfrac{1}{n+1} > 0$ by (a).

(c) The left sum of $\int_1^{n+1} \frac{1}{t} dt = \ln(n+1)$ with partition $\{1, 2, \ldots, n+1\}$ would be $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, since $\frac{1}{t}$ is increasing, the left sum is less than the actually integral, namely

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$$

(d) By (c), we know

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n > \ln(n+1) - \ln(n) > 0$$

(e) We use Bolzano-Weierstrass theorem, by (b), we know that $\{a_n\}_{n=1}^{\infty}$ is decreasing, and by (d), we know that $\{a_n\}_{n=1}^{\infty}$ is bounded, thus $\{a_n\}_{n=1}^{\infty}$ converges.

□

1.20 Worksheet for Section 9.4 and 9.5

1. (a) Give a careful definition of a sequence, and the definition of a sequence being a convergent sequence. Then give a careful definition of a series, and the definition of a series being a convergent series.

   (b) Suppose $\lim_{n\to\infty} a_n = 0$. Give two examples of $\sum_{n=1}^{\infty} a_n$, one which converges, and the other which diverges.

   (c) Give two different proofs that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   (d) Determine whether $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges, or whether it diverges. Support your answer.

2. (a) Show that $\dfrac{2}{(n+1)(n+3)} = \dfrac{1}{n+1} - \dfrac{1}{n+3}$ for all $n \geq 1$.

   (b) Write out the 5th partial sum of $\sum_{n=1}^{\infty} \left(\dfrac{1}{n+1} - \dfrac{1}{n+3}\right)$. Then find its numerical value.

   (c) Find the sums of the series $\sum_{n=1}^{\infty} \dfrac{2}{(n+1)(n+3)}$ and $\sum_{n=12}^{\infty} \dfrac{2}{(n+1)(n+3)}$.

3. (a) Write $0.636363\cdots$ as a geometric series, and find its numerical value.
   (*Hint:* $0.636363\cdots = 0.63 + 0.0063 + 0.000063 + \cdots$.)

   (b) Write $0.999\cdots$ as a geometric series, and find the numerical value of the geometric series.

   (c) Does the answer in (b) indicate that $0.999\cdots = 1$? Discuss with your group.

4. From Exercise 48(a) on p. 602, we know that the numerical value of $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

   (a) Using this information, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} \dfrac{1}{n^{1.9}}$ converges. If $\sum_{n=1}^{\infty} \dfrac{1}{n^{1.9}}$ converges, what are the relative sizes of $\sum_{n=1}^{\infty} \dfrac{1}{n^{1.9}}$ and of $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$?

(b) Similarly, explain in a complete sentence why we can, or cannot, conclude that $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$ converges. If $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$ converges, what are the relative sizes of $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{2}}$?

Solution.

1. (a) A sequence is an ordered list of numbers $\{a_n\}_{n=1}^{\infty}$, a_n converges to a if $\lim_{n\to\infty} = a$.

   A series is an infinite sum $\sum_{n=1}^{\infty} a_n$, the n-th partial sum of the series $S_n = \sum_{k=1}^{n} a_n$ is the sum of the first n terms of the series, $\{S_n\}_{n=1}^{\infty}$ forms a sequence. The series converges if S_n converges.

   (b) If $a_n = \frac{1}{n}$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is called the *harmonic series* which diverges. If $a_n = \frac{1}{n^2}$, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Basel problem, as in 4.) converges.

   (c) i. First notices that

$$\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} > \overbrace{\frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}}^{2^{n+1}-2^n \text{ times}} = 2^n \cdot \frac{1}{2^{n+1}} = \frac{1}{2}$$

   So

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}} \right) \geq 1 + \sum_{n=1}^{\infty} \frac{1}{2} = \infty$$

   ii. $\int_1^{\infty} \frac{1}{x} dx = \infty$.

   (d) i. $\int_1^{\infty} \frac{1}{2x-1} dx = \infty$

   ii. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots > \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = \infty.$

2. (a)

$$\frac{1}{n+1} - \frac{1}{n+3} = \frac{(n+3)-(n+1)}{(n+1)(n+3)} = \frac{2}{(n+1)(n+3)}$$

   (b) Its 5th partial sum is

$$\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{8}\right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{7} - \frac{1}{8} = \frac{95}{168}$$

   (c)

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{2}{(n+1)(n+3)} \\
&= \lim_{m\to\infty} \left[\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \cdots \right. \\
&\left. + \left(\cancel{\frac{1}{m}} - \cancel{\frac{1}{m+1}}\right) + \left(\cancel{\frac{1}{m+1}} - \frac{1}{m+2}\right) + \left(\cancel{\frac{1}{m+1}} - \frac{1}{m+3}\right) \right] \\
&= \lim_{m\to\infty} \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{m+2} - \frac{1}{m+3} \right] \\
&= \frac{5}{6}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=12}^{\infty} \frac{2}{(n+1)(n+3)} &= \lim_{m\to\infty} \sum_{n=1}^{m} \frac{2}{(n+1)(n+3)} \\
&= \lim_{m\to\infty} \left[\left(\frac{1}{13} - \cancel{\frac{1}{15}}\right) + \left(\frac{1}{14} - \cancel{\frac{1}{16}}\right) + \left(\cancel{\frac{1}{15}} - \cancel{\frac{1}{17}}\right) + \cdots \right. \\
&\left. + \left(\cancel{\frac{1}{m}} - \cancel{\frac{1}{m+1}}\right) + \left(\cancel{\frac{1}{m+1}} - \frac{1}{m+2}\right) + \left(\cancel{\frac{1}{m+1}} - \frac{1}{m+3}\right) \right] \\
&= \lim_{m\to\infty} \left[\frac{1}{13} + \frac{1}{14} - \frac{1}{m+2} - \frac{1}{m+3} \right] \\
&= \frac{27}{182}
\end{aligned}$$

3. (a)

$$0.636363\cdots = \frac{63}{100} + \frac{63}{100^2} + \frac{63}{100^3} + \cdots = 63 \sum_{n=1}^{\infty} \left(\frac{1}{100}\right) = 63 \frac{\frac{1}{100}}{1 - \frac{1}{100}} = \frac{63}{99} = \frac{7}{11}$$

(b)

$$0.999\cdots = 9 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 9 \frac{\frac{1}{10}}{1 - \frac{1}{10}} = 1$$

(c) Yes.

4. (a) No. Since by comparison $\frac{1}{n^2} < \frac{1}{n^{1.9}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it doesn't tell you much about $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$.

(b) Yes. Since by comparison $\frac{1}{n^2.1} < \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so is $\sum_{n=1}^{\infty} \frac{1}{n^{2.1}}$.

□

1.21 Worksheet for Section 9.6

1. **Warmup:** Fill in the blanks with short, reasonable answers. Please fill them out by yourself, and then check your answers with the others in your group.

   (a) A sequence is a ____________ and a series is a ____________

   (b) The 6th partial sum of $\sum_{n=2}^{\infty} a_n$ is ____________.

   (c) The series $\sum_{n=1}^{\infty} cr^n$ converges and equals ____________ if and only if ____________.

   (d) If $\sum_{n=1}^{\infty} a_n$ diverges, then what we know about $\lim_{n\to\infty} a_n$ is ____________.

2. $\sum_{n=2}^{\infty} \frac{1}{n^p}$ is called a p-series, for $p > 0$. Show that the Ratio Test *cannot* be used to determine the convergence or the divergence of a p-series. (The p-series converges if $p > 1$ and diverges if $0 < p \leq 1$.)

3. Let $m \geq 0$, and let $\sum_{n=m}^{\infty} a_n$ be a positive series. Show that $\sum_{n=m}^{\infty} a_n$ converges (so partial sums converge to a limit), or the partial sums s_j approach ∞ as $j \to \infty$ (and we associate $\sum_{n=m}^{\infty} a_n$ with ∞).

   Comparison Test (Theorem 9.13): Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be positive series.

i. If $\sum_{n=m}^{\infty} a_n$ converges and $0 < b_n \leq a_n$ for all $n \geq m$, then $\sum_{n=m}^{\infty} b_n$ converges.

ii. If $\sum_{n=m}^{\infty} a_n$ diverges and $0 < a_n \leq b_n$ for all $n \geq m$, then $\sum_{n=m}^{\infty} b_n$ diverges.

4. Use the Comparison Test, and the Ratio Test or Root Test where applicable, to determine whether each of the following series converges, or diverges.
(a) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ (b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ (c) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(n+1)}$ (*Hint:* Exercise 2 might help.)

5. Use the Ratio Test to determine if the series $\sum_{n-1}^{\infty} \frac{2^n (n!)^2}{(2n)!}$ converges.

6. Let $a_{2n} = 1/2^n$ and $a_{2n+1} = 1/3^n$ for $n \geq 1$. Show that $\lim_{n\to\infty} (a_{n+1}/a_n)$ does note exist, but nevertheless by the Comparison Test, $\sum_{n=1}^{\infty} a_n$ converges.

Solution.

1. (a) A sequence is an ordered list $\{a_n\}_{n=1}^{\infty}$ and a series is an infinite sum $\sum_{n=1}^{\infty} a_n$.

(b) The 6th partial sum of $\sum_{n=2}^{\infty} a_n$ is $a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

(c) The series $\sum_{n=1}^{\infty} cr^n$ converges and equals $\frac{cr}{1-r}$ if and only if $|r| < 1$.

(d) If $\sum_{n=1}^{\infty} a_n$ diverges, then what we know about $\lim_{n\to\infty} a_n$ is not much.

2. You cannot use the Ratio test because $\lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^p = 1$.

3. Consider the partial sums $\{s_n\}_{n=m}^{\infty}$. Since $a_n > 0$, $\{s_n\}$ strictly increases. If it is bounded, then by Bolzano-Weierstrass theorem, it has a limit, so the series converges, if $\{s_n\}$ is unbounded, then $s_j \to \infty$.

4. (a) We can use the Ratio test $\lim_{n\to\infty} \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} = \lim_{n\to\infty} \frac{n}{2(n+1)} = \frac{1}{2}$, thus the series converges.

(b) We can use the Comparison test since $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$ and $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$ diverges, so is $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$.

(c) We can use the Comparison test since $\frac{1}{(2n+1)(n+1)} < \frac{1}{(2n)(n)}$ and $\sum_{n=2}^{\infty} \frac{1}{2n^2}$ converges, so is $\sum_{n=2}^{\infty} \frac{1}{(2n+1)(n+1)}$.

5.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{2^{n+1}((n+1)!)^2}{(2(n+1))!}}{\frac{2^n(n!)^2}{(2n)!}} &= \lim_{n\to\infty} \left[\frac{2^{n+1}}{2^n}\left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)!}\right] \\
&= \lim_{n\to\infty} \left[\frac{2^{n+1}}{2^n}\left(\frac{(n+1)!}{n!}\right)^2 \frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right] \\
&= \lim_{n\to\infty} 2\cdot(n+1)^2 \frac{1}{(2n+2)(2n+1)} = \frac{1}{2}
\end{aligned}$$

Therefore the series converges.

6. Note that there is a parity subtlety.

$$\lim_{n\to\infty} \frac{a_{2n+1}}{a_{2n}} = \lim_{n\to\infty} \frac{1/3^n}{1/2^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0, \quad \lim_{n\to\infty} \frac{a_{2n+2}}{a_{2n+1}} = \lim_{n\to\infty} \frac{1/2^{n+1}}{1/3^n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^n = \infty$$

So $\lim\limits_{n\to\infty}(a_{n+1}/a_n)$ does not exist. On the other hand, by Comparison test, Consider series $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ such that $b_{2n} = 1/2^n$, $b_{2n+1} = 0$ and $c_{2n} = 0$, $c_{2n+1} = 1/3^n$. Then $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ and $\sum\limits_{n=1}^{\infty} c_n = \sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ both converge, so is $\sum\limits_{n=1}^{\infty}(b_n + c_n)$, we know that $a_n \le b_n + c_n$, therefore $\sum\limits_{n=1}^{\infty} a_n$ also converges.

$\square$

1.22 Worksheet for Section 9.7 and 9.8a

1. (a) In order to use the Alternating Series Test for a given series $\sum\limits_{n=1}^{\infty} a_n$, what properties must the terms a_n of the series have?

   (b) It is a fact that $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. Find the smallest positive integer $j > 0$ for which the Alternating Series Test guarantees that $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ differs from $\pi/4$ by less than 0.001.

2. Suppose that $\sum\limits_{n=0}^{\infty} a_n 3^n$ converges.

   (a) Find $\lim\limits_{n\to\infty}(a_n 3^n)$, giving reasons.

   (b) Prove that $(a_n)^{1/n} \le 1/3$, for all large values of n.

3. (a) Give an example of a series $\sum\limits_{n=1}^{\infty} a_n$ that converges, but such that $\sum\limits_{n=1}^{\infty} a_n^2$ diverges.

   (b) Suppose that a given series $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely. Must $\sum\limits_{n=1}^{\infty} a_n^2$ converge? Support your answer.

4. Suppose that the radius of convergence of $\sum\limits_{n=0}^{\infty} a_n x^n$ is precisely 4. Which of the following numbers is necessarily in the interval of convergence of the series, and why?

   a) 3.9 b) 4.1 c) -3 d) -5 e) 4 f) -4

5. (a) Determine whether the series $\sum\limits_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{3^n n!}$ converges absolutely, or converges conditionally, or diverges. Support your answer.

   (b) Find the radius of convergence R of the power series $\sum\limits_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{3^n n!} x^n$.

Solution.

1. (a) It should be of the form $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ where a_n decreases and $\lim\limits_{n\to\infty} a_n = 0$.

   (b) Recall there error estimate for alternating series $\sum\limits_{n=1}^{\infty}(-1)^n a_n$

$$\left|\sum_{n=1}^{\infty}(-1)^n a_n - \sum_{n=1}^{j}(-1)^n a_n\right| = E_j < |a_{j+1}|$$

Therefore we want

$$\left|\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} - \frac{\pi}{4}\right| < \frac{1}{2(j+1)+1} \leq 0.001 \Rightarrow j \geq 498.5$$

So the smallest positive integer j would be 499.

2. (a) $\lim_{n\to\infty} a_n 3^n = 0$ since $\sum_{n=0}^{\infty} a_n 3^n$ converges.

   (b) By (a), we know that for n large, $(a_n 3^n) \leq 1$, taking $1/n$ power on both sides we get $a_n^{1/n} 3 \leq 1 \Rightarrow a_n^{1/n} \leq 1/3$.

3. (a) If $a_n = \dfrac{(-1)^n}{\sqrt{n}}$, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ converges but $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

   (b) Yes. For large n, $|a_n| \leq 1$ since $\lim_{n\to\infty} a_n = 0$, so $a_n^2 < |a_n|$, but $\sum_{n=1}^{\infty} |a_n|$ converges since $\sum_{n=1}^{\infty} a_n$ converges absolutely, so is $\sum_{n=1}^{\infty} a_n^2$.

4. (a) and (c)

5. Note that $a_{n+1} = \dfrac{1\cdot 3\cdot 5\cdots(2(n+1)-3)}{3^{n+1}(n+1)!}$ which really is $a_{n+1} = \dfrac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{3^{n+1}(n+1)!}$, so the series converges absolutely since

$$\lim_{n\to\infty} \frac{\frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{3^{n+1}(n+1)!}}{\frac{1\cdot 3\cdot 5\cdots(2n-3)}{3^n n!}} = \lim_{n\to\infty} \frac{2n-1}{3(n+1)} = \frac{2}{3} < 1$$

6. We need that $R = \left(\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|\right)^{-1} = \left(\dfrac{2}{3}\right)^{-1} = \dfrac{3}{2}$.

□

1.23 Worksheet for Section 9.9

1. (a) Define carefully: "Taylor series." How does a Taylor series relate to a general power series?

   (b) What is the difference between the radius of convergence R of a power series $\sum_{n=0}^{\infty} c_n x^n$ and the interval of convergence I of the power series?

2. Let f be a function such that $f(n)(0)$ exists for all $n \geq 0$. Let $g(x) = \sum_{n=0}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$.

   (a) Suppose $f(0) = 3$ and $f^{(n+1)}(0) = 6f^{(n)}(0)$ for all $n \geq 0$. Does this guarantee that $g(3)$ exists? Explain your answer. (Hint: What is the relationship between $f^{(n)}(0)$ and $f(0)$?)

   (b) Suppose $f^{(n+1)}(0) = (n+1)f^{(n)}(0)$ for all $n \geq 0$. What can you say about the domain of g? Explain your answer.

3. Notice that the Taylor series for $\sin x$ is $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)!} x^{2n+1}$, which is valid for all x.

   (a) Write down the Taylor series for $\sin(2x)$.

   (b) For $\sin(2x)$, write down the formulas for the Taylor polynomials $p_{100}(x)$ and $p_{101}(x)$.

   (c) Find $\lim_{x\to\infty} p_{101}(x)$. What can you say about $|p_{101}(x) - \sin(2x)|$ for really large values of x?

   (d) Let $f(x) = \sin(2x)$. Show that $|f^{(n)}(x)| \leq 2^n$ for each real x and each $n > 0$.

(e) Let $M > 0$. Use the Lagrange Remainder Formula (equation (11) in Section 9.9) to show that

$$|r_n(x)| \leq \frac{2^{n+1}}{(n+1)!} M^{n+1} \text{ for all } x \text{ in } [M, M]$$

(f) By Example 7 in Section 9.7, $\lim\limits_{n\to\infty} (2^{n+1}M^{n+1}/(n+1)!) = 0$, so that for each x, (e) implies that $\lim\limits_{n\to\infty} r_n(x) = 0$. From (c) and the preceding remark, convince yourself that

i. for each x, the Taylor series for $\sin(2x)$ converges to $\sin(2x)$,

ii. the Taylor series of $\sin(2x)$ *does equal* $\sin(2x)$ for all numbers x,

iii. for each n, the Taylor polynomial $p_n(x)$ is in no way like $\sin(x)$ for really large values of x.

4. (a) Write down the power series for $\dfrac{1}{1-x}$, and tell why the radius of convergence of the power series is 1.

(b) Use the fact that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)$ is $\dfrac{1}{(1-x)^2}$ to find the power series for $\dfrac{1}{(1-x)^2}$. Then determine the radius of convergence of the power series both by using the Ratio Test and by citing the Differentiation Theorem for power series (Theorem 9.25).

(c) Evaluate $\sum\limits_{n=1}^{\infty} n\left(\dfrac{1}{2}\right)^{n-1}$. (*Hint:* The series found in (b) might be helpful.)

Solution.

1. (a) A Taylor series of a function $f(x)$ at $x = a$ is defined to be the power series $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$.

(b)

2. Try the Ratio Test

$$\lim_{n\to\infty}\left|\frac{\frac{f^{(n+1)}(0)}{(n+1)!}x^{n+1}}{\frac{f^{(n)}(0)}{n!}x^n}\right| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(0)}{f^{(n)}(0)}\right|\left|\frac{x^{n+1}}{x^n}\right|\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{f^{(n+1)}(0)}{f^{(n)}(0)}\right|\left|\frac{x}{n+1}\right|$$

(a) The ratio above becomes

$$\lim_{n\to\infty}\left|\frac{6f^{(n)}(0)}{f^{(n)}(0)}\right|\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}\left|\frac{6x}{n+1}\right| = 0$$

So $g(x)$ converges for any x, including 3.

(b) The above ratio becomes

$$\lim_{n\to\infty}\left|\frac{(n+1)f^{(n)}(0)}{f^{(n)}(0)}\right|\left|\frac{x}{n+1}\right| = \lim_{n\to\infty}|x|$$

So the series converges iff $|x| < 1$, thus the radius of convergence is $R = 1$, and the domain of g is $(-1, 1)$.

3. Note that here

(a)

$$f(x) = \sin(2x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2x)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+1}$$

(b)

$$p_{100}(x) = \sum_{n=0}^{49}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+1}, \qquad p_{101}(x) = \sum_{n=0}^{50}\frac{(-1)^n 2^{2n+1}}{(2n+1)!}x^{2n+1}$$

The subscript of p indicates the degree of the Taylor polynomial.

(c) $\lim_{x\to\infty} p_{101}(x) = \infty$ because the limit of any nonconstant polynomial is $\pm\infty$ when x approaches ∞. On the other hand, since $\sin(2x)$ is bounded by 1, so $|p_{101}(x) - \sin(2x)|$ also approaches ∞ for as x approaches ∞.

(d) Notice $f^{(0)}(x) = \sin(2x)$, $f^{(1)}(x) = 2\cos(2x)$, $f^{(2)}(x) = -2^2\sin(2x)$, $f^{(3)}(x) = -2^3\cos(2x)$, due to chain rule, we see that $|f^{(n)}(x)| \leq 2^n$

(e) Recall Lagrange Remainder Theorem $r_n(x) = \dfrac{f^{(n+1)(c)}x^{n+1}}{(n+1)!}$, where c is some number between 0 and x. By (d), we know that $|f^{(n+1)}(c)| \leq 2^{n+1}$, and since $x \in [-M, M]$, $|x| \leq |M|$, so

$$|r_n(x)| = \left|\frac{f^{(n+1)(c)}x^{n+1}}{(n+1)!}\right| \leq \frac{2^{n+1}}{(n+1)!}M^{n+1}$$

(f)

4. (a) $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$ is the geometric series whose radius of convergence is 1.

(b)

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \sum_{n=0}^{\infty}\frac{d}{dx}(x^n) = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$$

Which again has radius of convergence 1.

(c)

$$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^{n-1} = \frac{1}{(1-\frac{1}{2})^2} = 4$$

□

1.24 Worksheet on Complex Numbers

Definition: A *complex number* has the form $a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$, or equivalently, $i^2 = -1$. Numbers of the form bi or ib are called *pure imaginary numbers*.

1. Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$
2. Multiplication: $(a + bi)(c + di) = (ac + bidi) + (adi + bci) = (ac - bd) + (ad + bc)i$

Definition: The *complex plane* has the horizontal axis with the real numbers, and vertical axis with the pure imaginary numbers. The complex number $a + bi$ corresponds to the point (a, b) in the complex plane.

1. In the following, find each complex number in the form $a + bi$, and plot each in the complex plane.
(a) $(2 - 3i)(5 + 6i)$ (b) $(1 + i)^2$ (c) $(1 + i)^3$

2. For the given equation, find all solutions, and graph the solutions in the complex plane.
(a) $z^2 - 3z + 6 = 0$ (b) $z^3 + 8z = 0$

3. The *conjugate* of the complex number $z = a + bi$ is $z = a - bi$. Also, $|z| = \sqrt{a^2 + b^2}$

   (a) What is the (geometric) relationship between z and $\bar{z}$ in the complex plane?

   (b) Consider the equation $z^2 + az + b = 0$, with a and b real numbers. Suppose there are two distinct solutions to the equation, and one of them is a non-real number z_0. Then show that the other solution is $\overline{z_0}$. Was that true of the solutions of $z^2 - 3z + 6 = 0$ in Problem 2(a)?

   (c) Suppose $z_1 = 3 + 4i$ and $z_2 = 3 - 4i$. Find z_1 and z_2 in the complex plane, and show that $|z_1| = |z_2|$.

4. The complex power series for e^z, $\sin z$, and $\cos z$ are:

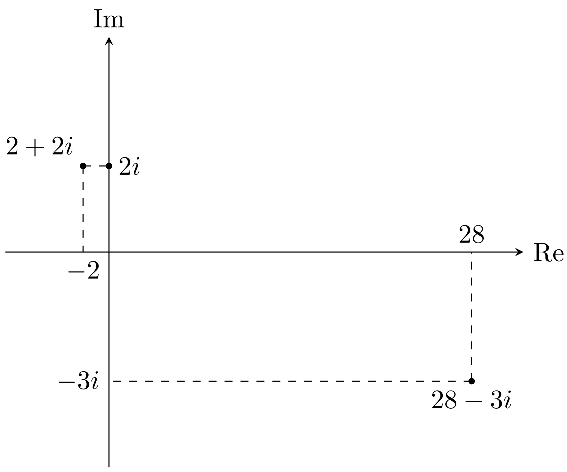
$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \qquad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

   (a) Write out the first several terms of the power series for e^{iz} (that is, when z is replaced by iz), eliminating all powers of i^n for which $n > 1$ (using the fact that $i^2 = -1$, $i^4 = -1$, etc.).
   (b) Use the formulas for $\sin z$ and $\cos z$ to show that $e^{iz} = \cos z + i \sin z$.
   (c) Replace z in (b) by the real number θ (which could represent an angle), which yields the formula $e^{i\theta} = \cos\theta + i\sin\theta$.
      i. Show that $|e^{i\theta}| = 1$.
      ii. Show that $e^{i\pi} = -1$. (Note: This is a celebrated formula in mathematics, in which 4 of the most important numbers $(-1, \pi, e, i)$ are in one equation.)

5. Find the 3 3rd roots of $-27i$, and graph them in the complex plane.

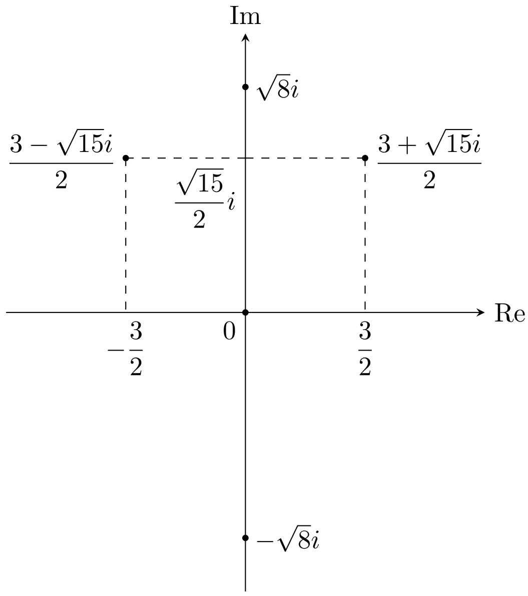
Solution.

1.



   (a) $(2-3i)(5+6i) = 10 - 3i - 18i^2 = 10 - 3i - 18 \cdot (-1) = 28 - 3i$
   (b) $(1+i)^2 = 1 + 2i + i^2 = 1 + 2i + (-1) = 2i$
   (c) $(1+i)^3 = 1 + 3i + 3i^2 + i^3 = 1 + 3i + 3 \cdot (-1) + i \cdot (-1) = -2 + 2i$

2.



(a) Recall the *quadratic formula:* the solutions of quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.24.1}$$

Where $\Delta = b^2 - 4ac$ is called the *discriminant*.

- If $\Delta > 0$, then these two roots are both real.
- If $\Delta = 0$, then these two roots coincide and is real.
- If $\Delta < 0$, then these two roots are both complex numbers.

In this case, we know that

$$z = \frac{(-) - 3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 6}}{2} = \frac{3 \pm \sqrt{-15}}{2} = \frac{3 \pm \sqrt{15}\sqrt{-1}}{2} = \frac{3 \pm \sqrt{15}i}{2}$$

(b) $z^3 + 8z = 0 \iff z(z^2 + 8) = 0 \iff \iff \begin{cases} z = 0 \text{ or} \\ z^2 + 8 = 0 \end{cases} \iff \begin{cases} z = 0 \text{ or} \\ z = \pm\sqrt{8}i \end{cases}$

3. (a) They are reflections of each other across the real axis.

(b)

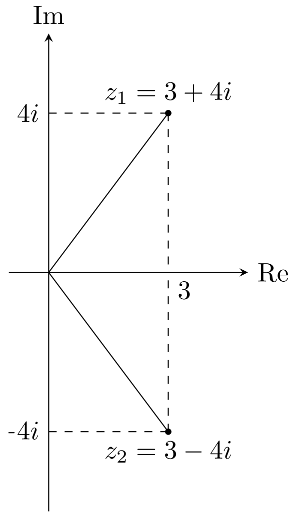
Method I: Note that two complex roots in (1.24.1) are always conjugate of each other.

Method II: Since z_0 is a solution of $z^2 + az + b = 0$ and a, b are real,

$$0 = \overline{z_0^2 + az_0 + b} = \overline{z_0^2} + \overline{az_0} + \overline{b} = \overline{z_0}^2 + \overline{az_0} + \overline{b} = \overline{z_0}^2 + a\overline{z_0} + b$$

Therefore $\overline{z_0}$ is the other solution.

(c) $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, $|z_1| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$. Being reflection of each other, they are of the same length.



4. (a)

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n z^n}{n!} = 1 + iz + \frac{i^2z^2}{2!} + \frac{i^3z^3}{3!} + \frac{i^4z^4}{4!} + \frac{i^5z^5}{5!} + \cdots$$
$$= 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \frac{z^4}{24} + \frac{iz^5}{120} + \cdots$$

(b) Let's divide the terms in the Taylor series into even powers and odd powers

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{k=0}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i^{2k+1}z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(i^2)^k z^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{i(i^2)^k z^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} + i\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} = \cos z + i \sin z \end{aligned}$$

(c) i. $|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

ii. $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

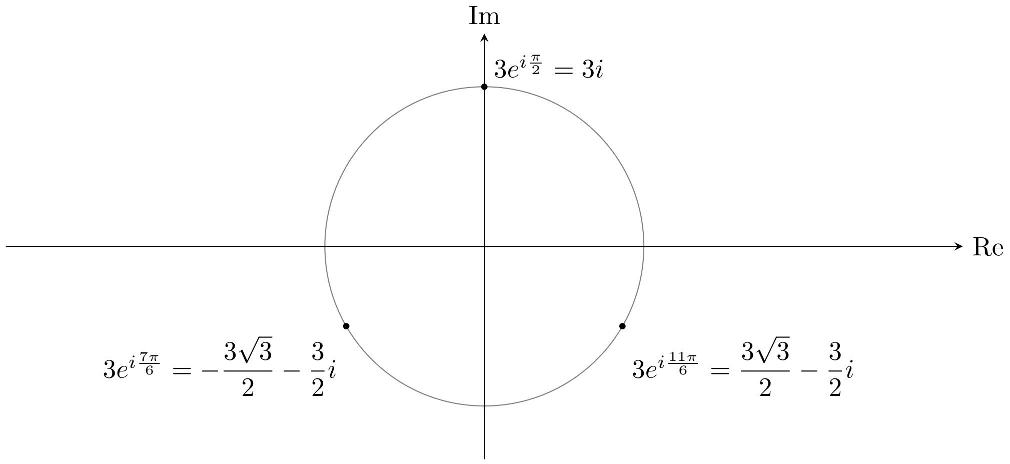
5. Recall to find $\sqrt[n]{z}$, first we write $z = re^{i\theta}$ in polar coordinates, where $r = |z| \geq 0$, θ is the angle of z. Then

$$\sqrt[n]{z} = \sqrt[n]{r}e^{i\frac{\theta}{n} + k\frac{2\pi}{n}i} = \sqrt[n]{r}e^{i\left(\frac{\theta+2k\pi}{n}\right)}, \quad k = 0, 1, 2, \cdots, n-1$$

Note that these roots are all on the circle of radius $\sqrt[n]{r}$ centered at origin, and the angle between two consecutive roots is $\dfrac{2\pi}{n}$, so they divide the circle into n equal parts.

Here $-27i = 27e^{\frac{3\pi}{2}}$, so the 3 cubic roots are

$$\sqrt[3]{-27i} = \begin{cases} 3e^{i\frac{\pi}{2}}, & k=0 \\ 3e^{i\frac{7}{6}}, & k=1 \\ 3e^{i\frac{11\pi}{6}}, & k=2 \end{cases}$$



□

1.25 Sample Exam 4 MATH 141 - Calculus II

Instructions: Please put everything away and no calculators are permitted on this exam. Legibly print your full name on each answer sheet, and write and sign the honor pledge on the first page. Write your solutions neatly on the answer sheets, doing problem 1 on the first page, problem 2 on the second page, and so on. Continue your work onto the back of the page, if necessary. When you have finished, arrange the answer sheets in order and turn them in to your TA.

1. (a) (10 points) Evaluate the limit as a number, ∞, or $-\infty$.

$$\lim_{n\to\infty}\left(1-\frac{4}{3n}\right)^{3n}$$

(b) (10 points) Find all values of $(27i)^{1/3}$, and plot them in the complex plane.

Solution.

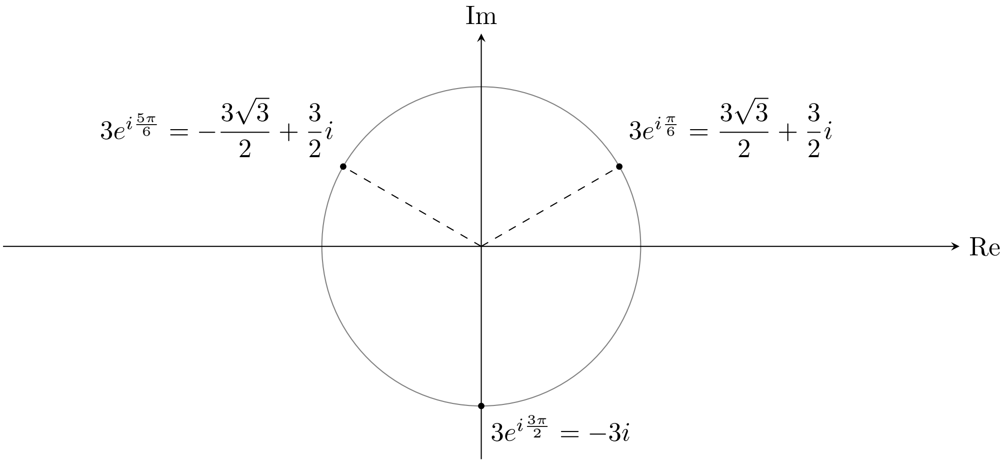
i. Let $y = \left(1-\frac{4}{3n}\right)^{3n}$, then $\ln y = 3n\ln\left(1-\frac{4}{3n}\right)$, and

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty}\frac{\ln\left(1-\frac{4}{3n}\right)}{\frac{1}{3n}} \overset{\text{l'H }\frac{0}{0}}{=\!=\!=} \lim_{n\to\infty}\frac{\frac{1}{1-\frac{4}{3n}}\cdot\frac{4}{3n^2}}{-\frac{1}{3n^2}} = \lim_{n\to\infty}-\frac{4}{1-\frac{4}{3n}} = -4$$

So $\lim\limits_{n\to\infty} y = e^{-4}$.

ii. First we write $27i = 27e^{i\frac{\pi}{2}}$, so

$$(27i)^{1/3} = 3e^{i\frac{\pi}{6}+k\frac{2\pi}{3}i} = \begin{cases} 3e^{i\frac{\pi}{6}}, & k=0 \\ 3e^{\frac{5\pi}{6}}, & k=1 \\ 3e^{\frac{3\pi}{2}}, & k=2 \end{cases}$$



□

2. Determine whether the series converges or diverges. If it converges, find its sum.

   (a) (10 points)

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{n-1}}$$

   (b) (10 points)

$$\sum_{n=2}^{\infty} \frac{4}{n^2-1}$$

Solution.

(a) Recall $\sum_{n=m}^{\infty} ar^n = a \sum_{n=m}^{\infty} r^n = a\frac{r^m}{1-r}$ for any $|r| < 1$.

$$\sum_{n=3}^{\infty} \frac{2^{n+1}}{3^{n-1}} = \sum_{n=3}^{\infty} \frac{2^n \cdot 2}{3^n \cdot 3^{-1}} = \sum_{n=3}^{\infty} 6\left(\frac{2}{3}\right)^n = 6\frac{\left(\frac{2}{3}\right)^3}{1-\frac{2}{3}} = \frac{16}{3}$$

(b)

$$\sum_{n=2}^{\infty} \frac{4}{n^2-1} = \sum_{n=2}^{\infty} \frac{4}{(n-1)(n+1)} = \sum_{n=2}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n+1}\right)$$
$$= \left(\frac{2}{1} - \cancel{\frac{2}{3}}\right) + \left(\frac{2}{2} - \cancel{\frac{2}{4}}\right) + \left(\cancel{\frac{2}{3}} - \cancel{\frac{2}{5}}\right) + \cdots = \frac{2}{1} + \frac{2}{2} = 3$$

□

3. Determine whether the series converges or diverges. State any tests that you use.

   (a) (10 points)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$$

   (b) (10 points)

$$\sum_{n=1}^{\infty} \frac{n^{2n}}{(2n)!}$$

Solution.

(a) We can use comparison test $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2+n+1} \le \sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n^2} = \sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ converges.

(b) We can use the ratio test (keep in mind that $2 < e < 3$)

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{\frac{(n+1)^{2(n+1)}}{(2(n+1))!}}{\frac{n^{2n}}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^{2(n+1)}}{n^{2n}} \frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty} \frac{(n+1)^{2n+2}}{n^{2n}} \frac{(2n)!}{(2n+2)!} \\
&= \lim_{n\to\infty} \frac{(n+1)^{2n}(n+1)^2}{n^{2n}} \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \lim_{n\to\infty} \frac{(n+1)^{2n}}{n^{2n}} \frac{(n+1)^2}{(2n+2)(2n+1)} \\
&= \lim_{n\to\infty} \frac{(n+1)^{2n}}{n^{2n}} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n} \frac{(n+1)}{2(2n+1)} \\
&= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^n\right]^2 \lim_{n\to\infty} \frac{(n+1)}{2(2n+1)} = e^2 \cdot \frac{1}{4} = \frac{e^2}{4} > \frac{2^2}{4} = 1
\end{aligned}
$$

Therefore this series diverges.

□

4. (a) (10 points) Show the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n+1}$$

converges, and give an upper bound for the fifth truncation error E_5.

(b) (10 points) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n3^{n+1}}$$

diverges, converges conditionally, or converges absolutely. State any tests that you use.

Solution.

(a) $\dfrac{1}{2^n+1}$ is positive, decreasing and $\lim\limits_{n\to\infty} \dfrac{1}{2^n+1} = 0$, so the series converges.

$$E_5 < |a_6| = \frac{1}{2^6+1} = \frac{1}{65}$$

(b) Consider the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(-2)^{n+1}}{(n+1)3^{(n+1)+1}}}{\frac{(-2)^n}{n3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}}{(-2)^n} \frac{n}{n+1} \frac{3^{n+1}}{3^{n+2}} \right| = \lim_{n\to\infty} \left| \frac{(-2)n}{3(n+1)} \right| = \frac{2}{3} < 1$$

Therefore the series converges absolutely.

□

5. (a) (10 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x-1)^n}{\sqrt{n}}$$

(b) (10 points) Find the Taylor series of $f(x) = x\sin(x/2)$ about $a = 0$.

Solution.

(a) Use the ratio test

$$\lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{(-2)^n(x-1)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}}{(-2)^n}\frac{\sqrt{n}}{\sqrt{n+1}}\frac{(x-1)^{n+1}}{(x-1)^n}\right|$$
$$= \lim_{n\to\infty}\left|(-2)\sqrt{\frac{n}{n+1}}(x-1)\right| = 2|x-1|$$

The series converges if $2|x-1|<1 \iff \dfrac{1}{2}<x<\dfrac{3}{2}$. As for the endpoints

- If $x=\frac{1}{2}$, $\sum_{n=0}\frac{(-2)^n\left(\frac{1}{2}-1\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{\left((-2)\cdot\left(-\frac{1}{2}\right)\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ diverges due to the p series test.
- If $x=\frac{3}{2}$, $\sum_{n=0}\frac{(-2)^n\left(\frac{3}{2}-1\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{\left((-2)\cdot\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}$ converges by the alternating series test.

So the interval of convergence is $\left(\frac{1}{2},\frac{3}{2}\right]$.

(b) Recall the Taylor series of f about a is

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f^{(3)}(a)}{3!}(x-a)^3+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Recall that $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$, so

$$\sin(x/2)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(2n+1)!}x^{2n+1}$$

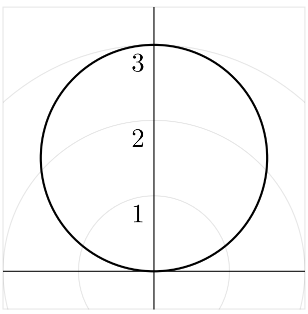
$$f(x)=x\sin(x/2)=x\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(2n+1)!}x^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}(2n+1)!}x^{2n+2}$$

□

1.26 Worksheet for Section 10.1

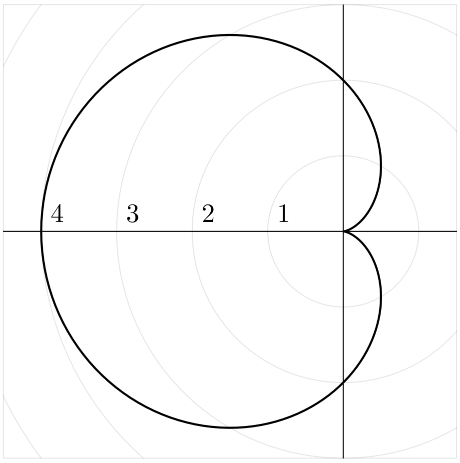
Identify each polar graph by name and equation, using information from the labels given on the graphs

1.



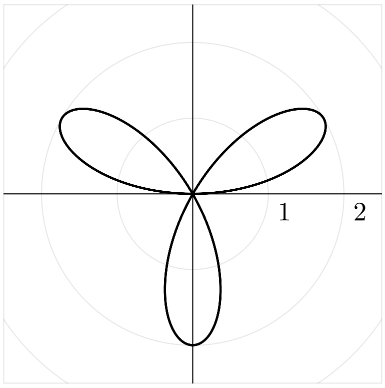
Solution. This is a circle: $r=3\sin\theta$. □

2.



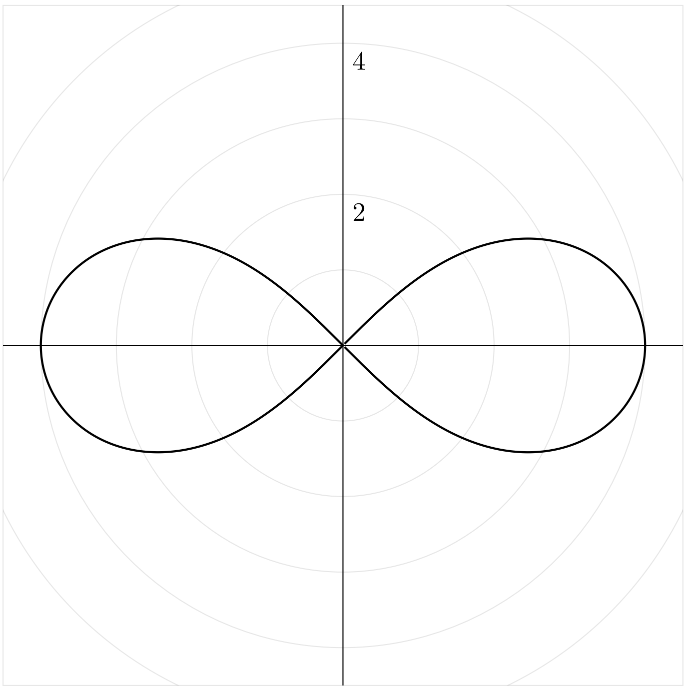
Solution. This is a cardioid: $r = 2 - 2\cos\theta$. □

3.



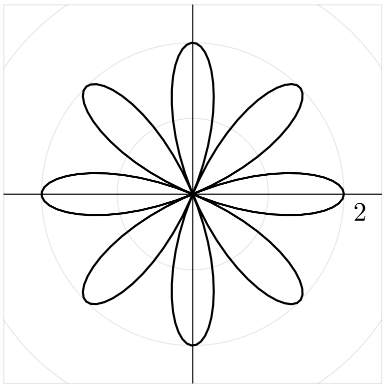
Solution. This is a 3-leaved rose: $r = 2\sin(3\theta)$. □

4.



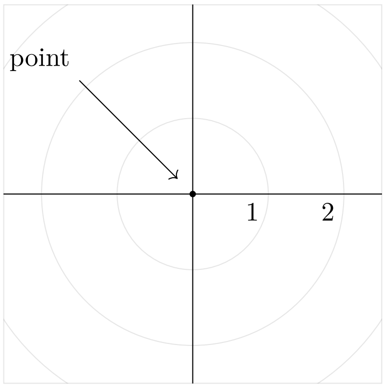
Solution. This is a lemniscate: $r^2 = 16\cos(2\theta)$. ☐

5.



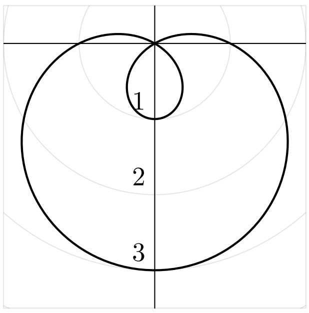
Solution. This is a 8-leaved rose: $r = 2\cos(4\theta)$. ☐

6.



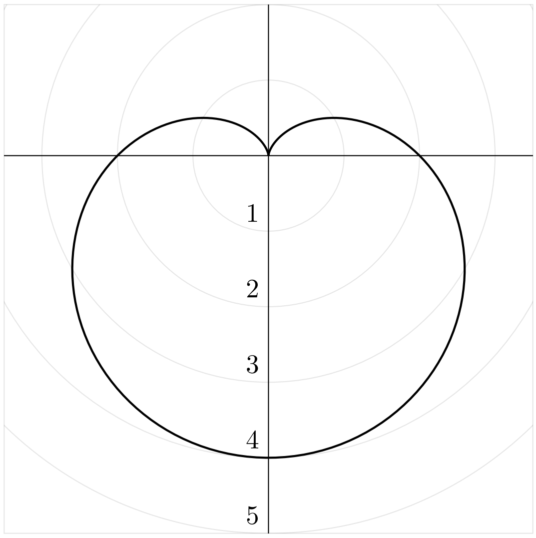
Solution. This is a point: $r = 0$. ☐

7.



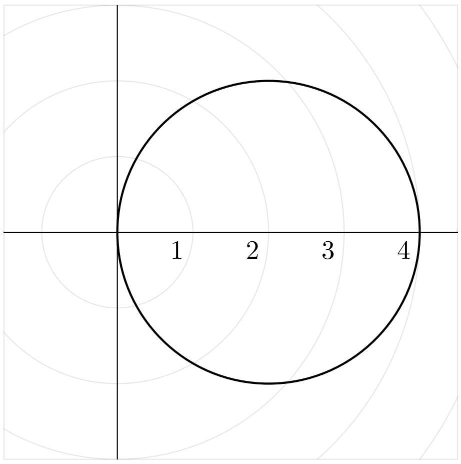
Solution. This is a (looped) limacon: $r = 1 - 2\sin\theta$. ☐

8.



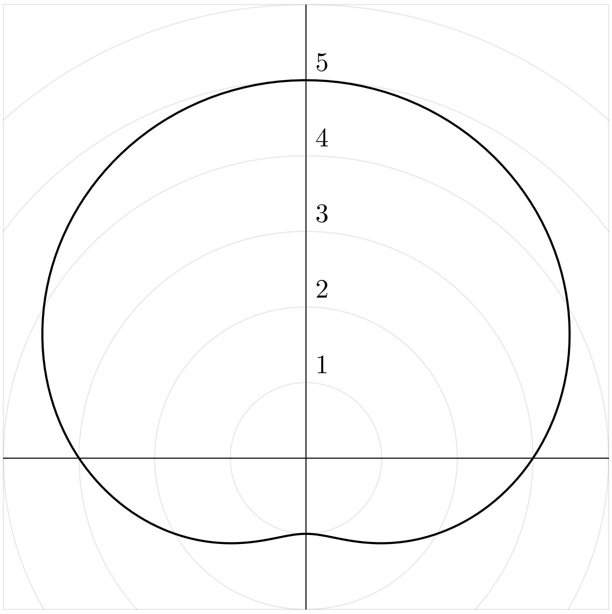
Solution. This is a cardioid: $r = 2 - 2\sin\theta$. $\square$

9.



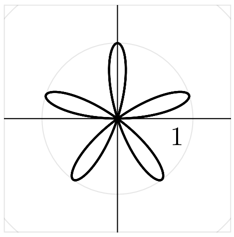
Solution. This is a circle: $r = 4\cos\theta$. $\square$

10.



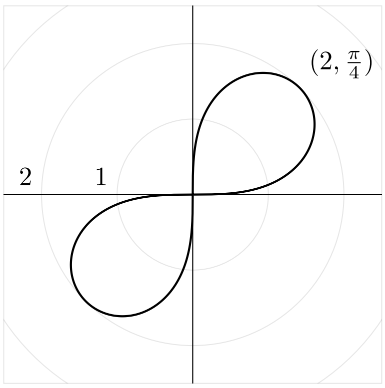
Solution. This is a (dimpled) limacon: $r = 3 + 2\sin\theta$. □

11.



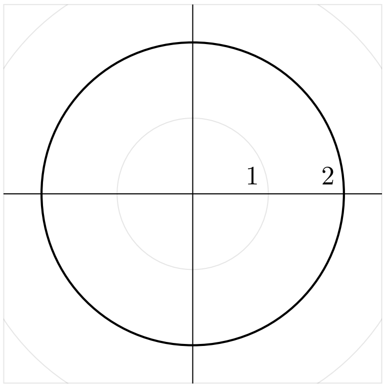
Solution. This is a 5-leaved rose: $r = \sin(5\theta)$. □

12.



Solution. This is a lemniscate: $r^2 = 4\sin(2\theta)$. □

13.



Solution. This is a circle: $r = 2$. □

1.27 Worksheet on Polar Coordinates

1. Consider the point $P = (x, y)$ with rectangular coordinates $(1, -\sqrt{3})$. Let (r, θ) be polar coordinates of P, with $r > 0$ and $0 \le \theta < 2\pi$.

   (a) Draw P in the usual Cartesian plane. Then find the the values of r and θ, giving reasons.

   (b) For the values of r and θ in (a), draw on the same large graph (different from the graph in part (a)) the points with the following polar coordinates, and label each:

   $$(r, -\theta), \qquad (r, \pi - \theta), \qquad (r, \pi + \theta), \qquad (-r, \theta), \qquad (-r, \pi - \theta), \qquad (-r, \pi/2 - \theta)$$

2. (a) Plot the graphs of $r = \sin(7\theta)$ and $r = \sin(4\theta)$, for $0 \le \theta \le 2\pi$. Which rose is traced out twice, and which rose is traced out once, as θ increases from 0 to 2π?

   (b) Let n be a positive integer, and consider the graph of $r = \sin(n\theta)$. Determine the number of leaves when n is odd, and the number of leaves when n is even. (The reason there is a difference between even n and odd n is interesting. Can you provide the reason?)

   (c) Find a polar equation for a 12-leaved rose.

3. Suppose you are asked to sketch the graph of $r = 2\cos(5\theta)$, without the help of a calculator. Discuss how you would proceed. Indicate what values of θ you would use to assist you in plotting significant points on the graph, and indicate what happens on the graph between successive significant points. Then sketch the graph of $r = 2\cos(5\theta)$.

4. Consider the lemniscate $r^2 = 4\sin(2\theta)$, for $0 \le \theta \le 2\pi$.

   (a) Sketch the lemniscate.

   (b) For what values of θ in $[0, 2\pi]$ is there no real value of r? Indicate these values of θ on the graph in part (a).

   (c) Find $\displaystyle\int_0^{2\pi} \frac{1}{2}(4\sin(2\theta))d\theta$. Evidently the value of the integral is not the area A of the region enclosed by the lemniscate. Why? What is wrong with the given integral.

   (d) Find the area A enclosed by the lemniscate. (*Hint:* Choose the limits of integration carefully.)

5. Write down a formula in polar coordinates for a function whose graph has the given symmetry, and draw the graph of the function.

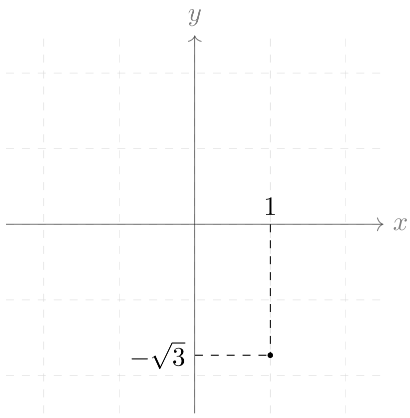
   (a) symmetry with respect to only the x axis.

   (b) symmetry with respect to only the y axis.

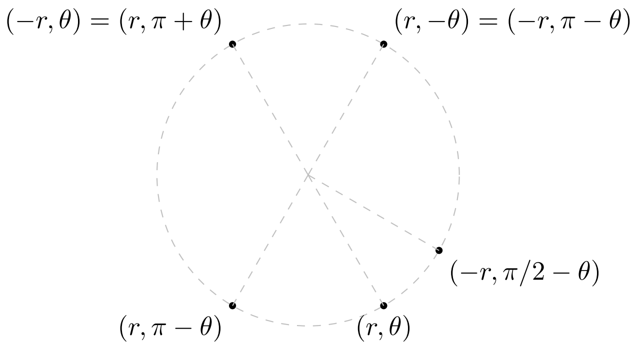
   (c) symmetry with respect to only the origin.

Solution.

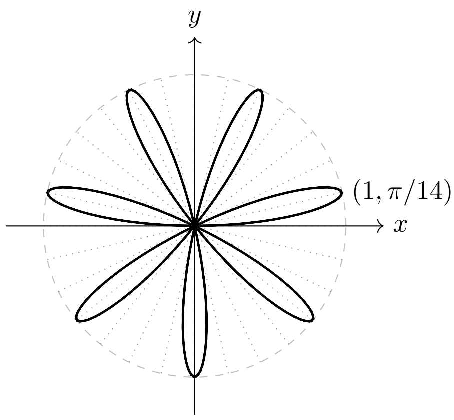
1. (a) $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$. $\tan\theta = \dfrac{-\sqrt{3}}{1} \Rightarrow \theta = \tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$. Since $0 \leq \theta < 2\pi$, so $\theta = -\dfrac{\pi}{3} + 2\pi = \dfrac{5\pi}{3}$.



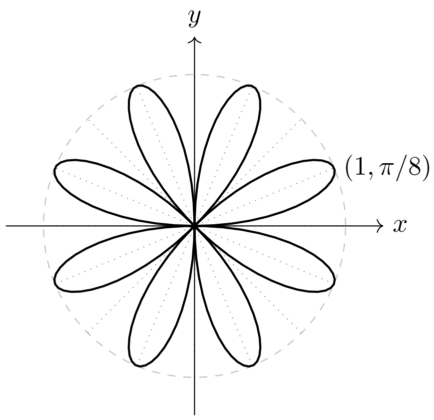
(b) The plot would look like



2. (a) The 7-leaved rose is traced out twice and the 8-leaved rose is traced out once.



(a) $r = \sin(7\theta)$

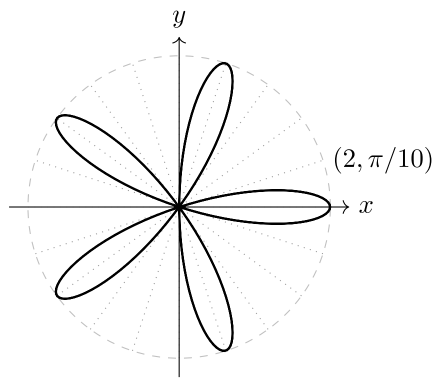


(b) $r = \sin(4\theta)$

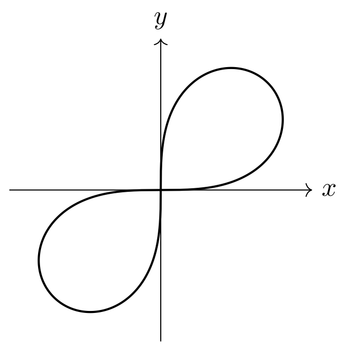
(b) If n is odd, each pedal is traced out twice. If n is even, each pedal is traced out once. Therefore, the number of leaves are n or $2n$ for n odd or even.

(c) $r = \sin(6\theta)$.

3.



4. (a)



   (b) For $\frac{\pi}{2} < \theta < \pi$ and $\frac{3\pi}{2} < \theta < 2\pi$, $r^2 = 4\sin(2\theta) < 0$, so r has no real solutions. The graph is on the first and third quadrant, but not on the second and fourth quadrant.

   (c) $\displaystyle\int_0^{2\pi} \frac{1}{2}(4\sin(2\theta))d\theta = 0$ which cannot possibly be the area A as A should be positive. The reason being, the integration takes in account of the second and fourth quadrant.

   (d) Notice symmetry between the first and third quadrant, it suffice to double the area on the first quadrant $A = 2\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2}(4\sin(2\theta))d\theta = 2$.

5. (a) $r = \cos\theta$.

   (b) $r = \sin\theta$.

   (c) $r^2 = \sin(2\theta)$.

$\square$

2 Exams

2.1 Exam 1

(1) (10 points)
Let $f(x) = x^2$ and $g(x) = 2 - x^2$, and let Ω be the region in between these two curves.

(a) Draw a picture of the region and a sketch of the solid generated by revolving Ω about the x-axis.

(b) Write down an expression for the volume of this solid of revolution. DO NOT EVALUATE THE INTEGRAL.

Solution.

(a)

(b) First solve $x^2 = 2 - x^2 \Rightarrow x = \pm 1$, so

$$V = \int_{-1}^{1} \pi[(2 - x^2)^2 - x^4]dx$$

□

(2) (10 points)
Consider the region Ω from Question 1, and assume that its area is 8/3.

(a) Write down the integrals that give the moments M_y and M_x about the y- and x-axis.

(b) Write down expressions for the center of gravity $(\bar{x}, \bar{y})$ in the terms of these moments.

(c) Find the values of $\bar{x}$ and $\bar{y}$ (with or without the integrals.)

Solution.

(a)

$$M_y = \int_{-1}^{1} x(2 - 2x^2)dx, \qquad , M_x = \frac{1}{2}\int_{-1}^{1} [(2 - x^2)^2 - x^4)dx$$

(b)

$$\bar{x} = \frac{M_y}{A}, \qquad \bar{y} = \frac{M_x}{A}$$

(c) We can use symmetry for both to see that

$$(\bar{x}, \bar{y}) = (0, 1)$$

□

(3) (10 points)
Consider a tank of water in the form of a half-cylinder on its side (see the picture), where the semi-circle has radius 2 feet, and the tank is 20 feet long (so a horizontal cross-section is a rectangle). A pump is placed at the top of a full tank of water, and the water is pumped out of the tank until the tank has depth 1 foot. How much work is done on the water in this process? Water weighs 62.5 lb/ft^3. DO NOT EVALUATE THE INTEGRAL.

Hint: Place $x = 0$ at the bottom of the tank.

Solution.

$$W = \int_{1}^{2} 62.5 \cdot 20(16 - 4x^2)(2 - x)dx$$

□

(4) (10 points)
Find the length of the curve $y = \frac{2}{3}x^{3/2} + 5$ from $x = 0$ to $x = 3$. You do not need to simplify your answer.

Solution. First evaluate $f'(x) = x^{1/2}$, so

$$L = \int_0^1 \sqrt{1 + f'(x)^2}dx = \int_0^1 \sqrt{1+x}dx \xlongequal{u=1+x} \int_1^2 \sqrt{u}du = \frac{2}{3}\left(u^{3/2}\right)\Big|_1^2 = \frac{14}{3}$$

$\square$

(5) (10 points)
Consider a particle that is moving in a circular path given parametrically by

$$\begin{cases} x(t) = 3 + 2\cos(2t) \\ y(t) = 2 - 2\sin(2t) \end{cases} \qquad \text{for } 0 \le t \le 4\pi$$

Find an expression for the speed $v(t)$ of the particle at time t. Determine the following information:

- Center of the circle.
- Radius of the circle.
- How many revolutions of the circle the particle makes.
- Which direction (clockwise or counterclockwise) the particle moves around the circle.

Solution. First we have $x'(t) = -4\sin(2t)$, $y'(t) = -4\cos(2t)$, and $v(t) = \sqrt{x'(t)^2 + y'(t)^2} = 4$ $\square$

- The center of the circle is $(3, 2)$.
- The radius of the circle is 2.
- 4 revolutions.
- Clockwise.

2.2 Exam 2

(1) (10 points) Let $f(x) = x^5 + 3x^3 + 2x - 2$.

(a) Show that $f(x)$ has an inverse on the entire real line.
(b) Find $f^{-1}(-2)$ by inspection, and then find $(f^{-1})'(-2)$.
Note. you can do this without solving part (a).

Solution.

(a) $f'(x) = 5x^4 + 9x^2 + 2 \ge 2 > 0$, so f is strictly increasing, and it has an inverse on the entire real line.
(b) By inspection, $f(0) = -2$, so

$$(f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{2}$$

$\square$

(2) (10 points) Find the derivative of

$$f(x) = (\arctan(x))^{2x} + 3^{7x+4} + \log_2(x^3(x+3)^9)$$

Do not simplify your answer.

Solution. First we simplify f

$$f(x) = e^{2x\ln(\arctan(x))} + 3^{7x+4} + 3\log_2(x) + 9\log_2(x+3)$$

Therefore

$$f'(x) = (\arctan x)^{2x}\left[2\ln(\arctan x) + \frac{2x}{\arctan(x)(x^2+1)}\right] + (\ln 3)3^{7x+4}\cdot 7 + \frac{3}{x\ln 2} + \frac{9}{(x+3)\ln 2}$$

□

(3) (10 points)

(a) Evaluate $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$.

(b) Solve the equation $\sec(\arctan(x)) = 1$ for x.

Solution.

(a) First we complete the squares $8x - x^2 = 16 - (x-4)^2$ Thus

$$\int \frac{dx}{\sqrt{16-(x-4)^2}} \overset{u=x-4}{=\!=\!=} \int \frac{du}{\sqrt{4^2-u^2}} = \arcsin\left(\frac{u}{4}\right) + C = \arcsin\left(\frac{x-4}{4}\right) + C$$

(b) Suppose $\theta = \arctan x$, then $x = \tan\theta$, so

$$\sec(\arctan(x)) = \sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2} = 1 \Rightarrow x = 0$$

□

(4) (10 points)

(a) Which of the following are indeterminate forms? $0^\infty, \infty^0, \infty^\infty, 1^\infty, \infty^1, 1^0, 0^1, 1^1, 0^0$.

(b) Find the limit: $\lim\limits_{x\to\infty} x^{(e^{-x})}$ *(that is: x raised to the power of e^x)*

Solution.

(a) The indeterminate forms are $\infty^0, 1^\infty, 0^0$.

(b)

$$\lim_{x\to\infty} x^{(e^{-x})} = e^{\lim_{x\to\infty}\frac{\ln x}{e^x}} \overset{\text{l'H } \frac{\infty}{\infty}}{=\!=\!=} e^{\lim_{x\to\infty}\frac{1}{xe^x}} = e^{\frac{1}{\infty}} = e^0 = 1$$

□

(5) (10 points)

(a) Evaluate $\displaystyle\int x^2e^{2x}\,dx$.

(b) Evaluate $\displaystyle\int_0^1 x^2e^{2x}\,dx$.

Solution.

(a)

$$\begin{aligned}
\int x^2e^{2x}dx &\xlongequal[du=2xdx,v=\frac{1}{2}e^{2x}]{u=x^2,dv=e^{2x}dx} \frac{x^2e^{2x}}{2}-\int xe^{2x}dx\\
&\xlongequal[du=dx,v=\frac{1}{2}e^{2x}]{u=x,dv=e^{2x}dx} \frac{x^2e^{2x}}{2}-\frac{xe^{2x}}{2}+\frac{1}{2}\int e^{2x}dx\\
&=\frac{x^2e^{2x}}{2}-\frac{xe^{2x}}{2}+\frac{e^{2x}}{4}+C
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^1 x^2e^{2x}dx &= \left(\frac{x^2e^{2x}}{2}-\frac{xe^{2x}}{2}+\frac{e^{2x}}{4}\right)\Big|_0^1\\
&=\left(\frac{e^2}{2}-\frac{e^2}{2}+\frac{e^2}{4}\right)-\frac{1}{4}\\
&=\frac{e^2-1}{4}
\end{aligned}$$

□

2.3 Exam 3

(1) (10 points)

(a) Evaluate $\int \sin^5 x\cos^4 x dx$.

(b) Write down what substitution you'd make for $\int \sqrt{x^2-4}dx$. Make the substitution and simplify it to an integral of trig functions, but DO NOT EVALUATE further.

Solution.

(a)

$$\begin{aligned}
\int \sin^5 x\cos^4 xdx &\xlongequal[du=-\sin xdx]{u=\cos x} -\int(1-u^2)^2u^4du=-\int(1-2u^2+u^4)u^4du\\
&=-\int(u^4-2u^6+u^8)du=\int(-u^4+2u^6-u^8)du\\
&=-\frac{u^5}{5}+\frac{2u^7}{7}-\frac{u^9}{9}+C=-\frac{\cos x^5}{5}+\frac{2\cos x^7}{7}-\frac{\cos x^9}{9}+C
\end{aligned}$$

(b)

$$\int\sqrt{x^2-4}dx \xlongequal[dx=2\sec u\tan udu]{x=2\sec u} \int\sqrt{4\tan^2 u}2\sec u\tan udu=4\int\sec u\tan^2 udu$$

□

(2) (10 points) Use partial fractions to evaluate $\int \frac{4x^2-x+1}{(x-1)(x^2+1)}dx$.

Solution. First we know

$$\frac{4x^2-x+1}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

Multiply $(x-1)(x^2+1)$ on both sides we get

$$4x^2-x+1=A(x^2+1)+(Bx+C)(x-1)$$

Take $x = 1$, we have
$$4 = 2A \Rightarrow A = 2$$
So
$$4x^2 - x + 1 = 2(x^2+1) + (Bx + C)(x-1) = (2+B)x^2 + (-B+C)x + (2-C)$$
Comparing the coefficients we have $B = 2, C = 1$, so
$$\begin{aligned}\int \frac{4x^2 - x + 1}{(x-1)(x^2+1)}dx &= \int \frac{2}{x-1} + \frac{2x+1}{x^2+1}dx \\ &= 2\int \frac{1}{x-1}dx + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx \\ &= 2\ln|x-1| + \ln|x^2+1| + \tan^{-1}(x) + C\end{aligned}$$
□

(3) (10 points)

(a) We want to use the Trapezoid Rule to estimate $\int_0^2 \frac{2}{x+1}dx$. Find the smallest integer value of n that guarantees that our error E_n^T will be at most $\frac{4}{75}$. Your answer should be exact (without any square roots). Recall that $E_n^T \leq \frac{K_T}{12n^2}(b-a)^3$.

(b) Use Simpson's Rule with $n = 6$ to approximate $\int_0^3 (x+1)^{-2}dx$. You do not need to simplify your answer.

Solution.

(a) Suppose $f(x) = 2(x+1)^{-1}$ $f''(x) = 4(x+1)^{-3}$ is decreasing on $[0,2]$, so $K_T = \max_{x\in[0,2]} |f''(x)| = |f''(0)| = 4$ and
$$E_n^T \leq \frac{4}{12n^2}(2-0)^3 \leq \frac{4}{57} \Rightarrow n^2 \geq \sqrt{50}$$
Therefore $n = 8$.

(b) Suppose $f(x) = (x+1)^{-2}$, the approximation should be
$$\frac{3-0}{3\cdot 6}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3)\right]$$
Which is
$$\frac{1}{6}\left[1 + \left(\frac{3}{2}\right)^{-2} + 2^{-2} + \left(\frac{5}{2}\right)^{-2} + 3^{-2} + \left(\frac{7}{2}\right)^{-2} + 4^{-2}\right]$$
□

(4) (10 points)

(a) Consider the integral
$$\int_1^\infty \frac{1}{x^2}dx$$
Explain why it's improper, and in what way(s). Use limits to show that the integral converges, and what it converges to.

(b) Rewrite the improper integral $\int_{-1}^1 \frac{1}{x^{2/3}}dx$ using limits, but DO NOT EVALUATE.

Solution.

(a) This integral is improper since it involves ∞

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1$$

(b) Note that $\frac{1}{x^{2/3}}$ is undefined at 0, so

$$\int_{-1}^1 \frac{1}{x^{2/3}}dx = \lim_{b\to 0^-}\int_{-1}^b \frac{1}{x^{2/3}}dx + \lim_{a\to 0^+}\int_a^1 \frac{1}{x^{2/3}}dx$$

□

(5) (10 points) Let $f(x) = \cos x$. Find the 4^{th} Taylor polynomial $p_4(x)$ centered at $x = 0$ for $f(x)$. Use this to write down an approximation of $\cos(1)$; you don't need to simplify your answer.

Solution. We can write a table

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1

So $p_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$ and $\cos(1) \approx p_4(1) = 1 - \frac{1}{2} + \frac{1}{24} = \frac{13}{24}$ □

2.4 Exam 4

Some useful power series:

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!} \quad \forall x$$

$$\sin x = \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \forall x$$

$$\cos x = \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{(2k)!} \quad \forall x$$

$$\ln(1+x) = \sum_{k=1}^\infty \frac{(-1)^k x^k}{k} \quad \text{for } |x| < 1$$

$$\frac{1}{1-x} = \sum_{k=0}^\infty x^k \quad \text{for } |x| < 1$$

(1) (10 points)

(a) Evaluate $\lim_{n\to\infty}\frac{5n^2+1}{4-3n^2}$.

(b) Let $z = 1+i$ and let $w = 4+4i$. Evaluate zw and find all three cube roots of zw. You can leave your answers in the format $re^{i\theta}$, where $r \geq 0$ and $0 \leq \theta < 2\pi$.

Solution.

(a)

$$\lim_{n\to\infty} \frac{5n^2+1}{4-3n^2} \overset{\text{l'H } \frac{\infty}{\infty}}{=\!=\!=} \lim_{n\to\infty} \frac{10n}{-6n} = \lim_{n\to\infty} \frac{10}{-6} = -\frac{5}{3}$$

(b) $zw = (1+i)(4+4i) = 4+8i+4i^2 = 4+8i-4 = 8i = 8e^{i\frac{\pi}{2}}$. Suppose $re^{i\theta}$ is a cubic root of zw, then

$\square$

(2) (10 points)

(a) Show that $\sum_{k=2}^{\infty} \frac{-1}{k(k-1)}$ converges, and find its sum.

(b) Find an upper bound for the 8^{th} truncation error E_8 when calculating $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k(k-1)}$.

Solution.

(a)

$$\sum_{k=2}^{\infty} \frac{-1}{k(k-1)} = \sum_{k=2}^{\infty} \left(\frac{1}{k} - \frac{1}{k-1}\right) = \left(\cancel{\frac{1}{2}} - \frac{1}{1}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{3}}\right) + \cdots = -1$$

(b) Note that this is an alternating series, so

$$E_8 < |a_{8+1}| = |a_9| = \left|\frac{1}{9(9-1)}\right| = \frac{1}{72}$$

$\square$

(3) (10 points)

(a) Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(-1)^k 9^k}{k} x^{2k}$.

(b) Write down the power series representation of $f(x) = x^4 e^{2x}$.

Solution.

(a) Use Ratio test

$$\lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}9^{k+1}}{k+1} x^{2(k+1)}}{\frac{(-1)^k 9^k}{k} x^{2k}} \right| = \lim_{k\to\infty} \left| 9x^2 \frac{k}{k+1} \right| = 9|x|^2$$

The series converges if $9|x|^2 < 1 \iff |x|^2 < \frac{1}{9} \iff |x| < \frac{1}{3} \iff -\frac{1}{3} < x < \frac{1}{3}$.

(b) Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so

$$f(x) = x^4 e^{2x} = x^4 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} x^4 \frac{2^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+4}$$

$\square$

(4) (10 points)

(a) Let $f(x) = x^2 + 4x + 9$. Find the Taylor series of $f(x)$ centered at $a = -1$.

(b) Find the smallest n that guarantees that the error when using n^{th} Taylor polynomial $p_n(x)$ of e^x to approximate e^1 is at most 0.01. You can assume that $e < 3$.

Solution.

(a) We can write a table

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	$x^2 + 4x + 9$	6
1	$2x + 4$	2
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

So the Taylor series is $f(x) = 6 + 2(x+1) + (x+1)^2$

(b) Recall The Lagrange Remainder Theorem

$$f(x) - p_n(x) = r_n(x) = \frac{f^{(n+1)(c)}(a)}{(n+1)!}(x-a)^{n+1}$$

For $f(x) = e^x$, $f^{(n)}(x) = e^x$, and

$$|r_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right| \leq \frac{e^c}{(n+1)!}|x|^{n+1} \leq \frac{e^1}{(n+1)!}(1)^{n+1} = \frac{e}{(n+1)!} < \frac{3}{(n+1)!} \leq 0.01$$

Note that $\dfrac{3}{(4+1)!} > 0.01$ but $\dfrac{3}{(5+1)!} \leq 0.01$, so the smallest integer $n = 5$.

□

(5) (10 points) For each of the following series, determine whether the series converges or diverges. If you use a convergence test, say which one.

(a) $\sum\limits_{k=4}^{\infty} \dfrac{k-3}{2k^3+1}$

(b) $\sum\limits_{k=1}^{\infty} \dfrac{5k^2+1}{4-3k^2}$

Solution.

(a) We use comparison test

$$\sum_{k=4}^{\infty} \frac{k-3}{2k^3+1} \leq \sum_{k=4}^{\infty} \frac{k}{2k^3} = \sum_{k=4}^{\infty} \frac{1}{2k^2}$$

Converges, so does $\sum\limits_{k=4}^{\infty} \dfrac{k-3}{2k^3+1}$

(b) Consider the term test

$$\lim_{k\to\infty} \frac{5k^2+1}{4-3k^2} = -\frac{5}{3} \neq 0$$

So $\sum\limits_{k=4}^{\infty} \dfrac{5k^2+1}{4-3k^2}$ diverges.

□

2.5 Final Exam