MATH240 Summer 2023

Haoran Li

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1 Worksheets

2 Exams

2.1 Final

(1) (a) (10) Let $f(x) = x^2$ and $g(x) = 8 - x^2$, and let R be the region in between these two curves. Write down an expression for the volume of hte solid generated by revolving R about the x-axis. DO NOT EVALUATE THE INTEGRAL.

(b) (10) Find the length of the curve $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$.

(2) (a) (15) A certain conical container has height 10 feet and base radius 3 feet. The container is placed on the ground with tis point facing up, and is filled with water to the top. How much work is required to pump out the top 5 feet of water to a level 1 foot above the tank? Set up the integral but DO NOT EVALUATE. You may assume that water weighs 62.5 pounds per cubic foot.

(b) Let $f(x) = x^2 - 8$ and $g(x) = -x^2$, and let R be the region in between these two curves. Assume that its area is 64/3.

(i) (10) Write down expressions using integrals for the center of gravity $(\overline{x}, \overline{y})$. Do not evaluate the integrals.

(ii) (10) By evaluating the integrals or otherwise, find the values of $\overline{x}$ and $\overline{y}$.

(3) (a) (10) Let $f(x) = e^{2x} + \sqrt{2x+1}$. find the largest interval I for which f^{-1} exists. Be sure to justify why the inverse exists.

(b) (10) Suppose $h(x) = 2x + \sqrt{x}$. calculate $(h^{-1})'(3)$. Justify your answer.

(4) (a) (10) Simplify the expression $\tan(\cos^{-1} 3x^2)$ so that no trigonometric function appers. Show your work.

(b) (10) Evaluate $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^{4x}$. Show the steps in your solution.

(5) For each integral, if you use a substitution of integration by parts, indicated it

(a) (10) Evaluate

$$\int \frac{\ln x}{x^2} dx$$

(b) (10) Evaluate

$$\int_0^{\pi/3} \tan x \sec^{3/2} x dx$$

(6) (a) (10) Evaluate $\int \frac{x+1}{x(x^2)+4} dx$.

(b) (10) Determine whether the improper integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} \frac{1}{x(\ln x)^2} dx$$

(7) (a) (10) Find the limit

$$\lim_{n\to\infty} (e^{3n} + n)^{2/n}$$

(b) (10) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n n^n}{(2n)!}$ converges absolutely, converges conditionally, or diverges.

(8) Suppose that $f(x) = \sum_{n=0}^{\infty} \frac{4^{n+1}}{\pi^{n+2}} x^{n+3}$.

(a) (10) Find the radius of convergence R and interval of convergence I of the power series of $f(x)$.

(b) (10) Find the power series for $\int f(x)dx$, and its radius of convergence R_0.

(9) (a) (10) Find all values, real and complex, of $(-64)^{1/3}$, and sketch them in the complex plane.

(b) (15) Consider the circle $r = \sin\theta$ and the cardioid $r = w + 2\sin\theta$. On one graph, draw the graphs of both. Then find the area of the region that is inside of the cardioid and outside ofthe circle.

Solution.

(1) (a) (10)

$$V = \int_{-2}^{2} \pi(g(x)^2 - f(x)^2)dx = \int_{-2}^{2} \pi((8-x^2)^2 - (x^2)^2)dx$$

(b) (10) $y' = -\dfrac{x}{\sqrt{4-x^2}}$, so the length is

$$L = \int_0^2 \sqrt{1+(y')^2}dx = \int_0^2 \sqrt{1+\frac{x^2}{4-x^2}}dx = \int_0^2 \sqrt{\frac{4}{4-x^2}}dx = \int_0^2 \frac{2}{\sqrt{4-x^2}}dx$$
$$\overset{x=2\sin t}{=\!=\!=} \int_{\arcsin(0)}^{\arcsin(1)} \frac{2}{\sqrt{4-4\sin^t}} 2\cos t dt = \int_0^{\pi/2} 2dt = \pi$$

(2) (a) (15)

$$W = \int_5^{10} 62.5\pi \left(\frac{3}{10}(10-x)\right)^2 (11-x)dx$$

(b) (i) (10)

$$\overline{x} = \frac{M_y}{A} = \frac{3}{64}\int_{-2}^{2} x(-x^2 - (x^2-8))dx$$
$$\overline{y} = \frac{M_x}{A} = \frac{3}{64}\int_{-2}^{2} \frac{1}{2}((-x^2)^2 - (x^2-8)^2)dx$$

(ii) (10) By symmetry, $(\overline{x}, \overline{y}) = (0, -4)$. Can also be found by computing the integrals above.

(3) (a) (10) $f'(x) = 2e^{2x} + \dfrac{1}{\sqrt{2x+1}} > 0$ for $x > -\dfrac{1}{2}$. Therefore f is increasing on $[-\frac{1}{2}, \infty)$. Note this is the domain of f. Hence $I = [-\frac{1}{2}, \infty]$.

(b) (10) Guess that $h(1) = 3$, and that $h'(x) = 2 + \frac{1}{2\sqrt{x}}$, so

$$(h^{-1})'(3) = \frac{1}{h'(1)} = \frac{1}{2+\frac{1}{2\sqrt{1}}} = \frac{2}{5}$$

(4) (a) (10) Assume $\theta = \cos^{-1}(3x^2)$, then $\cos\theta = 3x^2$, so $\tan(\cos^{-1}(3x^2)) = \tan\theta = \dfrac{\sqrt{1-9x^4}}{3x^2}$

(b) (10)

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{4x} = \lim_{x\to\infty} \frac{1}{\left(\frac{x+1}{x}\right)^{4x}} = \lim_{x\to\infty} \frac{1}{\left[\left(1+\frac{1}{x}\right)^x\right]^4} = \frac{1}{e^4} = e^{-4}$$

(5) (a) (10)

$$\int \frac{\ln x}{x^2}dx \; \underset{du=\frac{1}{x}dx, v=-\frac{1}{x}}{\overset{u=\ln x, dv=\frac{1}{x^2}dx}{=\!=\!=\!=\!=}} \; -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

(b) (10)

$$\int_0^{\pi/3} \tan x \sec^{2/3} x dx \xlongequal{u=\sec x} \int_{\sec(0)}^{\sec(\pi/3)} u^{1/2} du = \frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{3}(2^{2/3} - 1)$$

(6) (a) (10) Use partial fractions

$$\frac{x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow x+1 = A(x^2+4) + (Bx+C)x$$

Take $x = 0$ we get $1 = 4A \Rightarrow A = 1/4$. Therefore we have $x+1 = (1/4+B)x^2+Cx+4A$, comparing the coefficients we know $B = -1/4$, $C = 1$.

$$\int \frac{x+1}{x(x^2+4)} = \int \left(\frac{1}{4x} - \frac{1}{4}\frac{x}{x^2+4} + \frac{1}{x^2+4}\right) dx = \frac{1}{4}\ln|x| - \frac{1}{8}\ln|x^2+4| + \frac{1}{2}\tan^{-1}(x/2) + C$$

(b) (10)

$$\int_2^b \frac{1}{x(\ln x)^2} dx \xlongequal{u=\ln x} \int_{\ln 2}^{\ln b} \frac{1}{u^2} du = -\frac{1}{u}\Big|_{\ln 2}^{\ln b} = \frac{1}{\ln 2} - \frac{1}{\ln b}$$

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b\to\infty} \frac{1}{\ln 2} - \frac{1}{\ln b} = \frac{1}{\ln 2}$$

(7) (a) (10)

$$\lim_{n\to\infty} \ln((e^{3n}+n)^{2/n}) = \frac{2\ln(e^{3n}+n)}{n} = \frac{2(3e^{3n}+1)}{e^{3n}+n} = 6$$

Therefore $\lim_{n\to\infty}(e^{3n}+n)^{2/n} = e^6$.

(b) (10) Use Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^{n+1}}{(2(n+1))!}}{\frac{(-1)^n n^n}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \frac{(n+1)^{n+1}}{n^n} \frac{(2n)!}{(2n+2)!} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{1}{2n} = e \cdot 0 = 0 < 1$$

So the series converges absolutely.

(8) (a) (10) Use geometric series

$$\lim_{n\to\infty} \left| \frac{\frac{4^{(n+1)+1}}{\pi^{(n+1)+2}} x^{(n+1)+3}}{\frac{4^{n+1}}{\pi^{n+2}} x^{n+3}} \right| = \lim_{n\to\infty} \left| \frac{4^{n+2}}{4^{n+1}} \frac{x^{n+4}}{x^{n+3}} \frac{\pi^{n+2}}{\pi^{n+3}} \right| = \left| \frac{4x}{\pi} \right| < 1 \Rightarrow |x| < \frac{\pi}{4}$$

Therefore $R = \frac{\pi}{4}$. If

- $f(\pi/4) = \sum_{n=0}^{\infty} \frac{\pi}{16}$ which diverges
- $f(-\pi/4) = \sum_{n=0}^{\infty} \frac{(-1)^{n+3}\pi}{16}$ which diverges

Hence $I = (-\pi/4, \pi/4)$.

(b) (10)

$$\int f(x) dx = \int \sum_{n=1}^{\infty} \frac{4^{n+1}}{\pi^{n+2}} x^{n+3} = \sum_{n=1}^{\infty} \int \frac{4^{n+1}}{\pi^{n+2}} x^{n+3} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{\pi^{n+2}} \frac{x^{n+4}}{n+4}$$

$R_0 = R = \frac{\pi}{4}$.

(9) (a) (10) Note that $-64 = 64e^{i\pi}$

$$(-64)^{1/3} = 4e^{i\left(\frac{\pi}{3}+k\frac{2\pi}{3}\right)} = \begin{cases} 4e^{i\frac{\pi}{3}}, & k=0 \\ 4e^{i\pi} = -4, & k=1 \\ 4e^{i\frac{5\pi}{3}}, & k=2 \end{cases}$$

(b) (15) The circle has area π, the cardioid has area

$$\begin{aligned} \int_0^{2\pi} \frac{1}{2}(2+2\sin\theta)^2 d\theta &= \int_0^{2\pi} \frac{1}{2}\left(4+8\sin\theta+4\sin^2\theta\right) d\theta \\ &= \int_0^{2\pi} (2+4\sin\theta+1-\cos(2\theta))\, d\theta \\ &= \left(2\theta - 4\cos\theta + \theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = 6\pi \end{aligned}$$

Therefore the desired area is $6\pi - \pi = 5\pi$.

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