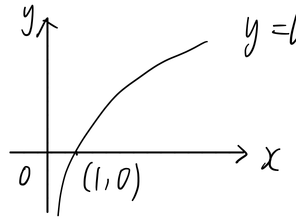
Review:

I find a lot of students don't know this, $\ln x$ only has zero at $x=1$, and when $x \downarrow 0$, $\ln x \downarrow -\infty$, $x \nearrow +\infty$, $\ln x \nearrow +\infty$ as suggested by the graph



Problem: solve $y' = \frac{t}{y}$, $y(0) = -1$

Answer: Use separation of variables $\frac{dy}{dt} = y' = \frac{t}{y} \Rightarrow y\,dy = t\,dt \xrightarrow{\text{integrate}}$
$\int y\,dy = \int t\,dt \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}t^2 + C$, plug in $y(0) = -1$, get
$\frac{1}{2}(-1)^2 = \frac{1}{2}0^2 + C \Rightarrow \frac{1}{2} = C$, plug C back, get $\frac{1}{2}y^2 = \frac{1}{2}t^2 + \frac{1}{2} \Rightarrow y^2 = t^2 + 1$
$\Rightarrow y = \sqrt{t^2+1}$ or $y = -\sqrt{t^2+1}$, since $y(0) = -1$, $y = \sqrt{t^2+1}$ is discarded, hence the solution is $y = -\sqrt{t^2+1}$

Problem: Solve $y' = 3t - y$

Answer: Use integrating factors

Rewrite as $y' + y = 3t$, identify $a(t) = 1$, $b(t) = 3t$, $A(t) = \int a(t)dt = t$
then the integrating factor is e^t, thus we have
$[ye^t]' = y'e^t + y(e^t)' = y'e^t + ye^t = (y'+y)e^t = 3te^t$, integrate
$ye^t = \int [ye^t]'dt = \int 3te^t dt = 3\int te^t dt = 3\int t(e^t)'dt$
$= 3\left[te^t - \int e^t t' dt\right] = 3te^t - 3\int e^t dt = 3te^t - 3e^t + C$
divide e^t, get $y = 3t - 3 + Ce^{-t}$
this DE doesn't have constant solutions, suppose it has $y = C$ as a constant solution, then $0 = C' = 3t - C \Rightarrow C = 3t$ which is a contradiction

Problem: Determine whether $\int_0^\infty \cos x\,dx$ is convergent

Answer: $\int_0^\infty \cos x\,dx = \lim_{b\to+\infty} \int_0^b \cos x\,dx = \lim_{b\to+\infty} \sin x\Big|_0^b = \lim_{b\to\infty}[\sin b - 0]$
doesn't converge, even though $\sin b$ is bounded $|\sin b| \le 1$ not tends to infinity, but it oscillates thus not convergent

Problem

A person planning for her retirement arranges to make continuous deposits into a savings account at the rate of \$3700 per year. The savings account earns 5% interest compounded continuously.

(a) Set up a differential equation that is satisfied by $f(t)$, the amount of money in the account at time t.

(b) Solve the differential equation in part (a), assuming that $f(0)=0$, and determine how much money will be in the account at the end of 20 years

Solution:

(a) $$y' = 0.05y + 3700$$

$$\begin{bmatrix}\text{rate of change} \\ \text{of } y\end{bmatrix} = \begin{bmatrix}\text{rate at which is} \\ \text{added}\end{bmatrix} + \begin{bmatrix}\text{rate at which money} \\ \text{is deposited}\end{bmatrix}$$

Here $y(t) = y = f(t)$

(b) rewrite as $y' - 0.05y = 3700$, $a(t) = -0.05$, $b(t) = 3700$

$A(t) = \int a(t)\,dt = -0.05t$, thus $[ye^{-0.05t}]' = 3700e^{-0.05t}$, integrate

$ye^{-0.05t} = \int [ye^{-0.05t}]'\,dt = \int 3700e^{-0.05t} = -74000e^{-0.05t} + C$

Plug in $f(0)=0$, $0 = y(0)e^{-0.05\cdot 0} = -74000e^{-0.05\cdot 0} + C = -74000 + C$

$\Rightarrow C = 74000$, plug back, get $ye^{-0.05t} = -74000e^{-0.05t} + 74000$

$\times e^{0.05t}$ on both sides $y = e^{0.05t}(-74000e^{-0.05t} + 74000) = -74000 + 74000e^{0.05t}$

The money in the account at the end of 20 years is by definition

$y(20) = -74000 + 74000 \times e^{-0.05\cdot 20} = -74000 + 74000 \times e^{-1}$

Prob: The fish population in a pond with capacity of 200 fish is modeled by the logistic equation $\frac{dN}{dt} = \frac{0.2}{2000} N(2000-N)$. Here N denotes the number of fish at time t in years. When the number of fish reached 500, the owner of the pond decided to remove 61 fish per year

(a) Modify the DE to model the fish population from the time it reached 500

$$\frac{dN}{dt} = \frac{0.2}{2000} N(2000-N) - 61$$

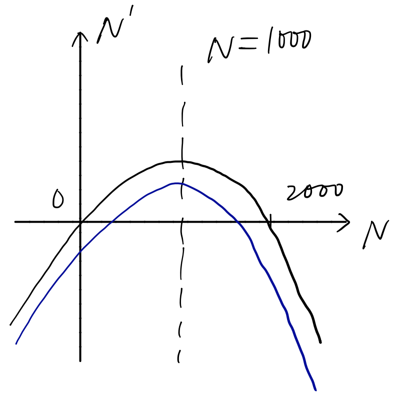
$$\begin{bmatrix}\text{rate of change} \\ \text{in number of} \\ \text{fish per year}\end{bmatrix} = \begin{bmatrix}\text{rate of change due to} \\ \text{the system itself (the} \\ \text{number of fish)}\end{bmatrix} - \begin{bmatrix}\text{rate of change due to} \\ \text{outer interference (artificial} \\ \text{interference)}\end{bmatrix}$$

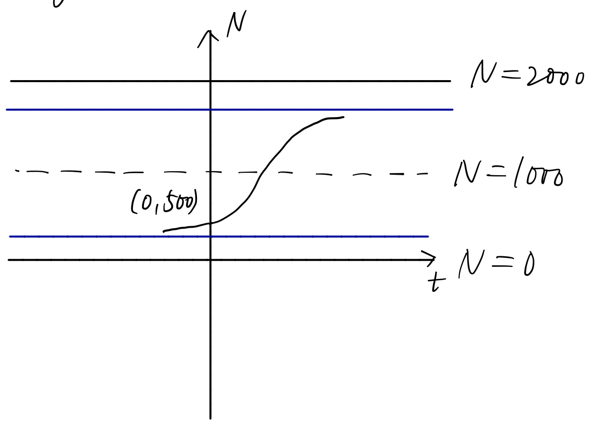
$N(0) = 500$

(b) Plot the solution curve of the new equation with $N(0)=500$

First: $\frac{0.2}{2000} N(2000-N) - 61 = -0.0001N^2 + 0.2N - 61$

you can always compute the zero, symmetry axis and vertex by using quadratic formula, but here let's do it in another way, notice





-61 has the effect of moving down the graph by 61, in order to if the solution curve is increasing, you only need to plug in $N(0)=500$, get $N'(0) = \frac{0.2}{2000} 500(2000-500) - 61 = 75 - 61 = 14 > 0$, you get a logistic curve

(d) Is the practice of catching 61 fish per year sustainable, or will it deplete the fish population in the pond? YES!!

Remark: as you K, r change (due to natural causes), or IC $N(0)$ changes, or the pratice changes (artificial interference, put in or take out) see how you can "control" the system

Challenge: What is the maximal profit you can have annually if you can change the IC and practice such that it is still sustainable