## MATH 241 Groupwork 1 - 09/04/2023

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1. **Discuss:** Ask your TA which branch of mathematics they are studying (or planning to study) in grad school and why.

   **Answer:**

2. **Discuss:** Given three points $P$, $Q$, and $R$ in 3D space, how could you figure out if they form a right triangle or not?

   **Answer:** Use pathagorean theorem to decide whether

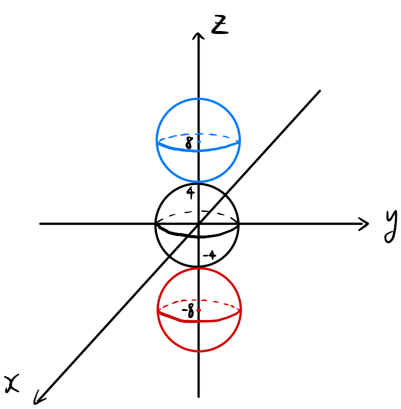
   $$|PQ|^2 = |PR|^2 + |QR|^2$$
   $$|PR|^2 = |PQ|^2 + |QR|^2$$
   $$|QR|^2 = |PQ|^2 + |PR|^2$$

3. **Discuss:** Consider the two spheres:

   $$S_1: \quad x^2 + y^2 + z^2 = 16$$
   $$S_2: \quad x^2 + y^2 + (z - A)^2 = 16$$

   These spheres may or may not intersect, depending on the value of $A$. Discuss the possible options and how each option corresponds to various values of $A$.

   **Answer:**




$S_1$ is the sphere centered at $(0,0,0)$ with radius 4

$S_2$ is the sphere centered at $(0,0,A)$ with radius 4

For $S_2$ to intersect $S_1$, it shuo be in between the blue and red sphere (inclusive), So

$-8 \le A \le 8$ : intersect

$A < -8$ or $A > 8$ : not intersect

4. **Discuss:** If $P$ and $Q$ are points, explain in words what the following equality means, geometrically speaking:

$$|PQ| = \left\|\overrightarrow{PQ}\right\|$$

**Answer:** The magnitude of $\overrightarrow{PQ}$ is equal to the distance between P and Q

5. **Discuss:** Suppose $\bar{\boldsymbol{v}}$ and $\bar{\boldsymbol{w}}$ are nonzero vectors. If they are parallel, does this mean that the following is true and why:

$$\frac{\bar{\boldsymbol{v}}}{||\bar{\boldsymbol{v}}||} = \pm\frac{\bar{\boldsymbol{w}}}{||\bar{\boldsymbol{w}}||}$$

**Answer:** $\frac{\vec{v}}{\|\vec{v}\|}$ and $\frac{\vec{w}}{\|\vec{w}\|}$ are unit vectors of the same directions as $\vec{v}$ and $\vec{w}$ respectively

Since $\vec{v} \parallel \vec{w}$, $\vec{v}$ and $\vec{w}$ are of the same direction or the opposite direction

hence $\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{w}}{\|\vec{w}\|}$ or $\frac{\vec{v}}{\|\vec{v}\|} = -\frac{\vec{w}}{\|\vec{w}\|}$

6. **Discuss:** Suppose $\bar{\boldsymbol{v}}$ has length 5. Construct a vector pointing in the opposite direction from $\bar{\boldsymbol{v}}$ which has length 7.

**Answer:** Use above (5.). Suppose we want to construct $\vec{w}$

Since $\vec{v}$ and $\vec{w}$ are of the opposite direction

$$\frac{\vec{v}}{\|\vec{v}\|} = -\frac{\vec{w}}{\|\vec{w}\|} \Rightarrow \frac{\vec{v}}{5} = -\frac{\vec{w}}{7} \Longrightarrow \vec{w} = -\frac{7}{5}\vec{v}$$

**MATH 241 Quiz 1 - 09/07/2022** **Your Name:**
**Your Section:**

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1. Suppose $P = (1, 2, 3)$ and $Q = (4, 5, 6)$. Suppose $R = (x, y, z)$ is such that the distance from $R$ to $P$ equals the distance from $R$ to $Q$. Write down an equation which represents this fact. You do not need to do anything with this equation! [2 pt]

   **Answer:** We need $|PR| = |QR|$, which is

   $$\sqrt{(x-1)^2+(y-2)^2+(z-3)^2} = \sqrt{(x-4)^2+(y-5)^2+(z-6)^2}$$

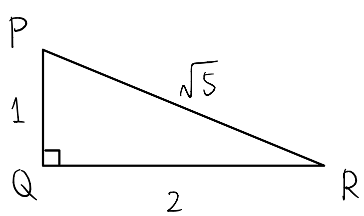
2. Show, using only the distance formula from class, that the three points $(1, 1, 1)$, $(1, 1, 2)$, and $(1, 3, 2)$ form a right triangle. [3 pt]

   **Answer:** Assume $P=(1,1,1)$, $Q=(1,1,2)$, $R=(1,3,2)$

   $$|PQ| = \sqrt{(1-1)^2+(1-1)^2+(2-1)^2} = \sqrt{0+0+1} = 1$$
   $$|PR| = \sqrt{(1-1)^2+(3-1)^2+(2-1)^2} = \sqrt{0+4+1} = \sqrt{5}$$
   $$|QR| = \sqrt{(1-1)^2+(3-1)^2+(2-2)^2} = \sqrt{0+4+0} = 2$$

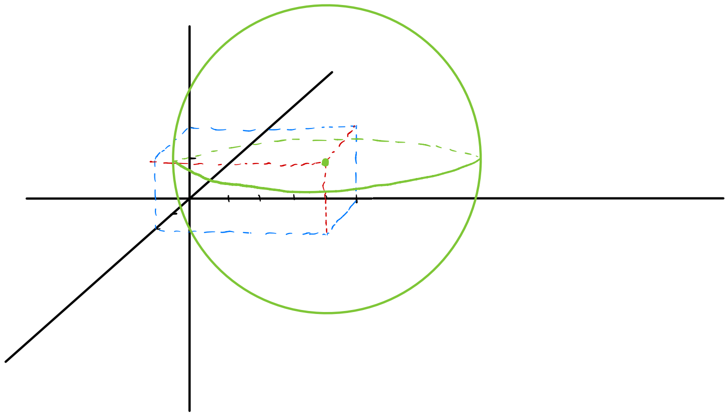
   Hence $|PQ|^2 + |QR|^2 = |QR|^2$

   $P, Q, R$ form a right triangle



3. Sketch the graph of $(x-2)^2 + (y-5)^2 + (z-2)^2 = 16$. [2 pt]

**Answer:**



4. Show that it is not possible to find $\beta$ such that $\mathbf{v}$ and $\mathbf{w}$ are parallel, where: [3 pt]

$$\mathbf{v} = 4\,\mathbf{i} + \beta^2\,\mathbf{j} + \beta^2\,\mathbf{k}$$
$$\mathbf{w} = 1\,\mathbf{i} + 2\,\mathbf{j} + 1\,\mathbf{k}$$

**Answer:**

Since $\vec{v} \parallel \vec{w}$, $\vec{v}$ is a scalar multiple of $\vec{w}$

and we know the first components are 4, 1 respectively,

thus $\vec{v} = 4\vec{w}$, so $\begin{cases} \beta^2 = 4 \times 2 = 8 \\ \beta^2 = 4 \times 1 = 4 \end{cases}$ $\#$ but this is impossible!!!

**MATH 241 Groupwork 2 - 09/12/2023**

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1. **Discuss/Calculate:** Imagine a regular 2D pre-calc line with formula $y = mx + b$. Suppose we move this into 3D at $z = 0$. What would the parametric equations be?

   **Answer:** This line would pass through $(0, b, 0)$ and pointing at the direction $(1, m, 0)$, therefore the parametric equation should be $\begin{cases} x = t \\ y = b + mt \\ z = 0 \end{cases}$

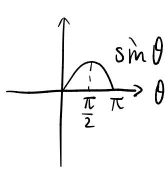
2. **Discuss:** Is it ever possible, for nonzero vectors $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$, to have $\mathrm{Proj}_{\bar{\boldsymbol{v}}}\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}$ or $\mathrm{Proj}_{\bar{\boldsymbol{v}}}\bar{\boldsymbol{u}} = \bar{\boldsymbol{v}}$? Explain.

   **Answer:** Yes. $\mathrm{Proj}_{\vec{v}}\vec{u} = \vec{u}$ if and only if $\vec{u} \parallel \vec{v}$

   $\mathrm{Proj}_{\vec{v}}\vec{u} = \vec{v}$ if $\vec{u} = \vec{v} + \vec{w}$ where $\vec{w} \perp \vec{v}$

3. **Discuss:** What happens to the length of $\bar{\boldsymbol{u}} \times \bar{\boldsymbol{v}}$ as the angle between them changes and as the vectors themselves get longer or shorter? Discover as much as you can.

   **Answer:** Recall $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$, $\theta$ being the angle between $\vec{u}, \vec{v}$




$\|\vec{u} \times \vec{v}\| = 0$ at $\theta = 0$, and increases to $\|\vec{u}\|\|\vec{v}\|$ at $\theta = \frac{\pi}{2}$, then decreases down to $0$ at $\theta = \pi$

$\|\vec{u} \times \vec{v}\|$ increases or decreases as $\|\vec{u}\|, \|\vec{v}\|$ increases or decreases

4. **Discuss:** If you were given the parametric equations of a line and the equation of a sphere, and if you wanted to know where the line contacted the sphere, how could you do it?

   **Answer:** Suppose $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ is the sphere equation with center $C=(a,b,c)$

   and $\begin{cases} x = x_0+\alpha t \\ y = y_0+\beta t \\ z = z_0+\gamma t \end{cases}$, $\vec{L}=(\alpha,\beta,\gamma)$ is the parametric equation of the line, $P=(x_0,y_0,z_0)$

   Plug the parametric line equations into the sphere equation and solve for $t$

   We can also use the distance formula $\dfrac{\|\overrightarrow{PC}\times\vec{L}\|}{\|\vec{L}\|}$ and decide how many intersection points there are

5. **Discuss:** Suppose you had the parametric equations of two lines and wished to know if and where the lines met. How might you go about this? This is not nearly as obvious as you might first think!

   **Answer:** Suppose these parametric equations are

   $\begin{cases} x = x_1+a_1 t \\ y = y_1+b_1 t \\ z = z_1+c_1 t \end{cases}$, $\vec{L_1}=(a_1,b_1,c_1)$, $P_1=(x_1,y_1,z_1)$

   $\begin{cases} x = x_2+a_2 t \\ y = y_2+b_2 t \\ z = z_2+c_2 t \end{cases}$, $\vec{L_2}=(a_2,b_2,c_2)$, $P_2=(x_2,y_2,z_2)$

   Plug one set of equations into the other and solve for $t$

**MATH 241 Quiz 2 - 09/04/2023** **Your Name:**
**Your Section:**

1. Given the two vectors:

$$\mathbf{a} = 0\,\mathbf{i} + 8\,\mathbf{j} - 2\,\mathbf{k}$$
$$\mathbf{b} = 5\,\mathbf{i} - 3\,\mathbf{j} + 7\,\mathbf{k}$$

   (a) Find and simplify $\mathbf{a} \cdot \mathbf{b}$. [1 pt]

   **Answer:**

$$\begin{aligned}\vec{a} \cdot \vec{b} &= 0 \cdot 5 + 8 \cdot (-3) + (-2) \cdot 7 \\ &= 0 - 24 - 14 \\ &= -38\end{aligned}$$

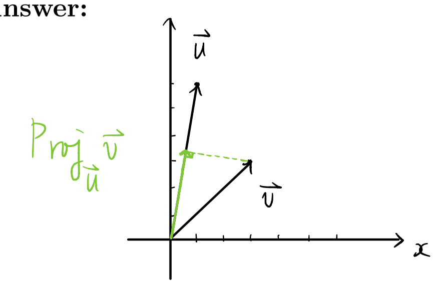
   (b) Find and simplify $\mathbf{a} \times \mathbf{b}$. [2 pt]

   **Answer:**

$$\begin{aligned}\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 8 & -2 \\ 5 & -3 & 7 \end{vmatrix} &= [8 \cdot 7 - (-2)(-3)]\,\vec{i} \\ &\quad - [0 \cdot 7 - (-2) \cdot 5]\,\vec{j} \\ &\quad + [0 \cdot (-3) - 8 \cdot 5]\,\vec{k} \\ &= 62\,\vec{i} - 10\,\vec{j} - 40\,\vec{k}\end{aligned}$$

2. Given the vectors $\mathbf{u} = 1\,\mathbf{i} + 6\,\mathbf{j}$ and $\mathbf{v} = 3\,\mathbf{i} + 3\,\mathbf{j}$, calculate $\text{Proj}_{\mathbf{u}}\mathbf{v}$ and then sketch all three together in 2D on one $xy$-plane. [3 pt]

**Answer:**




$$\text{Proj}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\,\vec{u}$$

$$= \frac{1 \cdot 3 + 6 \cdot 3}{1^2 + 6^2}\,\vec{u} = \frac{21}{37}\,\vec{u}$$

$$= \left(\frac{21}{37}, \frac{126}{37}\right)$$

3. Find the points where the line with parametric equations: [4 pt]

$$x = 2t + 1$$
$$y = t$$
$$z = 3$$

hits the sphere with equation:

$$x^2 + y^2 + z^2 = 11$$

**Answer:**

Plug $\begin{cases} x = 2t+1 \\ y = t \\ z = 3 \end{cases}$ into $x^2+y^2+z^2=11$ we get

$$(2t+1)^2 + t^2 + 3^2 = 11 \Rightarrow 4t^2 + 4t + 1 + t^2 + 9 = 11$$

$$\Rightarrow 5t^2 + 4t - 1 = 0$$

$$\Rightarrow (5t-1)(t+1) = 0$$

$$\Rightarrow t = \frac{1}{5} \quad \text{or} \quad t = -1$$

Hence these 2 intersection points are $(\frac{7}{5}, \frac{1}{5}, 3)$ and $(-1, -1, 3)$

**MATH 241 Groupwork 2 - 09/19/2023**

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1. **Discuss:** Given two planes, in what ways could they meet or not meet? If you were given the equations of the planes, how could you determine this?

   **Answer and Explain:** $\Pi_1 : a_1 x + b_1 y + c_1 z = d_1$, $\vec{n_1} = (a_1, b_1, c_1)$

   $\Pi_2 : a_2 x + b_2 y + c_2 z = d_2$, $\vec{n_2} = (a_2, b_2, c_2)$

   $$\begin{cases} \text{parallel} \Leftrightarrow \vec{n_1} \times \vec{n_2} = \vec{0} & \begin{cases} \text{overlap} \Leftrightarrow \text{Pick } P \in \Pi_1 \Rightarrow P_2 \in \Pi_2 \\ \text{non-intersecting} \Leftrightarrow \text{Pick } P \in \Pi_1 \Rightarrow P \notin \Pi_2 \end{cases} \\ \text{intersect in a line} \Leftrightarrow \vec{n_1} \times \vec{n_2} \neq \vec{0} & \begin{cases} \text{perpendicular} \Leftrightarrow \vec{n_1} \cdot \vec{n_2} = 0 \\ \text{not perpendicular} \Leftrightarrow \vec{n_1} \cdot \vec{n_2} \neq 0 \end{cases} \end{cases}$$

2. **Discuss:** If you are given a line in symmetric form and a plane how could you figure out where the line intersects the plane, assuming it does? Where would your method fail if the line did not intersect the plane?

   **Answer and Explain:**

   $$\ell : \frac{x - x_0}{a_0} = \frac{y - y_0}{b_0} = \frac{z - z_0}{c_0}, \quad \vec{L} = (a_0, b_0, c_0)$$

   $$\Pi : a_1 x + b_1 y + c_1 z = d_1, \quad \vec{n} = (a_1, b_1, c_1)$$

   $$\begin{cases} \ell \parallel \Pi \Leftrightarrow \vec{n} \cdot \vec{L} = 0 & \begin{cases} \ell \text{ lies on } \Pi \Leftrightarrow \text{Pick } P \in \ell \Rightarrow P \in \Pi \\ \ell \text{ doesn't lies on } \Pi \Leftrightarrow \text{Pick } P \in \ell \Rightarrow P \notin \Pi \end{cases} \\ \ell \nparallel \Pi \Leftrightarrow \vec{n} \cdot \vec{L} \neq 0 & \begin{cases} \ell \perp \Pi \Leftrightarrow \vec{L} \times \vec{n} = \vec{0} \\ \ell \not\perp \Pi \Leftrightarrow \vec{L} \times \vec{n} \neq \vec{0} \end{cases} \end{cases}$$

   Actually find the intersections: Rewrite $\ell$ in parametric form

   $$\vec{r}(t) : \begin{cases} x = x_0 + a_0 t \\ y = y_0 + b_0 t \\ z = z_0 + c_0 t \end{cases}$$ plug into $\Pi$ and solve for $t$

3. **Discuss:** If you are given the location of an object as $\bar{\boldsymbol{r}}(t)$ and the equation of a plane, assuming that the object hits the plane exactly once, how could you find the location and velocity of that object at that instant?

   **Answer and Explain:**

   plug $\vec{r}(t)$ into the plane equation and solve for $t$

   the compute $\vec{r}'(t)$ and $\vec{r}''(t)$

4. **Discuss:** Without solving anything, explain how many times $\bar{\boldsymbol{r}}(t) = 3\cos t\,\hat{\boldsymbol{\imath}} + 3\sin t\,\hat{\boldsymbol{\jmath}} + 2\,\hat{\boldsymbol{k}}$ hits the plane $y = 1$ and why.

   **Answer and Explain:** $3\sin t = 1$ has infinitely many solutions

   so hits infinitely many times.

**MATH 241 Quiz 3 - 09/21/2023** **Your Name:**

1. For each of the following pairs of planes, explain the way in which they meet, or don't meet. No calculation is necessary, just write a single sentence for each answer!

   (a) $2x + y - z = 10$ and $4x + 2y - 2z = 10$ [2 pts]

   **Answer:**

   $\vec{n}_1 = (2, 1, -1)$, $\vec{n}_2 = (4, 2, -2)$, $\vec{n}_2 = 2\vec{n}_1$

   $(0, 0, -10)$ is on $2x + y - z = 10$, not $4x + 2y - 2z = 10$

   (b) $2x + y - z = 10$ and $4x + 2y - 2z = 20$ [2 pts]

   **Answer:**

   $\vec{n}_1 = (2, 1, -1)$, $\vec{n}_2 = (4, 2, -2)$, $\vec{n}_2 = 2\vec{n}_1$

   $(0, 0, -10)$ is on both two planes

   (c) $2x + y - z = 10$ and $4x + 2y - z = 10$ [2 pts]

   **Answer:**

   $\vec{n}_1 = (2, 1, -1)$, $\vec{n}_2 = (4, 2, -1)$, $\vec{n}_1 \nparallel \vec{n}_2$

2. Write down two parameterizations of the circle $x^2 + z^2 = 4$ in the plane $y = 5$. [2 pts]

   **Answer:**

$$\begin{cases} x = 2\cos t \\ y = 5 \\ z = 2\sin t \end{cases} \quad \text{or} \quad \begin{cases} x = -2\sin t \\ y = 5 \\ z = 2\cos t \end{cases}$$

$$0 \le t < 2\pi$$

3. Suppose an object's location is given by: $\mathbf{r}(t) = t^2\,\mathbf{i} + (2t+1)\,\mathbf{j} + t^2\,\mathbf{k}$. At what time $t$ does the object strike the plane $2x + 3y - 2z = 15$? What is the object's position, velocity, speed and acceleration at that instant? [4 pt]

**Answer:**

plug $\begin{cases} x = t^2 \\ y = 2t+1 \\ z = t^2 \end{cases}$ into $2x+3y-2z = 15$ we get

$$2t^2 + 3(2t+1) - 2t^2 = 15 \Leftrightarrow 6t+3 = 15 \Leftrightarrow t = 2$$

$$\vec{r}'(t) = 2t\vec{i} + 2\vec{j} + 2t\vec{k}$$

$$\vec{r}''(t) = 2\vec{i} + 2\vec{k}$$

so $P = \vec{r}(2) = 4\vec{i} + 5\vec{j} + 4\vec{k}$

$\vec{v} = \vec{r}'(2) = 4\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{a} = \vec{r}''(2) = 2\vec{i} + 2\vec{k}$

**MATH 241 Groupwork 4 - 09/26/2023**

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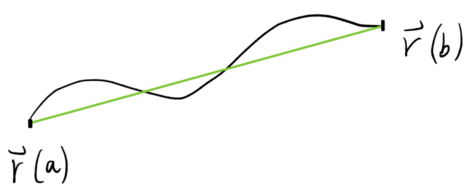
1. **Discuss:** Suppose $\bar{\boldsymbol{r}}(t)$ for $a \le t \le b$ describes the location of an object at time $t$. Explain the difference between these two: [3 pt]

$$\int_a^b ||\bar{\boldsymbol{r}}'(t)||\,dt \qquad \text{and} \qquad ||\bar{\boldsymbol{r}}(b) - \bar{\boldsymbol{r}}(a)||$$

Draw an example (no math, just a nice picture) where these values would be the same and an example (no math, just a nice picture) where these values would be different.

**Answer and Explain:**




$\int_a^b \|\vec{r}'(t)\| dt$ is the length traveled by $\vec{r}(t)$ from a to b

$\|\vec{r}(b) - \vec{r}(a)\|$ is the distance between endpoints

Therefore $\int_a^b \|\vec{r}'(t)\| dt \geq \|\vec{r}(b) - \vec{r}(a)\|$

And equality holds if and only if $\vec{r}(t)$ is a straight line

2. **Discuss:** Which two things do you need in order to write down the equation of a plane?

**Answer and Explain:** Given $P = (x_0, y_0, z_0)$, $\vec{n} = (a, b, c)$

The equation of plane passing through P and perpendicular to $\vec{n}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

3. **Discuss:** Which two things do you need in order to write down any type of equation of a line?

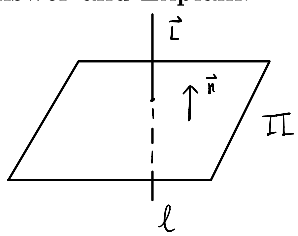
**Answer and Explain:** Given $P = (x_0, y_0, z_0)$, $\vec{L} = (a, b, c)$

The equation of line passing through P and parallel to $\vec{L}$ is

$$\begin{cases} x = (x_0 + at) \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

4. **Discuss:** If a line is perpendicular to a plane, what does this say about the line's $\bar{\boldsymbol{L}}$ and the plane's $\bar{\boldsymbol{n}}$?

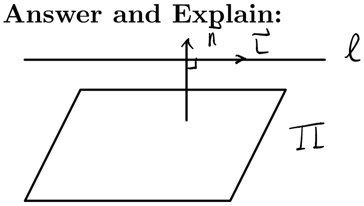
   **Answer and Explain:**




$$\vec{L} \parallel \vec{n} \iff \ell \perp \Pi$$

5. **Discuss:** If a line is parallel to a plane, what does this say about the line's $\bar{\boldsymbol{L}}$ and the plane's $\bar{\boldsymbol{n}}$?

   **Answer and Explain:**




$$\ell \parallel \Pi \iff \vec{n} \cdot \vec{L} = 0$$

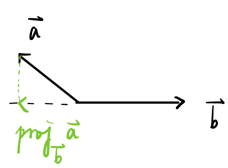
6. **Discuss:** What happens to the value of $\bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}}$ as the angle between them changes and as the vectors themselves get longer or shorter? Discover as much as you can.

   **Answer and Explain:**

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

7. **Discuss:** Could $\mathrm{Proj}_{\bar{\boldsymbol{b}}}\bar{\boldsymbol{a}}$ be a negative multiple of $\bar{\boldsymbol{b}}$?

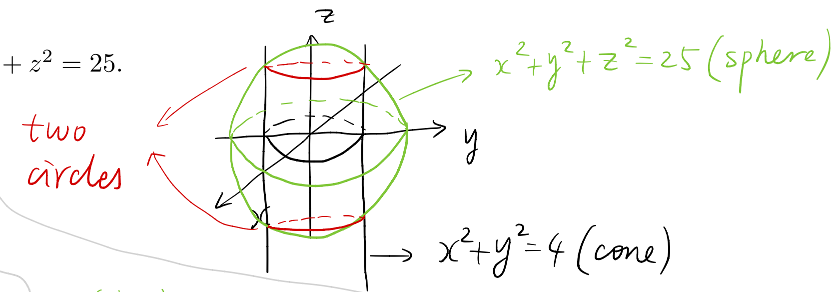
   **Answer and Explain:** Yes!



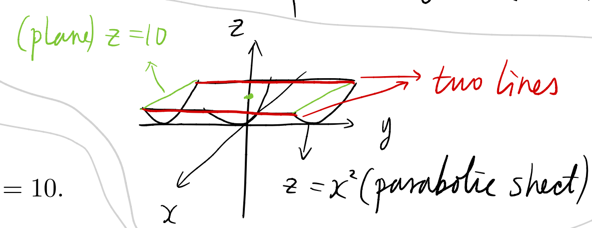
## MATH 241 Groupwork 5 - 10/03/2023

1. **Discuss:** For each of the following pairs of surfaces describe (with words!) as best you can what the intersection between the two given surfaces looks like. Each lends itself to a simple-ish description.

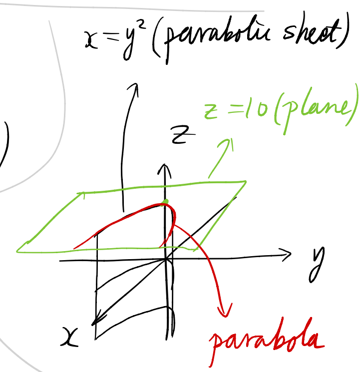
   (a) $x^2 + y^2 = 4$ and $x^2 + y^2 + z^2 = 25$.
   **Answer:**



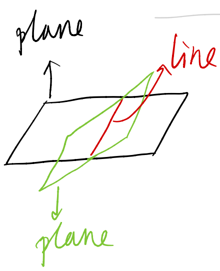

   (b) $z = x^2$ and $z = 10$.
   **Answer:**




   (c) $x = y^2$ and $z = 10$.
   **Answer:**




   (d) $x + y + z = 10$ and $x + y + 2z = 10$.
   **Answer:**




   $\vec{n}_1 = (1,1,1)$, $\vec{n}_2 = (1,1,2)$, $\vec{n}_1 \not\parallel \vec{n}_2$

   (e) $x^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 16$.
   **Answer:** similar to (a)

2. **Discuss:** Can you think of a function $f(x, y)$ which has $f_x(x, y) = 2xy + 1$ and $f_y(x, y) = x^2$?
   **Answer and Explain:**

$$f(x,y) = \int f_x(x,y)\,dx = \int 2xy+1\,dx = x^2y + x + g(y), \text{ so}$$

$$x^2 = f_y(x,y) = x^2 + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = C \text{ is an arbitrary const}$$

$$\text{so } f(x,y) = x^2y + x + C$$

3. **Discuss:** Can you think of a function $f(x, y)$ which has $f_x(x, y) = 2xy + y$ and $f_y(x, y) = x^2$?
   **Answer and Explain:**

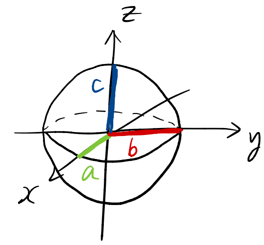
Method I: $f(x,y) = \int f_x(x,y)\,dx = \int 2xy + y\,dx = x^2y + xy + g(y)$, so

$x^2 = f_y(x,y) = x^2 + x + g'(y) \Rightarrow g'(y) = -x$ which is impossible

Method II:

$2x + 1 = f_{xy} \neq f_{yx} = 2x$ which is impossible if such an $f$ exists

4. **Discuss:** Suppose $a, b, c$ are unknown constants. Consider the shape:




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(a) Suppose $a = b = c = 1$. What is the shape?

**Answer:** $x^2 + y^2 + z^2 = 1$ is a sphere

(b) In general what are the roles of $a$, $b$, and $c$?

**Answer:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is in general a ellipsoid with radius a, b, c in the x, y, z - axis

(c) Suppose $a = c = 1$ and $b \to 0$. What does the shape become in the limit? Draw some pictures to clarify.

**Answer:** when $b \to 0$, it becomes a disk $\begin{cases} x^2+z^2=1 \\ y=0 \end{cases}$

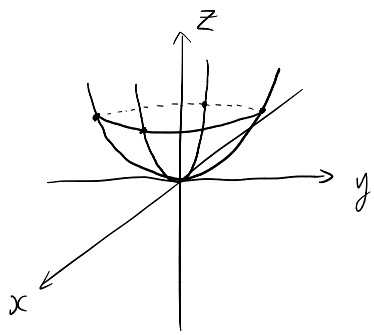
(d) Suppose $a = c = 1$ and $b \to \infty$. What does the shape become in the limit? Draw some pictures to clarify.

**Answer:** when $b \to \infty$, it becomes a cone: $x^2+z^2=1$

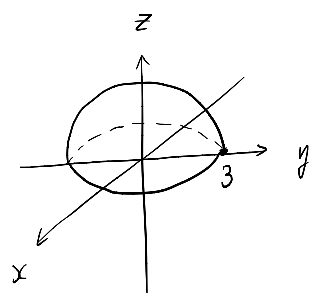
**MATH 241 Quiz 5 - 10/05/2023** **Your Name:**

1. Sketch the following in 3D:

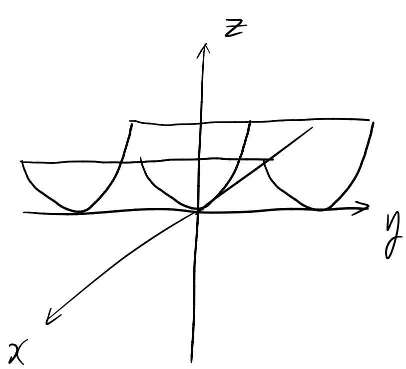
   (a) $f(x, y) = x^2 + y^2$ paraboloid [2 pts]




   (b) $f(x, y) = \sqrt{9 - x^2 - y^2}$ upper hemisphere of radius 3 [2 pts]




   (c) $z = x^2$ parabolic sheet [2 pts]



2. Suppose $f(x, y) = x^2y - x + y$ gives the temperature of the plane (distance measured in meters) in degrees Celsius. Find each of the following and, for each, write a full sentence explaining the meaning of each.

(a) $f(2, 1)$ $= 2^2 \cdot 1 - 2 + 1 = 3$ is the temperature at $(2,1)$ [1 pt]

$f_x(x,y) = 2xy - 1$

(b) $f_x(2, 1)$ $= 2 \cdot 2 \cdot 1 - 1 = 3$ is the rate of change along x-axis at $(2,1)$ [1.5 pt]

$f_y(x,y) = x^2 + 1$

(c) $f_y(2, 1)$ $= 2^2 + 1 = 5$ is the rate of change along y-axis at $(2,1)$ [1.5 pt]

Recall: If $\vec{u}$ is a unit vector, and $\theta$ is the angle between $\vec{u}$ and $\nabla f$

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \|\vec{u}\| \cdot \|\nabla f\| \cdot \cos\theta = \|\nabla f\| \cos\theta$$

**MATH 241 Groupwork 10 - 10/10/2023**

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1. A small heat-sensitive robot is exploring the $xy$-plane. The temperature at $(x, y)$ in $°C$ is $f(x, y) = xy + x^2(y - 2)$. where $x$ and $y$ are in meters.

   (a) **Discuss:** Suppose the robot is at $(2, 1)$. Its software is having issues and it is moving carelessly in the direction $3\,\hat{\imath} + 4\,\hat{\jmath}$. What instantaneous temperature change does it experience?

   **Answer:** $\vec{u} = \dfrac{3\hat{i} + 4\hat{j}}{\|3\hat{i} + 4\hat{j}\|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$, $\nabla f = (f_x, f_y) = (y + 2xy - 4x,\ x + x^2)$

   $$\nabla f(2,1) = (1 + 2\cdot 2\cdot 1 - 4\cdot 2,\ 2 + 2^2) = (-3, 6)$$

   the temperature change along $\vec{u}$ at $(2,1)$ is

   $$D_{\vec{u}} f(2,1) = \vec{u} \cdot \nabla f = \left(\frac{3}{5}, \frac{4}{5}\right) \cdot (-3, 6) = -\frac{9}{5} + \frac{24}{5} = \frac{15}{5} = 3$$

   (b) **Discuss:** Suppose the robot is at $(2, 1)$. Suppose the software is upgraded so that it can scan instantaneous temperature changes in all directions. In which direction should it go in order to experience the maximum instantaneous temperature increase?

   **Answer:** For $D_{\vec{u}} f$ to attain maximum, $\cos\theta$ must be $1 \Rightarrow \theta = 0$

   so $\vec{u}$ has be of the same direction of $\nabla f$.

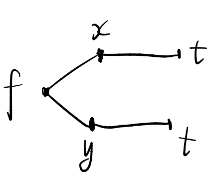
   Therefore along $\nabla f(2,1) = -3\hat{i} + 6\hat{j}$, $f$ experience maximum instantaneous temperature increase.

   (c) **Discuss:** Now the robot is following the VVF $\vec{\boldsymbol{r}}(t) = t\,\hat{\imath} + t^2\,\hat{\jmath}$ where $t$ is in seconds. At the instant when $t = 2$ how fast is the temperature changing with respect to time?

   **Answer:**

   $$\vec{r}(t) = t\hat{i} + t^2\hat{j} \iff \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \Rightarrow \begin{cases} x'(t) = 1 \\ y'(t) = 2t \end{cases}$$

   so the temperature changing rate with respect to time is




   $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ and

   $$\begin{aligned} \left.\frac{df}{dt}\right|_{t=2} &= f_x(x(2), y(2))\,x'(2) + f_y(x(2), y(2))\,y'(2) \\ &= f_x(2,4)\cdot 1 + f_y(2,4)\cdot 4 \\ &= (4 + 2\cdot 2\cdot 4 - 4\cdot 2)\cdot 1 + (2 + 2^2)\cdot 4 \\ &= 36 \end{aligned}$$

2. **Discuss:** Suppose you have a function $f(x, y)$ and you believe that $\nabla f(1, 2) = 4\,\hat{\imath} - 3\,\hat{\jmath}$. Your friend tells you that they believe that they have a unit vector $\bar{\boldsymbol{u}}$ with $D_{\bar{\boldsymbol{u}}} f(1, 2) = 6$. Explain why you cannot both be correct.

   **Answer and Explain:** Suppose such $\vec{u}$ does exist, then

   $$6 = D_{\vec{u}} f(1,2) = \|\nabla f(1,2)\| \cos\theta = 5\cos\theta \Rightarrow \cos\theta = \frac{6}{5}$$

   which is impossible because $\cos\theta \leq 1$

3. **Discuss:** Let $f(x, y, z) = x^3 + 3xy + xe^z$, and suppose $x(t)$, $y(t)$, and $z(t)$ are functions satisfying $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $x'(0) = 1$, $y'(0) = 2$, and $z'(0) = 3$. Calculate the following:

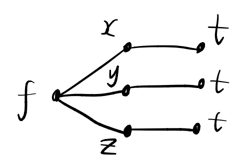
   $$\left.\frac{df}{dt}\right|_{t=0}$$

   Note: This was a question on the Fall 2021 Final Exam.

   **Answer and Explain:**

   $$f_x = 3x^2 + 3y + e^z, \quad f_y = 3x, \quad f_z = xe^z$$

   $$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$




   $$\left.\frac{df}{dt}\right|_{t=0} = f_x(x(0), y(0), z(0))\,x'(0) + f_y(x(0), y(0), z(0))\,y'(0) + f_z(x(0), y(0), z(0))\,z'(0)$$

   $$= f_x(1,0,0)\cdot 1 + f_y(1,0,0)\cdot 2 + f_z(1,0,0)\cdot 3$$

   $$= (3\cdot 1^2 + 3\cdot 0 + e^0)\cdot 1 + (3\cdot 1)\cdot 2 + (1\cdot e^0)\,3$$

   $$= 4 + 6 + 3$$

   $$= 13$$

1. Suppose $f(x, y) = \frac{x}{y} + x^2y - y$ and $\bar{u} = \frac{1}{\sqrt{10}}\,\mathbf{i} + \frac{3}{\sqrt{10}}\,\mathbf{j}$.

   (a) Calculate and simplify $D_{\bar{u}}f(2, 3)$. [2 pts]

   **Solution:**

   $$\nabla f = (f_x, f_y) = \left(\frac{1}{y} + 2xy,\ -\frac{x}{y^2} + x^2 - 1\right)$$

   $$\nabla f(2,3) = \left(\frac{1}{3} + 2\cdot 2\cdot 3,\ -\frac{2}{3^2} + 2^2 - 1\right) = \left(\frac{37}{3},\ \frac{25}{9}\right)$$

   $$D_{\vec{u}} f(2,3) = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)\cdot\left(\frac{37}{3}, \frac{25}{9}\right) = \frac{37}{3\sqrt{10}} + \frac{25}{3\sqrt{10}} = \frac{62}{3\sqrt{10}}$$

   (b) Calculate $||\nabla f(2, 3)||$. [2 pts]

   **Solution:**

   $$\|\nabla f(2,3)\| = \sqrt{\left(\frac{37}{3}\right)^2 + \left(\frac{25}{9}\right)^2} =$$

   (c) Explain in words why there is no unit vector $\bar{v}$ with $D_{\bar{v}}f(2, 3)$ larger than your answer to (b). [2 pts]

   **Solution:**

   For any unit vector $\vec{v}$,

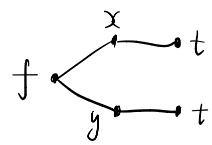
   $$D_{\vec{v}} f(2,3) = \|\vec{v}\|\|\nabla f(2,3)\|\cos\theta = \|\nabla f(2,3)\|\cos\theta \leq \|\nabla f(2,3)\|$$

2. A small heat-sensitive robot is exploring the $xy$-plane. The temperature at $(x, y)$ in °$C$ is $f(x, y) = xy + x^2(y - 2)$. where $x$ and $y$ are in meters. Now the robot is following the VVF $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$ where $t$ is in seconds. At any instant $t$ how fast is the temperature changing with respect to time? [4 pts]

**Solution:**

$$\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} \rightsquigarrow \begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \Longrightarrow \begin{cases} x'(t) = 1 \\ y'(t) = 2t \end{cases}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$




$$= (y + 2xy - 4x)\cdot 1 + (x + x^2)\cdot 2t$$

$$= (t^2 + 2\cdot t\cdot t^2 - 4\cdot t) + (t + t^2)\cdot 2t$$

$$= t^2 + 2t^3 - 4t + 2t^2 + 2t^3$$

$$= 4t^3 + 3t^2 - 4t$$

## MATH 241 Groupwork 7 - 10/17/2023

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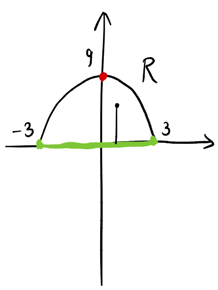
1. It's a common belief that when finding the maximum and minimum of a function on a constrained region $R$ that the maximum and minimum must occur at corners. Demonstrate that this is false by discussing the following situations and determing where the minimum and maximum occur.

   **Note: For these problems you don't need to calculate anything, you can figure these out by intuition and understanding.**

   (a) **Discuss:** The function $f(x, y) = |y|$ on the region $R$ between the parabola $y = 9 - x^2$ and the $x$-axis.
   Hint: $f(x, y)$ yields the distance from $(x, y)$ to the $x$-axis.
   **Answer and Explain:**

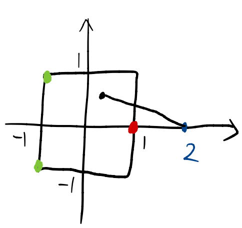



$f(x,y) = |y| = \text{dist}\big((x,y),\ x\text{-axis}\big)$

max: $f(0,9)=9$

min: $f(x,0)=0, \quad -3 \leq x \leq 3$

   (b) **Discuss:** The function $f(x, y) = \sqrt{(x-2)^2 + y^2}$ on the region $R$ which is the square with corners $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$.
   Hint: $f(x, y)$ yields the distance from $(x, y)$ to what?




$f(x,y) = \sqrt{(x-2)^2+y^2} = \sqrt{(x-2)^2+(y-0)^2} = \text{dist}\big((x,y),(2,0)\big)$

max: $f(1,0)=1$

min: $f(-1,-1) = f(-1,1) = \sqrt{10}$

Recall: Relative min/max of $f$ under constraint $g=c$ is such that

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases}$$

2. For each of the following, discuss whether you believe each $f$ has a max and/or min with the constraint given.

(a) **Discuss:** $f(x,y) = x^2$ with $x = 2y+1$.

**Answer:**

$f(x,y) = x^2$, $g(x,y) = x - 2y = 1$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases} \Rightarrow \begin{cases} 2x = \lambda \cdot 1 & ① \\ 0 = \lambda \cdot (-2) & ② \\ x = 2y+1 & ③ \end{cases}$$

② $\Rightarrow \lambda = 0 \longrightarrow$ ① $\Rightarrow x = 0$

$\rightarrow$ ③ $\Rightarrow y = -\frac{1}{2}$

so $f(0,-\frac{1}{2}) = 0$ is a critical pt

Observe $f(x,y) = x^2 \geq 0$

$f(0,-\frac{1}{2}) = 0$ is min

(b) **Discuss:** $f(x,y) = 10 - y$ with $y = x^2$.

**Answer:**

$f(x,y) = 10 - y$, $g(x,y) = y - x^2 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases} \Rightarrow \begin{cases} 0 = \lambda \cdot (-2x) & ① \\ -1 = \lambda \cdot 1 & ② \\ y = x^2 & ③ \end{cases}$$

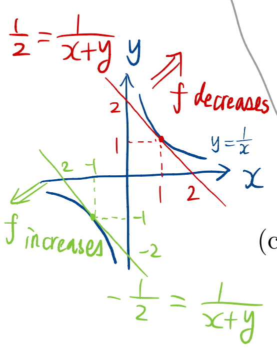
② $\Rightarrow \lambda = -1 \rightarrow$ ① $\Rightarrow x = 0$

$\rightarrow$ ③ $\Rightarrow y = 0$

so $f(0,0) = 10$ is a critical pt

observe $f(x,y) = 10 - y = 10 - x^2 \leq 10$

$f(0,0) = 10$ is max




(c) **Discuss:** $f(x,y) = 1/(x+y)$ with $y = 1/x$

**Answer:**

$f(x,y) = \frac{1}{x+y}$, $g(x,y) = xy = 1$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = c \end{cases} \Rightarrow \begin{cases} -\frac{1}{(x+y)^2} = \lambda y & ① \\ -\frac{1}{(x+y)^2} = \lambda x & ② \\ y = \frac{1}{x} & ③ \end{cases}$$

RHS ① = LHS ① = LHS ② = RHS ②

$\Rightarrow \lambda y = \lambda x \Rightarrow \lambda(y-x) = 0 \Rightarrow \boxed{\cancel{\lambda = 0}}$ $\rightarrow$ impossible for ①, ②

or $y = x \rightarrow$ ③ $\Rightarrow x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

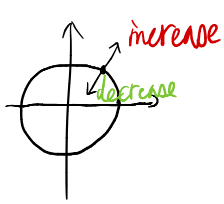
so $f(-1,-1) = -\frac{1}{2}$, $f(1,1) = \frac{1}{2}$ are critical pts. Note that the level curves are

$f(x,y) = c \Leftrightarrow \frac{1}{x+y} = c \Leftrightarrow x+y = \frac{1}{c}$, so $f(-1,-1)$ is min

$f(1,1)$ is max

3. Suppose you obtain a maximum at $(x_0, y_0)$ for a Lagrange Multipliers problem with objective function $f(x,y)$ and constraint $g(x,y) = 0$. Does this mean that $f(x,y)$ has a maximum at $(x_0, y_0)$ without the constraint? Explain.

**Answer and Explain:** False.




counter-example: $f(x,y) = \sqrt{x^2+y^2}$, $g(x,y) = x^2+y^2 = r^2$

$f = r$ on $g$, so every pt is max & min

but without $g$, $f$ increases/decreases as the radius increases/decreases

**MATH 241 Groupwork 8 - 10/24/2023**

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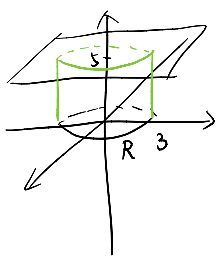
1. Each of the following double integrals can be evaluated by discussing and understanding the volume which is being calculated. No actual integration is necessary:

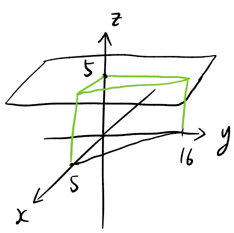
   (a) **Discuss:** $\iint_R 5\,dA$ where $R$ is the disk $x^2+y^2 \leq 9$.
   **Answer and Explain:**




$z = f(x,y) = 5$

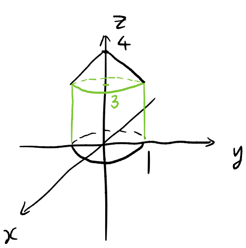
$$\iint_R 5\,dA = \text{Vol}(\text{cylinder}) = \pi \cdot 3^2 \cdot 5 = 45\pi$$

   (b) **Discuss:** $\iint_R 5\,dA$ where $R$ is the triangle with corners $(0,0)$, $(5,0)$ and $(0,16)$.
   **Answer and Explain:**




$$\iint_R 5\,dA = \text{Vol}(\text{triangular prism})$$
$$= \frac{1}{2}\cdot 5 \cdot 16 \cdot 5 = 200$$

   (c) **Discuss:** $\iint_R 4-\sqrt{x^2+y^2}\,dA$ where $R$ is the disk $x^2+y^2 \leq 1$.
   **Answer and Explain:**

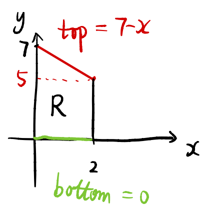



$$\iint_R 4-\sqrt{x^2+y^2}\,dA = \text{Vol}(\text{cone} + \text{cylinder})$$
$$= \text{Vol}(\text{cone}) + \text{Vol}(\text{cylinder})$$
$$= \frac{1}{3}\cdot \pi \cdot 1^2 \cdot (4-3) + \pi \cdot 1^2 \cdot 3$$
$$= \frac{\pi}{3} + 3\pi$$
$$= \frac{10\pi}{3}$$

2. **Discuss:** Draw a picture of the region $R$ in the $xy$ plane corresponding to the iterated integral:

$$\int_0^2 \int_0^{7-x} xy\, dy\, dx$$

**Answer and Explain:**

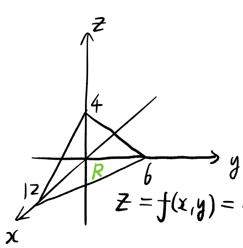



3. **Discuss:** If you wanted to find the volume below the plane $x + 2y + 3z = 12$ and in the first octant, draw a picture of the corresponding $R$ in the $xy$-plane.

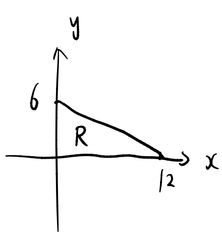
**Answer and Explain:**

$(12,0,0)$, $(0,6,0)$, $(0,0,4)$ are the $x, y, z$ - intercepts

$$Vol = \iint_R f(x,y)\, dA = Vol\ (\text{tetrahedron}) = \frac{1}{3} \cdot 6 \cdot 12 \cdot 4 = 96$$



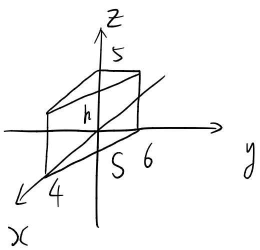

$z = f(x,y) = 4 - \frac{2}{3}y - \frac{1}{3}x$



**MATH 241 Quiz 8 - 10/26/2023** **Your Name:**

1. Each of the following double integrals can be evaluated to a number by discussing and understanding the volume which is being calculated. No actual integration is necessary. Give the value and explain briefly (pictures are fine) how you came to that conclusion.

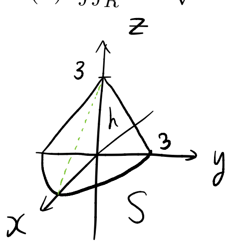
   (a) $\iint_R 5\,dA$ where $R$ is the triangle with corners $(0,0)$, $(4,0)$ and $(0,6)$. [2 pts]




$$\iint_R 5\,dA = Sh = \left(\tfrac{1}{2}\cdot 4\cdot 6\right)\cdot 5 = 60$$

   (b) $\iint_R 3 - \sqrt{x^2+y^2}\,dA$ where $R$ is the half-disk $x^2+y^2 \le 9$ with $x \ge 0$. [2 pts]

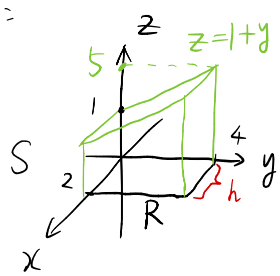
half cone




$$\iint_R 3-\sqrt{x^2+y^2}\,dA = \frac{1}{3}Sh = \frac{1}{3}\cdot\left(\frac{1}{2}\cdot\pi\cdot 3^2\right)\cdot 3 = \frac{9\pi}{2}$$

   (c) $\iint_R 1+y\,dA$ where $R$ is the rectangle with corners $(0,0)$, $(2,0)$, $(0,4)$ and $(2,4)$. [2 pts]

Method I:



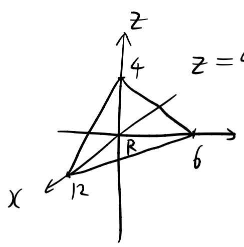

$$\iint_R 1+y\,dA = Sh = \underbrace{\left(\frac{(1+5)\cdot 4}{2}\right)}_{\text{area of the trapezoid}}\cdot \underset{\text{height}}{2} = 24$$

Method II:

$$\iint_R 1+y\,dA = \int_0^2\int_0^4 1+y\,dy\,dx = \int_0^2\left(y+\frac{y^2}{2}\Big|_0^4\right)dx = \int_0^2\left(4+\frac{4^2}{2}\right)dx$$
$$= \int_0^2 12\,dx = 24$$

2. Set up the double iterated integral in rectangular coordinates for the volume below the plane $x + 2y + 3z = 12$ and in the first octant. Do not evaluate! [4 pts]

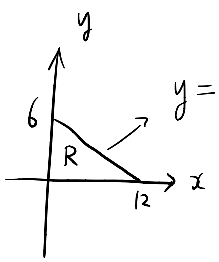
$(12, 0, 0)$, $(0, 6, 0)$, $(0, 0, 4)$ are the $x, y, z$ - intercepts




$z = 4 - \frac{2}{3}y - \frac{1}{3}x$

$$\text{Vol} = \iint_R z \, dA$$

$$= \int_0^{12} \int_0^{6 - \frac{1}{2}x} \left(4 - \frac{2}{3}y - \frac{1}{3}x\right) dy \, dx$$




$y = 6 - \frac{1}{2}x$

**MATH 241 Groupwork 9 - 10/30/2023**

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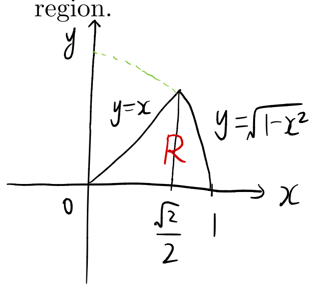
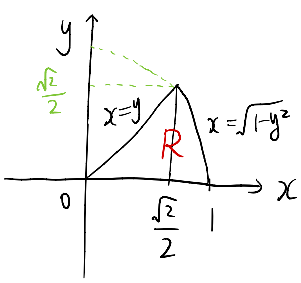
1. Consider the sum of the two vertically simply iterated integrals:

$$I = \int_0^{\sqrt{2}/2} \int_0^x x^2 + y^2 \, dy \, dx + \int_{\sqrt{2}/2}^1 \int_0^{\sqrt{1-x^2}} x^2 + y^2 \, dy \, dx$$

(a) Together these represent a double integral over one region $R$ in the $xy$-plane. Draw this region.




(b) Rewrite the double integral as just one horizontally simple iterated integral.




$$I = \int_0^{\frac{\sqrt{2}}{2}} \int_y^{\sqrt{1-y^2}} x^2 + y^2 \, dx \, dy$$

(c) Rewrite the double integral as just one polar iterated integral.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$I = \int_0^{\frac{\pi}{4}} \int_0^1 r^2 \cdot r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^1 r^3 \, dr \, d\theta$$

$$= \left(\int_0^1 r^3 \, dr\right)\left(\int_0^{\frac{\pi}{4}} 1 \, d\theta\right)$$

$$= \frac{1}{4} \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{16}$$

2. Fill in the blanks with the name of the quantity being measured. Each of the following may be used: mass (twice), area (once), volume (twice)

   (a) If $f(x,y)$ is density then $\iint_R f(x,y)\,dA$ measures: mass

   (b) If $f(x,y)$ is height at $(x,y)$ then $\iint_R f(x,y)\,dA$ measures: volume

   (c) If $f(x,y)=1$ then $\iint_R f(x,y)\,dA$ measures: area

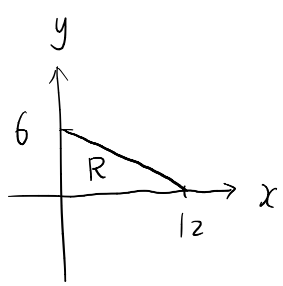
   (d) If $f(x,y,z)$ is density then $\iiint_D f(x,y,z)\,dV$ measures: mass

   (e) If $f(x,y,z)=1$ then $\iiint_D f(x,y,z)\,dV$ measures: volume

3. **Discuss:** If you wanted to find the volume below the plane $x+2y+3z=12$ and in the first octant, draw a picture of the corresponding $R$ in the $xy$-plane.

   **Answer and Explain:**

The $x, y, z$ - intercepts are $(12,0,0)$, $(0,6,0)$, $(0,0,4)$

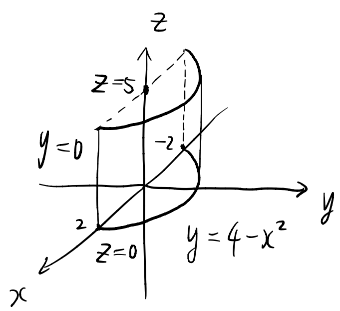


**MATH 241 Quiz 9 - 11/02/2023** **Your Name:**

1. Set up but do not evaluate the iterated triple integral in rectangular coordinates for the mass of $D$, where $D$ is the solid bounded by the planes $z = 0$, $z = 5$, $y = 0$ and the parabolic sheet $y = 4 - x^2$, and where the density is $f(x, y, z) = y$ grams/cm$^3$. [5 pts]

   **Solution:**

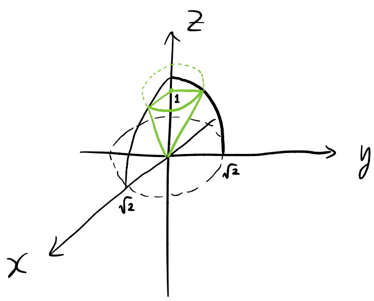
$$\text{mass of } D = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{5} y \, dz \, dy \, dx$$



2. Set up but do not evaluate the iterated triple integral in cylindrical coordinates for the volume of $D$, where $D$ is the solid in the first octant, below the paraboloid $z = 2 - x^2 - y^2$ and above the cone $z = \sqrt{x^2 + y^2}$. $\rightsquigarrow z = r$ [5 pts]

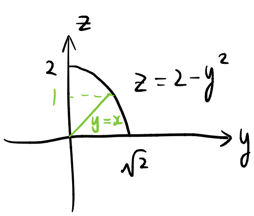
$z = 2 - r^2$

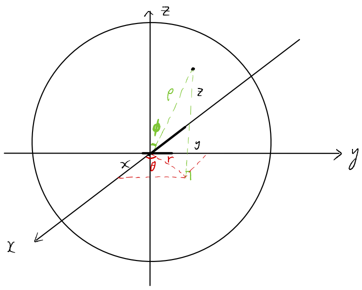
**Solution:**




$$\mathrm{Vol}(D) = \int_0^{\frac{\pi}{2}} \int_0^1 \int_z^{\sqrt{2-z}} r \, dr \, dz \, d\theta$$

$\big\downarrow$ $yz$ − plane






$$\begin{cases} r = \rho\sin\phi \rightarrow \begin{cases} x = r\cos\theta = \rho\sin\phi\cos\theta \\ y = r\sin\theta = \rho\sin\phi\sin\theta \end{cases} \\ z = \rho\cos\phi \end{cases}$$

**MATH 241 Groupwork 10 - 11/07/2023**

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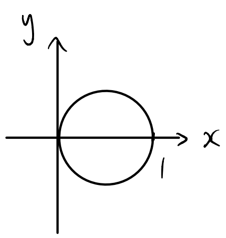
1. **Discuss:** The spherical function $\rho = \cos\phi$ was not discussed in class. It might remind you of the polar function $r = \cos\theta$.

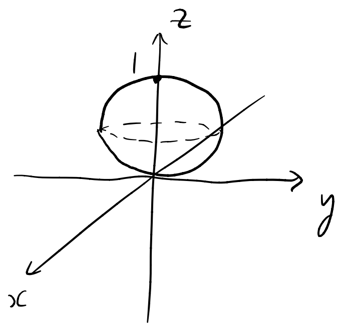
   Discuss what the second looks like and see if you can figure out what the first looks like.

   **Answer and Explain:**

$r = \cos\theta \Longleftrightarrow r^2 = r\cos\theta \Longleftrightarrow x^2 + y^2 = x$



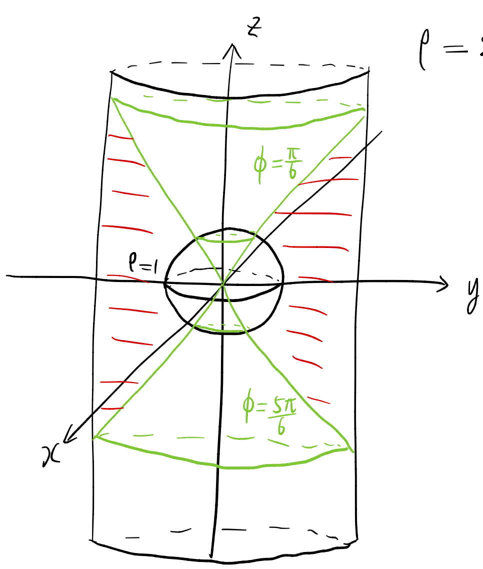

$\rho = \cos\phi \Longleftrightarrow \rho^2 = \rho\cos\phi \Longleftrightarrow x^2+y^2+z^2 = z$




2. **Discuss:** Draw the (challenging!) object described in spherical coordinates as follows. If you're struggling to draw it, explain what it is by explaining the shapes involved and words like "inside" and "outside".

$$0 \le \theta \le 2\pi$$
$$\frac{\pi}{6} \le \phi \le \frac{5\pi}{6}$$
$$1 \le \rho \le 2\csc\phi \text{ (these are the near and far functions)}$$

   **Answer and Explain:**




$\rho = 2\csc\phi$

$\rho = 2\csc\phi = \frac{2}{\sin\phi}$

$\Rightarrow \rho\sin\phi = 2$

$\Rightarrow r = 2$

A ring: inside: $\rho = 1$ sphere

outside: $x^2+y^2 = 4$ cylinder

top: $z = \sqrt{3(x^2+y^2)}$ cone

bottom: $z = -\sqrt{3(x^2+y^2)}$ cone

3. Consider the four curves in the first quadrant.

| | |
|---|---|
| (a) | $y = 1/x$ |
| (b) | $y = 2/x$ |
| (c) | $y = x$ |
| (d) | $y = 3x$ |

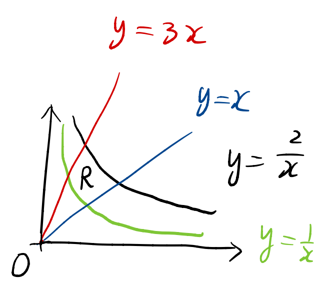
(a) **Discuss:** Rewrite the first two so that you can substitute $u = xy$.
**Answer and Explain:**

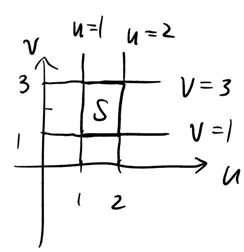
with $u = xy$, $\begin{cases} y = \frac{1}{x} \rightarrow xy = 1 \rightarrow u = 1 \\ y = \frac{2}{x} \rightarrow xy = 2 \rightarrow u = 2 \end{cases}$

(b) **Discuss:** Rewrite the second two so that you can substitute $v = y/x$.
**Answer and Explain:**

with $v = y/x$, $\begin{cases} y = x \rightarrow \frac{y}{x} = 1 \rightarrow v = 1 \\ y = 3x \rightarrow \frac{y}{x} = 3 \rightarrow v = 3 \end{cases}$

(c) **Discuss:** If $R$ is the region in the $xy$-plane bounded by the four original curves, draw a picture of $R$ and of the new region $S$ in the $uv$-plane bounded by the new curves you get after the substitution.
**Answer and Explain:**




(d) **Discuss:** Calculate the Jacobian. This is a bit challenging!
**Answer and Explain:**

$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Rightarrow \begin{cases} x = \sqrt{u/v} \\ y = \sqrt{uv} \end{cases}$$

$$J(u,v) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right|$$

$$= \left| \frac{1}{2\sqrt{uv}} \cdot \frac{1}{2}\sqrt{\frac{u}{v}} - \left(-\frac{1}{2}\right) \cdot \frac{1}{v}\sqrt{\frac{u}{v}} \cdot \frac{1}{2}\sqrt{\frac{v}{u}} \right| = \left| \frac{1}{4v} + \frac{1}{4v} \right| = \frac{1}{2v} \quad \text{since } v > 0$$