# MATH240 Summer 2022

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## Contents

# 1 Lectures

## 1.1 Lecture 1 - 05/31/2022

**Example 1.1.1** (Example in the ancient Chinese mathematics book titled *The Nine Chapters on the Mathematical Art*)**.** In a cage there are chicken and rabbits. The total number of heads is 10 and the total number of legs is 26. Question: how many rabbits and chicken are there?

**Definition 1.1.2.** A linear equation in the variables $x_1, x_2, x_3, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \tag{1.1.1}$$

where $b$ and the coefficients $a_1, a_2, a_3, \cdots, a_n$ are real or complex numbers, usually known in advance.

*Remark.* We adopt the following notations

- Natural numbers $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$.
- Integers $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.
- Rational numbers $\mathbb{Q}$ is the set of fractions.
- Real numbers $\mathbb{R}$ contains irrational numbers (like $\sqrt{2}, \sqrt[3]{3}$) and transcendental numbers (like $\pi, e$)
- Complex numbers $\mathbb{C} = \{a+bi | a, b \in \mathbb{R}\}$, here $\in$ means belong to, $i = \sqrt{-1}$ is the imaginary number such that $i^2 = -1$.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- $\mathbb{R}^n = \{(r_1, r_2, \cdots, r_n) | r_1, r_2, \cdots, r_n \in \mathbb{R}\}$ is the set of all $n$-tuples of real numbers. Geometrically, it is the $n$-dimensional Euclidean space. For example
  - $\mathbb{R}^1 = \mathbb{R}$ is a line
  - $\mathbb{R}^2$ is a plane
  - $\mathbb{R}^3$ is our usual three dimensional space
  - $\vdots$

**Example 1.1.3** (Examples and non-examples of linear equations)**.**

- $x_1 + \frac{1}{2}x_2 = 2$, ✓
- $\pi(x_1 + x_2) - 9.9x_3 = e$ ✓. Because we can expand to get $\pi x_1 + \pi x_2 - 9.9x_3 = e$ in which case $a_1 = \pi, a_2 = \pi, a_3 = -9.9, b = e$ as in the form of (1.1.1)
- $|x_2| - 1 = 0$, ✗
- $x_1 + x_2^2 = 9$, ✗
- $\sqrt{x_1} + \sqrt{x_2} = 1$, ✗

*Question.* Why do we use the word “linear”? What do they mean geometrically?

*Answer.* A linear equation uniquely characterizes a hyperplane. A hyperplane is a one-dimension less subspace, for example

- In $\mathbb{R}^1$, it is a point.
- In $\mathbb{R}^2$, it is a line.
- In $\mathbb{R}^3$, it is a plane.
- In $\mathbb{R}^4$, it is a hyperplane.

- $\vdots$

**Definition 1.1.4.** A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables, say $x_1, x_2, \cdots, x_n$.

**Example 1.1.5.** In Example 1.1.1, if we assume the number of chickens and rabbits are $x_1$ and $x_2$, then we can use the information we know about heads and legs to get the linear system

$$\begin{cases} x_1 + x_2 = 10 \\ 2x_1 + 4x_2 = 26 \end{cases} \tag{1.1.2}$$

**Definition 1.1.6.** A solution of the system is $\begin{cases} x_1 = s_1 \\ x_2 = s_2 \\ x_3 = s_3 \\ \vdots \\ x_n = s_n \end{cases}$ which makes each equation true.

The set of all possible solutions is called the solution set of the linear system. To solve a linear system is to find all its solutions.

*Question.* How many solutions could a linear system have?

**Example 1.1.7.** There are three different possibilities

- In Example (1.1.2), there is a unique solution $\begin{cases} x_1 = 7 \\ x_2 = 3 \end{cases}$
- $\begin{cases} x_1 + 2x_2 = 3 \\ 2x_1 + 4x_2 = 6 \end{cases}$ has uncountably infinitely many solutions since the two equations describes the same line. (the second equation is just twice of the first one)
- $\begin{cases} x_1 + 2x_2 = 3 \\ -2x_1 - 4x_2 = 6 \end{cases}$ has no solutions since these two equations defines two parallel, non-intersecting lines.

*Answer.* A linear system consists of two equations with two variables has

- infinitely many solutions if these two lines are overlapping.
- a unique solution if these two lines intersect.
- no solutions if these two lines are parallel and non-intersecting.

*Question.* How should we solve a linear system?

*Answer.* We use Gaussian elimination which is illustrated in the following example

**Example 1.1.8.** Let's solve the linear system $\begin{cases} 2x_1 + 2x_2 = 20 & ① \\ 2x_1 + 4x_2 = 26 & ② \end{cases}$ in Example 1.1.5

Step 1. Multiply ① by 2, we get $\begin{cases} 2x_1 + 2x_2 = 20 & ③ \\ 2x_1 + 4x_2 = 26 & ② \end{cases}$

Step 2. Replace ① by ③ − ②, we get $\begin{cases} \quad - 2x_2 = -6 & ④ \\ 2x_1 + 4x_2 = 26 & ② \end{cases}$

Step 3. Divide ④ by −2, we get $\begin{cases} \quad x_2 = 3 & ⑤ \\ 2x_1 + 4x_2 = 26 & ② \end{cases}$

Step 4. Replace ② by ② − 4⑤, we get $\begin{cases} \quad\quad x_2 = 3 & ⑤ \\ 2x_1 \quad\quad = 14 & ⑥ \end{cases}$

Step 5. Divide ⑥ by 2, we get $\begin{cases} \quad\quad x_2 = 3 & ⑤ \\ x_1 \quad\quad = 7 & ⑦ \end{cases}$

Step 6. Interchange ⑤ and ⑦, we finally have the solution $\begin{cases} x_1 \quad\quad = 7 & ⑦ \\ \quad\quad x_2 = 7 & ⑤ \end{cases}$

**Definition 1.1.9.** We say a linear system is consistent it has solution(s), and inconsistent if it has none

## 1.2 Lecture 2 - 06/01/2022

**Definition 1.2.1.** A $m$ by $n$ (or $m \times n$) matrix is a rectangular array of numbers of $m$ rows and $n$ columns, we use the $(i, j)$-th entry to mean the entry on the $i$-th row and $j$-column. A matrix is

- a zero matrix is a matrix with all entries zeros.
- a square matrix is a matrix with the same number of rows and columns, i.e. $m = n$.
- a vector if it only has one column, i.e. $n = 1$
- the identity matrix if it is a square matrix with diagonal elements 1's, and 0's otherwise. Here the diagonal are the $(i, i)$-th entries

**Example 1.2.2.**

- A general $m$ by $n$ matrix looks like $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, here $a_{ij}$'s are numbers
- A general $n \times n$ square matrix looks like $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, here $a_{ij}$'s are numbers
- A general vector looks like $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, here $a_i$'s are numbers
- A zero matrix looks like $\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$
- The $n$ by $n$ identity matrix looks like $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

**Definition 1.2.3.** Soon we will be getting tired of writing all these equations in a linear system, instead we write down its augmented matrix, obtained by omitting $x_i$'s, pluses, and equal signs. If we delete the last column, we will get the coefficient matrix

**Example 1.2.4.** For (1.1.2), its augmented matrix and coefficient matrix are

$$\begin{bmatrix} 1 & 1 & 10 \\ 2 & 4 & 26 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

For $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 \quad\quad - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \end{cases}$, its augmented matrix and coefficient matrix are

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

In general, a linear system of $m$ equations in $n$ variables has a $m$ by $(n+1)$ augmented matrix and a $m$ by $n$ coefficient matrix.

**Definition 1.2.5.** Look more carefully at Example 1.1.8, we define the following three elementary row operations

- Replacement: Replace one row by the sum of itself and a multiple of another row.
- Interchange: Interchange two rows.
- Scaling: Multiply all entries in a row by a nonzero constant.

We say matrices $A, B$ are row equivalent ($A \sim B$) if $B$ can obtained by applying a sequence of elementary row operations to $A$ (or vise versa)

**Example 1.2.6.** Let's rewrite the process in Example 1.1.8

$$\begin{bmatrix} 1 & 1 & 10 \\ 2 & 4 & 26 \end{bmatrix} \overset{2R1}{\sim} \begin{bmatrix} 2 & 2 & 20 \\ 2 & 4 & 26 \end{bmatrix} \overset{R1 \to R1 - R2}{\sim} \begin{bmatrix} 0 & -2 & -6 \\ 2 & 4 & 26 \end{bmatrix} \overset{\frac{R1}{-2}}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 4 & 26 \end{bmatrix}$$
$$\overset{R2 \to R2 - 4R1}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 14 \end{bmatrix} \overset{\frac{1}{2}R2}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 7 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\sim} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 3 \end{bmatrix}$$

**Example 1.2.7.** $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 \quad\quad - x_3 = 1 \\ x_1 + x_2 + x_3 = 3 \end{cases}$, first we write out its augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \overset{R2 \to R2 - 2R1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \overset{R3 \to R3 - R1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$
$$\overset{R3 \to R3 - R2}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix} \overset{\frac{R3}{3}}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{\substack{R1 \to R1 - R3 \\ R2 \to R2 + 3R3}}{\sim} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
$$\overset{\frac{1}{2}R2}{\sim} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \overset{R1 \to R1 + R2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This gives the unique solution $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$

**Definition 1.2.8.** A leading entry of a row refers to the leftmost nonzero entry (in a nonzero row). A matrix is of row echelon form, REF if it is of a "staircase shape". The leading entries of an REF matrix are called pivots, the position of pivots are called pivot positions, and the column pivots are in are called pivot columns. An REF of reduced row echelon form (RREF) if all its pivots are 1's and in each column, every entry except the pivot are 0's

$$\overset{\text{REF}}{\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}}, \quad \overset{\text{RREF}}{\begin{bmatrix} 1 & * & 0 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & * & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}}$$

**Example 1.2.9.** In Example 1.2.7, $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & 0 & 2 \end{bmatrix}$ is not an REF. $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ is an REF, but not an RREF. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is an RREF

**Theorem 1.2.10.** Every matrix is row equivalent to some REF matrix (which is not in general unique), and it is row equivalent to some unique RREF matrix.

*Remark.* This ensures that the pivot positions are well-defined, i.e. you won't get different pivot positions if you applied different row operations

**Example 1.2.11.** In Example 1.2.7, $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ is an REF row equivalent to the original matrix $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is the unique row equivalent RREF matrix.

*Remark.* A linear system has a unique solution if and only if its RREF deleting the last column gives the identity matrix.

**Example 1.2.12.** $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \\ x_1 + x_2 - 2x_3 = 1 \end{cases}$, we write down its augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \underset{R3\to R3-R1}{\overset{R2\to R2-2R1}{\sim}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 2 & -3 & 0 \end{bmatrix} \overset{R3\to R3-R2}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You might notice that the last row is a bit weird which indeed is true since it represents $0x_1 + 0x_2 + 0x_3 = 0 = 1$, this is a contradiction, hence the linear system is inconsistent, i.e. it has no solutions

*Remark.* This only happens if and only if the last pivot column is the last column

**Example 1.2.13.** $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 - x_3 = 1 \end{cases}$, we write down its augmented matrix

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 1 \end{bmatrix} \overset{R2\to R2-2R1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$
$$\overset{\frac{R2}{2}}{\sim} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \overset{R1\to R1+R2}{\sim} \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

This gives the solution set

$$\begin{cases} x_1 - \frac{1}{2}x_3 = \frac{1}{2} \\ x_2 - \frac{3}{2}x_3 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{2}x_3 + \frac{1}{2} \\ x_2 = \frac{3}{2}x_3 - \frac{1}{2} \end{cases} \tag{1.2.1}$$

Let's formalize the row reduction algorithm

Step 1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3. Use row replacement operations to create zeros in all positions below the pivot.

Step 4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Step 5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Here steps 1-4 are call Forward phase, and step 5 is called backward phase.

**Definition 1.2.14.** The variables corresponding to pivot columns in a matrix are called basic variables, the other variables are called free variables. In a solution set, basic variables are expressed in terms of free variables, and a free variable can take any value.

**Example 1.2.15.** In Example 1.2.13, $x_1, x_2$ are basic variables and $x_3$ is a free variable. And we formally write our solution set as $\begin{cases} x_1 = \frac{1}{2}x_3 + \frac{1}{2} \\ x_2 = \frac{3}{2}x_3 - \frac{1}{2} \\ x_3 \text{ is free} \end{cases}$

**Exercise 1.2.16.** Find the general solution of the system $\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 4x_3 + 8x_4 = 2 \end{cases}$

*Solution.*

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -4 & 8 & 2 \end{bmatrix} \xrightarrow[R3 \to R3 - 3R1]{R2 \to R2 + 2R1} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \xrightarrow{(-1)\cdot R3} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{R2 \to R2 - 3R3} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 0 & -2 & 9 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & -2 & 9 \end{bmatrix}$$

$$\overset{\frac{R3}{-2}}{\sim} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xrightarrow[R1 \to R1 - 3R3]{R2 \to R2 - R3} \begin{bmatrix} 1 & -2 & -1 & 0 & \frac{27}{2} \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix} \xrightarrow{R1 \to R1 + R2} \begin{bmatrix} 1 & -2 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 0 & 1 & -\frac{9}{2} \end{bmatrix}$$

Write this as solution set, we get

$$\begin{cases} x_1 - 2x_2 = 16 \\ x_3 = \frac{5}{2} \\ x_4 = -\frac{9}{2} \end{cases} \Rightarrow \begin{cases} x_1 = 2x_2 + 16 \\ x_2 \text{ is free} \\ x_3 = \frac{5}{2} \\ x_4 = -\frac{9}{2} \end{cases}$$

□

**Theorem 1.2.17.** Consider the RREF matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ equivalent to the augmented matrix of a linear system, then the linear system has

- no solutions $\iff$ the last column of $M$ is a pivot column, i.e. has a pivot
- infinitely many solutions $\iff$ the last column is not a pivot column and there are free variables
- a unique solution $\iff$ $M$ deleting the last column is an identity matrix

## 1.3 Lecture 3 - 06/02/2022

**Definition 1.3.1.** Recall a vector is a matrix with one column, the zero vector is a vector with all entries zero. For scalar (i.e. a number) $c$, and vectors $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, we could define

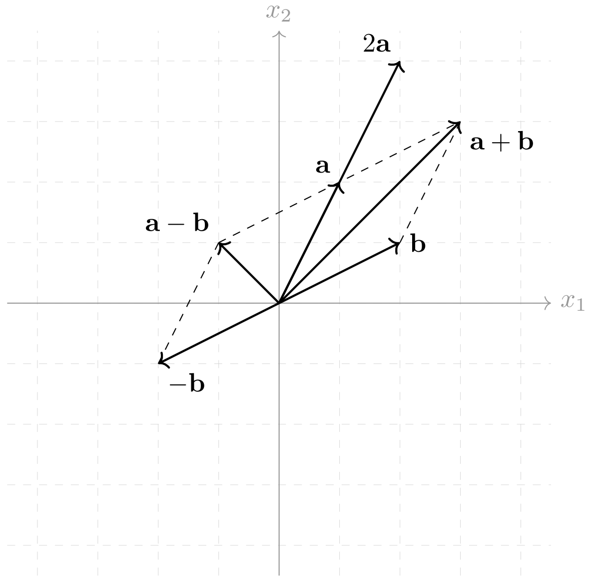
- Addition $\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$
- Scalar multiplication $c\mathbf{a} = c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$
- Subtraction $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1)\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_n - b_n \end{bmatrix}$

A linear combination of vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ means a sum $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$ for some scalars $c_1, \cdots, c_n$

*Remark.* In handwritings, we use $\vec{v}$ or $\overrightarrow{v}$ to denote a vector, while in printing materials we often use $\mathbf{v}$

**Example 1.3.2.** Let $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

- $\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$
- $2\mathbf{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- $-\mathbf{b} = (-1)\mathbf{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$
- $\mathbf{a} - \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$




We can now rewrite a linear system as a vector equation

**Example 1.3.3.** (1.1.2) can be written as $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$

$$x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}x_1\\2x_1\end{bmatrix} + \begin{bmatrix}x_2\\4x_2\end{bmatrix} = \begin{bmatrix}x_1+x_2\\2x_1+4x_2\end{bmatrix} = \begin{bmatrix}10\\26\end{bmatrix}$$

Here $\mathbf{a}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}1\\4\end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix}10\\26\end{bmatrix}$

*Remark.* In general, vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{1.3.1}$$

can be thought of as a generalization of (1.1.1)

**Example 1.3.4.** $\begin{cases} x_1 + x_3 = 1 \\ 2x_1 + x_3 = 2 \end{cases}$ can be written as

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\2\end{bmatrix} + x_3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

**Definition 1.3.5.** The span of $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is the set of all its linear combinations, which we denote $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$. We see that (1.3.1) has solution(s) $\iff$ $\mathbf{b}$ is in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ (i.e. $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$) $\iff$ the linear system is consistent

**Exercise 1.3.6.** Let $\mathbf{a}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Is $\mathbf{b}$ in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

*Solution.* This is equivalent of asking if the following linear system is consistent $\begin{cases} x_1 - x_2 + x_3 = 1 \\ 2x_1 \qquad - x_3 = 1 \\ x_1 + x_2 - 2x_3 = 1 \end{cases}$, we find its RREF

$$\begin{bmatrix}1&-1&1&1\\2&0&-1&1\\1&1&-2&1\end{bmatrix} \xrightarrow[R3\to R3-R1]{R2\to R2-2R1} \begin{bmatrix}1&-1&1&1\\0&2&-3&-1\\0&2&-3&0\end{bmatrix} \xrightarrow{R3\to R3-R2} \begin{bmatrix}1&-1&1&1\\0&2&-3&-1\\0&0&0&1\end{bmatrix}$$

Since there is a pivot in the last column, by Theorem 1.2.17, the linear system is inconsistent, hence $\mathbf{b}$ is not in $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ $\square$

## 1.4 Lecture 4 - 06/03/2022

**Definition 1.4.1.** Let's use $M_{m\times n}(\mathbb{R})$ to denote the set of all (real-valued) matrices of dimension $m$ by $n$

**Definition 1.4.2.** Suppose $A, B$ are $m \times n$ matrices, $c$ is a scalar (i.e. a number), then we can define

- Addition $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$

- Scalar multiplication $c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}$

*Remark.* Note that in the case of vectors (i.e. $n = 1$), these recovers addition and scalar multiplication of vectors.

**Example 1.4.3.** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$, $2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

**Definition 1.4.4.** Suppose $A$ is a $m \times n$ matrix, and $B$ is a $n \times p$ matrix, we can define matrix multiplication $AB$ to be the $m \times p$ matrix, computed via the rule-column rule: The $(i, j)$-entry is to multiply the $i$-row and $j$-th column

$$\begin{bmatrix} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} = \begin{bmatrix} \\ & \blacksquare & \\ \\ \end{bmatrix}$$

Where $\blacksquare = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$.

If $A$ is a square matrix, then we could define matrix power $A^k$ to be simply $\overbrace{AA\cdots A}^{k \text{ times}}$

**Example 1.4.5.**

- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$

- $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 9 & 7 \end{bmatrix}$

- $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_1 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_1 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \tag{1.4.1}$$

- $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{1.4.2}$$

**Fact 1.4.6.** Suppose $A, B, C, D$ are matrices, $c$ is a scalar, 0 is the zero matrix, $I$ is the identity matrix. we have the following facts

a) Matrix multiplication is not commutative, i.e. $AB \neq BA$

b) Matrix multiplication is associative, i.e. the order of multiplication doesn't matter, in other words $(AB)C = A(BC)$, so there is no confusion in writing successive multiplication $A_1A_2A_3 \cdots A_n$

c) Scalar multiplication and matrix multiplication commutes, $A(cB) = c(AB) = (cA)B$.

d) Matrix multiplication is distributive over addition, i.e. $A(B+C) = AB+AC$, $(A+B)C = AC + BC$

e) $A + 0 = 0 + A = A$, $A0 = 0A = 0$

f) $IA = AI = A$

g) Even if $A \neq 0$, $B \neq 0$, $AB$ could still be 0, take (1.4.2) for an example

h) $AB = AC$ does not imply $B = C$

*Remark.* Some of the properties of matrices are really similar to that of numbers, so we dub this the name of matrix algebra

**Definition 1.4.7.** $A$ is a partitioned (or block) matrix if is divided into smaller submatrix by some horizontal and vertical lines. And the submatrices are the blocks

$$\left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right] = \left[\begin{array}{cc|cccc|c} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * \\ \hline * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{array}\right]$$

Here the blocks are $A_{11} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$, $A_{12} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \end{bmatrix}$, $A_{13} = \begin{bmatrix} * \\ * \end{bmatrix}$, $A_{21} = \begin{bmatrix} * & * \end{bmatrix}$, $A_{22} = \begin{bmatrix} * & * & * & * \end{bmatrix}$, $A_{23} = \begin{bmatrix} * \end{bmatrix}$, $A_{31} = \begin{bmatrix} * & * \\ * & * \\ & * \end{bmatrix}$, $A_{32} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$, $A_{33} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$

**Fact 1.4.8.** Suppose $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \vdots & \vdots & & \vdots \\ B_{q1} & B_{q2} & \cdots & B_{qr} \end{bmatrix}$ are partitioned matrices, the number of columns of submatrix $A_{1k}$ is equal to the number of rows of submatrix $B_{k1}$ (i.e. all submatrices multiplications make sense). Then the usual matrix multiplication is the same as the first treating these submatrix as numbers, do the row-column rule and then multiply them out

**Example 1.4.9.** Consider $\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 1 \\ \hline 2 & 1 & 1 \end{array}\right]$, $\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 2 & 2 & 1 \\ \hline 2 & 1 & 1 \end{array}\right]$, then we have

$$\left[\begin{array}{c|cc} 5 & 3 & 7 \\ 8 & 5 & 11 \\ \hline 7 & 4 & 10 \end{array}\right] = AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$$

Now we are able to write the vector equation (1.3.1) as a matrix equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix} = A\mathbf{x} = \mathbf{b}$$

Here $A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n\end{bmatrix}$ as a partitioned matrix, $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$

**Example 1.4.10.** Follow Example 1.3.4, we can rewrite the vector equation as a matrix equation as in (1.4.1)

$$\begin{bmatrix}1 & 0 & 1\\0 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\cdot x_1 + 0\cdot x_2 + 1\cdot x_3\\0\cdot x_1 + 2\cdot x_2 + 1\cdot x_3\end{bmatrix} = x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\2\end{bmatrix} + x_3\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

## 1.5 Lecture 5 - 06/06/2022

*Question.* Suppose $A$ is $m \times n$, when does the matrix equation $A\mathbf{x} = \mathbf{b}$ always has a solution for any $\mathbf{b}$ in $\mathbb{R}^n$

**Theorem 1.5.1.** Let $A$ be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular $A$, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

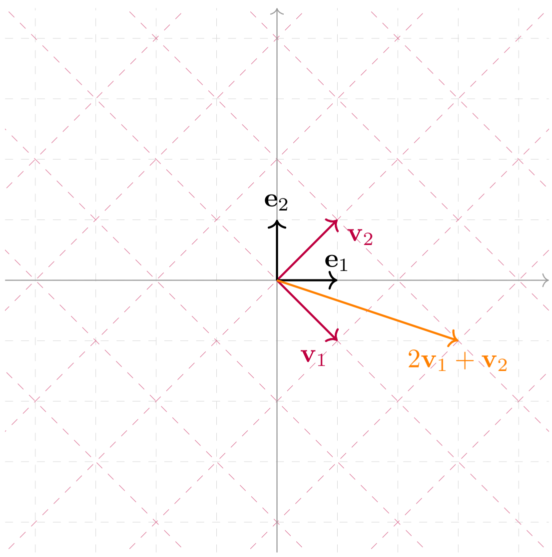
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of $A$.

c. The columns of $A$ span $\mathbb{R}^m$.

d. $A$ has a pivot position in every row. (Equivalently, in the last row)

*Question.* What is the geometric meaning of spans

**Example 1.5.2.** Consider $\mathbf{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$




What is Span$\{\mathbf{v}_1, \mathbf{v}_2\}$, find $c_1, c_2$ such that $\mathbf{e}_1 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$

*Answer.* Apply Theorem 1.5.1 to $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ which have a pivot on each row, so the column vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $A$ span the whole plane $\mathbb{R}^2$, i.e. $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathbb{R}^2$. To solve $c_1, c_2$, let's consider

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Hence $\mathbf{e}_1 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$

**Definition 1.5.3.** $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is linearly dependent if some $\mathbf{v}_i$ can be written as a linear combination of the others, or equivalently, if there is a non-trivial solution $c_1, \cdots, c_n$ (i.e. not all $c_i$'s are 0) to the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \tag{1.5.1}$$

(1.5.1) is refered to as a linear dependence between $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. If (1.5.1) has only the trivial solution (i.e. $c_1, \cdots, c_n$ are all 0, which is of course always a solution), $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is said to be linearly independent

*Remark.* Equivalence between two different definitions of linear dependence

- If $\mathbf{v}_i = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, then $c_1\mathbf{v}_1 + \cdots + (-1)\mathbf{v}_i + \cdots + c_n\mathbf{v}_n = \mathbf{0}$
- If $c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ and $c_i \neq 0$ (since not all $c_i$'s are zero, we may assume $c_i$ is nonzero), then $\mathbf{v}_i = -\frac{c_1}{c_i}\mathbf{v}_1 - \cdots - \frac{c_n}{c_i}\mathbf{v}_n$

**Theorem 1.5.4.** To determine the linear dependence of $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, we may consider $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$, then the solution to $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ is the solution to the RREF form of $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$. In other words, $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ is linearly independent $\iff$ each column of the RREF of $A$ is a pivot column.

**Example 1.5.5.** In Example 1.5.2, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1\}$ is linearly dependent since

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{e}_1 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Note that this computation is essentially the same as in Example 1.5.2, the $\mathbf{0}$ in the augmented matrix doesn't change when performing row reductions. We get

$$\begin{cases} x_1 = -\frac{1}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ is free} \end{cases}$$

Choose $x_3 = -1$ we get the linear dependence $\frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 - \mathbf{e}_1 = \mathbf{0}$. On the other hand, $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent since

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Which says that $\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$, the trivial solution.

**Definition 1.5.6.** Suppose $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$ (meaning a subset of vectors in $\mathbb{R}^m$) is linearly independent and $\text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_n\} = \mathbb{R}^m$, then $n$ necessarily equals to $m$. We call such a set $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ basis for $\mathbb{R}^m$, and the maximal number of independent vectors (in this case $m$) the dimension of $\mathbb{R}^m$. (This only has to do with the space $\mathbb{R}^m$ it self, not the choice of basis.)

**Example 1.5.7.** In Example 1.5.2, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$, if we let $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\{\mathbf{e}_1, \mathbf{e}_2\}$ is another basis for $\mathbb{R}^2$. In fact, any two vectors in $\mathbb{R}^2$ that are not parallel form a basis.

**Definition 1.5.8.** The standard basis for $\mathbb{R}^m$ is the set of vectors $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$, where

$$\mathbf{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th entry}$$

**Example 1.5.9.** $\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is the standard basis for $\mathbb{R}^3$, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

**Theorem 1.5.10.** $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is a basis $\iff$ the RREF of $\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix}$ is the identity matrix

**Theorem 1.5.11.** If $\{\mathbf{v}_1, \cdots, \mathbf{v}_m\}$ is a basis for $\mathbb{R}^m$, then $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} \sim I$. For any vector $\mathbf{b}$ in $\mathbb{R}^m$, $\mathbf{v}$ can be uniquely written as some linear combination of the basis vectors (since $A\mathbf{x} = \mathbf{b}$ always has a unique solution)

## 1.6 Lecture 6 - 06/07/2022

A linear system is homogeneous if it has matrix equation $A\mathbf{x} = \mathbf{0}$, note that this always have the zero solution, called the trivial solution. $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\iff$ it has at least one free variable $\iff$ columns of $A$ are linearly dependent.
We can express the solution set to a linear system in parametric vector form

**Example 1.6.1.** In Example 1.2.13, the solution set can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 + \frac{1}{2} \\ \frac{3}{2}x_3 - \frac{1}{2} \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

In Example 1.2.16, the solution set can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_2 + 16 \\ x_2 \\ \frac{5}{2} \\ -\frac{9}{2} \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \\ \frac{5}{2} \\ -\frac{9}{2} \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 16 \\ 0 \\ \frac{5}{2} \\ -\frac{9}{2} \end{bmatrix}$$

**Exercise 1.6.2.** Suppose the augmented matrix of a linear system is equivalent to the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Write down the solution set in parametric vector form

*Solution.*

$$\begin{cases} x_1 + x_2 + 2x_4 = 3 \\ x_3 - 2x_4 = 2 \\ x_5 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 3 - x_2 - 2x_4 \\ x_2 \text{ is free} \\ x_3 = 2 + 2x_4 \\ x_4 \text{ is free} \\ x_5 = 1 \end{cases}$$

So the solution in parametric vector form would be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 - x_2 - 2x_4 \\ x_2 \\ 2 + 2x_4 \\ x_4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_4 \\ 0 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

□

*Question.* What is the relation of the RREF of $A$ and the RREF of $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$

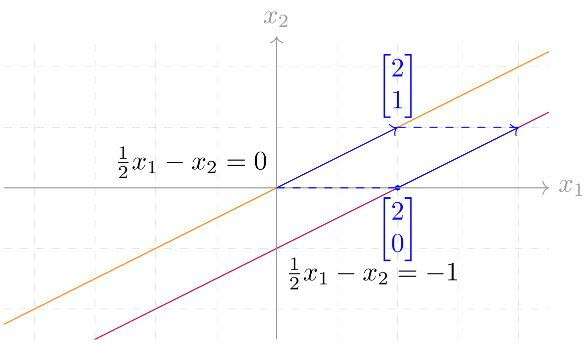
*Answer.* Suppose $\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \sim \begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ is the RREF, then $U$ will be the RREF of $A$. Note that the solutions of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$ differs by $\mathbf{d}$, i.e.

$$\mathbf{d} + \{\textsf{solutions of } A\mathbf{x} = \mathbf{0}\} = \{\textsf{solutions of } A\mathbf{x} = \mathbf{b}\}$$

*Question.* What is the geometric meaning of a solution set in parametric vector form

*Answer.* It is the hyperplane (solutions of $A\mathbf{x} = \mathbf{b}$) translated from the hyperplane through the origin (solutions of $A\mathbf{x} = \mathbf{0}$) by $\mathbf{d}$

**Example 1.6.3.** $\frac{1}{2}x_1 - x_2 = -1$ has solution set $\begin{cases} x_1 = -2 + 2x_2 \\ x_2 \text{ is free} \end{cases}$ which is $\begin{bmatrix} -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in parametric vector form, and $\frac{1}{2}x_1 - x_2 = 0$ has solution set $\begin{cases} x_1 = 2x_2 \\ x_2 \text{ is free} \end{cases}$ which is $x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in parametric vector form.




**Definition 1.6.4.** A Linear transformation (or mapping) $T$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ satisfying

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$
- $T(c\mathbf{u}) = cT(\mathbf{u})$ for any scalar $c$ and any $\mathbf{u}$ in $\mathbb{R}^n$

**Definition 1.6.5.** A matrix transformation is a linear transformation defined via matrix multiplication, i.e. $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix. It is linear thanks to Fact 1.4.6 c),d) since

- $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$

*Question.* How to know a linear transformation $T$?

*Answer.* Just need to know its effect on any basis! Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is a basis of $\mathbb{R}^n$, then any vector $\mathbf{v}$ can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, and by linearality, we have

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) \tag{1.6.1}$$

Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ as the standard basis as in Definition 1.5.8, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then any $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, by (1.6.1)

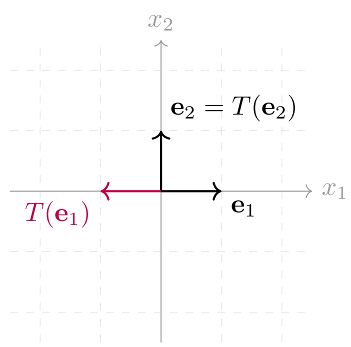
$$T(\mathbf{x}) = x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{1.6.2}$$

So $A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$ (which is called the standard matrix for the linear transformation $T$) is the unique matrix satisfies $T(\mathbf{x}) = A\mathbf{x}$

**Example 1.6.6.** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation

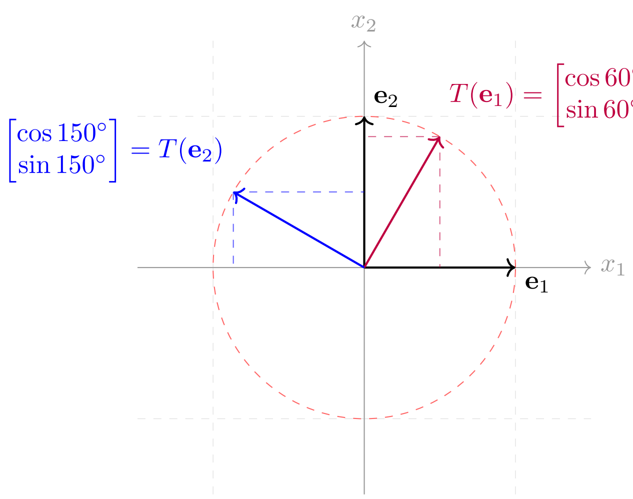
a) Assume $T$ is the reflection over $x_2$-axis, then the standard matrix for $T$ is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$




b) Assume $T$ is the rotation by $60°$ counter-clockwise, then the standard matrix for $T$ is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$




## 1.7 Lecture 7 - 06/08/2022

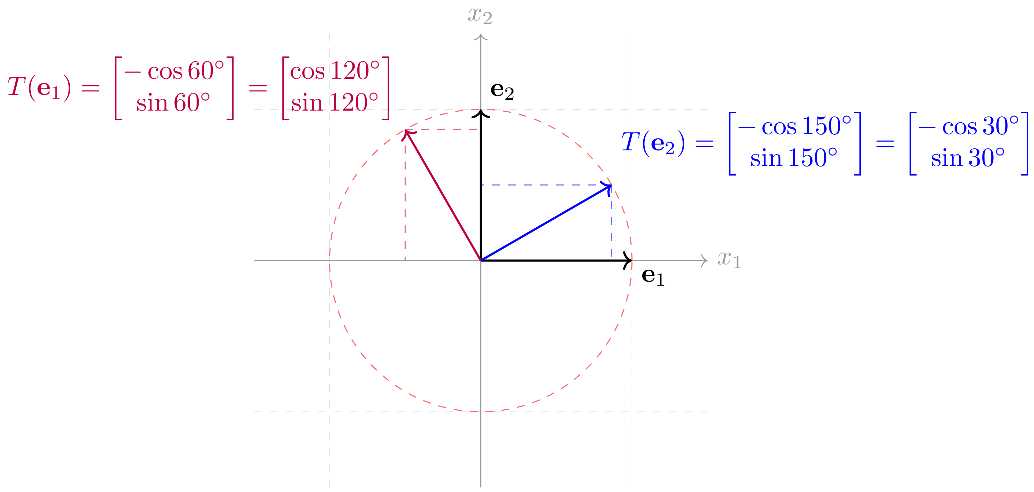
**Exercise 1.7.1.** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 + 1 \\ x_2 \end{bmatrix}$. Is $T$ is a linear mapping?

*Solution.* $T$ is not a linear transformation since $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\mathbf{x} + \mathbf{p}$ is not of the matrix transformation form (1.6.2) ($\mathbf{p} \neq 0$) $\square$

**Exercise 1.7.2.** $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation that rotate $60°$ counter-clockwise and then reflects over $x_2$-axis, what is its standard matrix?

*Solution.* The standard matrix for $T$ is

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$




□

**Definition 1.7.3.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a mapping. We call

- $\mathbb{R}^n$ the domain of $T$
- $\mathbb{R}^m$ the codomain of $T$
- $T(\mathbf{x})$ the image of $\mathbf{x}$ under $T$
- $T^{-1}(\mathbf{b}) = \{\mathbf{x} | T(\mathbf{x}) = \mathbf{b}\}$ the preimage of $\mathbf{b}$ under $T$
- the set of images $\{T\mathbf{x} | \mathbf{x} \in \mathbb{R}^n\}$ the range of $T$

**Exercise 1.7.4.** Suppose the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$, what is the standard matrix of $T$? What is the image $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, what is the preimage of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, what is the range?

*Solution.* The standard matrix is $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, the image $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ under $T$ is

$$T(\mathbf{x}) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - 0 + 1 \\ 2 \cdot 2 - 1 \\ 2 + 0 + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

The preimage of $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ under $T$ is the solution set to $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$ (this is Example 1.2.7), which is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. And since there is a pivot in each row, by Theorem 1.5.1, the range of $T$ is $\mathbb{R}^3$ □

**Definition 1.7.5.** A mapping $T$ is said to be onto $\mathbb{R}^m$ if each $b \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$.

Codomain is larger than the range if $T$ is not onto

**Definition 1.7.6.** A mapping $T$ is said to be one-to-one if each $b \in \mathbb{R}^m$ is the image of at most one $x \in \mathbb{R}^n$.

**Theorem 1.7.7.** Suppose $A$ is the standard matrix for linear transformation $T$ (i.e. $T(\mathbf{x}) = A\mathbf{x}$), then

- $T$ is one-to-one $\iff$ $A\mathbf{x} = \mathbf{0}$ has a unique solution $\iff$ $A\mathbf{x} = \mathbf{b}$ has at most a unique solution $\iff$ RREF of $A$ has a pivot in each column $\iff$ columns of $A$ are linearly independent.
- $T$ is onto $\iff$ the columns of $A$ span $\mathbb{R}^m$ $\iff$ RREF of $A$ has a pivot in each row.

**Exercise 1.7.8.** Suppose the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ 2x_1 - x_3 \end{bmatrix}$, Is $T$ onto? Is $T$ one-to-one?

*Solution.* This is the Example 1.2.13. The standard matrix for $T$ is $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \end{bmatrix}$, since there is a pivot in each row but not in each column, it is onto but not one-to-one $\square$

*Question.*

- If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, could it be one-to-one?
- If $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation, could it be onto?

*Answer.* Both no! Due to Theorem 1.7.7

- Since $A$ is a $2 \times 3$ matrix, there will be at most 2 pivots (only 2 rows), so there won't be enough pivots to fill all columns.
- Since $A$ is a $3 \times 2$ matrix, there will be at most 2 pivots (only 2 columns), so there won't be enough pivots to fill all rows.

## 1.8 Lecture 8 - 06/09/2022

*Question.* What happens if we compose two linear transformation (say $T_1 : \mathbb{R}^p \to \mathbb{R}^n$, $T_2 : \mathbb{R}^n \to \mathbb{R}^m$)?

**Definition 1.8.1.** We have
$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "T_1"] \arrow[rr, bend right, "T_2 \circ T_1"'] & \mathbb{R}^n \arrow[r, "T_2"] & \mathbb{R}^m \end{tikzcd}$$
, Here $\circ$ means composition which you compose from the left. suppose the standard matrices for $T_1, T_2$ are $A_1, A_2$ respectively, then $A_1$ is $n \times p$, $A_2$ is $m \times n$, and for $\mathbf{x} \in \mathbb{R}^p$, $(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x})) = A_2(T_1(\mathbf{x})) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x}$. So we have concluded that the standard matrix for $T_2 \circ T_1$ is the $m \times p$ matrix $A_2A_1$

**Example 1.8.2.** Consider Example 1.6.6., If we let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ to denote the rotation by $60°$ counter-clockwise, $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ to denote reflection over $x_2$-axis, and their standard matrices are

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then look at Exercise 1.7.2, this is the composition $T_2 \circ T_1$, which has the standard matrix

$$A_2A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

*Question.* Suppose $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ is the standard matrix for the linear transformation of rotating $60°$ counter-clockwise (Example 1.6.6). What is $A^7$?

*Answer.* $A^7 = AAAAAAA$ is the standard matrix for composition of linear transformations $T \circ T \circ T \circ T \circ T \circ T \circ T$ which is rotate $7 \times 60° = 420°$, but that is the same as rotating $420° - 360° = 60°$ which is the same linear transformation as $T$, so $A^7 = A$

$$\begin{tikzcd} A^7 = AAAAAAA \arrow[r, equal] & A \\ T^{\circ 7} = T \circ T \circ T \circ T \circ T \circ T \circ T \arrow[u, "\text{Standard matrix}"] \arrow[r, equal, "\text{same effect}"'] & T \arrow[u, "\text{Standard matrix}"'] \end{tikzcd}$$

**Definition 1.8.3.** Suppose $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is a $m \times n$ matrix, we define its transpose by flipping it over the diagonal, and this is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

**Example 1.8.4.** Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

**Theorem 1.8.5.** Here are some properties of matrix transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

**Definition 1.8.6.** For any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we can define the dot product to be $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = v_1 w_1 + \cdots + v_n w_n$. $\|\mathbf{v}\| = \sqrt{\mathbf{v}^T \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}$ is the length $\mathbf{v}$

Elementary row operations on matrix $A$ can be realized as elementary matrices $E$ multiplication on $A$.

**Exercise 1.8.7.** Let $A$ be a 3 by 3 matrix, write down the elementary matrix $E$ such that

1. $EA$ acts as subtracting 3 times row 3 from row 1
2. $AE$ acts as adding 2 times column 3 to column 1
3. $EA$ acts as scaling the third row by 2
4. $AE$ acts as scaling the third column by 3
5. $EA$ acts as interchanging row 1 and row 3
6. $AE$ acts as interchanging column 1 and column 2

*Solution.*

1. $E = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix} = \begin{bmatrix} R1 - 3R3 \\ R2 \\ R3 \end{bmatrix}$

2. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $AE = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 + 2\mathbf{a}_3 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$

3. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix} = \begin{bmatrix} R1 \\ 2R2 \\ R3 \end{bmatrix}$

4. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $AE = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & 3\mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$

5. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $EA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix} = \begin{bmatrix} R3 \\ R2 \\ R1 \end{bmatrix}$

6. $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $AE = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{bmatrix}$

□

### 1.9 Lecture 9 - 06/10/2022

**Definition 1.9.1.** Suppose linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is both onto and one-to-one (i.e. every vector $\mathbf{b}$ in the codomain $\mathbb{R}^m$ is a unique image, which we denote as $T^{-1}(\mathbf{b})$), and $A$ is the standard matrix for $T$, then $m$ necessary equal $n$ as showned in Question 1.7, so $A$ must be a square matrix. We know $T(\mathbf{x}) = \mathbf{b}$ always has a unique solution which is $T^{-1}(\mathbf{b})$, it can be shown that $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ as mapping is actually also a linear transformation (Why? See if you can figure this out). Then the standard matrix of $T^{-1}$ is defined to be $A^{-1}$ the inverse matrix of $A$, since

$$(T \circ T^{-1})(\mathbf{b}) = T(T^{-1}(\mathbf{b})) = T(\mathbf{x}) = \mathbf{b}$$
$$(T^{-1} \circ T)(\mathbf{x}) = T^{-1}(T(\mathbf{x})) = T^{-1}(\mathbf{b}) = \mathbf{x}$$

Note that $T \circ T^{-1}$, $T^{-1} \circ T$ work like the identity map, so $AA^{-1} = A^{-1}A = I$. In this case, we see that $A$ is equivalent to the identity matrix (because of Theorem 1.5.2, $A$ has a pivot in each row and column).

*Remark.* Because we can write elementary row operations as left elementary matrix multiplications, so we know there are elementary matrices $E_1, E_2, \cdots, E_k$ such that $E_k E_{k-1} \cdots E_2 E_1 A = I$. If we multiply $A^{-1}$ on the right on both sides, we get $E_k E_{k-1} \cdots E_2 E_1 = A^{-1}$

Using the remark above, we can deduce an algorithm for computing matrix inverses. Let's consider the RREF of the following partitioned matrix

$$\left[\ A \mid I\ \right] \sim \left[\ E_k E_{k-1} \cdots E_2 E_1 A \mid E_k E_{k-1} \cdots E_2 E_1 I\ \right] = \left[\ I \mid A^{-1}\ \right]$$

### 1.10 Lecture 10 - 06/13/2022

**Exercise 1.10.1.** Find the inverse of the following matrices.

a) $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

*Solution.*

a)

$$\left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

b)

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2 - 3R1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right]$$
$$\xrightarrow{R1 \to R1 + 2R2} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & -1 & -3 & 1 \end{array}\right] \xrightarrow{(-1)R2} \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

c)

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3 \to R3 - R1]{R2 \to R2 - 2R1} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & -1 & 1 \end{array}\right] \xrightarrow{R2 \to R2 + R3} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 & -1 & 1 \end{array}\right]$$
$$\xrightarrow[R3/3]{R2/2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \xrightarrow{R1 \to R1 + R2 - R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

□

**Exercise 1.10.2.** What is $(A^T)^{-1}$ in Exercise 1.10.1, c)?

*Solution.* Use Theorem 1.10.9, we know

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$$

□

**Definition 1.10.3.** A square matrix $A$ is invertible (or non-singular) if it has an inverse $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$. $A$ is called singular if $A$ is not invertible.

**Theorem 1.10.4.** Suppose $T$ is a linear transformation with standard matrix $A$, then

$$T \text{ is invertible with inverse } T^{-1} \iff A \text{ is invertible with inverse } A^{-1} \iff A \sim I$$

**Theorem 1.10.5.** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, here $\det A = ad - bc$

**Example 1.10.6.** If $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, then

$$A^{-1} = \frac{1}{\frac{1}{2}\frac{1}{2} - \frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2}\right)} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

**Theorem 1.10.7.** If $A$ is invertible, then the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} = A^{-1}\mathbf{b}$

**Example 1.10.8.** Let's consider (1.1.2), in which case $A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 10 \\ 26 \end{bmatrix}$, then $A^{-1} = \dfrac{1}{1 \cdot 4 - 1 \cdot 2} \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$, and

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 26 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 - \frac{1}{2} \cdot 26 \\ -10 + \frac{1}{2} \cdot 26 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

**Theorem 1.10.9.** Here are some properties of matrix inverse

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

**Exercise 1.10.10.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix $A$.

- If $A$ is invertible, then $A$ has $n$ pivots. ✓
- If $T$ is one-to-one, then $A$ is invertible. ✓
- If columns of $A$ span $\mathbb{R}^n$, then $A$ is invertible. ✓
- If $A$ is invertible, $A\mathbf{x} = \mathbf{0}$ only has the trivial solution. ✓
- If $T$ is onto, then $T$ is one-to-one. ✓
- If $T$ is one-to-one, then $T$ is onto. ✓

## 1.11 Lecture 11 - 06/14/2022

**Exercise 1.11.1.** Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Is $A^{-1}$ invertible? Is $T$ one-to-one? Is $A^T$ invertible? If so, what is $(A^T)^{-1}$? If so, what is $(A^{-1})^{-1}$. Is $T$ invertible (i.e. does $T^{-1}$ exist)? What is the standard matrix of $T^{-1}$? Is $T$ onto?

*Solution.*

$$\left[\ A \mid I\ \right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R1 \to R1 - 3R3 \\ R2 \to R2 + 2R3}}{\leadsto} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{(-1)R2}{\leadsto} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{R1 \to R1 - 2R2}{\leadsto} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\ I \mid A^{-1}\ \right]$$

So $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. By Theorem 1.10.9, we know

$$(A^{-1})^{-1} = A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

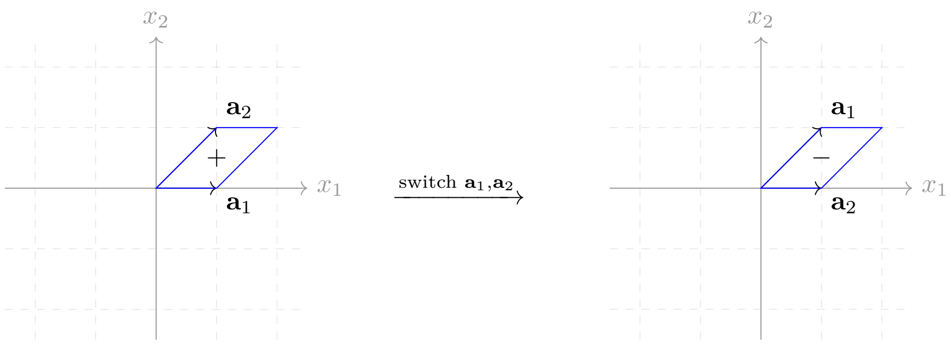
Therefore we know $A^{-1}$ and $A^T$ are invertible.
In general, if $T$ is invertible, then $A$ is invertible, so $A^{-1}$ will be the standard matrix for $T^{-1}$ as $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$, in more explict terms, we have

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + x_3 \\ -x_2 - 2x_2 \\ x_3 \end{bmatrix}$$

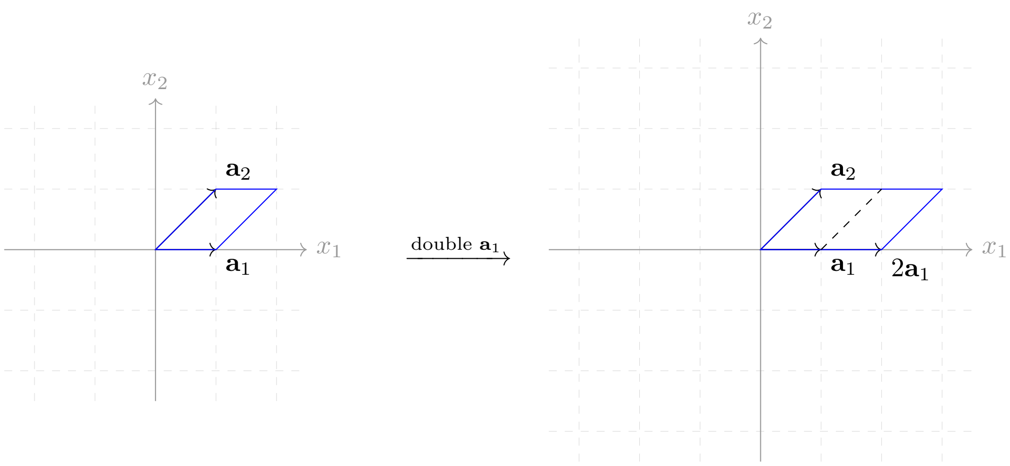
□

Now let's talk about determinants (ONLY for square matrices!!!): Consider the parallelepiped $P$ with edges $\mathbf{a}_1, \cdots, \mathbf{a}_n$ in $\mathbb{R}^n$. We want to think of determinant of $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ (Usually denoted $\det A$ or $|A| = \begin{vmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{vmatrix}$, replacing brackets with vertical lines) as signed volumes of $P$ . Therefore we have $\text{Vol}(P) = |\det A|$, i.e. actual volume is the absolute value of the determinant. Note that the determinant/signed volume has these following three properties (Take $n = 2$ for an example, in this case, volume is really area):

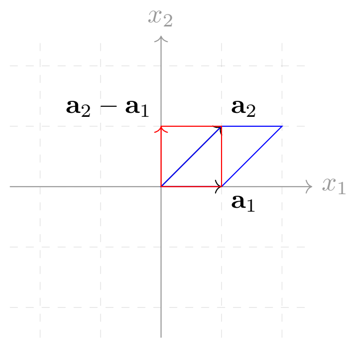
1. The following corresponds to interchange in elementary row operations, this changes the sign of the determinant/signed volume



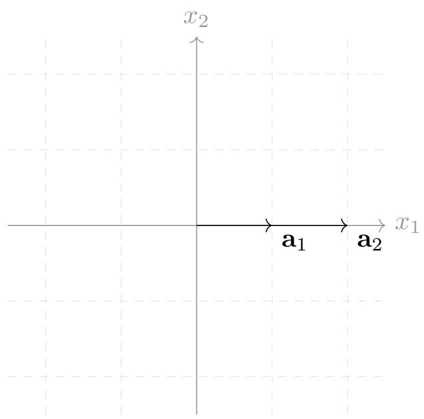

2. The following corresponds to scaling in elementary row operations, this scales the determinant/signed volume

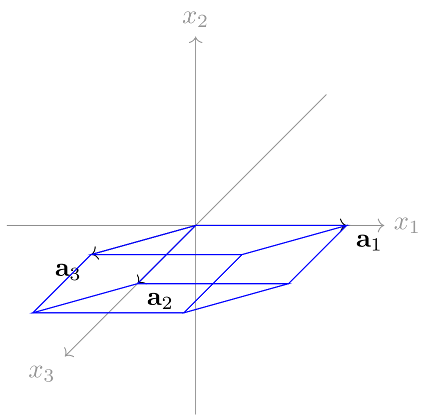



3. The following corresponds to replacement in elementary row operations, this doesn't change the determinant/signed volume

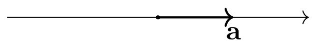



*Remark.* If $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ is linearly dependent, then $A$ is singular, i.e. not invertible, then the determinant will be zero, since the parallelepiped will be constraint in a hyperplane which has zero volume. Take $n = 2$ and 3 for examples



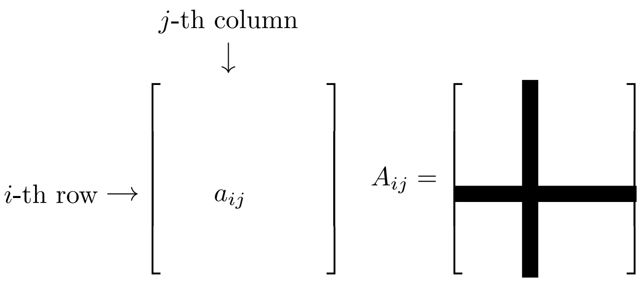



With geometric intepretation of determinants, the case where $n = 1$ is rather straightforward. Suppose $A = [a]$ is a 1 by 1 matrix, then it corresponds to the signed length of the number $a$ on the real line $\mathbb{R}^1$, which is really the number $a$ itself! In other words, we know $\det A = a$.




For $n \geq 2$, we use the following cofactor expansion as an inductive definition.

**Definition 1.11.2.** We use $a_{ij}$ to be denote the $(i,j)$-th entry of the matrix $A$, and $A_{ij}$ to denote the submatrix of $A$ by deleting the $i$-th row and the $j$-th column




We define the $(i,j)$-cofactor to be $C_{ij} = (-1)^{i+j} \det A_{ij}$ . The cofactor expansion across the $i$-th row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion across the $j$-th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$\begin{bmatrix} \\ \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \\ \\ \end{bmatrix} \qquad \begin{bmatrix} & a_{1j} & \\ & a_{2j} & \\ & \vdots & \\ & a_{nj} & \end{bmatrix}$$

**Exercise 1.11.3.** Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is $A_{11}, A_{12}, A_{21}, A_{22}$? What is $C_{11}, C_{12}, C_{21}, C_{22}$. Write down the cofactor expansion of $A$ across the

- 1st row
- 2nd row
- 1st column
- 2nd column

*Solution.* $A_{11} = [d], A_{12} = [c], A_{21} = [b], A_{22} = [a]$ are all 1 by 1 matrices. $C_{11} = (-1)^{1+1} \det A_{11} = d, C_{12} = (-1)^{1+2} \det A_{21} = -c, C_{21} = (-1)^{2+1} \det A_{21} = -b, C_{22} = (-1)^{2+2} \det A_{22} = a$. So the cofactor expansions are

- $\det A = aC_{11} + bC_{12} = ad - bc$
- $\det A = cC_{21} + dC_{22} = -bc + ad$
- $\det A = aC_{11} + cC_{21} = ad - bc$
- $\det A = bC_{12} + dC_{22} = -bc + ad$

Note that all of the above calculations show that $\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. □

## 1.12 Lecture 12 - 06/15/2022

**Exercise 1.12.1.** Suppose $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Please find the cofactor expansion of $A$ across the

a) 1st row

b) 2nd column

And evaluate determinant of $A$.

*Solution.*

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}(-1)^{1+1} \det A_{11} + a_{12}(-1)^{1+2} \det A_{12} + a_{13}(-1)^{1+3} \det A_{13} \\ &= 1 \cdot (-1)^{1+1} \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} + (-1) \cdot (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 1 \cdot (-1)^{1+3} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \\ &= 1 \cdot (0 \cdot 1 - (-1) \cdot 1) + (-1) \cdot (2 \cdot 1 - (-1) \cdot 1) + 1 \cdot (-1)(2 \cdot 1 - (-1) \cdot 1) \\ &= 6 \end{aligned}$$

$$\begin{aligned} \det A &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{11}(-1)^{1+1} \det A_{11} + a_{12}(-1)^{1+2} \det A_{12} + a_{13}(-1)^{1+3} \det A_{13} \\ &= (-1) \cdot (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1 \cdot (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ &= (-1) \cdot (-1)(2 \cdot 1 - (-1) \cdot 1) + 0 \cdot (1 \cdot 1 - 1 \cdot 1) + 1 \cdot (-1)(1 \cdot (-1) - 1 \cdot 2) \\ &= 6 \end{aligned}$$

□

**Exercise 1.12.2** (Quiz 1). Suppose $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix}$. Write out the cofactor expansion of $A$ across the second row, and evaluate the determinant $\det A$.

*Solution.*

$$\begin{aligned}
\det A &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\
&= a_{21}(-1)^{2+1}\det A_{21} + a_{22}(-1)^{2+2}\det A_{22} + a_{23}(-1)^{2+3}\det A_{23} \\
&= 1 \cdot (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + 0 \cdot (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (-1) \cdot (-1)^{2+3} \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\
&= 1 \cdot (-1)(2 \cdot 1 - 1 \cdot 2) + 0 \cdot (1 \cdot 1 - 1 \cdot (-1)) + (-1) \cdot (-1)(1 \cdot 2 - 2 \cdot (-1)) \\
&= 4
\end{aligned}$$

□

*Remark.* When use the cofactor expansion, we want to apply it to rows/columns with more 0's

## 1.13 Lecture 13 - 06/16/2022

**Theorem 1.13.1.** Cofactor expansion across a row in $A$ is the same as cofactor expansion across a column in $A^T$. From this obeservation we conclude inductively $\det(A) = \det(A^T)$

**Definition 1.13.2.** We say a square matrix $A$ is upper triangular if it only has zeros to the left of the diagonal

$$\begin{bmatrix} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$A$ is lower triangular if it only has zeros to the right of the diagonal

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & * \end{bmatrix}$$

$A$ is diagonal if $A$ only has nonzero entries on the diagonal

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

A diagonal matrix is both upper triangular and lower triangular

**Exercise 1.13.3.** Suppose $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Please find use cofactor expansion to find the $\det A$

*Solution.* Note that $A$ is upper triangular, so we could do cofactor expansions across first columns multiple times

$$\begin{vmatrix} 2 & -1 & 3 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} -2 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2\cdot(-2)(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 2\cdot(-2)\cdot 3\cdot(-1)^{1+1}1 = -12$$

□

We could summerize this into the following

**Theorem 1.13.4.** If $A$ is triangular, then $\det A$ is the product of the diagonal entries.

**Exercise 1.13.5.** Suppose $I$ is the $n \times n$ identity matrix, what is $\det I$, $\det(-I)$, $\det(2I)$ and $\det(aI)$?

*Solution.* Note that $I$ is a diagonal matrix. $\det I = 1$, $\det(-I) = (-1)^n$, $\det(2I) = 2^n$, and in general $\det(aI) = a^n$ □

As discussed in lecture 1.11, thinking in terms of signed volume, we know that

1. For column interchangement, determinant pick up a negtive sign
$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_3 & \mathbf{a}_2\end{bmatrix} = -\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$$
2. For column scaling, determinant scales correspondingly
$$\det\begin{bmatrix}\mathbf{a}_1 & c\mathbf{a}_2 & \mathbf{a}_3\end{bmatrix} = c\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$$
3. For column replacement, determinant doesn't change
$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 + c\mathbf{a}_1 & \mathbf{a}_3\end{bmatrix} = \det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$$

With the help of Theorem 1.13.1 we may conclude an algorithm for computing determinants using elementary row operations (instead elementary column operations discussed above). Let $A$ be a square matrix.

1. If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det B = \det A$.
$$\det\begin{bmatrix} R1 + cR3 \\ R2 \\ R3 \end{bmatrix} = \det\begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix}$$
2. If two rows of $A$ are interchanged to produce $B$, then $\det B = -\det A$.
$$\det\begin{bmatrix} R1 \\ R3 \\ R2 \end{bmatrix} = -\det\begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix}$$
3. If one row of $A$ is multiplied by $k$ to produce $B$ , then $\det B = k\det A$.
$$\det\begin{bmatrix} R1 \\ kR2 \\ R3 \end{bmatrix} = k\det\begin{bmatrix} R1 \\ R2 \\ R3 \end{bmatrix}$$

**Example 1.13.6.** Use elementary row operations to evaluate the following i.

$$\begin{vmatrix}1 & -1 & 1\\ 2 & 0 & -1\\ 1 & 1 & 1\end{vmatrix} \xlongequal[R3\to R3-R1]{R2\to R2-2R1} \begin{vmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 0 & 2 & 0\end{vmatrix} \xlongequal{\text{factor } R3} 2\begin{vmatrix}1 & -1 & 1\\ 0 & 2 & -3\\ 0 & 1 & 0\end{vmatrix} \xlongequal{R2\to R2-2R3} 2\begin{vmatrix}1 & -1 & 1\\ 0 & 0 & -3\\ 0 & 1 & 0\end{vmatrix}$$

$$\xlongequal{R2\leftrightarrow R3} (-1)\cdot 2\begin{vmatrix}1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & -3\end{vmatrix} = (-1)\cdot 2\cdot 1\cdot 1\cdot(-3) = 6$$

ii.

$$\begin{vmatrix}1 & 2 & 1\\ 2 & 0 & -1\\ -1 & 2 & 2\end{vmatrix} \xlongequal[R3\to R3+R1]{R2\to R2-2R1} \begin{vmatrix}1 & 2 & 1\\ 0 & -4 & -3\\ 0 & 4 & 3\end{vmatrix} \xlongequal{R3\to R3+R2} \begin{vmatrix}1 & 2 & 1\\ 0 & -4 & -3\\ 0 & 0 & 0\end{vmatrix} = 1\cdot(-4)\cdot 0 = 0$$

iii.

$$\begin{vmatrix}1 & 2 & 3 & 0\\ 0 & 3 & -1 & 0\\ -1 & 2 & 1 & 2\\ 2 & -3 & 1 & 0\end{vmatrix} \xlongequal{\text{cofactor expansion across last column}} 2(-1)^{3+4}\begin{vmatrix}1 & 2 & 3\\ 0 & 3 & -1\\ 2 & -3 & 1\end{vmatrix}$$

$$\xlongequal{R3\to R3-2R1} (-2)\begin{vmatrix}1 & 2 & 3\\ 0 & 3 & -1\\ 0 & -7 & -5\end{vmatrix} \xlongequal{\text{cofactor expansion across first column}} (-2)\cdot 1(-1)^{1+1}\begin{vmatrix}3 & -1\\ -7 & -5\end{vmatrix}$$

$$= (-2)(3(-5)-(-1)(-7)) = 44$$

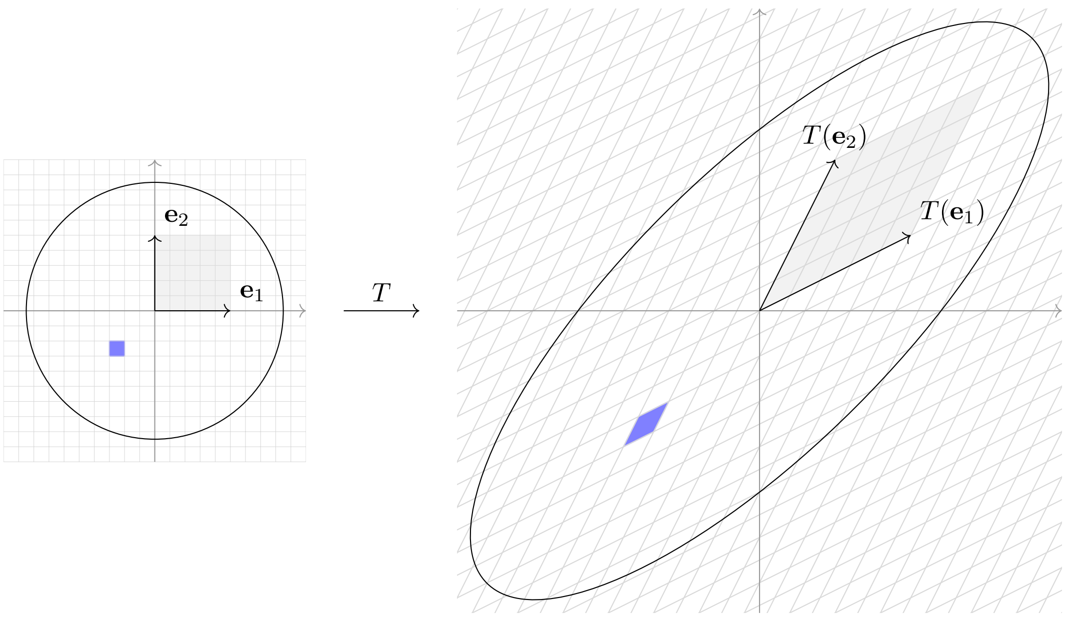
*Remark.* The REF of a square matrix $A$ is upper triangular, and $\det A = 0$ if $A$ has less then $n$ pivots.

**Theorem 1.13.7.** Suppose $A, B$ are $n\times n$ matrices, then $\det(AB) = (\det A)(\det B)$

If $A$ is invertible, then $A^{-1}$ is well-defined, then $1 = \det I = \det(AA^{-1}) = (\det A)(\det(A^{-1}) \Rightarrow \det(A^{-1}) = \dfrac{1}{\det A}$, so $\det A \neq 0$. Conversely, if $\det A \neq 0$, $A$ would have $n$ pivots, so a pivot in each row and column, thus $A$ will be invertible. Therefore we have the following theorem

**Theorem 1.13.8.** *$A$ is invertible $\iff$ $\det A \neq 0$. In addition, $\det(A^{-1}) = \dfrac{1}{\det A}$.*

Geometric meaning of the determinant of a linear transformation $T$ (with standard matrix $A = \begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix}$)



Since every small blue squares has been deformed into small parallelograms which are congruent to the gray square, parallelogram respectively and are of the same (signed) ratio $\det A$, so any shape under $T$ gets scaled by $\det A$.

## 1.14 Lecture 14 - 06/22/2022

To motivate the definition of a vector space, let's consider the following example

**Example 1.14.1.** Let $\mathbb{P}_n$ denote the set of (real) polynomials of degree less or equal to $n$. For example $\mathbb{P}_0 = \mathbb{R}$ is just the set of real numbers, and

$$\begin{aligned}
\mathbb{P}_1 &= \{a_0 + a_1 t | a_0, a_1 \in \mathbb{R}\} \\
\mathbb{P}_2 &= \{a_0 + a_1 t + a_2 t^2 | a_0, a_1, a_2 \in \mathbb{R}\} \\
\mathbb{P}_3 &= \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 | a_0, a_1, a_2, a_3 \in \mathbb{R}\} \\
&\vdots \\
\mathbb{P}_n &= \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n | a_0, a_1, a_2, \cdots, a_n \in \mathbb{R}\}.
\end{aligned}$$

You may soon realize that $\mathbb{P}_n$ can be identified with $\mathbb{R}^{n+1}$.

$$\begin{tikzcd}
\mathbb{P}_1 \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^2 \arrow[d, equal] \\
\{a_0 + a_1 t\} \arrow[r, leftrightsquigarrow] & \left\{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}\right\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{P}_2 \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^3 \arrow[d, equal] \\
\{a_0 + a_1 t + a_2 t^2\} \arrow[r, leftrightsquigarrow] & \left\{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right\}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{P}_n \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^{n+1} \arrow[d, equal] \\
\{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n\} \arrow[r, leftrightsquigarrow] & \left\{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}\right\}
\end{tikzcd}$$

More concrete examples could be

1. For $\mathbb{P}_1 \cong \mathbb{R}^2$, $1 + 2t \leftrightsquigarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
2. For $\mathbb{P}_2 \cong \mathbb{R}^3$, $3t^2 - 1 \leftrightsquigarrow \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

If we consider addition and scalar multiplication, we have

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad 2 \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$(1 + 2t^2) + (2 + t) = 3 + t + 2t^2 \qquad 2 \cdot (1 + 2t^2) = 2 + 4t^2$$

So we may conclude that addition and scalar multiplication in $\mathbb{P}_n$ can be identically translated to addition and scalar multiplication in $\mathbb{R}^{n+1}$

*Remark.* We call $\{1, t, t^2, \cdots, t^n\}$ the *standard basis* of $\mathbb{P}_n$, corresponding to the standard basis for $\mathbb{R}^{n+1}$

**Example 1.14.2.** $\{1, t, t^2\}$ is the standard basis for $\mathbb{P}_2$, and

$$p(t) = a_0 + a_1 t + a_2 t^2 = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot t^2$$

**Example 1.14.3.** Let's denote $M_{m\times n}(\mathbb{R})$ the set of $m \times n$ matrices. For example

$$\begin{aligned}
M_{2\times 2}(\mathbb{R}) &= \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \middle| a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\} \\
M_{3\times 2}(\mathbb{R}) &= \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \middle| a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32} \in \mathbb{R} \right\} \\
M_{2\times 3}(\mathbb{R}) &= \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \middle| a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} \in \mathbb{R} \right\} \\
&\vdots \\
M_{m\times n}(\mathbb{R}) &= \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \middle| a_{11}, \cdots, a_{1n}, \cdots, a_{m1}, \cdots, a_{mn} \in \mathbb{R} \right\}.
\end{aligned}$$

You may realize that $M_{m\times n}(\mathbb{R})$ can be identified with $\mathbb{R}^{mn}$

$$\begin{tikzcd}
M_{2\times 2}(\mathbb{R}) \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^4 \arrow[d, equal] \\
\left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\} \arrow[r, leftrightsquigarrow] & \left\{ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} \right\}
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{3\times 2}(\mathbb{R}) \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^6 \arrow[d, equal] \\
\left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \right\} \arrow[r, leftrightsquigarrow] & \left\{ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ a_{31} \\ a_{32} \end{bmatrix} \right\}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{2\times 3}(\mathbb{R}) \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^6 \arrow[d, equal] \\
\left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \right\} \arrow[r, leftrightsquigarrow] & \left\{ \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} \right\}
\end{tikzcd}$$

$$\begin{tikzcd}
M_{m\times n}(\mathbb{R}) \arrow[r, equal, "\sim"] \arrow[d, equal] & \mathbb{R}^{mn} \arrow[d, equal] \\
\left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \right\} \arrow[r, leftrightsquigarrow] & \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{1n} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}
\end{tikzcd}$$

In more concrete terms, addition and scalar multiplication can be identified as the following

$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}-1 & 0\\2 & 3\end{bmatrix} = \begin{bmatrix}0 & 0\\4 & 6\end{bmatrix} \qquad 2\cdot\begin{bmatrix}-1 & 0\\2 & 3\end{bmatrix} = \begin{bmatrix}-2 & 0\\4 & 6\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + \begin{bmatrix}-1\\0\\2\\3\end{bmatrix} = \begin{bmatrix}0\\0\\4\\6\end{bmatrix} \qquad 2\cdot\begin{bmatrix}-1\\0\\2\\3\end{bmatrix} = \begin{bmatrix}-2\\0\\4\\6\end{bmatrix}$$

So we may conclude that addition and scalar multiplication in $M_{m\times n}(\mathbb{R})$ can be identically translated to addition and scalar multiplication in $\mathbb{R}^{mn}$

*Remark.* We call $\{E_{ij}\}$ the *standard basis* of $M_{m\times n}(\mathbb{R})$, corresponding to the standard basis for $\mathbb{R}^{mn}$. Here $E_{ij}$ is the $m\times n$ matrix that only has a single 1 in the $(i,j)$-th spot, but 0's elsewhere.

**Example 1.14.4.** $\left\{E_{11} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, E_{12} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, E_{21} = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, E_{22} = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is the standard basis for $M_{2\times 2}(\mathbb{R})$, and

$$\begin{aligned}
\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix} &= \begin{bmatrix}a_{11} & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & a_{12}\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\a_{21} & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & a_{22}\end{bmatrix}\\
&= a_{11}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + a_{12}\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + a_{21}\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + a_{22}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\\
&= a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}
\end{aligned}$$

**Definition 1.14.5.** A (real) vector space is a set $V$ of objects, called *vectors*, on which are defined two operations, called *addition* $\boldsymbol{+}$ and *(left) scalar multiplication* $\bullet$, subject to axioms

0. $\boldsymbol{u} \boldsymbol{+} \boldsymbol{v}$ and $c \bullet \boldsymbol{v}$ are still in $V$
1. $\boldsymbol{u} \boldsymbol{+} \boldsymbol{v} = \boldsymbol{v} \boldsymbol{+} \boldsymbol{u}$
2. $(\boldsymbol{u} \boldsymbol{+} \boldsymbol{v}) \boldsymbol{+} \boldsymbol{w} = \boldsymbol{u} \boldsymbol{+} (\boldsymbol{v} \boldsymbol{+} \boldsymbol{w})$
3. There is a *zero vector* $\mathbf{0}$ such that $\boldsymbol{u} \boldsymbol{+} \mathbf{0} = \boldsymbol{u}$
4. For each $\boldsymbol{u}$ in $V$, there is a vector $\boldsymbol{-u}$ in $V$ such that $\boldsymbol{u} \boldsymbol{+} (\boldsymbol{-u}) = \mathbf{0}$
5. $c \bullet (\boldsymbol{u} \boldsymbol{+} \boldsymbol{v}) = c \bullet \boldsymbol{u} \boldsymbol{+} c \bullet \boldsymbol{v}$
6. $(c + d) \bullet \boldsymbol{u} = c \bullet \boldsymbol{u} \boldsymbol{+} d \bullet \boldsymbol{u}$
7. $c \bullet (d \bullet \boldsymbol{u}) = (cd) \bullet \boldsymbol{u}$
8. $1 \bullet \boldsymbol{u} = \boldsymbol{u}$

**Example 1.14.6.** Set $V$ to be $\mathbb{R}^n$, $\boldsymbol{+}$ to be addition $+$ for vectors, $\bullet$ to be scalar multiplication $\cdot$ for vectors, then this is a vector space

**Example 1.14.7** (non-example)**.** Suppose $V = \mathbb{R}$, $a \boldsymbol{+} b = a + b + 1$, $c \bullet a = c \cdot a = ca$, we can check

0. $a \boldsymbol{+} b = a + b + 1 \in \mathbb{R}$, $c \bullet a = ca \in \mathbb{R}$
1. $a \boldsymbol{+} b = a + b + 1 = b + a + 1 = b \boldsymbol{+} a$
2. $(a \boldsymbol{+} b) \boldsymbol{+} c = (a + b + 1) + c + 1 = a + (b + c + 1) + 1 = a \boldsymbol{+} (b \boldsymbol{+} c)$
3. There is a *zero vector* $\mathbf{0} = -1$ such that $a \boldsymbol{+} \mathbf{0} = a + (-1) + 1 = a$
4. For each $a$, we have $\boldsymbol{-}a = -a - 2$ such that $a \boldsymbol{+} (\boldsymbol{-}a) = a + (-a - 2) + 1 = -1 = \mathbf{0}$

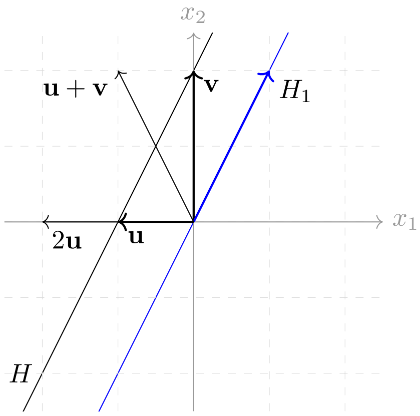
However $2 \bullet (a \boldsymbol{+} b) = 2(a + b + 1) \neq 2a + 2b + 1 = 2 \bullet a \boldsymbol{+} 2 \bullet b$. Therefore, this is not a vector space

## 1.15 Lecture 15 - 06/23/2022

**Definition 1.15.1.** Suppose $V$ is a vector space with addition $\boldsymbol{+}$ and scalar multiplication $\bullet$. A subspace $H$ is a non-empty subset which closed under addition and scalar multiplication, i.e. for any $\boldsymbol{u}, \boldsymbol{v} \in H$, $c \in \mathbb{R}$, $\boldsymbol{u} \boldsymbol{+} \boldsymbol{v}, c \bullet \boldsymbol{u} \in H$

*Remark.* It is easy to check that a subspace $H$ is again a vector space.

**Example 1.15.2.** Consider the vector space $V = \mathbb{R}^2$, and $H$ is the set of solutions to the linear equation $2x_1 - x_2 + 2 = 0$, then $H$ is not a subspace. For example, if we choose $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, then $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is not in $H$, nor is $2\mathbf{u} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



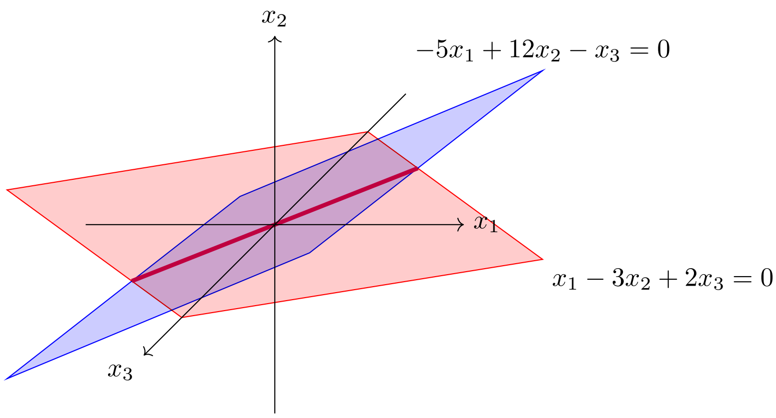

The reason is that $H$ is not homogeneous. If we consider $H_1$ to be solution set of the homogeneous equation $2x_1 - x_2 = 0$, we see that $H_1$ is a subspace as it is the span of a single vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

**Definition 1.15.3.** Suppose $A$ is a $m \times n$ matrix, we define the null space of $A$ to be $\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$. Note that the solution set to linear system $A\mathbf{x} = \mathbf{0}$ is the intersection of $m$ hyperplanes (one for each homogeneous equation) that pass through the origin.

**Example 1.15.4.** $A = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 12 & -1 \end{bmatrix}$, the to find the $\operatorname{Nul} A$ is equivalent to solve $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \xrightarrow{R2 \to R2 + 5R1} \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -3 & 9 & 0 \end{bmatrix} \xrightarrow{R2/(-3)} \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix} \xrightarrow{R1 \to R1 + 3R2} \begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix}$$

Hence the solution set is $\begin{cases} x_1 = 7x_3 \\ x_2 = 3x_3 \\ x_3 \text{ is free} \end{cases}$, in parametric form, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$, which describes a line in $\mathbb{R}^3$ that passes through the origin, and this line is the intersection of planes $x_1 - 3x_2 + 2x_3 = 0$ and $-5x_1 + 12x_2 - x_3 = 0$



*Remark.* As discussed in Example 1.15.2, in general, the solution set of $A\mathbf{x} = \mathbf{b}$ is not a subspace of $\mathbb{R}^n$ unless $\mathbf{b} = \mathbf{0}$. And in fact, any subspace of $\mathbb{R}^n$ is the null space for some $m \times n$ matrix $A$, i.e. the intersection of hyperplanes passing through the origin

### 1.16 Lecture 16 - 06/24/2022

**Definition 1.16.1.** Suppose $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ is an $m \times n$ matrix, then the column space (denote as $\operatorname{Col} A$) is the subspace $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ in $\mathbb{R}^m$. Suppose $A = \begin{bmatrix} R1 \\ R2 \\ \vdots \\ Rm \end{bmatrix}$, then the row space (denote as $\operatorname{Row} A$) is the subspace spaned by row vectors $\operatorname{Span}\{R1, R2, \cdots, Rm\}$ in $\mathbb{R}^n$ written horizontally.

*Remark.* Suppose column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ in $A$ has some linear dependence $A\mathbf{x} = \mathbf{0}$, then, after elementary row reduction $A \sim EA$, $(EA)\mathbf{x} = E(A\mathbf{x}) = E\mathbf{0} = \mathbf{0}$ again has the same linear dependence. In other words, the linear dependence of columns of $A$ is preserved by row equivalence.

*Remark.* Since row elementary operations can be reversed, so $\operatorname{Row} A$ preserved under row equivalence

We may conclude the following algorithm for finding basis for $\operatorname{Nul} A$, $\operatorname{Col} A$, $\operatorname{Row} A$ by simply use elementary row reductions!!!

**Theorem 1.16.2.** Suppose $A$ is a $m \times n$ matrix, $A \sim U$ is of RREF form

- The solution set of $\begin{bmatrix} U & \mathbf{0} \end{bmatrix}$ in parametric vector form gives a basis for $\operatorname{Nul} A$. Note that $\dim \operatorname{Nul} A =$ the number of free variables.
- A basis for $\operatorname{Col} A$ could be the set of pivot columns in $A$. Note that $\dim \operatorname{Col} A =$ the number of pivots
- A basis for $\operatorname{Row} A$ could be the set of non-zero row vectors in $U$ (Or any REF of $A$ actually). Note that $\dim \operatorname{Row} A =$ the number of pivots

**Definition 1.16.3.** $\dim \operatorname{Nul} A$ is also name the nullity of $A$. The number of pivots of $A$ (which is equal to both $\dim \operatorname{Col} A$ and $\dim \operatorname{Row} A$) is called the rank of $A$

**Theorem 1.16.4** (Rank-Nullity theorem)**.** Notice that the number of columns in $A$ (say a $m \times n$ matrix) is equal to the number of free variables and the number of pivot columns, thus we have

$$n = \text{nullity} + \text{rank}$$

**Example 1.16.5.** $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$, which is an REF and an RREF respectively. There is only one free variable $x_3$, so the nullity is 1, and the 1st, 2nd columns are pivot columns, so the rank is 2. We see that Theorem 1.16.4 holds as $3 = 1 + 2$, and

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right\}$$

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\operatorname{Row} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & -\frac{3}{2} \end{bmatrix} \right\} \text{ or } \operatorname{Span}\left\{ \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \right\}$$

**Exercise 1.16.6.** $A = \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Note that here we have 2 free variables $x_2, x_4$, so the nullity is 2, and the 1st, 3rd, 5th columns are pivot columns, so the rank is 3. We see that Theorem 1.16.4 holds as $5 = 2 + 3$, and

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \end{bmatrix} \right\}$$

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ -9 \\ 0 \end{bmatrix} \right\}$$

$$\operatorname{Row} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 5 & -7 & 8 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & -9 \end{bmatrix} \right\}$$

*Question.* If you have a set $\mathcal{S}$ of vectors in $\mathbb{R}_m$, how do you find a subset of $\mathcal{S}$ that is a basis for $\operatorname{Span}\{\mathcal{S}\}$ (i.e. remove linear dependences)?

*Answer.* Collect these vectors as the column vectors of a matrix, and then find its columns space.

**Exercise 1.16.7.** Recall that $M_{2\times 2}(\mathbb{R})$ is the set of 2 by 2 matrices, and that a square matrix $A$ is *symmetric* if $A^T = A$. Consider a subset $V$ consists of 2 by 2 symmetric matrices, i.e. $V = \{A \in M_{2\times 2}(\mathbb{R}) \big| A^T = A\}$

1. Show that $V$ is a vector space.
2. Find a basis for $V$.

*Solution.*

1. For any $A, B \in V$, $c \in \mathbb{R}$, by definition we know that $A^T = A$, $B^T = B$, we want to show that $A + B \in V$, $cA \in V$ (condition for subspace), i.e. $(A+B)^T = A^T + B^T$, $(cA)^T = cA$. This is true because

$$(A+B)^T = A^T + B^T = A + B, \qquad (cA)^T = cA^T = cA$$

Therefore $V$ is a subspace of $M_{2\times 2}(\mathbb{R})$, and thus a vector space

2. Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$, then $a_{12} = a_{21}$, so we may conclude that

$$V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in M_{2\times 2}(\mathbb{R}) \middle| a, b, c \in \mathbb{R} \right\}$$

Note that

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.16.1}$$

And that linear combination (1.16.1) is the zero matrix $\iff a = b = c = 0$, thus $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $V$

□

## 1.17 Lecture 17 - 06/27/2022

**Definition 1.17.1.** Suppose $V, W$ are vector spaces, a *linear transformation* $T : V \to W$ is such that

- $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
- $T(c \cdot u) = c \cdot T(\boldsymbol{u})$

Just as before, we call $V$ the *domain* of $T$, $W$ the *codomain* of $V$, the *image* of $\boldsymbol{u}$ under $T$ is $T(\boldsymbol{u})$, the set of images $\{T(\boldsymbol{u})|\boldsymbol{u} \in V\}$ the *range* (denoted as $\operatorname{Range} T$), and the set $\{\boldsymbol{u}|T(\boldsymbol{w}) = \boldsymbol{u}\}$ the preimage of $\boldsymbol{w}$ under $T$. We still say that $T$ is *one-to-one* if any $\boldsymbol{w} \in W$, there is at most one $\boldsymbol{u} \in V$ such that $T(\boldsymbol{u}) = \boldsymbol{w}$. $T$ is *onto* the range is the codomain. $T$ is said to be *invertible* if $T$ has a inverse (this happens if and only if $T$ is both one-to-one and onto), in this case we also call $T$ an isomorphism.

**Definition 1.17.2.** We call $\{\boldsymbol{u}|T(\boldsymbol{u}) = \boldsymbol{0}\}$ the kernel (or null space) of $T$

**Example 1.17.3.** The identification

$$T : \mathbb{P}_2 \to \mathbb{R}^3, \qquad T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

in Eaxmple 1.14.1 is an invertible linear transformation with inverse linear transformation

$$T^{-1} : \mathbb{R}^3 \to \mathbb{P}_2, \qquad T^{-1}\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = a_0 + a_1 t + a_2 t^2$$

**Example 1.17.4.** The identification

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4, \qquad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

in Eaxmple 1.14.3 is an invertible linear transformation with inverse linear transformation

$$T^{-1} : \mathbb{R}^4 \to M_{2\times 2}(\mathbb{R}), \qquad T^{-1}\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

**Theorem 1.17.5.** Suppose $T : V \to W$ is a linear transformation between vector spaces, then

- $\ker T$ is a subspace of $V$.
- $\operatorname{Range} T$ is a subspace of $W$.

*Proof.*

- Suppose $\boldsymbol{u}, \boldsymbol{v} \in \ker T$, then by definition $T(\boldsymbol{u}) = \boldsymbol{0}$, $T(\boldsymbol{v}) = \boldsymbol{0}$, so $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}) = \boldsymbol{0}$. And for any $c \in \mathbb{R}$, $T(c \cdot \boldsymbol{u}) = c \cdot T(\boldsymbol{u}) = \boldsymbol{0}$. In other words, we have shown that $\boldsymbol{u} + \boldsymbol{v}, c \cdot \boldsymbol{u} \in \ker T$, so $\ker T$ is a subspace.
- For any $T(\boldsymbol{u}), T(\boldsymbol{v}) \in \operatorname{Range} T$, $T(\boldsymbol{u}) + T(\boldsymbol{v}) = T(\boldsymbol{u} + \boldsymbol{v}) \in \operatorname{Range} T$, and for any $c \in \mathbb{R}$, $c \cdot T(\boldsymbol{u}) = T(c \cdot \boldsymbol{u}) \in \operatorname{Range} T$, so $\operatorname{Range} T$ is a subspace

□

*Remark.* Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$ is a matrix transformation, then $\ker T = \operatorname{Nul} A$, $\operatorname{Range} T = \operatorname{Col} A$

**Example 1.17.6.** Suppose $T : \mathbb{P}_2 \to \mathbb{R}$ takes the sum of coefficients, i.e. $T(a_0+a_1t+a_2t^2) = a_0+a_1+a_2$. $T$ is a linear transformation, since for any $p(t) = a_0+a_1+a_2, q(t) = b_0+b_1t+b_2t^2 \in \mathbb{P}_2$, $c \in \mathbb{R}$, we have

$$\begin{aligned} T(p+q) &= T((a_0+b_0)+(a_1+b_1)t+(a_2+b_2)t^2) = (a_0+b_0)+(a_1+b_1)+(a_2+b_2) \\ &= (a_0+a_1+a_2)+(b_0+b_1+b_2) = T(p)+T(q) \end{aligned}$$

$$T(cp) = T((ca_0)+(ca_1)t+(ca_2)t^2) = (ca_0)+(ca_1)+(ca_2) = c(a_0+a_1+a_2) = cT(p)$$

So $V = \ker T$ is a subspace of $V$ by Theorem 1.17.5

## 1.18 Lecture 18 - 06/28/2022

**Theorem 1.18.1** (Unique representation theorem)**.** Suppose $\mathcal{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is a basis of a vector sapce $V$, then any vector $\boldsymbol{v} \in V$ can be uniquely represented as a linear combination $x_1 \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} x_n \bullet \boldsymbol{b}_n$

*Remark.* Here $[\boldsymbol{v}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is called the $\mathcal{B}$-coordinate vector (or coordinate vector relative to the basis $\mathcal{B}$) of $\boldsymbol{v}$

*Proof.* Suppose $\boldsymbol{v} = c_1 \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} c_n \bullet \boldsymbol{b}_n = d_1 \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} d_n \bullet \boldsymbol{b}_n$ are two linear combinations that express $\boldsymbol{v}$, then we have $(c_1-d_1) \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} (c_n-d_n) \bullet \boldsymbol{b}_n = \boldsymbol{0}$, since $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is linearly independent, we have $c_1 = d_1, \cdots, c_n = d_n$. Therefore the expression is unique. □

**Definition 1.18.2.** We call $[\ \ ]_{\mathcal{B}} : V \to \mathbb{R}^n$ the coordinate mapping

**Theorem 1.18.3.** The coordinate mapping $[\ \ ]_{\mathcal{B}} : V \to \mathbb{R}^n$ is a linear transformation

**Example 1.18.4.** The identifications in Example 1.14.1 and Example 1.14.3 are coordinate mappings $[\ \ ]_{\mathcal{E}} : \mathbb{P}_2 \to \mathbb{R}^3$ with standard basis $\mathcal{E} = \{1, t, t^2\}$ and $[\ \ ]_{\mathcal{E}} : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$ with standard basis $\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, respectively

**Example 1.18.5.** $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Solve linear system $\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we get $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

*Question.* How should we study linear transformations via matrices in general?

Assume $T : V \to W$ is a linear transformation between vector spaces, $\{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is a basis for $V$ and $\{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_m\}$ is a basis for $W$, then for any

$$\boldsymbol{v} = x_1 \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} x_n \bullet \boldsymbol{b}_n = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix} [\boldsymbol{v}]_{\mathcal{B}}$$

We have

$$\begin{aligned} T(\boldsymbol{v}) &= T(x_1 \bullet \boldsymbol{b}_1 \boldsymbol{+} \cdots \boldsymbol{+} x_n \bullet \boldsymbol{b}_n) = x_1 \bullet T(\boldsymbol{b}_1) \boldsymbol{+} \cdots \boldsymbol{+} x_n \bullet T(\boldsymbol{b}_n) = x_1 \bullet T(\boldsymbol{b}_1) \boldsymbol{+} \cdots \boldsymbol{+} x_n \bullet T(\boldsymbol{b}_n) \\ &= \begin{bmatrix} T(\boldsymbol{b}_1) & \cdots & T(\boldsymbol{b}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} T(\boldsymbol{b}_1) & \cdots & T(\boldsymbol{b}_n) \end{bmatrix} [\boldsymbol{v}]_{\mathcal{B}} \end{aligned}$$

By Theorem 1.18.1, we can write

$$\begin{aligned} T(\boldsymbol{b}_1) &= a_{11} \bullet \boldsymbol{c}_1 + a_{21} \bullet \boldsymbol{c}_2 + \cdots + a_{m1} \bullet \boldsymbol{c}_m \\ T(\boldsymbol{b}_2) &= a_{12} \bullet \boldsymbol{c}_1 + a_{22} \bullet \boldsymbol{c}_2 + \cdots + a_{m2} \bullet \boldsymbol{c}_m \\ &\ \vdots \\ T(\boldsymbol{b}_n) &= a_{1n} \bullet \boldsymbol{c}_1 + a_{2n} \bullet \boldsymbol{c}_2 + \cdots + a_{mn} \bullet \boldsymbol{c}_m \end{aligned}$$

Therefore we have

$$\begin{bmatrix} T(\boldsymbol{b}_1) & \cdots & T(\boldsymbol{b}_n) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{bmatrix} A \tag{1.18.1}$$

Where

$$A = \begin{bmatrix} [T(\boldsymbol{b}_1)]_{\mathcal{C}} & \cdots & [T(\boldsymbol{b}_n)]_{\mathcal{C}} \end{bmatrix} \tag{1.18.2}$$

is called the matrix of $T$ relative to bases $\mathcal{B}$ and $\mathcal{C}$, thus

$$T(\boldsymbol{v}) = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{bmatrix} A[\boldsymbol{v}]_{\mathcal{B}}$$

On the other hand, we should have

$$T(\boldsymbol{v}) = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{bmatrix} [T(\boldsymbol{v})]_{\mathcal{C}}$$

So we may also conclude that

$$[T(\boldsymbol{v})]_{\mathcal{C}} = A[\boldsymbol{v}]_{\mathcal{B}} \tag{1.18.3}$$

The above discussion can be summerized by the following commutative diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ \ ]_{\mathcal{B}}}"'] & W \arrow[d, "{[\ \ ]_{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\boldsymbol{v}) \arrow[d, mapsto] \\ {[\boldsymbol{v}]_{\mathcal{B}}} \arrow[r, mapsto] & A[\boldsymbol{v}]_{\mathcal{B}} = [T(\boldsymbol{v})]_{\mathcal{C}} \end{tikzcd} \tag{1.18.4}$$

*Remark.* The philosophy here is that any statement about a general linear transformation can be converted to a corresponding statement about matrix transformation.

**Example 1.18.6.** Suppose $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ with bases $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$, $\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$, then (1.18.1) reads $A = \begin{bmatrix} T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$ is the standard matrix

**Example 1.18.7.** Consider linear transformation

$$T : \mathbb{P}_2 \to \mathbb{R}^3, \qquad T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

and $\mathcal{B} = \{1 + t, t + t^2, 1 + t^2\}$ is a basis for $\mathbb{P}_2$, $\mathcal{E}$ is the standard basis for $\mathbb{R}^3$, then matrix of $T$ relative to bases $\mathcal{B}$ and $\mathcal{E}$ can be read from (1.18.2) as

$$A = \begin{bmatrix} T(1+t) & T(t+t^2) & T(1+t^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

**Example 1.18.8.** Consider linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$, $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\mathcal{C} = \left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}\right\}$ are bases for $\mathbb{R}^2$. Then the matrix of $T$ relative to bases $\mathcal{B}$ and $\mathcal{C}$ can be read from (1.18.1)

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} A = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$$

*Remark.* Suppose $T : V \to W$ and $S : W \to U$ are linear transformations between vector spaces, $\mathcal{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ is a basis for $V$, $\mathcal{C} = \{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n\}$ is a basis for $W$ and $\mathcal{D} = \{\boldsymbol{d}_1, \cdots, \boldsymbol{d}_n\}$ is a basis for $U$, then the matrix of $S \circ T$ relative to bases $\mathcal{B}$, $\mathcal{D}$ is $BA$.

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[rr, bend left, "S\circ T"] \arrow[d, "{[\ ]_{\mathcal{B}}}"'] & W \arrow[r, "S"] \arrow[d, "{[\ ]_{\mathcal{C}}}"] & U \arrow[d, "{[\ ]_{\mathcal{D}}}"] \\
\mathbb{R}^n \arrow[r, "A"] \arrow[rr, bend right, "BA"'] & \mathbb{R}^m \arrow[r, "B"] & \mathbb{R}^p
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{v} \arrow[r, mapsto] \arrow[d, mapsto] & T(\boldsymbol{v}) \arrow[r] \arrow[d, mapsto] & S(T(\boldsymbol{v})) = (S\circ T)(\boldsymbol{v}) \arrow[d, mapsto] \\
[\boldsymbol{v}]_{\mathcal{B}} \arrow[r, mapsto] & A[\boldsymbol{v}]_{\mathcal{B}} = [T(\boldsymbol{v})]_{\mathcal{C}} \arrow[r, mapsto] & BA[\boldsymbol{v}]_{\mathcal{B}} = [S(T(\boldsymbol{v}))]_{\mathcal{D}}
\end{tikzcd}$$

If $T$ is invertible, then the matrix of $T^{-1}$ relative to bases $\mathcal{C}$, $\mathcal{B}$ is $A^{-1}$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "{[\ ]_{\mathcal{B}}}"'] & W \arrow[l, bend left, "T_{-1}"] \arrow[d, "{[\ ]_{\mathcal{C}}}"] \\
\mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \arrow[l, bend left, "A^{-1}"]
\end{tikzcd}$$

## 1.19 Lecture 19 - 06/29/2022

Now let's talk about change of basis. Suppose $V$ is a vector space with two basis $\mathcal{B} = \{\boldsymbol{b}_1, \cdots, \boldsymbol{b}_n\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \cdots, \boldsymbol{c}_n\}$, and $\mathrm{id}_V : V \to V$, $\mathrm{id}_V(\boldsymbol{v}) = \boldsymbol{v}$ is the identity mapping. Diagram (1.18.4) becomes

$$\begin{tikzcd}
V \arrow[r, equal, "\mathrm{id}_V"] \arrow[d, "{[\ ]_{\mathcal{B}}}"'] & V \arrow[d, "{[\ ]_{\mathcal{C}}}"] \\
\mathbb{R}^n \arrow[r, "\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}"] & \mathbb{R}^n
\end{tikzcd} \tag{1.19.1}$$

Where equation (1.18.2) and equation (1.18.3) become

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\boldsymbol{b}_1]_{\mathcal{C}} & \cdots & [\boldsymbol{b}_n]_{\mathcal{C}} \end{bmatrix}$$

$$[\boldsymbol{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\boldsymbol{v}]_{\mathcal{B}}$$

which is the matrix of $\mathrm{id}_V$ relative to basis $\mathcal{B}$ and $\mathcal{C}$, and we call this the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. Also remark 1.18 gives us

$$\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\right)\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right), \quad \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$$

**Example 1.19.1.** Continue Example 1.16.7, we have shown that $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\} = \{B_1, B_2, B_3\}$. Let's show that $\mathcal{C} = \left\{\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\right\} = \{C_1, C_2, C_3\}$ is another basis for $V$. First note that

$$C_1 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = B_1 + B_2$$

$$C_2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = B_1 + B_3$$

$$C_3 = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = B_2 + B_3$$

So $\{[C_1]_{\mathcal{B}}, [C_2]_{\mathcal{B}}, [C_3]_{\mathcal{B}}\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$, and it is easy to show that this is a basis for $\mathbb{R}^3$, hence $\mathcal{C}$ is a basis for $V$. Also we know that

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} [C_1]_{\mathcal{B}} & [C_2]_{\mathcal{B}} & [C_3]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$$

And

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\right)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Note that here the diagram (1.19.1) becomes

$$\begin{tikzcd} V \arrow[r, equal, "\mathrm{id}_V"] \arrow[d, "{[\ \ ]_{\mathcal{B}}}"'] & V \arrow[d, "{[\ \ ]_{\mathcal{C}}}"] \\ \mathbb{R}^3 \arrow[r, "\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}"] & \mathbb{R}^3 \end{tikzcd}$$

**Example 1.19.2.** Continue Example 1.17.6, let's find a basis for $V = \ker T$ which is supposed to correspond to $\mathrm{Nul}\, A$, where $A$ is the matrix relative to both standard bases $\mathcal{E} = \{1, t, t^2\}$ for $\mathbb{P}_2$ and $\mathcal{E} = \left\{ \mathbf{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$ which can be read from (1.18.3)

$$A = \begin{bmatrix}[T(1)]_{\mathcal{E}} & [T(t)]_{\mathcal{E}} & [T(t^2)]_{\mathcal{E}}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

So we get

$$\mathrm{Nul}\, A = \mathrm{Span}\left\{ \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix} \right\}$$

Which gives a basis $\{-1+t, -1+t^2\}$ for $V$. Note that here the diagram (1.18.4) becomes

$$\begin{tikzcd} \mathbb{P}_2 \arrow[r, "T"] \arrow[d, "{[\ \ ]_{\mathcal{E}}}"'] & \mathbb{R} \arrow[d, equal, "{[\ \ ]_{\mathcal{E}}}"] \\ \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R} \end{tikzcd}$$

## 1.20 Lecture 20 - 06/30/2022

An algorithm for computing $A^{-1}B$

$$\left[\ A \mid B\ \right] \sim \left[\ I \mid A^{-1}B\ \right]$$

**Exercise 1.20.1.** Suppose $\mathcal{B} = \{2t^2-1, 3t+1-t^2, 3-t\}$ and $\mathcal{C} = \{1+t, t^2, -t\}$ are both bases for $\mathbb{P}_2$. Please find the change of basis matrix form $\mathcal{B}$ to $\mathcal{C}$

*Solution.* First let's find the change of basis matrices from $\mathcal{B}$ and $\mathcal{C}$ to the standard basis $\mathcal{E} = \{1, t, t^2\}$. We have

$$\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}[-1+2t^2]_{\mathcal{E}} & [1+3t-t^2]_{\mathcal{E}} & [3-t]_{\mathcal{E}}\end{bmatrix} = \begin{bmatrix} -1 & 1 & 3 \\ 0 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

And

$$\underset{\mathcal{E}\leftarrow\mathcal{C}}{P} = \begin{bmatrix}[1-t]_{\mathcal{E}} & [t]_{\mathcal{E}} & [t^2]_{\mathcal{E}}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Hence we have

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\right)\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right) = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1}\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right)$$

Which can computed via the above algorithm

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 3 \\ 1 & 0 & -1 & 0 & 3 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 4 \end{array}\right]$$

$$
\begin{tikzcd}
\mathbb{P}_2 \arrow[r, equal, "\mathrm{id}_V"] \arrow[d, "{[\ \ ]_{\mathcal{B}}}"'] & \mathbb{P}_2 \arrow[r, equal, "\mathrm{id}_V"] \arrow[d, "{[\ \ ]_{\mathcal{E}}}"'] & \mathbb{P}_2 \arrow[d, "{[\ \ ]_{\mathcal{C}}}"] \\
\mathbb{R}^3 \arrow[r, "\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}"'] \arrow[rr, bend right=50, "\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}"'] & \mathbb{R}^3 \arrow[r, bend left, "\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}"] & \mathbb{R}^3 \arrow[l, "\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}"]
\end{tikzcd}
$$

□

**Example 1.20.2.** Suppose $\mathbb{R}^2$ has bases $\mathcal{B} = \left\{\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}\right\}$, $\mathcal{C} = \left\{\mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \left(\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}\right)^{-1}\left(\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}\right)$, which can be computed via

$$
\left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]
$$

So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, equal] \arrow[d, "{[\ \ ]_{\mathcal{B}}}"'] & \mathbb{R}^2 \arrow[r, equal] \arrow[d, "{[\ \ ]_{\mathcal{E}}}"'] & \mathbb{R}^2 \arrow[d, "{[\ \ ]_{\mathcal{C}}}"] \\
\mathbb{R}^2 \arrow[r, "\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}"'] \arrow[rr, bend right=50, "\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}"'] & \mathbb{R}^2 \arrow[r, bend left, "\underset{\mathcal{C}\leftarrow\mathcal{E}}{P}"] & \mathbb{R}^2 \arrow[l, "\underset{\mathcal{E}\leftarrow\mathcal{C}}{P}"]
\end{tikzcd}
$$

**Theorem 1.20.3.** Let's generalize the rank-nullity theorem 1.16.4, with the remark 1.17, we have

$$\dim V = \dim \operatorname{Range} T + \dim \ker T$$

**Definition 1.20.4.** Suppose $A$ is a square $n \times n$ matrix. A $\lambda$-eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda$. A scalar $\lambda$ is called an eigenvalue of $A$ has $\lambda$-eigenvectors.

## 1.21 Lecture 21 - 07/01/2022

**Definition 1.21.1.** We say that a vector space $V$ is trivial if $V = \{0\}$ is the zero vector space.

*Question.* How to decide whether a nonzero vector $\mathbf{x}$ is an eigenvector?

*Answer.* We can evaluate $A\mathbf{x}$ and see if it is a multiple of $\mathbf{x}$

**Example 1.21.2.** $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, determine whether $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are eigenvectors of $A$.

$$
A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = (-4)\mathbf{u}
$$

Hence $\mathbf{u}$ is a $(-4)$-eigenvector of $A$.

$$
A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}
$$

Which is not a multiple of $\mathbf{v}$, so $\mathbf{v}$ is not an eigenvector.

*Question.* How to decide if $\lambda$ is an eigenvalue of $A$?

*Answer.* By definition, we know that $\lambda$ is an eigenvalue of $A \iff A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution $\iff \lambda\mathbf{x} - A\mathbf{x} = \lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\iff \operatorname{Nul}(\lambda I - A)$ is nontrivial $\iff \lambda I - A$ is invertible $\iff \det(\lambda I - A) = 0$

**Example 1.21.3.** $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Determine whether $\lambda = 2$ is an eigenvalue of $A$

$$\det(2I - A) = \det\left(2\begin{bmatrix} 1 & 10 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}\right) = \begin{vmatrix} 2 & -1 & -6 \\ 2 & -1 & -6 \\ 2 & -1 & -6 \end{vmatrix} = 0$$

**Definition 1.21.4.** From above criterion, we see that the set of $\lambda$-vectors is $\mathrm{Nul}(\lambda I - A)$ which is a subspace of $\mathbb{R}^n$, we call this the $\lambda$-eigenspace.

**Example 1.21.5.** Continue Example 1.21.3. Let's find a basis for the 2-eigenspace of $A$, which is equivalent of finding a basis for $\mathrm{Nul}(2I - A)$, since $\left[\begin{array}{c|c} 2I - A & \mathbf{0} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Hence a basis could be $\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

*Question.* How could we find all the eigenvalues of $A$?

**Definition 1.21.6.** From above discussion, we see that $\lambda$ is an eigenvalue of $A \iff \det(\lambda I - A) = 0$, this motivates the following definition. We call $\det(tI - A)$ the characteristic polynomial of $A$, and $\det(tI - A) = 0$ the characteristic equation. And the roots of the characteristic polynomial are the eigenvalues of $A$.

**Example 1.21.7.** Suppose $A = \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}$, then the characteristic polynomial of $A$ would be

$$\det(tI - A) = \det\left(t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ 4 & 2 \end{bmatrix}\right) = \begin{vmatrix} t-1 & 4 \\ -4 & t-2 \end{vmatrix} = (t-1)(t-2) - 4 \cdot (-4) = t^2 - 3t + 18$$

And the characteristic equation is $t^2 - 3t + 18 = 0$. Since $\Delta = (-3)^2 - 4 \cdot 1 \cdot 18 = -63 < 0$, this equation has no (real) solutions, $A$ doesn't have (real) eigenvalues

*Note.* Recall that the quadratic equation $ax^2 + bx + c = 0$ has no (real) solutions $\iff$ the discriminant $\Delta = b^2 - 4ac < 0$.

**Example 1.21.8.** Suppose $A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, then the characteristic polynomial of $A$ would be

$$\det(tI - A) = \det\left(t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\right) = \begin{vmatrix} t-3 & 2 \\ -2 & t+1 \end{vmatrix} = (t-3)(t+1) - 2 \cdot (-2) = t^2 - 2t + 1$$

And the characteristic equation is $t^2 - 2t + 1 = (t-1)^2 = 0$. Hence the eigenvalues of $A$ is 2 with algebraic multiplicity 2

**Example 1.21.9.** Suppose $A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, then the characteristic polynomial of $A$ would be

$$\det(tI - A) = \begin{vmatrix} t-3 & 2 & -3 \\ 0 & t+1 & 0 \\ -6 & -7 & t+4 \end{vmatrix} \xlongequal{\text{cofactor expansion across the 2nd row}} (t+1)(-1)^{2+2}\begin{vmatrix} t-3 & -3 \\ -6 & t+4 \end{vmatrix}$$
$$= (t+1)((t-3)(t+4) - (-3) \cdot (-6)) = (t+1)(t^2 + t - 30) = (t+1)(t+6)(t-5)$$

So we see that the eigenvalues of $A$ are $-1, -6, -5$, all with multiplicity 1.

**Example 1.21.10.** Suppose $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then characteristic polynomial is

$$\det(tI - A) = \begin{vmatrix} t & -1 & 1 \\ -1 & t & -1 \\ -1 & -1 & t \end{vmatrix} \xlongequal[R2 \to R2 - R3]{R2 \to R1 + t \cdot R2} \begin{vmatrix} 0 & -1-t & 1+t^2 \\ 0 & t+1 & -1-t \\ -1 & -1 & t \end{vmatrix}$$

$$\xlongequal{\text{cofactor expansion across the 1st column}} (-1)(-1)^{3+1} \begin{vmatrix} -1-t & 1+t^2 \\ t+1 & -1-t \end{vmatrix}$$

$$= (-1)((-1-t)^2 - (1+t^2)(t+1)) = (-1)(t - t^3) = t(t+1)(t-1)$$

Thus the eigenvalues are $0, 1, -1$, with multiplicities $1, 1, 1$.

**Example 1.21.11.** Suppose

$$A = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

is an $n \times n$ triangular matrix, then the characteristic polynomial of $A$ is

$$\det(tI - A) = \begin{vmatrix} t - \lambda_1 & * & \cdots & * \\ 0 & t - \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t - \lambda_n \end{vmatrix} = (t - \lambda_1)(t - \lambda_2) \cdots (t - \lambda_n)$$

so the eigenvalues are the diagonal elements $\lambda_1, \lambda_2, \cdots, \lambda_n$. Note that $\lambda_i$'s might not be distinct, so there might be some multiplicities.

**Example 1.21.12.** Suppose $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, then the characteristic polynomial is

$$\det(tI - A) = \begin{vmatrix} t-1 & -2 & 0 & 1 \\ 0 & t-2 & -1 & -3 \\ 0 & 0 & t-3 & -4 \\ 0 & 0 & 0 & t-1 \end{vmatrix} = (t-1)^2(t-2)(t-3)$$

And the eigenvalues are $1, 2, 3$, with multiplicities $2, 1, 1$ respectively.

**Definition 1.21.13.** $A$ and $B$ are said to be similar if there exists an invertible matrix $P$ such that $A = PBP^{-1}$. Note that this definition is *symmetric* in the sense that we also have $B = P^{-1}AP = (P^{-1})A(P^{-1})^{-1}$. Similarity is *transitive* in the sense that if $A, B$ are similar, $B, C$ are similar, then so do $A, C$. The reason is that suppose $A = PBP^{-1}$, $B = RCR^{-1}$, we would have $A = PBP^{-1} = PRCR^{-1}P^{-1} = (PR)C(PR)^{-1}$.

**Theorem 1.21.14.** Similar matrices have the same

- determinant
- characteristic polynomial

*Proof.* Suppose $A, B$ are similar, $A = PBP^{-1}$, then

- $\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) = \det(P)\det(B)\det(P)^{-1} = \det(B)$
- Note that $tI - A = tPIP^{-1} - PBP^{-1} = P(tI - A)P^{-1}$, so $tI - A$ and $tI - B$ are similar, so they have the same determinant which is the characteristic polynomial.

□

## 2 Online Assignments

### 2.1 Online Assignment 1

**Problem 2.1.1.** Rewrite the following linear systems as augmented matrices and then solve them, show all your work

1. $\begin{cases} 5x_1 + x_2 = 2 \\ 3x_1 - x_2 = 6 \end{cases}$

2. $\begin{cases} x_1 + x_2 + x_3 = 6 \\ x_1 - x_2 + x_3 = 2 \\ -x_1 + x_2 + x_3 = 4 \end{cases}$

*Solution.*

1. The augmented matrix is

$$\begin{bmatrix} 5 & 1 & 2 \\ 3 & -1 & 6 \end{bmatrix} \xrightarrow{5R2} \begin{bmatrix} 5 & 1 & 2 \\ 15 & -5 & 30 \end{bmatrix} \xrightarrow{R2 \to R2 - 3R1} \begin{bmatrix} 5 & 1 & 2 \\ 0 & -8 & 24 \end{bmatrix} \xrightarrow{R2/(-8)} \begin{bmatrix} 5 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$
$$\xrightarrow{R1 \to R1 - R2} \begin{bmatrix} 5 & 0 & 5 \\ 0 & 1 & -3 \end{bmatrix} \xrightarrow{R1/5} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

Thus the solution to this linear system is $\begin{cases} x_1 = 1 \\ x_2 = -3 \end{cases}$

2. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 1 & 4 \end{bmatrix} \xrightarrow[R3 \to R3 + R1]{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 2 & 2 & 10 \end{bmatrix} \xrightarrow{R3 \to R3 + R2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$
$$\xrightarrow[R3/3]{R2/(-2)} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R1 \to R1 - R2 - R3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus the solution to this linear system is $\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$

□

**Problem 2.1.2.** How many solutions does the following linear systems of equations have

1. $\begin{cases} 5x_1 + 7x_2 = 3 \\ -10x_1 - 14x_2 = -3 \end{cases}$

2. $\begin{cases} 2x_1 - x_2 = 4 \\ x_1 - \frac{1}{2}x_2 = 2 \end{cases}$

*Solution.*

1. The augmented matrix is

$$\begin{bmatrix} 5 & 7 & 3 \\ -10 & -14 & -3 \end{bmatrix} \xrightarrow{R2 \to R2 + 2R1} \begin{bmatrix} 5 & 7 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Since the last column has pivot, by Theorem 1.2.17, this linear system has no solutions

2.

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix} \xrightarrow{R2 \to R2 - \frac{1}{2}R1} \begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the last column has no pivot and there is a free variable $x_2$, by Theorem 1.2.17, this linear system has infinitely solutions

□

**Problem 2.1.3.** Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

1. Which columns are the pivot columns of $A$?
2. Write down the RREF of the this matrix

*Solution.*

1. The pivot columns of $A$ are columns 1,3,4.
2. $$A \xrightarrow[R3/2]{R2/(-1)} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[R2 \to R2+2R3]{R1 \to R1-3R3} \begin{bmatrix} 1 & 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R1 \to R1-2R2} \begin{bmatrix} 1 & 2 & 0 & 0 & -11 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

□

**Problem 2.1.4.** Determine which of the following statements are true

1. The following matrix is of row reduced echelon form
$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix}$$
2. The following two matrices are equivalent
$$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & 4 & 1 & 2 & 4 \\ 1 & 1 & -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix}$$

*Solution.*

1. False. $(3,3)$, $(2,4)$-th entries are pivots which breaks the "staircase shape"
$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix}$$
2. False. Because
$$\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & 4 & 1 & 2 & 4 \\ 1 & 1 & -3 & 2 & 1 \end{bmatrix} \xrightarrow[R3-R1]{R2 \to R2-2R1} \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 6 & -1 & -4 & 0 \\ 0 & 2 & -4 & -1 & -1 \end{bmatrix}$$
$$\xrightarrow{R2 \to R2-3R3} \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 0 & 11 & 14 & 3 \\ 0 & 2 & -4 & -1 & -1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 0 & 2 & -4 & -1 & -1 \\ 0 & 0 & 11 & 14 & 3 \end{bmatrix}$$

has pivot columns 1,2,3, and

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -1 & 2 & 1 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

has pivot columns 1,3,4. By Theorem 1.2.10, they are not equivalent, otherwise they would have the same RREF, which implies same pivot columns.

□

**Problem 2.1.5.** Determine the value(s) of $h$ such that the matrix is the augmented matrix of a consistent linear system $\begin{bmatrix} 1 & h & 1 \\ 2 & 4 & 4 \end{bmatrix}$

*Solution.* First consider

$$\begin{bmatrix} 1 & h & 1 \\ 2 & 4 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - 2R1} \begin{bmatrix} 1 & h & 1 \\ 0 & 4 - 2h & 2 \end{bmatrix}$$

By Theorem 1.2.17, the linear system has solutions $\iff$ the last column is not a pivot column $\iff$ $4 - 2h \neq 0$ $\iff$ $h \neq 2$ □

**Problem 2.1.6.** Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

*Solution.* Note that a common point of intersection would be a solution to the linear system $\begin{cases} x_1 - 4x_2 = 1 \\ 2x_1 - \phantom{4}x_2 = -3 \\ -x_1 - 3x_2 = 4 \end{cases}$, consider its augmented matrix

$$\begin{bmatrix} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{bmatrix} \xrightarrow[R3 \to R3 + R1]{R2 \to R2 - 2R1} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & -7 & 5 \end{bmatrix} \xrightarrow{R3 \to R3 + R2} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

By Theorem 1.2.17, since the last column is not a pivot column, the linear system is consistent, i.e. these three lines has comon point(s) of intersection. □

## 2.2 Online Assignment 2

**Problem 2.2.1.** Consider the following statements

1. For any four distinct vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbb{R}^3$, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \mathbb{R}^3$

2. Suppose we know that the augmented matrix of the linear system

$$\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = d_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = d_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = d_3 \end{cases}$$

   has two pivot columns, then how many solutions could the linear system have?

3. Consider matrix equation $A\mathbf{x} = \mathbf{0}$ where $A$ is a 3 by 4 matrix, then it always has more than one solution (obviously $\mathbf{x} = \mathbf{0}$ will be a solution)

*Solution.*

1. False. A counter-example would be $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$,$\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$,$\mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$,$\mathbf{v}_4 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, then $\mathbf{b} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ is not in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ since

$$\begin{bmatrix}1 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

has pivot in the last row, and apply Theorem 1.2.17

2. The number of solutions could be none or infinitely many. There are in total 6 possible cases (apply Theorem 1.2.17)

Case i: The pivot columns are 1,2, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}1 & 0 & * & *\\ 0 & 1 & * & *\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have infinitely many solutions and $x_3$ is a free variable.

Case ii: The pivot columns are 1,3, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}1 & * & 0 & *\\ 0 & 0 & 1 & *\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have infinitely many solutions. and $x_2$ is a free variable.

Case iii: The pivot columns are 1,4, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}1 & * & * & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have no solutions.

Case iv: The pivot columns are 2,3, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}0 & 1 & 0 & *\\ 0 & 0 & 1 & *\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have infinitely many solutions. and $x_1$ is a free variable.

Case v: The pivot columns are 2,4, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}0 & 1 & * & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have no solutions.

Case vi: The pivot columns are 3,4, then augmented matrix is equivalent to the RREF matrix $\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$, so the linear system have no solutions.

3.

$$\begin{bmatrix}* & * & * & * & 0\\ * & * & * & * & 0\\ * & * & * & * & 0\end{bmatrix} = \begin{bmatrix}A & \mathbf{0}\end{bmatrix} \sim \begin{bmatrix}U & \mathbf{0}\end{bmatrix}$$

Since it doesn't have a pivot in the last column, it must have a solution (indeed at least the trivial solution) by Theorem 1.2.17, and since $A$ has more columns that rows, there must be a free variable, therefore it has infinitely many solutions

□

**Problem 2.2.2.** Answer the following questions

1. Determine whether $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is in the span of $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix} \right\}$, why?

2. Assume $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Find constants $c_1, c_2$ such that $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 2\mathbf{v}_3$. Show your work on how you found $c_1, c_2$.

3. Suppose
$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$$
Is it true that for any vector $\mathbf{b}$ in $\mathbb{R}^4$, matrix equation $A\mathbf{x} = \mathbf{b}$ always has solution(s)? If it is, please give your reason. If it is not, please find one such $\mathbf{b}$ and justify your answer.

*Solution.*

1. Consider
$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b} \end{bmatrix} = \begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -2 & 1 \\ 2 & 6 & -4 & 2 \end{bmatrix} \xrightarrow{R3 \to R3 - 2R1} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 2 & -2 & 0 \end{bmatrix}$$
$$\xrightarrow{R3 \to R3 - R2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$
By Theorem 1.2.17, $A\mathbf{x} = \mathbf{b}$ has no solution, i.e. $\mathbf{b}$ is not in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. So we should consider augmented matrix
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & -2 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow{R2 \to R2 - R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & -3 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow{R2/3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \xrightarrow[R1 \to R1 - R2]{R3 \to R3 + 4R2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
Hence $c_1 = 2$, $c_2 = -1$.

2. It is equivalent to solving $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w} - 2\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

3. 
$$A \xrightarrow[R4 \to R4 - 2R1]{R2 \to R2 + R1} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \xrightarrow[R4 \to R4 + 3R2]{R3 \to R3 + 2R2} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow{R3 \leftrightarrow R4} \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Which doesn't have pivot in the last row, so $A\mathbf{x} = \mathbf{b}$ doesn't always have a solution by Theorem 1.5.1. To find one such $\mathbf{b}$, you can just try (Since the range of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the span of the columns of $A$ which is a hyperplane in $\mathbb{R}^4$, so choosing an arbitrary point is most likely not on that hyperplane, i.e. $A\mathbf{x} = \mathbf{b}$ not

solvable!). Here we just try $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, then

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 2 & -8 & 1 \\ 2 & 0 & 3 & -1 & 0 \end{bmatrix} \underbrace{\xrightarrow{}}_{\substack{R2 \to R2 + R1 \\ R4 \to R4 - 2R1}} \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & -4 & 2 & -8 & 1 \\ 0 & -6 & 3 & -7 & 0 \end{bmatrix}$$

$$\underbrace{\xrightarrow{}}_{\substack{R3 \to R3 + 2R2 \\ R4 \to R4 + 3R2}} \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix} \underbrace{\xrightarrow{}}_{R3 \leftrightarrow R4} \begin{bmatrix} 1 & 3 & 0 & 3 & 0 \\ 0 & 2 & -1 & 4 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

□

## 2.3 Online Assignment 3

**Problem 2.3.1.** Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors in $\mathbb{R}^4$, if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, and $\{\mathbf{v}_1, \mathbf{v}_3\}$ is linearly independent, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent

*Solution.* False. For example $\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ □

**Problem 2.3.2.** Suppose $A$ is a $m$ by $n$ matrix, and the matrix equation $A\mathbf{x} = \mathbf{b}$ always has solution for any vector $\mathbf{b}$ in $\mathbb{R}^m$, the columns of $A$ are linearly independent

*Solution.* False. There could be free variables □

**Problem 2.3.3.** Solve the linear system $\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 2x_1 - x_2 + x_3 - x_4 = -1 \end{cases}$ and express its solution set in parametric vector form

*Solution.* Write the augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & -1 & -1 \end{bmatrix} \underbrace{\xrightarrow{}}_{R2 \to R2 - 2R1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -3 & -3 \end{bmatrix} \underbrace{\xrightarrow{}}_{R2/(-3)} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} & 1 & 1 \end{bmatrix}$$

$$\underbrace{\xrightarrow{}}_{R1 \to R1 - R2} \begin{bmatrix} 1 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 1 & 1 \end{bmatrix}$$

So the solution set is

$$\begin{cases} x_1 + \frac{2}{3}x_3 = 0 \\ x_2 + \frac{1}{3}x_3 + x_4 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{2}{3}x_3 \\ x_2 = 1 - \frac{1}{3}x_3 - x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

Its parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}x_3 \\ 1 - \frac{1}{3}x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{2}{3}x_3 \\ -\frac{1}{3}x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -x_4 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

□

**Problem 2.3.4.** Suppose $\mathbf{v}_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}-2\\3\\-11\end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix}0\\0\\-3\end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent? If so, please give your reason, if not, please find a linear dependence (i.e. some linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_2\mathbf{v}_3 + c_4\mathbf{v}_4 = 0$, $c_1, c_2, c_3, c_4$ not all zero)

*Solution.* Consider

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{0}\end{bmatrix} = \begin{bmatrix}1 & 0 & -2 & 0 & 0\\0 & -1 & 3 & 0 & 0\\1 & 1 & -11 & -3 & 0\end{bmatrix} \xrightarrow{R3 \to R3 - R1} \begin{bmatrix}1 & 0 & -2 & 0 & 0\\0 & -1 & 3 & 0 & 0\\0 & 1 & -9 & -3 & 0\end{bmatrix}$$

$$\xrightarrow{R3 \to R3 + R2} \begin{bmatrix}1 & 0 & -2 & 0 & 0\\0 & -1 & 3 & 0 & 0\\0 & 0 & -6 & -3 & 0\end{bmatrix} \xrightarrow[R3/(-6)]{(-1)R2} \begin{bmatrix}1 & 0 & -2 & 0 & 0\\0 & 1 & -3 & 0 & 0\\0 & 0 & 1 & \frac{1}{2} & 0\end{bmatrix} \xrightarrow[R2 \to R2 + 3R3]{R1 \to R1 + 2R3} \begin{bmatrix}1 & 0 & 0 & 1 & 0\\0 & 1 & 0 & \frac{3}{2} & 0\\0 & 0 & 1 & \frac{1}{2} & 0\end{bmatrix}$$

So the solution set is

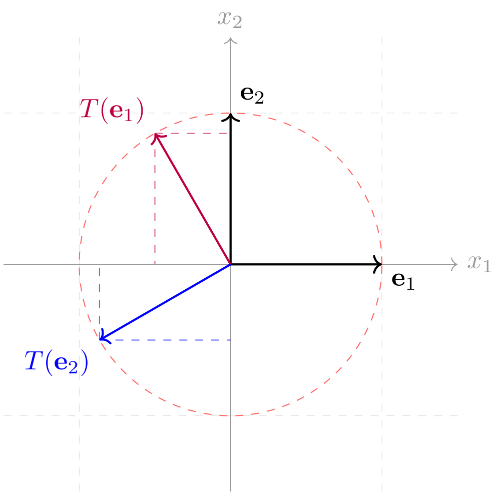
$$\begin{cases} x_1 = -x_4 \\ x_2 = -\frac{3}{2}x_4 \\ x_3 = -\frac{1}{2}x_4 \\ x_4 \text{ is free} \end{cases}$$

We can choose $x_4 = -2$, then $x_1 = 2, x_2 = 3, x_3 = 1$ so that we know $2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 - 2\mathbf{v}_4 = \mathbf{0}$ is a linear dependence. $\square$

**Problem 2.3.5.** Suppose linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates the plane $\mathbb{R}^2$ counter-clockwise by $120°$, what is the standard matrix for the this linear transformation?

*Solution.* the standard matrix for $T$ is

$$A = \begin{bmatrix}T(\mathbf{e}_1) & T(\mathbf{e}_2)\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$




$\square$

**Problem 2.3.6.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix}1 & 2 & 3\\0 & 2 & 1\end{bmatrix}.$$

What is $n$ and $m$? Find the preimage of $\begin{bmatrix}1\\2\end{bmatrix}$ under $T$. Is $T$ one-to-one? Is $T$ onto?

*Solution.* $m = 2$, $n = 3$, the preimage of $\mathbf{b} = \begin{bmatrix}1\\2\end{bmatrix}$ under $T$ will be the solution to $A\mathbf{x} = \mathbf{b}$, so we have

$$\begin{bmatrix}A & \mathbf{b}\end{bmatrix} = \begin{bmatrix}1 & 2 & 3 & 1\\0 & 2 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2 & -1\\0 & 1 & \frac{1}{2} & 1\end{bmatrix}$$

The solution set is given in parametric vector form

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-2\\-\frac{1}{2}\\1\end{bmatrix}$$

Therefore $T$ is not one-to-one but onto. □

## 2.4 Online Assignment 4

**Problem 2.4.1.** Suppose $A = \begin{bmatrix}1 & 2 & 1\\2 & 1 & 2\\0 & 2 & 1\end{bmatrix}$.

- Determine if $A$ is invertible, if not, explain why, if it is, find the inverse matrix $A^{-1}$
- Find $A^T$, the transpose of $A$. Is $A^T$ invertible? If yes, please evaluate $(A^T)^{-1}$

Please show all your work.

*Solution.*

- $A$ is invertible

$$\left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\2 & 1 & 2 & 0 & 1 & 0\\0 & 2 & 1 & 0 & 0 & 1\end{array}\right] \underset{R2\to R2-2R1}{\sim} \left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\0 & -3 & 0 & -2 & 1 & 0\\0 & 2 & 1 & 0 & 0 & 1\end{array}\right]$$

$$\underset{R2/(-3)}{\sim} \left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0\\0 & 2 & 1 & 0 & 0 & 1\end{array}\right] \underset{R3\to R3-2R2}{\sim} \left[\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0\\0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0\\0 & 0 & 1 & -\frac{4}{3} & \frac{2}{3} & 1\end{array}\right]$$

$$\underset{R3\to R3-2R2-R3}{\sim} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & -1\\0 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & 0\\0 & 0 & 1 & -\frac{4}{3} & \frac{2}{3} & 1\end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix}1 & 0 & -1\\\frac{2}{3} & -\frac{1}{3} & 0\\-\frac{4}{3} & \frac{2}{3} & 1\end{bmatrix}$

- $A^T = \begin{bmatrix}1 & 2 & 0\\2 & 1 & 2\\1 & 2 & 1\end{bmatrix}$, and $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix}1 & \frac{2}{3} & -\frac{4}{3}\\0 & -\frac{1}{3} & \frac{2}{3}\\-1 & 0 & 1\end{bmatrix}$

□

**Problem 2.4.2.** Suppose $A = \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$ is the standard matrix for the linear transformation of rotating 120° counter-clockwise. Evaluate $A^{24}$ and explain why.

*Solution.* Note that $A^{24}$ is the standard matrix for the composition of linear transformations $\overbrace{T \circ T \circ \cdots \circ T}^{24}$ which is rotate $24 \times 120° = 8 \times 360°$, which is the same as rotate $0°$, so we should have $A^{24} = I$, the identity matrix. □

**Problem 2.4.3.** We say $n$ is the order of a square matrix $A$ if $n$ is the smallest positive integer such that $A^n = I$, where $I$ is the identity matrix. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation of reflecting over $x_1$-axis, and $A$ is the standard matrix of $T$, the order of $A$ is

*Solution.* Note that $T \circ T$ is reflecting over $x_2$-axis twice, which amounts to doing nothing, so $A^2$ as the standard matrix of $T \circ T$ is the identity matrix, i.e. the order of $A$ is 2. $\square$

**Problem 2.4.4.** If $A, B$ are both invertible, then $A + B$ is also invertible

*Solution.* False. For example we can take $B = -A = -I$, then $A + B = 0$ which is not invertible. $\square$

**Problem 2.4.5.** We call $q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ a quadratic form, $a, b, c$ are constants. Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, try to rewrite $q(x_1, x_2)$ as the form of a matrix multiplication $\mathbf{x}^T A\mathbf{x}$, where $A$ is a symmetric matrix (i.e. $A^T = A$). Please find $A$.

*Solution.* We should choose $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, then

$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 & bx_1 + cx_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2bx_1x_2 + cx_2^2 = q(x_1, x_2)$$

$\square$

**Problem 2.4.6.** Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ 2x_2 + 3x_3 \\ x_1 - x_3 \end{bmatrix}$. Please find $T^{-1}$, then standard matrix for $T^{-1}$. Is $T^{-1}$ onto? Is $T^{-1}$ is one-to-one? Show all your work.

*Solution.* Note that $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$, use the algorithm to find $A^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R3 \to R3 - R1} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3 - R2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -5 & -1 & -1 & 1 \end{array}\right] \xrightarrow[R3/(-5)]{R2/2} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R2 \to R2 - \frac{3}{2}R3} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \xrightarrow{R1 \to R1 + 2R2 - R3} \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{4}{5} \\ -\frac{3}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$, so we have $T^{-1}\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{5}y_1 + \frac{1}{5}y_2 + \frac{4}{5}y_3 \\ -\frac{3}{10}y_1 + \frac{1}{5}y_2 + \frac{3}{10}y_3 \\ \frac{1}{5}y_1 + \frac{1}{5}y_2 - \frac{1}{5}y_3 \end{bmatrix}$. $T^{-1}$ is both onto and one-to-one. $\square$

**Problem 2.4.7.** An affine transformation is a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, Using partitioned matrix, note that $A\mathbf{x} + \mathbf{b} = \begin{bmatrix} A & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$. To make thing even nicer, we can use the trick of "adding" 1 at the end of the coordinates, which gives us $\begin{bmatrix} A\mathbf{x} + \mathbf{b} \\ 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$.

- Suppose $T_1(\mathbf{x}) = A_1\mathbf{x} + \mathbf{b}_1$ and $T_2(\mathbf{x}) = A_2\mathbf{x} + \mathbf{b}_2$ are both affine transformations. What is $T_2 \circ T_1$? What happens if you try the trick?

- Suppose $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ is an affine transformation and $A$ is invertible. What is $T^{-1}$? What happens if you try the trick?

*Solution.*

- $(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x})) = A_2(T_1(\mathbf{x})) + \mathbf{b}_2 = A_2(A_1\mathbf{x} + \mathbf{b}_1) + \mathbf{b}_2 = A_2A_1\mathbf{x} + (A_2\mathbf{b}_1 + \mathbf{b}_2)$, this can be explained use the trick as follows

$$\begin{bmatrix} A_2 & \mathbf{b}_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 & \mathbf{b}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_2A_1 & A_2\mathbf{b}_1 + \mathbf{b}_2 \\ 0 & 1 \end{bmatrix}$$

- The inverse should be $T^{-1}(\mathbf{y}) = A^{-1}\mathbf{y} - A^{-1}\mathbf{b}$, via the trick we can interpreted it as

$$\begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\mathbf{b} \\ 0 & 1 \end{bmatrix}$$

Since

$$\begin{bmatrix} A^{-1} & -A^{-1}\mathbf{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \mathbf{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A^{-1} & -A^{-1}\mathbf{b} \\ 0 & 1 \end{bmatrix}$$

□

## 2.5 Online Assignment 5

**Problem 2.5.1.** Suppose $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{bmatrix}$. Use cofactor expansion of $A$ across the last row to evaluate the determinant of $A$

*Solution.*

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{vmatrix} &= (-1)\cdot(-1)^{3+1}\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} + 2\cdot(-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 0 \\ &= (-1)(2\cdot 3 - 2\cdot 1) + (-2)(1\cdot 3 - 2\cdot 2) = -2 \end{aligned}$$

□

**Problem 2.5.2.** Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 2 & 0 & 0 \\ 3 & 5 & 2 & 3 \\ 2 & 1 & 0 & -1 \end{bmatrix}$. Is $A$ invertible?

*Solution.*

$$\begin{vmatrix} 1 & 2 & 2 & -1 \\ 0 & 2 & 0 & 0 \\ 3 & 5 & 2 & 3 \\ 2 & 1 & 0 & -1 \end{vmatrix} \xlongequal{\text{cofactor expansion second row}} 2(-1)^{2+2}\begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & -1 \end{vmatrix} \xlongequal[R3\to R3-2R1]{R2\to R3-3R1} 2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & -4 & 1 \end{vmatrix}$$

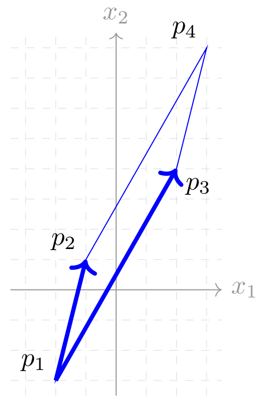
$$\xlongequal{R3\to R3-R2} 2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & -5 \end{vmatrix} = 2\cdot 1\cdot(-4)\cdot(-5) = 40$$

$A$ is invertible since $\det A \neq 0$ □

**Problem 2.5.3.** Consider the parallelgram $P$ with vertices $(-2,-3), (-1,1), (2,4), (3,8)$, use determinants to evaluate the area of $P$

*Solution.* Name these four points $p_1, p_2, p_3, p_4$, and the vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$. And we consider $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$, then area of $P$ would be

$$|\det A| = |(1\cdot 7 - 4\cdot 4))| = 9$$




□

**Problem 2.5.4.** $\det(A - B) = \det A - \det B$

*Solution.* This is false, for example, we could just take $A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = -I$, then $4 = \det(2I) = \det(A - B) \neq \det A - \det B = 1 - 1 = 0$ □

**Problem 2.5.5.** If $A$ is a 3 by 3 matrix, then $\det(2A) = 8(\det A)$

*Solution.* This is true. □

**Problem 2.5.6.** Suppose $A, B$ are both 3 by 3 matrices, and $\det A = 2$, $\det B = \frac{1}{3}$, then the determinant of $A^T B^{-1}$ is

*Solution.* $\det(A^T B^{-1}) = \det(A^T)\det(B^{-1}) = (\det A)(\det B)^{-1} = 2 \cdot 3 = 6.$ □

**Problem 2.5.7.** Suppose $A$ is a 3 by 3 matrix with entries integers, and $A^3 = I$ is the identity matrix. Then the determinant of $A$ has to be

*Solution.* $\det A$ must be some real number. $1 = \det I = \det(A^3) = (\det A)^3 \Rightarrow \det A = \sqrt[3]{1} = 1$. □

## 2.6 Online Assignment 6

**Problem 2.6.1.** Suppose $A = \begin{bmatrix} 1 & 2 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 3 & 1 & -2 \end{bmatrix}$

a) Find a basis for the null space of $A$

b) Find a basis for the column space of $A$

c) Find a basis for the row space of $A$

*Solution.* First realize

$$A \xrightarrow{R2 \leftrightarrow R3} \begin{bmatrix} 1 & 2 & 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Hence the solution in parametric form is $x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. And the pivot columns are the 1st, 3rd and 5th columns

a) A basis for $\operatorname{Nul} A$ could be $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

b) A basis for $\operatorname{Col} A$ could be $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

c) A basis for $\operatorname{Row} A$ could be

$$\left\{ \begin{bmatrix} 1 & 2 & 0 & -1 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \right\}$$

$\square$

**Problem 2.6.2.** Suppose $A = \begin{bmatrix} 3 & -1 & -5 \\ 1 & 1 & -1 \\ -2 & 2 & 4 \end{bmatrix}$

a) Determine whether $\mathbf{u} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$ is in the null space of $A$. Explain your reasoning.

b) Determine whether $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}$ is in the column space of $A$. Explain your reasoning.

*Solution.*

a) $\mathbf{u} \in \operatorname{Nul} A$ since $A\mathbf{u} = \mathbf{0}$

b) $\mathbf{b} \in \operatorname{Col} A$ since linear system $A\mathbf{x} = \mathbf{b}$ is consistent

$\square$

**Problem 2.6.3.** Recall $\mathbb{P}_2 = \{a_0 + a_1 t + a_2 t^2 | a_0, a_1, a_2 \in \mathbb{R}\}$ is the set of polynomials of degree less or equal to 2. Let $V$ be the subset of $\mathbb{P}_2$ consists of polynomials that evalute to 0 at $t = 1$ (i.e. polynomial $p(t)$ is in $V$ if $p(1) = 0$ and of degree less or equal to 2). With the usual addition and scalar multiplication for polynomials

a) Show that $V$ is a subspace.

b) Find a basis of $V$.

c) Cosider $T : V \to \mathbb{R}^2$ that maps polynomial $p(t)$ to $\begin{bmatrix} p(-1) \\ p(2) \end{bmatrix}$, show $T$ is a linear transformation

*Solution.*

a) Realize that $V = \ker S$ for the linear transformation $S : \mathbb{P}_2 \to \mathbb{R}$, $S(a_0 + a_1 t + a_2 t^2) = a_0 + a_1 + a_2$, thus $V$ is a subspace of $\mathbb{P}_2$

b) This is precisely Example 1.19.2

c) For any $p(t), q(t) \in V$ and $c \in \mathbb{R}$, we have

$$T(p+q) = \begin{bmatrix} (p+q)(-1) \\ (p+q)(2) \end{bmatrix} = \begin{bmatrix} p(-1) + q(-1) \\ p(2) + q(2) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(2) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(2) \end{bmatrix} = T(p) + T(q)$$

$$T(cp) = \begin{bmatrix} (cp)(-1) \\ (cp)(2) \end{bmatrix} = \begin{bmatrix} c \cdot p(-1) \\ c \cdot p(2) \end{bmatrix} = c \begin{bmatrix} p(-1) \\ p(2) \end{bmatrix} = cT(p)$$

Therefore $T : V \to \mathbb{R}^2$ is a linear transformation

$\square$

**Problem 2.6.4.** We say a square matrix $A$ is anti-symmetric if $A^T = -A$. Denote the set of $3 \times 3$ anti-symmetric matrices $V$.

a) Show that $V$ is a vector space.

b) What is the dimension of $V$?

c) Find a basis of $V$.

d) Show that

$$\mathcal{B} = \left\{ B_1 = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \right\}$$

form a basis for $V$.

*Solution.*

a) For any $A, B \in V$ and $c \in \mathbb{R}$, by definition $A^T = -A$, $B^T = -B$, so

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$
$$(cA)^T = cA^T = c(-A) = -(cA)$$

hence $A+B, cA \in V$, i.e. $V$ is a closed under addition and scalar multiplication. Therefore $V$ is a subspace of $M_{3\times 2}(\mathbb{R})$, and consequently a vector space.

b) It is not hard to realize

$$V = \left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \in M_{3\times 3}(\mathbb{R}) \middle| a, b, c \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

Thus $\dim V = 3$

c) Note that

$$\begin{aligned} \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -b \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & c & 0 \end{bmatrix} \\ &= a \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned} \tag{2.6.1}$$

So we know that

$$\mathcal{E} = \{E_1, E_2, E_3\} = \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

is a basis for $V$, this is linearly independent since the linear combination (2.6.1) is equal to zero $\iff a = b = c = 0$

d) The coordinate vectors $\{[B_1]_{\mathcal{E}}, [B_2]_{\mathcal{E}}, [B_3]_{\mathcal{E}}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$, so $\mathcal{B}$ form a basis for $V$

$\square$

# 3 Exams

## 3.1 Exam 1

**Problem 3.1.1.**

**a) (15 pt)** Write linear system

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 \qquad\quad = 2 \\ -x_1 + 3x_2 + \ \ x_3 + 2x_4 = 1 \\ 3x_1 + 2x_2 - \ \ x_3 + 2x_4 = -1 \end{cases}$$

as a matrix equation, and then solve it, write your solution in parametric vector form.

**b) (5 pt)** Is $\left\{ \mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \mathbf{a}_4 = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$ linearly independent? Why or why not.

**c) (4 pt)** What is the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$

*Solution.*

**a)** We could rewrite the linear system as

$$A\mathbf{x} = \begin{bmatrix} 2 & 4 & 2 & 0 \\ -1 & 3 & 1 & 2 \\ 3 & 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \mathbf{b}$$

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} \xrightarrow[R3 \to R3+3R2]{R1 \to R1+2R2} \begin{bmatrix} 0 & 10 & 4 & 4 & 4 \\ -1 & 3 & 1 & 2 & 1 \\ 0 & 11 & 2 & 8 & 2 \end{bmatrix} \xrightarrow{R3 \to R3-R1} \begin{bmatrix} 0 & 10 & 4 & 4 & 4 \\ -1 & 3 & 1 & 2 & 1 \\ 0 & 1 & -2 & 4 & -2 \end{bmatrix}$$

$$\xrightarrow[(-1)R2]{R1 \to R1-10R3} \begin{bmatrix} 0 & 0 & 24 & -36 & 24 \\ 1 & -3 & -1 & -2 & -1 \\ 0 & 1 & -2 & 4 & -2 \end{bmatrix} \xrightarrow[\text{move } R3 \text{ to the third row}]{R1/24} \begin{bmatrix} 1 & -3 & -1 & -2 & -1 \\ 0 & 1 & -2 & 4 & -2 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 \end{bmatrix}$$

$$\xrightarrow{R2 \to R2+2R3} \begin{bmatrix} 1 & -3 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 \end{bmatrix} \xrightarrow{R1 \to R1+3R2+R3} \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 1 \end{bmatrix}$$

So the solution is

$$\begin{cases} x_1 = \frac{1}{2}x_4 \\ x_2 = -x_4 \\ x_3 = 1 + \frac{3}{2}x_4 \\ x_4 \text{ is free} \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

**b)** Note the RREF of $A$ is $\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$ Since the RREF of $A$ doesn't have pivots in each column, $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is linearly dependent.

**c)** Since the RREF $A$ has a pivot in each row, the columns of $A$ (i.e. $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$) span $\mathbb{R}^3$

□

**Problem 3.1.2.** Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 2x_3 \\ 2x_1 + 3x_3 \\ 3x_1 + 5x_2 + 6x_3 \end{bmatrix}$. Denote the standard matrix for $T$ as $A$.

**a)** **(3 pt)** Evaluate $A$.

**b)** **(4 pt)** Is $T$ is onto? Is $T$ one-to-one?

**c)** **(17 pt)** Is $T$ is invertible? If so, what is the standard matrix for $T^{-1}$?

**d)** **(15 pt)** Find $A^T$ ,$(A^T)^{-1}$.

*Solution.*

**a)** The standard matrix for $T$ is $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 3 \\ 3 & 5 & 6 \end{bmatrix}$

**b)** $T$ is onto and one-to-to.

**c)** $T$ is invertible. And the standard matrix of $T^{-1}$ is $A^{-1}$

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \underbrace{\substack{R2\to R2-2R1 \\ R3\to R3-3R1}} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -4 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\underbrace{\substack{R2\to R2-4R3 \\ R3\to R1+2R3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -5 & 0 & 2 \\ 0 & 0 & -1 & 10 & 1 & -4 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array}\right] \underbrace{\substack{(-1)R2 \\ (-1)R3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -5 & 0 & 2 \\ 0 & 0 & 1 & -10 & -1 & 4 \\ 0 & 1 & 0 & 3 & 0 & -1 \end{array}\right]$$

$$\underbrace{R2\leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -5 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & -10 & -1 & 4 \end{array}\right] \underbrace{R1\to R1-2R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 15 & 2 & -6 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & -10 & -1 & 4 \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 15 & 2 & -6 \\ 3 & 0 & -1 \\ -10 & -1 & 4 \end{bmatrix}$

**d)** $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 5 \\ 2 & 3 & 6 \end{bmatrix}$, $(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 15 & 3 & -10 \\ 2 & 0 & -1 \\ -6 & -1 & 4 \end{bmatrix}$

□

**Problem 3.1.3.**

**a)** **(5 pt)** Suppose $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & h & 0 & k & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ is of reduced row echelon form (RREF), could we know what $h, k$ are? If not, please explain why, if so, please give their values.

**b)** **(3 pt)** `TRUE` or `FALSE`. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a one-to-one linear transformation, then it is also onto.

**c)** **(3 pt)** `TRUE` or `FALSE`. If $A$ is a 3 by 3 matrix and $A^3$ is invertible, then so is $A$.

**d)** **(3 pt)** `TRUE` or `FALSE`. If $A$ is a 4 by 3 matrix, then $A\mathbf{x} = \mathbf{b}$ cannot have non-trivial solution(s)

*Solution.*

**a)** $h = 1$, $k = 0$. Since $h$ has be in a pivot position, and $k$ is in a pivot column.

**b)** `TRUE`. Since its standard matrix would have a pivot in each column, that is three pivots, so there must be a pivot in each row also, hence $T$ is also onto.

**c)** `TRUE`. Since $A^3$ is invertible, $0 \neq \det(A^3) = (\det A)^3 \Rightarrow \det A \neq 0$, hence $A$ is invertible.

**d)** `FALSE`. Take $A$ to be $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{b} = \mathbf{0}$, and the linear system have two free variables.

□

**Problem 3.1.4.** Suppose $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & 0 & 6 & 1 \\ 2 & 1 & 2 & -4 \\ 3 & 0 & 2 & 1 \end{bmatrix}$.

**a)** **(16 pt)** Evaluate $\det A$.

**b)** **(3 pt)** Is $A$ invertible?

**c)** **(4 pt)** What is $\det(-2A)$?

*Solution.*

**a)**

$$\begin{vmatrix} 1 & 0 & 3 & 0 \\ 2 & 0 & 6 & 1 \\ 2 & 1 & 2 & -4 \\ 3 & 0 & 2 & 1 \end{vmatrix} \xlongequal{\text{cofactor expansion across second column}} 1(-1)^{3+2} \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$\xlongequal[R3 \to R3 - 3R1]{R2 \to R2 - 2R1} (-1) \begin{vmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & -7 & 1 \end{vmatrix} \xlongequal{R2 \leftrightarrow R3} (-1)(-1) \begin{vmatrix} 1 & 3 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-1)(-1)1(-7)1 = -7$$

**b)** $A$ is invertible since $\det A = -7 \neq 0$

**c)** Since $A$ is a 4 by 4 matrix, $\det(-2A) = (-2)^4 \det A = 16 \cdot (-7) = -112$

□

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