MATH121 - Elementary Calculus II

Haoran Li

Fall 2019

Contents

1 Introduction

My name is Haoran Li, or Harrison Li, my office is at MATH4423, and my office hours are 2-3pm on Mondays and Wednesdays.

Computations on tests will involve only fairly easy numbers, and an exact answer will be required rather than a decimal approximation. For example

Exact	Not exact
$\sqrt{2}$	1.414
π	3.14159265358979323846
$\frac{2}{3}$	0.67
$\ln 6$	1.79
e^2	7.389

No calculators will be allowed on any of the tests!

2 Review

The following statements are mathematically equivalent:
a) Find the slope of the line tangent to the graph of f at a point (x,y)
b) Find $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$
c) Find the first derivative of $f(x)$
d) Find $f'(x)$
e) Find $\dfrac{dy}{dx}$
Recall, however, that the first derivative is itself a function, which has its own domain and graph. Since it is a function, it has its own derivative. Given a function f, we can calculate the first derivative f' or $\dfrac{dy}{dx}$. We can then calculate the derivative of f', also called the second derivative of f, symbolically f'' or $\dfrac{d^2y}{dx^2}$

Important note: Just like $\frac{dy}{dx}$ is not a fraction, but is a notation for the first derivative, $\frac{d^2y}{dx^2}$ is also not a fraction but a notation. There is no multiplication involved! Rather, you need to interpret it this way: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ which means "the derivative of $\frac{dy}{dx}$", the derivative of a derivative.

Example A:
Given $f(x) = x^3 - 8x + 2$, find $f'(x), f''(x), f(-1), f'(-1)$ and $f''(-1)$
Answers:
$3x^2 - 8, 6x, 9, -5, -6$

Example B:
Given $f(x) = \left(5x^4 - 1\right)^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, then find y when $x = -1$, $\left.\frac{dy}{dx}\right|_{x=-1}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=-1}$
Answers:
$200x^6 - 40x^3, 1400x^6 - 120x^2, 16, -160, 1280$

Example C:
Public health officials use rates of change to quantify the spread of an epidemic into an equation, which they then use to determine the most effective measures to counter it. A recent measles epidemic followed the equation $45t^2 - t^3$ where y = the number of people infected and t = time in days. a) What is the domain of this function? b) How many people are infected after 5 days? c) What is the rate of spread after 5 days? d) After how many days does the number of cases reach its maximum? e) Use the above to sketch the graph of y
Answers:
$0 \le x \le 45$, 1000 people, 375 cases per day, 30 days

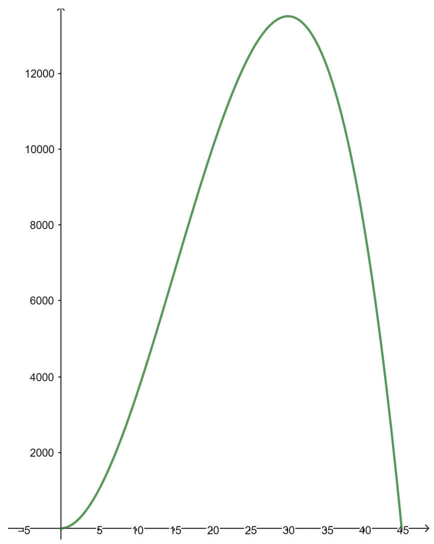

Figure 1: Example C, e)

Example D:
Optimization does not always involve a maximum. The fuel, maintenance and labor costs (in dollars per mile) of operating a truck on an interstate highway are described as a function of the truck's velocity (miles per hour) by the algebraic rule $C(v) = 78 + 1.2v + 5880v^{-1}$. What speed should the driver maintain on a 600 mile haul to minimize costs?
Answers:
70mph

Example E:
Given $f(x) = \sqrt{2x+1}(\sqrt{x} - 1)$, find $\frac{dy}{dx}$
Answers:
$\frac{4x + 1 - 2\sqrt{x}}{2\sqrt{x(2x+1)}}$

Example F:
Given $h(x) = \frac{3x+1}{x-2}$, find h'
Answers:
$\frac{-7}{(x-2)^2}$

Example G:
Determine whether $\sqrt[3]{x+\sqrt{2x}}$ has any extrema, either relative or absolute
Answers:
Absolute minimum at $(0,1)$, no maximum

Example H:
Given $h(x)=e^{x^2-x}$, find the first derivative and determine the location of any relative extrema
Answers:
$x=\frac{1}{2}$

Example I:
Given $f(x)=\ln\left(x^2e^x\right)$, find the first and second derivatives.
Answers:
$\frac{2}{x}+1, -\frac{2}{x^2}$
Note that domain is not an issue. For f and both derivatives, x can be any real number except 0

Example J:
The number of units a new worker can produce on an assembly line after t days on the job is given by the formula $N(t)=40-40e^{-0.35t}$. This function is called a learning curve. a) How many units can the worker make when she or he first begins? b) What is the worker's rate of production? c) What is the maximum number he or she can be expected to make?
Answers:
0 units, $14e^{-0.35t}$ units per day, 40 units

Example K:
Find $\int\left(3x^{-6}-2e^{5x}+4x^{-1}-7\right)dx$
Answers:
$-\frac{3}{5}x^{-5}-\frac{2}{5}e^{5x}+4\ln|x|-7x+C$
Note that domain is an issue. For f and its integral, x can be any real number except 0. Example L:
Find the area under the curve $y=e^x+e^{-x}$, on the interval $0\le x\le\ln(8)$
Answers:
$\frac{63}{8}$

3 Lecture 08/26

MATH121-Lesson1

Haoran Li

August 2019

1 Introduction

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Given $f(x) = \left(5x^4 - 1\right)^2$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$, then find y when $x = -1$, $\left.\dfrac{dy}{dx}\right|_{x=-1}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=-1}$

Answers:

$200x^{7} - 40x^3, 1400x^6 - 120x^2, 16, -160, 1280$

$$\frac{dy}{dx} = 2(5x^4-1)(20x^3) = 40x^3(5x^4-1) = 200x^7 - 40x^3$$

$$\frac{d^2y}{dx^2} = 40\left[(3x^2)(5x^4-1) + x^3(20x^3)\right]$$

$$= 40\left[15x^6 - 3x^2 + 20x^6\right]$$

$$= 40(35x^6 - 3x^2)$$

$$= 40x^2(35x^4 - 3) = 1400x^6 - 120x^2$$

Example C:
Public health officials use rates of change to quantify the spread of an epidemic into an equation, which they then use to determine the most effective measures to counter it. A recent measles epidemic followed the equation $45t^2 - t^3$ where $y =$ the number of people infected and $t =$ time in days. a) What is the domain of this function? b) How many people are infected after 5 days? c) What is the rate of spread after 5 days? d) After how many days does the number of cases reach its maximum? e) Use the above to sketch the graph of y

Answers:
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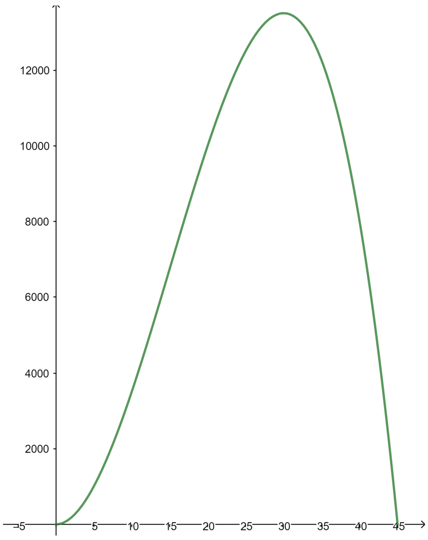


Figure 1: Example C, e)

(a) $0 \leq y = 45t^2 - t^3 = t^2(45-t) \Rightarrow 0 \leq t \leq 45$

(b) take $t=5$, $y = 5^2(45-5) = 25 \times 40 = 1000$

(c) $\frac{dy}{dt} = y' = (45t^2 - t^3)' = 90t - 3t^2 = 3t(30-t)$

take $t=5$, $y' = 90 \times 5 - 3 \times 5^2 = 450 - 75 = 375$

(d) Amounts to find the maximum, first find the extrema which is given by the zeros of equation

$y' = 0 \Leftrightarrow 3t(30-t) = 0 \Rightarrow t=0$ or $t = 30$

and if $t < 0$, $y' > 0$

if $0 < t < 30$, $y' > 0$

if $t > 30$, $y' < 0$

hence when $t = 30$, y reaches its maximum

Example D:
Optimization does not always involve a maximum. The fuel, maintenance and labor costs (in dollars per mile) of operating a truck on an interstate highway are described as a function of the trucks velocity (miles per hour) by the algebraic rule $C(v) = 78 + 1.2v + 5880v^{-1}$. What speed should the driver maintain on a 600 mile haul to minimize costs?
Answers:
70mph

$$C'(v) = 1.2 - \frac{5880}{v^2}$$

$$C'(v) = 0 \Rightarrow v^2 = \frac{5880}{1.2} = \frac{5880}{\frac{6}{5}} = \frac{\overset{980}{\cancel{5880}} \times 5}{\cancel{6}} = 4900$$

$$\Rightarrow v = 70$$

[$v = -70$ is eliminated since speed v should be a non-negtive number]

Example E:

Given $f(x) = \sqrt{2x+1}(\sqrt{x}-1)$, find $\dfrac{dy}{dx}$

Answers:

$\dfrac{4x+1-2\sqrt{x}}{2\sqrt{x(2x+1)}}$

$$\frac{dy}{dx} = \left(\frac{2}{2\sqrt{2x+1}}\right)(\sqrt{x}-1) + (\sqrt{2x+1})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\sqrt{x}-1}{\sqrt{2x+1}} + \frac{\sqrt{2x+1}}{2\sqrt{x}}$$

$$= \frac{(\sqrt{x}-1)2\sqrt{x}}{2\sqrt{x(2x+1)}} + \frac{2x+1}{2\sqrt{x(2x+1)}}$$

$$= \frac{2x-2\sqrt{x}+2x+1}{2\sqrt{x(2x+1)}}$$

$$= \frac{4x+1-2\sqrt{x}}{2\sqrt{x(2x+1)}}$$

Example F:
Given $h(x) = \dfrac{3x+1}{x-2}$, find h'
Answers:
$\dfrac{-7}{(x-2)^2}$

You can use the quotient rule, or

$$\frac{3x+1}{x-2} = \frac{3x - 3\times 2 + 3\times 2 + 1}{x-2}$$

$$= \frac{3(x-2) + 6 + 1}{x-2}$$

$$= 3 + \frac{7}{x-2}$$

$$h' = -\frac{7}{(x-2)^2}$$

Example G:
Determine whether $\sqrt[3]{x+\sqrt{2x}}$ has any extrema, either relative or absolute
Answers:
Absolute minimum at $(0,0)$, no maximum

First notice the domain should be $x \geq 0$

Let $h = \sqrt[3]{x+\sqrt{2x}} = (x+\sqrt{2x})^{\frac{1}{3}}$

$$h' = \frac{1}{3}(x+\sqrt{2x})^{-\frac{2}{3}}\left(1+\frac{2}{2\sqrt{2x}}\right)$$

$$= \frac{1}{3}(x+\sqrt{2x})^{-\frac{2}{3}}\left(1+\frac{1}{\sqrt{2x}}\right)$$

$h' = 0 \Leftrightarrow x = 0$

and $h' > 0$ when $x > 0$

hence h has absolute minimum at $(0,0)$ and no maximum

Example H:
Given $h(x) = e^{x^2-x}$, find the first derivative and determine the location of any relative extrema
Answers:
$x = \frac{1}{2}$

$h' = (2x-1)e^{x^2-x}$

$h' = 0 \Leftrightarrow x = \frac{1}{2}$ since e^x is always greater than zero

Example I:
Given $f(x) = \ln(x^2 e^x)$, find the first and second derivatives.
Answers:
$\frac{2}{x} + 1, -\frac{2}{x^2}$
Note that domain is not an issue. For f and both derivatives, x can be any real number except 0

$$f'(x) = \frac{2\cancel{x}\cancel{e^x} + x^{\cancel{2}}\cancel{e^x}}{x^{\cancel{2}}\cancel{e^x}} = \frac{2+x}{x} = \frac{2}{x} + 1$$

$$f''(x) = -\frac{2}{x^2}$$

Example J:
The number of units a new worker can produce on an assembly line after t days on the job is given by the formula $N(t) = 40 - 40e^{-0.35t}$. This function is called a learning curve. a) How many units can the worker make when she or he first begins? b) What is the workers rate of production? c) What is the maximum number he or she can be expected to make?
Answers:
0 units, $14e^{-0.35t}$ units per day, 40 units

(a) $N(0) = 40 - 40 \times 1 = 0$

(b) $N'(t) = -40 \cdot (-0.35)\, e^{-0.35t}$

$= 14e^{-0.35t}$

(c) as $t \to +\infty$, $e^{-0.35t} \to 0$

hence the maximum expected is 40

but never quite

Example K:

Find $\int \left(3x^{-6} - 2e^{5x} + 4x^{-1} - 7\right) dx$

Answers:

$-\frac{3}{5}x^{-5} - \frac{2}{5}e^{5x} + 4\ln|x| - 7x + C$

Note that domain is an issue. For f and its integral, x can be any real number except 0.

$$\int \left(3x^{-6} - 2e^{5x} + 4x^{-1} - 7\right) dx$$

$$= \frac{3}{-5}x^{-5} + 4\ln|x| - 7x - 2\int e^{5x}\,dx$$

$$= -\frac{3}{5}x^{5} + 4\ln|x| - 7x - \frac{2}{5}\int e^{5x}\,d(5x)$$

$$= -\frac{3}{5}x^{5} + 4\ln|x| - 7x - \frac{2}{5}e^{5x} + C$$

where C is an arbitrary constant

Example L:
Find the area under the curve $y = e^x + e^{-x}$, on the interval $0 \leq x \leq \ln(8)$
Answers:
$\frac{63}{8}$

Fun fact: the graph of this function is called a catenary, matching the shape of a hanging chain

$$\int_0^{\ln 8} \left(e^x + e^{-x}\right) dx = \int_0^{\ln 8} e^x dx + \int_0^{\ln 8} e^{-x} dx$$

$$= e^x\Big|_0^{\ln 8} + \left[-e^{-x}\right]\Big|_0^{\ln 8}$$

$$= (8-1) + \left[\left(-\frac{1}{8}\right) - (-1)\right]$$

$$= 7 + \frac{7}{8}$$

$$= \frac{63}{8}$$

$$e^{-\ln 8} = \left\{ \begin{array}{c} \frac{1}{e^{\ln 8}} \\ e^{\ln\left(\frac{1}{8}\right)} \end{array} \right\} = \frac{1}{8}$$

4 Lecture 08/28

Section 8.1

Conversion between degrees and radians

On the unit circle, the radian of an angle is the length of the corresponding arc with the sign

Examples: $90° = \frac{\pi}{2}$ rad $\quad 270° = \frac{3\pi}{2}$, $\quad 450° = \frac{5\pi}{2}$, $\quad -360° = -2\pi$

generally $\quad d° = d \times \frac{2\pi}{360} = d \times \frac{\pi}{180} = r$

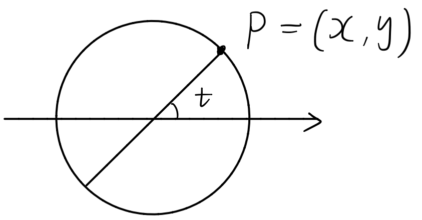
so conversely we get $\quad d = \frac{180}{\pi} r$

Exercise: $\frac{3\pi}{4}$, $-\frac{2\pi}{3}$, $75°$

Homework: 1-17 odd for 8.1
1-33 odd for 8.2

Section 8.2

Given an angle of t radians, let the angle be on the unit circle, then $(\cos t, \sin t) = (x, y)$, $\tan t = \frac{\sin t}{\cos t}$



hence we have

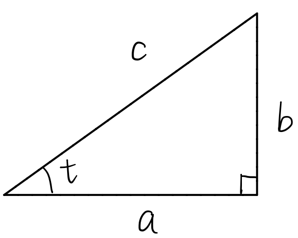
$$\cos^2 t + \sin^2 t = x^2 + y^2 = 1$$

$\cos(t \pm 2\pi) = \cos t$, $\sin(t \pm 2\pi) = \sin t$, $\cos(-t) = \cos t$

$\sin(-t) = -\sin t$, $\sin(s+t) = \sin s \cos t + \cos s \sin t$

For example
$$\begin{aligned} \sin(75^\circ) &= \sin(30^\circ + 45^\circ) \\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\ &= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{4}(1+\sqrt{3}) \end{aligned}$$

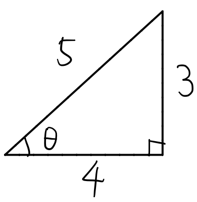
Connection with right triangle:



$a^2 + b^2 = c^2$ (Pythagorean identity)

$\sin t = \frac{b}{c}$, $\cos t = \frac{a}{c}$, $\tan t = \frac{b}{c}$

Example:



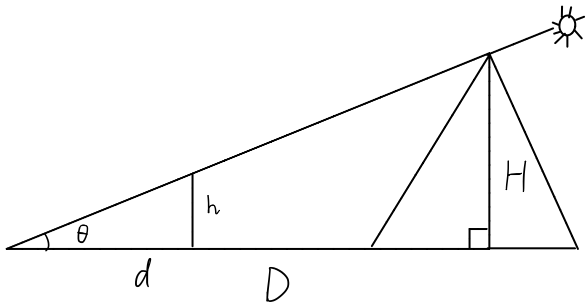
$\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

Real life applications:

The king nant to measure the height of Pyramid

a clever man put a pole and wait the shows of them

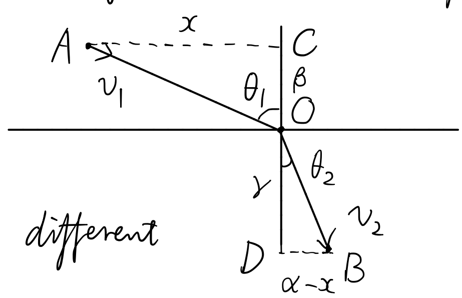
coincide



then measure h, d, D, but $\tan\theta = \frac{h}{d} = \frac{H}{D} \Rightarrow D = \frac{Hd}{h}$

Refraction: Férmat's Principle: light travels the path

which takes least time



A, B are two fixed points on different

medium which light travels at speeds v_1, v_2

where should O be such that it takes least time

to get B from A, which will take time

$$T = \frac{\sqrt{x^2+\beta^2}}{v_1} + \frac{\sqrt{(\alpha-x)^2+\gamma^2}}{v_2}$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2+\beta^2}} - \frac{\alpha-x}{v_2\sqrt{(\alpha-x)^2+\gamma^2}}$$

$$\frac{dT}{dx} = 0 \Rightarrow \quad \frac{v_1}{v_2} = \frac{x}{\sqrt{x^2+\beta^2}} \Big/ \frac{\alpha-x}{\sqrt{(\alpha-x)^2+\gamma^2}} = \frac{\sin\theta_1}{\sin\theta_2}$$

5 Lecture 08/30

Review: $150^\circ = \frac{5}{6}\pi$, $72^\circ = \frac{2\pi}{5}$

$\frac{\pi}{7} = \frac{180^\circ}{7}$, $-1 = -\frac{180^\circ}{\pi}$

$$\sin\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2}\right)\cos x - \cos\left(\frac{\pi}{2}\right)\sin x = \cos x$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

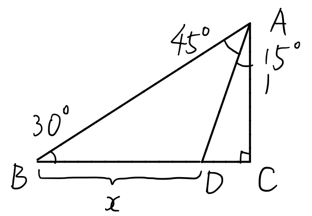
divide $\cos x \cos y$ $= \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$$\tan\left(x + \frac{\pi}{2}\right) = -\frac{1}{\tan x} = -\cot x$$

$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6} + 2k\pi$, $\frac{5\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

$\cos t = -1 \Rightarrow t = -\pi + 2k\pi$, $k \in \mathbb{Z}$

$\cos t = 0 \Rightarrow t = k\pi$, $k \in \mathbb{Z}$



$$\frac{1}{BC} = \frac{AC}{BC} = \tan 30^\circ = \frac{\sqrt{3}}{3} \Rightarrow BC = \sqrt{3}$$

$$DC = \frac{DC}{AC} = \tan 15^\circ = \tan(45^\circ - 30^\circ)$$

$$= \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \times \frac{\sqrt{3}}{3}}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} = \frac{\left(1 - \frac{\sqrt{3}}{3}\right)^2}{\left(1 + \frac{\sqrt{3}}{3}\right)\left(1 - \frac{\sqrt{3}}{3}\right)} = \frac{1 + \frac{1}{3} - \frac{2\sqrt{3}}{3}}{1 - \frac{1}{3}}$$

$$= \frac{\frac{4}{3} - \frac{2\sqrt{3}}{3}}{\frac{2}{3}} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

$$x = BD = BC - DC = \sqrt{3} - (2 - \sqrt{3}) = 2\sqrt{3} - 2$$

Section 8.3

$\frac{d}{dt}\sin t = \cos t$, $\frac{d}{dt}\cos t = -\sin t$

$(\sin 3t)' = 3\cos 3t$, $(\cos(t^3))' = -3t^2 \sin t^3$

$$(t^2 \cos 3t)' = (t^2)'\cos 3t + t^2(\cos 3t)'$$
$$= 2t\cos 3t - 3t^2 \sin 3t$$

$\int \cos t\, dt = \sin t + C$, $\int \sin t = -\cos t + C$

$$\int \cos 3t \, dt = \frac{1}{3}\int \cos 3t \, d(3t) = \frac{1}{3}\sin 3t + C$$

$$\int_0^{\pi} \sin t \, dt = \left[-\cos t\right]\Big|_0^{\pi} = \left[-(-1) - (-1)\right] = 2$$

$$\lim_{h\to 0} \frac{\sin h}{h} = \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} = \cos 0 = 1$$

$$\int \tan t \, dt = \int \frac{\sin t}{\cos t} dt = -\int \frac{1}{\cos t} d(\cos t) = -\ln|\cos t| + C$$

$$\int \cos \frac{x-2}{2} dx = \int \cos \frac{x-2}{2} d(x-2) = 2\int \cos \frac{x-2}{2} \, d\frac{x-2}{2} =$$

$$2\sin \frac{x-2}{2} + C$$

6 Lecture 09/04

$$\int \cos 3t \, dt = \frac{1}{3}\int \cos 3t \, d(3t) = \frac{1}{3}\sin 3t + C$$

$$\int_0^{\pi} \sin t \, dt = \left[-\cos t\right]\Big|_0^{\pi} = \left[-(-1)-(-1)\right] = 2$$

$$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h} = \cos 0 = 1$$

$$\int \tan t \, dt = \int \frac{\sin t}{\cos t}\, dt = -\int \frac{1}{\cos t}\, d(\cos t) = -\ln|\cos t| + C$$

$$\int \cos\frac{x-2}{2}\, dx = \int \cos\frac{x-2}{2}\, d(x-2) = 2\int \cos\frac{x-2}{2}\, d\,\frac{x-2}{2} =$$

$$2\sin\frac{x-2}{2} + C$$

Example 7 from Section 8.3 (Textbook)

$N(t) = 5000 + 2000\cos\left(\frac{2\pi t}{36}\right)$, evaluate

$$\int_0^{144} N(t)\, dt = \int_0^{144} 5000 + 2000\cos\left(\frac{\pi t}{18}\right) dt$$

$$= 1000\left[\int_0^{144} 5 + 2\cos\left(\frac{\pi t}{18}\right) dt\right]$$

$$= 1000 \cdot \frac{18}{\pi}\left[\int_0^{144} 5 + 2\cos\left(\frac{\pi t}{18}\right) d\left(\frac{\pi t}{18}\right)\right]$$

$$= \frac{18000}{\pi}\left[\int_0^{144\cdot\frac{\pi}{18}} 5 + 2\cos(u)\, du\right]$$

$$= \frac{18000}{\pi}\int_0^{8\pi}\left(5 + 2\cos u\right) du$$

$$= \frac{18000}{\pi}\left(\int_0^{8\pi} 5\, du + 2\int_0^{8\pi}\cos u\, du\right)$$

$$= \frac{18000}{\pi}\left(5u\Big|_0^{8\pi} + 2\sin u\Big|_0^{8\pi}\right)$$

$$= \frac{18000}{\pi}\left[(40\pi - 0) + 2(0-0)\right]$$

$$= \frac{18000}{\pi}\cdot 40\pi = 72000$$

$\sin(2t+1) = -\frac{\sqrt{3}}{2}$, $t = ?$

$2t+1 = -\frac{\pi}{3} + k\cdot 2\pi$ or $-\frac{2\pi}{3} + k\cdot 2\pi$

$\Rightarrow 2t = -\frac{\pi}{3} - 1 + k\cdot 2\pi$ or $-\frac{2\pi}{3} - 1 + k\cdot 2\pi$ $\qquad k \in \mathbb{Z}$

$\Rightarrow t = -\frac{\pi}{6} - \frac{1}{2} + k\cdot\pi$ or $-\frac{\pi}{3} - \frac{1}{2} + k\cdot\pi$

Review of basic laws of differentiation and integration

$(x^n)' = nx^{n-1}$ (including constants) $\quad (\ln x)' = \frac{1}{x} = x^{-1}$

$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C & (n \neq -1) \\ \ln|x| + C & (n = -1) \end{cases}$$

C is an arbitrary constant

$(e^x)' = e^x$, $\int e^x dx = e^x + C$

$(\sin x)' = \cos x$, $\int \cos x\,dx = \sin x + C$

$(\cos x)' = -\sin x$, $\int \sin x\,dx = -\cos x + C$

$$\frac{df}{dx} = f'(x)$$
$$\Updownarrow$$
$$df = f'(x)\,dx$$
$$\int f'(x)\,dx = \int df = f + C$$

differentiation and integration respect linear combination

examples: $(2\cos x - e^x)' = [2\cos x + (-1)e^x]' = 2(\cos x)' + (-1)(e^x)'$

$= 2(-\sin x) + (-1)e^x = -2\sin x - e^x$

$\int \sin x - \frac{1}{x}\,dx = \int \sin x\,dx - \int \frac{1}{x}\,dx$

$= -\cos x - \ln|x| + C$

$$\frac{d(c_1 f_1 + c_2 f_2)}{dx} = c_1\frac{df_1}{dx} + c_2\frac{df_2}{dx}$$
$$\Downarrow$$
$$d(c_1 f_1 + c_2 f_2) = c_1 df_1 + c_2 df_2$$

product rule for differentiation

$(\sin x\cos x)' = (\sin x)'\cos x + \sin x(\cos x)'$

$= \cos x\cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x = \cos 2x$

$$\frac{d(f_1 f_2)}{dx} = \frac{df_1}{dx} f_2 + f_1\frac{df_2}{dx}$$
$$\Downarrow$$
$$d(f_1 f_2) = f_2\,df_1 + f_1\,df_2$$

$(e^x\sin x)' = (e^x)'\sin x + e^x(\sin x)' = e^x\sin x + e^x\cos x$

composition law for differentiation

$[f(g(x))]' = f'(g(x))\,g'(x)$

$= f'(g)\,g'(x)$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

$df = \frac{df}{dg}dg = \frac{df}{dg}\frac{dg}{dx}dx = f'(g)g'(x)dx$

$$(e^{2x})' \xlongequal{y=2x} (e^y)' = e^y \cdot y' = e^{2x}(2x)' = 2e^{2x}$$

$$(\cos x)' = \left(\sin\left(\frac{\pi}{2}-x\right)\right)' \xlongequal{y=\frac{\pi}{2}-x} (\sin y)' = \cos y \cdot y'$$

$$= \cos\left(\frac{\pi}{2}-x\right)\left(\frac{\pi}{2}-x\right)'$$

$$= (0\cdot\cos x + 1\cdot\sin x)(-1) = -\sin x$$

$$(e^{\sin x})' \xlongequal{y=\sin x} (e^y)' = e^y\cdot y' = e^{\sin x}(\sin x)' = e^{\sin x}\cos x$$

$$\left[\sin(e^{x^2})\right]' \xlongequal{y=e^{x^2}} (\sin y)' = \cos y\cdot y' = \cos(e^{x^2})\cdot(e^{x^2})' \xlongequal{u=x^2}$$

$$\cos(e^{x^2})\cdot(e^u)' = \cos(e^{x^2})e^u\cdot u' = \cos(e^{x^2})\cdot e^{x^2}\cdot(x^2)'$$

$$= \cos(e^{x^2})\cdot e^{x^2}\cdot 2x = 2xe^{x^2}\cos(e^{x^2})$$

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{du}\frac{du}{dx}$$

quotient rule for differentiation

$$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x)\cdot\frac{1}{g(x)}\right]' = f'(x)\cdot\frac{1}{g(x)} + f'(x)\left[\frac{1}{g(x)}\right]$$

$$= \frac{f'(x)}{g(x)} + f'(x)\left[-\frac{g'(x)}{g(x)^2}\right.$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

example $(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= 1 + \tan^2 x$$

$$d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$$

7 Lecture 09/06

Exercise day:

Some review: $\frac{r}{2\pi} = \frac{d}{360}$ $\nearrow d = r\cdot\frac{180}{\pi}$, $\searrow r = d\cdot\frac{\pi}{180}$ $\quad 42^\circ = \quad -\frac{2\pi}{5} =$

$P_t = (\cos t, \sin t)$ is the terminal point

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad \cos(-t) = \cos t \quad \sin(-t) = -\sin t$

$\tan\left(\frac{5\pi}{12}\right) = \tan(30^\circ + 45^\circ) = \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

define $\sec t = \frac{1}{\cos t}$, $\csc t = \frac{1}{\sin t}$, notice the following formula

$$1 + \tan^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

$\sec(\pi t + 2) = 2 \Rightarrow t = ?$

$(af+bg)' = af' + bg'$, $(fg)' = f'g + fg'$, $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, $[f(g)]' = f'(g)g'$

$(\tan t)' = \sec^2 t$, $\int_0^{\frac{\pi}{4}} \sec^2 t\, dt = \tan t\Big|_0^{\frac{\pi}{4}} = 1$

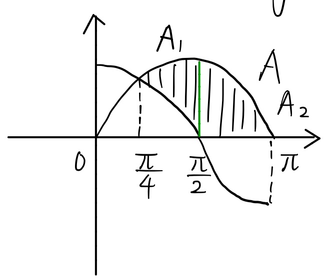
$$\left(e^{3x^2}\sin(2x)\right)' = 6xe^{3x^2}\sin 2x + 2e^{3x^2}\cos 2x = e^{3x^2}(6x\sin 2x + 2\cos 2x)$$

$$\left(\sqrt[3]{x^2}\right)' = \left((x^2)^{\frac{1}{3}}\right)' = \left(x^{\frac{2}{3}}\right)' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$\int (2 + \tan^2 x)\,dx = x + \tan x + C$$

$$\left(\sin^4 e^{3x}\right)' = 4\sin^3 e^{3x}\cdot\cos e^{3x}\cdot 3e^{3x}$$

The Area of the following



$$A = A_1 + A_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)\,dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$$

$$= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \left[-\cos x\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \left[(-0-1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right] + \left[(-(-1)) - (-0)\right]$$

$$= (-1 + \sqrt{2}) + 1 = \sqrt{2}$$

$\int \sin^2 x \cos x\, dx$, $\int \cos^3 x\, dx$, $\int \frac{1}{t\ln t}\, dt$,

8 Lecture 09/09

Integration by Substitution: $\int \sin^2 x \cos x \, dx$, $\int \cos^3 x \, dx$, $\int \frac{1}{t \ln t} dt$,
$\int (x^2+1) x \, dx$, $\int 2x e^{x^2} dx$, $\int 3x^2 \sqrt{x^3+1} \, dx$, $\int \frac{2x}{x^2+1}$, $\int \frac{2-x}{\sqrt{2x^2-8x+1}}$
$\int \frac{e^{\frac{3}{x}}}{x^2} dx$, $\int (2x-1)^7 dx$, $\int \frac{1}{\sqrt{2x+1}} dx$, $\int \frac{8x}{e^{x^2}}$
$\int \frac{(1+\ln x)^3}{x}$, $\int \frac{1}{x \ln x^2} dx$, $\int \frac{x^2-2x}{x^3-3x^2+1}$, $\int \frac{e^{-x}}{1-e^{-x}} dx$, $\int \frac{dx}{1+e^x}$

$\int \frac{e^{2x}-1}{e^{2x}+1}$, $\int \tan x \sec^2 x \, dx$, $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx$

$$\int u\,dv = \int [d(uv) - v\,du] = uv - \int v\,du$$

$$\int u(x) v'(x) \, dx = u(x) v(x) - \int v(x) u'(x) \, dx$$

$\int x e^x dx$, $\int \ln x \, dx$, $\int \frac{\ln(\ln x)}{x} dx$

$$\int \frac{dt}{(t+A)(t+B)}$$

9 Lecture 09/11

① $\int (1-2x)^5 dx = \frac{1}{-2}\int (1-2x)^5 d(-2x) = -\frac{1}{2}\int (1-2x)^5 d(1-2x)$

$\xlongequal{u=1-2x} -\frac{1}{2}\int u^5 du = -\frac{1}{2}\cdot\frac{1}{6}u^6 + C = -\frac{1}{12}u^6 + C$

$= -\frac{1}{12}(1-2x)^6 + C$

② $\int \frac{e^{\frac{3}{x}}}{x^2} dx = -\int e^{\frac{3}{x}}\cdot\left(-\frac{1}{x^2}dx\right) = -\int e^{\frac{3}{x}} d\left(\frac{1}{x}\right)$

$= -\frac{1}{3}\int e^{\frac{3}{x}} d\left(\frac{3}{x}\right) \xlongequal{u=\frac{3}{x}} -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{\frac{3}{x}} + C$

③ $\int \frac{8x}{e^{x^2}} dx = 4\int \frac{2x dx}{e^{x^2}} = 4\int \frac{1}{e^{x^2}} dx^2 = 4\int e^{-x^2} dx^2 = -4\int e^{-x^2} d(-x^2)$

$\xlongequal{u=-x^2} -4\int e^u du = -4e^u + C = -4e^{-x^2} + C$

④ $\int \frac{1}{x\ln x^2} dx = \int \frac{1}{\ln x^2} d\ln x = \int \frac{1}{2\ln x} d\ln x$

$\xlongequal{u=\ln x} \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln u = \frac{1}{2}\ln(\ln x)$

⑤ $\int \frac{\ln\sqrt{x}}{x} dx = \int \ln x^{\frac{1}{2}} d\ln x = \frac{1}{2}\int \ln x \, d\ln x \xlongequal{u=\ln x} \frac{1}{2}\int u\, du$

$= \frac{1}{2}\cdot\frac{1}{2}u^2 + C = \frac{1}{4}\ln^2 x + C$

⑥ $\int \frac{e^x}{1+e^x} dx = \int \frac{1}{1+e^x} de^x \xlongequal{u=e^x} \int \frac{1}{1+u} du = \int \frac{1}{1+u} d(u+1)$

$\xlongequal{v=u+1} \int \frac{1}{v} dv = \ln v + C = \ln(u+1) + C = \ln(e^x+1) + C$

⑦ $\int \frac{e^{-x}}{1+e^{-x}} dx = -\int \frac{e^{-x}}{1+e^{-x}} d(-x) \xlongequal{y=-x} -\int \frac{e^y}{1+e^y} dy \xlongequal{\text{by ⑥}}$

$-\ln(e^y+1) + C = -\ln(e^{-x}+1) + C$

⑧ $\int \frac{1}{1+e^x} dx = \int \frac{\frac{1}{e^x}}{\frac{1+e^x}{e^x}} dx = \int \frac{e^{-x}}{1+e^{-x}} dx \xlongequal{\text{by ⑦}} -\ln(e^{-x}+1) + C$

Or $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx = \int\left(\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right)dx$

$= \int\left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx \underset{=\!=}{\overset{\text{by } ⑥}{=\!=\!=}}$

$x - \ln(e^x+1) + C$

⑨ $\int \frac{e^x - e^{-x}}{e^x+e^{-x}}dx$, notice $d(e^x+e^{-x}) = de^x + de^{-x}$

$= e^x dx + (-e^{-x})dx = e^x dx - e^{-x}dx = (e^x - e^{-x})dx$

hence $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \int \frac{1}{e^x+e^{-x}} d(e^x - e^{-x}) \underset{=\!=}{\overset{u=e^x+e^{-x}}{=\!=\!=}}$

$\int \frac{1}{u}du = \ln u + C = \ln(e^x+e^{-x}) + C$

⑩ $\int \frac{\sin x - \cos x}{\sin x + \cos x}dx$, notice $d(\sin x + \cos x) = d\sin x + d\cos x$

$= \cos x\,dx + (-\sin x)dx = (\cos x - \sin x)dx$

hence $\int \frac{\sin x - \cos x}{\sin x + \cos x}dx = -\int \frac{1}{\sin x + \cos x} d(\sin x + \cos x)$

$\underset{=\!=}{\overset{u=\sin x+\cos x}{=\!=\!=}} -\int \frac{1}{u}du = -\ln u + C = -\ln(\sin x + \cos x) + C$

10 Lecture 09/13

Integration by parts

Since $d(fg) = fdg + gdf$, $\quad fdg = d(fg) - gdf$

$\int \ln x\, dx = \int \left(d(x\ln x) - x\, d\ln x\right) = \int d(x\ln x) - \int x\, d\ln x$

$= x\ln x - \int x \cdot \frac{1}{x} dx = x\ln x - \int dx$

$= x\ln x - x + C$

$\int x\cos x\, dx = \int x\, d\sin x = \int \left(d(x\sin x) - \sin x\, dx\right)$

$= x\sin x - \int \sin x\, dx = x\sin x + \cos x + C$

$\int x e^x dx = \int x\, de^x = \int \left(d(xe^x) - e^x dx\right) = xe^x - \int e^x dx$

$= xe^x - e^x + C$

$\int x^2 e^x dx = \int x^2 de^x = \int \left(d(x^2e^x) - e^x dx^2\right) = x^2e^x - \int e^x dx^2$

$= x^2e^x - \int 2xe^x dx = x^2e^x - 2(xe^x - e^x) + C$

$= x^2e^x - 2xe^x + 2e^x + C$

$\int x^3 e^{x^2} dx = \frac{1}{2}\int x^2 e^{x^2} 2x\, dx = \frac{1}{2}\int x^2 e^{x^2} dx^2 \overset{u=x^2}{=\!=\!=} \frac{1}{2}\int ue^u du$

$= \frac{1}{2}(ue^u - e^u) + C = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + C$

$\int x\sin(2x)dx = \frac{1}{4}\int (2x)\sin(2x)d(2x) \overset{u=2x}{=\!=\!=} \frac{1}{4}\int u\sin u\, du$

$= -\frac{1}{4}\int u(-\sin u\, du) = -\frac{1}{4}\int u\, d\cos u = -\frac{1}{4}\int \left(d(u\cos u - \cos u\, du)\right)$

$= -\frac{1}{4}\left(u\cos u - \int \cos u\, du\right) = -\frac{1}{4}(u\cos u - \sin u) + C$

$= -\frac{1}{4}\left(2x\cos(2x) - \sin(2x)\right) + C = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x)$

$\int x\sqrt{2-x}\, dx$, let $u = \sqrt{2-x}$, then $u^2 = 2-x \Rightarrow x = 2-u^2$, hence

$\int (2-u^2)u\, d(2-u^2) = \int (2u-u^3)(-2u\, du) = \int (2u^4 - 4u^2)du$

$= \frac{2}{5}u^5 - \frac{4}{3}u^3 + C = \frac{2}{5}(2-x)^{\frac{5}{2}} - \frac{4}{3}(2-x)^{\frac{3}{2}} + C$

11 Lecture 09/16

$\int f'(x)dx = f(x) + C$ indefinite integral

$\int_a^b f'(x)dx = f(b) - f(a)$ definite integral (Fundamental theorem of Calculus)

Example: $\int x^2 dx = \frac{1}{3}x^3 + C$ indefinite integral

$$\int_1^2 x^2 dx = \left(\frac{1}{3}x^3\right)\Big|_1^2 = \left(\frac{1}{3}2^3 - \frac{1}{3}1^3\right) = \frac{1}{3}(2^3 - 1^3) = \frac{1}{3}(8-1) = \frac{7}{3}$$

$$\int_1^3 x^2 e^{x^3} dx = \frac{1}{3}\int_1^3 e^{x^3}(3x^2 dx) = \frac{1}{3}\int_1^3 e^{x^3} dx^3 \overset{u=x^3}{=\!=\!=} \frac{1}{3}\int_{1^3}^{3^3} e^u du$$

$$= \frac{1}{3}\int_1^{27} e^u du = \frac{1}{3}(e^{27} - e^1)$$

$$\int_0^\pi \sin x\, dx = (-\cos x)\Big|_0^\pi = (-\cos(\pi) - (-\cos(0))) = (-(-1) - (-1)) = 1 + 1 = 2$$

$$\int_2^6 \frac{1}{\sqrt{4x+1}} dx = \frac{1}{4}\int_2^6 \frac{1}{\sqrt{4x+1}} d4x = \frac{1}{4}\int_2^6 \frac{1}{\sqrt{4x+1}} d(4x+1) \overset{u=4x+1}{=\!=\!=} \frac{1}{4}\int_{4\times 2+1}^{4\times 6+1} \frac{1}{\sqrt{u}} du$$

$$= \frac{1}{4}\int_9^{25} u^{-\frac{1}{2}} du = \frac{1}{2}\int_9^{25} \frac{1}{2}u^{-\frac{1}{2}} du = \frac{1}{2} u^{\frac{1}{2}}\Big|_9^{25} = \frac{1}{2}(\sqrt{25} - \sqrt{9}) = \frac{1}{2}(5-3) = 1$$

$$\int_0^\pi e^{\sin x}\cos x\, dx = \int_0^\pi e^{\sin x} d\sin x \overset{u=\sin x}{=\!=\!=} \int_{\sin(0)}^{\sin(\pi)} e^u du = \int_0^0 e^u du = e^u\Big|_0^0$$

$$= e^0 - e^0 = 0$$

$$\int_0^1 \frac{x}{x^2+3} dx = \frac{1}{2}\int_0^1 \frac{1}{x^2+3}(2x dx) = \frac{1}{2}\int_0^1 \frac{1}{x^2+3} dx^2 = \frac{1}{2}\int_0^1 \frac{1}{x^2+3} d(x^2+3)$$

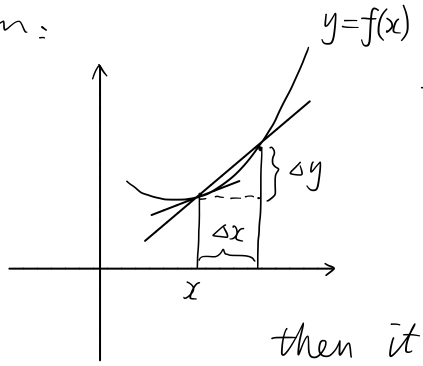
$$\overset{u=x^2+3}{=\!=\!=} \frac{1}{2}\int_{0^2+3}^{1^2+3} \frac{1}{u} du = \frac{1}{2}\int_3^4 \frac{1}{u} du = \frac{1}{2}\ln u\Big|_3^4 = \frac{1}{2}(\ln 4 - \ln 3) = \frac{1}{2}\ln\frac{4}{3}$$

12 Lecture 09/18

Geometric and physical intepretation of differentiation and integration

Differentiation:

Geometric:



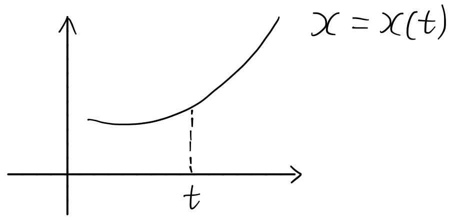
$\frac{\Delta y}{\Delta x}$ is the slope of secant line

as Δx gets smaller and smaller

being infinitesimally small, i.e. dx,

then it becomes $\frac{dy}{dx}$ which is the slope of the tangent line (which is also $f'(x)$)

physical: let t be time, x be the distance from the origin



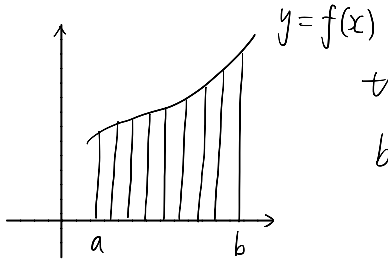
$\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$ is the velocity at time t

$$\frac{dx}{dt} = x'(t) = v(t)$$

$v'(t) = a(t)$ is acceleration

Integration:

Geometric:



the area under the graph of f between a and b can be approximated by Riemann sum

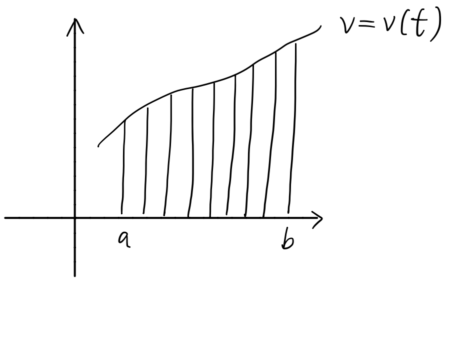
$f(x_1)\Delta x + \cdots + f(x_n)\Delta x$, dividing interval (a,b) evenly into n equal parts with length Δx, and x_i is inside the i-th subinterval

as Δx goes to infinitesimally small, i.e. dx, the sum $S f(x_i)\Delta x$ (S stands for sum) gets more accurate, hence the area is

$\int_a^b f(x)\,dx$ (the integral sign $\int$ is a modified version of S, d and $\int$ are inventions of Leibniz), and this can be calculated using the fundamental theorem of Calculus $\int_a^b f(x)\,dx = F(b) - F(a)$

F being any antiderivative of f

Physical: t be time, v be velocity



$dx = x'dt = v\,dt$ is the infinitesimally small change of distance (can be negtive) since infinitesimally, it is a constant speed v

If you sum all these infinitesimally small changes in distance, accumulatively you get the change of distane between time a and b, in other words:

$$\int_a^b v\,dt = \int_a^b x'\,dt = \int_a^b dx(t) = \int_{x(a)}^{x(b)} dx$$

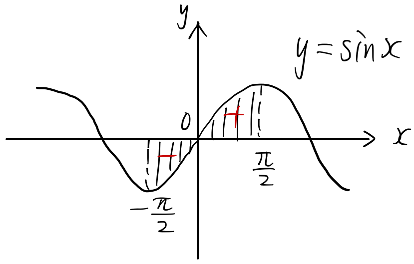
$$\int_a^b dx(t) = x(t)\Big|_a^b = x(b) - x(a)$$

$$\int_{x(a)}^{x(b)} dx = x\Big|_{x(a)}^{x(b)} = x(b) - x(a)$$

Remark: It can be considered as a physical proof of fundamental theorem of Calculus

Be aware: definite integrals are "algebraic", it has signs and not necessarily positive, for example, when $v(t)$ is negative, it will be going backwards, or if the graph is under the x-axis, the integral becomes negative of the actual area, since the actual area should be $\int_a^b |f(x)|\,dx$, $|f(x)|$ is the magnitude of $f(x)$

Example: the shaded area is



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx = -\int_{-\frac{\pi}{2}}^{0} \sin x\,dx + \int_0^{\frac{\pi}{2}} \sin x\,dx$$

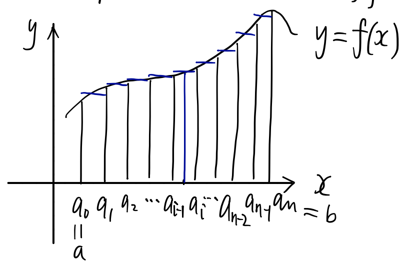
$$= -\left[-\cos x\right]\Big|_{-\frac{\pi}{2}}^{0} + \left[-\cos x\right]\Big|_0^{\frac{\pi}{2}} = 1 + 1 = 2$$

Approximations of definite integrals

Motivation: the integrand f of $\int_a^b f(x)\,dx$ in real life is not possible to find an elementary expression of its antiderivative, hence it is in general not practical to use fundamental theorem of Calculus, therefore it goes back to the Riemann sum

So now the question is what is a good choice for the approximation of each piece, we introduce three approximations

① midpoint rule: approximate each piece by a rectangle



divide interval evenly into n equal parts of width $\Delta x = \frac{a_n - a_0}{n}$

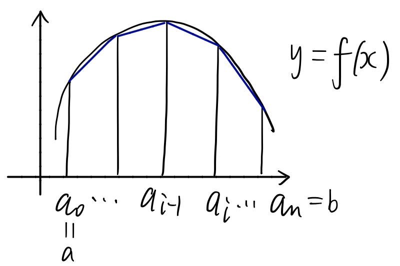
$x_i = \frac{a_{i-1}+a_i}{2}$ be the midpoint between a_{i-1} and a_i (of the i-th interval (a_{i-1}, a_i))

the approximation for the i-th piece is $f(x_i)\Delta x$

thus the approximation for the definite integral $\int_a^b f(x)\,dx$ is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \Delta x\big(f(x_1) + \cdots + f(x_n)\big)$$

② trapezoidal rule: approximate each piece by a trapezoid

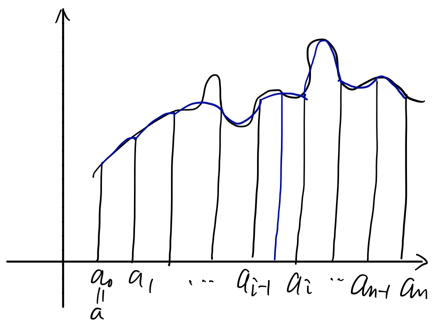


the approximation for the i-th piece is $\frac{(f(a_{i-1})+f(a_i))\Delta x}{2}$, thus

$$\int_a^b f(x)\,dx \approx \frac{(f(a_0)+f(a_1))\Delta x}{2} + \frac{(f(a_1)+f(a_2))\Delta x}{2} + \cdots + \frac{(f(a_{n-1})+f(a_n))\Delta x}{2}$$

$$= \frac{\Delta x}{2}\Big[f(a_0) + 2f(a_1) + \cdots + 2f(a_{n-1}) + f(a_n)\Big]$$

③ Simpson's Rule: approximate by pieces with parabola boundaries



the approximation for the i-th piece is

$\frac{1}{6}\big[f(a_{i-1}) + 4f(x_i) + f(a_i)\big]\Delta x$, $x_i = \frac{a_{i-1}+a_i}{2}$

[to be proved in the lemma]

hence $\int_a^b f(x)\,dx \approx \frac{\Delta x}{6}\left[f(a_0)+4f(x_0)+f(a_1)\right] + \cdots + \frac{\Delta x}{6}\left[f(a_{n-1})+4f(x_n)+f(a_n)\right]$

$$= \frac{\Delta x}{6}\left[f(a_0)+4f(x_0)+2f(a_1)+\cdots+2f(a_{n-1})+4f(x_n)+f(a_n)\right]$$

Lemma: Let the unique parabola passing through $(a_0, f(a_0))$, $(x_1, f(x_1))$ and $(a_1, f(a_1))$ be $\alpha x^2+\beta x+\gamma$

where $x_1 = \frac{a_0+a_1}{2}$ is the midpoint

the area of this piece with this parabola as boundary is

$$\int_{a_0}^{a_1} (\alpha x^2+\beta x+\gamma)\,dx = \left[\frac{\alpha}{3}x^3+\frac{\beta}{2}x^2+\gamma x\right]\Big|_{a_0}^{a_1}$$

$$= \left[\left(\frac{\alpha}{3}a_1^3+\frac{\beta}{2}a_1^2+\gamma a_1\right)-\left(\frac{\alpha}{3}a_0^3+\frac{\beta}{2}a_0^2+\gamma a_0\right)\right]$$

$$= \left[\frac{\alpha}{3}(a_1^3-a_0^3)+\frac{\beta}{2}(a_1^2-a_0^2)+\gamma(a_1-a_0)\right]$$

$$= \left[\frac{\alpha}{3}(a_1-a_0)(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1-a_0)(a_1+a_0)+\gamma(a_1-a_0)\right]$$

$$= (a_1-a_0)\left[\frac{\alpha}{3}(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1+a_0)+\gamma\right]$$

notice $a_1-a_0=\Delta x$

On the other hand,

$$\frac{1}{6}\left[f(a_0)+4f(x_1)+f(a_1)\right] = \frac{1}{6}\left[(\alpha a_0^2+\beta a_0+\gamma)+4(\alpha x_1^2+\beta x_1+\gamma)+(\alpha a_1^2+\beta a_1+\gamma)\right]$$

$$= \frac{1}{6}\left[(\alpha a_0^2+\beta a_0+\gamma)+4\left(\alpha\left(\frac{a_0+a_1}{2}\right)^2+\beta\frac{a_0+a_1}{2}+\gamma\right)+(\alpha a_1^2+\beta a_1+\gamma)\right]$$

$$= \frac{1}{6}\left[\alpha a_0^2+\beta a_0+\gamma+4\left(\frac{\alpha}{4}(a_0^2+2a_0a_1+a_1^2)+\frac{\beta}{2}(a_0+a_1)+\gamma\right)+\alpha a_1^2+\beta a_1+\gamma\right]$$

$$= \frac{1}{6}\left[2\alpha(a_0^2+a_1^2)+2\alpha a_0a_1+3\beta(a_1+a_0)+6\gamma\right]$$

$$= \frac{\alpha}{3}(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1+a_0)+\gamma$$

QED

13 Lecture 09/20

Section 9.5: Some Applications of the integral

Present value of an income stream

If you have 100 dollars, its value will decrease in 10 years, namely, you can't buy as much as you can now, another way to put it is that A dollars you make in t years only has present value $P = Ae^{-rt}$ (this expression is coming from Section 5.2)

Now suppose the annual rate of income is $K(t)$

hence in a infinitesimally small amount of time dt at time t, you can make $K(t)dt$, and its present value is $e^{-rt}K(t)dt$

hence the present value of the stream of income between $t=0$ to $t=T$, is to integrate ("sum") all these infinitesimally amount of present value which is $\int_0^T K(t)e^{-rt}dt$

Example: If r (interest rate) is $50\% = 0.5$, and $K(t) = 1+t$ (you can make money faster and faster), $T=10$, then

$$\int_0^{10}(1+t)e^{-0.5t}dt = \frac{1}{-0.5}\int_0^{10}(1+t)e^{-0.5t}(-0.5dt)$$

$$= -2\int_0^{10}(1+t)e^{-0.5t}d(-0.5t) \overset{u=-0.5t}{=\!=\!=} -2\int_{-0.5\times 0}^{-0.5\times 10}(1+(-2u))e^u du$$

$$= -2\int_0^{-5}(1-2u)e^u du = -2\int_0^{-5}(1-2u)de^u$$

$$= -2\left[\left((1-2u)e^u\right)\Big|_0^{-5} - \int_0^{-5}e^u d(1-2u)\right]$$

$$= -2\left[\left((1-2\times(-5))e^{-5}\right) - \left((1-2\times 0)e^0\right) - \int_0^{-5}e^u\cdot(-2)du\right]$$

$$= -2\left[11e^{-5} - 1 + 2\int_0^{-5}e^u du\right]$$

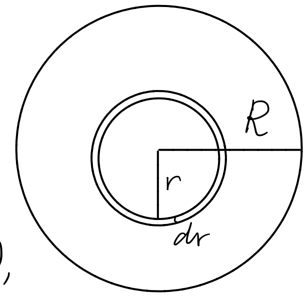
$$= -2\left[11e^{-5} - 1 + 2e^u\Big|_0^{-5}\right]$$

$$= -2\left[11e^{-5} - 1 + 2(e^{-5} - e^0)\right]$$

$$= -2\left[11e^{-5} - 1 + 2e^{-5} - 2\right]$$

$$= -2(13e^{-5} - 3) = 6 - 26e^{-5}$$

Demographic Model



In a disk shaped city, on each infinesimally narrow (width dr) circular band with radius r has population density $D(r)$, the whole disk is of radius R, then the infinitesimally amount of population on this circular band is

$$D(r)\cdot \boxed{\underbrace{2\pi r}_{\text{circumference}} \cdot \underbrace{dr}_{\text{width}}}^{\text{area}} = 2\pi r D(r)\, dr$$

then the total population would be to integrate ("sum") all these infinitesimally amount of people

$$\int_0^R 2\pi r D(r)\, dr = 2\pi \int_0^R r D(r)\, dr$$

Example: $D(r) = \frac{1}{1+r}$ (less people away from the city center), $R = 2$ (say miles), then total population is

$$2\pi \int_0^2 \frac{r}{1+r}\, dr = 2\pi \int_0^2 \frac{r}{r+1}\, d(r+1) \overset{u=r+1}{=\!=\!=} 2\pi \int_{0+1}^{2+1} \frac{u-1}{u}\, du$$

$$= 2\pi \int_1^3 \left(1 - \frac{1}{u}\right) du = 2\pi \left[u - \ln u\right]\Big|_1^3$$

$$= 2\pi \left[(3 - \ln 3) - (1 - \ln 1)\right] = 2\pi(2 - \ln 3)$$

Example with Simpon's rule: say $n = 4$, want to approximate $\int_4^6 x^2 dx$, now check the previous definition you will see $f(x) = x^2$,

$a_0 = 4$, $a_n = a_4 = 6$, $\Delta x = \frac{a_n - a_0}{n} = \frac{6-4}{4} = \frac{2}{4} = 0.5$

$a_1 = a_0 + \Delta x = 4 + 0.5 = 4.5$, $a_2 = a_1 + \Delta x = 5$, $a_3 = a_2 + \Delta x = 5.5$

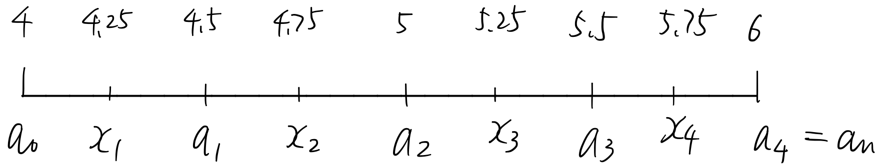
$x_1 = \frac{a_0 + a_1}{2} = \frac{4 + 4.5}{2} = 4.25$, $x_2 = \frac{a_1 + a_2}{2} = 4.75$, $x_3 = \frac{a_2 + a_3}{2} = 5.25$, $x_4 = \frac{a_3 + a_4}{2} = 5.75$

generally $x_i = \frac{a_{i-1} + a_i}{2}$ is the midpoint between points a_{i-1} and a_i

then $\int_4^6 x^2 dx \approx \frac{\Delta x}{6}\big[f(a_0) + 4f(x_1) + 2f(a_1) + 4f(x_2) + 2f(a_2) + 4f(x_3) + 2f(a_3) +$
$4f(x_4) + f(a_4)\big]$

$$= \frac{0.5}{6}\left[4^2 + 4\times 4.25^2 + 2\times 4.5^2 + 4\times 4.75^2 + 2\times 5^2 + 4\times 5.25^2 + 2\times 5.5^2 + 4\times 5.75^2 + 6^2\right]$$

$$= \cdots$$



14 Lecture 09/23

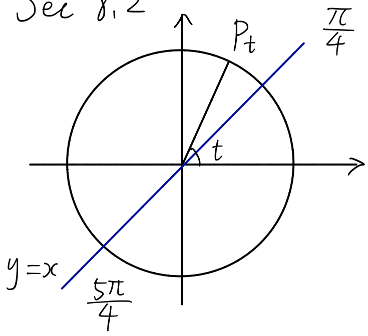
Review:

Sec 8.1 Conversion between radians and degrees

Examples: $\frac{7\pi}{12}$ rad $= \frac{7\pi}{12} \times \frac{180}{\pi} = 105^\circ$

$2^\circ = 2 \times \frac{\pi}{180} = \frac{\pi}{90}$ rad

Sec 8.2



$P_t = (\cos t, \sin t)$, $\tan t = \frac{\sin t}{\cos t}$ = slope of P_t

If $\sin t = \cos t$, what is t?

P_t will be on the intersection of the unit circle and the line $y = x$

thus $t = \frac{\pi}{4} + k2\pi$ or $\frac{5\pi}{4} + k2\pi$ (or rather $\frac{\pi}{4} + k\pi$), $k \in \mathbb{Z}$

Sec 8.3

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\qquad$ $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ $\qquad$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ $\qquad$ $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\sin(-x) = -\sin x$ $\qquad$ $\cos(-x) = -\cos x$ $\qquad$ $\tan(-x) = -\tan x$

Example: $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) = \sin\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$

$$\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

and $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

Sec 8.4 $\sec t = \frac{1}{\cos t}$, $(\tan t)' = \sec^2 t$, $\cos^2 t + \sin^2 t = 1$, $1 + \tan^2 t = \sec^2 t$

Chapter 9

Foremost: $(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x}$, $(\sin x)' = \cos x$

$(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

Note $\left(\frac{1}{x^n}\right)' = (x^{-n})' = -n x^{-n-1}$ and

$\left(\sqrt[n]{x^m}\right)' = \left(x^{\frac{m}{n}}\right)' = \frac{m}{n} x^{\frac{m}{n} - 1}$

$(te^t)' = t'e^t + t(e^t)' = e^t + te^t$

$(\sin(e^{t^2}))' \xlongequal{u=e^{t^2}} (\sin u)' = \cos u \cdot u' = \cos(e^{t^2})(e^{t^2})' \xlongequal{v=t^2} \cos(e^{t^2})(e^v)'$

$= \cos(e^{t^2})e^v v' = \cos(e^{t^2})e^{t^2}(t^2)' = \cos(e^{t^2})e^{t^2}2t = 2te^{t^2}\cos(e^{t^2})$

$\int \frac{\ln(2x)}{x}dx = \int \frac{\ln(2x)}{2x}d(2x) \xlongequal{u=2x} \int \frac{\ln u}{u}du = \int \ln u \frac{du}{u} = \int \ln u \, d\ln u$

$\xlongequal{v=\ln u} \int v dv = \frac{1}{2}v^2 + C = \frac{1}{2}\ln^2 u + C = \frac{1}{2}\ln^2(2x) + C$

$\int_0^1 2x^3 e^{x^2}dx = \int_0^1 x^2 e^{x^2} \cdot 2x dx = \int_0^1 x^2 e^{x^2} dx^2 \xlongequal{u=x^2} \int_{0^2}^{1^2} ue^u du$

$= \int_0^1 ue^u du = \int_0^1 u de^u = ue^u\Big|_0^1 - \int_0^1 e^u du = [1 \cdot e^1 - 0] - e^u\Big|_0^1$

$= e - (e^1 - e^0) = e - (e-1) = 1$

$\int te^{t^2}\cos(e^{t^2})dt = \frac{1}{2}\int e^{t^2}\cos(e^{t^2}) \cdot 2tdt = \frac{1}{2}\int e^{t^2}\cos(e^{t^2})dt^2 \xlongequal{v=t^2}$

$\frac{1}{2}\int e^v\cos(e^v)dv = \frac{1}{2}\int \cos(e^v) \cdot e^v dv = \frac{1}{2}\int \cos(e^v)de^v \xlongequal{u=e^v} \frac{1}{2}\int \cos u du$

$= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(e^v) + C = \frac{1}{2}\sin(e^{t^2}) + C$

Some other examples:

Given $\tan t = \frac{1}{2}$, what is $\frac{1}{\sin^2 t}$?

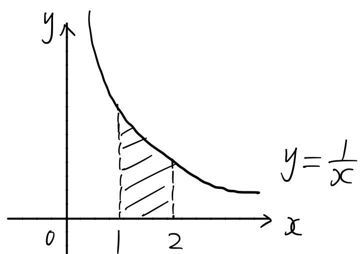
$\frac{\sin t}{\cos t} = \tan t \Rightarrow \sin t = \tan t \cos t = \frac{1}{2}\cos t \Rightarrow \sin^2 t = (\frac{1}{2}\cos t)^2 = \frac{1}{4}\cos^2 t$

also notice $\frac{1}{\cos^2 t} = \sec^2 t = 1 + \tan^2 t = 1 + (\frac{1}{2})^2 = \frac{5}{4} \Rightarrow \cos^2 t = \frac{4}{5}$

thus $\sin^2 t = \frac{1}{5}$, $\frac{1}{\sin^2 t} = \frac{1}{5}$ Or $\frac{1}{\sin^2 t} = \frac{\sin^2 t + \cos^2 t}{\sin^2 t} = 1 + \frac{\cos^2 t}{\sin^2 t}$

$= 1 + \frac{1}{\frac{\sin^2 t}{\cos^2 t}} = 1 + \frac{1}{\tan^2 t} = 1 + \frac{1}{(\frac{1}{2})^2} = 1 + \frac{1}{\frac{1}{4}} = 1 + 4 = 5$

Computes the shaded area



By the definition of definite integral, we have

$\text{Area} = \int_1^2 \frac{1}{x}dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2$

15 Lecture 09/30

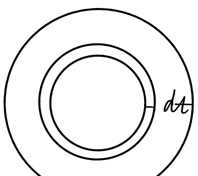
7th problem on Ch 09.1–5 Review test

A volcano erupts and spreads lava in all directions, the density of the deposits at a distance t kilometers from the center is $D(t)$ thousand tons per square kilometers and is determined by the following formula

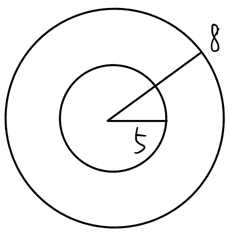
$$D(t) = 11\,(t^2+14)^{-2}$$

Find the tonnage of lava deposited between the distances of 5 and 8 kilometers from the center

Solution: consider the tonnage of lava deposited in an infinitesimally small circular band which should be



$$D(t)\cdot 2\pi t\,dt \quad (\text{density} \times \text{circumference} \times \text{width})$$
$$\|$$
$$11\,(t^2+14)^{-2}\cdot 2\pi t\,dt$$



Hence the tonnage of lava deposited between the distance of 5 and 8 kilometers from the center is to "sum" (integrate) over all these lava deposited in each infinitesimally small circular band

$$\int_5^8 11\,(t^2+14)^{-2}\cdot 2\pi t\,dt \overset{dt^2 = 2t\,dt}{=\!=\!=} \int_5^8 11\,(t^2+14)^{-2}\cdot \pi\,dt^2$$

$$\overset{u=t^2}{=\!=\!=} \int_{5^2}^{8^2} 11\,(u+14)^{-2}\,\pi\,du = 11\pi\int_{25}^{64}(u+14)^{-2}\,du \overset{d(u+14)=du}{=\!=\!=}$$

$$11\pi\int_{25}^{64}(u+14)^{-2}\,d(u+14) \overset{v=u+14}{=\!=\!=} 11\pi\int_{25+14}^{64+14} v^{-2}\,dv$$

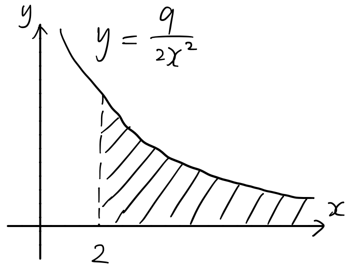
$$= 11\pi\int_{39}^{78} v^{-2}\,dv = \frac{11\pi}{(-1)}\int_{39}^{78}(-1)\,v^{-2}\,dv \overset{(v^{-1})' = (-1)v^{-2}}{=\!=\!=}$$

$$\frac{11\pi}{(-1)}\, v^{-1}\Big|_{39}^{78} = \frac{11\pi}{(-1)}\left[78^{-1} - 39^{-1}\right]$$

$$= -11\pi\left[\frac{1}{78} - \frac{1}{39}\right] = \frac{11\pi}{78}$$

Section 9.6 Improper integral

Find the area under the graph of $y = \frac{9}{2x^2}$ for $x \geq 2$



The area is $\int_2^\infty \frac{9}{2x^2} dx = \lim_{b\to\infty} \int_2^b \frac{9}{2x^2} dx$

here $\lim_{b\to\infty} \int_2^b \frac{9}{2x^2} dx$ by definition is a number that $\int_2^b \frac{9}{2x^2} dx$ converges to as $b \to +\infty$

$$\int_2^b \frac{9}{2x^2} dx = \frac{9}{2}\int_2^b x^{-2} dx = \frac{9}{2\cdot(-1)}\int_2^b (-1)x^{-2} dx$$

$$\overset{(x^{-1})'=(-1)x^{-2}}{=\!=\!=} \frac{9}{2\cdot(-1)} \; x^{-1}\Big|_2^b = -\frac{9}{2}\left[b^{-1}-2^{-1}\right]$$

$= -\frac{9}{2}\left[\frac{1}{b}-\frac{1}{2}\right]$, as $b \to \infty$, $\frac{1}{b} \to 0$, hence

$-\frac{9}{2}\left[\frac{1}{b}-\frac{1}{2}\right] \to -\frac{9}{2}\left[0-\frac{1}{2}\right] = \frac{9}{4}$

We say limit $\lim_{b\to\infty}\int_2^b \frac{9}{2x^2} dx$ excist and $\int_2^\infty \frac{9}{2x^2} dx$ is convergent

How about $\int_1^\infty \frac{x^6}{x^7+5} dx$?

$\int_1^\infty \frac{x^6}{x^7+5} dx = \lim_{b\to\infty}\int_1^b \frac{x^6}{x^7+5} dx$ is by definition $\int_1^b \frac{x^6}{x^7+5} dx \to ?$ as $b \to \infty$

$$\int_1^b \frac{x^6}{x^7+5} dx = \frac{1}{7}\int_1^b \frac{7x^6}{x^7+5} dx \overset{dx^7 = 7x^6 dx}{=\!=\!=} \frac{1}{7}\int_1^b \frac{dx^7}{x^7+5} \overset{u=x^7}{=\!=\!=}$$

$$\frac{1}{7}\int_{1^7}^{b^7} \frac{du}{u+5} = \frac{1}{7}\int_1^{b^7} \frac{du}{u+5} \overset{d(u+5)=du}{=\!=\!=} \frac{1}{7}\int_1^{b^7} \frac{d(u+5)}{u+5} \overset{v=u+5}{=\!=\!=} \frac{1}{7}\int_{1+5}^{b^7+5} \frac{dv}{v}$$

$$= \frac{1}{7}\int_6^{b^7+5} \frac{1}{v} dv \overset{(\ln v)'=\frac{1}{v}}{=\!=\!=} \frac{1}{7}\ln v\Big|_6^{b^7+5} = \frac{1}{7}\left[\ln(b^7+5)-\ln(6)\right]$$ but

as $b\to\infty$, $b^7\to\infty$, $b^7+5\to\infty$, $\ln(b^7+5)\to\infty$, $\ln(b^7+5)-\ln(6)\to\infty$,

$\frac{1}{7}\left[\ln(b^7+5)-\ln(6)\right] \to \infty$

We say $\lim_{b\to\infty}\int_1^b \frac{x^6}{x^7+5} dx$ doesn't exist and $\int_1^\infty \frac{x^6}{x^7+5} dx$ is divergent

16 Lecture 10/02

The capital value of an asset is sometimes defined as the present value of all future net earnings. The capital value of the asset may be written in the form [capital value] $= \int_0^{\infty} K(t)e^{-rt}\,dt$, where r is the annual rate of interest, compounded continuously. Find the capital value of an asset that generates income at the rate of \$8000 per year, assuming an interest rate of 8%

Solution: $K(t)$ is the rate of income generated by the asset

In this case $K(t) = 8000$, $r = 8\%$

hence capital value $= \int_0^{\infty} 8000e^{-0.08t}\,dt = \lim_{b\to+\infty}\int_0^b 8000e^{-0.08t}\,dt$

$\int_0^b 8000e^{-0.08t}\,dt = \frac{8000}{(-0.08)}\int_0^b (-0.08)e^{-0.08t}\,dt$ $\quad \underline{\underline{(e^{-0.08t})' = (-0.08)e^{-0.08t}}}$

$\frac{8000}{(-0.08)}\, e^{-0.08t}\Big|_0^b = -100000\left[e^{-0.08b} - e^0\right] = -100000\left[\frac{1}{e^{0.08b}} - 1\right]$

when $b\to+\infty$, $0.08b\to+\infty$, $e^{0.08b}\to+\infty$, $\frac{1}{e^{0.08b}}\to 0$,

$\int_0^b 8000e^{-0.08t}\,dt = -100000\left[\frac{1}{e^{0.08b}} - 1\right] = -100000[0-1] = 100000$

Interestingly, even though you get \$8000 dollars every year, but throughout eternity, you can only get a finite amount of present value

$\int_8^{\infty}\frac{x}{\sqrt{5+x^2}}\,dx = \lim_{b\to\infty}\int_8^b \frac{x}{\sqrt{5+x^2}}\,dx$, $\int_8^b\frac{x}{\sqrt{5+x^2}}\,dx$ $\quad \frac{u=5+x^2}{du = d(5+x^2) = (5+x^2)'dx = 2x\,dx}$

$\downarrow \div 2$

$\frac{1}{2}du = x\,dx$

$= \int_{5+8^2}^{5+b^2}\frac{1}{\sqrt{u}}\cdot\frac{1}{2}du = \int_{69}^{5+b^2}\frac{1}{2}u^{-\frac{1}{2}}\,du$

$\underline{\underline{(u^{\frac{1}{2}})' = \frac{1}{2}u^{-\frac{1}{2}}}}$ $\quad u^{\frac{1}{2}}\Big|_{69}^{5+b^2} = \left[(5+b^2)^{\frac{1}{2}} - 69^{\frac{1}{2}}\right] = \sqrt{5+b^2} - \sqrt{69}$

when $b\to+\infty$, $b^2\to+\infty$, $5+b^2\to+\infty$, $\sqrt{5+b^2}\to+\infty$, $\sqrt{5+b^2}-\sqrt{69}\to+\infty$

hence $\int_8^{\infty}\frac{x}{\sqrt{5+x^2}}\,dx$ is divergent

$\int_2^{\infty}\frac{12}{x\ln x}\,dx = \lim_{b\to\infty}\int_2^b\frac{12}{x\ln x}\,dx$, $\int_2^b\frac{12}{x\ln x}\,dx$ $\quad \frac{u=\ln x}{du = d\ln x = \frac{1}{x}dx}$ $\quad \int_{\ln 2}^{\ln b}\frac{12}{u}\,du$

$\underline{\underline{(\ln u)' = \frac{1}{u}}}$ $\quad 12\ln u\Big|_{\ln 2}^{\ln b} = 12\left[\ln(\ln b) - \ln(\ln 2)\right]$ as $b\to+\infty$,

$\ln b\to+\infty$, $\ln(\ln b)\to+\infty$, $\left[\ln(\ln b) - \ln(\ln 2)\right]\to+\infty$, hence

$12\left[\ln(\ln b) - \ln(\ln 2)\right]\to+\infty$, $\int_2^{\infty}\frac{12}{x\ln x}\,dx$ diverges

$$\int_0^{\infty} 2x\left(3x^2+1\right)^{-\frac{5}{4}}dx = \lim_{b\to\infty}\int_0^b 2x\left(3x^2+1\right)^{-\frac{5}{4}}dx$$

$$\int_0^b 2x(3x^2+1)^{-\frac{5}{4}}dx \xlongequal[du=d(3x^2+1)=(3x^2+1)'dx=6xdx]{u=3x^2+1} \int_{3\cdot 0^2+1}^{3b^2+1} u^{-\frac{5}{4}}\frac{1}{3}du$$

$$\downarrow \div 3$$

$$\frac{1}{3}du = 2xdx$$

$$= \frac{1}{3}\int_1^{3b^2+1} u^{-\frac{5}{4}}du = \frac{\frac{1}{3}}{(-\frac{1}{4})}\int_1^{3b^2+1}(-\tfrac{1}{4})u^{-\frac{5}{4}}du \xlongequal{(u^{-\frac{1}{4}})'=(-\frac{1}{4})u^{-\frac{5}{4}}} \frac{\frac{1}{3}}{(-\frac{1}{4})}\, u^{-\frac{1}{4}}\Big|_1^{3b^2+1}$$

$$= -\frac{4}{3}\left[(3b^2+1)^{-\frac{1}{4}} - 1^{-\frac{1}{4}}\right] = -\frac{4}{3}\left[\frac{1}{(3b^2+1)^{\frac{1}{4}}} - 1\right]$$

when $b\to+\infty$, $3b^2+1\to+\infty$, $(3b^2+1)^{\frac{1}{4}}\to+\infty$, $\frac{1}{(3b^2+1)^{\frac{1}{4}}}\to 0$,

$$\int_0^b 2x(3x^2+1)^{-\frac{5}{4}}dx = -\frac{4}{3}\left[\frac{1}{(3b^2+1)^{\frac{1}{4}}}-1\right] = -\frac{4}{3}[0-1] = \frac{4}{3}$$

hence $\int_0^{\infty} 2x(3x^2+1)^{-\frac{5}{4}}dx$ converges to $\frac{4}{3}$

Differential equations:

Definition: a differential equation is an equation involving $t, y, y', y'', \cdots$

for example: $y''-4y'-5y=0$, $y'=t^4y+2t^4$

Definition: a solution is a function $y=f(t)$ satisfying the equation, for example, $y=f(t)=Ce^{5t}$ (C being arbitsary constant) is a solution of equation $y''-4y'-5y=0$ since $(Ce^{5t})''-4(Ce^{5t})'-5(Ce^{5t}) = 25Ce^{5t}-20Ce^{5t}-5Ce^{5t}=0$

17 Lecture 10/04

(DE)

Definition: a differential equation is an equation involving $t, y, y', y'', \cdots$

for example: $y''-4y'-5y=0$ (1) $\quad y'=t^4y+2t^4$ (2)

Definition: a solution is a function $y=f(t)$ satisfying the equation, for example,

$y=f(t)=Ce^{5t}$ (C being arbitsary constant) is a solution of equation

$y''-4y'-5y=0$ since $(Ce^{5t})''-4(Ce^{5t})'-5(Ce^{5t})=25Ce^{5t}-20Ce^{5t}-5Ce^{5t}=0$

Constant solutions are solutions that are constants, for example

Find constant solutions of $y'=t^4y+2t^4$

suppose $y=C$ is a solution, then $0=y'=Ct^4+2t^4=(C+2)t^4 \Rightarrow C+2=0$

$\Rightarrow C=-2$, hence $y=-2$ is the only constant solution

DE with initial conditions (IC)

Example: $\begin{cases} y''-4y'-5y=0 \\ y(0)=3 \end{cases}$, $\quad y=3e^{5t}$ is a solution since $y(0)=3$

however, $y=4e^{5t}$ is not even though it solves the DE because it doesn't satisfy IC since $y(0)=4$ now

Problem:

If $f(t)$ is a solution of the initial value problem $y'=3y-1$, $y(0)=3$, find $f(0)$ and $f'(0)$

Solution: Since $f(t)$ is a solution, $f'(t)=3f(t)-1$ and $f(0)=3$, take $t=0$,

$f'(0)=3f(0)-1=8$

Let $f(t)$ be the balance in a savings account at the end of t years and suppose $y=f(t)$ satisfies the differential equation $y'=-0.04y+8000$

(a) Suppose that after one year the balance is \$220000, Is the balance increasing or decreasing at that time?

Answer: we are given $f(1)=220000$, hence $f'(1)=-0.04f(1)+8000=-800<0$

hence the balance is decreasing

Remark: $f'(t)=-0.04f(t)+8000=-0.04(f(t)-200000)$

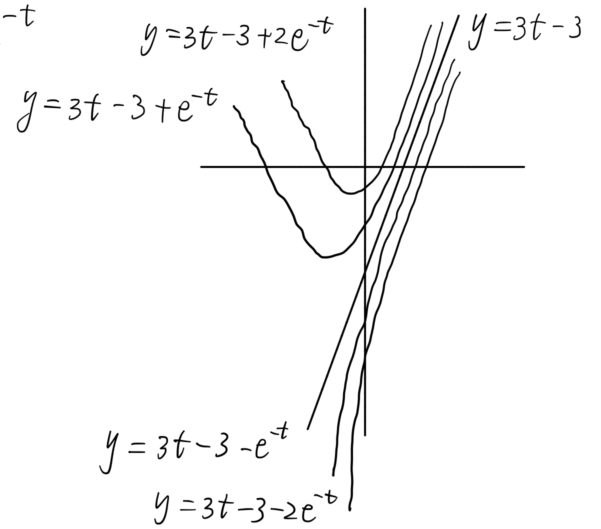
If pocess > \$200000 at t years, i.e. $f(t)>200000$, then $f'(t)<0$, the balance is decreasing, if $f(t)<200000$, then $f'(t)>0$, the balance is increasing, if $f(t)=200000$, $f'(t)=0$, $f(t)$ is "stable" $\Leftrightarrow$ constant solutions

(Direction field)

Slope field: $y' = \frac{dy}{dt} = 3t - y$, suppose you know $y(t)$ is a solution, and it passes point $(0,-3)$ [meaning $(0,-3)$ is on the solution curve $y(t)$, i.e. $y(0) = -3$ equivalent to giving an IC], then the slope of the tangent to $y(t)$ at $(0,-3)$ is $y'(0) = 3\times 0 - y(0) = 3$, thus the tangent line has the form $y = 3x + b$, since it passes $(0,-3)$, $-3 = 3\times 0 + b \Rightarrow b = -3$, hence the equation of the tangent line is $y = 3x - 3$, take a small piece of it at $(0,-3)$, similarly, for each point on the yt-plane, it gives an IC, and you can compute the slope of the tangent to the solution curve at that point, for example: $(0,1)$ gives $\begin{cases} y' = 3t - y \\ y(0) = 1 \end{cases} \Rightarrow y'(0) = -1$, $(2,5)$ gives

$\begin{cases} y' = 3t - y \\ y(2) = 5 \end{cases} \Rightarrow y'(0) = 1$ and all these small pieces form the slope field as showed in the graph below: each point (can be considered as an IC) is on a solution curve, the slope (direction) of each small piece tells you the slope (tangent) of the solution curve

the general solution is $y = 3t - 3 + Ce^{-t}$



18 Lecture 10/07

Separation of variables

$\frac{dy}{dt} = y' = 3y$, we separate y and t on two sides by $\div y$ and $\times dt$ on both sides, then we get $\frac{dy}{y} = 3dt$, integrate on both sides

$\Rightarrow \int \frac{dy}{y} = \int 3dt \Rightarrow \ln|y| = 3t + C \xrightarrow{\text{take exponential}} |y| = e^{3t+C}$

$\Rightarrow y = e^{3t+C}$ or $-e^{3t+C}$ $\quad C$ is a constant

Note that in dividing y, we missed a solution $y=0$

$\frac{dy}{dt} = y' = 12te^{-2y} - 19e^{-2y}$, $y(0) = 8$ $\quad$ DE with IC

$\frac{dy}{dt} = (12t-19)e^{-2y} \xrightarrow{\times e^{2y} dt} e^{2y} dy = (12t-19)dt$

$\Rightarrow \int e^{2y} dy = \int (12t-19)dt \Rightarrow \frac{1}{2}e^{2y} = 6t^2 - 19t + C$

plug in $y(0)=8$, $\frac{1}{2}e^{16} = \frac{1}{2}e^{2\cdot 8} = \frac{1}{2}e^{2\cdot y(0)} = 6\cdot 0^2 - 19\cdot 0 + C = C$, hence

$\frac{1}{2}e^{16} = C$, thus $\frac{1}{2}e^{2y} = 6t^2 - 19t + \frac{1}{2}e^{16} \xrightarrow{\times 2} e^{2y} = 12t^2 - 38t + e^{16}$

$\xrightarrow{\text{take } \ln} 2y = \ln(12t^2 - 38t + e^{16}) \xrightarrow{\div 2} y = \frac{1}{2}\ln(12t^2 - 38t + e^{16})$

Solve $y^5 y' = t\cos t$, $y(0) = 2$

$y^5 \frac{dy}{dt} = t\cos t \xrightarrow{\times dt} y^5 dy = t\cos t\, dt \Rightarrow \int y^5 dy = \int t\cos t\, dt$

$\int t\cos t\, dt = \int t(\sin t)' dt = \int [(t\sin t)' - t'\sin t] dt = \int (t\sin t)' dt - \int \sin t\, dt$

$= t\sin t + \cos t + C$

$\Rightarrow \frac{1}{6}y^6 = t\sin t + \cos t + C$, plug in $y(0)=2$, $\frac{32}{3} = \frac{1}{6}\times 2^6 = 0\times\sin(0) + \cos(0) + C$

$= 1 + C \Rightarrow C = 1 - \frac{32}{3} = \frac{29}{3}$, hence $\frac{1}{6}y^6 = t\sin t + \cos t + \frac{29}{3} \xrightarrow{\times 6} y^6 = 6t\sin t + 6\cos t + 54$

$y = \sqrt[6]{6t\sin t + 6\cos t + 54}$, not choosing the negative part because $y(0) = 2 > 0$

10.3 consider DE $y' = 6y + t$, can't really separate the variables

rewrite as $y' - 6y = t$ key observation: $[ye^{-6t}]' = y'e^{-6t} + y(e^{-6t})' = y'e^{-6t} - 6ye^{-6t} = (y' - 6y)e^{-6t}$, thus we $\times e^{-6t}$ on both sides to get $[ye^{-6t}]' = (y - y')e^{-6t} = te^{-6t}$, then $\times dt$ and integrate $\int [ye^{-6t}]' dt = \int te^{-6t} dt$

$\Rightarrow ye^{-6t} = -\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C$, C is constant, then $\div e^{-6t}$ / or $\times e^{6t}$ on both sides, we have $y = -\frac{1}{6}t - \frac{1}{36} + Ce^{6t}$

Generalization: Consider DE $y' + a(t)y = b(t)$

Step I: Find an antiderivative of $a(t)$, denote as $A(t)$, $A'(t) = a(t)$, notice there is choice of constant C from $\int a(t)dt$, you can pick any C

In correspondence, $a(t) = -6$, $A(t) = -6t$, $A'(t) = a(t)$

key observation is $[ye^{A(t)}]' = y'e^{A(t)} + y(e^{A(t)})' = y'e^{A(t)} + ye^{A(t)}A'(t)$

$= y'e^{A(t)} + a(t)ye^{A(t)} = (y' + a(t)y)e^{A(t)}$

Step II: multiplying integrating factor $e^{A(t)}$ on both sides, get

In correspondence, multiply e^{-6t} on both sides

$[ye^{A(t)}]' = (y' + a(t)y)e^{A(t)} = b(t)e^{A(t)}$

Step III: $\times dt$ and the integrate, $\int [ye^{A(t)}]' dt = \int b(t)e^{A(t)} dt$

$\Rightarrow ye^{A(t)} = \int b(t)e^{A(t)} dt$

In correspondence, $ye^{-6t} = \int te^{-6t} dt = -\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C$

Step IV: $\div e^{A(t)}$, $y = e^{-A(t)} \int b(t)e^{A(t)} dt$

In correspondence, $y = \dfrac{(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C)}{e^{-6t}} = \dfrac{(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C)}{\frac{1}{e^{6t}}}$

$= e^{6t}(-\frac{1}{6}te^{-6t} - \frac{1}{36}e^{-6t} + C) = -\frac{1}{6}t - \frac{1}{36} + Ce^{6t}$

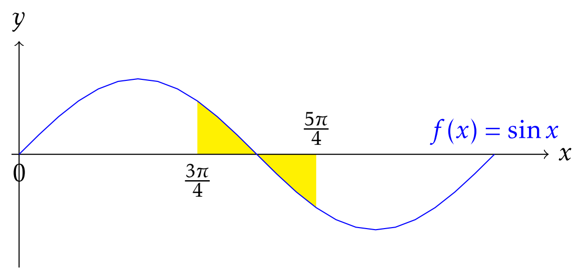
19 Review of chapter 8 and 9

Review of chapters 8 and 9

Problems:

(1) Find the slope of the curve $y = x\sin\left(\pi\sqrt{1-5x^2}\right)$ at $x = 0$

(2) Find the area of the following



(3) Let $f(x) = \sin\left(2x + \frac{\pi}{3}\right)$, for what values of $-\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$ does f has a tangent of slope 1

(4) What is the third Taylor polynomial of $f(x) = \sin\left(x^2\right)$ centered about $x = 0$

(5) Evaluate $\int x^2 \ln\left(x^3\right) \mathrm{d}x$

(6) Evaluate $\int \frac{\cos x}{1 + \sin x} \mathrm{d}x$

(7) Evaluate $\int \sin^3 x \mathrm{d}x$

20 Review of chapter 10 and 11

Review of chapters 10 and 11

Problems:

Review the definition of Taylor polynomials

(1) What is the Taylor series(expension) of $\frac{x^2+2}{e^{x^3}}$ at $x = 0$, at least four terms

Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

Review the definition of geometric series

(2) A patient is receiving medication, and he will take one pill per day, and each pill consists of 5 mg, and and 10% of the drug will be absorbed into the body, determine the equilibrium amount of drug in the body of that patient. That is, after a lenthy period of time, how much medication would we expect to be in patient's body

(3) Find the constant solutions of the differential equation $y' - y^2 = 4y - 5$

(4) $yy' = \frac{t}{y}$

(5) Solve the differential equation $y' - e^{\ln x} y = x$

(6) Use the Trapezoidal Rule with $n = 3$ partitions to approximate the area under the curve y^4 on the interval $1 \leq x \leq 3$

21 Section 11.3a

11.3a Infinite Series

Algebra fact: $1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$ where $r \neq 1$

Definition: An (infinite) series is an infinite addition of numbers, for example

$$1 + 1 + 1 + 1 + \cdots \tag{1}$$

$$1 - 1 + 1 - 1 + \cdots \tag{2}$$

$$\text{Harmonic series: } 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{3}$$

$$\text{Geometric series: } 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \tag{4}$$

$$\text{Basel problem: } 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \tag{5}$$

In general

$$a_1+a_2+a_3+a_4+\cdots$$

Definition: The sum of the first n terms of an infinite series is called its $\boldsymbol{n}$th partial sum, denoted S_n, i.e. $S_n=a_1+a_2+\cdots+a_n$, if the partial sums converges to $S(S_n\to S)$, we say S is the sum of the infinite series, for example, for (2), then partial sums are $S_1=1, S_2=0, S_3=1, S_4=1,\cdots$ alternate between 0 and 1 and do not approach a limit, for (4), use the algebra fact, $S_n=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}=1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots+\left(\frac{1}{2}\right)^{n-1}=\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$, from the expression, we know as $n\to\infty$, $\left(\frac{1}{2}\right)^n\to 0$, thus $S_n\to\frac{1-0}{1-\frac{1}{2}}=2$, we say (2) is divergent(not convergent), and (4) is convergent

Question: How about (1), (3) and (5), are they convergent, why? and what is the sum if it converges?
For (1), $S_n=n$, thus it is divergent
For (3), even though the term tends to zero, it is divergent
For (5), it is convergent, and surprisingly $S=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$

Definition:

$$a+ar+ar^2+ar^3+ar^4+\cdots \tag{6}$$

is called a geometric series with ratio $\boldsymbol{r}$(The "ratio" of consecutive terms is r)

Theorem: (6) converges if $|r|<1$, and the sum is $\frac{a}{1-r}$

For example $a=1, r=\frac{1}{2}$ is (4), note that $a=r=1$ is (1), and $a=1, r=-1$ is (2), but they are convergent geometric series sicce $|r|=1$

Problems: Determine the following infinite series if they are geometric series, and if they are, what are the sums?

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots \tag{7}$$

$$8+\frac{8}{3}+\frac{8}{9}+\frac{8}{27}+\frac{8}{81}+\cdots \tag{8}$$

$$2+\frac{2^2}{5}+\frac{2^3}{5^2}+\frac{2^4}{5^3}+\frac{2^5}{5^4}+\cdots \tag{9}$$

$$\frac{1}{5}+\frac{2}{5^2}+\frac{2^2}{5^3}+\frac{2^3}{5^4}+\frac{2^4}{5^5}+\cdots \tag{10}$$

$$2+1+0.5+0.25+0.125+\cdots \tag{11}$$

What is a general way of determining if it is a geometric series?
Determine if consecutive terms have ratio r with $|r|<1$, and if it is, a is the first term, the ratio is also known, so you can compute the sum

Conversions between rational numbers and geometric series

Recall: $17.152365\overline{917}=17.152365917917917917\cdots$, $\overline{917}$ means repeating cycle 917, rational numbers can always be converted into a terminating decimal or a repeating decimal(which is called its decimal expansion) and vice versa(for any base, normally in base 10, irrational numbers corresponds exactly to non-terminating, non-repeating decimals)

Question: What rational number has decimal expansion $0.6363\overline{63}$?
Answer: $0.\overline{63}=0.63+0.0063+0.000063+0.00000063+\cdots=\frac{63}{100}+\frac{63}{100^2}+\frac{63}{100^3}+\frac{63}{100^4}+\cdots=\frac{\frac{63}{100}}{1-\frac{1}{100}}=\frac{7}{11}$

What about $0.988\overline{988}$?

Here is another variant of such question: what rational number has decimal expansion $5.4256\overline{256}$, (Hint: $5.4256\overline{256} = 5.4256 + 0.0000\overline{256}$)

22 Section 11.3b

11.3b Infinite Series

The Multiplier Effect in economics

Suppose that the federal government enacts an income tax cut of \$8 billion, Assume that each person will spend 90% of all resulting extra income and save the rest. Estimate the total amount of new spending created by the tax cut

$8 \cdot 0.9$ extra billion dollars are going to be spent from the tax cut, and then these dollars become income and $8 \cdot 0.9 \cdot 0.9 = 8 \cdot (0.9)^2$ billion dollars are going to be spent, and then these dollars become income $\cdots$, hence the total extra money being spent is given by geometric series

$$8 \cdot 0.9 + 8 \cdot (0.9)^2 + 8 \cdot (0.9)^3 + 8 \cdot (0.9)^4 + \cdots \tag{12}$$

Identify that $a = 8 \times 0.9 = 7.2$, $r = 0.9$, thus the sum(total amount) $S = \dfrac{7.2}{1-0.9} = \dfrac{7.2}{0.1} = 72$, therefore, the total amount of new spending created by the tax cut is 72 billion dollars

This illustrates the multiplier effect, and the proportion of each extra dollar that a person will spend is called the marginal propensity to consume, denoted as MPC, notice that

$$8 \cdot 0.9 + 8 \cdot (0.9)^2 + 8 \cdot (0.9)^3 + 8 \cdot (0.9)^4 + \cdots = \frac{8 \cdot 0.9}{1-0.9} = 8 \cdot \frac{0.9}{1-0.9} = \text{tax cut} \cdot \frac{\text{MPC}}{1-\text{MPC}} \tag{13}$$

We say $\dfrac{\text{MPC}}{1-\text{MPC}}$ is the "multiplier"

Sigma nototaion

When studying series, it is often convenient to use the Greek capital letter sigma to indicate summation, for example

$\sum_{k=3}^{10} a_k := a_3 + a_4 + \cdots + a_{10}$ read as "the sum of a sub k from k equals 3 to 10"

The n-th partial sum S_n of the series $a_1 + a_2 + a_3 + \cdots$ can be written as $S_n = \sum_{k=1}^{n} a_k := a_1 + a_2 + a_3 + \cdots + a_n$, the letter k here is called the index of summation, the index can also starts at 0, $\sum_{k=0}^{4} a_k := a_0 + a_1 + a_2 + a_3 + a_4$, or starts at a negative number $\sum_{k=-2}^{1} a_k := a_{-2} + a_{-1} + a_0 + a_1$, also the letters can change, $\sum_{i=0}^{3} b_i := b_0 + b_1 + b_2 + b_3$, or $\sum_{j=1}^{3} \alpha_j := \alpha_1 + \alpha_2 + \alpha_3$, finally, the sum of the previous series can be written as $S = \sum_{k=1}^{\infty} a_k := a_1 + a_2 + a_3 + \cdots = \lim_{n\to\infty} S_n$ or simply as $\sum_{1}^{\infty} a_k$

In particular, in the case of the geometric series $a + ar + ar^2 + ar^3 + \cdots$, $S_n = \sum_{k=0}^{n-1} ar^k := a + ar + ar^2 + \cdots + ar^{n-1}$, $(ar^0 = a)$ and $S = \sum_{k=0}^{\infty} ar^k := a + ar + ar^2 + \cdots = \dfrac{a}{1-r}$ if $|r| < 1$ and $\sum_{k=0}^{\infty} ar^k$ is divergent if $|r| \geq 1$

Examples: $1+2+\cdots+17=\sum_{k=1}^{17} k$, $2+4+6+\cdots+18=\sum_{k=1}^{9} 2k$, $e^{2x}\sin x+e^{3x}\sin 2x+e^{4x}\sin 3x+\cdots=\sum_{k=2}^{\infty} e^{kx}\sin((k+1)x)+\sum_{k=1}^{\infty} e^{(k+1)x}\sin(kx)$, $f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

Problems

A patient receives 11 milligrams of a certain drug daily. Each day the body eliminates 10% of the amount of the drug present in the system. Estimate the total amount of the drug that should be present after extended treatment, immediately after a dose is given

Answer:

$11+11\cdot 0.9+11\cdot(0.9)^2+\cdots$, $a=11, r=0.9$, $S=\frac{11}{1-0.9}=11$ mg

Here is a variant of the question, where you have to solve the problem conversely!

A patient receives M milligrams of a certain drug each day. Each day the body eliminates 30% of the amount of the drug present in the system. Determine the value of the maintenance dose M such that after many days approximately 20 milligrams of the drug is present immediately after a dose is given (Hint: Try to figure out what equation M has to satisfy and then solve for M)

23 Section 11.4

11.4 Series with positive terms Remark:

$$\sum_{k=1}^{\infty} a_k \text{ is convergent/ divergent} \Leftrightarrow \sum_{k=N}^{\infty} a_k \text{ is convergent/divergent}$$

$$\int_1^{\infty} f(x)dx \text{ is convergent/divergent} \Leftrightarrow \int_A^{\infty} f(x)dx \text{ is convergent/divergent}$$

Integral test: Let $f(x)$ be a continuous, nonincreasing, and nonnegative function for $x\geq 1$, Then the infinite series $\sum_{k=1}^{\infty} f(k)$ is convergent or divergent if the improper integral $\int_1^{\infty} f(x)dx$ is convergent or divergent

There are nice geometric explanations!

Example: Consider the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k}$, we can find the corresponding function $f(x)=\frac{1}{x}$(check it is indeed continuous, nonincreasing and nonnegative), then notice that $\int_1^{\infty}\frac{1}{x}dx$ is divergent, hence the series is also divergent

Example: Consider the series in Basel problem $1+\frac{1}{4}+\frac{1}{9}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k^2}$, we can find the corresponding function $f(x)=\frac{1}{x^2}$(check it is indeed continuous, nonincreasing and nonnegative), then notice that $\int_1^{\infty}\frac{1}{x^2}dx$ is convergent, hence the series is also convergent

Problems: Use the integral test to determine if the following series or improper integrals are convergent

$$\sum_{k=3}^{\infty}\frac{2}{7k\sqrt{\ln k}}$$

$$\sum_{k=2}^{\infty}\frac{k}{(2k^2+9)^{\frac{4}{3}}}$$

$$\int_1^{\infty}\frac{3^x}{10^x}dx$$

24 Section 11.5a

11.5a Taylor Series Definition: Series of the form $a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+\cdots=\sum_{k=0}^{\infty}a_k(x-a)^k$, notice that if $a=0$, then it has the form $a_0+a_1x+a_2x^2+a_3x^3+\cdots=\sum_{k=0}^{\infty}a_kx^k$, and geometric series $a+ar+ar^2+ar^3+\cdots=\sum_{k=0}^{\infty}ar^k$ is a special case of power series

Definition: The Taylor series(Taylor expansion) of $f(x)$ at $x=a$ is a power series $f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$, if $a=0$, then it has the form $f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$

Remark: Not very rigorously, you can think of $f(x)$ equals to its Taylor series, i.e. $f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$

Crucial examples:

1: What is the Taylor series for $\frac{1}{1-x}$ at $x=0$

The direct(or brutal way): if $f(x)=\frac{1}{1-x}=(1-x)^{-1}$, then $f^{(n)}(x)=n!(1-x)^{n+1}$, $f^{(n)}(0)=n!$, $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}\frac{k!}{k!}x^k=\sum_{k=0}^{\infty}x^k$

A smarter way: consider geometric series $\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\cdots=\frac{1}{1-x}$ which of course coincides beautifully with previous formula

Note: the equality only holds if $|x|<1$

2: What is the Taylor series for $\ln(1-x)$ at $x=0$

The direct(or brutal way): if $f(x)=\ln(1-x)$, $f'(x)=-\frac{1}{1-x}$, then $f^{(n)}(x)=-(n-1)!(1-x)^n$ for $n\geq 1$ by the previous example, $f^{(n)}(0)=-(n-1)!$ for $n\geq 1$ and $f(0)=0$, $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+\sum_{k=1}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=1}^{\infty}\frac{-(k-1)!}{k!}x^k=-\sum_{k=1}^{\infty}\frac{x^k}{k}$

A smarter way: $\ln(1-x)=-\int\frac{1}{1-x}dx=-\int\left(\sum_{k=0}^{\infty}x^k\right)dx=\sum_{k=0}^{\infty}\int x^k dx=-\sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)}=-\sum_{k=1}^{\infty}\frac{x^k}{k}$ which coinsides with the formula above nicely!

Note: the equality only holds if $|x|<1$(actually $x=-1$ is also fine)

Also $\ln(1+x)=\ln(1-(-x))=-\sum_{k=1}^{\infty}\frac{(-x)^k}{k}=-\sum_{k=1}^{\infty}\frac{(-1)^kx^k}{k}=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$

Note: the equality only holds if $|x|<1$(but actually $x=1$ is also fine)

Remark: You can integrate the series term by term! 3: What is the Taylor series for e^x at $x=0$

If we let $f(x)=e^x$, because $(e^x)'=e^x$, $f^{(n)}(x)=e^x$, $f^{(n)}(0)=e^0=1$, then $\dfrac{f^{(n)}(0)}{n!}=\dfrac{1}{n!}$, thus the Taylor series is $\displaystyle\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=\sum_{k=0}^{\infty}\frac{x^k}{k!}$

Notice this makes perfect sense, since $\displaystyle (e^x)'=\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}x^k\right)'=\left(1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\right)'=\sum_{k=1}^{\infty}\left(\frac{x^k}{k!}\right)'=\sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!}=\sum_{k=0}^{\infty}\frac{x^k}{k!}=e^x$

Remark: You can differentiate the series term by term! 4: What is the Taylor series for $\sin x$ at $x=0$

If we let $f(x)=\sin x$, then $f'(x)=\cos x$, $f''(x)=-\sin x$, $f^{(3)}(x)=-\cos x$, $f^{(4)}(x)=\sin x$, then it repeats periodically with period 4, actually $f^{(n)}(x)=\sin\left(x+\dfrac{n\pi}{2}\right)$, $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f^{(3)}(0)=-1$, $f^{(4)}(0)=0$, and repeating periodically with period 4, we get the Taylor series $\displaystyle =x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

5: What is the Taylor series for $\cos x$ at $x=0$

The direct(or brutal way): Mimic what we have done above

A smarter way: we can differentiate the series for $\sin x$ term by term, $\displaystyle \cos x=(\sin x)'=\left(\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)'=\sum_{k=0}^{\infty}\left(\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)'=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$, how magical!

Problems: Find the Taylor series(First few terms, that also is what will be needed in real life because Taylor polynomials(truncates of Taylor series) is an approximation of the original function, then more terms(the higher the degree of the Taylor polynomial), the more accurate, normally, a few terms is accurate enough) of the following functions at $x=0$

1: $5\left(e^{-2x}-3\right)$

Notice $\displaystyle e^{-2x}=\sum_{k=0}^{\infty}\frac{(-2x)^k}{k!}=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}x^k=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots$

$e^{-2x}-3=1-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots-3=-2-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots$

$5\left(e^{-2x}-3\right)=5\left(-2-2x+2x^2-\frac{4}{3}x^3+\frac{2}{3}x^4+\cdots\right)=-10-10x+10x^2-\frac{20}{3}x^3+\frac{10}{3}x^4+\cdots$ 2; $\dfrac{6}{(1+2x)^2}$

Notice $\displaystyle \frac{1}{1+2x}=\frac{1}{1-(-2x)}=\sum_{k=0}^{\infty}(-2x)^k=\sum_{k=0}^{\infty}(-2)^k x^k=1-2x+4x^2-8x^3+\cdots$, thus

$$\begin{aligned}
\frac{1}{(1+2x)^2}&=\left(\frac{1}{1-(-2x)}\right)^2=\left(1-2x+4x^2-8x^3+\cdots\right)^2\\
&=\left(1-2x+4x^2-8x^3+\cdots\right)\left(1-2x+4x^2-8x^3+\cdots\right)\\
&=1\cdot\left(1-2x+4x^2-8x^3+\cdots\right)\\
&-2x\cdot\left(1-2x+4x^2-8x^3+\cdots\right)\\
&+4x^2\cdot\left(1-2x+4x^2-8x^3+\cdots\right)\\
&-8x^3\cdot\left(1-2x+4x^2-8x^3+\cdots\right)\\
&+\cdots\\
&=1-2x+4x^2-8x^3+\cdots\\
&-2x+4x^2-8x^3+16x^4+\cdots\\
&+4x^2-8x^3+16x^4-32x^5+\cdots\\
&-8x^3+16x^4+-32x^5+64x^6+\cdots\\
&+\cdots\\
&=1-4x+12x^2-32x^3+\cdots
\end{aligned}$$

Hence $\dfrac{6}{(1+2x)^2}=6\left(1-4x+12x^2-32x^3+\cdots\right)=6-24x+72x^2-192x^3+\cdots$

This may not seem much, BUT try this! 2: $\dfrac{6}{(1+2x^9)^2}$

Just replace x with x^9, thus the Taylor series is $6-24x^9+72\left(x^9\right)^2-192\left(x^9\right)^3+\cdots=6-24x^9+72x^{18}-192x^{27}+\cdots$

25 Section 11.5b

11.5a Taylor Series More examples: 1: Evaluate approximately $\int_0^1 \sin x^{11}\,dx$

$$\begin{aligned}\int_0^1 \sin x^{11}\,dx &= \int_0^1 \left(x^{11} - \frac{\left(x^{11}\right)^3}{3!} + \frac{\left(x^{11}\right)^5}{5!} - \frac{\left(x^{11}\right)^7}{7!} + \cdots\right)dx\\
&= \int_0^1 \left(x^{11} - \frac{x^{33}}{3!} + \frac{x^{55}}{5!} - \frac{x^{77}}{7!} + \cdots\right)dx\\
&= \int_0^1 x^{11}\,dx - \int_0^1 \frac{x^{33}}{3!}\,dx + \int_0^1 \frac{x^{55}}{5!}\,dx - \int_0^1 \frac{x^{77}}{7!}\,dx + \cdots\\
&= \left.\frac{x^{12}}{12}\right|_0^1 - \frac{1}{3!}\left.\frac{x^{34}}{34}\right|_0^1 + \frac{1}{5!}\left.\frac{x^{56}}{56}\right|_0^1 - \frac{1}{7!}\left.\frac{x^{78}}{78}\right|_0^1 + \cdots\\
&= \frac{1}{12} - \frac{1}{3!\cdot 34} + \frac{1}{5!\cdot 56} - \frac{1}{7!\cdot 78} + \cdots\end{aligned}$$

2: Find the Taylor series(First few terms) of $\int \frac{3}{1+2x^3}\,dx$

$$\begin{aligned}\int \frac{3}{1+2x^3}\,dx &= 3\int \frac{1}{1-(-2x^3)}\,dx\\
&= 3\int \left(1 + \left(-2x^3\right) + \left(-2x^3\right)^2 + \left(-2x^3\right)^3 + \cdots\right)dx\\
&= 3\int \left(1 - 2x^3 + 4x^6 - 8x^9 + \cdots\right)dx\\
&= 3\left(\int 1\,dx - 2\int x^3\,dx + 4\int x^6\,dx - 8\int x^9\,dx + \cdots\right)\\
&= 3\left(x - 2\cdot\frac{x^4}{4} + 4\cdot\frac{x^7}{7} - 8\cdot\frac{x^{10}}{10} + \cdots\right)\\
&= 3x - \frac{3x^4}{2} + \frac{12x^7}{7} - \frac{12x^{10}}{5} + \cdots\end{aligned}$$

3: Find the Taylor series(First few terms) of $\int 4e^{-x^2}\,dx$

4: Using the Taylor series of $\frac{1+x^2}{1-x}$ at $x=0$ to find $f^{(4)}(0)$ where $f(x) = \frac{1+4x^{10}}{1-2x^5}$

5: Find the Taylor series of $\frac{1}{\cos x - x^2}$ at $x=0$

6: Find the Taylor series of $\ln(\cos x)$

26 Section 12.1

Section 12.1 Motivating example: Throwing dice

The outcome X is a random variable, X takes value from 1 to 6, with probability $\frac{1}{6}$ for each outcome, and each time you throw the dice is called an experiment, and the result of each experiment is called an outcome, the following table of all possible outcomes with corresponding probility is called a probability table

outcome	1	2	3	4	5	6
probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

if you throw the dice, what will be your expected value, which motivate the definition of E(X):=expected value(or average or mean): $\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 2 + \frac{1}{6}\cdot 3 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 5 + \frac{1}{6}\cdot 6 = 3.5$ meaning for each $\frac{1}{6}$ chance, the outcome maybe $1,\cdots,6$, so this is like a weighted sum, notice the expected value may not be an possible outcome

how to measure the variance, how severely the result is deviated away from the averageV(X):=Variance is defined as follows, subtract the mean from each possible outcome, and then square, and then take weighted sum, namely, $V(X) = \frac{1}{6}(1-3.5)^2 + \frac{1}{6}(2-3.5)^2 + \frac{1}{6}(3-3.5)^2 + \frac{1}{6}(4-3.5)^2 + \frac{1}{6}(5-3.5)^2 + \frac{1}{6}(6-3.5)^2 = \frac{8.75}{3}$, why squared?(to avoid cancellation), for example $V(X) = \frac{1}{6}(1-3.5) + \frac{1}{6}(2-3.5) + \frac{1}{6}(3-3.5) + \frac{1}{6}(4-3.5) + \frac{1}{6}(5-3.5) + \frac{1}{6}(6-3.5) = 0$, since we have squared, let's take a square root, the standard deviation $\sigma(X) := \sqrt{V(X)} = \sqrt{\frac{8.75}{3}}$

In general, let X be a random variable, suppose its possible outcomes are $a_1, \cdots, a_n$, with probability $p_1, \cdots, p_n$, the probability table look like

X	a_1	a_2	a_3	$\cdots$	a_n
P	p_1	p_2	p_3	$\cdots$	p_n

Note that we should certainly have $0 \leq p_i \leq 1$ and $p_1 + \cdots + p_n = 1$

the expected value is $E(X) := a_1 p_1 + \cdots + a_n p_n$, sometimes people denotes $\bar{a} = E(X)$, then variance $V(X) := (a_1 - \bar{a})^2 p_1 + \cdots + (a_n - \bar{a})^2 p_n$, and the standard deviation $\sigma(X) := \sqrt{V(X)}$

Problems:

1: The number of accidents per week at a busy intersection was recorded for a year. There were 5 weeks with no accidents, 30 weeks with one accident, 15 weeks with two accidents, and 2 weeks with three accidents. A week is to be selected at random and the number of accidents noted. Let X be the outcome. Then X is a random variable taking on the values 0, 1, 2, and 3

(a) Write out a probability table for X

(b) Compute $E(X)$

(c) Interpret $E(X)$

(d) How about $V(X), \sigma(X)$

2: Consider a circle with radius 1

(a) What percentage of the points lie within $\frac{3}{4}$ unit of the center? (b) Let c be a constant with $0 < c < 1$. What percentage of the points lies within c units of the center?

3: A citrus grower anticipates a profit of $\$100,000$ this year if the nightly temperatures remain mild. Unfortunately, the weather forecast indicates a 35% chance that the temperatures will drop below freezing during the next week. Such freezing weather will destroy 10% of the crop and reduce the profit to $\$90,000$. However, the grower can protect the citrus fruit against the possible freezing (using smudge pots, electric fans, and so on) at a cost of$\$5,000$. Should the grower spend the $\$5,000$ and thereby reduce the profit to $\$95,000$?

27 Section 12.2

Section 12.2 Motivating example: Randomly picking point on the unit interval

Let $L = [0,1]$, be the unit interval from 0 to 1, now randomly pick a point(each point has the same "probability" of being picked), what is the probability of $\frac{1}{2}$ being picked? Seems like it should be 0, but then the probability of picking any point is also 0, so the sum of probabilities is 0?

The problem is that probability 0 doesn't mean it is impossible, and since there are continuously many different possible outcomes, we can't sum them to get 1, we have to integrate, notice that we can calculate the probability of picking a number from any sub interval, for example $\left[\frac{1}{4},\frac{3}{4}\right]$, which is $\frac{\frac{3}{4}-\frac{1}{4}}{1-0} = \frac{1}{2}$, and this is where the mathematical model of using calculus comes in

So we randomly pick a point from L, and let X be the coordinate of that point, X is called a continuous random variable(meaning the possible value/outcome is within a range of continuously many choice), and we define probability density function to help us define probability, in this case, the probability define density function $f(x) = \begin{cases} 1, x \in [0,1] \\ 0, \text{elsewhere} \end{cases}$, and define the probability of picking a number from interval $[a,b], 0 \le a \le b \le 1$ to be $Pr(a \le X \le b) = \frac{b-a}{1-0} = b - a$ which is also the area under $f(x)$ between a and b, notice $Pr\left(X = \frac{1}{2}\right) = Pr\left(\frac{1}{2} \le X \le \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2} = 0$

Generally: Suppose X is a continuous random variable taking values within A and B, where A can be $-\infty$ and B can be ∞, we define the probability through a density function $Pr(a \le X \le b) = \int_a^b f(x)dx, A \le a \le b \le B$ where f is called the probability density function satisfying $f(x) \ge 0$ for $A \le x \le B$(since the probability has to be nonnegative) and $\int_A^B f(x)dx = 1$(since the probability has to integrate to 1) for $A \le x \le B$, we want yet another notion:

cumulative distribution function $F(x) := Pr(A \leq X \leq x) = \int_A^x f(t)dt$(here use t because x is already taken), then we have $F'(x) = f(x)$ and thus by fundamental theorem of calculus we have $Pr(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$

For example, in the example above, $A = 0,\ B = 1,\ F(x) = \int_0^x f(t)dt = \int_0^x 1dt = t|_0^x = x - 0 = x$ for $0 \leq x \leq 1$

Problems:

1: $f(x) = \frac{1}{50}x, 0 \le x \le 11$, is it a probability density function?

2: $f(x) = kx^2, 0 \le x \le 4$ is a probability density function, what is k

3: $f(x) = \frac{1}{10}, 1 \le x \le 11$ is a probability density function, calculate $Pr(X \le 5)$

4: $F(x) = \frac{1}{2}\sqrt{x-2}, 2 \le x \le 6$ is a cumulative distribution function, what is the probability density function $f(x)$

5: $f(x) = \frac{1}{2}(4-x), 2 \le x \le 4$ is a probability density function, what is the cumulative distribution function $F(x)$

6: Suppose that the lifetime X(in hours) of a certain type of flashlight battery is a random variable on the interval $88 \le x \le 103$ with probability density function $f(x) = \frac{1}{80}$, find the probability that a battery selected at random will last at least 96 hours

7: Let X be a continuous random variable with values between $A = 1$ and $B = \infty$, and with the density function $f(x) = -3x^4$

a: Find the corresponding cumulative distribution function $F(x)$

b: Compute $Pr(1 \le X \le 6)$ and $Pr(X \ge 6)$

28 Section 12.3

Section 12.3 Definition: Let X be a continuous random variable whose possible values lie between A and B, and let $f(x)$ be the pdf(probability density function) for X. Then the expected value(or mean) of X is defined to be $E(X) := \int_A^B xf(x)dx$

The Variance of X is defined to be $Var(X) := \int_A^B [x - E(X)]^2 f(x)dx$

Useful simplification: $Var(X) = \int_A^B x^2 f(x)dx - E(X)^2$

If we give a more general definition: $E(g(X)) := \int_A^B g(x)f(x)dx$, notice expected value corresponds to when $g(x) = x$, we have $Var(X) = E(X^2) - E(X)^2$, similar formula holds true for discrete random variable also

Problems:

1: A newspaper publisher estimates that the proportion X of space devoted to news on a given day is a random variable with the beta probability density $f(x) = 30x^2(1-x)^2, 0 \le x \le 1$

(a) Find the cdf(cumulative distribution function) $F(x)$ for X

(b) Find the probability that less than 25% of the newspaper's space on a given day contains news

(c) Find expected value $E(X)$

(d) Compute variance $Var(X)$

2: The useful life (in hundreds of hours) of a certain machine component is a random variable X with the cumulative distribution function $F(x) = \frac{1}{1521}x^2, 0 \le x \le 39$

(a) Find expected value $E(X)$

(b) Compute variance $Var(X)$

3: The amount of time (in minutes) that a person spends reading the editorial page of the newspaper is a random variable with the density function, $f(x) = \frac{1}{50}x, 0 \le x \le 10$. Find the average time spent reading the editorial page

A bit more for the review test of chapter 10(involving section 10.2):

Problem 3: Use three repetitions of the Newton-Raphson algorithm to approximate the zero of $e^x + 8x - 3$ near $x_0 = 0$

Solution:

Step I: Identify your equation with a corresponding function, in this particular case, $f(x) = e^x + 8x - 3$

Step II: Iteration by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$, $\cdots$, $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, in this particular case, $f'(x) = e^x + 8$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.2222222222$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.2193433075$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.2193427483$$

Thus the zero of $e^x + 8x - 3$ near $x_0 = 0$ is approximately 0.21934

Problem 4: Suppose that an investment of \$590 yields returns of \$100, \$200, and \$300 at the end of the first, second, and third months, respectively. Determine the internal rate of return on this investment

Solution:

Step I: Suppose the internal rate of return is i, then we should have

$$590 = 100(1+i)^{-1} + 200(1+i)^{-2} + 300(1+i)^{-3}$$
$$\xRightarrow{\div 10} 59 = 10(1+i)^{-1} + 20(1+i)^{-2} + 30(1+i)^{-3}$$
$$\xRightarrow{\times(1+i)^3} 59(1+i)^3 = 10(1+i)^2 + 20(1+i) + 30$$
$$\Rightarrow 59(1+i)^3 - 10(1+i)^2 - 20(1+i) - 30 = 0$$

Step II: Construct a problem for Newton-Raphson algorithm, in this particular case, we use the Newton-Raphson algorithm to approximate the zero of $59x^3 - 10x^2 - 20x - 30$ near $x_0 = 1$

Step III: Using Newton-Raphson algorithm, $f(x) = 59x^3 - 10x^2 - 20x - 30$, $f'(x) = 177x^2 - 20x - 20$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.00729927$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.007235299$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.007235294$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.007235294$$

Thus $1 + i \approx 1.0072 \Rightarrow i \approx 0.0072 = 0.72\%$

29 Section 12.4

Section 12.4a Motivating example:

Definition: $f(x) = ke^{-kx} (k > 0)$ for $x \geq 0$ is called an exponential density function, if X is a random variable with an exponential density function, it is called an exponential random variable, cdf of X will be $F(x) = \int_0^x f(t)dt = \int_0^x ke^{-kt}dt = (-e^{-kt})\Big|_0^x = (-e^{-kx}) - (-1) = 1 - e^{-kx}$, using integration by parts $E(X) = \frac{1}{k}$, and $Var(X) = \frac{1}{k^2}$

We say $X \sim Exp(k)$ meaning X is a random variable satisfying exponential distribution with parameter k

Problems:

1: Find the expected value and variance of the exponential random variable with the pdf given by $1.7e^{-1.7x}$

2: Suppose that in a large factory there is an average of two accidents per day and the time between accidents has an exponential density function with expected value of $\frac{1}{2}$ day. Find the probability that the time between two accidents will be more than $\frac{1}{2}$ day and less than 1 day

3: The amount of time required to serve a customer at a fast food restaurant has an exponential density function with mean 5 minutes. Find the probability that a customer is served in less than 3 minutes

4: The amount of time required to serve a customer at a bank has an exponential density function with mean 4 minutes. Find the probability that a customer is served in more than 13 minutes

5: During a certain part of the day, the time between arrivals of automobiles at the tollgate on a turnpike is an exponential random variable with expected value 10 seconds. Find the probability that the time between successive arrivals is more than 20 seconds

6: Suppose that the average life span of an electronic component is 84 months and that the life spans are exponentially distributed

(a) Find the probability that a component lasts for more than 72 months

(b) The reliability function $r(t)$ gives the probability that a component will last for more than t months. Compute $r(t)$ in this case

30 Section 12.4b

Section 12.4b Remark: When we say random variable X satisfy some distribution, we mean its cdf(cumulative distribution function is of that form)

Motivating example: Galton Board $\to$ binomial distribution $\xrightarrow[\text{De Moivre-Laplace theorem}]{\text{in large scale}}$ Normal distribution

Definition: $X \sim N(\mu, \sigma^2)$, we say X is continuous random variable satisfying a normal distribution with mean μ, and variance σ^2(which means the standard deviation is $\sigma > 0$), where the pdf(probability density function) is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $\sigma > 0$, notice here $A = -\infty, B = +\infty$, $E(X) = \mu, Var(X) = \sigma^2, \sigma(X) = \sigma$, the graph of f is called a normal curve(sometimes bell curve), with symmetric axis $x = \mu$, if σ is greater, variance is bigger, the normal curve is shorter and wider, if σ is smaller, the variance is smaller, the normal curve is taller and narrower(centered around the mean)

The most special case is when $\mu = 0, \sigma = 1$, namely(for some psychological reasons, people often use Z,z,ϕ,Φ), $Z \sim N(0, 1^2) = N(0, 1)$(meaning Z is normally distributed with mean 0 and variance 1, sometimes also called the standard normal distribution), the correspponding pdf is $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ still with $A = -\infty, B = +\infty$, and $E(Z) = 0$, $Var(Z) = 1$, $\sigma(Z) = 1$, the cdf(cumulative distribution function) of $Z \sim N(0, 1)$ is not possible expressed in finite terms and compositions of those elementary functions(which has a mathematically rigorous proof!), instead we just invent a name to denote this function, define $\Phi(z) := Pr(Z \leq z) = \int_{-\infty}^{z} f(t)dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$, corresponding we would have $Pr(a \leq Z \leq b) = \int_a^b f(z)dz = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dx = \Phi(b) - \Phi(a)$, $Pr(Z > a) = 1 - Pr(Z \leq a) = 1 - \Phi(a)$, and informally $\Phi(-\infty) = 0, \Phi(+\infty) = 1$, note that $\phi(z)$ is symmetric over the y axis, thus $\Phi(0) = Pr(Z \leq 0) = Pr(Z \geq 0) = 1 - \Phi(0) \Rightarrow Pr(Z \leq 0) = Pr(Z \geq 0) = \Phi(0) = \frac{1}{2}$, use this symmetry, people created the table for areas under the standard normal curve(z table) to help with calculations(especially in the past when all calculations are done purely by hand!), from the table, $A(z)$ is the area between 0 and z(here z can be negative), we have $A(z)$ is symmetric over y axis, meaning $A(z) = A(-z)$, and $A(0) = 0$, thus for $z \geq 0$, $\Phi(z) = \frac{1}{2} + A(z)$, $z < 0$, $\Phi(z) = \frac{1}{2} - A(z)$

Standard normal random variable is of particular interest because every normal random variable can be normalized(translate and scale) into a standard normal random variable as follow, suppose $X \sim N(\mu, \sigma^2)$, then $Z := \frac{X - \mu}{\sigma} \sim N(0, 1)$

Problems:

1. Find the expected value and the standard deviation(by inspection) of the normal random variable with the density function $\frac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-6}{4}\right)^2}$

2. Let Z be a standard normal random variable. Calculate the following probabilities using the table for areas under the standard normal curve(z table)
$Pr(-1.5 \le Z \le 0) = 0.4332,\ Pr(0.83 \le Z) = 0.2033,\ Pr(-1 \le Z \le 2.5) = 0.8351,\ Pr(Z \le 1) = 0.8413$

3. Suppose that the life span of a certain automobile tire is normally distributed with $\mu = 28,000$ miles, $\sigma = 2500$ miles
(a) Find the probability that a tire will last between $33,000$ and $35,500$ miles

(b) Find the probability that a tire will last more than $33,000$ miles

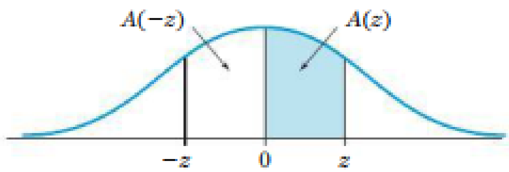


TABLE 1 Areas under the Standard Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3820
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

31 Section 12.5

Section 12.5 Definition: Let X be a discrete random variable that takes value $0, 1, 2, 3, \cdots$, and let $p_n = Pr(X = n)$, then we necessarily have $p_i \geq 0$ and $\sum_{i=0}^{\infty} p_i = p_0 + p_1 + p_2 + p_3 + \cdots = 1$, define the expected value to be $\overline{X} = E(X) = \sum_{i=0}^{\infty} i p_i = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + \cdots$, variance to be $Var(X) = \sum_{i=0}^{\infty} (i - \overline{X})^2 p_i = (0 - \overline{X})^2 \cdot p_0 + (1 - \overline{X})^2 \cdot p_1 + (2 - \overline{X})^2 \cdot p_2 + (3 - \overline{X})^2 \cdot p_3 + \cdots$,

and standard deviation to be $\sigma(X) = \sqrt{Var(X)}$
We use notation $X \sim P(\lambda)$ to mean X satisfies Poisson distribution(X is a Poisson random variable, etc), concretely means that $p_n = \frac{\lambda^n}{n!}e^{-\lambda}$, we can compute $E(X) = Var(X) = \lambda$
We use notation $X \sim G(p)$ to mean X satisfies geometric distribution(X is a geometric random variable, etc), concretely means that $p_n = p^n(1-p)$, we can compute $E(X) = \frac{p}{1-p}, Var(X) = \frac{p}{(1-p)^2}$
Modeling based on Poisson and Geometric distribution:
Suppose a company runs by waiting for phone calls, let X be the number of phone calls in a minute, and suppose the average of phone calls in a minute is λ, then $X \sim P(\lambda)$, and $Pr(X = n) = \frac{\lambda^n}{n!}e^{-\lambda}$
Suppose you have a uneven coin with probability of showing heads p, and therefore with probability of showing tails $1-p$, let X be the number of heads before the first tail, then $X \sim G(p)$, $Pr(x = n) = p^n(1-p)$
Problems:

(1) The monthly number of fire insurance claims filed with the Firebug Insurance Company is Poisson distributed with $\lambda = 13$
(a) What is the probability that in a given month no claims are filed?
(b) What is the probability that in a given month no more than two claims are filed?
(c) What is the probability that in a given month three or more claims are filed?

(2) A bakery makes gourmet cookies. For a batch of 6000 oatmeal and raisin cookies, how many raisins should be used so that the probability of a cookie having no raisins is 0.03? Assume the number of raisins in a random cookie has a Poisson distribution

(3) In a certain town, there are two competing taxicab companies, Red Cab and Blue Cab. The taxis mix with downtown traffic in a random manner. There are five times as many Red taxis as Blue taxis. Suppose you stand on a downtown street and count the number X of Red taxis before the first Blue taxi appears
(a) Determine the formula for $Pr(X = n)$
(b) What is the likelihood of observing at least four Red taxis before the first Blue taxi?
(c) What is the average number of consecutive Red taxis prior to the appearance of a Blue taxi?

(4) Suppose that a large number of persons become infected by a particular strain of a bacteria that is present in food served by a restaurant and that the germ usually produces a certain symptom in 12% of the persons infected. What is the probability that when customers are examined the first person to have the symptom is the fifth customer examined?

32 common formula

32.1 Notations

$\pm$ means $+$ or $-$, $\mp$ means $-$ or $+$, for example $1 \pm \sqrt{2}$ means $1 + \sqrt{2}$ or $1 - \sqrt{2}$

Composition: $f \circ g(x) := f(g(x))$, for example if $f(x) = \sqrt{x}, g(x) = e^x$, then $f(g(x)) = \sqrt{g(x)} = \sqrt{e^x}$, if $f(x) = \frac{1}{x}, g(x) = \sin x$, then $f(g(x)) = \frac{1}{g(x)} = \frac{1}{\sin x}$, if $f(x) = \sqrt{x}, g(x) = \frac{1}{x}, h(x) = e^x$, then $f \circ g \circ h(x) = f(g(h(x))) = \sqrt{g(h(x))} = \sqrt{\frac{1}{h(x)}} = \sqrt{\frac{1}{e^x}} = \sqrt{e^{-x}}$

$n! = 1 \times 2 \times \cdots \times (n-1) \times n$ is called the n-th factorial, and define $0! = 1$, for example, $1! = 1, 2! = 1 \times 2 = 2, 3! = 1 \times 2 \times 3 = 6$

$f^{(n)}(x)$ means the n-th derivative, for example $f^{(0)}(x) = f(x), f^{(1)}(x) = f'(x), f^{(2)}(x) = f''(x) = (f'(x))'$

32.2 Fractions

Scaling: $\frac{a}{b} = \frac{ac}{bc}, c \neq 0$, for example $\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$

Summation(Subtraction): $\frac{a}{b} \pm \frac{c}{d} = \frac{ad}{bd} \pm \frac{bc}{db} = \frac{ad \pm bc}{bd}$, for example $\frac{7}{5} + \frac{2}{3} = \frac{7 \cdot 3}{5 \cdot 3} + \frac{2 \cdot 5}{3 \cdot 5} = \frac{7 \cdot 3 + 2 \cdot 5}{5 \cdot 3} = \frac{21 + 10}{15} = \frac{31}{15}$, $\frac{2}{5} - 2 = \frac{2}{5} - \frac{2}{1} = \frac{2}{5} - \frac{2 \cdot 5}{1 \cdot 5} = \frac{2}{5} - \frac{10}{5} = \frac{2 - 10}{5} = \frac{-8}{5} = -\frac{8}{5}$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, for example $\frac{7}{5} \cdot \frac{2}{3} = \frac{7 \cdot 2}{5 \cdot 3} = \frac{14}{15}$, $\frac{4}{7} \cdot 3 = \frac{4}{7} \cdot \frac{3}{1} = \frac{4 \cdot 3}{7 \cdot 1} = \frac{12}{7}$

Division: $\frac{a}{b} \Big/ \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$, for example $\frac{7}{5} \Big/ \frac{2}{3} \frac{7}{5} \cdot \frac{3}{2} = \frac{7 \cdot 3}{5 \cdot 2} = \frac{21}{10}$, $\frac{4}{7} \Big/ 3 = \frac{4}{7} \Big/ \frac{3}{1} = \frac{4}{7} \cdot \frac{1}{3} = \frac{4 \cdot 1}{7 \cdot 3} = \frac{4}{21}$

32.3 Exponential

$A^0 = 1, A \neq 0$

$A^{-a} = \frac{1}{A^a}$, for example $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$A^{\frac{1}{a}} = \sqrt[a]{A}$ for example $\sqrt{3} = 3^{\frac{1}{2}}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$

$A^a \cdot A^b = A^{a+b}$, for example $2^3 \cdot 2^{-1} = 2^{3+(-1)} = 2^{3-1} = 2^2 = 4$, $A^a/A^b = A^a \cdot \frac{1}{A^b} = A^a \cdot A^{-b} = A^{a-b}$

$(A^a)^b = A^{ab}$, for example $\left(3^2\right)^3 = 3^{2\cdot 3} = 3^6 = 729$

$(AB)^a = A^a \cdot B^a$, for example $(3\cdot 5)^{\frac{1}{2}} = 3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}}$, $\left(\frac{A}{B}\right)^a = \left(A \cdot B^{-1}\right)^a = A^a \cdot \left(B^{-1}\right)^a = A^a \cdot B^{-a} = \frac{A^a}{B^a}$

32.4 Logarithms

$\log_e(x)$ is often denoted as $\ln(x)$ or simply $\ln x$, so remember $\ln x$ is a function of x

$\log_a(x) = y \Leftrightarrow a^y = x$, for example $\log_e 1 = \ln 1 = 0$ because $e^0 = 1$

$\log_a(xy) = \log_a(x) + \log_a(b)$, $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$, for example $\ln(2\cdot 3) = \ln 2 + \ln 3$, $\ln\left(\frac{5}{7}\right) = \ln 5 - \ln 7$

$\log_a\left(x^b\right) = b\log_a x$, for example $\ln\sqrt{x} = \ln\left(x^{\frac{1}{2}}\right) = \frac{1}{2}\ln x$

32.5 Derivatives

Summation(Subtraction): $(af(x) \pm bg(x))' = af'(x) \pm bg'(x)$, derivative is distributive,for example $(e^x - 2\sin x)' = (1\cdot e^x + (-2)\cdot \sin x)' = 1\cdot (e^x)' + (-2)(\sin x)' = (e^x)' - 2(\sin x)'$, or simply $(e^x - 2\sin x)' = (e^x)' - 2(\sin x)' = e^x + 2\cos x$, if we take $b = 0$, then $(af(x))' = af'(x)$

Product: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, for example $(e^x \sin x)' = (e^x)'\sin x + e^x(\sin x)' = e^x\sin x + e^x\cos x$

Composition(Chain rule): $[f(g(x))]' = f'(g(x))g'(x)$, for example $\left[\ln(1+5x^2)\right]'$, let $f(x) = \ln x, g(x) = 1+5x^2$, then $f'(x) = \frac{1}{x}, g'(x) = 10x$, $\left[\ln(1+5x^2)\right]' = [f(g(x))]' = f'(g(x))g'(x) = \frac{1}{g(x)}g'(x) = \frac{1}{1+5x^2}\cdot 10x = \frac{10x}{1+5x^2}$

Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, notice quotient is a product with composition $\frac{f(x)}{g(x)} = f(x)\cdot\frac{1}{g(x)}$

$(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x} = x^{-1}$,$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

32.6 Differentials

Summation(Subtraction): $d(af \pm bg) = adf \pm bdg$, where a,b are constants, in other words, d is distributive, if we take $b = 0$, then $d(af) = adf$

Product: $d(fg) = gdf + fdg$

Composition(Chain rule): $df = \frac{df}{dg}dg = f'(g)dg$

Quotient: $d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$

$\frac{df}{dx} = f'(x) \Rightarrow df = f'(x)dx \Rightarrow \int df = \int f'(x)dx = f(x)$

$dc = c'dx = 0 \Rightarrow d(af + c) = adf$, where a, c are constants

$dx^n = nx^{n-1}dx$, $de^x = e^x dx$, $d\ln x = \frac{1}{x}dx = \frac{dx}{x}$,$d\sin x = \cos x dx$, $d\cos x = -\sin x dx$, $d\tan x = \sec^2 x dx$

32.7 Integrations

$\int (af(x) \pm bg(x))dx = a\int f(x)dx \pm b\int g(x)dx$, $\int$ is distributive, for example $\int (e^x + 2\cos x)dx = \int e^x dx + 2\int \cos x dx = e^x + 2\sin x + C$. Note that adding C is to cover all antiderivatives, if we take $b = 0$, then $\int af(x)dx = a\int f(x)dx$

$\int f(g(x))g'(x)dx \overset{u=g(x),du=g'(x)dx}{=\!=\!=} \int f(u)du$, this is integration by substitution, for example $\int e^{\sin x}\cos x dx \overset{u=\sin x,du=\cos x dx}{=\!=\!=} \int e^u du = e^u + C = e^{\sin x} + C$

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$, this is integration by parts, for example $\int xe^x dx = \int x(e^x)'dx = xe^x - \int e^x x'dx = xe^x - \int e^x dx = xe^x - e^x + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n+1 \neq 0$, $\int x^{-1} dx = \int \frac{1}{x} dx = \ln x + C$, $\int e^x dx = e^x + C$, $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int \sec^2 x dx = \tan x + C$

32.8 Trigonometry identities

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$, for example $\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) = \sin\left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$

$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$

$\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$, $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$,

32.9 Trigonometry table

θ	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\times$

32.10 Approximation to definite integrals

Given $a = a_0, b = a_n$, and divide interval (a,b) into n equal parts $(a_0,a_1),(a_1,a_2),\cdots,(a_{n-2},a_{n-1}),(a_n,a_n)$ with $\Delta x = a_i - a_{i-1}$ being the length of each subinterval, and let $x_i = \frac{a_{i-1}+a_i}{2}$ be the midpoint of a_{i-1} and a_i

Midpoint rule: $\int_a^b f(x)dx \approx [f(x_1)+f(x_2)+\cdots+f(x_{n-1})+f(x_n)]\Delta x$

Trapezoidal rule: $\int_a^b f(x)dx \approx [f(a_0)+2f(a_1)+2f(a_2)+\cdots+2f(a_{n-1})+f(a_n)]\frac{\Delta x}{2}$

Simpson's rule: $\int_a^b f(x)dx \approx [f(a_0)+4f(x_1)+2f(a_1)+4f(x_2)+2f(a_2)+\cdots+2f(a_{n-1})+4f(x_n)+f(a_n)]\frac{\Delta x}{6}$

32.11 Method of Integrating factors

Always starts with a DE(differential equation) of the form $y' + a(t)y = b(t)$. Note that if your DE is $t^2y' + y = t^3$, then first you need to divide t^2 on both sides, then you would have $y' + \frac{1}{t^2}y = t$ with $a(t) = \frac{1}{t^2}, b(t) = t$

Then take any antiderivative $A(t) = \int a(t)dt$, such that $A'(t) = a(t)$

Then we get the integrating factor $e^{A(t)}$, for example if $a(t) = 2$, then $A(t)$ could be $2t$, then $e^{A(t)} = e^{2t}$

Then we have $\left[ye^{A(t)}\right]' = b(t)e^{A(t)}$

Then we integrate on both sides to get $ye^{A(t)} = \int ye^{A(t)}dt = \int b(t)e^{A(t)}dt$

At last, we divide $e^{A(t)}$ on both sides(which is equivalent to multiplying $e^{-A(t)}$ on both sides), we get $y = e^{-A(t)}\int b(t)e^{A(t)}dt$

For example, suppose the DE we have is $y' + 2y = t$

First we identify $a(t) = 2, b(t) = t$

Find one antiderivative of $a(t)$, $A(t) = \int 2dt = 2t$, notice that the reason for omitting C is that we only need to find one such antiderivative

The integrating factor is $e^{A(t)} = e^{2t}$

We have $\left[ye^{2t}\right]' = te^{2t}$

We have $ye^{2t} = \int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$

Dividing e^{2t}(or equivalently, multiplying e^{-2t}), we have $y = \frac{1}{2}t - \frac{1}{4} + Ce^{-2t}$

32.12 Taylor polynomials

The n-th Taylor polynomial of $f(x)$ at $x = a$ is the polynomial $p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$, note that $p_n(a) = f(a), p_n'(a) = f'(a), p_n''(a) = f''(a), \cdots, p_n^{(n)}(a) = f^{(n)}(x)$, for example $(e^x)^{(n)} = e^x$, thus the n-th Taylor polynomial of e^x at $x = 0$ is $p_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$

Taylor series(expansion) of $f(x)$ at $x = a$ means $f(x) \sim f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$, for example the Taylor series of $f(x) = \ln(1+x)$ at $x = 0$, since $(\ln(1+x))^{(n)} = (-1)^{n+1}(n-1)!(1+x)^{-n}$ for $n \geq 1$, $\ln(1+x) \sim x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots$

32.13 Random variable and probability

Let X be a discrete random variable, having possible outcomes $a_1, \cdots, a_n$ with corresponding probabilities $p_1, \cdots, p_n$, the probability table is as follows

X	a_1	a_2	a_3	$\cdots$	a_n
P	p_1	p_2	p_3	$\cdots$	p_n

The expected value is defined as $\bar{a} = E(X) = a_1p_1 + \cdots a_np_n$, and the variance is defined as $Var(X) = (a_1 - \bar{a})^2p_1 + \cdots + (a_n - \bar{a})^2p_n$, and the standard deviation is defined as $\sigma(X) = \sqrt{Var(X)}$

33 Formula sheet

33.1 Derivatives

Summation(Subtraction): $(af(x) \pm bg(x))' = af'(x) \pm bg'(x)$, for example $(e^x - 2\sin x)' = (1 \cdot e^x + (-2) \cdot \sin x)' = 1 \cdot (e^x)' + (-2)(\sin x)' = (e^x)' - 2(\sin x)'$, or simply $(e^x - 2\sin x)' = (e^x)' - 2(\sin x)' = e^x + 2\cos x$, if we take $b = 0$, then $(af(x))' = af'(x)$

Product: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$, for example $(e^x \sin x)' = (e^x)' \sin x + e^x(\sin x)' = e^x \sin x + e^x \cos x$

Composition: $[f(g(x))]' = f'(g(x))g'(x)$, for example $\left[\ln(1+5x^2)\right]'$, let $f(x) = \ln x, g(x) = 1 + 5x^2$, then $f'(x) = \frac{1}{x}, g'(x) = 10x$, $\left[\ln(1+5x^2)\right]' = [f(g(x))]' = f'(g(x))g'(x) = \frac{1}{g(x)}g'(x) = \frac{1}{1+5x^2} \cdot 10x = \frac{10x}{1+5x^2}$

Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, notice quotient is a product with composition $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$

Derivative of some elementary functions

$(x^n)' = nx^{n-1}$, $(e^x)' = e^x$, $(\ln x)' = \frac{1}{x} = x^{-1}$,$(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

33.2 Integrations

Summation(Subtraction): $\int (af(x) \pm bg(x))\,dx = a\int f(x)dx \pm b\int g(x)dx$, for example $\int (e^x + 2\cos x)\,dx = \int e^x dx + 2\int \cos x dx = e^x + 2\sin x + C$. Note that adding constant C is to cover all antiderivatives, if we take $b = 0$, then $\int af(x)dx = a\int f(x)dx$

$\int f(g(x))g'(x)dx \overset{u=g(x),du=g'(x)dx}{=\!=\!=} \int f(u)\,du$, this is integration by substitution, for example $\int e^{\sin x}\cos x dx \overset{u=\sin x,du=\cos x dx}{=\!=\!=} \int e^u du = e^u + C = e^{\sin x} + C$

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$, this is integration by parts, for example$\int xe^x dx = \int x(e^x)'dx = xe^x - \int e^x x' dx = xe^x - \int e^x dx = xe^x - e^x + C$

Integrations of some elementary functions

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ when $n+1 \neq 0$, $\int x^{-1}dx = \int \frac{1}{x}dx = \ln x + C$, $\int e^x dx = e^x + C$, $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int \sec^2 x dx = \tan x + C$

33.3 Trigonometry identities

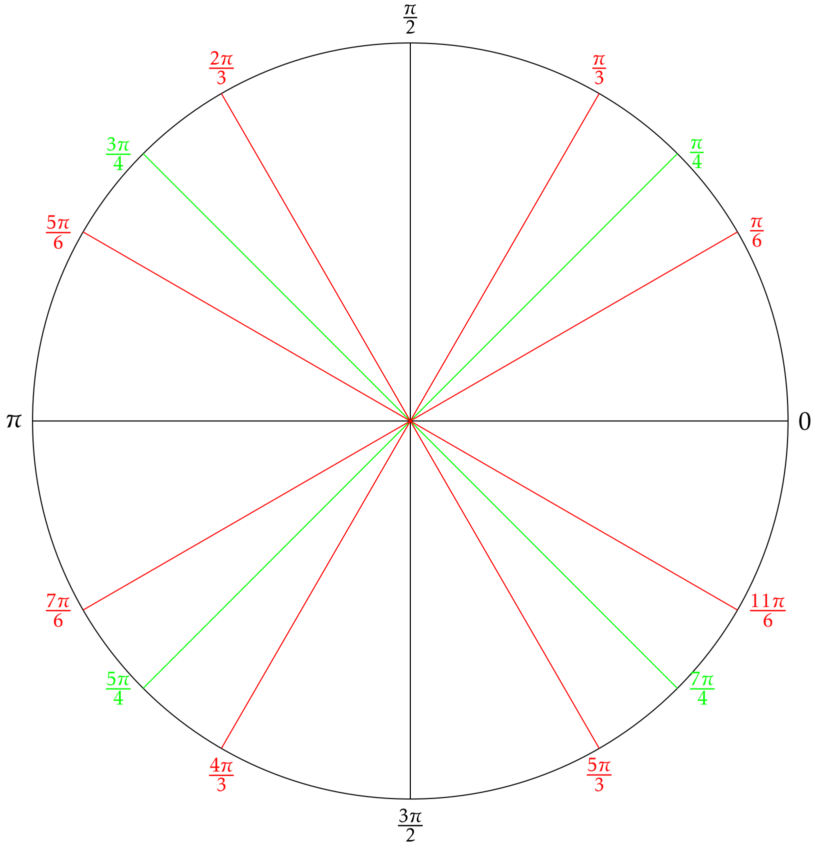
$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$, $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$,

33.4 Trigonometry table

θ	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$\times$



33.5 Approximation to definite integrals

Given $a = a_0, b = a_n$, and divide interval (a, b) into n equal parts $(a_0, a_1), (a_1, a_2), \cdots, (a_{n-2}, a_{n-1}), (a_n, a_n)$ with $\Delta x = a_i - a_{i-1}$ being the length of each subinterval, then $a_i = a_{i-1} + \Delta x$

Trapezoidal rule: $\int_a^b f(x)dx \approx [f(a_0) + 2f(a_1) + 2f(a_2) + \cdots + 2f(a_{n-1}) + f(a_n)]\frac{\Delta x}{2}$

33.6 Method of Integrating factors

Always starts with a DE(differential equation) of the form $y' + a(t)y = b(t)$. Note that if your DE is $t^2y' + y = t^3$, then first you need to divide t^2 on both sides, then you would have $y' + \frac{1}{t^2}y = t$ with $a(t) = \frac{1}{t^2}, b(t) = t$

Then take any antiderivative $A(t) = \int a(t)dt$, such that $A'(t) = a(t)$

Then we get the integrating factor $e^{A(t)}$, for example if $a(t) = 2$, then $A(t)$ could be $2t$, then $e^{A(t)} = e^{2t}$
Then we have $\left[ye^{A(t)}\right]' = b(t)e^{A(t)}$

Then we integrate on both sides to get $ye^{A(t)} = \int ye^{A(t)}dt = \int b(t)e^{A(t)}dt$

At last, we divide $e^{A(t)}$ on both sides(which is equivalent to multiplying $e^{-A(t)}$ on both sides), we get $y = e^{-A(t)}\int b(t)e^{A(t)}dt$

For example, suppose the DE we have is $y' + 2y = t$
First we identify $a(t) = 2, b(t) = t$

Find one antiderivative of $a(t)$, $A(t) = \int 2dt = 2t$, notice that the reason for omitting C is that we only need to find one such antiderivative

The integrating factor is $e^{A(t)} = e^{2t}$
We have $\left[ye^{2t}\right]' = te^{2t}$

We have $ye^{2t} = \int te^{2t}dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$

Dividing e^{2t}(or equivalently, multiplying e^{-2t}), we have $y = \frac{1}{2}t - \frac{1}{4} + Ce^{-2t}$

33.7 Geometric series and integral test

A geometric series is $a + ar + ar^2 + ar^3 + \cdots$, where $|r| < 1$, the sum is given by $\frac{a}{1-r}$

Suppose $f(x)$ is a continuous, positive, decreasing function on $x \geq n$, then $\int_n^\infty f(x)dx$ is convergent if and only if $\sum_{k=n}^{\infty} f(k)$ is convergent, which doesn't mean the have the same value!

33.8 Taylor polynomials

The n-th Taylor polynomial of $f(x)$ at $x = a$ is the polynomial $p_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$, note that $p_n(a) = f(a), p_n'(a) = f'(a), p_n''(a) = f''(a), \cdots, p_n^{(n)}(a) = f^{(n)}(x)$, for example $(e^x)^{(n)} = e^x$, thus the n-th Taylor polynomial of e^x at $x = 0$ is $p_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$

33.9 Random variable and probability

Suppose X is a continuous random variable taking values within A and B, where A can be $-\infty$ and B can be ∞, we define the probability through a density function $Pr(a \leq X \leq b) = \int_a^b f(x)dx, A \leq a \leq b \leq B$ where f is called the probability density function satisfying

(1) $f(x) \geq 0$ for $A \leq x \leq B$(since the probability has to be nonnegative)

(2) $\int_A^B f(x)dx = 1$(since the integral of all possibilities should be 1)

We also define cumulative distribution function $F(x) := Pr(A \leq X \leq x) = \int_A^x f(t)dt$(here use t because x is already taken), then we have $F'(x) = f(x)$ and thus by fundamental theorem of calculus we have $Pr(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$

So given a probability density function of a random variable X, we can find its cumulative distribution function by definition, given its cumulative distribution function, we can get its probability distribution function by taking derivative

Let X be a continuous random variable whose possible values lie between A and B, and let $f(x)$ be the probability density function for X. Then the expected value(or mean) of X is defined to be $E(X) := \int_A^B xf(x)dx$, and variance to be $Var(X) = \int_A^B x^2 f(x)dx - E(X)^2$

Exponential distribution: $X \sim E(k)$, pdf: $f(x) = ke^{-kx}(k > 0)$ for $x \geq 0$, $E(X) = \frac{1}{k}$, $Var(X) = \frac{1}{k^2}$

Normal distribution: $X \sim N(\mu, \sigma^2)$, pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $\sigma > 0$, $E(X) = \mu$, $Var(X) = \sigma^2$, a standard normal distribution is $Z \sim N(0, 1^2)$, pdf: $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $E(Z) = 0$, $Var(Z) = 1$, and for any $X \sim N(\mu, \sigma^2)$, we can normalize it to $Z = \frac{X - \mu}{\sigma} \sim N(0, 1^2)$ which is a standard normal distribution, know how to use the z-table

Poisson distribution: $X \sim P(\lambda)$, $p_n = \frac{\lambda^n}{n!} e^{-\lambda}$, $E(X) = Var(X) = \lambda$

Geometric distribution: $X \sim G(p)$, $p_n = p^n(1-p)$, $E(X) = \frac{p}{1-p}, Var(X) = \frac{p}{(1-p)^2}$