My Mathematical Universe

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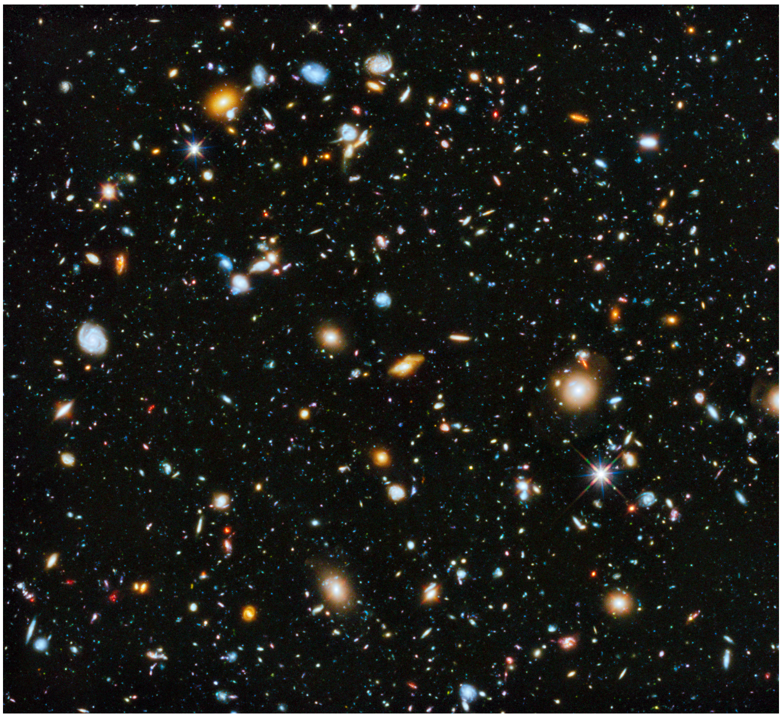

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Contents

Part I

Set theory

Chapter 1

Set theory

Definition 1.0.1. $\{A_i\} \subseteq \mathcal{P}(X)$, $X \xrightarrow{f} Y$ is a map. f **separates** A_i if $\bigcap_i f(A_i) = \varnothing$. f **completely separates** A_i if $f(A_i) = f(a_i)$ for some distinct $a_i \in A_i$. f **perfectly separates** A, B if $A_i = f^{-1}(a_i)$ for some $a_i \in A_i$

Zorn's lemma

Lemma 1.0.2 (Zorn's lemma)**.** P is a nonempty poset and every chain has an upper bound, then P contains a maximal element

Theorem 1.0.3 (Schröder–Bernstein theorem)**.** $A \xrightarrow{f} B$ and $B \xrightarrow{g} A$ are injective, then there exists $A \xrightarrow{h} B$ bijective

Inclusion-exclusion principle

Theorem 1.0.4 (Inclusion-exclusion principle)**.** $A_1, \cdots, A_n \subseteq S$ are of finite cardinality, then

$$\left| \bigcup_{i=1}^{n} A_n \right| = \sum_{k=1}^{n} (-1)^k \sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|$$

Definition 1.0.5. A **lattice** is a partially ordered set in which the supremum and infinimum of any two elements exists uniquely

Lemma 1.0.6. Trees are bipartite

Proof. Take some $v \in T$ as the root, and label the nodes that are even distance away from 2 and odd distance away from 1 $\square$

Definition 1.0.7. A **Hasse diagram** is a mathematical diagram used to represent a partially ordered set

$$\begin{tikzcd}
 & \{x,y,z\} & & 2 \\
\{x,y\} \arrow[ur] & \{x,z\} \arrow[u] & \{y,z\} \arrow[ul] & 1 \\
\{x\} \arrow[u] \arrow[ur] & \{y\} \arrow[ul] \arrow[ur] & \{z\} \arrow[u] \arrow[ul] & 0 \\
 & \{\} \arrow[ul] \arrow[u] \arrow[ur] & & -1
\end{tikzcd}$$

Definition 1.0.8. The *Bermoulli numbers* B_n are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}$$

For instance, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, etc.

Chapter 2

Graph theory

2.1 Graph

Definition 2.1.1. A **complete graph** K_n consists of n vertices and all possible edges **Com-**

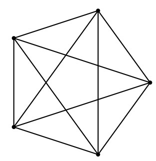

Figure 2.1.1: K_5

K5 complete graph

plete bipartite graphs are $K_{n,m}$ with n, m vertices on each side and all possible edges between them

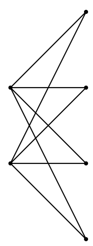

Figure 2.1.2: $K_{2,4}$

K2,4 complete bipartite graph

Definition 2.1.2. G is k vertex connected if G has more than k vertices and remain connected when removing less than k vertices

Proposition 2.1.3. Every convex polytope can be represented by a 3 vertex connected planar graph

Remark 2.1.4. A graph embedded in the plane through different ways may have different dual graphs However, if the graph is 3 vertex connected, then the dual graph will be canonical

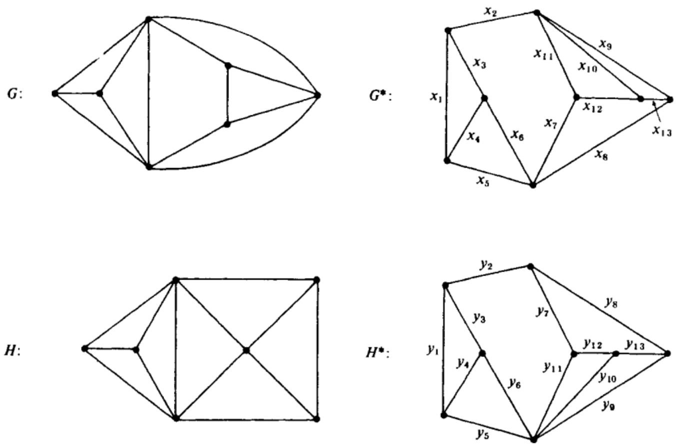

Figure 2.1.3: Different dual graphs for different planar embeddings of the same graph

Different dual graphs

Part II

Abstract Algebra

Chapter 3

Category theory

3.1 Category

Definition 3.1.1. A *semicategory* $\mathscr{C}$ consists of

- A class of *objects* $\mathtt{Ob}\,\mathscr{C}$
- A class of *morphisms* $\mathtt{Hom}\,\mathscr{C}$
- Compositions $\mathtt{Hom}(A,B)\times\mathtt{Hom}(B,C)\to\mathtt{Hom}(A,C)$

Such that

- Compositions are associative: $(hg)f = h(gf)$

Definition 3.1.2. A *category* $\mathscr{C}$ consists of

- A class of *objects* $\mathtt{Ob}\,\mathscr{C}$
- A class of *morphisms* $\mathtt{Hom}\,\mathscr{C}$
- Compositions $\mathtt{Hom}(A,B)\times\mathtt{Hom}(B,C)\to\mathtt{Hom}(A,C)$

Such that

- Compositions are associative: $(hg)f = h(gf)$
- $\mathtt{Hom}(A,A)$ contains *identity* 1_A: $1_A f = f$, $g1_A = g$

Note. 1_A, f^{-1} are unique

Note. A category is a semicategory with identities

Definition 3.1.3. A *functor* $F:\mathscr{C}\to\mathscr{D}$ consists of maps

- $\mathtt{Ob}\,\mathscr{C}\to\mathtt{Ob}\,\mathscr{D}$
- $\mathtt{Hom}_{\mathscr{C}}(A,B)\to\mathtt{Hom}_{\mathscr{D}}(F(A),F(B))$

Such that it

- Preserves identities: $F(1_A)=1_{F(A)}$
- Preserves compositions: $F(g\circ f)=F(g)\circ F(f)$

A *contravariant functor* $F:\mathscr{C}\to\mathscr{D}$ consists of maps

- $\mathtt{Ob}\,\mathscr{C}\to\mathtt{Ob}\,\mathscr{D}$
- $\mathtt{Hom}_{\mathscr{C}}(A,B)\to\mathtt{Hom}_{\mathscr{D}}(F(B),F(A))$

Such that

- $F(1_A)=1_{F(A)}$
- $F(g\circ f)=F(f)\circ F(g)$

Note. Functors are also called *covariant fucntors*

Definition 3.1.4. The *empty category* is the category without any objects nor morphisms

Definition 3.1.5. The *dual category* $\mathscr{C}^{op}$ of $\mathscr{C}$ consists of the same objects and morphisms but with morphisms reversed. A *contravariant functor* $F:\mathscr{C}\to\mathscr{D}$ is a functor $F:\mathscr{C}^{op}\to\mathscr{D}$

Definition 3.1.6. $A\xrightarrow{f}B$ is a *monomorphism* if $fg_1 = fg_2 \Rightarrow g_1 = g_2$, is an *epimorphism* if $g_1 f = g_2 f \Rightarrow g_1 = g_2$, is, is a *bimorphism* is both monic and epi, is an *isomorphism* if it is invertible. Monomorphism and epimorphism are dual notions. Isomorphisms are bimorphisms. A category is *balanced* if bimorphisms are isomorphisms

Remark 3.1.7. A bimorphism is not necessary an isomorphism. In the category of rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a bimorphism because $\mathbb{Q} = \mathbb{Z}_{(0)}$ is a localization and the universal property of localization

Definition 3.1.8. A *natural transformation* is a family of morphisms $\eta_A : F(A) \to G(A)$ making the following diagram commute

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

For contravariant functors

$$\begin{tikzcd} F(B) \arrow[r, "F(f)"] \arrow[d, "\eta_B"] & F(A) \arrow[d, "\eta_A"] \\ G(B) \arrow[r, "G(f)"] & G(A) \end{tikzcd}$$

η is a *natural isomorphism* if η_A are isomorphisms

Definition 3.1.9. $\mathscr{C}$ is a *small categroy* if $ob(\mathscr{C})$ and $Hom(\mathscr{C})$ are sets, otherwise *large*. $\mathscr{C}$ is a *locally small category* if $Hom(a, b)$ are sets

Definition 3.1.10. A *subcategory* $\mathscr{S}$ is a category consists of subclasses of objects and morphisms with the same composition map

Definition 3.1.11. we say categories $\mathscr{C}, \mathscr{D}$ are *isomorphic* if there are functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $G \circ F = 1_{\mathscr{C}}$, $F \circ G = 1_{\mathscr{D}}$ and we say $\mathscr{C}, \mathscr{D}$ are *equivalent* if $G \circ F$ is naturally isomorphic to $1_{\mathscr{C}}$ and $F \circ G$ is naturally isomorphic to $1_{\mathscr{D}}$

Definition 3.1.12. Suppose $\mathscr{C}, \mathscr{D}$ are categories, define the *functor category* $[\mathscr{C}, \mathscr{D}]$ or $\mathscr{D}^{\mathscr{C}}$ has all functors from $\mathscr{C}$ to $\mathscr{D}$ as objects, and natural transformations as morphisms

Definition 3.1.13. $\mathscr{C} \times \mathscr{D}$ is the *product category* with $ob\mathscr{C} \times D = ob\mathscr{C} \times ob\mathscr{D}$, $Hom_{\mathscr{C}\times\mathscr{D}}(A \times B, C \times D) = Hom_{\mathscr{C}}(A, C) \times Hom_{\mathscr{D}}(B, D)$

Definition 3.1.14. Suppose $\mathscr{C}, \mathscr{D}$ are locally small categories, F is *faithful* if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is injective, F is *full* if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is surjective, F is *fully faithful* if $Hom_{\mathscr{C}}(X, Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$ is bijective, F is *essentially surjective* if $\forall d \in ob\mathscr{D}$, $\exists c \in ob\mathscr{C}$ such that $Fc \cong d$

A functor F is an equivalence iff it is fully faithful and essentially surjective

Theorem 3.1.15. $F : \mathscr{C} \to \mathscr{D}$ is an equivalence iff F is fully faithful and essentially surjective

Proof. If F is an equivalence, there exist functor $G : \mathscr{D} \to \mathscr{C}$ and natural isomorphisms $\eta : 1_{\mathscr{C}} \to GF$, $\xi : 1_{\mathscr{D}} \to FG$, $\forall d \in \mathscr{C}$, $\xi_d : d = 1_{\mathscr{D}}(d) \to FG(d) = F(Gd)$ is an isomorphism, i.e. F is essentially surjective, similarly, so is G
The composition of

$$Hom(c, c') \xrightarrow{F} Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc'), \quad f \mapsto Ff \mapsto GFf$$

Is the same as

$$Hom(c, c') \xrightarrow{\eta} Hom(GFc, GFc'), \quad f \mapsto \eta'_c f \eta_c^{-1}$$

By Exercise **??**, this is bijective, thus $Hom(c, c') \xrightarrow{F} Hom(Fc, Fc')$ is injective, i.e. F is faithful. Similarly, consider the composition

$$Hom(Fc, Fc') \xrightarrow{G} Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$$

We know $Hom(GFc, GFc') \xrightarrow{F} Hom(FGFc, FGFc')$ is surjective, but we also have the following diagram

$$\begin{tikzcd} Hom(c,c') \arrow[r, "F"] \arrow[d, "\eta"'] & Hom(Fc,Fc') \arrow[d, "\xi"] \\ Hom(GFc,GFc') \arrow[r, "F"] & Hom(FGFc,FGFc') \end{tikzcd}$$

Since η, ξ are bijective, $Hom(c,c') \xrightarrow{F} Hom(Fc,Fc')$ is surjective, i.e. F is full
Conversely, suppose F is fully faithful and essentially surjective, then for any $d \in \mathscr{D}$, there exists c and an isomorphism $d \xrightarrow{\xi_d} Fc$, denote this c as Gd, we can define a functor $G : \mathscr{D} \to \mathscr{C}$, $d \mapsto Gd$(Here we have used the axiom of choice), $d \xrightarrow{f} d' \mapsto c \xrightarrow{Gf} c'$ where $FGf = \xi_d^{-1} f \xi_{d'}$ since F is fully faithful

$$\begin{tikzcd} d \arrow[r, "f"] \arrow[d, "\xi_d"'] & d' \arrow[d, "\xi_{d'}"] \\ FGd \arrow[r, "FGf"] & FGd' \\ Gd \arrow[r, "Gf"] \arrow[u, "F"] & Gd' \arrow[u, "F"'] \end{tikzcd}$$

$\xi : 1_{\mathscr{D}} \to FG$ is a natural isomorphism
Since F is fully faithful, there are unique $\eta_c : c \to GFc$, $F(\eta_c) = \xi_{Fc}$
If $f, g : c \to c'$ such that $\eta_{c'} f = \eta_{c'} g$, then $\xi_{Fc'} Ff = \xi_{Fc'} Fg \Rightarrow Ff = Fg \Rightarrow f = g$
If $f, g : c \to c'$ such that $f\eta_c = g\eta_c$, then $Ff\xi_{Fc} = Fg\xi_{Fc} \Rightarrow Ff = Fg \Rightarrow f = g$

$$\begin{tikzcd} c \arrow[r] \arrow[d] \arrow[dd, bend right, "\eta_c"'] & c' \arrow[d] \arrow[dd, bend left, "\eta_{c'}"] \\ Fc \arrow[r] \arrow[d, "G"'] \arrow[dd, bend right, "\xi_{Fc}"'] & Fc' \arrow[d, "G"] \arrow[dd, bend left, "\xi_{Fc'}"] \\ GFc \arrow[r] \arrow[d, "F"'] & GFc' \arrow[d, "F"] \\ FGFc \arrow[r] & FGFc' \end{tikzcd}$$

$\eta : 1_{\mathscr{C}} \to GF$ is a natural isomorphism $\square$

Definition 3.1.16. $A \xrightarrow{f} B$ is a *constant morphism* if $fg = fh$ for any g, h, f is a *coconstant morphism* if $gf = hf$ for any g, h, f is a *zero morphism* if it is both a constant and a coconstant morphism

Definition 3.1.17. Suppose $u : S \to A$, $v : T \to A$ are morphisms, v filter through s means there is a morphism $w : T \to S$ such that $v = u \circ w$, then mutually filter defines an equivalence relation on monomorphisms(or equivalent by saying that w is an isomorphism), the equivalence classes are called *subobjects* of A, the dual notion is called *quotient objects*

Proposition 3.1.18. Direct limit is an exact functor

Definition 3.1.19. An *injective object* Q is such that for any monomorphism $f : X \to Y$ and morphism $g : X \to Q$, there is a morphism $h : Y \to Q$ such that $g = h \circ f$

$$\begin{tikzcd} X \arrow[r, hook, "f"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "\exists h"] \\ Q & \end{tikzcd}$$

the dual notion is called a *projective object* P, such that for any epimorphism $f : X \to Y$, and morphism $g : P \to Y$, there is a morphism $h : P \to X$ such that $g = f \circ h$

$$\begin{tikzcd} & P \arrow[d, "g"] \arrow[dl, dashed, "\exists h"'] \\ X \arrow[r, two heads, "f"] & Y \end{tikzcd}$$

Definition 3.1.20. A functor $F : \mathscr{C} \to Set$ is called a *representable* functor if there is an object A in $\mathscr{C}$ such that $\Phi : Hom(A, -) \to F$ is a natural isomorphism

Definition 3.1.21. Let $\mathscr{C}$ be a category, we can define *quotient category* by moding out a congruence relation $\sim$, here $\sim$ is an equivalence relation on $Hom(X, Y)$ for any X, Y and it respects composition, i.e. suppose $f_1 \sim f_2 : X \to Y$, $g_1 \sim g_2 : Y \to Z$, then $g_1 \circ f_1 \sim g_2 \circ f_2$, thus $Hom_{\mathscr{C}/\sim}(X, Y) = Hom_{\mathscr{C}}(X, Y)/\sim$

Definition 3.1.22. $\mathscr{C}$ is *concretizable* if there is a faithful functor $F : \mathscr{C} \to Set$. A morphism $f : X \to Y$ is an *embedding* if $F(f)$ is injective, and for any $F(Z) \xrightarrow{\phi} F(X)$, $Z \xrightarrow{h} Y$ such that $F(Z) \xrightarrow{F(h)} F(Y)$, $F(h) = F(f) \circ \phi$, $\phi = F(g)$ for some $Z \xrightarrow{g} X$

Note. $\mathscr{C}$ may have different concretization

Definition 3.1.23. W is a class of morphisms of $\mathscr{C}$, the *localization* of $\mathscr{C}$ with respect to W, denoted $\mathscr{C}[W^{-1}]$, satisfies universal property

- Any functor $F : \mathscr{C} \to \mathscr{D}$ sending morphisms in W to isomorphisms in $\mathscr{D}$ uniquely factors through the *localization functor* $\mathscr{C} \to \mathscr{C}[W^{-1}]$

Construction 3.1.24. Add formal inverses of the morphisms in $\overline{W}$

Note. $\mathrm{Hom}_{\mathscr{C}[W^{-1}]}(A, B)$ consists of *roofs* $A \leftarrow C \to B$ and $A \to D \leftarrow B$

Definition 3.1.25. A *skeleton of a category* $\mathscr{D}$ of $\mathscr{C}$ is a full subcategory such that no two objects in $\mathscr{D}$ are isomorphic and for every object in $\mathscr{C}$ is isomorphic to some object in $\mathscr{D}$, the functor $\mathscr{D} \hookrightarrow \mathscr{C}$ is an equivalence of categories

Definition 3.1.26. $\mathscr{C}$ is *connected* if there is a finite sequence of morphisms connecting any two objects

Example 3.1.27. $C \leftarrow A \to B$ is a connected even there is no morphism between B, C

Definition 3.1.28. Suppose $\mathscr{C}$ is a category, a *filtered object* X is an object with a *filtration* of X, a descending filtration

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots \to X$$

Or an ascending filtration

$$X \to \cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

Definition 3.1.29. Suppose $\mathscr{C}$ is a category, $f : X \to Y$ is a morphism, the *image* of f is a monomorphism $m : I \to Y$ such that there is a morphism $e : X \to I$ such that the following diagram commutes and satisfies the universal property

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "e"] \arrow[ddr, bend right, "e'"'] & & Y \\ & I \arrow[ur, hook, "m"] \arrow[d, dashed, "\exists_1 v"] & \\ & I' \arrow[uur, hook, bend right, "m'"'] & \end{tikzcd}$$

Definition 3.1.30. A *quiver* in $\mathscr{C}$ is a functor from $\begin{tikzcd} \bullet \arrow[loop left] \arrow[r, shift left] \arrow[r, shift right] & \bullet \arrow[loop right] \end{tikzcd}$ to $\mathscr{C}$. Equivalently, a directed graph allowing multiple arrows and loops

Definition 3.1.31. The *free category* generated by quiver Q has objects vertices in Q and morphisms paths in Q with empty path the identity

Definition 3.1.32. $f \in End(A)$ is an *involution* if $f^2 = 1_A$

Definition 3.1.33. $A \xrightarrow{i} B$ has *left lifting property* or *LLP* and $X \xrightarrow{p} Y$ has *right lifting property* or *RLP* for each other in this diagram if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

i, p are *orthogonal* if the lifting is unique

Definition 3.1.34. A class of morphisms **M** of $\mathscr{C}$ satisfies *2 out of 3* if any two of $f, g, f \circ g$ are in **M**, so is the third. **M** is clearly closed under composition
A class of *weak equivalences* is a class of morphisms **W** containing isomorphisms and satisfies 2 out of 3. The class of isomorphisms **I** is a class of weak equivalences

3.2 Yoneda lemma

Yoneda lemma

Lemma 3.2.1 (Yoneda lemma)**.** $\mathscr{C}$ is locally small

$$Hom_{Set^{\mathscr{C}}}(Hom_{\mathscr{C}}(c,-),F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

$$Hom_{Set^{\mathscr{C}^{op}}}(Hom_{\mathscr{C}}(-,c),F) \xrightarrow{\cong} F(c), \eta \mapsto \eta_c(1_c)$$

If $F = Hom(-,d)$ or $F = Hom(d,-)$, then

$$Hom_{Set^{\mathscr{C}}}(Hom_{\mathscr{C}}(c,-),Hom_{\mathscr{C}}(d,-)) \xrightarrow{\cong} Hom_{\mathscr{C}}(d,c)$$

$$Hom_{Set^{\mathscr{C}^{op}}}(Hom_{\mathscr{C}}(-,c),Hom_{\mathscr{C}}(-,d)) \xrightarrow{\cong} Hom_{\mathscr{C}}(c,d)$$

$c \to Hom_{\mathscr{C}}(c,-)$ gives an fully faithful embedding of $\mathscr{C}^{op}$ into $Set^{\mathscr{C}}$, viewing $\{Hom(c,-)\}$ as a subcategory of $Set^{\mathscr{C}}$, $c \to Hom_{\mathscr{C}}(-,c)$ gives an fully faithful embedding of $\mathscr{C}$ into $Set^{\mathscr{C}^{op}}$, viewing $\{Hom(-,c)\}$ as a subcategory of $Set^{\mathscr{C}^{op}}$

Proof.

$$\begin{tikzcd}
Hom_{\mathscr{C}}(c,c) \arrow[rrr, "f"] \arrow[ddd, "\eta_c"'] & & & Hom_{\mathscr{C}}(c,x) \arrow[ddd, "\eta_x"] \\
& 1_c \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& u \arrow[r, mapsto] & Ff(u) = \eta_x(f) & \\
F(c) \arrow[rrr, "Ff"'] & & & F(x)
\end{tikzcd}$$

The natural transformation η is determined by the element u in $F(c)$ □

Remark 3.2.2. Functor $Hom(-,c)$ is called **Yoneda embedding**, here embedding in the sense of a fully faithful functor, which is injective on objects up to isomorphism as in Lemma **??**
Yoneda lemma tells us that if $Hom(c,-)$ and $Hom(d,-)$ are naturally isomorphic or $Hom(-,c)$ and $Hom(-,d)$ are naturally isomorphic, so are c and d, thus if we know where c goes to or what goes to c, we can determine c up to isomorphism, in other words, an object is determined by the morphisms that interact with it, this explains the uniqueness in universal construction

3.3 Limits

Definition 3.3.1. A **diagram** is a functor $D : J \to \mathscr{C}$, J is called the **indexed category**, the diagram D can be thought of as indexing a collection of objects and morphisms in $\mathscr{C}$ patterned on J, we say D is a diagram in $\mathscr{C}$ shaped J

Let $F : J \to \mathscr{C}$ be a diagram in $\mathscr{C}$, N be an object in $\mathscr{C}$, then a **cone** from N to F is a family of morphisms ψ_X such that the following diagram commutes, a **cocone** from F to N is a family of morphisms ψ_X such that the following diagram commutes

$$\begin{tikzcd}
 & N \arrow[dl, "\psi_X"'] \arrow[dr, "\psi_Y"] & \\
F(X) \arrow[rr, "Ff"] & & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N & \\
F(X) \arrow[ur, "\psi_X"] \arrow[rr, "Ff"] & & F(Y) \arrow[ul, "\psi_Y"']
\end{tikzcd}$$

A **limit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : N \to L$ such that the following diagram commutes, a **colimit** of the diagram F is cone (L, ϕ) such that for any other cone (N, ψ) there is a unique $u : L \to N$ such that the following diagram commutes

$$\begin{tikzcd}
 & N \arrow[d, dashed, "\exists_1 u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
 & L \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\
F(X) \arrow[rr, "Ff"] & & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N & \\
 & L \arrow[u, dashed, "\exists_1 u"'] & \\
F(X) \arrow[ur, "\phi_X"] \arrow[uur, bend left, "\psi_X"] \arrow[rr, "Ff"] & & F(Y) \arrow[ul, "\phi_Y"'] \arrow[uul, bend right, "\psi_Y"']
\end{tikzcd}$$

Limits may also be characterized as terminal objects in the category of cones to F, thus unique up to isomorphism, so is colimits, a category contains all limits is called **complete**, and is called **cocomplete** if containing all colimits

The **equaliser** $Eq(f, g)$ is defined to be the limit of the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, the **coequaliser** is the colimit

Remark 3.3.2. Direct limit and inverse limit are defined on directed set, thus limit and colimit are more general

Definition 3.3.3. A **directed set** X is a set with a preorder $\leq$ and any pair of elements has an upper bound, i.e., $\forall x, y \in X, \exists z \in X$ such that $x \leq z, y \leq z$

Definition 3.3.4. Given a directed set I, we can define a **direct(inductive) system**, with modules A_i, and functions $f_{ij} : A_i \to A_j$, $f_{ii} = 1_{A_i}$, $f_{jk} \circ f_{ij} = f_{ik}, i \leq j \leq k$, we can also define an **inverse system**, with module A_i, and functions $f_{ij} : A_j \to A_i$, $f_{ii} = 1_{A_i}$, $f_{ij} \circ f_{jk} = f_{ik}$

We can define morphism betweem direct and inverse systems and modules

Suppose A_i is a direct system, then a morphism $g_i : A_i \to B$ is such that $g_j \circ f_{ij} = g_i, i \leq j$ or $g_i : B \to A_i$ is such that $g_j = f_{ij} \circ g_i, i \leq j$. Suppose A_i is an inverse system, then a morphism $g_i : A_i \to B$ is such that $g_i \circ f_{ij} = g_j, i \leq j$ or $g_i : B \to A_i$ is such that $g_i = f_{ij} \circ g_j, i \leq j$, We can define morphisms between direct and inverse systems

Suppose A_i, B_i are both direct systems, a morphism $g_i : A_i \to B_i$ is a family of maps such that $g_j \circ f_{ij} = f_{ij} \circ g_i, i \leq j$. Suppose A_i, B_i are both inverse systems, a morphism $g_i : A_i \to B_i$ is a family of maps such that $g_i \circ f_{ij} = f_{ij} \circ g_j, i \leq j$.

Definition 3.3.5. The **direct limit** of a direct system is a module A_∞ and morphisms $\iota_i : A_i \to A$ with the universal property: given any morphism $g_i : A_i \to B$, it induces a unique $g_\infty : A_\infty \to B$ such that $g \circ \iota_i = g_i$, there is a concrete construction: define the direct limit $\varinjlim A_i = \bigsqcup_{i \in I} A_i / \sim$, where $a_i \sim a_j, a_i \in A_i, a_j \in A_j$ if there is an upper bound k such that $f_{ik}(a_i) = f_{jk}(a_k)$, or equivalently, $a_i \sim f_{ij}(a_i), i \leq j$

Definition 3.3.6. The **inverse limit** of an inverse system is a module A_∞ and morphisms $\pi_i : A \to A_i$ with the universal property: given any morphism $g_i : B \to A_i$, it induces a unique $g_\infty : B \to A_\infty$ such that $\pi_i \circ g = g_i$, there is a concrete construction: define the inverse limit $\varprojlim A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \middle| a_i = f_{ij}(a_j), i \leq j \right\}$,

Remark 3.3.7. Direct limit and inverse limit are dual to each other in the categorical sense

Definition 3.3.8. Product, coproduct The **biproducts** $(\bigoplus_i A_i, p_i, \iota_i)$ of A_i is such that $(\bigoplus_i A_i, p_i)$ is the product and $(\bigoplus_i A_i, \iota_i)$ is the coproduct

Definition 3.3.9. An **initial object** $\varnothing$ is for every X, there is a unique $\varnothing \to X$, a **final object** $*$ is for every X, there is a unique $X \to *$, a **zero object** is an object which is both initial and final. A **pointed category** is a category with zero object

Remark 3.3.10. The initial and final object are the limit and colimit of empty diagram
In the category of sets, the initial object is $\varnothing$ and a terminal object is $\{*\}$

3.4 Adjunction

Definition 3.4.1. Let $L : \mathscr{D} \to \mathscr{C}$, $R : \mathscr{C} \to \mathscr{D}$ be functors, and there is a natural isomorphism $\Phi_{X,Y}$, $X \in \mathscr{C}, Y \in \mathscr{D}$

$$\begin{tikzcd}
Hom_{\mathscr{C}}(LX, Y) \arrow[r, "\Phi_{X,Y}"] \arrow[d, "(Lf,g)"'] & Hom_{\mathscr{D}}(X, RY) \arrow[d, "(g,Rf)"] \\
Hom_{\mathscr{C}}(LX', Y') \arrow[r, "\Phi_{X',Y'}"] & Hom_{\mathscr{D}}(X', RY')
\end{tikzcd}$$

Here $f : X' \to X$, $g : Y \to Y'$, $Hom_{\mathscr{C}}(Lf, g)(h) = h \circ g \circ Lf$
We say L is the **left adjoint** of R and R is the **right adjoint** of L

Example 3.4.2. Let $G : Group \to Set$ be the forgetful functor, then the functor $F : Set \to Group$, sending S to $F(S)$ is the left adjoint of G
In the category of R-modules Mod, consider functor $F := - \otimes B$ and functor $G := Hom(B, -)$, then F is the left adjoint to G, i.e. $Hom(A \otimes B, C) \cong Hom(A, Hom(B, C))$

3.5 Pushout and pullback

Definition 3.5.1. The **pullback** of $f : X \to Z$, $g : Y \to Z$ is $(X \times_Z Y, p_X, p_Y)$ satisfying the universal property

$$\begin{tikzcd}
W \arrow[rrd, bend left, "\phi"] \arrow[rd, dashed, "\exists_1 h"] \arrow[rdd, bend right, "\psi"'] & & \\
& X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

p_X is the **base change** of g along f, p_Y is the base change of f along g
If f is an epimorphism, so is p_X
More generally, we can also define the puullback of $f_i : X \to Y_i$

Definition 3.5.2. The **pushout** of $f : Z \to X$, $g : Z \to Y$ is $(X \cup_Z Y, \iota_X, \iota_Y)$ satisfying the universal property

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\iota_X"] \arrow[rdd, bend left, "\phi"] & \\
Y \arrow[r, "\iota_Y"'] \arrow[rrd, bend right, "\psi"'] & X \cup_Z Y \arrow[rd, dashed, "\exists_1 h"] & \\
& & W
\end{tikzcd}$$

ι_X is the **cobase change** of g along f, ι_Y is the cobase change of f along g
If f is a monomorphism, so is ι_X
More generally, we can also define the pushout of $f_i : Z \to X_i$

Proposition 3.5.3. Pushout preserve epimorphisms and isomorphisms and in the category of sets, pushout preserve injection
Pullback preserve monomorphisms and isomorphisms and in the category of sets, pullback preserve surjection

3.6 Filtered category

Definition 3.6.1. A category J is called **filtered** if it is not empty, and for any two objects $j, j' \in J$, there is an object $k \in J$ and morphisms $f : j \to k$ and $f' : j' \to k$, for any two morphisms $u, v : i \to j$, there is an object $k \in J$ and a morphism $w : j \to k$ such that $w \circ u = w \circ v$

A filtered colimit is the colimit of a functor $F : J \to \mathscr{C}$ where J is a filtered category, direct limit is a special case of a filtered colimit

The dual notion is called **cofiltered**

3.7 Comma category

Definition 3.7.1. Consider functors $S : \mathscr{A} \to \mathscr{C}$, $T : \mathscr{B} \to \mathscr{C}$(for source and target), define **comma category** $(S \downarrow T)$ with objects (A, B, h), $A \in \mathscr{A}, B \in \mathscr{B}$ are objects, $h : S(A) \to T(B)$ is a morphism, and with morphisms $(f, g) : (A, B, h) \to (A', B', h')$ where $f : A \to A', g : B \to B'$ are morphisms such that the following diagram commutes

$$\begin{tikzcd} S(A) \arrow[r, "S(f)"] \arrow[d, "h"'] & S(A') \arrow[d, "h'"] \\ T(B) \arrow[r, "T(g)"] & T(B') \end{tikzcd}$$

Definition 3.7.2. Consider the comma category of $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$, $T : * \to \mathscr{A}$ which we call **slice category**, sometimes denoted as $(\mathscr{A} \downarrow A_*)$ where $A_* = T(*)$, the objects of the slice category are $A \xrightarrow{\pi_A} A_*$ and morphisms are

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\pi_A"'] & & A' \arrow[dl, "\pi_{A'}"] \\ & A_* & \end{tikzcd}$$

Its dual notion, the comma category of $S : * \to \mathscr{B}$ $1_{\mathscr{B}} : \mathscr{B} \to \mathscr{B}$, which we call **coslice category**, sometimes denoted as $(B_* \downarrow \mathscr{B})$ where $B_* = S(*)$, the objects of the slice category are $B_* \xrightarrow{\pi_B} B$ and morphisms are

$$\begin{tikzcd} & B_* \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_{B'}"] & \\ B \arrow[rr, "g"] & & B' \end{tikzcd}$$

The comma category of $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, $1_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ which we call **arrow category**, sometimes denoted as $\mathscr{C}^{\to}$ the objects of the arrow category are just the morphisms(arrows) $A \xrightarrow{f} A'$, and morphisms are

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & A' \arrow[d, "h'"] \\ B \arrow[r, "g"] & B' \end{tikzcd}$$

Definition 3.7.3. A right inverse are called a **section**, a left inverse is called a **retraction**

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, equal] & Y \arrow[d, "g"] \\ & X \end{tikzcd}$$

f is a section of g, g is a retraction of f

3.8 Sheaves

Definition 3.8.1. $\mathscr{A}$ is an abelian category, open subsets of X form a category τ under inclusion. A *presheaf* is a functor $\tau^{op} \xrightarrow{F} \mathscr{A}$, $F(U \hookrightarrow V) = \mathtt{res}_{UV}$ are *restriction maps*
A *morphism of presheaves* $F \xrightarrow{\phi} G$ is a natural transformation, i.e. the following diagram commutes

$$\begin{tikzcd} F(V) \arrow[r, "\mathtt{res}_{UV}"] \arrow[d, "\phi_V"'] & F(U) \arrow[d, "\phi_U"] \\ G(V) \arrow[r, "\mathtt{res}_{UV}"] & G(U) \end{tikzcd}$$

Definition 3.8.2. $U \subseteq X$ is an open subset, F is a presheaf over X, the *restricted presheaf* $F|_U$ is given by $F|_U(V) = F(U \cap V)$

Definition 3.8.3. $X \xrightarrow{f} Y$ is a continuous map, F is a presheaf over X, the *pushforward presheaf* f_*F of F under f is a presheaf over Y given by $f_*F(V) = F(f^{-1}(V))$

Definition 3.8.4. F is a presheaf, $x \in X$, open subsets containing x is full subcategory $\tau(x)$, the *stalk* F_x is the colimit $\varinjlim_{x\in U} F(U)$, elements in F_x are called *germs*, denote the germ of f at x as f_x

Lemma 3.8.5. $B(f, U) = \{f_x | x \in U, f \in F(U)\}$ form a basis on the *étalé space* $|F| = \bigcup F_x$. $|F| \to X$, $f_x \mapsto x$ is a local homeomorphism

Sheaf

Definition 3.8.6. Presheaf F is a *sheaf* if

$$F(U) \xrightarrow{\mathtt{res}_{U_i,U}} \prod_i F(U_i) \underset{\mathtt{res}_{U_i\cap U_j,U_j}}{\overset{\mathtt{res}_{U_i\cap U_j,U_i}}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

Is an equaliser. Equivalently, F satisfying

1. If $U = \bigcup_i U_i$, $f, g \in F(U)$, $f|_{U_i} = g|_{U_i}$, then $f = g$
2. If $U = \bigcup_i U_i$, $f_i \in F(U_i)$, $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$, then there exists $f \in F(U)$ such that $f|_{U_i} = f_i$, here f has to be unique because of 1

$Sh(X)$ is the category of sheaves over X

Proposition 3.8.7. $F \xrightarrow{\phi} G$ is a monomorphism or a epimorphism iff $F_x \xrightarrow{\phi_x} G_x$ is injective or surjective on each stalk

Definition 3.8.8 (Sheafification)**.** F is a presheaf over X, the sheaf of sections $X \to |F|$ is the *sheafification*

Definition 3.8.9. The *constant presheaf* $\underline{A}$ given by $\underline{A}(U) = A$, $\mathtt{res}_{UV} = 1_A$
F is a *locally constant sheaf* if for any $x \in X$, there exists $U \ni x$ such that $F|_U$ is a constant sheaf. $F : \Pi_1 X \to \mathscr{A}$ is a functor. The category of locally constant sheaves is equivalent to the category of covering spaces of X

Definition 3.8.10. Functor $\Gamma : Sh(X) \to \mathscr{A}$, $F \mapsto F(X)$ is a left exact functor, the *sheaf cohomology* is the right derived functor $R^i\Gamma$, i.e. $R^i\Gamma(F) = H^i(X, F)$

Definition 3.8.11. A *ringed space* $(X, \mathcal{O})$ is a topological space X and a sheaf of rings over X, $\mathcal{O}$ is the *structure sheaf*. $(X, \mathcal{O})$ is a *locally ringed space* if each stalk is a local ring

Definition 3.8.12. A morphism between ringed spaces is $(X, \mathcal{O}_X) \xrightarrow{(f,\phi)} (Y, \mathcal{O}_Y)$, $X \xrightarrow{f} Y$ is a continuous map, $\mathcal{O}_Y \xrightarrow{\phi} f_*\mathcal{O}_X$ is a morphism of sheaves. A morphism between locally ringed spaces require ϕ is a local ring homomorphism between stalks

Definition 3.8.13. $(X, \mathcal{O})$ is a ringed space, a sheaf of $\mathcal{O}$ *modules* F is $F(U)$ which are $\mathcal{O}(U)$ modules such that $\mathtt{res}_{UV}(rm) = \mathtt{res}_{UV}(r)\,\mathtt{res}_{UV}(m)$

Definition 3.8.14. A *fine sheaf* F is one with "partition of unities", more precisely, for any open cover, there is a family of endomorphisms such that each endomorphism is zero outside some element of the cover

Example 3.8.15. The de Rham complex is a resolution of the locally constant sheaf $\mathbb{R}$, although not injective but fine sheaves. Thus the de Rham cohomology coincides with the sheaf cohomology

Definition 3.8.16. An *acyclic sheaf* F if its higher sheaf cohomologies vanishes

Definition 3.8.17. A *soft sheaf* F is one that any section over a closed subset can be extended to a global section

Definition 3.8.18. A *flasque sheaf* or *flabby sheaf* F is one that the restriction maps are surjective

3.9 Exponential object

Definition 3.9.1. Y is an object such that all binary products $X \times Y$ exist, the **exponential object** is Z^Y together with morphism $Z^Y \times Y \xrightarrow{\mathtt{eval}} Z$ satisfying universal property

$$\begin{tikzcd}
X \times Y \arrow[d, dashed, "\exists_1 \tilde{f} \times 1_Y"'] \arrow[dr, "f"] & \\
Z^Y \times Y \arrow[r, "\mathtt{eval}"'] & Z
\end{tikzcd}$$

Proposition 3.9.2. $Hom(X \times Y, Z) \to Hom(X, Z^Y)$ is an adjunction

3.10 Factorization system

Definition 3.10.1. A **factorization system** $(\mathbf{E}, \mathbf{M})$ for category $\mathscr{C}$ is two classes of morphisms such that
1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. $\mathbf{E}, \mathbf{M}$ are closed under composition and contain all isomorphisms
3. Factorization is functorial, i.e. for any u, v such that $vme = m'e'u$, there exists a unique w such that the following diagram commutes

$$\begin{tikzcd} \bullet \arrow[r, "e"] \arrow[d, "u"] & \bullet \arrow[r, "m"] \arrow[d, dashed, "\exists_1 w"] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, "e'"] & \bullet \arrow[r, "m'"] & \bullet \end{tikzcd}$$

Example 3.10.2. $\mathbf{E}, \mathbf{M}$ being epi and mono in Set is a factorization system

Definition 3.10.3. A **weak factorization system** $(\mathbf{E}, \mathbf{M})$ for category $\mathscr{C}$ is two classes of morphisms such that
1. Any morphism f can be decomposed as $f = me$, $m \in \mathbf{M}$, $e \in \mathbf{E}$
2. $\mathbf{E}$ are exactly those morphisms having left lifting property for all morphisms in $\mathbf{M}$
3. $\mathbf{M}$ are exactly those morphisms having right lifting property for all morphisms in $\mathbf{E}$

3.11 Abelian category

Definition 3.11.1. $\mathscr{C}$ is a *preadditive category* or an **Ab**-*category* if

- $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are abelian groups
- Addition distributes over composition

$$f \circ (g + h) = f \circ g + f \circ h, (f + g) \circ h = f \circ h + g \circ h$$

Note. $0 \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a zero morphism

Definition 3.11.2. $F : \mathscr{C} \to \mathscr{D}$ is a functor between preadditive cateogories is *additive* if F is are group homomorphisms on $\mathrm{Hom}(F(A), F(B))$. i.e. $F(f + g) = F(f) + F(g)$

Definition 3.11.3. Preadditive category $\mathscr{C}$ is an *additive category* if any finite set of objects has a biproduct

Note. In particular, $\mathscr{C}$ has a zero object, the empty biproduct

Definition 3.11.4. An additive category is called a *preabelian category* if every morphism has a kernel and a cokernel, where kernels and cokernels means the equalisers and coequalisers of the morphism $f : X \to Y$ and the zero morphism $0 : X \to Y$

Definition 3.11.5. A preabelian category is called an *abelian category* if every monomorphisms is normal and every epimorphisms is conormal, a morphism is *normal* if it is a kernel, *conormal* if it is a cokernel and *binormal* if it is both a kernel and a cokernel

Definition 3.11.6. For a morphism $A \xrightarrow{f} B$, define its image $\mathrm{im}f$ by the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[d, dashed, "\exists_1"'] \arrow[dr, "f"] & & & \\
0 \arrow[r] & \mathrm{im}f \arrow[r, hook] & B \arrow[r, two heads] & \mathrm{coker}f \arrow[r] & 0
\end{tikzcd}$$

The image satisfies universal property

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr] \arrow[ddr, bend right] & & B \\
 & \mathrm{im}f \arrow[ur, hook] \arrow[d, dashed, "\exists_1"] & \\
 & I \arrow[uur, hook, bend right] &
\end{tikzcd}$$

Example 3.11.7. A ring R can be thought of as a preadditive category with a single object and morphisms $r \in R$ The category of left R modules can be thought of as the functor category $[R, Ab]$, where Ab the category of abelian groups

Proposition 3.11.8. In an abelian category $\mathscr{A}$, the equaliser of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the isomorphic to the kernel of $f - g$

Definition 3.11.9. Let $\mathscr{A}$ be an abelian category, a ($\mathbb{Z}$-graded) *chain complex* $C_\bullet$ is

$$\cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to \cdots$$

Such that $\partial_n \circ \partial_{n+1} = 0$, ∂_i are called *boundary maps(differentials)*

We can define chain maps, chain homotopy, boundaries, cycles, and homology groups, and we say the chain complex is exact if each homology groups is zero, the chain complexes form the *category of chain complexes* $Ch\mathscr{A}$

The *homotopy category of chain complexes* often denoted as $K(\mathscr{A})$ is the quotient category with chain maps modulo chain homotopy equivalence as morphisms

a chain map is called a *quasi-isomorphism* if it induces isomorphisms on homology groups

Lemma 3.11.10. An alternative definition of an exact functor F could be that F preserve exactness, i.e. $F(A) \to F(B) \to F(C)$ is exact for any short exact sequence $A \to B \to C$

Definition 3.11.11. The *direct sum* $(C \oplus D)_\bullet$ of chain complexes $C_\bullet, D_\bullet$ is

$$\cdots \to C_1 \oplus D_1 \xrightarrow{\partial_1^C \oplus \partial_1^D} C_0 \oplus D_0 \xrightarrow{\partial_0^C \oplus \partial_0^D} C_{-1} \oplus D_{-1} \to \cdots$$

Definition 3.11.12. A *double complex* $C_{*,*}$ is $\{C_{p,q}\}_{p,q\in\mathbb{Z}}$ two differentials $\partial' : C_{p,q} \to C_{p-1,q}$, $\partial'' : C_{p,q} \to C_{p,q-1}$ such that $(\partial')^2 = (\partial'')^2 = 0$ and $\partial'\partial'' + \partial''\partial' = 0(\partial', \partial''$ anticommutes)

The *total chain complexes* are $(Tot^{\oplus})_n = \bigoplus_{p+q=n} C_{p,q}$ and $(Tot^{\Pi})_n = \prod_{p+q=n} C_{p,q}$ with $\partial = \partial' + \partial''$

Example 3.11.13. $C_* \otimes D_*$ is the total complex of double complex $C_{p,q} := C_p \otimes D_q$, $\partial' := \partial^C \otimes 1$, $\partial'' := (-1)^p 1 \otimes \partial^D$

Definition 3.11.14. A *filtered chain complex* is a filtered object in $Ch\mathscr{A}$

$$\cdots \to F_{p+1}C_\bullet \to F_pC_\bullet \to \cdots \to C_\bullet$$

Snake lemma

Lemma 3.11.15 (Snake lemma)**.** Given the following commutative diagram with exact rows, then we have an exact sequence

$$\begin{tikzcd}
 & \ker a \arrow[r, "u_*"] \arrow[d] & \ker b \arrow[r, "v_*"] \arrow[d] & \ker c \arrow[d] \arrow[dddll, "\delta"] & \\
D \arrow[ur, "w_*"] \arrow[r] & A \arrow[r, "u"] \arrow[d, "a"] & B \arrow[r, "v"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "u'"] \arrow[d] & B' \arrow[r, "v'"] \arrow[d] & C' \arrow[r] \arrow[d] & D' \\
 & \operatorname{coker} a \arrow[r, "u'_*"] & \operatorname{coker} b \arrow[r, "v'_*"] & \operatorname{coker} c \arrow[ur] &
\end{tikzcd}$$

Five lemma

Lemma 3.11.16 (Five lemma)**.** If b and d are monic and a is an epi, then c is monic. Dually, if b and d are epis and e is monic, then c is an epi. In particular, if a, b, d and e are iso, then c is also an iso

$$\begin{tikzcd}
A' \arrow[r, "u'"] \arrow[d, "a"', "\cong"] & B' \arrow[r, "v'"] \arrow[d, "b"', "\cong"] & C' \arrow[r, "w'"] \arrow[d, "c"'] & D' \arrow[r, "x'"] \arrow[d, "d"', "\cong"] & E' \arrow[d, "e"', "\cong"] \\
A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & D \arrow[r, "x"] & E
\end{tikzcd}$$

Horseshoe lemma

Lemma 3.11.17 (Horseshoe lemma)**.** Suppose $P_\bullet \xrightarrow{\varepsilon} M$, $Q_\bullet \xrightarrow{\eta} N$ are projective resolutions, then any exact sequence $0 \to M \xrightarrow{f} A \xrightarrow{g} N \to 0$ can be extended into commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

With $(P \oplus Q)_\bullet$ being a projective resolution, every row and column are exact

Proof. Since $A \xrightarrow{g} N$ is epi and Q_0 is projective, we get $Q_0 \xrightarrow{s_0} A$ such that $gs_0 = \partial_0$ which gives us $P_0 \oplus Q_0 \xrightarrow{\begin{pmatrix} f\partial_0 & s_0 \end{pmatrix}} A$, by Lemma 3.11.15, this is epi, and we get an exact sequence $0 \to Z_0 P \to \ker i_0 \to Z_0 Q \to 0$, similarly, we can construct $Q_1 \xrightarrow{s_1} \ker i_0$, then $P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} \iota_0\partial_0 & s_1 \end{pmatrix}} \ker i_0$ is again epi by Lemma 3.11.15, inductively, we can construct the commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d, "\partial_1"'] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d, "\partial_1"] \arrow[dl, "s_1"] & 0 \\
0 \arrow[r] & Z_0 P \arrow[r, "\iota_0"] \arrow[d] & \ker i_0 \arrow[r] \arrow[d] & Z_0 Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "\partial_0"'] & P_0 \oplus Q_0 \arrow[r] \arrow[d, "i_0"] & Q_0 \arrow[r] \arrow[d, "\partial_0"] \arrow[dl, "s_0"] & 0 \\
0 \arrow[r] & M \arrow[r, "f"'] \arrow[d] & A \arrow[r, "g"'] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

□

Lemma for universal coefficient theorem for cohomology

Lemma 3.11.18. If $A \xrightarrow{f} B \xrightarrow{g} C$ is a sequence and there is a homomorphism(retraction) $C \xrightarrow{r} B$ such that $rg = 1_B$, then there is an exact sequence $0 \to \mathrm{coker} f \to \mathrm{coker}(gf) \to \mathrm{coker} g \to 0$

Proof. First observe that we have $0 \to \mathrm{im} g/\mathrm{im}(gf) \to C/\mathrm{im}(gf) \to C/\mathrm{im} g \to 0$, $B \to \mathrm{im} g$, $\mathrm{im} f \to \mathrm{im}(gf)$, thus $B/\mathrm{im} f \to \mathrm{im} g/\mathrm{im}(gf)$, since $rg = 1_B$, $B/\mathrm{im} f \cong \mathrm{im} g/\mathrm{im}(gf)$, therefore, $0 \to B/\mathrm{im} f \to C/\mathrm{im}(gf) \to C/\mathrm{im} g \to 0$ □

Lemma 3.11.19. Suppose $\mathscr{A}$ is abelian category, then $\mathrm{im} f = \ker \mathrm{coker} f = \mathrm{coker} \ker f$

Definition 3.11.20. Pick $p \in \mathbb{Z}$, define the *translation* of X by p is $X_\bullet[p]$ where $(X_\bullet[p])_n = X_{p+n}$, differential $X_\bullet[p]_n \to X_\bullet[p]_{n-1}$ is given by $(-1)^p \partial$ The *translation functor* $T : Ch(\mathscr{A}) \to Ch(\mathscr{A})$, $X \mapsto X_\bullet[1]$ is an auto morphism of $Ch(\mathscr{A})$

Acyclic model theorem

Theorem 3.11.21 (Acyclic model theorem)**.** [1] *Model* $\mathcal{M} = \{M_j\}$ is a subclass(possibly with repetition) of objects in $\mathscr{C}$, $F, G : \mathscr{C} \to Ch_{\geq 0}$ are functors, $H_n(G(M_j)) = 0$ for any $n \neq 0$, $M_j \in \mathcal{M}$. For any C, there exist $m_j \in F_k M_j$ such that $F_k(C)$ is free with basis $\left\{ F_k(\sigma)(m_j) \middle| M_j \xrightarrow{\sigma} C \right\}$

[1] Consult Theorem 9.12 of [?] or `https://amathew.wordpress.com/2010/09/11/the-method-of-acyclic-models/`

Universal coefficient theorem for cohomology

Theorem 3.11.22 (Universal coefficient theorem for cohomology)**.** There is an exact sequence

$$0 \to Ext^1(H_{n-1}, A) \to H^n(C; A) \to Hom(H_n, A) \to 0$$

Proof. Since C_n is a free group, so are subgroups B_n, Z_n, exact sequence

$$0 \to Z_n \hookrightarrow C_n \to B_{n-1} \to 0$$

Splits, i.e. we have a splitting homomorphism $B_{n-1} \xrightarrow{s} C_n$, $C_n \cong Z_n \oplus B_{n-1}$, thus exact sequence

$$0 \to H_n = Z_n/B_n \to C_n/B_n \to C_n/Z_n \cong B_{n-1} \to 0$$

Induces exact sequence

$$Hom(B_{n-1}, A) \to Hom(C_n/B_n, A) \to Hom(H_n, A) \to Ext^1(B_{n-1}, A) = 0$$

$Hom(H_n, A)$ is the cokernel of $Hom(B_{n-1}, A) \to Hom(C_n/B_n, A)$
Note that

$$H^n(C; A) = Z^n(C; A)/B^n(C; A) = \frac{\ker(Hom(C_n, A) \to Hom(C_{n+1}, A))}{\operatorname{im}(Hom(C_{n-1}, A) \to Hom(C_n, A))}$$

$C_n \xrightarrow{\phi} A \in Z^n(C; A) \Leftrightarrow \phi\partial = 0 \Leftrightarrow \phi \in Hom(C_n/B_n, A)$, thus $Hom(C_n/B_n, A) \cong Z^n(C; A)$

$$\begin{tikzcd} C_{n+1} \arrow[r, "\partial"] \arrow[dr] & C_n \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

$C_n \xrightarrow{\psi} A \in B^n(C; A) \Leftrightarrow \psi = \phi\partial$ for some $C_{n-1} \xrightarrow{\phi} A \Leftrightarrow \psi = \phi\partial$ for some $Z_{n-1} \xrightarrow{\phi} A$, and since $B^n(C; A) \subseteq Z^n(C; A) \cong Hom(C_n/B_n, A)$, we have $B^n(C; A) \cong \operatorname{im}(Hom(Z_{n-1}, A) \to Hom(C_n/B_n, A))$

$$\begin{tikzcd} C_n \arrow[r, "\partial"] \arrow[dr, "\psi"'] & C_{n-1} \arrow[d, "\phi"] \\ & A \end{tikzcd}$$

Exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to B_{n-1}/Z_{n-1} = H_{n-1} \to 0$$

Induces exact sequence

$$Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A) \to Ext^1(H_{n-1}, A) \to Ext^1(Z_{n-1}, A) = 0$$

$Ext^1(H_{n-1}, A)$ is the cokernel of $Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A)$
Since composition $B_{n-1} \xrightarrow{s} C_n \to C_n/B_n \xrightarrow{\partial} B_{n-1}$ is identity, we have a homomorphism $r : Hom(C_n/B_n, A) \to Hom(B_{n-1}, A)$ induced by $B_{n-1} \to C_n/B_n$ such that composition $Hom(B_{n-1}, A) \to Hom(C_n/B_n, A) \xrightarrow{r} Hom(B_{n-1}, A)$ is identity
Apply Lemma 3.11.18 to $Hom(Z_{n-1}, A) \to Hom(B_{n-1}, A) \to Hom(C_n/B_n, A)$, we get an exact sequence

$$0 \to Ext^1(H_{n-1}, A) \to H^n(C; A) \to Hom(H_n, A) \to 0$$

□

Remark 3.11.23. B_n is not necessarily a direct summand of C_n, a map $B_n \xrightarrow{\phi} A$ may not be possible to extended to $C_n \xrightarrow{\phi} A$, however a map $Z_n \xrightarrow{\phi} A$ can always be extended to $C_n \xrightarrow{\phi} A$

Algebraic Kunneth formula

Theorem 3.11.24 (Algebraic Künneth formula). C, D are free chain complexes, then

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor_1(H_p(C), H_q(D)) \to 0$$

Is exact

Proof. If D has trivial differentials, then $H_q(D) = D_q$ is free, hence

$$H_n(C \otimes D) = \bigoplus_{p+q=n} H_p(C \otimes D_q) = \bigoplus_{p+q=n} H_p(C) \otimes D_q = \bigoplus_{p+q=n} H_p(C) \otimes H_q(D)$$

In general, consider exact sequence $0 \to Z \xrightarrow{i} D \xrightarrow{\partial} B[-1] \to 0$, $0 \to B \xrightarrow{i} Z \to H(D) \to 0$, then $0 \to C \otimes Z \to C \otimes D \to C \otimes B[-1] \to 0$ is exact since C_k are free, this gives us long exact sequence

$$\cdots \to H_n(C \otimes Z) \xrightarrow{1\otimes i} H_n(C \otimes D) \xrightarrow{1\otimes\partial} H_n(C \otimes B[-1]) \xrightarrow{1\otimes i} H_{n-1}(C \otimes Z) \to \cdots$$

$Z, B[-1]$ have trivial differentials, hence the connecting homomorphism is just

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(B[-1]) = \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(B) \xrightarrow{1\otimes i} \bigoplus_{p+q=n-1} H_p(C) \otimes H_q(Z)$$

Then we have

$$0 \to \operatorname{coker}(1 \otimes i) \to H_n(C \otimes D) \to \ker(1 \otimes i) \to 0$$

We also have

$$0 \to Tor_1(H_p(C), H_q(D)) \to H_p(C) \otimes B_q \xrightarrow{1\otimes i} H_p(C) \otimes Z_q \to H_p(C) \otimes H_q(D) \to 0$$

Therefore, we have exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C \otimes D) \to \bigoplus_{p+q=n-1} Tor_1(H_p(C), H_q(D)) \to 0$$

□

Definition 3.11.25. A *composition series* of A is a sequence of subobjects

$$A = A_n \supsetneq \cdots \supsetneq A_1 \supsetneq A_0 = 0$$

With *composition factors* H_{i+1}/H_i simple and *composition length* $\ell(A) = n$

Lemma 3.11.26. $\ell(A)$ is indepent of the composition series

3.12 Spectral sequences

Definition 3.12.1. Suppose $\mathscr{A}$ is an abelian category, a **spectral sequence** consists of objects $\{E_r\}_{r\geq r_0}$(r_0 is mostly 0), and morphisms $d_r : E_r \to E_r$ such that $d_r \circ d_r = 0$ and $E_{r+1} \cong H(E_r) = \ker d_r/\mathrm{im} d_r$

Definition 3.12.2. Suppose $\mathscr{A}$ is an abelian category, an **exact couple** is (D, E, i, j, k)

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Such that it is exact at each term, define differential $d = jk$, then $d^2 = jkjk = j(kj)k = 0$, we can define the **derived couple**

$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Where $D' = i(D)$, $E' = \ker k/\mathrm{im} j$, $i'(a) = i(a)$, $j'(i(a)) = \overline{j(a)}$, $k'(b) = \overline{k(b)}$, then the derived couple is again an exact couple, thus we can carry this process indefinitely, giving the n-th derived couple $(D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$

Example 3.12.3. Suppose $\cdots \subseteq F_{p-1}C_\bullet \subseteq F_pC_\bullet \subseteq \cdots$ is a filteration of chain complex $C_\bullet$(or **filtered chain complex**), exact sequence $0 \to F_{p-1}C_\bullet \to F_pC_\bullet \to (grC_\bullet)_p \to 0$ give a long exact sequence

$$\cdots \to H_n(F_{p-1}C_\bullet) \xrightarrow{i_*} H_n(F_pC_\bullet) \xrightarrow{j_*} H_n(F_pC_\bullet/F_{p-1}C_\bullet) \xrightarrow{k_*} H_{n-1}(F_{p-1}C_\bullet) \to \cdots$$

If we write $D^1_{p,q} := H_{p+q}(F_pC_\bullet)$, $E^1_{pq} := H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet)$, then the long exact sequence become

$$\cdots \to D^1_{p,q} \to D^1_{p+1,q-1} \to E^1_{p,q} \to D^1_{p,q-1} \to \cdots$$

Consider $D^1 = \bigoplus D^1_{p,q}$, $E^1 = \bigoplus E^1_{p,q}$, then $(D^1, E^1, i_*, j_*, k_*)$ form an exact couple, note that $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$

Remark 3.12.4. $grC_\bullet = \bigoplus_p F_pC_\bullet/F_{p-1}C_\bullet$ is called the **associated graded complex**
If X is a CW complex, we can take $F_pC_\bullet = C_\bullet(X^p)$, here X^p is the p-th skeleton of X

Definition 3.12.5. A **double cochain complex** $C^{\bullet,\bullet}$ is bigraded with anticommuting differetials d_h, d_v, i.e. $(d_h)^2 = 0$, $(d_v)^2 = 0$, $d_hd_v + d_vd_h = 0$

$$\begin{tikzcd} & \vdots & \vdots & \\ \cdots \arrow[r] & C^{0,1} \arrow[u] \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[u] \arrow[r] & \cdots \\ \cdots \arrow[r] & C^{0,0} \arrow[u, "d_v^{0,0}"] \arrow[r, "d_h^{0,0}"] & C^{1,0} \arrow[u, "d_v^{1,0}"'] \arrow[r] & \cdots \\ & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd}$$

Define the **total cochain complex** to be $C^n = \bigoplus\limits_{p+q=n} C^{p,q}$, with total differential $d = d_h + d_v$, this is indeed a differential since $d^2 = (d_h + d_v)^2 = (d_h)^2 + d_hd_v + d_vd_h + (d_v)^2 = 0$

We can define the **horizontal filtration** of the total cochain complex $(F_p^h C)^n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C^{k,l}$ and the **vertical filtration** of the total cochain complex $(F_q^h C)^n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C^{k,l}$

A **double chain complex** $C_{\bullet,\bullet}$ is bigraded with anticommuting differetials ∂^h, ∂^v, i.e. $(\partial^h)^2 = 0$, $(\partial^h)^2 = 0$, $\partial^h \partial^v + \partial^v \partial^h = 0$

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C_{1,1} \arrow[r, "\partial^h_{1,1}"] \arrow[d, "\partial^v_{1,1}"] & C_{0,1} \arrow[r] \arrow[d, "\partial^v_{0,1}"] & \cdots \\
\cdots \arrow[r] & C_{1,0} \arrow[r, "\partial^h_{1,0}"] \arrow[d] & C_{0,0} \arrow[r] \arrow[d] & \cdots \\
& \vdots & \vdots &
\end{tikzcd}$$

Define the **total chain complex** to be $C_n = \bigoplus_{p+q=n} C_{p,q}$, with total differential $\partial = \partial_h + \partial_v$

We can define the **horizontal filtration** of the total chain complex $(F_p^h C)_n = \bigoplus_{\substack{k+l=n \\ k \leq p}} C_{k,l}$ and the **vertical filtration** of the total chain complex $(F_q^h C)_n = \bigoplus_{\substack{k+l=n \\ l \leq q}} C_{k,l}$

Remark 3.12.6. If d_h, d_v commutes instead of anticommuting, then $C^{\bullet,\bullet}$ can be viewed as a cochain complex of cochain complexes, the total differential becomes $d^n(c) = d_h^p c + (-1)^p d_v^q c$ for any $c \in C^{p,q}$, this is indeed a differential since

$$\begin{aligned}
d^{n+1} d^n(c) &= d^{n+1}(d_r^p c + (-1)^p d_v^q c) \\
&= d^{n+1} d_h^p c + (-1)^p d^{n+1} d_v^q c \\
&= d_h^{p+1} d_h^p c + (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c + (-1)^{2p} d_v^{q+1} d_v^q c \\
&= (-1)^{p+1} d_v^q d_h^p c + (-1)^p d_h^p d_v^q c \\
&= (-1)^p (d_h^p d_v^q - d_v^q d_h^p) c \\
&= 0
\end{aligned}$$

However, these two types of definitions are equivalent

Proposition 3.12.7. Let $E^r_{p,q}$ be the spectral sequence corresponds to the horizontal filtration
(1) $E^0_{p,q} \cong C^{p,q}$
(2) $E^1_{p,q} \cong H_q(C_{p,\bullet})$
(3) $E^0_{p,q} \cong H_p(H^v_q(C))$
(4) If $C_{p,q}$ vanishes outside the first quadrant, i.e. $C_{p,q} = 0$ for any $p < 0$ or $q < 0$, then the spectral sequence converges to the homology of the total chain complex $E^r_{p,q} \Rightarrow H_{p+q}(C)$, i.e. $E^\infty_{p,q} \cong H_{p+q}(C)$

Proof. **(1)** By definition $E^0_{p,q} := (F_p^h C)_{p+q} / (F^h_{p-1} C)_{p+q} \cong C^{p,q}$
(2) $E^1_{p,q} = H_{p+q}(F_p^h C / F^h_{p-1} C) \cong H_{p+q}(C_{p,\bullet})$
(3) □

3.13 Monoidal category

Definition 3.13.1. A category $\mathscr{C}$ is *monoidal* if there are

- A *tensor product* or *monoidal product* $\mathscr{C} \times \mathscr{C} \xrightarrow{\otimes} \mathscr{C}$ with a tensor unit I
- *Associator* $(x \otimes y) \otimes z \xrightarrow{\alpha_{x,y,z}} x \otimes (y \otimes z)$ which is natural isomorphism
- *Left* and *right unitor* $I \otimes x \xrightarrow{\lambda_x} x$, $x \otimes I \xrightarrow{\rho_x} x$ which are natural isomorphisms

Such that the following diagrams commute

$$\begin{tikzcd}
(x \otimes 1) \otimes y \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes I"'] & & x \otimes (I \otimes y) \arrow[dl, "1 \otimes \lambda"] \\
& x \otimes y &
\end{tikzcd}$$

$$\begin{tikzcd}
((w \otimes x) \otimes y) \otimes z \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & (w \otimes x) \otimes (y \otimes z) \arrow[d, "\alpha"] \\
(w \otimes (x \otimes y)) \otimes z \arrow[r, "\alpha"] & w \otimes ((x \otimes y) \otimes z) \arrow[r, "\alpha"] & w \otimes (x \otimes (y \otimes z))
\end{tikzcd}$$

$\mathscr{C}$ is *strictly monoidal* if α, λ, ρ are identities

Example 3.13.2. R is a commutative ring. The category of R-modules is a monoidal category

- R is the tensor unit with $\otimes = \otimes_R$
- $(A \otimes B) \otimes C \xrightarrow{\alpha} A \otimes (B \otimes C)$, $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$
- $R \otimes A \xrightarrow{\lambda} A$, $r \otimes a \mapsto ra$
- $A \otimes R \xrightarrow{\rho} A$, $a \otimes r \mapsto ra$

$$\begin{tikzcd}
(A \otimes R) \otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes 1"'] & & A \otimes (R \otimes B) \arrow[dl, "1 \otimes \lambda"] \\
& A \otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
(a \otimes r) \otimes b \arrow[rr, mapsto] \arrow[dr, mapsto] & & a \otimes (r \otimes b) \arrow[dl, mapsto] \\
& (ra) \otimes b = a \otimes (rb) &
\end{tikzcd}$$

Definition 3.13.3. A *monoid* in a monoidal category $\mathscr{C}$ is an object M with

- Multiplication $\mu : M \otimes M \to M$
- Unit $\eta : I \to M$

Such that following diagrams commute

$$\begin{tikzcd}
I \otimes M \arrow[r, "\eta \otimes 1"] \arrow[dr, "\lambda"'] & M \otimes M \arrow[d, "\mu"] & M \otimes I \arrow[l, "1 \otimes \eta"'] \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\alpha"] \arrow[d, "\mu \otimes 1"'] & & M \otimes (M \otimes M) \arrow[d, "1 \otimes \mu"] \\
M \otimes M \arrow[r, "\mu"'] & M & M \otimes M \arrow[l, "\mu"]
\end{tikzcd}$$

A *comonoid* C is a monoid in $\mathscr{C}^{\mathrm{op}}$, with

- Comultiplication $\Delta : C \to C \otimes C$
- Counit $\epsilon : C \to I$

Such that following diagrams commute

$$\begin{tikzcd}
 & C \otimes C \arrow[dl, "1\otimes\epsilon"'] \arrow[dr, "\epsilon\otimes 1"] & \\
C \otimes I \arrow[r, "\rho"'] & C \arrow[u, "\Delta"] & I \otimes C \arrow[l, "\lambda"]
\end{tikzcd}$$

$$\begin{tikzcd}
C \otimes C \arrow[d, "1\otimes\Delta"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] & C \otimes C \arrow[d, "\Delta\otimes 1"] \\
C \otimes (C \otimes C) & & (C \otimes C) \otimes C \arrow[ll, "\alpha"]
\end{tikzcd}$$

A *bimonoid* B is both a monoid and a comonoid satisfying following compatibility commutative diagrams

1. Multiplication μ and comultiplication Δ

$$\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
(B \otimes B) \otimes (B \otimes B) \arrow[d] & & (B \otimes B) \otimes (B \otimes B) \arrow[u, "\mu\otimes\mu"'] \\
B \otimes (B \otimes B) \otimes B \arrow[rr, "1\otimes\tau\otimes 1"] & & B \otimes (B \otimes B) \otimes B \arrow[u]
\end{tikzcd}$$

   Here $\tau(x \otimes y) = y \otimes x$

2. Multiplication μ and counit ϵ

$$\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & B \arrow[d, "\epsilon"] \\
I \otimes I \arrow[r] & I
\end{tikzcd}$$

3. Comultiplication Δ and unit η

$$\begin{tikzcd}
B \otimes B & B \arrow[l, "\Delta"'] \\
I \otimes I \arrow[r] \arrow[u, "\eta\otimes\eta"] & I \arrow[u, "\eta"']
\end{tikzcd}$$

4. Unit η and counit ϵ

$$\begin{tikzcd}
I \arrow[r, "\eta"] \arrow[dr, equal] & B \arrow[d, "\epsilon"] \\
 & I
\end{tikzcd}$$

Example 3.13.4. Ring R is a monoid of the category of abelian groups, i.e. R is an abelian group with

- Multiplication $R \otimes_{\mathbb{Z}} R \to R$ gives the ring multiplication satisfying distribution

- Unit $\mathbb{Z} \to R$ gives the multiplicative identity

Example 3.13.5. R is a commutative ring. R-algebra A is a monoid of the category of R-modules, i.e. A is an R-module with

- Multiplication $A \otimes_R A \to A$ gives the ring multiplication satisfying distribution
- Unit $R \to A$ gives the multiplicative identity

3.14 Derived category

Definition 3.14.1. $F : \mathscr{A} \to \mathscr{B}$ is a left exact functor, the derived functor $\mathbb{R}F : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ is given as follows

- Take any injective resolution $I^\bullet$ quasi-isomorphic to $C^\bullet \in D^+(\mathscr{A})$, $\mathbb{R}F(C^\bullet) = F(I^\bullet)$

$G : \mathscr{A} \to \mathscr{B}$ is a right exact functor, the derived functor $\mathbb{L}F : D^-(\mathscr{A}) \to D^-(\mathscr{B})$ is given as follows

- Take any projective resolution $P_\bullet$ quasi-isomorphic to $C_\bullet \in D^-(\mathscr{A})$, $\mathbb{L}F(C_\bullet) = F(P_\bullet)$

As cohomology and homology arise from derived functors, hypercohomology and hyperhomology arise from hyper-derived functors

Remark 3.14.2. By definition, $\mathbb{R}F(C^\bullet) = \mathbb{R}F(D^\bullet)$ if $C^\bullet, D^\bullet$ are quasi-isomorphic. Let $A^\bullet$ is the chain complex with only A centered at zero, then R^iF is the composition of the following

$$\mathscr{A} \to D^+(\mathscr{A}) \xrightarrow{\mathbb{R}F} D^+(\mathscr{B}) \xrightarrow{H^i} \mathscr{B}$$

i.e. $H^i(\mathbb{R}F(A^\bullet)) = R^iF(A)$. Denote $H^i \circ \mathbb{R}F$ as $\mathbb{R}^iF$. Since any resolution $0 \to A \to C^\bullet$ is a quasi-isomorphism between $A^\bullet$ and $C^\bullet$, $R^iF(A) = \mathbb{R}^iF(A^\bullet) = \mathbb{R}^iF(C^\bullet)$. Hyper-derived functor gives a way of computing derived functor using any resolution instead of only those "nice" resolutions

Example 3.14.3. The derived functors of $\Gamma : \mathrm{Sh}(X) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}(X)$ define sheaf cohomology

$$H^i(X, \mathcal{F}) = (R^i\Gamma)(\mathcal{F})$$

The hyper-derived functors of Γ define sheaf hypercohomology

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = (R^i\Gamma)(\mathcal{F}^\bullet)$$

The derived functors of $F : RMod \to Ab$, $M \mapsto M \otimes_{RG} R$ define group homology

$$H_i(G, M) = (L_iF)(M)$$

The hyper-derived functors of F define group hyperhomology

$$\mathbb{H}_i(G, M^\bullet) = (L_iF)(M^\bullet)$$

Proposition 3.14.4.