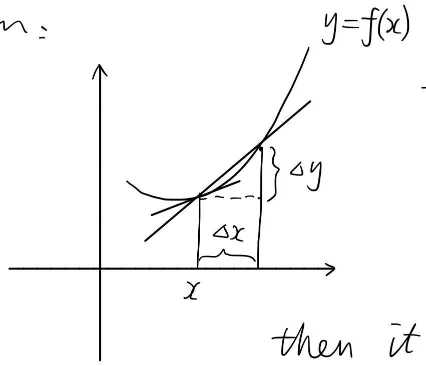
Geometric and physical intepretation of differentiation and integration

Differentiation:

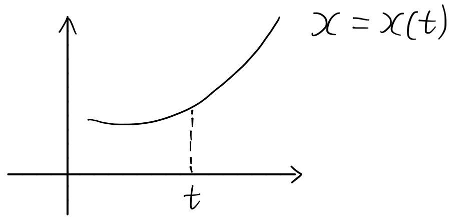
Geometric:



$\frac{\Delta y}{\Delta x}$ is the slope of secant line

as Δx gets smaller and smaller being infinitesimally small, i.e. dx, then it becomes $\frac{dy}{dx}$ which is the slope of the tangent line (which is also $f'(x)$)

physical: let t be time, x be the distance from the origin



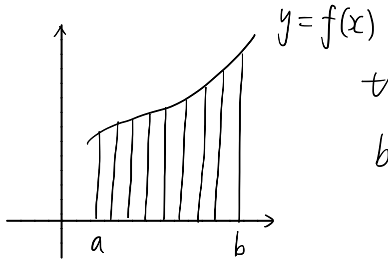
$\frac{\Delta x}{\Delta t} \rightarrow \frac{dx}{dt}$ is the velocity at time t

$$\frac{dx}{dt} = x'(t) = v(t)$$

$v'(t) = a(t)$ is acceleration

Integration:

Geometric:

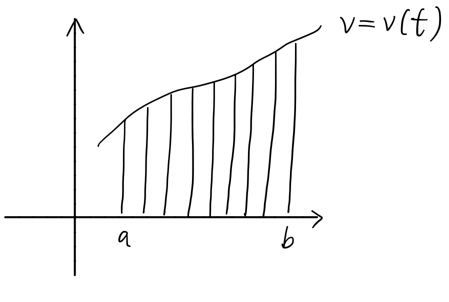


the area under the graph of f between a and b can be approximated by Riemann sum

$f(x_1)\Delta x + \cdots + f(x_n)\Delta x$, dividing interval (a,b) evenly into n equal parts with length Δx, and x_i is inside the i-th subinterval as Δx goes to infinitesimally small, i.e. dx, the sum $S f(x_i)\Delta x$ (S stands for sum) gets more accurate, hence the area is $\int_a^b f(x)\,dx$ (the integral sign $\int$ is a modified version of S, d and $\int$ are inventions of Leibniz), and this can be calculated using the fundamental theorem of Calculus $\int_a^b f(x)\,dx = F(b) - F(a)$

F being any antiderivative of f

Physical: t be time, v be velocity



$dx = x'dt = v\,dt$ is the infinitesimally small change of distance (can be negtive) since infinitesimally, it is a constant speed v

If you sum all these infinitesimally small changes in distance, accumulatively you get the change of distance between time a and b, in other words:

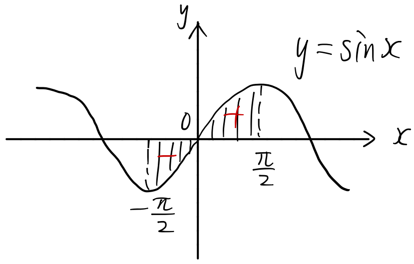
$$\int_a^b v\,dt = \int_a^b x'\,dt = \int_a^b dx(t) = \int_{x(a)}^{x(b)} dx$$

$$\int_a^b dx(t) = x(t)\Big|_a^b, \qquad \int_{x(a)}^{x(b)} dx = x\Big|_{x(a)}^{x(b)}, \qquad x(t)\Big|_a^b = x\Big|_{x(a)}^{x(b)} = x(b) - x(a)$$

Remark: It can be considered as a physical proof of fundamental theorem of Calculus

Be aware: definite integrals are "algebraic", it has signs and not necessarily positive, for example, when $v(t)$ is negative, it will be going backwards, or if the graph is under the x-axis, the integral becomes negative of the actual area, since the actual area should be $\int_a^b |f(x)|\,dx$, $|f(x)|$ is the magnitude of $f(x)$

Example: the shaded area is



$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx = -\int_{-\frac{\pi}{2}}^{0} \sin x\,dx + \int_0^{\frac{\pi}{2}} \sin x\,dx$$

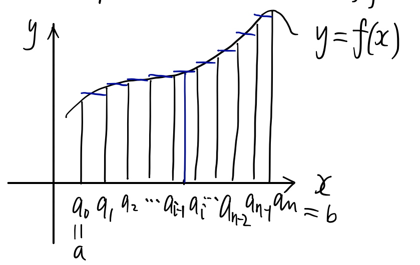
$$= -\left[-\cos x\right]\Big|_{-\frac{\pi}{2}}^{0} + \left[-\cos x\right]\Big|_0^{\frac{\pi}{2}} = 1 + 1 = 2$$

Approximations of definite integrals

Motivation: the integrand f of $\int_a^b f(x)\,dx$ in real life is not possible to find an elementary expression of its antiderivative, hence it is in general not practical to use fundamental theorem of Calculus, therefore it goes back to the Riemann sum

So now the question is what is a good choice for the approximation of each piece, we introduce three approximations

① midpoint rule: approximate each piece by a rectangle



divide interval evenly into n equal parts of width $\Delta x = \frac{a_n - a_0}{n}$

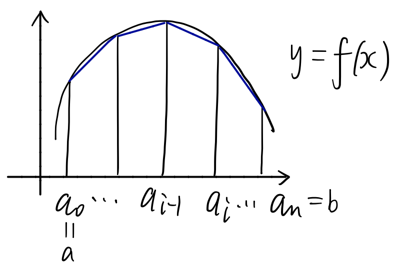
$x_i = \frac{a_{i-1}+a_i}{2}$ be the midpoint between a_{i-1} and a_i (of the i-th interval (a_{i-1}, a_i))

the approximation for the i-th piece is $f(x_i)\Delta x$

thus the approximation for the definite integral $\int_a^b f(x)\,dx$ is

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \Delta x\left(f(x_1) + \cdots + f(x_n)\right)$$

② trapezoidal rule: approximate each piece by a trapezoid

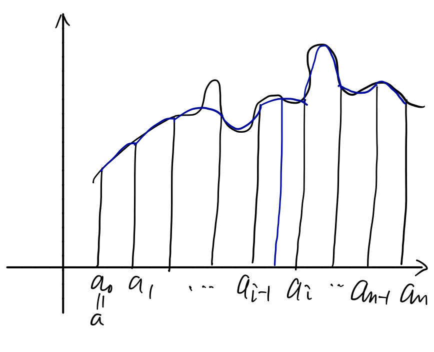


the approximation for the i-th piece is $\frac{(f(a_{i-1})+f(a_i))\Delta x}{2}$, thus

$$\int_a^b f(x)\,dx \approx \frac{(f(a_0)+f(a_1))\Delta x}{2} + \frac{(f(a_1)+f(a_2))\Delta x}{2} + \cdots + \frac{(f(a_{n-1})+f(a_n))\Delta x}{2}$$

$$= \frac{\Delta x}{2}\left[f(a_0) + 2f(a_1) + \cdots + 2f(a_{n-1}) + f(a_n)\right]$$

③ Simpson's Rule: approximate by pieces with parabola boundaries



the approximation for the i-th piece is

$\frac{1}{6}\left[f(a_{i-1}) + 4f(x_i) + f(a_i)\right]\Delta x$, $x_i = \frac{a_{i-1}+a_i}{2}$

[to be proved in the lemma]

hence $\int_a^b f(x)\,dx \approx \frac{\Delta x}{6}\left[f(a_0)+4f(x_0)+f(a_1)\right] + \cdots + \frac{\Delta x}{6}\left[f(a_{n-1})+4f(x_n)+f(a_n)\right]$

$$= \frac{\Delta x}{6}\left[f(a_0)+4f(x_0)+2f(a_1)+\cdots+2f(a_{n-1})+4f(x_n)+f(a_n)\right]$$

Lemma: Let the unique parabola passing through $(a_0, f(a_0))$, $(x_1, f(x_1))$ and $(a_1, f(a_1))$ be $\alpha x^2+\beta x+\gamma$

where $x_1 = \frac{a_0+a_1}{2}$ is the midpoint

the area of this piece with this parabola as boundary is

$$\int_{a_0}^{a_1} (\alpha x^2+\beta x+\gamma)\,dx = \left[\frac{\alpha}{3}x^3+\frac{\beta}{2}x^2+\gamma x\right]\Big|_{a_0}^{a_1}$$

$$= \left[\left(\frac{\alpha}{3}a_1^3+\frac{\beta}{2}a_1^2+\gamma a_1\right)-\left(\frac{\alpha}{3}a_0^3+\frac{\beta}{2}a_0^2+\gamma a_0\right)\right]$$

$$= \left[\frac{\alpha}{3}(a_1^3-a_0^3)+\frac{\beta}{2}(a_1^2-a_0^2)+\gamma(a_1-a_0)\right]$$

$$= \left[\frac{\alpha}{3}(a_1-a_0)(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1-a_0)(a_1+a_0)+\gamma(a_1-a_0)\right]$$

$$= (a_1-a_0)\left[\frac{\alpha}{3}(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1+a_0)+\gamma\right]$$

notice $a_1-a_0=\Delta x$

On the other hand,

$$\frac{1}{6}\left[f(a_0)+4f(x_1)+f(a_1)\right] = \frac{1}{6}\left[(\alpha a_0^2+\beta a_0+\gamma)+4(\alpha x_1^2+\beta x_1+\gamma)+(\alpha a_1^2+\beta a_1+\gamma)\right]$$

$$= \frac{1}{6}\left[(\alpha a_0^2+\beta a_0+\gamma)+4\left(\alpha\left(\frac{a_0+a_1}{2}\right)^2+\beta\frac{a_0+a_1}{2}+\gamma\right)+(\alpha a_1^2+\beta a_1+\gamma)\right]$$

$$= \frac{1}{6}\left[\alpha a_0^2+\beta a_0+\gamma+4\left(\frac{\alpha}{4}(a_0^2+2a_0a_1+a_1^2)+\frac{\beta}{2}(a_0+a_1)+\gamma\right)+\alpha a_1^2+\beta a_1+\gamma\right]$$

$$= \frac{1}{6}\left[2\alpha(a_0^2+a_1^2)+2\alpha a_0a_1+3\beta(a_1+a_0)+6\gamma\right]$$

$$= \frac{\alpha}{3}(a_1^2+a_1a_0+a_0^2)+\frac{\beta}{2}(a_1+a_0)+\gamma$$

QED