关于二维和三维变换的李群与李代数

1. SO(3)：表示三维空间中的旋转

1.1 表示

SO(3)这个群表示三维空间的旋转矩阵。旋转矩阵是正交的，其逆与转置相等。

$$R \in SO(3) \tag{1}$$

$$R^{-1} = R^{T} \tag{2}$$

李代数 so(3)对应着一个3×3的反对称矩阵，李代数ω是一个三维的向量，用$\omega_{\times}$表示对应的反对称矩阵。

1.2 指数映射

由李代数构成的反对称矩阵的指数映射，实际就是矩阵在不同指数下的线性组合：

$$\exp(\omega_{\times}) \equiv \exp\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \tag{3}$$

$$= I + \omega_{\times} + \frac{1}{2!}\omega_{\times}^2 + \frac{1}{3!}\omega_{\times}^3 + \ldots \tag{4}$$

我们还可以写作：

$$\exp(\omega_{\times}) = I + \sum_{i=0}^{\infty}\left[\frac{\omega_{\times}^{2i+1}}{(2i+1)!} + \frac{\omega_{\times}^{2i+2}}{(2i+2)!}\right] \tag{5}$$

我们知道反对称矩阵有如下的特性：

$$\omega_{\times}^3 = -(\omega^T\omega)\,\omega_{\times} \tag{6}$$

我们做如下的变换：

$$\theta^2 \equiv \omega^T\omega \tag{7}$$

$$\omega_{\times}^{2i+1} = (-1)^i\,\theta^{2i}\,\omega_{\times} \tag{8}$$

$$\omega_{\times}^{2i+2} = (-1)^i\,\theta^{2i}\,\omega_{\times}^2 \tag{9}$$

根据上面的这些特性，我们可以重新整理指数映射的泰勒展开式。

$$\exp\left(\omega_\times\right)=I+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+1)!}\right)\omega_\times+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)\omega_\times^2 \tag{10}$$

$$=I+\left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}+\dots\right)\omega_\times+\left(\frac{1}{2!}-\frac{\theta^2}{4!}+\frac{\theta^4}{6!}+\dots\right)\omega_\times^2 \tag{11}$$

$$=I+\left(\frac{\sin\theta}{\theta}\right)\omega_\times+\left(\frac{1-\cos\theta}{\theta^2}\right)\omega_\times^2 \tag{12}$$

由此我们实现了从 so(3) 到 SO(3) 的转换。如果要想从 SO(3) 反求 so(3) 可以通过下面的公式计算：

$$R\in SO(3) \tag{13}$$

$$\theta=\arccos\left(\frac{tr(R)-1}{2}\right) \tag{14}$$

$$\ln(R)=\frac{\theta}{2\sin\theta}\left(R-R^T\right) \tag{15}$$

这里$\ln(R)=\omega_\times$，由此我们可以求出ω。

1.3 伴随

在李群中，我们经常需要将一个切向量从一个切平面转换到另一个切平面。下面就利用伴随矩阵来实现这一转化。若 X 属于李群，那么伴随阵记为 Adj_X。

$$\omega\in so(3),R\in SO(3) \tag{16}$$

$$\mathrm{R}\cdot\exp(\omega)=\exp\left(Adj_R\cdot\omega\right)\cdot R \tag{17}$$

所以，

$$\exp\left(Adj_R\cdot\omega\right)=R\cdot\exp(\omega)\cdot R^{-1} \tag{18}$$

在不失一般性的情况下，令$\omega=t\cdot v$，$t\in R$，在$t=0$处求导，

$$\frac{d}{dt}|_{t=0}\exp\left(Adj_R\cdot t\cdot v\right)=\frac{d}{dt}|_{t=0}\left[R\cdot\exp(t\cdot v)\cdot R^{-1}\right] \tag{19}$$

$$\frac{d}{dt}|_{t=0}\left[I+\left(Adj_R\cdot t\cdot v\right)_\times+O\left(t^2\right)\right]=R\cdot\frac{d}{dt}|_{t=0}\left[I+(t\cdot v)_\times+O\left(t^2\right)\right]\cdot R^{-1} \tag{20}$$

$$\left(Adj_R\cdot v\right)_\times=R\cdot v_\times\cdot R^{-1} \tag{21}$$

$$\left(Adj_R\cdot v\right)_\times=(Rv)_\times \tag{22}$$

$$Adj_R = R \tag{23}$$

我们可以根据上述关系交换乘积顺序。

1.4 雅克比

考虑 $R \in SO(3), x \in R^3$。向量 x 通过 R 的旋转表示如下：

$$y = f(R, x) = R \cdot x \tag{24}$$

变换结果对向量 x 求导得，

$$\frac{\partial y}{\partial x} = R \tag{25}$$

如果变换结果对旋转矩阵求导则推导如下：

$$\omega = [\omega_1, \omega_2, \omega_3]$$

$$\omega_\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\frac{\partial \omega_\times}{\partial \omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = G_1 \tag{26}$$

$$\frac{\partial \omega_\times}{\partial \omega_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = G_2 \tag{27}$$

$$\frac{\partial \omega_\times}{\partial \omega_3} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = G_3 \tag{28}$$

$$\frac{\partial y}{\partial R} = \frac{\partial}{\partial \omega}|_{\omega=0} \left(\exp(\omega) \cdot R\right) \cdot x \tag{29}$$

$$= \frac{\partial}{\partial \omega}|_{\omega=0} \exp(\omega)(R \cdot x) \tag{30}$$

$$= \frac{\partial}{\partial \omega}|_{\omega=0} \exp(\omega) y \tag{31}$$

$$= \frac{\partial}{\partial \omega}|_{\omega=0} \left(I + \omega_\times + O(\omega^2)\right) y \tag{32}$$

$$= (G_1 y, G_2 y, G_3 y) \tag{33}$$

$$=\begin{bmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{bmatrix}=-y_{\times} \tag{34}$$

下面我们讨论输入扰动量与输出扰动量之间的关系。我们首先建立两者的关系式：

$$\exp(\varepsilon)\cdot f(g)=f(\exp(\delta)\cdot g) \tag{35}$$

根据导数的定义，∂f 对 ∂g 求导等于因变量的变化量 $\partial\varepsilon$ 除以自变量的变化量 $\partial\delta$，

因为是小量所以把 exp 直接去掉了。

$$\frac{\partial f}{\partial g}\equiv\frac{\partial\varepsilon}{\partial\delta}\bigg|_{\delta=0} \tag{36}$$

那么根据式(35)可得，

$$\varepsilon=\log\left(f(\exp(\delta)\cdot g)\cdot f(g)^{-1}\right) \tag{37}$$

所以，

$$\frac{\partial f}{\partial g}\equiv\frac{\partial\log\left(f(\exp(\delta)\cdot g)\cdot f(g)^{-1}\right)}{\partial\delta}\bigg|_{\delta=0} \tag{38}$$

式(38)提供了一个从自变量的左切平面空间扰动到因变量的左切平面空间扰动的线性映射关系。

下面我们考虑这样一个用例，

$$R_2=f(R_0)\equiv R_1\cdot R_0 \tag{39}$$

首先建立关于输入扰动与输出扰动之间的等式。

$$\exp(\varepsilon)\cdot R_2=R_1\cdot\exp(\omega)\cdot R_0 \tag{40}$$

根据上面推导的公式有：

$$\frac{\partial R_2}{\partial R_0}=\frac{\partial\log\left(R_1\exp(\omega)R_0(R_1R_0)^{-1}\right)}{\partial\omega}\bigg|_{\omega=0} \tag{41}$$

$$=\frac{\partial\log\left(R_1\exp(\omega)R_1^{-1}R_1R_0(R_1R_0)^{-1}\right)}{\partial\omega}\bigg|_{\omega=0} \tag{42}$$

$$=\frac{\partial\log\left(\exp\left(Adj_{R_1}\omega\right)\right)}{\partial\omega}\bigg|_{\omega=0} \tag{43}$$

$$=\frac{\partial}{\partial\omega}\bigg|_{\omega=0}\left(Adj_{R_1}\omega\right)=R_1 \tag{44}$$

2.SE(3)三维空间的刚体变换

2.1 表示

$$R \in SO(3), t \in R^3$$

$$C = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in SE(3) \tag{45}$$

两个变换矩阵相乘与变换矩阵求逆的公式如下：

$$C_1, C_2 \in SE(3)$$

$$C_1 \cdot C_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_2 & t_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ 0 & 1 \end{bmatrix} \tag{46}$$

$$C_1^{-1} = \begin{bmatrix} R_1^T & -R_1^T t \\ 0 & 1 \end{bmatrix} \tag{47}$$

三维坐标乘转移矩阵可表示为：

$$x = (x, y, z, w)^T \in RP^3, (\lambda x \simeq x, \forall \lambda \in R)$$

$$C \cdot x = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \cdot x = \begin{pmatrix} R(x, y, z)^T + \omega t \\ \omega \end{pmatrix} \tag{48}$$

$w = 1$时，x是笛卡尔点。

转移矩阵对应的李代数 se(3)有六个变量，因此它所对应的四维矩阵对每一个变量求偏导可得：

$$G_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_4=\begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix}$$

$$G_5=\begin{bmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{bmatrix}$$

$$G_6=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

2.2 指数映射

$$\delta=(u,\omega)\in se(3)$$

$$\exp(\delta)=\exp\begin{pmatrix}\omega_\times & u\\0&0\end{pmatrix}=I+\begin{bmatrix}\omega_\times & u\\0&0\end{bmatrix}+\frac{1}{2!}\begin{bmatrix}\omega_\times^2 & \omega_\times u\\0&0\end{bmatrix}+\frac{1}{3!}\begin{bmatrix}\omega_\times^3 & \omega_\times^2 u\\0&0\end{bmatrix}+\dots \tag{49}$$

我们可以分块求解该矩阵。

$$\exp\begin{pmatrix}\omega_\times & u\\0&0\end{pmatrix}=\begin{bmatrix}\exp(\omega_\times) & Vu\\0&1\end{bmatrix} \tag{50}$$

$$V=I+\frac{1}{2!}\omega_\times+\frac{1}{3!}\omega_\times^2+\dots \tag{51}$$

$$V=I+\sum_{i=0}^{\infty}\left[\frac{\omega_\times^{2i+1}}{(2i+2)!}+\frac{\omega_\times^{2i+2}}{(2i+3)!}\right] \tag{52}$$

$$=I+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)\omega_\times+\left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+3)!}\right)\omega_\times^2 \tag{53}$$

展开得，

$$V=I+\left(\frac{1}{2!}-\frac{\theta^2}{4!}+\frac{\theta^4}{6!}+\dots\right)\omega_\times+\left(\frac{1}{3!}-\frac{\theta^2}{5!}+\frac{\theta^4}{7!}+\dots\right)\omega_\times^2 \tag{54}$$

$$=I+\frac{1-\cos\theta}{\theta^2}\omega_\times+\frac{\theta-\sin\theta}{\theta^3}\omega_\times^2 \tag{55}$$

另外，可由下式反解u，

$$V^{-1} = I - \frac{1}{2}\omega_\times + \frac{1}{\theta^2}\left(1 - \frac{A}{2B}\right)\omega_\times^2 \tag{56}$$

$$u = V^{-1}t \tag{57}$$

2.3 伴随矩阵推导

$$\delta = (u,\omega)^T \in se(3), C = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in SE(3)$$

$$C \cdot \exp(\delta) = \exp(Adj_C \cdot \delta) \cdot C \tag{58}$$

$$\exp(Adj_C \cdot \delta) = C \exp(\delta) C^{-1} \tag{59}$$

$$Adj_C \cdot \delta = C\left(\sum_{i=1}^{6} \delta_i G_i\right) C^{-1} \tag{60}$$

$$Adj_C \cdot \delta = \begin{pmatrix} Ru + t \times R\omega \\ R\omega \end{pmatrix} \tag{61}$$

$$Adj_C = \begin{bmatrix} R & t_\times R \\ 0 & R \end{bmatrix} \in R^{6\times 6} \tag{62}$$

2.4 雅克比

已知 $C = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \in SE(3), x \in R^3$，则，

$$y = f(C, x) = Rx + t \tag{63}$$

根据上文的推导有，

$$\frac{\partial y}{\partial x} = R \tag{64}$$

$$\frac{\partial y}{\partial C} = (G_1 y \,|\, G_2 y \,|\, G_3 y \,|\, G_4 y \,|\, G_5 y \,|\, G_6 y) = (I \,|\, -y_\times) \tag{65}$$

另外还有，

$$C \equiv C_1 C_0 \tag{66}$$

$$\frac{\partial C}{\partial C_0} = \frac{\partial}{\partial \delta}\left[C_1 \exp(\delta) C_0\right] = Adj_{C_1} \tag{67}$$

3.Sim(3)相似性变换

3.1 表示

三维相似性变换结合了三维刚体变换与尺度因子。

$$R \in SO(3), t \in R^3, s \in R$$

$$T = \begin{bmatrix} R & t \\ 0 & s^{-1} \end{bmatrix} \in Sim(3) \tag{68}$$

另外有，$T_1, T_2 \in Sim(3)$

$$T_1 \cdot T_2 = \begin{bmatrix} R_1 & t_1 \\ 0 & s_1^{-1} \end{bmatrix} \cdot \begin{bmatrix} R_2 & t_2 \\ 0 & s_2^{-1} \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 t_2 + s_2^{-1} t_1 \\ 0 & (s_1 s_2)^{-1} \end{bmatrix} \tag{69}$$

$$T_1^{-1} = \begin{bmatrix} R_1^T & -s_1 R_1^T t \\ 0 & s_1 \end{bmatrix} \tag{70}$$

当 $x = (x, y, z, w)^T \in RP^3, (\lambda x \simeq x, \forall \lambda \in R)$ 时，有，

$$Tx = \begin{bmatrix} R & t \\ 0 & s^{-1} \end{bmatrix} x = \begin{pmatrix} R(x, y, z)^T + wt \\ s^{-1} w \end{pmatrix} \simeq \begin{pmatrix} s\left(R(x, y, z)^T + wt\right) \\ w \end{pmatrix} \tag{71}$$

3.2 指数映射

与前文的方法相一致，$\delta = (u, \omega, \lambda) \in sim(3)$，

$$\exp(\delta) = \exp\begin{pmatrix} \omega_\times & u \\ 0 & -\lambda \end{pmatrix} \tag{72}$$

$$= I + \begin{bmatrix} \omega_\times & u \\ 0 & \lambda \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \omega_\times^2 & \omega_\times u - \lambda u \\ 0 & \lambda^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} \omega_\times^3 & \omega_\times^2 u - \lambda \omega_\times u + \lambda^2 u \\ 0 & -\lambda^3 \end{bmatrix} \tag{73}$$

有，

$$\exp\begin{pmatrix} \omega_\times & u \\ 0 & 0 \end{pmatrix} = \begin{bmatrix} \exp(\omega_\times) & V_u \\ 0 & \exp(-\lambda) \end{bmatrix} \tag{74}$$

$$V = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\omega_\times^{n-k} (-\lambda)^k}{(n+1)!} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{\omega_x^{n-k} (-\lambda)^k}{(n+1)!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\omega_\times^j (-\lambda)^k}{(j+k+1)!} \tag{75}$$

展开有，

$$V=\left(\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(k+1)!}\right)I+\sum_{k=0}^{\infty}(-\lambda)^k\sum_{i=0}^{\infty}\left[\frac{\omega_\times^{2i+1}}{(2i+k+2)!}+\frac{\omega_\times^{2i+2}}{(2i+k+3)!}\right] \tag{76}$$

$$=\left(\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(k+1)!}\right)I+\left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+2)!}\right)\omega_\times+\left(\sum_{k=0}^{\infty}\sum_{i-0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+3)!}\right)\omega_\times^2 \tag{77}$$

令，

$$V=AI+B\omega_\times+C\omega_\times^2 \tag{78}$$

其中，$A=\frac{1-\exp(-\lambda)}{\lambda}$，$B=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+2)!}$，$C=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+3)!}$。

先考虑参数 B，

$$B=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+2)!}=\sum_{i=0}^{\infty}\left[(-1)^i\theta^{2i}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(2i+k+2)!}\right] \tag{79}$$

$$=\sum_{i=0}^{\infty}\left[\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{k=0}^{\infty}\frac{(-\lambda)^{2i+k}}{(2i+k+2)!}\right] \tag{80}$$

$$=\sum_{i=0}^{\infty}\left[\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=2i}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right] \tag{81}$$

$$=\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}-\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right]\right) \tag{82}$$

$$=\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right]\right)-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right) \tag{83}$$

令 $L\equiv\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right]\right)$，

$$B=L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right)=L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{p=0}^{i-1}\left[\frac{(-\lambda)^{2p}}{(2p+2)!}+\frac{(-\lambda)^{2p+1}}{(2p+3)!}\right]\right) \tag{84}$$

$$=L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{p=0}^{i-1}(-\lambda)^{2p}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) \tag{85}$$

$$=L-\sum_{p<i}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}(-\lambda)^{2p}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) \tag{86}$$

$$=L-\sum_{p<i}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2(i-p)}}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) \tag{87}$$

$$=L-\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\left(\frac{(-1)^{q}\theta^{2q}}{\lambda^{2q}}\left[\frac{(-1)^{p}\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^{p}\theta^{2p}}{(2p+3)!}\right)\right]\right) \tag{88}$$

$$=L-\left(\sum_{q=1}^{\infty}\frac{(-1)^{q}\theta^{2q}}{\lambda^{2q}}\right)\left(\sum_{p=0}^{\infty}\left[\frac{(-1)^{p}\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^{p}\theta^{2p}}{(2p+3)!}\right)\right]\right) \tag{89}$$

$$=\left(\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{2i}}{\lambda^{2i}}\right)\left(\sum_{m=0}^{\infty}\frac{(-\lambda)^{m}}{(m+2)!}\right)-\left(\sum_{q=0}^{\infty}\frac{(-1)^{q}\theta^{2q}}{\lambda^{2q}}-1\right)\left(\sum_{p=0}^{\infty}\left[\frac{(-1)^{p}\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^{p}\theta^{2p}}{(2p+3)!}\right)\right]\right) \tag{90}$$

令，

$$B=\alpha\cdot\beta-(\alpha-1)\cdot\gamma=\alpha\cdot(\beta-\gamma)+\gamma \tag{91}$$

其中，

$$\alpha\equiv\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{2i}}{\lambda^{2i}} \tag{92}$$

$$\beta\equiv\sum_{m=0}^{\infty}\frac{(-\lambda)^{m}}{(m+2)!} \tag{93}$$

$$\gamma\equiv\sum_{p=0}^{\infty}\left[\frac{(-1)^{p}\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^{p}\theta^{2p}}{(2p+3)!}\right)\right] \tag{94}$$

由泰勒展开式的形式得：

$$\alpha=\frac{\lambda^{2}}{\lambda^{2}+\theta^{2}} \tag{95}$$

$$\beta=\frac{\exp(-\lambda)-1+\lambda}{\lambda^{2}} \tag{96}$$

$$\gamma=\frac{1-\cos\theta}{\theta^{2}}-\lambda\left(\frac{\theta-\sin\theta}{\theta^{3}}\right) \tag{97}$$

接下来再求C，

$$C=\alpha\cdot(\mu-v)+v \tag{98}$$

$$\mu=\frac{1\text{-}\lambda+\frac{1}{2}\lambda^{2}\text{-}\exp(-\lambda)}{\lambda^{2}} \tag{99}$$

$$v=\frac{\theta-\sin\theta}{\theta^{3}}-\lambda\left(\frac{\cos\theta-1+\frac{\theta^{2}}{2}}{\theta^{4}}\right) \tag{100}$$

最后统一整理得，

$$(u,\omega,\lambda)^T \in sim(3) \tag{101}$$

$$\theta^2 = \omega^T \omega \tag{102}$$

$$X = \frac{\sin\theta}{\theta} \tag{103}$$

$$Y = \frac{1-\cos\theta}{\theta^2} \tag{104}$$

$$Z = \frac{1-X}{\theta^2} \tag{105}$$

$$\omega = \frac{\frac{1}{2}-Y}{\theta^2} \tag{106}$$

$$\alpha = \frac{\lambda^2}{\lambda^2+\theta^2} \tag{107}$$

$$\beta = \frac{\exp(-\lambda)-1+\lambda}{\lambda^2} \tag{108}$$

$$\gamma = Y - \lambda Z \tag{109}$$

$$\mu = \frac{1-\lambda+\frac{1}{2}\lambda^2-\exp(-\lambda)}{\lambda^2} \tag{110}$$

$$v = Z - \lambda\omega \tag{111}$$

$$A = \frac{1-\exp(-\lambda)}{\lambda} \tag{112}$$

$$B = \alpha(\beta-\gamma)+\gamma \tag{113}$$

$$C = \alpha(\mu-v)+v \tag{114}$$

$$R = I + X\omega_\times + Y\omega_\times^2 \tag{115}$$

$$V = AI + B\omega_\times + C\omega_\times^2 \tag{116}$$

$$\exp\begin{pmatrix} u \\ \omega \\ \lambda \end{pmatrix} = \begin{bmatrix} R & Vu \\ 0 & \exp(-\lambda) \end{bmatrix} \tag{117}$$

3.3 伴随矩阵

直接给出求解公式:

$$\delta = (u, \omega, \lambda)^T \in sim(3), T = \begin{bmatrix} R & t \\ 0 & s^{-1} \end{bmatrix} \in Sim(3)$$

$$T \cdot \exp(\delta) = \exp(Adj_T \cdot \delta) \cdot T \tag{118}$$

$$\exp(Adj_T \cdot \delta) = T \exp(\delta) T^{-1} \tag{119}$$

$$Adj_T \cdot \delta = T\left(\sum_{i=1}^{7} \delta_i G_i\right) T^{-1} = \begin{pmatrix} s(Ru + t_\times R\omega - st) \\ R\omega \\ -\lambda \end{pmatrix} \tag{120}$$

$$Adj_T = \begin{pmatrix} sR & st_\times R & -st \\ 0 & R & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R^{7\times 7} \tag{121}$$