Week 10: Tutorial Handout on series

- (Stirling's formula) We have $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ (i.e. $\lim\limits_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1$).
- A series converges uniformly when $\left\|\sum^N f_k - \sum f_k\right\|_\infty = \left\|\sum_{k\geq N} f_k\right\|_\infty \to 0$.
- (Weierstrass M-test) If $\|f_k\|_\infty \leq M_k$ and $\sum M_k < \infty$, then $\sum f_k(x)$ converges uniformly.
- (Hadamard's theorem) For power series $\sum a_k x^k$ if $\limsup\limits_{k\to\infty} |a_k|^{1/k} = \frac{1}{R}$, then the radius of convergence is R. If $\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < \infty$, then $\lim\limits_{k\to\infty} |a_k|^{1/k} = \lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$.
- (Calculus with power series) For power series $f(x) := \sum a_k x^k$ with interval of convergence $x \in (-R, R)$ we have

$$f'(x) = \sum a_k k x^{k-1} \text{ and } \int_0^x f(t)dt = \sum \frac{a_k}{k+1} x^{k+1}.$$

8.4 Series

1. Show that $\sum\limits_{k=1}^{\infty} x^k e^{-kx}$ converges uniformly in [0,A] ? What about for $A = \infty$?
2. Does $\sum\limits_{k=1}^{\infty} \frac{1}{x^2+n^2}$ converge uniformly on $\mathbb{R}$?

8.4 Power series

3. Find the radius of convergence for the following power series:
   - $\sum\limits_{k=1}^{\infty} k^3 x^k$
   - $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$
   - $\sum\limits_{k=1}^{\infty} \frac{k!}{k^k} x^k$.
4. Compute $f(x) = \sum\limits_{k=1}^{\infty} (\frac{1}{(k-1)!} + k) x^{k-1}$.
5. Compute $f(x) = \sum\limits_{k=1}^{\infty} (k+1) x^k$ and $\sum\limits_{k=1}^{\infty} \frac{k}{3^k}$. Justify your method.