Theorem $W_1 \cup W_2 \not\subset V$

Proof Take $W_1 = span\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}, W_2 = span\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}, \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} \notin W_1 \cup W_2$

Theorem $W_1 \cap W_2 \subseteq V$

Proof $(0 \in W_1 \wedge 0 \in W_2) \rightarrow 0 \in W_1 \cap W_2$

Take $c \in \mathbb{R}, w, w' \in W_1 \cap W_2$

Since $w, w' \in W_1, cw + w' \in W_1, w, w' \in W_2, cw + w' \in W_2$. $cw + w' \in W_1 \cap W_2$

Theorem $W_1 + W_2 \subseteq V$

Proof Take $c \in \mathbb{R}, w, w' \in W_1 + W_2$

Then take $w_1, w_1' \in W_1, w_2, w_2' \in W_2$ s.t. $w_1 + w_2 = w, w_1' + w_2' = w'$

Hence $cw + w' = c(w_1 + w_2) + w_1' + w_2' = (cw_1 + w_1') + (cw_2 + w_2') \in W_1 + W_2$

Definition $W_1, W_2 \subseteq V$, then V is the direct sum of W_1, W_2, denote $V = W_1 \oplus W_2$, if $\forall v \in V, \exists!\, w_1 \in W_1, w_2 \in W_2, v = w_1 + w_2$

Theorem $V = W_1 \oplus W_2$ IFF($V = W_1 + W_2 \wedge W_1 \cap W_2 = \{0\}$)

Definition the set $S \neq \emptyset$ is a basis for V if $V = span(S)$ and S is linearly independent set

Alternatively, a basis is the minimal spanning set of a vector space

Theorem S is a basis for V if $\forall v \in V$, v can be written uniquely as a linear combination of vectors in S

Definition Dimension is the minimum number of vectors required to span V

Theorem If V spanned by n vectors, any set of more than n vectors from V must be linearly dependent, if V is spanned by m vectors, any linearly independent set in V must contain $\leq m$ vectors

The two different bases must be the same number of vectors

$$\dim W = \dim V \text{ IFF } W = V$$

Any linearly independent subset of a vector space can be expanded to a basis for the vector space

Any linearly dependent set of a vector space can be reduced to a basis for the vector space

$$\mathrm{U}, \mathrm{V} \subseteq \mathrm{W} \dim U + V = \dim U + \dim V - \dim(U \cap V)$$

Definition $T: V \rightarrow W$ is linearly IFF $T(cx + y) = cT(x) + T(y)$

Theorem T are uniquely defined by their values on any bases for V

Take $v \in V = \{v_1, \dots, v_n\}, v = c_1 v_1 + \cdots + c_n v_n$

$$T(v) = c_1 T(v_1) + \cdots + c_n T(v_n)$$

Theorem T maps subspace to subspace

$$T(x_1), T(x_2) \in W \rightarrow T(cx_1 + x_2) = cT(x_1) + T(x_2) \in W$$

Theorem $T(0) = 0$

Theorem Linear transformation that outputs subspace, its pre-image is also a subspace

Definition The image of a linear transformation T. $im(T) = T(V) = \{ T(x) \mid x \in V \} \subseteq W$

The kernel space $\ker T = T^{-1}(\{0\}) = \{ x \in V \mid T(x) = 0 \} \subseteq V$

Theorem $T: V \rightarrow W. \dim V = \dim \ker T + \dim imT$

Proof Let $\dim V = n$

Let $\{v_1, \dots v_k\}$ be a basis for $\ker T$

Then $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ be basis for V

Since $im(T) = \{v \in T(V)\} = span\{T(v_1), \dots, T(v_n)\}$

However, $v_1, \dots, v_k$ are in kerT, which contributes nothing to $im(T)$

Hence, $imT = span\{T(v_{k+1}), \dots, T(n)\}$

Let $T(d_{k+1} v_{n+1} + \cdots + d_n v_n) = d_{k+1} T(v_{n+1}) + \cdots + d_n T(v_n) = 0$

$$(d_{k+1} v_{n+1} + \cdots + d_n v_n) \in \ker T$$

Take $c_1 v_1 + \cdots + c_k v_k = (d_{k+1} v_{n+1} + \cdots + d_n v_n)$

$$c_1 v_1 + \cdots + c_k v_k + d_{k+1} v_{k+1} + \cdots + d_n v_n = 0$$

Since $\{v_1, \dots, v_n\}$ basis for $V, c, d's = 0$

$\{T(v_{k+1}), \dots, T(v_n)\}$ linearly independent and span imT

Theorem $T: V \rightarrow W$ *inj IFF* $\ker T = \{0\}$ AND *surj IFF* $\dim imT = \dim W$

Theorem $T: V \to W$ linear, then $\dim V = \dim \ker T + \dim imT$

Definition $T: V \to W$ *injective IFF* $\dim \ker T = 0$ $(\ker T = \{0\})$, *surjective IFF* $\dim imT = \dim W$

Example $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ defined $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$, is T inj or surj?

$$p(x) = ax^2 + bx + c \in \ker T \;\; IFF \; T(p(x)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a+b+c=0 \\ 4a+2b+c=0 \\ 9a+3b+c=0 \end{pmatrix}$$, the only

solution is $a = b = c = 0$

Hence $\ker T = \{0\} \to inj$

$$\dim imT = \dim V - \dim \ker T = 3 - 0 = 3 = \dim W \to surj$$

$$T \; is \; bij$$

Example $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$ defined $T(p(x)) = xp'(x)$

Not inj: $\forall c \in \mathbb{R}. c' = 0$

Not surj: $\dim \ker T > 0. \dim imT < n + 1.$ while $\dim W = n + 1 \neq \dim imT$

Theorem $(T: V \to W \; inj) \to$
$(\{v_1, \dots, v_n\} \; linearly \; independent) \to (\{T(v_1), \dots, T(k)\} \; linearly \; independent)$

Injective linear transformation maps linear independent sets to linear independent sets

Proof Assume $T: V \to W \; inj$

Assume $(\{v_1, \dots, v_n\} \; linearly \; independent)$

take $c_1, \dots, c_k \in \mathbb{R} \; s.t. \; c_1T(v_1) + \dots + c_kT(v_k) = 0$

Because T is linear, $c_1T(v_1) + \dots + c_kT(v_k) = T(c_1v_1 + \dots + c_kv_k) = 0$

Hence $c_1v_1 + \dots + c_kv_k \in \ker T$

Because $T \; inj, \ker T = \{0\}$, $c_1v_1 + \dots + c_kv_k = 0$

Because $\{v_1, \dots, v_k\}$ linearly independent, $c_1 = \dots = c_k = 0$

$\{T(v_1), \dots, T(v_k)\}$ linearly independent

Theorem $T: V \to W \; linear \; \{v_1, \dots, v_k\}$ basis for V, $(\{T(v_1) \dots T(v_k)\}$ linear independent $\to T$ inj)

Proof Assume $\{T(v_1) \dots T(v_k)\}$ linear independent

Take $c_1, \dots, c_k \in \mathbb{R}$. $T(c_1v_1 + \dots + c_kv_k) = 0 \in \ker T$

Because T is linear $T(c_1v_1 + \dots + c_kv_k) = c_1T(v_1) + \dots + c_kT(v_k) = 0$

Because $\{T(v_1) \dots T(v_k)\}$ linear independent, $c_1 = \dots = c_k = 0$ and there is no other solutions

$$\ker T = \{0\}$$

T inj

Theorem $T: V \to W$ linear. $\dim V > \dim W \to T \; NOT \; inj$. $\dim V < \dim W \to T \; NOT \; surj$

Proof By dimension theorem $\dim V = \dim \ker T + \dim imT$

Because $\dim imT < \dim W$, $\dim \ker > 0$, hence $\dim \ker T \neq 0$

Because $\dim imT \leq \dim V < \dim W$, hence $\dim imT \neq \dim W$

Definition If $T: V \to W$ is bijective, T is an isomorphism. If there exists an isomorphism $T: V \to W, V, W$ are isomorphic vector spaces

Theorem V, W are isomorphic IFF $\dim V = \dim W$

Proof

Assume V, W are isomorphic, take isomorphism $T: V \to W$, then T is bijective

Hence $\dim \ker T = 0\ (inj)\ \dim W = \dim imT = \dim V - 0\ (surj)$

Assume $\dim W = \dim V$, let $\{v_1, \dots, v_n\}$ be basis for V, $\{w_1, \dots, w_n\}$ be basis for W

Let isomorphism $T: V \to W$ defined $T(v_i) = w_i$ for $i = 1, \dots, n$;

Let $x \in \ker T$, notice $x \in V$, hence take $c_1, \dots, c_n \in \mathbb{R}. c_1 v_1 + \cdots + c_n v_n = x$

$$T(x) = 0$$

$$T(c_1 v_1 + \cdots + c_n v_n) = 0$$

$c_1 T(v_1) + \cdots c_n T(v_n) = c_1 w_1 + \cdots + c_n w_n = 0$ (T is linear)

Because $w_1, \dots, w_n$ are linearly independent, $c_1 = \cdots = c_n = 0$

$x = 0$ is the only element in $\ker T$, $\dim \ker T = 0$ inj

$$\dim imT = \dim V - 0 = \dim V = \dim W \ \ surj$$

Theorem $T: V \to W$ is isomorphism IMPLIES T maps a basis of V to a basis of W

Proof above

Example $T: P_2(\mathbb{R}) \to \mathbb{R}^3$ defined $T(p(x)) = \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$ is an isomorphism, hence $P_2(\mathbb{R})$, $\mathbb{R}^3$ are isomorphic

Theorem any n-dimension vector space V is isomorphic to R^n

Proof Let $\{v_1, \dots, v_n\}$ be any basis for V

Let $T: V \to \mathbb{R}^n$ defined $T(v_i) = e_i$, then $\forall x \in V. x = c_1 v_1 + \cdots c_n v_n, c_1, \dots, c_n \in \mathbb{R}, T(x) = \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix}$ T is an isomorphism

Definition V vector space, $\alpha = \{v_1, \dots, v_n\}$ be any basis for V, $\forall x \in V. x = c_1 v_1 + \cdots c_n v_n, c_1, \dots, c_n \in \mathbb{R}$, then $(c_1, \dots, c_n)$ is called coordinates of x relative to α, $[x]_\alpha = \begin{pmatrix} c_1 \\ \cdots \\ c_n \end{pmatrix}$ called the coordinate vecotr for x relative to α

Theorem $[x + y]_\alpha = [x]_\alpha + [y]_\alpha, [cx]_\alpha = c[x]_\alpha$

Proof By linearity

NOTICE For α, α' be two different bases of V, in most caes $[x]_\alpha \neq [x]_{\alpha'}$

Definition say W vector space, $\beta = \{w_1, \dots, w_2\}$ basis for W, say $T: V \to W$ linear

$$[T(x)]_\beta = [T(c_1v_1+\cdots+c_nv_n)]_\beta = c_1[T(v_1)]_\beta+\cdots+c_n[T(v_n)]_\beta$$
$$= \left[[T(v_1)]_\beta \dots [T(v_n)]_\beta\right]\begin{pmatrix} c_1 \\ \dots \\ c_n \end{pmatrix}$$

$\left[[T(v_1)]_\beta \dots [T(v_n)]_\beta\right]$ is the matrix of T with respect to α and β, denote $[T]_\alpha^\beta$

Example $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined $T\big(p(x)\big) = xp(x)$, $\alpha = \{1-x, 1-x^2, x\}, \beta = \{1, 1+x, 1+x+x^2, 1-x^3\}$, find $[T]_\alpha^\beta$

$$T(1-x) = x - x^2 = -1 + 2(x+1) - (x^2+x+1) + 0(1-x^3)$$
$$[T(1-x)]_\beta = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, [T(1-x^2)]_\beta = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, [T(x)]_\beta = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, [T]_\alpha^\beta$$
$$= \begin{pmatrix} -1 & -2 & 0 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Imagine For every $T: V \to W$:

$$\begin{matrix} V & \to_T & W \\ \updownarrow_{[v]\alpha} & linear & \updownarrow_{[T(v)]\beta} \\ \mathbb{R}^n & \to_{[T]_\alpha^\beta} & \mathbb{R}^m \end{matrix}$$

Theorem $x \in \ker T\ \ IFF\ T(x) = 0\ IFF\ [T(x)]_\beta = [0]_\beta\ IFF\ [T]_\alpha^\beta [x]_\alpha = 0 \in \mathbb{R}^m\ IFF\ [x]_\alpha \in null\left([T]_\alpha^\beta\right)$

$$w \in imT\ IFF\ w = T(x \in V)\ IFF\ [w]_\beta = [T(x)]_\beta = [T]_\alpha^\beta [x]_\alpha = col\left([T]_\alpha^\beta\right)$$

Dimension theorem IFF rank-nullity theorem

Example $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined $T(a+bx+cx^2) = \begin{pmatrix} c & -c \\ a-c & a+c \end{pmatrix}, \alpha = \{x^2-x, x-1, x^2+1\}, \beta = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$, find $[T]_\alpha^\beta, \ker T, imT$

$$T(x^2-x) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \dots$$
$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Because $null\left([T]_{\alpha}^{\beta}\right) = span\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$ (by null-dim theory, because the matrix has 2 pivots, hence $\dim col\left([T]_{\alpha}^{\beta}\right) = 2,\ 3 - 2 = 1,$ there will only be one vector in the null space)

Hence $\ker T = span\{-(x^2 - x) + (x - 1) + x^2 + 1\} = span\{x\}$

Because $col\left([T]_{\alpha}^{\beta}\right) = span\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}\right\},$

$$imT = span\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\} = spam\left\{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right\}$$

Theorem $T_1, T_2: V \to W$ linear. $\forall x \in V. (T_1 + T_2)(x) = T_1(x) + T_2(x)$

$\forall c \in \mathbb{R}.\ (cT_1): V \to W,\ (cT_1)(x) = cT_1(x)$

Similarly, let α be basis of V, β be basis of W.

$$[T_1 + T_2]_{\alpha}^{\beta} \coloneqq [T_1]_{\alpha}^{\beta} + [T_2]_{\alpha}^{\beta}$$

$$[cT_1]_{\alpha}^{\beta} \coloneqq c[T_1]_{\alpha}^{\beta}$$

Composition $T: V \to W,\ S: W \to U$ then $S \circ T: V \to U \coloneqq \forall x \in V.\ S \circ T\ (x) = S\big(T(x)\big)$

If S, T linear, then $S \circ T$ linear

Proof $S \circ T(ax + by) = S\big(T(ax + by)\big)$

$= S\big(aT(x) + bT(y)\big)$ because T linear

$= aS\big(T(x)\big) + bS\big(T(x)\big)$ because S linear

$= a(S \circ T)(x) + b(S \circ T)(y)$ by definition of composition

Example $T: P_3(\mathbb{R}) \to P_2(\mathbb{R}) \coloneqq T\big(p(x)\big) = p'(x),\ S: P_2(\mathbb{R}) \to P_3(\mathbb{R}) \coloneqq S\big(p(x)\big) = xp(x)$

Then, $S \circ T: P_3(\mathbb{R}) \to P_3(\mathbb{R}) \coloneqq S \circ T\big(p(x)\big) = S\big(p'(x)\big) = xp'(x)$

$T \circ S: P_2(\mathbb{R}) \to P_2(\mathbb{R}) \coloneqq T \circ S\big(p(x)\big) = T\big(xp(x)\big) = p(x) + xp'(x)$

Matrix of composition Let α, β, γ be basis for V, W, U respectively, known $[T]_{\alpha}^{\beta}, [S]_{\beta}^{\gamma}$

Then, $\forall x \in V. [S \circ T]_{\alpha}^{\gamma}[x]_{\alpha} = [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}[x]_{\alpha} = [S]_{\beta}^{\gamma}[T(x)]_{\beta} = [S \circ T(x)]_{\gamma}$,

$$[S \circ T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$$

Inverse Transformation

Definition $T: V \to W$ isomorphism IFF $\exists S: W \to V. (\forall w \in W. T \circ S(w) = w$ AND $\forall v \in V. S \circ T(v) = v)$

S is called inverse of T $(S = T^{-1})$

Proof Let $S: W \to V$, assume $(\forall w \in W. T \circ S(w) = w$ AND $\forall v \in V. S \circ T(v) = v)$

Because $\forall w \in W. T \circ S(w) = w$, then $\forall w \in W. T\big(S(w)\big) = w$, all w in W is defined by T, T is surj

Suppose $T(x_1) = T(x_2)$, because $\forall v \in V. S \circ T(v) = v, S\big(T(x_1)\big) = S\big(T(x_2)\big) = x_1 = x_2$, T is inj

$\exists S: W \to V. (\forall w \in W. T \circ S(w) = w$ AND $\forall v \in V. S \circ T(v) = v)$ IMPLIES $T: V \to W$ isomorphism

Let $T: V \to W$ isomorphism, let $S: W \to V \coloneqq \forall v \in V.\ \forall w \in W. S(w) = v$ IFF $T(v) = w$

Because T is bijective, S is bijective $(\forall w \in W. T \circ S(w) = w$ AND $\forall v \in V. S \circ T(v) = v)$

$T: V \to W$ isomorphism IMPLIES $\exists S: W \to V. (\forall w \in W. T \circ S(w) = w$ AND $\forall v \in V. S \circ T(v) = v)$

Theorem $T: V \to W$ isomorphism IFF bijective IFF invertible IFF $\exists T^{-1}: W \to V$ is linear

Proof Show $T^{-1}(aw_1 + bw_2) = aT^{-1}(w_1) + bT^{-1}(w_2)$

$T^{-1}(w_1) = x_1$ unique such that $T(x_1) = w_1$ (because T bijective and by definition of inverse)

$$T^{-1}(w_2) = x_2 \ s.t.\ T(x_2) = w_2$$

$$T^{-1}(aw_1 + bw_2) = x \ s.t.\ T(x) = aw_1 + bw_2$$

Hence $T(x) = aT(x_1) + bT(x_2)$, since T bijective $x = ax_1 + bx_2, T^{-1}(aw_1 + bw_2) = aT^{-1}(w_1) + bT^{-1}(w_2)$

Example $T: P_2(\mathbb{R}) \to P_2(\mathbb{R}) \coloneqq T(a + bx + cx^2) = (a + 2b + c) + (2a + 3b + 2c)x + (a + 3b + 2c)x^2$

Find $T^{-1}(a + bx + cx^2)$

Method1 Known $T^{-1}(a + bx + cx^2)$ linear, hence $T^{-1}(a + bx + cx^2) = aT^{-1}(1) + bT^{-1}(x) + cT^{-1}(x^2)$

Because $T(1) = T(1 + 0x + 0x^2) = 1 + 2x + x^2, T(x) = 2 + 3x + 3x^2$, $T(x^2) = 1 + 2x + 2x^2$

$T^{-1}(x^2) = T^{-1}\big(1 + 2x + 2x^2 - (1 + 2x + x^2)\big) = x^2 - 1$, similarly $T^{-1}(x) = 1 - x + x^2, T^{-1}(1) = 2x - 3x^2$

$$T^{-1}(a + bx + cx^2) = 2ax - 3ax^2 + b - bx + bx^2 + cx^2 - c$$

$$= (b - c) + (2a - b)x + (c + b - 3a)x^2$$

Theorem If $T: V \to W$ isomorphism and α, β basis V, W respectively, then $[T]_\alpha^\beta$ is invertible $[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$

Method 2 Consider basis $\alpha = \{1, x, x^2\}$

$$[T]_\alpha^\alpha = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 3 & 2 \end{pmatrix}, [T^{-1}]_\alpha^\alpha = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

$$[T^{-1}(a + bx + cx^2)]_\alpha = [T^{-1}]_\alpha^\alpha [a + bx + cx^2]_\alpha = \begin{pmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= (b - c) + (2a - b)x + (c + b - 3a)x^2$$

Definition α, α' are two basis for V, $\forall v \in V. [I]_{\alpha}^{\alpha'}[x]_{\alpha} = [x]_{\alpha'}$, $[I]_{\alpha}^{\alpha'}$ is the identity transformation, or change of basis matrix transformation from a to a'.

Proof $I(x) = x$

$$[I(x)]_{\alpha'} = [x]_{\alpha'} = [I]_{\alpha}^{\alpha'}[x]_{\alpha}$$

Example $\alpha = \{1, 1+x, 1+x+x^2\}, a' = \{1-x^2, 1+x, 1\}$

$$[I]_{\alpha}^{\alpha'}[a+bx+cx^2]_{\alpha} = [a+bx+cx^2]_{\alpha'}$$

$$a+bx+cx^2 = (a-b)1 + (b-c)(1+x) + c(1+x+x^2),\ [a+bx+cx^2]_{\alpha} = \begin{pmatrix} a-b \\ b-c \\ c \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{\alpha} \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{a'}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{a} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}_{a'}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{a} \to \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}_{a'}$$

$$[I]_{\alpha}^{\alpha'}[a+bx+cx^2]_{\alpha} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} a-b \\ b-c \\ c \end{pmatrix} = \begin{pmatrix} -c \\ b \\ a-b+c \end{pmatrix} = [a+bx+cx^2]_{\alpha'}$$

$$= -c + cx^2 + b + bx + a - b + c = a + bx + cx^2$$

Theorem If α, β basis for V, $I: V \to V, [I]_{\alpha}^{\beta}[x]_{\alpha} = [x]_{\beta}$, $[I]_{\beta}^{\alpha}[x]_{\beta} = [x]_{\alpha}$, then $[I]_{\alpha}^{\beta} = \left([I]_{\alpha}^{\beta}\right)^{-1}$

Example $\alpha = \{x^2, 1+x, x+x^2\}$, β are two basis for V, $[I]_{\alpha}^{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, $[p(x)]_{\beta} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, find $p(x)$, β

$$[I]_{\beta}^{\alpha} = \left([I]_{\alpha}^{\beta}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix},\ [p(x)]_{\alpha} = [I]_{\beta}^{\alpha}[p(x)]_{\beta} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$p(x) = x^2 - (1+x) + 2(x+x^2) = 3x^2 + x - 1$$

$$\beta = \{x^2 - (1+x) + x + x^2, 1+x, (-1+x) + x + x^2\} = \{-1+2x^2, 1+x, x^2-1\}$$

Theorem $T: V \to W$ linear, a, a' basis for V, β, β' basis for W, then $[T]_{\alpha'}^{\beta'} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[I]_{\alpha}^{\alpha'}$

Proof Known $T = ITI$, Let $v \in V$ be arbitrary

$$[I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta'}[I]_{\alpha}^{\alpha'}[v]_{a} = [I]_{\beta}^{\beta'}[T]_{\alpha}^{\beta}[v]_{\alpha'} = [I]_{\beta}^{\beta'}[T(v)]_{\beta} = [T(v)]_{\beta'}$$

Theorem $T: V \to V$, then $[T]_{\alpha'}^{\alpha'} = [T]_{\alpha}^{\alpha}$

Proof $\left([I]_{\alpha}^{\alpha'}\right)^{-1}[T]_{\alpha'}^{\alpha'}[I]_{\alpha}^{\alpha'} = [T]_{\alpha}^{\alpha}$

Theorem Say A, B are similar if exists invertible matrix P, $B = P^{-1}AP$

Two matrix represents the same linear transformation T relative to different bases IFF they're similar

Example $\alpha = \{(1,1,1), (1,1,0), (1,0,0)\}$ for $\mathbb{R}^3, T: \mathbb{R}^3 \to \mathbb{R}^3 \coloneqq T(1,1,1) = (2,2,2), T(1,1,0) = (3,3,0), T(1,0,0) = (-1,0,0)$, find $[T]_{\beta}^{\beta}$ be a standard basis for $\mathbb{R}^3$

Method 1: $[T]_{\beta}^{\beta} = [I]_{\alpha}^{\beta}[T]_{\alpha}^{\alpha}[I]_{\beta}^{\alpha} = [I]_{\alpha}^{\beta}[T]_{\alpha}^{\alpha}\left([I]_{\alpha}^{\beta}\right)^{-1} = \cdots$

Method 2: $T(1,0,0) = (-1,0,0); T(0,1,0) = (3,3,0) - (-1,0,0) = (4,3,0); T(0,0,1) = (2,2,2) - (3,3,0) = (-1,-1,2)$. $[T]_{\beta}^{\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 4 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}$

Definition linear operator $T: V \to V$ is diagonalizable if $\exists\beta$ basis for V s.t. $[T]_\beta$ is a diagonal matrix

Equivalently, $T(v_i) = \lambda_i v_i$ for some $i \in \mathbb{N}$. $\lambda \in \mathbb{R}$. $v \in V$ or $[T]_\beta =$

$$\begin{pmatrix} \lambda_1 & \dots & 0 \\ \dots & \lambda_i & \dots \\ 0 & \dots & \lambda_n \end{pmatrix}$$

Definition linear operator $T: V \to V$, a non-zero $v \in V$ is an eigenvector of T if $T(x) = \lambda x$, $\lambda \in \mathbb{R}$. λ is called the eigenvalue of T corresponding to x

Theorem $T: V \to V$ linear operator is disgonalizable IFF $\exists\beta$ consisting of eigenvalues of T.

If T is diagonalizable, the diagonal entries of $[T]_\beta$ are corresponding eigenvalues of T.

Definition $\det T = \det[T]_\alpha$, $\forall\alpha$: The determinant of linear operator is independent of any choice of basis

Proof Let $T: V \to V$ be a linear operator, take a, a' be two different bases for V.

Because $B = PAP^{-1}$, $\det B = \det(PAP^{-1}) = \det P \det A \det P^{-1} = \det P \det P^{-1} \det A = \det A$

Theorem λ is an eigenvalue of T iff $\det(T - \lambda I) = 0$

Proof λ is an eigenvalue of T iff exists non-zero $x \in V$ is an eigenvector of T where $T(x) = \lambda x$, take such x

Hence $T(x) - \lambda x = 0$, $(T - \lambda I)(x) = 0$

Because $x \neq 0, x \in \ker(T - \lambda I)$, hence $\ker(T - \lambda I) \neq \{0\}$, hence $T - \lambda I$ not *inj*, hence not invertible,

$\det(T - \lambda I) = 0$

Definition The characteristic polynomial of T $P_T(\lambda) = \det(T - \lambda I) = 0$

Theorem λ is an eigenvalue of T iff it's a root of $P_T(\lambda)$

Theorem $T: V \to V$ linear operator λ eigenvalue of T, x is an eigenvector of T corresponding to λ iff $x \neq \vec{0}, x \in \ker(T - \lambda I)$. the eigenspace of T corresponding to λ, $E_\lambda = \ker(T - \lambda I) \subseteq V$

Proof Let $x \in \ker(T - \lambda I)$ be arbitrary, hence $(T - \lambda I)(x) = 0$, $T(x) - \lambda I(x) = 0$, Since $x \neq \vec{0}, T(x) = \lambda x$

Let $x \in V$ such that $T(x) = \lambda x$, hence $T(x) - \lambda x = 0, (T - \lambda I)x = 0, x \in \ker(T - \lambda I)$

Example Choose any basis a for V, then x is an eigenvector of T corresponding to λ IFF $[x]_a$ is an eigenvector for $[T]_a^a$ corresponding to λ

$$T(x) = \lambda x,\ [T(x)]_a = [\lambda x]_a,\ [T]_a^a [x]_a = \lambda [x]_a$$

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ linear operator that has $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ with respond to basis $a = \{x^2, x-2, x+1\}$

$$P_T(\lambda) = \det(T - \lambda I)$$

$$= \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 2 & 3 & 1-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)^2$$

$$\lambda_1 = 1,\ E_1 = null(A - I) = null\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{pmatrix} = span\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = span\{x+1\}$$

$$\lambda_2 = 2, E_2 = null(A - 2I) = null\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 3 & -1 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right\}$$

$$= span\{x^2 + 2x + 2\}$$

Example $T(p(x)) = p(x) + (x+1)p'(x)$

Consider the standard basis $a = \{1, x, x^2\}$

$T(1) = 1, T(x) = x + (x+1) = 1 + 2x,\ T(x^2) = x^2 + (x+1)2x = 2x + 3x^2$

$[T]_a^a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is upper triangular, hence

$$\lambda_1 = 1, E_1 = null\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} = span\{1\}$$

$$\lambda_2 = 2, E_2 = null\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\} = span\{1 + x\}$$

$$\lambda_3 = 3, E_3 = null\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right\} = span\{1 + 1 + 2x + 2x + 3x^2\}$$

$$= span\{1 + 2x + x^2\}$$

Theorem λ_0 is an eigenvalue of linear operator T, then $(\lambda - \lambda_0)^{\dim E_{\lambda_0}} \mid P_T(\lambda)$

Proof Let $\{v_1, \dots, v_k\}$ be basis for E_{λ_0}, since eigenspaces are subspace of V, it can extends to basis $a = \{v_1, v_k, v_{k+1}, \dots, v_n\}$ for V, then $[T]_a^a = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ $A_{k*k} = \begin{pmatrix} \lambda_0 & \dots & 0 \\ \dots & \lambda_0 & \dots \\ 0 & \dots & \lambda_0 \end{pmatrix}, B_{(n-k)*(n-k)} = \begin{pmatrix} 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}, C_{(n-k)*(n-k)}, D_{k*k}$ are some matrices with real number entries

$$P_T(\lambda) = \det\begin{pmatrix} A-\lambda I_k & C \\ B & D-\lambda I_{n-k}\end{pmatrix} = \det(A-\lambda I_k)\det(D-\lambda I_{n-k})$$
$$= (\lambda_0-\lambda)^k \det(D-\lambda I_{n-k})$$
$$(\lambda_0-\lambda) \mid (\lambda_0-\lambda)^k \det(D-\lambda I_{n-k})$$

Definition The multiplicity of λ_0 is the number of times $(\lambda-\lambda_0)$ appears as a factor in $P_T(\lambda)$

Theorem $1 \le \dim E_{\lambda_0} \le m =$ multiplicity of λ_0, $m = 1 \to \dim E_{\lambda_0} = 1$

Example $P_T(\lambda) = (\lambda+3)^4(\lambda-1)^7(\lambda-2)$, $\dim E_2 = 1, 1 \le \dim E_1 \le 7$, $1 \le \dim E_{-3} \le 4$

Theorem $\lambda_1, \dots, \lambda_k$ are distinct and the set of eigenvectors corresponding to its eigenvalues are linear independent.

Theorem $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of $T: V \to V$, suppose $P_T(\lambda) = (\lambda-\lambda_1)^{m_1} \dots (\lambda-\lambda_k)^{m_k}$, then T is diagonalizable IFF $\forall i = 1,2,\dots,k$, $\dim E_i = m_i$

Proof Assume $\forall i = 1,2,\dots,k$, $\dim E_i = m_i$

Let $E_{\lambda_1} = \{v_{11}, v_{21}, \dots, v_{m_1 1}\}$, $\dots, E_{\lambda_k} = \{v_{1k}, \dots, v_{m_k k}\}$

Then take $c_{ij} \in \mathbb{R}$, $\underbrace{(c_{11}v_{11} + \dots + c_{m_1 1}v_{m_1 1})}_{in\ E_{\lambda_1}} + \dots + \underbrace{(c_{1k}v_{1k} + \dots + c_{m_k k}v_{m_k k})}_{in\ E_{\lambda_k}} = 0$

This is a sum of k vectors, each one from a distinct eigenspace, hence they are linearly indepedent, which for each vector, their sum is 0. Hence $c_{11}v_{11} + \dots + c_{m_1 1}v_{m_1 1} = \dots = c_{1k}v_{1k} + \dots + c_{m_k k}v_{m_k k} = 0$.

Since $\{v_{11}, v_{21}, \dots, v_{m_1 1}\}, \dots, \{v_{1k}, \dots, v_{m_k k}\}$ are all linearly independent sets,

All c_{ij}'s are 0, the set of all eigenvectors are linearly independent

Assume T is diagonalizable

Then $\exists$ basis for V consisting of eigenvectors of T and the matrix of T relative to this basis has the pattern which its diagonal is $\left(\underbrace{\lambda_1, \dots, \lambda_1}_{m_1\ terms}, \underbrace{\lambda_2, \dots, \lambda_2}_{m_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{m_k}\right)$, all the other entries are 0

$$\dim E_{\lambda_i} \ge m_i \dim E_{\lambda_i} \le m_i \dim E_{\lambda_i} = m_i$$

Example $\alpha = \{1, x, x^2\}, [T]_\alpha^\alpha = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, check if diagonalizable

$$T(1) = 1, T(x) = 1 + x + x^2, T(x^2) = 1 + 2x^2$$

$$P_T(\lambda) = \det([T]_\alpha^\alpha - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda)$$

$\dim E_2 = 1$ guarented

$$\dim E_1 = \left(3 - \dim \operatorname{Im}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right) = 3 - 1 = 2$$

Therefore diagonalizable

Definition A field is a set Ftogether with addition and multiplication that satisfy

1. $\forall x, y \in F.\ x + y = y + x$
2. $\forall x, y \in F.\ x + y + z = (x + y) + z$
3. $\exists 0 \in F,\ \forall x \in F,\ x + 0 = x$
4. $\forall x \in F,\ \exists - x \in F.\ x + (-x) = 0$
5. $\forall x, y, z \in F.\ xy = yx$
6. $\forall x, y \in F.\ xyz = x(yz)$
7. $\forall x, y, z \in F. x(y + z) = xy + xz$
8. $\exists 1 \in F.\ 1x = x$
9. $\forall x \in F, x \neq 0.\ \exists x^{-1} \in F. xx^{-1} = 1$

Example $\mathbb{R}$ is a field, $\mathbb{Z}$ is not (no inverse), $\mathbb{Q}$ is a field

Definition Complex number ($\mathbb{C}$): the set of order pairs of real numbers together with

1. addition $(a, b) + (c, d) = (a + c, b + d)$
2. Multiplication $(a, b)(c, d) = (ac - bd, ad + bc)$

Convention any complex number whose 2nd component is 0 (ex. $(a, 0)$) is identified as real number a

Example $a \in \mathbb{R}\, IFF\ (a, 0) \in \mathbb{C},\ (a, 0) + (b, 0) = (a + b, 0) \in \mathbb{C} \to a + b \in \mathbb{R},\ (a, 0)(b, 0) = (ab, 0) \in \mathbb{C} \to ab \in \mathbb{R}$

$\mathbb{R} \subset \mathbb{C}$ when restrict operations $(\in \mathbb{C})$ to subset $\mathbb{R}$, get usual addtion and multiplication in $\mathbb{R}$

$$(a, b) = a(1,0) + b(0,1) = a + b(0,1)$$

Note $(0,1)^2 = (-1,0) = -1$, hence write $i = (0,1),\ i^2 = -1$

Therefore $a + b(0,1) = a + ib, \mathbb{C} = \{\, a + ib \mid a, b \in \mathbb{R} \text{ AND } i^2 = -1 \,\}$

Addition and multiplication are usual as in real numbers together with $i^2 = -1$

$$(a, b) + (c, d) = (a + c, b + d) \to a + ib + c + id = (a + c) + i(b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc) \to (a + ib)(c + id) = ac + iad + ibc + i^2bd$$

$$= (ac - bd) + i(ad + bc)$$

$$\forall z \in \mathbb{C}. z = a + ib = (a, b)$$

$\exists 1 - 1$ correspondence between $\mathbb{C}\ \&\ \mathbb{R}^2$: $(Re(z) = a, Im(z) = b)$ Re: the real part, Im: the imagery part

$$\forall w, z \in \mathbb{C}. w = z \text{ IFF } Re(z) = Re(w) \text{AND } Im(z) = Im(w)$$

Proof $\mathbb{C}$ is a field

Easy to prove 1)2)4)5)6)7)

$$\text{Let } z = a + ib \in \mathbb{C}$$

Let $0 \in \mathbb{C}.\ 0 + z = (0 + a) + i(0 + b) = a + ib$

Let $-z = -a - ib. z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$

Let $1 \in \mathbb{C},\ 1 = 1 + i0,\ 1z = (a1 - 0) + i(b1 + 0) = a + ib$

Let $z^{-1} \in \mathbb{C}.\ z^{-1} = \frac{1}{a+ib}\frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\left(-\frac{b}{a^2+b^2}\right).\ zz^{-1} =$

$\left(\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2}\right) + i\left(\frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right) = 1 + i0$

Example $(2-3i)^{-1} = \frac{2+3i}{(2-3i)(2+3i)} = \frac{2}{13} + i\frac{3}{13}$

If $z = a + ib$ comjugate of $z, \bar{z} = a - ib$, then if $z \neq 0, z^{-1} = \frac{\bar{z}}{z\bar{z}}$

$$z^2 + 1 \in \mathbb{C} \to z^2 - (-1) = (z-i)(z+i),\ z^2 + 2 = (z-2i)(z+2i)$$

Vector Space over a field

Example $F^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \middle| \, x_1 \dots x_n \in F \right\}$ just as $\mathbb{C}^n = \left\{ \begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z^n \end{pmatrix} \middle| \, z_1 \dots z_n \in \mathbb{C} \right\}$ just as $\mathbb{R}^n$

Add and multiply by scalar coordinate-wise by a basis for $\mathbb{C}^n$, $\dim \mathbb{C}^n = n$
Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{C}^n$

$$\begin{pmatrix} 1+i \\ 2i \end{pmatrix} = (1+i)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} i \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2i \end{pmatrix}$$

Theorem A field follows all the behaviors in a vector space as real numbers scalars, including things such as basis, linear dependency, inverse, eigenvector/value/space, subspaces …

Example $\left\{ \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ i \\ 1 \end{pmatrix} \right\}$ is linearly dependent since $i\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix}$

Example $T: \mathbb{C}^3 \to \mathbb{C}^3 \coloneqq T(z_1, z_2, z_3) = \begin{pmatrix} (1+i)z_1 \\ -2iz_1 + (1+i)z_2 + 2iz_3 \\ z_3 + iz_1 \end{pmatrix}$ λ & E_{λ_i}?

Let $\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ be a basis for $\mathbb{C}^3$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+i \\ -2i \\ i \end{pmatrix}, T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1+i \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2i \\ 1 \end{pmatrix}$$

$[T]_\alpha^\alpha = \begin{bmatrix} 1+i & 0 & 0 \\ -2i & 1+i & 2i \\ i & 0 & 1 \end{bmatrix}$ $P_T(\lambda) = \det\begin{pmatrix} 1+i-\lambda & 0 & 0 \\ -2i & 1+i-\lambda & 2i \\ i & 0 & 1-\lambda \end{pmatrix} =$
$(1+i-\lambda)^2(1-\lambda)$

$$E_{\lambda_1} = E_{i+1} = Ker\begin{pmatrix} 0 & 0 & 0 \\ -2i & 0 & 2i \\ i & 0 & -i \end{pmatrix} = span\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$E_{\lambda_2} = E_1 = Ker\begin{pmatrix} i & 0 & 0 \\ -2i & i & 2i \\ i & 0 & 0 \end{pmatrix} = span\left\{ \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$$

T is diagonalizable and $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$ is a basis for $\mathbb{C}^2$ corresponding of

$$[T]_{\beta}^{\beta} = \begin{pmatrix} i+1 & & \\ & i+1 & \\ & & 1 \end{pmatrix}$$

Theorem $T: V \to V$, β basis for V. Let $w_i =$ span of the first i vectors in β, then $[T]_\beta^\beta$ is in upper triangular IFF $\forall i \leq \dim V, T(w_i) \subset w_i$

Definition $T: V \to V$ linear operator, a subspace W of $V$is called invariant under T (T-invariant) if $T(W) \subset W$

Theorem $[T]_\beta^\beta$ is upper triangular IFF each of subspace $\forall i \leq k$, $w_i = span\{w_1, \ldots, w_k\}$ is T-invariant

Example $T: V \to V$ linear

1. V, $\{0\}$ since $T(0) = 0$
2. $\ker T$, $T(\ker T) = 0 \in \ker T$ and it is a subspace, hence contain 0
3. $im\, T$, $T(imT) \in imT$ since $imT \subseteq V$
4. E_λ, since $\forall x \in E_\lambda$, $T(x) = \lambda x \in E_\lambda$ since E_λ closed under multiplication

Example $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = \begin{pmatrix} 3x + 2y \\ y - z \\ 4x + 2y - z \end{pmatrix}$, $W = \{(x, y, x) \mid x, y \in \mathbb{R}\}$ is T-invariant

Proof Let $x, y \in \mathbb{R}$, $T(x, y, x) = \begin{pmatrix} 3x + 2y \\ y - x \\ 3x + 2y \end{pmatrix} \in W$

Definition $T: V \to V$ is triangalizeable if $\exists \beta\ s.t.\ [T]_\beta^\beta$ is triangular

Note: if $T: V \to V$ is triangular, then it's easy to see their eigenvalues, which is on its diagonal, while non-0 entries aren't uniquely determined since it depends on the choices of β

Example $T: \mathbb{R}^4 \to \mathbb{R}^4: T(x, y, z, w) = (zx + w, zy - w, -x + y + 2z, 2w)$, let α be the standard basis, then $[T]_a^a = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & -1 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$$P_T(\lambda) = (\lambda - 2)^4$$

Then, to produce invariant subspace, take $\{0\} \subset W_1 \subset W_2 \subset \cdots \subset W_4$

Notice $\{0\} \subset \underbrace{\ker([T]_a^a - 2I)}_{\dim 2} \subset \underbrace{\ker^2([T]_a^a - 2I)}_{\dim 3} \subset \underbrace{\ker^3([T]_a^a - 2I)}_{\substack{\dim 4 \\ [0]}}$

$$\begin{pmatrix} & & & 1 \\ & & & -1 \\ -1 & 1 & & \\ & & & \end{pmatrix} \qquad \begin{pmatrix} & & & \\ & & & \\ & & & -2 \\ & & & \end{pmatrix}$$

So find basis $\{x_1, x_2\}$ for $\ker([T]_a^a - 2I)$, extends it to $\{x_1, x_2, x_3\}$ for $\ker^2([T]_a^a - 2I)$, extend to $\{x_1, x_2, x_3, x_4\}$ for $\mathbb{R}^4$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow [T]_\beta^\beta = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$c = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow [T]_c^c = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Notice the two matrix are different

Theorem $T: V \to V$ linear over F, if $P_T(\lambda)$ has $\dim V$ roots in F, then $\exists \beta\ s.t.\ [T]_\beta$ is upper triangular

Guaranteed if $F = \mathbb{C}$ since $\mathbb{C}$ is algebracally closed

Proof known any linear transformation $T: V \to V$ whose eigenvalues all have multiplicity 1 is necessary diagonalizable, so any non-diagonalizable T must have an eigenvalue whose multiplicity > 1

1. Suppose $T: V \to V$ such that all $\dim V$ $\lambda = 0$
2. Suppose $T: V \to V$ such that all $\dim V$ $\lambda = \lambda_1 \neq 0$, then $T(x) = \lambda_1 x \Rightarrow (T - \lambda_1 I)x = 0$

   ...

k. Suppose $T: V \to V$ such that there are muliple eigenvalues $\lambda_1, \dots \lambda_k$, then it can be the direct sum of 2)

Theorem V is a complex vector space $T: V \to V$ has only $\lambda = 0$ IFF $T^k = \{0\}$ for some $k \in \mathbb{Z}^+$

Proof Suppose $T^k = \{0\}$ for some $k \in \mathbb{Z}^+$

$$T(x) = \lambda x \text{ AND } x \neq 0 \to T^2(x) = \lambda^2 x, \dots, T^k(x) = \lambda^k(x) = 0 \to \lambda = 0$$

Suppose the only λ of T is 0

Known $\exists \beta$ basis for V s.t. matrix of T relative to the basis is upper triangular with 0's on its diagonal, multiply itself $\leq \dim V$ times, then it eventually becomes $[0]$

Definition $T: V \to V$ is called nilpotent if $T^k = 0$ for some $k \in \mathbb{Z}^+$, the minimum k s.t. $T^k = 0$ is called order of T

Example $T: P_n(\mathbb{C}) \to P_n(\mathbb{C}) := T\big(p(x)\big) = p'(x)$. T nilpotent, order $= n + 1$

$T: P_4(\mathbb{C}) \to P_4(\mathbb{C}) := T\big(p(x)\big) = p''(x) + p'''(x)$. T nilpotent, order $= 3$

Theorem If $T^{k-1} \neq 0$ but $T^k = 0$, then $\{T^{k-1}(x), T^{k-2}(x), \dots T(x), x\}$ is linear independent

Proof take $c_{k-1}, \dots, c_0\ s.t.\ c_{k-1}T^{k-1}(x) + \dots + c_1 T(x) + c_0 x = 0$

Then $T^{k-1}\big(c_{k-1}T^{k-1}(x) + \dots + c_1 T(x) + c_0 x\big) = 0$

Since $T^k = 0, \forall n \geq k, T^n = 0$, then $c_0 T^{k-1}(x) = 0$, $c_0 = 0$

Then, $T^{k-1}\left(c_{k-1}T^{k-1}(x) + \dots + c_1 T(x)\right) = 0 \to c_1 = 0$

...

All $c = 0$

Theorem $T: V \to V$ is nilpotent of oder $\mathrm{n} = \dim V$, then $\exists x \in V\ s.t.\ \beta =$

$\{T^{n-1}(x), \dots, T(x), x\}$ is a basis for V and $[T]_\beta^\beta = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

Proof Since $T^n = 0$ but $T^{n-1} \neq 0$, $\exists x \in V\ s.t.\ T^n(x) = 0 \text{ AND } T^{n-1}(x) \neq 0$

Theorem Show that if $T: V \to V$ nilpotent of order between 1 and $\dim V$, then there is a matrix Trelative to some basis in the form

$$\begin{pmatrix} [J_{m_1}] & & & \\ & [J_{m_2}] & & \\ & & \ddots & \\ & & & [J_{m_k}] \end{pmatrix} [J_{m_i}] \in M(m_i \times m_i) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

To find such matrix, consider the chain $0 \subset \ker T \subset \ker^2 T \subset \ker^3 T \subset \cdots \subset \ker^{\mathrm{k}} T = V$

Theorem $T: V \to V$ nilpotent of order k, say $W \subset \ker^{\mathrm{k}} T$ $s.t.$ $W \cap \ker^{\mathrm{k}-1} T = \{0\} \dim T^i(W) = \dim W, T(\ker^k T) = T\left(T(\ker^{k-1} T)\right)$

Proof Say $\{w_1, \ldots, w_s\}$ basis for $W, \dim W = s$

Known $span\{T^i(w_1), \ldots, T^i(w_s)\} = T^i(W)$

Take $c_1, \ldots, c_s \in \mathbb{R}, c_1 T^i(w_1) + \cdots + c_s T^i(w_s) = 0$

$$T^{k-1-i}\left(c_1 T^i(w_1) + \cdots + c_s T^i(w_s)\right) = T^{k-1}(c_1 w_1) + \cdots + T^{k-1}(c_s w_s) = 0$$

Since $W \cap \ker^{\mathrm{k}-1} T = \{0\}$, $c_1 = \cdots = c_s = 0$

Example Represent such relationship with a tableau:

$$\begin{array}{cccc} X & X & X & X \\ X & X & X & X \\ X & X & & \\ X & & & \\ \ker T & \ker T^2 & \ker T^3 & \ker T^4 \end{array}$$

$\ker T^4$ $\ker T^3$only in 0, let $\{x_1, x_2\}$ be its basis, if apply T, T^2, T^3 $\ker T^3$, $\ker T^2$, $\ker T$, respectively

Let $W_1 = span\{x_1\} \ker T^2$, then Tmaps W_1 $\ker T$ (notice $T^3(x_1), T^2(x_1), T(x_1)$ linear independent, by construction)

Similarly, construct $x_2, x_3, x_4 x_5$

$$\begin{array}{cccc} T^3(x_1) & T^2(x_1) & T(x_1) & x_1 \\ T^3(x_2) & T^2(x_1) & T(x_2) & x_2 \\ T(x_3) & x_3 & & \\ x_4 & & & \\ x_5 & & & \end{array}$$

Then, the basis relative to the matrix is the tableau, count from left to right, then top to bottom

And the length of each row represents a part of the matrix, its corresponding matrix is

$$\begin{pmatrix} [J_4] & & & & \\ & [J_4] & & & \\ & & [J_2] & & \\ & & & [J_1] & \\ & & & & [J_1] \end{pmatrix}$$ the matrix is called the canonical form of T, the basis is called canonical basis

Example $T: P_4(\mathbb{R}) \to P_4(\mathbb{R}) \coloneqq T(p(x)) = p''(x) + p'''(x)$, find canonical matrix and canonical basis

Since $T^2(p(x)) = p^{(4)}(x)$, $T^3(p(x)) = 0, T$ is nilpotent of order3

$$\dim \ker T = \dim\{1, x\} = 2, \dim \ker T^2 = \dim\{1, x, x^2, x^3\} = 4, \dim \ker T^3 = \dim\{1, x, x^2, x^3, x^4\} = 5$$

Hence, the tableau is $\begin{matrix} T^2(x_1) & T(x_1) & x_1 \\ T(x_2) & x_2 & \end{matrix}$, the canonical form for $T$is $\begin{pmatrix} [J_3] & \\ & [J_2] \end{pmatrix}$

Then, the canonical basis is:

Since $T^2(x_1) \in \ker T$ and $T^2(x_1) \in imT^2$, choose to take $T^2(x_1) = 1$

Then, $x_1 = \frac{x^4}{24}$ by solving x_1

Then, $T(x_1) = \frac{x^2}{2} + x$

Since $T(x_2) \in \ker T$ and $T(x_2) \in imT$, choose to take $T(x_2) = x$

Then, solve $x_2 = \frac{x^3}{6} - \frac{x^2}{x}$

Therefore, the canonical basis is $\left\{1, \frac{x^2}{2} + x, \frac{x^4}{24}, x, \frac{x^3}{6} - \frac{x^2}{2}\right\}$

Theorem Nilpotent transformation are similar iff they have the same canonical form

Theorem Canonical form for $T: V \to V$ that only have single eigenvalue, then Tcan be represented by a canonical matrix with its diagonal entries replaced by λ

Proof Take such T, then $(T - \lambda I)$ is nilpotent, hence has canonical form matrix, then the matrix Tcan be obtained by $[T - \lambda I] + \lambda I$

Example $T: \mathbb{R}^4 \to \mathbb{R}^4 \coloneqq A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ relative to $e$for $\mathbb{R}^4$, find canonical matrix and basis

$P_T(\lambda) = (\lambda - 2)^4$, hence $(T - 2I)$ nilpotent

$$\dim(A - 2I) = \dim\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2, \dim(A - 2I)^2 = \dim\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= 3, \dim \mathbb{R}^4 = 4$$

Tableau: $\quad (T-2I)^2(x) \quad (T-2I)(x) \quad x$

$\qquad\qquad y$

$(T-2I)^2(x) \in \ker(T-2I),\ (T-2I)^2(x) \in im(T-2I)^2$

Choose $(T-2I)^2(x) = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$, then $x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, $(T-2I)(x) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, choose

$y = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Theorem $\forall T: V \to V$ linear, it can write V into direct sum of two invariant subspace such that on one subspace T has only single eigenvalue λ and the other has no eigenvalue of T is λ

Definition For $T: V \to V$, The generalized eigenspace corresponding to eigenvalue λ.

$$k_\lambda = \{x \in V \mid (T - \lambda I)^i(x) = 0 \text{ for some } i \in \mathbb{Z}^+\}$$

Theorem1 $k_\lambda = \ker(T - \lambda \mathrm{I})^k$ for some $k \in \mathbb{Z}^+$

Proof consider the chain $\{0\} \subset \ker(T - \lambda I) \subset \ker(T - \lambda I)^2 \subset \cdots \subset V$

Hence there exists a smallest k such that $\forall \mathrm{l} > \mathrm{k}.\ \ker(T - \lambda I)^k = \ker(T - \lambda I)^l$

Theorem2 k_λ of T is invariant ($\forall v \in k_\lambda . T(v) \in k_\lambda$)

Proof Let $v \in k_\lambda$, then let $i \in \mathbb{Z}^+$ sucht that $(T - \lambda I)^i(v) = 0$

Consider $(T - \lambda I)^{i+1}(v)$, by theorem 1, $(T - \lambda I)^{i+1}(v) = 0$

$$(T - \lambda I)^{i+1}(v) = (T - \lambda I)^i (T - \lambda I)(v) = (T - \lambda I)^i T(v) - \lambda (T - \lambda I)^i(v)$$

Because $(T - \lambda I)^i(v) = 0$, $\lambda (T - \lambda I)^i(v) = \lambda 0 = 0$

Therefore, $(T - \lambda I)^i T(v) = 0$,

Theorem3 The only eigenvalue of T on k_λ is λ $(T(v) = uv \to \lambda = u)$

Proof Let $v \in k_\lambda . v \neq 0$. $(T - \lambda I)^k(v) = (u - \lambda)^k(v) = 0$

Since $v \neq 0, u = \lambda$

Theorem4 $V = \ker(T - \lambda I)^k \oplus \mathrm{Im}(T - \lambda I)^k$

Proof Since $\dim V = \dim \ker(T - \lambda I)^k + \dim Im(T - \lambda I)^k$

Let $v \in \left(\ker(T - \lambda I)^k \cap Im(T - \lambda I)^k\right)$

$$\exists w \in V . v = (T - \lambda I)^k(w) \in \ker(T - \lambda I)^k$$

Hence $(T - \lambda I)^k(v) = (T - \lambda \mathrm{I})^k (T - \lambda I)^k(w) = (T - \lambda I)^{2k}(w) = 0, w \in \ker(T - \lambda I)^{2k}$

Since $2k > k$, by Theorem1, $\ker(T - \lambda I)^{2k} = \ker(T - \lambda I)^k$, $w \in \ker(T - \lambda I)^k$

Hence, $v = (T - \lambda I)^k(w) = 0$

$$\left(\ker(T - \lambda I)^k \cap Im(T - \lambda I)^k\right) = \{0\}$$

Theorem5 There is no eigenvalue in $T\mid_{Im(T - \lambda I)^k}$ is λ (eigenvalue of $T\mid_{\ker(T - \lambda I)^k}$)

Proof To obtain contradiction, Let $v \in Im(T - \lambda I)^k \wedge T(v) = \lambda v \wedge v \neq 0$

Since $v \in Im(T - \lambda I)^k$, take $w \in V.\ v = (T - \lambda I)^k(w)$

Since $T(v) = \lambda v, T(T - \lambda I)^k(w) = \lambda (T - \lambda I)^k(w)$

Then, $(T - \lambda I)^k(w) \in E_\lambda \in \ker(T - \lambda I)^k$

Where $v \in \ker(T - \lambda I)^k$, since $v \in Im(T - \lambda I)^k$ and by Theorem4, $\left(\ker(T - \lambda I)^k \cap Im(T - \lambda I)^k\right) = \{0\}$

$$v = 0$$

Contradiction

Theorem6 $T: V \to V$ linear where eigenvalue λ of T has multiplicity m, then $\dim \ker(T - \lambda I)^k = \dim k_\lambda = m$

Proof By Theorem4, let a be basis for $\ker(T-\lambda I)^k$, β be basis for $Im(T-\lambda I)^k$, then $\gamma = a \cup \beta$ be basis for V

$$[T]_\gamma = \begin{bmatrix} [T \mid_{\ker(T-\lambda I)^k}]_a & [0] \\ [0] & [T \mid_{\mathrm{Im}(T-\lambda I)^k}]_\beta \end{bmatrix}, P_T(x) = P_{T|\ker(T-\lambda I)^k}(x) P_{T|\mathrm{Im}(T-\lambda I)^k}(x)$$

Since multiplicity of λ is m, $P_T(x) = (x-\lambda)^m q(x)$ where $q(x) \neq 0$

Also, since λ is the only eigenvalue for $[T \mid_{\ker(T-\lambda I)^k}]_a$, $P_{T|\ker(T-\lambda I)^k}(x) = (x-\lambda)^l$

where $l \in \mathbb{Z}^+$

By Theorem5, since there is no eigenvalue in $T \mid_{Im(T-\lambda I)^k}$ is $\lambda, l = m$

Theorem For any vector space V over $\mathbb{C}$, $T: V \to V$ with distinct eigenvalues $\lambda_1, \dots, \lambda_l$

$P_T(\lambda) = (\lambda-\lambda_1)^{m_1} \dots (\lambda-\lambda_l)^{m_l}$*, then V=k_(λ_1) ⊕...⊕k_(λ_l)*

Proof *V=k_(λ_1)⊕Im(T−λI)^k*, by Induction hypothesis, $Im(T-\lambda I)^k$ can be written as the direct sum of generalized eigenspaces, hence by strong induction, the theorem holds

Definition Jordon canonical form of T is $\begin{bmatrix} [B_{\lambda_1}] & & \\ & \ddots & \\ & & [B_{\lambda_l}] \end{bmatrix}$ for some bases of T

Where $[B_{\lambda_i}]$ is in form $\begin{bmatrix} [J_{m_1}(\lambda_i)] & & \\ & \ddots & \\ & & [J_{m_i}(\lambda_i)] \end{bmatrix}$

Where $[J_{m_i}(\lambda_i)]$ is $m_i \times m_i$ matrix in form $\begin{bmatrix} \lambda_i & 1 & \\ & \ddots & 1 \\ & & \lambda_i \end{bmatrix}$

JCF is unique up to the ordering of Jordon blocks $[B_{\lambda_i}]$ and two matrix are similar IFF they have the same JCF

Example $T: \mathbb{R}^4 \to \mathbb{R}^4$ has matrix $A = \begin{pmatrix} 2 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ relative to e, find JCF and basis

$P_T(\lambda) = (\lambda-2)^3(\lambda-1)$, hence $\mathbb{R}^4 = k_2 \oplus k_1$

Consider k_2, $\dim\ker(T-2I) = \dim\ker\begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$

Tableau $\begin{matrix} (T-2I)(x) & x \\ y & \end{matrix}$, Find $(T-2I)(x) \in Im(T-2I) \wedge \in \ker(T-2I)$, pick $(T-2I)(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Then, pick $x = \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$

Find $y \in \ker(T-2I) \wedge$ independent of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, pick $y = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Consider k_1, since $\dim k_1 = 1$,

Pick $z \in \ker(T-I) = \ker \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, pick $z = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Then, the basis is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}, JCF \begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}$

Example $T: \mathbb{R}^6 \to \mathbb{R}^6$ has matrix $\begin{pmatrix} 2 & & & & & \\ 1 & 2 & & & & \\ 1 & 0 & 2 & & & \\ 1 & 1 & 1 & 2 & & \\ 1 & 1 & 1 & 1 & 3 & \\ 1 & 1 & 1 & 1 & 1 & 3 \end{pmatrix}$, find JCF

$$P_T(\lambda) = (\lambda-2)^4(\lambda-3)^2$$

Consider k_4, since $\ker(T-2I) = 2, \dim(T-2I)^2 = 3$,

The tableau is in form $\begin{matrix} X & X & X \\ X & & \end{matrix}$

Consider k_3, since $\ker(T-3I) = 1$

The tableau is in form $X \quad X \quad X$

Hence JCF $\begin{pmatrix} 2 & 1 & & & & \\ & 2 & 1 & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{pmatrix}$