STA257: Probability and Statistics 1

Instructor: Katherine Daignault

Department of Statistical Sciences
University of Toronto

Week 4

Outline

Outline

Limits

- Last week's refresher on limits included
  - Definition: $\lim_{x \to a} f(x) = L$
  - We can take limits from either side: $x \to a^+$ (right) and $x \to a^-$ (left)
  - Convergence: $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$
- This week, you should also be familiar the notion of a function being right-continuous:

$$\lim_{\delta \to 0^+} F(x + \delta) = F(x)$$

Derivatives

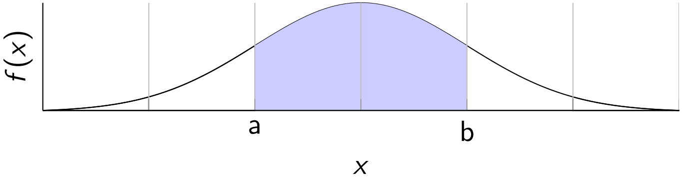
- We will be using a lot of derivatives in this course and you will be expected to know how to compute many types of derivatives.
- Notation: the first derivative of a function $f(x)$ is denoted by $f'(x) = \frac{d}{dx}f(x)$
- Some useful results:
  - if $f(x) = x^r$ then $f'(x) = rx^{r-1}$
  - $f(x) = e^x = f'(x)$ but if $f(x) = a^x$ then $f'(x) = a^x ln(a)$
  - if $f(x) = ln(x)$ then $f'(x) = {}^1\!/_x$
- Rules of derivatives:
  - Product rule: $(fg)' = f'g + g'f$
  - Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
  - Chain rule: if $f(x) = h(g(x))$ then $f'(x) = h'(g(x)) \times g'(x)$

Definite Integrals

- Again you will need to be very comfortable with taking definite integrals of functions.
- Definition: $\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$
- Some useful properties:
  - Reversing limits: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
  - Additivity: $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
- Some useful results (written as indefinite integrals):
  - $\int adx = ax + C$
  - $\int x^a dx = \frac{x^{a+1}}{a+1} + C$
  - $\int ({}^1\!/\!{}_x)dx = ln(x) + C$
  - $\int e^{ax} dx = ({}^1\!/\!{}_a)e^{ax} + C$
  - $\int a^x dx = {}^{a^x}\!/\!{}_{ln(a)} + C$
  - $\int ln(x)dx = xln(x) - x + C$

Area under the Curve and FTC

- The interpretation of an integral that we will be concerned with is that it is measuring the area under the curve of some function $f(x)$ between points a and b.



- It will also helpful to remember the Fundamental Theorem of Calculus (FTC):
  - Let f be a continuous real-valued function defined on $[a, b]$. Let F be the function defined for all x in $[a, b]$, by
    $$F(x) = \int_a^x f(t)dt$$
    then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in (a, b).

Integration by Parts

- You will also need to be able to solve complicated integrals using either substitution or integration by parts (IBP).
- If $u = u(x)$ and $du = u'(x)dx$, while $v = v(x)$ and $dv = v'(x)dx$, then IBP states that

$$\int_a^b u(x)v'(x)dx = \left[u(x)v(x)\right]_a^b - \int_a^b u'(x)v(x)dx$$

or as a short-hand

$$\int_a^b u(x)v'(x)dx = [uv]_a^b - \int_a^b vdu$$

Outline

Continuous Random Variables

- We have previously introduced random variables (RVs) that can only take on finite or countably infinite number of values.
- These discrete RVs may not always be appropriate for the type of data being collected.
- **Continuous random variables** can take on values from a continuum , i.e. decimals, real numbers, any value within an interval
- As with discrete RVs, we may also define functions to represent the probabilities that the random variable takes on certain values.

Density Functions for Continuous Variables

- For the discrete case, the **probability mass function** was used to represent the probability for every value of a random variable.
- We also had a cumulative distribution function (CDF) to represent $P(X \leq x)$.
- The analogous functions for a continuous RV are the **probability density function** (PDF) as well as a cumulative distribution function (CDF).
- To distinguish the PMF and the PDF, we will use $f(x)$ to denote the density function for the continuous RV, and $p(x)$ for the discrete PMF.
- Before we can give a formal definition of a continuous random variable, we must discuss these distribution functions.

Properties of PDFs

- For discrete RVs, the PMF was basically a step function, only taking non-zero probabilities for values the RV was defined over.
- This was because the discrete RV only took on discrete values.
- If a random variable can take on values on a continuum, this can be seen as adding more bars for all possible values between those discrete points.

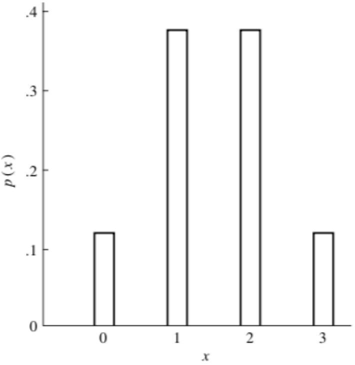


FIGURE 2.1 A probability mass function.

Properties of PDFs

- As we add bars for every real number between two discrete values, we end up with a smooth curve.
- The main difference between a PMF and PDF is that a PDF, f, is a **continuous** function, rather than a stepwise function.
- Another way to look at this is that $f(x)$ must be a Riemann-integrable function.

Definition of a PDF

$f(x)$, a Riemann-integrable function, is a PDF if and only if

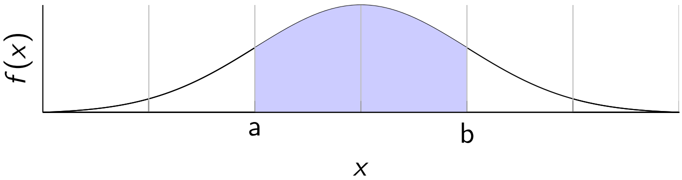
1. $f(x) \geq 0$ for all $x \in \mathbb{R}$
2. $\int_{\infty}^{\infty} f(x)dx = 1$

Properties of PDFs

- Note that this definition is analogous to that of a PMF, where rather than summing over probabilities, we are now integrating.
- For a continuous RV, X, with a density function f, and for $a < b$, we may find the probability that X falls in the interval (a, b) as

$$P(a < X < b) = \int_a^b f(x)dx$$

- This means that the probability of being within an interval corresponds to finding the **area under the curve** in this interval.



Example: Uniform Random Variable

This is a simple example of a continuous RV. A **Uniform random variable** is a continuous RV defined on a closed interval $[a, b]$. This variable corresponds to the situation "choose a number at random between a and b". What is the PDF?

- Any real number between a and b is a possible outcome for this RV.
- This RV has the property that, since all values are equally likely, the probability that X is in any subinterval h is proportional to the size of h.
- Our interval is $[a, b]$ so the size of this interval is $b - a$ and therefore the PDF is

$$f(x) = \begin{cases} 1/(b-a), & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}$$

Properties of PDFs

- A consequence of how a PDF is defined, in the continuous case, is that it is not possible to compute $P(X = c)$.
- This is because

$$P(X = c) = P(c \leq X \leq c) = \int_c^c f(x)dx = 0$$

- Since we are dealing with a variable taking values on a continuum, if every possible value had positive probability, we would be summing probabilities over a countably infinite set.
- This results in the sum being infinite which contradicts the definition of a probability function $\Rightarrow$ an infinite sum violates definition of PDF!
- However it means that the following are equivalent (for continuous RVs only):

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$$

Cumulative Distribution Function

- The CDF for a continuous random variable is defined in the same way as the discrete case: $F(x) = P(X \leq x)$
- In the discrete case, we would find $P(X \leq x)$ by summing $P(X = x)$ for all values satisfying $X \leq x$.
- For continuous RVs, summation becomes integration:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

- From the fundamental theorem of calculus, if f is continuous at x, then $f(x) = \frac{d}{dx}F(x) = F'(x)$
- We can now evaluate probabilities of the form

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx = F(b) - F(a)$$

Example: Uniform Random Variable

Find the CDF for the Uniform random variable defined on the interval $[a, b]$.

- Recall the PDF of the Uniform RV is

$$f(x) = \begin{cases} 1/(b-a), & a \leq x \leq b \\ 0, & x < a \text{ or } x > b \end{cases}$$

- To find the CDF, we must take the integral of $f(x)$ since

$$f(x) = F'(x) \Rightarrow \int f(x)dx = F(x)$$

- We must also do this for each set of x's where the probability jumps.

Example continued

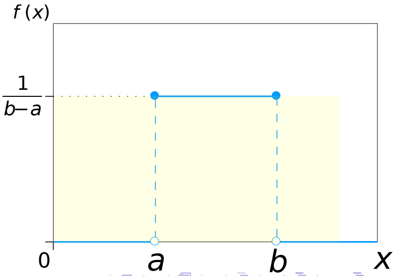
- First we consider $P(X < x)$ when $x < a$:

$$P(X \leq x) = \int_{-\infty}^{x} f(u)du = \int_{-\infty}^{x} 0du = 0$$

- Then we move to when $a \leq x \leq b$:

$$P(X \leq x) = \int_{-\infty}^{x} f(u)du = \int_{a}^{x} \frac{1}{b-a}du = \frac{x-a}{b-a}$$

- Finally we consider when $x > b$, which involves using the previous two pieces, since the CDF counts everything that is below my cutoff b:



Example continued

- Thus we need to consider that $P(X \leq x)$ for $x > b$ involves considering all of
  1. $F(x)$ for $x < a$,
  2. $F(x)$ for $a \leq x \leq b$, and
  3. $F(x)$ up $x > b$
- So we have:

$$\begin{aligned} P(X \leq x) &= \int_{-\infty}^{x} f(u)du \\ &= \int_{-\infty}^{a} f(u)du + \int_{a}^{b} f(u)du + \int_{b}^{x} f(u)du \\ &= \int_{-\infty}^{a} 0du + \int_{a}^{b} \frac{1}{b-a}du + \int_{b}^{x} 0du \\ &= 0 + \frac{b-a}{b-a} + 0 = 1 \end{aligned}$$

Example continued

- To give a complete specification of the CDF, we need to collect all of the $F(x)$'s for each set of x's we considered:

$$F(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b \end{cases}$$

Finding Quantiles from the CDF

- When F is CDF for a continuous RV, is strictly increasing on some interval, and that $F = 0$ to the left of the interval and $F = 1$ to the right, the **inverse function** F^{-1} is defined as

$$x = F^{-1}(y) \text{ if } y = F(x)$$

- We may use this to find the pth **quantile** of distribution F.

Definition of pth quantile

The pth quantile of the distribution F is the value x_p such that $F(x_p) = p$, or $P(X \leq x_p) = p$.

Example: Finding Quantiles

Suppose $F(x) = x^2$ for $0 \leq x \leq 1$ is a CDF. Find the median, and the first and third quartiles.

- First, realize that we do have a strictly increasing function because our variable X is defined on the interval $[0, 1]$.
- Find the inverse function:

$$y = F(x) = x^2 \Rightarrow x = F^{-1}(y) = \sqrt{y}$$

- The **median** is the 50th quantile, so we are looking for x such that $y = 0.5$: $x_{0.5} = \sqrt{0.5} = 0.707$
- The **first quartile** is the 25th quantile, so we want x such that $y = 0.25$: $x_{0.25} = \sqrt{0.25} = 0.50$
- Similarly the **third quartile** is the 75th quantile, so we want x such that $y = 0.75$: $x_{0.75} = \sqrt{0.75} = 0.866$

Exercise - Give it a try!

Suppose that X has PDF $f(x) = I_{[0,1]}(x) \times cx^2$, for some $c > 0$. (*Remember:* $I_{[0,1]}(x)$ *takes value 1 when* $x \in [0,1]$ *and 0 otherwise*)

1. Find the value of c.
2. Find the CDF.

Outline

Exponential Random Variable

- Like discrete RVs, we have certain named continuous RVs that are important.
- We have previously seen the Uniform distribution (in example)
- Now we will consider the **Exponential** random variable.
- Setup: represents lifetimes or waiting times between events.
  - $X =$ time between events
- Like the Poisson random variable, the behaviour of this RV is determined by a single parameter, $\lambda > 0$, which is the rate at which events occur.
  - Notation: $X \sim Exp(\lambda)$

Exponential Distribution Functions

- The PDF of an Exponential random variable is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- Obviously, since X denotes time, the Exponential distribution has non-zero probability for $x \geq 0$.

- Since λ determines the rate at which events occur, and X models the time between these events, as λ increases, the probability drops more rapidly.
- R demonstration

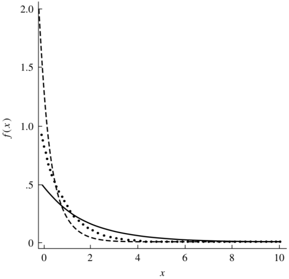


FIGURE 2.9 Exponential densities with $\lambda = .5$ (solid), $\lambda = 1$ (dotted), and $\lambda = 2$ (dashed).

Exponential Distribution Functions

- The CDF of an Exponential random variable is found simply by integrating over the PDF:

$$F(x) = \int_{-\infty}^{x} f(u)du = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- As in the general CDF case, quantiles of the Exponential distribution can be found by finding the inverse of the CDF:
  - For example, let $\eta = x_{0.5}$ denote the median.
  - We must solve for η in the following expression:

$$F(\eta) = 0.5 \Rightarrow 1 - e^{-\lambda\eta} = 0.5$$

  - So we have $\eta = \frac{\log 2}{\lambda}$

Memoryless Property of Exponential

Memoryless Property

The Exponential distribution is characterized by

$$P(X > x + s \mid X > x) = P(X > s), \;\; s > 0.$$

- This property follows directly from the Poisson process.
- Makes the Exponential a good choice for modelling lifetimes of electronics, but not lifetimes of humans.
- Says probability of lasting more than s time past x, if I know it's already lasted up to x, is the same as just considering probability of lasting longer than s time units.

Memoryless Property

Proof:

$\square$

Example: Carbon Monoxide Concentrations

The concentration of CO as assessed at an air quality station at a given time of day is found to follow approximately an Exponential distribution with parameter 0.278. Find the probability that the concentration exceeds 9ppm.

- Let X = CO concentration, then $X \sim Exp(0.278)$
- We are asked for the probability the concentration exceeds 9ppm, so $P(X > 9)$.
- We can use the Exponential CDF for this:

$$P(X > 9) = 1 - P(X \leq 9) = 1 - \left[1 - e^{-9 \times 0.278}\right] \approx 0.08$$

Exercise - Give it a try!

The lifetime in hours of an electronic component is distributed as $Exp(0.01)$. Three of these components operate independently in a piece of equipment. The equipment fails if at least 2 of the components fail. What is the probability that the equipment will operate for at least 200 hours without failure?

Outline

Gamma Random Variables

- The Exponential random variable can be seen as a specific case of our next named distribution, **Gamma** distribution.
- Setup: In some situations, represents the sum of independent and identically distributed *Exp*(λ) random variables.
- By this logic, the behaviour of a Gamma random variable is determined by 2 parameters, α and λ.
  - Unlike the Exponential, λ no longer represents a rate, but instead is a **scale** parameter, i.e. like changing from minutes to seconds
  - The second parameter, α, is a **shape** parameter.
  - Both $\alpha, \lambda > 0$.
    - When α is an integer, then X can be seen as the sum of IID Exponential RVs, and α is the number of RVs we are summing

Gamma Distribution Functions

- The Gamma is a very flexible distribution for modelling positive real numbers
- The shape parameter dictates how the PDF will look.
- Note: When $\alpha = 1$ we see in (a) below that we have the Exponential PDF.
- R demonstration

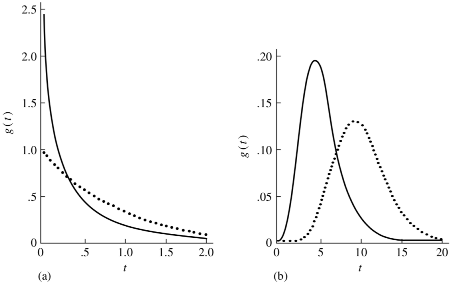


FIGURE **2.11** Gamma densities, (a) $\alpha = .5$ (solid) and $\alpha = 1$ (dotted) and (b) $\alpha = 5$ (solid) and $\alpha = 10$ (dotted); $\lambda = 1$ in all cases.

Gamma Distribution Functions

- Because the Gamma RV can sometimes be seen as a summation of times, X is defined on non-negative numbers by

$$f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

- Here $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du,\ \alpha > 0$ is a Gamma function, which has the following properties:
  - $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
  - $\Gamma(1) = 1$
  - $\Gamma(\alpha + 1) = \alpha!$
  - $\Gamma(1/2) = \sqrt{\pi}$
- There is no closed form expression for the CDF of a Gamma! However we can still compute probabilities using integration.

Example: Rainfall in Midwest U.S.

Four-week summer rainfall totals (in inches) in a section of the Midwest United States have approximate Gamma distribution with $\alpha = 2$ and $\beta = 0.5$. What is the probability that the four-week rainfall total exceeds 4 inches?

- First start by writing out the PDF for this random variable:

$$f(x) = \frac{0.5^2}{\Gamma(2)} x^{2-1} e^{-0.5x},\ x \geq 0$$

- We are asked to find $P(X > 4)$, so we integrate over the sample space required

$$P(X > 4) = \int_4^\infty \frac{0.5^2}{\Gamma(2)} x e^{-0.5x} dx$$

- This will require us to use software or integration by parts

Example continued

- Define our u and v functions:

$$\begin{array}{l|l} u = x & v = \frac{-1}{0.5}e^{-0.5x} = -2e^{-0.5x} \\ du = 1 & dv = e^{-0.5x} \end{array}$$

- Then we compute $P(X > 4) = uv|_4^\infty - \int_4^\infty vdu$:

$$\begin{aligned} P(X > 4) &= \int_4^\infty \frac{0.5^2}{\Gamma(2)} xe^{-0.5x} dx = \frac{0.5^2}{1} \int_4^\infty xe^{-0.5x} dx \\ &= \frac{1}{4}\left[-2xe^{-0.5x} - \int_4^\infty -2e^{-0.5x} dx\right]_4^\infty \\ &= \frac{1}{4}\left[-2xe^{-0.5x} + (-4e^{-0.5x})\right]_4^\infty \\ &= \frac{1}{4}[0 - (-8e^{-2}) + (0 + 4e^{-2})] \\ &= 2e^{-2} + e^{-2} = 3e^{-2} = 0.41 \end{aligned}$$

Exercise - Give it a try!

The magnitude of earthquakes in a region can be modelled with a Gamma distribution with $\alpha = 3$ and $\lambda = 1/4$. What is the probability that the magnitude of an earthquake will exceed 3.0?

Outline

Normal Random Variable

- **Normal** random variables, and their distribution functions, play an important role in probability and statistics, which will become apparent later in the course.
- Also sometimes referred to as the Gaussian distribution, after Gauss who proposed it as a model for measurement errors.
- Setup: represents an approximate distribution of scaled average of independent and identically distributed RVs (to be discussed later)
  - X can take on values from the entire real line.
- Normal RVs can thus be used to model many diverse phenomena, such as height, IQ scores, velocities, etc.

Normal Distribution Functions

- The PDF of a Normal random variable is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

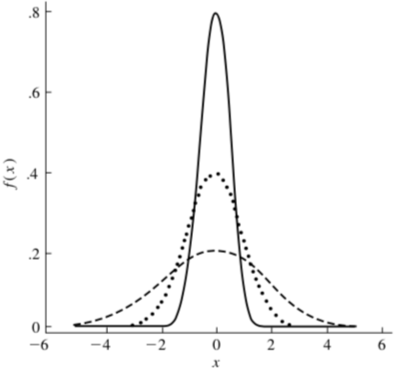
- Behaviour is dictated by two parameters:
  - $\mu \in \mathbb{R}$: mean, represents the centre of the distribution
  - $\sigma > 0$: standard deviation, represents the spread/variability of the distribution
  - Often use $X \sim N(\mu, \sigma^2)$ as a shorthand
- A special case is the **Standard Normal** distribution, where $\mu = 0$ and $\sigma = 1$, where we sometimes use $Z \sim N(0, 1)$ to distinguish from other Normals

Normal Distribution Functions

- One key property of the Normal distribution is its shape: bell-shaped and symmetric around the mean μ, namely

$$f(\mu - x) = f(\mu + x)$$

- The PDF $f(x)$ thus takes its maximum at μ.
- The rate at which the function drops on either side of μ is determined by σ
- R demonstration



Normal Distribution Functions

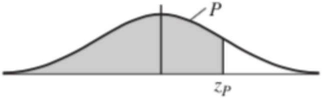
- Like the Gamma distribution, the Normal distribution does not have a closed-form expression for the CDF.
- However, we may consider the CDF of the standard Normal distribution:

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

  as an integral over the standard Normal PDF.
- Again this has no closed-form solution, but it has been evaluated using numerical techniques.
- Thus computing probabilities using the CDF $\Phi(z)$ is done through tables of standard Normal values (see Appendix B of the textbook).

Standard Normal CDF Table

TABLE 2 Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable. The value of P for $-z_p$ equals 1 minus the value of P for $+z_p$; for example, the P for -1.62 equals $1 - .9474 = .0526$.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879

Example: Standard Normal Table

Let Z denote a Normal random variable with mean 0 and standard deviation 1.

A. Find $P(Z > 2)$.

- Because the table of Standard Normal CDF values represents $\leq$ probabilities, start with the complement rule:

$$P(Z > 2) = 1 - P(Z \leq 2)$$

- Now find the area under the curve below 2, by looking up 2.00 in the table: $P(Z < 2) = 0.9772$
- Plug this into the first step and finish:

$$P(Z > 2) = 1 - P(Z \leq 2) = 1 - 0.9772 = 0.0228$$

- This tells us that $z = 2$ is the 97.72% quantile of the standard Normal distribution

Example continued

B. Find $P(0 \leq Z \leq 1.73)$.

- We are looking for the area under the curve between the points 0 and 1.73.
- Since our table only gives $P(Z \leq z)$ values, we need to do:

$$P(0 \leq Z \leq 1.73) = P(Z \leq 1.73) - P(Z \leq 0)$$

- Now we look up the first value in the table: $P(Z \leq 1.73) = 0.9582$
- We don't need the table for $P(Z \leq 0)$ because 0 is the mean and thus we know by symmetry that $P(Z \leq 0) = 0.5$.
- Then

$$P(Z \leq 1.73) - P(Z \leq 0) = 0.9582 - 0.5 = 0.4582$$

Exercise - Give it a try!

C. Find $P(-2 \leq Z \leq 2)$.

Computing Probabilities from the Normal Distribution

- So obviously, if we could move from a Normal RV X to a standard Normal RV Z, then we would be able to use the standard Normal table to compute probabilities
- This can be done, but we won't get into it until later.
- The Normal distribution can also be used to derive other important distributions that we have not yet discussed.
- We will get to this later in the course.

Outline

Beta Random Variables

- Our last named continuous random variable for now is the **Beta** random variable.
- Setup: Often used to model proportions, or other quantities taking continuous values between 0 and 1.
  - Can also be used, when scaled and translated, to model task time in product management
- Since we may talk about the Beta RV in terms of scaling and translating, it makes sense that the behaviour is dictated by two parameters: $a, b > 0$ are both shape parameters
- R demonstration

Beta Distribution Functions

- The PDF of the Beta distribution is

$$f(x) = \begin{cases} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- The collection of Gamma functions has no intuitive meaning, other than it ensures that $\int_0^1 f(x)dx = 1$, i.e. that it is a density function.
- This gives us a useful result that will be used frequently later in the course:

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

Beta Distribution Functions

- Once again, there is no closed form expression for the CDF of a Beta random variable.
- We can write out the integral formulation of the CDF as

$$F(x) = \int_0^x \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1} du$$

- It turns out that when a, b are positive integers, the CDF is related to the Binomial distribution function:

$$F(x) = \int_0^x \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1}(1-u)^{b-1} du = \sum_{i=a}^{n} \binom{n}{i} x^i (1-x)^{n-i}$$

where $n = a + b - 1$.

Example: Gasoline Wholesaler

A distributer has bulk storage tanks that hold fixed supplies that are filled every Monday. Interest is in proportion of supply sold every week. This proportion can be modelled with a Beta(4, 2). Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

- The PDF of this Beta is $f(x) = \frac{\Gamma(4+2)}{\Gamma(4)\Gamma(2)} x^3(1-x)$
- We want $P(X > 0.9)$, which we can find by integrating over the relevant piece of the PDF:

$$\begin{aligned}
P(X > 0.9) &= \int_{0.9}^{1} 20(x^3 - x^4)dx \\
&= 20 \int_{0.9}^{1} x^3 dx - 20 \int_{0.9}^{1} x^4 dx \\
&= 20 \frac{x^4}{4}\Big|_{0.9}^{1} - 20 \frac{x^5}{5}\Big|_{0.9}^{1} \\
&= 20(0.004) = 0.08
\end{aligned}$$

Exercise - Give it a try!

The percentage of impurities per batch in a chemical product is a random variable X with density function

$$f(x) = 12y^2(1-y),\ 0 \leq x \leq 1.$$

A batch with more than 40% impurities cannot be sold. What is the probability that a randomly selected batch cannot be sold because of impurities?