Outline: Weeks 13 and 14

Weierstrass

1. 10.2.A: we have $p_n(t) \to g(t)$ and so $p_n(\frac{x-a}{b-a}) \to f(x)$.

2. 10.2.F,G: set $p_{n,0} := \int p_{n,1}$ where $p_{n,1} \rightrightarrows f$.

3. Suppose that $f \in C([0,1])$ and $f(0) = 0$ then there exists q_n such that $h_n := x^n q_n \rightrightarrows f$ such that $h_n'(0) = 0$.

   - lemma 1: any even function $g \in C([-1,1])$ can be approximated by even polynomials. Proof: approximate $q_n \rightrightarrows g$ and then let $p_n(x) := \frac{q_n(x)+q_n(-x)}{2}$. Then p_n are even and satisfy $p_n \rightrightarrows g$.

   - Extend f to the even function

   $$F(x) := \begin{cases} f(x) & x \in [0,1] \\ f(-x) & x \in [-1,0] \end{cases}.$$

   - Then even $p_n \rightrightarrows F$ on $[-1,1]$. Now let $h_n(x) := p_n(x) - p_n(0)$. This is also even and it satisfies $h_n(0) = 0$. It also satisfies $h_n'(x) = \frac{1}{2}(h_n'(x) - h_n'(-x))$ and so $h_n'(0) = 0$.

   - So we get

   $$\sup_{x\in[0,1]} |f(x) - h_n(x)| \leq \sup_{x\in[-1,1]} |F(x) - p_n(x)| + |f(0) - p_n(0)| + |f(0)|.$$

   The first term goes to zero. The second term goes to zero because $p_n \rightrightarrows f$ in [0,1]. The third term is zero.

   - An alternative proof is to approximate $f(\sqrt{x})$.

4. vector form: Weierstrass theorem $p_n(x_1, ..., x_d) \rightrightarrows f(x_1, ..., x_d)$.

5. Every function that can be uniformly approximated by polynomials over $\mathbb{R}$ is a polynomial in itself.

   - We have that $p_n \rightrightarrows f$ over $\mathbb{R}$. So p_n is also Cauchy. For every $\varepsilon > 0$ there exists $N > 0$ such that $\forall n, m \geq N$ we have

   $$\|p_n - p_m\| \leq \varepsilon.$$

- However $q_n(x) := p_n(x) - p_m(x) = a_l x^l + ... + a_0$ is a polynomial in itself. Polynomials go to infinity $\lim_{x\to\infty} |q_n(x)| = \infty$ iff $a_1 \neq 0$. But since $|q_n(x)| \leq \varepsilon := 1$,for $n \geq N_1$, we get that $|q_n(x)| := |a_{0,m,n}| \leq 1$.
- By Bolzano-Weierstrass $\lim_{n_k\to\infty} a_{0,m,n_k} =: a_{0,m}$
- Therefore, if we write

$$f(x) = f(x) - p_{n_k}(x) + p_n(x) - p_m(x) + p_m(x) = f(x) - p_{n_k}(x) + q_{n_k}(x) + p_m(x)$$

then we get

$$f(x) = f(x) - p_{n_k}(x) + a_{0,m,n_k} + p_m(x).$$

Then take limit in n_k to get

$$f(x) = a_{0,m} + p_m(x),$$

which is a polynomial.

Banach fixed point

Theorem 0.0.1. *Let $X \subset V$ be a closed subset of a complete normed space $(V, \|\cdot\|)$ and $T : X \to X$ is a contraction $\|Tx - Ty\| \leq r\|x - y\|$ for $r < 1$ then there exists $x_* \in X$ such that for all $x \in X$ we have:*

$$\|T^n x - x_*\| \to 0.$$

1. Let $x_n := T^n(x)$ then

$$\|x_n - x_{n+1}\| \leq r^n \|x_1 - x_0\|.$$

2. Therefore, by triangle inequality

$$\|x_n - x_{n+m}\| \leq \sum_{k=0}^{m} r^{n+k} \|x_1 - x_0\| = r^n \|x_1 - x_0\| \frac{1 - r^{m+1}}{1 - r} = cr^n,$$

which goes to zero.

3. So $x_n \in X \subset V$ is Cauchy and it converges to $x_* \in V$. But by closed $x_* \in X$.
4. By T's continuity we find

$$Tx_* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x_*.$$

ODEs and Fixed points

1. $f^{(3)} = \varphi(x, f(x), f'(x), f''(x))$ with IC $f(0) = \gamma_0, f'(0) = \gamma_1, f''(0) = \gamma_2$.

   - Example a) $y' = xy$ $y(0) = 1$. Via separation of variables we have $y(x) := e^{\frac{1}{2}x^2}$.
   - Example b)$y'' = -y - \sqrt{y^2 + (y')^2}$ for $y(0) = 0, y'(0) = 1$.

2. Let $F(x) := (f(x), f'(x), f''(x))$ and $\Phi(x, y_0, y_1, y_2) = (y_1, y_2, \varphi(x, y_0, y_1, y_2))$. So the above problem is

$$F'(x) = \Phi(x, f(x), f'(x), f''(x)) = \Phi(x, F(x))$$

   with $F(0) = \Gamma := (\gamma_0, \gamma_1, \gamma_2)$.

   - Example a):for $y' = xy$ we have $F(x) := f(x)$ and $\Phi(x, y_0) := xy_0$. Indeed, $F'(x) = f'(x) = xf(x) = \Phi(x, f(x))$.
   - Example b)$y'' = -y - \sqrt{y^2 + (y')^2}$ we have $F = (f(x), f'(x))$ and $\Phi(x, y_0, y_1) := (y_1, -y_0 - \sqrt{y_0^2 + (y_1)^2})$. Indeed

$$F' = (f', f'') = (f', -f - \sqrt{f^2 + (f')^2}) = \Phi(x, f, f').$$

3. Consider the operator

$$TF(x) := \Gamma + \int_0^x \Phi(t, F(t))dt.$$

4. If we can find an $F_0(x)$ s.t. $TF_0 = F_0$ then

$$F_0(x) = TF_0(x) = \Gamma + \int_0^x \Phi(t, F_0(t))dt$$

   and so $F_0'(x) = \Phi(x, F_0(x))$.

ODEs and Existence

We will prove that if $\Phi(t, y)$ is Lipschitz over the second coordinate y, $[a, b] \times \mathbb{R}^n$ then there exists solution F_0 to

$$F_0(x) = \Gamma + \int_0^x \Phi(t, F_0(t))\mathrm{d}t.$$

Lemma 1

Suppose that $F, G \in C([a,b] \times \mathbb{R}^n)$ satisfy

$$\|F(x) - G(x)\|_2 \leq M\frac{(x-a)^k}{k!}$$

then

$$\|TF(x) - TG(x)\|_2 \leq LM\frac{(x-a)^{k+1}}{(k+1)!}.$$

Proof.

$$\|TF(x) - TG(x)\|_2 = \left\| \int_a^x \Phi(t, F(t)) - \Phi(t, G(t))\mathrm{d}t \right\|_2$$

by triangle

$$\leq \int_a^x \|\Phi(t, F(t)) - \Phi(t, G(t))\|_2 \mathrm{d}t$$

by Lipschitz of Φ we find

$$\begin{aligned} &\leq L\int_a^x \|F(t) - G(t)\|_2 \mathrm{d}t \\ &\leq L\int_a^x M\frac{(t-a)^k}{k!}\mathrm{d}t \\ &= LM\frac{(x-a)^{k+1}}{(k+1)!}. \end{aligned}$$

□

First proof of existence

1. Let $F_{k+1} := TF_k$ and $F_0 := \Gamma$. We have that
$$\|F_1(x) - F_0(x)\|_2 = \left\| \int_a^x \Phi(t, \Gamma)\mathrm{d}t \right\|_2 \leq \max_{t\in[a,b]}(\|\Phi(t,\Gamma)\|_2)\frac{(x-a)}{1!}$$
and let $M := \max_{t\in[a,b]}(\|\Phi(t,\Gamma)\|_2)$.

2. So by the previous lemma and induction we find
$$\|F_{k+1}(x) - F_k(x)\|_2 \leq M\frac{L^k(x-a)^{k+1}}{(k+1)!}.$$

3. So by triangle inequality we find
$$\|F_{k+m}(x) - F_k(x)\|_2 \leq \sum_{n=k}^{k+m} \|F_{n+1}(x) - F_n(x)\|_2 \leq M\sum_{n=k}^{k+m} \frac{L^n(x-a)^{n+1}}{(n+1)!}.$$
The sum $\sum_{n=k}^{k+m} \frac{(L(x-a))^{n+1}}{(n+1)!}$ goes to zero because $\sum_{n=0}^{\infty} \frac{(L(x-a))^n}{(n)!} = e^{L(x-a)} < e^{L(b-a)} < \infty$.

4. Therefore F_{k+m} is Cauchy and by completeness of continuous functions we get continuous limit F_*.

5. By continuity of T we get $TF_* = \lim_{n\to\infty} TF_n = \lim_{n\to\infty} F_{n+1} = F_*$.

0.0.1 Stone Weierstrass

- Let $\mathcal{A} \subset C([0,1])$ be a subspace (i.e. closed under linear operations) and also closed under products. We call this a function algebra.
- We say that $\mathcal{A}$ **vanishes nowhere** if there is no point $p \in [0,1]$ such that $f(p) = 0$ for all $f \in \mathcal{A}$.
- We say that $\mathcal{A}$ **separates points** if $x \neq y \in [0,1]$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 0.0.2. *The function algebra $\mathcal{A} \subset C([0,1])$ that vanishes nowhere and separates points is uniformly approximates any element in $C([0,1])$.*

Lemma 1

Lemma 0.0.3. *For any $c_1, c_2 \in \mathbb{R}$ and $p_1, p_2 \in [0,1]$ there exists a function $f \in \mathcal{A}$ such that $f(p_i) = c_i$.*

1. Since $\mathcal{A}$ vanishes nowhere there exists $g_1, g_1 \in \mathcal{A}$ such that $g_1(p_1) \neq 0$ and $g_2(p_2) \neq 0$.

2. Since $\mathcal{A}$ separates points there exists h such that $h(p_1) \neq h(p_2)$.

3. Consider the matrix system

$$\begin{pmatrix} g_1(p_1) & g_2(p_2)h(p_1) \\ g_1(p_1) & g_2(p_2)h(p_2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

4. The determinant is

$$det(H) = g_1(p_1)g_2(p_2)(h(p_1) - h(p_2)) \neq 0$$

and so the above system has a solution $a_0, b_0 \in \mathbb{R}$.

5. Thus, let $f(x) := a_0 g_1(p_1) + b_0 g_2(p_2)h(x) \in \mathcal{A}$. This satisfies $f(p_i) = a_0 g_1(p_1) + b_0 g_2(p_2)h(p_i) = c_i$.

Lemma 2

Lemma 0.0.4. *If $f \in \bar{\mathcal{A}}$ then $|f| \in \bar{\mathcal{A}}$. Hence if $f, g \in \bar{\mathcal{A}}$ then* $\max(f, g), \min(f, g) \in \bar{\mathcal{A}}$ *since* $\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$ *and* $\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$.

1. We will show that for every $\varepsilon > 0$ there exists $g_n \in \bar{\mathcal{A}}$ such that
$$|g_n(x) - |f(x)|| \leq \varepsilon.$$

2. The function $h(y) := |y|$ is continuous for $y \in [-\|f\|_\infty, \|f\|_\infty]$. Therefore, by Weierstrass approximation there exists polynomial $p_n(y) := a_{k,n}y^k + ... + a_{1,n}y + a_{0,n}$ such that
$$\|p_n(y) - h(y)\|_\infty = \sup_{y \in [-\|f\|_\infty, \|f\|_\infty]} |p_n(y) - h(y)| \leq \varepsilon.$$

3. Let $g_n(x) := a_{k,n}f(x)^k + ... + a_{1,n}f(x)$, this is in $g_n(x) \in \bar{\mathcal{A}}$ since $f \in \bar{\mathcal{A}}$ and $\bar{\mathcal{A}}$ is closed under products.

4. Since the range of y is $[-\|f\|_\infty, \|f\|_\infty]$, for every $x \in [0, 1]$ there exists a $y_x = f(x)$ and so we find
$$|g_n(x) - |f(x)|| = |p_n(y_x) - h(y_x)| < \varepsilon.$$

Stone Weierstrass

Let $h \in C([0, 1])$. We will show that for every $\varepsilon > 0$ there exists $G \in \bar{\mathcal{A}}$ such that
$$|G(x) - h(x)| \leq \varepsilon.$$

1. Fix arbitrary points $p, q \in [0, 1]$. Then by lemma 1, there exists $G_{p,q} \in \bar{\mathcal{A}}$ such that for $c_1 := h(p)$ and $c_2 := h(q)$ we find
$$G_{p,q}(p) = h(p) \text{ and } G_{p,q}(q) = h(q).$$

2. The function $G_{p,q}(x) - h(x)$ is uniformly continuous and has zeroes at p, q. Using the zero at q and continuity we have some $\delta_{\varepsilon,q} > 0$ such that for all $x \in B_{\delta_{\varepsilon,q}}(q)$:
$$|G_{p,q}(x) - h(x)| \leq \varepsilon \Rightarrow G_{p,q}(x) > h(x) - \varepsilon.$$

3. By compactness the open cover $\{B_{\delta_{\varepsilon,q}}(q)\}_{q \in [0,1]}$ has a finite subcover $\{B_{\delta_{\varepsilon,q_i}}(q_i)\}_{i=1}^M$.

4. Consider the continuous function

$$G_p(x) := \max_{i=1,\dots,M}(G_{p,q_1}(x), ..., G_{p,q_M}(x)),$$

which is in $\bar{\mathcal{A}}$ by lemma 2. This max still satisfies

$$G_p(x) > h(x) - \varepsilon$$

but now for all $x \in [0,1]$.

5. It also satisfies $G_p(p) - h(p)$. So by continuity of $G_p(x) - h(x)$ we have some $\delta_{\varepsilon,p} > 0$ such that for all $x \in B_{\delta_{\varepsilon,p}}(p)$:

$$|G_p(x) - h(x)| \leq \varepsilon \Rightarrow G_p(x) < h(x) + \varepsilon.$$

6. Therefore, as above by compactness the open cover $\{B_{\delta_{\varepsilon,p}}(p)\}_{q\in[0,1]}$ has a finite subcover $\{B_{\delta_{\varepsilon,p_i}}(p_i)\}_{i=1}^{L}$.

7. Consider the continuous function

$$G(x) := \min_{i=1,\dots,L}(G_{p_1}(x), ..., G_{p_L}(x)),$$

which is in $\bar{\mathcal{A}}$ by lemma 2. This min still satisfies

$$G(x) < h(x) + \varepsilon$$

but now for all $x \in [0,1]$. However, for every x it attains one the minima, and so it also satisfies

$$G(x) - h(x) = G_{p_k}(x) - h(x) > -\varepsilon.$$

So together we find

$$|G(x) - h(x)| \leq \varepsilon.$$

Trigonometric polynomials

The function algebra $\mathcal{A}_{trig} := \{a_0 + \sum_{k=1}^{m} a_k cos(kx) + b_k sin(kx) : a_k, b_k \in \mathbb{R}, m \in \mathbb{N}\}$ uniformly approximates continuous periodic functions in $C([-\pi,\pi]$, where we identify the endpoints $-\pi \equiv \pi$.

1. First we must show that it forms a function algebra. The subspace properties are clear. For the product we use that

   $cos(a)cos(b) = \frac{cos(a+b)+cos(a-b)}{2}$, $sin(a)cos(b) = \frac{sin(a+b)+sin(a-b)}{2}$ and $sin(a)sin(b) = \frac{c}{}$

   to get that $\mathcal{A}_{trig}$ is closed under products.

2. Next we show that it vanishes nowhere. Suppose that there was a point p such that

   $$a_0 + \sum_{k=1}^{m} a_k cos(kp) + b_k sin(kp) = 0$$

   for all $a_k, b_k \in \mathbb{R}, m \in \mathbb{N}$. We zero all coefficients for $k \neq 1$ and for $k = 1$ we choose them so that

   $$cos(p) + sin(p) = 0 \text{ and } cos(p) - sin(p) = 0,$$

   which is a contradiction.

3. Next we show that it separates points. In $[-\pi, \pi]$ we have $cos(a) = cos(b)$ iff $a = b$ or $a = -b$. So

   - if $x \neq -y$ then cosine separates them $cos(x) \neq cos(y)$.
   - If $x = -y \neq \pi$ then sine separates them because $sin(x) = -sin(y) \neq sin(y) \neq 0$.
   - the endpoints were identified so there is no third case of $x = -y = \pi$.

4. Therefore, Stone Weierstrass applies.