05 June, 2019

- ***Administration items***
  - *Pick up midterm papers during TA office hours tomorrow (Thursday)*
  - *Ask remark during office hours tomorrow*
  - *Remaining topics*
    - i. *June 05—Transfer function noise model + VAR*
    - ii. *June 10—Cointegration + Bootstrapping time series*
    - iii. *June 12—GARCH/Multivariate GARCH + Review of final exam*

- Transfer function noise model (Chapter 14, Wei's book)
    - Distributed lag model

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + a_t, \qquad a_t \sim NID(0, \sigma_a^2)$$

$x_t \sim ARMA(p,q)\ model$:

$$\phi(B)x_t = \theta(B)e_t, \quad e_t \sim NID(0, \sigma_e^2)$$

$$cov(a_t, e_s) = 0, \qquad \forall t, s$$

$v_i = \phi^i(1-\phi)$ from example of Geltner's model

$$y_t = \sum_{i=0}^{m} v_i x_{t-i} + a_t$$

- <u>Pre-whitening:</u>

$$y_t = \sum_{i=0}^{m} v_i x_{t-i} + a_t,$$

$$y_t = \underbrace{(v_0 B^0 + v_1 B + v_2 B^2 + \cdots + v_m B^m)}_{v(B)} x_t + a_t$$

$$\phi(B) x_t = \theta(B) e_t, \qquad cov(a_t, e_s) = 0, \qquad \forall t, s$$

$$\underbrace{\frac{\phi(B)}{\theta(B)}}_{\sum_{i=0}^{\infty} \pi_i B^i} x_t = e_t$$

$$\underbrace{\frac{\phi(B)}{\theta(B)} y_t}_{\tau_t} = v(B) \underbrace{\frac{\phi(B)}{\theta(B)} x_t}_{e_t} + \underbrace{\frac{\phi(B)}{\theta(B)} a_t}_{\tilde{a}_t = \sum \pi_i a_{t-i}}, \qquad v(B) = \sum_{i=0}^{m} v_i B^i$$

$$\tau_t = v(B)e_t + \tilde{a}_t$$

$$\underbrace{E(e_t\tau_t)}_{cov(e_t,\tau_t)=\gamma_{e\tau}(0)} = \overbrace{\underbrace{E[(v_0e_t + v_1e_{t-1} + \cdots + v_me_{t-m})e_t]}_{v_0\sigma_e^2=v_0\gamma_e(0)}}^{\sum_{i=0}^m v_iE(e_{t-i}e_t)=\sum_{i=0}^m v_i\cdot\gamma_e(i)} + \underbrace{E(e_t\tilde{a}_t)}_{=0}$$

$$\rho_{e\tau}(0)\cdot\sigma_e\sigma_\tau = cov(e_t,\tau_t) = v_0\gamma_e(0) = v_0\cdot\sigma_e^2 \to v_0 = \rho_{e\tau}(0)\cdot\frac{\sigma_\tau}{\sigma_e}\backslash\!\propto \rho_{e\tau}(0)$$

$$\overbrace{\underbrace{E(e_{t-1}\tau_t)}_{\gamma_{e\tau}(1)}}^{\rho_{e\tau}(1)\cdot\sigma_e\sigma_\tau} = \overbrace{\underbrace{E[(v_0e_t + v_1e_{t-1} + \cdots + v_me_{t-m})e_{t-1}]}_{v_1\gamma_e(1)=v_1\sigma_e^2}}^{\sum_{i=0}^m v_iE(e_{t-i}e_{t-1})=\sum_{i=0}^m v_i\gamma_e(i-1)} + \underbrace{E(e_{t-1}\tilde{a}_t)}_{=0}$$

$$\rho_{e\tau}(k)\cdot\sigma_e\sigma_\tau = \gamma_{e\tau}(k) = v_k\sigma_e^2 \to v_k = \rho_{e\tau}(k)\cdot\frac{\sigma_\tau}{\sigma_e} \propto \rho_{e\tau}(k)$$

$$co2_t = v_3gas_{t-3} + \cdots + v_7gas_{t-7} + \xi_t$$

$$\boxed{v(B) = \frac{\delta(B)}{\omega(B)}}$$

$$y_t = v(B)x_t + \xi_t = \frac{\delta(B)}{\omega(B)}x_t + \xi_t$$

$$\rightarrow \underbrace{\omega(B)}_{1-\omega_1 B-\omega_2 B^2} y_t = \underbrace{\delta(B)}_{\delta_0+\delta_1 B+\delta_2 B^2} x_t + \omega(B)\xi_t$$

- <u>Box-Tiao Transformation:</u>

$$y_t = v(B)x_t + n_t, \qquad \phi_n(B)n_t = \theta_n(B)a_t$$

$$\frac{\phi_n(B)}{\theta_n(B)} n_t = a_t$$

$$\underbrace{\frac{\phi_n(B)}{\theta_n(B)} y_t}_{\tilde{y}_t} = v(B) \underbrace{\frac{\phi_n(B)}{\theta_n(B)} x_t}_{\tilde{x}_t} + \underbrace{\frac{\phi_n(B)}{\theta_n(B)} n_t}_{a_t}$$

$$\rightarrow \tilde{y}_t = v(B)\tilde{x}_t + a_t$$

Diagnostic test on $\{a_t\}$

<u>Diagnostic test of a transfer function noise model</u>

Vector autoregressive model

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + a_t$$

$$\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ x_{2t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}^{(2)} & \phi_{12}^{(2)} \\ \phi_{21}^{(2)} & \phi_{22}^{(2)} \end{bmatrix} \begin{bmatrix} x_{1t-2} \\ x_{2t-2} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

$$x_{1t} = \phi_{11}^{(1)} x_{1,t-1} + \phi_{12}^{(1)} x_{2,t-1} + \phi_{11}^{(2)} x_{1,t-2} + \phi_{12}^{(2)} x_{2,t-2} + a_{1t}$$

We can test stationarity of a VAR(1) model by checking the eigenvalues of its autoregressive coefficient matrix.

Granger causality:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \phi_{11}^{(1)} & \phi_{12}^{(1)} \\ \cancel{0} & \phi_{22}^{(1)} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \phi_{11}^{(2)} & \phi_{12}^{(2)} \\ \cancel{0} & \phi_{22}^{(2)} \end{bmatrix} \begin{bmatrix} y_{t-2} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} a_t \\ e_t \end{bmatrix}$$