Week 11: Tutorial Handout

- A family $\mathcal{F} \subset C(S, \mathbb{R})$ is called **equicontinuous** at $x_0 \in S$ when for $\varepsilon > 0$ $\forall f \in \mathcal{F}$ we have a $\delta_{\mathcal{F}}(x_0, \varepsilon)$ s.t.
$$|x - y| \leq \delta_{\mathcal{F}}(x_0, \varepsilon) \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$
If there is a global such delta for all $x_0 \in S$, then $\mathcal{F}$ is **uniformly equicontinuous** in S.

- A family $\mathcal{F}$ is **bounded** if $\forall f \in \mathcal{F}$ we have $\|f\|_\infty < B$ for some $B > 0$.

- A family $\mathcal{F}$ is **closed** if for all $\{f_n\} \in \mathcal{F}$ we have that $\|f_n - f\|_\infty \to 0$ implies $f \in \mathcal{F}$.

- (AA theorem) If $K \subset \mathbb{R}$ is compact, then a family $\mathcal{F} \subset C(K, \mathbb{R})$ is compact iff it is closed, bounded and equicontinuous.

Compactness and Equicontinuity

1. Which of the following families are equicontinuous or uniformly equicontinuous? Let $n \geq 1$.

   - $f_n(x) := x^n$ in [0,1].
   - $f_n(x) := (\frac{x}{2})^n + (\frac{1}{x})^n + x$ in $(\frac{5}{4}, \frac{7}{4})$.
   - $f_n(x) := (\frac{x}{2})^n + (\frac{1}{x})^n + x^2$ in $(\frac{5}{4}, \frac{7}{4})$.
   - $f_n(x) := nsin(x/n) + x^{\frac{1}{2}} \ln(x)$ in $[0, \pi]$.
   - sequence $f_n(x)$ that are C_n-Lipschitz and $C_n \to C$ in [0,1].

2. Which of the following families are compact? Let $n \geq 1$.

   - $\{x^n\} \cup \{0\}$ in $[0, 1]$.
   - $\{x^n\} \cup \{0\}$ in $[0, \frac{1}{2}]$.
   - $\{nsin(x/n) + x^{\frac{1}{2}} \ln(x)\} \cup \{1 + x^{\frac{1}{2}} \ln(x)\}$ in $[0, \pi]$.
   - $\{e^{x+\frac{1}{n}}\} \cup \{e^x\}$ in $(0, \infty)$.
   - sequence $f_n(x)$ that are C_n-Lipschitz and $C_n \to C$ in [0,1].