1. (8 points) Circle the letter of **at least one** correct answer for each of the questions below. More than one answers could be true but we only ask you to pick **one** of them.

(a) (2 points) Epsilon definition of sup/inf , the Bolzano-Weierstrass (BW) and Monotone convergence theorem (MCT). Let $A \subset \mathbb{R}$ be non-empty and $\epsilon > 0$. Let $\{x_n\}_{n\geq 1} \in [c, d]$ be a sequence and $\{x_{n_k}\}_{k\geq 1}$ a subsequence of it. Choose at least one correct answer:

A. $\exists a_\epsilon \in A \cap \mathbb{R}$ s.t. $\inf_{a\in A}(a) > a_\epsilon + \epsilon$. BW states that if $\{x_n\} \subset [c, d]$, then $\exists x_{n_k} \to L \in [c, d]$.

B. $\exists a_\epsilon \in A$ s.t. $\inf_{a\in A}(a) > a_\epsilon - \epsilon$. MCT states that i $\{x_n\} \subset [c, d]$ and $x_{n+1} \leq x_n$, then $x_n \to L \in [c, d]$.

C. $\exists a_\epsilon \in A \cap \mathbb{R}$ s.t. $\sup_{a\in A}(a) < a_\epsilon - \epsilon$. MCT states that i $\{x_n\} \subset [c, d]$ and $x_{n+1} \geq x_n$, then $x_n \to L \in [c, d]$.

D. $\exists a_\epsilon \in A$ s.t. $\sup_{a\in A}(a) < a_\epsilon + \epsilon$. BW states that i $\{x_n\} \subset [c, d]$, then $\exists x_{n_k} \to L \in [c, d]$.

(b) (2 points) Let $\{a_n\}_{n\geq 1} \in \mathbb{R}$ represent a Cauchy sequence and $f : \mathbb{R} \to \mathbb{R}$ a continuous function. Choose at least one correct answer:

A. $\forall \epsilon > 0, \exists \delta_\epsilon > 0$ s.t. $\forall n, m \geq 1$ we have

$$|a_n - a_m| < \delta_\epsilon \Rightarrow |f(a_n) - f(a_m)| < \epsilon.$$

B. $\forall \epsilon > 0, \exists \delta_\epsilon > 0$ s.t. $\forall n, m \geq 1$ we have

$$|a_n - a_m| < \delta_\epsilon.$$

Completeness states that "every Cauchy sequence converges".

C. If a subsequence convergences $a_{n_k} \to L$ as $k \to \infty$, then $f(a_n) \to f(L)$ as $n \to \infty$. Completeness is equivalent to "every Cauchy sequence has a converging subsequence but the full sequence might not converge".

D. $\forall \epsilon > 0, \exists N_\epsilon > 0$ s.t. $\forall n, m \geq N_\epsilon$ we have

$$|a_n - a_m| < \epsilon.$$

Completeness's corrolary is that every Cauchy sequence has a converging subsequence.

(c) (2 points) Heine-Borel, Extreme value theorem (EVT) and Intermediate value theorem (IVT). Let $K \subset \mathbb{R}$ be a subset and $f : K \to \mathbb{R}$ a continuous function.Choose at least one correct answer:

A. The set $f(K)$ is compact if K is closed and bounded.

B. The set K is compact if and only if K is closed and bounded. If $[c, d] \subset K$ then $f([c, d]) = [f(c), f(d)]$.

C. The set $f(K)$ is closed if K is closed. If $[c, d] \subset K$ then $f([c, d]) = [\inf_{x\in[c,d]} f(x), \sup_{x\in[c,d]} f(d)]$.

D. The set K is compact if and only if K is closed and bounded. If $[c, d] \subset K$ then $f([c, d]) = [\min_{x\in[c,d]} f(x), \max_{x\in[c,d]} f(d)]$.

(d) (2 points) Uniform continuity and Heine-Cantor theorem. Let $K \subset \mathbb{R}$ be a subset and $f : K \to \mathbb{R}$ a continuous function. Choose at least one correct answer:

A.f is not uniformly continuous if $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta, y_\delta \in K$,s.t.

$$|x_\delta - y_\delta| \leq \delta \Rightarrow |f(x) - f(y)| \geq \epsilon_0.$$

Also,f is uniformly continuous if the set K is closed.

B.f is uniformly continuous if $\forall \epsilon > 0$ and $x \in K$, $\exists \delta(\epsilon, x) > 0$ s.t.

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon.$$

Also, f is uniformly continuous if $K \subset [c, d]$ and K^c is open.

C.f is uniformly continuous if $\forall \epsilon > 0$ $\exists \delta(\epsilon) > 0$ s.t.

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon.$$

Also, f is uniformly continuous if if $K \subset [c, d]$ and K contains its limit points.

D. f is not uniformly continuous if $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta, y_\delta \in K$,s.t.

$$|x_\delta - y_\delta| \leq \delta \Rightarrow |f(x) - f(y)| \leq \epsilon_0.$$

Also, f is uniformly continuous if K is closed and bounded.

2. (10 points) Show that $\{a_n\}_{n\geq 1}$ converges to $L \in \mathbb{R}$ if and only if $\lim_{n\to\infty} \sup_{k\geq n} a_k = L = \lim_{n\to\infty} \inf_{k\geq n} a_k$.

   (Hint: use epsilon definition of sup/inf and squeeze theorem)

   (a) (5 points) The necessary condition: $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} \sup_{k\geq n} a_k = L = \lim_{n\to\infty} \inf_{k\geq n} a_k$.

   We will only do the limsup case due to their similarity. Fix $\epsilon > 0$ WTS that there exists $N > 0$ s.t. $\forall n \geq N$ we have

$$\left|\sup_{k\geq n} a_k - L\right| \leq \epsilon.$$

   By the limit definition we have $N_\epsilon > 0$ s.t.$\forall n \geq N_\epsilon$ we have

$$|a_n - L| \leq \frac{\epsilon}{2}.$$

   By the epsilon definition of sup, there exists a_m for some $m \geq N_\epsilon$ s.t.

$$\sup_{k\geq N_\epsilon} a_k - \frac{\epsilon}{2} \leq a_m.$$

   Next we put this two facts together.

$$-\epsilon < a_m - L \sup_{k\geq N_\epsilon} a_k - L \leq a_m - L + \frac{\epsilon}{2} < \epsilon \Rightarrow$$

$$\left|\sup_{k\geq n} a_k - L\right| \leq \epsilon.$$

   (b) (5 points) The sufficient condition: $\lim_{n\to\infty} \sup_{k\geq n} a_k = L = \lim_{n\to\infty} \inf_{k\geq n} a_k \Rightarrow \lim_{n\to\infty} a_n = L$.

   Fix $\epsilon > 0$ WTS that there exists $N > 0$ s.t. $\forall n \geq N$ we have

$$|a_n - L| \leq \epsilon.$$

   By the limit definition we have $N_\epsilon := \max(N_{\epsilon,1}, N_{\epsilon,2}) > 0$ s.t.$\forall n \geq N_\epsilon$ we have

$$\left|\sup_{k\geq N_\epsilon} a_k - L\right| \leq \epsilon \text{ and } \left|\inf_{k\geq N_\epsilon} a_k - L\right| \leq \epsilon.$$

   Next we put this two facts together.

$$-\epsilon \leq \inf_{k\geq N_\epsilon} a_k - L \leq a_n - L \leq \sup_{k\geq N_\epsilon} a_k - L \leq \epsilon \Rightarrow$$

$$|a_n - L| \leq \epsilon.$$

3. (8 points) If $\{x_n\}_{n\geq 1}$ is Cauchy, find a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that

$$\sum_{k\geq 1} \left|x_{n_k} - x_{n_{k+1}}\right| < \infty.$$

This is similar to the diagonalization argument (see handout) where for each ϵ_k we find a representative x_{n_k} from the kth row. We will simply need some ϵ_k and subsequence x_{n_k} that satisfy

$$\sum_{k\geq 1} \left|x_{n_k} - x_{n_{k+1}}\right| \leq \sum_{k\geq 1} \epsilon_k < \infty.$$

For concreteness lets work with $\epsilon_k := 2^{-k}$.

Starting elements For $\epsilon_1 := 2^{-1}$ the Cauchy property gives that there $\exists N_1$ s.t. for all $n, m \geq N_1$ we have

$$|x_n - x_m| \leq \epsilon_1 := 2^{-1}.$$

Therefore, let $x_{n_1} := x_{N_1}$. For $\epsilon_2 := 2^{-2}$ the Cauchy property gives that there $\exists N_2 > N_1$ s.t. for all $n, m \geq N_2$ we have

$$|x_n - x_m| \leq \epsilon_2 := 2^{-2}.$$

Therefore, let $x_{n_2} := x_{N_2}$. Note that since $N_2 > N_1$ we also have

$$|x_{n_1} - x_{n_2}| = |x_{N_1} - x_{N_2}| \leq \epsilon_1 := 2^{-1}.$$

General element For $\epsilon_k := 2^{-k}$ the Cauchy property gives that there $\exists N_k > N_{k-1}$ s.t. for all $n, m \geq N_k$ we have

$$|x_n - x_m| \leq \epsilon_k := 2^{-k}.$$

Therefore, let $x_{n_k} := x_{N_k}$. Note that since $N_k > N_{k-1}$ we also have

$$\left|x_{n_k} - x_{n_{k-1}}\right| = \left|x_{N_k} - x_{N_{k-1}}\right| \leq \epsilon_{k-1} := 2^{-(k-1)}.$$

Therefore,

$$\sum_{k\geq 1} \left|x_{n_k} - x_{n_{k+1}}\right| \leq \sum_{k\geq 1} 2^{-k} = 1 < \infty.$$

4. (17 points) Let $a_0 = 0$ and $a_{n+1} = cos(a_n)$ for $n \geq 0$.

   (a) (7 points) Show that cos(x)=x has a unique solution in [0,1] using intermediate value theorem and derivative sign.

   We have $f(0) = cos(0) - 0 = 1 >$ and $f(1) = cos(1) - 1 < 0$. So there is at least one solution in [0,1]. The derivative is $f'(x) = -sin(x) - 1 < 0$ for all x and so f is strictly decreasing and so there is only one solution.

   (b) (10 points) Assume $a_n \in [cos(1), 1]$. Show that a_n is Cauchy.

   We start by controlling the difference $|a_n - a_{n+1}|$.

   - Consider the following interval $I_n \subset [cos(1), 1]$

   $$I_n := \begin{cases} (a_n, a_{n+1}) & \text{,if n is even} \\ (a_{n+1}, a_n) & \text{,if n is odd} \end{cases}.$$

   - Since $a_n \in [cos(1), 1]$, we have by MVT some $c_n \in I_n \subset [cos(1), 1]$ s.t.

   $$|a_{n+1} - a_n| = |cos(a_n) - cos(a_{n-1})| = |sin(c_n)| \, |a_n - a_{n-1}| \, .$$

   - Since $c_n \in [cos(1), 1]$ and sin(x) is increasing in that interval, we have that $sin(c_n) \leq sin(1) =: r < 1$.

   $$|a_{n+1} - a_n| \leq r \, |a_n - a_{n-1}| \, .$$

   - Therefore, recursively we find the bound

   $$|a_{n+1} - a_n| \leq r \, |a_n - a_{n-1}| \leq r^2 \, |a_{n-1} - a_{n-2}| \leq ... \leq r^n \, |a_2 - a_1| =: cr^n.$$

   Now we control the difference $|a_n - a_{n+m}|$ for any $n, m > 0$. We write the telescoping sum

   $$\begin{aligned} |a_n - a_{n+m}| &= |a_n - a_{n+1} + a_{n+1} - a_{n+2} + ... + a_{n+m-1} - a_{n+m}| \\ &\leq |a_n - a_{n+1}| + |a_{n+1} - a_n| + ... + |a_{n+m-1} - a_{n+m}| \end{aligned}$$

   and bound it by

   $$c \sum_{k=0}^{m-1} r^{n+k} = cr^n \frac{1 - r^m}{1 - r}.$$

   Since $r < 1$ this bound goes to zero as $n \to +\infty$. Therefore, given fixed $\epsilon > 0$ we can pick large enough $N > 0$ s.t. for all $n, m \geq N$ we have

   $$\epsilon > cr^N \frac{1 - r^N}{1 - r} \geq cr^n \frac{1 - r^m}{1 - r} \geq |a_n - a_{n+m}| \, .$$

Since the sequence a_n is Cauchy, it converges to some number $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} cos(a_{n-1}) = cos(\lim_{n\to\infty} a_{n-1}) = cos(L)$. Therefore, L satisfies the fixed point relation $L = cos(L)$. In other words, it is the intersection of the graphs of $y_1 = cos(x)$ and $y_2 = x$.

5. (15 points) Showing closed and compact.

   (a) (4 points) Is the following set compact: the rationals $\mathbb{Q} \cap [0, 1]$?
   It is bounded but not closed because it doesn't contain the irrational limit points (Hw1 problem).

   (b) (4 points) Is the following set compact: the set $S := \{(e^{-x}cos(x), e^{-x}sin(x)) : x \geq 0\} \cup \{(x, 0) : x \in [0, 1]\}$.
   Yes. It is bounded because $|e^{-x}cos(x)| \leq 1$ and $|e^{-x}sin(x)| \leq 1$. Next we show closed. Take converging sequence

$$(a_n, b_n) := (e^{-x_n}cos(x_n), e^{-x_n}sin(x_n)) \to (a, b),$$

   WTS that $(a, b) \in S$. If $x_n \to x < \infty$, then the result follows by the continuous mapping theorem. If $x = \infty$, then $(a, b) = (0, 0)$, which is inside $\{(x, 0) : x \in [0, 1]\}$.

   (c) (7 points) Prove that the sum $A + B$ of a closed set A and a compact set B is also closed.
   Take converging sequence $a_n + b_n \to z$, WTS that $z \in A + B$. By compactness we have converging subsequence $b_{n_k} \to b \in B$. Therefore,

$$a_{n_k} = a_{n_k} + b_{n_k} - b_{n_k} \to z - b.$$

   Since A is closed, we have that $z - b \in A$ i.e. $z - b = a$ for some $a \in A$ and so $z = a + b$.

6. (15 points) Suppose that $f : [0, \infty)$ is continuous and $\lim_{x\to\infty} f(x) = f(0)$. Prove that f attains its maximum.

If f(0) is the global maximum, then we are done since f will attains its maximum at 0. So suppose otherwise, that there exists large enough $x_0 > 0$ s.t. $f(0) + \delta \leq f(x_0)$ for some $\delta > 0$.

- By $\lim_{x\to\infty} f(x) = f(0)$ we set $\epsilon := \frac{\delta}{2}$ and obtain $\exists N_{\delta/2}$ s.t. $\forall x \geq N_{\delta/2}$ we have

$$|f(x) - f(0)| \leq \epsilon := \frac{\delta}{2}.$$

  Therefore, for $x \in [N_{\delta/2}, \infty)$ we have

$$f(x) \leq f(0) + \frac{\delta}{2} < f(0) + \delta \leq f(x_0).$$

- Apply EVT to the complement interval $[0, N_{\delta/2}]$ to obtain some $M_{N_{\delta/2}}$ s.t. for all $x \in [0, N_{\delta/2}]$ we have a maximum:

$$f(x) \leq f(M_{N_{\delta/2}}).$$

- Finally, let $f(M) := \max(f(x_0), f(M_{N_{\delta/2}}))$. Then for $x \in [0, \infty) = [0, N_{\delta/2}] \cup [N_{\delta/2}, \infty)$ we have that

$$f(x) \leq \max(f(x_0), f(M_{N_{\delta/2}})) = f(M)$$

  irrespective of whether $x \in [0, N_{\delta/2}]]$ or in the tail $x \in [N_{\delta/2}, \infty)$.

7. (17 points) Show that $\frac{sin(x^3)}{x}$ is uniformly continuous (u.c.) on $[0, \infty)$. You can use the following results if you want:

- If f is continuous on (a, c) and u.c. on (a,b] and [b,c), then it is u.c. on (a,c) (with c possibly infinite).
- If $\lim_{x \to 0^+} f(x)$ exists then $f(x)$ is u.c. on (0,1].

8. (10 points) (New question) Studying absolute continuity. A function $f : [a,b] \to \mathbb{R}$ is called **absolutely continuous** if $\forall \epsilon > 0$ there exists $\delta_{global}(\epsilon) > 0$ so that given any finite collection of disjoint subintervals $\{(a_k, b_k)\}_{k=1}^{M} \subset [a,b]$ that satisfy

$$\text{If } \sum_{k=1}^{M} b_k - a_k < \delta, \text{ then } \sum_{k=1}^{M} |f(b_k) - f(a_k)| < \epsilon.$$

Such functions can be shown to be differentiable almost everywhere. Show that $\frac{sin(x)}{x}$ is absolutely continuous in $[0,1]$. One suggestion is to study the derivative of $\frac{sin(x)}{x}$ for $x \in [0,\epsilon)$ for any small $\epsilon > 0$.

The derivative of $\frac{sin(x)}{x}$ is $\frac{xcos(x)-sin(x)}{x^2}$ which by L'Hopital is bounded at zero

$$\lim_{x\to 0^+} \frac{xcos(x) - sin(x)}{x^2} = \lim_{x\to 0^+} \frac{-xsin(x)}{2x} = 0.$$

Therefore, the function $\frac{sin(x)}{x}$ is 1-Lipschitz and we get for $\delta := \frac{\epsilon}{C}$

$$\text{If } \sum_{k=1}^{M} b_k - a_k < \delta, \text{ then } \sum_{k=1}^{M} |f(b_k) - f(a_k)| < C\sum_{k=1}^{M} b_k - a_k < C\delta = \epsilon.$$