Week 6: Tutorial Handout

A function $\langle , \rangle : V \times V \to \mathbb{R}$ is called an inner product if it satisfies

- (positive definiteness) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.
- (symmetry) $\langle x, y\rangle = \langle y, x\rangle$
- (bilinear) $\langle a_1x_1 + a_2x_2, z\rangle = a_1\langle x_1, z\rangle + a_2\langle x_2, z\rangle$ and $\langle w, a_1x_1 + a_2x_2\rangle = a_1\langle w, x_1\rangle + a_2\langle w, x_2\rangle$ for any $a_1, a_2 \in \mathbb{R}$ and $x_1, x_2, z, w \in V$.

The **Fourier** series for a continuous function $f : [-1, 1] \to \mathbb{R}$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n cos(\pi nx) + b_n sin(\pi nx),$$

where the **Fourier coefficients** are defined to be

$$a_n := \int_{-1}^{1} f(t)cos(\pi nt)\mathrm{d}t \text{ and } b_n := \int_{-1}^{1} f(t)sin(\pi nt)\mathrm{d}t.$$

Inner products

1. (D&D 7.4.C Cauchy-Schwartz inequality) Find the minimum point t_* of the function $f(t) := \|x\|^2 - 2t\langle x, y\rangle + t^2\|y\|^2$ and deduce the CS-inequality

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

2. (D&D 7.4.I) Let A be $n \times n$ matrix and consider the form

$$\langle x, y\rangle_A := \langle Ax, Ay\rangle_2,$$

where $\langle u, v\rangle_2 := u \cdot v = \sum_{i=1}^{n} u_iv_i$ is the dot product. This form is an inner product over $\mathbb{R}^n$ iff A is an invertible matrix.

Fourier series

3. Find the Fourier series for the step function $f(x) := \begin{cases} 1 & , if\ x \in [0, \frac{1}{2}] \\ 0 & , otw \end{cases}$, for $x \in [-1, 1]$.
4. Find the Fourier series for $|x|$ for $x \in [-1, 1]$.
5. Show that if $f \in C([-1, 1])$ is an even function (i.e. $f(-x) = f(x)$), then the Fourier coefficients $b_n = 0$.
6. (D&D 7.6.E) Find the Fourier series for $|sin(\pi x)|$ for $x \in [-1, 1]$.