Orthogonal Bases / Complement

$$x, y \in R^n, Proj_y x = \frac{xy}{\|y\|^2} y, Proj_y x = cx, (x - Proj_y x) \perp y$$

Set of vectors $\{v1, \dots, vk\}$ is orthogonal if none of vectors are 0 and $vi\ vj = 0, \forall i \neq j$ on a orthogonal set that's also a basis that is called an orthogonal basis.

Standard basis for R^3 is orthogonal basis for R^3

$e.x. show \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ is an orthogonal basis for R^3

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

Since $dimR^3 = 3$, any set of 3 are linear independent vectors for R^3 is a basis for R^3

check $rank \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = 3$

Theorem: $\{v1, \dots, vk\}$ is orthogonal basis for $R^n, x \in R^n, x = c1v1 + \cdots + ckvk, c_j = \frac{xivj}{\|vj\|^2}$

Proof:

$$x = c1v1 + \cdots + ckvk$$
$$x\ vi = (c1v1 + \cdots + ckvk)vi$$
$$\frac{xvi}{\|vi\|^2} = ci$$

If any orthogonal set is linear independent, then any orthogonal set of n vectors from R^n is an orthogonal basis for R^n

Generalized Pythagorean Theorem: if $\{x1, \dots, xk\}$ orthogonal then

$$\begin{aligned} \|c1v1 + \cdots + ckvk\|^2 &= (c1v1 + \cdots + ckvk)(c1v1 + \cdots + ckvk) \\ &= c1v1(c1v1 + \cdots + ckvk) + \cdots ckvk(c1v1 + \cdots + ckvk) \\ &= c1^2\|v1\|^2 + \cdots + ck^2\|vk\|^2 \end{aligned}$$

Proof: any orthogonal set is lin.indep., WTS $c1v1 + \cdots + ckvk = 0 \to c1, \dots, ck = 0$

$$\|c1v1 + \cdots + ckvk\|^2 = \|0\|^2 = 0 = c1^2\|v1\|^2 + \cdots + ck^2\|vk\|^2 = 0 + \cdots + 0$$

Set of vectors $\{v1, \dots, vk\}$ is orthonormal if it's orthogonal and $\|vi\| = 1, i = 1, \dots, k$

$\{v1, \dots, vk\}$ orthogonal then $\left\{\frac{v1}{\|v1\|}, \dots, \frac{vk}{\|vk\|}\right\}$ is orthonormal

$$e.x. \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \to \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}, then\ Proj_y x = \frac{xy}{1} y$$

Orthogonal Complements

Projection onto a subspace W of R^n, $Proj_W x$ is defined by $i)\ Proj_W x = cx$, $ii)\ (x - Proj_W x) \perp y$

W subspace of R^n, the orthogonal complement of W ($W^\perp$) is the set of vectors in R^n that is orthogonal to all vectors in W. $W^\perp = \{x \in R^n | xw = 0, \forall w \in W\}$

$e.x. in\ R^3, W = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}, W^\perp = R^3,$

$W = span\left\{\begin{pmatrix}a1\\a2\\a3\end{pmatrix}\right\}, W^\perp$ is a plain through the origin with the normal $\begin{pmatrix}a1\\a2\\a3\end{pmatrix}$, verse versa

How to compute $Proj_W x, \{w1, \dots, wk\}$ be orthogonal basis for W,

$$\exists c1, \dots, ck \in R^n, Proj_W x = xc1w1 + \cdots + xckwk$$

Because $(x - Proj_W x) \in W^\perp, W^\perp Wi = 0,$

$$(x - Proj_W x)wi = \big((x - c1w1) + \cdots + (x - ckwk)\big)wi$$
$$0 = x\ wi - ci\|wi\|^2$$
$$ci = \frac{xwi}{\|wi\|^2}$$
$$Proj_W x = \frac{xw1}{\|w1\|^2}w1 + \cdots \frac{xwk}{\|wk\|^2}wk = Proj_{w1}x + \cdots + Proj_{wk}x$$

$e.x. W = span\left\{\begin{pmatrix}-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\3\\2\end{pmatrix}\right\}, find\ Proj_W\begin{pmatrix}1\\2\\2\end{pmatrix}$

Because $\begin{pmatrix}-1\\1\\-1\end{pmatrix}\begin{pmatrix}1\\3\\2\end{pmatrix} = 0$, W is an orthogonal basis for W

$$Proj_W\begin{pmatrix}1\\2\\2\end{pmatrix} = Proj_{\begin{pmatrix}-1\\1\\-1\end{pmatrix}}\begin{pmatrix}1\\2\\2\end{pmatrix} + Proj_{\begin{pmatrix}1\\3\\2\end{pmatrix}}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Gran-Schmidt Process

Given $V = \{v1, \dots, vk\}$, but not orthogonal, want $W = \{w1, \dots, wk\}, spanV = spanW$

Let $w1 = v1, W1 = span\{w1\}$, then

$w2 = (v2 - Proj_{v2}W1) = (v2 - Proj_{v2}w1)$ is orthogonal to $w1$ and $w2$ is in span(v1, v2)

$$wi = \big(vi - Proj_{vi}W(i-1)\big) = \big(vi - Proj_{vi}w1 - Proj_{vi}w2 - \cdots - Proj_{vi}w(i-1)\big)$$

$$W = \left\{\begin{pmatrix}1\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\-1\end{pmatrix}\right\}, find\ W^\perp = \{x \in R^4 \mid xw = 0, \forall w \in W\}$$

$\left(\begin{array}{cccc|c}1&1&-1&1&0\\1&1&1&-1&0\end{array}\right) \gg \left(\begin{array}{cccc|c}1&1&0&0&0\\0&0&1&-1&0\end{array}\right), x = \begin{pmatrix}-s\\s\\t\\t\end{pmatrix}, s, t \in R, W^\perp =$

$span\left\{\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$

Basic properties of $W^\perp$

$W^\perp \cap W = \{0\}$: if $x \in W^\perp \wedge x \in W, meaning\ x.x = 0, \|x\|^2 = 0, x = 0$

$(W^{\perp})^{\perp} = W$

Theorem: If W is a subspace of R^n, every $x \in R^n$ can be written uniquely as $x = w1 + w2, w1 \in W, w2 \in W^{\perp}$

Proof:

Existence: $w1 + w2 = Proj_W x + x - Proj_W x = x$

Uniqueness: Assume $w_1 + w_2 = w_1' + w_2'$, WTS $w_1 = w_1'\ and\ w_2' = w_2$

$$w_1 + w_2 = w_1' + w_2'$$

$$w_1 + w_1' = w_2' - w_2, w_1 - w_1' \in W, w_2' - w_2 \in W^{\perp}$$

$$W \cap W^{\perp} = \{0\}, w_1 - w_1' = w_2' - w_2 = 0$$

$$w_1 = w_1'\ and\ w_2' = w_2$$

Consequences

$$W\ subspace\ R^n, dimW + dimW^{\perp} = dimR^n = n$$

Note: W is a subspace of R^n, every $x \in R^n$ can be written uniquely as $x = w1 + w2, w1 \in W, w2 \in W^{\perp}$ meaning R^n is the direct sum of $W \& W^{\perp}, R^n = W \oplus W^{\perp}$

$e.x. A\ m \times n: (rowA)^{\perp} = nullA, (colA)^{\perp} = nullA^{\intercal}$

Consequences II

$$R^n = nullA + (nullA)^{\perp} = nullA + rowA = nullA + colA^{\intercal}$$