June 12, 2019

- TA tutorial sessions tomorrow and next Tuesday @ LM158, 6-9pm
- Final exam questions (topics taught after midterm):
    1) Forecast (Ch 5)
    2) Transfer function noise model
    3) Multivariate time series—VAR & cointegration…
    4) GARCH model
    5) Modeling appraisal returns…
    6) Bootstrap time series

Generalized autoregressive conditional heteroskedastic (GARCH) model:

The first order of autoregressive conditional heteroskedastic (ARCH(1)) process:

$$a_t \sim NID(0, \sigma_t^2)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2$$

On defining $v_t = a_t^2 - \sigma_t^2$, the model can also be written as

$$\boldsymbol{a_t^2} = \alpha_0 + \alpha_1 \boldsymbol{a_{t-1}^2} + v_t.$$

Since $E(v_t | x_{t-1}, x_{t-2}, \ldots) = 0$, the model corresponds directly to an AR(1) model for the squared error a_t^2.

ARCH(q) process:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2,$$

with

$$\alpha_0 \geq 0, \alpha_i > 0, \sum_{i=1}^{q} \alpha_i < 1.$$

Or equivalently,

$$a_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 + v_t = a_0 + \alpha(B) a_{t-1}^2 + v_t,$$

where

$$\alpha(B) = \alpha_1 + \alpha_2 B + \cdots + \alpha_q B^{q-1}.$$

<u>The (generalized) ARCH(GARCH) process:</u>

Consider a GARCH(p,q) model

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 + \sum_{i=1}^{q} \alpha_i a_{t-i}^2 = \alpha_0 + \alpha(B) a_{t-1}^2 + \beta(B) \sigma_{t-1}^2.$$

Provided that $\alpha_0 \geq 0$, $\alpha(B)$ and $\beta(B)$ have no common roots and that the roots of $1 - \beta(B)$ all less than unity.

Review of forecast:

$$y_t = 0.5y_{t-1} + a_t, \qquad a_t \sim NID(0,1).$$

1. Optimal l-step ahead forecast (conditional expectation):

$$E_t(y_{t+1}) = 0.5E_t(y_t) + E_t(a_{t+1})$$

$$\rightarrow \hat{y}_t(1) = 0.5y_t$$

$$y_{t+l} = 0.5y_{t+l-1} + a_{t+l}$$

$$\rightarrow \hat{y}_t(l) = 0.5\hat{y}_t(l-1), \qquad l > 1$$

2. Optimal l-step ahead forecast in terms of a_t:

   1) Express y_{t+l} as a causal process:

$$y_{t+l} = \sum_{i=0}^{\infty} 0.5^i a_{t+l-i}. \qquad (1)$$

   2) Optimal MSE forecast a time t is

$$\hat{y}_t(l) = \sum_{i=0}^{\infty} \tilde{\phi}_i a_{t-i}. \qquad (2)$$

Finding $\tilde{\phi}_i$:

$$\min_{\tilde{\phi}_j} E[y_{t+l} - \hat{y}_t(l)]^2 = E\left[\sum_{i=0}^{l-1} 0.5^i a_{t+l-i} + \sum_{j=0}^{\infty}\left(0.5^{j+l} - \tilde{\phi}_j\right)a_{t-j}\right]^2$$

$$= \underbrace{\sigma_a^2}_{1} \sum_{i=0}^{l-1} 0.25^i + \sum_{j=0}^{\infty}\left(0.5^{j+l} - \tilde{\phi}_j\right)^2$$

That is, the optimal $\tilde{\phi}_j$'s are found as $\tilde{\phi}_j = 0.5^{j+l}$, $j = 0,1,2 \ldots$

3. The variance of the forecast error:

$$var\left(y_{t+l} - \hat{y}_t(l)\right) = \sum_{i=0}^{l-1} 0.25^i$$

4. The forecast error for a transfer function noise model

$$z_t = \beta f_t + y_t, \qquad f_t = 0.6 f_{t-1} + e_t, \qquad cov(e_t, a_s) = 0$$

l-step ahead forecast error of z_t is the sum of:

$$\beta^2 var\left(f_{t+l} - \hat{f}_t(l)\right) + var\left(y_{t+l} - \hat{y}_t(l)\right)$$

$$f_t = \sum_{i=0}^{m} v_i x_{t-i} \quad \text{(temporal aggregation)}$$

Modeling appraisal returns:

Consider appraisal returns satisfying:

$$y_t = a_t + \theta a_{t-1}, \qquad a_t \sim NID(0,1).$$

According to Getmansky et al. (2005), we can define

$$y_t = \overbrace{\frac{1}{(1+\theta)}}^{w_0} \underbrace{[(1+\theta)a_t]}_{r_t} + \overbrace{\frac{\theta}{(1+\theta)}}^{w_1} \underbrace{[(1+\theta)a_{t-1}]}_{r_{t-1}}$$

$$r_t = (1+\theta)a_t$$

Suppose $\theta = 0.5$

$$var(y_t) = (1 + .25)\sigma_a^2 = 1.25$$

$$var(r_t) = (1 + 0.5)^2\sigma^2 = 2.25$$

$$\frac{var(r_t)}{var(y_t)} = \frac{2.25}{1.25} = 1.8$$