STA257: Probability and Statistics 1

Instructor: Katherine Daignault

Department of Statistical Sciences
University of Toronto

Week 1

Outline

Introduction to Probability (Chapter 1.1-1.3)
- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)
- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Outline

Introduction to Probability (Chapter 1.1-1.3)
- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)
- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

The term 'Probability'

- In everyday use, 'probability' is used to reflect a believe in the occurrence of a future event, e.g.
  - weather: probability of precipitation
  - insurance: dealing with the probability of an early death
  - medicine: probability of surgical complications during procedure
- The theory of probability is used in many disciplines, but originated in the study of games of chance.
- Random events cannot be predicted with any certainty, but the frequency of events from a long series of such trials is often quite stable

Example: Gambler

A gambler wishes to determine the balance of a die. A die is balanced if the chance of rolling each number is equal, i.e. $1/6$. How can he/she check this?

- if he/she had an infinite amount of time, the gambler could simply roll the die **infinitely** many times and record the frequency of each number
- in practice, use the **scientific method**:
  1. make a hypothesis, i.e. *the die is balanced*
  2. run an experiment: seek observations/data to contradict the hypothesis
- the gambler rolls the dice 10 times and gets all 1s ☹
- concludes hypothesis was wrong, i.e. *the die is not balanced*.

Role of Probability in Inference

How could the gambler make such a conclusion?

- 10 rolls is obviously not the same as *infinitely many* rolls, so maybe the gambler was just unlucky?
- Hypothesis rejected **not** because getting all 1s was *impossible* if the die is balanced, but because it was *improbable*.
  - based on the probability of observing the data he collected
- Statistical inference uses models of the probability of events occurring to determine whether data collected contradicts some hypothesis.
- *All of statistics relies on us being able to specify the probability of some event!*

Outline

Introduction to Probability (Chapter 1.1-1.3)
- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)
- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Sample spaces

Probability theory is focused on situations in which the outcomes occur randomly.

- Such situations are called **experiments**, e.g. flipping 2 coins.
- The set/collection of all possible outcomes of an experiment is called the **sample space**.
  - e.g. all possible results of flipping 2 coins: $\{TT, TH, HT, HH\}$
- Mathematically, we denote the sample space by Ω, and an element of the sample space by ω, so that $\omega \in \Omega$.
  - e.g. $\Omega = \{TT, TH, HT, HH\}$ and the outcome $\omega = \{HH\} \in \Omega$
- We denote the size of the sample space by $|\Omega|$
  - e.g. in our experiment, $|\Omega| = 4$

Example A: Commuting to work

Driving to work, a commuter passes through a sequence of three intersections with traffic lights. At each light, he/she either stops, s, or continues, c.

- The experiment is what the commuter does at all three intersections.
- One option is to stop at all three lights: $\omega_1 = \{sss\}$
- Another option is to continue through all three lights: $\omega_2 = \{ccc\}$
- The sample space Ω is all possible combinations of actions at these three lights:

$$\Omega = \{ccc, ccs, css, csc, sss, ssc, scc, scs\}$$

- With 2 possible actions at 3 different lights, $|\Omega| = 2^3 = 8$

Example B: Print Queue

The number of jobs in a print queue of a computer may be modeled as random (i.e. we can develop a probability model). What is the sample space?

- Not necessarily as straightforward...
- First, we are counting 'jobs in a queue', so it makes no sense for ω_i to be negative or non-integers
- Therefore, our sample space is $\Omega = \{0, 1, 2, 3, \ldots\}$
- Is this realistic?
  - Most computers probably cannot handle an infinite number of printing jobs...
  - So we may define our sample space to instead be $\Omega = \{0, 1, 2, \ldots, N\}$ for some upper bound N.

Subsets of Ω

- Often we are interested in computing the probability of a particular subset of Ω.
- We refer to this subset of Ω as an **event**.
- In Example A, we may be interested in the event that our commuter stops at the first light.
  - This event, let's call it A, is the set of all outcomes in Ω whose first digit is s:

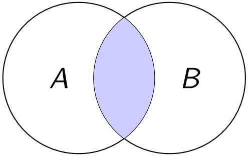
$$A = \{sss, ssc, scc, scs\} \subset \Omega$$

- We can define multiple events from the same sample space of an experiment
  - e.g. let B denote the event that our commuter stopped at exactly 2 lights: $B = \{ssc, scs, css\}$

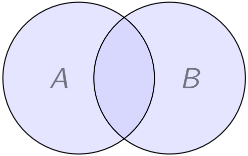
Algebra of Set Theory

All of the traditional algebra in set theory carries over to probability theory.

- Event C is the **union** of A and B when either A or B or both occur: $C = A \cup B$

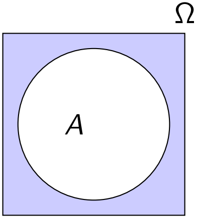


- Event C is the **intersection** of A and B when both A and B occur: $C = A \cap B$

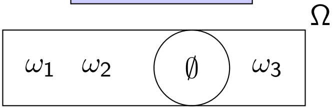


Algebra of Set Theory

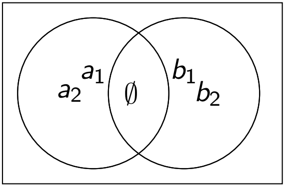
- The **complement** of an event, A^c, is the event that A does not occur and thus consists of elements in Ω that are not in A.



- The **empty set** is a set that has no elements, usually denoted by $\emptyset$.



- Two events are **disjoint** if their intersection is empty: $A \cap B = \emptyset \Rightarrow A, B$ are disjoint



Example: Commuter

If A is the event that the commuter stops at the first light, $A = \{sss, ssc, scc, scs\}$, and B is the event that the commuter stops at the third light, $B = \{sss, scs, ccs, css\}$, then list the elements of the following events:

- stops at first or third light, $C = A \cup B$:
  $C = \{sss, ssc, scc, scs, ccs, css\}$
- stops at first or third light, $C = A \cap B$: $C = \{sss, scs\}$
- doesn't stop at first light, $C = A^c$: $C = \{css, csc, ccc, ccs\}$
- stops at first light and continues through all three lights:
  $C = \emptyset$

Exercise - Give it a try!

$A = \{FF\}$ $B = \{MM\}$ $C = \{MF, FM, MM\}$

A family has 2 children of different ages and we are interested in the sex (F or M) of these children. Let a pair such as FM denote the older child is female and the younger is male. If A denotes no make children, B denotes two males, and C denotes at least one male child, list the elements of the following:

- $A \cap B$: $\emptyset$
- $A \cup B$: $\{FF, MM\}$
- $A \cap C$: $\emptyset$
- $A \cup C$: $\{MM, FF, MF, FM\}$
- $B \cap C$: $\{MM\}$
- $B \cup C$: C
- $C \cap B^c$: $\{MF, FM\}$

Laws of Set Theory

There are 3 laws of set theory that may come in handy in proofs of probability theory:

1. **Commutative Laws**: the order of events in a union or intersection don't matter
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

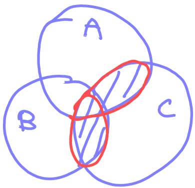
2. **Associative Laws**: an extension of the commutative laws
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

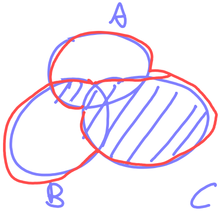
Laws of Set Theory

3. **Distributive Laws**: can distribute an event kind of like multiplication of a sum

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Let's use a Venn diagram to see why this works:





Outline

Introduction to Probability (Chapter 1.1-1.3)
- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)
- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Probability Measures

- Up to this point, we haven't really done anything with *probability*.
- We've talked a lot about defining elements of our experiment, and how we refer to outcomes that we are interested in.
- We've also discussed how to use set theory to isolate exactly the subset of events we want.
- But how do we go from *sets* to calculating *probabilities of events*?

Probability – Just a function...

- We need to go from elements in Ω, which can be defined on any space, to a single number, which we call the probability (a real number).
- This can be done by using a function, called a **measure**, which maps one set onto another set.
- Specifically, we say a function P that maps subsets of Ω to the real numbers $\mathbb{R}$ is a **probability measure** when the following axioms hold:
  1. $P(\Omega) = 1$.
  2. If $A \subset \Omega$, then $P(A) \geq 0$.
  3. If $A_1, A_2, \ldots, A_n, \ldots$ are mutually disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The Axioms of Probability

- The axioms themselves are what determine whether or not a specific function is a probability measure.
- So what are they saying?
  1. If Ω is all possible outcomes, then $P(\Omega) = 1$, i.e. probabilities have an upper bound.
  2. This just says probabilities are non-negative, i.e. have a lower bound.
  3. Probability measures have an additive property, namely, if events have no outcome in common then you can simply add the probabilities of the individual events.
- The axioms are not proved since they <u>define</u> what a probability is $\rightarrow$ but they can be used to prove other properties!

Properties of Probability Measures

- These properties are extensions of the set theory shown earlier:

Property A - Complements

If A is an event in Ω, then $P(A^c) = 1 - P(A)$.

Proof.

We know from axiom 2 that $P(\Omega) = 1$. Therefore we can write

$$\begin{aligned} 1 &= P(\Omega) && \text{(axiom 2)} \\ &= P(A \cup A^c) && \text{(definition of } A^c\text{)} \\ &= P(A) + P(A^c) && \text{(axiom 3)} \\ &\Rightarrow P(A^c) = 1 - P(A) \end{aligned}$$

$\square$

Properties of Probability Measures

Property B - Empty set

For an empty set $\emptyset$, $P(\emptyset) = 0$.

Proof

Let A be an event

$P(A \cup \emptyset) = P(A)$

$P(A \cap \emptyset) = P(A)$

$P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$

$P(\emptyset) = 0$

$A_1 = \Omega \quad A_2, A_3 = \emptyset$

so $A_1 \cap A_2 = \emptyset$ We can apply axiom 3

so $1 \overset{(a_1)}{=} P(\Omega) = P(A_1 \cup A_2 \cdots) \overset{def.}{=} P(A_1) + \cdots$

$\overset{(a3)}{=} PA + P(A_2 \cup A_3) \overset{(a_3)}{=} P(A) + P(\emptyset)$

def.

$= 1 + P(\emptyset)$ (by axiom 1)

$\Rightarrow P(\emptyset) = 0$

$\square$

Properties of Probability Measures

Property C - Subsets

If A and B are events in Ω and $A \subset B$, then $P(A) \leq P(B)$.

Proof

$A \subset B \Rightarrow P(A^B) \geq 0$

$P(A) \geq 0 \quad P(B) \geq 0$

$P(A) + P(A^B) = P(B)$

$P(B) - P(A) = P(A^B) \geq 0$

$\square$

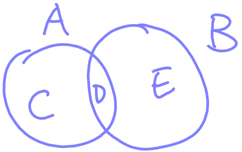
Properties of Probability Measures

Property D - Addition Law

If A and B are events in Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof



$$P(A \cup B) = P(C) + P(D) + P(E)$$

$$= P(C) + P(D) + P(D) + P(E) - P(D)$$

$$= P(A) + P(B) - P(A \cap B)$$

$\square$

Example: Tossing a coin

Suppose a fair coin is thrown twice. Let A denote the event of heads on the first toss, and let B denote the event of heads on the second toss. What is the probability that heads comes up on the first or the second toss?

- The sample space is $\Omega = \{hh, ht, th, tt\}$.
- Fair coin means events are equally likely, i.e. $P(\omega_i) = 0.25$.
- Event C: Heads on first <u>or</u> second toss $\Rightarrow$ union of events.
- Is C a disjoint union, i.e. can both A and B happen?
- Use Property D - Additive Law:

$$\begin{aligned} P(C) &= P(A) + P(B) - P(A \cap B) \\ &= P(\{hh, ht\}) + P(\{hh, th\}) - P(\{hh\}) \\ &= 0.5 + 0.5 - 0.25 = 0.75 \end{aligned}$$

Exercise - Give it a try!

Out of donors at a blood clinic, 1 in 3 had O^+ blood, 1 in 15 have O^-, 1 in 3 have A^+ and 1 in 16 have A^-. The name of a blood donor was selected from the records. What is the probability that the donor selected has

A. type O^+ blood? $\frac{1}{3}$ #

B. type O blood? $\frac{6}{15} = \frac{2}{5}$ #

C. type A blood? $\frac{1}{3} + \frac{1}{16} = \frac{3+16}{48} = \frac{19}{48}$ #

D. neither type A nor type O blood?

$1 - \frac{2}{5} - \frac{19}{48} = 1 - \frac{96+95}{240} = \frac{181}{240}$ #

Outline

Introduction to Probability (Chapter 1.1-1.3)

- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)

- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Computing Probabilities: The Sample-Point Method

- Probabilities are easy to compute for finite sample spaces, where $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ and $P(\omega_i) = p_i$.
- To compute the probability of event A, simply add the p_i corresponding to the ω_i in A.
- Alternatively, when the ω_i are all equiprobable, and event A can occur in any of n mutually exclusive ways, then

$$P(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of outcomes}} = \frac{n}{N}$$

- This only works if all outcomes are equally likely to occur.

Example: Tossing a coin

Recall that we tossed a fair coin twice and recorded the results. Our sample space was

$$\Omega = \{hh, ht, th, tt\}.$$

Let A be the event that at least one head is thrown. Then $A = \{hh, ht, th\}$.

- We can use both ways to calculate this probability:
  1. Each $\omega_i \in \Omega$ has probability 0.25 of occurring, so we add the probabilities of the sample points in A:

     $$P(\{hh\}) + P(\{ht\}) + P(\{th\}) = 0.25 + 0.25 + 0.25 = 0.75$$

  2. We have $n = 3$ ways to get "at least one head", and a total of $N = 4$ outcomes in Ω, so

     $$P(A) = {}^{n}/_{N} = {}^{3}/_{4} = 0.75$$

Example: Tossing a coin

- Note that we could run the same experiment, but if we now want to record the number of heads as our outcome (rather than the results of the tosses), we would have

$$\Omega = \{0, 1, 2\}.$$

- In this case, the ω_i are **not** equiprobable:

ω_i	0	1	2
$P(\omega_i)$	0.25	0.5	0.25

- We would not be able to use $P(A) = {}^{n}/_{N}$ (but we could still sum the $P(\omega_i)$ for $\omega_i \in A$).
- So how you define the sample space matters!

Example: Simpson's Paradox - Game 1

A black urn contains 5 red and 6 green balls, and a white urn contains 3 red and 4 green balls. You choose an urn and then choose a ball at random from that urn. If it's red, you win! Which urn should you choose?

- We have two possible sample spaces: Ω_{black_1} and Ω_{white_1}
- Within each urn, we have equal chances of picking any ball.
- If event A is that we pick a red ball, then
  - from Ω_{black_1}: $n = 5$ and $N = 11$ so $P(A) = 5/11 = 0.455$.
  - from Ω_{white_1}: $n = 3$ and $N = 7$ so $P(A) = 3/7 = 0.429$.
- So you should pick from the black urn to increase your chances of winning

Example: Simpson's Paradox - Game 2

Now, for game 2, we have a second black urn with 6 red and 3 green balls, and a second white urn with 9 red and 5 green balls. Which of these new urns should I choose?

- Now we have two new sample spaces: Ω_{black_2} and Ω_{white_2}
- We still have equiprobability within each urn.
- The probability of a red ball is:
  - from Ω_{black_2}: $n = 6$ and $N = 9$ so $P(A) = 6/9 = 0.667$.
  - from Ω_{white_2}: $n = 9$and $N = 5$ so $P(A) = 9/14 = 0.643$.
- You should still pick from the black urn.

Example: Simpson's Paradox - Game 3

In the last game, we combine the contents of two black urns into one, and combine the content of the white urns into one. Now which urn should you choose from?

- Your intuition might tell you that, since both black urns in the previous games had the higher chance of drawing a red ball, you should still choose from the black urn.
- BUT... The sample spaces you are working with have now changed!
  - $\Omega_{\text{black}_{1+2}}$ has 11 red and 9 green
  - $\Omega_{\text{white}_{1+2}}$ has 12 red and 9 green
- So the probability of drawing a red ball is now
  - from $\Omega_{\text{black}_{1+2}}$: $n = 11$ and $N = 20$ so $P(A) = {}^{11}/_{20} = 0.55$.
  - from $\Omega_{\text{white}_{1+2}}$: $n = 12$and $N = 21$ so $P(A) = {}^{12}/_{21} = 0.571$.
- You should pick the white urn!

Outline

Introduction to Probability (Chapter 1.1-1.3)
- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)
- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Computing Probabilities: The Multiplication Principle

- The multiplication principle isn't itself used to compute probabilities
- It is useful in the determination of the total number of outcomes from multiple experiments.
- This is often necessary for the denominator (N) of some counting problems.

Multiplication Principle

If one experiment has m outcomes and another experiment has n outcomes, then there are mn possible outcomes for the two experiments.

Computing Probabilities: The Multiplication Principle

Proof.

- Denote outcome from first experiment by $a_1, \ldots, a_m$, and outcomes from second experiment by $b_1, \ldots, b_n$.
- The outcomes for both experiments are the ordered pairs (a_i, b_j).
- These pairs can be considered entries of an $m \times n$ rectangular array/matrix:

$$\begin{bmatrix} (a_1, b_1) & (a_1, b_2) & \cdots & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & \cdots & (a_2, b_n) \\ \vdots & \vdots & \ddots & \vdots \\ (a_m, b_1) & (a_m, b_2) & \cdots & (a_m, b_n) \end{bmatrix}$$

- There are mn entries in this array.

$\square$

Examples

1. Playing cards have 13 face values and 4 suits. How many face-value/suit combinations are there?
   - Using the multiplication principle, there are $13 \times 4 = 52$ combinations.

2. A class has 12 boys and 18 girls. The teacher selects 1 boy and 1 girls to act as representatives to the student government. How many ways can this be done?
   - Again using the multiplication principle, there are $12 \times 18 = 216$ different ways.

Extended Multiplication Principle

Extended Multiplication Principle

If there are p experiments and the first has n_1 possible outcomes, the second n_2, $\ldots$, and the pth has n_p possible outcomes, then there are a total of $n_1 \times n_2 \times \cdots \times n_p$ possible outcomes for the p experiments.

Proof

This can be proved from the multiplication principle by induction...

Previous theorem holds for $p=2$. Assume T for $p=q\geq 2$ i.e. $\exists\, n_1 \times \cdots n_q$ possible outcomes for first q. show holds for $q+1=p$ we have $\underbrace{(n_1 \times \cdots)}_{q} \times n_{q+1}$

$= n_p$

$\square$

Example: DNA

A DNA molecule is a sequence of 4 types of nucleotides (A, G, C, T). For a molecule 1 million (10^6) units long, how many different possible sequences are there?

- Each unit has 4 possible nucleotide options, and there are 10^6 units, so we have 4^{10^6} possible sequences.

An amino acid is coded for by a sequence of 3 nucleotides. How many different codes are there?

- Each spot in the sequence has 4 possible nucleotides types, and there are 3 spots, so there are 4^3 different codes.

Outline

Introduction to Probability (Chapter 1.1-1.3)
Motivation
Sample Spaces and Set Notation
Probability Measures

Counting Method of Probability (Chapter 1.4)
Sample-Point Method
Multiplication Principle
Permutations
Combinations

Computing Probabilities: Permutations

- A **permutation** is an ordered arrangement of objects.
- Often we will have a set of elements, $C = \{c_1, c_2, \ldots, c_n\}$, and we will be choosing a subset of them, of size r, and will list them in some order
  - e.g. think of balls being drawn from an urn
- How many ways can this be done?
- It depends on how we are sampling the elements:
  - **Sampling without replacement** means we are not allowing the same element to be chosen more than once
  - **Sampling with replacement** means the chosen element is returned and may be chosen again
- Either way, we will end up with r elements, listed in the order they were chosen.

Sampling with Replacement

- The easier sampling scheme of the two.
- If we have n elements (e.g. balls in the urn), and we want to choose r of them, how do we proceed?
  1. Choose first ball: this can be done in n ways
  2. Return the ball to the urn: the urn still has n balls in it
  3. Choose the second ball: this can still be done in n ways
  4. Return it and continue until r balls have been chosen.
- We can use the extended multiplication principle to count the number of different ordered samples possible from these n elements:

$$n \times n \times \cdots \times n = n^r$$

Sampling without Replacement

- The procedure for choosing r items out of n is similar, but now we no longer replace the items after selecting them:
  1. Choose the first ball: this can be done n ways
  2. Choose the second ball: this can now be done $n-1$ ways
  3. Choose the third ball: this can be done $n-2$ way
  4. Continue until you have r balls.
- From this process, we have two important results:
  1. For a set of size n and a sample of size r, there are
  $$n(n-1)(n-2)\cdots(n-r-1)$$
  different ordered samples **without** replacement.
  2. The total number of orderings of n elements is
  $$n(n-1)(n-2)\cdots(1) = n!$$

Exercises - Give them a try!

A. How many ways can five children be lined up?

$$5! = 5 \times 4 \times 3 \times 2 \times 1$$
$$= 120$$

B. Suppose that from ten children, five are to be chosen and lined up. How many different lines are possible?

$$^{10}C_5 \times 5! = 30240$$

Exercises (cont.)

C. In some states, license plates have six characters: 3 letters followed by 3 numbers. How many distinct such plates are possible?

$$26^3 \times 10^3 = 17576000$$

D. If all sequences of six characters are equally likely, what is the probability that the license plate for a new car will contain no duplicate letters or numbers?

$${}^{26}P_3 \times {}^{10}P_3 = 15600 \times 720$$
$$= 11232000$$

Outline

Introduction to Probability (Chapter 1.1-1.3)

- Motivation
- Sample Spaces and Set Notation
- Probability Measures

Counting Method of Probability (Chapter 1.4)

- Sample-Point Method
- Multiplication Principle
- Permutations
- Combinations

Computing Probabilities: Combinations

- With permutations, the order in which elements are selected matters, but what if we don't want it to.
- Now we are interested in: how many different samples are possible if we select r objects from a set of n **without replacement and disregarding order they were selected**?
- We know from the multiplication principle that

$$\begin{aligned}
\begin{array}{c}\#\text{ ordered samples}\\ \text{(permutation)}\end{array} &= \begin{array}{c}\#\text{ unordered}\\ \text{samples}\end{array} \times \begin{array}{c}\#\text{ ways to}\\ \text{order them}\end{array}\\
n(n-1)\cdots(n-r+1) &= \begin{array}{c}\#\text{ unordered}\\ \text{samples}\end{array} \times r!\\
\Rightarrow \begin{array}{c}\#\text{ unordered}\\ \text{samples}\end{array} &= \frac{n(n-1)\cdots(n-r+1)}{r!}
\end{aligned}$$

Combinations

- Therefore the number of unordered samples of size r from a set of size n is found by

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

 which is called a **combination**

- It is based off the following result:

Binomial Coefficient

The numbers $\binom{n}{k}$, called the binomial coefficient, occur in the expansion

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

In particular, $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Example: Lottery Numbers

A lottery player can win the jackpot by choosing the 6 numbers from 1 to 49 that were subsequently chosen at random by lottery officials. How many possible combinations of numbers are there?

- There are 49 numbers to choose from, and we are not allowing duplicate numbers, so we get $\binom{49}{6} = 13,983,816$ possibilities.

Since jackpots were being won too often, the rules were changed so that players must select 6 numbers out of the numbers 1 to 53. How many combinations are there now?

- Now there are 53 numbers to choose from, so we have $\binom{53}{6} = 22,957,480$ possibilities.

Exercise - Give it a try!

In quality control, only a fraction fo the output of a manufacturing process is sampled and examined. Suppose that n items are in a lot and a sample of r are taken. There are $\binom{n}{r}$ such samples. Now suppose that the lot contains k defective items. What is the probability that the sample contains exactly m defective items? (*Hint: use the multiplication principle.*)

$$\frac{\binom{n-k}{r-m} \times \binom{k}{m}}{\binom{n}{r}}$$

!

Multinomial Coefficient

- Suppose we are interested in knowing how many ways there are to group n objects into r classes of sizes $n_1, \ldots, n_r$.
- We can use an extension of the binomial coefficient:

Multinomial Coefficient

The number of ways that n objects can be grouped into r classes with n_i in the ith class, $i = 1, \ldots, r$, $\sum_{i=1}^{r} n_i = n$ is

$$\binom{n}{n_1 n_2 \cdots n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Multinomial Coefficient

Proof of Multinomial Coefficient

$$\binom{n}{n_1}\binom{n}{n_2}\cdots\binom{n}{n_k} = \frac{n!}{n_1!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}$$

$$\cdots \frac{(n-n_1\cdots-n_{k-1})!}{n_k!(n-n_1\cdots-n_k)!} = \frac{n!}{n_1!n_2!\cdots n_k!0!}$$

$$= \frac{n!}{n_1!\cdots n_k!}$$

□

Examples

A. A committee of 7 members is to be divided into subcommittees of size three, two, and two. How many ways can this be done?

- Straight application of multinomial coefficient:

$$\binom{7}{322} = \frac{7!}{3!2!2!} = 210$$

B. In how many ways can the set of nucleotides $\{A, A, G, G, G, G, C, C, C\}$ be arranged in a sequence of nine letters?

- Note that this is the same as asking the number of ways that 9 positions can be divided into groups of size two, four and three...

$$\binom{9}{243} = \frac{9!}{2!4!3!} = 1260$$

Exercise - Give it a try!

A. In how many ways can $n = 2m$ people be paired **and** assigned to m courts for the first round of a tennis tournament?

$$\frac{(2m)!}{(2!)^m}$$

$n = 2m$

$n_i = 2$

B. In how many ways can $n = 2m$ people be paired **without** assigning the pairs to courts?

$$\frac{(2m!)}{(2!)^m \times m!}$$