Strong and weak induction

Weak Induction

- $\forall n \in \mathbb{N}. P(n)$
  For all $n \in \mathbb{N}$, Let $P: \mathbb{N} \to \{T, F\}$ be a predicate
  Let $n = 0$
  ...
  $P(0)$
  Let $n \in \mathbb{N}$ be arbitrary
  Assume $P(n)$
  ...
  $P(n+1)$
  $P(n)$ IMPLIES $P(n+1)$ direct proof
  $\forall n \in \mathbb{N}. P(n)\text{IMPLIES } P(n+1)$ generalization
  $\forall n \in \mathbb{N}. P(n)$ induction

- $\forall n \in \mathbb{N}. \left(n \geq b \text{ IMPLIES } P(n)\right)$
  $P(b)$
  $\forall n \in \mathbb{N}. \left((n > b)\text{AND } P(n)\right)\text{IMPLIES } P(n+1)$

- $\forall n \in \mathbb{N}. \text{even}(n)$ IMPLIES $P(n)$
  Let $Q(k) = P(2k)$
  Prove $Q(k)$
  Or
  $P(0)$
  $\forall n \in \mathbb{N}. \left(\text{even}(n)\text{AND } P(n)\right)\text{IMPLIES } P(n+2)$

Strong Induction

- $\forall n \in \mathbb{N}. \forall m \in \mathbb{N}. \left((m < n)\text{IMPLIES } P(m)\right)\text{IMPLIES } P(n)$
  $\forall n \in \mathbb{N}. P(n)$ by strong induction
- $\forall n \in \mathbb{N}. P(i)$
  Let $i \in \mathbb{N}$ be arbitrary
  Assume $\forall j \in \mathbb{N}. \left(j < i \text{ IMPLIES } P(j)\right)$
  ...various cases...
  $P(i)$
  $\left(\forall j \in \mathbb{N}. j < i \text{ IMPLIES } P(j)\right)$ IMPLIES $P(i)$
  $\forall i \in \mathbb{N}. \forall j \in \mathbb{N}. \left((j < i)\text{IMPLIES } P(j)\right)$ IMPLIES $P(i)$
  $\forall i \in \mathbb{N}. P(i)$

- Prove $\forall m \in \mathbb{N}. \forall n \in \mathbb{N}. P(m, n)$
  Method 1 Let $Q(m) = \forall n \in \mathbb{N}. P(m, n)$.prove P is equivalent to prove Q
  Method 2 Let $(m, n) \in \mathbb{N} \times \mathbb{N}$ be arbitrary
  Assume $\forall(i, j) \in \mathbb{N} \times \mathbb{N}. \left(i \leq m \text{ AND } j \leq n \text{ AND } (i < m \text{ OR } j < n)\right)$ IMPLIES $P(i, j)$
  ...
  $$\left(\begin{array}{c}\forall(i, j) \in \mathbb{N} \times \mathbb{N}. \left(i \leq m \text{ AND } j \leq n \text{ AND } (i < m \text{ OR } j < n)\right) \\ \text{IMPLIES } P(i, j)\end{array}\right)$$
  IMPLIES $P(m, n)$

Thrm every integer greater than 1 is a product of primes
Proof For $n \in \mathbb{N}$, let $P(n)$="n is a product of prime"
Let n be arbitrary
Suppose $n > 1$ AND $\left(\forall i \in \mathbb{N}. 0 < i < n \text{ IMPLIES } P(i)\right)$

If n is prime, then n is the product of 1 and n by definition of prime numbers,
Otherwise, by definition of prime, $\exists k \in \mathbb{N}. \exists m \in \mathbb{N}. m > 1$ AND $k > 1$ AND $mk = n$
Since $k < n$ AND $m < n$, by induction hypothesis, k and m are both products of primes, hence $n = km$ is a product of primes

$P(n)$

$\forall i \in \mathbb{N}. (1 < i < n)$IMPLIES $P(n)$ direct proof

$\forall n \in \mathbb{N}. \forall i \in \mathbb{N}. (1 < i < n)$IMPLIES $P(n)$ generalization

$\forall n \in \mathbb{N}. \big(n > 1 \text{ IMPLIES } P(n)\big)$ strong induction

Binary relation induction

- Let $Q(m) = \forall n \in \mathbb{N}. P(m, n)$, then to $Q(m)$
  Strong induction within strong induction
- Let (m, n) be arbitrary
  Assume $\forall(i, j) \in \mathbb{N} \times \mathbb{N}. \Big(i \leq m \text{ AND } j \leq n \text{ AND } \big(i < m \text{ OR } j < n\big)\Big) \text{ IMPLIES } P\big(i, j\big)$
  …
  $\bigg(\forall\big(i, j\big) \in \mathbb{N} \times \mathbb{N}. \Big(i \leq m \text{ AND } j \leq n \text{ AND } \big(i < m \text{ OR } j < n\big)\Big) \text{ IMPLIES } P\big(i, j\big)\bigg) \text{ IMPLIES } P(m, n)$

  $P(m, n)$

Thrm consider any square chessboard whose size is $2^n \times 2^n$, if any 1 square is removed, the result shape can be tiled using only 3 square L-shaped tiles.

Proof For all $n \in \mathbb{N}$, let C_n denote the set of all $2^n \times 2^n$ chessboard with 1 square removed.
Let $P(n) = \forall c \in C_n$, c can be tiled using only L-tiles (3 square L-shaped tiles)
Let $n = 0, c = 2^0 \times 2^0 - 1$ can be tiled by 0 L-tiles
$P(0)$
Let $n \in \mathbb{N}$ be arbitrary
Suppose $P(n)$
Let $c \in C_{n+1}$ be arbitrary
Divide c into 4 sub chessboard ($2^n \times 2^n$), one of the chessboard can be tiled using L-tiles (by IH), then remove the center-most square from each sub chessboard, which can form a L-tile, the other 3 sub chessboard with one square removed can be tiled using only L-tiles (by IH). Hence, the whole shape is tiled by L-tiles
P(n+1)
$\forall n \in \mathbb{N}. P(n)$IMPLIES $P(n + 1)$
$\forall n \in \mathbb{N}. P(n)$ weak induction

Thrm all square chessboards whose size is $2^n \times 2^n$ and with one 2×2 square removed from the center can be tiled using L-tiles
Hard to prove inductive process
Sometimes induction is used to prove some more abstract proofs then prove specific theorems

Thrm $\forall \in \mathbb{N}. 2n + 1 \leq 2^n$
The statement is false since $P(1)$, $P(2)$ is false

Thrm $\forall n \in \mathbb{N}. n \geq 3$ IMPLIES $2n + 1 \leq 2^n$
Proof For all $n \in \mathbb{N}$, Let $P(n) = 2n + 1 \leq 2^n$
Let $n = 3, 2(3) + 1 = 7 < 8 = 2^3$
$P(3)$
Let $n \in \mathbb{N}, n \geq 3$ be arbitrary, assume $P(n)$
$2(n + 1) + 1 = 2n + 1 + 2 \leq 2^n + 2 \text{ (IH)} \leq 2^n + 2^n(2^3 = 8 > 2) \leq 2^{n+1}$
P(n) IMPLIES P(n+1)
$\forall n \in \mathbb{N}. P(n)$IMPLIES $P(n + 1)$

Thrm Define arithmetic mean $\frac{\left(\sum_{i=1}^{n} a_i^2\right)}{n}$, geometric mean $\left(\Pi_{i=1}^{n} a^i\right)^{\frac{1}{n}}$

$\forall n \in \mathbb{Z}^+, \forall a_1, \dots, a_n \in \mathbb{R}^+. \left(\Pi_{i=1}^{n} a^i\right)^{\frac{1}{n}} \leq \sum_{i=1}^{n} a^i$

Proof For all $n \in \mathbb{Z}^+$, let $P(n) = \forall a_1, \dots, a_n \in \mathbb{R}^+. \left(\Pi_{i=1}^{n} a^i\right)^{\frac{1}{n}} \leq \sum_{i=1}^{n} a^i$

Let $n = 2$

Let $a_1, a_2 \in \mathbb{R}^+$ be arbitrary

Then $0 \le (a_1 - a_2)^2 - a_1^2 - 2a_1a_2 + a_2^2$

$a_1^2 + a_2^2 \ge 2a_1a_2$

$\left(\frac{a_1 + a_2}{2}\right)^2 = \frac{a_1^2 + a_2^2 + 2a_1a_2}{4} > \frac{2a_1a_2 + 2a_1a_2}{4} = a_1a_2$

P(2)

Let $n \in \mathbb{N}, n \ge 2$, assume P(n)

Let $a_1, \dots a_n \in \mathbb{R}^+$ be arbitrary, let $b_i = a_i$ for $i = 1, \dots, n-1$, let $b_n = \frac{\sum_{i=1}^{n-1} a_i}{n-1}$

Be specialization of P(n), $\Pi_{i=1}^{n} b_i \le \left(\frac{\sum_{i=1}^{n} b_i}{n}\right)^n = \left(\frac{\sum_{i=1}^{n-1} a_i + b_n}{n}\right)^n = \left(\frac{(n-1)b_n + b_n}{n}\right)^n = b_n^n$

So $\Pi_{i=1}^{n-1} a_i = \Pi_{i=1}^{n-1} a_i = \frac{\Pi_{i=1}^{n} b_i}{b_n} \le \frac{b_n^n}{b_n} = b_n^{n-1} = \left(\frac{\sum_{i=1}^{n-1} a_i}{n-1}\right)^{n-1}$

P(n)IMPLIES P(n − 1)

Let $b_1 = \frac{\sum_{i=1}^{n} a_i}{n}$ and $b_2 = \frac{\sum_{i=n+1}^{2n} a_i}{n}$

By specialization of P(n) $\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n \ge \Pi_{i=1}^{n} a_i$ AND $\left(\frac{\sum_{i=n+1}^{2n} a_i}{n}\right)^n \ge \Pi_{i=n+1}^{2n} a_i$

By specialization of P(2) $b_1b_2 \le \left(\frac{b_1 + b_2}{2}\right)^2$

$\Pi_{i=1}^{2n} a_i \le \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^n \left(\frac{\sum_{i=n+1}^{2n} a_i}{n}\right)^n = b_1^n b_2^n \le \left(\frac{b_1 + b_2}{2}\right)^2 = \left(\frac{\sum_{i=1}^{n} a_i}{2n} + \frac{\sum_{i=n+1}^{2n} a_i}{2n}\right)^2 = \left(\frac{\sum_{i=1}^{2n} a_i}{2n}\right)^2$

P(n)IMPLIES P(2n)

$\forall n \in \mathbb{N}. P(n)$