Week 13: Tutorial Handout

- If $y^{(n)}(x) = \varphi(x, y(x), ..., y^{(n-1)}(x))$ and $\Gamma := (y(0), ..., y^{(n)}(0))$ we consider the vector valued function

$$F(x) := (f(x), ..., f_{n-1}(x)) \text{ with } F(0) = (y(0), ..., y^{(n-1)}(0)).$$

- Let the associated integral operator be

$$TF(x) := \Gamma + \int_0^x \Phi(t, F(t))\mathrm{d}t,$$

where $\Phi(t, F(t)) = (f(t), ..., f_{n-1}(t), \varphi(t, f(t), ..., f_{n-1}(t)))$. Then we are looking for a fixed point F s.t. $TF(x) = F(x)$.

ODEs and Fixed point

1. Convert the following DEs into a first-order vector valued F and write the fixed-point problem TF=F:
   - $y^{(3)} + y'' - x(y')^2 = e^x$, with $y(0) = 1, y'(0) = -1, y''(0) = 0$.
   - $y' = xy$ with $y(0) = 1$. Also solve this equation by separating variables.
2. Compute $f_2(x) = Tf_1(x)$ and show that the following TF mappings are contractions over $(C([0,1]), \|\cdot\|_\infty)$ (i.e. show $\|Tf - Tg\|_\infty \leq r\|f - g\|_\infty$ for some $r < 1$).
   - $Tf(x) := 1 + \int_0^x tf(t)$ starting from $f(0) = 1$.
   - $Tf(x) := 1 + \int_0^x \frac{1}{t^{1/2}} f(t)$ starting from $f(0) = 1$.

Global solution

3. For the following DEs show whether the corresponding maps $\Phi(t, f_0, ..., f_{n-1})$ are Lipschitz and thus Global Picard give unique solution:
   - the DE $y' = xy + 1$ starting from $y(0) = 0$ in [-1,1]
   - the DE $y'' + y + \sqrt{y^2 + (y')^2} = 0$ with $y(0) = y(0) = 1$. (Hint: use $\sqrt{a} - \sqrt{b} \leq \sqrt{a-b}$).