Recurrence

Example 1 Let $T: \mathbb{Z}^+ \to \mathbb{N} \coloneqq \begin{cases} 0 \mid n = 1(\text{initial condition}) \\ 4 + T(n-1) \mid n > 1(\text{self referential part}) \end{cases}$

Recurrence: inductively defined functions
Solving a recurrence: finding a non-recursive description

Methods:

0. Look it Up
1. Guess and verify
   a. Generate a table of values of function
   b. Look for a pattern and guess for a solution
   c. Prove the guess if correct using induction

   For **Example 1**

   Proof For $n \in \mathbb{Z}^+$, let $P(n) \coloneqq "T(n) = 4(n-1)"$

   Let $n = 1, 4(1-1) = 0 = T(0)$

   Let $n \in \mathbb{Z}^+$ be arbitrary, assume $P(n)$

   $4(n+1-1) = 4(n-1) + 4 = T(n) + 4 = T(n+1)$

   $\forall n \in \mathbb{Z}^+. P(n)$
2. Plug and Chug / repeated substitution and verify

n	T(n)
1	0
2	4
3	8
4	12

   a. Apply the recurrence into subproblems
   b. Simplify the result
   c. Repeat a, b until finding a pattern
   d. Verify the pattern
   e. Write the pattern using early terms with known values

Example For $n \in \mathbb{Z}^+. M(n) = \begin{cases} c \mid n = 1 \\ M\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + M\left(\left\lceil \frac{n}{2} \right\rceil\right) + dn \mid n > 1 \end{cases}, c, d \in \mathbb{N}$

When n is a power of 2

$$M(n) = \begin{cases} c \mid n = 1 \\ 2M\left(\frac{n}{2}\right) + dn \mid n > 1 \end{cases}$$

Let $n \in \mathbb{Z}^+$, n is a power of 2

$$\begin{aligned} M(n) &= 2M\left(\frac{n}{2}\right) + dn \\ &= 2(2M(n/4) + dn/2) + dn = 4M(n/4) + 3dn \\ &= 4\left(2M\left(\frac{n}{8}\right) + \frac{dn}{4}\right) + 3dn = 8M\left(\frac{n}{8}\right) + 4dn \\ &= \cdots \\ &= 2^i M\left(\frac{n}{2^i}\right) + idn \quad \text{Known } M(1) = c, \text{let } \frac{n}{2^i} = 1, i = \lg n \\ &= 2^{\lg n} c + \lg n \, dn \\ &= cn + dn \lg n \end{aligned}$$

Lemma 1 $\forall k \in \mathbb{N}. M(2^k) = c2^k + dk2^k$

Proof For all $k \in \mathbb{N}$, let $Q(k) \coloneqq "M(2^k) = c2^k + dk2^k"$

Let $k = 0, M(1) = c$

Let $k \in \mathbb{N}$, assume $Q(k)$

$c2^{k+1} + d(k+1)(2^{k+1}) = 2c2^k + 2dk2^k + 2d2^k = M(2^k) + d2^{k+1} = M(k+1)$

$\forall k \in \mathbb{N}. Q(k)$

Theorem $M(n) \subseteq Q(n \log n)$

Lemma 2 $\forall n \in \mathbb{Z}^+. \forall m \in \mathbb{Z}^+. m < n$ IMPLIES $M(m) \leq M(n)$

Proof For $n \in \mathbb{Z}^+$, let $R(n) := "\forall m \in \mathbb{Z}^+, m < n$ IMPLIES $M(m) \leq M(n)"$

Let $n \in \mathbb{Z}^+$, assume $\forall i \in \mathbb{Z}^+, 1 < i < n$ IMPLIES $R(i)$

Assume $\forall n' \in \mathbb{N}. 1 < n' < n$ IMPLIES $R(n')$

$R(1)$ vacuously true

$M(2) = 2c + 2d > c = M(1)$

$R(2)$

Consider $n > 2, 1 \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lceil \frac{n}{2} \right\rceil \leq n - 1 < n$, thus by induction hypothesis, $R\left(\left\lfloor \frac{n}{2} \right\rfloor\right), R\left(\left\lceil \frac{n}{2} \right\rceil\right)$

Let $m \in \mathbb{Z}^+$ be arbitrary and suppose $m < n$

Suppose $m = n - 1$

Suppose n is odd, $\left\lceil \frac{n-1}{2} \right\rceil < \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$

$M\left(\left\lceil \frac{n-1}{2} \right\rceil\right) \leq M\left(\left\lceil \frac{n}{2} \right\rceil\right)$ by specialization of IH, $M\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) = M\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$ by substitution

$M(m) = M(n-1) = M\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + d(n-1) \leq M\left(\left\lceil \frac{n}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + dn = M(n)$

Suppose n is even, $\left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil, \left\lfloor \frac{n-1}{2} \right\rfloor < \left\lfloor \frac{n}{2} \right\rfloor$

Similarly, $M(m) = M(n-1) = M\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + d(n-1)$

$\leq M\left(\left\lceil \frac{n}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + dn = M(n)$

$M(m) = M(n-1) = M\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + d(n-1) \leq M\left(\left\lceil \frac{n}{2} \right\rceil\right) + M\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + dn = M(n)$ i)

Suppose $m < n - 1$

$M(m) < M(n-1) \leq M(n)$ by specializaiton of IH and i)

$\forall n \in \mathbb{Z}^+. R(n)$ generalization and strong induction

Proof Let $n \in \mathbb{Z}^+$ be arbitrary, let $k \in \mathbb{N}, n \leq 2^k < 2n$

$M(n) \leq M(2^k) = c2^k + dk2^k$ by lemma 2, 1

$< c2n + d\lg(2n)\,2n = 2cn + 2dn(\lg n + 1) \in O(n \log n)$

Master Theorem

For $n \in \mathbb{Z}^+, T(n) = \begin{cases} c \mid n < B \\ a_1 T\left(\left\lceil \frac{n}{b_1} \right\rceil\right) + a_2 T\left(\left\lfloor \frac{n}{b_2} \right\rfloor\right) + dn^l \mid n > B \end{cases}, a_1, a_2, b \in \mathbb{N}, b_1, b_2 > 1, a_1 + a_2 \geq 1, c, d, l \in \mathbb{R}^+ \cup \{0\}$

$$T(n) \in \begin{cases} \Theta(n^l \lg n) \mid a = b^l \\ \Theta(n^l) \mid a < b^l \\ \Theta(n^{\log_b a}) \mid a > b^l \end{cases}$$

3. Transformation

k	H(k)
0	0
1	1
2	4
3	11
4	26

Example $G(n) = \begin{cases} 2G\left(\frac{n}{2}\right) + \lg n \mid n > 1 \\ 0 \mid n = 1 \end{cases}$

Let $n = 2^k, H(k) = G(2^k) = \begin{cases} 2G(2^{k-1}) + k \mid k > 0 \\ 0 \mid k = 0 \end{cases}$

$H(k) = \begin{cases} 2H(k-1) + k \mid k > 0 \\ 0 \mid k = 0 \end{cases}$

$2H(k-1) + k$
$= 2\big(2H(k-2) + (k-1)\big) + k = 4H(k-2) + 3k - 2$
$= 4\big(2H(k-3) + (k-2)\big) + 3k - 2 = 8H(k-3) + 7k - 10$
$= 8\big(2H(k-4) + (k-3)\big) + 7k - 10 = 16H(k-4) + 15k - 34$
$= 16\big(2H(k-5) + (k-4)\big) + 15k - 34 = 32H(k-5) + 31k - 98$
$= \cdots$
$= 2^i H(k-i) + (2^i - 1)k - \sum_{n=1}^{i} n2^n = 2^{k+1} - k - 2$
$G(n) = 2n - \lg n - 2$ when n is a power of 2

Domain Transformation
$A(n) = \begin{cases} 3A(n-1)^2 \mid n > 0 \\ 1 \mid n = 0 \end{cases}$
Let $B(n) = \lg A(n) = \begin{cases} \lg 3 + 2\lg\big(A(n-1)\big) = \lg\big(3 + 2B(n-1)\big) \mid n > 0 \\ 0 \mid n = 0 \end{cases}$
By plug and chug, $B(n) = (2^n - 1)\lg 3$, $A(n) = 2^{B(n)} = 2^{(2^n-1)\lg 3} = 3^{2^n - 1}$