Outline: Week 4 TR

Uniform continuity

1. uniform continuity definition also write the continuity definition and compare them.

2. Examples: a)any differentiable function (using MVT) with bounded derivative or over compact set,b)Holder functions eg. x^p for $0 < p \leq 1$ c) $\frac{1}{x}$ is uniformly continuous at $(b,1)$ for $b > 0$ because $\left|\frac{1}{x} - \frac{1}{y}\right| \leq \frac{1}{b^2}|x-y|$. d)$f(x) := |x|$ on [-1,1] is u.c. but not differentiable. e)$f(x) := \sum a^n cos(b^n \pi x)$ with $ab > 1 + \frac{3}{2}\pi$ is not differentiable but it is u.c. (eg. $f(x) := \sum \frac{1}{2^n} cos(2^n \pi x)$ has period $d = 2$)

3. If it is not uniformly continuous, use the sequential criterion: A function $f : A \to \mathbb{R}$ is not uniformly continuous if there exists $\varepsilon_0 > 0$ and sequences x_n, y_n s.t.

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| > \varepsilon_0 > 0.$$

   Just the negation of the uniform continuity statement

4. Examples: a)$\frac{1}{x}$ in (0,1]; the sequences are $x_n = 0$ and $y_n = \frac{1}{n}$. b)

5. Heine–Cantor theorem (Abott theorem 4.4.8): if $f : M \to \mathbb{R}$ is a continuous on M compact, then f is uniformly continuous.

   - Take sequences $x_n - y_n \to 0$ but $|f(x_n) - f(y_n)| > \varepsilon_0 > 0$
   - By BW, they have converging subsequence with $x_{n_k} \to x$ and thus $y_{n_k} \to x$.
   - By continuity $f(x_{n_k}) - f(y_{n_k}) \to x - x = 0$, which is a contradiction.

6. Open cover proof: let $\Delta(x) := \{\delta > 0 : |x-y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{2}\}$ then $C := \{B_{\frac{\delta}{2}}(x) : x \in K, \delta \in \Delta(x)\}$ is an open cover. Then $\delta := \frac{1}{2}\min(\delta_k)$. Take $|x-y| \leq \delta$ then $|x_k - y| \leq \delta + \frac{\delta_k}{2} \leq \delta_k$. Therefore,

$$|f(x) - f(y)| \leq |f(x) - f(x_k)| + |f(x_k) - f(y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

7. 5.5 I: Let $f(0) := \lim\limits_{x \to 0} f(x)$. For $\varepsilon > 0$ we have $|x| \leq \delta \Rightarrow |f(0) - f(x)| \leq \varepsilon$. In $[\delta, 1]$, the Heine-Cantor gives $\widetilde{\delta}$. So take $\delta_{global} := \min(\delta, \widetilde{\delta})$.

8. IVT: Consider $A_z := \{x \in [a,b] : f(x) < z\}$. Take $c - \frac{1}{n} \leq a_n \leq c$. So $f(c) = \lim_{n \to \infty} f(a_n) \leq z < f(b)$. So $c \neq b$ otherwise $f(c) = z = f(b)$. So take $b_n \geq c$ going to c, then $z \leq f(b_n) \to f(c) \geq z$.

9. A set S is connected if you cannot write it as $S = A \sqcup B$ for open A,B and $A \cap B = \varnothing$ eg. $S = (0,1) \cup (2,3)$. By IVT, continuous image of connected is also connected.