Linear combination: $let\ v1, v2, \dots, vk\ \in R^n, c1, c2, \dots, ck \in R.$
$the\ linear\ combinaiton\ is\ c1v1 + c2v2 + \cdots + ckvk$

$$Ex: \quad 2\begin{pmatrix}1\\-1\\2\end{pmatrix} - 3\begin{pmatrix}4\\-5\\-7\end{pmatrix} = \begin{pmatrix}2-12\\-2+15\\4+21\end{pmatrix} = \begin{pmatrix}-10\\13\\25\end{pmatrix}$$

Span: $let\ v1, v2, \dots, vk\ \in R^n,$
$the\ set\ of\ all\ linear\ combination\ of\ v1, v2, \dots, vk\ is\ called\ the\ span\ of\ v1, v2, \dots, vk$

$span\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\} = \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ origin

$span\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\right\} = \left\{c\begin{pmatrix}1\\2\\1\end{pmatrix} | c \in R\right\}$ a line through origin containing the direction vector (1 2 1)

$span\ \left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}2\\0\\7\end{pmatrix}\right\} = \left\{c1\begin{pmatrix}1\\2\\1\end{pmatrix} + c2\begin{pmatrix}2\\0\\7\end{pmatrix}\right\} |\ c1, c2 \in R$ a plain through the origin containing (1 2 1) and (2 0 7)

Dot product: $x.y = x1y1 + x2y2 + \cdots + xnyn$

Norm in R^n/ length in R^2 or R^3: $\|x\| = \sqrt{x.x} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Distance between x, y: $d(x-y) = \|x-y\|$

Angles between vectors:

$$\begin{aligned} In\ R^2 \cos\theta &= \cos(A-B) \\ &= cosAcosB - sinAsinB \\ &= \frac{y1}{\|y\|}\frac{x1}{\|x\|} + \frac{y2}{\|y\|}\frac{x2}{\|x\|} \end{aligned}$$

$$Expand\ to\ R^n, defined\ \cos\theta = \frac{x.y}{\|x\|\|y\|}$$

$$-1 \le \cos\theta \le 1\ \rightarrow -1 \le \frac{x.y}{\|x\|\|y\|} \le 1$$

Thrm: Cauchy Schwarz equality: $\forall x, y \in R^n, |x.y| \le \|x\|\|y\|$

Pf: $\forall C \in R. let\ C = \frac{x.y}{\|y\|^2}$

$a)$
$$\begin{aligned} \|x - Cy\|^2 &= (x - Cy).(x - Cy) \\ &= x.x - 2C\ x.y + C^2 y.y \\ &= \|x\|^2 - 2C(x.y) + C^2\|y\|^2 \end{aligned}$$

$b)\ 0 \le \|x - Cy\|^2$

$Therefore$: $\|x\|^2 - 2C(x.y) + C^2\|y\|^2 \ge 0$

$$Substitute\ C: \|x\|^2 - 2\left(\frac{x.y}{\|y\|^2}\right)(x.y) + \left(\frac{x.y}{\|y\|^2}\right)^2\|y\|^2 \ge 0$$

$$\|x\|^2 - \frac{2(x.y)^2}{\|y\|^2} + \frac{(x.y)^2}{\|y\|^4}\|y\|^2 \ge 0$$

$$\|x\|^2 - \frac{(x.y)^2}{\|y\|^2} \ge 0$$

$$\|x\|^2\|y\|^2 \ge (x.y)^2$$

$$Take\ out\ squares\ \|x\|\|y\| \geq |x.y|$$

When $x = cy, c \in R, \|x\|\|y\| = |x.y|$

Consequences: extend formula for angle between vector to R^n. $\forall r \in R, -1 \leq r \leq 1, \exists! \theta \in [0,\pi]\ s.t. \cos\theta = r. \cos\theta = \frac{x.y}{\|x\|\|y\|}$ take this as the definition of angle θ between x, y, $x, y \in R^n, x, y \neq 0$

Orthogonal (R^n)

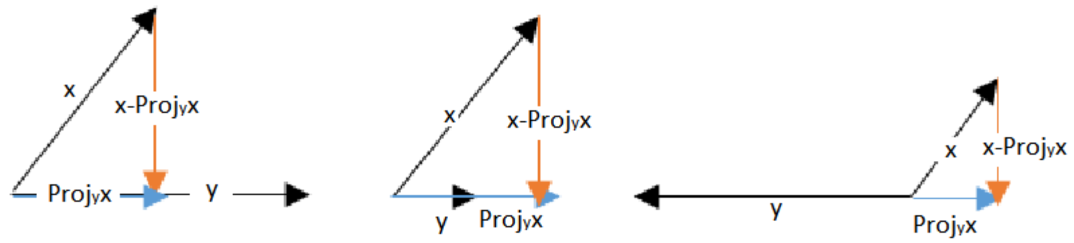
In R^2 R^3, perpendicular: if $x \perp y \leftrightarrow x, y = 0$

Extend to R^n, orthogonal: $x \perp y \leftrightarrow x, y = 0, x, y \in R^n$. Note $(x.y = 0) \neq (x = 0 \vee y = 0)$

e.x. $x = (1\ 2\ 3\ 4), y = (-3\ 0\ 1\ 0), x.y = 0$

according to the definition, $\forall x \in R^n, x \perp 0^n$

Projections:



$$Proj_y x\ is\ defined\ via{:}\ 1. Proj_y x = Cy, c \in R\ 2. (x - Proj_y x) \perp y$$

$$(x - Proj_y x) \perp y \rightarrow (x - Proj_y x).y = 0\ (2)$$

$$(x - cy).y = 0\ (1)$$

$$x.y - cy.y = 0$$

$$c = \frac{x.y}{y.y} = \frac{x.y}{\|y\|^2}$$

$$c{:}\ the\ component\ of\ Proj_y x$$

Vector equation:

Definition: the line through the point Q_o in direction of d. as:

$$Q = Q_0 + td, or \begin{pmatrix} x \\ y \\ z \\ \ldots \end{pmatrix} = \begin{pmatrix} x_o \\ y_o \\ z_o \\ \ldots \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \\ \ldots \end{pmatrix}\quad t \in R$$

$$ex.\ find\ the\ shortest\ distance\ from\ (2,1,3) to\ line\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

$$let\ x\ be\ the\ vector\ from\ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\ to\ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, which\ \begin{pmatrix} 2-1 \\ 1-0 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

$$y\ be\ the\ direction\ of\ the\ line$$

$$\min(d) = \left\| \begin{pmatrix}1\\1\\2\end{pmatrix} - Proj_{\begin{pmatrix}-3\\2\\1\end{pmatrix}} \begin{pmatrix}1\\1\\2\end{pmatrix} \right\| = \left\| \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\2\end{pmatrix} . \begin{pmatrix}-3\\2\\1\end{pmatrix}}{\left\| \begin{pmatrix}-3\\2\\1\end{pmatrix} \right\|^2} \begin{pmatrix}-3\\2\\1\end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{1}{14} \begin{pmatrix}-3\\2\\1\end{pmatrix} \right\| = \left\| \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}-\frac{3}{14}\\ \frac{2}{14}\\ \frac{1}{14}\end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix}\frac{17}{14}\\ \frac{12}{14}\\ \frac{27}{14}\end{pmatrix} \right\| = \sqrt{\frac{17^2 + 12^2 + 27^2}{14^2}} = \frac{\sqrt{1162}}{14}$$

$$point\ on\ the\ line = \begin{pmatrix}1\\0\\1\end{pmatrix} + Proj_{\begin{pmatrix}-3\\2\\1\end{pmatrix}} \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \begin{pmatrix}-\frac{3}{14}\\ \frac{2}{14}\\ \frac{1}{14}\end{pmatrix} = \begin{pmatrix}\frac{11}{14}\\ \frac{1}{7}\\ \frac{15}{14}\end{pmatrix}$$