ARIMA models, unit root tests, and modeling seasonality

JEN-WEN LIN, PhD, CFA

06 January, 2019

Review classical decomposition of time series

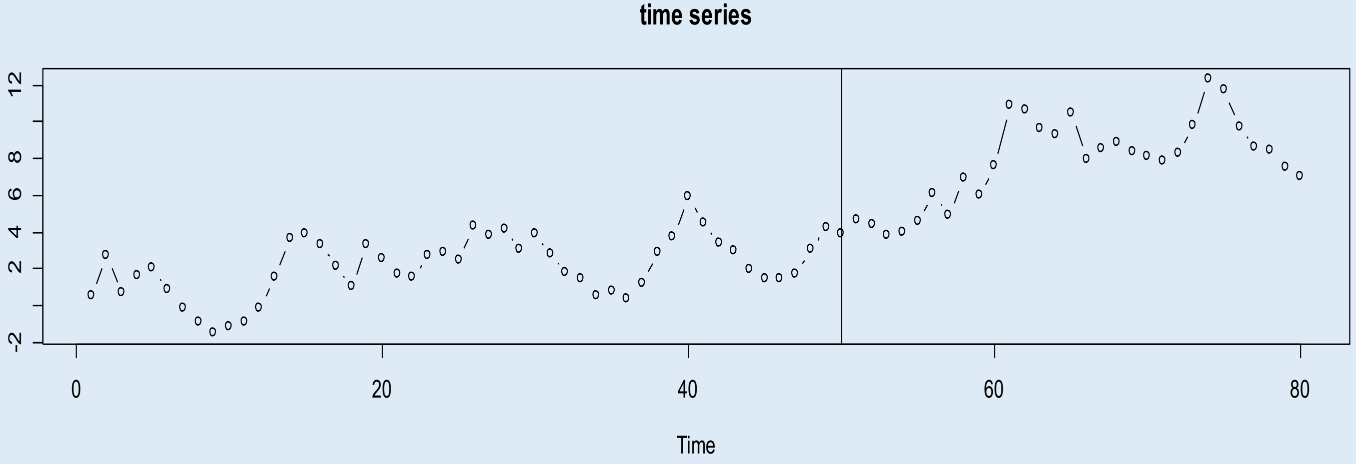
Seasonal variation	Time series exhibit variation that is annual in period (or every 12 units of time).
	For example, the sales of electronic companies in the second quarter are typically the lowest.
Cyclical variation	Time series exhibit variation at a fixed period due to some other physical cause.
	Examples are daily variation in temperature and business cycles.
Trend	This may be loosely defined as 'long-term change in the mean level'.

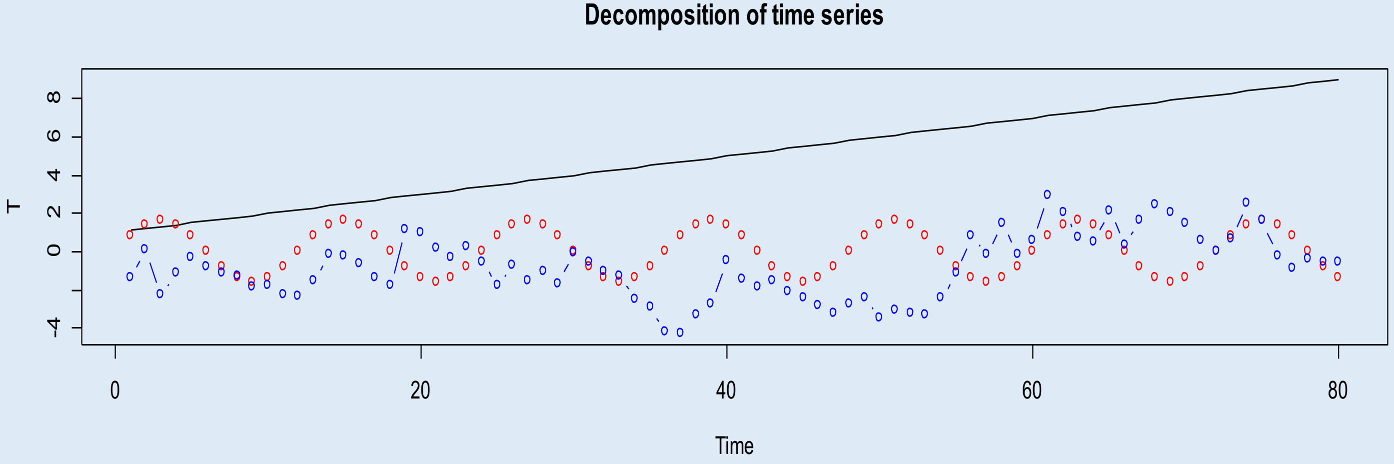
Simulated example

Trend (black): $T_t = 1 + 0.1 \cdot t$

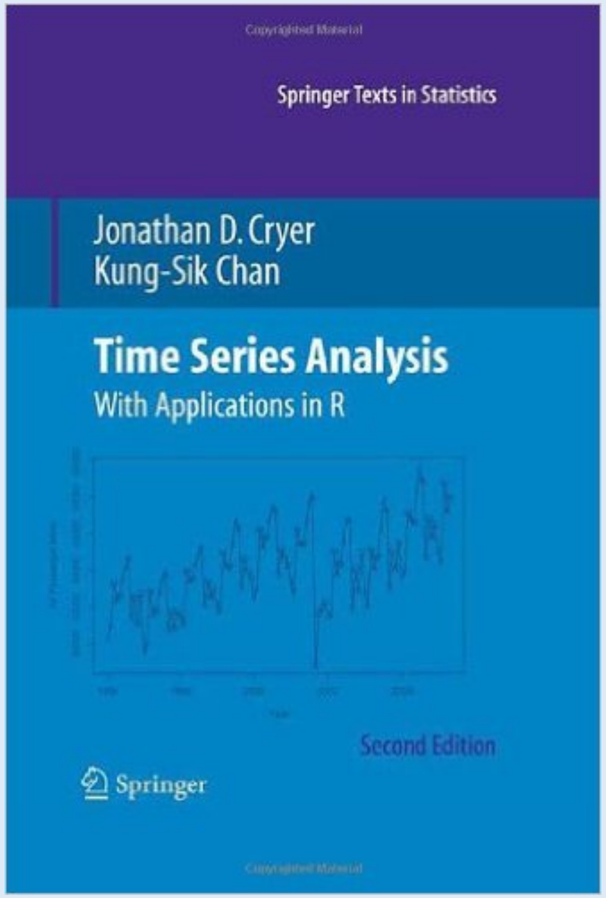
Seasonal (red): $S_t = 1.6 \cdot \sin(\frac{t\pi}{6})$

Irregular (blue): $I_t = 0.7 \cdot I_{t-1} + \varepsilon_t$





Time series reference book for R



- Cryer and Chan (2010), *Time Series Analysis: With Applications in R*, Second Edition, Springer.

- [Amazon link: http://www.amazon.ca/Time-Analysis-Applications-Jonathan-Cryer](http://www.amazon.ca/Time-Analysis-Applications-Jonathan-Cryer)

Regression methods to remove time trend

- *Example*: linear regression to remove linear time trend

$$Y_t = \mu_t + X_t,$$

where $\mu_t = \beta_0 + \beta_1 t, \quad t = 1, \dots, n.$

- Least squared estimation:

$$Q(\beta_0, \beta_1) = \sum_{t=1}^{n} [Y_t - (\beta_0 + \beta_1 t)]^2$$

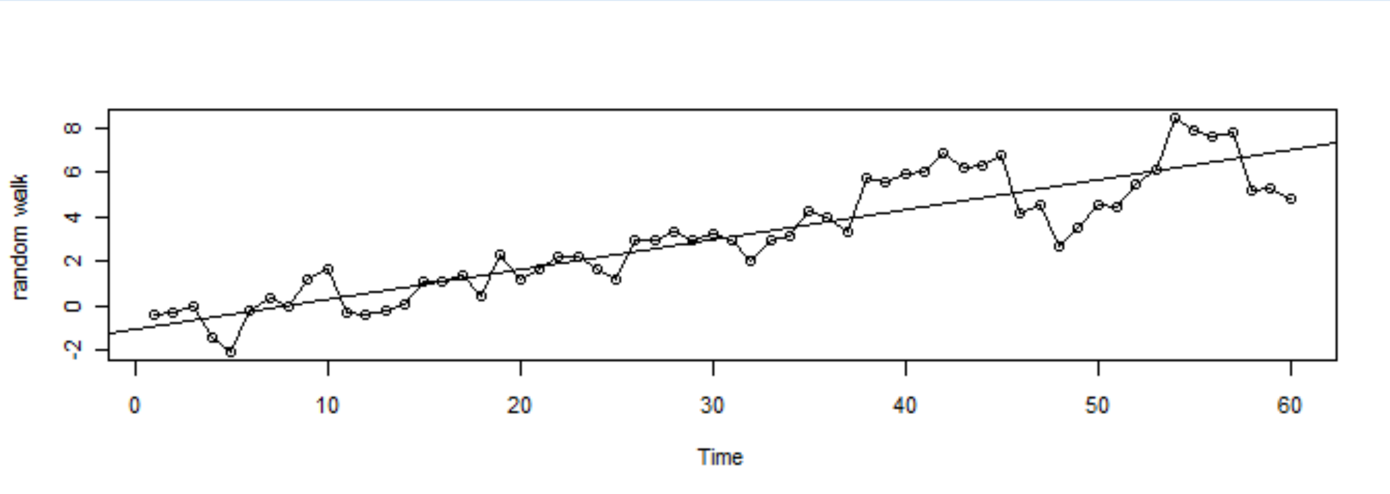
- Estimator:

$$\hat{\beta}_1 = \sum_{t=1}^{n} (Y_t - \bar{Y})(t - \bar{t}) / \sum_{1}^{n} (t - \bar{t})$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{t}, \qquad \bar{t} = \frac{n+1}{2}.$$

Linear and quadratic trends in time

R code:
•library(TSA) •data(rwalk) •mod_timetr<-lm(rwalk~time(rwalk)) •summary(mod_timetr) •win.graph(height=2.5, pointsize=8) •plot(rwalk, type='o', ylab="random walk") •abline(mod_timetr) # add the fitted regression line



Regression methods to remove seasonality

- *Example*: Monthly mean model:

$$Y_t = \mu_t + X_t, \qquad E(X_t) = 0, \forall t,$$

  where X_t denotes the stationary irregular component, and μ_t is monthly data with 12 constants (parameters) which gives the expected value for each of the 12 months.
- Specifically, we may write

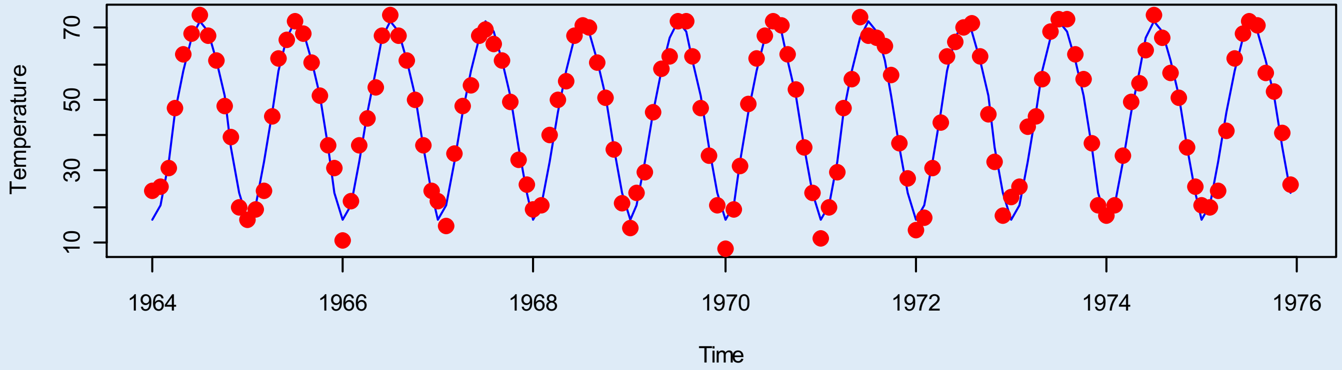
$$\mu_t = \begin{cases} \beta_1, & t = 1,13,25,\dots \\ \beta_2, & t = 2,14,26,\dots \\ & \vdots \\ \beta_{12}, & t = 12,24,36,\dots \end{cases}.$$

Regression methods to remove seasonality

R code:

```
•data(tempdub)
•month.<-season(tempdub)
•mod_cyctr<-lm(tempdub~month.); temp<-fitted(mod_cyctr)
•win.graph(height=2.5, pointsize=8)
•plot(ts(temp,freq=12,start=c(1964,1)), ylab='Temperature', type="l"
 ,col=4, ylim=range(c(temp,tempdub)))
•points(tempdub,col=2, lwd=4)
```

Monthly average temperature (in degrees Fahrenheit) recorded in Dubuque 1/1964 - 12/1975.



Regression methods to remove seasonality

- *Example*: Monthly mean model:

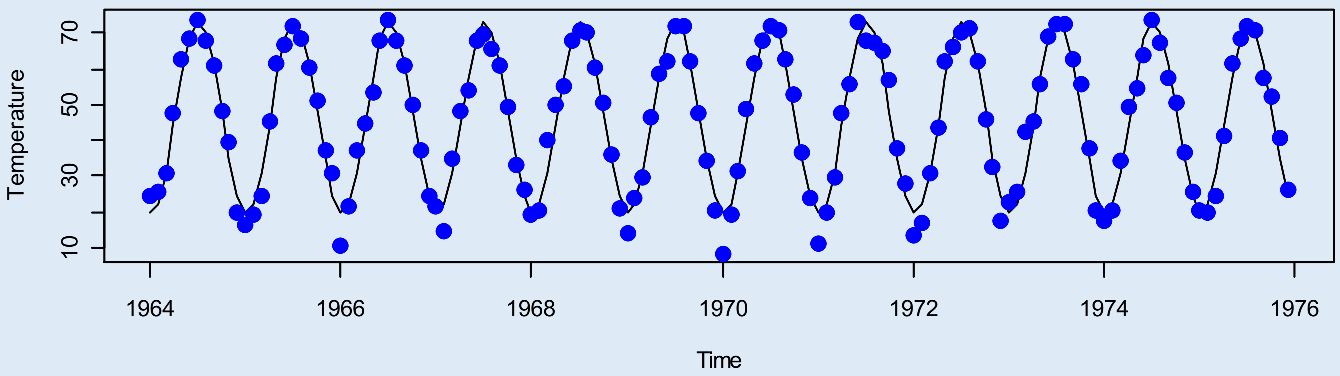
$$Y_t = \mu_t + X_t, \qquad E(X_t) = 0, \forall t,$$
$$\mu_t = \beta_1 \cos(2\pi f t) + \beta_2 \sin(2\pi f t),$$

  Where X_t denotes the stationary irregular component, and $1/f$ is called the period.
- For example, monthly data with time index as 1,2, ..., has $f = 1/12$ because such sinusodial function will repeat itself every 12 months. In this case, the period is 12.
- Least square estimation: use $\cos(2\pi f t)$ and $\sin(2\pi f t)$ as predictor variables.

Regression methods to remove seasonality

R code:

```
• har.<-harmonic(tempdub,1)
• mod_costr<-lm(tempdub~har.); temp_<-fitted(mod_costr)
• win.graph(height=2.5, pointsize=8)
• plot(ts(temp_,freq=12,start=c(1964,1)),
• ylab='Temperature', type="l" ,ylim=range(c(temp_,tempdub)))
• points(tempdub,col=2, lwd=4)
```



Autoregressive integrated moving average (ARIMA) models

- For nonstationary time series, Box-Jenkins (1970) suggested applying difference operators repeatedly to the data $\{X_t\}$ until the differenced observations resemble a realization of some stationary process $\{W_t\}$.
- $\{X_t\}$ is said to follow an ARIMA model of order (p, d, q) if $W_t = (1-B)^d X_t$ is a stationary $ARMA$ model. Mathematically, we have

$$(1-B)^d \phi(B) X_t = \theta(B) a_t, \qquad a_t \sim N(0, \sigma^2)$$

where

$$\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p,$$

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_q B^q.$$

Differencing to remove time trend

- Let $Y_t = a + bt + ct^2 + X_t$, where X_t is a stationary time series. Consider the following transformation:
- Backward operator B: $By_t = y_{t-1}, Bt = t - 1, Bc = c$.
  - Notation: $\nabla^d = (1-B)^d, \nabla^2 = (1-B)(1-B)$
  - $(1-B)^2 a = (1-2B+B^2)a = a - 2a + a = 0$
  - $(1-B)^2 bt = \nabla(1-B)bt = \nabla[bt - b(t-1)] = \nabla b = 0$
  - $(1-B)^2 ct^2 = \nabla(1-B)ct^2 = \nabla[ct^2 - c(t-1)^2] = \nabla[ct^2 - ct^2 + 2ct + c] = \nabla(2ct + c) = 2c$
- *Question*: whether $(1-B)^2 X_t$ is stationary

Differencing to remove seasonal component

- The technique of differencing that we applied to trend-stationary data can be adapted to deal with seasonality of period d by introducing the lag-d difference operator ∇_d defined by $\nabla_d X_t = X_t - X_{t-d} = \left(1 - B^d\right) X_t$
- This operator should not be confused with the operator $\nabla^d = (1 - B)^d$ defined earlier.
  - Applying the ∇_d operator to the classical decomposition model, $X_t = m_t + s_t + Y_t$, where s_t has period d, we have
$$\nabla_d X_t = m_t - m_{t-d} + Y_t - Y_{t-d}.$$
  - $m_t - m_{t-d}$ is a trend component and $Y_t - Y_{t-d}$ is a noise term.

Nonstationarity in variance

- Differencing can be used to transform a nonstationary time series due to the unstable mean level over time to a stationary (trend-stationary) time series.
- Many nonstationary time series, however, are not due to their time dependent means but their time-dependent variance and autocovariances.
  - We refer to time-dependent unconditional second moments rather than conditional second moments.
- To reduce these types of nonstationarity, we need to different transformations other than differencing.

Nonstationarity in variance

- Power transformation by Box and Cox (1964):

$$T(X_t) = (X_t^{\lambda} - 1)/\lambda \,.$$

- We can incorporate the Box-Cox transformation into model estimation. For example, we can include λ as one of the parameters

$$\phi(B)\left(X_t^{(\lambda)} - \mu\right) = \theta(B)a_t, \qquad a_t \sim NID(0, \sigma^2),$$

and choose the values of λ as well as $\{\phi_i\}_{i=1}^{p}$ and $\{\theta_i\}_{i=1}^{q}$ that give the minimum residual mean square error (RMSE).

- A variance stabilizing transformation, if needed, should be performed before any analysis such as differencing.

Some remarks

- In the preliminary analysis, one can use an AR model to obtain the value of λ through an AR fitting that minimizes the RMSE on a grid of λ values.
- Frequently, the transformations also improve the approximation of the distribution by a normal distribution.
- Finally, it is worth noting that the variance stabilizing transformations are defined by positive series. The definition is not restrictive as it seems because a constant can always be added to the series without affecting the correlation structure of the series.

I(d) process and Dickey-Fuller unit root test

- A series follows a stationary *ARMA* model after differencing d times is said to be integrated of order d, or $I(d)$ process.
- The Dickey-Fuller test is used to test $I(1)$ processes. Consider

$$X_t = \phi X_{t-1} + a_t, \qquad a_t \sim NID(0, \sigma^2).$$

$$\Delta X_t = (\phi - 1) X_{t-1} + a_t = \pi X_{t-1} + a_t$$

$$H_0 \colon \pi = 0 \ or\ X_t \sim I(1)\ ; H_a \colon \pi > 0 \ or\ X_t \sim I(0).$$

  - Remark: Under $H_0 \colon X_t \sim I(1)$, the OLS estimate of π does not follow a Student-t distribution.

More on Dickey Fuller test

- The general Dickey-Fuller test may contain an intercept and a deterministic time trend as

$$\Delta X_t = a + \tau^T DR_t + \pi X_{t-1} + a_t,$$

where a denotes the regression intercept and DR_t are deterministic independent variables, τ is the corresponding coefficient vector, and $a_t \sim NID(0, \sigma^2)$

Issues on Dickey-Fuller test

- The Dickey-Fuller test considers only a single unit root.
- Correct model specification
  - Correct specification of time trend and intercept
  - The DGP may contain both autoregressive and moving average terms
  - There might be structural breaks in the data

Augmented Dickey-Fuller test

- Dickey and Fuller (1981) have suggested the encompassing Augmented Dickey-Fuller test equation:

$$\Delta X_t = \tau^T DR_t + \pi X_{t-1} + \sum_{j=1}^{k} \gamma_j \cdot \Delta X_{t-j} + a_t \,,$$

where $k = p - 1$. The above equation use the autoregression to take into account the presence of serial correlated errors.

Selection of the lag length

- **Autoregression approximation:** Said and Dickey (1984) later on show that an unknown $ARIMA(p,1,q)$ process can often be approximated by an $ARIMA(n,1,0)$ autoregression of order n where $n \leq T^{\frac{1}{3}}$.
- **General-to-specific** methodology:
  - Start with a relatively long lag length and pare down the model by the usual t-test or F-test.

General-to-specific methodology

- For example, let's start with a lag length p^*. If the t-statistic of lag p^* is insignificant at some specified critical value, re-estimate the regression using the length $p^* - 1$.
- Repeat the process until the last lag is significant different from zero.
- In the pure autoregressive case, such a procedure will yield the true lag length with an asymptotic probability of unity, provided the initial choice of lag length include the true length.

More on selection of lag Length

- Once a tentative lag length has been determined, diagnostic checking should be conducted.
  - Residual autocorrelation plot
  - Portmanteau tests on regression residuals
- If the regression equation does not omit a deterministic regressor in the data-generating process, it is possible to perform lag-length test using t-tests or F-tests. (Sims, Stock, and Watson, 1990)

Spurious regression revisited

- Consider a simple regression on two random walks

$$y_t = \alpha + \beta x_t + \epsilon_t,$$

  where $x_t = x_{t-1} + a_t$ and $y_t = y_{t-1} + e_t$ with a_t and e_t are mutually independent. For simplicity, let's assume that all error terms $\{\epsilon_t, a_t, e_t\}$ are IID random variables.
- What statistical inference can we know about a conventional simple regression?
  - $\hat{\beta} \to 0$ in probability
  - $R^2 \to 0$ in probability
  - $t_\beta = \frac{\hat{\beta}-0}{se(\hat{\beta}\ )}$ converges to Student t-distribution

False statistical inference

- What if x_t and y_t are both random walks?
  - The absolute value of t_β tends to become larger and larger as the series length T increases;
  - Therefore, we will eventually rejects the null hypothesis that $\beta = 0$ with probability one as $T \to \infty$.
  - Additionally, R^2 does not converge to zero but to a random, positive number that varies from sample to sample.

False statistical inference

- When a regression model appears to find relationship that do not really exist, it is called spurious regression.
- We have discuss in class that spurious regression can occur even when all variables are stationary. The risk can be far from negligible with stationary series that exhibit substantial series correlation.

R-squared and spurious regression

A spurious regression is usually characterized by a high R-square (R^2)

Rule of thumb:

- A model is suspicious if the R^2 is greater than the Durbin-Watson statistics.

Simulation example

- library(lmtest)
- set.seed(1112)
- e1 <- rnorm(500)
- e2 <- rnorm(500)
- y1 <-cumsum(e1)
- y2 <-cumsum(e2)
- sr.reg<- lm(y1 ~ y2)
- sr.dw <- dwtest(sr.reg1)$statistic
- R-square is 0.58 and the Durbin-Watson statistic 0.0507 is close to zero, as expected.