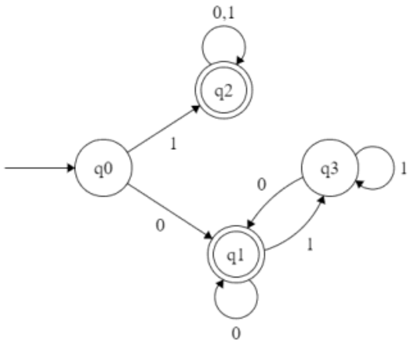
Finite Automata

Definition

$Q =$ the finite set of states
$q_0 =$ the initial state
$F \subset Q$ the final/accepting state
$\Sigma =$ the finite alphabet
Each direct edge represents a transition from a state to another, the letter on the edge causes the transition
$\delta: Q \times \Sigma \to Q, \forall q \in Q. \forall a \in \Sigma. \exists! \, e$ labelled by a and going out of q



Example

$Q = \{q_0, q_1, q_2, q_3\}$, q_0 initial, $q_1, q_2 \in F, \Sigma = \{0,1\}. \delta =$ edges
input 0110, then $q_0 \to q_1 \to q_3 \to q_3 \to q_1$

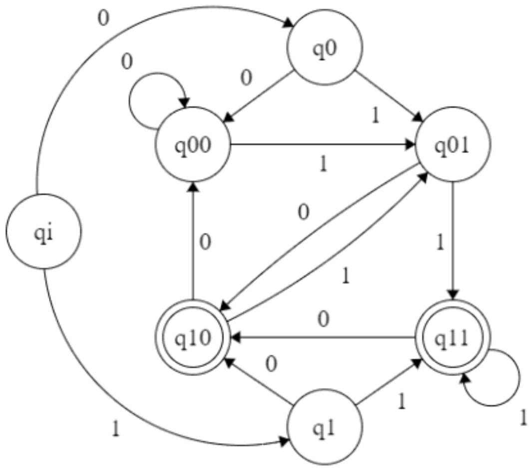
Definition Deterministic finite automata (DFA) is a 5-tuple, $M = (Q, \Sigma, \delta, q_0, F)$
After the entire input string had been read, M accepts x IFF the resulting state is in F.
If M DFA, $\mathcal{L}(M) = \{x \in \Sigma^* \mid M \text{ accepts } x\}$ is the language accepted by M

Example For the example above, $\mathcal{L}(M) = \left\{ x \in \Sigma^* \,\middle|\, \mathcal{L}\left((1(0+1)^*) + ((0+1)^*0)\right) \right\}$

Definition Extended transition function is the state reached when starting in initial state and reading each letters of string s

$\delta^*: Q \times \Sigma^* \to Q \coloneqq \delta^*(q, s) = \begin{cases} q & \mid s = \lambda \\ \delta(\delta^*(q, x), a) & \mid s = xa \end{cases}$ or $\delta^*(q, s) = \begin{cases} q & \mid s = \lambda \\ \delta^*(\delta(q, a), x) & \mid s = ax \end{cases}$
Then, $L(M) = \{x \in \Sigma^* \mid \delta^*(q_0, x) \in F\}$ is the set of all languages accepted by machine M

Example $L_2 = \mathcal{L}((0+1)^*1(0+1)) = \{x \in \{0,1\}^* \mid \text{the second letter is } 1\}$
Then, $\mathcal{L}(M) = L_2$, find M
Consider M, the last two letters are 10 or 11,
Take $\{q_{00}, q_{01}, q_{10}, q_{11}\}$ where q_{ij}: i, j are 2 last symbols read, hence $q_{10}, q_{11} \in F$



Proof Let

$L_q = \{ x \in \Sigma^* \mid \delta^*(q_0, x) = q \}$

$L_{q_{ab}} = \{ x \in \Sigma^* \mid \text{ab is a suffix of x} \}$

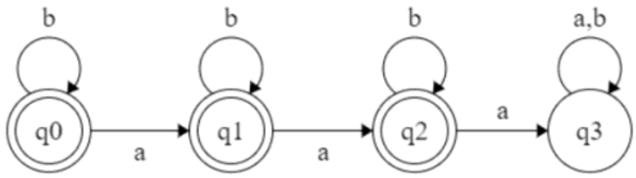
$L_{q_a} = \{a\}$

$L_{q_\lambda} = \{\lambda\}$

$\forall n \in \mathbb{N}. \forall x \in \Sigma^*. |x| = n \text{ IMPLIES } \left(x \in \mathcal{L}(M) \text{ IFF } x \in L_2\right)$

Example $L = \{ x \in \{a, b\}^* \mid \text{x contains at most 2 a's} \}$

$r = b^*(a + \lambda)b^*(a + \lambda)b^*$



$A = \left(\{q_0, q_1, q_2, q_3\}, \{a, b\}, \delta, q_0, \{q_0, q_1, q_2\}\right)$

$\delta(q_i, b) = q_i \mid i = \{0,1,2,3\}$

$\delta(q_i, a) = q_i \mid i = \{0,1,2\}$

$\delta(q_3, a) = q3$

Proof Call $L_i = \{ x \in \{a, b\}^* \mid \text{x has exactly i a's} \}$

For all $x \in \{a, b\}^*, P(x) \coloneqq \left(\delta^*(q_0, x) = q_3 \text{ IFF } x \notin (L_0 \cup L_1 \cup L_2)\right) \text{ AND } \left(\delta^*(q_0, x) = q_i \text{ IFF } x \in (L_0 \cup L_1 \cup L_2)\right)$

Suppose $x = \lambda$, then $x \in L_0, \delta^*(q_0, \lambda) = q_0$

Let x be an arbitrary string, suppose P(x),

Suppose $x \in L_0$, then $\delta^*(q_0, x) = q_0, xa \in q_1, xb \in q_2$

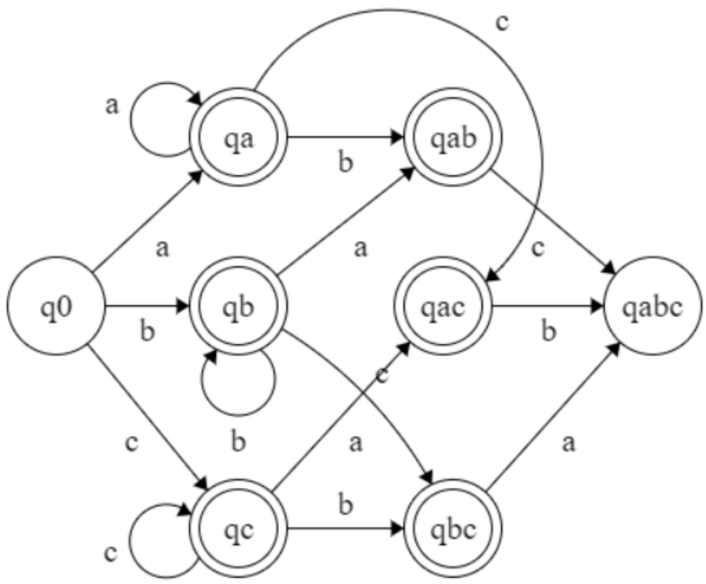
$\delta^*(q_0, xb) = \delta(\delta^*(q_0, x), b) = \delta(q_0, b) = q_0, \delta^*(q_0, xa) = \delta(\delta^*(q_0, x), a) = \delta(q_0, b) = q_1$

Similarly prove for other 3 cases $\left(x \in L_1, x \in L_2, x \notin (L_0 \cup L_1 \cup L_3)\right)$

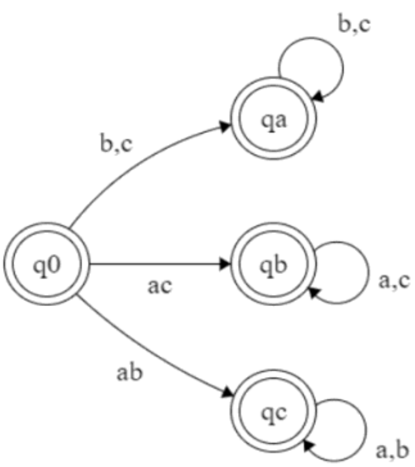
Example $L = \{ x \in \{a, b, c\}^* \mid \text{x doesn't contain all letters} \}$

$r = (a + b)^* + (a + c)^* + (b + c)^*$

DFA


c
a
qa
qab
b
a
a
c
q0
qb
qac
qabc
b
b
c
c
b
a
a
c
qc
b
qbc

NFA


b,c
qa
b,c
q0
qb
ac
a,c
ab
qc
a,b