Week 12: Tutorial Handout

- (Taylor's theorem) If $f \in C^k([a,b])$ and $\left|f^{(n+1)}(x)\right| \leq M$ for some $M > 0$ then for $x_0, x \in [a,b]$ we have $\left|f(x) - \sum_{k=0}^{n} \frac{f_k(x_0)}{k!}(x-x_0)^k\right| \leq M\frac{|x-x_0|^{n+1}}{(n+1)!}$.
- (Weierstrass approximation) If $f \in C([a,b])$, then there exists polynomials $p_k(x) := a_{n_k}x^{n_k} + ... + a_0$ such that $\|f - p_k\| \to 0$.

Taylor series

1. For n=2, find $\frac{f_k(x_0)}{k!}$ and the Taylor approximation around x_0 and estimate the error at the point x.
   - $tan(x)$ around $x_0 = \frac{\pi}{4}$ and error for $x = 0.75$.
   - $\sqrt{1+x^2}$ around $x_0 = 0$ and error for $x = 0.1$.
   - x^4 around $x_0 = 1$ and error for $x = 0.99$.
   - $sinh(x)$ around $x_0 = 0$ and error for $x = 0.003$.

Weierstrass polynomial approximation

2. If $f \in C^1([0,1])$ show that there exists polynomials $p_n(x)$ s.t. both $p_n \rightrightarrows f(x)$ and $p_n' \rightrightarrows f'(x)$.
3. If $0 \notin [a,b]$ then for $f \in C([a,b])$ there exists polynomials p_n s.t. $x^n p_n(x) \rightrightarrows f(x)$.
4. If $x_0, ..., x_n \in [a,b]$ and $a_0, ..., a_n \in \mathbb{R}$ show that there is a polynomial p s.t. $p(x_i) = a_i$ for $0 \leq i \leq n$.