Week 5: Tutorial Handout

A form $\|\cdot\|$ is called a norm over V if any $x, y \in V$ satisfy:

- (Positive definite) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
- (Homogeneous) $\|ax\| = |a|\|x\|$ for any $a \in \mathbb{R}$.
- (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$

A subset $U \subset V$ is called **open** if $\forall x \in U$, there exists $\delta > 0$ s.t. $B_\delta(x) := \{y : \|x - y\| \leq \delta\} \subset U$. A subset $C \subset V$ is called **closed** if for any sequence $x_n \in C$ with limit point $\|x_n - x\| \to 0$, we have that $x \in C$.

Norms

1. Prove that the form $\|\mathbf{x}\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$ is a norm on $\mathbb{R}^n$ for $p \geq 1$ (called the p-norm). (Hint: Concave functions $f(x)$ are subadditive $f(x+y) \leq f(x) + f(y)$).
2. For $n \times n$ matrices A and vectors $x \in \mathbb{R}^n$ with the standard Euclidean norm $\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$, prove that the form
$$\|A\| := \max_{\|x\|_2 \leq 1} \|Ax\|_2$$
is a norm over the space of $n \times n$ matrices. It is called the operator norm.
3. The set $\{f \in C([0,1]) : f(x) > 0 \;\; \forall x \in [0,1]\}$ is open wrt to the sup norm $\|f\|_{\infty,[0,1]} := \sup_{x \in [0,1]} |f(x)|$.
4. The set $\{f \in C([0,1]) : f(0) = 0\}$ is closed wrt to the sup norm $\|f\|_{\infty,[0,1]} := \sup_{x \in [0,1]} |f(x)|$.
5. The set $\{f \in C^1([0,1]) : f(x) > 0 \;\; \forall x \in [0,1], \|f'\|_\infty < 1, \}$ is open wrt to the sup norm $\|f\|_{1,\infty,[0,1]} := \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)|$.
6. The set $\{f \in C^2([0,1]) : f(x) > 0 \;\; \forall x \in [0,1], \|f'\|_\infty < 1, |f''(0)| > 2\}$ is open wrt to the sup norm $\|f\|_{2,\infty,[0,1]} := \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)| + + \sup_{x \in [0,1]} |f''(x)|$.