29 May, 2019

- Forecast (Chapter 5, Wei's book) :
  - Causal and invertible ARMA model:

$$\phi(B)X_t = \theta(B)a_t, a_t \sim NID(0, \sigma_a^2)$$

$$X_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \rightarrow X_{t+h} = \sum_{j=0}^{\infty} \psi_j a_{t+h-j}$$

  - Forecast: origin and lead time

$$\underbrace{x_{t+h}}_{forecast} \ \ldots . \ \underbrace{x_t}_{you} \ , x_{t-1}, \ldots$$

  - Mean square error forecast: $\hat{X}_t(h)$ as our forecaster

$$E\left[\left(\underbrace{X_{t+h} - \hat{X}_t(h)}_{forecast\ error}\right)^2\right]$$

$$X_{t+h} = \sum_{j=0}^{\infty} \psi_j a_{t+h-j}$$

$$\hat{X}_t(h) = \sum_{i=0}^{\infty} \hat{\psi}_i a_{t-i}$$

$$\min_{\hat{\psi}_i} E\left[\left(\underbrace{X_{t+h} - \hat{X}_t(h)}_{forecast\ error}\right)^2\right] = E\left(\sum_{j=0}^{\infty} \psi_j a_{t+h-j} - \sum_{i=0}^{\infty} \hat{\psi}_i a_{t-i}\right)^2$$

$$= E\left(\sum_{j=0}^{h-1} \psi_j a_{t+h-j} + \sum_{i=0}^{\infty} (\psi_{h+i} - \hat{\psi}_i) a_{t-i}\right)^2$$

$$= \underbrace{\sigma_a^2 \sum_{j}^{h-1} \psi_j^2}_{variance\ of\ forecast\ error} + \underbrace{\sigma_a^2 \sum_{i}^{\infty} \left(\psi_{h+i} - \hat{\psi}_i\right)^2}_{=0\ for\ \psi_{h+i} = \hat{\psi}_i\ \forall i=0,1,2\ldots}$$

- $h = 1$

$$\psi_0 a_{t+1} + \psi_1 a_t + \psi_2 a_{t-1} + \cdots$$
$$\hat{\psi}_0 a_t + \hat{\psi}_1 a_{t-1} + \hat{\psi}_2 a_{t-2} + \cdots$$

- Mean square error forecast = conditional expectation for linear stochastic process (ARMA model)
  - Rule of calculating conditional expectation:

1. $E(X_{t+h}|X_t, X_{t-1}, \dots) = E_t(X_{t+h}) = \hat{X}_t(h), h > 0$
2. $E_t(X_{t-h}) = X_{t-h}, h > 0$
3. $E_t(a_{t+h}) = E(a_{t+h}) = 0, h > 0$
4. $E_t(a_{t-h}) = X_{t-h} - \underbrace{\hat{X}_{t-h-1}(1)}_{E_{t-h-1}(X_{t-h})}$

$$y_t = a_t + 0.3a_{t-1}$$

Example: Wei's book (page 103)

$$X_t = 0.5X_{t-1} + a_t$$

- $X_t(1)$:

$$\underbrace{E_t(X_{t+1})}_{\hat{X}_t(1)} = 0.5\,\underbrace{E_t(X_t)}_{X_t} + \underbrace{E_t(a_{t+1})}_{0}$$

- $X_t(l)$: $l \geq 1$

$$\underbrace{E_t(X_{t+l})}_{\hat{X}_t(l)} = 0.5\,\underbrace{E_t(X_{t+l-1})}_{\hat{X}_t(l-1)} + \underbrace{E_t(a_{t+l})}_{0}$$

$$\hat{X}_t(l) = 0.5\hat{X}_t(l-1)$$

- Transfer function noise model:

$$X_t = \underbrace{f}_{linear\ function}\left(\underbrace{X_{t-1}, X_{t-2}, \dots}_{past\ observations\ of\ X_t}, \underbrace{Z_t, Z_{t-1}, \dots}_{exogenous\ variable}\right)$$

Dynamic regression, ARMAX model. A special case—distributed lag model

$$X_t = \sum_{i=0}^{\infty} \beta_i Z_{t-i} + a_t$$

$$y_t = \beta x_t + e_t$$

$$y_t = var(e_t)$$

$$var(y_t) = var(E(y_t|x_t)) + E[var(y_t|x_t)]$$

- *Application of ARMA models in investments*

Alternative assets modeling:

Two popular approaches to model appraisal returns:

1) Geltner method: y_t and r_t denote appraisal and economic returns, respectively

$$y_t = \phi y_{t-1} + \underbrace{(1-\phi)r_t}_{a_t} = \sum_{j=0}^{\infty} \phi^j \underbrace{(1-\phi)r_{t-j}}_{a_{t-j}}$$

2) Getmansky, Low, & Markorov (2005, Journal of financial economics):

$$y_t = \sum_{i=0}^{q} w_i r_{t-i}\, , w_i \in (0,1), \sum w_i = 1.$$