Outline: Week 10

8.4 Series of Functions

1. Cauchy implies uniform: Cauchy completeness of reals gives pointwise $S_n(x_0) \to S(x_0)$. Then uniform gives:

$$|S(x) - S_k(x)| \leq |S(x) - S_n(x)| + |S_n(x) - S_k(x)| \leq \varepsilon_n + \varepsilon.$$

   So taking limit in n gives

$$|S(x) - S_k(x)| \leq \varepsilon.$$

2. pointwise versus uniform:

   - For $\sum \frac{x^n}{n!}$ we have pointwise and uniform in $[-R, R]$

$$\left\| \sum_{k=n}^{n+m} \frac{x^k}{k!} \right\| \leq \sum_{k=n}^{n+m} \frac{R^k}{k!} \to 0$$

     as $n \to \infty$.

   - For $\sum \frac{x^2}{n(n+x^2)}$ we have pointwise. Suppose we have uniform, then $\left\| \sum_{k=n}^{n+m} \frac{x^2}{k(k+x^2)} \right\| \leq \varepsilon$, which is a contradiction by taking x large enough in order to give $\frac{x^2}{k(k+x^2)} \sim \frac{1}{k}$ the harmonic series.

3. Weierstrass M-test: $\|S_n - S_{n+m}\| \leq \sum_{k=n+1}^{n+m} M_k \to 0$ as $n \to \infty$.

4. example uniform in $[-r, r]$: $\sum (-x^2)^k = \frac{1}{1+x^2}$. Not uniform in $(-1, 1)$ because for any $n \geq N$ and $n + m$ being even we have

$$\sup_{x \in [-1,1]} \left| \sum_{k=n}^{n+m} (-x^2)^k \right| \geq \frac{1}{2} \left| \sum_{k=n}^{n+m} (-1)^k \right| = \frac{1}{2},$$

   where we used that $(\frac{1}{2})^{\frac{1}{2N}} \in [-1, 1]$. In $[-r, r]$ we have bound by $M_k = r^{2k}$ and so we get uniform by M-test.

5. Swap integral and sum: If $S_n \rightrightarrows S$, then by ICT $\int \sum f_k = \sum \int f_k$.

6. Example: in $[-r, r]$ we integrate to get

$$arctan(x) = \int_0^x \frac{1}{1+s^2} ds = \sum \frac{(-1)^n}{2n+1} x^{2n+1}.$$

We in fact have uniform convergence in $[-1,1]$. For alternating series $\sum_{k\geq n}(-1)^k a_k$ with decreasing $a_k \geq a_{k+1} > 0$, we have

$$\left|\sum_{k\geq n}(-1)^k a_k\right| \leq a_n.$$

Therefore,

$$\sup_{x\in[-1,1]}\left|\sum_{k=n}^{n+m}\frac{(-1)^k}{2k+1}x^{2k+1}\right| \leq \frac{1}{2n+1}.$$

7. Swap derivative and sum: $\frac{\mathrm{d}}{\mathrm{d}x}\sum f_k = \sum \frac{\mathrm{d}}{\mathrm{d}x} f_k$ when both $S_k \rightrightarrows S$ and $\frac{\mathrm{d}}{\mathrm{d}x}S_k \rightrightarrows \frac{\mathrm{d}}{\mathrm{d}x}S$. By FTC and ICT

$$f_n = c_n + \int f_n' \to f = c + \int g ds$$

then $f' = g$.

8.5 Power Series

1. Ratio test to $\sum \frac{x^n}{n!}$ and $\sum \frac{x^n}{n}$. We have $\frac{|x|}{n+1} \to 0$ and $|x|\frac{n}{n+1} \to |x|$. So we get $(-\infty,\infty)$ and $[-1,1)$ respectively.
2. HADAMARD'S THEOREM:
   - we have $L = |x|\frac{1}{R}$.
   - we have $[a,b] \subset [-c,c]$ and so we apply M-test for $M_k := a_k c^k$ to get uniform convergence.
3. if ratio test limit is defined then

$$\lim \frac{|a_{n+1}|}{|a_n|} = \frac{1}{R}.$$

4. Examples:
   - for $\sum \frac{x^n}{n!}$ we have $\frac{1}{R} = 0$.
   - given two series $f(x) = \sum a_k x^k$ and $g(x) = \sum b_k x^k$, their sum $f+g$ and product $fg = \sum_{n\geq 0}\sum_{k=0}^{n} a_k b_{n-k} x^n$ have

$$\sqrt[n]{|a_n + b_n|} \leq \frac{1}{R_f} + \frac{1}{R_g}$$

and

$$\sqrt[n]{\sum_{k=0}^{n} a_k b_{n-k}} \geq max(\frac{1}{R_f}, \frac{1}{R_g}).$$

Proof: Suppose wlog that $R_f \geq R_g$. Then use that

$$|a_n| \leq M(\frac{1}{R_f} + \varepsilon)^n \text{ and } |b_n| \leq M(\frac{1}{R_g} + \varepsilon)^n$$

for $n \geq 0$ and M and small $\varepsilon > 0$.

- for $\sum \frac{2^{2n}}{n^2} x^n$ we have $R = \frac{1}{4}$.Next we check the endpoints: we get $\sum \frac{1}{n^2}$ and $\sum \frac{(-1)^n}{n^2}$.
- for $\sum \frac{(-1)^n}{\sqrt{n}}(x-5)^2$ we get $R = 1$ and so $|x-5| < 1 \Rightarrow (4,6)$. Next we check the endpoints: we get $\sum \frac{1}{\sqrt{n}} = \infty$ and $\sum \frac{(-1)^n}{\sqrt{n}} < \infty$.

5. Swap derivative and sum: the coefficient na_n has same radius R. So we have u.c. and so $S_k(x) = \int_0^x S_k'(s)ds \to \int_0^x g(s) = S(x) \Rightarrow S'(x) = g(x)$.
6. in fact we have smooth inside $(-R,R)$: by the ratio test we get $\frac{n+1}{n-k}\frac{a_{n+1}}{a_n} \to \frac{1}{R}$.
7. Swap integral and sum: the coefficient $\frac{a_n}{n+1}$ has same radius R. So by ICT we can swap.
8. Examples:

- for $\sum_{n\geq 1} n(n-1)t^{2n}$

$$\sum_{n\geq 1} n(n-1)t^n = t^2 \frac{\mathrm{d}}{\mathrm{d}t}^2 \sum_{n\geq 0} t^n = t^2 \frac{\mathrm{d}}{\mathrm{d}t}^2 \frac{1}{1-t} = t^2 \frac{2}{(1-t)^3}.$$

- for $\ln(1+x)$ for $|x| < 1$ we have

$$\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \sum \frac{(-1)^k}{k} x^k.$$