Outline: Week 12

Taylor series

1. Let $P_{n,a}(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$

2. Taylor's theorem assuming $f^{(n)}$ but not $f^{(n+1)}$, we have $f(x) = P_{n,a}(x) + g_n(x)(x-a)^{n+1}$. This follows quickly from L'Hopital

$$\lim_{h\to 0} \frac{f(a+h) - P_n(h)}{h^n} = ... = \lim_{h\to 0} \frac{f^{(n-1)}(a+h) - f^{(n-1)}(a)}{n!h} = \frac{f^{(n)}(a)}{n!}.$$

3. Specifying coefficient when $f^{(n+1)}$ exists: We will show that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$.

   - Let constant K be such that $f(x) = T_n(x) + K(x-a)^{n+1}$.
   - Consider $F(y) := f(y) - T_n(y) - K(y-a)^{n+1}$ for $y \in [a, x]$. We will show that $K = \frac{f^{(n+1)}(c)}{(n+1)!}$ for some $c \in [a, x]$
   - We have $F(a) = 0$ and $F(x) = 0$. So by Rolle's theorem we have c_1 s.t. $F'(c_1) = 0$.
   - We have $F'(a) = 0$ and $F(c_1) = 0$. So by Rolle's theorem we have c_2 s.t. $F''(c_2) = 0$.
   - So on and so forth till we get $F^{(n)}(a) = 0$ and $F^{(n)}(c_n) = 0$, to get $F^{(n+1)}(c_{n+1}) = 0$.
   - Since T_n is an n-degree polynomial we are left with $F^{(n+1)}(y) = f^{(n+1)}(y) - (n+1)!K$. Thus, $K = \frac{f^{(n+1)}(c_{n+1})}{(n+1)!}$

4. Corollary: $\frac{f(a+h)-P_n(h)}{(h^n} \leq M\frac{h^{n+1}}{h^n} \to 0$ as $h \to 0$.

5. Examples: a)$sin(x) = \sum \frac{(-1)^k}{(2k+1)!}x^{2k+1}$ and then by differentiating $cos(x) = \sum \frac{(-1)^{k+1}}{(2k)!}x^{2k}$ b)for e^x.

6. expanding sin(x) around $\pi/2$ we get

$$sin(x) = 1 - 1/2(x-\pi/2)^2 + 1/24(x-\pi/2)^4 - 1/720(x-\pi/2)^6 + O((x-\pi/2)^7).$$

7. cosh(x) around $x_0 = 0$

$$cosh(x) = 1 + x^2/2 + x^4/24 + x^6/720 + O(x^7).$$

8. Example (smooth but not analytic): The function $\frac{1}{1+x^2}$ has all its derivatives on $\mathbb{R}$ but the geometric series equality

$$\sum a_k x^k = \sum (-1)^k x^{2k} = \frac{1}{1+x^2}$$

is only true for $|x| < 1$. We have $\left|\frac{1}{1+x^2} - \sum^n (-1)^k x^{2k}\right| \le \frac{x^{2(n+1)}}{1+x^2}$.

9. $arctan(x) = \int_0^x \frac{1}{1+s^2} \mathrm{d}s = \sum \frac{(-1)^k}{2k+1} x^{2k+1}$. The radius is $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = 1$.

so

$$\left| arctan(x) - \sum^n \frac{(-1)^k}{2k+1} x^{2k+1} \right| \le \frac{|x|^{2n+1}}{2n+1} < \frac{1}{2n+1}.$$

Weierstrass approximation theorem

1. Bernstein polynomials $P_k^n(x) := \binom{n}{k} x^k (1-x)^{n-k}$ and consider the functional

$$(B_n f)(x) := \sum_{k=0}^n f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}.$$

   - linear
   - monotone i.e. $|f|(x) \le g(x) \Rightarrow B_n|f| \le B_n g$.

2. Moments of Berstein polynomials

   - $B_n 1 = 1$
   - $B_n x = x$. Differentiate to get $\sum k \binom{n}{k} a^k b^{n-k} = an(a+b)^{n-1}$
   - $B_n x^2 = x^2 + \frac{x-x^2}{n}$. Differentiate to get $\sum k^2 \binom{n}{k} a^k b^{n-k} = a^2 n(n-1)(a+b)^{n-2} + an(a+b)^{n-1}$.

3. By uniform continuity for $\varepsilon/2 > 0$ there exists $\delta > 0$ s.t.

$$|x-y| \le \delta \Rightarrow |f(x) - f(y)| \le \varepsilon/2.$$

4. For $|x-y| > \delta$ we use that

$$|f(x) - f(y)| \le 2M \le 2M\left(\frac{|x-y|}{\delta}\right)^2,$$

where $M := \max_{x\in[0,1]} |f(x)|$.

5. So together for all $x, y \in [0,1]$ we get:

$$|f(x) - f(y)| \le 2M\left(\frac{|x-y|}{\delta}\right)^2 + \varepsilon/2,$$

6. By treating $c = f(y)$ as constant and using monotonicity we find

$$|B_n f(x) - f(y)| = |B_n(f(x) - c)| \leq B_n\left(2M\left(\frac{|x-y|}{\delta}\right)^2 + \varepsilon/2\right)$$
$$= \varepsilon/2 + \frac{2M}{\delta^2}\left((x-y)^2 + \frac{x - x^2}{n}\right)$$

let $x = y$ to get

$$= \varepsilon/2 + \frac{2M}{\delta^2}\frac{x(1-x)}{n} \leq \varepsilon/2 + \frac{2M}{\delta^2}\frac{2}{n},$$

So if we choose $n \geq N$ we get

$$\leq \varepsilon/2 + \frac{2M}{\delta^2}\frac{2}{N}$$

and then pick N large we find

$$\leq \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon.$$

7. 10.2.A: we have $p_n(t) \to g(t)$ and so $p_n(\frac{x-a}{b-a}) \to f(x)$.

8. 10.2.F,G: set $p_{n,0} := \int p_{n,1}$ where $f' - p_{n,1} \to 0$.