Let W be subspace of R^n, set S be vectors from R^n is basis of W if $W = span(S)$ and S is linear independent.
The dimension of W is the smallest number of vectors to span W
$\dim W = k$: if there exists set of k vectors from W that span W, but no set containing $k - 1$ vectors, or fewer from W can span W.
Recall: known in R^n, any set of more than n vectors from R^n must be linear dependent (consider the columns of I), same applies to all subspaces of R^n

Fundamental Theorem: $\dim W = k$, any set of more than k vectors must be linear dependent
Proof: WTS: $W2, W3, \dots Wn$ be linear dependent, take $c1, c2, c3, \dots, cn$ that not all of them are 0 and satisfy $c1W1 + c2W2 + \cdots + cnWn = 0$,

take $v1, v2, \dots, vn$ span W, such

$$W1 = a_{11}v1 + a_{12}v2 + \cdots + a_{1k}vk$$
$$Wi = a_{i1}v1 + a_{i2}v2 + \cdots + a_{ik}vk$$
$$c1W1 + \cdots + cnWn = c1(a_{11}v1 + a_{12}v2 + \cdots + a_{1k}vk) + \cdots + cn(a_{m1}v1 + a_{m2}v2 + \cdots + a_{mk}vk) *$$
$$= v1(c1a_{11} + \cdots + cna_{m1}) + \cdots + vk(c1a_{1k} + cna_{mk}) ** = 0$$

$$\left[\begin{array}{ccc|c} a_{11} & \dots & a_{m1} & 0 \\ \dots & \dots & \dots & 0 \\ \underbrace{a_{1k} \quad \dots \quad a_{mk}}_{c1 \quad \dots \quad cm} & & & 0 \end{array}\right]$$

If $c1, c2, c3, \dots, cn$ solve *, they also solve **, this homogenous system with more variables than equations (m>k), which is guaranteed to have non-trivial solutions, so $W2, W3, \dots Wm$ are linear dependent

$$e.x. W = \left\{ \begin{pmatrix} a+b+3c \\ 2a-b \\ 2a+b+4c \end{pmatrix} \middle| a, b, c \in R \right\} \text{ find } \dim W$$

W is in subspace of R^4,

$$W = a\begin{pmatrix}1\\2\\0\\2\end{pmatrix} + b\begin{pmatrix}1\\-1\\0\\1\end{pmatrix} + c\begin{pmatrix}3\\0\\0\\4\end{pmatrix}$$

$$= span\left\{\begin{pmatrix}1\\2\\0\\2\end{pmatrix}\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\begin{pmatrix}3\\0\\0\\4\end{pmatrix}\right\} = span\left\{\begin{pmatrix}1\\2\\0\\2\end{pmatrix}\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\right\}\left(\begin{pmatrix}3\\0\\0\\4\end{pmatrix} = \begin{pmatrix}1\\2\\0\\2\end{pmatrix} + 2\begin{pmatrix}1\\-1\\0\\1\end{pmatrix}\right) \dim W = 2$$

$\dim W > 1$ since the set of more than 1 vector from W could be linear independent by fundamental theorem

Theorem: $\dim W = k \rightarrow$ any basis of W must contain k vectors, $\dim W = \#\ of\ vectors\ in\ any\ basis$
Proof: take W has a basis containing m vectors, take $k \le m$, since fundamental theorem, but it's impossible that $m > k$ since any set containing more than k vectors must be linear dependent, which is not a basis, therefore $k = m$

$e.x. \dim R^n = n \; since \; col(I^n) = \{e1 \; e2 \ldots en\}$ is the basis

Theorem: $\dim W = n$ if the set of n vectors from W that span W must be linear independent, hence the basis; any set of n linear independent vectors from W spans W and the set is a basis for W