Linear Recurrence

Definition **homogeneous linear recurrence** 's self-referential part is a linear combination of some fixed number of preceding terms

$$f(n) = \sum_{i=1}^{d} a_i f(n-i)\,, d, a_i \in \mathbb{R}$$

Solving linear recurrences: guess $f(n) = cx^n$, where c, x are parameters

Example Fibonacci sequences: $f(n) = \begin{cases} 0 \mid n = 0 \\ 1 \mid n = 1 \\ f(n-1) + f(n+2) \mid n > 1 \end{cases}$

Guessing $f(n) = x^n = x^{n-1} + x^{n-2}$

Dividing by $x^{n-2}, x^2 = x + 1$, which solves the roots $x = \frac{1 \pm \sqrt{5}}{2}$

Let $g(n) = \left(\frac{1-\sqrt{5}}{2}\right)^n, h(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n$

Theorem If $f(n)$ and $g(n)$ are both solutions to a homogeneous linear recurrence, then $\forall s, t \in \mathbb{R}. sf(n) + tg(n)$ is all a solution

Proof $sf(n) + tg(n) = s\big(a_1 f(n-1) + \cdots a_d f(n-d)\big) + t\big(a_1 g(n-1) + \cdots a_d g(n-d)\big) = a_1\big(sf(n-1) + tg(n-1)\big) + \cdots + a_d\big(sf(n-d) + tg(n-d)\big)$

Then $f(n) = c_1 \left(\frac{1-\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1+\sqrt{5}}{2}\right)^n$

Solving $\begin{cases} f(0) = 0 = c_1 + c_2 \\ f(1) = 1 = c_1(\frac{1-\sqrt{5}}{2}) + c_2 \left(\frac{1+\sqrt{5}}{2}\right) \end{cases} \begin{matrix} c_1 = \frac{1}{\sqrt{5}} \\ c_2 = \frac{-1}{\sqrt{5}} \end{matrix}$

$$f(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

General Procedure of solving homogenous linear recurrence

1. Guess that $f(n) = x^n$ is a solution for the recurrence, then $f(n) = cx^n = \sum_{i=1}^{d} a_i x^{n-i}$
2. Divides all the terms by $x^{n-d}, x^d = \sum_{i=1}^{d} a_i x^{d-i}$ is the characteristic equation of the recurrence
3. Suppose this equation has d distinct roots $r_1, \dots r_d$, then $f(n) = c_1 r_1^n + \cdots + c_d r_d^n$
4. Substituting them into base cases and solve for $c_1, \dots, c_d$
5. Verifying by induction

Definition **non-homogenous linear recurrence** is a linear recurrence with an extra function $g(n)$

$$f(n) = \sum_{i=1}^{d} a_i f(n-i) + g(n)\,, d, a_i \in \mathbb{R}$$

Example $H(n) = \begin{cases} 0 \mid n \le 1 \\ H(n-1) + H(n-2) + 4 \mid n > 1 \end{cases}$

By range transformation, let $G(n) = H(n) + 4$, then $G(0) = G(1) = 4, G(n) = H(n) + 4 = H(n-1) + H(n-2) + 4 + 4 = G(n-1) + G(n-2)$

Solving $G(n) = 2\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right), H(n) = 2\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) - 4$

General Procedure of solving non-homogenous linear recurrence

1. Solving its homogenous part (homogeneous solution)
2. Find a single solution to the homogenous linear recurrence ignoring the boundary conditions (particular solution)

3. Add 1. and 2. to get the general solution
4. Substitute into the boundary conditions
5. Verify by induction

Example $f(n) = \begin{cases} 4f(n-1) + 3^n \mid n > 1 \\ 1 \mid n \leq 1 \end{cases}$

Guess $f(n) = cx^n$, then $f(n) = cx^n = 4cx^{n-1} = 4f(n-1), x = 4, f(n) = c4^n$ (homogeneous solution)

Guess $f(n) = c'3^n$ is a solutoin to the inhomogenous solution

$f(n) = c'3^n = 4\left(c'3^{n-1}\right) + 3^n, 3c' = 4c' + 3. c' = -3, f(n) = -3^{n+1}$

$f(n) = c4^n - 3^{n+1}, 1 = 4c - 3^2, c = \frac{5}{2}$

$f(n) = \frac{5(4^n)}{2} - 3^{n+1}$