Variance Stablizing Transformations

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1. Problem:

It is very common for the variance of a nonstationary process to change as its level change. Thus,

$$var(Z_t) = c \cdot f(\mu_t), \qquad (1)$$

for some postive constant c and function f.

2. How do we find a function T so that the transformed series, $T(Z_t)$ has a constant variance?

Consider a first-order Taylor series about μ_t

$$T(Z_t) \approx T(\mu_t) + T'(\mu_t)(Z_t - \mu_t), \qquad (2)$$

where $T'(\mu_t)$ is the first derivative of $T(Z_t)$ evaluated at μ_t. Now, take the variance operator on Equation (2) and substitue $Var(Z_t)$ with Equation (1)

$$var[T(Z_t)] \approx [T'(\mu_t)]^2 var(Z_t) = c \cdot [T'(\mu_t)]^2 f(\mu_t). \qquad (3)$$

In order for the variance of $T(Z_t)$ to be constant, the variance stablizing transformation $T(Z_t)$ must be chosen so that

$$T'(\mu_t) = \frac{1}{\sqrt{f(\mu_t)}}. \qquad (4)$$

Equation (4) implies that

$$T(\mu_t) = \int \frac{1}{\sqrt{f(\mu_t)}} d\mu_t. \qquad (5)$$

3. Examples:

1. If the standard devation of a series is proportional to the level so that $var(Z_t) = c\mu_t^2$, then

$$T(\mu_t) = \int \frac{1}{\sqrt{\mu_t^2}} d\mu_t = ln(\mu_t). \qquad (6)$$

   Therefore, $ln(Z_t)$ will have a constant variance.

2. If the variance of a series is proportional to the level so that $var(Z_t) = c\mu_t$, then

$$T(\mu_t) = \int \frac{1}{\sqrt{\mu_t}} d\mu_t = 2\sqrt{\mu_t}. \qquad (7)$$

   Therefore, $\sqrt{Z_t}$ will have a constant variance.

3. If the standard devation of a series is proportional to the square of the level so that $var(Z_t) = c\mu_t^4$, then

$$T(\mu_t) = \int \frac{1}{\sqrt{\mu_t^4}} d\mu_t = -\frac{1}{\mu_t}. \qquad (8)$$

Therefore, a desired transformation will be $1/Z_t$.

4. Power transformation

More generally, we can use the power transformation

$$T(Z_t) = \frac{Z_t^\lambda - 1}{\lambda}, \qquad (9)$$

introduced by Box and Cox (1964).

5. Reference

- William W.S. Wei (2006), *Time Series Analysis: Univariate and Multivariate Methods*, 2nd edition.