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- Forecast (Chapter 5, Wei's book) :

- Causal and invertible ARMA model:

$$\phi(B)X_t = \theta(B)a_t, a_t \sim NID(0, \sigma_a^2)$$

$$X_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \rightarrow X_{t+h} = \sum_{j=0}^{\infty} \psi_j a_{t+h-j}$$

- Forecast: **origin** and **lead time**

$$\underbrace{x_{t+h}}_{\text{forecast}} \dots \underbrace{x_t}_{\text{you}}, x_{t-1}, \dots$$

- Mean square error forecast: $\hat{X}_t(h)$ as our forecaster

$$E \left[\left(\underbrace{X_{t+h} - \hat{X}_t(h)}_{\text{forecast error}} \right)^2 \right]$$

$$X_{t+h} = \sum_{j=0}^{\infty} \psi_j a_{t+h-j}$$

$$\hat{X}_t(h) = \sum_{i=0}^{\infty} \hat{\psi}_i a_{t-i}$$

$$\begin{aligned} \min_{\hat{\psi}_i} E \left[\left(\underbrace{X_{t+h} - \hat{X}_t(h)}_{\text{forecast error}} \right)^2 \right] &= E \left(\sum_{j=0}^{\infty} \psi_j a_{t+h-j} - \sum_{i=0}^{\infty} \hat{\psi}_i a_{t-i} \right)^2 \\ &= E \left(\sum_{j=0}^{h-1} \psi_j a_{t+h-j} + \sum_{i=0}^{\infty} (\psi_{h+i} - \hat{\psi}_i) a_{t-i} \right)^2 \\ &= \underbrace{\sigma_a^2 \sum_j^{h-1} \psi_j^2}_{\text{variance of forecast error}} + \underbrace{\sigma_a^2 \sum_i^{\infty} (\psi_{h+i} - \hat{\psi}_i)^2}_{=0 \text{ for } \psi_{h+i} = \hat{\psi}_i \ \forall i=0,1,2\dots} \end{aligned}$$

- $h = 1$

$$\begin{aligned} &\psi_0 a_{t+1} + \psi_1 a_t + \psi_2 a_{t-1} + \dots \\ &\hat{\psi}_0 a_t + \hat{\psi}_1 a_{t-1} + \hat{\psi}_2 a_{t-2} + \dots \end{aligned}$$

- Mean square error forecast = conditional expectation for linear stochastic process (ARMA model)
 - Rule of calculating conditional expectation:

$$1. E(X_{t+h} | X_t, X_{t-1}, \dots) = E_t(X_{t+h}) = \hat{X}_t(h), h > 0$$

$$2. E_t(X_{t-h}) = X_{t-h}, h > 0$$

$$3. E_t(a_{t+h}) = E(a_{t+h}) = 0, h > 0$$

$$4. E_t(a_{t-h}) = X_{t-h} - \underbrace{\hat{X}_{t-h-1}(1)}_{E_{t-h-1}(X_{t-h})}$$

$$y_t = a_t + 0.3a_{t-1}$$

Example: Wei's book (page 103)

$$X_t = 0.5X_{t-1} + a_t$$

- $X_t(1)$:

$$\underbrace{E_t(X_{t+1})}_{\hat{X}_t(1)} = 0.5 \underbrace{E_t(X_t)}_{X_t} + \underbrace{E_t(a_{t+1})}_0$$

- $X_t(l): l \geq 1$

$$\underbrace{E_t(X_{t+l})}_{\hat{X}_t(l)} = 0.5 \underbrace{E_t(X_{t+l-1})}_{\hat{X}_t(l-1)} + \underbrace{E_t(a_{t+l})}_0$$

$$\hat{X}_t(l) = 0.5\hat{X}_t(l-1)$$

- Transfer function noise model:

$$X_t = \underbrace{f}_{\text{linear function}} \left(\underbrace{X_{t-1}, X_{t-2}, \dots}_{\text{past observations of } X_t}, \underbrace{Z_t, Z_{t-1}, \dots}_{\text{exogenous variable}} \right)$$

Dynamic regression, ARMAX model. A special case—distributed lag model

$$X_t = \sum_{i=0}^{\infty} \beta_i Z_{t-i} + a_t$$

$$y_t = \beta x_t + e_t$$

$$y_t = \text{var}(e_t)$$

$$\text{var}(y_t) = \text{var}(E(y_t|x_t)) + E[\text{var}(y_t|x_t)]$$

- *Application of ARMA models in investments*

Alternative assets modeling:

Two popular approaches to model appraisal returns:

- 1) Geltner method: y_t and r_t denote appraisal and economic returns, respectively

$$y_t = \phi y_{t-1} + \underbrace{(1 - \phi)r_t}_{a_t} = \sum_{j=0}^{\infty} \phi^j \underbrace{(1 - \phi)r_{t-j}}_{a_{t-j}}$$

- 2) Getmansky, Low, & Markorov (2005, Journal of financial economics):

$$y_t = \sum_{i=0}^q w_i r_{t-i}, w_i \in (0,1), \sum w_i = 1.$$