Eigenvalue & Eigenvalues

Geometrically, a matrix function $T: R^n \to R^n = Ax$ rotates and stretch a vector

For $A\, n \times n$, a number λ is an eigenvalue of A if $\exists x \in R^n, x \neq 0, s.t. Ax = \lambda$x, x is called eigenvector

Finding Eigenvalue/vectors

$$Ax = \lambda \text{x}$$
$$Ax - \lambda \text{x} = 0$$
$$(A - \lambda I)\text{x} = 0$$

Solving $(A - \lambda I)\text{x} = 0$ has non-trivial solutions iff $\det(A - \lambda I) = 0$

$$e.x. A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \det A = \det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0$$
$$(3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = 0$$
$$\lambda = 2, \qquad \lambda = 4$$

If $A\, n \times n$, characteristic polynomial of A, $C_A(\lambda) = \det(A - \lambda I)$, note $C_A(\lambda)$ is polynomial of degree n in λ and eigenvalues of A are roots of $C_A(\lambda)$

λ is the eigenvalue of $A\, n \times n$ the eigenspace corresponding to λ is $E_\lambda = null(A - \lambda\,\text{I})$

Note: E_λ is a subspace of R^n and contains 0, eigenvectors are all vectors in E except for 0

$$E_2 = null(A - 2I) = null \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = span\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$
$$E_4 = null(A - 4I) = null \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$e.x. T: R^2 \to R^2. T(x) = Proj_{x=0} x$

$$E_1 = span\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, E_0 = span\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

$A = \begin{pmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{pmatrix}$ find eigenvalues and basis for each eigenspaces

$$C_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ -4 & -4 & -11-\lambda \end{pmatrix}$$
$$= det \begin{pmatrix} 5-\lambda & 8 & 16 \\ 4 & 1-\lambda & 8 \\ 0 & -3-\lambda & -3-\lambda \end{pmatrix} = det \begin{pmatrix} 5-\lambda & -8 & 16 \\ 4 & -7-\lambda & 8 \\ 0 & 0 & -3-\lambda \end{pmatrix}$$
$$= (-3-\lambda)\big((5-\lambda)(-7-\lambda) + 32\big) = (-3-\lambda)\big((-3-\lambda)(\lambda-1)\big) = -(\lambda-1)(\lambda+3)^2$$
$$\lambda = 1, \ \lambda = -3$$
$$E_{-3} = null(A + 3\lambda I) = null \begin{pmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{pmatrix} = span\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\right\}$$
$$E_1 = null(A - \lambda I) = null \begin{pmatrix} 4 & -8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & -12 \end{pmatrix} = span\left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$$