Irregular Language

Example prove $L = \{a^i b^i \mid i \geq 1\}$ is not regular
Proof Suppose L is regular, then it's accepted by some DFA $M = \{Q, \{a, b\}, \delta, q_0, F\}$
Let $|Q| = n$, call the state $q_i = \delta^*(q_0, a^i)$ for $i = 0,1, \dots, n$
By pigeonhole principle, $\exists 0 \leq i < j \leq n.\, q_i = q_j$
Then $\delta^*(q_0, a^i b^i) = \delta^*(\delta^*(q_0, a^i), b^i) = \delta^*(q_i, b^i) = \delta^*\left(q_j, b^i\right) = \delta^*(\delta^*(q_0, a^j), b^i) = \delta^*(q_0, a^j b^i)$
Then $a^j b^i \in \mathcal{L}(M)$, while $a^j b^i \notin L$
By generalization and contradiction, $L = \{a^i b^i \mid i \geq 1\}$ is not regular

Theorem Pumping lemma
For every regular language $L \subseteq \Sigma^*$, there exists $n \in \mathbb{Z}^+$ such that

$$\forall x \in L.\left(|x| \geq n \text{ IMPLIES } \left(\exists u, v, w \in \Sigma^*.\, v \neq \lambda \text{ AND } |uv| \leq n \text{ AND } x = uvw \text{ AND } \left(\forall k \in \mathbb{N}.\, uv^k w \in L\right)\right)\right)$$

Proof Let L be an arbitrary language, since L is regular, $L = \mathcal{L}(M)$ for some DFA $M = (Q, \Sigma, \delta, q_0, F)$
Let $n = |Q| \in \mathbb{Z}^+$.
Let $x \in L$ be arbitrary and suppose $|x| = m \geq n$
Say $x = x_1 x_2 \dots x_n$ where $x_1, x_2, \dots, x_n \in \Sigma$
Consider the states $q_i = \delta(q_0, x_1 x_2 \dots x_i)$ for $i = 1,2, \dots, i$
By pigeonhole principle, $\exists i, j \in \mathbb{N}.\, 0 \leq i < j \leq n$ s.t. $q_i = q_j$
Take $u = x_1 \dots x_i, v = x_{i+1} \dots x_j, w = x_{j+1},,, x_m,$
$x = uvw$
Since $i < j, v = x_{i+1} \dots x_j \neq \lambda$
Since $j \leq n, |uv| = \left|x_1 \dots x_j\right| \leq n$
Let $k \in \mathbb{N}$ be arbitrary
$\delta^*(q_0, uv^k w) = \delta^*(\delta^*(q_0, u), v^k w) = \delta^*(\delta^*(q_i, v^k), w) = \delta^*\left(\delta^*\left(q_j, v^{k-1}\right), w\right) = \cdots = \delta^*\left(q_j, w\right)$
$= \delta^*(q_i, w) = \delta^*(q_0, uvw)$
By generalization, this is true $\forall k \in \mathbb{N}$
By construction and generalization, the lemma is true

Example $L = \{z \in \{0,1\}^* \mid z = z^R\}$ is not regular
Proof Suppose L is regular, then by the pumping lemma,
$\exists n \in \mathbb{Z}^+.\, \forall x \in L.\left(|x| \geq n \text{ IMPLIES } \left(\exists u, v, w \in \{0,1\}^*.\, v \neq \lambda \text{ AND } |uv| \leq n \text{ AND } x = uvw \text{ AND } \forall k \in \mathbb{N}.\, uv^k w \in L\right)\right)$
Instantiate n, consider $\underline{x = 0^n 1 0^n \in L}$
By specialization and modue ponens, take $u, v, w \in \Sigma^*$ $v \neq \lambda$ AND $|uv| \leq n$ AND $x = uvw$ AND $\forall k \in \mathbb{N}.\, uv^k w \in L$
Note that $\underline{u = 0^i, v = 0^j, w = 0^{n-i-j} 1 0^n}$ where $i > 0, j > 1, i + j \leq n$
Take $\underline{k = 2, uv^2 w = 0^i 0^{2j} 0^{n-i-j} 1 0^n = 0^{n+j} 1 0^n \notin L}$
By contradiction, L is not regular

Example $L = \{a^m b^r \mid m \neq r\}$ is not regular
Proof consider $x = a^{n!} b^{(n+1)!}$, since $n \in \mathbb{Z}^+.\, n! < (n + 1)!.\, x \in L$
Note that $u = a^i, v = a^j$ where $i \geq 0, j > 0, i + j \leq n, w = a^{n!-i-j} b^{(n+1)!}$
Take $k = 1 + \frac{n(n!)}{j}$, since $j \leq n, 1 + \frac{n(n!)}{j} \in \mathbb{N}$
$uv^k w = a^m b^{(n+1)!}$ where $m = i + j\left(1 + \frac{n(n!)}{2}\right) + n! - i - j = n! + j\left(\frac{n(n!)}{j} + 1 - 1\right) = n! + n(n!) = (n + 1)!$

Example $L = \{ a^p \mid \text{prime}(P) \}$ is not regular
Proof consider $x = a^p$ where $p \geq n$ is prime
Note that $u = a^i, v = a^j$ where $i \geq 0, j > 0, i + j \leq n, w = a^{p-i-j}$
Take $k = p + 1$
$uv^k w = a^m$ where $m = i + j(p+1) + p - i - j = p + jp = (j+1)p$ is a composite

Example $L = \{ a^m b^r \mid m > r \}$ is not regular
Proof consider $x = a^{n+1} b^n$
Note that $u = a^i, v = a^j$ where $i \geq 0, j > 0, i + j \leq n$
Take $k = 0$,
$uv^k w = a^i \lambda a^{n+1-i-j} b^n = a^{n-j} b^n$, since $j > 0, n - j \leq n$