STA457 Time series analysis assignment (Fall 2018)

Statistical properties of (moving-average) rule returns

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1. Introduction

Technical indicator is widely used to generate trading signals by practitioners to make trading decisions. The usual rule is to trade with the trend. In this case, the trader initiates a position early in the trend and maintains that position as long as the trend continues.

In this assignment, you are asked to study the statistical properties of returns for applying the oldest and most widely used method in technical indicators—moving averages.[1]

The structure of this paper is given as follows. Section 2 defines the trading rule (or strategy). In Section 3 and 4, we formulate the trading return based on a given trading rule and state the corresponding statistical properties, respectively. The questions for you to answer are listed in Section 5. Finally, references and appendix are given in Section 6 and 7, respectively.

2. Trading rule

Suppose that at each time t, market participants predict the direction of the trend of asset prices using a price-based forecast F_t, where F_t is a function of past asset prices

$$F_t = f(P_t, \ldots, P_{t-m+1}, \ldots).$$

[1] *The simplest rule of this family is the single moving average which says when the rate penetrates from below (above) a moving average of a given length, a buy (sell) signal is generated.*

The above predictor is then converted to buy and sell trading signals B_t: buy (+1) and sell (-1) using, i.e.

$$\begin{cases} \text{"}Sell\text{"} & \Leftrightarrow \quad B_t = -1 \quad \Leftrightarrow \quad F_t < 0 \\ \text{"}Buy\text{"} & \Leftrightarrow \quad B_t = +1 \quad \Leftrightarrow \quad F_t > 0 \end{cases}$$

Note that the signal of a trading rule is completely defined by one of the inequalities giving a sell or buy order (if the position is not short, it is long).

For example, consider a trading rule based on the moving average of order five rule ($m = 5$). In this case, f is given by

$$f(P_t, \ldots, P_{t-m+1}) = P_t - \frac{\sum_{i=0}^{4} P_{t-i}}{5}.$$

In this case, we buy the asset ($B_t = +1$) at time $t + 1$when

$$F_t > 0 \Leftrightarrow P_t > \frac{\sum_{i=0}^{4} P_{t-i}}{5};$$

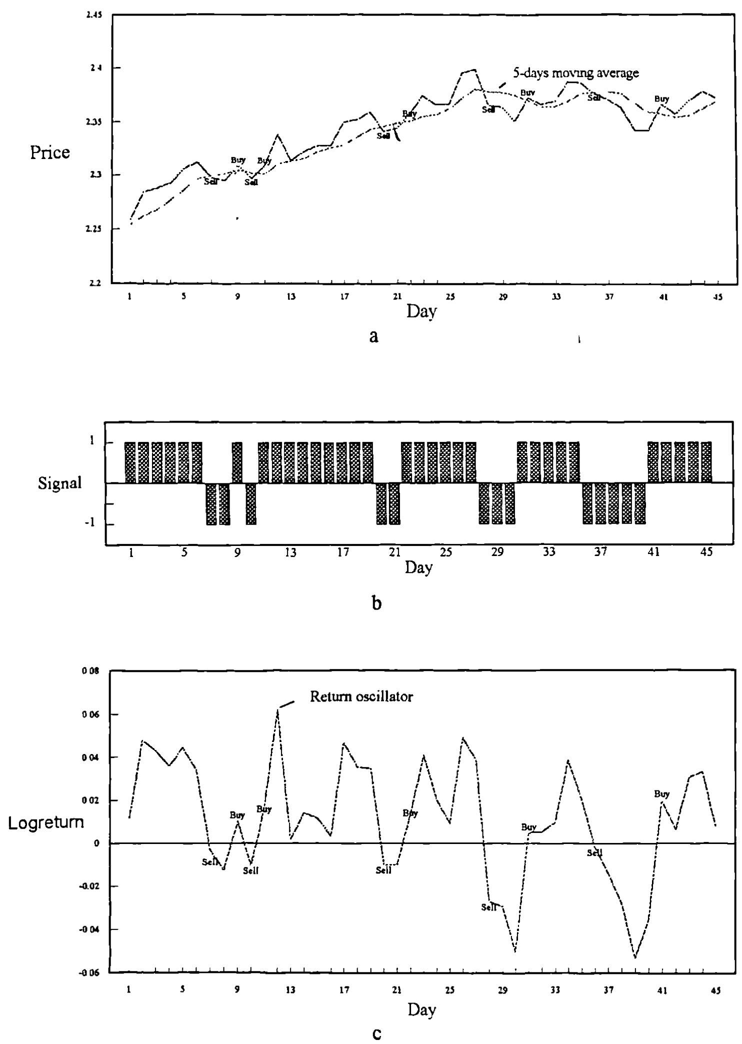
and sell the asset ($B_t = -1$) when

$$F_t < 0 \Leftrightarrow P_t < \frac{\sum_{i=0}^{4} P_{t-i}}{5}.$$

The Figure below illustrates the dynamics of the above 5-periods moving average method—when the rate penetrates from below (above) the moving average of order five, a buy (sell) signal is generated.

Simple moving average method

5-days moving average



For your assignment, we consider F_t based on a moving-average technical indicator. In general, for a given moving-average indicator, F_t may be expressed as (a function of log returns):

$$F_t = \delta + \sum_{j=0}^{m-2} d_j X_{t-j}\,, \qquad (1)$$

where $X_t = ln(P_t/P_{t-1})$, δ and d_j are defined by a given trading rule (See Appendix for more details). For this assignment, we assume $\delta = 0$.

3. **Rule returns**

For the period $[t-1,t)$, a trader following a given technical rule establishes a position (long or short) at time $t-1$, B_{t-1}. The returns at time t made by applying such a decision rule is called "ruled returns" and denoted as R_t. Their value can be expressed as

$$R_t = B_{t-1}X_t \Leftrightarrow \begin{Bmatrix} R_t = -X_t & if & B_{t-1} = -1 \\ R_t = +X_t & if & B_{t-1} = +1 \end{Bmatrix},$$

where $X_t = ln(P_t/P_{t-1})$ denote the logarithm return over this period (assume no dividend payout during period t).

<u>Remark</u>*:* R_t is unconditional and unrealized returns. By unrealized we mean that rule returns are recorded every day even if the position is neither closed nor reversed, but simply carries on.

<u>Remark</u>*:* We may define the realized returns as

$$\tilde{R}_t = \sum_{D=1}^{n} R_{t+D},$$

where D represents the stochastic duration of the position which will last n days if

$$\{D = n\} \Leftrightarrow \{B_{t-1} \neq B_t, B_t = B_{t+1} = \cdots = B_{t+n-1}, B_{t-n+1} \neq B_{t+n}\}.$$

4. Statistical properties of rule returns

Under the assumption that X_t follows a stationary Gaussian process, several statistical properties of rule returns can be derived:

1. Unconditional expected return:

$$E(R_t) = \sqrt{\frac{2}{\pi}}\sigma_X \cdot corr(X_t, F_{t-1}) \cdot exp\left\{-\frac{\mu_F^2}{2\sigma_F^2}\right\} + \mu_X\left(1 - 2\Phi\left[-\frac{\mu_F}{\sigma_F}\right]\right), \qquad (2)$$

where $\Phi(h) = \int_{-\infty}^{h}\left(\sqrt{2\pi}\right)^{-1} exp\{-x^2/2\}\, dx$, $\mu_X = E(X_t), \sigma_X = var(X_t)$, $\mu_F = E(F_t)$, and $\sigma_F^2 = var(F_t)$.

2. Unconditional variance:

$$var(R_t) = E(X_t^2) - E(R)^2 = \sigma_X^2 + \mu_X^2 - E(R_t)^2.$$

Additionally, Kedem (1986) shows that the expected zero crossing rate for a stationary process as the expected zero-crossing rate for a discrete-time, zero-mean, stationary Gaussian sequence Z_t is given by

$$\frac{1}{\pi}cos^{-1}\rho_Z(1),$$

where $\rho_Z(1)$ denotes the autocorrelation function of $\{Z_t\}$ at lag one. Using the same assumption, we can show that F_t is stationary. Using this result, we may approximate the expected length of the holding period[2] for a given trading rule as

$$H = \frac{\pi}{cos^{-1}\rho_F(1)}. \qquad (3)$$

[2] *Intuitively, the longer holding period, the larger the expected return on a trading rule.*

5. Questions

1. Derive the variance of the predictor F_t given in Equation (1).

   Hint: $\sigma_F^2 = var(\sum_{i=0}^{m-2} d_i X_{t-i})$.

2. Derive the expectation of the predictor F_t.

   Hint: $\mu_F = E(\sum_{i=0}^{m-2} d_i X_{t-i})$.

3. Derive the autocorrelation function at lag one for the predictor.

   Hint: $\rho_F(1) = corr(F_t, F_{t-1})$.

4. Write a R function to calculate the expectation of the rule return for a given **double MA** trading rule (See Appendix) and the expected length of the holding period.

   Hint: Given asset price time series and a pair of integers, m and r (function arguments), your function calculates the expected rule return $E(R_t)$ and the expected length of holding periods H.

5. Use a R function to download daily, weekly S&P500 index from **Oct/01/2009** to **Sep/30/2018** from yahoo finance

   Hint: adjusted Close and `R quantmod library`.

6. Write a R function to choose the **optimal** daily and weekly double MA trading rules (that maximize the expected rule returns) for S&P500 index.

   Hint: Find the m and r pair that has the highest $E(R_t)$. For simplicity, let the maximum values of m be 250 and 52 for daily and weekly data, respectively.

7. Write a R function to calculate the in-sample trading statistics (**cumulative return** and **holding time**) of your choice and compare them with your theoretical results.

   Hint: Use the ratio of the cumulative return over the number of trading periods as the estimate $E(R_t)$.

8. ~~(Optional) Run and back-test your daily trading rule using six months of rolling window. Show the empirical trading statistics and show the difference between the theoretical results.~~

6. Reference

1. Acar, E. (1993). Economic evaluation of financial forecasting. (Unpublished Doctoral thesis, City University London.)
2. Acar E. (200?), “Advanced trading rule”, Second edition. (Chapter 4. Expected returns of directional forecasters).
3. Kedem (1986), “Spectral analysis and discrimination by zero-crossings”, Proceedings of IEEE, Vol 74, No. 11, page 1477-1493.

7. Appendix

Table 1: Return/Price signals equivalence

Rule	Parameter(s)	Price sell signals	Return sell signals
Simple order		$P_t < \sum_{j=0}^{m-1} a_j P_{t-j}$	$\sum_{j=0}^{m-2} d_j X_{t-j} < 0$
Simple MA	$m \geq 2$	$a_j = \frac{1}{m}$	$d_j = (m - j - 1)$
Weighted MA	$m \geq 2$	$a_j = \frac{m-j}{[m(m-1)]/2}$	$d_j = \frac{(m-j)(m-j-1)}{2}$
Exponential MA	$1 > a > 0, m \geq 2$	$a_j = a(1-a)^j$	see generalization
Momentum	$m \geq 2$	$a_j = 1$ for $j = m-1$, $a_j = 0$ for $j \neq m-1$	$d_j = 1$
Double orders		$\sum_{j=0}^{r-1} b_j P_{t-j} < \sum_{j=0}^{m-1} a_j P_{t-j}$	$\sum_{j=0}^{m-2} d_j X_{t-j} < 0$
Double MA	$m > r \geq 1$	$b_j = \frac{1}{r}, \quad a_j = \frac{1}{m}$	$d_j = (m-r)(j+1)$ for $0 \leq j \leq r-1$ $d_j = r(m-j-1)$ for $r \leq j \leq m-1$
Generalization		$\sum_{j=0}^{m-1} a_j P_{t-j} < 0$	$\delta + \sum_{j=0}^{m-2} d_j X_{t-j} < 0$, with: $d_j = -\sum_{i=j+1}^{m-1} a_i$ and $\delta = \sum_{j=0}^{m-1} a_j$