Outline: Week 5 T

IVT

1. IVT: Consider $A_z := \{x \in [a,b] : f(x) < z\}$. Take $c - \frac{1}{n} \leq a_n \leq c$. So $f(c) = \lim_{n\to\infty} f(a_n) \leq z < f(b)$. So $c \neq b$ otherwise $f(c) = z = f(b)$. So take $b_n \geq c$ going to c, then $z \leq f(b_n) \to f(c) \geq z$.
   Generalization in Rn with path in $\mathbb{R}^n$ by taking $g(x) := f(\gamma(x))$ for $x \in [0,1]$.

2. A set S is connected if you cannot write it as $S = A \sqcup B$ for open A,B and $A \cap B = \varnothing$ eg. $S = (0,1) \cup (2,3)$. By IVT, continuous image of connected is also connected.

3. 5.6.C: x-(2sin(x)+3cos(x)) is positive for $x = -\frac{\pi}{2}$.

Norms

1. Definition of norm (a)positive and identically zero (b)|ax|=|a||x| (c)triangle inequality

2. the dot product

3. The p-norm is a norm for $p \geq 1$.

4. the uniform norm

5. the norm $\sum_{k=1}^{m} \|f^{(k)}\|_\infty$ for derivatives over $C^k([a,b])$ kth-continuously differentiable functions.
   The function $f(x) := |x|^{k+1} \in C^k([-a,a])$ but it is not in $C^{k+1}([-a,a])$.

6. Convergence is defined in terms of the norm. Same for Cauchy sequence. A complete normed space is called Banach space.

7. The set $\{f \in C([0,1]) : f(x) > 0\}$ is open in the sup norm. The set $\{f \in C([0,1]) : f(0) = 1\}$ is closed in the sup norm but is not bounded because $f_n(x) := (1+x)^n$ has supnorm equal to 2^n.

8. The set $\{f \in C([0,1]) : f(x) > 0, \|f'\|_\infty\}$ is open in the sup norm of f plus sup norm of f'.

9. Exercise 7.4.I: their sup norm is 1 and we can use that C(K) is complete if K is compact.

Detailed proof for 5.6.C

Let $f(x) := x - (2sin(x) + 3cos(x))$. We have that $f(x) > 0$ for $x > 5$ and $f(x) < 0$ for $x < -5$. So any zeroes of f will be contained in $[-5, 5]$. We will show that f has 3 solutions. We only expect you to be able to show at least three solutions. You are not expected to show that it has exactly three but we included the proof anyhow.

At least three roots

- At $x = 0$, we have $f(0) = -3 <$. Therefore, by IVT there exists $c_1 \in [0, 5]$ s.t. $f(c_1) = 0$.
- At $x = -\frac{\pi}{2}$, we have $f(-\frac{\pi}{2}) = -\frac{\pi}{2} - (2sin(-\frac{\pi}{2}) + 3cos(-\frac{\pi}{2})) = -\frac{\pi}{2} - 2 \cdot 1 - 3 \cdot 0 = 2 - \frac{\pi}{2} > 0$. Therefore, by IVT there exists $c_2 \in [-\frac{\pi}{2}, 0]$ s.t. $f(c_2) = 0$.
- However, $f(-6) < 0$ and so by IVT there exists $c_3 \in [-6, -\frac{\pi}{2}]$ s.t. $f(c_3) = 0$.

Only three roots

The derivative is

$$f'(x) = 1 - 2cos(x) + 3sin(x).$$

We will show that f' has at most two roots in $[-5, 5]$. This will imply that f has two most two maximum or minimum and so at most three roots.
So we will start by solving the trigonometric equation

$$acos(x) + bsin(x) = c$$

for arbitrary $a, b, c \in \mathbb{R}$.

Lemma 0.0.1. *The solutions of the trigonometric equation*

$$acos(x) + bsin(x) = c$$

for arbitrary $a, b, c \in \mathbb{R}$ *are the following:*

$$x = 2\pi k + arcsin(\frac{c}{\sqrt{a^2 + b^2}}) - \theta_{a,b} \text{ for } k \in \mathbb{Z}$$

and $\theta_{a,b} := arcsin(\frac{a}{\sqrt{a^2+b^2}})$.

Proof.

We have

$$acos(x) + bsin(x) = \sqrt{a^2+b^2}[\frac{a}{\sqrt{a^2+b^2}}\cos(x) + \frac{b}{\sqrt{a^2+b^2}}\sin(x)].$$

Take $\theta_{a,b} \in [0, \frac{2}{\pi}]$ s.t.

$$sin(\theta_{a,b}) = \frac{a}{\sqrt{a^2+b^2}}$$

then using $cos^2(\theta_{a,b}) = 1 - sin^2(\theta_{a,b})$ we find that

$$cos(\theta_{a,b}) = \frac{b}{\sqrt{a^2+b^2}}.$$

Therefore, we rewrite

$$\begin{aligned} acos(x) + bsin(x) =& \sqrt{a^2+b^2}[\frac{a}{\sqrt{a^2+b^2}}\cos(x) + \frac{b}{\sqrt{a^2+b^2}}\sin(x)] \\ =& \sqrt{a^2+b^2}[sin(\theta_{a,b})\cos(x) + cos(\theta_{a,b})\sin(x)] \end{aligned}$$

using summation formula for sine we find

$$= \sqrt{a^2+b^2}sin(\theta_{a,b}+x).$$

Therefore, the solution is

$$\begin{aligned} \sqrt{a^2+b^2}sin(\theta_{a,b}+x) =& acos(x) + bsin(x) = c \Rightarrow \\ \sqrt{a^2+b^2}sin(\theta_{a,b}+x) =& c \Rightarrow \\ sin(\theta_{a,b}+x) =& \frac{c}{\sqrt{a^2+b^2}} \Rightarrow \\ \theta_{a,b} + x =& 2\pi k + arcsin(\frac{c}{\sqrt{a^2+b^2}}) \text{ for } k \in \mathbb{Z}. \end{aligned}$$

□

In our case we obtain

$$\begin{aligned} sin(\theta_{a,b}+x) =& \frac{-1}{\sqrt{3^2+2^2}} = \frac{-1}{\sqrt{12}} \Rightarrow \\ \theta_{a,b} + x =& 2\pi k + arcsin(\frac{-1}{\sqrt{12}}) \text{ for } k \in \mathbb{Z}. \end{aligned}$$

In our case we have $x \in [-5, 5] \subset [-2\pi, 2\pi]$ and so we obtain only two solutions

$$\begin{aligned} x =& 2\pi + arcsin(\frac{-1}{\sqrt{12}}) - \theta_{a,b} \\ x =& -2\pi + arcsin(\frac{-1}{\sqrt{12}}) - \theta_{a,b}. \end{aligned}$$