$A_{(n\times n)}, \exists B_{(n\times n)}, AB = I \rightarrow A$ is invertible (non-singular) $B = A^{-1}$

Fundamental properties: $xAA^{-1} = x$

Not every matrix has inverse (ex. $A = \vec{0}$)

If a homogeneous system $Ax = 0$ has non-trivial solution, A's not invertible

$Ax = 0 \; not \; imply \; x \neq 0$

$$ex. A = \begin{pmatrix} 2 & 5 \\ 4 & 10 \end{pmatrix} \left(\begin{array}{cc|c} 2 & 5 & 0 \\ 4 & 10 & 0 \end{array}\right) \gg \begin{pmatrix} 5t \\ -2t \end{pmatrix}$$

i) Equivalently $Ax = 0$ has only trivial solution, A is invertible.

$$A \; has \; an \; inverse \; \rightarrow \exists! A^{-1} \; (uniqueness)$$

Proof: assume B & C are inverse of A, show $B = C$

$$B, C \; are \; inverse \; of \; A, means \; BA = I = AB, CA = I = AC$$

$$B = BI = B(AC) = (BA)C = IC = C$$

ii) $(A^{-1})^{-1} = A$

iii) $A_{(n\times n)}, \exists B_{(n\times n)}, s.t. AB = I \rightarrow BA = I \; \wedge A = B^{-1} \wedge B = A^{-1}$

iv) A one-sided inverse is automatically a two-sided inverse

Proof: $Bx = 0 \rightarrow A(Bx) = 0, A(0), (AB)x = 0, Ix = 0, x = 0, B \; is \; invertible$

Show A's inverse of B, given AB = 1

$$AI = A(BB^{-1}) = IB^{-1} = B^{-1}$$

Finding $A^{-1} \; if \; exists$

$A \; is \; invertible \; \leftrightarrow \forall b \in R^n, Ax = b, \rightarrow x = A^{-1}b$

$Ax = b = A(A^{-1}b), \exists x \in R^n, s.t. Ax = b$

Multiple by A expresses in term of $AA^{-1}b = Ib = b$, multiple A^{-1} expresses x in terms of b.

$e.x. find \; A^{-1} \; of \; A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 4 & 1 \end{pmatrix} \; if \; exists$ (using $AA^{-1} = I$)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \gg \begin{pmatrix} R2 - R1 \\ R3 - 2R1 \end{pmatrix} \gg \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right)$$

$$\gg \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 13 & -2 & 5 \\ 0 & 1 & 0 & -7 & 1 & 3 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$

Process fails iff A can't convert to I (if # of pivots in $A < n$)

$$A \; is \; invetible \; iff \; \# \; pivots \; in \; A \; is \; n$$

$e.x. A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ show $(I - A)^3 = 0$ and use this to find A^{-1} without using row reduction

$$I - A = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \; (I - A)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; (I - A)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$0 = I^3 - 3I^2A + 3IA^2 - A^3$$

$$I = 3A - 3A^2 + A^3 = A(A^2 - A + 3I)$$

$$A^{-1} = A^2 - A + 3I = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse of a product:

If *A, B* are invertible, then *AB* is invertible: $(AB)^{-1} = B^{-1}A^{-1}$

Check $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

Inverse of transpose A is invertible $\rightarrow$ A^{T} is invertible: $(A^T)^{-1} = (A^{-1})^T$

Check $A^T(A^{-1})^T = (AA^{-1})^T = I^T = I$

Inverse of scalar multiple: $c \in R,\ (cA)^{-1} = \frac{A^{-1}}{c}$

$A\ n \times n$ invertible iff:

1) $Ax = b$ has unique solution for all b
2) $Ax = 0$ has only trivial solution
3) $A \rightarrow I$ by row reduction
4) A has n pivots
5) Columns of A *span* R^n
6) Columns of A are linear independent
7) Matrix transformation $T: R^n \to R^2$ defined by $T(x) = Ax$ is onto and 1-1

Elementary Matrices and inverses

An elementary matrix operation is a single elementary row operation on I

There are 3 types of elementary row operations hence 3 types of elementary matrices

Thrm: Any elementary row operations on matrix A can be done by multiply A on the left by elementary matrix E. (EA is A after an elementary row operation)

Proof: $A\ m \times n, B = EA$, WTS: $B = Rj + kRi\ of\ A, Rj(B) = Rj(A) + kRi(A)$

$$Rj(B) = Rj(EA) = Rj(E)A = \big(Rj(1) + kRi(1)\big)A = Rj(I)A + kRi(I)A = Rj(A) + kRi(A)$$

Since elementary row operations are reversible, every elementary matrix is invertible

Swap: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Multiply a row by x $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{x} \end{pmatrix}$

Add a multiple of a row to another $\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Elementary matrices and invertibility

Thrm: $A\ n \times n$ is invertible iff A can be written as a product of elementary matrices

Proof:

i) $A\ n \times n$ is invertible $\leftarrow$ A can be written as a product of elementary matrices

$$A = E_1E_2E_3 \ldots E_k \rightarrow \exists {E_1}^{-1}{E_2}^{-1}{E_3}^{-1} \ldots {E_k}^{-1} \rightarrow A^{-1}$$

ii) $A\ n \times n$ is invertible $\rightarrow$ A can be written as a product of elementary matrices

$$E_kE_{k-1}E_{k-2} \ldots E_1A = I$$

$$A^{-1} = E_kE_{k-1}E_{k-2} \ldots E_1$$

$$A = {E_1}^{-1}{E_2}^{-1}{E_3}^{-1} \ldots {E_k}^{-1}$$

$e.x. A = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ shown A is invertible through the product of elementary matrices

$$\begin{pmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \gg (R2 + 2R1) \gg \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} \gg \begin{pmatrix} R1 - 2R3 \\ R2 - 4R3 \end{pmatrix} \gg I$$

$$E1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E2 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = E3E2E1$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}$$

LU decomposition: In $n \times n$ matrix *A* is upper or lower triangular if all entries below or above diagonals are zero, (ref is an upper triangular matrix, I is both)

$$e.x. \left(\begin{array}{ccc|c} 1 & 2 & 1 & b1 \\ 1 & 3 & 4 & b2 \\ 1 & 7 & 8 & b3 \end{array}\right) \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 1 & 7 & 8 & 0 & 0 & 1 \end{array}\right) \gg\gg \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -3 & 1 & -3 & 1 \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

Solving $Ax = b$ is equivalent to solving $Yx = Lb$

To solve $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, Lb = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 2 \\ 1 & -3 & 1 & 3 \end{array}\right) = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, then\ solve \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -3 & -2 \end{array}\right)$

In general $(A \mid I) \rightarrow (Y \mid L), Ax = Ib \rightarrow Yx = Lb \rightarrow Yx = LAx \rightarrow Y = LA \rightarrow A = L^{-1}Y$

$$E1E2E3\, A = Y$$

$$A = (E_3^{-1}E_2^{-1}E_1^{-1})Y$$

$e.x.$ find LU factorization of $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \gg \begin{pmatrix} R2 - 2R1 \\ R3 - R1 \\ R3 + 3R2 \end{pmatrix} \gg \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{pmatrix}$$

$$A = (E_1^{-1}E_2^{-1}E_3^{-1})U$$

$$L = (E_1^{-1}E_2^{-1}E_3^{-1}) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{pmatrix}$$

In term of solving systems: solve $Ax = b$, since $A = LU$, $LU x = b$

Solve in 2 steps: find w, s.t. $Lw = b$, then solve $Ux = w$

$$e.x.\,solve \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \; given\; U = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{pmatrix}$$

$$Lw = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\dfrac{1}{3} & 1 \end{pmatrix} w = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$Ux = w, \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, x = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$