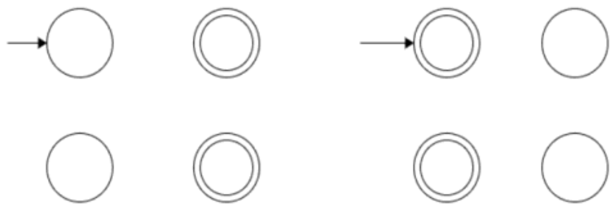
FA and Regex

Theorem Suppose $L_1, L_2 \subseteq \Sigma^*$ are accepted by FA, then so are $\Sigma^* - L_1, L_1 \cap L_2, L_1 \cup L_2, L_1L_2, L_1^*, L_1^+$

Proof Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1), M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be two DFA's such that $\mathcal{L}(M_1) = L_1, \mathcal{L}(M_2) = L_2$

Without losing the generality, assume $Q_1 \cap Q_2 = \emptyset$

1. $\Sigma^* - L_1$

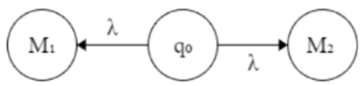


$M' = (Q_1, \Sigma, \delta_1, q_1, Q - F_1)$

Proof Let $x \in \Sigma^*$, then $x \in \mathcal{L}(M') \equiv \delta_1^*(q_1, x) \in Q_1 - F_1 \equiv \delta_1^*(q_1, x) \notin F_1 \equiv x \notin \mathcal{L}(M_1) = x \in (\Sigma^* - L_1)$

This construction mightn't work if M_1 NFA since $\delta_1^*(q_1, x) \cap Q_1 - F_1 \neq \emptyset$ AND $\delta_1^*(q_1, x) \cap F_1 \neq \emptyset$ can both be true

2. $L_1 \cup L_2$

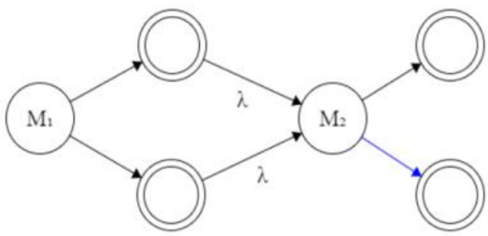


OR $M' = (Q_1 \times Q_2, \Sigma, \delta', \{q_1, q_2\}, F_1 \times Q_2 \cup F_2 \times Q_1)$

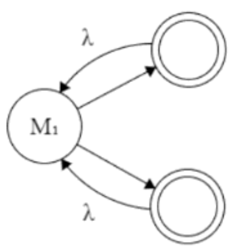
3. $L_1 \cap L_2 = \overline{\overline{L_1} \cap \overline{L_2}}$

OR $M'' = (Q_1 \times Q_2, \Sigma, \delta', \{q_1, q_2\}, F_1 \times Q_2 \cap F_2 \times Q_1)$

4. $L(M_1)L(M_2)$



5. $(L(M_1))^+$



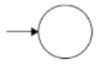
6. $(L(M_1))^* = (L(M_1))^+$ with $q_0 \in F$

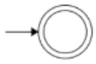
Theorem All regular language can be accepted by a FA

Proof Base case: $\mathcal{L}(\phi) = \emptyset$

$\mathcal{L}(\lambda) = \lambda$

$\forall a \in \Sigma. \mathcal{L}(a) = \{a\}$

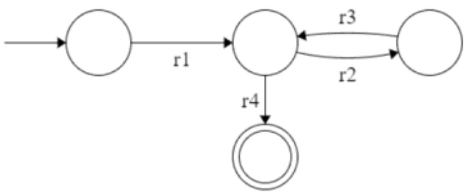




Constructor case:

Suppose $\mathcal{L}(r'), \mathcal{L}(r'')$ accepted by FA, then
$\mathcal{L}(r'r'') = \mathcal{L}(r')\mathcal{L}(r'')$
$\mathcal{L}(r' \cup r'') = \mathcal{L}(r') \cup \mathcal{L}(r'')$
$\mathcal{L}\left((r')^*\right) = \left(\mathcal{L}(r')\right)^*$
By closure results, they are all accepted by FA

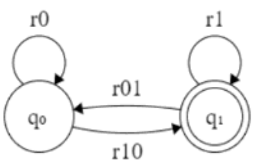
Definition generalized transition graph:
A NFA with 1 initial state and in which edges are labelled by regex instead of subset of Σ or $\Sigma \cup \{\lambda\}$
Then, a string is accepted by a GTG IFF $\exists k \in \mathbb{N}.\ (x = x_1 \dots x_k$ where$x_1, \dots, x_k \in \Sigma^*$ and $\exists$path $e_1 \dots e_k$ from q_0 to $f \in F$ s.t. $x_i \in \mathcal{L}(r_i)$ for $i = 1, \dots, k$ and r_i is the regex labelled on e_i
Example



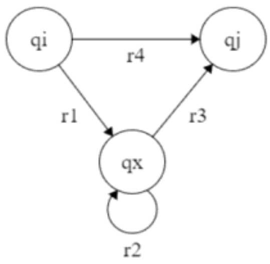
If $x = x_1x_2x_3x_4$ where $x_i \in \mathcal{L}(r_i)$, then M accepts x

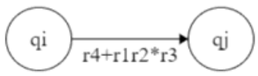
Theorem $\forall A \in FA, \exists r \in \text{regex}.\ \mathcal{L}(A) = \mathcal{L}(r)$
Discussion without lose generality, take a GTG has exactly one final state that is not q_0
Also assume $\forall q, q' \in Q$ there exists edge between, if not add an edge labelled ϕ
Then, prove by induction on the number of states in $M = (Q, \Sigma, \delta, q_0, F)$
To prove if $L = \mathcal{L}(M)$ for a GTG M, then $L = \mathcal{L}(r)$ for a regex r
By previous theorem, $L = \mathcal{L}(M')$ for FA M'
By definition, all NFA is a GFG

Example
Proof base case: $n = 2$



Then, $\mathcal{L}(M) = \mathcal{L}\left(r_0^* r_{01}\left(r_1 + r_{10} r_0^* r_{01}\right)^*\right)$
Suppose $n > 2$, constrcut a GFG M' s.t. $\mathcal{L}(M') = \mathcal{L}(M)$ which has 1 fewer state, then by IH, exists $r, \mathcal{L}(M') = \mathcal{L}(r)$
To construct such GFG, take arbitrary $q_x \in Q - \{q_0\} - F$, let $Q' = Q - \{q_x\}$
Without lose generality, assume q_x has edge connected to q_i, q_j and have paths as labelled below.


qi
qj
r4
r1
r3
qx
r2


qi
qj
r4+r1r2*r3