Outline: Week 3 T

Section 4.3 from D& D

1. Definition of limit points and closed sets. Examples: [a,b], $\mathbb{R}^n$,
2. Continuous mapping theorem: If $x_n \to x$, then $f(x_n) \to f(x)$. The corollary is that if C is closed and bounded, then $f(C)$ is also closed (and bounded).
3. Finite union of closed sets is closed.
4. Definition of open sets. Examples $(0,1)$. Show $\{x : |f(x)| < r\}$ is open for continuous function f. In general, if U is open then the preimage $f^{-1}(U)$ is also open.
5. A is open iff A^c is closed:
   - Assume A is open and sequence $x_n \in A^c$ going to $x_n \to x$. WTS that $x \in A^c$. Assume not i.e. $x \notin A^c \Leftrightarrow x \in A$. Obtain a contradiction using a ball around x that is fully contained in A.
   - Assume A^c is closed and pick $x \in A$. WTF ball $B_\delta(x) \subset A$. Assume not i.e. for every $\delta > 0$, the $B_\delta(x) \nsubseteq A \Leftrightarrow B_\delta(x) \cap A^c \neq \varnothing$. So we have a sequence $x_n \in B_\delta(x) \cap A^c$ that converges to $x_n \to x$ (Why is that? use limit definition). Since $x_n \to x$ and A^c is closed we obtain $x \in A^c$, which contradicts $x \in A$ ($A \cap A^c = \varnothing$).
6. Finite intersection of open is open. We used DeMorgan's law
7. Exercise 4.3.F: A set S has empty interior if no ball is fully contained in the set i.e. $B(x;\delta) \nsubseteq S$.

Finite intersection of open is open

We will use that "Finite union of closed sets is closed" and DeMorgan's law

$$(\bigcap_{i\geq 1}^{M} U_i)^c = \bigcup_{i\geq 1}^{M} U_i^c.$$

Let $A := U_1$ and $B := \bigcap_{i\geq 2}^{M} U_i$, we will prove this by bootstrapping. We will prove $(A \cap B)^c = A^c \cup B^c$. Here is a word proof. We will show that one is contained in the other.

First we show $(A \cap B)^c \subset A^c \cup B^c$

$$
\begin{aligned}
& x \in (A \cap B)^c \Leftrightarrow \\
& x \notin A \cap B \Leftrightarrow \\
& x \notin A \text{ or } x \notin B \Leftrightarrow \\
& x \in A^c \text{ or } x \in B^c.
\end{aligned}
$$

Next, we show $A^c \cup B^c \subset (A \cap B)^c$ to finish the claim.

$$
\begin{aligned}
& y \in A^c \cup B^c \Leftrightarrow \\
& y \in A^c \text{ or } y \in B^c \Leftrightarrow \\
& y \notin A \text{ or } y \notin B \Leftrightarrow \\
& y \notin A \cap B \Leftrightarrow \\
& y \in (A \cap B)^c.
\end{aligned}
$$

Here is a set-notation proof. We have

$$x \in (A \cap B)^c = \mathbb{R} \setminus (A \cap B),$$

which we can split as union of three sets

$$\mathbb{R} \setminus (A \cap B) = (B \setminus A) \sqcup (A \setminus B) \sqcup (\mathbb{R} \setminus (A \cup B))$$

because $\mathbb{R} \setminus A = (B \setminus A) \sqcup (\mathbb{R} \setminus (A \cup B))$ we find

$$
\begin{aligned}
&= (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B) \\
&= A^c \cup B^c.
\end{aligned}
$$

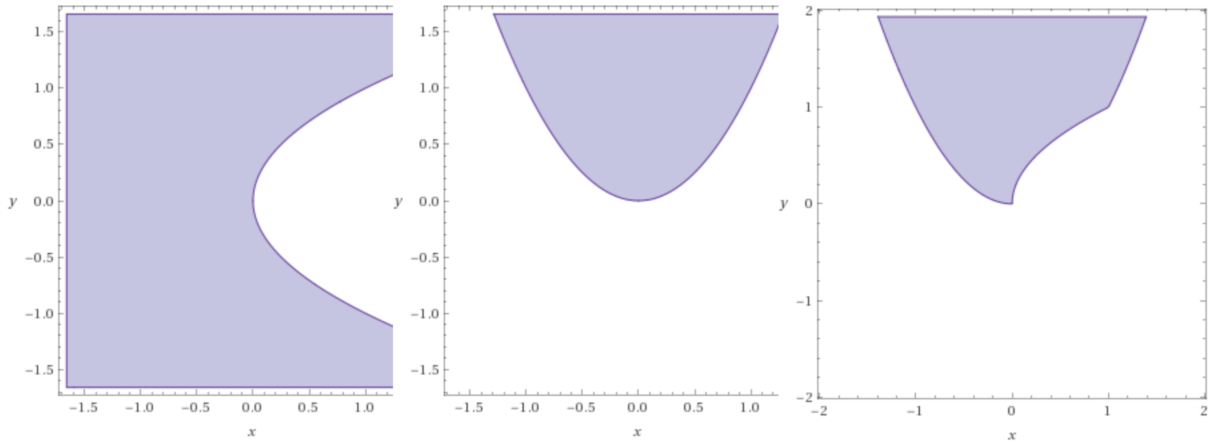
0.0.1 Exercise 4.3.F

A set S has empty interior if no ball is fully contained in the set i.e. $B(x; \delta) \not\subseteq S$.

- A has empty interior because we can approximate the first coordinate by irrationals. B has empty interior because we can approximate the first coordinate by rationals.

- Next we study the union $A \cup B$. We draw the two parabolas

$$A_{full} := \{(x,y) : y^2 \geq x\} \text{ and } B_{full} := \{(x,y) : y \geq x^2\}$$



$y^2 \geq x$ $\quad$ $y \geq x^2$ $\quad$ Their intersection $A_{full} \cap B_{full}$

- We write

$$A_{full} \cup B_{full} = A_{full} \setminus B_{full} \sqcup A_{full} \cap B_{full} \sqcup B_{full} \setminus A_{full}.$$

- The region $A_{full} \setminus B_{full}$ will have an empty interior if we restrict the x-coordinate to be rational (by approximating with irrationals).

- The region $B_{full} \setminus A_{full}$ will have an empty interior if we restrict the x-coordinate to be irrational (as argued above by approximating with rationals).

- The region $A_{full} \cap B_{full}$ will have an x coordinate that is free to be either rational or irrational.

- Therefore, the interior of $A \cup B$ will be the region

$$A_{full} \cap B_{full} := \{(x,y) : x \in \mathbb{R}, y^2 \geq x \text{ and } y \geq x^2\}.$$