Week 9: Tutorial Handout on Uniform and Integral convergence

- Pointwise convergence $f_n(x_0) \to f(x_0)$ is defined as $|f_n(x_0) - f(x_0)| \to 0$.
- Uniform convergence $f_n \rightrightarrows f$ is defined as $\|f_n - f\|_\infty \to 0$.

8.1 Limits of functions

1. (D&D 8.1.B) Let $f_n(x) := nx(1-x^2)^n$ on [0,1] for $n \geq 1$. Find the limit $\lim\limits_{n\to\infty} f_n(x)$. Is it uniform? Compare $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\mathrm{d}x$ with $\int_0^1 \lim\limits_{n\to\infty} f_n(x)\mathrm{d}x$.
2. (D&D 8.1.B) Let $f_n(x) := nxe^{-nx}$ on [0,R] for $n \geq 1$. Find the limit $\lim\limits_{n\to\infty} f_n(x)$. Is it uniform? Compare $\lim\limits_{n\to\infty} \int_0^R f_n(x)\mathrm{d}x$ with $\int_0^R \lim\limits_{n\to\infty} f_n(x)\mathrm{d}x$. What happens when $R \to +\infty$?

8.2 Uniform convergence properties

3. Find the pointwise limits of the following functions. Find an interval on which convergence is uniform and another on which it is not.
   - $f_n(x) := (\frac{x}{2})^n + (\frac{1}{x})^n$
   - $g_n(x) := \frac{nx}{2+5nx}$.

Integral convergence

4. (D&D 8.3.D): For $n \geq 1$ define f_n on $[0,\infty)$ by

$$f_n(x) := \begin{cases} e^{-x} & 0 \leq x \leq n \\ e^{-2n}(e^n + n - x) & n \leq x \leq e^n + n \\ 0 & e^n + n \leq x \end{cases}.$$

   (a) Find the pointwise limit f and show that the convergence is uniform on $[0,\infty)$.
   (b) compute $\lim\limits_{n\to\infty} \int_0^R f_n(x)\mathrm{d}x$ and $\int_0^R \lim\limits_{n\to\infty} f_n(x)\mathrm{d}x$ (Take $n > R$).
   (c) What happens as $R = \infty$? Why doesn't it contradict the integral convergence theorem?