MAT337 Real Analysis **Name:** ______________________

Summer 2019 **Student No:** ______________________

Midterm 2 **Time Limit: 180 Minutes**

This Crowdmark-Test contains 10 pages (including this cover page) and 7 questions. No aids allowed. The back pages are ONLY for rough work, they will be not be scanned. You are allowed to use any of theorems from the textbook unless we ask you to specifically prove it. You are not allowed to use homework problems.
Total of points is 100.

Grade Table (for grader use only)

Question	Points	Score
1	8	
2	10	
3	10	
4	12	
5	20	
6	20	
7	20	
Total:	100	

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1. (8 points) Circle the letter of **one** correct answer. More than one answers could be true but we only ask you to pick **one** of them.

   (a) (2 points) Properties about norms, norm topology and inner products.

   A. For $f \in C([0,1])$ the sup norm is $\|f\|_\infty := \sup_{x\in[0,1]} f(x)$.

   B. The set $\{f \in C([0,1]) : f(x) > 0\}$ is open in $C([0,1])$ with respect to the sup norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$.

   C. Every norm $\|\cdot\|$ is bilinear and thus can define an inner product.

   D. Let T be an $n \times n$ matrix, then $\langle Tx, Ty\rangle$ is an inner product on $\mathbb{R}^n$ iff T is surjective.

   (b) (2 points) Statements about uniform convergence $f_n \rightrightarrows f$.

   A. If the f_n are strictly monotone i.e. $x < y \Rightarrow f_n(x) < f_n(y)$ and $f_n \rightrightarrows f$, then f is monotone i.e. $x < y \Rightarrow f(x) \leq f(y)$.

   B. If $f_n \in C^1([0,1])$ and $f_n \rightrightarrows f$, then $f \in C^1([0,1])$.

   C. If f_n and f are continuous on $\mathbb{R}$, $f_n \leq f_{n+1}$ and $f_n(x) \overset{pointwise}{\to} f(x)$ for $x \in \mathbb{R}$, then $f_n \rightrightarrows f$ over $\mathbb{R}$.

   (c) (2 points) Statements about integral convergence theorem and Leibniz rule.

A. Suppose that $f_n \rightrightarrows f$ on $\mathbb{R}$ then

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{d}x.$$

B. Suppose that $f(x,t)$ and $\partial_x f(x,t)$ are both continuous over $[a,b]\times[c,d]$ then

$$\partial_x \int_c^d f(x,t)\mathrm{d}t = \int_c^d \partial_x f(x,t)\mathrm{d}t.$$

C. Suppose that f_n are continuously differentiable $C^1([0,1])$ and its derivatives uniformly converge to some g $f_n' \rightrightarrows g$, then $f_n \rightrightarrows f$ where $f'(x) = g(x)$.

(d) (2 points) Statements about series.

A. If the following limit exists. $\lim\limits_{n\to\infty} \frac{a_n}{a_{n+1}} = L$ then the radius of convergence is $\frac{1}{L}$.

B. Suppose that $f_k : S \subset \mathbb{R} \to \mathbb{R}$ satisfy $\sup_{x\in S} |f_k(x)| \le M_k$ where $\sum_{k\ge 1} M_k < \infty$, then the series $\sum_{k\ge1} f_k(x)$ converges uniformly on S.

C. Suppose that the series $\sum a_k x^k$ has radius of convergence equal to R but diverges at the endpoints i.e. the interval is $(-R,R)$, then its derivative $\sum a_k k x^{k-1}$ has the same interval of convergence $(-R,R)$.

D. Suppose that the series $\sum a_k x^k$ has radius of convergence equal to R but diverges at the endpoints i.e. the interval is $(-R,R)$, then its integral $\sum a_k \frac{x^{k+1}}{k+1}$ has the same interval of convergence $(-R,R)$.

2. (10 points) Here we will show that $C([0,1])$ with the L^2 norm $\|f\|_{L^2} := \int_0^1 f^2 \mathrm{d}x$ is not complete. Consider the sequence

$$f_n(x) := \begin{cases} 0 & , x \in [0, \frac{1}{2} - \frac{1}{n}] \\ 1 + n(x - \frac{1}{2}) & , x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ 1 & , x \in [\frac{1}{2}, 1] \end{cases}.$$

Show that the sequence is $\|\cdot\|_{L^2}$ –Cauchy but $\|\cdot\|_{L^2}$ –convergent to a function $f \notin C([0,1])$.

1. We show Cauchy for $n > m > N$ for large enough N

$$\|f_n - f_m\|_{L^2} = \int_0^1 (f_n(x) - f_m(x))^2 \mathrm{d}x$$
$$= \int_0^1 f_n^2(x)\mathrm{d}x - 2\int_0^1 f_n(x) f_m(x)\mathrm{d}x + \int_0^1 f_m^2(x)\mathrm{d}x$$
$$= \frac{1}{2} + \frac{1}{2n} - 2\int_0^1 f_n(x) f_m(x)\mathrm{d}x + \frac{1}{2} + \frac{1}{2n}.$$

We split the middle integral into

$$\int_0^1 f_n(x) f_m(x)\mathrm{d}x = \int_0^{\frac{1}{2}-\frac{1}{m}} f_n(x) f_m(x)\mathrm{d}x + \int_{\frac{1}{2}-\frac{1}{m}}^{\frac{1}{2}-\frac{1}{n}} f_n(x) f_m(x)\mathrm{d}x$$
$$+ \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} f_n(x) f_m(x)\mathrm{d}x + \int_{\frac{1}{2}}^{1} f_n(x) f_m(x)\mathrm{d}x$$
$$= 0 + 0 + \int_{\frac{1}{2}-\frac{1}{n}}^{\frac{1}{2}} \left(1 + n(x - \frac{1}{2})\right)\left(1 + m(x - \frac{1}{2})\right)\mathrm{d}x + \frac{1}{2}$$
$$= \frac{1}{n} + \frac{nm}{3n^3} - \frac{n+m}{2n^2} + \frac{1}{2}.$$

So returning to our difference we find the bound

$$\|f_n - f_m\|_{L^2} \leq \frac{2}{n} + \frac{nm}{3n^3} - \frac{n+m}{2n^2},$$

which is less than ϵ for large enough N.

2. First we have pointwise convergence to $\chi_{[\frac{1}{2},1]} \notin C([0,1])$ because $[\frac{1}{2}-\frac{1}{n},\frac{1}{2}] \to \{\frac{1}{2}\}$ (by the Cantor intersection theorem). But we also have it in L2:

$$\begin{aligned}\left\|f_n - \chi_{[\frac{1}{2},1]}\right\|_{L^2} &= \int_0^1 f_n^2(x)\mathrm{d}x - 2\int_0^1 f_n(x)\chi_{[\frac{1}{2},1]}(x)\mathrm{d}x + \int_0^1 \chi^2_{[\frac{1}{2},1]}(x)\mathrm{d}x \\ &= \frac{1}{2} + \frac{1}{2n} - 2\int_0^{\frac{1}{2}} f_n(x)(x)\mathrm{d}x + \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{2n} - 1 + \frac{1}{n} + \frac{1}{2} \\ &= \frac{3}{2n}.\end{aligned}$$

So $f_n \overset{L2}{\to} \chi_{[\frac{1}{2},1]}$.

3. (10 points) Show that $g(x) := \sum_{n=1}^{\infty} x^n e^{-nx}$ converges uniformly on $[0, A]$ for each $A > 0$. Does it converge uniformly on $[0, \infty)$?

We have by a quick derivative computation that $\frac{x}{e^x} \leq \frac{1}{e}$ for all $x \geq 0$, therefore

$$\sum_{n=1}^{\infty} x^n e^{-nx} \leq \sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{1 - \frac{1}{e}} - 1.$$

So by the M-test we have uniform convergence for $[0, A]$. In particular we have uc over $\mathbb{R}$ because

$$\sum_{n=N}^{\infty} x^n e^{-nx} \leq \sum_{n=N}^{\infty} \frac{1}{e^n} = \frac{e^{-N}}{1 - \frac{1}{e}} \to 0$$

for all $x \in \mathbb{R}$ uniformly.

4. (12 points) Prove whether the following set

$$D := \{f \in C^2([0,1]) : f(x)f'(x) > 0 \text{ for } 0 < x < 1\}$$

is open in $C^2([0,1])$ with respect to the norm $\|f\|_{C^2} := \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty$.

This set is not open. Let $f_0(x) := x$ and $g_\delta(x) := x - \delta$ for $\delta > 0$. Then

$$|f_0(x) - g_\delta(x)| = |\delta| \leq \delta$$

and

$$|f_0'(x) - g_\delta'(x)| = 0 \leq \delta.$$

So they are $\delta-$close in $\|\cdot\|_{C^2}$

$$\|f_0 - g_\delta\|_{C^2} \leq \delta.$$

However,

$$g_\delta(x)g_\delta'(x) = (x - \delta) = x - \delta < 0$$

whenever $x < \delta$.

5. (20 points) For $n \geq 1$, define the functions f_n on $[0, M]$ by

$$f_n(x) := \begin{cases} e^{-x} & , \text{ for } 0 \leq x \leq n \\ e^{-2n}(e^n + n - x) & , \text{ for } n \leq x \leq n + e^n \\ 0 & , \text{ for } n + e^n \leq x \leq M \end{cases} .$$

Compute $\lim\limits_{n\to\infty} \int_0^M f_n(x)\mathrm{d}x$.

We have uniform convergence to e^{-x} because at the first interval they agree, at the second interval we get

$$\sup_{n \leq x \leq n+e^n} \left| e^{-2n}(e^n + n - x) - e^{-x} \right|$$

attains its supremum at $x = 2n \in [n, n + e^n]$ and so

$$\leq e^{-2n}(e^n - n - 1) \to 0$$

and at the third integral we get

$$\sup_{n \leq x \leq n+e^n} \left| e^{-x} \right| \leq e^{-n} \to 0.$$

So by ICT for compact $[0, M]$ we get

$$\lim_{n\to\infty} \int_0^M f_n(x)\mathrm{d}x = \int_0^M \lim_{n\to\infty} f_n(x)\mathrm{d}x = \int_0^M e^{-x}\mathrm{d}x = 1 - e^{-M}.$$

6. (20 points) Suppose that $f_n : [0,1] \to \mathbb{R}$ are continuously differentiable functions such that

   1. pointwise convergence: $f_n(x_0) \to f(x_0)$ for all $x_0 \in [0,1]$,
   2. bounded derivative: $\|f_n'(x)\| \leq M$.

   then we have uniform convergence $f_n \rightrightarrows f$.

   Since such functions are Lipschitz, the same proof is true as in PS5. First we will show that $\{f_n\}$ is a Cauchy sequence in $(C([0,1]), \|\cdot\|_\infty)$.

   First, split [0,1] in intervals $\{I_k\}_{k=1}^M$ of size $|I_k| := \frac{\epsilon}{3L}$ and pick a representative $x_k \in I_k$ from each one such as the midpoint. Second by pointwise continuity there exists N_k s.t. $\forall n, m \geq N_k$

$$|f_m(x_k) - f_n(x_k)| \leq \frac{\epsilon}{3}.$$

   So now take $N := \max_{k=1,\ldots,M} N_k$ and $n \geq N$. Any given x falls inside some interval $x \in I_q$. By triangle inequality we have for $n \geq N$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_q)| + |f_m(x_q) - f_n(x_q)| + |f_n(x_q) - f_n(x))|$$

   the first and third term are bounded by Lipschitz and the second term by pointwise continuity

$$\begin{aligned} &\leq L\,|x - x_q| + \frac{\epsilon}{3} + L\,|x_q - x| \\ &\leq L\frac{\epsilon}{3L} + \frac{\epsilon}{3} + L\frac{\epsilon}{3L} \\ &= \epsilon \Rightarrow \\ &|f_m(x) - f_n(x)| \leq \epsilon \end{aligned}$$

   for all $n \geq N$. Therefore $\{f_n\}$ is $\|\cdot\|_\infty$-Cauchy. By completeness of $(C([0,1]), \|\cdot\|_\infty)$ we obtain some continuous limit $f \in C([0,1])$.

   The Lipschitz property for f follows quickly by triangle inequality

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq \epsilon_n + L\,|x - y| + \epsilon_n,$$

   where due to uniform convergence the ϵ_n does not depend on x,y and so by taking limit in $n \to \infty$ on both sides we find

$$|f(x) - f(y)| \leq L\,|x - y|.$$

7. (20 points) (New question) Consider $f \in C([0,\pi])$ with norm $\|f\|_{L^2} := \frac{2}{\pi}\int_0^\pi f^2 \mathrm{d}x$. For the coefficients

$$a_0 := \frac{2}{\pi}\int_0^\pi f(x)\frac{1}{\sqrt{2}}\mathrm{d}x, a_n := \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\mathrm{d}x \text{ and } b_n := \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\mathrm{d}x,$$

let

$$S_N(x) := a_0\frac{1}{\sqrt{2}} + \sum_{n\geq 1}^{N} a_n\cos(nx) + b_n\sin(nx).$$

If the coefficients are absolutely summable: $\sum_{n\geq 1}|a_n|+|b_n| < \infty$, then prove that we have uniform convergence $S^N \rightrightarrows f$ i.e.

$$\sup_{x\in[0,\pi]} |S_N(x) - f(x)| \to 0 \text{ as } N \to +\infty.$$

By the M-test we have uniform convergence of $S_N \to g$. By completeness of continuous functions we have that g is continuous. It remains to prove that $f = g$. By orthogonality, f and g have the same Fourier coefficients

$$\int f(x)\cos(nx)\mathrm{d}x = a_n = \int g(x)\sin(nx)\mathrm{d}x.$$

Let $\widetilde{a}_n$ be the Fourier coefficient of $f - g$ i.e.

$$\widetilde{a}_n := \int (f(x) - g(x))\cos(nx)\mathrm{d}x = 0$$

and same for $\widetilde{b}_n$.By Parseval's identity we have

$$0 = \sum \widetilde{a}_n^2 + \widetilde{b}_n^2 = \int (f(x) - g(x))^2\mathrm{d}x.$$

So by continuity of $f - g$ we will show that $f(x) = g(x)$. Suppose that there exists x_0 such that $f(x_0) - g(x_0) > 0$ then for $\epsilon > 0$ there exists $\delta > 0$ such that we have the following for $B_\delta(x_0)$

$$0 = \int (f(x) - g(x))^2\mathrm{d}x > \int_{B_\delta(x_0)} (f(x) - g(x))^2\mathrm{d}x > \int_{B_\delta(x_0)} \epsilon^2\mathrm{d}x = \epsilon^2\delta > 0,$$

which is a contradiction. The argument is similar if $f(x_0) - g(x_0) < 0$.