Outline: Week 1 T

Reals

1. We defined the decimal expansion.
2. We proved that $x \in \mathbb{R}$ is rational iff its decimal expansion is terminating or repeating.
3. We proved that if $x < y$, then there exists a terminating decimal expansion r s.t. $x < r < y$.

Least upper bound

1. We defined sup(S) for bounded above set S and inf(C) for a bounded below set.
2. we proved that $sup(0, 1) = 1$.
3. We proved that for bounded above set S, we have $u = \sup(S)$ if and only if

$$u \geq s, \forall s \in S$$

and $\forall \varepsilon > 0$ there exists $s_\varepsilon \in S$ s.t.

$$u - \varepsilon \leq s_\varepsilon.$$

The proof of this is in Abbott lemma 1.3.7

Detailed proof of rational iff terminating or repeating

Proposition 0.0.1. *$x \in \mathbb{R}$ is rational iff its decimal expansion is terminating or repeating.*

First the necessary condition: we prove that if $x = \frac{l}{m}$ then its decimal expansion is terminating or repeating.

Proof. The proof is an inductive use of Euclidean division (ED).

1. By ED, there exists q and $r_0 < m$ s.t. $l = q{*}m + r_0$. This implies that $q \leq \frac{l}{m} = q + \frac{r_0}{m} \leq q + 1$

2. Repeat for $\frac{r_0}{m}$: $10r_0 = d_1 \cdot m + r_1$ and so we find

$$\frac{l}{m} = q + \frac{d_1}{10} + \frac{r_1}{10m}.$$

3. Induction Hypothesis: $\frac{l}{m} = q + \sum_{k=1}^{n} \frac{d_k}{10^k} + \frac{r_n}{10^n m}$. We repeat $10r_n = d_{n+1}m + r_{n+1} \Rightarrow \frac{r_n}{10^n m} = \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$

$$\frac{l}{m} = q + \sum_{k=1}^{n} \frac{d_k}{10^k} + \frac{d_{n+1}}{10^{n+1}} + \frac{r_{n+1}}{10^{n+1}m}$$

   So we got the n+1 case.

4. From $r_l \in \{0, ..., m-1\}$ we get that $r_n = r_k$ for some $k < n$. Therefore, by uniqueness of ED, we get $d_{k+1} = d_{n+1}$ and $r_{k+1} = r_{n+1}$.

5. if at some stage $r_n = 0$ we are done. Otherwise since $0 < r_k \leq m-1$, the remainders will repeat.

$\square$

Second the sufficient condition: we prove that if x's decimal expansion is terminating or repeating, then $x = \frac{l}{m}$.

Proof. Let $x = x_0.x_1....$ If the expansion is terminating then

$$x = x_0.x_1...x_n00... = x_0 + \sum_{k=1}^{n} \frac{x_k}{10^k},$$

which is a rational number. So suppose that the expansion is repeating:

$$x = x_0.x_1...x_k x_{k+1}...x_n x_{k+1}...x_n... = x_0.x_1...x_k\overline{x_{k+1}...x_n}.$$

1. It suffices to show that for any digits $0.\overline{d_1...d_n}$ is a rational number because

$$\begin{aligned} x =& x_0.x_1...x_k\overline{x_{k+1}...x_n} \\ =& x_0.x_1...x_k00... + 0.0...0\overline{x_{k+1}...x_n} \end{aligned}$$

   and we already know that $x_0.x_1...x_k00...$ is rational.

2. We have that

For general expansion we write

$$\begin{aligned} 0.d_1 d_2.... &= \sum_{k=1}^{\infty} \frac{d_k}{10^k} \\ &= \sum_{k=1}^{n} \frac{d_k}{10^k} + \sum_{k=n+1}^{2n} \frac{d_k}{10^k} + \sum_{k=2n+1}^{3n} \frac{d_k}{10^k} + ... \end{aligned}$$

However, by periodicity we have that for $k > n$ $d_k = d_{k-n} =$. Therefore,

$$= \sum_{k=1}^{n} \frac{d_k}{10^k} + \sum_{k=n+1}^{2n} \frac{d_{k-n}}{10^k} + \sum_{k=2n+1}^{3n} \frac{d_{k-2n}}{10^k} + ...$$

by shifting the index we write

$$= \sum_{k=1}^{n} \frac{d_k}{10^k} + \sum_{k=1}^{n} \frac{d_k}{10^{2k}} + \sum_{k=1}^{n} \frac{d_k}{10^{3k}} + ...$$

so we write it as one double sum

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{n} \frac{d_k}{10^{mk}}.$$

Next we use the geometric series

$$\sum_{m=1}^{\infty} x^m = \frac{1}{1-x} \text{ for } |x| < 1.$$

In particular,we swap the sums

$$= \sum_{k=1}^{n} d_k \sum_{m=1}^{\infty} (\frac{1}{10^k})^m$$

for $x = 10^{-k}$ we have

$$= \sum_{k=1}^{n} d_k \frac{1}{1-10^k}.$$

Therefore, we proved that

$$0.\overline{d_1 ... d_n} = \sum_{k=1}^{n} d_k \frac{1}{1-10^k},$$

which is a finite sum of rationals and thus rational.

$\square$