If $A = (a)\ is\ [x], \det A = a$, if $A = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix}\ is\ 2 \times 2, \det A = ad - bc$

Take $\det A, A\ n \times n$ reduce to a bunch of det of 2*2

$A\ n \times n = a_{ij}, \det A_{ij} : \det of\ n-1 \times n-1$ matrix obtained from A by deleting ith row and jth column.

The (*i, j*) cofactor of A, $C_{ij}(A) = (-1)^{i+j} \det A_{ij}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \det A_{23} = \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = -6, C_{23}(A) = (-1)^{2+3}(-6) = 6,$$

$$C_{22}(A) = (-1)^{2+2} \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} = -12$$

Laplace expansion:

$A\ n \times n = a_{ij}$, the Laplace expansion along row i of A for $\det A = a_{i1}C_{i1}(A) + \cdots + a_{in}C_{in}(A)$

Along column *j* of A $\det A = a_{1j}C_{1j}(A) + \cdots + a_{nj}C_{nj}(A)$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, expand\ along\ row\ 3$$

$$\begin{aligned} \det A &= a_{31}C_{31}(A) + a_{32}C_{32}(A) + a_{33}C_{33}(A) \\ &= 7(-1)^{3+1} \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} + 8(-1)^{3+2} \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} + 9(-1)^{3+3} \det \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \\ &= -21 + 48 - 27 = 0 \end{aligned}$$

$$expand\ along\ col\ 2$$

$$\det A = -2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 5 \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 8 \det \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = 12 - 60 + 48 = 0$$

$$A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 1 & 2 & 7 \\ 0 & -1 & 3 & 0 \\ 0 & 8 & 9 & -5 \end{pmatrix} expand\ along\ row\ 1$$

$$\det A = 3 \det \begin{pmatrix} 1 & 2 & 7 \\ -1 & 3 & 0 \\ 8 & 9 & -5 \end{pmatrix} expand\ along\ row\ 2$$

$$= 3\left(-(-1) \det \begin{pmatrix} 2 & 7 \\ 9 & -5 \end{pmatrix} + 3\det(\begin{matrix} 1 & 7 \\ 8 & -5 \end{matrix})\right) = -768$$

Laplace expansion on a ref matrix (upper triangular matrix): det = product of entries on matrix diagonal

Properties of det:

1) Row exchange property: if B is obtained from A by interchanging any two rows/cols of A, detB = -detA
2) Row scalar property: if B is obtained from A by multiplying any row/col of A by c, $c \in R, \det B = c \det A$
3) Row additive property, let $U, V, A2, \dots, An\ be\ all\ 1 \times n\ or\ all\ n \times 1$,

$$\det\begin{pmatrix} U+V \\ A2 \\ \dots \\ An \end{pmatrix} = \det\begin{pmatrix} U \\ A2 \\ \dots \\ An \end{pmatrix} + \det\begin{pmatrix} V \\ A2 \\ \dots \\ An \end{pmatrix}$$

$$e.x. given \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix} = -2, find \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 5 \end{pmatrix} = \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix} = -2 + 10 = 8$$

Reduction and det

If A has two identical cols/rows, then detA = 0.

If A has two identical rows, let B be matrix obtained from A by interchanging the two identical rows, then detA = detB, while detA = -detB, therefore detA = 0

Implication:

$$A = \begin{pmatrix} A1 \\ A2 \\ A3 \end{pmatrix}, \det B = \det\begin{pmatrix} A1 \\ A2 \\ A3 \end{pmatrix} + \det\begin{pmatrix} A1 \\ A2 \\ 2A1 \end{pmatrix} (B\ is\ the\ matrix\ after\ R3 + 2R1)$$

$$\det B = \det\begin{pmatrix} A1 \\ A2 \\ A3 \end{pmatrix} + 2\det\begin{pmatrix} A1 \\ A2 \\ A1 \end{pmatrix} = \det\begin{pmatrix} A1 \\ A2 \\ A3 \end{pmatrix} = \det A$$

Adding a multiple of one row to another does not change values of the det

$$A = \begin{pmatrix} 3 & 6 & 9 & 12 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix}$$

$$\det A = 3\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 2 & 1 \\ 3 & 5 & 2 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix} \left(\frac{1}{3}R1\right)$$

$$= 3\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -1 & -3 \\ 0 & -1 & -7 & -11 \\ 0 & 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} R2 - R1 \\ R3 - 3R1 \end{pmatrix}$$

$$= -3\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & -1 & -7 & -11 \\ 0 & 0 & -1 & -3 \end{pmatrix} (R2 \leftrightarrow R4)$$

$$= -3\det\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 0 & -1 \end{pmatrix} \left(R3 + \frac{R2}{2}\right)\left(R4 - \frac{1}{5}R3\right)$$

$$= -3(10) = -30$$

Determinants and invertibility:

$A\ n \times n$ invertible iff A can be reduced to I

Theorem: $A\ n \times n$ invertible iff $\det A \neq 0$

Proof: let R be ref of A, detA = non-zero multiple of detR

Properties:

$$A, B\ n \times n, \det(AB) = \det A \det B$$

$$A\ n \times n\ invertible\ \rightarrow \det A^{-1} = \frac{1}{\det A}$$

$$\det(cA) = c^n \det A \qquad c \in R, \det A = \det A^T$$