Outline: Week 2 TR

1. Intro to Cauchy sequence
2. cauchy sequence is bounded
3. Constructing the reals using Cauchy sequences (online resource)
4. The reals are complete. (2.8.5 theorem)
5. Exercise 2.8.H: the Dottie Number.

Detailed proof for the Dottie Number

We first we show that $a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1}$ for $n \geq 0$.

- Base case: $a_0 = 0, a_1 = cos(0) = 1, a_2 = cos(1), a_3 = cos(cos(1))$ so indeed
$$0 \leq cos(1) \leq cos(cos(1)) \leq 1.$$
The middle inequality follows because cos(x) is decreasing for $x \in [0, 1]$ so $cos(1) < 1 \Rightarrow cos(cos(1)) > cos(1)$. Also, note that they are all non-negative numbers and bounded by 1.

- IH: Assume we have $0 \leq a_{2n-4} \leq a_{2n-2} \leq a_{2n-1} \leq a_{2n-3} \leq 1$. We will show that $0 \leq a_{2n} \leq a_{2n+2} \leq a_{2n+3} \leq a_{2n+1} \leq 1$. First for $x \in [0, 1]$ we have $1 \geq cos(x) \geq 0$ and thus the bounds follow from the IH.

- Since cos(x) is decreasing for $x \in [0, 1]$, we also get the inequalities:
$$a_{2n-3} \geq a_{2n-1} \Rightarrow a_{2n-2} = cos(a_{2n-3}) \leq cos(a_{2n-1}) = a_{2n} \Rightarrow$$
$$a_{2n-1} = cos(a_{2n-2}) \geq cos(a_{2n}) = a_{2n+1} \Rightarrow a_{2n} = cos(a_{2n-1}) \leq cos(a_{2n+1}) = a_{2n+2}.$$

Therefore, for $n \geq 2$ we have $a_n \in [cos(1), 1]$. Next we show that the sequence a_n is Cauchy. We start by controlling the difference $|a_n - a_{n+1}|$.

- Consider the following interval $I_n \subset [cos(1), 1]$

$$I_n := \begin{cases} (a_n, a_{n+1}) & \text{,if n is even} \\ (a_{n+1}, a_n) & \text{,if n is odd} \end{cases}.$$

- Since $a_n \in [cos(1), 1]$, we have by MVT some $\xi_n \in I_n \subset [cos(1), 1]$ s.t.

$$|a_{n+1} - a_n| = |cos(a_n) - cos(a_{n-1})| = |sin(\xi_n)||a_n - a_{n-1}|.$$

- Since $\xi_n \in [cos(1), 1]$ and sin(x) is increasing in that interval, we have that $sin(\xi_n) \leq sin(1) =: r < 1$.

$$|a_{n+1} - a_n| \leq r|a_n - a_{n-1}|.$$

- Therefore, recursively we find the bound

$$|a_{n+1} - a_n| \leq r|a_n - a_{n-1}| \leq r^2|a_{n-1} - a_{n-2}| \leq ... \leq r^n|a_2 - a_1| =: cr^n.$$

Now we control the difference $|a_n - a_{n+m}|$ for any $n, m > 0$. We write the telescoping sum

$$\begin{aligned} |a_n - a_{n+m}| =& |a_n - a_{n+1} + a_{n+1} - a_{n+2} + ... + a_{n+m-1} - a_{n+m}| \\ \leq& |a_n - a_{n+1}| + |a_{n+1} - a_n| + ... + |a_{n+m-1} - a_{n+m}| \end{aligned}$$

and bound it by

$$c\sum_{k=0}^{m-1} r^{n+k} = cr^n \frac{1 - r^m}{1 - r}.$$

Since $r < 1$ this bound goes to zero as $n \to +\infty$. Therefore, given fixed $\varepsilon > 0$ we can pick large enough $N > 0$ s.t. for all $n, m \geq N$ we have

$$\varepsilon > cr^N \frac{1 - r^N}{1 - r} \geq cr^n \frac{1 - r^m}{1 - r} \geq |a_{2n} - a_{2n+m}|.$$

Since the sequence a_n is Cauchy, it converges to some number $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} cos(a_{n-1}) = cos(\lim_{n\to\infty} a_{n-1}) = cos(L)$. Therefore, L satisfies the fixed point relation $L = cos(L)$. In other words, it is the intersection of the graphs of $y_1 = cos(x)$ and $y_2 = x$.