Linear equations: In n variables/unknowns: x_1, x_2, x_3 … x_n, is an equation of the form of $a_1x_1+a_2x_2+a_3x_3+\ldots+a_nx_n=b$.	a, b are coefficients. $\{a\in R$ and $b\in R\}$
System of linear equations: system of n linear equations in n unknowns x_1, x_2, x_3 … x_n of the form: $$\begin{gathered} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2 \\ a_{31}x_1+a_{32}x_2+\cdots+a_{1n}x_n=b_3 \\ \ldots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n=b_n \end{gathered}$$	$\{a_{mn}\in R: m=1,2,3,\ldots,n, \mathrm{n}=1,2,3,\ldots,n\}$ $\{b\in R\}$ e.x. a_{mn} is the coefficient of x_n in the mth. equation.
Solution: numbers $s_1, s_2, s_3, \ldots, s_n$ s.t. if $x_1=s_1$, $x_2=s_2, \ldots, x_n=s_n$, all equations are satisfied. Goal: To find out if a system of linear equations has a solution, if it does, how to find all solutions	
A s.l.e can have and only have 1, infinite, or ∅ solutions.	

General Procedure for solving s.l.e: Eliminating variables to make the system simpler enough to solve

<table>
<tr><td>Equivalent systems: two s.l.e that have the same set of variables. Which means all operations in solving s.l.e are reversible</td><td>To eliminate x_1 except for the first equation, then x_2 except for the first two equations … till ref is reached</td></tr>
<tr><td>e.x.
$$\begin{cases} x_1+2x_2+x_4=7 \\ x_1+x_2+x_3-x_4=3 \\ 3x_1+x_2+5x_3-7x_4=1 \end{cases} \gg\gg \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right) \begin{array}{c} R1 \\ R2 \\ R3 \end{array}$$</td><td>Convert to a matrix</td></tr>
<tr><td>$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 1 & 1 & 1 & -1 & 3 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right) \gg\gg \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right)$$</td><td>$R_2 - R_1$
Eliminate x_1 from R_2</td></tr>
<tr><td>$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 3 & 1 & 5 & -7 & 1 \end{array}\right) \gg\gg \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right)$$</td><td>$R_3 - 3R_1$
Eliminate x_1 from R_3</td></tr>
<tr><td>$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right) \gg\gg \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) (ref)$$</td><td>$R_3 - 5R_2$
Eliminate x_2 from R_3</td></tr>
<tr><td>Matrix is said to be row-echelon form (ref) to tell if the system has one, infinite, or zero solution.
A matrix is at ref if:
a) Each leading entry is to the right of others.</td><td></td></tr>
</table>

<table>
<tr><td>b) Any row that consists entirely of zeros is at bottom of matrix.
Leading entries: the first non-zero entry from the left of ref</td><td></td></tr>
<tr><td>$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \ggg \left(\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
Anytime try to eliminate a variable, new variable appears</td><td>$R_1 + 2R_2$
To make x_2 in R_1 0.</td></tr>
<tr><td>$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \ggg \left(\begin{array}{cccc|c} 1 & 0 & 2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) (rref)$$</td><td>R_2 * -1</td></tr>
<tr><td>Matrix is said to be in reduced row-echelon form (rref) if
a) Each leading entry is 1
b) Each leading entry is the only non-zero entry in its column</td><td></td></tr>
<tr><td>Let $x_3 = s, x_4 = t; \{s, t \in R\}$
$$\begin{cases} x_1 = -1 - 2s + 3t \\ x_2 = 4 + s - 2t \\ x_3 = s \\ x_4 = t \end{cases}$$</td><td>s and t are parameters.
At rref, we can represent variables with the rest variables.</td></tr>
<tr><td colspan="2">Theorem: every s.l.e is equivalent to one in ref or rref, and the ref can be achieved through sequences of elementary row operations to original equations.
Elementary row operations include:
1) Adding a multiple (non-zero) of one row to another
2) Multiply one row by a non-zero constant
3) Interchange any two rows</td></tr>
</table>

Consistent system: a s.l.e that has at least one solution, otherwise it's in consistent.

Theorem: suppose an augmented matrix of s.l.e is in ref/rref.

i) System is inconsistent iff ref of the augmented matrix of s.l.e. has row of form (0,0,0, … ,0 | c), c≠0

ii) If the system is consistent, two possibilities exist

  a) The number of leading variables = number of variables in the system. The system has a unique solution.

  b) The number of leading variables < number of variables in the system. The system has infinite solutions.

Pf:

i) (B ⊨ A) if a row of form (0,0,0, … ,0 | c), c≠0 exists, this corresponds 0 = c, c≠0, which has no solution.
(¬B ⊨ ¬A) if a row of form (0,0,0, ... ,0 | c), c≠0 does not exist, then rows are of form (0,0,0, ... ,0 | 0), which satisfies 0 = 0, or satisfies for all rows that have a leading variable. Which can be solved by assigning values to non-leading variables

to determine values of leading variable.

ii) Assume the system is consistent

   a) (number of variables = number of leading variables) ╞ There's no free (non-leading) variables ╞ system has unique solution.
   b) (number of variables < number of leading variables) ╞ There are free (non-leading) variables ╞ system has infinite solution.