Outline: Week 4 T

Open cover equivalent definition of compactness

The definition of compactness in terms of open covers is "A set K is compact if every of its open covers, has a finite subcover". We showed that this implies that the set K is closed and bounded. Therefore, by Henei-Borel K is sequentially compact (any sequence has a converging subsequence). The reference is Abott theorem 3.3.8.

Extreme value theorem D& D 5.4

Suppose K is compact and $f : K \to \mathbb{R}$ continuous, then $f(K)$ is also compact.

1. By Heine-Borel it suffices to show that $f(K)$ is closed and bounded.

2. In the assignment you will show that $f(K)$ is bounded.

3. Closedness is an application of the continuous mapping theorem. Take $y_n \in f(K)$ with $f(x_n) = y_n \in y$, we want to show that $y \in f(K)$. By Bolzano-Weierstrass we have $x_{n_k} \to x \in K$. Therefore, $f(x) = y$ by continuous mapping theorem.

- We went over some examples: a)the bump function $exp\{-\frac{1}{1-x^2}\}1_{x\in[-1,1]}$ b)$-x^2$ on (0,1] has a maximum at 0 but doesn't attain it since $0 \notin (0, 1]$. Thus closedness is an essential assumption of EVT.

- Definition of local maximum is $f(x) > f(t)$ for all $t \in (x - \varepsilon, x + \varepsilon)$ for some small enough $\varepsilon > 0$.

- D&D 5.4 C: Suppose that f is a continuous function on [a,b] with no local maximum or local minimum.Prove that f is monotone

- D&D 5.4K

Uniform continuity

1. uniform continuity definition also write the pointwise continuity definition and compare them.

2. Examples : a)ax+b is uniformly continuous over $\mathbb{R}$ because for $\varepsilon > 0$ we set $\delta := \frac{\varepsilon}{|a|}$ b)x^2 is not uniformly continuous.

Details in D&D 5.4 C

- By applying EVT to [a,b] for function f, we have $M, m \in [a, b]$ s.t. $f(M) \geq f(x) \geq f(m)$ for all $x \in [a, b]$.

- However, since f has no local max/min, the maximum and minimum must be attained at the boundary points i.e. $M = a$ or b and $m = a$ or b.

- If instead we had $M, m \in (a, b)$, then by openness of (a, b) we would have δ_M, δ_m s.t.

$$(M - \delta_M, M + \delta_M) \subset (a, b) \text{ and } (m - \delta_m, m + \delta_m) \subset (a, b).$$

  Therefore, M, m would be local maximum and minimum, which is a contradiction since ,by assumption, f has no local max/min.

Okay so now WLOG assume that $a = m$ and $b = M$. Therefore, $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$. We will show that any $x, y \in [a, b]$ with $x < y$ implies that $f(x) \leq f(y)$.

- Suppose otherwise that there is some $x < y$ s.t. $f(x) > f(y)$. We will EVT to the intervals $[a, x]$ and $[x, y]$ to get a contradiction.

- Applying EVT to $[a, x]$, we showed above that the max/min are attained at the endpoints. Since $f(a) \leq f(x)$, we obtain $f(a) \leq f(c_1) \leq f(x)$ for all $c_1 \in [a, x]$.

- Applying EVT to $[x, y]$, we showed above that the max/min are attained at the endpoints. Since $f(y) \leq f(x)$, we obtain $f(y) \leq f(c_2) \leq f(x)$ for all $c_3 \in [x, y]$.

- Therefore, we claim that x is a local maximum. Indeed, we have that $f(x) \geq f(w)$ for all $w \in [a, x] \cup [x, y] = [a, y]$. So for $\delta := min(\frac{y-x}{2}, \frac{x-a}{2})$ we have $B_\delta(x) \subset [a, b]$ and $f(x) \geq f(w)$ for all $w \in B_\delta(x) \subset [a, b]$.

Details in D & D 5.4 K

If f(0) is the global maximum, then we are done since f will attains its maximum at 0. So suppose otherwise, that there exists large enough $x_0 > 0$ s.t. $f(0) + \delta \leq f(x_0)$ for some $\delta > 0$.

- By $\lim_{x\to\infty} f(x) = f(0)$ we set $\varepsilon := \frac{\delta}{2}$ and obtain $\exists N_{\delta/2}$ s.t. $\forall x \geq N_{\delta/2}$ we have

$$|f(x) - f(0)| \leq \varepsilon := \frac{\delta}{2}.$$

  Therefore, for $x \in [N_{\delta/2}, \infty)$ we have

$$f(x) \leq f(0) + \frac{\delta}{2} < f(0) + \delta \leq f(x_0).$$

- Apply EVT to the complement interval $[0, N_{\delta/2}]$ to obtain some $M_{N_{\delta/2}}$ s.t. for all $x \in M_{N_{\delta/2}}$ we have a maximum:

$$f(x) \leq f(M_{N_{\delta/2}}).$$

- Finally, let $f(M) := \max(f(x_0), f(M_{N_{\delta/2}}))$. Then for $x \in [0, \infty) = [0, N_{\delta/2}]] \cup [N_{\delta/2}, \infty)$ we have that

$$f(x) \leq \max(f(x_0), f(M_{N_{\delta/2}})) = f(M)$$

  irrespective of whether $x \in [0, N_{\delta/2}]]$ or in the tail $x \in [N_{\delta/2}, \infty)$.