Outline: Week 1 R

Least upper bound

1. We prove the least upper bounded principle (D&D 2.3) LEAST UPPER BOUND PRINCIPLE:every nonempty subset S of R that is bounded above has a supremum.

2. we prove uniqueness of supremum

3. We study exercise 1.3.5 from Abbott: Proving sup(cA)=csup(A) for $c > 0$. WTS:$\forall \varepsilon > 0$ find $s_\varepsilon \in cA$ s.t. $csup(A) - \varepsilon \leq s_\varepsilon$. We pick $\widetilde{\varepsilon} = \frac{\varepsilon}{c}$ to get for $v := sup(A)$
$$v - \frac{\varepsilon}{c} \leq a_{\varepsilon/c} \Rightarrow cv - \varepsilon \leq a_{\varepsilon/c}c$$
and thus let $s_\varepsilon := a_{\varepsilon/c}c$. By having sup(S)=-inf(-S) we have $sup(cA) = sup((-c)(-A)) = (-c)sup(-A) = c\inf(A)$ for $c < 0$.

Detailed proof for LUB for sup

We prove the least upper bounded principle (D&D 2.3) LEAST UPPER BOUND PRINCIPLE:every nonempty subset S of R that is bounded above has a supremum.

Construction of the sup

- Let M be the upper bound for S with decimal expansion $M := m_0.m_1....$ We have that $m_0 + 1$ upper bounds S.

- (0th step) Let $s \in S$ have expansion $s = s_0.s_1...$, then let $s_0 < a_0 < m_0 + 1$ be the lowest number so that $a_0 + 1$ is still an upper bound for S but a_0 is not: $a_0 + 1 \geq s$ for all $s \in S$. Therefore, there is $x_0 \in S$ s.t. $a_0 \leq x_0 \leq a_0 + 1$.

- (1st step) Let $y_1 = a_0 + \frac{a_1}{10}$ where $a_1 \in \{0, ..., 9\}$ is the lowest number so that $y_1 + 0.1$ is still an upper bound for S but y_1 is not. Therefore, we can pick $x_1 \in S$ s.t. $a_0.a_1 \leq x_1 \leq a_0.a_1 + 0.1$.

- (2nd step) let $y_2 = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2}$ where $a_2 \in \{0, ..., 9\}$ is the lowest number so that $y_2 + \frac{1}{10^2}$ is still an upper bound for S but y_2 is not. Therefore, we can pick $x_2 \in S$ s.t. $a_0.a_1a_2 \leq x_2 \leq a_0.a_1a_2 + 0.01$

- (kth step) Given $y_{k-1} = a_0.a_1...a_{k-1}$ and x_{k-1} with $y_{k-1} \leq x_{k-1} \leq y_{k-1} + 10^{-(k-1)}$, select $y_k = a_0.a_1...a_k$ with the lowest a_k so that $y_k + 10^{-k}$ is an upper bound for S but y_k is not. So there exists $x_k \in S$ with
$$y_k \leq x_k < y_k + 10^{-k}.$$
- We claim that $L := \lim_{n\to\infty} y_n = a_0.a_1...a_k...$ is the sup(S).

Upper bound step:

Start with abstract $s = s_0.s_1... \in S$.

- If $s_i = a_i$ for all i, then $s = L$.
- If $s_i = a_i$ for $i < k$ but $s_k > a_k$ for some k then we get a contradiction:

  Let
  $$\widetilde{y}_k := a_0.a_1...a_{k-1}s_k00...$$
  Then due to $s_k > a_k$ we have
  $$y_k + 10^{-k} \leq \widetilde{y}_k.$$
  Due to $s_i = a_i$ for $i < k$ we have
  $$\begin{aligned} y_k + 10^{-k} &< \widetilde{y}_k \\ &= a_0.a_1...a_{k-1}s_k00... \\ &= s_0.s_1...s_k0... \\ &\leq s_0.s_1...s_k s_{k+1}... \\ &= s. \end{aligned}$$
  However, we have that
  $$y_k + 10^{-k} \geq s, \quad \forall s \in S.$$
- So we are left with the case $s_i = a_i$ for $i < k$ but $s_k < a_k$ for some k, which implies
$$s = s_0.s_1...s_k... < a_0.a_1...a_k0... = y_k < L.$$

Subsequence step:

Given $\varepsilon > 0$ we have to find $x_\varepsilon \in S$ s.t.

$$L - \varepsilon \leq x_\varepsilon.$$

- Let n be so large that $\frac{1}{10^n} < \varepsilon$, then

$$L - \varepsilon < L - \frac{1}{10^n},$$

- But we have that

$$L - \frac{1}{10^n} = a_0.a_1...a_{n-1}(a_n - 1)a_{n+1}... < y_n.$$

- This is in turn less than x_n from above. So all together

$$L - \varepsilon < L - \frac{1}{10^n} < y_n < x_n.$$

- So we let $x_\varepsilon := x_n$.