Outline: Week 9

8.1

1. The focus is swapping limit and integral.
2. pointwise and uniform convergence
3. The growing steeple example each have area 1 yet they converge to the zero function. The $\|f_n - f\|_\infty = n + 1$.
4. the $\frac{sin(nx)}{n}$ example has uniform convergence in $[0, A]$ and both integrals are zero. However, the derivatives do not converge uniformly.
5. Dini's Theorem and MCT
   - use continuity to get $r := \delta_{\varepsilon/2}$.
   - if $\|f_n - f\|_\infty \to d$, then since they attain their supremum we have $f(x_n) - f_n(x_n) \to d$.
   - By BW $x_{n_k} \to x_0$ so $g(x_{n_k}) \to g(x_0)$. So $g(x_{n_k}) \leq \varepsilon < d$.

0.0.1 8.2

1. uniform convergence of continuous is continuous.
2. Example of $f_n(x) := (1 + \frac{x}{n})^n$ over any interval $[a, b]$. Integrate in [0,1]: $((n + x)((n + x)/n)^n)/(1 + n)$. Compare with integral of e^x in [0,1].
3. Example of $f_n(x) := (x)^n$ (has continuous limit somewhere but not everywhere). (eg. $x = (\frac{1}{2})^{1/n}$ gets mapped to $1/2$ and it is not bounded by ε).
4. $f_n g_n \rightrightarrows fg$.
5. C(K,R) is complete.
   - $f_n(x)$ sequence is cauchy in the reals and so it has limit f(x).
   - we take m,n large enough so that

$$|f(x) - f_m(x)| \leq \varepsilon_m$$

and

$$|f_n(x) - f_m(x)| \leq \varepsilon.$$

- Therefore together we find

$$|f(x) - f_n(x)| \leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)|$$
$$\leq \varepsilon_m + \varepsilon.$$

We take limit in m to find

$$|f(x) - f_n(x)| \leq \varepsilon.$$

- so convergence is uniform and so the limiting function is continuous.

0.0.2 8.3 integral convergence theorem

1. integral convergence theorem: $\|F - F_n\|_\infty \leq |b-a| \|f - f_n\|_\infty$
2. derivative: $f_n = \int f_n' \to \int g = f$.
3. Leibniz's rule:
$$\frac{F(x_0+h) - F(x_0)}{h} = \int_c^d \frac{f(x_0+h,t) - f(x_0,t)}{h} dt.$$
By MVT there exists $x(t)$ in $[x_0, x_0+h]$ s.t.
$$= \int_c^d f_1(x(t),t)dt.$$
By u.c. in [c,d], for $\frac{\varepsilon}{d-c} > 0$ we have $\delta > 0$ s.t.
$$|x-y| \leq \delta \Rightarrow |f_1(x,t) - f_1(y,t)| \leq \frac{\varepsilon}{d-c}.$$
So this applies to $x = x_0, y = x(t)$ for $|h| \leq \delta$:
$$\frac{F(x_0+h) - F(x_0)}{h} - \int_c^d f_1(x_0,t)dt = \int_c^d f_1(x(t),t) - f_1(x_0,t)dt$$
$$\leq (d-c) \sup_{t \in [c,d]} |f_1(x(t),t) - f_1(x_0,t)|$$
so by u.c. for $|x(t) - x_0| \leq |h| \leq \delta$ we find
$$\leq (d-c)\frac{\varepsilon}{d-c}$$
$$= \varepsilon.$$
Therefore, as $h \to 0$ we find
$$F'(x_0) = \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = \int_c^d f_1(x_0,t)dt.$$

4. (D&D 8.3.C): Since g is bounded we have

$$\|f_n g - fg\|_\infty \le \|g\|_\infty \|f_n - f\|_\infty \le B\|f_n - f\|_\infty \to 0.$$