Countability and diagonalization proof

Countable / non-countable

Definition a non-empty set C is countable if there exists a function $f: \mathbb{N} \to C$ surjective. An empty set is countable

Theorem Every non-empty finite set is countable $C = \{c_0, c_1, c_2, \ldots c_n\}, f: \mathbb{N} \to C \coloneqq f(i) = c_i$

Example $\mathbb{Z}$ is countable. Let $f: \mathbb{N} \to \mathbb{Z}$ be defined as $\begin{cases} f(0) = 0 \\ f(2i-1) = i & \text{for } i = 1,2,3,\ldots \\ f(2i) = -i \end{cases}$

Theorem

1) $\mathbb{N} \times \mathbb{N}$ countable $0 \to (0,0), 1 \to (0,1), 2 \to (1,0), 3 \to (0,2), \ldots$
2) A, B countable IMPLIES ($A \cup B$ countable AND $A \times B$ countable)
3) (A countable AND $B \subseteq A$) IMPLIES B countable
4) ($A \neq \emptyset$ AND A countable AND $\exists f: A \to B$ surj) IMPLIES B countable
5) $\mathbb{Q}^+ \cup \{0\}$ is countable

   Proof Let $f: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+ \coloneqq f(a,b) = \begin{cases} \frac{a}{b} \ (b \neq 0) \\ 0 \ (b = 0) \end{cases}$ f surj (notice f is not necessarily injective)
6) Binary string is countable

   Proof Let $f: \mathbb{N} \times \mathbb{N} \to \{0,1\}^* \coloneqq g(i,j) =$

   $\begin{cases} \text{jth lexicgraphically smallest string of length i if } 1 \leq j \leq i \\ \text{empty string otherwise} \end{cases}$
7) S be a finite set IMPLIES $\mathcal{P}(S)$ is countatble

Theorem $\mathcal{P}(\mathbb{N})$ is uncountable

Method 1: construct contradiction

Proof Suppose $\mathcal{P}(\mathbb{N})$ is countable

By definition, take $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be surjective

Let $D = \{ i \in \mathbb{N} \mid i \notin f(i) \} \in \mathcal{P}(\mathbb{N})$

Since $D \subseteq \mathbb{N}, \exists j \in \mathbb{N}, f(j) = D$

Then, $\forall i \in \mathbb{N}. i \in f(j)$ IFF $i \in D$ since $f(j) = D$

$\forall i \in \mathbb{N}. i \in f(j)$ IFF $i \notin f(i)$ by definition of D

Since $j \in \mathbb{N}$, by specialization

$(j \in f(j))$ AND $(j \notin f(j))$ contradiction

Method 2: construct diagonalization

Proof for any subset $S \in \mathbb{N}$, we can represent it by an infinite binary sequence where $s_i = 1\ (i \in S)$ OR $0\ (i \notin S)$

For example, $\{0\} = 1000\ldots, \{ x \in \mathbb{N} \mid odd(x) \} = 0101010101\ldots$

Suppose $\mathcal{P}(\mathbb{N})$ is countable, take $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be surjective

Characteristic vector→ ↓Subset of ℕ	s_0	s_1	s_2	…
f(0)	$f(0)_0$	$f(0)_1$	$f(0)_2$	
f(1)	$f(1)_0$	$f(1)_1$	$f(1)_2$	
f(2)	$f(2)_0$	$f(2)_1$	$f(2)_2$	
…				

Let $M: \mathbb{N} \times \mathbb{N} \to \{0,1\}\ M(i,j) \coloneqq f(i)_j = \begin{cases} 1\ (j \in f(i)) \\ 0\ (j \notin f(i)) \end{cases}$

Consider set $D = \{ i \in \mathbb{N} \mid i \notin f(i) \}$, then $\forall i \in D. M(i,i) = 0. f(i)_i$=0

Consider the characteristic vector of D: $i \in f(i)$ IMPLIES $f(i)_i = 1, D_i = 0, i \notin f(i)$ IMPLIES $f(i)_i = 0, D_i = 1$

Therefore, D is the complement of the diagonal of M, D can't be any of characteristic vectors of $f(\mathbb{N})$ since there is always one bit in chatacteristic vector $\left(f(i)_i\right)$ is different, contradiction.

Theorem There is no function $H: \text{ASCII} \times \text{ASCII} \to \{0,1\}$ such that $H(p, x) = \begin{cases} 1 & \text{p is syntactically correct} \\ & \text{and returns given x} \\ 0 & \text{otherwise} \end{cases}$

Method 1: Contradiction

Proof Suppose there is such a H and assume all inputs of H are syntactically correct

Consider program D: ASCII, $D(x) \coloneqq$ if $H(x, x)$ returns 1, then goes into an infinite loop, else return 1

Then, if $H(D, D)$ return 0, $D(D)$ return 1, if $H(D, D)$ return 0, $D(D)$ return 1

However, by the definition of H

If $H(D, D)$ return 0, $D(D)$ runs into infinite loop, if $H(D, D)$ return 1, $D(D)$ returns something

Contradiction

Method 2: diagonalization

$H(p, x)$ $\to x$ $\downarrow p$	λ	a	b...
λ	$H(\lambda, \lambda)$	...	...
a	...		
b	...		
...			

D is the complement of the diagonal, hence $D \notin p$, hence such H does not exists.