$T: R^n \to R^m$ means a rule that assigns to each vector $x \in R^n$ to a unique vector $b \in R^m$, it's said to be linear transformation

$$\forall T: R^n \to R^m$$
$$T(x+y) = T(x) + T(y)$$
$$T(cx) = cT(x) \;\; c \in R$$

T maps linear combinations in R^n to linear combinations in R^m. In the other words, T maps straight line segments, or single point in R^n to straight line segments, or single point in R^m.

$\forall x, y \in R^n$, line segment joining *x* to *y* is given by $x + t(y-x), 0 \le t \le 1$

$$T\big(x + t(y-x)\big) = T(x) + tT(y) - tT(x)$$
$$T(x) \ne T(y) \to line\ segment \mid T(x) = T(y) \to point$$

$ex. T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} x1 \\ x2 \end{pmatrix},$

$sketch\ the\ image\ of\ the\ transformation\ given\ (0,0), (1,0), (0,1), (1,1)$

$$T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \;\; T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$ex. let\ L\ be\ the\ line\ x2 = x1 + 1,$

$find\ the\ line\ after\ the\ transformation\ T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{pmatrix} x1 \\ x2 \end{pmatrix}$

$$T\begin{pmatrix} x1 \\ x1+1 \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{pmatrix} x1 \\ x1+1 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_1 + 1 \\ 4x_1 + 2x_1 + 2 \end{pmatrix} = \begin{pmatrix} t \\ 2t \end{pmatrix}, t = 3x + 1$$
$$Therefore, the\ line\ after\ transformation\ is\ x_2 = 2x_1$$

Matrix representation theorem:

$$R^n \leftrightarrow R^m\ is\ linear\ \to \exists!\, A \in [m \times n], T(x) = Ax, \forall x \in R^n$$

Every linear transformation is a matrix transformation

Proof: $\forall x = \begin{pmatrix} x1 \\ x2 \\ \dots \\ xn \end{pmatrix} \in R^n, x = x1\begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \\ e1 \end{pmatrix} + x2\begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \\ e2 \end{pmatrix} + \dots + xn\begin{pmatrix} 0 \\ 0 \\ \dots \\ 1 \\ en \end{pmatrix}$

$T(x1e1 + x2e2 + \cdots xnen) = x1Te1 + x2Te2 + \cdots + xnTen$

$$= (Te1\ Te2 \dots Te3)\begin{pmatrix} x1 \\ x2 \\ \dots \\ xn \end{pmatrix} = Ax$$

$ex. T: R^3 \to R^2\ is\ defined\ by\ T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 2x1 + 3x2 \\ x1 - x2 \\ x1 \end{pmatrix}, find\ A$

$method\ 1)\ A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$

$method\ 2)\ uniqueness\ T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 2x1+3x2 \\ x1-x2 \\ x1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x1 \\ x2 \end{pmatrix}$

$ex.\ T: R^3 \to R^2\ is\ defined\ by\ T\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, determine\ T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$Discussion: \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = c1\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + c2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = c1T\begin{pmatrix} 4 \\ 7 \\ 3 \end{pmatrix} + c2T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c3T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 1 \\ 7 & 1 & 0 & 2 \\ 3 & 0 & 0 & 1 \end{array}\right] \gg c1 = \frac{1}{3}, c2 = -\frac{1}{3}, c3 = 0$

$T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix}$

$determine\ A, s.t.\ T(x) = Ax, \forall x \in R^3$

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = T\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{3} \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{22}{3} \end{pmatrix}$

$A = \left[T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\ \ T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\ \ T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & -\frac{22}{3} \end{bmatrix}$

$ex.$ Let l be the line in R^3 through origin in direction of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ find 3×3 matrix $s.t.\ Proj_l x =$

$Proj_d x = Ax, \forall x \in R^3$

$Proj_d x = \frac{d.x}{\|d\|^2} d = \frac{d1x1 + d2x2 + d3x3}{d_1^2 + d_2^2 + d_3^2}\begin{pmatrix} d1 \\ d2 \\ d3 \end{pmatrix} = \frac{ax1 + bx2 + cx3}{a^2 + b^2 + c^2}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$= \frac{1}{a^2 + b^2 + c^2}\begin{pmatrix} aax1 + abx2 + acx3 \\ abx1 + bbx2 + bcx3 \\ cax1 + cbx2 + ccx3 \end{pmatrix} = \frac{1}{a^2 + b^2 + c^2}\begin{bmatrix} a^2 & ab & ab \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

Say $T: R^3 \to R^2$ linear $A = (a1\ a2\ ...\ an)$ is the matrix of T

Definition: $range(T) = \left\{\begin{array}{c} b \in R^m \mid b = Ax\ (x \in R^n) \\ = (a1\ a2\ ...\ an)\begin{pmatrix} x1 \\ x2 \\ ... \\ xn \end{pmatrix} \\ = a1x1 + a2x2 + \cdots + anxn \\ = span\{a1, a2, a3, ..., an\} \end{array}\right\}$

Asking if b is in the range is the same as asking if b is in the span of A.

$$ex. T: R^2 \to R^3 \ defined\ by\ T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} x1 + x2 \\ x1 - x2 \\ x1 \end{pmatrix}$$

$$range(T) = span\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right\}$$

$$Q: is\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} in\ range(T): no, \nexists k1, k2 \in R, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = k1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + k2\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Definition: $T: R^n \to R^m$ is said to be

1) $OneToOne \leftrightarrow T(x) = 0\ only\ have\ trivial\ solution$
2) $Onto \leftrightarrow \forall b \in R^m, \exists x, Ax = b \leftrightarrow in\ ref\ \#\ of\ leading\ variables = \#\ of\ rows$

$$ex.\ T: R^2 \to R^3\ defined\ by\ T\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} x1 + x2 \\ x1 - x2 \\ x1 \end{pmatrix}, is\ OneToOne(T), Onto(T)?$$

$Is\ OneToOne\ since\ it\ has\ trivial\ solution$

$Is\ not\ Onto\ since\ \#\ of\ leading\ variables < \#\ of\ rows$