Week 4: Tutorial Handout

Extreme value theorem

1. What can you say about the function $f(x) := -\frac{1}{1+|x|^2}$ for $x \in \mathbb{R}$? Is it bounded? Is it continuous? Does it attain its extrema?

2. If $A \subset \mathbb{R}$ is not bounded, construct a bounded continuous function $f : A \to \mathbb{R}$ that doesn't attain is maximum.

3. If $A \subset \mathbb{R}$ is not closed, construct a bounded continuous function $f : A \to \mathbb{R}$ that doesn't attain is maximum.

Uniform continuity

4. Which of the following functions are uniformly continuous? Explain. If it is, use epsilon-delta argument to prove it (or some result from class). If it is not uniformly continuous, use the sequential criterion: A function $f : A \to \mathbb{R}$ is not uniformly continuous if there exists $\varepsilon_0 > 0$ and sequences x_n, y_n s.t.

$$|x_n - y_n| \to 0 \text{ but } |f(x_n) - f(y_n)| > \varepsilon_0 > 0.$$

   (a) Any continuous function on a closed and bounded set.

   (b) $f_1(x) := ax + b$ for some $a, b \in \mathbb{R}$ and $x \in \mathbb{R}$.

   (c) $f_2(x) := ax^2 + b$ for some $a, b \in \mathbb{R}$ and $x \in \mathbb{R}$.

   (d) $f_3(x) := \frac{1}{x^2}$ on $(0, 1]$.

   (e) $f_4(x) := \frac{1}{x^2}$ on $(1, \infty)$.

   (f) $f_5(x) := \sqrt{x}$ on $[b, \infty)$ for any $b > 0$.

   (g) $f_6(x) := \sqrt{x}$ on $[0, \infty)$.