More closure properties

Reversal
Definition $x \in \Sigma^*. x^R$ is the reversal of x. $\left(x = x_1 x_2 \dots x_n \text{ IMPLIES } x^R = x_n x_{n-1} \dots x_1\right)$

$L^R = \left\{ x^R \mid x \in L \right\}$

Theorem $L \in \mathcal{R}$ IMPLIES $L^R \in \mathcal{R}$
Proof $\emptyset^R = \emptyset, \{\lambda\}^R = \{\lambda\}, \forall a \in \Sigma. \{a\}^R = \{a\}$

Let s, s' be regular expressions such that $\mathcal{L}(s) = \left(\mathcal{L}(r)\right)^R, \mathcal{L}(s') = \left(\mathcal{L}(r')\right)^R$

$\left(\mathcal{L}(r + r')\right)^R = \mathcal{L}(s + s')$

$\left(\mathcal{L}(rr')\right)^R = \mathcal{L}(s's)$

$\left(\mathcal{L}(r^*)\right)^R = \mathcal{L}(s^*)$

Homomorphism
Definition a homomorphism is a function $h: \Gamma^* \to \Sigma^*$ defined on strings in Γ^*, which replaces each letter $a \in \Gamma$ with a string $h(a) \in \Sigma^*$
Recursively define a homomorphism function:

$h(\lambda) = \lambda$

$h(xa) = h(x)h(a)$ for $x \in \Gamma^*, a \in \Gamma$

Example $h: \{0,1\}^* \to \{a, b\}^* \coloneqq h(0) = ab, h(1) = \lambda$

Then $h(0011) = h(0)h(0)h(1)h(1) = abab$

Definition $L \subseteq \Gamma^*$ IMPLIES $h(L) = \{ h(w) \mid w \in L \}$
Example $L = \{0,1,00\}, h(L) = \{ab, \lambda, abab\}$

Theorem $L \in \mathcal{R}$ IMPLIES $h(L) \in \mathcal{R}$
Proof $\emptyset^R = \emptyset, \{\lambda\}^R = \{\lambda\}, \forall a \in \Sigma. \{a\}^R = \{a\}$

$\left(\mathcal{L}(r + r')\right)^R = \mathcal{L}\left(h(r) + h(r')\right)$

$\left(\mathcal{L}(rr')\right)^R = \mathcal{L}\left(h(r)h(r')\right)$

$\left(\mathcal{L}(r^*)\right)^R = \mathcal{L}(h(r)^*)$

Proof irregular using closure properties and contradiction
Example $L = \{ a^m b^n \mid m \neq n \}$
Proof Suppose $L \in \mathcal{R}$

Then $(\{a, b\}^* - L) \in \mathcal{R}$ close under complement

Then $\left((\{a, b\}^* - L) \cap \mathcal{L}(aa^*bb^*)\right) \in \mathcal{R}$ close under intersection

Which $\left\{ a^i b^i \mid i \geq 1 \right\} \in \mathcal{R}$

While $\left\{ a^i b^i \mid i \geq 1 \right\} \notin \mathcal{R}$ as proven before

Example $L = \left\{ a^i b^j c^h \mid i, j, h \geq 0 \text{ AND } \left(i = 1 \text{ IMPLIES } j = h\right) \right\}$
Proof $L \cap \mathcal{L}(ab^*c^*) = \left\{ ab^j c^j \mid j \geq 0 \right\} \in \mathcal{R}$ close under intersection

Let $h: \{a, b\}^* \to \{a, b\}^* \coloneqq h(a) = \lambda, h(b) = a, h(c) = b$

$h\left(\left\{ ab^j c^j \mid j \geq 0 \right\}\right) = \left\{ a^j b^j \mid j \geq 0 \right\} \in \mathcal{R}$ close under homomorphism

Then $\left\{ a^j b^j \mid j \geq 0 \right\} \cap \mathcal{L}(aa^*bb^*) = \left\{ a^i b^i \mid i \geq 1 \right\} \in \mathcal{R}$ close under intersection

$\left\{ a^i b^i \mid i \geq 1 \right\} \notin \mathcal{R}$

Prove L is under pumping lemma
Proof Take $n = 2$, let $x \in L$ be arbitrary s.t. $|x| \geq 2, x = a^j b^j c^h$ where $i, j, h > 0, i = 1$ IMPLIES $j = h$

Case	i	j	u	v	w	uv^kw

1	0	0	λ	c	c^{h-1}	c^{k+h-1}
2	0	> 0	λ	b	$b^{j-1}c^h$	$b^{k+j-1}c^h$
3	1	$j = h$	λ	a	$b^j c^j$	$a^k b^j c^j$
4	2	$\mathbb{N}$	λ	aa	$b^j c^n$	$a^{2k} b^j c^n$
5	>2	$\mathbb{N}$	λ	a	$a^{i-1} b^j c^n$	$a^{k+j-1} b^j c^h$

In all cases, pumping lemma works.
Pumping lemma can only be used to prove that something is not regular, but can't say that everything satisfies pumping lemma is regular