1. Does it make sense?

   In each of the following sentences, decide which word is being used incorrectly and why. Do not worry about saying whether the statement is true or not.

   (a) $P_2(\mathbb{R})$ and $\mathbb{R}^3$ are isomorphic, since their bases $\{1, x, x^2\}$ and $\{e_1, e_2, e_3\}$ have the same dimension.

   (b) If a vector v is in the span of $\{v_1, \ldots, v_n\}$, then it is not independent of the other vectors.

   (c) If a linear map is one-to-one and onto, then it is isomorphic.

   (d) Suppose $T : V \to W$ is a linear map, and $\{v_1, \ldots, v_n\}$ is a basis for V, then the rank of T is the dimension of $\{T(v_1), \ldots, T(v_n)\}$.

   (e) The kernel of an $m \times n$ matrix A is the set of all possible solutions to $Ax = b$.

2. Let $T : V \to V$ be a linear transformation. Define the following:

   (a) Eigenvector.

   (b) Eigenvalue.

   (c) The subspace $W \subset V$ is an invariant subspace with respect to T.

   (d) T is nilpotent.

3. Let $T : V \to V$ be a linear transformation and let V be a finite dimensional vector space. Prove that T is invertible if and only if 0 is not an eigenvalue of T.

4. Prove that the only eigenvalue of a nilpotent linear transformation is 0.

5. Projections are nice! Recall that an operator T is called a projection if it satisfies $T^2 = T$. Prove the following statements.

   (a) If T is a projection on V, then $V = ker(T) \oplus Im(T)$.

   (b) If T is a projection on V, its only possible eigenvalues are 0 and 1.

   (c) If T is a projection on V and V is finite-dimensional, then T is diagonalizable.

6. If v is an eigenvector for T with eigenvalue λ, is v necessarily in the image of $T - \lambda I$? Under what conditions is it? What would this tell you about the Jordan form of T? What about being in the image of $(T - \lambda I)^2$, etc.?

7. Consider the following two operators on $\mathbb{R}^2$: $N(x, y) = (y, 0)$ and $R(x, y) = (-y, x)$. Neither of them can be diagonalized (over $\mathbb{R}$), but for slightly different reasons. Try to diagonalize them both, and explain the difference between the two cases.

8. Prove that an $n \times n$ matrix A with entries from $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized. Is the same statement true over any field $\mathbb{F}$?

9. A square matrix has characteristic polynomial $p(\lambda) = \lambda^3(\lambda-3)$ and its column space is two-dimensional. Find the Jordan form of this matrix.

10. Suppose that A is a 2×2 matrix with $A^2 = I$. Show that A is diagonalizable.

11. Find three distinct subspaces W_1, W_2, W_3 of $\mathbb{R}^2$ such that:
    - $\mathbb{R}^2 = W_1 \oplus W_2$;
    - $\mathbb{R}^2 = W_2 \oplus W_3$;
    - $\mathbb{R}^2 = W_1 \oplus W_3$;

12. Suppose that T sends a set of k vectors $B = \{x_1, \ldots, x_k\}$ to $T(B) = \{T(x_1), \ldots, T(x_k)\}$. Prove that if the latter is linearly independent, the former is linearly independent. Prove the converse is false.

13. Let A be the $n \times n$ matrix with a 1 in every entry. Find the eigenvalues of A. Is A diagonalizable? Explain your thinking.

14. Let $T : V \to W$ be a linear transformation. Prove that for all subspaces $S \subseteq W$, $T^{-1}(S)$ is a subspace of V.

15. Let $T : P_n(\mathbb{R}) \to \mathbb{R}^{n+1}$ be defined as:

$$T(f) = \begin{bmatrix} f(c_0) \\ f(c_1) \\ \vdots \\ f(c_n) \end{bmatrix}$$

    where $c_0, c_1, \ldots, c_n$ are $n+1$ distinct real numbers. Let $\alpha = \{1, x, \ldots, x^n\}$ be the standard basis for $P_n(\mathbb{R})$ and $\beta = \{e_1, e_2, \ldots, e_{n+1}\}$ be the standard basis for R^{n+1}.

    (a) Compute $[T]_\alpha^\beta$.

    (b) Is T invertible? Justify your answer.

    (c) Determine its eigenvalues and corresponding eigenspaces.

16. Determine whether the following subset S of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$:

$$S := \{x \in \mathbb{R}^n \ : \ Ax - \lambda^3 x = x\}$$

    where A is an $n \times n$ real matrix and λ is a real number.

17. Prove that for all $\theta \in \mathbb{R}$, the set

$$\left\{ \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \right\}$$

    is a linearly independent set in $\mathbb{R}^2$.

18. (a) If possible, write down a 5×5 real matrix with -1 as its only real eigenvalue and where the eigenspace with eigenvalue -1 has dimension 3.

    (b) If possible, write down a 5×5 real matrix with -1 as its only real eigenvalue and where the eigenspace with eigenvalue -1 has dimension 5.

    (c) If possible, write down a 5×5 real matrix with -1 as its only real eigenvalue and where the eigenspace with eigenvalue -1 has dimension 1 and the generalized eigenspace with eigenvalue -1 has dimension 3.

    (d) If possible, write down a 5×5 real matrix with -1 as its only real eigenvalue and where the eigenspace with eigenvalue -1 has dimension 1 and the generalized eigenspace with eigenvalue -1 has dimension 4.