Homogeneous System: $Ax = 0$.

- The vector equation will always have a solution x=0, which is the trivial solution
- Look for non-trivial solution

$$e.x.\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -3 & 1 & 0 & 0 \end{array}\right] \gg \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right] \gg \begin{pmatrix} x1 = -\frac{s}{2} - \frac{3t}{2} \\ x2 = \frac{s}{4} + \frac{5t}{4} \\ x3 = -\frac{s}{2} - \frac{t}{2} \\ x4 = s \\ x5 = t \end{pmatrix}$$

$$\gg s\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{3}{2} \\ \frac{5}{4} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \gg span\left\{\left(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ \frac{5}{4} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\right)\right\}$$

Theorem: Homogenous system with more variables than equations always have infinitely many non-trivial solutions.

$$e.x.\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 14 \\ 1 & 0 & 3 & 3 & 19 \end{array}\right) \; gives\ solution \begin{pmatrix} 4+3t \\ s \\ 5-2t \\ t \end{pmatrix} \gg \begin{pmatrix} 4 \\ 0 \\ 5 \\ 0 \end{pmatrix} + \left(s\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right)$$

$$consider \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 3 & 0 \end{array}\right) \gg \left(s\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 0 \\ -2 \\ 1 \end{pmatrix}\right)$$

Theorem: every solution to $Ax = b$ is of form $x = xp + x0$, where *xp* is a particular solution to *Ax=b* and *x0* is a solution to the associated homogeneous equation

Proof:

$$\begin{aligned} Ax &= A(xp + x0) \\ &= Axp + Ax0 \\ &= b + 0 \\ &= b \end{aligned}$$

$$\begin{aligned} Ax0 &= A(x - xp) \\ &= Ax - Axp \\ &= b - b \\ &= 0 \end{aligned}$$

Definition 1.1) vector $v1, v2, \dots, vk \in R^n$are linear dependent if

$$\exists scalar\ c1, c2, \ldots, ck\ \in R, are\ not\ all\ 0, s.t.\ c1v1 + c2v2 + \cdots + ckvk = 0$$

Definition 1.2) $v1, v2, \ldots, vk\ \in R^n$are linear dependent iff homogenous solution $(v1, v2, \ldots, vk|0)$ has non-trivial solutions

$$e.x. \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\} are\ lin.dep\ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$c1 = 1, c2 = 1, c3 = -1$$

$$also\ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Definition 1.3) $v1, v2, \ldots, vk\ \in R^n$are linear dependent iff at least one of $v1, v2, \ldots, vk$ can be written as a linear combination of remaining (k-1) vectors.

$$e.x. \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\} are\ lin.dep. \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

Definition 2) $v1, v2, \ldots, vk\ \in R^n$are linear independent if they are not lin. dep., which

2.1) $c1 = c2 = \cdots = ck = 0$

2.2) $(v1, v2, \ldots, vk|0)$ only have trivial solutions

e.x. For what value of c is the set of vectors lin.in.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} c \\ 1 \\ 1 \\ 2c \end{pmatrix} \right\} \gg \begin{pmatrix} 1 & 1 & 1 & c \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2c \end{pmatrix} \gg \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & c-1 \\ 0 & 0 & 1 & c+1 \end{pmatrix} \gg c \neq -1$$