Outline: Week 6

Inner products

1. Inner products definition
2. Examples: 1) Euclidean dot product 2)xAy for A positive definite symmetric $xAx = \sum \lambda_k v_k^2$ where $v := Px$.3) L2 over continuous fncs
3. Cauch-Schwartz inequality (done in tutorial handout) and triangle inequality
4. Complete inner product space is called Hilbert
5. 11:30 L2[S] is a complete inner product space. Detailed proof below.
6. 12:10 Definition of orthogonal and orthonormal
7. Any Hilbert space has an orthonormal basis. If $\{e_k\}$ is an orthonormal basis for a Hilbert space then
$$h := \sum_{k \geq 1} \langle h, e_k \rangle e_k$$
8. The elements of the $\{1, \sqrt{2}cos(n\theta), \sqrt{2}sin(n\theta)\}$ are orthonormal for $(C[-\pi, \pi], L^2)$ and L^2. This is homework.

Fourier series

1. The elements of the $\{1, \sqrt{2}cos(\frac{n\pi x}{L}), \sqrt{2}sin(\frac{n\pi x}{L})\}$ are orthonormal for $(C[-L, L], \|\cdot\|_{L^2})$.
2. But when do we have $f(x) = a_0 + \sum a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})$, where $a_0 := \frac{1}{2L}\int_{-L}^{L} f$ and $a_n := \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})$ and $b_n := \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})$?
3. Example of continuous that fails: Fejer example the continuous function $f(x) := \sum \frac{1}{k^2} sin((2^{k^3} + 1)\frac{x}{2})$ has a fourier series that blows up at 0. see Fejer example
4. Piecewise lipschitz continuous: f has finitely many discontinuities but it is Lipschitz everywhere else.

5. 14.2.6. THE DIRICHLET–JORDAN THEOREM. If $f : R \to R$ is piecewise Lipschitz continuous and periodic, then $S_k f(\theta) \to \frac{f(\theta_-)+f(\theta_+)}{2}$ and thus $S_k f(\theta) \to f(\theta)$ on continuity points θ.

6. Carleson-Hunt: For L^2 functions $f(x) = Sf$ almost everywhere.

7. If f satisfies a Holder condition $\alpha > \frac{1}{2}$, then its Fourier series converges uniformly.

8. If f is continuous and its Fourier coefficients are absolutely summable, then the Fourier series converges uniformly.

9. Example: For $f(x) = x^2$ in $[-2, 2]$ we have
   - $b_n = 0$ because f is even and set $x = 1$ and split into even and odd to get $b_n = 0$.
   - $a_0 := \frac{4}{3}$
   - $a_n = \frac{1}{2}\int_{-2}^{2} x^2 cos(\frac{n\pi x}{2}) = 16\frac{(-1)^n}{n^2\pi}$
   - Thus, $x^2 = \frac{4}{3} + \sum a_n cos(\frac{n\pi x}{2}) = \frac{4}{3} + \sum 16\frac{(-1)^n}{n^2\pi} cos(\frac{n\pi x}{2})$.
   - Thus at $x = 0$ we have
   $$\sum \frac{(-1)^n}{n^2} = \frac{\pi^2}{12}.$$

10. Example: For $f(x) = x^3$ in $[-2, 2]$ we have
   - $a_n = 0$ because f is odd.
   - $b_n = \int_0^2 x^3 sin(\frac{n\pi x}{2}) = 16 * 3 * 2 * \frac{(-1)^n}{(n\pi)^3}$
   - Thus, $x^3 = \sum 96\frac{(-1)^n}{(n\pi)^3} sin(\frac{n\pi x}{2})$.
   - Alternatively integrate and used dominated convergence theorem.

11. Example: The discontinuous function $f(x) := \begin{cases} -1 & , x \in [-2, 0] \\ 1 & , x \in [0, 2] \end{cases}$.
   - We have $a_n = 0$ because the function is odd
   - For $b_n := \frac{1}{2}\int_{-2}^{2} f(x) sin(\frac{n\pi x}{2}) = \frac{1}{2}[2\int_0^2 sin(\frac{n\pi x}{2})] = \frac{2}{n\pi}[-cos(\frac{n\pi x}{2})]_0^2 = \frac{2}{n\pi}(1 - cos(n\pi))$. So $b_{2n} = 0$ and $b_{2n+1} = \frac{4}{n\pi}$
   - Thus, $f(x) \sim \sum_{n\geq 1} b_{2n+1} sin(\frac{(2n+1)\pi x}{2})$ only a.e.. At $x = 0$ we have $\sum_{n\geq 1} b_{2n+1} sin(\frac{(2n+1)\pi x}{2}) = 0$. The Gibbs phenomenon: Disagreement at the discontinuities.
   - At $x = 1$ we have $\sum_{n\geq 1} b_{2n+1} sin(\frac{(2n+1)\pi}{2}) = \sum_{n\geq 1} \frac{4}{(2n+1)\pi}(-1)^n = 1 \Rightarrow \sum_{n\geq 1} \frac{1}{(2n+1)}(-1)^n = \frac{\pi}{4}$. So we have an approximation for π.

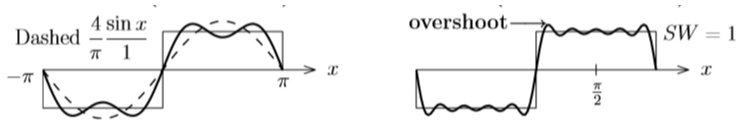

Figure 4.2: **Gibbs phenomenon**: Partial sums $\sum_1^N b_n \sin nx$ overshoot near jumps.

12. Parseval's identity for Hilbert spaces: $\sum |\langle v, e_k\rangle|^2 = \|v\|^2$. Therefore, in the $L^2[-L, L]$ case for for $f(x) := a_0 + \sum a_k cos(nx) + b_k sin(nx)$ we have:

$$a_0^2 + \sum_{k\geq 1} a_k^2 + b_k^2 = \|f\|_{L^2}^2 = \frac{1}{L}\int_{-L}^{L} |f|^2.$$

13. applied to the examples.

    - we have $\frac{1}{L}\int_{-L}^{L} |x^2|^2 = \frac{2}{L}\frac{L^5}{5} = \frac{L^4}{10}$ and

    $$(\frac{4}{3})^2 + \sum (16\frac{(-1)^n}{n^2\pi})^2 = \frac{2^4}{10} \Rightarrow \sum \frac{1}{n^4} = \frac{\pi^4}{90}.$$

    - we have $\frac{1}{L}\int_{-L}^{L} |x^3|^2 = \frac{2}{L}\frac{L^6}{6} = \frac{L^5}{12}$ and

    $$\sum (96\frac{(-1)^n}{(n\pi)^3})^2 = \frac{2^5}{12} \Rightarrow \sum \frac{1}{n^6} = \frac{\pi^6}{96^2}.$$

    - $1 = \sum_{n\geq 1} (\frac{4}{(2n+1)\pi})^2 \Rightarrow \sum_{n\geq 1} \frac{1}{(2n+1)^2} = (\frac{\pi}{4})^2$.

Detailed proof for L2[S] is a complete inner product space

Take cauchy sequence f_n. Then take subsequence such that $\left\|f_{n_k} - f_{n_{k+1}}\right\|_{L^2} \leq 2^{-k}$. We will show that $f_n \to f(x) := f_{n_1}(x) + \sum f_{n_{k+1}}(x) - f_{n_k}$ in L^2. First we will show that $f_{n_k} \overset{L^2}{} f$.

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- By triangle inequality we have for $S_K g(x) := |f_{n_1}(x)| + \sum^K |f_{n_{k+1}}(x) - f_{n_k}|$

$$\|S_k g\| \leq \|f_{n_1}\| + \sum^K \left\|f_{n_{k+1}}(x) - f_{n_k}\right\| \leq \|f_{n_1}\| + \sum^K 2^{-k} < \infty.$$

- Monotone convergence theorem for integrals: Consider sequence $\{h_n\}$ s.t. a)they are monotone $0 \leq h_k(x) \leq h_{k+1}(x)$ and b)converge pointwise to $h_k(x) \to h(x)$ then

$$\int h_n \to \int h.$$

- Therefore
$$\|g\| = \lim_{J\to\infty} \|S_K g\| \leq \lim_{k\to\infty} \|f_{n_1}\| + \sum^{K} 2^{-k} \leq \|f_{n_1}\| + 2 < \infty.$$
and so $|f| \leq |f_{n_1}(x)| + \sum |f_{n_{k+1}}(x) - f_{n_k}| \in L^2$. Therefore, $f_{n_k}(x) \to f(x)$ pointwise.

- Dominated convergence theorem for integrals:Suppose that the sequence $\{h_n\}$ a) is bounded by integrable $|h_n(x)| \leq B(x)$ and b)converge pointwise to $h_k(x) \to h(x)$ then
$$\int |h_n - h| \to 0.$$

- Since $|f_{n_K}| = |S_{K-1}f| \leq S_{K-1}g \leq g \in L^2$, we obtain
$$\int |f_{n_K} - f|^2 = \int |S_{K-1}f - f|^2 \to 0.$$

- Therefore, $f_{n_k} \overset{L^2}{f}$

Finally, we will show that $f_n \overset{L^2}{f}$. This follows from the Cauchy property: Given $\varepsilon > 0$ there exists $N_{\varepsilon/2}$ s.t.
$$\|f_n - f_m\|_{L^2} \leq \varepsilon/2$$
for all $n, m \geq N_{\varepsilon/2}$. Next choose an $n_{k_\varepsilon} > N_{\varepsilon/2}$ from the subsquence s.t.
$$\left\|f_{n_{k_\varepsilon}} - f\right\|_{L^2} \leq \varepsilon/2.$$

Therefore, by triangle inequality we find
$$\|f_n - f\|_{L^2} \leq \left\|f_n - f_{n_{k_\varepsilon}}\right\|_{L^2} + \left\|f_{n_{k_\varepsilon}} - f\right\|_{L^2} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \geq N_{\varepsilon/2}$.