Transfer function noise model and intervention analysis

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Transfer function noise model

Dynamic or time series regression

Optional reading: Chapter 14 of Wei (2005)

Transfer function noise (TFN) model

- A TFN model is a time series regression that predict values of a dependent variable based on both the current and lagged values of one or more explanatory variables.
- A distributed lag model in statistics and econometrics*
- E.g. sales and advertisement are the example of the dependent variable and the input or explanatory variable in a TFN model

- Mathematical form (causal single-input, single-output system)

$$y_t = v(B)x_t + n_t$$

where $v(B) = \sum_{j=0}^{\infty} v_j B^j$, and x_t and n_t are independent.

- The coefficients $v_0, v_1, \cdots$ are referred to as the impulse response function of the system.* (Distributed lag models require impulse response functions of the same sign.)

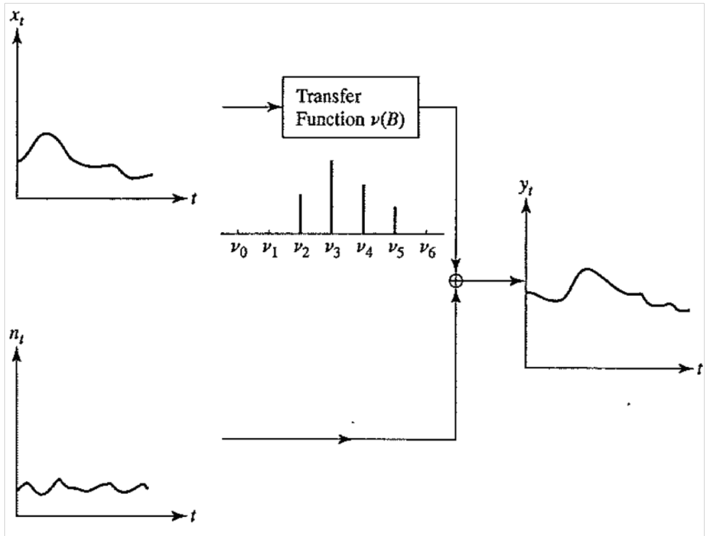


FIGURE 14.1 Dynamic transfer function system.

TFN Model

Transfer function noise model

- TFN model: $y_t = v(B)x_t + n_t$
- For the above equation to be *meaningful*, the impulse responses must be absolutely summable, i.e., $\sum_{j=0}^{\infty}\left|v_j\right| < \infty$.
  - In this case, the system is said to be stable.
- The value $g = \sum_{j=0}^{\infty} v_j$ is called the stead-state gain
  - It represents the impact on Y when X_{t-j} are held constant over time.

Model with infinite number of parameters

1. Unstructured estimation
2. Structured estimation/approximation
   - ❖Finite distributed lag model, e.g. Almon distributed lag model

$$v_j = \sum_{j=0}^{n} a_j i^j ,$$

   where $i = 0, \ldots, k$ and $n < k$.
   - ❖Rational (infinite) distributed lag model, e.g. Koyck distributed lag model

Koyck distributed lag model

$$y_t = \alpha + \sum_{i=0}^{\infty} \beta\lambda^i x_{t-i} + \sum_{i=0}^{\infty} \lambda^i \xi_{t-i} \, .$$

- That is, we have $v_i = \beta\lambda^i$. Suppose that $|\lambda| < 1$
- We can approximate the Koyck distributed lag model using the following ARX model

$$y_t = a + \lambda y_{t-1} + \beta x_t + \xi_t.$$

Rational distributed lag model

$$y_t = \sum_{i=0}^{\infty} v_i x_{t-i} + n_t$$

- Jorgenson (1966, Econometrica) proves that $v(B) = \sum_{i=0}^{\infty} v_i B^i$

can be approximated by a ratio of two polynomials

$$v(B) = \frac{\delta_0 + \delta_1 B + \cdots + \delta_r B^r}{1 - \vartheta_1 B - \cdots - \vartheta_s B^s} = \frac{\delta(B)}{\vartheta(B)}.$$

- Using the rational distributed lag function, we cam approximate y_t as

$$y_t = \frac{\delta(B)}{\vartheta(B)} x_t + n_t.$$

where we allows the error term ε_t to follow a stationary ARMA process. The above equation satisfies the form of a transfer function noise model.

Model building process

- The procedure of building the single input TFN model includes
  1. Preliminary identification of the impulse response coefficients ν_i*'s*;
  2. Specification of the noise term ε_t;
  3. Specification of the transfer function using a rational polynomial in *B if necessary*;
  4. Estimation of the TFN model specified in Step 2 and 3;
  5. Model diagnostic checks.
- See the supplement materials for Model diagnostic checks and estimation using the Box and Tiao approach.
- In practice, we may model the multiple inputs TFN model using vector autoregression.

Preliminary identification (prewhitening)

- Suppose that x follows an $ARMA$ model

$$\phi_x(B)x_t = \theta_x(B)\alpha_t, \qquad \alpha_t \sim NID(0, \sigma_\alpha^2).$$

- Apply the operator $\phi_x(B)/\theta_x(B)$ on both sides of the above equation

$$\tau_t = \frac{\phi_x(B)}{\theta_x(B)} y_t = \nu(B) \underbrace{\frac{\phi_x(B)}{\theta_x(B)} x_t}_{\alpha_t} + \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t = \nu(B)\alpha_t + n_t, \qquad (*)$$

where $\tau_t = \frac{\phi_x(B)}{\theta_x(B)} Y_t$ and $n_t = \frac{\phi_x(B)}{\theta_x(B)} \varepsilon_t$

- By design, $\{n_t\}$ is independent of $\{\alpha_t\}$.

Preliminary identification (prewhitening)

- Multiplying both sides of eqn. $(*)$ by α_{t-j} for $j \geq 0$, we have

$$\tau_t \alpha_{t-j} = \nu(B)\alpha_t \alpha_{t-j} + n_t \alpha_{t-j}.$$

- Taking expectation, we have

$$cov(\tau_t, \alpha_{t-j}) = \nu_j \cdot var(\alpha_{t-j}).$$

- By definition

$$\nu_j = \frac{cov(\tau_t, \alpha_{t-j})}{var(\alpha_t)} = corr(\tau_t, \alpha_{t-j}) \cdot \frac{se(\tau_t)}{se(\alpha_t)}.$$

- Thus, we can test the statistical significance of ν_j by examining the statistical significance of $corr(\tau_t, \alpha_{t-j})$.

Intervention analysis

Dummy variables in dynamic regression

Reading: Chapter 10 of Wei (2005)

What do intervention analysis study

- Given that a known intervention occurs at time T
    1. Is there any evidence of a change in the time series (such as the increase of mean level)?
    2. If so, by how much?

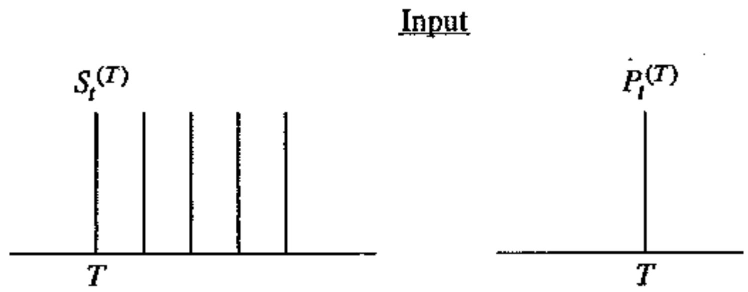
Two common types of intervention variables

- Step function:

$$S_t^{(T)} = \begin{cases} 0, & t < T \\ 1, & t \geq T \end{cases}$$

- Pulse function:

$$P_t^{(T)} = \begin{cases} 0, & t \neq T \\ 1, & t = T \end{cases}$$



- Relation between step and pulse functions:

$$P_t^{(T)} = S_t^{(T)} - S_{t-1}^{(T)} = (1 - B)S_t^{(T)}$$

Possible responses to intervention

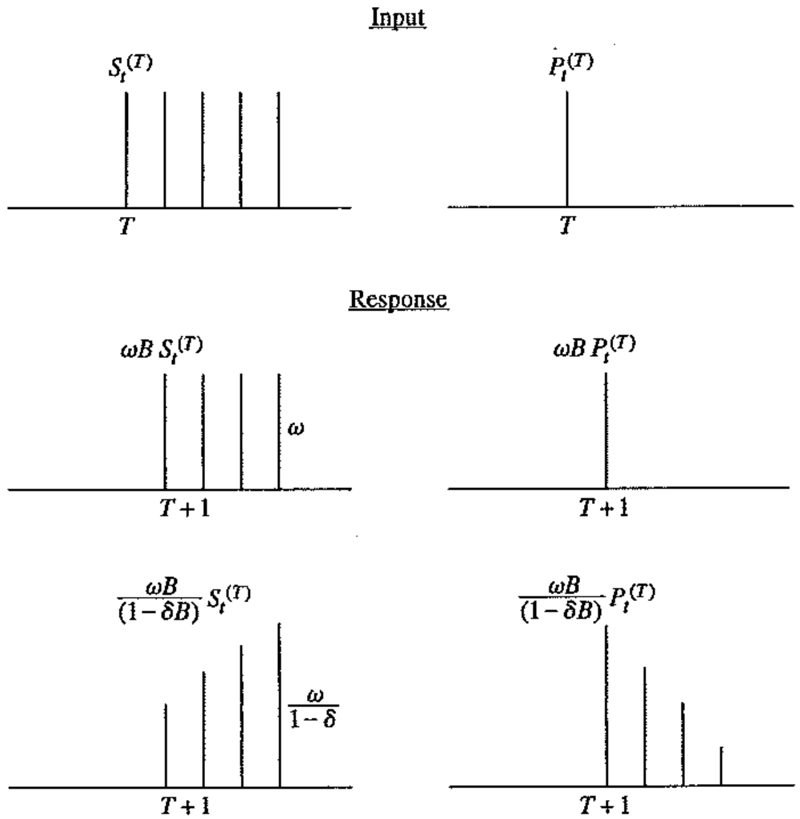


FIGURE 10.1 Responses to step and pulse inputs.

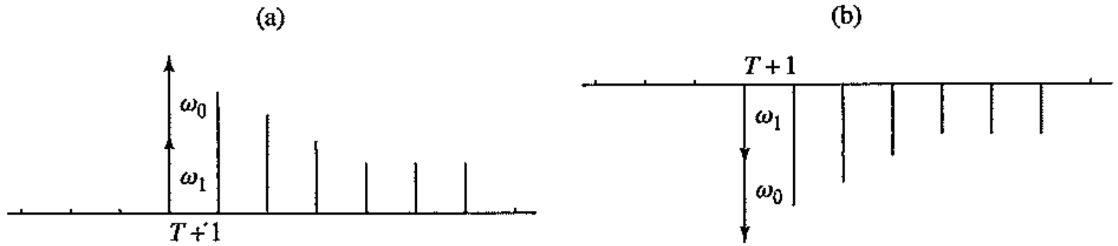


FIGURE 10.2 Response to combined inputs, $[\omega_0 B/(1-\delta B)]P_t^{(T)} + \omega_1 B S_t^{(T)}$. (a) $\omega_0 > 0$ and $\omega_1 > 0$. (b) $\omega_0 < 0$ and $\omega_1 < 0$.

General model for intervention analysis

- For multiple intervention inputs, we have the following general class of models:

$$Z_t = \theta_0 + \sum_{j=1}^{k} \frac{w_j(B)B_j^b}{\delta_j(B)} I_{jt} + \frac{\theta(B)}{\psi(B)} a_t$$

where I_{jt} are intervention variables

$$\omega_j(B) = w_{j0} - w_{j1}B - \cdots - w_{s_j}B^{s_j}$$

$$\delta_j(B) = \delta_{j0} - \delta_{j1}B - \cdots - \delta_{r_j}B^{r_j}$$

Time series outliers

- Time series observations are sometimes influenced by interruptive events, such as strikes, outbreaks of war, or sudden political or economic crises.
- The consequence of these interruptive events create spurious observations that are inconsistent with the rest of the series. Such observations are usually referred to as outliers.

Additive and innovational outliers

- Z_t denotes the observed serie
- X_t is the outlier-free series satisfying a stationary and invertible ARMA model

$$\phi(B)X_t = \theta(B)a_t, \qquad a_t \sim NID(0, \sigma_a^2).$$

- Additive outlier (AO):

$$Z_t = \begin{cases} X_t & t \neq T \\ X_t + \omega & t = T \end{cases} = X_t + \omega P_t^{(T)} = \frac{\theta(B)}{\phi(B)} a_t + \omega P_t^{(T)}, \qquad (1)$$

- Innovational outlier (IO):

$$Z_t = X_t + \frac{\theta(B)}{\phi(B)} \omega P_t^{(T)} = \frac{\theta(B)}{\phi(B)} \left(a_t + \omega P_t^{(T)} \right), \qquad (2)$$

Estimation when the timing is known

1. Define

$$e_t = \pi(B)Z_t,$$

where $\pi(B) = \phi(B)/\theta(B) = 1 - \pi_1 B - \pi_2 B^2 - \cdots$

2. From Equation (1) and (2), we have

$$\text{AO: } e_t = \omega\pi(B)P_t^{(T)} + a_t$$

$$\text{IO: } e_t = \omega P_t^{(T)} + a_t$$

AO model: $e_t = \omega\pi(B)P_t^{(T)} + a_t$

$$\begin{bmatrix} e_1 \\ \vdots \\ e_{T-1} \\ e_T \\ e_{T+1} \\ e_{T+2} \\ \vdots \\ e_n \end{bmatrix} = \omega \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\pi_1 \\ -\pi_2 \\ \vdots \\ -\pi_{n-T} \end{bmatrix} + \begin{bmatrix} a_1 \\ \vdots \\ a_{T-1} \\ a_T \\ a_{T+1} \\ a_{T+2} \\ \vdots \\ a_n \end{bmatrix}.$$

$$\hat{\omega}_{AT} = \frac{e_T - \sum_{j=1}^{n-T} \pi_j e_{T+j}}{\sum_{j=0}^{n-T} \pi_j^2}$$

$$\begin{aligned} \operatorname{Var}(\hat{\omega}_{AT}) &= \operatorname{Var}\left(\frac{\pi^*(F)e_T}{\tau^2}\right) \\ &= \frac{1}{\tau^4}\operatorname{Var}\left[\pi^*(F)a_T\right] \\ &= \frac{\sigma_a^2}{\tau^2}. \end{aligned}$$

$$\tau^2 = \sum_{j=0}^{n-T} \pi_j^2.$$

IO model: $e_t = \omega P_t^{(T)} + a_t$

$$\text{IO:} \quad \hat{\omega}_{IT} = e_T$$

$$\begin{aligned} \operatorname{Var}(\hat{\omega}_{IT}) &= \operatorname{Var}(e_T) = \operatorname{Var}(\omega I_t^{(T)} + a_T) \\ &= \sigma_a^2. \end{aligned}$$

Hypothesis testing

H_0: Z_T is neither an AO nor an IO

H_1: Z_T is an AO

H_2: Z_T is an IO.

The likelihood ratio test statistics for AO and IO are

$$H_1 \text{ vs. } H_0: \qquad \lambda_{1,T} = \frac{\tau \hat{\omega}_{AT}}{\sigma_a}$$

and

$$H_2 \text{ vs. } H_0: \qquad \lambda_{2,T} = \frac{\hat{\omega}_{IT}}{\sigma_a}.$$

Under the null hypothesis H_0, both $\lambda_{1,T}$ and $\lambda_{2,T}$ are distributed as $N(0, 1)$.