L. Sensitivity Analysis

In this section, we discuss the sensitivity analysis for the IPW estimators. The validity of the estimator depends on the unconfoundedness assumption (Assumption 1). Therefore, we propose a sensitivity analysis to address situations where Assumption 1 may be violated in real-world scenarios.

Assume there exists the unobserved time-varying confounding variable U_t, then we assume

$$Z_t \perp\!\!\!\perp H_{\leq t}, U_{\leq t} | h_{\leq t}, u_{\leq t}, \tag{14}$$

where $U_{\leq t}$ denotes all possible values of unobserved confounding variables up to time t, and $u_{\leq t}$ represents the realization of unobserved confounding variables up to time t. In Eq. (14), we assume that the unconfoundedness assumption only holds when conditioning on both the observed historical information and the realization of the unobserved confounding variables.

Under the new unconfoundedness assumption (Eq. (14)), the true propensity score is denoted as $e_t^*(z_t) = \mathbb{P}(Z_t = z_t | h_{\leq t}, u_{\leq t})$, so the propensity score used in this paper ($e_t(z_t) = \mathbb{P}(Z_t = z_t | h_{\leq t})$) becomes invalid.

Therefore, to perform the sensitivity analysis, we assume that the ratio between the propensity score used in this paper ($e_t(z_t)$) and the true propensity score ($e_t^*(z_t)$) is bounded:

$$\rho_t = \frac{e_t(z_t)}{e_t^*(z_t)} \in [\Gamma^{-1}, \Gamma], \tag{15}$$

where $\Gamma \geq 1$ is a real number. Obviously, a larger Γ represents a greater degree of violation of the unconfoundedness assumption.

Next, we will determine the bounds of the IPW estimator for each fixed value of Gamma. Let $\rho = (\rho_1, \rho_2, \ldots, \rho_T) \in [\Gamma^{-1}, \Gamma]$, then for all values in ρ, the objective is to derive a bound for:

$$\hat{N}_\rho(F_H) = \sum_{t=M}^{T} \prod_{j=t-M+1}^{t} \rho_j \ \hat{N}_t^\omega(F_H). \tag{16}$$

Obviously, $\prod_{j=t-M+1}^{t} \rho_j \in [\Gamma^{-M}, \Gamma^M]$. Let $\alpha \in [\Gamma^{-M}, \Gamma^M]$ denote the set of all vectors with length T, where the entry at position t can be represented as the product $\prod_{j=t-M+1}^{t} \rho_j$ for some values ρ_j satisfying $\Gamma^{-1} \leq \rho_j \leq \Gamma$ for all j. Then we have

$$\min_{\rho \in [\Gamma^{-1}, \Gamma]} \{\hat{N}_\rho(F_H)\} \geq \min_{\alpha \in [\Gamma^{-M}, \Gamma^M]} \sum_{t=M}^{T} \alpha_t \ \hat{N}_t^\omega(F_H) \tag{17}$$

$$\max_{\rho \in [\Gamma^{-1}, \Gamma]} \{\hat{N}_\rho(F_H)\} \leq \max_{\alpha \in [\Gamma^{-M}, \Gamma^M]} \sum_{t=M}^{T} \alpha_t \ \hat{N}_t^\omega(F_H). \tag{18}$$

We can compute the quantities on the right by applying the Charnes-Cooper transformation, which converts the linear fractional problem into a linear optimization problem [2] (Cooper & Charnes, 1962). Then, the resulting values provide conservative lower and upper bounds for our target quantity.

In summary, we derive a sensitivity analysis for the estimator under potential violations of the unconfoundedness assumption.

References

Cooper, A. C. W., & Charnes, W. (1962). Programming with linear fractional functionals. Naval Research logistics quarterly, 9(3), 181-186.

[2]Cooper et al. proved that the linear fractional problem can be transformed into a linear programming problem.