hw4

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1

All 4 of the integrals are equal to $\boxed{\text{zero}}$. None of them have singularities inside the curve of integration, and the Cauchy-Goursat Theorem tells us that an integral on a simple closed path is equal to zero if the interior is continuously differentiable (no singularities).

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a) $z = -1$

b) $z = 0, 1$

c) $z = 0$

d) $z = \pm i$

3

a)

$$H(z) = z^2 \implies \frac{z^2}{z-(-1)} \implies z^2\bigg|_{z=-1} = \boxed{1}$$

b)

$$H(z) = \frac{z^2+3z+1}{z} \implies \frac{\frac{z^2+3z+1}{z}}{z-(-1)} \implies \frac{z^2+3z+1}{z}\bigg|_{z=-1} = \boxed{1}$$

$$H(z) = \frac{z^2+3z+1}{z-1} \implies \frac{\frac{z^2+3z+1}{z-1}}{z-0} \implies \frac{z^2+3z+1}{z-1}\bigg|_{z=0} = \boxed{-1}$$

c)

$$H(z) = \cos(z) \implies \frac{\cos(z)}{z} \implies \cos(z)\bigg|_{z=0} = \boxed{1}$$

d)

$$H(z) = \frac{e^z}{z-i} \implies \frac{\frac{e^z}{z-i}}{z-(-i)} \implies \frac{e^z}{z-i}\bigg|_{z=-i} = \boxed{\frac{e^{-i}}{-2i}}$$

$$H(z) = \frac{e^z}{z+i} \implies \frac{\frac{e^z}{z+i}}{z-i} \implies \frac{e^z}{z+i}\bigg|_{z=i} = \boxed{\frac{e^{i}}{2i}}$$

4

The singularities are at $z = 0, 1, 2$. Let's find all the residues first.

$$\frac{\frac{z+1}{(z-1)(z-2)}}{z} \implies \left.\frac{z+1}{(z-1)(z-2)}\right|_{z=0} \implies 1/2$$

$$\frac{\frac{z+1}{z(z-2)}}{z-1} \implies \left.\frac{z+1}{z(z-2)}\right|_{z=1} \implies -2$$

$$\frac{\frac{z+1}{z(z-1)}}{z-2} \implies \left.\frac{z+1}{z(z-1)}\right|_{z=2} \implies 3/2$$

4.1 Case I

We only need to worry about the singularity at $z = 0$, since $z = 1, 2 \notin |z| = 1/2$ (the singularities at $z = 1, 2$ are not inside the current circle). Since the integral is about a circle of radius R (simple closed curve), the answer is just the residue at $z = 0$ multiplied by $2\pi i$. The answer is $\boxed{\pi i}$.

4.2 Case II

Sum up the residue at $z = 0, 1$, then multiply. The answer is $1/2 - 2 \implies \boxed{-3\pi i}$.

4.3 Case III

Sum up the residue at $z = 0, 1, 2$. The answer is $1/2 - 2 + 3/2 \implies \boxed{0}$.

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The Cauchy-Integral Theorem is

$$\oint_{\gamma} f(z)\, dz = 0$$

if γ is a simple closed path.

6

6.1 a)

$$\frac{1}{2\pi i}\int\limits_{|z-a|=r}\frac{g(z)}{z-a}\,dz=\frac{1}{2\pi i}\int\limits_0^{2\pi}\frac{g(a+re^{i\theta})}{re^{i\theta}-\cancel{a+a}}\cdot ire^{i\theta}\,d\theta$$

$$=\frac{1}{2\pi \cancel{i}}\int\limits_0^{2\pi}\frac{g(a+re^{i\theta})}{\cancel{re^{i\theta}}}\cdot\cancel{ire^{i\theta}}\,d\theta$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}g(a+re^{i\theta})\,d\theta$$

$\square$

6.2 b)

As r approaches 0, the integral

$$\frac{1}{2\pi}\int\limits_0^{2\pi}g(a+re^{i\theta})\,d\theta$$

turns into

$$\frac{1}{2\pi}\int\limits_0^{2\pi}g(a)\,d\theta$$

Since there's no theta inside the integral, we can simplify to

$$\frac{g(a)}{2\pi}\int\limits_0^{2\pi}d\theta=\frac{g(a)}{2\pi}\cdot 2\pi=g(a)$$

6.3 c)

The residual is equal to the average value around the singularity. But since it doesn't actually depend on how big the radius of this "around", we can just have the radius approach 0 and conclude that it's just the value at the singularity.

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We need to compute the sum of the residuals at $z = 1, 2$

$$\begin{aligned}
\oint\limits_{|z-1|=3} \frac{z^2}{(z-2)^2(z-1)}\,dz &= 2\pi i\left[Res_{z=1}\frac{z^2/(z-2)^2}{z-1} + Res_{z=2}\frac{z^2/(z-1)}{(z-2)^2}\right] \\
&= 2\pi i\left[\left[z^2/(z-2)^2\right]\Bigg|_{z=1} + \left[\frac{1}{1!}\frac{\mathrm{d}}{\mathrm{d}z}(z^2/(z-1))\right]\Bigg|_{z=2}\right] \\
&= 2\pi i\left[1 + \left[\left(\frac{2z(z-1)-z^2}{(z^2/(z-1))^2}\right)\right]\Bigg|_{z=2}\right] \\
&= 2\pi i\left[1 + \left[\left(\frac{2\cdot 2(2-1)-2^2}{(2^2/(2-1))^2}\right)\right]\right] \\
&= 2\pi i\left[1 + \left[\left(\frac{4-4}{(4)^2}\right)\right]\right] \\
&= 2\pi i\left[1+0\right] \\
&= \boxed{2\pi i}
\end{aligned}$$

8

8.1 a)

If a function $f(z)$ is analytic on a disk $|z-a|<R$, then

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

will converge to $f(z)$

8.2 b)

1.
$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + \cdots + \frac{1}{n!}z^n$$
Since e^z is analytic everywhere, so this Taylor expansion is valid everywhere.

2.
$$\cos(z) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$
The cosine function is also analytic everywhere, so the expansion is valid everywhere.

3.
$$1/(1+z) = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n$$
This is valid only on the disk with $R = 1$ because the function is centered at $z = 0$ is not analytic at $z = -1$, resulting in a disk with radius 1.

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The largest disk would be one of $\boxed{R = 1}$. This is because the point $z = 1 + i$ is 1 away from $z = 1$. The other point is disregarded because it is further away.

The largest disk would be one of $\boxed{R = \sqrt{4^2 + \frac{1}{2^2}} = 65/4}$. This is because the point $z = 1/2 + 4i$ is 65/4 away from both $z = 0, 1$.