Homework 6

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1

a) $f(z)$ needs to include:

1) $15e^{1/z}$

   $e^{1/z} = 1 + 1/z + 1/(2!z^2) + 1/(3!z^3) + \cdots$ has infinite terms with negative powers, making it an essential singularity. The residue is 1 because of the coefficient of $1/z$. To get a residue of 15, we can just multiply the entire thing by 15.

2) $\dfrac{16}{z - i}$

   $\dfrac{1}{z-i}$ has a simple pole at $z = i$, and a residue of 1. We can multiply by 16 to get the residue we want.

3) $\dfrac{1}{(z-(2+3i))^4} + \dfrac{17+22i}{z-(2+3i)}$

   The first term satisfies the pole order requirement, but the residue is 0 because we take derivatives of 1 (and multiply), which would be 0. So the second term fulfills the requirement of the residue, which equals $17 + 22i$

4) $\dfrac{\sin(z+2)}{z+2}$

   $\dfrac{\sin(z+2)}{z+2} = 1 - \dfrac{(z+2)^2}{3!} + \dfrac{(z+2)^2}{5!} + \cdots$, making it a removable singularity.

b) Removable singularities have a residue of 0.

2

First, find the distances of the singularities to the center of the circle.

point	distance
-10	$\sqrt{145}$
5	$\sqrt{10}$
3i	$\sqrt{8}$

(a) There are no singularities inside $R = 2$, so the result is 0, using the Cauchy Integral Theorem.

(b) The singularity at $3i$ needs to be accounted for. Using the residue theorem, the result is $2\pi i(6 + 7i)$.

(c) One new singularity appears at 5, so we just add it to the previous integral. The result is $2\pi i(6 + 12i)$

(d) All singularities needs to be accounted for now. The result is $2\pi i(21 + 12i)$

(e) We need to recalculate the distances.

point	distance
-10	$\sqrt{101}$
5	$\sqrt{26}$
3i	2

The smallest distance to a singularity is 2, so that's our radius.

(f)

point	distance
-10	10
5	5
3i	3

The smallest distance is 3.

3

(a) $z = \pm 1$

(b) The shortest distance from $z = 0$ to $z = \pm 1$ is 1, so the set of points is $|z| < 1$.

(c) $f(z) = \sum_{n=0}^{\infty} \frac{f^(n)(a)}{n!}(z-a)^n, |z-a| < R$

(d)

$$
\begin{aligned}
a_0 &\to h(z) && \to 0 \\
a_1 &\to (-2)(1-z^2)^{-3}(-2z) && \to 0 \\
a_2 &\to 4(1-z^2)^{-3}4z(-3)(1-z^2)^{-4}(-2z) && \to 0 \\
a_3 &\to 4(-3)(1-z^2)^{-4}(-2z) + 48z(1-z
\end{aligned}
$$

(e) $1 + 2z^2 + 3z^4 + 4z^6 + \cdots$

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= 2 \\
a_3 &= 0 \\
a_4 &= 3 \\
a_5 &= 0 \\
a_6 &= 4
\end{aligned}
$$

4

Because the function is even and it meets the power requirements, we can use Fact #11.

$$\int_0^\infty \frac{1}{(x^2+16)^3}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(x^2+16)^3}\,dx \qquad = \frac{1}{2}\oint_{Im(z)>0} \frac{1}{(z^2+16)^3}\,dz$$

We just need to find the relevant singularities and residues. The only singularity is at $z = +4i$. The result is

$$\frac{1}{2}2\pi i(\mathrm{Res}_{z=4i}\,\frac{(z+4i)^-3}{(z-4i)^3}) = \frac{3}{2^1 4}$$

5

(a)

$$\begin{aligned}\frac{1+2i}{1-2i} &= \frac{(1+2i)(1+2i)}{(1-2i)(1-2I)} \\ &= \frac{-3+4i}{5} \\ &= \frac{-3}{5} + i\frac{4}{5}\end{aligned}$$

(b)

$$\begin{aligned}e^{3+4i} &= e^3 e^{4i} \\ &= e^3(\cos(4) + i\sin(4)) \\ &= e^3\cos(4) + ie^3\sin(4)\end{aligned}$$

(c)

$$\begin{aligned}\sin(3+7i) &= \frac{e^{3i-7} - e^{-3i+7}}{2i} \\ &= \frac{e^{-7}(\cos(3) + i\sin(3)) - e^{7}(\cos(3) - i\sin(3))}{2i} \\ &= \frac{i\sin(3)(e^{-7} + e^{7}) + \cos(3)(e^{-7} - e^{7})}{2i} \\ &= \frac{\sin(3)(e^{-7} + e^{7})}{2} + i\frac{\cos(3)(e^{7} - e^{-7})}{2}\end{aligned}$$

6

Use Fact #12.

$$\int_0^{2\pi} \frac{1}{6+\sin(\theta)}\,d\theta = \oint_{|z|=1} \frac{1}{6+\dfrac{z+z^{-1}}{2i}}\frac{1}{iz}\,dz$$

$$\oint_{|z|=1} \frac{1}{6iz+\dfrac{z^2-1}{2}}\,dz$$

Find the singularities.

$$z^2+12iz-1=0$$
$$\frac{-12i\pm\sqrt{-144+4}}{2} = -6i \pm i\sqrt{35}$$

Only one of these singularities is inside our region. Now find the residue.

$$\begin{aligned}
\operatorname{Res}_{z=(-6+\sqrt{35})i} \frac{1}{6iz+(z^2-1)/2} \\
&= \frac{1}{\dfrac{\mathrm{d}6iz+(z^2-1)/2}{\mathrm{d}z}} \\
&= \frac{1}{6i+z} \\
&= \frac{1}{\sqrt{35}i}
\end{aligned}$$

The result is $\dfrac{2\pi}{\sqrt{35}}$

7

(a) Verify $u_{xx} + u_{yy} = 0$

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \\ u_{xx} &= 6x \end{aligned}$$

$$\begin{aligned} u_y &= -6xy \\ u_{yy} &= -6x \end{aligned}$$

$$6x - 6x = 0$$

$k(z)$ is harmonic.

(b) Set $u_x = v_y \wedge u_y = -v_x$

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \\ v_y &= A(-3x^2 + 3y^2) \end{aligned}$$

$$\begin{aligned} v_y &= -6xy \\ -v_x &= A(6xy) \end{aligned}$$

The only way this can work is if $A = -1$.

(c) Use the $y = 0 \implies z = x$ heuristic. We get $k(z) = z^3$

8

(a) The mapping cubes the radius and multiplies the angle by 3. Both of those are one-to-one functions, and $\pi/3$ multiplied by 3 is the upper half of the plane.

(b) $w = e^x e^{iy}$. $\mathrm{Im}(z)$ corresponds to the angle of w. The angle from 0 to π is just the upper half of the plane.

(c) $w = -1/z = -1/re^{i\theta} = 1/2e^{i(\pi-\theta)}$. Since we take the inverse of the radius, we satisfy $|w| < 1/5$. We get the upper half of the plane, just backwards starting from π to 0 (which is the same region).

(d) Find $\mathrm{Re}(1/z) > 1$

$$\begin{aligned}
\mathrm{Re}(1/z) &> 1 \\
\mathrm{Re}(1/z \cdot \bar{z}/\bar{z}) &> 1 \\
\mathrm{Re}(\bar{z}/|z|) &> 1 \\
x/|z| &> 1 \\
x &> |z| \\
x &> x^2 + y^2 \\
x^2 - x + y^2 < 0 & \\
(x - 1/2)^2 + y^2 < 1/2^2 &
\end{aligned}$$

This is a circle centered at $(1/2, 0)$ with $r = 1/2$, which is the same as $|z - 1/2| < 1/2$.