LA HW 3

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G1

(a)

$$\text{Inc} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Delta(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

(b)

$$\text{Deg} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Adj} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c) The eigenvalues are the row vectors.

$$\text{eigen}(\Delta(G)) = \begin{bmatrix} 1 & 1 & -3 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, [4, 3, 1, 0]$$

$$\text{eigen}(\text{Deg}) = I_4, [2, 2, 3, 1]$$

$$\text{eigen}(\text{Adj}) \approx \begin{bmatrix} 1.85 & 1.85 & 2.17 & 1 \\ 0.60 & 0.60 & -1.48 & 1 \\ -1 & 1 & 0 & 0 \\ -0.45 & -0.45 & 0.31 & 1 \end{bmatrix}, [-2.17, 1.48, 1, -0.311]$$

(d) Note: This question will only be answered once because it is not dependent on the graph. The Cheeger estimate tells us this. Alternatively, since the eigenvalues of a matrix are equal to that of its transpose, the eigenvalues of $\Delta(G)$ are real and non-negative because it's the product of a matrix and it's transpose. It's like how multiplying a value by itself makes it real and non-negative.

G2 (a)

$$\text{Inc} = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\Delta(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

(b)

$$\text{Deg} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\text{Adj} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(c)

$$\text{eigen}(G) = \begin{bmatrix} -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, [4, 4, 4, 0]$$

$$\text{eigen}(Deg) = [1, 1, 1, 1], [3]$$

$$\text{eigen}(Adj) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}, [3, 1, 1, 1]$$

G3 (a)

$$\text{Inc} = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\Delta(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

(b)

$$\text{Deg} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\text{Adj} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$\text{eigen}(G) = \begin{bmatrix} -2 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, [4, 4, 2, 0]$$

$$\text{eigen}(Deg) = I_4, [3, 2, 3, 2]$$

$$\text{eigen}(Adj) \approx \begin{bmatrix} 1.28 & 1 & 1.28 & 1 \\ -0.79 & 1 & -0.78 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, [3, 1, 1, 1]$$

2

(a)

$$\text{eigen}(\Delta(G)) = \begin{bmatrix} 1 & 1 & -3 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, [4, 3, 1, 0]$$

We can see that $4 \geq 3 \geq 1 \geq 0 = 0$

(b) a

$$\frac{2 \cdot 2}{1} = 4 > \lambda_2 = 1$$

b

$$\frac{2 \cdot 2}{1} = 4 > \lambda_2 = 1$$

c

$$\frac{2 \cdot 1}{1} > \lambda_2 = 1$$

(c)

$$\lambda_2 \leq \frac{2\|\#E(V_1, V_2)\|}{\min\{\|V_1\|, \|V_2\|\}}$$

(d) The determinant of the mesh of the graph is equal to the number of spanning trees. Coefficient of T in the characteristic polynomial is equal to the number of vertices times the number of spanning trees.

3

(G1) (a) There are 3 trees that span the graph.

(b)

$$\Delta(G) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The characteristic polynomial is $-12T + 19T^2 - 8T^3 + T^4$.

(c) Since we only care about the zeros of this polynomial, we can multiply it by -1. So we don't need to worry about the sign. The coefficient is equal to the number of connected nodes/vertices times the number of spanning trees. $4 \cdot 3 = 12$

(d) The tree that omits e_2 will be used.

The vector is $[1, -1, 1]$

$$[3]$$

(e) The determinant is 3. They are equal.

(G2) (a) There are 16 trees that span the graph.

(b)

$$\Delta(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The characteristic polynomial is $-64T + 48T^2 - 12T^3 + T^4$.

(c) Since we only care about the zeros of this polynomial, we can multiply it by -1. So we don't need to worry about the sign. The coefficient is equal to the number of connected nodes/vertices times the number of spanning trees. $16 \cdot 4 = 64$

(d) The tree that omits the 3 sides of the triangle will be used.

The vectors are $[-1, 1, -1], [-1, 1, -1], [-1, 1, -1]$

$$M = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

(e) The determinant is 16. They are equal.

(G3) (a) There are 8 trees that span the graph.

(b)

$$\Delta(G) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

The characteristic polynomial is $-21T + 32T^2 - 10T^3 + T^4$.

(c) Since we only care about the zeros of this polynomial, we can multiply it by -1. So we don't need to worry about the sign. The coefficient is equal to the number of connected nodes/vertices times the number of spanning trees. $16 \cdot 4 = 64$

(d) The tree that has the right and top edge omitted will be used.

The vectors are $[1, 1, 1], [1, 1, -1]$.

$$M = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

(e) The determinant is 8. They are equal.

4

(a) $R \approx 7.66, S \approx 22.48$. The general formulas are $R = \dfrac{\text{Sample Correlation}(X, Y)}{\text{Sample Variance}(X)}, S = \bar{Y} - \bar{X}R$

(b) Yes. It does by definition.

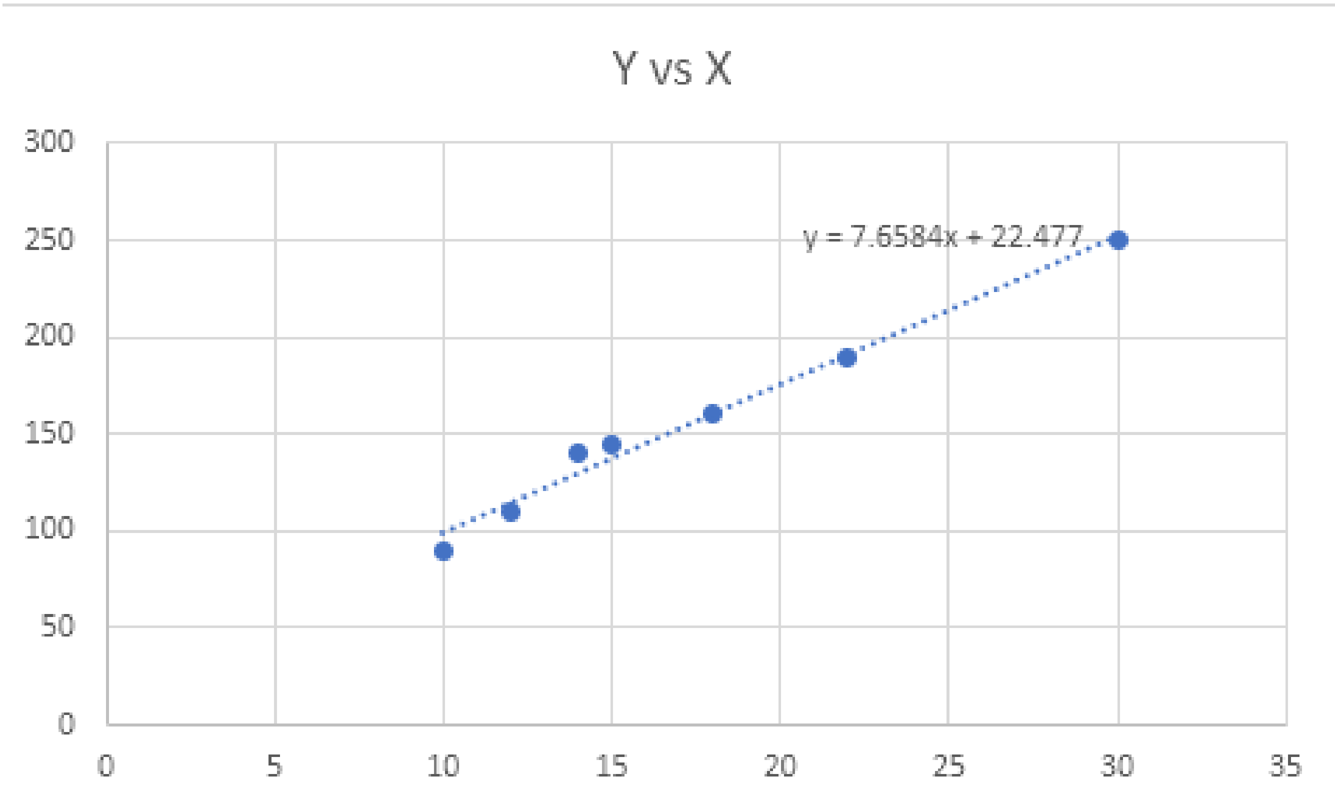

Figure 1: Excel screenshot

(c)

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$$A = \begin{bmatrix} 10 & 12 & 14 & 15 & 18 & 22 & 30 \\ 90 & 110 & 140 & 145 & 160 & 189 & 250 \end{bmatrix}$$

Subtract $\bar{X}$ element-wise from the first row, and subtract $\bar{Y}$ element-wise from the second row to get B. Note: The equation had to be shrunk to fit on the page.

$$B = \begin{bmatrix} 10 - 17.286 & 12 - 17.286 & 14 - 17.286 & 15 - 17.286 & 18 - 17.286 & 22 - 17.286 & 30 - 17.286 \\ 90 - 154.857 & 110 - 154.857 & 140 - 154.857 & 145 - 154.857 & 160 - 154.857 & 189 - 154.857 & 250 - 154.857 \end{bmatrix}$$

Now calculate $BB^\intercal$

$$BB^\intercal = \begin{bmatrix} 281.42 & 2155.28 \\ 2155.18 & 16780.68 \end{bmatrix}$$

This matrix's row eigenvectors are

$$\begin{bmatrix} 0.127 & 0.991 \\ 0.991 & -0.127 \end{bmatrix}$$

Our result is $0.127X + 0.991Y, 0.991X00.127Y$.

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Note: the matrix A given in the hw pdf is different from the one covered in class; the difference is the value of $A_{2,3}$. Since the matrix values covered in class results in cleaner numbers, I will be using those values.

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

Decompose into the form $U\Sigma V^\intercal$

Calculate $A^\intercal A$

$$A^\intercal A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Find the eigenvalues and eigenvectors.

$$\lambda_1 = 360, \lambda_2 = 90, \lambda_3 = 0$$

$$V = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

Find U

$$u_1 = Av_1/\sqrt{\lambda_1} = [3/\sqrt{10}, 1/\sqrt{10}]^\intercal$$
$$u_2 = Av_2/\sqrt{\lambda_2} = [1/\sqrt{10}, -3/\sqrt{10}]^\intercal$$

$$A = \left(\frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}\right)\left(\frac{1}{\sqrt{10}}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}^\intercal$$
$$= \frac{1}{10}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{bmatrix}^\intercal$$