hw2

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1

1.1 a)

$$\begin{aligned}
e^z &= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x \left(\cos(y) + i\sin(y)\right) \\
&= e^x \cos(y) + ie^x \sin(y)
\end{aligned}$$

$\boxed{X = e^x \cos(y), Y = e^x \sin(y)}$

1.2 b)

Let $z = x + iy, w = a + ib$. Addition and multiplication are commutative, so we can rearrange to get the real together, and the imaginaries together.

$$\begin{aligned}
e^z \cdot e^w &= e^{a+ib} \cdot e^{x+iy} \\
&= e^a \cdot e^i b \cdot e^x \cdot e^i y \\
&= e^{ax} \cdot e^{i(b+y)} \\
&= e^{a+ib+x+iy} \\
&= e^{z+w} \\
&\square
\end{aligned}$$

1.3 c)

Show $e^{2\pi in} = +1$

$$\begin{aligned} e^{2\pi in} &= \left(e^{2\pi i}\right)^n \\ &= (\cos 2\pi + i \sin 2\pi)^n \\ &= (1i \cdot 0)^n \\ &= 1^n \\ &= +1 \\ & \quad \square \end{aligned}$$

Show e^z is periodic with period $2\pi i$: $e^{z+2\pi in} = e^z$

$$\begin{aligned} e^{z+2\pi in} &= e^z \cdot e^{2\pi in} \\ &= e^z \cdot \left(e^{2\pi i}\right)^n \\ &= e^z \left(\cos 2\pi + i \sin 2\pi\right)^n \\ &= e^z \left(1 + 0\right)^n \\ &= e^z \\ & \quad \square \end{aligned}$$

1.4 d)

Compute $\mathrm{Re}\,(\ldots)$, $\mathrm{Im}\,(\ldots)$ of $e^{1+\pi i}, e^{3+i\pi/3}, e^{5+i\pi/4}$

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x \left(\cos y + i \sin y\right) \\ \mathrm{Re}\left(e^{x+iy}\right) &= e^x \cos y \\ \mathrm{Im}\left(e^{x+iy}\right) &= e^x \sin y \end{aligned}$$

1. $z = 1 + i\pi \implies \mathrm{Re}\left(e^z\right) = -e, \mathrm{Im}\left(e^z\right) = 0$
2. $z = 3 + i\pi/3 \implies \mathrm{Re}\left(e^z\right) = e^3/2, \mathrm{Im}\left(e^z\right) = e^3 \cdot \sqrt{3}/2$
3. $z = 5 + i\pi/4 \implies \mathrm{Re}\left(e^z\right) = e^5 \cdot \sqrt{2}/2, \mathrm{Im}\left(e^z\right) = e^5 \cdot \sqrt{2}/2$

2

2.1 a)

$$\begin{aligned}
\cos(x) &= \frac{e^{ix}+e^{-ix}}{2} \\
&= \frac{\cos(x)+i\sin(x)+\cos(-x)+i\sin(-x)}{2} \\
&= \frac{2\cos(x)}{2} && \cos(-x)=\cos(x), \sin(-x)=\sin(x) \\
&= \cos(x) \\
& \square
\end{aligned}$$

$$\begin{aligned}
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} \\
&= \frac{\cos(x)+i\sin(x)-\cos(-x)-i\sin(-x)}{2i} \\
&= \frac{2i\sin(x)}{2i} && \cos(-x)=\cos(x), \sin(-x)=\sin(x) \\
&= \sin(x) \\
& \square
\end{aligned}$$

2.2 b)

$$\begin{aligned}
\cos(2+3i) &= \frac{e^{2+3i}+e^{-2-3i}}{2} \\
&= (1/2)\left(e^2(\cos(3)+i\sin(3))+e^{-2}(\cos(-3)+i\sin(-3))\right) \\
&= (1/2)\left(e^2\cos(3)+ie^2\sin(3)+e^{-2}\cos(-3)+ie^{-2}\sin(-3)\right) \\
&= (1/2)\left(e^2\cos(3)+e^{-2}\cos(-3)+i\left(e^2\sin(3)+e^{-2}\sin(-3)\right)\right)
\end{aligned}$$

$$\begin{aligned}
\sin(2+3i) &= \frac{e^{2+3i}-e^{-2-3i}}{2i} \\
&= (1/2i)\left(e^2(\cos(3)+i\sin(3))-e^{-2}(\cos(-3)-i\sin(-3))\right) \\
&= (1/2i)\left(e^2\cos(3)+ie^2\sin(3)-e^{-2}\cos(-3)-ie^{-2}\sin(-3)\right) \\
&= (1/2i)\left(e^2\cos(3)-e^{-2}\cos(-3)-i\left(e^2\sin(3)+e^{-2}\sin(-3)\right)\right)
\end{aligned}$$

2.3 c)

$$
\begin{aligned}
\cos(z) &= \cos(x+iy) \\
&= \frac{e^{i(x+iy)} + e^{i(-x-iy)}}{2} \\
&= \frac{e^{ix-y} + e^{y-ix}}{2} \\
&= \frac{e^{-y}\cos(x) + ie^{-y}\sin(x) + e^{y}\cos(-x) + ie^{y}\sin(-x)}{2} \\
&= \frac{e^{-y}\cos(x) + e^{y}\cos(-x) + ie^{-y}\sin(x) + ie^{y}\sin(-x)}{2} \\
&= \frac{e^{-y}\cos(x) + e^{y}\cos(x) + ie^{-y}\sin(x) - ie^{y}\sin(x)}{2} \\
&= \frac{\cos(x)\left(e^{-y} + e^{y}\right) + i\sin(x)\left(e^{-y} - e^{y}\right)}{2} \\
&= \frac{\cos(x)\left(e^{-y} + e^{y}\right) - i\sin(x)\left(e^{y} - e^{-y}\right)}{2} \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \\
&\square
\end{aligned}
$$

2.4 d)

$$
\begin{aligned}
\sin(z) &= \sin(x+iy) \\
&= \frac{e^{i(x+iy)} - e^{i(-x-iy)}}{2i} \\
&= \frac{e^{ix-y} - e^{y-ix}}{2i} \\
&= \frac{e^{-y}\cos(x) + ie^{-y}\sin(x) - e^{y}\cos(-x) - ie^{y}\sin(-x)}{2i} \\
&= \frac{e^{-y}\cos(x) - e^{y}\cos(-x) + ie^{-y}\sin(x) - ie^{y}\sin(-x)}{2i} \\
&= \frac{e^{-y}\cos(x) - e^{y}\cos(x) + ie^{-y}\sin(x) + ie^{y}\sin(x)}{2i} \\
&= \frac{\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{-y} + e^{y}\right)}{2i} \\
&= \frac{\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{y} + e^{-y}\right)}{2i} \cdot \frac{i}{i} \\
&= \frac{-i\cos(x)\left(e^{-y} - e^{y}\right) + \sin(x)\left(e^{y} + e^{-y}\right)}{2} \\
&= \frac{i\cos(x)\left(e^{y} - e^{-y}\right) + \sin(x)\left(e^{y} + e^{-y}\right)}{2} \\
&= i\cos(x)\sinh(y) + \sin(x)\cosh(y) \\
&\square
\end{aligned}
$$

3

3.1 a)

$$\begin{aligned}
\cos(-z) &= \frac{e^{-iz} + e^{--iz}}{2} \\
&= \frac{e^{iz} + e^{-iz}}{2} \\
&= \cos(z) \\
&\square
\end{aligned}$$

$$\begin{aligned}
\sin(-z) &= \frac{e^{-iz} - e^{--iz}}{2} \\
&= -\frac{e^{iz} - e^{-iz}}{2} \\
&= -\sin(z) \\
&\square
\end{aligned}$$

3.2 b)

$$\begin{aligned}
\cos(\pi/2 - z) &= \frac{e^{i\pi/2 - iz} + e^{-i\pi/2 + iz}}{2} \\
&= \frac{\frac{e^{i\pi/2}}{e^{iz}} + \frac{e^{iz}}{e^{i\pi/2}}}{2} \\
&= (1/2)\frac{e^{i\pi} + e^{2iz}}{e^{iz + i\pi/2}} \\
&= (1/2)\left(\frac{e^{i\pi}}{e^{iz+i\pi/2}} + \frac{e^{2iz}}{e^{iz+i\pi/2}}\right) \\
&= (1/2)\left(e^{-iz+i\pi/2} + e^{iz-i\pi/2}\right) \\
&= (1/2)\left(e^{-iz}e^{i\pi/2} + e^{iz}e^{-i\pi/2}\right) \\
&= (1/2)\left(ie^{-iz} - ie^{iz}\right) && (e^{i\pi/2} = i, e^{-i\pi/2} = -i) \\
&= \frac{1}{2}\frac{i}{i}\left(ie^{-iz} - ie^{iz}\right) \\
&= \frac{1}{2i}\left(-e^{-iz} + e^{iz}\right) \\
&= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \sin(z) \\
&\square
\end{aligned}$$

$$
\begin{aligned}
\sin\left(\pi/2 - z\right) &= \frac{e^{i\pi/2 - iz} - e^{-i\pi/2 + iz}}{2i} \\
&= \frac{e^{i\pi/2}/e^{iz} - e^{iz}/e^{i\pi/2}}{2i} \\
&= \frac{\frac{e^{i\pi} - e^{2iz}}{e^{iz + i\pi/2}}}{2i} \\
&= \frac{e^{-iz + i\pi/2} - e^{iz - i\pi/2}}{2i} \\
&= \frac{ie^{-iz} + ie^{iz}}{2i} \\
&= \frac{e^{-iz} + e^{iz}}{2} \\
&= \cos\left(z\right) \\
&\square
\end{aligned}
$$

3.3 c)

$$
\begin{aligned}
\sin^2\left(z\right) + \cos^2\left(z\right) &= \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 \\
&= \frac{e^{2iz} - 2 + e^{-2iz}}{-4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} \\
&= \frac{\cancel{e^{2iz}} - 2 + \cancel{e^{-2iz}}}{-4} + \frac{\cancel{e^{2iz}} + 2 + \cancel{e^{-2iz}}}{4} \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1 \\
&\square
\end{aligned}
$$

3.4 d)

$$\begin{aligned}
\cos(2z) &= \frac{e^{2iz} + e^{-2iz}}{2} \\
&= 1 - 2\sin(z)^2 \\
&= 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 \\
&= 1 - 2\left(\frac{e^{2iz} - 2 + e^{-2iz}}{-4}\right) \\
&= 1 + \left(\frac{e^{2iz} - 2 + e^{-2iz}}{2}\right) \\
&= \cancel{1} + \left(\frac{e^{2iz} - \cancel{2} + e^{-2iz}}{2}\right) \\
&= \frac{e^{2iz} + e^{-2iz}}{2} && = \cos(2z)
\end{aligned}$$

$\square$

4

Let $n \in \mathbb{Z}$

$z = 1 + i\sqrt{3} = 2e^{i\pi/3}$

$w = -1 + i = \sqrt{2}e^{i3\pi/4}$

$\operatorname{Log}(z) = \operatorname{Log}(2) + i\pi/3$

$\log(z) = \operatorname{Log}(z) + 2\pi i n$

$\operatorname{Log}(w) = \operatorname{Log}(\sqrt{2}) + i3\pi/4$

$\log(w) = \operatorname{Log}(w) + 2\pi i n$

The difference is that Log has an unique solution, while log has infinitely many solutions.

5

$$\begin{aligned}
i^{1+i} &= \left(e^{i\pi/2}\right)^{1+i} \\
&= e^{i\pi/2\cdot(1+i)} \\
&= e^{i\pi/2-\pi/2} \\
&= \frac{e^{i\pi/2}}{e^{\pi/2}} \\
&= \frac{i}{e^{\pi/2}} \\
&= ie^{-\pi/2} \\
&\square
\end{aligned}$$

6

Let $A = e^{iz}$

$\arctan(2+i)$

$$
\begin{aligned}
\tan(z) &= 2+i \\
\frac{\sin(z)}{\cos(z)} &= 2+i \\
\cancel{1/2 \cdot \frac{1}{1/2}} \cdot 1/i \cdot \frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} &= 2+i \\
1/i \frac{A-A^{-1}}{A+A^{-1}} &= 2+i && (A = e^{iz}) \\
\frac{A-A^{-1}}{A+A^{-1}} &= 2i-1 \\
A - A^{-1} &= (2i-1)\left(A+A^{-1}\right) \\
A^2 - 1 &= (2i-1)\left(A^2+1\right) \\
A^2 - 1 &= 2iA^2 + 2i - A^2 - 1 \\
2A^2 - 2iA^2 &= 2i \\
A^2(2-2i) &= 2i \\
A^2 &= \frac{i}{1-i} \\
2\log(e^{iz}) &= \log(i) - \log(1-i) \\
2iz &= \log(e^{i\pi/2}) - \log(\sqrt{2}e^{-i\pi/4}) \\
&= i\pi/2 - \log(\sqrt{2}) + i\pi/4 \\
z &= \frac{i\pi/2 - \log(\sqrt{2}) + i\pi/4}{2i} \\
&= \frac{i3\pi/4 - \log(\sqrt{2})}{2i} \\
&= \boxed{\frac{3\pi/4 + i\log(\sqrt{2})}{2}}
\end{aligned}
$$

$\arccos\left(\dfrac{e^2+e^{-2}}{2}\right)$

$$\begin{aligned}
\cos(z) &= \frac{e^2+e^{-2}}{2} \\
\frac{A+A^{-1}}{2} &= && (A = e^{iz}) \\
A^2+1 &= A\left(e^2+e^{-2}\right) \\
A^2 - A\left(e^2+e^{-2}\right)+1 &= 0 \\
A &= e^2, e^{-2} \\
e^{iz} &= \\
iz &= 2, -2 \\
z &= \boxed{2i, -2i}
\end{aligned}$$

7

7.1 a)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

7.2 b)

$f(z)$ is analytic if $f'(z)$ exists.

7.3 c)

If the 4 partials are continuous and the Cauchy-Riemann equations are satisfied, then the function is analytic.

7.4 d)

$f'(z) = u_x(x, y) + iv_x(x, y)$, where subscripts are the shorthand for partial differentiation.

8

8.1 a)

$u(x, y) = u = e^x \cos(y), v(x, y) = v = e^x \sin(y)$

$u_x = e^x \cos(y) = v_y, u_y = -e^x \sin(y) = -v_x$

$f'(z) = u_x + iv_x = e^x \cos(y) + ie^x \sin(y) = e^z$

8.2 b)

$u = \sin(x) \cosh(y), v = \cos(x) \sinh(y)$

$u_x = \cos(x) \cosh(y) = v_y, u_y = \sin(x) \sinh(y) = -v_x$

$f'(z) = u_x + iv_x = \cos(x) \cosh(y) + i \sin(x) \sinh(y) = \cos(z)$

9

9.1 a)

$u_x = 3Ax^2 + By^2 + C = Dx^2 + 3Ey^2 + F = v_y, u_y = 2Bxy = -2Dxy = -v_x$

Since these 4 partials are polynomials with real coefficients, we know that they are continuous. After matching the terms up, we get $f(z)$ is analytic when $3A = D \land B = 3E \land C = F \land B = -D$.

9.2 b)

By setting $y = 0 \implies z = x$, we get

$$f(z) = Az^3 + Dz^2 + Bx + C$$

9.3 c)

$$f'(z) = 3Az^2 + 2Dz + B$$

10

10.1 a)

$$u_x = -e^{-y}\sin(x) = v_y, u_y = -e^{-y}\cos(x) = -v_x$$

These 4 partials are continuous because they are the product of 2 continuous functions.

10.2 b)

$$g'(z) = -e^{-y}\sin(x) + ie^{-y}\cos(x)$$

10.3 c)

$$g(z) = e^{-y+ix} = e^{i(x+iy)} = e^{iz}$$

10.4 d)

Verify $u_{xx} + u_{yy} = 0$

$$u_x = -e^{-y}\sin(x) \implies u_{xx} = -e^{-y}\cos(x)$$
$$u_y = -e^{-y}\cos(x) \implies u_{yy} = e^{-y}\cos(x)$$
$$u_{xx} + u_{yy} = 0$$