hw2

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1 Cartesian Form

1.1 a)

$$
\begin{aligned}
e^z &= e^{x+iy} \\
&= e^x e^{iy} \\
&= e^x \left(\cos\left(y\right) + i\sin\left(y\right)\right) \\
&= e^x \cos\left(y\right) + ie^x \sin\left(y\right)
\end{aligned}
$$

$\boxed{X = e^x \cos\left(y\right), Y = e^x \sin\left(y\right)}$

1.2 b)

Let $z = x + iy, w = a + ib$. Addition and multiplication are commutative, so we can rearrange to get the real together, and the imaginaries together.

$$
\begin{aligned}
e^z \cdot e^w &= e^{a+ib} \cdot e^{x+iy} \\
&= e^a \cdot e^i b \cdot e^x \cdot e^i y \\
&= e^{ax} \cdot e^{i(b+y)} \\
&= e^{a+ib+x+iy} \\
&= e^{z+w}
\end{aligned}
$$

1.3 c)

Show $e^{2\pi i n} = +1$

$$\begin{aligned} e^{2\pi i n} &= \left(e^{2\pi i}\right)^n \\ &= \left(\cos 2\pi + i \sin 2\pi\right)^n \\ &= \left(1i \cdot 0\right)^n \\ &= 1^n \\ &= +1 \\ &\square \end{aligned}$$

Show e^z is periodic with period $2\pi i$: $e^{z+2\pi i n} = e^z$

$$\begin{aligned} e^{z+2\pi i n} &= e^z \cdot e^{2\pi i n} \\ &= e^z \cdot \left(e^{2\pi i}\right)^n \\ &= e^z \left(\cos 2\pi + i \sin 2\pi\right)^n \\ &= e^z \left(1 + 0\right)^n \\ &= e^z \\ &\square \end{aligned}$$

1.4 d)

Compute $\operatorname{Re}(\ldots)$, $\operatorname{Im}(\ldots)$ of $e^{1+\pi i}, e^{3+i\pi/3}, e^{5+i\pi/4}$

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x \left(\cos y + i \sin y\right) \\ \operatorname{Re}\left(e^{x+iy}\right) &= e^x \cos y \\ \operatorname{Im}\left(e^{x+iy}\right) &= e^x \sin y \end{aligned}$$

1. $z = 1 + i\pi \implies \operatorname{Re}\left(e^z\right) = -e, \operatorname{Im}\left(e^z\right) = 0$
2. $z = 3 + i\pi/3 \implies \operatorname{Re}\left(e^z\right) = e^3/2, \operatorname{Im}\left(e^z\right) = e^3 \cdot \sqrt{3}/2$
3. $z = 5 + i\pi/4 \implies \operatorname{Re}\left(e^z\right) = e^5 \cdot \sqrt{2}/2, \operatorname{Im}\left(e^z\right) = e^5 \cdot \sqrt{2}/2$

2 Cosine and Sine

2.1 a)

$$
\begin{aligned}
\cos(x) &= \frac{e^{ix}+e^{-ix}}{2} \\
&= \frac{\cos(x)+i\sin(x)+\cos(-x)+i\sin(-x)}{2} \\
&= \frac{2\cos(x)}{2} && \cos(-x)=\cos(x), \sin(-x)=\sin(x) \\
&= \cos(x) \\
& \square
\end{aligned}
$$

$$
\begin{aligned}
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} \\
&= \frac{\cos(x)+i\sin(x)-\cos(-x)-i\sin(-x)}{2i} \\
&= \frac{2i\sin(x)}{2i} && \cos(-x)=\cos(x), \sin(-x)=\sin(x) \\
&= \sin(x) \\
& \square
\end{aligned}
$$

2.2 b)

$$
\begin{aligned}
\cos(2+3i) &= \frac{e^{2+3i}+e^{-2-3i}}{2} \\
&= (1/2)\left(e^2(\cos(3)+i\sin(3))+e^{-2}(\cos(-3)+i\sin(-3))\right) \\
&= (1/2)\left(e^2\cos(3)+ie^2\sin(3)+e^{-2}\cos(-3)+ie^{-2}\sin(-3)\right) \\
&= (1/2)\left(e^2\cos(3)+e^{-2}\cos(-3)+i\left(e^2\sin(3)+e^{-2}\sin(-3)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\sin(2+3i) &= \frac{e^{2+3i}-e^{-2-3i}}{2i} \\
&= (1/2i)\left(e^2(\cos(3)+i\sin(3))-e^{-2}(\cos(-3)-i\sin(-3))\right) \\
&= (1/2i)\left(e^2\cos(3)+ie^2\sin(3)-e^{-2}\cos(-3)-ie^{-2}\sin(-3)\right) \\
&= (1/2i)\left(e^2\cos(3)-e^{-2}\cos(-3)-i\left(e^2\sin(3)+e^{-2}\sin(-3)\right)\right)
\end{aligned}
$$

2.3 c)

$$
\begin{aligned}
\cos(z) &= \cos(x+iy) \\
&= \frac{e^{i(x+iy)} + e^{i(-x-iy)}}{2} \\
&= \frac{e^{ix-y} + e^{y-ix}}{2} \\
&= \frac{e^{-y}\cos(x) + ie^{-y}\sin(x) + e^{y}\cos(-x) + ie^{y}\sin(-x)}{2} \\
&= \frac{e^{-y}\cos(x) + e^{y}\cos(-x) + ie^{-y}\sin(x) + ie^{y}\sin(-x)}{2} \\
&= \frac{e^{-y}\cos(x) + e^{y}\cos(x) + ie^{-y}\sin(x) - ie^{y}\sin(x)}{2} \\
&= \frac{\cos(x)\left(e^{-y} + e^{y}\right) + i\sin(x)\left(e^{-y} - e^{y}\right)}{2} \\
&= \frac{\cos(x)\left(e^{-y} + e^{y}\right) - i\sin(x)\left(e^{y} - e^{-y}\right)}{2} \\
&= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \\
&\quad \square
\end{aligned}
$$

2.4 d)

$$
\begin{aligned}
\sin(z) &= \sin(x+iy) \\
&= \frac{e^{i(x+iy)} - e^{i(-x-iy)}}{2i} \\
&= \frac{e^{ix-y} - e^{y-ix}}{2i} \\
&= \frac{e^{-y}\cos(x) + ie^{-y}\sin(x) - e^{y}\cos(-x) - ie^{y}\sin(-x)}{2i} \\
&= \frac{e^{-y}\cos(x) - e^{y}\cos(-x) + ie^{-y}\sin(x) - ie^{y}\sin(-x)}{2i} \\
&= \frac{e^{-y}\cos(x) - e^{y}\cos(x) + ie^{-y}\sin(x) + ie^{y}\sin(x)}{2i} \\
&= \frac{\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{-y} + e^{y}\right)}{2i} \\
&= \frac{\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{y} + e^{-y}\right)}{2i} \cdot \frac{i}{i} \\
&= \frac{-i\cos(x)\left(e^{-y} - e^{y}\right) + \sin(x)\left(e^{y} + e^{-y}\right)}{2} \\
&= \frac{i\cos(x)\left(e^{y} - e^{-y}\right) + \sin(x)\left(e^{y} + e^{-y}\right)}{2} \\
&= i\cos(x)\sinh(y) + \sin(x)\cosh(y) \\
&\quad \square
\end{aligned}
$$

3 Complex Trigonometric Identities

3.1 a)