hw3

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1

$$\begin{aligned}
\left(-1-\sqrt{3}i\right)^{432} &= \left(2e^{-2i\pi/3}\right)^{432} \\
&= 2^{423}e^{-2i\pi/3\cdot 432} \\
&= 2^{423}e^{-288i\pi} \\
&\quad \square
\end{aligned}$$

2

$$
\begin{aligned}
(-1-i)^{1/737} &= \left(\sqrt{2}e^{-3i\pi/4}\right)^{1/737} \\
&= 2^{1/(2\cdot 737)}e^{-3i\pi/(4\cdot 737)}e^{2i\pi n/737}, n \in \{0, 1, \ldots, 736\} \\
&= 2^{1/(2\cdot 737)}\left(\cos\left(-3\pi/(4\cdot 737)+2\pi n/737\right)+i\sin\left(-3\pi/(4\cdot 737)+2\pi n/737\right)\right) \\
&\quad \square
\end{aligned}
$$

3

$$
\begin{aligned}
\left(1-\sqrt{3}i\right)^{(5+7i)} &= \left(2e^{i\pi/3}\right)^{5+7i} \\
&= \left(\operatorname{Log}(2) - i\pi/3 + 2i\pi n\right)^{5+7i} \\
&= e^{(\operatorname{Log}(2) - i\pi/3 + 2i\pi n)(5+7i)} \\
&= e^{5\operatorname{Log}(2) - 5i\pi/3 + 10i\pi n + 7i\operatorname{Log}(2) + 7\pi/3 - 14\pi n} \\
&= e^{5\operatorname{Log}(2) - 7(-\pi/3 + 2\pi n)} e^{i(7\operatorname{Log}(2) + 5(-\pi/3 + 2\pi n))} \\
&= e^{5\operatorname{Log}(2) - 7(-\pi/3 + 2\pi n)} \left(\cos\left(7\operatorname{Log}(2) + 5\left(-\pi/3 + 2\pi n\right)\right) + i\sin\left(7\operatorname{Log}(2) + 5\left(-\pi/3 + 2\pi n\right)\right)\right) \\
&\quad \square
\end{aligned}
$$

4

4.1

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

4.2

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

4.3

$$\tan(z) = 1/i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

5

$$
\begin{aligned}
\cos(4+3i) &= \frac{e^{4i-3}+e^{3-4i}}{2} \\
&= (1/2)\left(e^{4i}e^{-3}+e^{3}e^{-4i}\right) \\
&= (1/2)\left(e^{-3}\left(\cos(4)+i\sin(4)\right)+e^{3}\left(\cos(-4)+i\sin(-4)\right)\right) \\
&= (1/2)\left(\cos(4)\left(e^{3}-^{e-3}\right)+\sin(4)\left(e^{3}-e^{-3}\right)\right) \\
&= cos(4)\sinh(3)+\sin(3)\cosh(3) \\
&\quad \square
\end{aligned}
$$

$$\operatorname{Re}\left(\cos(4+3i)\right)=\cos(4)\sinh(3)+\sin(3)\cosh(3)$$

$$\operatorname{Im}\left(\cos(4+3i)\right)=0$$

6

$$
\begin{aligned}
4\cos^3(T) &= \cos(3T) + 3\cos(T) \\
\cos(3T) + 3\cos(T) &= 4\cos^3(T) \\
&= 4\left(\frac{e^{iT}+e^{-iT}}{2}\right)^3 \\
&= (1/2)\left(e^{3iT} + 3e^{2iT}e^{-iT} + 3e^{iT}e^{-2iT} + e^{-3iT}\right) && \text{(Binomial Expansion)} \\
&= (1/2)\left(e^{3iT} + 3e^{iT} + 3e^{-iT} + e^{-3iT}\right) \\
&= \cos(3T) + 3\cos(T) \\
&\quad \square
\end{aligned}
$$

7

$$
\begin{aligned}
\tan(z) &= \frac{2-i}{5} \\
(1/i)\frac{A-A^{-1}}{A+A^{-1}} &= \frac{2-i}{5} && \text{(A = exp(iz))} \\
\frac{A-A^{-1}}{A+A^{-1}} &= \frac{2i+1}{5} \\
\frac{A^2-1}{A^2+1} &= \frac{2i+1}{5} \\
A^2-1 &= (A^2+1)\frac{2i+1}{5} \\
A^2-1 &= (1/5)\left(2iA^2+A^2+2i+1\right) \\
5A^2-5 &= 2iA^2+A^2+2i+1 \\
5A^2-2iA^2-A^2 &= 2i+6 \\
4A^2-2iA^2 &= 2i+6 \\
A^2(4-2i) &= 2i+6 \\
A^2 &= \frac{2i+6}{4-2i} \\
A^2 &= 1+i && \text{(Hint)} \\
e^{2iz} &= \sqrt{2}e^{i\pi/4} \\
2iz &= \operatorname{Log}(2) + i\pi/4 + 2i\pi n, n \in \mathbb{Z} \\
z &= (1/2i)\operatorname{Log}(2) + \pi/8 + \pi n \\
&\square
\end{aligned}
$$

8

8.1 a)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

8.2 b)

$$f(z) = (Ax^2 + Cy^2 + Dx + F) + i(Bxy + Ey)$$

Match partials of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2Ax + D = \frac{\partial v}{\partial y} = Bx + E \wedge \frac{\partial u}{\partial y} = 2Cy - \frac{\partial v}{\partial x} - By$$
$$B = 2A = -2C \wedge D = E$$

8.3 c)

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$\frac{\mathrm{d}f}{\mathrm{d}z} = 2Ax + D + 2iCy$$

8.4 d)

Use the $y = 0$ trick:

$$f(z) = (Ax^2 + Dx + F)$$
$$f(z) = Az^2 + Dz + F$$