LA HW2

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1

(a)

$$V = \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

(b)

$$Av_1 = v_1\lambda_1, Av_2 = v_2\lambda_2 \implies A[v_1, v_2] = [v_2, v_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

(c)

$$\begin{aligned} A &= VDV^{-1} \\ \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \\ \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} &= \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \\ \begin{bmatrix} 5 & 8 \\ 5 & -6 \end{bmatrix} &= \begin{bmatrix} 5 & 8 \\ 5 & -6 \end{bmatrix} \end{aligned}$$

(d) The transformation matrix applied to one of it's eigenvectors is the same as multiplying the eigenvector by its respective eigenvalue.

(e)

$$\begin{aligned} A^{33} &= VD^{33}V^{-1} \\ &= \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^{33} \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5^{33} & 0 \\ 0 & (-2)^{33} \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}^{33} \begin{bmatrix} 1 & -4 \\ 1 & 3 \end{bmatrix}^{-1} \\ &= \frac{1}{7} \left(5^{33} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} + (-2)^{33} \begin{bmatrix} 4 & -4 \\ -3 & 3 \end{bmatrix} \right) \end{aligned}$$

2

(a) Distance preserving means that the length before and after the transformation are the same, or $\|v\| = \|Vv\|$, which is also the same as $\langle v, v\rangle = \langle Vv, Vv\rangle$.

$$\begin{aligned}
\langle v, v\rangle &= \langle Vv, Vv\rangle \\
&= \langle v, V^tVv\rangle && \text{Real matrix property} \\
&= \langle v, (Id)(v)\rangle && \text{Orthogonal matrix property} \\
&= \langle v, v\rangle && \text{Identity matrix property} \\
&\square
\end{aligned}$$

Angle preserving means that the angle before and after the transformation are the same. One way to express this is that the angle between v and Vv is 0, indicating no change. This is the same as the cosine of the angle being 1.

$$\begin{aligned}
\frac{\langle v, Vv\rangle}{\|v\|\|Vv\|} &= \frac{\langle v, Vv\rangle}{\|v\|^2} && \text{Distance preserving} \\
&= \frac{\|v\|^2}{\|v\|^2} \\
&= 1 \\
&\square
\end{aligned}$$

(b)

$$\begin{aligned}
VV^t &= Id \\
det(VV^t) &= det(Id) \\
det(V)det(V^t) &= det(Id) && det(XY) = det(X)det(Y) \\
det(V)det(V) &= det(Id) && det(X^t) = det(X) \\
det(V)^2 &= det(Id) && det(Id) = 1 \\
det(V)^2 &= 1 \\
det(V) &= \pm 1
\end{aligned}$$

(c) The eigenvector(s) would be on the specific axis, such as $(1, 0, 0)$ for the x-axis. This is because a rotation around an axis wouldn't affect the orientation of something that is on the axis, which makes sense because the transformation of an eigenvector is just a scalar multiple of itself.

(d) The Identity matrix has values of 1 on the diagonal and 0 elsewhere. This means $j = k \implies 1 \wedge j \neq k \implies 0$. The jth row of V^t is just the jth column of V, so we would get 1 when $j = k$ and 0 when $j \neq k$.

3

(a)

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \to \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$$

The lengths of the eigenvectors don't matter since we can just scale them. $\langle (1,1), (1,-1) \rangle = 0$, so our eigenvectors are orthogonal.

(b)

$$\begin{aligned} B &= VDV^t \\ \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^t \\ &= \frac{1}{2} \begin{bmatrix} 5 & 3 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^t \\ &= \frac{1}{2} \begin{bmatrix} 2 & 8 \\ 8 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} Id &= VV^t \\ &= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^t \\ &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= Id \end{aligned}$$

$$VV^{-1} = Id = VV^t \implies V^{-1} = V^t$$

(c) An orthogonal vector A has the property that $AA^t = Id$.

4

I did both parts in Jupyter Notebook (Python). Here is the code snippet.

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

def rrq(v, B): # Ritz-Rayleight Quotient
        # @ is matrix multiplication
        return np.dot(v, B@v)/(np.dot(v,v))

# generate 100 random vectors of size 2 in range (-10, 10)
v = np.random.uniform(-10,10,[100,2])

# our matrix
B = np.array([1,4,4,1]).reshape((2,2))

rrqs = [] # Ritz-Rayleigh Quotients
for i in range(v.shape[0]):
        rrqs.append(rrq(v[i], B))


plt.plot(rrqs)
plt.axhline(y = 5, color = 'b', linestyle = '-') # large eigenvalue
plt.axhline(y = -3, color = 'r', linestyle = '-') # small eigenvalue
plt.show()

for i in range(10):
        # output 10 vectors and their RRQs
        print(v[i], rrqs[i])
```

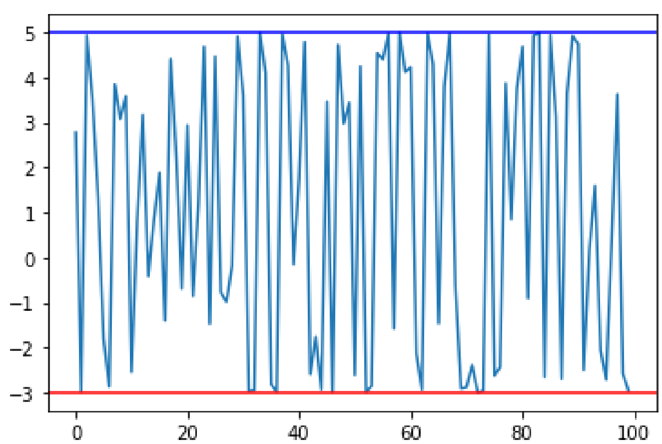

Figure 1: RRQ

10 vectors and their RRQs are :

$$\begin{aligned}
[8.79546278, 2.06447912] &\rightarrow 2.779715855809442 \\
[9.5063905, -8.90054609] &\rightarrow -2.9913427706469595 \\
[7.76486343, 6.4787083] &\rightarrow 4.935298822710415 \\
[8.980074, 3.18657326] &\rightarrow 3.521315522110756 \\
[-3.16092004, -0.13259838] &\rightarrow 1.3350048755291104 \\
[-6.54872245, 2.68558311] &\rightarrow -1.8084307457639526 \\
[-9.18640328, 7.06257587] &\rightarrow -2.8656246827096234 \\
[8.16544874, 3.41700323] &\rightarrow 3.8488780492756383 \\
[9.74184802, 2.73129675] &\rightarrow 3.079479954078956 \\
[8.5244294, 3.12348018] &\rightarrow 3.584346724232756
\end{aligned}$$

5

$$\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} \longrightarrow \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}^t \frac{1}{\sqrt{5}}$$

$$2X^2 + 7Y^2 = 1$$

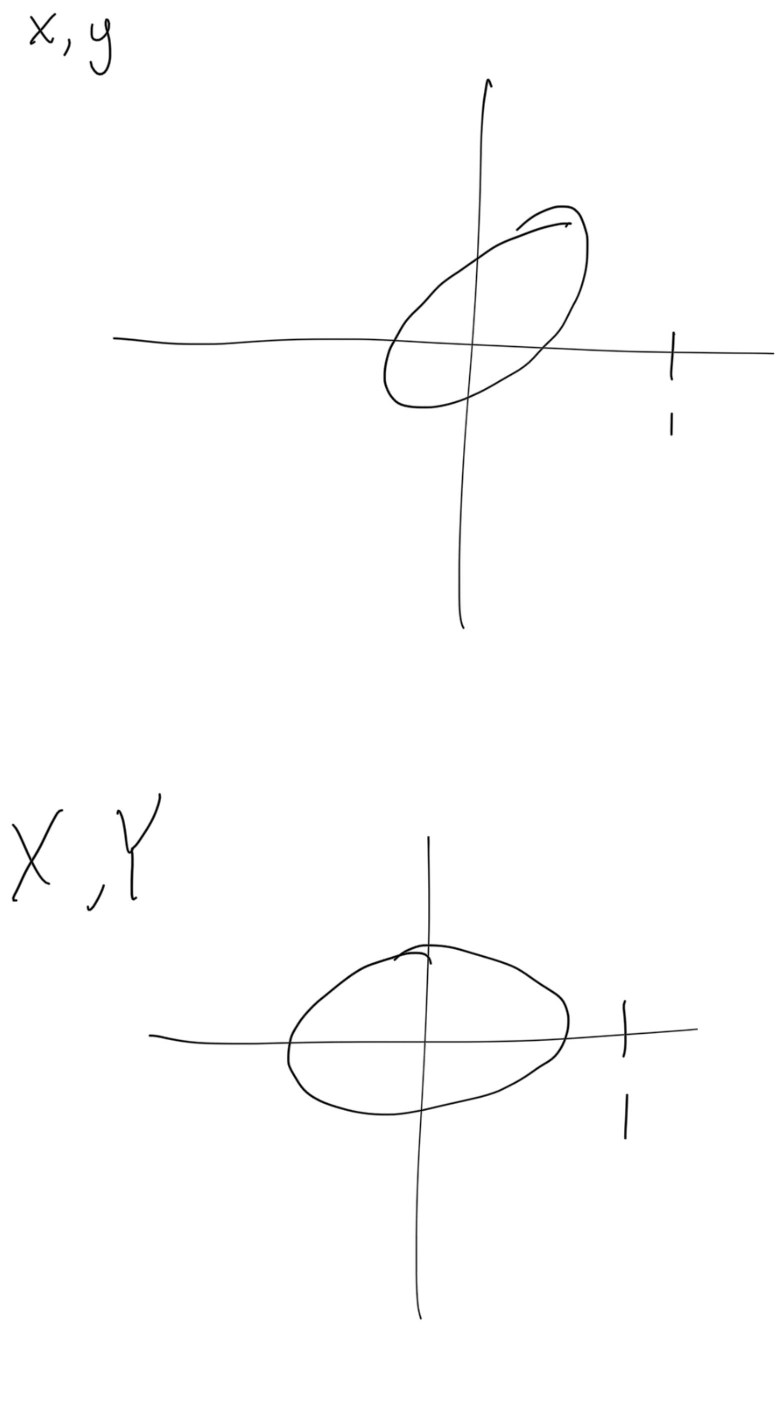

Figure 2: Before and After the transform

6

(a)

$$\begin{bmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{bmatrix} \longrightarrow \frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix}^t \frac{1}{3}$$

$$15X^2 + 9Y^2 + 3Z^2 = 1$$

(b)

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix}^t \frac{1}{3} = 1/9 \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = Id$$

7

(a) Verify that $\sigma_{\ldots}$ is equal to its conjugate transpose.

σ_x

$$\begin{bmatrix} 0 & 1+0i \\ 1+0i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1-0i \\ 1-0i & 0 \end{bmatrix}$$

σ_y

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

σ_z

$$\begin{bmatrix} 1+0i & 0 \\ 0 & -1+0i \end{bmatrix} = \begin{bmatrix} 1-0i & 0 \\ 0 & -1-0i \end{bmatrix}$$

(b) σ_x

$$0+0=0;\ 0\cdot 0 - 1\cdot 1 = -1$$

σ_y

$$0+0=0;\ 0\cdot 0 - -i^2 = -1$$

σ_z

$$1-1=0;\ 1\cdot -1 - 0\cdot 0 = -1$$

(c)

$(\sigma_x)^2 = Id$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(\sigma_y)^2 = Id$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(\sigma_z)^2 = Id$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\sigma_y \sigma_z = -\sigma_z \sigma_y = i\sigma_x$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\sigma_z \sigma_x = -\sigma_x \sigma_z = i\sigma_y$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

(d)

$$a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -bi \\ bi & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$$
$$\begin{bmatrix} c & a - bi \\ a + bi & -c \end{bmatrix}$$

The trace is $c - c = 0$.

$$\begin{aligned} det\left(\begin{bmatrix} c & a - bi \\ a + bi & c \end{bmatrix}\right) &= c^2 - (a - bi)(a + bi) \\ &= -c^2 - a^2 - b^2 \\ &= -(a^2 + b^2 + c^2) \\ &= -(1) \\ &= -1 \end{aligned}$$

The determinant is -1.

(f) [Note: Looks like you skipped (e)]

Let $A = \begin{bmatrix} a & b + 0i \\ b + 0i & -a \end{bmatrix}$, where a, b are real. The trace is $a - a = 0$, and it's also a Hermitian matrix because it's symmetric and real. The determinant is $-a^2 - b^2 = -1$. All Hermitian matrices meeting our requirements will be of this form. The eigenvalues of this matrix are

$$\begin{aligned} det\left(\begin{bmatrix} a - \lambda & b \\ b & -a - \lambda \end{bmatrix}\right) &= (a - \lambda)(-a - \lambda) - b^2 = 0 \\ &= -a^2 - b^2 + \lambda^2 = 0 \\ \lambda^2 &= a^2 + b^2 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned}$$

$\square$