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1

Matrix A (a)

$$\lambda_1 = 4, \lambda_2 = -2, v_1 = (1, 1), v_2 = (-1, 1)$$

(b)

$$1 + 1 = 1 + 1 = 2$$

(c)

$$4 \cdot -2 = -8$$

(d)

$$\arccos\left(\frac{\langle(1, 1), (-1, 1)\rangle}{\sqrt{1+1} \cdot \sqrt{1+1}}\right) = \arccos(0) = \pi/2$$

(e) The eigenvalues are real. The matrix is symmetric.

Matrix B (a)

$$\lambda_1 = 37, \lambda_2 = -15, v_1 = (3, 2), v_2 = (-2, 3)$$

(b)

$$37 - 15 = 21 + 1 = 22$$

(c)

$$37 \cdot -15 = -555$$

(d)

$$\arccos\left(\frac{\langle(3, 2), (-2, 3)\rangle}{\sqrt{3^2+2^2} \cdot \sqrt{3^2+2^2}}\right) = \arccos(0) = \pi/2$$

(e) The eigenvalues are real. The matrix is symmetric.

Matrix C (a)

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1, v_1 = (-1, 0, 1), v_2 = (1, 0, 1), v_3 = (0, 1, 0)$$

(b)

$$-1 + 1 + 1 = 0 + 1 + 0 = 1$$

(c)

$$-1 \cdot 1 \cdot 1 = -1$$

(d) Find the inner products. $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_1, v_3 \rangle = 0$. Since the inner products are all 0, then the angle is $\pi/2$

(e) The eigenvalues are real. The matrix is symmetric.

(f) Given that X has all real values, $X = X^T \implies \mathrm{Im}(v_i) = 0$

2

$$\theta = \arccos\left(\frac{\langle v, w\rangle}{\|v\|\|w\|}\right)$$

$$PQ = (2,1,0,3) - (1,2,0,1) = (1,-1,0,2)$$
$$PR = (0,1,1,0) - (1,2,0,1) = (-1,-1,1,-1)$$
$$RQ = (2,1,0,3) - (0,1,1,0) = (2,0,-1,3)$$

As long as we take the acute angle, it doesn't matter if a vector is backwards: PQ, QP are equivalent.

$$P = \arccos\left(\frac{\langle PQ, PR\rangle}{\|PQ\|\|PR\|}\right) = 114.095°$$

$$Q = \arccos\left(\frac{\langle PQ, RQ\rangle}{\|PQ\|\|RQ\|}\right) = 29.206°$$

$$R = 180 - \arccos\left(\frac{\langle PR, RQ\rangle}{\|PR\|\|RQ\|}\right) = 36.700°$$

The 3 sum to 180°.

3

$$
\begin{aligned}
\langle \vec{N}, \vec{X} - P \rangle &= 0 \\
\langle \vec{N}, \vec{X} \rangle - \langle \vec{N}, P \rangle &= 0 \\
(a_1 x_1 + \cdots + a_n x_n) - (r_1 x_1 + \cdots + c_n x_n) &= 0 && \text{(solves means evals to } b\text{)} \\
b - b &= 0
\end{aligned}
$$

4

$$\theta = \arccos\left(\frac{\langle(1,1,1,1),(2,1,3,7)\rangle}{\sqrt{1+1+1+1}\cdot\sqrt{2^2+1+3^2+7^2}}\right) = 35.023^\circ$$

5

$$\theta = \arccos\left(\frac{\langle (1,1,1,1), (2,3,0,5) \rangle}{\sqrt{1+1+1+1} \cdot \sqrt{2^2+3^2+0+5^2}}\right) = 35.800^\circ$$

6

(a)

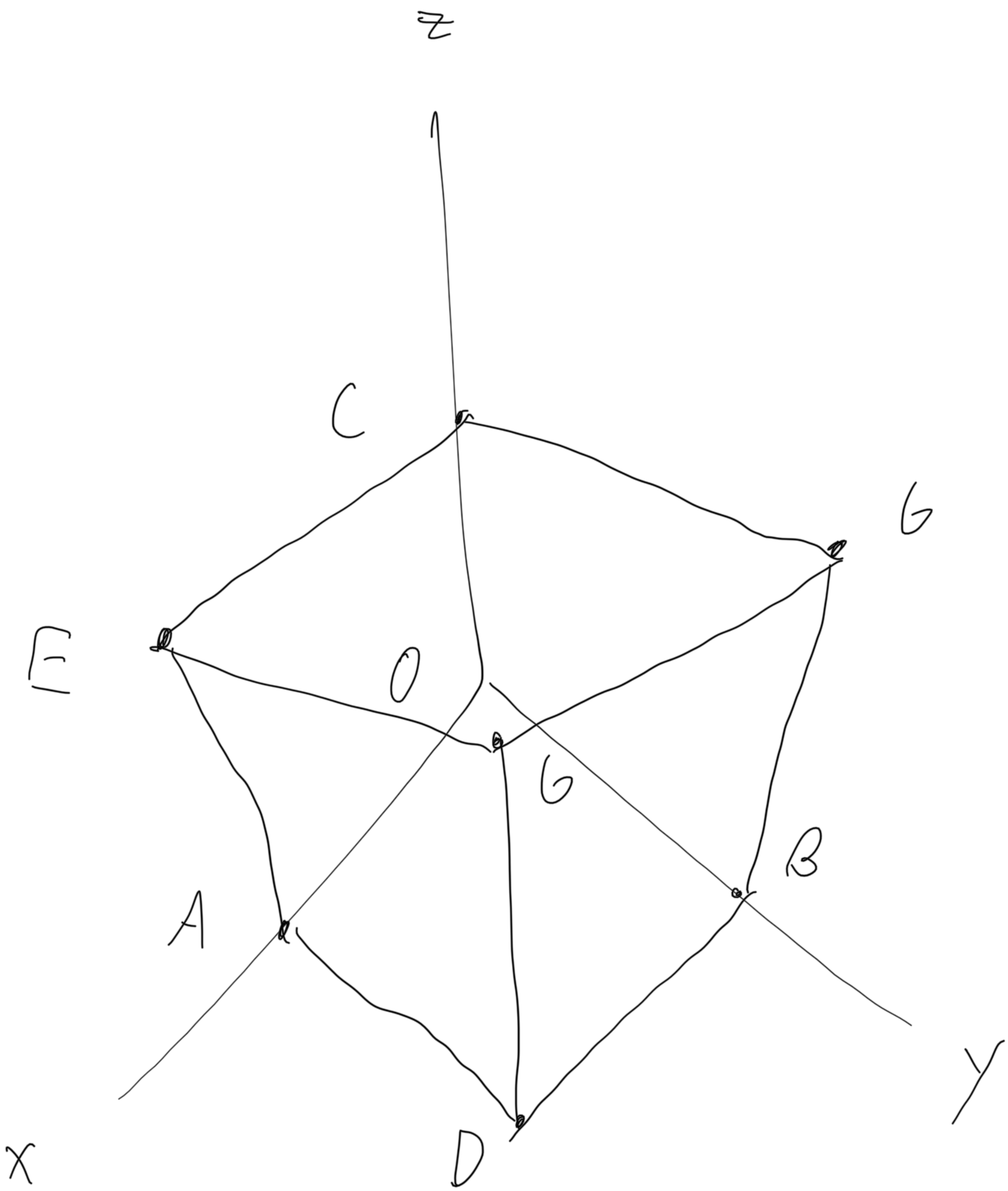

Figure 1: rough sketch

(b)

$$\sqrt{1+1+1} = \sqrt{3}$$

(c)

$$\arccos\left(\frac{\langle (1,0,0),(1,1,1)\rangle}{\sqrt{1}\cdot\sqrt{3}}\right) = 54.736^\circ$$

7

(a)

$$\|x^2\| = \sqrt{\langle x^2, x^2 \rangle} = \sqrt{\int_0^1 x^2 \cdot x^2 \, dx} = \sqrt{1/5}$$

$$\|x^3\| = \sqrt{\langle x^3, x^3 \rangle} = \sqrt{\int_0^1 x^3 \cdot x^3 \, dx} = \sqrt{1/7}$$

$$\theta = \arccos\left(\frac{\langle x^2, x^3 \rangle}{\sqrt{1/35}}\right) = \arccos\left(\frac{\int_0^1 x^2 x^3 \, dx}{\sqrt{1/35}}\right) = 9.594^\circ$$

(b)

$$\|\sin(m\pi x)\| = \sqrt{\langle \sin(m\pi x), \sin(m\pi x) \rangle} = \sqrt{\int_0^1 \sin(m\pi x)\sin(m\pi x)\, dx} = \frac{1}{2} - \frac{\sin(2m\pi)}{4\pi^2}$$

$$\|\sin(n\pi x)\| = \sqrt{\langle \sin(n\pi x), \sin(n\pi x) \rangle} = \sqrt{\int_0^1 \sin(n\pi x)\sin(n\pi x)\, dx} = \frac{1}{2} - \frac{\sin(2n\pi)}{4\pi^2}$$

$$\theta = \arccos\left(\frac{\langle \sin(m\pi x), \sin(n\pi x) \rangle}{\left(\frac{1}{2} - \frac{\sin(2n\pi)}{4\pi^2}\right)\left(\frac{1}{2} - \frac{\sin(2m\pi)}{4\pi^2}\right)}\right) = \arccos\left(\frac{\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx}{den}\right)$$

If $m = n$, then the angle between something that is itself would be 0.

8

(a) We get I_4. This tell us that the rows are independent.

(b) We get

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we have 2 rows of all 0, this means they are dependent. We would need to add $(0, 0, 1, 0, 0)$, $(0, 0, 0, 0, 1)$ because the third and fifth column do not have a leading 1.

(c) We get

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since we have a row of all 0, this means they are dependent. We would need to add $(0, 0, 0, 1)$ because the fourth column does not have a leading 1.

9

Use the hint.

$$\sum_{k=1}^{n} a_k \langle w_j, w_k \rangle = 0$$

can be simplified into

$$a_k \langle w_j, w_k \rangle = 0$$

because $j \neq k \implies \langle w_j, w_k \rangle = 0$. Only one pair $j = k$ evaluates to 1, so

$$a_k = 0$$

We can vary j to get that every $a_k = 0$.

10

(a)

$$\text{Proj}_v(w) = \frac{\langle (1,2,0,0,1),(1,1,1,0,1)\rangle}{1+4+1} \cdot (1,2,0,0,1) = \frac{1+2+1}{6} \cdot (1,2,0,0,1) = (2/3, 4/3, 0, 0, 2/3)$$

In this case, $\lambda = 2/3$

(b)

$$w = (\lambda \cdot v) + (w - \lambda \cdot v)$$

$$(1,1,1,0,1) = [2/3(1,2,0,0,1)] + [(1,1,1,0,1) - 2/3(1,2,0,0,1)]$$

$$\langle v, w - \lambda \cdot v\rangle = 0$$

$$\langle (1,2,0,0,1),(1,1,1,0,1) - 2/3(1,2,0,0,1)\rangle = 1 - 2/3 + 2 - 8/3 + 1 - 2/3 = 0$$

11

$$v = \{(1,2,0,1),(0,1,1,4),(1,1,1,1)\}$$

$$\begin{aligned}
v_1 &= \frac{(1,2,0,1)}{\sqrt{1+4+1}} \\
w_1 &= \frac{1}{\sqrt{6}}(1,2,0,1)
\end{aligned}$$

$$\begin{aligned}
r_2 &= (0,1,1,4) - \langle (0,1,1,4), 1/\sqrt{6}(1,2,0,1)\rangle 1/\sqrt{6}(1,2,0,1) \\
&= (0,1,1,4) - (2+4)1/\sqrt{6}\cdot 1/\sqrt{6}(1,2,0,1) \\
&= (0,1,1,4) - 6/6(1,2,0,1) \\
&= (-1,-1,1,3) \\
w_2 &= \frac{(-1,-1,1,3)}{\sqrt{1+1+1+9}} \\
w_2 &= \frac{(-1,-1,1,3)}{2\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
r_3 &= (1,1,1,1) - \langle (1,1,1,1), 1/\sqrt{6}(1,2,0,1)\rangle 1/\sqrt{6}(1,2,0,1)(-1,-1,1,3) - \langle v_3, w_2\rangle w_2 \\
&= (1,1,1,1) - (4/\sqrt{6})/\sqrt{6}(1,2,0,1) - \langle v_3, w_2\rangle w_2 \\
&= (1,1,1,1) - 2/3(1,2,0,1) - \langle v_3, w_2\rangle w_2 \\
&= 1/3(1,-1,3,1) - \langle v_3, w_2\rangle w_2 \\
&= 1/3(1,-1,3,1) - \langle (1,1,1,1), 1/(2\sqrt{3})(-1,-1,1,3)\rangle 1/(2\sqrt{3})(-1,-1,1,3) \\
&= 1/3(1,-1,3,1) - 1/(2\sqrt{3})(-1-1+1+3)1/(2\sqrt{3})(-1,-1,1,3) \\
&= 1/3(1,-1,3,1) - 1/12\cdot(2)(-1,-1,1,3) \\
&= 1/6(3,-1,5,-1) \\
w_3 &= \frac{(3,-1,5,-1)}{\sqrt{9+1+25+1}} \\
w_3 &= 1/6(3,-1,5,-1)
\end{aligned}$$

$$w = \left\{\frac{(1,2,0,1)}{\sqrt{6}}, \frac{(-1,-1,1,3)}{2\sqrt{3}}, \frac{(3,-1,5,-1)}{6}\right\}$$