Homework 5

Liheng Cao

March 20, 2021

1

(a) Rewrite as

$$\frac{z^2+2}{z-1}$$

The singularity is at $z=1$, the order is 1, and the residue is $1^2+2=3$.

(b) Rewrite as

$$\frac{z^3/2^3}{(z+1/2)^3}$$

The singularity is at $z=-1/2$, the order is 3, and the residue is $\left(\frac{\mathrm{d}^2\left(z^3/8\right)}{\mathrm{d}z^2}\right)_{-1/2}\cdot 1/2! = 6\cdot -1/2/8\cdot 1/2 = -3/16$

(c) Rewrite as

$$\frac{\exp(z)/(z+i\pi)}{z-i\pi}, \frac{\exp(z)/(z-i\pi)}{z+i\pi}$$

The poles are at $z=\pm i\pi$, the orders are both 1, and the residues are $\frac{\exp(i\pi)}{2i\pi}, \frac{\exp(-i\pi)}{-2i\pi} = \frac{1}{2i\pi}, \frac{1}{-2i\pi} = \frac{-i}{2\pi}, \frac{i}{2\pi} = \mp\frac{i}{2\pi}$

(d) Rewrite as

$$\frac{f(z)}{g(z)} = \frac{1}{\sin(z)} \longrightarrow \frac{f(z)}{g'(z)} = \frac{1}{\cos(z)}$$

The singularities are located at $\pi n, n\in\mathbb{Z}$, the order is 1, and the residue is $(-1)^n$

2

2.1 f(z)

(a)

$$\begin{aligned} z^3 \sin(1/z) &= z^3 \left(\frac{1}{z} - \frac{1}{z^3 \cdot 3!} + \frac{1}{z^5 \cdot 5!} + \cdots + (-1)^n \frac{1}{z^{2n+1} \cdot (2n+1)!} \right) \\ &= z^2 - \frac{1}{3!} + \frac{1}{z^2} + \cdots + (-1)^n \frac{1}{z^{2n-2} \cdot (2n+1)!} \end{aligned}$$

The principal part consists of the terms with negative powers of z. In this case, it is

$$\frac{1}{z^2} + \cdots + (-1)^n \frac{1}{z^{2n-2} \cdot (2n+1)!}$$

(b) The residual is 0 because the coefficient of the z^{-1} term is 0.

(c) Since there are an infinite amount of negatives power terms, this is an essential singularity.

2.2 g(z)

(a)

$$\begin{aligned} \frac{1}{z^4} e^z &= \frac{1}{z^4} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots + \frac{z^n}{n!} \right) \\ &= \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z \cdot 3!} + \frac{1}{4!} + \cdots + \frac{z^{z-4}}{n!} \end{aligned}$$

The principal part is

$$\frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z \cdot 3!}$$

(b) The residue is 3!.

(c) This is a pole of order 4 because there are 4 terms in the principal part (4 terms with negative powers).

3

(a) $R = |a|$ because the point a is $|a|$ away from $z = 0$. We use the Taylor expansion theorem.

(b)

$$a_n = \frac{\mathrm{d}^n 1/z}{\mathrm{d}z^n} / n!$$

$$\begin{aligned}
\frac{1}{z} &= a_0(z-a)^0/0! + a_1(z-a)^1/1! + a_2(z-a)^2/2! + \cdots + a_n(z-a)^n/n! \\
&= 1/a - \frac{1}{a^2}(z-a) + \frac{1}{a^3}(z-a)^2 + \cdots + a_n(z-a)^n/n!
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{a} \cdot \frac{1}{1-(-(z-a)/a)} \\
&= \frac{1}{a} \cdot \left((-(z-a)/a)^0 + (-(z-a)/a)^1 + (-(z-a)/a)^2 + \cdots + (-(z-a)/a)^n\right) \\
&= \frac{1}{a} - \frac{z-a}{a^2} + \frac{(z-a)^2}{a^3} + \cdots + \frac{(z-a)^n}{a^{n+1}}
\end{aligned}$$

4

Let's find the distances of the singularities to each point first. The distance from $1+i$ to $+i$ is 1. The distance from $1+i$ to $-i$ is $\sqrt{5} \approx 2.24$. The distance from $1+i$ to $\sqrt{3}i$ is 1.24. The distance from $1+i$ to $-\sqrt{3}i$ is 2.91.

(a) The integral evaluates to 0 because it has no singularities inside its bounds. (Cauchy-Goursat Theorem)

(b) The singularities are at $z = i, \sqrt{3}i$.

$$\operatorname{Res}_{z=i} \frac{z^3 / \left[(z+i)(z^2+3)\right]}{z-i} = i^3 / \left(2i \cdot (-1+3)\right) = 1/4$$

$$\operatorname{Res}_{z=\sqrt{3}i} \frac{z^3 / \left[(z^2+1)(z+\sqrt{3}i)\right]}{z-\sqrt{3}i} = 3/4$$

The integral is equal to $2\pi i(3/4 + 1/4) = 2\pi i$

(c) The singularities are those of the previous integral, plus the ones located at $z = -i, -\sqrt{3}i$

$$\operatorname{Res}_{z=-i} \frac{z^3 / \left[(z-i)(z^2+3)\right]}{z+i} = -1/4$$

$$\operatorname{Res}_{z=-\sqrt{3}i} \frac{z^3 / \left[(z^2+1)(z-\sqrt{3}i)\right]}{z+\sqrt{3}i} = 3/4$$

The integral is equal to the previous integral plus $2\pi i(3/4 - 1/4) = \pi i$. The result is $3\pi i$

(d) Since all the singularities have been accounted for, the result is the same as (b): $3\pi i$

(e) Same situation as (c): $3\pi i$

5

These functions are even, so

$$\int_0^\infty \cdots = \frac{1}{2}\int_{-\infty}^\infty \cdots$$

6

There is only 1 singularity with $Im(z) > 0$ at $z = i$

$$\operatorname{Res}_{z=i} \frac{1/(z+i)}{z-i} = 1/2i = -i/2$$

The integral would be equal to $1/2 \cdot 2\pi i \cdot -i/2 = \pi/2$

7

There is only 1 singularity with $Im(z) > 0$ at $z = i$

$$\operatorname{Res}_{z=i} \frac{1/(z+i)^2}{(z-i)^2} = \frac{\mathrm{d}(1/(z+i)^2)}{\mathrm{d}z}(i) = -i/4$$

The integral is equal to $1/2 \cdot 2\pi i \cdot -i/4 = \pi/4$

8

The eligible singularities are at $z = \sqrt{i}, z = -\sqrt{-i} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}i}$.

$$\operatorname{Res}_{z=\sqrt{i}} \frac{1/\left((z+\sqrt{i})(z^2+i)\right)}{z-\sqrt{i}} = \frac{-\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\operatorname{Res}_{z=-\sqrt{-i}} \frac{1/\left((x-\sqrt{-i})(z^2-i)\right)}{z+\sqrt{-i}} = \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

The integral is equal to $1/2 \cdot 2\pi i \cdot (\cancel{\frac{-\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i + \cancel{\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i) = \pi\sqrt{2}/4 = \frac{\pi}{2\sqrt{2}}$

9

The eligible singularities are at $z = i, 2i$

$$\operatorname{Res}_{z=i} \frac{z^2/\left((z+i)(z^2+4)\right)}{z-i} = i/6$$

$$\operatorname{Res}_{z=2i} \frac{z^2/\left((z^2+1)(z+2i)\right)}{z-2i} = -i/3$$

The integral is $1/2 \cdot 2\pi i \cdot (i/6 - i/3) = \pi i \cdot -i/6 = \pi/6$

10

The eligible singularities are at $z = 2i, 3i$

$$\text{Res}_{z=2i} \frac{z^2/\left((z+2i)^2(z^2+9)\right)}{(z-2i)^2} = -13i/200$$

$$\text{Res}_{z=3i} \frac{z^2/\left((z^2+4)^2(z+3i)\right)}{z-3i} = 3i/50$$

The integral is equal to $1/2 \cdot 2\pi i \cdot (-13i/200 + 3i/50) = \pi i \cdot -i/200 = \pi/200$

11

(a)

$$\begin{aligned}
\cdots &= \oint \frac{1}{5+4\left(\dfrac{z+z^{-1}}{2}\right)}\frac{dz}{iz} \\
&= \oint \frac{1}{5+2z+2z^{-1}}\frac{dz}{iz} \\
&= \oint \frac{-i}{2z^2+5z+2}dz
\end{aligned}$$

Find the singularities:

$$\begin{aligned}
2z^2+5z+2 &= 0 \\
\frac{-5\pm\sqrt{25-4\cdot2\cdot4}}{4} \\
\frac{-5\pm3}{4} \\
-2, -1/2 \\
&= (2z+1)(z+2)
\end{aligned}$$

Find the residuals:

$$\text{Res}_{z=-2}\frac{-i/(2z+1)}{z+2} = i/3\,\text{Res}_{z=-1/2}\frac{-i/(z+2)}{2z+1} = -2i/3$$

The integral is equal to $2\pi i\cdot(i/3+-2i/3) = 2\pi i\cdot -i/3 = 2\pi/3$

(b)

$$\begin{aligned}
\cdots &= \oint \frac{1}{1+\left(\dfrac{z-z^{-1}}{2i}\right)^2}\frac{dz}{iz} \\
&= \oint \frac{1}{1+\dfrac{z^2-2+z^{-2}}{-4}}\frac{dz}{iz} \\
&= \oint \frac{-4/i}{-4+z^2-2+z^{-2}}dz/z \\
&= \oint \frac{4i}{z^3-6z+z^{-1}}
\end{aligned}$$

Find the singularities:

$$\begin{aligned}
z^3 - 6z + z^{-1} &= 0 \\
z^4 - 6z^2 + 1 &= 0 \\
a^2 - 6a + 1 &= 0 && (a = z^2) \\
\frac{6 \pm \sqrt{36 - 4 \cdot 1 \cdot 1}}{2} \\
\frac{6 \pm \sqrt{32}}{2} \\
\frac{6 \pm 4\sqrt{2}}{2} \\
a &= 3 \pm 2\sqrt{2} \\
z &= \pm(\sqrt{2} \pm 1)
\end{aligned}$$

Only $\sqrt{2} - 1, -\sqrt{2} + 1$ are inside out unit circle. Calculate residues:

$\mathrm{Res}_{z=\sqrt{2}-1} \dfrac{-4/i}{z^3 - 6z + z^{-1}} = \dfrac{-4/i}{3z^2 - 6 - z^{-2}}\bigg|_{z=\sqrt{2}+1} = -\dfrac{i}{2\sqrt{2}}$ $\mathrm{Res}_{z=-\sqrt{2}+1} \dfrac{-4/i}{z^3 - 6z + z^{-1}} = \dfrac{-4/i}{3z^2 - 6 - z^{-2}}\bigg|_{z=-}$

The integral is equal to $2\pi i \cdot (-\frac{i}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}) = 2\pi i \cdot (-\frac{i}{\sqrt{2}}) = 2\pi/\sqrt{2} = \sqrt{2}\pi$