

hw3

Liheng Cao

March 5, 2021

1

$$\begin{aligned}(-1 - \sqrt{3}i)^{432} &= (2e^{-2i\pi/3})^{432} \\&= 2^{423}e^{-2i\pi/3 \cdot 432} \\&= 2^{423}e^{-288i\pi} \\&\quad \square\end{aligned}$$

2

$$\begin{aligned}(-1-i)^{1/737} &= \left(\sqrt{2}e^{-3i\pi/4}\right)^{1/737} \\&= 2^{1/(2\cdot 737)}e^{-3i\pi/(4\cdot 737)}e^{2i\pi n/737}, n \in \{0, 1, \dots, 736\} \\&= 2^{1/(2\cdot 737)}(\cos(-3\pi/(4\cdot 737) + 2\pi n/737) + i\sin(-3\pi/(4\cdot 737) + 2\pi n/737))\end{aligned}$$

□

3

$$\begin{aligned}\left(1 - \sqrt{3}i\right)^{(5+7i)} &= \left(2e^{i\pi/3}\right)^{5+7i} \\ &= (\text{Log}(2) - i\pi/3 + 2i\pi n)^{5+7i} \\ &= e^{(\text{Log}(2) - i\pi/3 + 2i\pi n)(5+7i)} \\ &= e^{5\text{Log}(2) - 5i\pi/3 + 10i\pi n + 7i\text{Log}(2) + 7\pi/3 - 14\pi n} \\ &= e^{5\text{Log}(2) - 7(-\pi/3 + 2\pi n)} e^{i(7\text{Log}(2) + 5(-\pi/3 + 2\pi n))} \\ &= e^{5\text{Log}(2) - 7(-\pi/3 + 2\pi n)} (\cos(7\text{Log}(2) + 5(-\pi/3 + 2\pi n)) + i \sin(7\text{Log}(2) + 5(-\pi/3 + 2\pi n)))\end{aligned}$$

□

4

4.1

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

4.2

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

4.3

$$\tan(z) = 1/i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

5

$$\begin{aligned}\cos(4 + 3i) &= \frac{e^{4i-3} + e^{3-4i}}{2} \\&= (1/2) (e^{4i}e^{-3} + e^3e^{-4i}) \\&= (1/2) (e^{-3}(\cos(4) + i\sin(4)) + e^3(\cos(-4) + i\sin(-4))) \\&= (1/2) (\cos(4)(e^3 - e^{-3}) + \sin(4)(e^3 + e^{-3})) \\&= \cos(4)\sinh(3) + \sin(4)\cosh(3) \\&\quad \square\end{aligned}$$

$$\operatorname{Re}(\cos(4 + 3i)) = \cos(4)\sinh(3) + \sin(4)\cosh(3)$$

$$\operatorname{Im}(\cos(4 + 3i)) = 0$$

6

$$\begin{aligned}4 \cos^3 (T) &= \cos (3T) + 3 \cos (T) \\ \cos (3T) + 3 \cos (T) &= 4 \cos^3 (T) \\ &= 4 \left(\frac{e^{iT} + e^{-iT}}{2} \right)^3 \\ &= (1/2) \left(e^{3iT} + 3e^{2iT} e^{-iT} + 3e^{iT} e^{-2iT} + e^{-3iT} \right) && \text{(Binomial Expansion)} \\ &= (1/2) \left(e^{3iT} + 3e^{iT} + 3e^{-iT} + e^{-3iT} \right) \\ &= \cos (3T) + 3 \cos (T) \\ &\square\end{aligned}$$

7

$$\begin{aligned}\tan(z) &= \frac{2-i}{5} \\ (1/i) \frac{A-A^{-1}}{A+A^{-1}} &= \frac{2-i}{5} & (A = \exp(iz)) \\ \frac{A-A^{-1}}{A+A^{-1}} &= \frac{2i+1}{5} \\ \frac{A^2-1}{A^2+1} &= \frac{2i+1}{5} \\ A^2-1 &= (A^2+1) \frac{2i+1}{5} \\ A^2-1 &= (1/5) (2iA^2 + A^2 + 2i + 1) \\ 5A^2 - 5 &= 2iA^2 + A^2 + 2i + 1 \\ 5A^2 - 2iA^2 - A^2 &= 2i + 6 \\ 4A^2 - 2iA^2 &= 2i + 6 \\ A^2(4-2i) &= 2i + 6 \\ A^2 &= \frac{2i+6}{4-2i} \\ A^2 &= 1+i & (\text{Hint}) \\ e^{2iz} &= \sqrt{2}e^{i\pi/4} \\ 2iz &= \text{Log}(2) + i\pi/4 + 2i\pi n, n \in \mathbb{Z} \\ z &= (1/2i) \text{Log}(2) + \pi/8 + \pi n \\ &\square\end{aligned}$$

8

8.1 a)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

8.2 b)

$$f(z) = (Ax^2 + Cy^2 + Dx + F) + i(Bxy + Ey)$$

Match partials of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \wedge \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2Ax + D = \frac{\partial v}{\partial y} = Bx + E \wedge \frac{\partial u}{\partial y} = 2Cy - \frac{\partial v}{\partial x} - By$$

$$B = 2A = -2C \wedge D = E$$

8.3 c)

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\frac{df}{dz} = 2Ax + D + 2iCy$$

8.4 d)

Use the $y = 0$ trick:

$$f(z) = (Ax^2 + Dx + F)$$

$$f(z) = Az^2 + Dz + F$$