Linear Programming

In this lecture we describe linear programming that is used to express a wide variety of different kinds of problems. LP can solve the max-flow problem and the shortest distance, find optimal strategies in games, and many other things.

We will primarily discuss the setting and how to code up various problems as linear programs.

$y \leq_p NP$

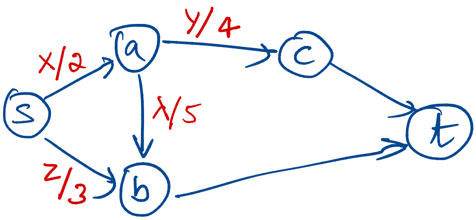
$y \leq_p LP$

$\downarrow$

Reduce problem to Linear programing

Intuition

NF



max-flow : The thing we are maximizing is called the Objective function : $\max(x+z)$

$$\left.\begin{array}{l} \left.\begin{array}{l} x \leq 2 \\ y \leq 4 \end{array}\right\} \forall \text{ edges} \\ x = y + \lambda \;\} \forall \text{ vertices} \end{array}\right\} \text{Constraints}$$

Reducing a computer science problem to linear algebra is linear programming. **Objective functions & constraints are linear**

A Production Problem

A company wishes to produce two types of souvenirs: type-A will result in a profit of \$1.00, and type-B in a profit of \$1.20.
To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II.
A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II.
There are 3 hours available on machine I and 5 hours available on machine II.
How many souvenirs of each type should the company make in order to maximize its profit?

We need to reduce the problem into a system of equations.

A Production Problem

	Type-A	Type-B	Time Available
Profit/Unit	\$1.00	\$1.20	
Machine I	2 min	1 min	180 min
Machine II	1 min	3 min	300 min

Let $x \geq 0$ be the number of type A
Let $y \geq 0$ be the number of type B

Objective function : $\max_{x,y} (x + 1.2y)$

A Linear Program

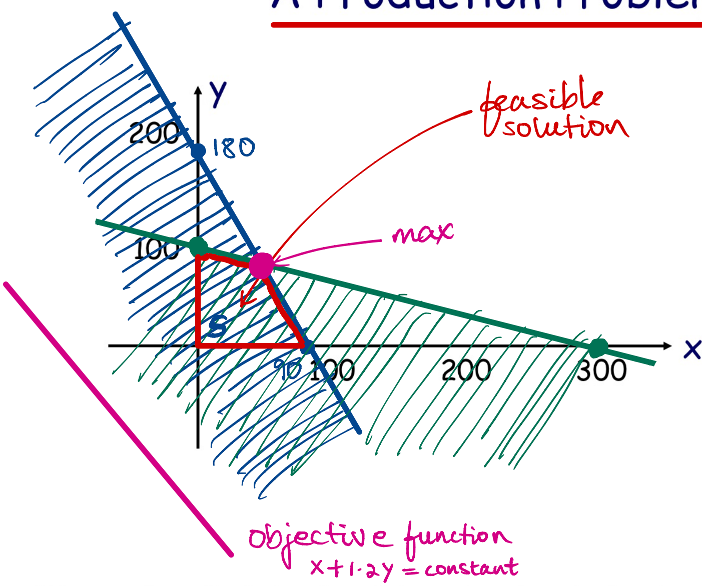
We want to maximize the objective function

$$\max(x + 1.2y)$$

subject to the system of inequalities:

$$2x + y \le 180$$
$$x + 3y \le 300$$
$$x \ge 0, \quad y \ge 0$$

A Production Problem



$2x + y \le 180$
$x = 0, \; y = 180$
$y = 0, \; x = 90$

Since it is ≤ 180, we need to shade towards the origin.

$x + 3y \le 300$
$x = 0, \; y = 100$
$y = 0, \; x = 300$

$x \ge 0, \; y \ge 0$

We need to find the feasible point that is farthest in the "objective" direction

P.T.O

With the help of calculus we know that the maximum can be found in the corner point.

– Victor (30:53 secs)

Fundamental Theorem

1. If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set S associated with the problem.

2. If the objective function P is optimized at *two* adjacent vertices of S, then it is optimized at *every point* on the line segment joining these vertices, in which case there are infinitely many solutions to the problem. In the previous diagram assume that the pink objective function line is parallel to the green line, then the whole green edge can be the solution.

Existence of Solution

Suppose we are given a LP problem with a feasible set S and an objective function P. There are 3 cases to consider

1. S is empty: example: $\max(x)$
   $x \leq -1$
   $x \geq 0$

   ∴ LP has no solution

2. S is unbounded: ex: $\max(x)$
   $x \geq 0$

   ∴ LP may or may not have a solution

③ S is bounded

$\therefore$ LP has a solution/solutions.

Standard LP form

We say that a maximization linear program with n variables is in standard form if for every variable x_k we have the inequality $x_k \geq 0$ and all other m linear inequalities.
An LP in standard form is written as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

vector (columns in computer science)

constants

$$\max\,(c_1x_1 + \ldots + c_nx_n)$$

$$c^T x = x^T c$$

subject to

$$\left.\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &\leq b_m \end{aligned}\right\} \quad A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1 \geq 0, \ldots, x_n \geq 0 \qquad \underline{\underline{Ax \leq b}}$$

Standard LP in Matrix Form

The vector c is the column vector $(c_1, \ldots, c_n)$.
The vector x is the column vector $(x_1, \ldots, x_n)$.
The matrix A is the $n \times m$ matrix of coefficients of the left-hand sides of the inequalities, and
$b = (b_1, \ldots, b_m)$ is the vector of right-hand sides of the inequalities.

$$\boxed{\begin{aligned} &\max\,(c^T x) \\ &\text{subject to} \\ &\quad Ax \leq b \\ &\quad \underbrace{x \geq 0}_{} \quad x \in \mathbb{R} \end{aligned}}$$

Exercise: Convert to Matrix Form

$\max(x_1 + 1.2\,x_2)$

$2x_1 + x_2 \le 180$ b

$x_1 + 3x_2 \le 300$

$x_1 \ge 0$

$x_2 \ge 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}$$

$$b = \begin{pmatrix} 180 \\ 300 \end{pmatrix}$$

$$\boxed{\text{Rule : } (Ab)^T = b^T A^T}$$

Algorithms for LP

The standard algorithm for solving LPs is the Simplex Algorithm, due to Dantzig, 1947.

This algorithm starts by finding a vertex of the polytope, and then moving to a neighbor with increased cost as long as this is possible. By linearity and convexity, once it gets stuck it has found the optimal solution.

Unfortunately, simplex does not run in polynomial time it does well in practice, but poorly in theory.

P.T.O

Algorithms for LP

In 1974 Khachian has shown that LP could be done in polynomial time by something called the Ellipsoid Algorithm (but it tends to be fairly slow in practice).

In 1984, Karmarkal discovered a faster polynomial-time algorithm called "interior-point". While simplex only moves along the outer faces of the polytope, "interior-point" algorithm moves inside the polytope.

Discussion Problem 1

$2x_1+x_2+3x_3 \le 100$

≤ 60

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

	Density	Volume	Price
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3

Write a linear program that optimizes revenue within the constraints.

Let x_1, x_2, x_3 be the volumes.

Objective Function: max $(1000x_1 + 2000x_2 + 12000x_3)$

Subject to:

$$2x_1 + x_2 + 3x_3 \le 100$$

$$x_1 + x_2 + x_3 \le 60$$

$$0 \le x_1 \le 40,\ 0 \le x_2 \le 30,\ 0 \le x_3 \le 20$$

? ? ?

In standard form we cannot have any upper bounds

- Victor

In matrix form:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad C = \begin{pmatrix} 1000 \\ 2000 \\ 12000 \end{pmatrix} \quad A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad b = \begin{pmatrix} 100 \\ 60 \\ 40 \\ 30 \\ 20 \end{pmatrix}$$

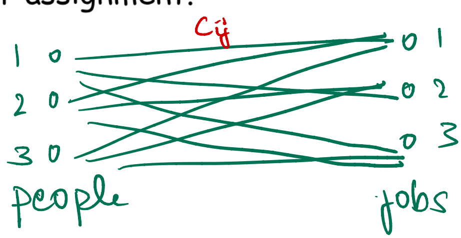
$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$

$$\left.\begin{array}{l} x_1 \leq 40 \rightarrow 1 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 \leq 40 \\ x_2 \leq 30 \rightarrow 0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 \leq 30 \\ x_3 \leq 20 \rightarrow 0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 \leq 20 \end{array}\right\} A$$

Discussion Problem 2

There are n people and n jobs. You are given a cost matrix, C, where c_{ij} represents the cost of assigning person i to do job j. You need to assign all the jobs to people and also only one job to a person. You also need to minimize the total cost of your assignment. Write a linear program that minimizes the total cost of your assignment.

min matching



① Define variables

Let x_{ij} be an assignment between i^{th} person and j^{th} job.

② Objective function : $\min \sum_{ij} x_{ij} c_{ij}$

③ Constraints

ⓐ Pick a person, $i = 1, 2, \ldots, n$

$$x_{i1} + x_{i2} + \cdots + x_{in} = \underline{\underline{1}}$$

The no of jobs a person has is max 1.

ⓑ pick a job, $j = 1, 2, \ldots n$

$$x_{1j} + x_{2j} + \cdots + x_{nj} = \underline{\underline{1}}$$

$$x_{ij} = \begin{cases} 1, \\ 0 \end{cases}$$

Integer Linear Programming (ILP), here all the variables $\in \mathbb{N}$

We do not know how to solve ILP in polynomial time.

Similar to 0-1 knapsack problem.

Discussion Problem 3

Convert the following LP to standard form

$$\max\ (5x_1 - 2x_2 + 9x_3)$$

$$3x_1 + x_2 + 4x_3 = 8$$

$$2x_1 + 7x_2 - 6x_3 \le 4$$

$$x_1 \le 0,\ x_3 \ge 1$$

$-3x_1 - x_2 - 4x_3 \le -8$

$3x_1 + x_2 + 4x_3 \le 8$

$z_1 = -x_1,\ z_1 \ge 0$

$z_3 = x_3 - 1,\ z_3 \ge 0$

x_2 ?

$-\infty < x_2 < +\infty$

free variable

$z_4 =$ ① $x_2^2 \ge 0$ / ② $|x_2| \ge 0$ — NOT POSSIBLE! ① Not linear ② Piecewise — non linear

We need to make use of 2 variables.

$x_2 = z_5 - z_6,$

$z_5 \ge 0,$

$z_6 \ge 0$

Write your LP in new variables : z, z_3, z_5, z_6

$x_1 = -z_1$

$x_2 = z_5 - z_6$

$x_3 = z_3 + 1$

Rewritten forms of each:

$\max(5x_1 - 2x_2 + 9x_3) \approx \max(-5z_1 - 2(z_5 - z_6) + 9(z_3 + 1))$

$= \max(-5z_1 - 2z_5 - 2z_6 + 9z_3 + 9)$

$3x_1 + x_2 + 4x_3 \overset{?}{\leq} 8 \approx -3z_1 + z_5 - z_6 + 4z_3 + 4 \leq 8$

$2x_1 + 7x_2 - 6x_3 \leq 4 \approx -2z_1 + 7z_5 - 7z_6 - 6z_3 - 6 \leq 4$

$z_1 \geq 0,\ z_3 \geq 0,\ z_5 \geq 0,\ z_6 \geq 0$

$$X = \begin{bmatrix} z_1 \\ z_3 \\ z_5 \\ z_6 \end{bmatrix} \quad C = \begin{bmatrix} -5 \\ 9 \\ -2 \\ -2 \end{bmatrix} \quad A = \begin{bmatrix} -3 & 4 & 1 & -1 \\ -2 & -6 & 7 & -7 \end{bmatrix}$$

$$b = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

Discussion Problem 4

Explain why LP <u>cannot</u> contain constrains in the form of *strong* inequalities.

$\max(7x_1 - x_2 + 5x_3)$

$x_1 + x_2 + 4x_3 < 8$

$3x_1 - x_2 + 2x_3 > 3$

$2x_1 + 5x_2 - x_3 \leq -7$

$x_1, x_2, x_3 \geq 0$

Example : $\max(x)$

$x < 1$

$x \geq 0$

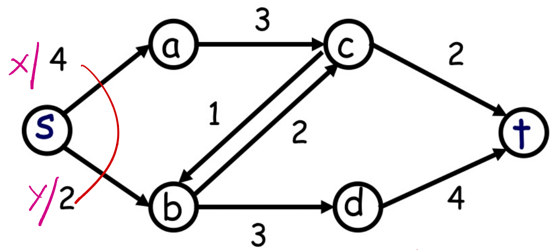
$x = 1$ is not a solution

$1 < 1 \Rightarrow$ FALSE

not feasible

Exercise: Max-Flow as LP

Write a max-flow problem as a linear program.



NF $\leq_p$ LP

See the first slide for the solving this.

max $(x+y)$

1. Capacity constraint: $0 \leq f(u,v) \leq c(u,v)$, for each edge $(u,v) \in E$
2. Conservation constraint: $\sum_u f(u,v) = \sum_w f(v,w)$, for each $v \in V - \{s,t\}$

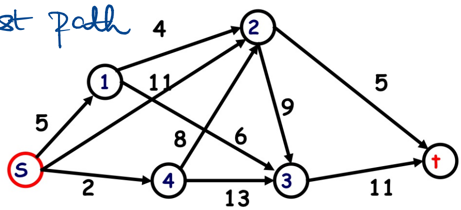
If we want to write this LP in the standard from, we need to change the equalities, $\sum_u f(u,v) = \sum_w f(v,w)$, into inequalities, $\sum_u -f(u,v) \leq \sum_w -f(v,w)$ and $\sum_u f(u,v) \leq \sum_w f(v,w)$.

Exercise: Shortest Path as LP

Write a shortest st-path problem as a linear program.

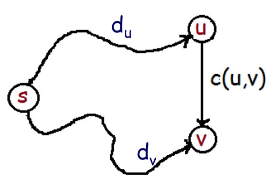
Let $d(v)$ be the shortest path from s to v.

$d(s) = 0$



$d(v) \leq d(u) + c(u,v)$

we enforce this rule for each edge.



$d(1) \leq d(s) + 5$

$d(4) \leq d(s) + 2$

$d(2) \leq d(1) + 4$ for each edge

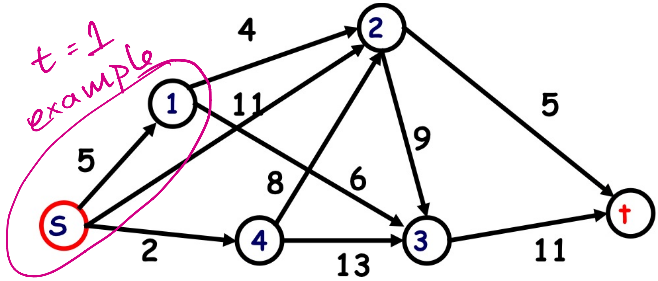
$d(2) \leq d(4) + 8$

Objective function : $\min d(t)$

constraints :

$d(1) \le d(s) + 5 = 5$

$d(1) \ge 0$



Objective function :

$\min d(1) \rightarrow d(1) = 0 \rightarrow$ Here when we see mathematically minimum d(1) will be 0 because the most trivial solution would be to not travel to t.

Hence we do this,

$$\boxed{\begin{array}{c} \max d(1) \\ d(1) \le 5 \\ d(1) \ge 0 \end{array}}$$

$\rightarrow d(1) = 5$

$\Rightarrow$ This is only for the given example & probably applies only for this

Discussion Problem 5

Write a 0-1 Knapsack Problem as a linear program.

0-1 knapsack $\le_p$ LP

Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. Put these items in a knapsack of capacity W to get the maximum total value in the knapsack.

$$\text{Given} \sum_{k=1}^{m} w_k \le W$$

$$\text{optimize} \sum_{k=1}^{m} v_k \rightarrow \max$$

① Variables:

$$X_K \begin{cases} 1, & K^{th} \text{ item is taken} \\ 0, & \text{otherwise} \end{cases}$$

② Objective function:

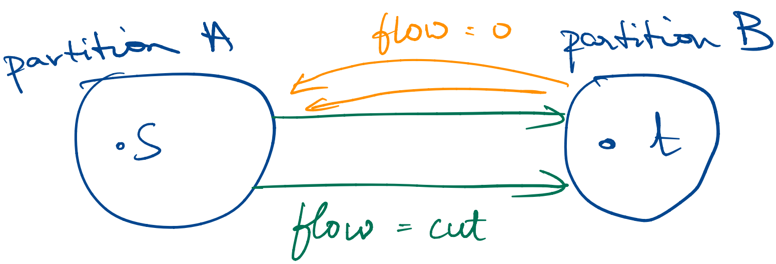
$$\max \sum_{i=1}^{n} X_i \cdot V_i$$

③ Constraints:

$$\sum_{i=1}^{n} X_i \cdot \omega_i \le W$$

Dual LP

To every linear program there is a dual linear program



$$\left.\begin{array}{l} ① \ \forall \text{ flow}, \forall \text{ cut} : \text{flow} \le \text{cut} \\ ② \ \exists \text{ flow}, \exists \text{ cut} : \text{flow} = \text{cut} \end{array}\right\} \text{maxflow-mincut}$$

Duality

<u>Definition.</u> The dual of the standard (primal) maximum problem

$\max c^T x$
$Ax \le b$ and $x \ge 0$

Primal

is defined to be the standard minimum problem

$\min b^T y$
$A^T y \ge c$ and $y \ge 0$

Dual

Exercise: duality

Consider the LP:

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 \le 8$$
$$3x_1 - x_2 + 2x_3 \le 3$$
$$2x_1 + 5x_2 - x_3 \le -7$$
$$x_1, x_2, x_3 \ge 0$$

primal
↓
matrix form
↓
dual

Write the dual problem.

primal LP		dual LP
$\max (c^T x)$ $A x \le b$ $x \ge 0$	⟷	$\min (b^T y)$ $A^T y \ge c$ $y \ge 0$

P.T.O

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad c = \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} \quad b = \begin{bmatrix} 8 \\ 3 \\ -7 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 5 & -1 \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 5 \\ 4 & 2 & -1 \end{bmatrix}$$

dual:

$$\min_y (8y_1 + 3y_2 - 7y_3)$$

$$y_1 + 3y_2 + 2y_3 \geq 7$$

$$y_1 - y_2 + 5y_3 \geq -1$$

$$4y_1 + 2y_2 - y_3 \geq 5$$

From Primal to Dual

Consider the max LP constrains

$$y_1(a_{11}x_1 + \ldots + a_{1n}x_n) \leq b_1 y_1$$

$$+ \quad \vdots$$

$$y_m(a_{m1}x_1 + \ldots + a_{mn}x_n) \leq b_m y_m$$

$y_i \geq 0$

new variables

1) Multiply each equation by a new variable $y_k \geq 0$.
2) Add up those m equations.
3) Collect terms wrt to x_k.
4) Choose y_k in a way such that $A^T y \geq c$.

$$x_1\underbrace{(y_1 \cdot a_{11} + y_2 \cdot a_{21} + \cdots + y_m \cdot a_{m1})}_{\geq c_1}$$

$$+ \cdots +$$

$$x_n\underbrace{(y_1 a_{1n} + y_2 a_{2n} + \cdots + \cancel{y}_m a_{mn})}_{\geq c_n}$$

$$\leq \cancel{y}_1 b_1 + \cancel{y}_2 b_2 + \cdots + \cancel{y}_m b_m$$

New constraints

$$\left.\begin{aligned} y_1 a_{11} + \cdots + y_m a_{m1} &\geq c_1 \\ &\cdots \\ y_1 a_{1n} + \cdots + y_m a_{mn} &\geq c_n \end{aligned}\right] \quad A^T y \geq c$$

Objective function

$$x_1 \cdot c_1 + x_2 \cdot c_2 + \cdots + x_n c_n \leq b_1 \cdot \cancel{y}_1 + b_2 \cancel{y}_2 + \cdots + b_m \cancel{y}_m$$

In matrix form

$$\text{max} \rightarrow \boxed{\underset{\text{primal}}{c^T x} \leq \underset{\text{dual}}{b^T y}} \leftarrow \text{min}$$

Discussion Problem 6

Consider the LP:

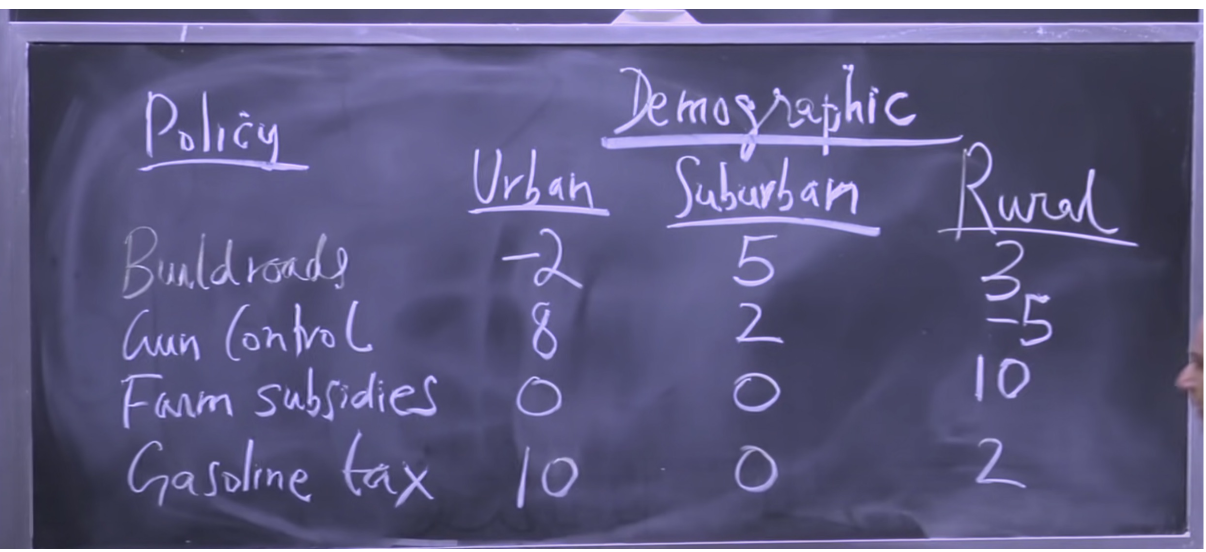
$$\min(3x_1 + 8x_2 + x_3)$$

$$x_1 + 4x_2 - 2x_3 \leq 20$$
$$x_1 + x_2 + x_3 \geq 7$$
$$x_2 + x_3 = 3$$
$$x_1, x_2, x_3 \geq 0$$

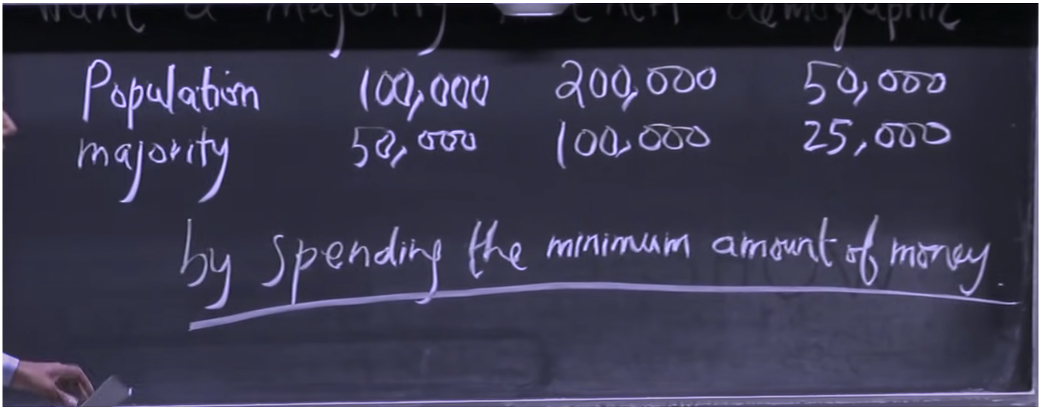
Write the dual problem.

NEED TO SOLVE ON YOUR OWN, VIKTOR doesnt cover it in the class.

MIT



Policy	Demographic: Urban	Suburban	Rural
Build roads	-2	5	3
Gun Control	8	2	-5
Farm subsidies	0	0	10
Gasoline tax	10	0	2



Let x_1, x_2, x_3, x_4 be the no of dollars spent on each issue.

Minimize $x_1 + x_2 + x_3 + x_4$

Subject to: $-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50000$

$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100000$

$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25000$

$x_1, x_2, x_3, x_4 \geq 0$

PTO Very Important

Variables $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

Objective fn : $\vec{c} \cdot \vec{x} = c_1 x_1 + \cdots + c_n x_n$

Inequalities : $A \vec{x} \leq \vec{b}$

max $\vec{c} \cdot \vec{x}$, st. and $\vec{x} \geq 0$

Goal is to maximize by following the cost & $\vec{x} \geq 0$

LP Duality

PRIMAL		DUAL
max $\vec{c}\,\vec{x}$ st. $A\vec{x} \leq \vec{b}$ $\vec{x} \geq 0$	identical $\equiv$	min $\vec{b} \cdot \vec{y}$ st $A^T \vec{y} \leq \vec{c}$ $\vec{y} \geq 0$

Convert to standard form | TO DO

① Minimize $-2x_1 + 3x_2$ ⇒ Negate to $2x_1 - 3x_2$ & Maximize

② Suppose x_j does not have a non negativity constraint

Take x_j & replace it with

$x_j' - x_j''$ where $x_j' \geq 0$ & $x_j'' \geq 0$

③ Suppose we have equality constraint

$$x_1 + x_2 = 7$$

$x_1 + x_2 \leq 7$ → First eq.

$-x_1 - x_2 \leq -7$ → second eq.

- flip signs
- make $\leq$

So 2 equations

④ $\geq$ (greater than equal to) constraint

translates to $\leq$ by (-1) multiply

Discussion Problem 6

Consider the LP:

$\min(3x_1 + 8x_2 + x_3) \Rightarrow \max(-3x_1 - 8x_2 - x_3)$

$x_1 + 4x_2 - 2x_3 \le 20$

$x_1 + x_2 + x_3 \ge 7 \Rightarrow -x_1 - x_2 - x_3 \le -7$

$x_2 + x_3 = 3 \Rightarrow x_2 + x_3 \le 3$

$-x_2 - x_3 \le -3$

$x_1, x_2, x_3 \ge 0$

$$c = \begin{bmatrix} -3 \\ -8 \\ -1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 4 & -2 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \quad b = \begin{bmatrix} 20 \\ -7 \\ 3 \\ -3 \end{bmatrix}$$

Write the dual problem.

$\max(c^T x) \Longrightarrow \min(b^T y)$

$Ax \le b \qquad A^T y \ge c$

$$A^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 4 & -1 & 1 & -1 \\ -2 & -1 & 1 & -1 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\min(b^T y) = \min(20y_1 - 7y_2 + 3y_3 - 3y_4)$$

$$A^T y \ge c = \quad y_1 - y_2 \ge -3$$

$$4y_1 - y_2 + y_3 - y_4 \ge -8$$

$$-2y_1 - y_2 + y_3 - y_4 \ge -1$$