Analysis of Algorithms

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Lecture 3

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Heaps

Reading: chapter 3

Amortized Analysis

In a <u>sequence</u> of operations the worst case does not necessarily occur in each operation - some operations may take different times.

Therefore, a traditional worst-case per operation analysis can give overly *pessimistic* bound.

Consider insertions into an array

some operations take $O(n)$, others – $O(1)$

if the current array is full, the cost of insertion is linear;
if it is not full, insertion takes a constant time.

Therefore, amortized analysis is an alternative to the traditional worst-case analysis. Namely, we perform a worst-case analysis on a sequence of operations.

The Aggregate Method

The amortized cost of an operation is given by $\frac{T(n)}{n}$, where $T(n)$ is the upper bound on the total cost of *n* operations.

Example: unbounded array (with a doubling-up resizing policy)

Insertions:	1, 2, 3, 4, 5, 6, 7, 8, 9, …, 2^n+1
Insertion Cost:	1, 1, 1, 1, 1, 1, 1, 1, 1, …, 1
Copy Cost:	0, 1, 2, 0, 4, 0, 0, 0, 8, … , 2^n

In lecture 2 we computed the average cost per insert: O(1)

It is important to realize that we achieve a great amortized cost just because we have implemented a clever resizing policy!

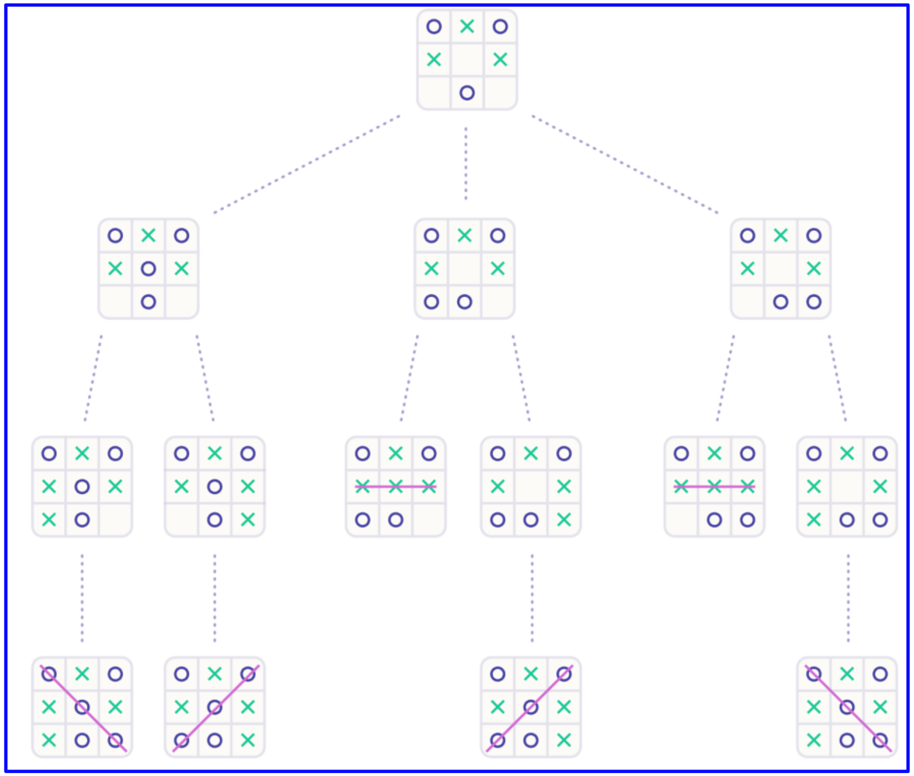
Review Questions

2. (**T/F**) Amortized analysis is used to determine the average runtime complexity of an algorithm.
3. (**T/F**) Compared to the worst-case analysis, amortized analysis provides a more accurate upper bound on the performance of an algorithm.
4. (**T/F**) The total amortized cost of a sequence of n operations gives a lower bound on the total actual cost of the sequence.
5. (**T/F**) Amortized constant time for a dynamic array is still guaranteed if we increase the array size by 5%.
6. (**T/F**) If an operation takes $O(1)$ expected time, then it takes $O(1)$ amortized time.
7. Suppose you have a data structure such that a sequence of n operations has an amortized cost of $O(n \log n)$. What could be the highest actual time of a single operation?

$O(n \log n)$

Heap and Priority Queue for Solving Optimization Problems

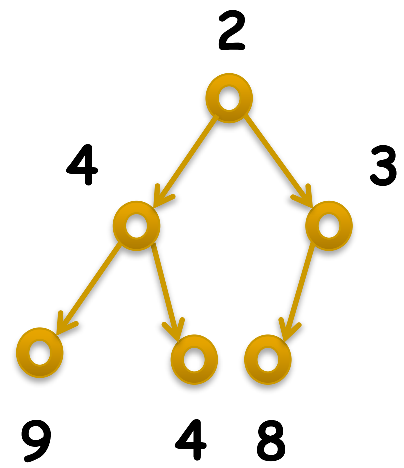
In this lecture, we will discuss a data structure that allows us to quickly access the highest priority element.



To win a game you cannot simply run a DFS/BFS, among all possible moves you have to choose the best move!

Binary min-Heap

A binary heap is a complete binary tree which satisfies the heap ordering property.



1. Structure Property

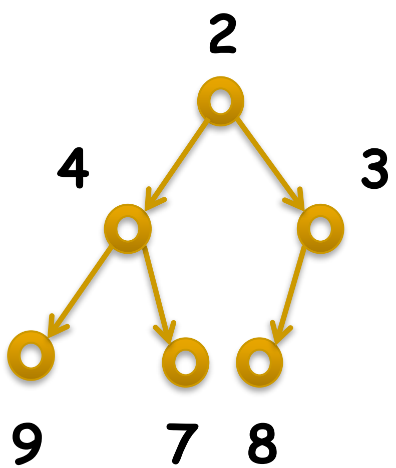
2. Ordering Property

0	1	2	3	4	5	6	7

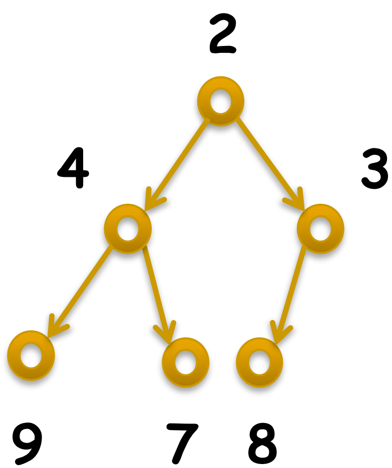
Consider k-th element of the array,

- its left child is located at 2*k index
- its right child is located at 2*k+1 index
- its parent is located at k/2 index

insert (tree reps)



insert (array reps)

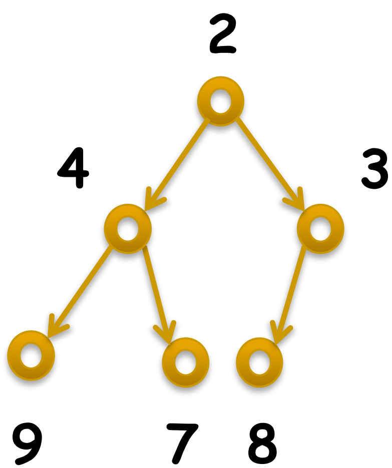


0	1	2	3	4	5	6	7

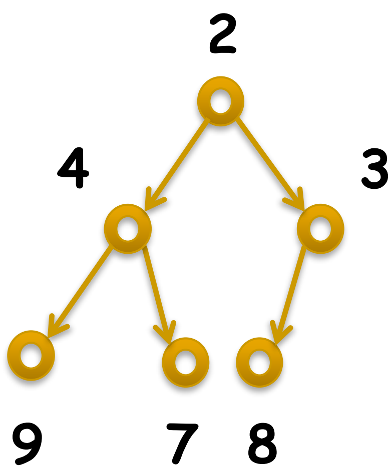
Discussion Problem 1

The values 1, 2, 3, . . . , 63 are all inserted (in any order) into an initially empty min-heap. What is the smallest number that could be a leaf node?

deleteMin (tree reps)



deleteMin (array reps)



0	1	2	3	4	5	6	7

Discussion Problem 2

Suppose you have two binary min-heaps, A and B, with a total of *n* elements between them. You want to discover if A and B have a key in common. Give a solution to this problem that takes time O(n log n). Do not use the fact that heaps are implemented as arrays, use only API operations: insert and deleteMin.

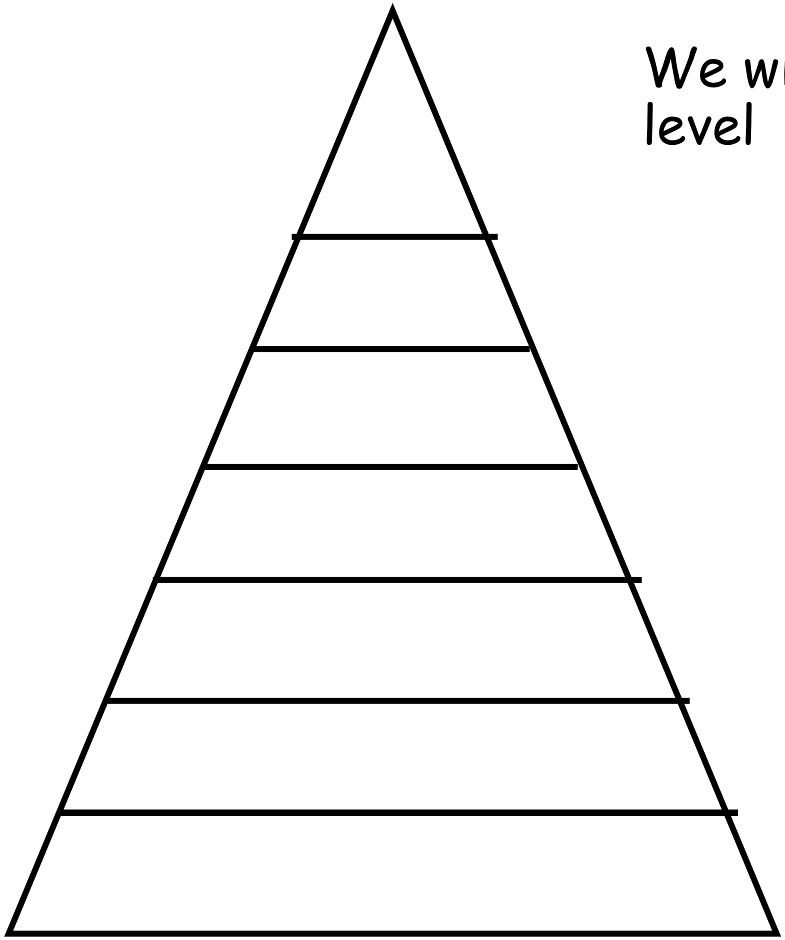
Build a Heap by Insertion

Given an array - turn it into a heap.
insert 7, 3, 8, 1, 4, 9, 4, 10, 2, 0 into an initially empty heap.

Build a Heap in O(n)

Heapify: 7, 3, 8, 1, 4, 9, 4, 10, 2, 0

Complexity of heapify



We will count the max number of swaps at each level

height	# of nodes	# of swaps
0		
1		
2		
...	...	...
h-1		

Complexity of heapify

Discussion Problem 3

How would you sort using a binary heap?

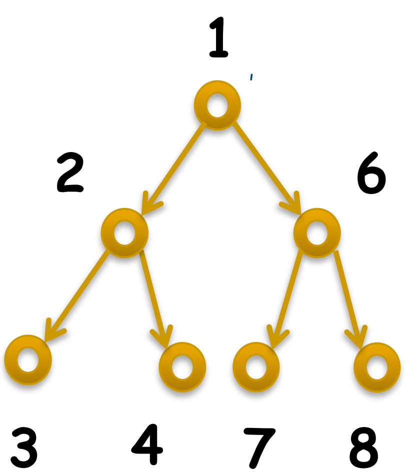
What is it runtime complexity?

HEAPSORT

Run delMin n-times

$O(n \log n)$

in-place
nonstable



0	1	2	3	4	5	6	7
	1	2	6	3	4	7	8

	2	3	6	8	4	7	1

	3	4	6	8	7	2	1

	4	8	6	7	3	2	1

Discussion Problem 4

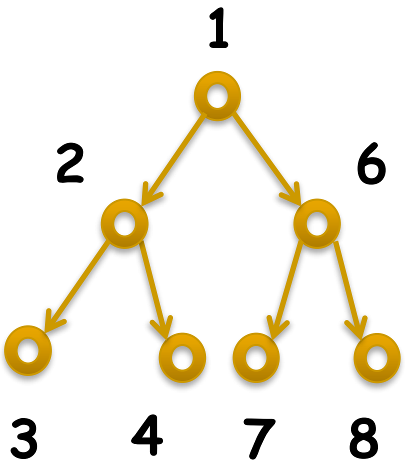
How would you merge two binary min-heaps?

What is it runtime complexity?

Discussion Problem 5

Devise a heap-based algorithm that finds k largest elements out of n elements. Assume that $n > k$. What is its runtime complexity?

decreaseKey



A new kind of heaps

We want to create a heap with a better amortized complexity of insertion. This example will demonstrate that binary heaps do not provide a better upper bound for the worst-case complexity.

Insert 7, 6, 5, 4, 3, 2, 1 into an empty binary min-heap.

Binomial Trees B_k

The binomial tree B_k is defined as

1. B_0 is a single node
2. B_k is formed by joining two B_{k-1} trees

Binomial Heaps

A binomial heap is a collection (a linked list or a queue) of at most Celling(log n) binomial trees (of unique rank) in increasing order of size where each tree has a heap ordering property.

Discussion Problem 6

Given a sequence of numbers: 3, 5, 2, 8, 1, 5, 2, 7.
Draw a binomial heap by inserting the above numbers reading them from left to right

Discussion Problem 7

How many binomial trees does a binomial heap with
25 elements contain?
What are the ranks of those trees?

Insertion

What is its worst-case runtime complexity?

What is its amortized runtime complexity?

Use an accounting method.

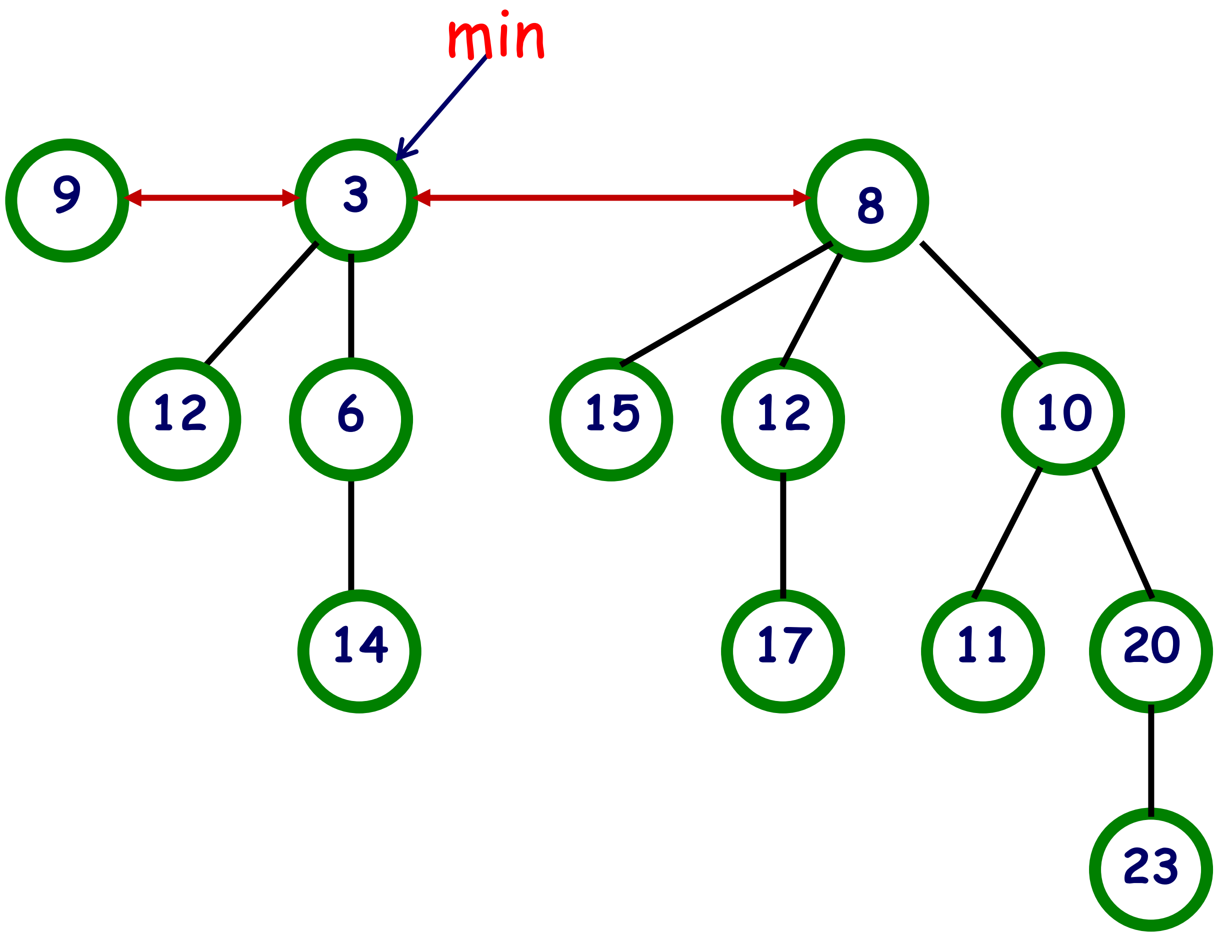
Building : Binomial vs Binary Heaps

The cost of inserting n elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case. This is an online algprithm.

If n is known in advance (an offile algorithm), we run heapify, so a binary heap can be constructed in time $\Theta(n)$.

The cost of inserting n elements into a binomial heap, one after the other, is $\Theta(n)$ (amortized cost), even if n is not known in advance.

deleteMin()



deleteMin()

Discussion Problem 8

Devise an algorithm for merging two binomial heaps and discuss its complexity. Merge $B_0B_1B_2B_4$ with B_1B_4.

Heaps

	Binary	Binomial	Fibonacci
findMin	$\Theta(1)$	$\Theta(1)$	
deleteMin	$\Theta(\log n)$	$\Theta(\log n)$	
insert	$\Theta(\log n)$	$\Theta(1)$ (ac)	
decreaseKey	$\Theta(\log n)$	$\Theta(\log n)$	
merge	$\Theta(n)$	$\Theta(\log n)$	

ac – amortized cost.

FIBONACCI HEAPS

Idea: *relaxed (lazy)* binomial heaps

Goal: decreaseKey in $O(1)$ ac.

The algorithm is outside of the scope of this course.

Heaps

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	Binary	Binomial	Fibonacci
findMin	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$
deleteMin	$\Theta(\log n)$	$\Theta(\log n)$	$O(\log n)$ (ac)
insert	$\Theta(\log n)$	$\Theta(1)$ (ac)	$\Theta(1)$
decreaseKey	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(1)$ (ac)
merge	$\Theta(n)$	$\Theta(\log n)$	$\Theta(1)$ (ac)