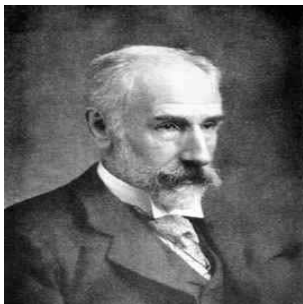


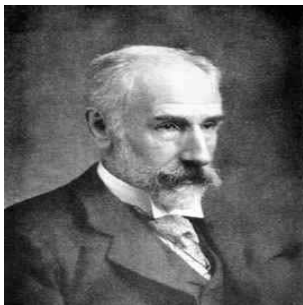
High Dimensional Edgeworth Expansion With Applications to Bootstrap and Its Variants

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Francis Ysidro Edgeworth
(1845 - 1926)



Peter Gavin Hall
(1951 - 2016)

1 Background

2 High Dimensional Settings

3 Main Results

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3 Main Results

- Given the data $\mathcal{X} = (X_1, \dots, X_n) \stackrel{i.i.d.}{\sim} F$;
- $X_i \in \mathbb{R}^p, \mathbb{E}X_i = 0, \mathbb{E}X_i X_i^T = V$;
- Sample mean $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$;
- Goal: approximate the distribution of \bar{X} , i.e. approx.

$$P\left(\sqrt{n}V^{-\frac{1}{2}}\bar{X} \in A\right)$$

for $A \in \mathcal{C}$ where \mathcal{C} denote the collection of all convex sets in \mathbb{R}^p .

Let Φ be the measure of $N(0, I_{p \times p})$. When the dimension p is fixed:

- Central Limit Theorem (CLT):

$$\sup_{A \in \mathcal{C}} |P(\sqrt{n}V^{-\frac{1}{2}}\bar{X} \in A) - \Phi(A)| = o(1);$$

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- Berry-Esseen Bound (with third-order moments):

$$\sup_{A \in \mathcal{C}} |P(\sqrt{n}V^{-\frac{1}{2}}\bar{X} \in A) - \Phi(A)| = O\left(n^{-\frac{1}{2}}\right);$$

Edgeworth Expansion with Fixed Dimensions

- Edgeworth Expansion (with $(\nu + 3)$ -order moments):

$$\sup_{A \in \mathcal{C}} \left| P \left(\sqrt{n} V^{-\frac{1}{2}} \bar{X} \in A \right) - \Phi(A) - \sum_{j=1}^{\nu} n^{-\frac{j}{2}} P_j(A) \right| = O \left(n^{-\frac{\nu+1}{2}} \right);$$

where $P_j(\cdot)$ are sign measures determined by the cumulants of F .

- When $\nu = 1$ and $p = 1$,

$$\sup_{A \in \mathcal{C}} \left| P \left(\sqrt{n} V^{-\frac{1}{2}} \bar{X} \in A \right) - \Phi(A) - n^{-\frac{1}{2}} P_1(A) \right| = O \left(n^{-1} \right)$$

where $P_1(A)$ has a density

$$p_1(x) = \frac{1}{6}(x^3 - x) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Edgeworth Expansion and Bootstrap

- Draw a bootstrap sample $X_1^*, \dots, X_n^* \stackrel{i.i.d.}{\sim} \hat{F}_n$ where \hat{F}_n is the ecdf of X_1, \dots, X_n ;

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- Heuristically, a first-order edgeworth expansion implies

$$\sup_{A \in \mathcal{C}} \left| P \left(\sqrt{n} (V^*)^{-\frac{1}{2}} (\bar{X}^* - \bar{X}) \in A \middle| \mathcal{X} \right) - \Phi(A) - n^{-\frac{1}{2}} P_1^*(A) \right| = O(n^{-1});$$

where $V^* = \text{Var}(X_1^*)$ and $P_1^*(\cdot)$ is determined by the cumulants of X_1^* .

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- Recall that

$$\sup_{A \in \mathcal{C}} \left| P \left(\sqrt{n} V^{-\frac{1}{2}} \bar{X} \in A \right) - \Phi(A) - n^{-\frac{1}{2}} P_1(A) \right| = O(n^{-1});$$

- The cumulants of F and those of \hat{F}_n are closed and thus

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- As a consequence,

$$\sup_{A \in \mathcal{C}} \left| P\left(\sqrt{n}(V^*)^{-\frac{1}{2}}(\bar{X}^* - \bar{X}) \in A \middle| \mathcal{X}\right) - P\left(\sqrt{n}V^{-\frac{1}{2}}\bar{X} \in A\right) \right| = O(n^{-1}).$$

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- This is called *Higher-Order Accuracy* (Hall, 1992).

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- Fundamental limit: $p = o(n^{\frac{2}{7}})$ for CLT to hold;

Edgeworth Expansion in High Dimensions

- In contrast to CLT, very few works on edgeworth expansion in high dimensions; Some results on Banach space but focus on $\mathbb{E}f(\bar{X})$ for smooth f instead of the law of \bar{X} (Gotze, 1981; Bentkus, 1984).

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Question: How fast can the dimension grow with n ?

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Theorem 1.

Let X_1, \dots, X_n be i.i.d. samples with zero mean and covariance matrix V . Assume that

- ① $\lambda_{\max}(V) = O(1), \lambda_{\min}(V) = \Omega(1);$
- ② $p = O(n^\gamma)$ for some $\gamma < \frac{1}{14};$
- ③ $|X_{ij}| \leq B = O(1);$

Then for any positive integer S ,

$$\sup_{A \in \mathcal{C}_n} \left| P(\sqrt{n}V^{-\frac{1}{2}}\bar{X} \in A) - \Phi(A) - \sum_{j=1}^{\nu} n^{-\frac{j}{2}} P_j(A) \right| = O\left(\left(\frac{p^9}{n}\right)^{\frac{\nu+1}{2}}\right).$$

Here \mathcal{C}_n includes all convex sets plus all sets with form

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As a corollary, for a smooth F ,

$$P(F(\sqrt{n}\bar{X}) \in A) \approx \Phi_{0,V}(F^{-1}(A)) + \sum_{j=1}^{\nu} n^{-\frac{j}{2}} P_j(V^{\frac{1}{2}} F^{-1}(A)).$$

This gives an edgeworth expansion for smooth transform of mean.

Theorem 2.

Let X_1, \dots, X_n be i.i.d. samples with zero mean and covariance matrix V and $X_1^*, \dots, X_n^* \stackrel{i.i.d.}{\sim} \hat{F}_n$. Assume that

- ① $\lambda_{\max}(V) = O(1), \lambda_{\min}(V) = \Omega(1)$;
- ② $p = \Theta(n^\gamma)$ for some $0 < \gamma < \frac{1}{17}$;
- ③ $|X_{ij}| \leq B = O(1)$.

Then with probability $1 - \exp\{-\Omega(n^\gamma)\}$,

$$\sup_{A \in \mathcal{C}_n} \left| P\left(V^{*- \frac{1}{2}}(\bar{X}^* - \bar{X}) \in A\right) - P\left(V^{- \frac{1}{2}}\bar{X} \in A\right) \right| \leq \frac{Cp^9}{n}.$$

This is strictly better than the bound given by CLT since

$$\frac{p^9}{n} \ll \frac{p^{\frac{7}{4}}}{\sqrt{n}}, \quad \text{when } p = O(n^{\frac{1}{17}}).$$

Thanks!

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