Inference For High Dimensional M-estimates: Fixed Design Results

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Setup

Observe $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$:

- response vector $Y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$;
- design matrix $X = (x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^{n \times p}$.

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Model:

- Linear Model: $Y = X\beta^* + \epsilon$;
- $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T \in \mathbb{R}^n$ being a random vector;

M-Estimator

M-Estimator: Given a convex loss function $\rho(\cdot) : \mathbb{R} \to [0, \infty)$,

$$\hat{\beta} = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \rho(y_i - x_i^T \beta).$$

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When ρ is differentiable with $\psi = \rho'$, $\hat{\beta}$ can be written as the solution:

$$\frac{1}{n} \sum_{i=1}^{n} \psi(y_i - x_i^T \hat{\beta}) = 0.$$

M-Estimator: Examples

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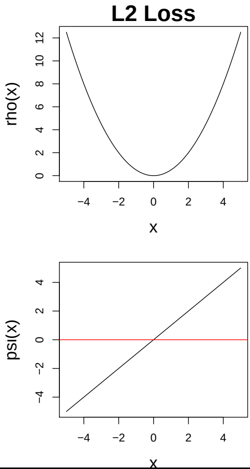

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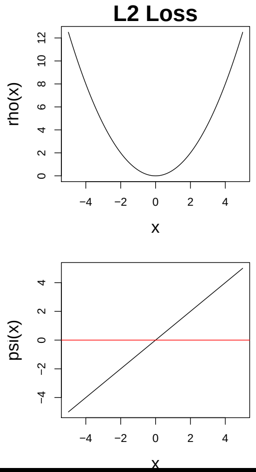

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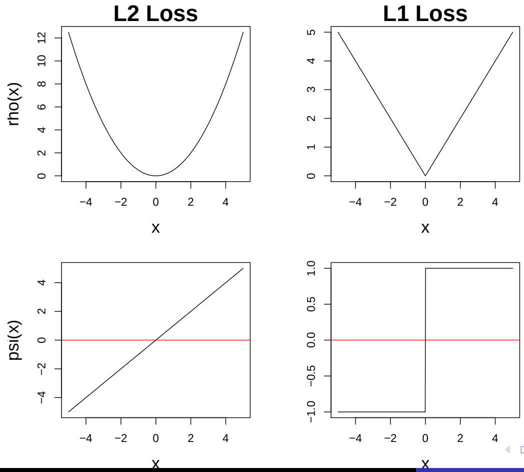

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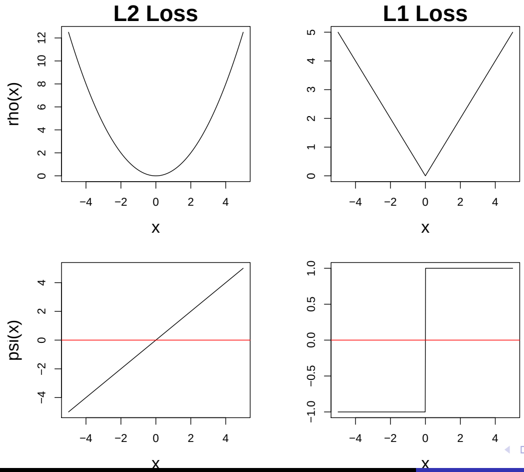
- $\rho(x) = \begin{cases} x^2/2 & |x| \le k \\ k(|x| - k/2) & |x| > k \end{cases}$ gives the Huber estimator.



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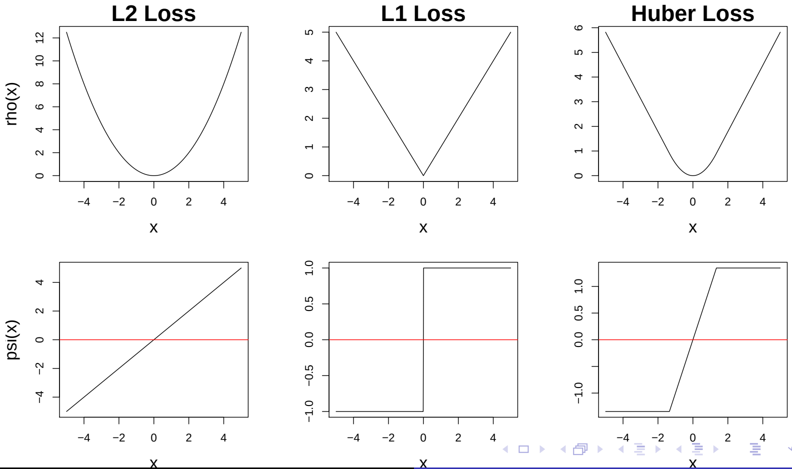

Goals (Informal)

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- the dimension p is **comparable to** the sample size n;
- and X is treated as **fixed**;
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- **without assumptions on** β^*.

- Consider β_1^* WLOG;
- Given X and $\mathcal{L}(\epsilon)$, $\mathcal{L}(\hat{\beta}_1)$ is uniquely determined;
- Ideally, we construct a 95% confidence interval for β_1^* as

$$\left[q_{0.025}\left(\mathcal{L}(\hat{\beta}_1)\right), q_{0.975}\left(\mathcal{L}(\hat{\beta}_1)\right)\right]$$

  where q_α denotes the α-th quantile;
- Unfortunately, $\mathcal{L}(\hat{\beta}_1)$ is complicated.

Asymptotic Arguments

Exact finite sample inference is hard. This motivates statisticians to resort to asymptotic arguments, i.e. find a distribution F s.t.

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- The limiting behavior of $\hat{\beta}$ when p is fixed, as $n \to \infty$,

$$\mathcal{L}(\hat{\beta}) \to N\left(\beta^*, (X^TX)^{-1}\frac{\mathbb{E}(\psi^2(\epsilon_1))}{[\mathbb{E}\psi'(\epsilon_1)]^2}\right);$$

- As a consequence, we obtain an approximate 95% confidence interval for β_1^*,

$$\left[\hat{\beta}_1 - 1.96\widehat{\mathrm{sd}}(\hat{\beta}_1), \hat{\beta}_1 + 1.96\widehat{\mathrm{sd}}(\hat{\beta}_1)\right]$$

where $\widehat{\mathrm{sd}}(\hat{\beta}_1)$ could be any consistent estimator of the standard deviation.

Asymptotic Arguments

In other words, to approximate $\mathcal{L}(\hat{\beta}_1)$, we consider a sequence of hypothetical problems, indexed by j, where the j-th problem has a sample size $n_j \to \infty$ and a dimension $p_j = p$.

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For j-th problem, denote by $\hat{\beta}^{(j)}$ the corresponding M-estimator, then the previous slide uses

$$\lim_{j \to \infty} \mathcal{L}(\hat{\beta}_1^{(j)}) \text{ to approximate } \mathcal{L}(\hat{\beta}_1).$$

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In general, p_j is not necessarily fixed and can grow to infinity.

Asymptotic Arguments

- Huber (1973) raised the question of understanding the behavior of $\hat{\beta}$ when both n and p tend to infinity;
- Huber (1973) showed the L_2 consistency of $\hat{\beta}$:

$$\|\hat{\beta} - \beta^*\|_2^2 \to 0$$

under the regime

$$\frac{p^3}{n} \to 0;$$

- Portnoy (1984) prove the L_2 consistency of $\hat{\beta}$ under the regime

$$\frac{p \log p}{n} \to 0;$$

Asymptotic Arguments

- Portnoy (1985) showed that $\hat{\beta}$ is **jointly asymptotically normal** under the regime

$$\frac{(p \log n)^{\frac{3}{2}}}{n} \to 0,$$

in the sense that for any sequence of vectors $a_n \in \mathbb{R}^p$,

$$\mathcal{L}\left(\frac{a_n^T(\hat{\beta} - \beta^*)}{\sqrt{\mathrm{Var}(a_n^T \hat{\beta})}}\right) \to N(0, 1)$$

p/n: A Measure of Difficulty

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n/p is **the number of samples per parameter**. Heuristically, a larger n/p would give an easier problem.

p/n: A Measure of Difficulty

Recall that the approximation can be seen as a sequence of hypothetical problems with sample size n_j and dimension p_j. If $n_j/p_j \to \infty$, the problems become increasingly easier as j grows.

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In other words, the hypothetical problem used for approximation is **much easier** than the original problem. Then the approximation accuracy might be compromised.

Instead, we can consider a sequence of hypothetical problems with p_j/n_j fixed to be the same as the original problem, i.e.

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Moderate p/n Regime

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In this case, the **difficulty** of the problem **is fixed**.

Moderate p/n Regime

Formally, we define **Moderate p/n Regime** as

$$p_j/n_j \to \kappa > 0.$$

A typical value for κ is p/n in the original problem.

Moderate p/n Regime: More Informative Asymptotics

Consider a set of small-sample problems where $n = 50$ and $p = n\kappa$ for $\kappa \in \{0.1, \ldots, 0.9\}$. For each pair (n, p),

Step 1 Generate $X \in \mathbb{R}^{n \times p}$ with i.i.d. $N(0, 1)$ entries;

Step 2 Fix $\beta^* = 0$ and sample $Y = \epsilon$ with

$$\epsilon_i \overset{i.i.d.}{\sim} N(0,1) \quad \text{or} \quad \epsilon_i \overset{i.i.d.}{\sim} \mathrm{t}_2;$$

Step 3 Estimate β_1^* by $\hat{\beta}_1$ with a Huber loss;

Step 4 Repeat Step 2 - Step 3 for 100 times and estimate $\mathcal{L}(\hat{\beta}_1)$.

Moderate p/n Regime: More Informative Asymptotics

Now consider two types of approximations:

- **Fixed-p Approx.**: $N = 1000$, $P = p$;
- **Moderate-p/n Approx.**: $N = 1000$, $P = 1000\kappa$;

Repeat Step 1-Step 4 for new pairs (N, P) and estimate

- $\mathcal{L}(\hat{\beta}_1^F)$ (Fixed p);
- $\mathcal{L}(\hat{\beta}_1^M)$ (Moderate p/n).

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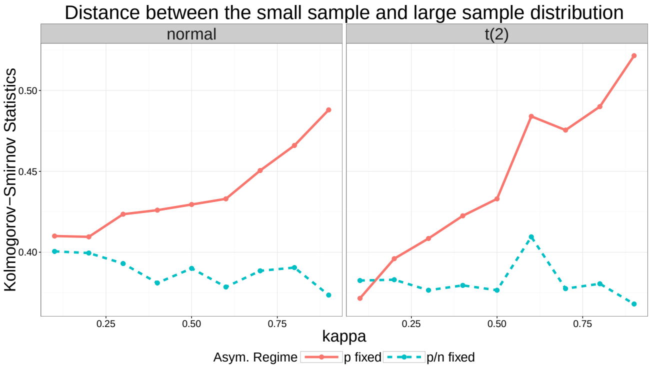
Measure the accuracy of two approximations by the Kolmogorov-Smirnov statistics

$$d_{KS}\left(\mathcal{L}(\hat{\beta}_1), \mathcal{L}(\hat{\beta}_1^F)\right) \text{ and } d_{KS}\left(\mathcal{L}(\hat{\beta}_1), \mathcal{L}(\hat{\beta}_1^M)\right)$$

Moderate p/n Regime: More Informative Asymptotics



Moderate p/n Regime: Negative Results

The moderate p/n regime has been widely studied in random matrix theory. In statistics:

- Huber (1973) showed that for least-square estimators there always exists a sequence of vectors $a_n \in \mathbb{R}^p$ such that

$$\mathcal{L}\left(\frac{a_n^T(\hat{\beta}^{LS} - \beta^*)}{\sqrt{\mathrm{Var}(a_n^T \hat{\beta}^{LS})}}\right) \not\to N(0,1).$$

- Bickel and Freedman (1982) showed that the bootstrap fails in the Least-Square case and the usual rescaling does not help;
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- Main reason: $\hat{F}_n$, the empirical distribution of the residuals, namely $R_i \triangleq y_i - x_i^T\hat{\beta}$, does not converge to $\mathcal{L}(\epsilon_i)$.

Moderate p/n Regime: Positive Results

If X is assumed to be a random matrix under regularity conditions,

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If X is assumed to be a random matrix under regularity conditions,

- Bean et al. (2013) showed that when X has i.i.d. Gaussian entries, for any sequence of $a_n \in \mathbb{R}^p$

$$\mathcal{L}_{X,\epsilon}\left(\frac{a_n^T(\hat{\beta}-\beta^*)}{\sqrt{\mathrm{Var}_{X,\epsilon}(a_n^T\hat{\beta})}}\right) \to N(0,1);$$

- The above result does not contradict Huber (1973) in that the randomness comes from both X and ϵ;
- El Karoui et al. (2011) showed that for general loss functions,

$$\|\hat{\beta}-\beta^*\|_\infty \to 0.$$

- Under weaker assumptions on X, El Karoui (2015) showed

$$\mathcal{L}_{X,\epsilon}\left(\frac{\hat{\beta}_1(\tau)-\beta_1^*-\mathrm{bias}(\hat{\beta}_1(\tau))}{\sqrt{\mathrm{Var}_{X,\epsilon}(\hat{\beta}_1(\tau))}}\right) \to N(0,1)$$

where $\hat{\beta}_1(\tau)$ is the ridge-penalized M-estimator.

- Provides a more accurate approximation of $\mathcal{L}(\hat{\beta}_1)$;

Moderate p/n Regime: Summary

- Provides a more accurate approximation of $\mathcal{L}(\hat{\beta}_1)$;
- Qualitatively different from the classical regimes where $p/n \to 0$;
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  - L_2-consistency of $\hat{\beta}$ no longer holds;
  - the residuals R_i behaves differently from ϵ_i;
  - fixed design results are different from random design results.
- Inference on the vector $\hat{\beta}$ is hard; but inference on the coordinate / low-dimensional linear contrasts of $\hat{\beta}$ is still possible.

Goals (Formal)

Our Goal (formal): Under the **linear model**

$$Y = X\beta^* + \epsilon,$$

Derive the asymptotic distribution of **coordinates** $\hat{\beta}_j$:

- under the **moderate p/n regime**, i.e. $p/n \to \kappa \in (0, 1)$;
- with a **fixed design** matrix X;
- **without assumptions on** β^*.

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Main Result (Informal)

Definition 1.

Let P and Q be two distributions on $\mathbb{R}^p$,

$$d_{\mathrm{TV}}(P, Q) = \sup_{A \subset \mathbb{R}^p} |P(A) - Q(A)|.$$

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Theorem.

Under appropriate conditions on the design matrix X, the distribution of ϵ and the loss function ρ, as $p/n \to \kappa \in (0, 1)$, while $n \to \infty$,

$$\max_j d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}}\right), N(0, 1)\right) = o(1).$$

Examples: Realization of i.i.d. Designs

We consider the case where X is **a realization** of a random design Z. The examples below are proved to **satisfy the technical assumptions with high probability** over Z.

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We consider the case where X is **a realization** of a random design Z. The examples below are proved to **satisfy the technical assumptions with high probability** over Z.

Example 1 Z has i.i.d. mean-zero sub-gaussian entries with $\mathrm{Var}(Z_{ij}) = \tau^2 > 0$;

Example 2 Z contains an intercept term, i.e. $Z = (\mathbf{1}, \tilde{Z})$ and $\tilde{Z} \in \mathbb{R}^{n\times(p-1)}$ has independent sub-gaussian entries with

$$\tilde{Z}_{ij} - \mu_j \stackrel{d}{=} \mu_j - \tilde{Z}_{ij}, \quad \mathrm{Var}(\tilde{Z}_{ij}) > \tau^2$$

for some arbitrary μ_j.

Examples: Realizations of Dependent Gaussian Designs

Example 3 Z is matrix-normal with $\mathrm{vec}(Z) \sim N(0, \Lambda \otimes \Sigma)$ and

$$\lambda_{\max}(\Lambda), \lambda_{\max}(\Sigma) = O\left(1\right), \quad \lambda_{\min}(\Lambda), \lambda_{\min}(\Sigma) = \Omega\left(1\right)$$

Example 4 Z contains an intercept term, i.e. $Z = (\mathbf{1}, \tilde{Z})$ and $\mathrm{vec}(\tilde{Z}) \sim N(0, \Lambda \otimes \Sigma)$ with Λ and Σ satisfy the above condition and

$$\frac{\max_i |(\Lambda^{-\frac{1}{2}}\mathbf{1})_i|}{\min_i |(\Lambda^{-\frac{1}{2}}\mathbf{1})_i|} = O\left(1\right).$$

A Counter-Example

Consider a one-way ANOVA situation. Each observation i is associated with a label $k_i \in \{1, \ldots, p\}$ and let $X_{i,j} = I(j = k_i)$. This is equivalent to

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It is easy to see that

$$\hat{\beta}_j = \arg\min_{\beta \in \mathbb{R}} \sum_{i : k_i = j} \rho(y_i - \beta_j).$$

This is a standard location problem.

A Counter-Example

Let $n_j = |\{i : k_i = j\}|$. In the least-square case, i.e. $\rho(x) = x^2/2$,

$$\hat{\beta}_j = \beta_j^* + \frac{1}{n_j} \sum_{i:k_i=j} \epsilon_i.$$

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Assume a balance design, i.e. $n_j \approx n/p$. Then $n_j \ll \infty$ and

- none of $\hat{\beta}_j$ is normal (unless ϵ_i are normal);
- holds for general loss functions ρ.

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Conclusion: some “non-standard” assumptions on X are required.

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Least Square Estimator

The L_2 loss, $\rho(x) = x^2/2$, gives the least-square estimator

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Let e_j denote the canonical basis vector in $\mathbb{R}^p$, then

$$\hat{\beta}_j^{LS} - \beta_j^* = e_j^T (X^T X)^{-1} X^T \epsilon.$$

Write $e_j^T (X^T X)^{-1} X^T$ as α_j^T, then

$$\hat{\beta}_j^{LS} - \beta_j^* = \sum_{i=1}^{n} \alpha_{j,i} \epsilon_i.$$

Least Square Estimator

Lindeberg-Feller CLT claims that in order for

$$\mathcal{L}\left(\frac{\hat{\beta}_j^{LS} - \beta_j^*}{\sqrt{\mathrm{Var}(\hat{\beta}_j^{LS})}}\right) \to N(0,1)$$

it is **sufficient and almost necessary** that

$$\frac{\|\alpha_j\|_\infty}{\|\alpha_j\|_2} \to 0. \tag{1}$$

Least Square Estimator

To see the necessity of the condition, recall the one-way ANOVA case. Let $n_j = |\{i : k_i = j\}|$, then

$$X^T X = \mathrm{diag}(n_j)_{j=1}^{p}.$$

This gives

$$\alpha_{j,i} = \begin{cases} \frac{1}{n_j} & \text{if } k_i = j \\ 0 & \text{if } k_i \neq j \end{cases}$$

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This gives

$$\alpha_{j,i} = \begin{cases} \frac{1}{n_j} & \text{if } k_i = j \\ 0 & \text{if } k_i \neq j \end{cases}$$

As a result, $\|\alpha_j\|_\infty = \frac{1}{n_j}, \|\alpha_j\|_2 = \frac{1}{\sqrt{n_j}}$ and hence

$$\frac{\|\alpha_j\|_\infty}{\|\alpha_j\|_2} = \frac{1}{\sqrt{n_j}}$$

However, in moderate p/n regime, there exists j such that $n_j \leq 1/\kappa$ and thus $\hat{\beta}_j^{LS}$ is not asymptotically normal.

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Let $\psi = \rho'$, it is the solution of

$$\frac{1}{n}\sum_{i=1}^{n}\psi(y_i - x_i^T\hat{\beta}) = 0$$

WLOG, assume $\beta^* = 0$, then

$$\frac{1}{n}\sum_{i=1}^{n}\psi(\epsilon_i - x_i^T\hat{\beta}) = 0.$$

M-Estimator

Write R_i for $\epsilon_i - x_i^T\hat{\beta}$ and define $D, \tilde{D}$ and G as

$$D = \mathrm{diag}(\psi'(R_i)), \tilde{D} = \mathrm{diag}(\psi''(R_i)), G = I - X(X^TDX)^{-1}X^TD.$$

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Lemma 2.

Suppose $\psi \in C^2(\mathbb{R}^n)$, *then*

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon^T} = e_j^T (X^T DX)^{-1} X^T D, \tag{2}$$

$$\frac{\partial \hat{\beta}_j}{\partial \epsilon \partial \epsilon^T} = G^T \mathrm{diag}(e_j^T (X^T DX)^{-1} X^T \tilde{D}) G. \tag{3}$$

Second-Order Poincaré Inequality

$\hat{\beta}_j$ is a smooth transform of a random vector, ϵ, with independent entries. A powerful CLT for this type of statistics is Second-Order Poincaré Inequality (Chatterjee, 2009).

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Definition 3.

For each $c_1, c_2 > 0$, let $L(c_1, c_2)$ be the class of probability measures on $\mathbb{R}$ that arise as laws of random variables like $u(W)$, where $W \sim N(0, 1)$ and $u \in C^2(\mathbb{R}^n)$ with

$$|u'(x)| \leq c_1 \text{ and } |u''(x)| \leq c_2.$$

For example, $u = \mathrm{Id}$ gives $N(0, 1)$ and $u = \Phi$ gives $U([0, 1])$

Proposition 1 (SOPI; Chatterjee, 2009).

Let $\mathscr{W} = (\mathscr{W}_1, \ldots, \mathscr{W}_n) \overset{indep.}{\sim} L(c_1, c_2)$. *Take any* $g \in C^2(\mathbb{R}^n)$ *and let* $U = g(\mathscr{W})$,

$$\kappa_0 = \left(\mathbb{E}\sum_{i=1}^{n}\left|\nabla_i g(\mathscr{W})\right|^4\right)^{\frac{1}{2}};$$

$$\kappa_1 = (\mathbb{E}\|\nabla g(\mathscr{W})\|_2^4)^{\frac{1}{4}};$$

$$\kappa_2 = (\mathbb{E}\|\nabla^2 g(\mathscr{W})\|_{op}^4)^{\frac{1}{4}}.$$

If U *has a finite fourth moment, then*

$$d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{U - \mathbb{E}U}{\sqrt{\mathrm{Var}(U)}}\right), N(0,1)\right) \preceq \frac{\kappa_0 + \kappa_1\kappa_2}{\mathrm{Var}(U)}.$$

Assume that

A1 $\rho(0) = \psi(0) = 0$ and for any $x \in \mathbb{R}$,

$$0 < K_0 \leq \psi'(x) \leq K_1, \quad |\psi''(x)| \leq K_2;$$

A2 ϵ has independent entries with $\epsilon_i \in L(c_1, c_2)$;

A3 Let λ_+ and λ_- be the largest and smallest eigenvalues of $X^T X / n$ and

$$\lambda_+ = O(1), \quad \lambda_- = \Omega(1).$$

Second-Order Poincaré Inequality on $\hat{\beta}_j$

Apply Second-Order Poincaré Inequality to $\hat{\beta}_j$, we obtain that

Lemma 4.

Let $D = \mathrm{diag}(\psi'(\epsilon_i - x_i^T \hat{\beta}))_{i=1}^n$, *and*

$$M_j = \mathbb{E}\|e_j^T (X^T D X)^{-1} X^T D^{\frac{1}{2}}\|_\infty.$$

Then under assumptions **A***1*-**A***3*,

$$\max_j d_{\mathrm{TV}} \left(\mathcal{L}\left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0,1) \right) = O_p\left(\frac{\max_j (n M_j^2)^{\frac{1}{8}}}{n \cdot \min_j \mathrm{Var}(\hat{\beta}_j)} \right),$$

The main result is obtained if we prove

$$M_j = o\left(\frac{1}{\sqrt{n}}\right), \quad \mathrm{Var}(\hat{\beta}_j) = \Omega\left(\frac{1}{n}\right).$$

Further Assumptions

Define the following quantities:

- **leave-one-predictor-out estimate** $\hat{\beta}_{[j]}$: the M-estimator obtained by removing the j-th column of X (El Karoui, 2013);
- **leave-one-predictor-out residuals** $r_{i,[j]} = \epsilon_i - x_{i,[j]}^T \hat{\beta}_{[j]}$ where $x_{i,[j]}^T$ is the i-th row of X after removing j-th entry;
- $h_{j,0} = (\psi(r_{1,[j]}), \ldots, \psi(r_{n,[j]}))^T$;
- $Q_j = \mathrm{Cov}(h_{j,0})$ be the covariance matrix of $\psi(r_{i,[j]})$.

Further Assumptions

Besides assumptions **A**1 - **A**3, we assume that

A4 $\min_j \frac{X_j^T Q_j X_j}{\mathrm{tr}(Q_j)} = \Omega(1)$.

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A4 $\min_j \frac{X_j^T Q_j X_j}{\mathrm{tr}(Q_j)} = \Omega(1)$.

- Q_j does not involve X_j;
- Assumption **A**4 guarantees

$$\mathrm{Var}(\hat{\beta}_j) = \Omega\left(\frac{1}{n}\right).$$

Further Assumptions

If X_j is a realization of a random vector Z_j with i.i.d. entries, then

$$\mathbb{E}Z_j^T Q_j Z_j = \mathrm{tr}(\mathbb{E}Z_j Z_j^T Q_j) = \mathbb{E}Z_{1,j}^2 \cdot \mathrm{tr}(Q_j).$$

If $Z_j^T Q_j Z_j$ concentrates around its mean, then

$$\frac{Z_j^T Q_j Z_j}{\mathrm{tr}(Q_j)} \approx \mathbb{E}Z_{1,j}^2 > 0.$$

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For example, when Z_j has i.i.d. sub-gaussian entries, the Hansen-Wright inequality implies the concentration.

$$P(|Z_j^T Q_j Z_j - \mathbb{E}Z_j^T Q_j Z_j| \geq t) \leq 2\exp\left\{-c\min\left\{\frac{t^2}{\|Q_j\|_F^2}, \frac{t}{\|Q_j\|_{op}}\right\}\right\}.$$

Further Assumptions

To describe the last assumption, we define the following quantities:

- $D_{[j]} = \mathrm{diag}(\psi'(r_{i,[j]}))$: leave-one-predictor-out version of D;
- $G_{[j]} = I - X_{[j]}(X_{[j]}^T D_{[j]} X_{[j]})^{-1} X_{[j]}^T D_{[j]}$;
- $h_{j,1,i}^T = e_i^T G_{[j]}$: the i-th row of $G_{[j]}$;
- $$\Delta_C = \max\left\{\max_j \frac{|h_{j,0}^T X_j|}{\|h_{j,0}\|_2}, \max_{i,j} \frac{|h_{j,1,i}^T X_j|}{\|h_{j,1,i}\|_2}\right\}.$$

Further Assumptions

The last assumption:

A5 $\mathbb{E}\Delta_C^8 = O\left(\mathrm{polyLog(n)}\right)$.

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A5 $\mathbb{E}\Delta_C^8 = O\left(\mathrm{polyLog(n)}\right)$.

It turns out that when $\rho(x) = x^2/2$,

$$\Delta_C \approx \max_j \frac{\|e_j^T(X^TX)^{-1}X^T\|_\infty}{\|e_j^T(X^TX)^{-1}X^T\|_2}.$$

Recall that for Least-Squares, $\hat{\beta}_j$ are all asymptotically normal iff the right-handed side tends to 0. This indicates that the assumption **A**5 is **not just an artifact of the proof**.

Let

$$\alpha_{j,0} = h_{j,0}/\|h_{j,0}\|_2, \quad \alpha_{j,1,i} = h_{j,1,i}/\|h_{j,1,i}\|_2.$$

Again, if X_j is a realization of a random vector Z_j with i.i.d. σ^2-sub-gaussian entries, then $\alpha_{j,0}^T Z_j$ and $\alpha_{j,1,i}^T Z_j$ are all σ^2-sub-gaussian.

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Again, if X_j is a realization of a random vector Z_j with i.i.d. σ^2-sub-gaussian entries, then $\alpha_{j,0}^T Z_j$ and $\alpha_{j,1,i}^T Z_j$ are all σ^2-sub-gaussian.

Then Δ_C is the maximum of $np + p$ sub-gaussian random variables and hence

$$\mathbb{E}\Delta_C^8 = O(\mathrm{polyLog(n)}).$$

Review of All Assumptions

A1 $\rho(0) = \psi(0) = 0$ and for any $x \in \mathbb{R}$,

$$0 < K_0 \leq \psi'(x) \leq K_1, \quad |\psi''(x)| \leq K_2;$$

A2 ϵ has independent entries with $\epsilon_i \in L(c_1, c_2)$;

A3 Let λ_+ and λ_- be the largest and smallest eigenvalues of $X^T X/n$ and

$$\lambda_+ = O(1), \quad \lambda_- = \Omega(1).$$

A4 $\min_j \frac{Z_j^T Q_j Z_j}{\mathrm{tr}(Q_j)} = \Omega\left(1\right)$.

A5 $\mathbb{E}\Delta_C^8 = O\left(\mathrm{polyLog(n)}\right)$.

Main Results

Theorem 5.

Under assumptions $\mathbf{A}1 - \mathbf{A}5$*, as* $p/n \to \kappa$ *for some* $\kappa \in (0,1)$ *while* $n \to \infty$*,*

$$\max_j d_{\mathrm{TV}}\left(\mathcal{L}\left(\frac{\hat{\beta}_j - \mathbb{E}\hat{\beta}_j}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}}\right), N(0,1)\right) = o(1).$$

A Corollary

If further assume that

A6 ρ is an even function and $\epsilon_i \stackrel{d}{=} -\epsilon_i$.

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Theorem 6.

Under assumptions $\mathbf{A}1 - \mathbf{A}6$*, as* $p/n \to \kappa$ *for some* $\kappa \in (0,1)$ *while* $n \to \infty$,

$$\max_j d_{\mathrm{TV}} \left(\mathcal{L}\left(\frac{\hat{\beta}_j - \beta_j^*}{\sqrt{\mathrm{Var}(\hat{\beta}_j)}} \right), N(0,1) \right) = o(1),$$

Table of Contents

Setup

Design matrix X:

- (i.i.d. design): $X_{ij} \overset{i.i.d.}{\sim} F$;
- (partial Hadamard design): a matrix formed by a random set of p columns of a $n \times n$ Hadamard matrix.

Entry Distribution F:

- $F = N(0, 1)$;
- $F = \mathrm{t}_2$.

Error Distribution $\mathcal{L}(\epsilon)$: ϵ_i are i.i.d. with

- $\epsilon_i \sim N(0, 1)$;
- $\epsilon_i \sim \mathrm{t}_2$.

Setup

Sample Size n: $\{100, 200, 400, 800\}$;

$\kappa = \mathbf{p}/\mathbf{n}$: $\{0.5, 0.8\}$;

Loss Function ρ: Huber loss with $k = 1.345$,

$$\rho(x) = \begin{cases} \frac{1}{2}x^2 & |x| \le k \\ kx - \frac{k^2}{2} & |x| > k \end{cases}$$

Asymptotic Normality of A Single Coordinate

For each set of parameters, we run 50 simulations with each consisting of the following steps:

(Step 1) Generate one design matrix X;

(Step 2) Generate the 300 error vectors ϵ;

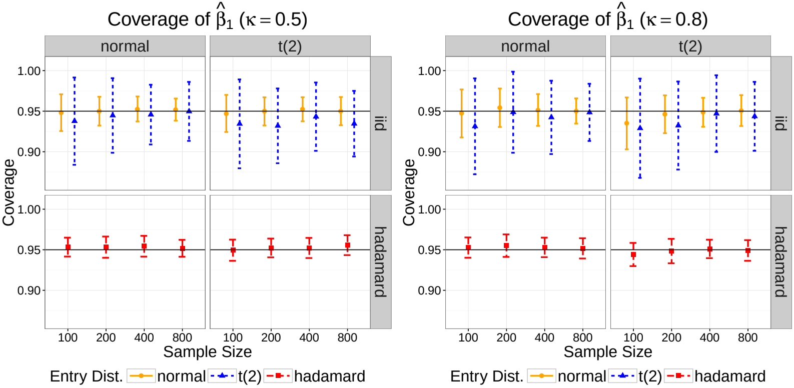
(Step 3) Regress each $Y = \epsilon$ on the design matrix X and end up with 300 random samples of $\hat{\beta}_1$, denoted by $\hat{\beta}_1^{(1)}, \ldots, \hat{\beta}_1^{(300)}$;

(Step 4) Estimate the standard deviation of $\hat{\beta}_1$ by the sample standard error $\hat{\mathrm{sd}}$;

(Step 5) Construct a confidence interval $\mathcal{I}^{(k)} = \left[\hat{\beta}_1^{(k)} - 1.96 \cdot \hat{\mathrm{sd}}, \hat{\beta}_1^{(k)} + 1.96 \cdot \hat{\mathrm{sd}}\right]$ for each $k = 1, \ldots, 300$;

(Step 6) Calculate the empirical 95% coverage by the proportion of confidence intervals which cover the true $\beta_1^* = 0$.

Asymptotic Normality of A Single Coordinate



Conclusion

- We establish the **coordinate-wise asymptotic normality** of the M-estimator for certain **fixed design matrices** under the **moderate p/n regime** under regularity conditions on $X, \mathcal{L}(\epsilon)$ and ρ but **no condition on** β^*;
- We prove the result by using the novel approach Second-Order Poincaré Inequality (Chatterjee, 2009);
- We show that the regularity conditions are satisfied by a broad class of designs.

Future Works

Future works for this project:

- Estimate $\mathrm{Var}(\hat{\beta}_j)$
- Relax the assumptions on $\mathcal{L}(\epsilon)$
- Relax the strong convexity of ρ
- Extend the results to GLM

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Future works for my dissertation:

- Distributional properties in high dimensions
- Resampling methods in high dimensions

Thank You!

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