Stochastically Controlled Stochastic Gradient (SCSG) Method

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joint works with Cheng Ju, Jianbo Chen and Michael Jordan

March 14, UC Davis

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Herbert Robbins (1915-2001)

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A STOCHASTIC APPROXIMATION METHOD[1]

By Herbert Robbins and Sutton Monro

University of North Carolina

1. Summary. Let $M(x)$ denote the expected value at level x of the response to a certain experiment. $M(x)$ is assumed to be a monotone function of x but is unknown to the experimenter, and it is desired to find the solution $x = \theta$ of the equation $M(x) = \alpha$, where α is a given constant. We give a method for making successive experiments at levels $x_1, x_2, \cdots$ in such a way that x_n will tend to θ in probability.

Working Horse of Modern Machine Learning

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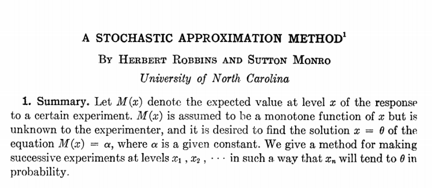

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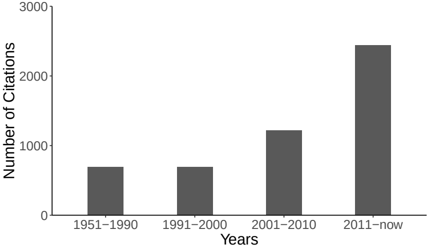
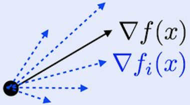


Robbins-Monro Algorithm/ Stochastic Gradient Descent

Finite sums

$$f(x) \stackrel{\text{def.}}{=} \frac{1}{n}\sum_{i=1}^{n} f_i(x)$$
$$\nabla f(x) = \frac{1}{n}\sum_i \nabla f_i(x)$$

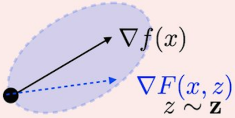


Draw $i \in \{1, \dots, n\}$ uniformly.

$$x_{k+1} = x_k - \tau_k \nabla f_i(x_k)$$

Expectation

$$f(x) \stackrel{\text{def.}}{=} \mathbb{E}_{\mathbf{z}}(f(x, \mathbf{z}))$$
$$\nabla f(x) = \mathbb{E}_{\mathbf{z}}(\nabla F(x, \mathbf{z}))$$



Draw $z \sim \mathbf{z}$

$$x_{k+1} = x_k - \tau_k \nabla F(x, z)$$

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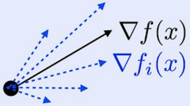

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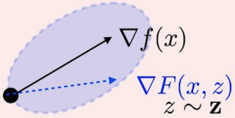

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Theorem 1 (Robbins and Monro, 1951).

Let $\sum_k \tau_k = \infty, \sum_k \tau_k^2 < \infty$. *Then under technical conditions,*

$$x_k \stackrel{a.s.}{\rightarrow} \arg\min f(x)$$

Optimization in Machine Learning

Assume $(y_i, z_i) \overset{i.i.d.}{\sim} G$. The goal is to learn a map $h(\cdot; x)$ from a function class parametrized by $x \in \mathbb{R}^d$, such that $h(z; x)$ is a good "guess" of y.

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Empirical Risk Minimization

$$\min_x \hat{f}(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, h(z_i; x))$$

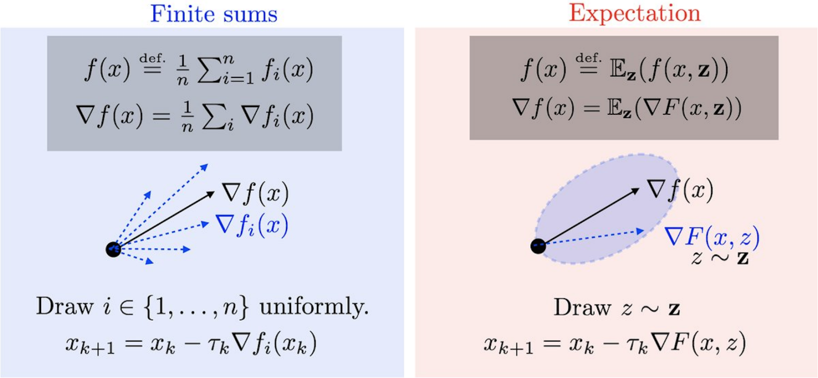
- batch learning;
- observed objective;
- training loss.

Stochastic Optimization

$$\min_x f(x) \triangleq \mathbb{E}_G \, \ell(Y, h(Z; x)).$$

- online/streaming learning;
- unobserved objective;
- testing loss.

Finite-Sum Optimization V.S. Stochastic Optimization



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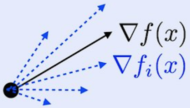

Draw $i \in \{1, \dots, n\}$ uniformly.

$$x_{k+1} = x_k - \tau_k \nabla f_i(x_k)$$

- can access each data for multiple times;
- full gradients can be computed with finite cost

Expectation

$$f(x) \stackrel{\text{def.}}{=} \mathbb{E}_{\mathbf{z}}(f(x, \mathbf{z}))$$

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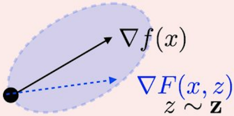

Draw $z \sim \mathbf{z}$

$$x_{k+1} = x_k - \tau_k \nabla F(x, z)$$

- must access a "fresh" sample at each step;
- full gradients cannot be computed with finite cost

Finite-Sum Optimization V.S. Stochastic Optimization

- Finite-sum optimization can be regarded as a special case of stochastic optimization:

$$\frac{1}{n}\sum_{i=1}^{n} f_i(x) = \mathbb{E}_{z\sim U([n])} f_z(x);$$

- Any algorithm that works for stochastic optimization also works for finite-sum optimization, with same complexity.

Finite-Sum Optimization V.S. Stochastic Optimization

- Finite-sum optimization has more structure and more applications than stochastic optimization;
- (y_i, z_i) are not i.i.d. or even not random:
  - ubiquitous in statistical inference for fixed designs;
  - stochastic optimization even not defined
- objective involving pairwise comparison:
  - $$\begin{aligned} f(x) &= \mathbb{E}F(x; (y_1, z_1), (y_2, z_2)) \\ &\approx \frac{1}{n(n-1)} \sum_{i \neq j} F(x; (y_i, z_i), (y_j, z_j)) \end{aligned}$$
  - metric learning, preference elicitation, sport analysis...

SGD: A Brief Overview

Algorithm (for finite-sum optimization and stochastic optimization):

$$\mathrm{SGD}: \; x_{t+1} = x_t - \eta_t g_t, \quad \mathbb{E} g_t = \nabla f(x_t)$$

Main assumption (smoothness):

$$\mu I \preceq \nabla^2 f(x) \preceq LI, \quad (L > 0)$$

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- strongly convex $(\mu > 0, \kappa = L/\mu)$;
- non-strongly convex $(\mu = 0)$;
- non-convex: $(\mu = -L)$.

type of objective	η_t	goal	complexity
strongly convex	$O\left(\frac{1}{\mu t}\right)$	$\mathbb{E}(f(x) - f(x^*)) \leq \epsilon$	$O\left(\frac{1}{\mu \epsilon}\right)$
convex	$O\left(\frac{1}{\sqrt{t}}\right)$	$\mathbb{E}(f(x) - f(x^*)) \leq \epsilon$	$O\left(\frac{1}{\epsilon^2}\right)$
non-convex	$O\left(\frac{1}{\sqrt{t}}\right)$	$\mathbb{E}\|\nabla f(x)\|^2 \leq \epsilon$	$O\left(\frac{1}{\epsilon^2}\right)$

SVRG: A Brief Overview

Algorithms (for finite-sum optimization):

$\mathrm{SAG}, \mathrm{SAGA}, \mathrm{SVRG}, \mathrm{SDCA}, \mathrm{APCG}, \mathrm{SPDC}, \mathrm{Katyusha}, \mathrm{Natasha} \ldots$

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SGD v.s. SVRG: An Brief Comparison

type of objective	complexity (SGD)	complexity (SVRG)
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- SVRG only works for finite-sums while SGD works for both;

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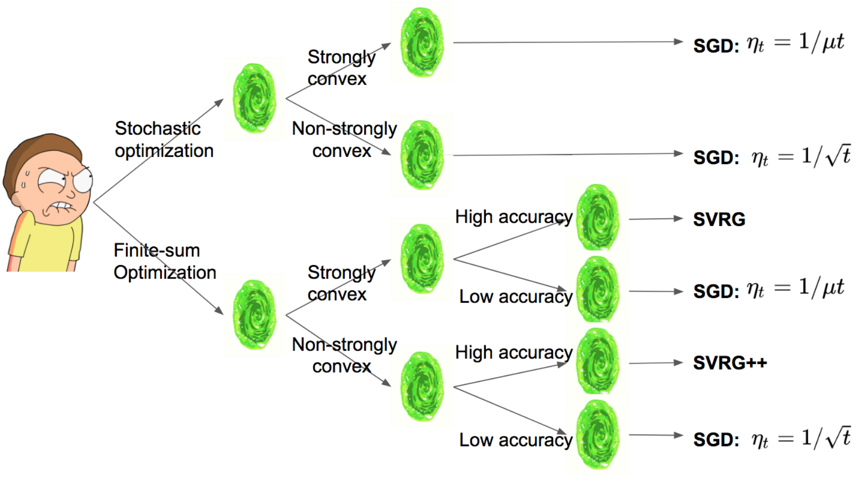
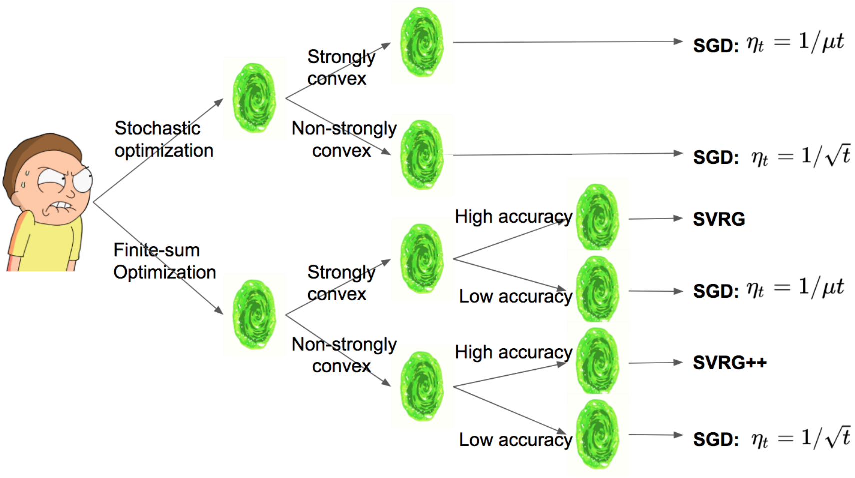
- SVRG only works for finite-sums while SGD works for both;
- Both SGD and SVRG need different settings for strongly/non-strongly/non-convex objectives;
- SVRG has better dependence on ϵ but may be worse than SGD for low accuracy computation where $\frac{1}{\epsilon} \ll n$.

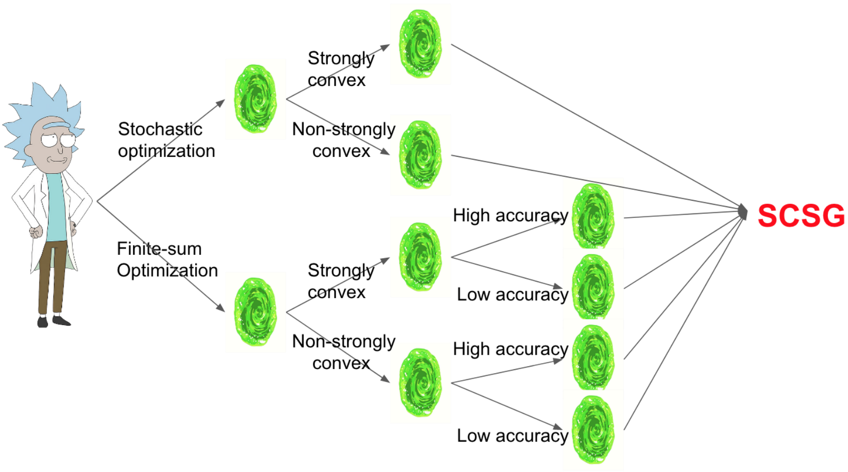

Finite-Sum optimization
Stochastic optimization
Strongly Convex
Non-strongly Convex
High Accuracy
Low Accuracy



Stochastic optimization
Finite-sum Optimization
Strongly convex
Non-strongly convex
Strongly convex
Non-strongly convex
High accuracy
Low accuracy
High accuracy
Low accuracy
SGD: $\eta_t = 1/\mu t$
SGD: $\eta_t = 1/\sqrt{t}$
SVRG
SGD: $\eta_t = 1/\mu t$
SVRG++
SGD: $\eta_t = 1/\sqrt{t}$



Oh Geez, why is life so complicated ?


Stochastic optimization
Finite-sum Optimization
Strongly convex
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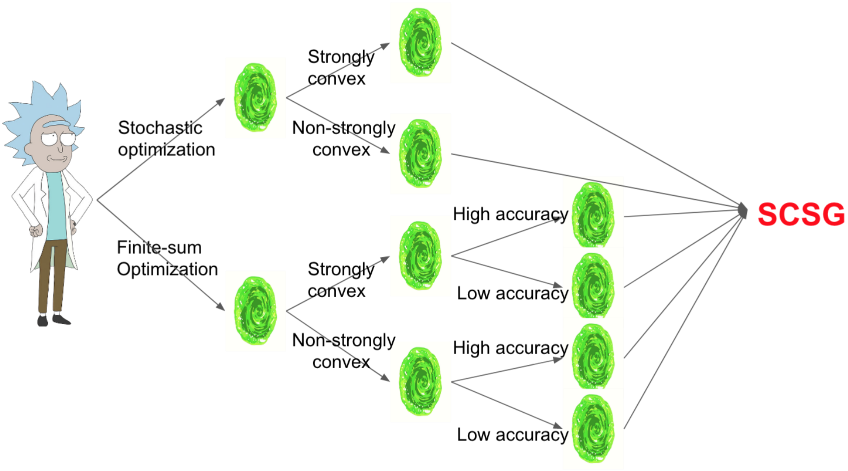
Strongly convex
Non-strongly convex
High accuracy
Low accuracy
High accuracy
Low accuracy
SCSG



Hey Morty, let's adventure in the new world!

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Stochastic Variance Reduced Gradient (SVRG) Method

SGD with constant stepsize:

$$x_{t+1} = x_t - \eta g_t, \quad \mathbb{E} g_t = \nabla f(x_t).$$

It does not converge because $\mathrm{Var}(x_{t+1} - x_t) = \eta^2 \, \mathrm{Var}(g_t) \not\to 0$.

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Idea: find an extra term h_t with

$$x_{t+1} = x_t - \eta(g_t - h_t), \quad \mathbb{E} h_t = 0, \quad \mathrm{Var}(g_t - h_t) \to 0.$$

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$$x_{t+1} = x_t - \eta(g_t - h_t), \quad \mathbb{E} h_t = 0, \quad \mathrm{Var}(g_t - h_t) \to 0.$$

SVRG: $h_t = g_{t'} - \mathbb{E} g_{t'}$ for some $t' \leq t$. Then

$$g_t - g_{t'} \to 0, \quad t, t' \to \infty.$$

SVRG

Consider finite-sum optimization:

$$\min_{x \in \mathbb{R}^d} f(x) = \frac{1}{n} \sum_{i \in [n]} f_i(x)$$

SVRG (Outer Loop)

Inputs: $\tilde{x}_0, \{\eta_j\}, \{m_j\}, T$

for $j = 1, 2, \cdots, T$ **do**
  $\tilde{x}_j \leftarrow$ **SVRGEpoch**$(\tilde{x}_{j-1}, \eta_j, m_j)$
end for

Output: $\tilde{x}_T$

SVRGEpoch (Inner Loop)

Inputs: x_0, η, m

$g \leftarrow \frac{1}{n} \sum_{i \in [n]} f_i'(x_0)$
Generate $N \sim U([m])$
for $k = 1, 2, \cdots, N$ **do**
  Randomly pick $i \in [n]$
  $\nu \leftarrow f_i'(x) - f_i'(x_0) + g$
  $x \leftarrow x - \eta\nu$
end for

Output: x

SVRG and Its Variants

type	algorithm	η_j	m_j	complexity
strongly convex	[JZ13]	$O\left(\frac{1}{L}\right)$	$O(\kappa)$	$O\left((n+\kappa)\log\left(\frac{1}{\epsilon}\right)\right)$
convex	[AZY15]	$O\left(\frac{1}{L}\right)$	2^j	$O\left(n\log\left(\frac{1}{\epsilon}\right)+\frac{1}{\epsilon}\right)$
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Theoretical concerns:

- SVRG does not work for stochastic optimization, in which the full gradient is inaccessible;
- SVRG outperforms SGD only if ϵ is small;
- SVRG requires the knowledge of κ to achieve the fast rate for strongly-convex objectives.

Practical Concern of SVRG

Computing full gradient is too costly!

SCSG in Finite-Sum Optimization

SVRGEpoch

Inputs: x_0, η, m

$\mathcal{I} \leftarrow [n]$

$g \leftarrow \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} f_i'(x_0)$

Gen. $N \sim U([m])$

for $k = 1, 2, \cdots, N$ **do**

  Randomly pick $i \in [n]$

  $\nu \leftarrow f_i'(x) - f_i'(x_0) + g$

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SCSGEpoch

Inputs: x_0, η, B, m

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Inputs: x_0, η, m

1: $\mathcal{I} \leftarrow [n]$

2: $g \leftarrow \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} f_i'(x_0)$

3: Gen. $N \sim U([m])$

4: **for** $k = 1, 2, \cdots, N$ **do**

5: Randomly pick $i \in [n]$

6: $\nu \leftarrow f_i'(x) - f_i'(x_0) + g$

7: $x \leftarrow x - \eta\nu$

8: **end for**

SCSGEpoch

Inputs: x_0, η, B, m

1: Randomly pick $\mathcal{I}$ with size B

2: $g \leftarrow \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} f_i'(x_0)$

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SCSGEpoch

Inputs: x_0, η, B, m

Randomly pick $\mathcal{I}$ with size B
$g \leftarrow \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} f_i'(x_0)$
Gen. $N \sim \mathrm{Geo}$ with mean m

$$N \sim \mathrm{Geo}(\gamma) \text{ iff } P(N = k) = (1-\gamma)\gamma^k \ (k \geq 0) \Longrightarrow \mathbb{E}N = \frac{\gamma}{1-\gamma}$$

SCSG in Finite-Sum Optimization

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SCSG in Stochastic Optimization

SCSGEpoch (finite-sum)

Obj.: $f(x) = \frac{1}{n}\sum_{i\in[n]} f_i(x)$

Inputs: x_0, η, B, m

Randomly pick $\mathcal{I}$ with size B

$g \leftarrow \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} f_i'(x_0)$

Gen. $N \sim \mathrm{Geo}$ with mean m

for $k = 1, 2, \cdots, N$ **do**

  Randomly pick $i \in [n]$

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  $x \leftarrow x - \eta\nu$

end for

SCSGEpoch (expectation)

Obj.: $f(x) = \mathbb{E}_{\xi\sim G} F_\xi(x)$

Inputs: x_0, η, B, m

Gen. $\{\xi_i\}_{i=1}^B \overset{i.i.d.}{\sim} G$

$g \leftarrow \frac{1}{|\mathcal{I}|}\sum_{i=1}^B F_{\xi_i}'(x_0)$

Gen. $N \sim \mathrm{Geo}$ with mean m

for $k = 1, 2, \cdots, N$ **do**

  Gen. $\xi \sim G$

  $\nu \leftarrow F_\xi'(x) - F_\xi'(x_0) + g$

  $x \leftarrow x - \eta\nu$

end for

SCSG: A Brief Summary

In non-convex optimization problems,

- SCSG strictly outperforms SGD in both finite-sum and stochastic optimization, for all accuracy levels;
- SCSG is never worse than SVRG in finite-sum optimization, for all accuracy levels.

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In convex optimization problems,

- SCSG is never worse than SGD and SVRG for all accuracy levels and for both finite-sum and stochastic optimization;
- SCSG does not need the knowledge of μ to achieve the same complexity for strongly convex objectives as SVRG.

Two Techniques

SCSGEpoch

Inputs: x_0, η, B, m

Randomly pick $\mathcal{I}$ with size B — Batching-VR

$g \leftarrow \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} f_i'(x_0)$

Gen. $N \sim \mathrm{Geo}$ with mean m — Geometrization

for $k = 1, 2, \cdots, N$ **do**

  Randomly pick $i \in [n]$

  $\nu \leftarrow f_i'(x) - f_i'(x_0) + g$

  $x \leftarrow x - \eta\nu$

end for

Two Techniques

Batching-VR

- First considered by [HAV+15]. However the analysis requires $B = O(n)$ and unrealistic assumptions (e.g. bounded domain).
- [HAV+15] only holds for strongly-convex objectives and requires the knowledge of μ;
- Also considered by [FGKS15]. However the analysis relies on stringent assumptions and the algorithm has extremely unrealistic settings.

Geometrization

- Implicitly considered by [HLLJM15] in a special setting. However, the analysis still relies on the strong convexity and does not show the gain.

Batching-VR + Geometrization work!

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Smooth Non-convex Optimization

Finite-Sum Optimization

$$\min_x f(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} f_i(x)$$

$$\text{Goal}: \mathbb{E}\|\nabla f(x)\|^2 \leq \epsilon$$

Assumptions:

A1 $-LI \preceq \nabla^2 f_i(x) \preceq LI$;

A2 $\sup \|\nabla f_i(x)\|^2 = O(1)$.

Complexity Results:

- SGD: $O\left(\frac{1}{\epsilon^2}\right)$;
- SVRG: $O\left(n + \frac{n^{2/3}}{\epsilon}\right)$;
- SCSG: $\tilde{O}\left(\frac{1}{\epsilon^{5/3}} \wedge \frac{n^{2/3}}{\epsilon}\right)$.

Stochastic Optimization

$$\min_x f(x) \triangleq \mathbb{E}_{\xi\sim G}\, F(x;\xi).$$

$$\text{Goal}: \mathbb{E}\|\nabla f(x)\|^2 \leq \epsilon$$

Assumptions:

A1 $-LI \preceq \nabla^2 F(x;\xi) \preceq LI$;

A2 $\sup \|\nabla F(x;\xi)\|^2 = O(1)$.

Complexity Results:

- SGD: $O\left(\frac{1}{\epsilon^2}\right)$;
- SVRG: not available;
- SCSG: $\tilde{O}\left(\frac{1}{\epsilon^{5/3}}\right)$.

Comparison in Finite-Sum Optimization

	General	$\epsilon \sim n^{-1/2}$	$\epsilon \sim n^{-1}$
Gradient Methods			
GD	$O\left(\frac{n}{\epsilon}\right)$	$O\left(n^{3/2}\right)$	$O\left(n^{2}\right)$
Best available	$\tilde{O}\left(\frac{n}{\epsilon^{5/6}}\right)$	$\tilde{O}\left(n^{17/12}\right)$	$\tilde{O}\left(n^{11/6}\right)$
Stochastic Gradient Methods			
SGD	$O\left(\frac{1}{\epsilon^2}\right)$	$O\left(n\right)$	$O\left(n^{2}\right)$
Best available	$O\left(n + \frac{n^{2/3}}{\epsilon}\right)$	$O\left(n^{7/6}\right)$	$O\left(n^{5/3}\right)$
SCSG	$\tilde{O}\left(\frac{1}{\epsilon^{5/3}} \wedge \frac{n^{2/3}}{\epsilon}\right)$	$\tilde{O}\left(n^{5/6}\right)$	$\tilde{O}\left(n^{5/3}\right)$

Parameter Settings in SCSG

SCSG (Outer Loop)

Inputs:
$\tilde{x}_0, \{\eta_j\}, \{B_j\}, \{m_j\}, T$

for $j = 1, 2, \cdots, T$ **do**
    $\tilde{x}_j \leftarrow$ **SCSGEpoch**$(\tilde{x}_{j-1}, \eta_j, B_j, m_j)$
end for

Output: $\tilde{x}_T$

SCSGEpoch (Inner Loop)

Inputs: x_0, η, B, m

Randomly pick $\mathcal{I}$ with size B
$g \leftarrow \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} f_i'(x_0)$
Gen. $N \sim \mathrm{Geo}$ with mean m
for $k = 1, 2, \cdots, N$ **do**
    Randomly pick $i \in [n]$
    $\nu \leftarrow f_i'(x) - f_i'(x_0) + g$
    $x \leftarrow x - \eta\nu$
end for

Ouput: x

Parameter Settings in SCSG

SCSG (Outer Loop)

Inputs:
$\tilde{x}_0, \{\eta_j\}, \{B_j\}, \{m_j\}, T$

1: **for** $j = 1, 2, \cdots, T$ **do**
2: $\quad\tilde{x}_j \leftarrow$ **SCSGEpoch**$(\tilde{x}_{j-1}, \eta_j, B_j, m_j)$
3: **end for**

Output: $\tilde{x}_T$

Parameters:

	option 1	option 2
B_j	$O\left(\frac{1}{\epsilon} \wedge n\right)$	$j^{3/2} \wedge n$
m_j	B_j	B_j
η_j	$\frac{1}{2LB_j^{2/3}}$	$\frac{1}{2LB_j^{2/3}}$

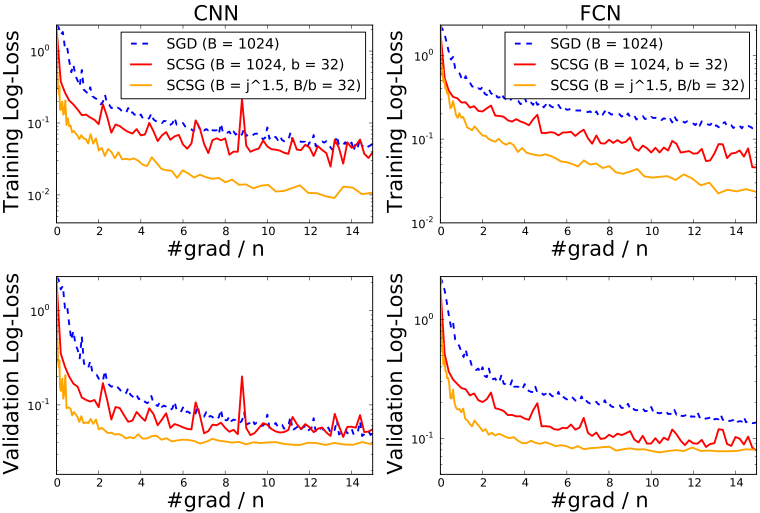
SCSGEpoch (Inner Loop)

Inputs: x_0, η, B, m

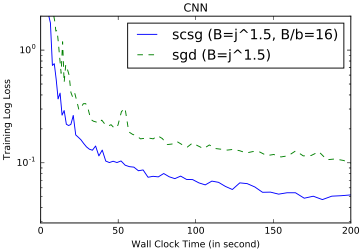
1: Randomly pick $\mathcal{I}$ with size B
2: $g \leftarrow \frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} f_i'(x_0)$
3: Gen. $N \sim \mathrm{Geo}$ with mean m
4: **for** $k = 1, 2, \cdots, N$ **do**
5: $\quad$Randomly pick $i \in [n]$
6: $\quad\nu \leftarrow f_i'(x) - f_i'(x_0) + g$
7: $\quad x \leftarrow x - \eta\nu$
8: **end for**

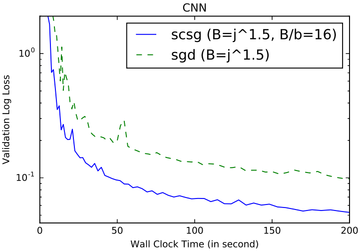
Ouput: x

SCSG for Training Neural Networks



SCSG for Training Neural Networks





Discussion

- Existing acceleration techniques include: **Variance Reduction**, **Momentum**, **Adaptive Gradient**:
  - **Momentum**: Momentum SGD;
  - **Adaptive Gradient**: AdaGrad;
  - **Momentum** + **Adaptive Gradient**: Adam;
  - **Variance Reduction**: SVRG/SAGA, but not in practice!
- The mechanisms of three techniques are different and might be "orthogonal"! Potential gain by combining all:

Variance Reduction + **Momentum** + **Adaptive Gradient**

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Assumption:

$$\mu I \preceq \nabla^2 f_i(x) \preceq LI,\ \ \mu \ge 0$$

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$$\min_x f(x) \triangleq \mathbb{E}_{\xi\sim G}\ F(x;\xi).$$

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Assumption:

$$\mu I \preceq \nabla^2 F(x;\xi) \preceq LI,\ \ \mu \ge 0;$$

Convex Optimization Theory is Weird

SGD (in convex stochastic optimization):

- Always different settings (of stepsizes) for strongly and non-strongly convex objectives:
- $\eta_t = O\left(\frac{1}{\sqrt{t}}\right)$ for non-strongly convex case; complexity $O\left(\frac{1}{\epsilon^2}\right)$;
- $\eta_t = O\left(\frac{1}{\mu t}\right)$ for strongly convex case; complexity $O\left(\frac{1}{\mu\epsilon}\right)$;

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- $\eta_t = O\left(\frac{1}{\mu t}\right)$ for strongly convex case; complexity $O\left(\frac{1}{\mu\epsilon}\right)$;
- Use $\eta_t = O\left(\frac{1}{\sqrt{t}}\right)$ for strongly convex case does not yield the better complexity $O\left(\frac{1}{\mu\epsilon}\right)$ (in general);
- Use $\eta_t = O\left(\frac{1}{\mu t}\right)$ for non-strongly convex case (with a wrong guess of μ) could yield a complexity as bad as $O\left(e^{\frac{1}{\epsilon}}\right)$;
- Users must know the property of the objective and must know μ to take advantage of strong convexity!

Convex Optimization Theory is Weird

SVRG (in convex finite-sum optimization):

- Also different settings for strongly and non-strongly convex objectives:
- Original SVRG only for strongly convex objectives with $m_j \equiv O(\kappa), \eta_j \equiv O\left(\frac{1}{L}\right)$; complexity $\tilde{O}(n+\kappa)$;
- In order to extend SVRG to non-strongly convex objectives,
  - [AZY15]: $m_j = 2^j$; complexity $\tilde{O}\left(n + \frac{1}{\epsilon}\right)$;
  - [RHS$^+$16]: $m_j = O(n), \eta_j = O\left(\frac{1}{L\sqrt{n}}\right)$; complexity $O\left(n + \frac{\sqrt{n}}{\epsilon}\right)$.
- Again, separate analyses for different settings.

Adaptivity Matters!

An popular hand-waving argument of strong convexity:

μ is always known in practice because an L_2 regularizer,
in the form of $\frac{\lambda}{2}\|x\|^2$, is always added so one can set $\mu = \lambda$.

Adaptivity Matters!

An popular hand-waving argument of strong convexity:

μ is always known in practice because an L_2 regularizer,
in the form of $\frac{\lambda}{2}\|x\|^2$, is always added so one can set $\mu = \lambda$.

No!

- λ is usually small, e.g. $\lambda \sim 10^{-6}$, in which case the condition number κ is too large to justify the gain of strong convexity: compare $O\left(\frac{1}{\epsilon^2}\right)$ with $O\left(\frac{10^6}{\epsilon}\right)$;
- λ is too conservative: the global strong convexity parameter might be way larger than λ and the local strong convexity parameter, around the optimum, could be even larger.

Adaptivity Matters!

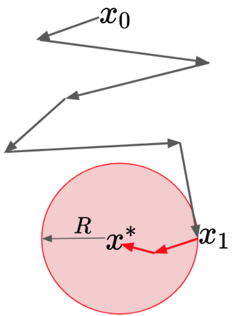
The degree of strong convexity forms a continuum.
An algorithm should depend on μ continuously without knowing it!

Adaptivity Matters!

The degree of strong convexity forms a continuum.
An algorithm should depend on μ continuously without knowing it!

Advantages of adaptive algorithms:

- Unified algorithm for both cases;
- Global adaptivity $\Longrightarrow$ local adaptivity.



Adaptivity Matters!

Knowing μ makes a difference in terms of oracle lower bounds:

- [AS16] proves the lower bound $\Omega\left((n+\sqrt{n\kappa})\log\left(\frac{1}{\epsilon}\right)\right)$, in terms of ϵ, for CLI algorithms in finite-sum optimization, if μ is known;
- [Arj17b] shows that the above bound is not achievable without knowing μ, in which case the lower bound is $O\left((n+\kappa)\log\left(\frac{1}{\epsilon}\right)\right)$, in terms of ϵ.

Existing Works on Adaptivity

- Deterministic gradient method [N+07]:
  - doubling/halving technique;
  - need to check conditions on the norm of gradients at each step, thus not applicable in stochastic algorithms.
- Adaptive SVRG [XLY17]:
  - doubling/halving technique;
  - achieves the complexity $\tilde{O}((n+\kappa)\log\left(\frac{1}{\epsilon}\right))$;
  - need a lower bound for μ and hence no guarantee for non-strongly convex objective;
  - parameters depend on ϵ.
- Hand-waving algorithms:
  - Ad-hoc approaches to adaptively estimate μ; extra overhead may dominate;
  - Restarting schemes; need the knowledge of μ to obtain theoretical guarantee.

Achieving Adaptivity Via SCSG

Randomized SVRG [LJ17]:

- A special case of SCSG;
- $B_j = m_j \equiv n, \eta_j = \frac{1}{3L}$ with complexity
$$\tilde{O}\left(\frac{n}{\epsilon} \wedge (n + \kappa)\right);$$
- need to record both the average (for the former) and the last iterate (for the latter);
- compared to SVRG: $B_j \equiv n, m_j \equiv m = O(\kappa), \eta_j \equiv \eta < \frac{1}{2L}$ with complexity
$$\tilde{O}\left(n + \kappa\right).$$

Achieving Adaptivity Via SCSG

SCSG+ (to appear soon):

- $B_j = B_0 \cdot 1.05^{2j} \wedge n, m_j = m_0 \cdot 1.05^j, \eta_j \equiv \eta = \frac{1}{4L}$ with complexity

$$\tilde{O}\left(\frac{1}{\epsilon^2} \wedge \left(n + \frac{1}{\epsilon}\right) \wedge \left(n + \kappa \left(\frac{1}{\epsilon\kappa}\right)^{0.05}\right)\right);$$

- The extra term $\left(\frac{1}{\epsilon\kappa}\right)^{0.05}$ is almost negligible. In addition, the exponent 0.05 can be made arbitrarily small by shrinking η; roughly $\log(1.05)/\log(1/\eta L)$;
- SGD with $\eta_t = \frac{1}{\sqrt{t}}$ achieves $\tilde{O}\left(\frac{1}{\epsilon^2}\right)$;
  SVRG^{++} achieves $\tilde{O}\left(n + \frac{1}{\epsilon}\right)$;
  SVRG achieves $\tilde{O}(n + \kappa)$ with known μ;
  SCSG almost achieves the best of them, without knowing μ!
- Adaptivity to both strong convexity and required accuracy.

Other Remarks on SCSG

- SGD relies on bounded gradient condition

$$\mathcal{H}^* \triangleq \sup_{i,x} \|\nabla f_i(x)\|^2 = O(1) \text{ (for finite-sum optimization)}$$

or $\mathcal{H}^* \triangleq \sup_{\xi,x} \|\nabla F(x;\xi)\|^2 = O(1)$ (for stochastic optimization).

- Unfortunately this even does not hold for least squares unless the domain is bounded and projection step is performed every step. But nobody uses that in practice!
- SCSG relies on a much weaker condition ($x^* = \arg\min f(x)$)

$$\mathcal{H} \triangleq \sup_{i} \|\nabla f_i(x^*)\|^2 = O(1) \text{ (for finite-sum optimization)}$$

or $\mathcal{H} \triangleq \sup_{\xi} \|\nabla F(x^*;\xi)\|^2 = O(1)$ (for stochastic optimization).

- Extensive discussion of $\mathcal{H}$ in [LJ16].

Other Remarks on SCSG

Refined rate of SCSG+

$$\tilde{O}\Bigg(\left(\frac{D}{\epsilon}\right)^2 \wedge \Bigg(\left(\frac{D_H}{\epsilon}\right)^2 + \kappa^2 \left(\frac{D_x}{\epsilon\kappa}\right)^{0.09} \Bigg) \wedge \left(n + \frac{D}{\epsilon}\right) \wedge \Bigg(n + \kappa \left(\frac{D_H}{\epsilon\kappa}\right)^{0.05}\Bigg) \Bigg);$$

where $\quad D_x = L \cdot \mathbb{E}\|\tilde{x}_0 - x^*\|^2, \quad D_H = \frac{\mathcal{H}}{L}, \quad D = \max\{D_x, D_H\}.$

- D_x measures the quality of initialization; D_H measures heterogeneity of the components;
- D_x is algorithm/user driven while D_H is intrinsic;
- SCSG+ shows adaptivity for large ϵ with more tolerance to bad initialization when $\kappa \ll \frac{1}{\epsilon}$ (same condition for SGD to take advantage of strong convexity).

Optimality?

- [AB14] proves the lower bound $\Omega\left(\frac{1}{\epsilon^2}\right)$;
- [Arj17a] proves the lower bound $\tilde{\Omega}\left(n+\kappa\right)$ for strongly-convex objectives;
- [Arj17a] proves the lower bound $\tilde{\Omega}\left(n+\sqrt{\frac{n}{\epsilon}}\right)$, achieved by Accelerate SDCA on Generalized Linear Models;
- [WS16] proves the lower bound $\Omega\left(\frac{1}{\mu\epsilon}\right)$ for strongly-convex objectives when μ is known.
- My conjecture: $\Omega\left(\frac{1}{\mu\epsilon}\right)$ is not achievable when μ is unknown.

Optimality?

The above results give a (possibly loose) lower bound as

$$\tilde{\Omega}\left(\frac{1}{\epsilon^2} \wedge \frac{1}{\mu\epsilon} \wedge \left(n + \sqrt{\frac{n}{\epsilon}}\right) \wedge (n+\kappa)\right)$$

Recall the bound of SCSG:

$$\tilde{O}\left(\frac{1}{\epsilon^2} \wedge \left(n + \frac{1}{\epsilon}\right) \wedge (n+\kappa)\right)$$

Summary

In non-convex optimization problems,

- SCSG has complexity $\tilde{O}\left(\frac{1}{\epsilon^{5/3}} \wedge \frac{n^{2/3}}{\epsilon}\right)$ to reach an ϵ-approximated first-order stationary point;
- SCSG strictly outperforms SGD, with complexity $O\left(\frac{1}{\epsilon^2}\right)$, in both finite-sum and stochastic optimization, for all accuracy;
- SCSG is never worse than SVRG, with complexity $O\left(n + \frac{n^{2/3}}{\epsilon}\right)$, in stochastic optimization, for all accuracy.

Summary

In convex optimization problems,

- SCSG has complexity $\tilde{O}\left(\frac{1}{\epsilon^2} \wedge \left(n + \frac{1}{\epsilon}\right) \wedge (n + \kappa)\right)$ to reach an ϵ-approximated solution;
- SCSG is never worse than SGD, with complexity and SVRG (SVRG^{++}, ...), for all accuracy and for both finite-sum and stochastic optimization;
- SCSG does not need the knowledge of μ to achieve the same complexity for strongly convex objectives as SVRG.

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THANKS!

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- With William Fithian
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  - knockoffs-based inference;
- With Alex D'amour, Peng Ding, Avi Feller and Jasjeet Sekhon
  - debiasing regression-adjustment in randomized experiments;
  - robust randomized designs;
  - justifying overlap condition in observational studies.