20200922 Homework

1.1. Calculate the entropy.

Let X be a random variable such that $\Pr\{X=i\}=2^{-i}$, $i=1,2,\cdots$, Then

$$H(X) = -\sum_{i=1}^{+\infty} 2^{-i}\log 2^{-i} = \sum_{i=1}^{+\infty} i\cdot 2^{-i} \quad ①$$

$$\frac{1}{2}H(X) = \sum_{i=1}^{+\infty} i\cdot 2^{-(i+1)} = \sum_{i=2}^{+\infty}(i-1)\cdot 2^{-i} \quad ②$$

$$① - ② = \frac{1}{2}H(X) = \frac{1}{2} + \sum_{i=2}^{+\infty} 2^{-i} = \frac{1}{2} + \frac{\frac{1}{4}\left(1-\lim\limits_{i\to+\infty}\frac{1}{2^i}\right)}{1-\frac{1}{2}} = 1$$

$\therefore H(X) = 2$

1.2. Let Y be a random variable which takes values in the subset of pairs of integers $\left\{(i,j): 1\le i<\infty \text{ and } 1\le j\le \frac{2^{2^i}}{2^i}\right\}$ such that $\Pr\{Y=(i,j)\}=2^{-2^i}$.

$$H(Y) = -\sum_{i=1}^{+\infty}\sum_{j=1}^{\frac{2^{2^i}}{2^i}} 2^{-2^i}\log 2^{-2^i} = \sum_{i=1}^{+\infty} 2^{2^i-2^i} = +\infty$$

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2. Try to prove the equivalence of three definitions of the Markov chain with n nodes.

~~① $p(X_1=x_1, X_2=x_2, X_3=x_3, \cdots, X_n=x_n) = p(X_1=x_1)p$~~

① $p(x_1,x_2,\cdots,x_n)p(x_2)p(x_3)\cdots p(x_{n-1}) = p(x_1,x_2)p(x_2,x_3)\cdots p(x_{n-1},x_n)$ symmetric form.

② $p(x_1,x_2,\cdots,x_n) = p(x_1)p(x_2|x_1)p(x_3|x_2)\cdots p(x_n|x_{n-1})$ jumping form

③ $p(x_t|x_1,x_2,\cdots,x_{t-1}) = p(x_t|x_{t-1})$ for $t\in[1,n]$ shortened form.

①⟺②:

将①式第2至第n项移至右边：$p(x_1,\cdots,x_n) = \frac{p(x_1,x_2)}{\cancel{p(x_1)}}\cdot\frac{p(x_2,x_3)}{p(x_2)}\cdots\frac{p(x_{n-1},x_n)}{p(x_{n-1})} = p(x_1)p(x_2|x_1)\cdot p(x_3|x_2)\cdots p(x_n|x_{n-1})$

for $p(x_1)p(x_2)\cdots p(x_{n-1})>0$. 从右到左即可证得②⟹①

②⟺③:

由②有：$p(x_1,\cdots,x_n) = \cancel{p(x_1)}\cdot\frac{p(x_1,x_2)}{\cancel{p(x_1)}}\cdot\frac{p(x_2,x_3)}{p(x_2)}\cdots\frac{p(x_{n-2},x_{n-1})}{p(x_{n-2})}\cdot p(x_n|x_{n-1})$ ----④

$\because X_1\to X_2\to\cdots\to X_{n-1}$ 构成马尔可夫链，由其定义①得 $(n\ge 4)$

④ $= p(x_1,x_2,\cdots,x_{n-1})p(x_n|x_{n-1})$ ----⑤

由④、⑤式即得 $p(x_t|x_1,\cdots,x_{n-1}) = \frac{p(x_1,\cdots,x_n)}{p(x_1,x_2,\cdots,x_{n-1})} = p(x_n|x_{n-1})$. 即③在 $n\ge 4$、$t\ge 4$ 时成立.

当 $t=3$ 时 $p(x_3|x_1,x_2) = \frac{p(x_1,x_2,x_3)}{p(x_1,x_2)} \xrightarrow{X_1\to X_2\to X_3} \frac{p(x_2,x_3)}{p(x_2)} = p(x_3|x_2)$. ③成立.$(t=3)$

当 $t=2$ 时 ③ $= p(x_2|x_1) = p(x_2|x_1)$ 成立.

当 $t=1$ 时 ③ $= p(x_1) = p(x_1)$ 成立.

$\therefore$ ②⟹③ 上述过程在 $p(x_1)p(x_2)\cdots p(x_{n-1})>0$ 时可逆. 故③⟹②

$\therefore$ ①⟺②⟺③

3. If $X_1\to X_2\to X_3\to X_4\to X_5$ is a Markov Chain, please show that $X_5\to X_3\to X_2$ is also a Markov chain by using the 3 equivalent definitions only.

解: 由 symmetric、reversibility: $X_5\to X_4\to X_3\to X_2$

$\therefore p(x_5,x_4,x_3,x_2)p(x_4)p(x_3) = p(x_5,x_4)p(x_4,x_3)p(x_3,x_2)$

$\therefore p(x_5,x_4,x_3,x_2) = p(x_5|x_4)p(x_4|x_3)p(x_3,x_2)$ --- ①

由 shortened form & $X_3\to X_4\to X_5$: $p(x_5|x_4,x_3) = p(x_5|x_4)$.

the right side of ① $= p(x_5|x_4,x_3)p(x_4|x_3)p(x_3,x_2)$

$= p(x_4,x_5|x_3)p(x_3,x_2)$

$\sum_{x_4} p(x_5,x_4,x_3,x_2) = \sum_{x_4} p(x_4,x_5|x_3)p(x_3,x_2)$

$\therefore p(x_5,x_3,x_2) = \frac{p(x_3,x_5)p(x_3,x_2)}{p(x_3)}$ (symmetric form).

$\therefore X_5\to X_3\to X_2$.

4. Present the definitions of

1) $I_1(X;Y|W,S=s,T)$ 2) $I_2(X;Y|W,S=s,T=t)$

3) $I_3(X;Y|W=w,S=s,T=t)$ 4) What are the relationships among the three measurements?

1): $I_1 = \sum_{x,y,w,t} p(x,y,w,t|s)\log\frac{p(x,y|w,s,t)}{p(x|w,s,t)p(y|w,s,t)}$.

2): $I_2 = \sum_{x,y,w} p(x,y,w|s,t)\log\frac{p(x,y|w,s,t)}{p(x|w,s,t)p(y|w,s,t)}$.

3) $I_3 = \sum_{x,y} p(x,y|w,s,t)\log\frac{p(x,y|w,s,t)}{p(x|w,s,t)p(y|w,s,t)}$

4) I_1 is the top level, I_2 is the medium level, I_3 is the buttom level.

$I_1 = \sum_t p(t) I_2 = \sum_t p(t)\left(\sum_w p(w) I_3\right)$

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