# Uncertainty Quantification Final Project

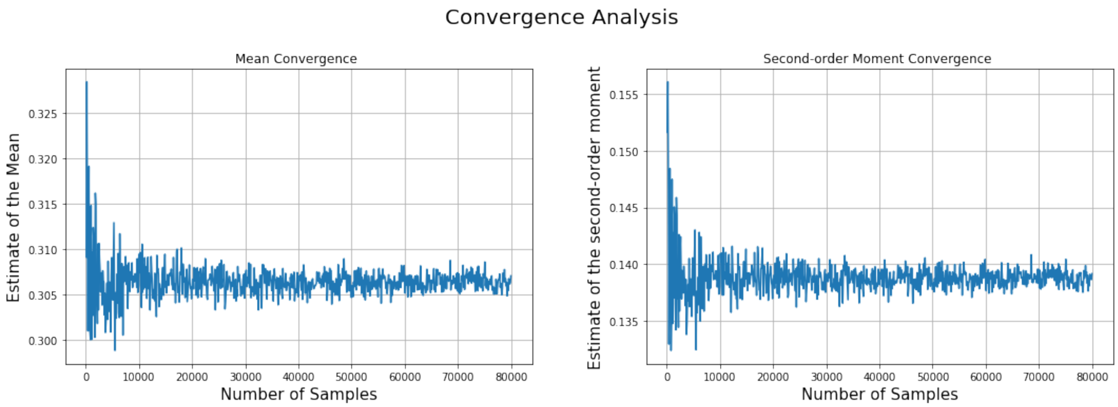
April 28, 2021

```
In [577]: import random
          import numpy as np
          import pandas as pd
          import matplotlib.pyplot as plt
          import seaborn as sns
          from sympy import *
          from sympy.stats import Normal, cdf
          from matplotlib.ticker import MaxNLocator
          import scipy.stats
          pd.set_option('display.expand_frame_repr', False)
```

## 0.1 Monte Carlo Solver

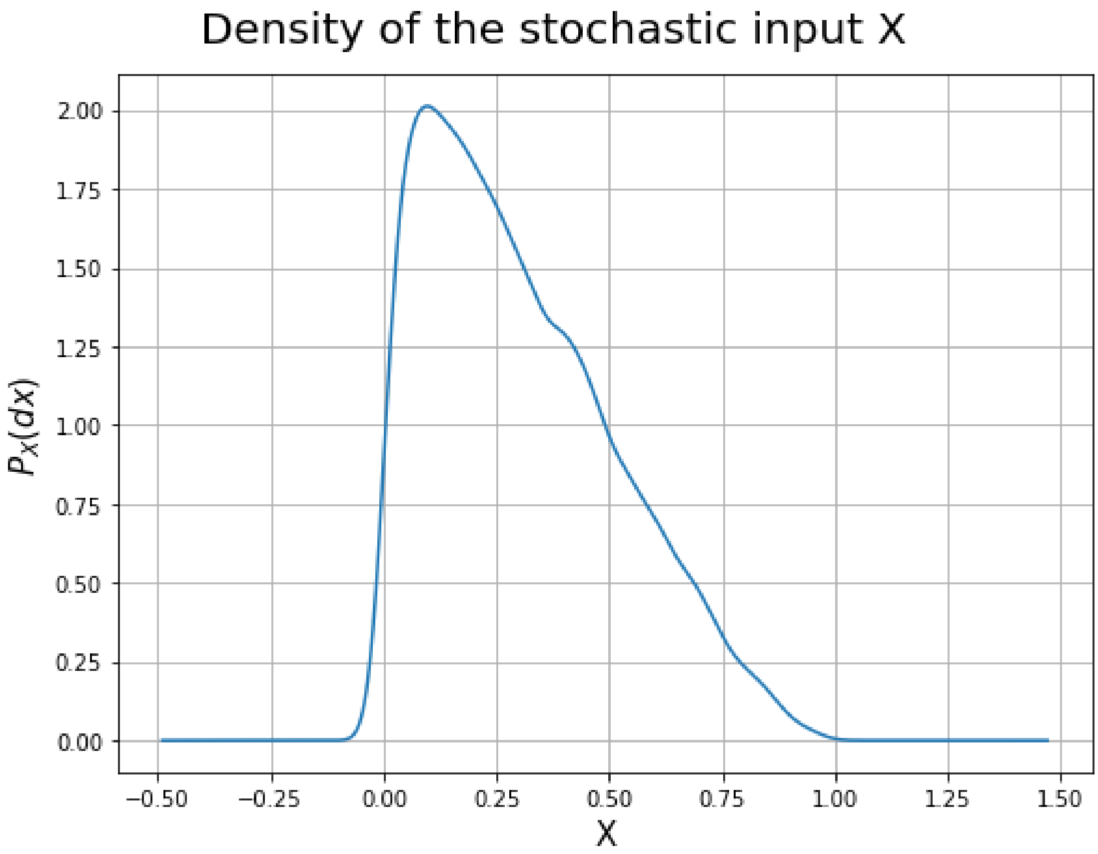
```
In [433]: alpha = 1.1444307615618827
          beta = 2.590929757651513

In [19]: def getbetarandomvars (alpha, beta, number):
             random.seed()
             ans = []
             for i in range(number):
                 ans.append(random.betavariate(alpha, beta))
             return ans

In [55]: CA = pd.DataFrame(columns={"number", "mean", "second order moment"})
         nums = np.linspace (100, 80000, 800).astype(int)
         for ind in range (len(nums)) :
             num = nums[ind]
             rvs = np.array( getbetarandomvars (alpha, beta, num))
             CA.loc [ind] = [num, rvs.mean(), (rvs*rvs).mean()]

In [117]: fig, (ax1, ax2) = plt.subplots(nrows=1, ncols=2,figsize=(16, 6));

          ax1.plot(CA['number'].values, CA['mean'].values);
          ax1.set_title('Mean Convergence ')
          ax1.set_xlabel('Number of Samples',fontsize=15)
          ax1.set_ylabel('Estimate of the Mean',fontsize=15)
          ax1.grid(True)
```

```
ax2.plot(CA['number'].values, CA['second order moment'].values);
ax2.set_title('Second-order Moment Convergence ')
ax2.set_xlabel('Number of Samples',fontsize=15)
ax2.set_ylabel('Estimate of the second-order moment',fontsize=15)
ax2.grid(True)

fig.suptitle('Convergence Analysis',fontsize=20)
fig.tight_layout(pad=5.0)
fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\convergence ar
```

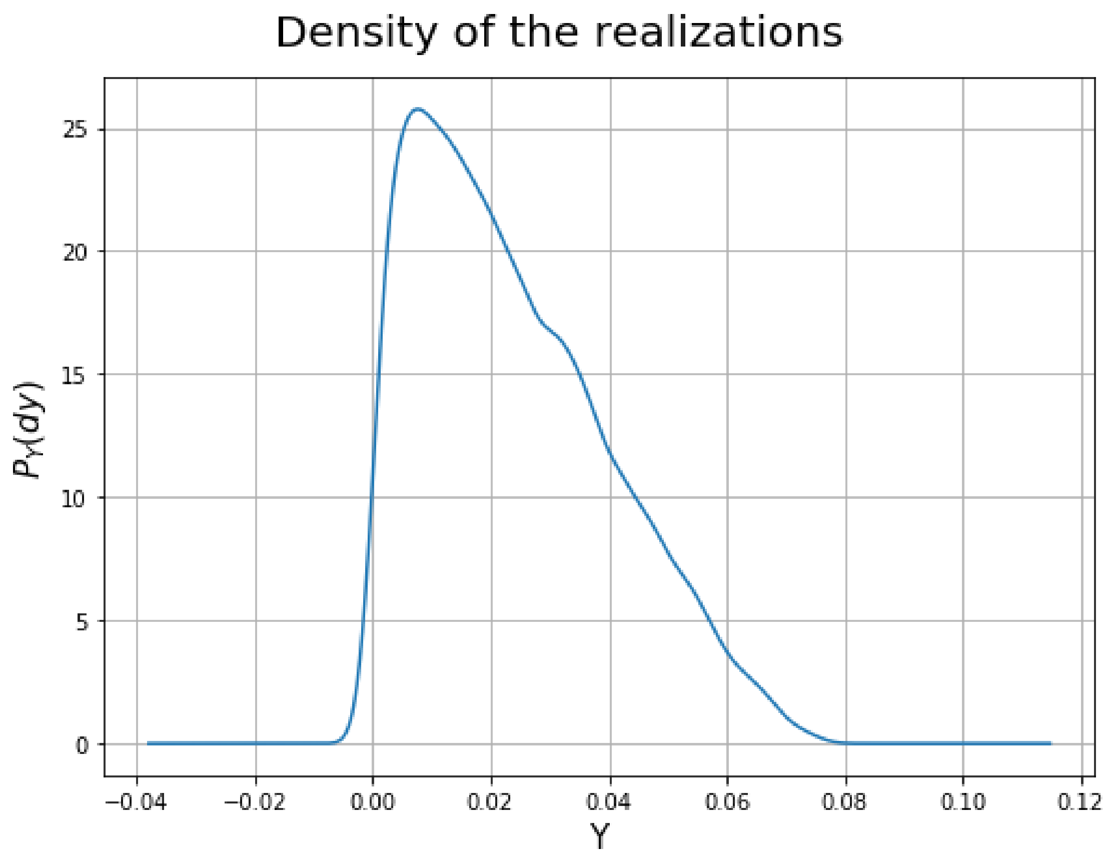



```
In [221]: S = 1
          K = 1
          r = 0.0063
          dT = 14/365
          iv = Symbol('\sigma')
          d1 = (log (S/K)+(r+iv*iv/2)*dT)/iv/sqrt(dT)
          d2 = d1 - iv * sqrt(dT)
          N = Symbol ('n')
          N = Normal ('n', 0, 1)
          C = S * simplify(cdf(N))(d1) - exp(-r * dT) * K * simplify(cdf(N))(d2)

In [92]: number = 20000
         X = np.array(getbetarandomvars (alpha, beta, number))

In [122]: number = 20000
          X = np.array(getbetarandomvars (alpha, beta, number))
          MC = pd.DataFrame(columns={"ramdom variable", "realization"})
          for ind in range (number) :
              rv = X[ind]
              realization = float(C.subs(iv,rv).evalf())
              MC.loc [ind] = [rv, realization]

In [325]: fig, ax = plt.subplots(nrows=1, ncols=1,figsize=(8, 6));
          ax = MC['ramdom variable'].plot.kde()
```

```
ax.set_xlabel('X',fontsize=15)
ax.set_ylabel('$P_X(dx)$',fontsize=15)
ax.grid(True)
fig.suptitle('Density of the stochastic input X',fontsize=20)
fig.tight_layout(pad=3.5)
fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\MC solver X po
```




```
In [324]: fig, ax = plt.subplots(nrows=1, ncols=1,figsize=(8, 6));
          ax = MC['realization'].plot.kde()
          ax.set_xlabel('Y',fontsize=15)
          ax.set_ylabel('$P_Y(dy)$',fontsize=15)
          ax.grid(True)
          fig.suptitle('Density of the realizations',fontsize=20)
          fig.tight_layout(pad=3.5)
          fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\MC solver Y po
```




### 0.2 Stochastic Modeling Through Series Expansions

Jacobi Polynomials

In [687]:
```
alpha = 1.1444307615618827
beta = 2.590929757651513
i = Symbol('i')
x = Symbol('x')
m = Symbol('m')
J = gamma(alpha+i)/factorial(i)/gamma(alpha+beta+i-1)*Sum(binomial(i,m)*gamma(alpha+b
```

In [606]:
```
# weight function
w = (1-x)**(alpha-1) * (1+x)**(beta-1)
# normalization factor
n = sqrt(gamma(i+alpha)*gamma(i+beta)/(2*i+alpha+beta-1)/gamma(i+alpha+beta-1)/factor
# normalization factor for Jacobi polynomial inner product
nJ = sqrt(2**(alpha+beta-1)/(2*i+alpha+beta-1)*gamma(i+alpha)*gamma(i+beta)/gamma(i+a
```

Check Jacobi polynomials properties

```
In [689]: check = pd.DataFrame(np.zeros((5,5)))
          for ii in range(5):
              for jj in range(5):
                  ans=0
                  for xx in np.linspace(-1,1,2001):
                      ans+=((J/nJ).subs(i,ii)*(J/nJ).subs(i,jj)*w).subs(x,xx).evalf()*0.001
                  check.iloc[ii,jj]=ans
          print(check)
```

```
                       0                      1                      2                      3
0      0.999789419002016  -0.000304472239597669  -0.000383338867566917  -0.000453670196368348
1  -0.000304472239597669      0.999559769606514  -0.000554252913307230  -0.000655958381653051
2  -0.000383338867566917  -0.000554252913307230      0.999302170885729  -0.000825841774924408
3  -0.000453670196368348  -0.000655958381653051  -0.000825841774924408      0.999022591974159
4  -0.000518151339530541  -0.000749164531746407  -0.000943248442348874   -0.00111624878145781
```

Check the normalization

```
In [690]: check = pd.DataFrame(np.zeros((5,5)))
          for ii in range(5):
              for jj in range(5):
                  ans=0
                  for xx in np.linspace(-1,1,2001):
                      ans+=((J/n).subs(i,ii)*(J/n).subs(i,jj)*0.5*P.subs(x,(1-x)/2)).subs(x,xx)
                  check.iloc[ii,jj]=ans
          print(check)
```

```
                       0                      1                      2                      3
0      0.999789419002016  -0.000304472239597765  -0.000383338867566863  -0.000453670196368365
1  -0.000304472239597765      0.999559769606514  -0.000554252913307099  -0.000655958381653016
2  -0.000383338867566863  -0.000554252913307099      0.999302170885729  -0.000825841774924314
3  -0.000453670196368365  -0.000655958381653016  -0.000825841774924314      0.999022591974159
4  -0.000518151339530545  -0.000749164531746435  -0.000943248442348878   -0.00111624878145791
```

h(x)

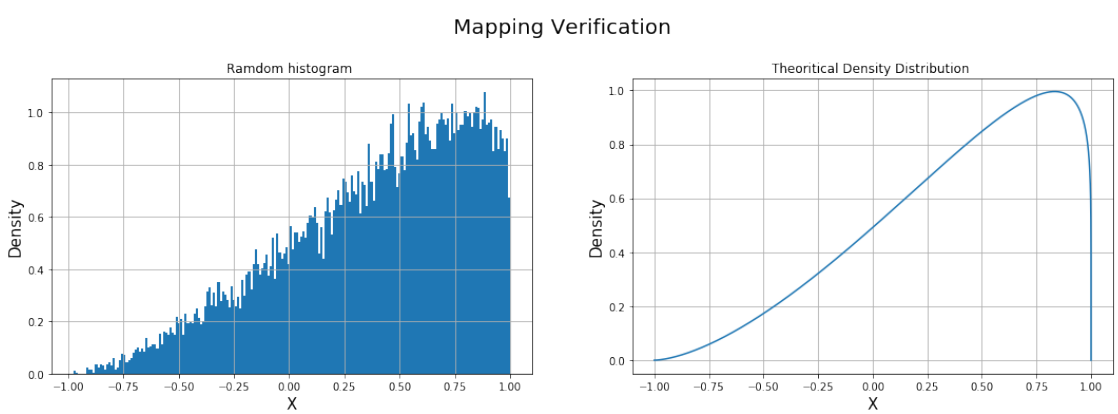
```
In [522]: S = 1
          K = 1
          r = 0.0063
          dT = 14/365
          x = Symbol('x')
          d1 = (log (S/K)+(r+x*x/2)*dT)/x/sqrt(dT)
          d2 = d1 - x * sqrt(dT)
          N = Symbol ('n')
          N = Normal ('n', 0, 1)
          h = S * simplify(cdf(N))(d1) - exp(-r * dT) * K * simplify(cdf(N))(d2)
```

P_X(dx) for Beta distribution

```
In [608]: P = x**(alpha-1) * (1-x)**(beta-1)/gamma(alpha)/gamma(beta)*gamma(alpha+beta)
```

Check mapping

```
In [646]: X = np.array(getbetarandomvars (alpha, beta, 20000))
          X = 1-X*2
          fig, (ax1, ax2) = plt.subplots(nrows=1, ncols=2,figsize=(16, 6));
          pdf_mapped = [0.5*P.subs(x,(1-x)/2).subs(x,xx) for xx in np.linspace(-1,1,20000)]

          ax1.hist(X, 200, density=1);
          ax1.set_title('Ramdom histogram ')
          ax1.set_xlabel('X',fontsize=15)
          ax1.set_ylabel('Density',fontsize=15)
          ax1.grid(True)

          ax2.plot(np.linspace(-1,1,20000), pdf_mapped);
          ax2.set_title('Theoritical Density Distribution ')
          ax2.set_xlabel('X',fontsize=15)
          ax2.set_ylabel('Density',fontsize=15)
          ax2.grid(True)

          fig.suptitle('Mapping Verification',fontsize=20)
          fig.tight_layout(pad=5.0)
          fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\mapping verifi
```




```
In [627]: X = np.array(getbetarandomvars (alpha, beta, 2000))
          X = 1-2*X
```

Then integrate from 0 to 1 to find yi.

```
In [635]: yi = []
          for ii in range(5):
```

```
    ans=0
    for xx in np.linspace(-1,1,2001):
        ans+=(h.subs(x,(1-x)/2)*(J/n).subs(i,ii)*0.5*P.subs(x,(1-x)/2)).subs(x,xx).ev
    yi.append(ans)
```

In [629]:
```
yi
```

Out[629]:
```
[0.0542393444813036,
 -0.115640573811862,
 0.142271972338891,
 -0.145694613572554,
 0.143758197099776]
```

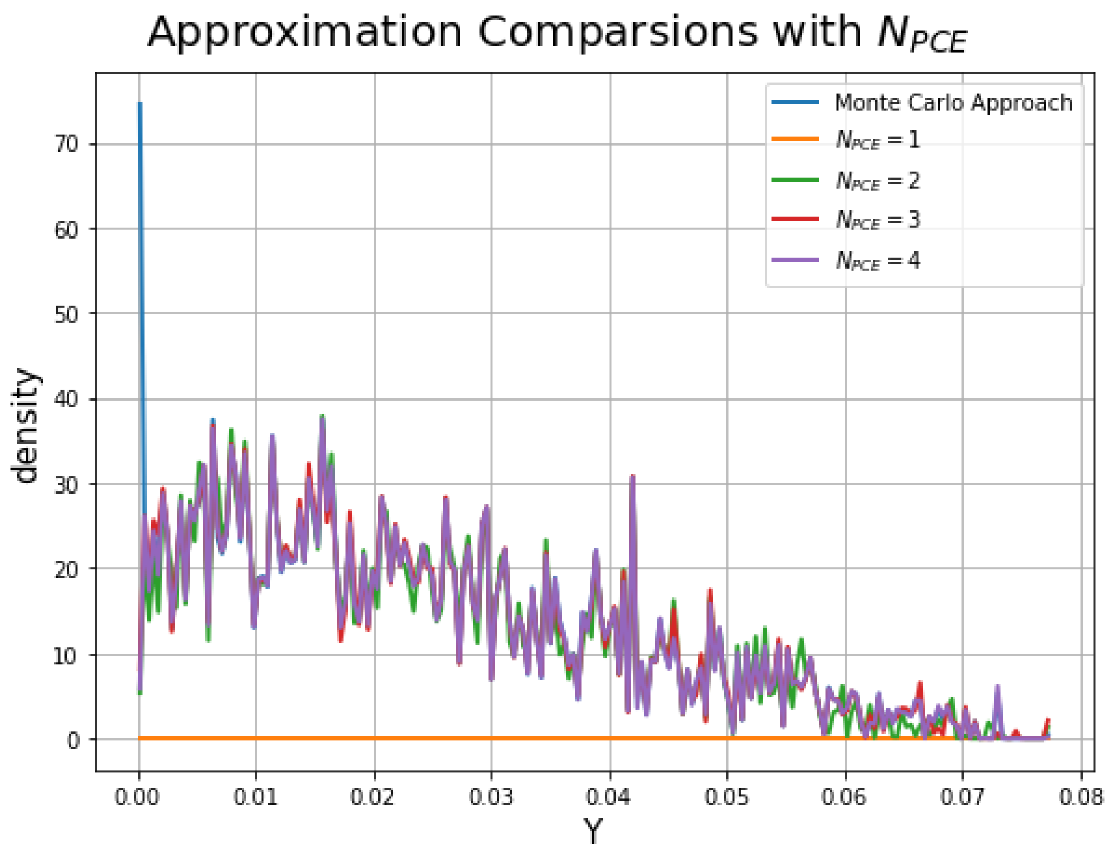
In [653]:
```
Poly0 = yi[0]*(J/n).subs(i,0)
Poly1 = yi[0]*(J/n).subs(i,0)+yi[1]*(J/n).subs(i,1)
Poly2 = yi[0]*(J/n).subs(i,0)+yi[1]*(J/n).subs(i,1)+yi[2]*(J/n).subs(i,2)
Poly3 = yi[0]*(J/n).subs(i,0)+yi[1]*(J/n).subs(i,1)+yi[2]*(J/n).subs(i,2)+yi[3]*(J/n)
Poly4 = yi[0]*(J/n).subs(i,0)+yi[1]*(J/n).subs(i,1)+yi[2]*(J/n).subs(i,2)+yi[3]*(J/n)
Poly = [Poly0, Poly1, Poly2, Poly3, Poly4]
```

In [654]:
```
Y_approx = [np.array([Poly[i].subs(x,xx).evalf() for xx in X]).astype(float) for ii i
```

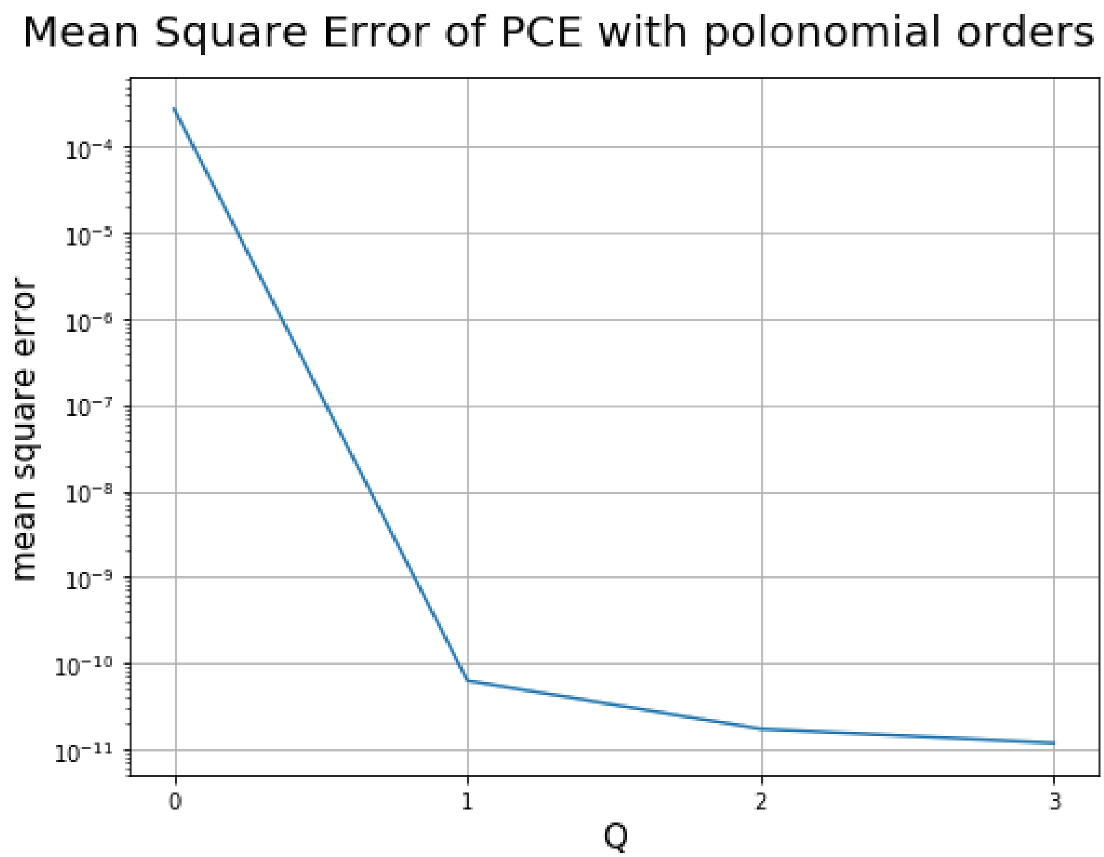
In [655]:
```
Y_real = np.array([h.subs(x,(1-xx)/2) for xx in X]).astype(float)
```

In [667]:
```
kde_real = scipy.stats.gaussian_kde(Y_real,bw_method=0.0005)
kde_approx = [scipy.stats.gaussian_kde(Y_approx[i],bw_method=0.0005) for  ii in range
Y_range = np.linspace(Y_real.min(),Y_real.max(),200)
```

In [670]:
```
fig, ax = plt.subplots(nrows=1, ncols=1,figsize=(8, 6));
ax.plot(t_range,kde_real(t_range),lw=2, label='Monte Carlo Approach')
for ii in range(4):
    ax.plot(t_range,kde_approx[ii](t_range),lw=2, label='$N_{PCE}=$'+str(ii+1))
ax.set_xlabel('Y',fontsize=15)
ax.set_ylabel('density',fontsize=15)
ax.grid(True)
ax.legend()
fig.suptitle('Approximation Comparsions with $N_{PCE}$',fontsize=20)
fig.tight_layout(pad=3.5)
fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\N_PCE compars
```




```
In [671]: mse=[]
          for ii in range(4):
              mse.append(((Y_approx[i]-Y_real)**2).mean())

In [675]: fig, ax = plt.subplots(nrows=1, ncols=1,figsize=(8, 6));
          ax.plot(mse)
          ax.set_xlabel('Q',fontsize=15)
          ax.set_ylabel('mean square error',fontsize=15)
          ax.set_yscale('log')
          ax.xaxis.set_major_locator(MaxNLocator(integer=True))
          ax.grid(True)
          fig.suptitle('Mean Square Error of PCE with polonomial orders',fontsize=20)
          fig.tight_layout(pad=3.5)
          fig.savefig(r"C:\DUKE\courses\Uncertainty Quantification\final project\PCE mse.jpg",
```


Mean Square Error of PCE with polonomial orders
mean square error
$10^{-4}$
$10^{-5}$
$10^{-6}$
$10^{-7}$
$10^{-8}$
$10^{-9}$
$10^{-10}$
$10^{-11}$
0
1
2
3
Q