The logic behind testing hypotheses

We toss a coin 10 times and get 7 tails. Is this sufficient evidence to conclude that the coin is biased?

The **null hypothesis, H_0**, states that "nothing extraordinary is going on". So in this case

$$H_0\colon\ P(T) = \tfrac{1}{2}$$

The **alternative hypothesis, H_A**, states that there is a different chance process that generates the data. Here we can take

$$H_A\colon\ P(T) \neq \tfrac{1}{2}$$

Hypothesis testing proceeds by collecting data and evaluating whether the data are compatible with H_0 or not (in which case one **rejects H_0**).

The logic behind testing hypotheses

A different example: A company develops a new drug to lower blood pressure. It tests it with an experiment involving 1,000 patients.
In this case "nothing extraordinary going on" means that the drug has no effect. So

H_0: no change in blood pressure $\qquad$ H_A: blood pressure drops

Note that in this case the company would like to reject H_0!

So the logic of testing is typically indirect: One assumes that nothing extraordinary is happening and then hopes to reject this assumption H_0.

Setting up a test statistic

A **test statistic** measures how far away the data are from what we would expect if H_0 were true.

The most common test statistic is the **z-statistic**:

$$z = \frac{\text{observed} - \text{expected}}{\text{SE}}$$

'Observed' is a statistic that is appropriate for assessing H_0. In the example of the 10 coin tosses, appropriate statistics would be the number of tails or the percent of tails.

'Expected' and SE are the expected value and the SE of this statistic, *computed under the assumption that H_0 is true.*

In the example: Using the formulas for the sum of 0/1 labels we get

'expected' $= 10 \times \frac{1}{2} = 5$ and SE $= \sqrt{10}\sqrt{\frac{1}{2} \times \frac{1}{2}} = 1.58$. So

$$z = \frac{7-5}{1.58} = 1.27$$

for binomial distribution, std is $\sqrt{np(1-p)}$

p-values measure the evidence agains H_0

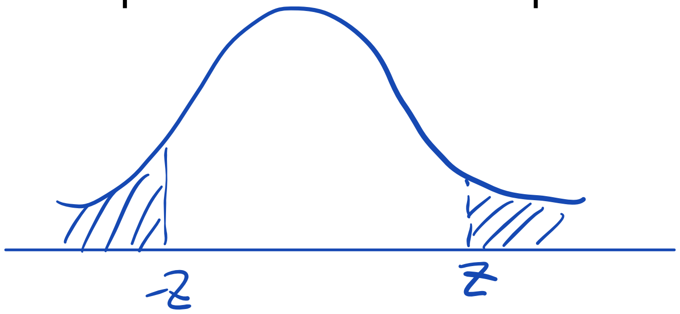
Large values of $|z|$ are evidence against H_0: The larger $|z|$ is, the stronger the evidence. The strength of the evidence is measured by the **p-value** (or: **observed significance level**):

$$z = \frac{\text{observed} - \text{expected}}{SE}$$

The p-value is the probability of getting a value of z as extreme or more extreme than the observed z, assuming H_0 is true.

standardize

But if H_0 is true, then z follows that standard normal curve, according to the central limit theorem, so the p-value can be computed with normal approximation:

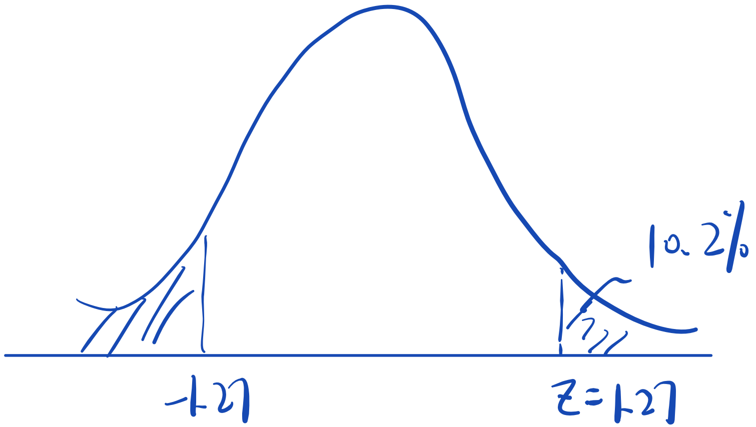


small shaded area is evidence against the null hypothesis.

The smaller the p-value, the stronger the evidence against H_0. Often the criterion for rejecting H_0 is a p-value smaller than 5%. Then the result is called **statistically significant**.

p-values measure the evidence against H_0

In the example:



p-value = 10.2% × 2 = 20.4% > 5%

Note that the p-value **does not** give the probability that H_0 is true, as H_0 is either true or not - there are no chances involved. Rather, it gives the probability of seeing a statistic as extreme, or more extreme, that the observed one, assuming H_0 is true.

Distinguishing Coke and Pepsi by taste

It has been said that it is difficult to distinguish Coke and Pepsi by taste alone, without the visual cue of the bottle or can.

In an experiment that I did in a class at Stanford, 10 cups were filled at random with either Coke or Pepsi. A student volunteer tasted each of the 10 cups and correctly named the conents of seven. Is this sufficient evidence to conclude that the student can tell apart Coke and Pepsi?

"Nothing extraordinary is going on" means that the student does not have any special ability to tell them apart and is just guessing. null hypothesis

To write this down formally we introduce 0/1 labels since we are counting correct answers: $1 =$ correct answer, $0 =$ wrong answer

H_0: $P(0) = P(1) = \frac{1}{2}$ $\quad$ H_A: $P(1) > \frac{1}{2}$

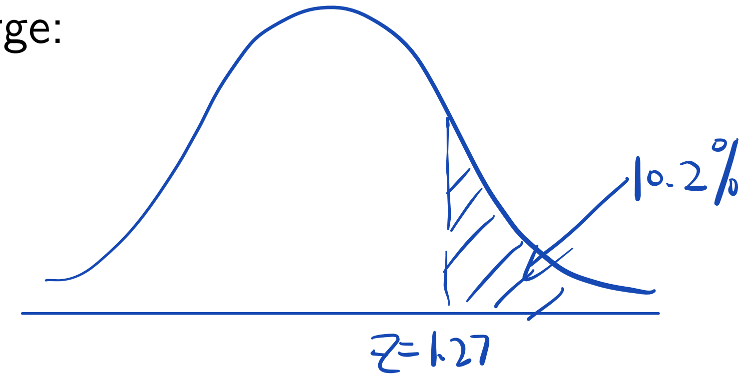
This is a **one-sided test**: the alternative hypothesis for for P(1) we are interested in is on one side of $\frac{1}{2}$.

Distinguishing Coke and Pepsi by taste

Since we are looking at the sum of ten 0/1 labels, the z-statistic is the same that we had for coin-tossing:

$$z = \frac{\text{observed sum} - \text{expected sum}}{\text{SE of sum}} = \frac{7-5}{1.58} = 1.27$$

But since we do a one-sided test instead of a two-sided test, the p-value is only half as large:



p-value = 10.2% > 5%

Since 10.2% is not smaller than 5%, we don't reject H_0: We are not convinced that the student can distinguish Coke and Pepsi.

Distinguishing Coke and Pepsi

A two-sided alternative might also be appropriate:

H_A: $\mathrm{P}(1) \neq \frac{1}{2}$

H_A corresponds to a student who is more likely than not to distinguish Coke and Pepsi, but who may confuse them. Such a student might get one correct answer (say).

One has to carefully consider whether the alternative should be one-sided or two-sided, as the p-value gets doubled in the latter case.

It is not ok to change the alternative afterwards in order to get the p-value below 5%.

declare ahead of time

The t-test

The health guideline for lead in drinking water is a concentration of not more than 15 parts per billion (ppb).

Five independent samples from a reservoir average 15.6 ppb. Is this sufficient evidence to conclude that the concentration μ in the reservoir is above the standard of 15 ppb?

Recall our model for measurements:

$$\text{measurement} = \mu + \text{measurement error}$$

So it may be that the concentration μ is below 15 ppb, but measurement error results in an average of 15.6 ppb.

H_0: $\mu = 15$ ppb $\quad$ H_A: $\mu > 15$ ppb

We can try a z-test for the average of the measurements:

$$z = \frac{\text{observed average} - \text{expected average}}{\text{SE of average}} = \frac{15.6 \text{ ppb} - 15 \text{ ppb}}{\text{SE of average}}$$

since the measurement error has expected value zero.

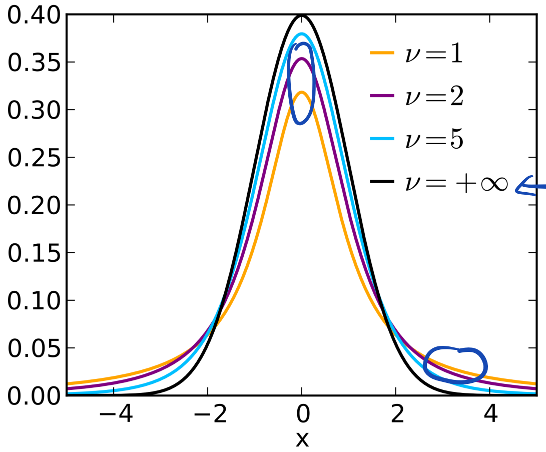
The t-test

use t-test when sample size is small

SE of average $= \frac{\sigma}{\sqrt{n}}$, but the standard deviation σ of the measurement error is unknown.

We can estimate σ by s, the sample standard deviation of the measurements. However:

If we estimate σ and n is small ($n \leq 20$), then the normal curve is not a good enough approximation to the distribution of the z-statistic. Rather, an appropriate approximation is **Student's t-distribution with $n-1$ degrees of freedom**:



standard normal distribution

fatter tails → extra uncertainty

The t-test

The fatter tails account for the additional uncertainty introduced by estimating σ by $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$.

Using the **t-test** in place of the z-test is only necessary for small samples: $n \leq 20$ (say).

In that case it is also better to replace the confidence interval $\bar{x} \pm z$ SE by

$$\bar{x} \pm t_{n-1}\text{SE}$$

More on testing

- *Statistically significant does not mean that the effect size is important*:

  Suppose the sample average shows a lead concentration that is only slightly above the health standard of 15 ppb: say the sample average is 15.05 ppb.

  That may not be of practical concern, even though the test may be highly signficant: Statistical significance convinces us that there is an effect, but it doesn't say how big the effect is.

  $$\text{z-score} = \frac{\text{observed} - \text{expected}}{\text{SE}}$$

  Reason: A large sample size n makes SE $= \frac{\sigma}{\sqrt{n}}$ small, so even a small exceedance over the limit by (say) 0.05 ppb may give a statistically significant result.

  Therefore it is helpful to complement a test with a confidence interval: In the above case a 95% confidence interval for μ might be [15.02 ppb, 15.08 ppb].

More on testing

- There is a general connection between confidence intervals and tests:

  A 95% confidence interval contains all values for the null hypothesis that will not be rejected by a two-sided test at a 5% significance level.
  (A 5% **significance level** means that the threshold for the p-value is 5%).

- There are two ways that a test can result in a wrong decision:

  H_0 is true, but was erroneously rejected $\rightarrow$ Type I error ('false positive')

  H_0 is false, but we fail to reject it $\rightarrow$ Type II error   false negative

  Rejecting H_0 if the p-value is smaller than 5% means P(type I error) $\leq 5\%$

The two-sample z-test

Last month, the President's approval rating in a sample of 1,000 likely voters was 55%. This month, a poll of 1,500 likely voters resulted in a rating of 58%. Is this sufficient evidence to conclude that the rating has changed?

We want to assess whether

$p_1 =$proportion of all likely voters approving last month

is equal to

$p_2 =$proportion of all likely voters approving this month

"nothing unusual is going on" means $p_1 = p_2$. It's common to look at the difference $p_2 - p_1$ instead:

$$\text{H}_0: \; p_2 - p_1 = 0 \qquad\qquad \text{H}_1: \; p_2 - p_1 \neq 0$$

p_1 is estimated by $\hat{p}_1 = 55\%$, p_2 by $\hat{p}_2 = 58\%$. The central limit theorem applies to the difference $\hat{p}_2 - \hat{p}_1$ just as it does to $\hat{p}_1$ and $\hat{p}_2$. So we can use a z-test:

The two-sample z-test

We can use a z-test for the difference $\hat{p}_2 - \hat{p}_1$:

$$z = \frac{\text{observed difference} - \text{expected difference}}{\text{SE of difference}} = \frac{(\hat{p}_2 - \hat{p}_1) - \overbrace{(p_2 - p_1)}^{=0}}{\text{SE of difference}}$$

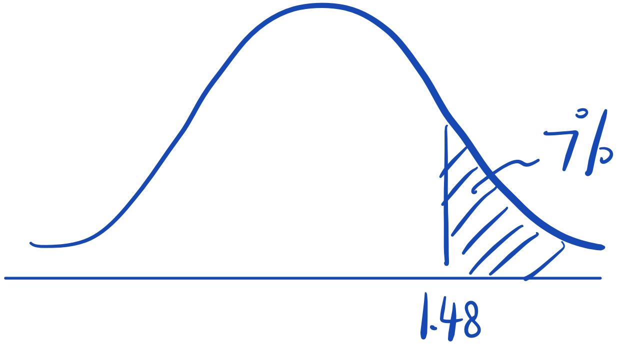
under the null hypothesis

An important fact is that if $\hat{p}_1$ and $\hat{p}_2$ are independent, then

$$\text{SE}(\hat{p}_2 - \hat{p}_1) = \sqrt{(\text{SE}(\hat{p}_1))^2 + (\text{SE}(\hat{p}_2))^2}. \qquad \text{So}$$

$$z = \frac{(\hat{p}_2 - \hat{p}_1) - 0}{\sqrt{\sqrt{\frac{p_1(1-p_1)}{1000}}^2 + \sqrt{\frac{p_2(1-p_2)}{1500}}^2}} = \frac{0.03}{0.0202} = 1.48$$

$$SE = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}$$



p-value

$= 14\% > 5\%$

cannot reject H_0

The two-sample z-test

The confidence interval for $p_2 - p_1$ is

observed $\pm$ z SE

98% confidence interval

$$(\hat{p}_2 - \hat{p}_1) \ \pm\ z\,\mathrm{SE}(\hat{p}_2 - \hat{p}_1) \ = \ [-1\%, 7\%] \quad \text{when } z = 2$$

We can improve the estimate of $\mathrm{SE}(\hat{p}_2 - \hat{p}_1)$ somewhat by using the fact that $p_1 = p_2$ on H_0. Since there is a common proportion we can estimate it by **pooling** the samples:

$0.55 \times 1000 = 550$ voters approve in the first sample, 870 in the second, so in total there are 1420 approvals out of 2500. So the **pooled estimate** of $p_1 = p_2$ is $\frac{1420}{2500} = 56.8\%$.

So we estimate $\mathrm{SE}(\hat{p}_2 - \hat{p}_1)$ by $\sqrt{\frac{0.568(1-0.568)}{1000} + \frac{0.568(1-0.568)}{1500}} = 0.02022$, which essentially gives the same answer in this case.

The two-sample z-test

The two-sample z-test is applicable in the same way to the difference of two sample means in order to test for equality of two population means.

If the two samples are independent, then again

$$\mathsf{SE}(\bar{x}_2 - \bar{x}_1) \;=\; \sqrt{(\mathsf{SE}(\bar{x}_1))^2 + (\mathsf{SE}(\bar{x}_2))^2}$$

and $\mathsf{SE}(\bar{x}_1) = \frac{\sigma_1}{\sqrt{n_1}}$ is estimated by $\frac{s_1}{\sqrt{n_1}}$.

If the sample sizes n_1, n_2 are not large, then the p-value needs to computed from the t-distribution.

The pooled standard deviation

If one has reason to assume that $\sigma_1 = \sigma_2$ (or if this has been checked), then one may use the **pooled estimate** for $\sigma_1 = \sigma_2$ given by

$$s^2_{\text{pooled}} = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

However, the advantages of using s^2_{pooled} are small and the analysis rests on the assumption that $\sigma_1 = \sigma_2$. For these reasons the pooled t-test is usually avoided.

All of the above two-sample tests require that the two samples are independent. They are also applicable in special situations where the samples are dependent, e.g. to compare the treatment effect when subjects are randomized into treatment and control groups.

The paired-difference test

Do husbands tend to be older than their wives?

The ages of five couples:

Husband's age	Wife's age	age difference
43	41	2
71	70	1
32	31	1
68	66	2
27	26	1

The two-sample t-test is not applicable since the two samples are not independent. Even if they were independent, the small differences in ages would not be significant since the standard deviations are large for husbands and also for the wives.

The paired-difference test

Since we have paired data, we can simply analyze the differences obtained from each pair with a regular t-test, which in this context of **matched pairs** is called **paired t-test**:

H_0: population difference is zero

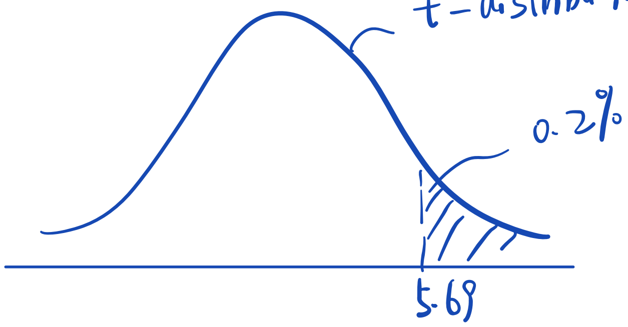
average of age difference

$t = \frac{\bar{d}-0}{\text{SE}(\bar{d})}$, where d_i is the age difference of the ith couple.

expected difference under H_0

$\text{SE}(\bar{d}) = \frac{\sigma_d}{\sqrt{n}}$. Estimate σ_d by $s_d = 0.55$. Then $t = \frac{1.4-0}{0.55/\sqrt{5}} = 5.69$

t-distribution with freedom 4



p-value = 0.4% < 5%

The independence assumption is in the sampling of the couples.

The sign test

What if didn't know the age difference d_i but only if the husband was older or not?

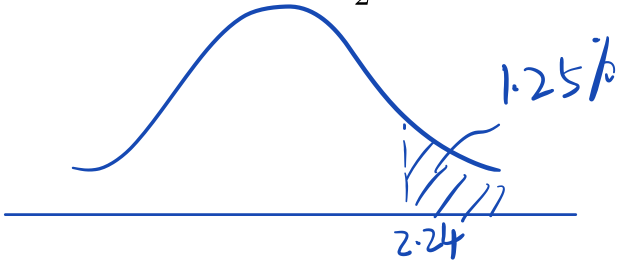
We can test

H$_0$: half the husbands in the population are older than their wives

using 0/1 labels and a z-test, just as we tested whether a coin is fair:

n=5

$$z = \frac{\text{sum of 1s} - \frac{n}{2}}{\text{SE of sum}} = \frac{5 - \frac{5}{2}}{\sqrt{5}\,\frac{1}{2}} = 2.24 \quad \text{since } \sigma = \frac{1}{2} \text{ on H}_0.$$



p-value = 2.5%

The p-value of this **sign-test** is less significant than that of the paired t-test. This is because the latter uses more information, namely the size of the differences. On the other hand, the sign test has the virtue of easy interpretation due to the analogy to coin tossing.

When do you think you could use a sign-test?

- [x] To test if a diet is working - compare the weight of subjects before and after the diet.

Correct
That's correct.

- [x] To test if COVID had any influence on students' scores - compare the scores of students before and during COVID.

Correct
That's correct.

- [x] To test if a new type of swimming suit has any influence on the performance of swimmers - compare the speed of swimming wearing the usual swimsuit and with the speed of swimming wearing the new type of swimsuit.

Correct
That's correct.

z-test: simple , $z = \dfrac{\text{observed} - \text{expected}}{SE}$

t-test: use when sample size is small $(n \leq 20)$

two-sample z-test: samples are independent

pair-difference test:

sign test

Quiz

1. Which of the following statements are true? (Select all that apply.)

- [x] The p-value depends on the data.
- [x] If the p-value is smaller than 5%, then there is less than a 5% chance that the null hypothesis is true.
- [x] If the null hypothesis is true, then there is less than a 5% chance to get a p-value that is smaller than 5%.
- [ ] If a data scientist does many tests, then even if all the null hypotheses are true, a certain proportion will be rejected in error.

This should not be selected
A p-value is a measure of the chance of witnessing the observed data assuming that the null hypothesis is true, and so can give evidence against our assumption if the p-value is small, but is not a measure of the chance that the null hypothesis is true. The truth status of the null hypothesis is not a random value: It is either true or false.

2. Read the first five paragraphs of the article "Online daters do better in the marriage stakes" by Regina Nuzzo in Nature News, 2013. [You can find it on the internet or here]. The main claim of the article is that there is a statistically significant difference in marital outcomes between couples that meet online and couples that meet in other ways. Is this finding is of practical relevance?

- ( ) yes
- (x) no

Correct
Because a result is statistically significant does not have to mean it is practically relevant: The difference between 92% and 94% is not practically relevant.

3. A fair coin is tossed 100 times.

$SE = \frac{\sigma}{\sqrt{n}}$ use $s = \sqrt{p(1-p)}$ to approximate σ

$SE = \sqrt{\frac{\frac{1}{2} \times \frac{1}{2}}{100}} = \frac{\frac{1}{2}}{10} = 5$

Which of the following statements are true? (Select all that apply.)

- [x] The standard error for the percentage of heads among the 100 tosses is 5%.
- [x] The standard error for the percentage of tails among the 100 tosses is 5%.
- [ ] The standard error for the quantity "percentage of heads - percentage of tails" is $\sqrt{0.05^2 + 0.05^2} = 7\%$.

Correct
Because the standard deviation of binomial experiment with probability of success p is $\sqrt{p(1-p)}$, the standard error for the proportion of successes in a run of n such experiments is $\frac{\sqrt{p(1-p)}}{\sqrt{n}}$. Thus in our case, the standard error for the proportion of heads in 100 tosses is

$$\frac{\sqrt{\frac{1}{2} \cdot \frac{1}{2}}}{\sqrt{100}} = \frac{1}{20} = 0.05,$$

and the corresponding standard error for the percentage of heads is 5%.

4. Is there a relationship between age and insomnia? A random sample of 184 people ages 18-29 was taken, and it was found that 26.1% suffer from insomnia and 73.9% do not. A separate random sample of 811 people ages 30 and over was taken, and it was found that 39.2% suffer from insomnia and 60.8% do not.

Which of the following four test statistics are appropriate for testing whether the prevalence of insomnia is different between the two age groups? (Select all that are.)

$$z = \frac{\text{observed} - \text{expected}}{SE}$$

H_0: no difference

expected = 0

observed = 0.261 − 0.392

- [ ] 2-sample z-test $z = \frac{0.261 - 0.739}{\frac{\sqrt{0.739(1-0.739)}}{\sqrt{184}}}$

- [ ] $z = \frac{0.261 - 0.739}{\frac{\sqrt{0.261(1-0.261)}}{\sqrt{184}}}$

- [x] $z = \frac{0.261 - 0.392}{\sqrt{\frac{0.261(1-0.261)}{184} + \frac{0.392(1-0.392)}{811}}}$

- [ ] $z = \frac{0.261 - 0.608}{\sqrt{\frac{0.261(1-0.261)}{184} + \frac{0.608(1-0.608)}{811}}}$

5. You want to test whether plain M&Ms really contain 24% blue M&Ms as claimed on the manufacturer's web site. You sample 500 plain M&Ms at random and count the fraction of blue M&Ms.

Which of the following tests is appropriate to address this question?

- [x] z-test
- [x] t-test
- [ ] 2-sample z-test
- [ ] sign test
- [ ] paired-difference test.

This should not be selected
The t-test could be used, but since the sample size is large and the null hypothesis provides us with a standard devation for the population, the z-test is simpler to apply.

use t-test when sample size is small

6. A high school principal wants to find out whether the average SAT score of this year's graduating class is higher than last year's. She samples 13 students from this year's graduating class at random and wants to compare their average SAT score to the average SAT score from last year's graduating class.

the question involves only one sample

- [x] z-test
- [ ] t-test
- [x] 2-sample z-test
- [x] sign test
- [x] or paired-difference test.

This should not be selected
Because the question doesn't suggest or imply a clear choice for the standard deviation of the population, it requires us to use the sample standard deviation. However, since the sample size is less than 20, our use of the sample standard deviation could lead to us to draw invalid conclusions from the normal approximation used to apply the z-test.

7. To investigate whether there is a difference in scholastic abilities between first-borns and second-born siblings, 600 families that have at least two children were randomly selected. The scholastic abilities of the first-born and the second-born siblings were assessed with a test and are to be compared.

the samples are not independent since they are formed from sibling pairs.

- [ ] z-test
- [ ] t-test
- [x] 2-sample z-test
- [x] ? sign test
- [x] paired-difference test.

Correct
Because the natural units of the question are sibling pairs, the paired-difference test is the most appropriate.