数值公式表 V4(最终版)

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说明

- 该版为最终版
- 排版工具：Microsoft® Word 2019 MSO (16.0.11629.20238)

版本

- V0
  - 收集、编写、校对公式
- V1
  - 联合排版
- V2
  - 微调了公式
- V3
  - 缩排
  - 删除了章标题，横线分割与章号取而代之
  - 加粗了公式标题
  - 微调了公式
  - 会产生歧义的地方添加了逗号或分号
  - 调整了部分公式顺序
- V4
  - 删除了页眉
  - 微调了公式

编写组

		收集&编写	校对
Chapter 1	Preliminaries	计 1 白晓曈	计 2 冯戴鹏
Chapter 2	Solution of Nonlinear Equations	计 1 白晓曈	计 2 冯戴鹏
Chapter 3	Solution of Linear Systems	计 4 林正青	计 5 汤禹
Chapter 4	Interpolation and Polynomial Approximation	计 4 梁萍佳	计 5 汤禹
Chapter 5	Curve Fitting	计 1 曾天宇	计 5 施晴
Chapter 6	Numerical Differentiation	计 2 贾丰帆	计 4 刘斯宇
Chapter 7	Numerical Integration	计 6 王程钥	计 4 林楠
Chapter 9	Solution of Differential Equations	计 1 陈泓璇 计 2 符尧	计 4 卢鹏

排版：计 4 梁济凡

1

Absolute Error $E_p = |p-\hat{p}|$

Relative Error $E_r = \left|\frac{p-\hat{p}}{p}\right|$

Significant Digits $\left|\frac{p-\hat{p}}{p}\right| < \frac{10^{1-d}}{2}$

Horner's Method

$P_n(x) := \sum_{i=0}^{n} a_i x^i, b_n := a_n, b_k := a_k + c b_{k+1}$

$\Rightarrow b_0 = P(c)$

2

Fixed Point Iteration $p_{i+1} = g(p_i)$

Fixed Point Theorem

$g \in C^2[a,b], p_0 \in (a,b), g(x) \in [a,b]$;

$\exists K > 0$ s.t. $|g'(x)| \le K < 1 \Rightarrow$ converge to p;

$|g'(x)| > 1 \Rightarrow$ not converge to p

Bisection Theorem $|r - c_n| \le \frac{b-a}{2^{n+1}}$

False Position Method $c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$

Newton-Raphson Theorem

$p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})}$;

$f \in C^2[a,b], p \in [a,b], f(p) = 0$;

$f'(p) \ne 0 \Rightarrow \exists \delta > 0 \forall p_0 \in [p_0-\delta, p_0+\delta]$ converge to p

Acceleration of Newton-Raphson Iteration

$p_k = p_{k-1} - \frac{M f(p_{k-1})}{f'(p_{k-1})}$, M is the order of root p

$|\mathrm{E}_{k+1}| \approx |\mathrm{E}_k|^{1.618} \left|\frac{f''(\mathrm{p})}{2f'(\mathrm{p})}\right|^{0.618}$

3

Back Substitution

$x_k = \frac{b_k - \sum_{j=k+1}^{N} a_{kj} x_j}{a_{kk}}, k = N, N-1, \dots, 1$

LU Factorization

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 8 & 6 \\ 3 & 10 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & -2 \\ 0 & 0 & -2 \end{bmatrix}$$

Jacobi Iteration

$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}}, i = 1,2,\dots,N$

Gauss-Seidel Iteration

$x_i^{(k+1)} = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}}{a_{ii}}, i = 1,2,\dots,N$

4

Taylor Series Expansions

$\sin x = \sum_{i=1}^{\infty} \frac{(-1)^{i+1} x^{2i-1}}{(2i-1)!}, \cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}$,

$\exp x = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \ln(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} x^i}{i}$

Taylor Polynomial Approximation

$f(x) \approx P_N(x) = \sum_{k=0}^{N} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$;

$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} (x-x_0)^{N+1}$

Lagrange Polynomial

$P_N(x) = \sum_{i=0}^{N} \left(y_i \frac{\prod_{j=0, j\ne i}^{N} (x - x_j)}{\prod_{j=0, j\ne i}^{N} (x_i - x_j)} \right)$

Divided Differences

$f[x_k, x_{k-1}, \dots, x_j] = \frac{f[x_k, x_{k-1}, \dots, x_{j+1}] - f[x_{k-1}, x_{k-2}, \dots, x_j]}{x_k - x_j}, k > j$

Newton Polynomial

$P_N(x) = a_0 + \sum_{i=0}^{N-1} \left(a_{i+1} \prod_{j=0}^{i} (x - x_j)\right)$,

$a_i := f[x_i, x_{i-1}, \dots, x_0]$

Lagrange / Newton Polynomial Error Term

$E_N(x) = \frac{\left(\prod_{i=0}^{N} (x - x_i)\right) f^{(N+1)}(c)}{(N+1)!}$

5

Least-Squares Line

$y = Ax + B$

$$\begin{cases} \left(\sum_{k=1}^{N} x_k^2\right) A + \left(\sum_{k=1}^{N} x_k\right) B = \sum_{k=1}^{N} x_k y_k \\ \left(\sum_{k=1}^{N} x_k\right) A + NB = \sum_{k=1}^{N} y_k \end{cases}$$

$E(A,B) = \sum_{k=1}^{N} (A x_k + B - y_k)^2 = \sum_{k=1}^{N} d_k^2$

Least-Squares Parabola

$y = f(x) = Ax^2 + Bx + C$

$$\begin{cases} \left(\sum_{k=1}^{N} x_k^4\right) A + \left(\sum_{k=1}^{N} x_k^3\right) B + \left(\sum_{k=1}^{N} x_k^2\right) C = \sum_{k=1}^{N} y_k x_k^2 \\ \left(\sum_{k=1}^{N} x_k^3\right) A + \left(\sum_{k=1}^{N} x_k^2\right) B + \left(\sum_{k=1}^{N} x_k\right) C = \sum_{k=1}^{N} y_k x_k \\ \left(\sum_{k=1}^{N} x_k^2\right) A + \left(\sum_{k=1}^{N} x_k\right) B + NC = \sum_{k=1}^{N} y_k \end{cases}$$

Root-Mean-Square Error

$E_2(f) = \sqrt{\frac{1}{N} \sum_{k=1}^{N} |f(x_k) - y_k|^2}$

Linearization

$y = Cx^A \Rightarrow \ln(y) = A\ln(x) + \ln(C)$

$\Rightarrow X = \ln(x), Y = \ln(y), C = e^B$;

$y = Ce^{Ax} \Rightarrow \ln(y) = Ax + \ln(C)$

$\Rightarrow X = x, Y = \ln(y), C = e^B$

Cubic Spline

$S(x) = S_k(x)$

$= s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3$,

$x \in [x_k, x_{k+1}], k = 0,1,\cdots,N-1$

$m_k = S''(x_k), h_k = x_{k+1} - x_k, d_k = \frac{y_{k+1} - y_k}{h_k}$,

$h_{k-1} m_{k-1} + 2(h_{k-1} + h_k) m_k + h_k m_{k+1} = 6(d_k - d_{k-1})$

for $k = 1,2,\dots,N-1$

$s_{k,0} = y_k, s_{k,1} = d_k - \frac{h_k(2m_k + m_{k+1})}{6}$,

$s_{k,2} = \frac{m_k}{2}, s_{k,3} = \frac{m_{k+1} - m_k}{6h_k}$

Natural Runout $m_0 = 0, m_N = 0$

Parabolic Runout $m_0 = m_1, m_N = m_{N-1}$

Cubic Runout

$m_0 = m_1 - \frac{h_0(m_2 - m_1)}{h_1}, m_N = m_{N-1} + \frac{h_{N-1}(m_{N-1} - m_{N-2})}{h_{N-2}}$

6

Central-Difference Formulas

$f'(x_i) = \frac{f_1 - f_{-1}}{2h} - \frac{h^2 f^{(3)}(c)}{6}$;

$f'(x_i) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4 f^{(5)}(c)}{30}$;

$f''(x_i) = \frac{f_1 - 2f_0 + f_{-1}}{h^2} - \frac{h^2 f^{(4)}(c)}{12}$;

$f''(x_i) = \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2} + O(h^4)$;

$f^{(3)}(x_i) = \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3} + O(h^2)$;

$f^{(3)}(x_i) = \frac{-f_3 + 8f_2 - 13f_1 + 13f_{-1} - 8f_{-2} + f_{-3}}{8h^3} + O(h^4)$;

$f^{(4)}(x_i) = \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4} + O(h^2)$;

$f^{(4)}(x_i) = \frac{-f_3 + 12f_2 - 39f_1 + 56f_0 - 39f_{-1} + 12f_{-2} - f_{-3}}{6h^4} + O(h^4)$

Forward-Difference Formulas

$f'(x_i) = \frac{f_1 - f_0}{h} - \frac{h}{2} f''(c)$;

$f'(x_i) = \frac{-3f_0 + 4f_1 - f_2}{2h} + O(h^2)$;

$f''(x_i) = \frac{f_0 - 2f_1 + f_2}{h^2} + O(h);$

$f''(x_i) = \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} + O(h^2);$

$f^{(3)}(x_i) = \frac{-f_0 + 3f_1 - 3f_2 + f_3}{h^3} + O(h);$

$f^{(3)}(x_i) = \frac{-5f_0 + 18f_1 - 24f_2 + 14f_3 - 3f_4}{2h^3} + O(h^2);$

$f^{(4)}(x_i) = \frac{f_0 - 4f_1 + 6f_2 - 4f_3 + f_4}{h^4} + O(h);$

$f^{(4)}(x_i) = \frac{3f_0 - 14f_1 + 26f_2 - 24f_3 + 11f_4 - 2f_5}{h^4} + O(h^2)$

Backward-Difference Formulas

$f'(x_i) = \frac{f_0 - f_{-1}}{h} + \frac{h}{2} f''(c);$

$f'(x_i) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + O(h^2);$

$f''(x_i) = \frac{f_0 - 2f_{-1} + f_{-2}}{h^2} + O(h);$

$f''(x_i) = \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2} + O(h^2);$

$f^{(3)}(x_i) = \frac{f_0 - 3f_{-1} + 3f_{-2} - f_{-3}}{h^3} + O(h);$

$f^{(3)}(x_i) = \frac{5f_0 - 18f_{-1} + 24f_{-2} - 14f_{-3} + 3f_{-4}}{2h^3} + O(h^2);$

$f^{(4)}(x_i) = \frac{f_0 - 4f_{-1} + 6f_{-2} - 4f_{-3} + f_{-4}}{h^4} + O(h);$

$f^{(4)}(x_i) = \frac{3f_0 - 14f_{-1} + 26f_{-2} - 24f_{-3} + 11f_{-4} - 2f_{-5}}{h^4} + O(h^2)$

7

Trapezoidal Rule

$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} f^{(2)}(c)$

Simpson's 1/3 Rule

$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{h^5}{90} f^{(4)}(c)$

Simpson's 3/8 Rule

$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3h^5}{80} f^{(4)}(c)$

Boole's Rule

$\int_{x_0}^{x_4} f(x)dx$

$= \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8h^7}{945} f^{(6)}(c)$

Newton-Cotes Open Formula

$\int_a^b f(x)dx = (b-a) f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f^{(2)}(c)$

Two-point Newton-Cotes Open Formula

$\int_a^b f(x)dx = \frac{b-a}{2}[f(x_1) + f(x_2)] + \frac{(b-a)^3}{108} f^{(2)}(c)$

Three-point Newton-Cotes Open Formula

$\int_a^b f(x)dx$

$= \frac{b-a}{3}[2f(x_1) - f(x_2) + 2f(x_3)] + \frac{7(b-a)^5}{23040} f^{(4)}(c)$

Composite Trapezoidal Rule $h = \frac{b-a}{M};$

$\int_a^b f(x)dx \approx \sum_{k=1}^{M} \frac{h}{2}\big(f(x_{k-1}) + f(x_k)\big)$

$= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{M-2} + 2f_{M-1} + f_2 M);$

$E_T(f,h) = \frac{-(b-a) f^{(2)}(c) h^2}{12}$

Composite Simpson's 1/3 Rule $h = \frac{b-a}{2M};$

$\int_a^b f(x)dx \approx \sum_{k=1}^{M} \frac{h}{3}\big(f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\big)$

$= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \cdots + 2f_{2M-2} + 4f_{2M-1} + f_2 M);$

$E_S(f,h) = \frac{-(b-a) f^{(4)}(c) h^4}{180}$

Romberg Integration

$h = \frac{b-a}{2^J}, K = 0,1,2 \ldots, J \ge K$

$R(J,0) = T(J)$ (Trapezoidal Rule)

$R(J,1) = S(J)$ (Simpson's 1/3 Rule)

$R(J,2) = B(J)$ (Boole's Rule)

$R(J,K) = \frac{4^K R(J,K-1) - R(J-1,K-1)}{4^K - 1}$

Precision of Romberg Integration

$\int_a^b f(x)dx = R(J,K) + c_1 h^{2K+2} + c_2 h^{2K+4} + \cdots$

$= R(J,K) + b_K h^{2K+2} f^{(2K+2)}(c_{J,K}), c_{J,K} \in [a,b]$

$= R(J,K) + O(h^{2K+2})$

Euler's Method

$y_{i+1} = y_i + hf(t_i, y_i), E_a = \frac{f'(t_i, y_i)}{2!} h^2 = O(h^2)$

Heun's Method

$y_{i+1}^0 = y_i + f(t_i, y_i)h, y_{i+1} = y_i + \frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)}{2} h$

RK2

$k_1 = f(t_i, y_i), k_2 = f(t_i + p_1 h, y_i + q_{11} k_1 h),$

$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h,$

$a_1 + a_2 = 1, a_2 p_1 = \frac{1}{2}, a_2 q_{11} = \frac{1}{2}$

Mid-Point $a_1 = 0, p_1 = \frac{1}{2}, q_{11} = \frac{1}{2}$

Ralston's Method $a_2 = \frac{2}{3}, p_1 = q_{11} = \frac{3}{4}$

9

Classical RK3

$k_1 = f(t_i, y_i), k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{k_1 h}{2}\right),$

$k_3 = f(t_i + h, y_i - k_1 h + 2k_2 h),$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$

3rd-Order Heun

$k_1 = f(t_i, y_i), k_2 = f\left(t_i + \frac{1}{3}h, y_i + \frac{1}{3}k_1 h\right),$

$k_3 = f\left(t_i + \frac{2}{3}h, y_i + \frac{2}{3}k_2 h\right), y_{i+1} = y_i + \frac{1}{4}(k_1 + 3k_3)h$

Classical RK4

$k_1 = f(t_i, y_i), k_2 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right),$

$k_3 = f\left(t_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right),$

$k_4 = f(t_i + h, y_i + k_3 h),$

$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$

Two ODE-IVPs

$y_{1,i+1} = y_{1,i} + hf_1(t_i, y_{1,i}, y_{2,i}),$

$y_{2,i+1} = y_{2,i} + hf_2(t_i, y_{1,i}, y_{2,i})$

Linear Shooting Method

$x'' = p(t)x'(t) + q(t)x(t) + r(t), \; x(a) = \alpha, x(b) = \beta;$

$u'' = p(t)u'(t) + q(t)u(t) + r(t), u(a) = \alpha, u'(a) = 0;$

$v'' = p(t)v'(t) + q(t)v(t), \quad v(a) = 0, v'(a) = 1;$

$x(t) = u(t) + \frac{\beta - u(b)}{v(b)} v(t)$

Finite-Difference Method

$\left(-\frac{h}{2}p_j - 1\right)x_{j-1} + (2 + h^2 q_j)x_j + \left(\frac{h}{2}p_j - 1\right)x_{j+1}$

$= -h^2 r_j$

for $j = 1,2,\ldots,N-1$ where $x_0 = \alpha, x_N = \beta$