DOMINATING SETS AND ITS VARIANTS IN PLANAR GRAPHS

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CERTIFICATE

This is to certify that the thesis titled ***Dominating Sets and its Variants in Planar Graphs***, submitted by ***Lijo M. Jose (Roll No. 112004005)*** for the award of the degree of ***Doctor of Philosophy*** of ***Indian Institute of Technology Palakkad***, is a record of bonafide work carried out by him under my guidance and supervision at ***Indian Institute of Technology Palakkad***. To the best of my knowledge and belief, the work presented in this thesis is original and has not been submitted, either in part or full, for the award of any other degree, diploma, fellowship, associateship or similar title of any university or institution.

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I hereby declare that the work reported in this thesis is original and was carried out by me. Further, this thesis has not formed the basis, neither has it been submitted for the award of any degree, diploma, fellowship, associateship or similar title of any university or institution.

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Lijo M. Jose
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To my dear parents, Jose Francis and Sali Jose.
Your unwavering love, sacrifices, and support have turned this dream into a reality.

ABSTRACT

This thesis investigates the concept of dominating sets and its variations in planar graphs, addressing a few significant conjectures and open problems in the field. Dominating sets have been widely studied in various graph classes, particularly planar graphs. Although there are many variations on the domination problem, this thesis primarily focuses on two important ones: total domination and independent domination. In addition, the connection between independence and domination is briefly explored.

Let G be a graph with the vertex set $V(G)$ and let $S \subseteq V(G)$. S is a *dominating set* of G if for any vertex $x \in V(G)$, x is either an element of S or x has an adjacent vertex in S. S is an *independent set* if no two vertices in S are adjacent. S is called an *independent dominating set* if it is both dominating and independent, while S is said to be a *total dominating set* if every element $x \in V(G)$ is adjacent to some vertex in S.

A *near-triangulation* is a planar graph embedded in the plane such that all its faces except possibly the outer one are bounded by three edges. A near-triangulation is called a *triangulated disc* when the boundary of the outer face of it is a simple cycle. Likewise, a near-triangulation is called a *triangulated planar graph* or simply a *triangulation* if the outer face is bounded by a triangle. A graph is a triangulation if and only if it is a *maximal planar graph*, which means that no more edges can be added without violating the planarity of the graph.

The research presented in this thesis includes three main contributions. The first contribution addresses and resolves a conjecture posed by Wayne Goddard and Michael A. Henning in *Journal of Graph Theory* in 2017, on total domination in triangulated discs. They conjectured that if G is a planar triangulation of order at least four, then there must exist two disjoint total dominating sets in G. Furthermore, they hypothesized that the same statement might hold for near-triangulations with minimum degree at least three. In our work, we confirmed both these conjectures by proving a slightly stronger result. The total domination problem is also studied in the literature as an equivalent graph coloring problem known as the coupon coloring problem. A graph G is said to have a k-*coupon coloring* if its vertex set can be partitioned into k color classes, such that every vertex in G has at least one neighbor in each of the k color classes. Note that these color classes need not be independent sets in the graph. Each color class in this coloring corresponds to a total dominating set in the graph. We viewed the problem from the perspective of coupon coloring and developed a method to color any

near-triangulation with minimum degree of three using two colors, resulting in a valid 2-coupon coloring. The result was published in *Journal of Graph Theory* in 2024.

The second contribution is on face-hitting dominating sets in planar triangulations. A *face-hitting set* of a plane graph G is a set of vertices in G that meets every face of G. In this work, we proved that the vertex set of every plane graph without isolated vertices and 2-length faces can be partitioned into two disjoint sets such that both sets are dominating and face-hitting. This result was a consequence of our attempt to address a question by Matheson and Tarjan from 1996. In *European Journal of Combinatorics*, 1996, they demonstrated that any triangulated disc on n vertices possesses a dominating set with a maximum size of $\frac{1}{3}n$. They further showed that this bound is optimal for triangulated discs. In the same paper, they conjectured that the domination number could be as small as $\frac{1}{4}n$ for sufficiently large triangulations. This conjecture has attracted significant attention in the graph theory community, with numerous efforts made to improve the bound or prove the conjecture. Despite these efforts, no improvement on the $\frac{1}{3}n$ bound for general triangulations was published until 2020, when Simon Špacapan [JCTB, 2020] provided a breakthrough by improving it to $17n/53$ for all $n > 6$. As a corollary of our theorem on face-hitting dominating sets in planar graphs, we established that every n-vertex simple plane triangulation has a dominating set of size at most $(1-\alpha)n/2$, where αn is the maximum size of an independent set in the triangulation. This result illustrates a connection between domination and independence in triangulations. Currently, the best known general bound for dominating sets in triangulations is by Christiansen, Rotenberg, and Rutschmann [SODA, 2024], who showed that every plane triangulation on $n > 10$ vertices has a dominating set of size at most $2n/7$. While their result represents a significant advancement, our corollary improves this bound for n-vertex plane triangulations containing a maximal independent set of size less than $2n/7$ or more than $3n/7$. We presented these results at WG 2024, where the paper was awarded the Best Student Paper Award.

For the third contribution, we proved a structural strengthening of the result by Matheson and Tarjan. As discussed previously, they proved that any triangulated disc on n vertices contains a dominating set with a maximum size of $\frac{1}{3}n$. In addition, Goddard and Henning [JAMC,2020] asked the question of whether it is possible to find three disjoint independent dominating sets in every triangulation. This would imply that for any triangulation G, the size of the smallest independent dominating set known as the *independence domination number* $\iota(G)$ is at most $n/3$. In 2023, Botler, Fernandes, and Gutiérrez [Electron. J. Comb. 2023] demonstrated that for any triangulation G on n vertices, $\iota(G) < 3n/8$. Moreover, they showed that if the minimum degree of the triangulation is at least five, then $\iota(G) \leqslant n/3$. Based on these findings, they conjectured that $\iota(G) \leqslant n/3$ for all triangulations. We confirmed this conjecture by proving that every planar triangulation on n vertices has a maximal independent set of size at most $n/3$.

In the concluding parts of the chapters in which we address our contributions, we have highlighted various open problems and existing research gaps associated with each contribution. In addition, we have proposed some new problem of our own. In the final chapter, we address some open problems in detail. In addition, we highlight some of the existing research gaps and touch upon a problem we are presently tackling, which involves the development of a linear-time algorithm for our second result.

Keywords: Dominating sets, Total dominating sets, Coupon coloring, Face-hitting sets, Polychromatic coloring, Independent sets, Independent dominating sets, Maximal independent set, Triangulations, Triangulated discs, Near-triangulation.

AMS Subject Classification: 05C10, 05C15, 05C69, 68R10, 68R05

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PUBLICATIONS

[1] P. Francis, Abraham M. Illickan, Lijo M. Jose, and Deepak Rajendraprasad. "Disjoint total dominating sets in near-triangulations." In: *Journal of Graph Theory* 105.1 (2024), pp. 68–77.

[2] P. Francis, Abraham M. Illickan, Lijo M. Jose, and Deepak Rajendraprasad. "Face-Hitting Dominating Sets in Planar Graphs." In: *Graph-Theoretic Concepts in Computer Science*. Ed. by Daniel Kráľ and Martin Milanič. Cham: Springer Nature Switzerland, 2024, pp. 211–219.

[3] P. Francis, Abraham M. Illickan, Lijo M. Jose, and Deepak Rajendraprasad. "Maximal Independent Sets in Planar Triangulations." Accepted for publication in Journal of Graph Theory. Revised version submitted. (under review).

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1 INTRODUCTION

The innate curiosity of humans often leads to remarkable scientific breakthroughs and progress. Graph theory, a relatively new branch of mathematics, traces its origins back to the early 18^{th} century. The keenness of Königsberg's residents to discover a path through the city that would traverse each of its seven bridges exactly once ignited the area of study now known as graph theory. In 1736 Leonhard Euler wrote a paper [49] which formally proved a negative resolution to the problem. This paper is often considered as the first literature in graph theory. Euler tackled the problem by converting it into a graph-theoretical model. He depicted the city's land areas as vertices and the bridges as edges connecting these vertices. This transformation enabled Euler to discuss the problem in terms of graph traversal, paving the way for what would become a fundamental topic in graph theory. His work demonstrated how theoretical mathematical methods could solve practical problems, laying the foundation for systematic graph studies. His method of addressing the issue not only resolved the particular challenge, but also established basic principles that have been further advanced and utilized in numerous disciplines. Two centuries later in the year 1936, Dénes König authored the first book entirely dedicated to the subject of graph theory. Although it started modestly, graph theory has grown to be applicable in various fields due to its capacity to represent intricate relationships between objects.

A variety of graph theory problems have their origins in fascinating puzzles. The first documented research on domination in graphs was motivated by an intriguing chessboard problem. In the mid-1800s, chess enthusiasts discovered two tricky chessboard puzzles. The first, which is popularly known as the *eight queens problem* is to place the maximum number of queens on a standard chessboard so that no two queens could attack each other. A queen on a chessboard can move any number of unoccupied squares along horizontal, vertical, or diagonal lines in a single move putting all such squares within her attacking range. Therefore, to prevent mutual attacks, only a single queen can occupy each row. Given that there are only eight rows on the chessboard, the highest possible number of queens that can participate in the solution is eight, and hence the name. German chess player, named Max Bezzel [15] in 1848, asked how many possible configurations are possible for the eight queens problem. This challenge can be generalized to any $n \times m$ rectangular board with any type of chess pieces. In 1910, Ahrens [1] demonstrated that for any positive integer $n \geqslant 4$, it is always feasible to position n queens on an $n \times n$ chessboard such that none of the queens attacks one another. Treat each square on an $n \times n$ chessboard as a vertex and connect two vertices with an edge if a queen can move between them in a single move; this graph is referred to as *queens graph* in the literature. A set of vertices in a graph forms an *independent set* if no two vertices in the set are connected by an edge. In graph theory terminology, the eight queens puzzle translates to identifying a largest possible independent set (*maximum independent set*) in the 8×8 queens graph. Figure 1.1 demonstrates one possible solution to the eight queens problem. In 1850, Franz Nauck [96, 97] provided 92 distinct solutions to the problem, and in 1874, Pauls [99] demonstrated that these are the only possible ones.

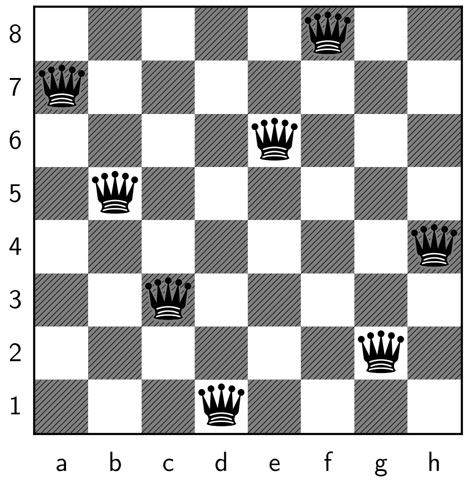

Figure 1.1: A solution to the eight queens problem

The second puzzle involves arranging the least number of queens on a chessboard such that every square is either occupied by a queen or can be attacked by one. This problem was popularized by C.F. de Jaenisch, a Finnish-Russian chess player and theorist, who documented it in 1862 [78]. It is called the *five-queens problem* due to the fact that any solution to this problem requires at least five queens. A set of vertices in a graph is a *dominating set* if every vertex outside this set is adjacent to at least one vertex from the set. While the challenge in the first problem was to place maximum number of queens here the challenge is to minimize the number of queens. Hence the five queens problem translates to identifying a smallest possible dominating set (*minimum dominating set*) in the 8×8 queens graph. A more generalized version of this problem considers the arrangement of queens on an $n \times m$ chessboard which is often referred as the *queens domination problem*. Jaenisch [78] correctly identified the least number of queens required to dominate an $n \times n$ chessboard for n up to eight. The corresponding values are $1, 1, 1, 2, 3, 3, 4, 5$ in order. Ahrens [1] in 1910 reported correctly that five queens are enough to dominate 9×9, 10×10 and 11×11 chessboards also. Jaenisch [78] also formulated another problem that merges the concepts of both puzzles. The task involves determining the smallest number of queens that can be placed on a chessboard in such a way that every square is either occupied by a queen or under attack by one, while ensuring that no queen attacks another. The generalization of this problem is known as the *queens independent domination problem*. In queens graph the challenge is to identify the smallest independent dominating set. It is important to observe that any resolution of the eight queens problem naturally results in the entire chessboard being within the attack range of at least one queen, thus forming a dominating set in the queens graph. However, the challenge of queens independent domination problem lies in finding the smallest sets that dominate while also being independent. Jaenisch [78] accurately determined the sizes of these sets for each $n \times n$ chessboard with n ranging up to eight. The corresponding values are $1, 1, 1, 3, 3, 4, 4, 5$. It indeed varies from the queen's domination problem. Another variant of the queens domination problem is the *queens total domination problem* introduced in 1892 by W.W. Rouse Ball [10]. In this case, the task is to determine the smallest number of queens that can be placed on an $n \times n$ chessboard such that every square, including those occupied by the queens, is under the attack range of some queen. In the independent domination problem, since no queen can be placed on a square that is under attack by another queen, a solution to the total domination problem cannot be used as a solution for the independent domination problem, and the reverse is also true. Figure 1.2 depicts a solution to the queens independent domination problem on 8×8 chessboard. Figure 1.3 gives a solution to the queens total domination problem on 8×8 chessboard. It should be noted that both of these solutions are acceptable for the queens domination problem.

While, discovering the largest independent set and the smallest dominating set are interesting problems on their own, there is another equally interesting challenge: partitioning the vertex set into the highest number of disjoint dominating sets and the least number of disjoint independent sets. Partitioning the vertex set is always feasible when dealing with independence and domination problems. For the independence set problem, even in the worst-case scenario, the vertex set of a graph with n vertices can be divided into n disjoint subsets, each containing a single vertex. Likewise, for the domination set problem, a single partition containing all n vertices is sufficient in the worst case. However, the situation becomes more complex with independent domination and total domination. In these cases, it might not always be possible to partition the entire vertex set so that each partition qualifies as an independent dominating set or a total dominating set. Yet, identifying the largest possible number of disjoint subsets of independent dominating sets or total dominating sets remains

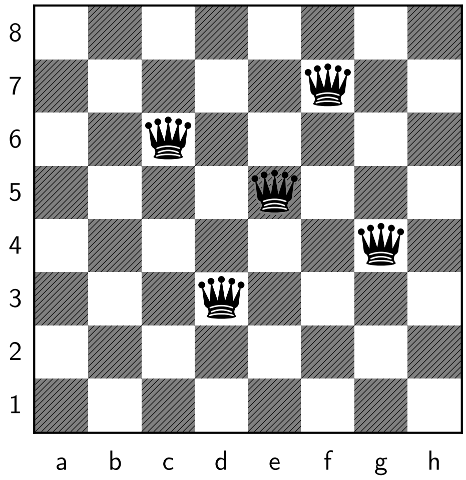

Figure 1.2: A solution to the queens independent domination problem on 8×8 chessboard. Note that this is also a solution to the queens domination problem but not a solution to the queens total domination problem.

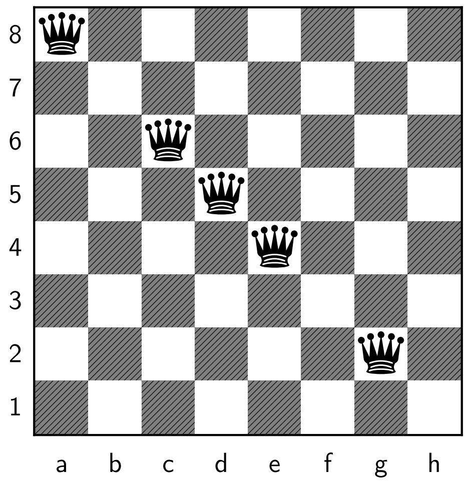

Figure 1.3: A solution to the queens total domination problem on 8×8 chessboard. Note that this is also a feasible solution to the queens domination problem but not a solution to the queens independent domination problem.

an equally significant problem. Thus, we can broaden the scope of dominating set and independent set problems to the wider category referred to as graph partitioning problems. In graph partitioning problems, we seek to divide the vertex set or the edge set of a graph into subsets according to some shared characteristics. A significant subset of graph partitioning problems is the set of graph coloring problems. The celebrated *Four Color Theorem* is the most well-known result in graph coloring. This problem, having a long historical background, also arose as an intriguing question. Instead of a chessboard, it was geographical maps that triggered the spark. Francis Guthrie, who was both a mathematician and a botanist, observed that any map could be colored with just four colors such that no two adjacent countries or regions would have the same color. He shared this observation with his brother, Frederick Guthrie, who was then studying at University College London under Augustus De Morgan. Frederick subsequently discussed the idea with De Morgan, and the earliest documented reference to this problem appears in a letter from De Morgan to William Rowan Hamilton dated 1852. This problem was referred to as the four color conjecture and became one of the most renowned and heavily researched challenges in the field of graph theory. To model the problem mathematically, one could represent the regions in the map as vertices and connect two vertices with an edge if the corresponding regions share a border. Now it becomes exactly a coloring problem in the corresponding graph, where the objective is to color the vertices with a maximum of four colors such that no two adjacent vertices get the same color. Figure 1.4 demonstrates how a graph derived from the map of India can be colored with four colors, thereby showcasing the four color problem on maps as an example of a graph coloring problem. The four color conjecture intrigued mathematicians for more than a century, leading to numerous unsuccessful attempts to prove it. The long journey to a proof is a testament to its difficulty. The eventual proof, achieved in 1976 by Kenneth Appel and Wolfgang Haken [6, 7] at the University of Illinois, was groundbreaking because it was the first major theorem to be proven using computer assistance. Among the notable attempts to solve the problem was one by British mathematician Alfred Kempe in 1879 [80]. Kempe's proof was widely accepted for more than a decade and earned him considerable recognition. However, in 1890, Percy Heawood [65] discovered a critical flaw in Kempe's argument, invalidating the proof. Despite this setback, Kempe's work was not in vain; his ideas laid the groundwork for important concepts in graph theory, and the techniques he introduced are still relevant today. Although he exposed the flaw in Kempe's proof, Percy Heawood demonstrated that Kembe's proof can be used to show that any map could be colored with no more than five colors, regardless of the complexity of the map. The five color theorem was simpler to prove and provided a partial solution to the map-coloring problem, offering a stepping stone toward the eventual proof of the four color theorem. The general version of this problem is coloring the vertices of a graph G with a minimum number of colors such that no two adjacent vertices share the same color. This number is known as the *chromatic number* and is denoted as $\chi(G)$. This coloring problem popularly known as the proper vertex coloring is studied on many graph classes. Observe that the proper vertex coloring problem is analogous to an independent set problem, where the objective is to partition the vertex set into the smallest number of independent sets. Here each independent set represents a unique

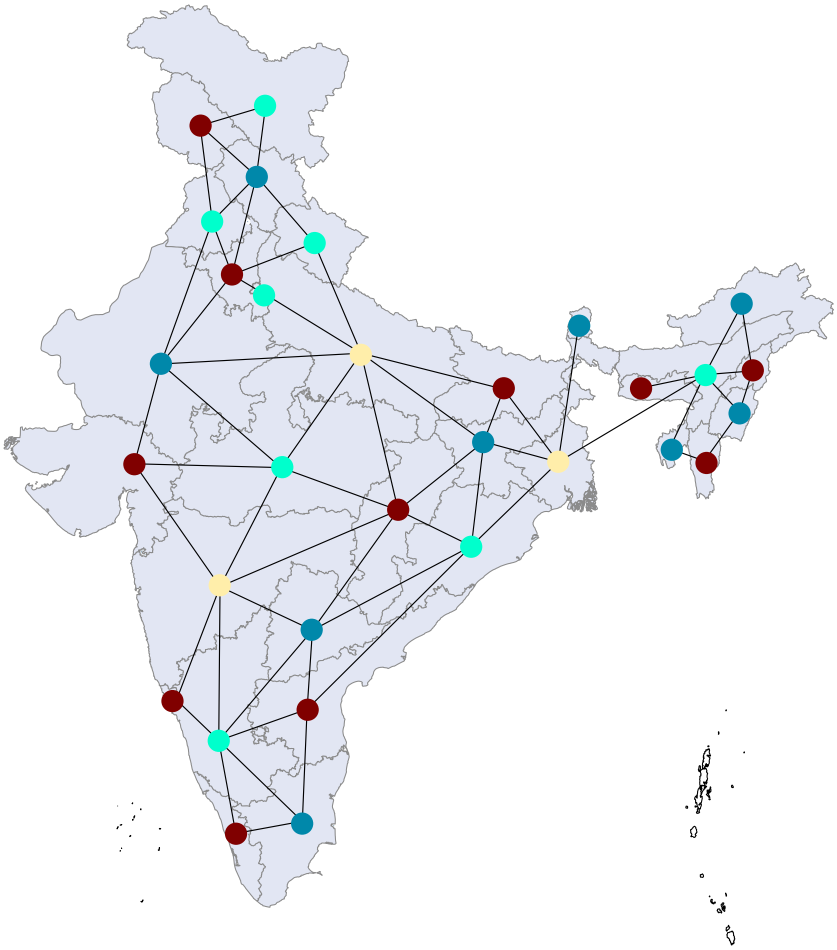

Figure 1.4: An example of how four coloring problem on maps can be studied as an equivalent graph coloring problem. Map of India with all 28 states and the union territories of Delhi NCR, Jammu and Kashmir and Ladakh colored using four colors.

color class. Figure 1.5 shows a proper coloring of the queens graph with nine colors as provided by M. R. Iyer and V. V. Menon [77] in 1966. They also researched the proper coloring of chessboards with various chess pieces. Different forms of the domination problem can also be examined by representing them as various coloring problems. This transformation sometimes simplifies the analysis. We make use of the four color theorem in several proofs in this thesis. Additionally, we examine certain domination problems by treating them as equivalent coloring problems.

Although Jaenisch's 1862 paper [78] is often considered as the first published research work on the topic of domination in graph theory, the mathematical concept of domination was formalized a century later, in 1962, by Berge [12] and Ore [98]. Ore provides the first use of the word "domination," for what König had previously referred to as *basis of the second kind* and Berge had referred to as *coefficient of external stability*. The first comprehensive book on domination in graphs was authored by Teresa W. Haynes,

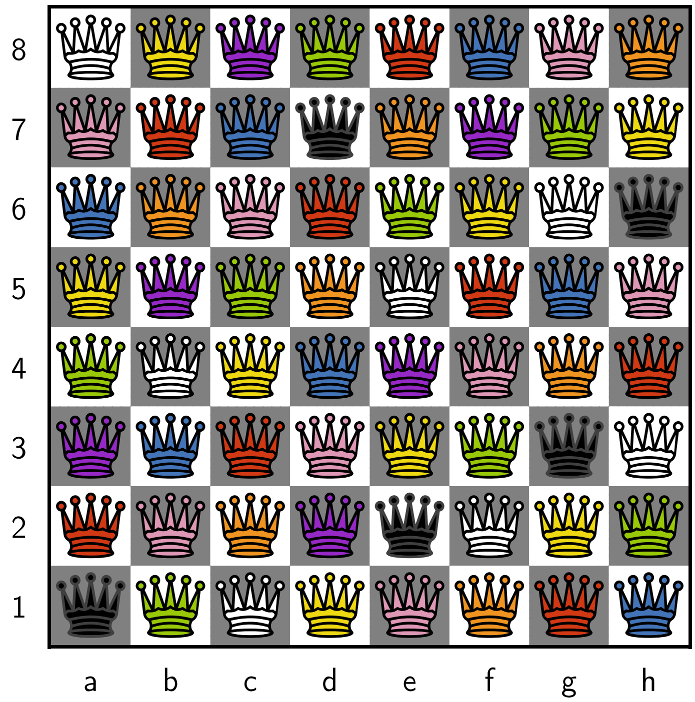

Figure 1.5: A partitioning of the queens graph using nine colors where each color class is an independent set i.e. queens in the same color class cant attack each other. [77]

Stephen Hedetniemi, and Peter Slater in 1998 [64]. Although the first documented research on domination emerged only after the mid-1800s, the roots of the concept can be traced back to the Roman Empire's defense strategies in the fourth century AD. In order to ensure the protection of their entire empire, the Romans had to deploy their troops strategically. However, due to the vast size of the empire and the limited number of available soldiers, it was impractical to assign one unit per area. Consequently, they had to develop a strategy to position their troops such that any unprotected region would have nearby troops that could be dispatched there if necessary. In 2004, Cockayne, Dreyer, Hedetniemi and Hedetniemi [38] formally characterized this defense strategy as a type of domination in graph theory. Domination theory has widespread applications across many fields, such as chemistry, communication networks, resource management, social networks, surveying, genetics, transportation, and electrical power systems, among others. Consequently, different variations of it have been investigated in different classes of graphs. We focus our work mainly on *planar graphs* and some subclasses of it. A graph is called planar if it can be embedded in a plane so that its edges intersect only at their endpoints and do not overlap elsewhere. When a planar graph is given a specific embedding in the plane, it is called a *plane graph*. The edges and vertices of a plane graph partition the plane into distinct regions. Each of these regions is referred to as a *face*. The four color theorem is one of the major results in planar graphs. The study of planar graphs traces its roots to the very inception of graph theory. In fact Euler's polyhedral formula from 1758 can be treated as a result on plane graphs. A *near-triangulation* is a planar graph embedded in the plane such that all its faces except possibly the outer one are bounded by three edges. When the boundary of the outer face of a near-triangulation is a simple cycle, then it is referred as a *triangulated disc*. Likewise if the outer face is bounded by a triangle then it is called as a *triangulated planar graph* or simply a *triangulation*. A graph is a triangulation if and only if it is a *maximal planar graph*, meaning that no more edges can be added without violating the planarity of the graph. Almost all the fundamental concepts such as connectivity, graph coloring, and duality, among others, have been extensively studied within these classes of graphs. The exploration of near-triangulations goes beyond abstract mathematics and into several practical areas. In computer graphics, these structures are instrumental in surface modeling and the creation of effective rendering and simulation algorithms. Within geographic information systems (GIS), triangulated planar graphs serve as the foundation for numerous algorithms dedicated to terrain modeling and map generation. Our research focuses on dominating sets and independent sets within near-triangulations.

1.1 TERMINOLOGY

A graph G can be formally described as a pair (V, E), where V represents a set of vertices and E consists of 2-element subsets of V called edges. Let $S \subseteq V$ be any subset of vertices of G. The *induced subgraph* $\langle S \rangle$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S. A graph G is referred to as a directed graph if its edges are represented by ordered pairs (directed edges); otherwise, it is termed an undirected graph. If both V and E are finite, then G is called a finite graph. All graphs considered in this thesis are finite and undirected. The vertex set and the edge set of G are denoted respectively by $V(G)$ and $E(G)$. In graph G the open neighborhood of a vertex v denoted as $N_G(v)$ is the set of all vertices adjacent to v, while the closed neighborhood of v denoted as $N_G[v]$ is $\{v\} \cup N_G(v)$. The *degree* $d_G(v)$ of a vertex v in G is the number of edges incident on v. The union of the open neighborhoods of all vertices in a set S is simply referred to as the neighborhood of S and is denoted $N_G(S)$. That is $N_G(S) = \bigcup_{x \in S} N_G(x)$. The subscripts may be omitted when the graph is clear from the context. In any plane graph we denote the set of vertices that lie on the boundary of a face f as $V(f)$. The minimum and maximum degree of G are denoted, respectively, by $\delta(G)$ and $\Delta(G)$. A vertex of degree exactly k, at least k and at most k in G are respectively termed k-*vertex*, k^+-*vertex* and k^--*vertex*. A cycle of length exactly k, at least k and at most k in G are respectively termed k-*cycle*, k^+-*cycle* and k^--*cycle*. Similarly, a face with exactly k, at least k and at most k edges in its boundary are termed respectively k-*face*, k^+-*face* and k^--*face*. We use the notations P_n, C_n, K_n and $K_{n,n}$ to denote a path on n vertices, a cycle on n vertices, a complete graph on n vertices, and a complete bipartite graph with n vertices in each part, respectively. The graph obtained by adding a universal vertex to P_n (resp. C_n) is called the n-*fan* F_n (resp. n-*wheel* W_n). A diamond is a 4-cycle (v_1, v_2, v_3, v_4) together with a chord v_1v_3. A 3-sun is a 6-cycle $(v_1, v_2, v_3, v_4, v_5, v_6)$ together with three chords that form a triangle (v_1, v_3, v_5). For a vertex v (resp. edge e) of G, the subgraph of G obtained by removing v (resp. e) from G is denoted by $G \setminus v$ (resp. $G \setminus e$). A *cut* in G is a partition of $V(G)$ into two disjoint subsets. An edge in G, which has one endpoint in each part of the cut, is said to *cross* the cut.

In a near-triangulation G, we refer to the cycle bounding the unbounded face as the *boundary* $B(G)$ of G and the vertices and edges in $B(G)$ as *boundary vertices* and *boundary edges* of G. The remaining vertices and edges are called *internal*. An internal edge between two boundary vertices is called a *chord*. Let f be a face in a plane graph G. If G is not connected, the boundary of f could be a disjoint union of closed walks. The *length* of f is the total length of all these walks, where the length of a walk is the number of edges in the walk (counting repetitions).

A set of vertices I of a graph G is called an *independent set* if no two vertices in I are adjacent. *Independence number* $\alpha(G)$ is the size of a maximum independent set in G. The set S is called *dominating* if $N(S) \cup S = V$ and *total dominating* if $N(S) = V$. A dominating set S of graph G is a *minimal dominating set* if no proper subset $S' \subset S$ is a dominating set of G. The minimum size of a dominating (resp. total dominating) set is called the *domination number* $\gamma(G)$ (resp. *total domination number* $\gamma_t(G)$) of G, while the maximum number of pairwise disjoint dominating (total dominating) sets in G is called the *domatic number* $\mathrm{dom}(G)$ (*total domatic number* $\mathrm{dom}_t(G)$) of G. The size of a largest minimal dominating set in G is often referred as the *upper domination number* $\Gamma(G)$. A set $S \subseteq V$ is an *independent dominating set* if S is both independent and dominating. The size of a smallest independent dominating set is called the *independent domination number* $\iota(G)$) of G. We use the notation $\beta(G)$ to denote the size of a smallest face-hitting set in G.

1.2 OUR WORK

This thesis focuses primarily on three investigations. The first explores total dominating sets in near-triangulations. The second delves into face-hitting dominating sets in planar graphs and the domination number of triangulations. The third examines independent dominating sets in triangulations. These three investigations were conducted together with P. Francis and Abraham Mathew Illickal, who, together with the author and advisor, contributed significantly.

1.2.1 *Total Dominating Sets in Near-triangulations*

Our research in this domain began with the examination of a conjecture proposed by Goddard and Henning [59] in 2017. They conjectured that if G is a planar triangulation of order at least four, then $\mathrm{dom}_t(G) \geqslant 2$. Furthermore, they hypothesized that the same might hold for triangulated discs with a minimum degree of at least three. We confirmed both these conjectures by proving a slightly stronger result (Theorem 3.1.1).

Result on total domination in near-triangulations

Theorem 3.1.1 Let G be a near-triangulation and V' be a subset of vertices of G containing all the vertices of degree at least three and at most two vertices of degree two. Then, there exist two disjoint subsets V_1 and V_2 of $V(G)$ such that each vertex $v \in V'$ has at least one neighbor each in V_1 and V_2.

In research, the above problem is also investigated as an analogous coloring challenge called the *coupon coloring* problem. We explored the conjecture by interpreting it as a coupon coloring problem. Graph coloring typically refers to proper vertex coloring, ensuring that no two adjacent vertices share the same color. The goal in the proper vertex coloring problem is to minimize the number of colors required to achieve a valid coloring. Coupon coloring, on the other hand, is a different vertex coloring challenge. A coloring of a graph with k colors is considered as a valid k-*coupon coloring* if every vertex's open neighborhood includes at least one vertex from each of the k color classes. In this case, the objective is to maximize the number of colors used. The *coupon coloring number* $\chi_c(G)$ is the maximum k for which the graph G is k-coupon colorable. Note that $\chi_c(G) \leqslant \delta(G)$ for every graph G. Bob Chen, Jeong Han Kim, Michael Tait, and Jacques Verstraete were the first to introduce the term coupon coloring [30]. They termed this problem as the coupon coloring problem, because one can visualize the colors as various types of coupons. After the coupons are distributed to the nodes, each node must gather all types of coupons from its neighbors. The coupon coloring of a graph can never be a proper coloring as the open neighborhood of any vertex v should also include a vertex with the same color as the color of v to satisfy the coupon coloring condition. It violates the requirement of proper coloring. Note that each color class in this coloring corresponds to a total dominating set in the graph. The coupon coloring problem has many direct applications, such as resource allocation problem in a network. The challenge arises in efficiently allocating resources to all nodes while ensuring that each node can only utilize the resources assigned to itself and its neighbors. Total domination plays a crucial role in maintaining network functionality even if a resource at one location fails. When viewed from the perspective of coupon coloring, the conjecture can be restated as follows: Does each planar triangulation with an order of at least four has a valid 2-coupon coloring? We developed a method to color any near-triangulation with a minimum degree of three using two colors, resulting in a valid 2-coupon coloring. Chapter 3 will discuss the proof in detail. These findings were published in the Journal of Graph Theory in 2023 [53].

1.2.2 *Face-hitting Dominating Sets in Planar Graphs*

A set of vertices S in a plane graph G is called a *face-hitting set* if every face in G has at least one vertex from S in its boundary. We proved that the vertex-set of every plane (multi-)graph without isolated vertices, self-loops or 2-faces can be partitioned into two disjoint sets so that both the sets are dominating and face-hitting (Theorem 4.1.1).

Result on face-hitting dominating sets in planar graphs

Theorem 4.1.1 Every plane graph G without isolated vertices, self-loops or 2-faces, has two disjoint subsets $V_1, V_2 \subseteq V(G)$, such that both V_1 and V_2 are dominating and face-hitting.

Lesley R. Matheson and Robert E. Tarjan in 1996 [92] demonstrated that any triangulated disc on n vertices possesses a dominating set with a maximum size of $\frac{1}{3}n$. They further showed that this is the optimal bound for triangulated discs by identifying a class of near-triangulations that require at least $\frac{1}{3}n$ vertices for domination. Thus, this bound is tight for triangulated discs. Furthermore, they conjectured in the same paper that the domination number could be as small as $\frac{1}{4}n$ for large enough triangulations. This bound cannot be reduced further. Consider numerous vertex disjoint tetrahedrons placed flat and interconnected to create a triangulation; such a graph would require at least $\frac{1}{4}n$ vertices for domination. Similarly, an octahedron, which is a triangulation on six vertices, needs at least two vertices for domination. This implies that n must exceed six. Despite multiple efforts to resolve the conjecture, an improvement on $\frac{1}{3}n$

for triangulations wasn't made until 2020 by Simon Špacapan[111]. However, there were notable findings on specific subclasses of triangulations. We began investigating this conjecture immediately after resolving the total domination conjecture by Goddard and Henning. Interestingly, when we received feedback on our first paper, a reviewer also recommended that we examine this conjecture. Unfortunately, we could not settle the conjecture and it still remains open. But as a corollary of our theorem on face-hitting dominating sets in planar graphs, we could show that every n-vertex simple plane triangulation has a dominating set of size at most $(1-\alpha)n/2$, where αn is the maximum size of an independent set in the triangulation (Corollary 4.3.1).

Result on dominating sets in triangulations

Corollary 4.3.1 Every n-vertex (simple) plane triangulation G with an independent set of size at least αn has $\gamma(G) \leqslant (1-\alpha)n/2$.

Currently, the best known general bound for domination in triangulations is by Aleksander BG Christiansen, Eva Rotenberg and Daniel Rutschmann [31] who showed that every plane triangulation on $n > 10$ vertices has a dominating set of size at most $2n/7$. Our corollary improves their bound for n-vertex plane triangulations which contain a maximal independent set of size less than $2n/7$ or more than $3n/7$. However, when we submitted the first draft of our results, the best known upper bound was $17n/53$ by Špacapan. Our corollary improves this bound for n-vertex plane triangulations which contain a maximal independent set of size either less than $17n/53$ or more than $19n/53$. We presented these results in WG 2024 and the paper won the best student paper award. Chapter 4 will provide details of this work.

1.2.3 *Independent Dominating Sets in Triangulations*

After putting considerable effort into tackling the 1996 conjecture by Matheson and Tarjan, we decided to investigate the possibility of any structural improvement to their result. As discussed earlier in [92] demonstrated that any triangulated disc on n vertices possesses a dominating set with a maximum size of $\frac{1}{3}n$. Goddard and Henning [60] asked whether there exist three disjoint independent dominating sets in every triangulation. This will imply that, for any triangulation G, the independence domination number $\iota(G) \leqslant n/3$. Fábio Botler and Cristina G Fernandes and Juan Gutiérrez [21] in 2023 proved that, for every triangulation G on n vertices, $\iota(G) < 3n/8$ and if the minimum degree is at least five, then $\iota(G) \leqslant n/3$. Based on these results, they conjectured $\iota(G) \leqslant n/3$ for every triangulation. We proved that every planar triangulation on n vertices has a maximal independent set of size at most $n/3$ affirming the conjecture (Theorem 5.1.1), and adding to the believe that the answer to Goddard and Henning's question might be positive.

Result on independent dominating sets in triangulations

Theorem 5.1.1 Every n-vertex triangulation has an independent dominating set of size at most $n/3$.

Goddard and Henning [60] also construct an infinite family of triangulations where the size of any independent dominating set is at least $6n/19$. Small triangulations such as the triangle and the octahedron have $\iota(G) = n/3$. However, it is not clear whether the upper bound of $n/3$ can be improved for large enough n. Chapter 5 provides a comprehensive examination of this work.

1.3 ORGANIZATION OF THE THESIS

The rest of this thesis is structured into five chapters. Chapter 2 is a literature review, offering a summary of relevant studies in this area. It consists of five sections: the initial section investigates research on dominating sets broadly; the second section concentrates particularly on domination within planar graphs; the third section delves into total domination; the fourth section looks into independent domination; and the last section investigates face-hitting dominating sets.

Chapter 3 is dedicated to our work on total dominating sets in near-triangulations. It begins by presenting the key result, along with its corollaries and an analysis of the tightness of the result. This is followed by a detailed proof of the key result. The chapter concludes by discussing open problems and conjectures related to total domination, highlighting potential directions for future research.

Chapter 4 focuses on face-hitting dominating sets in planar graphs and their application to solving the Matheson-Tarjan Conjecture in specific graph classes. The chapter is organized into three sections: the first discusses the key result and its tightness; the second provides a detailed proof of the result; and the third explores the application of this result to the Matheson-Tarjan Conjecture.

Chapter 5 presents our findings on independent dominating sets in triangulations. The chapter is divided into three sections: the first introduces the key result; the second provides a rigorous proof of the result; and the third concludes with remarks on the tightness of the result and a discussion of open problems, offering insights for further exploration in this area.

Lastly, Chapter 6 presents the concluding remarks of this thesis. It consists of two sections: the initial section investigates the computational complexity of the problems addressed in this research, providing insight into their algorithmic consequences; the subsequent section introduces a set of open issues, encapsulating possible research paths arising from this study.

2

LITERATURE REVIEW

This chapter reviews key findings in the area of graph domination, initially presenting some broad general results before concentrating on planar graphs. Important results related to two major variations of the domination problem, independent domination and total domination, are also explored. Additionally, the concept of face-hitting sets, which is also studied in the name of polychromatic coloring, is analyzed in the context of its relationship to domination problems. This overview sets the context for the contributions presented in this thesis. The areas of domination and independence have been thoroughly investigated within numerous graph classes. Conducting a comprehensive review of these topics is beyond the scope of this thesis; therefore, this literature survey discusses only key findings and focuses on those most relevant to our research. For an in-depth understanding on domination theory, the 2023 monograph by Haynes, Hedetniemi, and Henning [63] is suggested.

2.1 DOMINATING SETS

This section discusses some of the basic concepts in domination and some major results in the area. Haynes et al. [63]noted that more than 5000 articles on graph domination had been published when the book was written, and the field is growing rapidly. Different variations of the dominating set problem have been explored in the literature, but two variants that have attracted more focus are independent domination and total domination. Nearly every chapter in the above mentioned monograph has a separate section dedicated to independent domination and total domination. In his 1962 book, Ore [98] proved the following lemma, which essentially states that a vertex u is in a minimal dominating set S if and only if u is not dominated by any other vertex in S or there is a vertex in $V \setminus S$ dominated only by u in S.

Lemma 2.1.1. *A dominating set S of a graph G is a minimal dominating set if and only if, for any $u \in S$, $N(u) \subset V \setminus S$ or there exists a $v \in V \setminus S$ for which $N(v) \cap S = \{u\}$.*

Two significant research parameters within the study of domination in graphs are the domination number and the domatic number. The domination problem generally refers to a decision problem in which one must determine whether a graph G contains a dominating set with a size of at most k. Similarly, the total-domination and independent-domination problems are defined in an analogous manner. In their 1979 book [56], Garey and Johnson showed that for an arbitrary graph, this problem is NP-complete. They proved the result by showing that 3SAT can be reduced to the dominating set problem in polynomial time. In 1981, Dewdney demonstrated that the domination problem is NP-complete for both bipartite graphs and comparability graphs [43]. Booth and Johnson in 1982 [19] proved NP-completeness for the case of chordal graphs. In addition, the problem has shown to be NP-complete in numerous graph classes such as split graphs [14, 40], k-trees with arbitrary k [39] , and chordal bipartite graphs [94], among others. Although the domination problem is NP-complete for these graph classes, there exist certain classes of graphs for which finding a dominating set of size k can be done in polynomial time. Considerable research effort has been put into identifying classes of graphs in which the domination problem can be resolved within polynomial time. Cockayne, S. Goodman, S.T. Hedetniemi [34] in 1975 came up with an algorithm for trees which run in polynomial time (now known to have linear time algorithm). Similarly, polynomial-time algorithms have been developed for graph classes such as directed path graphs [19], and cocomparability graphs [83], among others. Some graph classes like permutation graphs [29], interval graphs [28], k-trees with fixed k etc. have linear time algorithms as well. Haynes et al. [63] observed that using the same proof technique used by Garey and Johnson in [56] for domination problem, it can be proved that the independent dominating set problem and the total dominating set problem are also NP-complete in general graphs.

The upper and lower bounds for the domination number in terms of different graph invariants are also studied extensively. According to Berge's observation in 1973 [13], if G is a graph of order n and v is any vertex in G, then v along with all vertices that are not its neighbors form a dominating set of G. Choosing v to be a vertex of maximum degree in G, the domination number satisfies $\gamma(G) \leqslant |V \setminus N(v)| = n - \Delta(G)$.

Theorem 2.1.2. *If G is a graph of order n, then $\gamma(G) \leqslant n - \Delta(G)$, and this bound is tight.*

This result is tight in the case of general graphs. For example, consider a star graph with k vertices and attach a pendant vertex to each of the k vertices. This construction results in a graph G with $2k$ vertices, where $\Delta(G) = k$ and $\gamma(G) = n - \Delta = 2k - k = k$. Some other notable results regarding the bounds on the domination number in relation to the maximum degree and the order of the graph are discussed below.

Walikar and Acharya [119] in 1979 proved that for any arbitrary graph G, $\gamma(G) \geqslant \frac{n}{1+\Delta(G)}$. They observed that within a minimum dominating set S of a graph G, each vertex in S dominates itself and also no more than $\Delta(G)$ additional vertices. Consequently, each vertex in S can dominate up to $\Delta(G) + 1$ vertices altogether. This leads to a maximum of $(\Delta(G) + 1)\,|S|$ unique vertices being dominated by the set S. They also demonstrated that the equality holds if and only if the closed neighborhoods of every vertex v in every minimum dominating set of G are pairwise disjoint and for each v, their degree $d(v) = \Delta(G)$. Note that the same observation can be used to prove the result for the total domination number $\gamma_t(G) \geqslant \frac{n}{\Delta(G)}$. In 1986 Marcu [91] gave the bound $\gamma(G) \leqslant \left\lfloor \frac{(n-\Delta(G)-1)(n-\delta(G)-2)}{n-1} \right\rfloor + 2$. In 1990, Flach and Volkmann [52] proved that for graphs without isolated vertices of, $\gamma(G) \leqslant \frac{1}{2}\left(n+1-(\delta(G)-1)\frac{\Delta(G)}{\delta(g)}\right)$. In 1980 Cockayne et al. [33] proved that if G is a connected graph that does not contain a dominating vertex, then $\gamma_t(G) \leqslant n - \Delta(G)$. Haviland [62] in 1991 proved a similar result for $\iota(G)$. They showed that if G is a graph of order n, then $\iota(G) \leqslant n - \Delta(G)$.

An alternative approach to constraining the domination number involves focusing on the order of the graph and its minimum degree. In 1962 Ore [98] proved that,

Lemma 2.1.3. *If G is a graph without isolated vertices and D is a minimum dominating set of G, then $V(G) - D$ is also a dominating set.*

As a direct consequence of Lemma 2.1.3 Ore [98] proved the following theorem,

Theorem 2.1.4. *If a graph G has no isolated vertices, then $\gamma(G) \leqslant \frac{n}{2}$.*

Payan and Xuong in 1982 [100] provided a description of graphs that achieve this bound. They showed that equality is achievable in Theorem 2.1.4 if and only if every component of G is a 4 cycle or a corona $H \circ K_1$ for some graph H. In 1973 Blank [17] proved that this bound can be improved to $2n/5$ if we restrict to connected graphs with $\delta(G) \geqslant 2$ and order at least eight. McCuaig and Shepherd [93] reproved this finding in 1989 and also provided a description of an infinite set of graphs that satisfy the equality. In 1996 Reed [105] proved that if G is a graph of order n with $\delta(G) \geqslant 3$, then $\gamma(G) \leqslant 3n/8$. In 2009, Shan et al. [109] improved Reed's proof to extend its applicability by reintroducing vertices of degree two. They proved that if G is a graph of order n with $\delta(G) \geqslant 2$ and n_2 is the number of vertices of degree 2 in G, then $\gamma(G) \leqslant \frac{3}{8}n + \frac{1}{8}n_2$. It is worth noting that for graphs with a minimum degree of two, if $n_2 < \frac{n}{5}$, this finding offers an improved bound of $\gamma(G) < 2n/5$, improving Blank's result. In 2009 Sohn and Xudong [110] gave an improved upper bound of $4n/11$ for graphs with minimum degree at least four. In 2021 Bujtás [23] gave a simpler proof for the same result. In the same paper she came up with an upper bound of $n/3$ for graphs with a minimum degree of at least 5. This was an improvement of the $5n/14$ upper bound by Xing et al. [123] for the same graph class. In 2016, for graphs where the minimum degree is at least 6, Bujtás and Klavžar [25] established the best known upper bound available until that point, which was $\frac{1702}{5389}n$. Subsequently, in 2021 Bujtás and Henning [24] improved this bound to $\frac{127}{148}n$. Bujtás use an approach of vertex weighting arguments and discharging methods, combined with a detailed case analysis for these proofs. Bujtás and Klavžar in [25] determined the upper bounds of the domination number for graphs having minimum degree of up to fifty. In addition, they formulated a general theorem to determine upper bounds on the domination number of a graph with a specified minimum degree. Alon in [4] gives a general upper bound for the domination number in terms of minimum degree. He proved that for any graph G, $\gamma(G) \leqslant \frac{n(1+\ln(\delta(G)+1))}{\delta(G)+1}$.

Substantial research has been conducted on the upper bound of the domination number in terms of the number of edges of the graph. In 1958 Berge [12] proved that if G is a graph with n vertices and m edges, then $\gamma(G) \geqslant n - m$, with equality if and only if each component of G is a star. In 1965, Vizing [118] established a result that provides a limit for the size of a graph based on its order and domination number.

Theorem 2.1.5. *If* G *is a graph on* n *vertices and* m *edges, then* $m \leqslant 1/2(n-\gamma(G))(n-\gamma(G)+2)$.

They also proved that this bound is tight by constructing a family of graphs that attains this equality. As a corollary to theorem 2.1.5 we can obtain the upper bound $\gamma(G) \leqslant n+1-\sqrt{1+2m}$.. There were several improvements on Vizing's theorem. In 1999 Rautenbach [104] proved that for every graph G without isolated vertices and of order n, the size $m \leqslant \Delta n - (\Delta+1)\gamma(G)$.

2.2 DOMINATION IN PLANAR GRAPHS

The domination problem is NP-hard even when restricted to planar graphs of maximum degree three [56]. In 1996, MacGillivray and Seyffarth [88] gave upper bounds on domination number for planar graphs with diameters of two and three, which are three and ten, respectively. This indicates that the domination number for these graphs can be computed in polynomial time. Additionally, they established that the domination number for outerplanar graphs with diameters of two and three is capped at two and three, respectively. In 2002 Goddard and Henning [57] proved that there is a unique planar graph of diameter two with domination number three, and all other planar graphs of diameter two have domination number at most two. They also proved that every planar graph of diameter three and of radius two has a domination number of at most six. Further, they proved that every sufficiently large planar graph of diameter three has a domination number at most seven. In 2006, Dorfling, Goddard, and Henning [44] enhanced these findings by showing that any planar graph with a diameter of three and a radius of two has a total domination number and consequently, domination number at most five. Further, they demonstrated that any adequately large planar graph with a diameter of three has domination number at most six, which is

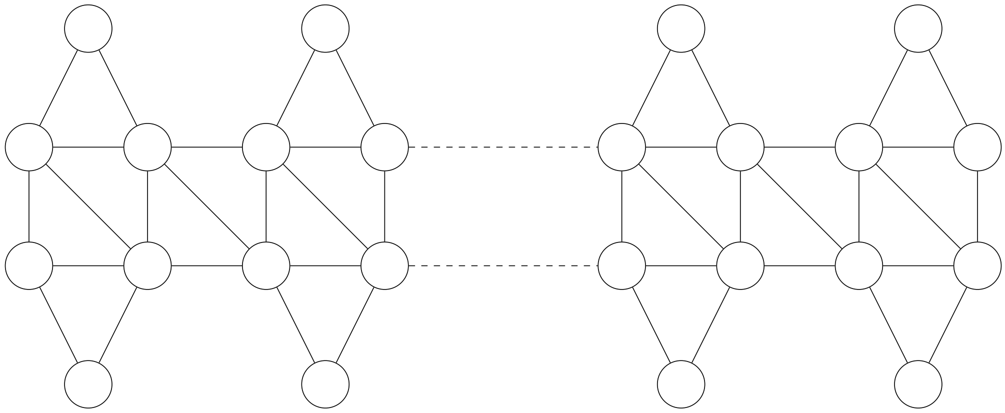

Figure 2.1: Class of triangulated discs that have no dominating sets of size less than $n/3$

the best possible bound. Additionally, they stated that if generalized to any planar graph with a diameter of three, then the upper bound is nine. Numerous researchers have explored the concept of domination number in planar graphs. Among these, the work by Matheson and Tarjan [92] on triangulated discs stands out as one of the most renowned contributions in the study of domination within planar graphs. In 1996, they investigated dominating sets within triangulated discs and established a precise lower bound for their domatic number. They proved the following:

Theorem 2.2.1. *If* G *is a triangulated disc, then* $\mathrm{dom}(G) \geqslant 3$*.*

As an immediate corollary to this theorem, we have an upper bound for the domination number.

Corollary 2.2.2. *If* G *is a triangulated disc of order* n*, then* $\gamma(G) \leqslant n/3$*.*

The proof method applied by Matheson and Tarjan was both novel and intriguing. They showed that every triangulated disc has an *outward numbering*. Outward numbering of a triangulated disc assigns numbers to the vertices from 1 to n. For each i, the vertices labeled from 1 to i form a triangulated disc. All other vertices fall outside this disc. This outward numbering technique has been used extensively to prove various results about triangulated discs. It also inspired the development of widely used techniques, such as the Schnyder realizer [107]. They presented a linear time algorithm that employs outward numbering to generate three disjoint dominating sets. The same paper also contains an example of a family of maximal outerplanar graphs which have no dominating sets smaller than $n/3$. However, they suspected that for triangulations with a sufficiently large number of vertices the bound could be smaller. They note that an octahedron (a triangulation on 6 vertices) has no dominating set of size smaller than 2. The following conjecture still remains an open problem even after 28 years.

Conjecture 2.2.3 ([92]). For sufficiently large n, any planar triangulation on n vertices has a dominating set that contains at most $1/4$ of the vertices.

This conjecture has attracted interest from numerous researchers worldwide. Despite improvements in certain subclasses, no progress was made in improving the upper bound of $n/3$ for domination number in general triangulations until 2020. In 2020 Špacapan [111], proved that every plane triangulation on $n > 6$ vertices has a dominating set of size at most $17n/53$. This upper bound was improved on triangulations with order at least 10 by Christiansen, Rotenberg and Rutschmann [31] to $2n/7$.

There were also some notable results in some subclasses of triangulated discs. King and Pelsmajer [81] in 2010 confirmed the conjecture on triangulations with a maximum degree of at most six. Liu and Pelmajer [86] in 2011 improved this result by proving that there is a constant c such that any n-vertex plane triangulation with maximum degree at most 6 has a dominating set of size at most $n/6 + c$. Extensive research has been conducted on outerplanar graphs, especially with regard to domination in *maximal outerplanar graphs*. Since a maximal outerplanar graph is also a triangulated disc, it can be easily observed from Corollary 2.2.2 that for any maximal outerplanar graph the domination number is at most $n/3$. However, this result was already solved before it was proved on triangulated discs. In 1978 Fisk [51] gave a proof for Chvátal's Watchman Theorem [32] on the famous art gallery problem, which also showed that the domination number for a maximum outerplanar graph of order $n \geqslant 3$ is at most $n/3$. In 2013 Campos and Wakabayashi [26] proved that if G is an n-vertex maximal outerplanar graph then $\gamma(G) \leqslant (n+t)/4$ where t is the number of 2-degree vertices in the graph. Later in the same year, Tokunaga [117] came up with the same result independently but with a much simpler proof using a coloring technique. He used a critical lemma that states that any maximal outerplanar graph G can be 4-colored so that every 4-cycle in it has all four colors. Let G be a maximal outerplanar graph with t 2-degree vertices. He constructs a supergraph G' by adding t additional vertices connecting them to the 2-degree vertices and one of its neighbors. G' will also be a maximal outerplanar graph, and each of the four color classes dominates all the vertices in G' that were originally in G. In 2016 Li et al. improved the result. They showed that for any maximal outerplanar graph of order at least three, $\gamma(G) \leqslant (n+k)/4$. Here k is the number of pairs of consecutive 2-degree vertices with distance at least 3 on the outer cycle. With the help of this theorem they also proved that for any Hamiltonian triangulation of order at least seven, $\gamma(G) \leqslant 5n/16$. Note that this result does not prove Conjecture 2.2.3 on Hamiltonian triangulations and the general bound provided by Christiansen et al is better. As every four connected triangulations are Hamiltonian [122], this conclusion also holds with four connected triangulations. Plummer, Ye, and Zha [102] proved that $\gamma(G) \leqslant \max\{\lceil 2n/7 \rceil, \lfloor 5n/16 \rfloor\}$ for every 4-connected plane triangulation G and later, in [103], that $\gamma(G) \leqslant 5n/16$ for every Hamiltonian triangulation on $n \geqslant 23$ vertices. In the same paper they also proved that if G is a maximal outerplanar graph on n vertices whose outer cycle contains k pairs of consecutive 2-degree vertices separated by distance at least 3 along the outer cycle of G, then $\gamma(G) \leqslant \lceil \frac{n+k}{4} \rceil$. In 2010, Hojo, Kawarabayashi and Nakamoto [74] extended the result of $dom(G) \geqslant 3$ to every triangulation on the torus and the Klein bottle.

2.3 TOTAL DOMINATION

Even though it was studied in various contexts, the concept of total domination in graphs was first formally introduced by Cockayne, Dawes, and Hedetniemi [33] in 1980. They introduced the terms total domatic number and total domination number for the first time and also gave some upper bounds for both. They observed that if S is a minimal total dominating set of a connected graph G then each vertex $v \in S$ has the property that either there exists a vertex $w \in (V \setminus S)$ such that $N(w) \cap S = v$ or the induced subgraph $\langle S \setminus \{v\} \rangle$ contains an isolated vertex. In the same paper they also proved that $\gamma_t(G) \leqslant 2n/3$ for any connected graph of order at least three. This bound is tight; In 2000 Brigham et al. [22] characterized the connected graphs that achieve equality in this upper bound. They showed that the upper bound is achieved if and only if the graph is C_3, C_6 or a graph obtained by connecting a 2-path to every vertex of any arbitrary connected graph F (also known as corona graph $F \circ P_2$). If we restrict the minimum degree of connected graphs then better upper bounds are possible for the total domination number as well. In 1995 Sun [113] proved that if G is any connected graph with $\delta(G) \geqslant 2$, then $\gamma_t(G) \leqslant \lfloor \frac{4}{7} \rfloor (n+1)$. This result was improved by Henning [69] in 2000, he proved that if G is connected graph of order at least 11 and with $\delta(G) \geqslant 2$, then $\gamma_t(G) \leqslant 4n/7$. He also characterized the graphs that meet the equality as well as those graphs with lower order that deviate from this upper bound. In the same year Favaron et al. showed that the upper bound $4n/7$ can be improved to $7n/13$ for connected graphs with a minimum degree of at least three. In 2004 this result was improved to $n/2$ by Archdeacon et al [9]. For minimum degrees four, five and six the best known upper bounds are $3n/7$ (Thomassé and Yeo [115]), $\frac{2453}{6500}n$ (Dorfling and Henning [45]) and $4n/13$ (Henning and Yeo [72]) respectively.

A total domatic partition is partitioning the vertex set of a graph into disjoint total domatic sets. The maximum number of total domatic partitions possible for a graph G is denoted as the total domatic number $dom_t(G)$. Note that if a graph has isolated vertices, then it does not have a total dominating set, and hence the total domatic partition makes sense only in isolate free graphs. There are at least two other ways in which the total domatic number is studied in the literature. A k-coloring of the vertices of a graph G is called a k-*coupon coloring* if every vertex sees at least one vertex of each color in its open neighborhood [30]. This is also known as *total domatic coloring* [54, 55] and as *thoroughly dispersed coloring* [59]. The *coupon chromatic number* $\chi_c(G)$ of a graph G is the maximum k for which G has a k-coupon coloring. It is easy to see that $\mathrm{dom}_t(G) = \chi_c(G)$ since every color class in a coupon coloring has to be a total dominating set. A graph G is k-coupon colorable if $\chi_c(G) \geqslant k$. We will present our proof for the theorem on total domination on near-triangulations in the form of coupon coloring (Theorem 3.2.1). Zelinka [125] observed that each part of a total domatic partition of a graph G is a total dominating set. Therefore, $\mathrm{dom}_t(G) \leqslant n/\gamma_t(G)$. Also, since at least two vertices should be in any total dominating set $\mathrm{dom}_t(G) \leqslant \lfloor n/2 \rfloor$. Haynes et al. [63] lists these tight upper bounds for the domatic number of graphs without isolated vertices; $\mathrm{dom}_t(G) \leqslant \delta(G)$, $\mathrm{dom}_t(G) \leqslant \mathrm{dom}(G) \leqslant 2\mathrm{dom}_t(G) + 1$. Zelinka [125] also proved that no minimum degree is sufficient to guarantee the existence of a 2-coupon coloring. He proved it by showing the existence of a bipartite graph G for any given minimum degree and $\chi_c(G) < 2$. Further in the same paper he also proved that $\mathrm{dom}_t(G) \geqslant \left\lfloor \frac{n}{n-\delta(G)+1} \right\rfloor$. Cockayne, Dawes and Hedetniemi [33] observed that, for a complete graph K_n with $n \geqslant 2$ vertices, the total domination number $\mathrm{dom}_t(K_n)$ is $\lfloor \frac{n}{2} \rfloor$. Similarly, for any complete bipartite graph $K_{m,n}$ where $1 \leqslant m \leqslant n$, $\mathrm{dom}_t(K_{m,n}) = m$. They further observed that a cycle C_n has $\mathrm{dom}_t(C_n) = 2$ if and only if n is a multiple of four. Zelinka identified that any wheel W_n with $n \geqslant 3$, has exactly two total domatic partitions. In [126], Zelinka obtained the characterization of r-regular bipartite graphs with $\mathrm{dom}_t(G) = r$. Akbari et al., [2] gave a criterion for cubic graphs to have total domatic number at least two. Heggernes and Telle [68] showed that the problem of deciding whether a graph has two disjoint total dominating sets in NP-complete even for bipartite graphs. Henning and Peterin [71] provided a constructive characterization of graphs that have two disjoint total dominating sets. In 2012 Aram et al. [8] showed that for a random r-regular graph G, $\frac{r}{3\ln(r)} \leqslant \mathrm{dom}_t(G) \leqslant r-1$. Chen et al., [30] shown that every r-regular graph has $\mathrm{dom}_t(G) \geqslant (1-o(1))r/\log r$ as $r \to \infty$, and the proportion of r-regular graphs for which $\mathrm{dom}_t(G) \leqslant (1+o(1))r/\log r$ tends to 1 as $|V(G)| \to \infty$. In [82], Koivisto et al., showed that it is NP-complete to decide whether $\mathrm{dom}_t(G) \geqslant 3$ where G is a bipartite planar graph of bounded maximum degree. Also, they have shown that if G is split or k-regular graph for $k \geqslant 3$, then it is NP-complete to decide whether $\mathrm{dom}_t(G) \geqslant k$.

The investigation of total domatic partitions in planar graphs is a well-explored field of research. Since for any graph G, $\mathrm{dom}_t(G) \leqslant \delta(G)$ and since the minimum degree of a planar graph is at most five it is trivial that total domatic number of any planar graph is at most five. Goddard and Henning [59], improved this upper bound to four.

Theorem 2.3.1. *If* G *is planar graph, then* $\mathrm{dom}_t \leqslant 4$.

They further identified specific cases where equality is achieved and broadened this proof to apply to toroidal graphs, for which the upper bound is five. They put forward a conjecture in the paper that if G is a planar triangulation of order at least four, then $\mathrm{dom}_t(G) \geqslant 2$. They also speculated that this could be true for near-triangulations with minimum degree at least three. As mentioned in Section 1.2 we began our research investigating these conjectures and were able to prove both of them. We discuss some notable attempts on this conjecture and some interesting partial results in Chapter 3.

2.4 INDEPENDENT DOMINATION

The exploration of the idea of independent sets in graphs began before the study of dominating sets. While the interesting challenge in domination involves finding a minimum dominating set, the independent set problem focuses on identifying maximum independent set. The independence problem is studied as many equivalent variants in the literature. Identifying independent sets within a graph is equivalent to identifying cliques in its complement graph. The survey paper by Dainyak and Sapozhenko [42] provides a comprehensive catalog of the results on independent sets. Determining the minimum number of independent partitions of the vertices of a graph corresponds to finding the chromatic number $\chi(G)$. Extensive literature exists on these subjects, which fall outside the scope of this thesis. One direction of research on this topic is bounding the independence number $\alpha(G)$. Caro [27] and Wei [120] independently proved that for any graph G, $\alpha(G) \geqslant \sum_{v \in V(G)} (1 + d_G(v))^{-1}$. There were some notable studies and improvements to this result in specific graph classes like triangle free graphs, r-colorable graphs, triangle free planar graphs, etc. An upper bound in terms of the number of edges e and the maximum degree Δ for the independence number is given as an exercise in [121]. The result credited to Kwok in the book proves that $\alpha(G) \leqslant n - e/\Delta(G)$.

By the Four Color Theorem, we know that for any planar graph G, $\alpha(G) \geqslant n/4$. This result is tight. In 1973, while the Four Color Theorem was still unproven, Erdős [13] conjectured that $\alpha(G) \geqslant n/4$ for any n-vertex planar graph G. This is an invitation to prove this without using the Four Color Theorem. Albertson [3] in 1976, without using the Four Color Theorem, proved that every n-vertex planar graph G has $\alpha(G) \geqslant 2n/9$. Cranston and Rabern [41] improved this to $3n/13$ in 2016. Interestingly, the maximum independent sets have received considerably more attention in triangle-free planar graphs than in triangulations. From Grötzsch's theorem that every triangle-free planar graph is 3-colorable [61], it follows that n-vertex triangle-free planar graphs have an independent set of size at least $n/3$. Heckman and Thomas [67] proved that every triangle-free planar graph on n vertices with maximum degree three has an independent set with size at least $3n/8$. Steinberg and Tovey [112] proved that every triangle-free planar graph has an independent set of size at least $(n+1)/3$ and showed that this lower bound is tight for an infinite family of graphs $\mathcal{G}$. Dvořák et al., [47] improved the same to $(n+2)/3$ except for the same infinite families of graphs $\mathcal{G}$. Finding the size of a maximum independent set is NP-complete in planar graphs and same is true even if we restrict to triangle-free or cubic planar graphs [90]. Dvořák and Mnich [48] proved that if a triangle-free planar graph of order n does not have an independent set larger than $(n+k)/3$, then its tree-width is $O(\sqrt{k})$.

Even though studied earlier, the term independent domination and the notation $\iota(G)$ for independent domination number were introduced by Cockayne and Hedetniemi in 1974 [35, 37]. The survey paper authored by Goddard and Henning [58] offers an excellent compilation of the results on independent domination in graphs. A maximal independent set is also a dominating set. Berge [12, 13] proved that a subset of vertices in a graph G is maximal independent if and only if it is independent and minimal dominating. They also proved that $\gamma(G) \leqslant \iota(G) \leqslant \alpha(G) \leqslant \Gamma(G)$. Alen and Lasker proved that if G is a claw-free graph, then $\gamma(G) = \iota(G)$. In 1979 Bollobás and Cockayne [18] extended this result to prove that, for $k \geqslant 3$, if G is $K_{1,k}$-free, then $\iota(G) \leqslant (k-2)\gamma(G) - (k-3)$. They also proved that if G is an isolate-free graph of order n, then $\iota(G) \leqslant n + 2 - \gamma(G) - \frac{n}{\gamma(G)}$. Favaron [50] in 1988 conjectured that if G is a graph of order n with $\delta(G) = \delta$, then $\iota(G) \leqslant n + 2\delta - 2\sqrt{\delta n}$. Sun and Wang [114] proved this to be true in 1999. Haviland [62] in 1991 gave an upper bound for independent dominating number in terms of the maximum degree of the graph. She proved that for any graph G with order n, $\iota(G) \leqslant n - \Delta(G)$. In the same paper, she also studied the independent domatic number of graphs with a bounded minimum degree. She proved that if $\frac{1}{4}n \leqslant \delta(G) \leqslant \frac{2}{5}n$ then, $\iota(G) \leqslant \frac{2}{3}n - \delta(G)$ and if $\frac{2}{5}n \leqslant \delta(G) \leqslant \frac{1}{2}n$ then, $\iota(G) \leqslant \delta(G)$. MacGillivray and Seyffarth in [89] proved that if G is a connected graph on n vertices with chromatic number $k \geqslant 3$, then $\iota(G) \leqslant (k-1)n/k - (k-2)$. Combining this with the Four Color Theorem [6, 7], we get $\iota(G) \leqslant 3n/4 - 2$ for any planar graph G. In the same paper, they also show that the upper bound can be improved to $\lceil n/3 \rceil$ if we restrict to planar graphs with diameter 2. In 2020 Goddard and Henning [60] proved that if G is a planar graph of order n with $\delta(G) \geqslant 2$, then $\iota(G) \leqslant n/2$. They also proved that if G is an outerplanar graph with minimum degree at least 2, then G has two disjoint independent dominating sets. In 2023 they improved this result on 2-connected outerplanar graphs. They proved if G is any 2-connected outerplanar graph of order n then $\iota(G) \leqslant (2n+1)/5$.

Although Cockayne and Hedetniemi introduced the concept of idomatic number (maximum number of pairwise disjoint independent dominating sets) $dom_i(G)$ in [36] the term was first used by Zelinka in [124]. Dunbar et al. [46] studied idomatic partition as a special proper coloring of graphs where every vertex sees all colors in its closed neighborhood. They called this coloring the *fall coloring*. Note that there are many graphs that do not have an idomatic partition: C_5 for example. The graphs with independent domatic partitions are known as *idomatic graphs*. Cockayne and Hedetniemi [36] characterized certain classes of graphs that are idomatic and those that are not. They showed that for any graph G, if $\chi(G) > dom(G)$ or if $\chi(G) > \delta(G)+1$ then G is not idomatic. In 1983 Zelinka proved that for any integers a and b with $2 \leqslant a \leqslant b$, there exists a graph G such that $dom_i(G) = a$ and $dom(G) = b$. Laskar and Lyle [84] proved that no hypercube has a fall 3-coloring, but for each $k \geqslant 2, k \neq 3$, there exist sufficiently large hypercubes with fall k-coloring. Lauri and Mitillos [85] observed that if G is a uniquely k-colorable graph, then G is fall k-colorable. They also proved that maximal outerplanar graph G has a fall k-coloring if and only if $k = 3$. Heggernes and Telle [68] showed that it is NP-hard to determine whether any graph G has $dom_i(G) = k$ for any fixed $k \geqslant 3$. Henning et al. [70] showed that it is NP-complete to decide whether a given graph has two disjoint independent dominating sets. Goddard and Henning [60] asked whether there exist three disjoint independent dominating sets in every triangulation. This will imply that for any triangulation G, $\iota(G) \leqslant n/3$. Goddard and Henning [60] also constructed an infinite family of triangulations where the size of any independent dominating set is at least $6n/19$. Small triangulations like the triangle and the octahedron have $\iota(G) = n/3$. However, it is not clear whether the upper bound of $n/3$ can be improved for large enough n. Botler, Fernandes and Gutiérrez [21] recently proved that, for every triangulation G on n vertices, $\iota(G) < 3n/8$ and if the minimum degree is at least five, then $\iota(G) \leqslant n/3$. Based on these results, they conjectured $\iota(G) \leqslant n/3$ for every triangulation. We proved these conjectures to be true and the findings are discussed in Chapter 5.

2.5 FACE-HITTING SETS

A k-coloring $\psi : V(G) \to [k]$ is called *polychromatic* if each color class of ψ is a face-hitting set of G. The polychromatic number $p(G)$ of G is the largest number k such that there is a polychromatic k-coloring of G. Bose et al. [20] used the idea of polychromatic colorings of triangulations in the context of guarding polyhedral terrains. They showed that there is an infinite family of plane triangulations where the smallest face-hitting set has size $\lfloor n/2 \rfloor$. For a plane graph G, let $g(G)$ denote the size (the number of edges that form the boundary) of a smallest face in G. Alon et al. [5] proved that for any plane graph G with $g(G)$ at least 3, $p(G) \geqslant \left\lfloor \frac{3g-5}{4} \right\rfloor$. They also showed that this bound is nearly tight, as there are plane graphs for which $p(G) \leqslant \left\lfloor \frac{3g+1}{4} \right\rfloor$. It is easy to observe that for every plane triangulation G, $p(G)$ is either 2 or 3. As a consequence of Heawood's theorem [66], $p(G) = 3$ for a plane triangulation G if and only if G is Eulerian (the degree of each vertex is even) [5]. Hoffmann and Kriegel [73] showed that every 2-connected bipartite plane graph admits a polychromatic 3-coloring by showing that every such graph can be transformed into an Eulerian triangulation by adding edges. They make use of a special walk in the dual of the graph called an S-walk for this. The decision problem whether a plane graph is polychromatic k-colorable is NP-complete when $k \in \{3, 4\}$ [5]. Horev and Krakovski [76] proved that every plane graph of degree at most 3, other than K_4 and a subdivision of K_4 on five vertices, admits a polychromatic 3-coloring. Horev et al. [75] proved that every 2-connected cubic bipartite plane graph admits a polychromatic 4-coloring. This result is tight, since any such graph must contain a face of size 4. If G' is a graph obtained by deleting an independent set from a triangulation G, then a face-hitting and dominating set in G' is a dominating set in the triangulation G. We studied polychromatic colorings in planar graphs to explore this possibility in addressing Conjecture 2.2.3. More discussion of this conjecture and related results is given in Chapter 4.

3 TOTAL DOMINATION IN NEAR-TRIANGULATIONS

This chapter demonstrates that any simple planar near-triangulation with a minimum degree of at least three has two disjoint total dominating sets. We introduce the main result as Theorem 3.1.1, and in the following section we provide a proof of the theorem that also leads to a polynomial-time algorithm. Furthermore, as discussed in previous chapters, the theorem allows us to confirm two conjectures put forth by Goddard and Henning in [59]. The question of whether there exist two disjoint total dominating sets in every triangulation was highlighted by them as "the most frustrating". In fact, it served as our primary target in this investigation. There were several attempts made on this conjecture which affirmed it on many interesting classes of triangulations. This includes Hamiltonian triangulations (Nagy [95]), triangulations with all odd-degree vertices, triangulations with a Hamiltonian dual (Goddard and Henning [59]), triangulations with at most two vertices of degree at most four, and triangulations with a 2-factor in which no cycle has length congruent to 2 modulo 4. (Bérczi and Gábor [11]). Some of these attempts also reformulated the conjecture in various equivalent and slightly stronger ways. The interested reader is invited to check [11] for a nice catalog.

3.1 KEY RESULT, COROLLARIES AND TIGHTNESS

Initially, our main objective was to address Goddard and Henning's conjecture regarding the existence of two disjoint total dominating sets within every triangulation. Upon resolving this for triangulations, we attempted to broaden the scope. The most comprehensive version of our finding is

Theorem 3.1.1. *Let G be a near-triangulation and V' be a subset of vertices of G containing all the vertices of degree at least three and at most two vertices of degree two. Then, there exists two disjoint subsets V_1 and V_2 of $V(G)$ such that each vertex $v \in V'$ has at least one neighbor each in V_1 and V_2.*

If we place a restriction on the minimum degree, the above result can be stated in the language of total domination. The next two corollaries follow by restricting Theorem 3.1.1 to the respective graph classes.

Corollary 3.1.2 (The Goddard-Henning Conjecture, Conjecture 30, [59])**.** *If* G *is a planar triangulation of order at least four, then* $\mathrm{dom}_t(G) \geqslant 2$*.*

Proof. In a planar triangulation G of order at least four, the neighborhood of every vertex forms a cycle. Since our triangulation is simple there are no 2-cycles, hence the degree of every vertex is at least three. The result follows from Theorem 3.1.1 by setting $V' = V(G)$. □

Corollary 3.1.3 (Speculated in [59])**.** *If* G *is a triangulated disc with minimum degree at least three, then* $\mathrm{dom}_t(G) \geqslant 2$*.*

Theorem 3.1.1 is tight in two senses. Firstly we cannot increase the number of degree two vertices in V'. The 3-sun, which is a graph obtained by adding a triangle of chords to a six-cycle, does not have two disjoint total dominating sets [59]. Secondly, the result cannot be extended to general graphs as shown by Zelinka [125]. He observed that for any positive integer k, the incidence graph of the complete k-uniform hypergraph H on n vertices with $n \geqslant 2k-1$ does not have two disjoint total dominating sets even though the minimum degree is k. But we do not know whether Theorem 3.1.1 can be extended to planar graphs which are not near-triangulations. Goddard and Henning [59] had also shown that $\mathrm{dom}_t(G) \leqslant 4$ for every planar graph G and hence conditions which ensure $\mathrm{dom}_t(G) \geqslant 3$ are equally interesting, but we haven't been able to make any progress there yet. Goddard and Henning [59] conjecture that every triangulation G with minimum degree four has $\mathrm{dom}_t(G) \geqslant 3$.

3.2 PROOF OF THE KEY RESULT

In this section, we prove the following theorem which is a restatement of Theorem 3.1.1 in the language of vertex coloring. However, this is stronger than Corollary 3.1.3 since we handle all near-triangulations. The strengthening helps us run a proof by structural induction since near-triangulations, unlike triangulated disks, is a family that is closed under deletion of vertices and edges from the boundary. On the other hand, if one restricts to triangulations (Corollary 3.1.2), then the initial observations in this section are unnecessary. We will say more about this simplification after Observation 3.2.9.

Theorem 3.2.1. *Let* G *be a near-triangulation. Let* T *be the set of all* 3^+*-vertices in* G *and let* S *be any subset of* 2*-vertices in* G *such that* $|S| \leqslant 2$*. There exists a two-coloring of* $V(G)$ *such that each vertex* $v \in T \cup S$ *sees both the colors in* $N(v)$*.*

Till we complete the proof of Theorem 3.2.1, we call near-triangulations which satisfy the theorem as *good* and others as *bad*. Given a near-triangulation G and a subset S of

2-vertices in G, a two-coloring which satisfies all the 3^+-vertices and the vertices in S is called a *good coloring* of (G, S). The vertices in S will be called *special*.

For the rest of this section, we fix $G = (V, E)$ to be a bad near-triangulation with the smallest $|V| + |E|$. We also fix S to be an arbitrary subset of 2-vertices of G such that $|S| \leqslant 2$. We will show that (G, S) has a good coloring, contradicting the existence of G. The minimality in the choice of G helps us to establish the following observations.

Observation 3.2.2. G is 2-connected.

Proof. If G is disconnected, then each component of G is smaller than G and hence good. In this case, G is easily seen to be good.

If G contains a bridge, then we can always choose a bridge $e = uv$ such that $d_G(v) \neq 2$ and $|S \cap V(G_v)| \leqslant 1$, where G_u and G_v are the two components of $G \setminus e$ containing u and v respectively. For $x \in \{u, v\}$, let $S_x = S \cap V(G_x)$. Let $S'_v = S_v \cup \{v\}$, if $d_{G_v}(v) = 2$ and $S'_v = S_v$ otherwise. By the minimality of G, both G_u and G_v are good. If $d_G(u) > 3$ or $d_G(u) = 1$ then a good coloring of (G_u, S_u) together with a good coloring of (G_v, S'_v) will be a good coloring of (G, S). If $d_G(u) \in \{2, 3\}$ then the above procedure will still give a good coloring of (G, S) provided we flip the coloring of G_v if u does not see both colors in the first coloring.

Suppose G is bridgeless but contains a cut vertex v. We can consider G as two smaller graphs G_1 and G_2 which share exactly one common vertex v. For $i \in \{1, 2\}$, let $S_i = S \cap V(G_i)$ and let $d_i = d_{G_i}(v)$. Since G is bridgeless, $d_i \geqslant 2$. Without loss of generality we can assume $|S_2| \leqslant 1$ and let $S'_2 = S_2 \cup \{v\}$ if $d_2 = 2$ and $S'_2 = S_2$ otherwise. A good coloring of (G_1, S_1) can be combined with a good coloring of (G_2, S'_2), flipping the coloring of G_2 if necessary to match the color of v in both colorings, to obtain a good coloring of (G, S). □

Remark. Insisting that a good coloring should satisfy a set S of 2-vertices along with all the 3^+-vertices not just strengthened Theorem 3.2.1 marginally, but also helped us critically in establishing Observation 3.2.2. In fact, we could not bypass this strengthening even if we restricted to 2-connected near-triangulations (triangulated disks). This is because we need closure of the graph class under deletion of boundary vertices and edges for some of the further observations (c.f. Observation 3.2.3) too.

Since G is a 2-connected near-triangulation, it is a triangulated disk. The only triangulated disks on at most 4 vertices are K_3, K_4 and the diamond. We can easily verify that all three of them are good. Henceforth we assume that G is a triangulated disk with at least 5 vertices.

Observation 3.2.3. There are no consecutive 4^+-vertices on the boundary of G.

Proof. Let e be the boundary edge between such a pair of consecutive vertices. A good coloring of $(G \setminus e, S)$, which exists by the minimality of G, will be a also a good coloring of (G, S). $\square$

Observation 3.2.4. No 2-vertex in G has a 3^--neighbor.

Proof. Let v be a 2-vertex in G with neighbors u and w. Since G is a simple triangulated disk on at least 5 vertices, all the three vertices u, v and w are on the boundary of G and there exists an edge uw which is an internal in G. Hence $\{u, w\}$ has a second common neighbor x and $\langle\{u, v, w, x\}\rangle$ is a diamond with uw as the chord. In particular, both u and w are 3^+-vertices. Let one of them, say w, be a 3-vertex. Then the edge wx is also a boundary edge of G and hence $H = G \setminus \{v, w\}$ is also a near-triangulation. We can extend a good coloring of $(H, S \setminus \{v\})$ to a good coloring of (G, S) by giving v the color different from that of x and w the color different from that of u. $\square$

Observation 3.2.5. G has no 2-vertices outside S.

Proof. Suppose G has a 2-degree vertex $v \notin S$ and let u and w be its neighbors. By Observation 3.2.4, both u and w are 4^+-vertices. Hence any good coloring of $(G \setminus \{v\}, S)$ can be extended to a good coloring of (G, S) by giving any color to v. Notice that we do not need to satisfy v since it is a 2-vertex outside S. $\square$

Due to Observation 3.2.5, we do not need to specify S separately anymore. S will be the set of all 2-vertices in the triangulated disk G. Moreover any good coloring of (G, S) is an ordinary 2-coupon coloring of G since it satisfies all the vertices of G.

Observation 3.2.6. There is no chord between two 3-vertices on the boundary of G.

Proof. Let uv be a chord between two 3-degree vertices u and v on the boundary of G. As uv is an internal edge, it will be a part of two triangular faces $\langle\{u, v, w_1\}\rangle$ and $\langle\{u, v, w_2\}\rangle$. Since u and v are 3-vertices, w_1 and w_2 are their only neighbors other than each other. In this case all the four edges of the cycle $C = (u, w_1, v, w_2, u)$ are on the boundary of G. Since G is a triangulated disk, (Observation 3.2.2), C is the entire boundary of G, and hence G is a diamond. This contradicts our assumption that G had at least 5 vertices. $\square$

Next we construct a special independent set I in G as follows.

Construction 3.2.7 (I)**.** *Start with $I = \emptyset$. In Round 1, for each 4^+-vertex on the boundary $B(G)$ of G, we add its clockwise next boundary vertex to I. In Round 2, we enlarge I to a maximal independent set of 3^--vertices from $B(G)$. In Round 3, we enlarge I to a maximal independent set of 4^--vertices from G.*

Observation 3.2.8. I is a maximal independent set of 4^--vertices in G which contains all the 2-vertices and none of the 4^+-vertices from the boundary of G.

Proof. By Observations 3.2.3 and 3.2.6, I is an independent set after Round 1. It remains so, by construction, after the subsequent two rounds. By Observation 3.2.4, all the 2-vertices in $B(G)$ are added to I in Round 1. By Observation 3.2.3, no 4^+-vertices in $B(G)$ are added to I in Round 1 and since they are all dominated by I after Round 1, by construction, they are not added to I in any of the two subsequent rounds. □

Observation 3.2.9. If $v \in I$, then every pair of vertices in $N(v)$ has a second common neighbor in G.

Proof. Let v be a 2-vertex in I with neighbors u and w. Since G is a triangulated disk with at least 5 vertices, uw is a chord of G and hence $\{u, w\}$ have a second common neighbor.

Let v be a boundary 3-vertex in I with neighbors u, v' and w, where u and w are in $B(G)$. Since v has degree exactly three, v' is a common neighbor of u and w. If uv' is a boundary edge of G, u would be a 2-vertex. By Observation 3.2.8, $u \in I$ in which case v would not have been in I. Hence uv' is an internal edge of G and hence $\{u, v'\}$ has a common neighbor other than v. The case for $\{v', w\}$ is similar.

Finally let v be any 3-degree or 4-degree internal vertex in I. Let u and w be two distinct vertices in $N(v)$. If u and w are non-adjacent in G, then v is a 4-vertex, and hence both the remaining vertices in $N(v)$ are common neighbors of $\{u, w\}$. If uw is an internal edge of G, then since G is a near-triangulation, $\{u, w\}$ has a common neighbor other than v. Suppose uw is a boundary edge of G, then by Observation 3.2.3 either u or w is a 3-degree vertex. Without loss of generality, let w be the 3-degree vertex. Then the cyclically next boundary neighbor of w other than u, say x is also a neighbor of v. At least one vertex in $\{u, w, x\}$ will be included in I by the end of Round 2. Hence v would not have been added to I.

Since these are the only types of vertices in I (Observation 3.2.8), the above cases are exhaustive. □

Remark. If G is a triangulation of order at least 4, Observation 3.2.9 can be directly established for any 4^--vertex v, since every edge is part of two triangles. In that case, we can pick I to be any maximal independent set of 4^--vertices in G and skip all previous observations made in this section. This would have sufficed if our aim was limited to affirming Corollary 3.1.2.

Observation 3.2.9 leads us to a simple idea which helped us unlock this problem.

Observation 3.2.10 (Key Observation)**.** If there exists a two-coloring f of $V(G)$ such that every vertex in $V(G) \setminus N(I)$ is satisfied, then f can be modified to a two-coloring which satisfies every vertex in G.

Proof. Suppose there is a vertex $v \in I$ such that a vertex in $N(v)$ is unsatisfied. Let $G_v = G \setminus \{v\}$ and f_v be f restricted to $V(G_v)$. Let $U \subset N(v)$ be the set of unsatisfied vertices in $N(v)$ under f_v. Note that this may contain vertices which were satisfied under f. If $|U| \geqslant 2$, by Observation 3.2.9, each pair of vertices in U have a common neighbor in G_v and hence miss the same color under f_v (the color different from that of the common neighbor). Since this is true for every pair in U, all the vertices in U miss the same color, say c, under f_v. If $|U| = 1$, we choose c to be the color missing for the single vertex $u \in U$ under f_v.

Recoloring v to c in f gives a new two-coloring f' of $V(G)$ which satisfies all the vertices in U and all the vertices originally satisfied by f. We can repeat this procedure till no vertex in I has an unsatisfied neighbor to get a 2-coupon coloring of G. $\square$

In view of Observation 3.2.10, we can focus on finding a two-coloring of G such that every vertex in $V' = V(G) \setminus N(I)$ is satisfied. The set V' consists of two types of vertices - those in I and those not dominated by I. Those in I are all 4^--vertices and those not dominated by I are 5^+-vertices in G and remain so even in $G \setminus I$. These two types of vertices pose different challenges which can be jointly addressed by a new type of coloring for a triangulation G' which contains $G \setminus I$ as a spanning subgraph. First we construct the graph G' and then define the new type of coloring.

Construction 3.2.11 ((G', P'))**.** *Let $v_1, \ldots, v_k$ be an arbitrary order of vertices in I and let $G_0 = G$ and $P_0 = \emptyset$. For each $i \in [k]$, the graph G_i is obtained from G_{i-1} by deleting v_i and adding any maximal set of missing edges between the neighbors of v_i in G_{i-1} maintaining planarity. Add all the edges of G_i between the neighbors of v_i in G_{i-1} to P_{i-1} to obtain P_i. Let $(G', P') = (G_k, P_k)$. We call the edges in P' and their endpoints* protected *and the remaining vertices* unprotected.

Observation 3.2.12. Every unprotected vertex in G' is a 5^+-vertex in G'.

Proof. Since I is a maximal independent set of 4^--vertices, every 4^--vertex v is in $I \cup N(I)$. If $v \in I$ it gets deleted and if $v \in N(I)$, it gets protected. Hence every unprotected vertex u in G' is a 5^+-vertex in G. Since $u \notin N(I)$, it remains a 5^+-vertex in $G \setminus I$ and in G' which is a supergraph of $G \setminus I$. $\square$

Observation 3.2.13. If $v \in I$ is a 3^+-vertex in G, $\langle N_G(v) \rangle$ contains a triangle T_v in G' and all the three edges of T_v are in P'. If $v \in I$ is a 2-vertex in G, the edge in $\langle N_G(v) \rangle$ is in P'.

Proof. For every vertex $v_i \in I$, $N(v_i) \subset V(G')$ since $V(G') = V(G) \setminus I$ and I is an independent set. If v_i is an internal 3-vertex of G, then $N_G(v_i)$ induces a triangle in G. If v_i is a boundary 3-vertex of G and $N_{G_{i-1}}(v_i)$ induced only a 2-length path, it can be completed to a triangle in G_i by connecting the two boundary neighbors of v_i through the outer face of G_{i-1}. Finally, if v_i is an internal 4-vertex, $N_{G_{i-1}}(v_i)$ contains at least one pair of non-adjacent vertices in G_{i-1} (since G_{i-1} is K_5-free) and they can be connected in G_i by an edge through the 4-face created by deleting v_i from G_{i-1}. Notice that no edge between two vertices in $N_G(v_i)$, whether originally present in G or added in one of the steps, gets deleted later since I is an independent set. The required memberships in P' follow from the construction. □

Definition 3.2.14 (Fair four-coloring)**.** Given a graph G and $P \subset E(G)$, a four-coloring of $V(G)$ is called a *fair four-coloring* of (G, P) if the endpoints of every edge in P gets different colors and every vertex v not spanned by P sees at least three colors in $N(v)$.

Lemma 3.2.15. *If Γ is a planar graph and $P \subseteq E(\Gamma)$ spans all 4^--vertices in Γ, then (Γ, P) has a fair four-coloring.*

Proof. Borrowing the terminology from Construction 3.2.11, we call the edges in P and their endpoints *protected* and the remaining vertices *unprotected*. Consider a cut (A, B) in Γ with maximum number of edges crossing the parts subject to the constraint that no protected edge crosses the cut. By the maximality of the cut, every unprotected vertex v will have at least half of its neighbors in the opposite part. Otherwise, we can shift v to the other part to get a larger cut without violating the constraint. Since all unprotected vertices in G are 5^+-vertices, they will have at least three neighbors on the other side.

We color $V(\Gamma)$ by coloring A and B independently, starting with A. Remove all the edges between the vertices in B. If there are more than three neighbors for any unprotected vertex $v \in B$, then arbitrarily delete some edges incident on v until exactly three edges remain. Call the resulting subgraph Γ'. Pick a planar drawing of Γ' and for each unprotected vertex $v \in B$, add any missing edge in the A-part between every pair of vertices in $N_{\Gamma'}(v)$. This can be done without violating planarity since each unprotected vertex in B has only three neighbors in Γ'. Let us call this graph H and the planar subgraph of H induced on A as H_A. By the four color theorem [6, 7], there exists a proper four-coloring f_A of H_A. Repeat the same procedure to find a coloring f_B of the vertices in B and combine them to get a coloring f of Γ.

For every unprotected vertex $v \in B$ (resp., $v \in A$), there will be a triangle induced in H_A (resp., H_B) by the three neighbors of v in H. Hence $N(v)$ will see 3 different colors in f. Since none of the edges in P crossed the cut and since all of them where present in either H_A or H_B, the endpoints of every protected edge gets different colors. Hence f is a fair four-coloring of (Γ, P). □

Remark. The idea of using a max-cut (without any constraint) was used by Bérczi and Gábor [11] to prove that triangulations with at most two vertices of degree at most four have a 2-coupon coloring.

By Lemma 3.2.15, the pair (G', P') obtained from Construction 3.2.11 has a fair four-coloring f. We construct a two-coloring of G from f as follows.

Construction 3.2.16 (f_2)**.** *Consider* f *as a partial four-coloring of* G. *Note that for each 2-vertex* $v \in I$, $N(v)$ *contains a protected edge (Observation 3.2.13) and hence* $N(v)$ *sees two colors under* f. *Group the four colors into two pairs so that for each 2-vertex* v *in* I, *the two colors on the neighbors of* v *go to different pairs. This is indeed possible since we have at most two 2-vertices in* G. *Now merge the two colors in a pair into a single color. Extend this partial two-coloring to a full two-coloring* f_2 *of* G *by giving the vertices in* I *any of the two colors arbitrarily.*

Observation 3.2.17. In the two-coloring f_2 obtained from Construction 3.2.16, every vertex in $V(G) \setminus N(I)$ is satisfied.

Proof. Recall that $V(G) \setminus N(I)$ consists of two types of vertices - the unprotected vertices in G' and the vertices in I. If v is an unprotected vertex, since f is a fair four-coloring, $N(v)$ contains vertices of at least three colors under f and hence two colors under f_2. If v is a 2-vertex in I, then it is satisfied due to the careful merging of colors in Construction 3.2.16. If v is a 3^+-vertex in I, then $N(v)$ in G' contains a triangle T_v, all of whose edges are protected (Observation 3.2.13). Hence $N(v)$ sees at least three colors under f and hence two colors under f_2. □

Observation 3.2.17 says that the f_2 satisfies the premise of Observation 3.2.10 and hence we can conclude that f_2 can be modified to a two-coloring which satisfies every vertex in G. Hence G is good. This completes the proof of Theorem 3.2.1 and equivalently Theorem 3.1.1.

Remark. It should be clear by now that key role played by the proper four-coloring of H_A and H_B is in ensuring that no triangle is monochromatic after the merger into a two-coloring. The existence of such two-colorings can be proved without resorting to the four-color theorem (c.f. Kaiser and Škrekovski, 2004 [79], Thomassen, 2008 [116]). Perhaps the easiest way (due to Barnette) is to use a stronger version of Petersen's theorem which asserts that every edge of a bridgeless cubic multigraph is contained in a 1-factor (Schönberger, 1934 [108]). Hence we could have bypassed the use of four-color theorem if we did not have to handle the two 2-vertices. In particular, we can prove Corollary 3.1.2 without using the four-color theorem.

3.3 CONCLUDING REMARKS

Our proof of Theorem 3.2.1 lends itself to a polynomial-time algorithm. Notice that even though finding a max-cut is NP-hard for general graphs, it is polynomial-time solvable for planar graphs. Moreover, we do not even need a max-cut for our purpose. As soon as we get a cut which cannot be improved by shifting one unprotected vertex to the opposite part, we are done. This can be done greedily and will terminate in at most as many rounds as the number of unprotected edges. Four coloring of a planar graph can be done in quadratic time [106].

While we were able to affirm two of the conjectures in [59], we could not solve a tantalizing strengthening which states that the vertex set of a triangulation G with at least four vertices can be partitioned into two total dominating sets - both of which induce a bipartite subgraph of G. Equivalently, there exists a proper four-coloring with color classes $\{V_1, V_2, V_3, V_4\}$ such that both $V_1 \cup V_2$ and $V_3 \cup V_4$ are total dominating sets (Conjecture 32, [59]). Our method seems to be limited in power when we need a proper coloring.

Since we have affirmed two conjectures in this chapter, we wish to restore the balance by posing two of our own. The first one stems out of the key technique we used in our proof and the second one comes out of our attempts to refute the original conjecture which we ended up proving.

Conjecture 3.3.1. Every near-triangulation has a four-coloring of its vertices such that every vertex v sees at least $\min\{d(v), 3\}$ different colors in $N(v)$.

Conjecture 3.3.1 will immediately give Theorem 3.2.1 via the color merger argument we used in Construction 3.2.16. We anticipate that, if proven, this may find many more applications than Theorem 3.2.1. Some of the earlier attempts to settle the Goddard-Henning conjecture on certain classes of triangulations can be modified to affirm Conjecture 3.3.1 for those classes. For example, once can see that triangulations with acyclic chromatic number at most four (this includes triangulations with all vertex degrees odd) satisfy Conjecture 3.3.1.

Conjecture 3.3.2. If G is a planar graph with minimum degree at least three, then $\mathrm{dom}_t(G) \geqslant 2$.

A look at the coloring part in the proof of Lemma 3.2.15 will show that if G is a planar graph which has a cut such that every vertex v has at least three neighbors in the opposite part, then $(G, \emptyset)$ has a fair four-coloring and hence G has a 2-coupon coloring. This suffices to confirm Conjecture 3.3.2 for all planar graphs with minimum degree at least five and all bipartite planar graphs.

4

FACE-HITTING DOMINATING SETS IN PLANE GRAPHS

In this chapter we show that the vertex-set of every plane (multi-)graph without isolated vertices, self-loops or 2-faces can be partitioned into two disjoint sets so that both the sets are dominating and face-hitting. We also show that all the three assumptions above are necessary for the conclusion. The novelty in our result is that we find a single 2-coloring which is both polychromatic and domatic. In fact, it is easy to show that every plane graph G in this class has a domatic 2-coloring and a polychromatic 2-coloring. Any proper 2-coloring of a spanning forest of G will be a domatic 2-coloring. Existence of a polychromatic 2-coloring for G is a solved exercise in Lovász's famous book - *Combinatorial Problems and Exercises* [87].

In the sections that follow, we will initially discuss the result and its application, followed by the proof for each.

4.1 KEY RESULT AND ITS TIGHTNESS

The main result of this chapter is that every plane graph without isolated vertices, self-loops or 2-faces has a 2-coloring which is simultaneously domatic and polychromatic.

Theorem 4.1.1. *Every plane graph G without isolated vertices, self-loops or 2-faces, has two disjoint subsets $V_1, V_2 \subseteq V(G)$, such that both V_1 and V_2 are dominating and face-hitting.*

The assumptions made on G in the above theorem are indeed necessary. It follows from Theorem 4.1.1 that every n-vertex plane graph G satisfying the premise of the theorem has a subset S of vertices of cardinality at most $n/2$ which is both dominating and face-hitting. It is easy to see that, if we allow isolated vertices or length-1 faces, then $\gamma(G)$ can be as large as n. If we allow 2-faces, then $\beta(G)$ (size of the smallest face hitting set) can be as large as $3n/4$. For the second observation, consider G to be a disjoint union of $n/4$ components, each component being a K_4 in which every edge is replaced with a 2-face. We cannot allow self-loops even if all faces have length at least 3 because, in this case, we can still have a 3^+-face which has only one vertex in its boundary. For example, consider a vertex v with three self-loops l_1, l_2, l_3 such that l_1 and l_2 have disjoint interiors, but both are inside of l_3. We then add a neighbor of v inside the self-loops l_1 and l_2 so that all the faces have length at least 3. Then the face bounded by l_1, l_2, l_3 has only one vertex in its boundary. It is clear that we cannot have a polychromatic 2-coloring in this case.

We can also see that the bound of $n/2$ is tight under these assumptions. A disjoint collection of edges (or 3-length paths, or 4-length cycles) have $\gamma(G) \geqslant |V(G)|/2$. Bose et al. [20] showed that there exists an infinite family of simple connected plane graphs with $\beta(G) \geqslant \lfloor |V(G)|/2 \rfloor$.

If we remove an independent set from a triangulation G and then apply Theorem 4.1.1 on the resultant graph G', we can show that the dominating and face-hitting sets V_1, V_2 of G' will both be dominating sets of G. This observation helps us make some progress towards a conjecture by Matheson and Tarjan. As a consequence of Theorem 4.1.1, we improve the $2n/7$ bound by Christiansen et al. for n-vertex plane triangulations which contain a maximal independent set of size either less than $2n/7$ or more than $3n/7$ and verify Matheson-Tarjan conjecture for n-vertex plane triangulations with an independent set of size $n/2$. .

4.2 PROOF OF THE KEY RESULT

In this section, we prove the following theorem, which is a restatement of Theorem 4.1.1 in the language of vertex coloring.

Theorem 4.2.1. *If* G *is a plane graph without isolated vertices, self-loops or 2-faces, then* G *has a 2-coloring which is simultaneously domatic and polychromatic.*

The main part of the chapter is devoted to proving the next lemma from which Theorem 4.2.1 will follow easily. We call a 2-coloring of a plane graph 3^+*-polychromatic* if every 3^+-face contains vertices of both colors.

Lemma 4.2.2. *If* G *is a connected plane graph without self-loops and with every vertex having at least two neighbors, then* G *has a 2-coloring which is simultaneously domatic and* 3^+*-polychromatic.*

To prove Lemma 4.2.2 we construct a supergraph G' from G by adding some 2-chords to G. A 2-chord of G is an edge whose addition creates a new facial triangle in G. We call the edges in $E(G)$ as *true edges* and those in $E(G') \setminus E(G)$ as *dummy edges*. Vertices u and v are called *true neighbors* in G' if uv is a true edge. We call a vertex v of G *happy* in G' if there exists a triangle uvw in G' where uv and vw are true edges. Otherwise we call v *unhappy* in G'. A face f in G is *happy* in G' if at least one of the faces of G' contained inside f is a 3-face. Otherwise we call it *unhappy* in G'. A *true angle at* v in G' is an angle between two cyclically consecutive true edges incident on v which connects v to two distinct vertices (ignoring any dummy edges between them). Let $\mathcal{G}$ be the family of all plane multigraphs that can be obtained by adding dummy edges to G which maximizes the sum of the number of happy vertices and the number of happy faces. Let G' be a graph in $\mathcal{G}$ with the smallest number of dummy edges. We make the following observations about G'.

Observation 4.2.3. Every 3^+-face of G is happy in G'.

Proof. Suppose f is a 3^+-face of G which is unhappy in G'. Then there is no dummy edge in f. Since G is connected, the boundary B of f is a single closed walk. Further since G does not have self loops and f has length at least three, $V(f)$ has size at least 3. Hence we can make f happy by adding a dummy edge inside f between two distinct vertices in $V(f)$ which are at distance two along the walk B. This supergraph G'' of G' is also a planar supergraph of G and the number of faces of G which are happy in G'' is one more than the number of faces of G which are happy in G'. This violates the membership of G' in $\mathcal{G}$. $\square$

Remark. Notice that both the assumptions on G are necessary for Observation 4.2.3. Recall that if we allow self-loops, we can have 3^+-faces bounded by a single vertex, and such faces cannot be made happy. If we allow disconnected graphs, then a 4-face bounded between two 2-cycles cannot be made happy by adding a dummy edge.

Unlike 3^+-faces, we cannot guarantee that all the vertices are happy in G'. The next observation, even though technical, illustrates precisely what happens in the neighborhood of an unhappy vertex.

Observation 4.2.4. If v is an unhappy vertex in G', then

1. there is exactly one dummy edge incident on v through every true angle at v, and
2. each of these dummy edges makes exactly one true neighbor of v happy.

Proof. Let D_v denote the set of dummy edges incident to v in G' and H_v denote the set of happy true neighbors of v. Since every vertex in G has at least two neighbors, there are at least two true angles at v. If there are no dummy edges incident on v through a true angle uvw, we can add a dummy edge uw without violating planarity, making v happy. This will increase the number of happy vertices contradicting the membership of G' in $\mathcal{G}$. Since the number of true angles at v is at least $|N_G(v)|$, we have $|D_v| \geqslant |N_G(v)|$. Further, since $|H_v| \leqslant |N_G(v)|$, we have $|D_v| \geqslant |H_v|$. Every vertex in H_v needs at most one edge from D_v to become happy. No vertex outside H_v can be made happy by an edge in D_v. Hence, either if $|D_v| > |H_v|$ or if $|D_v| = |H_v|$ and one of the edges in D_v is making two true neighbors of v happy, then at least one edge in D_v is redundant. That is, we can remove this edge and still leave all the vertices in H_v

happy. If this deletion does not leave a true angle at v without a dummy edge, then the happiness of the corresponding face is also intact. This contradicts the choice of G' as a smallest member in $\mathcal{G}$. On the other hand, if this deletion leaves a true angle at v without a dummy edge, we can make v happy as we did in the first case and restore the happiness of the corresponding face. This contradicts the membership of G' in $\mathcal{G}$. Hence $|D_v| = |H_v|$ and each edge in D_v makes exactly one vertex in H_v happy. Since $|N_G(v)|$ is sandwiched between $|H_v|$ and $|D_v|$, we also have $|H_v| = |D_v| = |N_G(v)|$. □

The next observation is an easy restatement of the equality $|H_v| = |N_G(v)|$ that we established in Observation 4.2.4.

Observation 4.2.5. Two unhappy vertices cannot be true neighbors in G'.

Observation 4.2.6 (Key Observation)**.** G' has at most one unhappy vertex.

Proof. Let v be an unhappy vertex and let u be any one of its true neighbors. From Observation 4.2.4, it is clear there is a dummy edge e incident to v which makes u happy (and helps no other vertex become happy). If we remove e, the only effect on happiness is that u becomes unhappy and the face f in G containing e may become unhappy. Moreover, one true angle at v, say uvw, becomes free. We can now add the dummy edge uw inside f to make v and f happy. So, the unhappiness of a vertex v can be shifted to any one of its true neighbors without creating any other unhappy vertices or faces. Note that this shifting does not increase the total number of dummy edges.

Suppose there were two unhappy vertices, say v, v' in G'. Since G is connected, there is a path $P = (v = x_0, \dots, x_k = v')$ in G. Then we can do the above shifting of unhappiness repeatedly from x_i to x_{i+1} for $0 \leqslant i \leqslant k-2$. This shifting will end when x_{k-1} and v' are unhappy. This new graph also qualifies to be G' and it contradicts Observation 4.2.5. Hence, there can be at most one unhappy vertex in G'. □

By the Four Color Theorem [6, 7], there exists a proper 4-coloring $\phi : V[G'] \to \{1, 2, 3, 4\}$ of G'. By Observation 4.2.6, we can assume without loss of generality that the unhappy vertex in G' (if one exists) gets color 1 and has a true neighbor of color 2. Obtain a 2-coloring ψ of G' by merging the color classes 1 and 3 to a single color class and 2 and 4 to a single color class different from the previous. This ensures that, in ψ, the unhappy vertex sees both colors of ψ in its closed neighborhood. Every happy vertex v is part of a triangle uvw in G' with uv, vw being true edges. Since u, v, w get three different colors in ϕ, $N_G[v] \supseteq \{u, v, w\}$ will contain vertices of both colors in ψ. By Observation 4.2.3, every 3^+-face in G contains a 3-face of G'. Hence every 3^+-face of G also sees both the colors in ψ. Hence ψ is a domatic as well as a 3^+-polychromatic 2-coloring of G. This completes the proof of Lemma 4.2.2.

To prove Theorem 4.2.1, we need to extend Lemma 4.2.2 to handle graphs with multiple components and pendant vertices. Let G be a plane graph without isolated vertices, self-loops or 2-faces. Trim the graph by recursively removing the vertices with exactly one neighbor to obtain a subgraph G' that no longer has such vertices. Since we remove vertices with exactly one neighbor at every stage, we do not create any new components. G' can have only two types of components - components with every vertices having at least two neighbors or isolated vertices (trivial components).

Consider a nontrivial component H of G'. Given a (partial) 2-coloring of the vertices of G, a vertex v (resp. face f) is said to be *satisfied*, if $N_G[v]$ (resp. $V(f)$) contains at least one vertex of each color. By Lemma 4.2.2 we know that there exists a 2-coloring of H satisfying every 3^+-face and vertices of H. Notice that H may have 2-faces even though G does not.

We start with such a coloring for every nontrivial component in G' and color the isolated vertices in G' with any of the two colors. We then add back the deleted vertices one at a time in the reverse order as we deleted them. Every time we add back a vertex, we color it with the color different from that of its parent. This makes sure that every vertex added back and its parent are satisfied. Since there are no isolated vertices in G, every vertex of G is satisfied in the resulting coloring of G.

Now consider the faces. We might create some new faces while adding back the vertices with degree more than one, but we are adding back vertices that have only one parent and we are coloring the newly added vertex with the opposite color of the parent. This makes sure that all newly formed faces are also satisfied. The other faces which were already in G' may have a larger boundary in G due to the new edges added inside them. Note that the vertices which were on the boundary of any such face in G' still remain in its boundary in G. So every face which was satisfied in G' is satisfied in this coloring of G. Every face in G' whose boundary contains a closed walk of length at least three is satisfied in G' since that closed walk is the boundary of a 3^+-face in one of the nontrivial components of G'. Let f be an unsatisfied face in G'. By the previous observation, f is bounded by a collection consisting of only 2-cycles and isolated vertices. Since G does not have any 2^--faces, f will either contain a pendant vertex of G inside it or $V(f)$ contains vertices from more than one component of G. In the first case, we colored this pendant vertex differently from its parent and hence f will be satisfied. In the second case, we can flip the colors of one of the interior components and all the components nested inside it, if needed, to ensure that f is satisfied.

This completes the proof of Theorem 4.2.1.

4.3 APPLICATION TO MATHESON-TARJAN CONJECTURE

As a corollary of Theorem 4.1.1 we can give a conditional upper bound on the domination number of plane triangulations.

Corollary 4.3.1. *Every n-vertex (simple) plane triangulation G with an independent set of size at least αn has $\gamma(G) \leqslant (1-\alpha)n/2$.*

Proof. The result can be easily verified when $n = 3$. Consider a plane triangulation G on n vertices, $n \geqslant 4$, with an independent set I of size at least αn. We delete I from G to get a plane graph G'. Note that since G is a triangulation and every vertex is part of a triangle, removing I from G will not create any isolated vertices in G'. Moreover, G has no 2-faces and since G is a simple triangulation, every vertex removed from G has degree more than two. Hence G' has no 2-faces. Corresponding to every deleted vertex v, a unique face f_v is formed in G' where $V(f_v) = N_G(v)$. Note that $|V(G')| \leqslant (1-\alpha)n$. By Theorem 4.1.1 we know that there is a face-hitting dominating set S in G' with $|S| \leqslant (1-\alpha)n/2$. Since S is a dominating set in G' and G' is a subgraph of G, S dominates all the vertices of $V(G') \setminus S = V(G) \setminus (I \cup S)$ in G as well. Since S is a face-hitting set in G', it dominates every vertex of I in G. Hence S is a dominating set of G. □

Corollary 4.3.1 proves Matheson-Tarjan Conjecture for plane triangulations which have an independent set of size at least $n/2$. Recall that Christiansen et al. [31] showed that every plane triangulation on $n > 10$ vertices has a dominating set of size at most $2n/7$. Corollary 4.3.1 improves this bound when $\alpha > 3/7$. This includes triangulations obtained by adding a new vertex inside every face of a planar graph whose average face length is slightly below $14n/3$ (and connecting it to the vertices on this face). Since a maximal independent set in a graph is a dominating set, our bound improves Christiansen et al.'s bound when G contains a maximal independent set of size either less than $2n/7$ or more than $3n/7$.

5

INDEPENDENT DOMINATING SETS IN TRIANGULATIONS

Recall that Matheson and Tarjan [92] in 1996 proved that there exist three disjoint dominating sets in any triangulated disc. Hence, for every triangulated disc G on n vertices, $\gamma(G) \leqslant n/3$. The result discussed in this chapter is a structural strengthening of Matheson and Tarjan's result for triangulations. We show that every planar triangulation on n vertices has a maximal independent set of size at most $n/3$. This affirms a conjecture by Botler, Fernandes and Gutiérrez [21], which in turn would follow if an open question of Goddard and Henning [60] which asks if every planar triangulation has three disjoint maximal independent sets were answered in the affirmative. The chapter follows the same format as the preceding ones, with the result presented in the next section and the proof provided subsequently.

5.1 KEY RESULT

In this chapter, we present the following result.

Theorem 5.1.1. *Every n-vertex triangulation has an independent dominating set of size at most $n/3$.*

Botler et al. [21] were able to prove that $\iota(G) \leqslant n/3$, for every triangulation G on n vertices, with minimum degree greater than or equal to five. We identified that a major challenge in extending this result to triangulations involves handling vertices of degree 4, which also presented notable obstacles in the proof of Theorem 3.1.1. Therefore, we devised a specialized approach to handle the 4-degree vertices and employed a counting strategy that leverages Euler's formula, analogous to the method used by Botler et al.

5.2 PROOF OF THE KEY RESULT

We will first show that a smallest (in terms of number of vertices) counterexample to Theorem 5.1.1 cannot contain facial cycles formed by one 4-vertex and two vertices of degree 4 or 5 (Lemma 5.2.2). We will then exploit this structure and the Four Color Theorem to obtain a partial proper four-coloring of this counterexample with the key property that every 5^{-}-vertex is guaranteed to see all four colors in its closed neighborhood (Lemma 5.2.3). We will then show that one color class from this partial coloring can be expanded into an independent dominating set of size less than $n/3$. Bounding the size of this extension requires a technical lemma (Lemma 5.2.5) and an easy consequence of Euler's formula (Observation 5.2.1). We will prove and set aside Observation 5.2.1 and then take up the lemmas in order.

Observation 5.2.1. Every independent set I of 6^+-vertices in an n-vertex triangulation G has size at most $(n-2)/3$.

Proof. Let H be a plane graph obtained by deleting I from G. H contains only 3-faces and 6^+-faces. Let f_3 and f_{6^+} denote the number of 3-faces and 6^+-faces respectively. By Euler's formula we have,

$$|V(H)| - 2 = |E(H)| - (f_3 + f_{6^+}). \tag{5.1}$$

By the standard double counting, $2|E(H)| \geqslant 3f_3 + 6f_{6^+}$. Also from the construction of H, we have $|V(H)| = n - |I|$ and $f_{6^+} = |I|$. Substituting these facts in equation (5.1) we get

$$|I| \leqslant \frac{n-2-f_3/2}{3} \leqslant \frac{n-2}{3}. \tag{5.2}$$

□

Now we begin the proof of Theorem 5.1.1. We will fix an arbitrary smallest counterexample (in terms of number of vertices) as G and analyze it in detail. The first task is to prove Lemma 5.2.2.

Lemma 5.2.2. *If G is a smallest counterexample to Theorem 5.1.1, then G does not contain a facial cycle (v_1, v_2, v_3) such that $4 = d(v_1) \leqslant d(v_2) \leqslant d(v_3) \leqslant 5$.*

Proof. The proof strategy is as follows. Suppose there exists a facial cycle $t = (v_1, v_2, v_3)$ in G with $4 = d(v_1) \leqslant d(v_2) \leqslant d(v_3) \leqslant 5$. We delete the set $T = \{v_1, v_2, v_3\}$ from G to obtain a subgraph $G - T$ with a new face f. We then triangulate f to obtain a simple triangulation G' on $|V(G)| - 3$ vertices. Finally, we show that an independent dominating set D' in G' can be extended to an independent dominating set D of G by adding at most one more vertex. This gives a contradiction since we can choose D' to be of size $(|V(G')| - 2)/3 \leqslant (|V(G)| - 2)/3 - 1$ and hence $|D| \leqslant (|V(G)| - 2)/3$. The selection of the additional vertex v_{sp} depends on f and $D' \cap V(f)$. These cases constitute the bulk of the proof.

Let B denote the closed walk bounding f and let $s = \sum_{i=1}^{3} d(v_i)$. Each $v_i \in T$ has exactly two neighbors from T itself in G. This contributes a total of six to s and hence

$s-6$ edges go from T to $V(B)$. If we contract the 3-face t to a single vertex v_t (removing the self loops but retaining multiple edges), we get $s-6$ faces incident to v_t of which exactly three are 2-faces (the faces which contained an edge of t) while the remaining are 3-faces. Since each step in the walk B contributes to one 3-face incident to v_t, the total length of B is $s-9$. There are only three possibilities for $(d(v_1), d(v_2), d(v_3))$, viz., $(4,4,4)$, $(4,4,5)$, and $(4,5,5)$.

Case 1 $d(v_1) = d(v_2) = d(v_3) = 4$. Here $s = 12$ and length l_B of the closed walk B is $12-9=3$. So B is a 3-cycle. Here $G-T$ itself is a triangulation which is smaller than G, so we choose $G' = G-T$. If a smallest independent dominating set D' of G' itself dominates G, then we are done. Otherwise, there exists at least one vertex $v \in T$ which is not dominated by D'. We choose v as v_{sp}.

Case 2 $d(v_1) = d(v_2) = 4$, $d(v_3) = 5$. In this case $s = 13$ and $l_B = 4$. Since B is a closed walk of length four bounding a face in a simple graph G, B can either be a 4-cycle or along a P_3. Note that a path cannot divide a plane into two regions. So in any planar graph, if a P_3 is the boundary of a face f, then f is the only face in the entire graph and P_3 is its only boundary. So $G-T$ is a P_3. Since T was the only set of vertices deleted from G, G has only six vertices. Hence v_3, being a 5-vertex, will dominate all of G.

If B is a 4-cycle, We obtain a simple triangulation G' by adding an edge uv to $G-T$. If D', a smallest dominating set in G' dominates G, then we are done. Otherwise, we consider two cases: when $D' \cap \{u, v\} = \emptyset$ and when it is not. In the first case, we select any undominated vertex $v_i \in T$ as v_{sp}. In the second case, without loss of generality, we assume $u \in D'$. This case is different from the former since v may not be dominated by D' in G. For any vertex $v_i \in T$, its neighbors in B are consecutive and since each v_i has at least two neighbors in B, either u or v is adjacent to it. If all the three vertices of T are adjacent to u, then it creates a K_4 which is impossible since every vertex of T has degree more than 3. Hence we can pick a vertex of T non-adjacent to u as v_{sp} and it will dominate v and all of T.

Case 3 $d(v_1) = 4$, $d(v_2) = d(v_3) = 5$. In this case, $s = 14$ and $l_B = 5$. Since B is a closed walk of length five bounding a face in a simple graph, it can be either a 5-cycle or along a 3-cycle with a pendent vertex attached to one of its vertices. When B is a 5-cycle, We add two chords uv and uw in the face f of $G-T$ to get a simple triangulation G'. If D', a smallest dominating set in G' dominates G, we are done. If $D' \cap \{u, v, w\} = \emptyset$ then choose one undominated vertex from T as v_{sp}. Otherwise, at most one vertex $x \in \{u, v, w\}$ is in D' since those three vertices form a triangle in G'.

Let y and y' be the two vertices of B which are non-adjacent to x along B. Let z be the common neighbor of y and y' in T. Since z has degree at most 5, it has at most three neighbors in B. Since these neighbors of z are consecutive along B, x is non-adjacent to z. We choose z as v_{sp} and it will dominate all of T and $\{u, v, w\} \setminus \{x\}$.

The remaining part of the third case is when B is along a 3-cycle B' with a pendent vertex x attached to one of its vertices. The vertex x is adjacent to exactly two vertices in T. This is because the degree of x is at least three and if x is adjacent to all three vertices it will create a K_4 in which case at least one vertex of T will have degree three. So, the induced graph on $\{x\} \cup T$ is a diamond U. Let $T = \{v_1, v_2, v_3\}$ and let v_i be the vertex in T that is not adjacent to x. We contract U to a single vertex x' to get a triangulation G'. This is equivalent to deleting T and triangulating f. If a minimum dominating set D' of G' contains x', then $D = D' \setminus \{x'\} \cup \{x, v_i\}$. We now show that x and v_i together dominate all three vertices in B'. Suppose a vertex $u \in B'$ is non-adjacent to both x and v_i, then since its neighbors in U are consecutive on the boundary of the diamond, u has at most one neighbor say v_j in D. But then v_j has to be adjacent to all three vertices of B'. Since v_j also has three neighbors in U, its degree will be at least 6, which is a contradiction. Hence $\{x, v_i\}$ dominates all of B' and U. If $x' \notin D'$, then D' contains exactly one vertex y from the B'. As all neighbors of y on the diamond are consecutive along its boundary, the undominated vertices form a connected component of at most three vertices, which can be dominated by a single vertex. □

With the help of Lemma 5.2.2 we will now define a partial proper 4-coloring ψ of the counterexample G in which each color class dominates all 5^-−vertices. For a coloring ψ of G, we denote the set of colors used in the closed neighborhood of a vertex v as $\psi[v]$.

Lemma 5.2.3. *If G is a smallest counterexample to Theorem 5.1.1, then there exists a partial proper 4-coloring ψ of G, such that*

1. *any uncolored vertex has degree exactly four,*
2. *for every vertex v, $|\psi[v]| \geqslant 3$,*
3. *for every 5^--vertex v, $|\psi[v]| = 4$.*

Proof. Since K_3 satisfies Theorem 5.1.1, G is a triangulation on at least 4 vertices and hence has minimum degree at least 3. Consider all 4-vertices in G which are not adjacent to any 3-vertex. Let I be a maximal independent set of such vertices. We construct a new triangulation G' (possibly non-simple) from G to start our coloring. If I is empty, G itself is considered as G'. Else we first delete I from G. Since I is an independent set of 4-vertices, deletion of each vertex in I creates a 4-face in $G - I$. Let f

be such a 4-face in $G - I$ and B be its boundary. We make two observations about B. By the choice of I, no 3-vertex of G is on B. Further, by Lemma 5.2.2, two 5^--vertices of G do not appear as adjacent vertices on B. Hence there exist two diagonally opposite 6^+-vertices on B. Triangulate $G - I$ by adding an edge joining two 6^+-vertices in every 4-face of $G - I$ to obtain a triangulation G'. We allow the possibility that G' may have multi-edges. A 4-cycle in G' corresponding to the boundary of a 4-face in $G - I$ is called *special*.

The Four Color Theorem [6] ensures a proper 4-coloring of G' exists. In a proper 4-coloring of G', each special 4-cycle gets either three or four colors. Among all the proper 4-colorings of G', let ψ be a coloring where the maximum number of special 4-cycles are 3-colored. Consider this same coloring ψ of $V(G) \setminus I$ in G. For every vertex u in I, if there are only three colors in $N_G(u)$, then extend ψ by giving u the fourth color. Else, leave it uncolored. Since the uncolored vertices (if any) are in I and I consists of only 4-vertices, ψ satisfies (1).

Since G is a triangulation, any vertex $v \in V(G)$ is part of a triangle uvw. If the vertices u, v and w are all colored then $|\psi[v]| \geqslant 3$. Since I is an independent set, there can be only at most one uncolored vertex in any triangle of G. If v itself is uncolored, then it was left so only because it saw all four colors in its neighborhood and hence $|\psi[v]| = 4$. The only remaining case is when u or w, say u, is uncolored. Vertex v is part of a 4-colored 4-cycle around u and thus sees three colors from this 4-cycle. Hence, ψ satisfies (2).

We call a vertex v *happy* if $|\psi[v]| = 4$. By choice of I, no vertex in the closed neighborhood of a 3-vertex is in I and hence every 3-vertex is part of a 4-colored K_4. Since all the four vertices of a 4-colored K_4 are happy, all 3-vertices and their neighbors are happy. The open neighborhood of any 4-vertex v in I has either four or three colors. In the former case, $N(v)$ itself has all four colors and in the latter case, v is colored with the fourth color and so v is happy. By the choice of I, any remaining 4-vertex is adjacent to a vertex in I. Next, we consider both 4-vertices and 5-vertices adjacent to I.

Let v be any 5^--vertex adjacent to a vertex u in I and let $B_u = (v, v_2, v_3, v_4)$ be the 4-cycle around u. The triangulation G' contains the edge v_2v_4 (since the endpoints of new edges are 6^+-vertices) and hence $\phi(v_2) \neq \phi(v_4)$. Thus v sees three colors from B_u. If u is colored, then v is happy. Otherwise v was not a part of any 3-colored special 4-cycle. If v did not have a neighbor with the fourth color in G', then recoloring v with the fourth color will give a proper coloring of G' with more 3-colored special 4-cycles, which is a contradiction to the choice of ψ. Hence v is happy in this case too.

The only 5^--vertices to be analyzed are the 5-vertices that are not adjacent to any vertex in I. If v is such a vertex, then it has a fully colored 5-cycle around it, hence v is happy. This concludes the proof that ψ satisfies (3). $\square$

Let ψ be a partial proper 4-coloring of G which satisfies the conclusions of Lemma 5.2.3. The coloring ψ partitions $V(G)$ into five subsets $C_1, C_2, C_3, C_4, \overline{C}$. Each C_i is the set of vertices with color i and $\overline{C}$ is the set of uncolored vertices. Let U_i, $i \in [4]$, be the set of vertices of G not in C_i and with no neighbor in C_i. Let $U = \cup_{i=1}^{4} U_i$ be the set of unhappy vertices. From Lemma 5.2.3 (1) and (3), it can be observed that $U \cap \overline{C} = \emptyset$. Similarly, from Lemma 5.2.3 (2), we know when $i \neq j$, $U_i \cap U_j = \emptyset$. We call the set of edges between U_i and U_j, $i \neq j$ as *bad edges* E_B. Let V_B be the set of all endpoints of bad edges. We call the graph $G_B = (V_B, E_B)$ the *bad subgraph* of G. We refer to the 4-cycle around an uncolored vertex as a *critical cycle*.

Observation 5.2.4. Every bad edge e of G is a part of two critical cycles.

Proof. Every edge in a triangulation is a part of two faces; hence e is also part of two faces, say uxv and uyv. If either x or y is colored then u and v see three common colors in their closed neighborhood which contradicts the fact that e is a bad edge. So e is a part of two critical cycles. □

Lemma 5.2.5. *Let G be a smallest counterexample to Theorem 5.1.1 and ψ be a partial proper 4-coloring satisfying the conclusions of Lemma 5.2.3 with $\overline{C}$ being the set of uncolored vertices. The bad subgraph G_B of G has a vertex cover of size at most $|\overline{C}|$.*

Proof. Let $H_1, \ldots, H_k$ be the connected components of G_B. For each H_l, $l \in [k]$, we prove the following claim. We say that a color i is *missing* at a vertex v if no vertex in $N[v]$ is colored i, i.e., $v \in U_i$.

Claim 5.1. H_l is bipartite with each part being the union of two color classes of ψ restricted to H_l.

Let x be a vertex in H_l. Let us call $\psi(x)$ as a, the color missing at x as c and the two remaining colors as b and d. Let $V_0 = (C_a \cap U_c) \cup (C_c \cap U_a)$ and $V_1 = (C_b \cap U_d) \cup (C_d \cap U_b)$. Let u be any vertex in V_0 and v be any neighbor of u in H_l. Since one of the colors in $\{a, c\}$ is $\psi(u)$ and the other is missing at u, v has to be colored b or d. The vertices u and v cannot be missing the same color (by the definition of a bad edge) and v cannot be missing $\psi(u)$. Hence v will be missing b or d. Thus v is in V_1. Similarly, we can show that for any vertex u in V_1 all its neighbors in H_l belong to V_0. Since H_l is connected, every vertex in H_l belongs to either V_0 or V_1 and every edge in H_l is between V_0 and V_1. Hence, H_l is bipartite with each part being a union of two color classes. Note that both these parts are vertex covers of H_l.

Let J be a vertex cover of G_B which is constructed by adding one part of each H_l.

Claim 5.2. $|J| \leqslant |\overline{C}|$.

Consider the bipartite subgraph Γ of G defined as follows. The two parts of Γ are J and $\overline{C}$. We retain an edge $xy \in E(G)$ from $\overline{C}$ to J only if y is the endpoint of a bad edge in the critical cycle around x. We ignore all other edges. Consider any vertex v in J. Since every bad edge is part of two critical cycles (Observation 5.2.4), v is adjacent to at least two vertices in $\overline{C}$. Now consider any vertex u in $\overline{C}$. The vertex u is only connected to end-points of bad edges in the critical cycle around it. Only one end-point of any bad edge is there in J, since it is constructed by adding one part of each H_l. So u has at most two neighbors in Γ. Since every vertex in J has at least two neighbors in $\overline{C}$ and every vertex in $\overline{C}$ has at most two in Γ, a simple double counting shows that $|J| \leqslant |\overline{C}|$. □

We proceed to prove Theorem 5.1.1. Let J be a vertex cover of G_B of size at most $|\overline{C}|$.

For each $i \in [4]$, consider a maximal independent set I_i of $U_i \setminus J$. Expand I_i to a maximal independent set I_i^+ of U_i by greedily adding vertices from $U_i \cap J$. Hence, the set $D_i = C_i \cup I_i^+$ is an independent dominating set of G.

Since $G[U \setminus J]$ does not have any bad edges, there are no edges between I_i (subset of U_i) and I_j (subset of U_j) when $i \neq j$. Hence, $I = \cup_{i=1}^4 I_i$ is an independent set. By Lemma 5.2.3 (3), we know that every vertex in U and hence I has degree at least six. From Observation 5.2.1, the size of I is at most $(n-2)/3$. So we have

$$\begin{aligned}
\sum_{i=1}^4 |D_i| &= \sum_{i=1}^4 |C_i| + \sum_{i=1}^4 |I_i^+| \\
&\leqslant \sum_{i=1}^4 |C_i| + \sum_{i=1}^4 |I_i| + \sum_{i=1}^4 |J \cap U_i| \\
&= n - |\overline{C}| + |I| + |J| \\
&\leqslant n - |\overline{C}| + \frac{n-2}{3} + |\overline{C}| \\
&= \frac{4n-2}{3}.
\end{aligned} \tag{5.3}$$

Hence, the smallest D_i has size less than $n/3$. This contradiction completes the proof of Theorem 5.1.1.

5.3 CONCLUDING REMARKS

The triangle and the octahedron are triangulations which satisfy $i(G) = |V(G)|/3$. However, we are unable to find an infinite family of triangulations with this property. Let $f(n)$ denote the maximum independent domination number among all n-vertex triangulations. Goddard and Henning [60] construct, for each positive integer k, a triangulation G on $n = 19k - 12$ vertices with $i(G) \geqslant 6k - 4$. This gives a lower bound on $f(n)$ which approaches $6n/19$ for large k. Note that [60] incorrectly reported it as $5n/19$ and Botler et al. [21] gave a different construction which showed that $f(n) \geqslant 2n/7$. Hence $\frac{6}{19} - o(1) \leqslant \frac{f(n)}{n} \leqslant \frac{1}{3}$. Closing this gap is an immediate task.

We can construct an infinite family of non-simple triangulations where $i(G) \geqslant |V(G)|/3$. Start with a 2k-vertex triangulation T, which contains a perfect matching M. Replace every edge xy of the matching M with a gadget made of a 2-cycle (x, y, x) and a 2-length path (x, z, y) inside it to obtain a new triangulation G on $n = 3k$ vertices. Any dominating set D of G should contain at least one vertex from each of the gadgets. Hence $|D| \geqslant k$. So we have $i(G) \geqslant \gamma(G) \geqslant n/3$. However, we do not have a matching upper bound for non-simple triangulations. This is another interesting problem.

The question by Goddard and Henning, whether there exist three disjoint independent dominating sets in every triangulation, remains unsettled and is quite intriguing. We also wonder whether the upper bound $i(G) \leqslant n/3$ can be extended to triangulated discs. If so, it will be a structural strengthening of the quantitatively tight result of Matheson and Tarjan that $\gamma(G) \leqslant n/3$ for any triangulated disc G.

6 CONCLUSION

In this thesis, we have plotted the summary of a research journey that explored the area of domination in planar graphs. Our primary focus was on three specific problems. Although our exploration involved numerous routes and methodologies, we have documented only the successful outcomes. We anticipate that our research will contribute significant insight to both structural and algorithmic aspects of domination on planar graphs and will pave the way for numerous future research directions. This chapter provides a discussion on the computational complexity of the works presented and highlights some of the open problems and research gaps that we found intriguing.

6.1 COMPUTATIONAL COMPLEXITY

While working on the problems presented in this thesis, our primary focus was on improving the bounds. The computational aspects of these proofs were not a central concern during our initial exploration. However, while presenting this work at various academic forums and conferences, some interesting questions emerged from an algorithmic perspective of the proofs. These inquiries inspired us to revisit the proofs with a computational lens, exploring their complexity and potential for algorithmic implementation. In fact, we are currently designing a linear-time algorithm to find two disjoint face-hitting and dominating sets in any plane graph G without isolated vertices, self-loops or 2-faces. In the proof of Theorem 4.2.1 we used the Four Color Theorem to color triangulations generated from the base graph and then merged two color classes to get a 2-coloring. However, a careful application of Peterson's theorem [101] allows for this 2-coloring to be executed in linear time. Peterson's theorem states that every cubic bridgeless graph contains a perfect matching. Biedl et al. in 2001 [16] presented a linear-time algorithm capable of identifying such a matching. As the dual of a triangulation meets these criteria, this approach can be used to determine a perfect matching in the dual of any planar triangulation. By removing the edges intersecting this matching in the primal graph, a quadrangulation is obtained, which is two-colorable. This new algorithm will only take the order of the degree sum of G if we use *Doubly Connected Edge List (DCEL)* as the data structure to represent G.

Regarding the total domination problem, the proof provides a pathway to developing a polynomial-time algorithm. The optimal run time attainable for this algorithm should be quadratic due to the application of four-coloring. However, by limiting our focus to triangulations, there might be potential for enhancement, possibly achieving a linear-time algorithm by leveraging Peterson's theorem in place of four-coloring. The proof for the independent domination problem is complex. It not only involves using a four-coloring of the graph but also requires selecting a specific coloring among all possible four-colorings. This suggests that the existing proof cannot be converted into a polynomial-time algorithm. However, we have not extensively considered the possibility of modifying the coloring algorithm to obtain the desired coloring without choosing one from all possible colorings.

6.2 OPEN PROBLEMS

There are several open problems related to our work that we find interesting. Some of these problems have already been mentioned in the corresponding chapters. This include two of our own conjectures from Chapter 3.

There are a couple of unsolved conjectures in Goddard and Henning's paper [59] which we find interesting. They conjectured that every planar triangulation with at least four vertices has a valid four-coloring consisting of color classes $\{V_1, V_2, V_3, V_4\}$ such that both $V_1 \cup V_2$ and $V_3 \cup V_4$ form total dominating sets. Our method seems somewhat limited in its ability to achieve a proper coloring that can be merged into total dominating sets. They also conjectured that if G is a planar triangulation with minimum degree at least four, then $\mathrm{dom}_t \geqslant 3$. If we restrict to near-triangulations with minimum degree at least three without 4-degree vertices then our Conjecture 3.3.1 can be proved with the same proof we have for Theorem 3.2.1. Dealing with the 4-degree vertices remains the primary challenge. Our proof method is applicable to all planar graphs that have a minimum degree of at least five. Therefore, in addressing our second conjecture (Conjecture 3.3.2), the difficulty lies similarly in managing the 4^--vertices.

Conjecture 2.2.3 by Matheson and Tarjan was one problem for which we had spent considerable time, the problem still remains open. Since this conjecture was proposed in 1995 and numerous efforts have been made to resolve it, we designate it as the most significant open problem addressed in this thesis. There is still a big gap between the best known upper bound of $2n/7$ by Christiansen et al [31] and the conjecture. Although there are limitations to our method of deleting the maximal independent set and finding a dominating and face-hitting set in the resultant graph, we have not actually explored the potential that Theorem 4.1.1 also works on multigraphs. There could be some scope for further exploration in this direction.

The most unsettling question for us from Chapter 5 is about bridging the gap between the lower and upper bounds for the independent domination number. The lower bound of $6n/19$ is very close to the upper bound of $n/3$ which we have proved. Proving this tightness is something that we wish to address immediately. We would also like to see if the upper bound $\iota(G) \leqslant n/3$ can be extended to triangulated discs. If so, it will be a true structural strengthening of the result of Matheson and Tarjan. We do not have an answer to the original question of Goddard and Henning. They asked whether there exist three disjoint independent dominating sets in any triangulation. Our proof fails when it comes to identifying disjoint sets.

We conclude this thesis with the hope that it will motivate more research on these topics and that most of the open questions mentioned above will be resolved soon.

BIBLIOGRAPHY

[1] Wilhelm Ahrens. *Mathematische Unterhaltungen und Spiele*. Leipzig-Berlin, 1910.

[2] Saieed Akbari, Mohammad Motiei, Sahand Mozaffari, and Sina Yazdanbod. "Cubic graphs with total domatic number at least two." In: *Discussiones Mathematicae. Graph Theory* 38 (2018), pp. 75–82.

[3] Michael O. Albertson. "A lower bound for the independence number of a planar graph." In: *Journal of Combinatorial Theory, Series B* 20.1 (1976), pp. 84–93.

[4] Noga Alon. "Transversal numbers of uniform hypergraphs." In: *Graphs and Combinatorics* 6.1 (1990), pp. 1–4.

[5] Noga Alon, Robert Berke, Kevin Buchin, Maike Buchin, Péter Csorba, Saswata Shannigrahi, Bettina Speckmann, and Philipp Zumstein. "Polychromatic colorings of plane graphs." In: *Discrete Computational Geometry* 42.3 (2009), pp. 421–442.

[6] Kenneth Appel and Wolfgang Haken. "Every planar map is four colorable. I. Discharging." In: *Illinois Journal of Mathematics* 21 (1977), pp. 429–490.

[7] Kenneth Appel, Wolfgang Haken, and John Koch. "Every planar map is four colorable. II. Reducibility." In: *Illinois Journal of Mathematics* 21 (1977), pp. 491–567.

[8] Hamideh Aram, Seyed Mahmoud Sheikholeslami, and Lutz Volkmann. "On the total domatic number of regular graphs." In: *Transactions on Combinatorics* 1 (2012), pp. 45–51.

[9] Dan Archdeacon, Joanna Ellis-Monaghan, David Fisher, Dalibor Froncek, Peter Lam, Suzanne Seager, Bing Wei, and Raphael Yuster. "Some remarks on domination." In: *Journal of Graph Theory* 46 (July 2004), pp. 207–210.

[10] W.W. Rouse Ball. *Mathematical Recreations and Problems of Past and Present Times*. MacMillan, London, 1892.

[11] Kristóf Bérczi and Hanna Gábor. "Total domatic number of triangulated planar graphs." In: *EGRES Quick-Proof* (2018), pp. 1–12.

[12] Claude Berge. *Théorie des graphes et ses applications*. Vol. 2. Universitaire de Mathématiques. Dunod, Paris, 1958.

[13] Claude Berge. *Graphs and hypergraphs*. North-Holland Mathematical Library, 1973.

[14] Alan A. Bertossi. "Dominating sets for split and bipartite graphs." In: *Information processing letters* 19.1 (1984), pp. 37–40.

[15] Max Bezzel. "Schachfreund." In: *Berliner Schachzeitung* (1848).

[16] Therese C. Biedl, Prosenjit Bose, Erik D. Demaine, and Anna Lubiw. "Efficient algorithms for Petersen's matching theorem." In: *Journal of Algorithms* 38.1 (2001), pp. 110–134.

[17] M. M. Blank. "An estimate of the external stability number of a graph without suspended vertices." In: *Prikladnaya Matematika i Programmirovanie* 10 (1973), pp. 3–11.

[18] Béla Bollobás and Ernest J. Cockayne. "Graph-theoretic parameters concerning domination, independence, and irredundance." In: *Journal of Graph Theory* 3.3 (1979), pp. 241–249.

[19] Kellogg S. Booth and J. Howard Johnson. "Dominating sets in chordal graphs." In: *SIAM Journal on Computing* 11 (1982), pp. 191–199.

[20] Prosenjit Bose, Thomas Shermer, Godfried Toussaint, and Binhai Zhu. "Guarding polyhedral terrains." In: *Computational Geometry* 7.3 (1997), pp. 173–185.

[21] Fábio Botler, Cristina G. Fernandes, and Juan Gutiérrez. "Independent Dominating Sets in Planar Triangulations." In: *The Electronic Journal of Combinatorics* (2024), P2–12.

[22] Robert Brigham, Julie Carrington, and Richard Vitray. "Connected graphs with maximum total domination number." In: *Journal of Combinatorial Mathematics and Combinatorial Computing* 34 (Jan. 2000), pp. 81–96.

[23] Csilla Bujtás. "Domination Number of Graphs with Minimum Degree Five." In: *Discussiones Mathematicae Graph Theory* 41 (Aug. 2021), p. 763.

[24] Csilla Bujtás and Michael A. Henning. "On the domination number of graphs with minimum degree six." In: *Discrete Mathematics* 344.8 (2021), p. 112449.

[25] Csilla Bujtás and Sandi Klavžar. "Improved upper bounds on the domination number of graphs with minimum degree at least five." In: *Graphs and Combinatorics* 32 (2016), pp. 511–519.

[26] Christiane N. Campos and Yoshiko Wakabayashi. "On dominating sets of maximal outerplanar graphs." In: *Discrete Applied Mathematics* 161.3 (2013), pp. 330–335.

[27] Yair Caro. *New results on the independence number*. Tech. rep. Tel-Aviv University, 1979.

[28] Maw-Shang Chang. "Efficient algorithms for the domination problems on interval and circular-arc graphs." In: *SIAM Journal on Computing* 27.6 (1998), pp. 1671–1694.

[29] H. S. Chao, Fang Rong Hsu, and Richard C. T. Lee. "An optimal algorithm for finding the minimum cardinality dominating set on permutation graphs." In: *Discrete Applied Mathematics* 102.3 (2000), pp. 159–173.

[30] Bob Chen, Jeong Han Kim, Michael Tait, and Jacques Verstraete. "On coupon colorings of graphs." In: *Discrete Applied Mathematics* 193 (2015), pp. 94–101.

[31] Aleksander B.G. Christiansen, Eva Rotenberg, and Daniel Rutschmann. "Triangulations Admit Dominating Sets of Size 2 n/7." In: *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM. 2024, pp. 1194–1240.

[32] Václav Chvátal. "A combinatorial theorem in plane geometry." In: *Journal of Combinatorial Theory, Series B* 18.1 (1975), pp. 39–41.

[33] Ernie J. Cockayne, R. M. Dawes, and Stephen T. Hedetniemi. "Total domination in graphs." In: *Networks. An International Journal* 10 (1980), pp. 211–219.

[34] Ernie J. Cockayne, Seymour E. Goodman, and Stephen T. Hedetniemi. "A linear algorithm for the domination number of a tree." In: *Information Processing Letters* (1975), pp. 41–44.

[35] Ernie J. Cockayne and Stephen T. Hedetniemi. "Independence graphs." In: *Congressus Numerantium, X* (1974), pp. 471–491.

[36] Ernie J. Cockayne and Stephen T. Hedetniemi. "Disjoint independent dominating sets in graphs." In: *Discrete Mathematics* 15.3 (1976), pp. 213–222.

[37] Ernie J. Cockayne and Stephen T. Hedetniemi. "Towards a theory of domination in graphs." In: *Networks. An International Journal* 7 (1977), pp. 247–261.

[38] Ernie J. Cockayne, Paul A. Dreyer Jr, Sandra M. Hedetniemi, and Stephen T. Hedetniemi. "Roman domination in graphs." In: *Discrete mathematics* 278.1-3 (2004), pp. 11–22.

[39] Derek G. Corneil and J. M. Keil. "A dynamic programming approach to the dominating set problem on k-trees." In: *SIAM Journal on Algebraic Discrete Methods* 8.4 (1987), pp. 535–543.

[40] Derek G. Corneil and Yehoshua Perl. "Clustering and domination in perfect graphs." In: *Discrete Applied Mathematics* 9.1 (1984), pp. 27–39.

[41] Daniel W. Cranston and Landon Rabern. "Planar graphs have independence ratio at least 3/13." In: *The Electronic Journal of Combinatorics* (2016), P3–45.

[42] Aleksandr B. Dainyak and Aleksandr A. Sapozhenko. “Independent sets in graphs.” In: *Discrete Mathematics and Applications* 26.6 (2016), pp. 323–346.

[43] Alexander K. Dewdney. *Fast Turing reductions between problems in NP: Reductions between NP-complete problems*. Tech. rep. Dept. of Computer Science, The University of Western Ontario, 1981.

[44] Michael Dorfling, Wayne Goddard, and Michael A. Henning. “Domination in planar graphs with small diameter II.” In: *Ars Combinatoria* 78 (2006), 237 – 255.

[45] Michael Dorfling and Michael A. Henning. “Transversals in 5-uniform hypergraphs and total domination in graphs with minimum degree five.” In: *Quaestiones Mathematicae* 38 (Apr. 2015), pp. 155–180.

[46] Jean E. Dunbar, Sandra M. Hedetniemi, Stephen T. Hedetniemi, David Jacobs, James A. Knisely, Renu Laskar, and Douglas F. Rall. “Fall Colorings of Graphs.” In: *Journal of Combinatorial Mathematics and Combinatorial Computing* 33 (Jan. 2000), pp. 257–273.

[47] Zdeněk Dvořák, Tomáš Masařík, Jan Musílek, and Ondřej Pangrác. “Triangle-free planar graphs with the smallest independence number.” In: *Journal of Graph Theory* 90.3 (2019), pp. 443–454.

[48] Zdeněk Dvořák and Matthias Mnich. “Large independent sets in triangle-free planar graphs.” In: *SIAM Journal on Discrete Mathematics* 31.2 (2017), pp. 1355–1373.

[49] Leonhard Euler. “Solutio problematis ad geometriam situs pertinentis.” In: *Commentarii academiae scientiarum Petropolitanae* (1736), pp. 128–140.

[50] Odile Favaron. “Two relations between the parameters of independence and irredundance.” In: *Discrete Mathematics* 70.1 (1988), pp. 17–20.

[51] Steve Fisk. “A short proof of Chvátal’s Watchman Theorem.” In: *Journal of Combinatorial Theory, Series B* 24.3 (1978), p. 374.

[52] Peter Flach and Lutz Volkmann. “Estimations for the domination number of a graph.” In: *Discrete Mathematics* 80.2 (Mar. 1990), 145–151.

[53] P. Francis, Abraham M. Illickan, Lijo M. Jose, and Deepak Rajendraprasad. “Disjoint total dominating sets in near-triangulations.” In: *Journal of Graph Theory* 105.1 (2024), pp. 68–77.

[54] P. Francis and Deepak Rajendraprasad. “On coupon coloring of Cartesian product of some graphs.” In: *Algorithms and discrete applied mathematics*. Vol. 12601. Lecture Notes in Comput. Sci. Springer, Cham, 2021, pp. 309–316.

[55] P. Francis and Deepak Rajendraprasad. "On domatic and total domatic numbers of product graphs." In: *arXiv preprint arXiv:2103.10713* (2021).

[56] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, Co., New York, 1979.

[57] Wayne Goddard and Michael A. Henning. "Domination in planar graphs with small diameter*." In: *Journal of Graph Theory* 40 (May 2002), pp. 1 –25.

[58] Wayne Goddard and Michael A. Henning. "Independent domination in graphs: A survey and recent results." In: *Discrete Mathematics* 313.7 (2013), pp. 839–854.

[59] Wayne Goddard and Michael A. Henning. "Thoroughly dispersed colorings." In: *Journal of Graph Theory* 88 (2018), pp. 174–191.

[60] Wayne Goddard and Michael A. Henning. "Independent domination, colorings and the fractional idomatic number of a graph." In: *Applied Mathematics and Computation* 382 (2020), p. 125340.

[61] Herbert Grötzsch. "Ein Dreifarbensatz fur dreikreisfreie Netze auf der Kugel." In: *Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle- Wittenberg, Math.-Naturwissenschaftliche Reihe* 8 (1959), pp. 109–120.

[62] Julie Haviland. "On minimal maximal independent sets of a graph." In: *Discrete Mathematics* 94.2 (Nov. 1991), 95–101.

[63] Teresa W. Haynes, Stephen T. Hedetniemi, and Michael A. Henning. *Domination in graphs: Core concepts*. Springer, 2023.

[64] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter Slater. *Fundamentals of Domination in Graphs*. CRC Press, 1998.

[65] Percy J. Heawood. "Map-color theorem." In: *Quarterly Journal of Pure and Applied Mathematics* 24 (1890), pp. 332–338.

[66] Percy J. Heawood. "On the four-colour map theorem." In: *Quarterly Journal of Pure and Applied Mathematics* 29 (1898), pp. 270–285.

[67] Christopher Carl Heckman and Robin Thomas. "Independent sets in triangle-free cubic planar graphs." In: *Journal of Combinatorial Theory, Series B* 96.2 (2006), pp. 253–275.

[68] Pinar Heggernes and Jan Arne Telle. "Partitioning graphs into generalized dominating sets." In: *Nordic Journal of Computing* 5 (1998), pp. 128–142.

[69] Michael A. Henning. "Graphs with large total domination number." In: *Journal of Graph Theory* 35.1 (2000), pp. 21–45.

[70] Michael A. Henning, Christian Löwenstein, and Dieter Rautenbach. "Remarks about disjoint dominating sets." In: *Discrete Mathematics* 309.23 (2009), pp. 6451–6458.

[71] Michael A. Henning and Iztok Peterin. "A characterization of graphs with disjoint total dominating sets." In: *Ars Mathematica Contemporanea* 16 (2019), pp. 359–375.

[72] Michael A. Henning and Anders Yeo. "A new upper bound on the total domination number in graphs with minimum degree six." In: *Discrete Applied Mathematics* 302 (2021), pp. 1–7.

[73] Frank Hoffmann and Klaus Kriegel. "A graph-coloring result and its consequences for polygon-guarding problems." In: *SIAM Journal on Discrete Mathematics* 9.2 (1996), pp. 210–224.

[74] Tatsuya Honjo, Ken-ichi Kawarabayashi, and Atsuhiro Nakamoto. "Dominating sets in triangulations on surfaces." In: *Journal of Graph Theory* 63.1 (2010), pp. 17–30.

[75] Elad Horev, Matthew J. Katz, Roi Krakovski, and Atsuhiro Nakamoto. "Polychromatic 4-coloring of cubic bipartite plane graphs." In: *Discrete Mathematics* 312.4 (2012), pp. 715–719.

[76] Elad Horev and Roi Krakovski. "Polychromatic colorings of bounded degree plane graphs." In: *Journal of Graph Theory* 60.4 (2009), pp. 269–283.

[77] M. R. Iyer and V. V. Menon. "On Coloring the $n \times n$ Chessboard." In: *The American Mathematical Monthly* 73.7 (1966), pp. 721–725.

[78] Carl Friedrich von Jaenisch. *Applications de L'Analyse Mathématique au Jeu des Échecs*. Dufour et cie, 1862.

[79] Tomá Kaiser and Riste Škrekovski. "Planar graph colorings without short monochromatic cycles." In: *Journal of Graph Theory* 46 (2004), pp. 25–38.

[80] Alfred B. Kempe. "On the Geographical Problem of the Four Colours." In: *American Journal of Mathematics* 2.3 (1879), pp. 193–200.

[81] Erika L.C. King and Michael J. Pelsmajer. "Dominating sets in plane triangulations." In: *Discrete Mathematics* 310.17 (2010), pp. 2221–2230.

[82] Mikko Koivisto, Petteri Laakkonen, and Juho Lauri. "NP-completeness results for partitioning a graph into total dominating sets." In: *Theoretical Computer Science* 818 (2020), pp. 22–31.

[83] Dieter Kratsch and Lorna Stewart. "Domination on Cocomparability Graphs." In: *SIAM Journal on Discrete Mathematics* 6.3 (1993), pp. 400–417.

[84] Renu Laskar and Jeremy Lyle. “Fall colouring of bipartite graphs and cartesian products of graphs.” In: *Discrete Applied Mathematics* 157.2 (2009), pp. 330–338.

[85] Juho Lauri and Christodoulos Mitillos. “Complexity of fall coloring for restricted graph classes.” In: *Theory of Computing Systems* 64 (2020), pp. 1183–1196.

[86] Hong Liu and Michael J. Pelsmajer. “Dominating sets in triangulations on surfaces.” In: *Ars mathematica contemporanea* 4.1 (2011), pp. 177–204.

[87] László Lovász. *Combinatorial problems and exercises*. North Holland & Akadémiai Kiadó, 1979.

[88] Gary MacGillivray and Karen Seyffarth. “Domination Numbers of Planar Graphs.” In: *Journal of Graph Theory* 22.3 (1996). Cited by: 40, 213 – 229.

[89] Gary MacGillivray and Karen Seyffarth. “Bounds for the independent domination number of graphs and planar graphs.” In: *Journal of Combinatorial Mathematics and Combinatorial Computing* 49 (2004), pp. 33–56.

[90] C.E. Veni Madhavan. “Approximation algorithm for maximum independent set in planar traingle-free graphs.” In: *International Conference on Foundations of Software Technology and Theoretical Computer Science*. Springer. 1984, pp. 381–392.

[91] Dánuţ Marcu. “An upper bound on the domination number of a graph.” In: *Mathematica Scandinavica* 59 (1986), pp. 41–44.

[92] Lesley R. Matheson and Robert E. Tarjan. “Dominating sets in planar graphs.” In: *European Journal of Combinatorics* 17 (1996), pp. 565–568.

[93] William McCuaig and Bruce Shepherd. “Domination in graphs with minimum degree two.” In: *Journal of Graph Theory* 13.6 (1989), pp. 749–762.

[94] Haiko Müller and Andreas Brandstädt. “The NP-completeness of Steiner tree and dominating set for chordal bipartite graphs.” In: *Theoretical Computer Science* 53.2-3 (1987), pp. 257–265.

[95] Zoltán Lóránt Nagy. “Coupon-coloring and total domination in Hamiltonian planar triangulations.” In: *Graphs and Combinatorics* 34 (2018), pp. 1385–1394.

[96] Franz Nauk. “Briefwechseln mit allen für alle.” In: *Illustrirte Zeitung* (1850).

[97] Franz Nauk. “Schach: eine in das gebiet der mathematik fallende aufgabe von herrn dr. Nauck in Schleusingen.” In: *Illustrirte Zeitung* (1850).

[98] Oystein Ore. *Theory of graphs*. American Mathematical Society, 1962.

[99] E. Pauls. “Das maximal problem der Damen auf dem Schachbrett.” In: *Deutsche Schachzeitung* 29 (1874), 129–134.

[100] Charles Payan and Nguyen Huy Xuong. "Domination-balanced graphs." In: *Journal of Graph Theory* 1 (1982), pp. 23–32.

[101] Julius Petersen. "Die Theorie der regulären graphs." In: *Acta Mathematica* (1891).

[102] Michael D. Plummer, Dong Ye, and Xiaoya Zha. "Dominating plane triangulations." In: *Discrete Applied Mathematics* 211 (2016), pp. 175–182.

[103] Michael D. Plummer, Dong Ye, and Xiaoya Zha. "Dominating maximal outerplane graphs and Hamiltonian plane triangulations." In: *Discrete Applied Mathematics* 282 (2020), pp. 162–167.

[104] Dieter Rautenbach. "A linear Vizing-like relation between the size and the domination number of a graph." In: *Journal of Graph Theory* 31.4 (1999), pp. 297–302.

[105] Bruce Reed. "Paths, stars and the number three." In: *Combinatorics, Probability and Computing* 5.3 (1996), pp. 277–295.

[106] Neil Robertson, Daniel P. Sanders, Paul Seymour, and Robin Thomas. "Efficiently four-coloring planar graphs." In: *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*. ACM, New York, 1996, pp. 571–575.

[107] Walter Schnyder. "Embedding planar graphs on the grid." In: *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*. SODA '90. 1990, pp. 138–148.

[108] T. Schönberger. "Ein beweis des petersenschen graphensatzes." In: *Acta Scientiarum Mathematicarum Szeged* 7 (1934), pp. 51–57.

[109] Er Fang Shan, Moo Young Sohn, Xu Dong Yuan, and Michael A. Henning. "Domination Number in Graphs with Minimum Degree Two." In: *Acta Mathematica Sinica-english Series* 25 (Aug. 2009), pp. 1253–1268.

[110] Moo Young Sohn and Yuan Xudong. "Domination in graphs of minimum degree four." In: *Journal of The Korean Mathematical Society* 46 (July 2009), pp. 759–773.

[111] Simon Špacapan. "The domination number of plane triangulations." In: *Journal of Combinatorial Theory, Series B* 143 (2020), pp. 42–64.

[112] Richard Steinberg and Craig A. Tovey. "Planar ramsey numbers." In: *Journal of Combinatorial Theory, Series B* 59.2 (1993), pp. 288–296.

[113] Liang Sun. "An upper bound for total domination number." In: *Journal of Beijing Institute of Technology* 4.2 (1995), pp. 111–113.

[114] Liang Sun and Jianfang Wang. "An upper bound for the independent domination number." In: *Journal of Combinatorial Theory, Series B* 76.2 (1999), pp. 240–246.

[115] Stéphan Thomassé and Anders Yeo. "Total Domination of Graphs and Small Transversals of Hypergraphs." In: *Combinatorica* 27 (July 2007).

[116] Carsten Thomassen. "2-list-coloring planar graphs without monochromatic triangles." In: *Journal of Combinatorial Theory. Series B* 98 (2008), pp. 1337–1348.

[117] Shin ichi Tokunaga. "Dominating sets of maximal outerplanar graphs." In: *Discrete Applied Mathematics* 161.18 (2013), pp. 3097–3099.

[118] Vadim Georgievich Vizing. "An estimate of the external stability number of a graph." In: *Doklady Akademii Nauk*. Vol. 164. 4. Russian Academy of Sciences. 1965, pp. 729–731.

[119] H.B. Walikar, B. Devadas Acharya, and E. Sampathkumar. "Recent developments in the theory of domination in graphs." In: *MRI lecture notes in mathematics* 1 (1979).

[120] Victor K. Wei. *A lower bound on the stability number of a simple graph*. Tech. rep. Bell Laboratories Technical Memorandum Murray Hill, NJ, USA, 1981.

[121] Douglas Brent West. *Introduction to graph theory*. Vol. 2. Prentice hall Upper Saddle River, 2001.

[122] Hassler Whitney. "A Theorem on Graphs." In: *Annals of Mathematics* 32.2 (1931), pp. 378–390.

[123] Hua-Ming Xing, Liang Sun, and Xue-Gang Chen. "Domination in graphs of minimum degree five." In: *Graphs and Combinatorics* 22 (2006), pp. 127–143.

[124] Bohdan Zelinka. "Adomatic and idomatic numbers of graphs." In: *Mathematica Slovaca* 33.1 (1983), pp. 99–103.

[125] Bohdan Zelinka. "Total domatic number and degrees of vertices of a graph." In: *Mathematica Slovaca* 39 (1989), pp. 7–11.

[126] Bohdan Zelinka. "Regular totally domatically full graphs." In: *Discrete Mathematics* 86 (1990), pp. 71–79.