Cluster structures on spinor helicity and momentum twistor varieties

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Spinor helicity varieties

- In the study of scattering amplitudes in quantum field theories for massless scattering the kinematic space modeling particle interactions is represented by two matrices λ and $\tilde{\lambda}$ of size $2 \times n$ with property that $\lambda\tilde{\lambda}^T$ is zero.
- The 2×2-minors of λ are denoted by $\langle ij \rangle$ and the 2×2-minors of $\tilde{\lambda}$ by $[ij]$ for $1 \leq i < j \leq n$.
- Each class of minors satisfy quadratic Plücker relations: for $1 \leq i < j < k < l \leq n$

$$[ij][kl] - [ik][jl] + [il][jk] = 0 \quad \text{and} \quad \langle ij \rangle\langle kl \rangle - \langle ik \rangle\langle jl \rangle + \langle il \rangle\langle jk \rangle = 0$$

and momentum conservation: for all $1 \leq i, j \leq n$,

$$\sum_{s=1}^{n} \langle is \rangle [sj] = 0.$$

Spinor helicity varieties

- Denote by $\mathcal{SH}_n$ the variety determined by the above equations. It is called a spinor helicity variety.
- Mazzouz, Pfister, and Sturmfels 2024 showed that $\mathcal{SH}_n$ is isomorphic to the partial flag variety $\mathcal{F}\ell_{2,n-2;n}$ with isomorphism given by

$$\langle ij \rangle \mapsto P_{ij}, \quad \text{and} \quad [ij] \mapsto (-1)^{i+j-1} P_{[n]-\{i,j\}}. \tag{1}$$

Momentum twistor varieties

- The scattering process may also be described in terms of momentum twistors $Z_1, \ldots, Z_n \in \mathbb{C}P^3$ representing the particles in Minkowski space.
- Interpreting the coordinates of Z_i as column vectors we obtain a $4 \times n$ matrix.
- Denote by $\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l)$ a Plücker coordinate P_{ijkl} on the Grassmannian $\mathrm{Gr}_{4,n}$.
- If the system does not have dual conformal symmetry, i.e. it is *non-dual conformal invariant* (or NDCI for short) we add an infinity twistor in form of a line ℓ_∞.

Momentum twistor varieties

- The line ℓ_∞ may be understood as the line spanned by two additional points $Z_{n+1}, Z_{n+2} \in \mathbb{C}P^3$.
- We consider only those minors $\langle ijkl \rangle$ that satisfy

$$n+1, n+2 \in \{i,j,k,l\} \quad \text{or neither} \quad n+1, n+2 \notin \{i,j,k,l\}.$$

- We identify

$$\langle i,j,n+1,n+2 \rangle \mapsto P_{ij}, \quad \text{and} \quad \langle i,j,k,l \rangle \mapsto P_{ijkl}, \tag{2}$$

  where $i,j,k,l \in [n]$.
- We define the **momentum twistor variety** denoted by $\mathcal{MT}_n$ as the subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{4}-1}$ as the vanishing set of the Plücker relations. Hence $\mathcal{MT}_n$ coincides with the partial flag variety $\mathcal{F}\ell_{2,4;n}$.

Spinor helicity varieties and momentum twistor varieties

- Physicists know the change of parametrization between spinor helicity and momentum twistor variables. Part of the map is given by

$$\langle i-1, i, j, j+1\rangle \mapsto [ij]\langle ij\rangle\langle i-1, i\rangle\langle j, j+1\rangle.$$

We have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{SH}_n \arrow[d] & \mathcal{F}\ell_{2,n-2;n} \arrow[l, "\sim"'] \arrow[d] \\
\mathcal{MT}_n & \mathcal{F}\ell_{2,4;n} \arrow[l, "\sim"']
\end{tikzcd}$$

Spinor helicity varieties and momentum twistor varieties

- We explore the cluster structures of the spinor helicity variety $\mathcal{SH}_n \cong \mathcal{F}\ell_{2,n-2;n}$ and the momentum twistor variety $\mathcal{MT}_n \cong \mathcal{F}\ell_{2,4;n}$ and show how they can be obtained from the cluster structure on Grassmannians.
- The cluster structure on momentum twistor variety can be obtained from the initial cluster of the Grassmannain $\mathrm{Gr}_{4,n+2}$ by two mutations followed by freezing one cluster variable.
- The cluster structure on spinor helicity variety $\mathcal{F}\ell_{2,n-2;n}$ can be obtained from the initial cluster of the Grassmannain $\mathrm{Gr}_{n-2,2n-4}$ by $\frac{n(n-4)(n-5)}{2}$ mutations followed by freezing $n-5$ cluster variables.

Partial flag varieties

- Denote by SL_n the special linear group of $n \times n$ matrices with entries in a field $\mathbb{k}$.
- We fix the Borel subgroup of upper triangular matrices $B \subset SL_n$ and the subgroup unipotent matrices $U \subset B$.
- For a positive integer i write $[i] := \{1, \ldots, i\}$ and for $j > i$ write $[i, j] := \{i, i+1, \ldots, j\}$. Let n be a positive integer and $1 \leq d_1 < d_2 < \cdots < d_k < n$. To this data we associate the variety of partial flags of subspaces in $\mathbb{k}^n$, $\mathbb{k}$ a field of characteristic zero,

$$\mathcal{F}\ell_{d_1,\ldots,d_k;n} := \{0 \in V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{k}^n : \dim V_i = d_i\}. \quad (3)$$

- If $k = 1$ the associated partial flag variety is a Grassmannian $\mathrm{Gr}_{d,n}$.

Partial flag varieties

- Points in $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$ can be represented by matrices in $\mathbb{C}^{d_k\times n}$ where the first d_i rows span the i^{th} vector space in the flag.
- A Plücker coordinate $P_{i_1,\ldots,i_r} \in \mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ ($i_1 < \cdots < i_r$, $r \in \{d_1,\ldots,d_k\}$): for a $d_k \times n$ matrix $x = (x_{ij})_{d_k\times n}$, $P_{i_1,\ldots,i_r}(x)$ is the minor of x with 1st, ..., rth rows and i_1th, ..., i_rth columns.
- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{k}[\mathrm{Gr}_{k,n}]$ ($i_1 < \cdots < i_k$): for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, ..., kth rows and i_1th, ..., i_kth columns.

Partial flag varieties

- We define a morphism from the Grassmannian to the flag variety via its pullback on the homogeneous coordinate rings:

$$\varphi^* : \mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}] \hookrightarrow \mathbb{k}[\mathrm{Gr}_{d_k;n+d_k-d_1}], \tag{4}$$

determined by the images of the Plücker coordinates:

$$P_{I_1} \mapsto P_{I_1 \cup [n+1, n+d_k-d_1]}, P_{I_2} \mapsto P_{I_2 \cup [n+d_2-d_1+1, n+d_k-d_1]}, \ldots, P_{I_k} \mapsto P_{I_k}.$$

Here I_j is a subset of $[n]$ of cardinality d_j for all $1 \leq j \leq k$.

Conjecture

The algebra embedding (4) *maps cluster variables of* $\mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ *to cluster variables of* $\mathbb{k}[\mathrm{Gr}_{d_k;n+d_k-d_1}]$.

An initial seed for a Grassmannian cluster algebra

- The coordinate ring $\mathbb{C}[\mathrm{Gr}_{k,n}]$ has a cluster algebra structure with an initial seed given by Plücker coordinates, [Scott 2006].

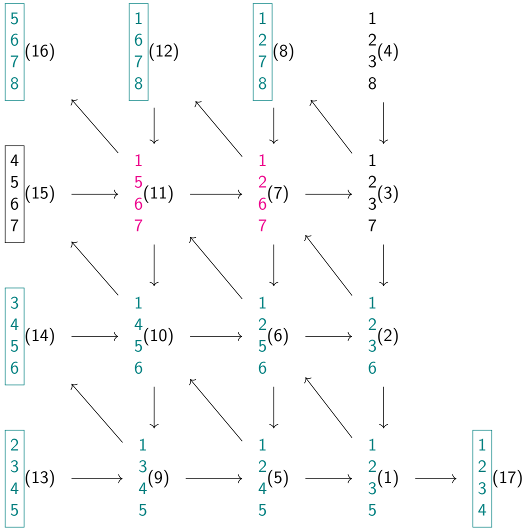

Figure: An initial seed for the Grassmannian $\mathrm{Gr}_{4;8}$.

Cluster algebra structure on partial flag varieties

- The coordinate ring $\Bbbk[U]$ has a cluster algebra structure due to Berenstein, Fomin and Zelevisnky in 1996, 2005.
- Geiss, Leclerc and Schröer 2008 generalized their work to the case of $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ and they gave a cluster structure on $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$.

Pseudoline arrangements

- We describe the pseudoline arrangement $\mathcal{P}_{d_1,\ldots,d_k;n}$ for $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$.
- Let $\sigma \in S_n$ be a permutation whose one-line presentation is $\sigma = [d_1, d_1-1, \ldots, 1, d_2, d_2-1, \ldots, d_1+1, \ldots, d_k, d_k-1, \ldots, d_{k-1}+1]$.
- Draw nodes labelled by $1, \ldots, n$ (left to right) on a horizontal line and n nodes labelled $\sigma(1), \ldots, \sigma(n)$ on a vertical line as in a positive orthant.
- Start drawing pseudolines ℓ_i connecting i with $i = \sigma(j)$ (for some j) in the following way: the lines $\ell_{d_i+1}, \ell_{d_i+2} \ldots, \ell_{d_{i+1}}$ do not cross; they start vertically and then turn once they reach the height of the vertical nodes $d_i + 1, \ldots, d_{i+1}$.

From pseudoline arrangements to quivers

- mutable vertices of $Q_{d_1,\ldots,d_k;n}$ correspond to bounded faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- there are two types of frozen vertices: $n-1$ of them correspond to the unbounded faces on the left end of $\mathcal{P}_{d_1,\ldots,d_k;n}$; additionally there are k frozen vertices denoted by $v_{d_1},\ldots,v_{d_k}$.

From pseudoline arrangements to quivers

There are 4 types of arrows.

- from left to right perpendicular to a vertical straight lines segment connecting adjacent faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- from top to bottom perpendicular to a horizontal straight line segment connecting adjacent faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- diagonally from bottom right to top left through a crossing of straight line segments connecting faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$ that share a vertex;
- there are arrows to and from the extra frozen vertices $v_{d_1}, \ldots, v_{d_k}$: there is an arrow from the face bounded by ℓ_{d_i-1}, ℓ_{d_i} vertically and $\ell_{d_i+1}, \ell_{d_i+2}$ horizontally to the vertex v_{d_i}, and an arrow from v_{d_i} to the face bounded by ℓ_{d_i} on the left ℓ_{d_i+1} on the top and right (this is where ℓ_{d_i+1} bends) and by ℓ_{d_i+2} on the bottom.

Pseudoline arrangements

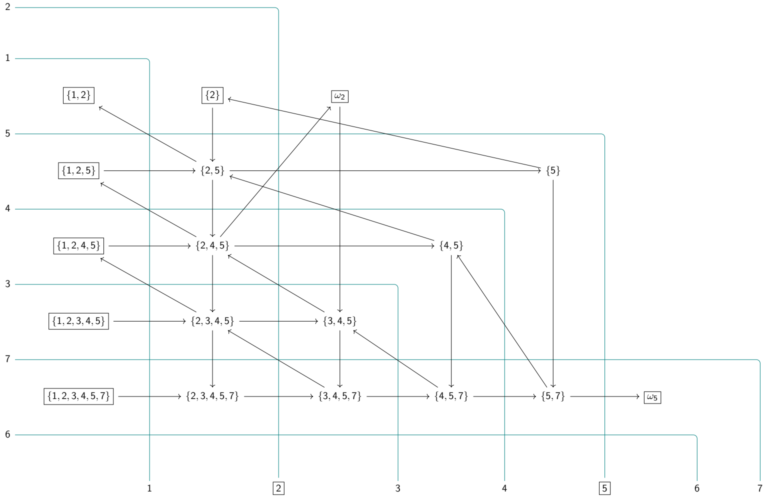

Figure: The pseudoline arrangement $\mathcal{P}_{2,5;7}$ and its quiver $Q_{2,5;7}$

Minors at the vertices of the quiver

- Every face of the pseudoline arrangement $\mathcal{P}_{d_1,\dots,d_k;n}$ can be associated with a minor in $\Bbbk[U]$, [BFZ96].
- The minors are of form form $D_{I,J}$ with $I, J \subset [n]$ of the same size. The column index set J is always $\{n-|I|-1,\dots,n\}$. The index set I is given by the indices of lines passing north east of the face.
- Given a face F of $\mathcal{P}_{d_1,\dots,d_k;n}$ we set

$$D_{I_F} := D_{I_F,\{n-|I_F|+1,\dots,n\}} \quad \text{where} \quad I_F := \{i : \ell_i \text{ passes north-east of } F\}.$$

Dual canonical basis elements as tableaux

- For $k, m \in \mathbb{Z}_{\geq 1}$, denote by $\mathrm{SSYT}_{\leq k,m}$ the set of all semi-standard Young tableaux with less or equal to k rows and with entries in $[m]$.
- The dual canonical basis of $\Bbbk[U]$ is parametrized by the set $\mathrm{SSYT}_{\leq n-1,n,\sim}$ of equivalence classes of semistandard Young tableaux and there is an explicit formula for the dual canonical basis elements [L. 2024].
- The dual canonical basis of $\Bbbk[SL_k/U]$ is parametrized by the set $\mathrm{SSYT}_{\leq n-1,n}$ [L. 2024]. The algebra $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ is a subalgebra of $\Bbbk[SL_k/U]$. Its dual canonical basis is parametrized by $\mathsf{SSYT}_{d_1,\ldots,d_k;n}$ which is the set of semistandard tableaux which may have columns with $d_1, \ldots, d_k$ many rows and entries in $[n]$.
- We denote by $\mathrm{SSYT}_{k,n}$ the subset of $\mathrm{SSYT}_{\leq k,n}$ consisting of all semistandard Young tableaux of rectangular shapes with k rows.
- The dual canonical basis of $\Bbbk[\mathrm{Gr}_{k;n}]$ is parametrized by the set $\mathrm{SSYT}_{k,n}$ of semistandard Young tableaux and there is an explicit formula for the dual canonical basis elements [CDFL2020, DGL2024].

Partial order on tableaux

- Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, with $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$, $\mu_1 \geq \cdots \geq \mu_\ell \geq 0$, be partitions. Then $\lambda \leq \mu$ in the dominance order if
$$\sum_{j \leq i} \lambda_j \leq \sum_{j \leq i} \mu_j \text{ for all } 1 \leq i \leq \ell.$$
- For $T \in \mathrm{SSYT}_{\leq k,m}$ and $i \in [m]$, denote by $T[i]$ the sub-tableau obtained from T by restriction to the entries in $[i]$.
- For a tableau T, let $\mathrm{sh}(T)$ denote the shape of T.
- If $T, T' \in \mathrm{SSYT}_{\leq k,m}$ are of the same shape, then $T \leq T'$ in the dominance order if for every $i \in [m]$, $\mathrm{sh}(T[i]) \leq \mathrm{sh}(T'[i])$ in the dominance order on partitions.

Mutation of cluster variables in terms of tableaux

- Mutations of cluster variables in the cluster algebras $\mathbb{k}[U]$ and $\mathbb{k}[SL_n/U]$ can be described in terms of tableaux [L20].
- Starting from an initial seed of $\mathbb{k}[U]$ (or $\mathbb{k}[SL_n/U]$), each time we perform a mutation at a cluster variable $\mathrm{ch}(T_r)$, we obtain a new cluster variable $\mathrm{ch}(T'_r)$ determined by

$$\mathrm{ch}(T'_r)\mathrm{ch}(T_r) = \prod_{i \to r} \mathrm{ch}(T_i) + \prod_{r \to i} \mathrm{ch}(T_i), \tag{5}$$

  where $\mathrm{ch}(T_i)$ is the cluster variable at the vertex i. The two tableaux $\cup_{i\to r} T_i$, $\cup_{r\to i} T_i$ are always comparable under the dominance order and T'_r is determined by

$$T'_r = T_r^{-1} \max\{\cup_{i\to r} T_i, \cup_{r\to i} T_i\}. \tag{6}$$

- The same mutation rule works for $\mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$.

Initial cluster variables as tableaux

Proposition

Consider an arbitrary flag variety $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$ *and an arbitrary initial minor* $\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]}$ *with* $1 \leq i_j \leq d_j < i_{j+1} \leq d_{j+1} \leq n$ *and* $0 \leq j < k$ *(recall, that* $d_0 := 0, d_{k+1} := n$*). Set* $\ell = n - d_j - d_{j+1} + i_j + i_{j+1} - 1$*. Then*

$$\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} = \sum_{J \in \binom{[\ell,n]}{d_j - i_j + 1},\ J' = [\ell,n]\setminus J} (-1)^{\Sigma(i_j,d_j,J)} P_{[i_j-1]\cup J} P_{[i_{j+1}-1]\cup J'} \quad (7)$$

where $\Sigma(i_j, d_j, J) := \sum_{q=i_j}^{d_j} q + \sum_{j\in J} j$*.*

Initial cluster variables as tableaux

The tableau corresponding to the initial cluster variable $\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} \in \mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ is

$$\begin{array}{cc} 1 & 1 \\ \vdots & \vdots \\ \vdots & i_j - 1 \\ i_{j+1} - 1 & n - d_{j+1} - d_j + i_{j+1} + i_j - 1. \\ n - d_{j+1} + i_{j+1} & \vdots \\ \vdots & n - d_{j+1} + i_{j+1} - 1 \\ n & \end{array} \tag{8}$$

Initial cluster variables as tableaux

Example

In the case of $\mathcal{F}\ell_{2,4;6}$, there are two cluster variables (including frozen variables) which are with two-columns. They are

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = P_{1234}P_{56} - P_{1235}P_{46} + P_{1236}P_{45},$$

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = P_{1236}P_{15} - P_{1235}P_{16}.$$

Initial cluster variables as tableaux

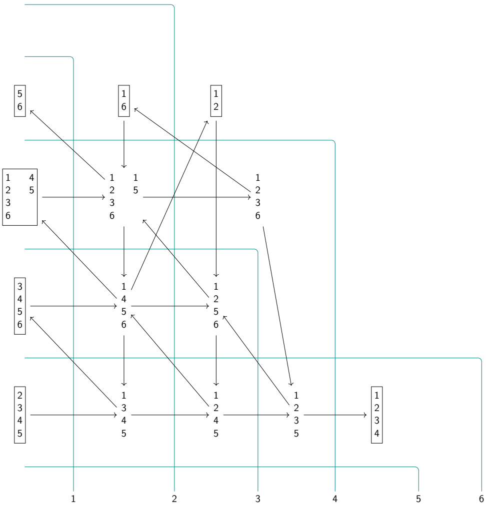

Figure: The initial seed for the partial flag variety $\mathcal{F}\ell_{2,4;6}$.

From Grassmannian cluster algebras to cluster algebras of partial flag varieties

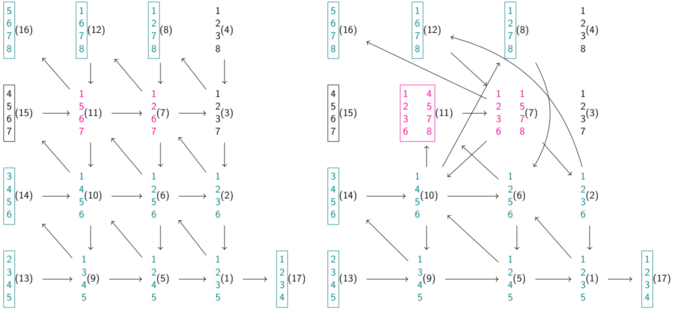

Figure: LHS: The initial seed for the Grassmannian $\mathrm{Gr}_{4;8}$. Mutation at (7), then (11) followed by freezing (11) yields the full subquiver depicted on the RHS on all vertices but (3),(4),(15) that coincides with the initial seed for $\mathcal{F}\ell_{2,4;6}$.

From Grassmannian cluster algebras to cluster algebras of partial flag varieties

- For general n, an initial seed of $\mathcal{MT}_n \cong \mathcal{F}\ell_{2,4;n}$ could also be obtained from a Grassmannian cluster algebra using two mutations similar to the mutations in the previous page.
- We also obtain a mutation sequence which gives an initial seed for $\mathcal{SH}_n \cong \mathcal{F}\ell_{2,n-2;n}$ starting from the initial seed of a Grassmannian cluster algebra.

Thank you!