Quantum affine algebras and KLR algebras

Joint with Elie Casbi

Motivation

Recently, Baumann-Kamnitzer-Knutson introduced a remarkable algebra morphism

$$\overline{D}: \mathbb{C}[N] \to \mathbb{C}(\alpha_1, \cdots, \alpha_n)$$

in their proof of a conjecture of Muthiah about MV basis of $\mathbb{C}[N]$.

Here N is a maximal unipotent subgroup of a semisimple Lie group of type ADE. $\alpha_1, \cdots, \alpha_n$ are simple roots of the root system of G.

$\mathbb{C}[N]$ has a monoidal categorification using representations of quantum affine alg introduced by Hernandez and Leclerc (2010). They introduced a category $\mathcal{C}^{\leq \xi}$ and conjectured that $\mathcal{C}^{\leq \xi}$ is a monoidal categorification of the cluster alg $K_0(\mathcal{C}^{\leq \xi})$.

This Conjecture was proved by Kang-Kashiwara-Kim-Oh 2018

Qin 2017

Kashiwara-Kim-Oh-Park 2021.

$\mathbb{C}[N]$ is a subalg of $K_0(\mathcal{C}^{\leq \xi})$.

With E. Casbi, we defined an alg morphism

$\widetilde{D} : K_0(\mathcal{C}^{\leq \xi}) \longrightarrow \mathbb{C}(\alpha_1, \dots, \alpha_n)$

and proved that $\widetilde{D}\big|_{\mathbb{C}[N]} = \overline{D}$.

Moreover, using $\widetilde{D}$ and $\overline{D}$, we can recover information of q-character of reps of $U_q(\hat{\mathfrak{g}})$ using ungraded character of modules of KLR algebras and vice versa.

Notation

$q \in \mathbb{C}^\times$ not a root of unity.

$\mathfrak{g}$ the Lie alg of G ③

$U_q(\hat{\mathfrak{g}})$ the quantum affine algebra associated to $\hat{\mathfrak{g}}$.

I set of vertices of the Dynkin diagram of $\mathfrak{g}$.

P the abelian group generated by $Y_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^\times$.

P^+ submonoid of P generated by $Y_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$.

Fix $a \in \mathbb{C}^\times$, denote $Y_{i,s} = Y_{i,aq^s}$, $i \in I$, $s \in \mathbb{Z}$.

$\mathcal{C}$ category of finite dimentional $U_q(\hat{\mathfrak{g}})$-modules.

For an object V in $\mathcal{C}$, its q-character (Frenkel-Reshetikin 98) is

$V_\gamma = \{ v \in V \mid \exists k \in \mathbb{N}, \forall i \in I, m \geq 0, (\phi^{\pm}_{i,\pm s} - \gamma^{\pm}_{i,\pm s})^k v = 0 \}$ $\Big|$ $\phi^{\pm}_{i,s} \in U_q(\hat{g})$

$\gamma = (\gamma^{\pm}_{i,\pm s})_{i \in I, s \in \mathbb{Z}_{\geq 0}}$ γ is called an l-weight.

$\gamma_i^{\pm}(u) = \sum_{m=0}^{\infty} \gamma_{i,\pm s} u^{\pm s} = q_i^{\deg Q_i - \deg R_i} \dfrac{Q_i(uq_i^{-1}) R_i(uq_i)}{Q_i(uq_i) R_i(uq_i^{-1})}$, $Q_i(u) = \prod_{a \in \mathbb{C}^\times} (1-ua)^{w_{i,a}}$, $R_i(u) = \prod_{a \in \mathbb{C}^\times} (1-ua)^{x_{i,a}}$ 4

identify γ with $m = \prod_{i \in I, a \in \mathbb{C}^\times} Y_{i,a}^{w_{i,a} - x_{i,a}}$.

$$\chi_q(V) = \sum_{m \in P} \dim(V_m)\, m$$

V_m is the l-weight space of V with l-weight $m \in P$.

Simple objects in $\mathcal{C}$ are of the form $L(m)$, $m \in P^+$. (Chari–Pressley).

$L(Y_{i,s})$ is called a fundamental module.

$L(Y_{i,s} Y_{i,s+2} \cdots Y_{i,s+2k})$ is called a Kirillov–Reshetikhin module. (in type A)

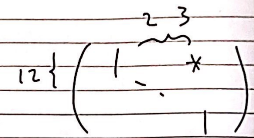
cluster structure of $\mathbb{C}[N]$.

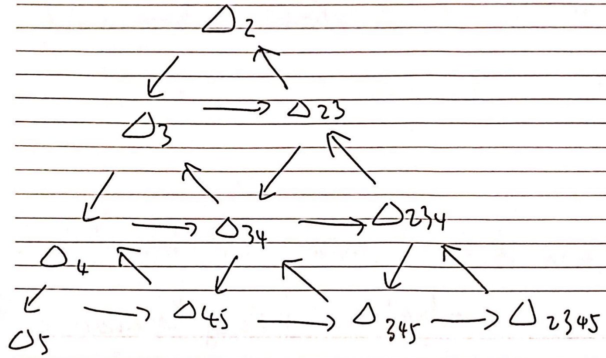
type A_4, $N \subset SL_5$.

⑤ initial seed

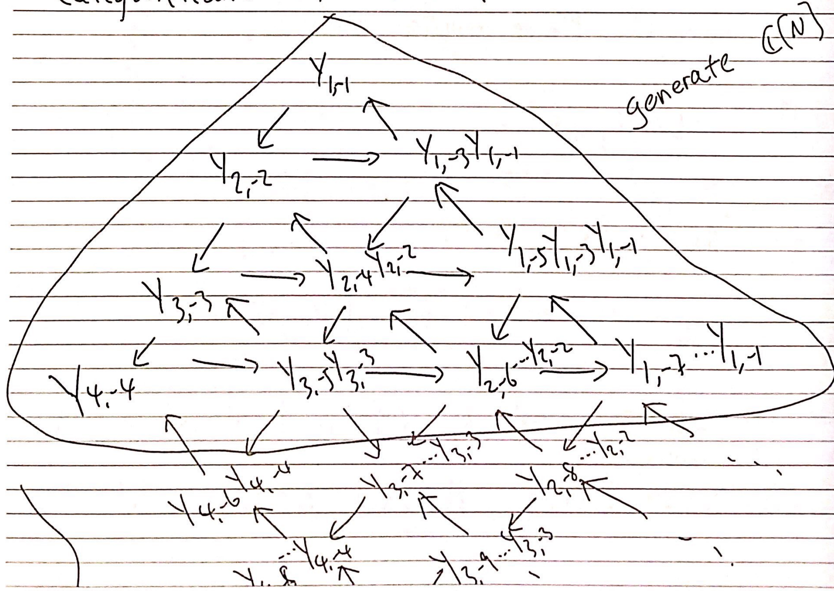
Δ_{23} means the minor

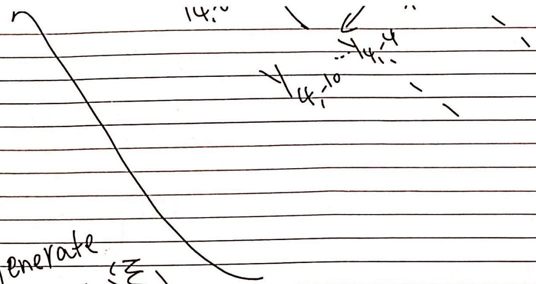
$\Delta_{12,23}$





Categorification of $\mathbb{C}[N]$ (Hernandez–Leclerc).





generate $K_0(\mathcal{C}^{\le \xi})$

Categorification of $\mathbb{C}[N]$ using modules of KLR algebras (Kang-Kashiwara-Kim-Oh 2018)

M = set of finite words over I.

$Q^+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$,

$j = (j_1, \dots, j_d) \in M$.

$wt(j) = \sum_{i \in I} \#\{k \mid j_k = i\}\alpha_i \in Q^+$.

KLR alg is a family $\{R(\beta) \mid \beta \in Q^+\}$ of associative $\mathbb{C}$-alg. For each $\beta \in Q^+$, $R(\beta)$ is

7 generated by 3 kinds of generators, $x_1, \cdots, x_n$, $\tau_1, \cdots, \tau_{n-1}$, $e(j)$. $j \in Seq(\beta) = \{ j \in M \mid wt(j) = \beta \}$

$R(\beta)$-mod = Category of finite dim $R(\beta)$-modules.

$$R\text{-mod} = \bigoplus_{\beta} R(\beta)\text{-mod}$$

$i = (i_1, \cdots, i_N)$ reduced word of w_0.

$$\gamma_j^* = D(s_{i_1} \cdots s_{i_j} w_{i_j}, s_{i_1} \cdots s_{i_{j-1}} w_{i_j}), \quad 1 \le j \le N.$$

dual root vector.

$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in R^+$, set of positive roots.

McNamara proved that $\exists$ simple modules S_{β_j} in R-mod, unique up to isomorphism, such that $[S_{\beta_j}] = \gamma_j^*$ for $j \in \{1, \cdots, N\}$. S_{β_j} are called cuspidal modules.

$x \in F$, $\max(x)$ = largest word in x.

A word $i \neq \emptyset$ is called Lyndon if $x <$ all its proper right factors.

Every word i has a unique factorization $i = i^{(1)} \cdots i^{(k)}$ such that $i^{(1)} \geq \cdots \geq i^{(k)}$. $i^{(j)}$ are Lyndon. highest weight of simple modules $\leftrightarrow$ good words (dominant). heighest weight,

Generalizing Leclerc's algorithm, Brundan-Kleshchev McNamara described a procedure producing a word of cuspidal modules $\leftrightarrow$ good Lyndon words.

An irreducible module $L \in Rep(R_\beta)$ is called cuspidal if its heighest weight is a good Lyndon word.

$j_\beta \in M$ for each $\beta \in R^+$, j_β is called a good Lyndon word.

(Kleshchev-Ram, McNamara, Brundan-Kleshchev-McNamara)

$\exists$ bijection

$$\{\text{isomorphism classes of simple objects in } R\text{-mod}\} \longleftrightarrow \cancel{\mathbb{N}^{R^+}} \text{ dominant words}.$$

$$L(j_{\beta_N}^{c_N} \cdots j_1^{c_1}) \longleftrightarrow \text{dominant words} = \{\text{products of good Lyndon words}\}$$

$$:= \mathrm{hd}(S_{\beta_N}^{\circ c_N} \circ \cdots \circ S_{\beta_1}^{\circ c_1}) \longleftrightarrow (c_1, \cdots, c_N) \in \mathbb{N}^{R^+}.$$

maximal simple quotient

$j_{\beta_N}^{c_N} \cdots j_1^{c_1}$ are called dominant words.

$$M \circ N = R(\beta+\gamma) \otimes_{R(\beta)\otimes R(\gamma)} M \boxtimes N$$

(Khovanov-Lauda 2009, Rouquier 2012)

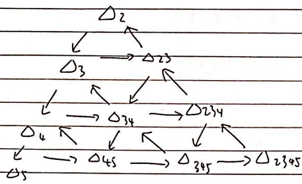
$\exists$ algebra isomorphism

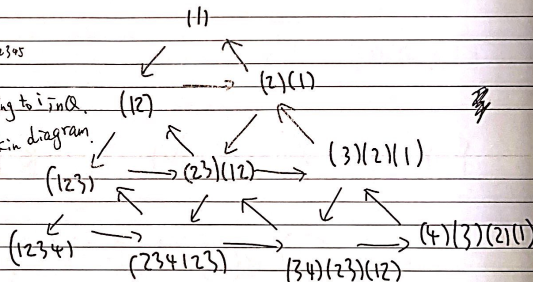
$$\mathbb{C} \otimes K_0(R\text{-mod}) \simeq \mathbb{C}[N].$$

(Rouquier 2012, Varagnolo-Vasserot 2011)

The above isomorphism induces a bijection

$\{$classes of simple objects in R-mod$\}$ $\longleftrightarrow$ dual canonical basis of $\mathbb{C}[N]$.





$s_i(Q)$ = reverse arrows connecting to i in Q.

Q orientation of Dynkin diagram.

$s_{i_1}\cdots s_{i_k}$ is adapted to Q.

if i_1 is source of Q,

i_2 is source of $s_{i_1}(Q)$

$\cdots$

i_k is source of $s_{i_{k-1}}\cdots s_{i_1}(Q)$.

123   1 — 2 — 3

~~[Nao's talk yesterday]~~

$\exists$ unique Coxeter element having reduced expression adapted to Q.

Generalized quantum affine Schur Weyl duality

(Chari-Pressley type A, Kang-Kashiwara-Kim 2018)

reduced expression $w = s_{i_1}\cdots s_{i_k}\cdots s_{i_m}$ ~~gives order of positive root~~.   choose total order on $\{1,2,3,4\}$.

$\tau = s_{i_1}\cdots s_{i_k}$. τ adapt to Q.   Leclerc's algorithm gives a convex order on ~~positive roots~~ good Lyndon words.

A_4. choose $\xi(i) = -i$, $i = 1,2,3,4$.

fundamental monomial   good Lyndon word   positive root   gives positive roots using weights.

~~determine~~   Leclerc's alg gives good Lyndon words. $\alpha_1, \alpha_1+\alpha_2, \alpha_2,$ $(12) \mapsto \alpha_1+\alpha_2$

$Y_{1,-1}$ — (1) — α_1 positive roots by taking weights. $\alpha_1+\alpha_2+\alpha_3$, $\alpha_2+\alpha_3$,

$\gamma_i = \sum_{j \in B(i)} \alpha_j$

$Y_{1,-3}$ — (2) — α_2 α_3, $\alpha_1+\alpha_2+\alpha_3+\alpha_4$,

$B(i) = \{ j \mid \exists$ path $j \to i \}$.

$Y_{1,-5}$ — (3) — α_3 $\gamma_3 = \alpha_1+\alpha_2+\alpha_3$ $\alpha_2+\alpha_3+\alpha_4$, $\alpha_3+\alpha_4$,

$\gamma_4 = \alpha_1+\alpha_2+\alpha_3+\alpha_4$ α_4

$Y_{i,s} \longleftrightarrow \tau^{(\xi_i - s)/2}(\gamma_i)$ positive root. $1 \to 2 \to 3 \to 4$. $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_1+\alpha_2$ Coxeter element $\tau = s_1 s_2 s_3 s_4$

$Y_{1,-7}$ — (4) — α_4

$Y_{2,-2}$ — (12) — $\alpha_1+\alpha_2$

$Y_{2,-4}$ — (23) — $\alpha_2+\alpha_3$

$Y_{2,-6}$ — (34) — $\alpha_3+\alpha_4$

$Y_{3,-3}$ — (123) — $\alpha_1+\alpha_2+\alpha_3$

$Y_{3,-5}$ — (234) — $\alpha_2+\alpha_3+\alpha_4$

$Y_{4,-4}$ — (1234) — $\alpha_1+\alpha_2+\alpha_3+\alpha_4$.

The map $\overline{D}$.

For $\beta \in R^+$, M in $R(\beta)$-mod.

$$M = \bigoplus_{j \in Seq(\beta)} e(j)\cdot M \quad \text{weight space decomposition},$$

$$\overline{D}(M) = \sum_{j=(j_1,\cdots,j_d)\in Seq(\beta)} \dim(e(j)\cdot M)\,\frac{1}{\alpha_{j_1}(\alpha_{j_1}+\alpha_{j_2})\cdots(\alpha_{j_1}+\cdots+\alpha_{j_d})}.$$

(Casbi-L) $\qquad I^{\leq \xi} = \{(i,p) \mid i \in I,\ p \in \xi(i)+2\mathbb{Z}_{\leq 0}\}$

$\tilde{c}_{ij}(s-p-1)$

(11)

$$\tilde{D}(Y_{i,p}) = \prod_{(j,s)\in I_{\le\xi}} \left(\varepsilon_{j,s}\, \tau_Q^{(\xi(j)-s)/2}(\gamma_j)\right)^{-\tilde{c}_{ij}(s-p+1)}$$

Q is an orientation of Dynkin diagram. τ_Q a certain coxeter element

$\left(\tilde{C}_{ij}(z)\right)$ is the inverse of the quantum Cartan matrix. $\tilde{C}_{ij}(z) = \sum_{m\geq 1} \tilde{C}_{ij}(m) z^m$.

$\varepsilon_{j,s}$ is some element in $\{1,-1\}$.

$$\begin{tikzcd}
\mathbb{C}[N] \arrow[r, "\sim"] \arrow[dr, "\overline{D}"'] & \mathbb{C}\otimes K_0(\mathcal{C}_Q) \arrow[r, "\tilde{\chi}_q"] \arrow[d, phantom, "\circlearrowright"] & \mathbb{C}\otimes \mathcal{Y}_Q \arrow[dl, "\tilde{D}_Q"] \\
& \mathbb{C}(\alpha_i \mid i\in I) &
\end{tikzcd}$$

$\tilde{\chi}_q$ is the truncated q-character map.

$\mathcal{Y}_Q = \mathbb{Z}[Y_{i,p}^{\pm 1} \mid (i,p)\in I_Q]$.

Corollary

For any simple $U_q(\hat{g})$ module $L(M)$,

we have

$$\sum_{m'} \dim(V_{m'}) \tilde{D}_Q(m') = \sum_{j=(j_1,\cdots,j_d)} \dim\left((\mathcal{F}_Q(L(M)))_j\right)\bar{D}_j$$

$\mathcal{F}_Q$ is the generalized quantum affine

Schur Weyl dual functor.

$V_{m'}$'s are l-weight subspaces.

$(\mathcal{F}_Q(L(M)))_j$ is a weight subspace of $\mathcal{F}_Q(L(M))$

$$\bar{D}_j = \frac{1}{\alpha_{j_1}(\alpha_{j_1}+\alpha_{j_2})\cdots(\alpha_{j_1}+\cdots+\alpha_{j_d})}.$$

Using this formula, we can obtain

information of q-character of $L(M)$ ⑬

and ungraded character $F_Q(L(M))$ from each other.

For example, by computing the q-character of $L(Y_{2,-2}Y_{2,-4}Y_{1,-7})$ and apply the map $\tilde{D}$, we have

$$\tilde{D}(L(Y_{2,-2}Y_{2,-4}Y_{1,-7})) = \alpha_2\alpha_4(\alpha_1+\alpha_2)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$

$$F_Q(L(Y_{2,-2}Y_{2,-4}Y_{1,-7})) = L(\underset{\substack{\uparrow \\ Y_{1,-7}}}{(4)}\underset{\substack{\uparrow \\ Y_{2,-4}}}{(23)}\underset{\substack{\uparrow \\ Y_{2,-2}}}{(12)})$$

write

$$ch(L(42312)) = c_1(42312) + c_2(42132) + \cdots$$

(25 terms)

shuffles of 42312

$$\bar{D}(ch(L(42312))) = \tilde{D}(L(Y_{2,-2}Y_{2,-4}Y_{1,-7}))$$

$c_1 \; \alpha_4(\alpha_2+\alpha_4)(\alpha_2+\alpha_3+\alpha_4)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$
$(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)$

$+ \cdots = $

$\Rightarrow \; ch(L(42312)) = (42312)+(42132)+(24312)$
$+(21432)+(24132).$

$w = s_{i_1}\cdots s_{i_k}$

reduced expression adapt to Q: i_1 is a source of Q, i_2 is a source of $s_{i_1}(Q)$, $\cdots$, i_k is a source of $s_{i_{k-1}}\cdots s_{i_1}(Q)$.

$s_i(Q)$ change arrows connecting ~~edges~~ i. ~~incident~~

$\exists$ unique Coxeter element ~~connect to~~ c adapt to Q.