Cluster structures on spinor helicity and momentum twistor varieties

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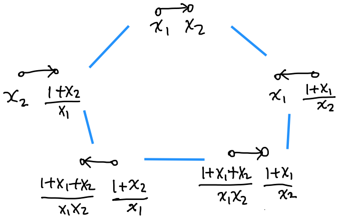
Cluster algebras

- Cluster algebras are a class of commutative algebras introduced by Fomin and Zelevinsky.
- Each cluster algebra is defined using some initial data called a seed and using a procedure called mutations.

Example (cluster algebra of type A_2)

initial seed $\quad x_1 \longrightarrow x_2 \qquad (x, Q),\ x = (x_1, x_2),\ Q = \circ\!\longrightarrow\!\circ$



$$A(x, Q) = \mathbb{C}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right].$$

mutate at vertex k, $\mu_k(Q)$ is obtained from Q by:

(1) for every $i \to k \to j$, add $i \longrightarrow j$.

(2) change  to ,

(3) erase every .

$(x_1, \cdots, x_n)$ is changed to $(x_1, \cdots, x_k', \cdots, x_n)$, $x_k' = \frac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{k \to j} x_j\right)$.

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned} \mathrm{Gr}(k, n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations}. \end{aligned}$$

- Scott (arXiv:math/0311148) showed that the coordinate ring $\mathbb{C}[\mathrm{Gr}(k, n)]$ has a cluster algebra structure.
- The algebra $\mathbb{C}[\mathrm{Gr}(k, n)]$ is called a Grassmannian cluster algebra.

Grassmannian cluster algebras and physics

Recently, Grassmannian cluster algebras are found to be an important tool in scattering amplitudes in physics, see for examples,

- Arkani-Hamed, Lam, Spradlin, J. High Energ. Phys. 2021, 65 (2021),
- Chicherin, Henn, Papathanasiou, Phys.Rev.Lett. 126 (2021) 9, 091603,
- Drummond, Foster, Gürdoğan, Kalousios, J. High Energ. Phys. 2020, 146 (2020),
- Golden, Paulos, Spradlin, Volovich, J. Phys. A: Math. Theor. 47 474005,
- Henke, Papathanasiou, J. High Energ. Phys. 2021, 7 (2021).

An initial seed for a Grassmannian cluster algebra

- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{C}[\mathrm{Gr}(k,n)]$ ($i_1 < \cdots < i_k$): for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, ..., kth rows and i_1th, ..., i_kth columns.
- The coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ has a cluster algebra structure with an initial seed given by Plücker coordinates, [arXiv:math/0311148, Scott].

An initial cluster for $\mathbb{C}[\mathrm{Gr}(5,10)]$

$$
\begin{tikzcd}
\boxed{P_{1,2,3,4,5}} & & & & \\
P_{1,2,3,4,6} \arrow[u] \arrow[dr] & P_{1,2,3,5,6} \arrow[l] \arrow[dr] & P_{1,2,4,5,6} \arrow[l] \arrow[dr] & P_{1,3,4,5,6} \arrow[l] \arrow[dr] & \boxed{P_{2,3,4,5,6}} \arrow[l] \\
P_{1,2,3,4,7} \arrow[u] \arrow[dr] & P_{1,2,3,6,7} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,5,6,7} \arrow[l] \arrow[u] \arrow[dr] & P_{1,4,5,6,7} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{3,4,5,6,7}} \arrow[l] \\
P_{1,2,3,4,8} \arrow[u] \arrow[dr] & P_{1,2,3,7,8} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,6,7,8} \arrow[l] \arrow[u] \arrow[dr] & P_{1,5,6,7,8} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{4,5,6,7,8}} \arrow[l] \\
P_{1,2,3,4,9} \arrow[u] \arrow[dr] & P_{1,2,3,8,9} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,7,8,9} \arrow[l] \arrow[u] \arrow[dr] & P_{1,6,7,8,9} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{5,6,7,8,9}} \arrow[l] \\
\boxed{P_{1,2,3,4,10}} \arrow[u] & \boxed{P_{1,2,3,9,10}} \arrow[u] & \boxed{P_{1,2,8,9,10}} \arrow[u] & \boxed{P_{1,7,8,9,10}} \arrow[u] & \boxed{P_{6,7,8,9,10}}
\end{tikzcd}
$$

An example of exchange relation

- The relation

$$P_{13456}P_{24567} = P_{14567}P_{23456} + P_{12456}P_{34567}$$

  is an example of exchange relation (corresponding to mutation at the vertex P_{13456} in the initial quiver). It is a Plücker relation.
- In general, exchange relations are more complicated than Plücker relations.

Monoid $\mathrm{SSYT}(k, [n])$ of semi-standard Young tableaux

- $\mathrm{SSYT}(k, [n])$ = the set consisting of the empty tableau and semi-standard Young tableaux of rectangular shape with k rows and with entries in $[n]$.
- For $A, B \in \mathrm{SSYT}(k, [n])$, $A \cup B$ is the semi-standard tableau with k rows and the elements in the ith row are the union of elements in the ith row of A and B, $i \in [k]$.

Example

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 7 \\ \hline 6 & 8 \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 2 & 9 \\ \hline 8 & 10 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 7 \\ \hline 2 & 2 & 7 & 9 \\ \hline 6 & 8 & 8 & 10 \\ \hline \end{array}.$$

Dual canonical basis of a Grassmannian cluster algebra

Dual canonical basis elements (in particular, cluster monomials) of $\mathbb{C}[\mathrm{Gr}(k, n)]$ correspond to semistandard Young tableaux in $\mathrm{SSYT}(k, [n])$ [Chang-Duan-Fraser-L. 2020].

Mutations of tableaux

- The exchange relations become simpler when written in terms of tableaux.
- Mutation rule:

$$T'_k = T_k^{-1} \max\{\cup_{i \to k} T_i, \cup_{k \to i} T_i\}.$$

- The following is an exchange relation in $\mathbb{C}[\mathrm{Gr}(3,8)]$:

$$\begin{aligned}
&\mathrm{ch}\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline 4&7&8\\\hline\end{array}\right) = \mathrm{ch}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|}\hline 3&4\\\hline 5&6\\\hline 7&8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)\\
&\qquad + \mathrm{ch}\left(\begin{array}{|c|}\hline 3\\\hline 4\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|}\hline 2&4\\\hline 5&6\\\hline 7&8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right).
\end{aligned}$$

Spinor helicity varieties

- In the study of scattering amplitudes in quantum field theories for massless scattering the kinematic space modeling particle interactions is represented by two matrices λ and $\tilde{\lambda}$ of size $2 \times n$ with property that $\lambda\tilde{\lambda}^T$ is zero.
- The 2×2-minors of λ are denoted by $\langle ij \rangle$ and the 2×2-minors of $\tilde{\lambda}$ by $[ij]$ for $1 \leq i < j \leq n$.
- Each class of minors satisfy quadratic Plücker relations: for $1 \leq i < j < k < l \leq n$

$$[ij][kl] - [ik][jl] + [il][jk] = 0 \quad \text{and} \quad \langle ij \rangle \langle kl \rangle - \langle ik \rangle \langle jl \rangle + \langle il \rangle \langle jk \rangle = 0$$

and momentum conservation: for all $1 \leq i, j \leq n$,

$$\sum_{s=1}^{n} \langle is \rangle [sj] = 0.$$

Spinor helicity varieties

- Denote by $\mathcal{SH}_n$ the space determined by the above equations. It is called a spinor helicity variety.
- Mazzouz, Pfister, and Sturmfels 2024 showed that $\mathcal{SH}_n$ is isomorphic to the partial flag variety $\mathcal{F}\ell_{2,n-2;n}$ with isomorphism given by

$$\langle ij \rangle \mapsto P_{ij}, \quad \text{and} \quad [ij] \mapsto (-1)^{i+j-1} P_{[n]-\{i,j\}}. \tag{1}$$

Momentum twistor varieties

- The scattering process may also be described in terms of momentum twistors $Z_1, \ldots, Z_n \in \mathbb{C}P^3$ representing the particles in Minkowski space.
- Interpreting the coordinates of Z_i as column vectors we obtain a $4 \times n$ matrix.
- Denote by $\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l)$ a Plücker coordinate P_{ijkl} on the Grassmannian $\mathrm{Gr}_{4,n}$.
- If the system does not have dual conformal symmetry, i.e. it is *non-dual conformal invariant* (or NDCI for short) we add an infinity twistor in form of a line ℓ_∞.

Momentum twistor varieties

- The line ℓ_∞ may be understood as the line spanned by two additional points $Z_{n+1}, Z_{n+2} \in \mathbb{C}P^3$.
- We consider only those minors $\langle ijkl \rangle$ that satisfy

$$n+1, n+2 \in \{i,j,k,l\} \quad \text{or neither} \quad n+1, n+2 \notin \{i,j,k,l\}.$$

- We identify

$$\langle i,j,n+1,n+2 \rangle \mapsto P_{ij}, \quad \text{and} \quad \langle i,j,k,l \rangle \mapsto P_{ijkl}, \tag{2}$$

  where $i,j,k,l \in [n]$.
- We define the **momentum twistor variety** denoted by $\mathcal{MT}_n$ as the subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{4}-1}$ as the vanishing set of the Plücker relations. Hence $\mathcal{MT}_n$ coincides with the partial flag variety $\mathcal{F}\ell_{2,4;n}$.

- Physicists know the change of parametrization between spinor helicity and momentum twistor variables. Part of the map is given by

$$\langle i-1, i, j, j+1\rangle \mapsto [ij]\langle ij\rangle\langle i-1, i\rangle\langle j, j+1\rangle.$$

We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{SH}_n \arrow[d] & \mathcal{F}\ell_{2,n-2;n} \arrow[l, "\sim"'] \arrow[d] \\ \mathcal{MT}_n & \mathcal{F}\ell_{2,4;n} \arrow[l, "\sim"'] \end{tikzcd}$$

Spinor helicity varieties and momentum twistor varieties

- We explore the cluster structures of the spinor helicity variety $\mathcal{SH}_n \cong \mathcal{F}\ell_{2,n-2;n}$ and the momentum twistor variety $\mathcal{MT}_n \cong \mathcal{F}\ell_{2,4;n}$ and show how they can be obtained from the cluster structure on Grassmannians.
- The cluster structure on momentum twistor variety can be obtained from the initial cluster of the Grassmannain $\mathrm{Gr}_{4,n+2}$ by two mutations followed by freezing one cluster variable.
- The cluster structure on spinor helicity variety $\mathcal{F}\ell_{2,n-2;n}$ can be obtained from the initial cluster of the Grassmannain $\mathrm{Gr}_{n-2,2n-4}$ by $\frac{n(n-4)(n-5)}{2}$ mutations followed by freezing $n-5$ cluster variables.

Partial flag varieties

- Denote by SL_n the special linear group of $n \times n$ matrices with entries in a field $\mathbb{k}$.
- We fix the Borel subgroup of upper triangular matrices $B \subset SL_n$ and the subgroup unipotent matrices $U \subset B$.
- For a positive integer i write $[i] := \{1, \ldots, i\}$ and for $j > i$ write $[i, j] := \{i, i+1, \ldots, j\}$. Let n be a positive integer and $1 \leq d_1 < d_2 < \cdots < d_k < n$. To this data we associate the variety of partial flags of subspaces in $\mathbb{k}^n$, $\mathbb{k}$ a field of characteristic zero,

$$\mathcal{F}\ell_{d_1,\ldots,d_k;n} := \{0 \in V_1 \subset V_2 \subset \cdots \subset V_k \subset \mathbb{k}^n : \dim V_i = d_i\} . \quad (3)$$

- If $k = 1$ the associated partial flag variety is a Grassmannian $\mathrm{Gr}_{d,n}$.

Partial flag varieties

- Points in $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$ can be represented by matrices in $\mathbb{C}^{d_k \times n}$ where the first d_i rows span the i^{th} vector space in the flag.
- A Plücker coordinate $P_{i_1,\ldots,i_r} \in \mathbb{C}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ ($i_1 < \cdots < i_r$, $r \in \{d_1, \ldots, d_k\}$): for a $d_k \times n$ matrix $x = (x_{ij})_{d_k \times n}$, $P_{i_1,\ldots,i_r}(x)$ is the minor of x with 1st, …, rth rows and i_1th, …, i_rth columns.
- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{k}[\mathrm{Gr}_{k,n}]$ ($i_1 < \cdots < i_k$): for a $k \times n$ matrix $x = (x_{ij})_{k \times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, …, kth rows and i_1th, …, i_kth columns.

Partial flag varieties

- We define a morphism from the Grassmannian to the flag variety via its pullback on the homogeneous coordinate rings:

$$\varphi^* : \Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}] \hookrightarrow \Bbbk[\mathrm{Gr}_{d_k;n+d_k-d_1}], \tag{4}$$

determined by the images of the Plücker coordinates:

$$P_{I_1} \mapsto P_{I_1\cup[n+1,n+d_k-d_1]}, P_{I_2} \mapsto P_{I_2\cup[n+d_2-d_1+1,n+d_k-d_1]}, \ldots, P_{I_k} \mapsto P_{I_k}.$$

Here I_j is a subset of $[n]$ of cardinality d_j for all $1 \leq j \leq k$.

Theorem

The algebra embedding (4) maps cluster variables of $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ to cluster variables of $\Bbbk[\mathrm{Gr}_{d_k;n+d_k-d_1}]$.

An initial seed for a Grassmannian cluster algebra

- The coordinate ring $\mathrm{Gr}_{k,n}$ has a cluster algebra structure with an initial seed given by Plücker coordinates, [Scott 2006].

An initial cluster for $\mathrm{Gr}_{4,8}$

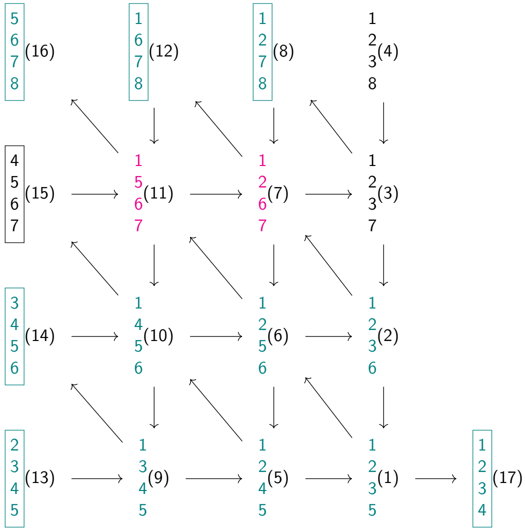


Figure: An initial seed for the Grassmannian $\mathrm{Gr}_{4;8}$.

Cluster algebra structure on partial flag varieties

- The coordinate ring $\Bbbk[U]$ has a cluster algebra structure due to Berenstein, Fomin and Zelevisnky in 1996, 2005.
- Their work was generalized to the case of $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ by Geiss, Leclerc and Schröer 2008 and is the basis for the cluster structure on partial flag varieties

Pseudoline arrangements

- We describe the pseudoline arrangement $\mathcal{P}_{d_1,\ldots,d_k;n}$ for $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$.
- Let $\sigma \in S_n$ be a permutation whose one-line presentation is $\sigma = [d_1, d_1{-}1, \ldots, 1, d_2, d_2{-}1, \ldots, d_1{+}1, \ldots, d_k, d_k{-}1, \ldots, d_{k-1}{+}1]$.
- Draw nodes labelled by $1, \ldots, n$ (left to right) on a horizontal line and n nodes labelled $\sigma(1), \ldots, \sigma(n)$ on a vertical line as in a positive orthant.
- Start drawing pseudolines ℓ_i connecting i with $i = \sigma(j)$ (for some j) in the following way: the lines $\ell_{d_i+1}, \ell_{d_i+2} \ldots, \ell_{d_{i+1}}$ do not cross; they start vertically and then turn once they reach the height of the vertical nodes $d_i + 1, \ldots, d_{i+1}$.

From pseudoline arrangements to quivers

- mutable vertices of $Q_{d_1,\ldots,d_k;n}$ correspond to bounded faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- there are two types of frozen vertices: $n-1$ of them correspond to the unbounded faces on the left end of $\mathcal{P}_{d_1,\ldots,d_k;n}$; additionally there are k frozen vertices denoted by $v_{d_1},\ldots,v_{d_k}$.

From pseudoline arrangements to quivers

There are 4 types of arrows.

- from left to right perpendicular to a vertical straight lines segment connecting adjacent faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- from top to bottom perpendicular to a horizontal straight line segment connecting adjacent faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$;
- diagonally from bottom right to top left through a crossing of straight line segments connecting faces of $\mathcal{P}_{d_1,\ldots,d_k;n}$ that share a vertex;
- there are arrows to and from the extra frozen vertices $v_{d_1}, \ldots, v_{d_k}$: there is an arrow from the face bounded by ℓ_{d_i-1}, ℓ_{d_i} vertically and $\ell_{d_i+1}, \ell_{d_i+2}$ horizontally to the vertex v_{d_i}, and an arrow from v_{d_i} to the face bounded by ℓ_{d_i} on the left ℓ_{d_i+1} on the top and right (this is where ℓ_{d_i+1} bends) and by ℓ_{d_i+2} on the bottom.

Pseudoline arrangements

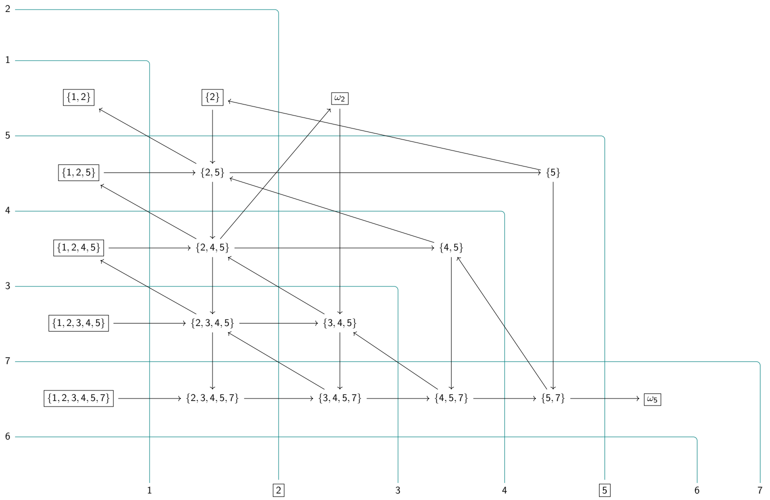


Figure: The pseudoline arrangement $\mathcal{P}_{2,5;7}$ and its quiver $Q_{2,5;7}$

Minors at the vertices of the quiver

- Every face of the pseudoline arrangement $\mathcal{P}_{d_1,\ldots,d_k;n}$ can be associated with a minor in $\Bbbk[U]$, [BFZ96].
- The minors are of form form $D_{I,J}$ with $I, J \subset [n]$ of the same size. The column index set J is always $\{n - |I| - 1, \ldots, n\}$. The index set I is given by the indices of lines passing north east of the face.
- Given a face F of $\mathcal{P}_{d_1,\ldots,d_k;n}$ we set

$$D_{I_F} := D_{I_F,\{n-|I_F|+1,\ldots,n\}} \quad \text{where} \quad I_F := \{i : \ell_i \text{ passes north-east of } F\}.$$

Dual canonical basis elements as tableaux

- For $k, m \in \mathbb{Z}_{\geq 1}$, denote by $\mathrm{SSYT}_{\leq k,m}$ the set of all semi-standard Young tableaux with less or equal to k rows and with entries in $[m]$.
- The dual canonical basis of $\Bbbk[U]$ is parametrized by the set $\mathrm{SSYT}_{\leq n-1,n,\sim}$ of equivalence classes of semistandard Young tableaux and there is an explicit formula for the dual canonical basis elements [L. 2024].
- The dual canonical basis of $\Bbbk[SL_k/U]$ is parametrized by the set $\mathrm{SSYT}_{\leq n-1,n}$ [L. 2024]. The algebra $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ is a subalgebra of $\Bbbk[SL_k/U]$. Its dual canonical basis is parametrized by $\mathsf{SSYT}_{d_1,\ldots,d_k;n}$ which is the set of semistandard tableaux which may have columns with $d_1, \ldots, d_k$ many rows and entries in $[n]$.
- We denote by $\mathrm{SSYT}_{k,n}$ the subset of $\mathrm{SSYT}_{\leq k,n}$ consisting of all semistandard Young tableaux of rectangular shapes with k rows.
- The dual canonical basis of $\Bbbk[\mathrm{Gr}_{k;n}]$ is parametrized by the set $\mathrm{SSYT}_{k,n}$ of semistandard Young tableaux and there is an explicit formula for the dual canonical basis elements [CDFL2020, DGL2024].

Partial order on tableaux

- Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell)$, with $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$, $\mu_1 \geq \cdots \geq \mu_\ell \geq 0$, be partitions. Then $\lambda \leq \mu$ in the dominance order if
$$\sum_{j \leq i} \lambda_j \leq \sum_{j \leq i} \mu_j \text{ for all } 1 \leq i \leq \ell.$$
- For $T \in \mathrm{SSYT}_{\leq k,m}$ and $i \in [m]$, denote by $T[i]$ the sub-tableau obtained from T by restriction to the entries in $[i]$.
- For a tableau T, let $\mathrm{sh}(T)$ denote the shape of T.
- If $T, T' \in \mathrm{SSYT}_{\leq k,m}$ are of the same shape, then $T \leq T'$ in the dominance order if for every $i \in [m]$, $\mathrm{sh}(T[i]) \leq \mathrm{sh}(T'[i])$ in the dominance order on partitions.

Mutation of cluster variables in terms of tableaux

- Mutations of cluster variables in the cluster algebras $\mathbb{k}[U]$ and $\mathbb{k}[SL_n/U]$ can be described in terms of tableaux [L20].
- Starting from an initial seed of $\mathbb{k}[U]$ (or $\mathbb{k}[SL_n/U]$), each time we perform a mutation at a cluster variable $\mathrm{ch}(T_r)$, we obtain a new cluster variable $\mathrm{ch}(T'_r)$ determined by

$$\mathrm{ch}(T'_r)\mathrm{ch}(T_r) = \prod_{i \to r} \mathrm{ch}(T_i) + \prod_{r \to i} \mathrm{ch}(T_i), \tag{5}$$

  where $\mathrm{ch}(T_i)$ is the cluster variable at the vertex i. The two tableaux $\cup_{i\to r} T_i$, $\cup_{r\to i} T_i$ are always comparable under the dominance order and T'_r is determined by

$$T'_r = T_r^{-1} \max\{\cup_{i\to r} T_i, \cup_{r\to i} T_i\}. \tag{6}$$

- The same mutation rule works for $\mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$.

Initial cluster variables as tableaux

Proposition

Consider an arbitrary flag variety $\mathcal{F}\ell_{d_1,\ldots,d_k;n}$ *and an arbitrary initial minor* $\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]}$ *with* $1 \le i_j \le d_j < i_{j+1} \le d_{j+1} \le n$ *and* $0 \le j < k$ *(recall, that* $d_0 := 0, d_{k+1} := n$*). Set* $\ell = n - d_j - d_{j+1} + i_j + i_{j+1} - 1$*. Then*

$$\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} = \sum_{J \in \binom{[\ell,n]}{d_j - i_j + 1},\ J' = [\ell,n]\setminus J} (-1)^{\Sigma(i_j,d_j,J)} P_{[i_j-1]\cup J} P_{[i_{j+1}-1]\cup J'} \quad (7)$$

where $\Sigma(i_j, d_j, J) := \sum_{q=i_j}^{d_j} q + \sum_{j\in J} j$*.*

Initial cluster variables as tableaux

The tableau corresponding to the initial cluster variable $\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} \in \mathbb{k}[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ is

$$\begin{array}{cc} 1 & 1 \\ \vdots & \vdots \\ \vdots & i_j - 1 \\ i_{j+1} - 1 & n - d_{j+1} - d_j + i_{j+1} + i_j - 1 \\ n - d_{j+1} + i_{j+1} & \vdots \\ \vdots & n - d_{j+1} + i_{j+1} - 1 \\ n & \end{array}. \tag{8}$$

Initial cluster variables as tableaux

Example

In the case of $\mathcal{F}\ell_{2,4;6}$, there are two cluster variables (including frozen variables) which are with two-columns. They are

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = P_{1234}P_{56} - P_{1235}P_{46} + P_{1236}P_{45},$$

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 5 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = P_{1236}P_{15} - P_{1235}P_{16}.$$

Initial cluster variables as tableaux

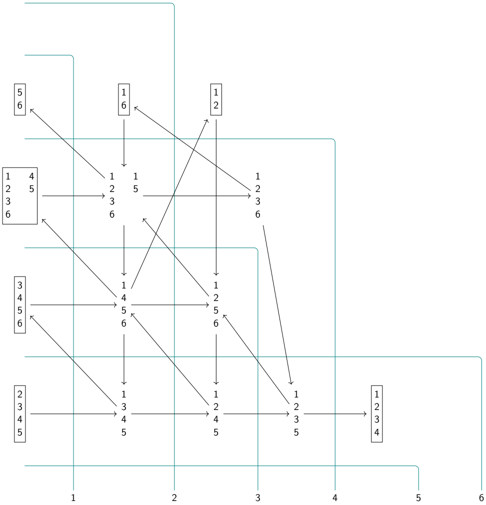


Figure: The initial seed for the partial flag variety $\mathcal{F}\ell_{2,4;6}$.

From Grassmannian cluster algebras to cluster algebras of partial flag varieties

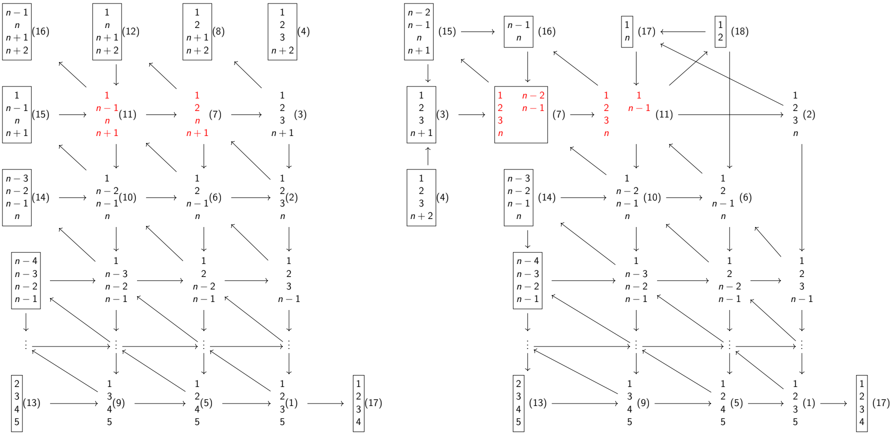


Figure: LHS: The initial seed for Grassmannian $\mathrm{Gr}_{4;n+2}$. RHS: a seed obtained by mutating (11), (7), (11) of the seed on LHS and freeze (7). The full subquiver on RHS on all vertices but (3), (4), (15) coincides with the initial seed for $\mathcal{F}\ell_{2,4;n}$.

From Grassmannian cluster algebras to cluster algebras of partial flag varieties

For more general flag varieties, we also a mutation sequence which gives an initial seed for $\Bbbk[\mathcal{F}\ell_{d_1,\ldots,d_k;n}]$ starting from the initial seed of the Grassmannian cluster algebra $\Bbbk[\mathrm{Gr}_{d_k;n+d_k-d_1}]$.

Thank you!