Grassmannian cluster algebras and machine learning

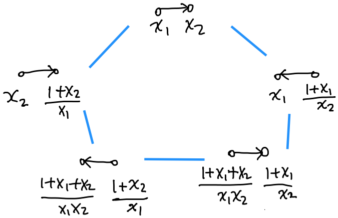
Jian-Rong Li
University of Vienna
joint work with Wen Chang, Bing Duan, Chris Fraser
and with Man-Wai Cheung, Pierre-Philippe Dechant, Yang-Hui He, Elli Heyes, Edward Hirst

Cluster algebras

- Cluster algebras are a class of commutative algebras introduced by Fomin and Zelevinsky.
- Each cluster algebra is defined using some initial data called a seed and using a procedure called mutations.

Example (cluster algebra of type A_2)

initial seed $\underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ}$ (x, Q), $x = (x_1, x_2)$, $Q = \circ \longrightarrow \circ$



$$A(x, Q) = \mathbb{C}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right].$$

mutate at vertex k, $\mu_k(Q)$ is obtained from Q by:

(1) for every $i \to k \to j$, add $i \longrightarrow j$.

(2) change  to ,

(3) erase every .

$(x_1, \cdots, x_n)$ is changed to $(x_1, \cdots, x_k', \cdots, x_n)$, $x_k' = \frac{1}{x_k}\left(\prod_{j \to k} x_j + \prod_{k \to j} x_j\right)$.

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned} \mathrm{Gr}(k,n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations}. \end{aligned}$$

- Scott (arXiv:math/0311148) showed that the coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ has a cluster algebra structure.
- The algebra $\mathbb{C}[\mathrm{Gr}(k,n)]$ is called a Grassmannian cluster algebra.

Grassmannian cluster algebras and physics

Recently, Grassmannian cluster algebras are found to be an important tool in scattering amplitudes in physics, see for examples,

- Arkani-Hamed, Lam, Spradlin, J. High Energ. Phys. 2021, 65 (2021),
- Chicherin, Henn, Papathanasiou, Phys.Rev.Lett. 126 (2021) 9, 091603,
- Drummond, Foster, Gürdoğan, Kalousios, J. High Energ. Phys. 2020, 146 (2020),
- Golden, Paulos, Spradlin, Volovich, J. Phys. A: Math. Theor. 47 474005,
- Henke, Papathanasiou, J. High Energ. Phys. 2021, 7 (2021).

Topics in this talk

I will talk about the following

- connections of $\mathbb{C}[\mathrm{Gr}(k, n)]$, representations of quantum affine algebras and representations of p-adic groups.
- classification of cluster variables using tools from machine learning.

An initial seed for a Grassmannian cluster algebra

- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{C}[\mathrm{Gr}(k,n)]$ $(i_1 < \cdots < i_k)$: for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, $\ldots$, kth rows and i_1th, $\ldots$, i_kth columns.
- The coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ has a cluster algebra structure with an initial seed given by Plücker coordinates, [arXiv:math/0311148, Scott].

An initial cluster for $\mathbb{C}[\mathrm{Gr}(5,10)]$

$$
\begin{tikzcd}
\boxed{P_{1,2,3,4,5}} & & & & \\
P_{1,2,3,4,6} \arrow[u] \arrow[dr] & P_{1,2,3,5,6} \arrow[l] \arrow[dr] & P_{1,2,4,5,6} \arrow[l] \arrow[dr] & P_{1,3,4,5,6} \arrow[l] \arrow[dr] & \boxed{P_{2,3,4,5,6}} \arrow[l] \\
P_{1,2,3,4,7} \arrow[u] \arrow[dr] & P_{1,2,3,6,7} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,5,6,7} \arrow[l] \arrow[u] \arrow[dr] & P_{1,4,5,6,7} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{3,4,5,6,7}} \arrow[l] \\
P_{1,2,3,4,8} \arrow[u] \arrow[dr] & P_{1,2,3,7,8} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,6,7,8} \arrow[l] \arrow[u] \arrow[dr] & P_{1,5,6,7,8} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{4,5,6,7,8}} \arrow[l] \\
P_{1,2,3,4,9} \arrow[u] \arrow[dr] & P_{1,2,3,8,9} \arrow[l] \arrow[u] \arrow[dr] & P_{1,2,7,8,9} \arrow[l] \arrow[u] \arrow[dr] & P_{1,6,7,8,9} \arrow[l] \arrow[u] \arrow[dr] & \boxed{P_{5,6,7,8,9}} \arrow[l] \\
\boxed{P_{1,2,3,4,10}} \arrow[u] & \boxed{P_{1,2,3,9,10}} \arrow[u] & \boxed{P_{1,2,8,9,10}} \arrow[u] & \boxed{P_{1,7,8,9,10}} \arrow[u] & \boxed{P_{6,7,8,9,10}}
\end{tikzcd}
$$

An example of exchange relation

- The relation

$$P_{13456}P_{24567} = P_{14567}P_{23456} + P_{12456}P_{34567}$$

  is an example of exchange relation (corresponding to mutation at the vertex P_{13456} in the initial quiver). It is a Plücker relation.
- In general, exchange relations are more complicated than Plücker relations.

Representations of quantum affine algebras

- $\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$, I = the set of vertices of the Dynkin diagram of $\mathfrak{g}$.
- $U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra associated to $\mathfrak{g}$, $q \in \mathbb{C}^\times$ is not a root of unity. It is generated by $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z} \setminus \{0\}$), $c^{\pm 1}$, subject to certain relations.
- (Chari-Pressley 1994) Every finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-module is a highest l-weight (loop-weight) module.
- The highest l-weight of a finite dimensional simple module corresponds to a unique I-tuple $(P_i(u))_{i \in I}$ of polynomials $P_i(u) \in \mathbb{C}[u]$, $i \in I$, the constant term of $P_i(u)$ is 1. These polynomials are called Drinfeld polynomials.

Finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- $\mathcal{P}$ = the free abelian group generated by $Y_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^\times$.
- $\mathcal{P}^+$ = the submonoid of $\mathcal{P}$ generated by $Y_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$.
- Elements in $\mathcal{P}^+$ are called dominant monomials.
- Every I-tuple of Drinfeld polynomials can be identified with a monomial in $\mathcal{P}^+$. For example, $((1-au)(1-bu), 1, 1-cu)$ is identified with $Y_{1,a}Y_{1,b}Y_{3,c}$.
- Finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-module with highest l-weight $M \in \mathcal{P}^+$ is denoted by $L(M)$.

q-characters of finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- Frenkel and Reshetikhin introduced the theory of q-characters of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules.
- (Frenkel-Reshetikhin 1998) The q-character of V is defined by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

  where V_m is the l-weight space with l-weight m.
- The q-character map $\chi_q : K_0(\mathcal{C}) \to \mathbb{Z}\mathcal{P}$ is an injective homomorphism, where $K_0(\mathcal{C})$ is the Grothendieck ring of the category $\mathcal{C}$ of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

Examples of q-characters

In the case of $\mathfrak{g} = \mathfrak{sl}_3$,

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1} Y_{2,aq} + Y_{2,aq^3}^{-1} Y_{3,aq^2} + Y_{3,aq^4}^{-1},$$

$$\begin{aligned}\chi_q(L(Y_{2,a})) = {} & Y_{2,a} + Y_{1,aq} Y_{2,aq^2}^{-1} Y_{3,aq} + Y_{1,aq^3}^{-1} Y_{3,aq} \\ & + Y_{1,aq} Y_{3,aq^3}^{-1} + Y_{1,aq^3}^{-1} Y_{2,aq^2} Y_{3,aq^3}^{-1} + Y_{2,aq^4}^{-1}.\end{aligned}$$

Hernandez and Leclerc's category $\mathcal{C}_\ell$

- Hernandez and Leclerc in 2010 introduced a subcategory $\mathcal{C}_\ell$ of $\mathcal{C}$ and they proved that the Grothendieck ring $K_0(\mathcal{C}_\ell)$ of $\mathcal{C}_\ell$ has a cluster algebra structure.
- From now on, we take $\mathfrak{g} = \mathfrak{sl}_k$, $I = [k-1] = \{1, \ldots, k-1\}$.
- We fix $a \in \mathbb{C}^\times$ and denote $Y_{i,s} = Y_{i,aq^s}$, $i \in I$, $s \in \mathbb{Z}$.
- $\mathcal{P}_\ell^+ =$ the submonoid of $\mathcal{P}^+$ generated by $Y_{i,i-2r-2}$, $i \in I$, $r \in [0, \ell]$.
- Simple modules in $\mathcal{C}_\ell$ are of the form $L(m)$, $m \in \mathcal{P}_\ell^+$.

Cluster algebra structure on the Grothendieck ring of $\mathcal{C}_\ell$

- For $i \in I$, $r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{Z}$, denote $X_{i,r}^{(s)} = Y_{i,s} Y_{i,s+2} \cdots Y_{i,s+2r-2}$. $L(X_{i,r}^{(s)})$ is called a Kirillov-Reshetikhin modules. $L(Y_{i,s})$ is called a fundamental module.

Theorem (Hernandez-Leclerc 2010)

The ring $K_0(\mathcal{C}_\ell)$ has a cluster algebra structure. The cluster variables in the initial seed of the cluster algebra are certain Kirillov-Reshetikhin modules.

The initial cluster for $K_0(\mathcal{C}_4)$

This is the initial cluster in the case of $U_q(\widehat{\mathfrak{sl}_5})$, $\ell = 4$.

$$
\begin{tikzcd}
\boxed{\mathbb{C}} & & & & \\
X_{1,1}^{(-1)} \arrow[u] \arrow[dr] & X_{2,1}^{(0)} \arrow[l] \arrow[dr] & X_{3,1}^{(1)} \arrow[l] \arrow[dr] & X_{4,1}^{(2)} \arrow[l] \arrow[dr] & \boxed{\mathbb{C}} \arrow[l] \\
X_{1,2}^{(-3)} \arrow[u] \arrow[dr] & X_{2,2}^{(-2)} \arrow[l] \arrow[u] \arrow[dr] & X_{3,2}^{(-1)} \arrow[l] \arrow[u] \arrow[dr] & X_{4,2}^{(0)} \arrow[l] \arrow[u] \arrow[dr] & \boxed{\mathbb{C}} \arrow[l] \\
X_{1,3}^{(-5)} \arrow[u] \arrow[dr] & X_{2,3}^{(-4)} \arrow[l] \arrow[u] \arrow[dr] & X_{3,3}^{(-3)} \arrow[l] \arrow[u] \arrow[dr] & X_{4,3}^{(-2)} \arrow[l] \arrow[u] \arrow[dr] & \boxed{\mathbb{C}} \arrow[l] \\
X_{1,4}^{(-7)} \arrow[u] \arrow[dr] & X_{2,4}^{(-6)} \arrow[l] \arrow[u] \arrow[dr] & X_{3,4}^{(-5)} \arrow[l] \arrow[u] \arrow[dr] & X_{4,4}^{(-4)} \arrow[l] \arrow[u] \arrow[dr] & \boxed{\mathbb{C}} \arrow[l] \\
\boxed{X_{1,5}^{(-9)}} \arrow[u] & \boxed{X_{2,5}^{(-8)}} \arrow[u] & \boxed{X_{3,5}^{(-7)}} \arrow[u] & \boxed{X_{4,5}^{(-6)}} \arrow[u] & \boxed{\mathbb{C}}
\end{tikzcd}
$$

An example of exchange relations

- The following is an example of exchange relations in the cluster algebra $K_0(\mathcal{C}_4)$:

$$\chi_q(L(Y_{4,2}))\chi_q(L(Y_{4,0})) = \chi_q(L(Y_{4,2}Y_{4,0})) + \chi_q(L(Y_{3,1})).$$

  This is a relation in the T-system of type A. T-system relations are certain relations satisfied by the q-characters of Kirillov-Reshetikhin modules.

- In general, exchange relations are more complicated than relations in T-systems.

Isomorphism of $\mathbb{C}[\mathrm{Gr}(k,n,\sim)]$ and $K_0(\mathcal{C}_\ell)$

- Define $\mathbb{C}[\mathrm{Gr}(k,n,\sim)] = \mathbb{C}[\mathrm{Gr}(k,n)]/\langle P_{i,i+1,\ldots,i+k-1} - 1, i \in [n-k+1]\rangle$.
- Denote $P^{(a,b,c)} = P_{j_1,\ldots,j_k}$, $j_1 = b$, $j_r = j_{r-1} - 1$, $r \in [2,a] \cup [a+2,k]$, $j_{a+1} - j_a = c$.

Theorem (Hernandez-Leclerc 2010)

The assignments $L(X_{i,t+1}^{(i-2t-2)}) \mapsto P^{(k-i,1,t+2)}$, $i \in I$, $t \in [0,\ell]$, extends to a ring isomorphism $\Phi : K_0(\mathcal{C}_\ell^{\mathfrak{sl}_k}) \to \mathbb{C}[\mathrm{Gr}(k,n,\sim)]$, $n = k+\ell+1$.

Under the map Φ, Kirillov-Reshetikhin modules are sent to certain Plücker coordinates. A natural question is: what are the images of the simple modules. To answer the question, we use rectangular tableaux with k rows.

Monoid $\mathrm{SSYT}(k,[n])$ of semi-standard Young tableaux

- $\mathrm{SSYT(k,[n])}$ = the set consisting of the empty tableau and semi-standard Young tableaux of rectangular shape with k rows and with entries in $[n]$.
- For $A, B \in \mathrm{SSYT(k,[n])}$, $A \cup B$ is the semi-standard tableau with k rows and the elements in the ith row are the union of elements in the ith row of A and B, $i \in [k]$.

Example

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 7 \\ \hline 6 & 8 \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 2 & 9 \\ \hline 8 & 10 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 7 \\ \hline 2 & 2 & 7 & 9 \\ \hline 6 & 8 & 8 & 10 \\ \hline \end{array}.$$

Monoid $\mathrm{SSYT}(k, [n])$ of semi-standard Young tableaux

- We say that $A \in \mathrm{SSYT}(\mathrm{k}, [\mathrm{n}])$ is a trivial tableau if either A is empty or $A = \cup_j T_{i_j}$, where T_{i_j} is a one column tableau with entries $i_j, i_j + 1, \ldots, i_j + k - 1$, $i_j \in \mathbb{Z}_{\geq 1}$.

  The tableau $\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}$ is a trivial tableau.

- For $A \in \mathrm{SSYT}(k, [n])$, denote by $\mathrm{red}(A) \subset A$ the semi-standard Young tableau with minimum number of columns such that $A = \mathrm{red}(A) \cup A'$ for some trivial tableau A'.

Monoid $\mathrm{SSYT}(k,[n])$ of semi-standard Young tableaux

- For $A, B \in \mathrm{SSYT}(k,[n])$, define $A \sim B$ if either A, B are trivial tableaux or $\mathrm{red}(A) = \mathrm{red}(B)$.

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} \sim \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline 4 & 5 & 6 \\ \hline \end{array}.$$

- Denote $\mathrm{SSYT}(k,[n],\sim) = \mathrm{SSYT}(k,[n])/\sim$.

Lemma

$\mathrm{SSYT}(k,[n])$ and $\mathrm{SSYT}(k,[n],\sim)$ are commutative cancellative monoids under the multiplication "$\cup$".

Isomorphism of monoids $\mathcal{P}^+_{\ell,\mathfrak{sl}_k} \to \mathrm{SSYT}(k,[n],\sim)$

Theorem (Chang-Duan-Fraser-L. 2020)

The isomorphism $\Phi : K_0(\mathcal{C}_\ell^{\mathfrak{sl}_k}) \to \mathbb{C}[\mathrm{Gr}(k,n,\sim)]$, $n = k+\ell+1$, induces an isomorphism of monoids $\widetilde{\Phi} : \mathcal{P}^+_{\ell,\mathfrak{sl}_k} \to \mathrm{SSYT}(k,[n],\sim)$.

$$\widetilde{\Phi}(Y_{1,-1}Y_{2,-4}Y_{1,-7}Y_{2,-6}Y_{1,-9}) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline 4 & 7 & 8 \\ \hline \end{array},$$

$$\widetilde{\Phi}(Y_{1,-1}Y_{1,-3}Y_{1,-5}Y_{2,-4}Y^2_{1,-7}Y_{2,-6}Y^2_{1,-9}) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline 7 & 8 & 8 \\ \hline \end{array}.$$

Cluster monomials in a Grassmannian cluster algebra

- Recall that $T_M = \widetilde{\Phi}(M)$ and $M_T = \widetilde{\Phi}^{-1}(T)$.

Definition

For a semi-standard tableau $T \in \mathrm{SSYT}(k, [k+\ell+1], \sim)$, $n \in \mathbb{Z}_{\geq 2}$, $\ell \in \mathbb{Z}_{\geq 1}$, we define $\mathrm{ch}(T) \in \mathbb{C}[\mathrm{Gr}(k, k+\ell+1, \sim)]$ by $\mathrm{ch}(T) = \Phi(L(M_T))$.

Corollary

The isomorphism $\Phi : K_0(\mathcal{C}_\ell^{\mathfrak{sl}_k}) \to \mathbb{C}[\mathrm{Gr}(k, k+\ell+1, \sim)]$ *sends a module* $L(M)$ *to* $\mathrm{ch}(T_M)$ *and* $\Phi^{-1}(\mathrm{ch}(T)) = L(M_T)$.

Hernandez and Leclerc's conjecture about cluster monomials and real modules

- A simple $U_q(\widehat{\mathfrak{g}})$-module M is called real if $M \otimes M$ is simple.
- A simple $U_q(\widehat{\mathfrak{g}})$-module M is called prime if $M \cong M_1 \otimes M_2$ implies that M_1 or M_2 is trivial.
- A cluster monomial is a product of non-negative powers of cluster variables belonging to the same cluster.
- Hernandez and Leclerc in 2010 conjectured that

$$\begin{aligned} &\{\text{cluster monomials in } K_0(\mathcal{C}_\ell^{\mathfrak{g}})\} \\ = &\{[L(M)] : \ L(M) \text{ is a real module in } \mathcal{C}_\ell^{\mathfrak{g}}\}. \end{aligned} \tag{1}$$

- Qin in 2017 and Kang-Kashiwara-Kim-Oh in 2018 that

$$\begin{aligned} &\{\text{cluster monomials in } K_0(\mathcal{C}_\ell^{\mathfrak{g}})\} \\ \subset &\{[L(M)] : \ L(M) \text{ is a real module in } \mathcal{C}_\ell^{\mathfrak{g}}\}. \end{aligned}$$

- The other direction of (1) is still open.

We call T prime (resp. real) if $L(M_T)$ is prime (resp. real).

Theorem (Chang-Duan-Fraser-L. 2020)

Every cluster monomial (resp. cluster variable) in $\mathbb{C}[\mathrm{Gr}(k,n,\sim)]$ is of the form $\mathrm{ch}(T)$ *for some real tableau (resp. prime real tableau)* $T \in \mathrm{SSYT}(k,[n],\sim)$.

We will explain how to compute $\mathrm{ch}(T)$ in the following.

Representations of p-adic groups

- F is a non-archimedean local field with a normalized absolute value $|\cdot|$.
- Consider complex, smooth, finite length representations of $GL_n(F)$.
- Let $\mathrm{Irr} = \cup_{n \geq 0} \mathrm{Irr} GL_n(F)$ be the equivalence classes of irreducible representations.
- For representations π_1, π_2 of $GL_{n_1}(F)$, $GL_{n_2}(F)$ respectively, $\pi_1 \times \pi_2$ is the representation of $GL_{n_1+n_2}(F)$ parabolically induced from $\pi_1 \otimes \pi_2$.
- Denote by $\mathrm{soc}(\pi)$ the socle of π, i.e., the sum of the irreducible subrepresentations of π.

Representations of p-adic groups

- Fix a supercuspidal representation $\rho \in \mathrm{Irr}$. For $a \le b$, we write $[a, b] = \{\nu^a \rho, \ldots, \nu^b \rho\}$, where ν is the character $\nu(g) = |\det(g)|$, and $[a, b]$ is called a segment.
- A multisegment is a formal finite sum $\mathbf{m} = \sum_{i=1}^{p} \Delta_i$ of segments.
- For $\Delta = \{\nu^a \rho, \ldots, \nu^b \rho\}$, denote $\mathrm{Z}(\Delta) = \mathrm{soc}(\nu^a \rho \times \cdots \times \nu^b \rho)$.
- For a multisegment $\mathbf{m} = \sum_{i=1}^{p} \Delta_i$ ($\Delta_1, \ldots, \Delta_p$ are ordered in a certain way), denote $\zeta(\mathbf{m}) = \mathrm{Z}(\Delta_1) \times \cdots \times \mathrm{Z}(\Delta_p)$ and $\mathrm{Z}(\mathbf{m}) = \mathrm{soc}(\zeta(\mathbf{m}))$.
- The map $\mathbf{m} \mapsto Z(\mathbf{m})$ defines a bijection between multisegments and Irr (Bernstein-Zelevinsky 1977, Zelevinsky 1980).

Representations of p-adic groups

- For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$, denote by S_λ the subgroup of S_r consisting of elements σ such that $\lambda_{\sigma(i)} = \lambda_i$.
- For $\mu = (\mu_1, \ldots, \mu_r)$, $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$, we denote $\mathbf{m}_{\mu,\lambda} = \sum_{i=1}^r [\mu_i, \lambda_i]$.
- For any $w, w' \in S_r$ and any $\mu, \lambda \in \mathbb{Z}^r$, $\mathbf{m}_{w' \cdot \mu, \lambda} = \mathbf{m}_{w \cdot \mu, \lambda}$ if and only if $w' \in S_\lambda w S_\mu$.
- For a multisegment $\mathbf{m}$ with r terms, there exist unique weakly decreasing tuples $\mu_{\mathbf{m}}, \lambda_{\mathbf{m}} \in \mathbb{Z}^r$ and unique permutation of maximal length $w_{\mathbf{m}} \in S_r$ such that $\mathbf{m} = \mathbf{m}_{w_{\mathbf{m}} \cdot \mu_{\mathbf{m}}, \lambda_{\mathbf{m}}}$.
- The element $w_{\mathbf{m}} \in S_r$ is also the unique permutation of maximal length in $S_{\lambda_{\mathbf{m}}} w_{\mathbf{m}} S_{\mu_{\mathbf{m}}}$.

Arakawa-Suzuki functor

Arakawa-Suzuki functor (Arakawa-Suzuki 98) is a functor from category $\mathcal{O}$ to the category of finite-dimensional representations of a graded affine Hecke algebra.
Arakawa-Suzuki functor implies the following result.

Theorem (Arakawa-Suzuki 98, see also Henderson 07, Barbasch-Ciubotaru 15, Lapid-Minguez 18)

For a multisegment $\mathbf{m}$ *with* r *terms,*

$$\begin{aligned}[\mathrm{Z}(\mathbf{m})] &= [\mathrm{Z}(\mathbf{m}_{w_{\mathbf{m}}\cdot\mu_{\mathbf{m}},\lambda_{\mathbf{m}}})] \\ &= \sum_{w'\in S_r} (-1)^{\ell(w'w_{\mathbf{m}})} p_{w'w_0,w_{\mathbf{m}}w_0}(1)[\zeta(\mathbf{m}_{w'\cdot\mu_{\mathbf{m}},\lambda_{\mathbf{m}}})],\end{aligned} \tag{1}$$

where $p_{y,y'}(q)$ $(y, y' \in S_r)$ *is the Kazhdan-Lusztig polynomial and* w_0 *is the longest word in* S_r*.*

Equivalence of categories

- (Chari-Pressley 1996) For $N \leq k-1$, the category of finite dimensional representations of the affine Hecke algebra $\widehat{H}_N(q^2)$ is equivalent to certain subcategory of finite dimensional representations of $U_q(\widehat{\mathfrak{sl}_k})$. The correspondence between multisegments and dominant monomials is given by

$$[a,b] \mapsto Y_{b-a+1,a+b-1}, \qquad Y_{i,s} \mapsto [\frac{s-i+2}{2}, \frac{s+i}{2}].$$

- Denote by $M_{\mathbf{m}}$ the monomial corresponding to a multisegment $\mathbf{m}$ and $\mathbf{m}_M$ the multisegment corresponding to a monomial M.
- Let $M = Y_{2,0}Y_{1,-3}Y_{2,-2}Y_{1,-5}Y_{2,-6}Y_{2,-8}$. Then

$$\mathbf{m}_M = [0,1] + [-1,0] + [-1,-1] + [-2,-2] + [-3,-2] + [-4,-3].$$

- We write $\lambda_M = \lambda_{\mathbf{m}}$, $\mu_M = \mu_{\mathbf{m}}$, $w_M = w_{\mathbf{m}}$, where $\mathbf{m} = \mathbf{m}_M$.

q-character formula

- For any r-tuples $(\mu, \lambda) \in \mathbb{Z}^r \times \mathbb{Z}^r$, we define a multi-set:

$$\mathrm{Fund}_M(\mu, \lambda) = \{M_{[\mu_i, \lambda_i]} : i \in [r]\}.$$

- Translating Formula (1) to the language of q-characters, we have that for any simple $U_q(\widehat{\mathfrak{sl}_k})$-module $L(M)$,

$$\chi_q(L(M)) = \sum_{w' \in S_r} (-1)^{\ell(w' w_M)} p_{w' w_0, w_M w_0}(1) \prod_{M' \in \mathrm{Fund}_M(w' \mu_M, \lambda_M)} \chi_q(L(M')).$$

where r is the number of factors $Y_{i,s}$ in M (count with multiplicities).

A formula for ch(T)

- We say that a tableau T is a fundamental tableau if T has one column and the entries of T are $\{i, i+1, \ldots, j, j+2, \ldots, k+i-1\}$ for some $i \leq j \in [n]$.
- Let $T = \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_r \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array}$ be a tableau with r columns and every column of T is a fundamental tableau.
- Define $i_T = (i_1, \ldots, i_r)$, $l_T = (l_1, \ldots, l_r)$, where l_a is defined by $[i_a, i_a + k] \setminus \{l_a\}$ is the set of entries of the ath column of T.
- Define $j_T = (j_1, \ldots, j_r)$ to be the reordering of $l_1, \ldots, l_r$ such that $j_1 \leq \ldots \leq j_r$.
- For a tableau T, define $P_T = P_{T_1} \cdots P_{T_r}$, where $T_1, \ldots, T_r$ are columns of T. P_T is called the standard monomial of T.

A formula for ch(T)

- Let T be a tableau with r columns and each column is a fundamental tableau. For $u \in S_r$, we define $P_{u;T}$ as follows.
- If $j_a \in [i_{u(a)}, i_{u(a)} + k]$ for all $a \in [r]$, then define a tableau $\alpha(u; T)$ to be the semi-standard tableau whose columns have entries $[i_{u(a)}, i_{u(a)} + k] \setminus \{j_a\}$, $a \in [r]$ and define $P_{u;T} = P_{\alpha(u;T)}$. Otherwise define $P_{u;T} = 0$.
- There exists a unique element $w_T \in S_r$ with maximal length such that $P_{w_T;T} = P_T$.
- For any tableau $T \in \mathrm{SSYT}(k, [n])$, there is a unique tableau $T' \in \mathrm{SSYT}(k, [n])$ whose columns are fundamental tableaux such that $T \sim T'$. Define $w_T = w_{T'}$.

A formula for ch(T)

- For $T \in \mathrm{SSYT}(k, [n])$,

$$\mathrm{ch}(T) = \sum_{u \in S_r} (-1)^{\ell(u w_T)} p_{u w_0, w_T w_0}(1) P_{u; T'} \in \mathbb{C}[\mathrm{Gr}(k, n, \sim)]$$

  where $T' \in \mathrm{SSYT}(k, [n])$ is the unique tableau whose columns are fundamental tableaux such that $T \sim T'$, and r is the number of columns of T'.

- For example,

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}\right) = P_{124}P_{356} - P_{123}P_{456},$$

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}\right) = P_{145}P_{236} - P_{123}P_{456}.$$

Mutations of tableaux

- The exchange relations become simpler when written in terms of tableaux.
- Mutation rule:

$$T_k' = T_k^{-1} \max\{\cup_{i \to k} T_i, \cup_{k \to i} T_i\}.$$

- The following is an exchange relation in $\mathbb{C}[\mathrm{Gr}(3,8)]$:

$$\begin{aligned} \mathrm{ch}\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&6\\\hline 4&7&8\\\hline\end{array}\right) &= \mathrm{ch}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|}\hline 3&4\\\hline 5&6\\\hline 7&8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right) \\ &+ \mathrm{ch}\left(\begin{array}{|c|}\hline 3\\\hline 4\\\hline 8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|c|}\hline 2&4\\\hline 5&6\\\hline 7&8\\\hline\end{array}\right)\mathrm{ch}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right). \end{aligned}$$

Real modules and non-real modules

- We call a semi-standard tableau T real if the corresponding module $L(M_T)$ is real.
- Let $T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$. Then $\mathrm{ch}(T)$ is a cluster variable and T is real.
- The following are smallest prime non-real tableaux for $\mathrm{Gr}(3,9)$ and $\mathrm{Gr}(4,8)$.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 5 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}.$$

Cluster variables

- Not all tableaux are cluster variables.
- We use the mutation rule for tableaux to generate all cluster variables in $\mathbb{C}[\mathrm{Gr}(3,12)]$ up to 6 columns, in $\mathbb{C}[\mathrm{Gr}(4,10)]$ up to 6 columns, in $\mathbb{C}[\mathrm{Gr}(4,12)]$ up to 4 columns.

Numbers of cluster variables

r	1	2	3	4	5	6	7	8	9	10
$N_{3,3,r}$	1	0	0	0	0	0	0	0	0	0
$N_{3,4,r}$	4	0	0	0	0	0	0	0	0	0
$N_{3,5,r}$	10	0	0	0	0	0	0	0	0	0
$N_{3,6,r}$	20	2	0	0	0	0	0	0	0	0
$N_{3,7,r}$	35	14	0	0	0	0	0	0	0	0
$N_{3,8,r}$	56	56	24	0	0	0	0	0	0	0
$N_{3,9,r}$	84	168	225	288	372	414	522	594	612	744
$N_{3,10,r}$	120	420	1170	3280	8200	19140				
$N_{3,11,r}$	165	924	4455	20504	77957	256553				
$N_{3,12,r}$	220	1848	13860	92980	486172	2061132				
$N_{4,4,r}$	1	0	0	0	0	0	0	0	0	0
$N_{4,5,r}$	5	0	0	0	0	0	0	0	0	0
$N_{4,6,r}$	15	0	0	0	0	0	0	0	0	0
$N_{4,7,r}$	35	14	0	0	0	0	0	0	0	0
$N_{4,8,r}$	70	120	174	208	296	304	420	416	536	480
$N_{4,9,r}$	126	576	2421	8622	27054	69390				
$N_{4,10,r}$	210	2040	17665	117930	597500	2353760				
$N_{4,11,r}$	330	5940	90563	980100						
$N_{4,12,r}$	495	15048	367479	5963856						

Table: Number of cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ of rank r.

Conjectures of cluster variables

According to the data set, we obtain the following conjectures

Conjecture

$$N_{3,n,3} = 24\binom{n}{8} + 9\binom{n}{9},$$

$$N_{3,n,4} = 288\binom{n}{9} + 400\binom{n}{10} + 264\binom{n}{11} + 48\binom{n}{12},$$

$$N_{4,n,3} = 174\binom{n}{8} + 855\binom{n}{9} + 1285\binom{n}{10} + 693\binom{n}{11} + 123\binom{n}{12}.$$

Conjecture

For any tableau $T \in \mathrm{SSYT}(k, [n])$ with entries $a_1 < \ldots < a_r$ and any function $f : \{a_1, \ldots, a_r\} \to [n']$, $n' \geq n$, such that $f(a_1) < \ldots < f(a_r)$, we have that T is a cluster variable in $\mathbb{C}[\mathrm{Gr}(k,n)]$ if and only if $f(T)$ is a cluster variable in $\mathbb{C}[\mathrm{Gr}(k,n')]$.

Apply machine learning to classify cluster variables

- We generate all semistandard Young tableaux in $\mathrm{SSYT}(k,[n])$ for $(k,n)=(3,12),(4,10),(4,14)$ up to $6,6,4$ columns respectively as numpy arrays in python.
- Using the dataset of the cluster variables, we obtain the dataset of non-cluster variables for $(k,n)=(3,12),(4,10),(4,14)$ up to $6,6,4$ columns respectively.
- We take a part of the cluster variables and non-cluster variables as training set and the other part as testing set.

Apply machine learning to classify cluster variables

- We apply Support Vector Machines (SVM) and dense feed-forward Neural Networks (NN) to train the machine and use it to predict cluster variables.
- Accuracy is the proportion of predictions that are correctly classified.
- SVM and NN can determine the cluster variables from the full sets of semistandard Young tableaux with high accuracies, around 0.92 and 0.94 respectively.

Thank you!