Tropical geometry, quantum affine algebras, and scattering amplitudes

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Representations of quantum affine algebras

- $\mathfrak{g}$ = simple Lie algebra over $\mathbb{C}$, I = the set of vertices of the Dynkin diagram of $\mathfrak{g}$.
- $U_q(\widehat{\mathfrak{g}})$ is the quantum affine algebra associated to $\mathfrak{g}$, $q \in \mathbb{C}^\times$ is not a root of unity. It is generated by $x_{i,r}^{\pm}$ ($i \in I$, $r \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,r}$ ($i \in I$, $r \in \mathbb{Z} \backslash \{0\}$), $c^{\pm 1}$, subject to certain relations.
- (Chari-Pressley 1994) Every finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-module is a highest l-weight (loop-weight) module.
- The highest l-weight of a finite dimensional simple module corresponds to a unique I-tuple $(P_i(u))_{i \in I}$ of polynomials $P_i(u) \in \mathbb{C}[u]$, $i \in I$, the constant term of $P_i(u)$ is 1. These polynomials are called Drinfeld polynomials.

Finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- $\mathcal{P}$ = the free abelian group generated by $Y_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^\times$.
- $\mathcal{P}^+$ = the submonoid of $\mathcal{P}$ generated by $Y_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$.
- Elements in $\mathcal{P}^+$ are called dominant monomials.
- Every I-tuple of Drinfeld polynomials can be identified with a monomial in $\mathcal{P}^+$. For example, $((1-au)(1-bu), 1, 1-cu)$ is identified with $Y_{1,a}Y_{1,b}Y_{3,c}$.
- Finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-module with highest l-weight $M \in \mathcal{P}^+$ is denoted by $L(M)$.

q-characters of finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- Frenkel and Reshetikhin introduced the theory of q-characters of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules.
- (Frenkel-Reshetikhin 1998) The q-character of a finite dimensional module V is defined by

$$\chi_q(V) = \sum_{m \in \mathcal{P}} \dim(V_m) m \in \mathbb{Z}\mathcal{P},$$

  where V_m is the l-weight space with l-weight m defined as follows.
- Consider $\phi^{\pm}_{i,\pm n}$ defined by

$$\sum_{n=0}^{\infty} \phi^{\pm}_{i,\pm n} u^{\pm n} = k_i^{\pm} \exp(\pm(q - q^{-1}) \sum_{m=1}^{\infty} h_{i,\pm m} u^{\pm m}).$$

q-characters of finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- The module V can be decomposed as $V = \oplus V_\gamma$, where

$$V_\gamma = V_{(\gamma_{i,n}^{\pm})} = \{x \in V : \exists p, (\phi_{i,n}^{\pm} - \gamma_{i,n}^{\pm})^p \cdot x = 0, \forall i \in I, n \in \mathbb{Z}\},$$

  is the l-weight space of V with l-weight $\gamma = (\gamma_{i,n}^{\pm})$.
- Frenkel and Reshetikhin proved that

$$\sum_{n=0}^{\infty} \gamma_{i,n}^{\pm} u^{\pm n} = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1}) R_i(uq_i)}{Q_i(uq_i) R_i(uq_i^{-1})},$$

  where $Q_i(z) = \prod_{r=1}^{k_i}(1 - za_{ir})$, $R_i(z) = \prod_{s=1}^{l_i}(1 - zb_{is})$ are some polynomials.

q-characters of finite dimensional simple $U_q(\widehat{\mathfrak{g}})$-modules

- The l-weight $\gamma = (\gamma_{i,n}^{\pm})$ can be identified with

$$m_\gamma = \prod_{i \in I} \left(\prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1} \right)$$

and the l-weight space V_γ can be identified with V_{m_γ}.

- The q-character of V is

$$\chi_q(V) = \sum_{\gamma} \dim(V_\gamma) e^{\gamma} = \sum_{m} \dim(V_m) m.$$

- The q-character map $\chi_q : K_0(\mathcal{C}) \to \mathbb{Z}\mathcal{P}$ is an injective homomorphism, where $K_0(\mathcal{C})$ is the Grothendieck ring of the category $\mathcal{C}$ of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules.

Examples of q-characters

In the case of $\mathfrak{g} = \mathfrak{sl}_4$,

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1} Y_{2,aq} + Y_{2,aq^3}^{-1} Y_{3,aq^2} + Y_{3,aq^4}^{-1},$$

$$\begin{aligned}\chi_q(L(Y_{2,a})) = Y_{2,a} + Y_{1,aq} Y_{2,aq^2}^{-1} Y_{3,aq} + Y_{1,aq^3}^{-1} Y_{3,aq} \\ + Y_{1,aq} Y_{3,aq^3}^{-1} + Y_{1,aq^3}^{-1} Y_{2,aq^2} Y_{3,aq^3}^{-1} + Y_{2,aq^4}^{-1}.\end{aligned}$$

Prime modules of quantum affine algebras

- A simple $U_q(\widehat{\mathfrak{g}})$-module $L(m)$ is called prime if $L(m)$ is not isomorphic to $L(m') \otimes L(m'')$ for any non-trivial $U_q(\widehat{\mathfrak{g}})$-modules $L(m')$, $L(m'')$.
- Classification of prime modules of $U_q(\widehat{\mathfrak{g}})$ is a difficult open problem. It is only known in the case of $\mathfrak{g} = \mathfrak{sl}_2$ by Chari and Pressley 1994.
- A simple $U_q(\widehat{\mathfrak{sl}_2})$-module $L(m)$ is prime if m is of the form $Y_{1,a}Y_{1,aq^2} \cdots Y_{1,aq^{2r-2}}$ for some $r \in \mathbb{Z}_{\geq 1}$. Prime $U_q(\widehat{\mathfrak{sl}_2})$-modules are Kirillov-Reshetikhin modules.

Hernandez and Leclerc's work

- Hernandez and Leclerc in 2010 made a breakthrough in constructing prime modules. They applied the theory of cluster algebras (introduced by Fomin and Zelevinsky 2000) to study $U_q(\widehat{\mathfrak{g}})$-modules.
- A simple $U_q(\widehat{\mathfrak{g}})$-module $L(m)$ is called real if $L(m) \otimes L(m)$ is simple.
- For each $\ell \geq 0$, Hernandez and Leclerc 2010 introduced certain a subcategory $\mathcal{C}_\ell$ of the category $\mathcal{C}$ of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules and they proved that the Grothendieck ring $K_0(\mathcal{C}_\ell)$ has a cluster algebra structure.
- Hernandez and Leclerc 2010 conjectured (proved by Qin 2017, Kashiwara-Kim-Oh-Park 2020, 2021) that cluster monomials are real modules, cluster variables are prime and real modules. Therefore their method can produce a very large family of prime modules.

Prime modules which are not cluster variables

- On the other hand, there are prime modules which are not cluster variables. These modules are also important in applications. For example, these modules can be used to construct algebraic letters (cluster variables correspond to rational letters) in the computations of Feymann integrals in scattering amplitudes in physics.
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- Drummond, Foster, Gürdoğan, Kalousios, J. High Energ. Phys. 2020, 146 (2020).
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Newton polytopes approach to classify prime modules

- In [Early-L. 2023], for every quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, we construct a sequence of Newton polytopes. We given an explicit construction of simple $U_q(\widehat{\mathfrak{g}})$-modules from the facets of these Newton polytopes.
- We conjecture that simple modules corresponding to facets of the Newton polytopes are prime. Conversely, for every prime module, there is a Newton polytope such that the prime module corresponds to a facet of the Newton polytope.
- I will explain how to define the Newton polytopes and applications of prime modules to scattering amplitudes in physics.

Hernandez and Leclerc's category $\mathcal{C}_\ell$

- Hernandez and Leclerc in 2010 introduced a subcategory $\mathcal{C}_\ell$ ($\ell \in \mathbb{Z}_{\geq 0}$) of $\mathcal{C}$ and they proved that the Grothendieck ring $K_0(\mathcal{C}_\ell)$ of $\mathcal{C}_\ell$ has a cluster algebra structure.
- Fix $a \in \mathbb{C}^*$ and denote $Y_{i,s} = Y_{i,aq^s}$, $i \in I$, $s \in \mathbb{Z}$.
- For $\ell \in \mathbb{Z}_{\geq 0}$, denote by $\mathcal{P}_\ell$ the subgroup of $\mathcal{P}$ generated by $Y_{i,\xi(i)-2r}^{\pm 1}$, $i \in I$, $r \in [0, d\ell]$, where d is the maximal diagonal element in the diagonal matrix D, and $\xi : I \to \mathbb{Z}$ is a certain function called height function. D is the diagonal matrix such that DC is symmetric.
- Denote by $\mathcal{P}_\ell^+$ the submonoid of $\mathcal{P}^+$ generated by $Y_{i,\xi(i)-2r}$, $i \in I$, $r \in [0, d\ell]$.
- Simple modules in $\mathcal{C}_\ell$ are of the form $L(M)$, where $M \in \mathcal{P}_\ell^+$.

Cluster algebra structure on the Grothendieck ring of $\mathcal{C}_\ell$

- For $i \in I$, $r \in \mathbb{Z}_{\geq 1}$, $s \in \mathbb{Z}$, denote $X_{i,r}^{(s)} = Y_{i,s} Y_{i,s+2d_i} \cdots Y_{i,s+2rd_i-2}$. $L(X_{i,r}^{(s)})$ is called a Kirillov-Reshetikhin modules, where $d_1, \ldots, d_n$ are diagonal elements in D.
- $L(Y_{i,s})$ is called a fundamental module.

Theorem (Hernandez-Leclerc 2010)

The ring $K_0(\mathcal{C}_\ell)$ has a cluster algebra structure. The cluster variables in the initial seed of the cluster algebra are certain Kirillov-Reshetikhin modules.

The initial cluster for $K_0(\mathcal{C}_2^{A_2})$

This is the initial cluster in the case of $U_q(\widehat{\mathfrak{sl}_3})$, $\ell = 2$.

$$\begin{tikzcd}
Y_{1,-1} \arrow[rd] & Y_{2,0} \arrow[l] \\
Y_{1,-3}Y_{1,-1} \arrow[u] \arrow[rd] & Y_{2,-2}Y_{2,0} \arrow[l] \arrow[u] \\
\color{red}{Y_{1,-5}Y_{1,-3}Y_{1,-1}} \arrow[u] & \color{red}{Y_{-2,-4}Y_{2,-2}Y_{2,0}} \arrow[l] \arrow[u]
\end{tikzcd}$$

Newton polytope of a polynomial

- Consider the polynomial $1 + y_1$ in y_1, y_2.
- We fix an order y_1, y_2 of the variables. Then $1 + y_1 = y_1^0 y_2^0 + y_1^1 y_2^0$.
- Take the exponents $(0, 0), (1, 0)$ as vertices and the Newton polytope is the convex hull of the vertices $(0, 0), (1, 0)$.

Newton polytopes for a quantum affine algebra

- Denote by $\mathcal{M}^{(0)}$ the set of isomorphism classes of Kirillov-Reshetikhin modules in $\mathcal{C}_\ell$.
- In [Early-L. 2023], we define

$$\mathbf{N}_{\mathfrak{g},\ell}^{(0)} = \mathrm{Newt}(\prod_{L(m)\in\mathcal{M}^{(0)}} \widetilde{\chi}_q(L(m))/m).$$

- Let $\mathcal{M}^{(d+1)}$ ($d \geq 0$) be the set of simple modules which correspond to facets of $\mathbf{N}_{\mathfrak{g},\ell}^{(d)}$ (I will explain how to construct a simple module from a facet later).
- Define

$$\mathbf{N}_{\mathfrak{g},\ell}^{(d)} = \mathrm{Newt}(\prod_{L(m)\in\mathcal{M}^{(d)}} \widetilde{\chi}_q(L(m))/m).$$

g-vectors and simple modules

- g-vectors were introduced by Fomin and Zelevinsky 2007.
- For any simple $U_q(\widehat{\mathfrak{g}})$-module $L(M)$, the dominant monomial M can be written as $M = \prod_{i,s} Y_{i,s}^{a_{i,s}}$ for some non-negative integers $a_{i,s}$. On the other hand, M can also be written as $M = \prod_j M_j^{g_j}$ for some integers g_j, where the product runs over all initial cluster variables and frozen variables $L(M_j)$ in $\mathcal{C}_\ell$. The g_j's are the unique solution of $\prod_{i,s} Y_{i,s}^{a_{i,s}} = \prod_j M_j^{g_j}$ (Hernandez-Leclerc 2016). With a chosen order, g_j's form the g-vector of $L(M)$.
- Every simple module in $\mathcal{C}_\ell$ has a g-vector in the above sense even if it is not a cluster monomial.

From facets of Newton polytopes to simple modules

- Frenkel-Reshetikhin 98 and Frenkel-Mukhin 2000 proved that for every simple $U_q(\widehat{\mathfrak{g}})$-module $L(m)$, $\widetilde{\chi}_q(L(m))/m$ is a polynomial in $A_{i,s}^{-1}$ with constant term 1, where

$$A_{i,s} = Y_{i,s+d_i} Y_{s-d_i} \Big(\prod_{j; C_{ji}=-1} Y_{j,s}^{-1} \Big) \times \\ \times \Big(\prod_{j; C_{ji}=-2} Y_{j,s-1}^{-1} Y_{j,s+1}^{-1} \Big) \Big(\prod_{j; C_{ji}=-3} Y_{j,s-2}^{-1} Y_{j,s}^{-1} Y_{j,s+2}^{-1} \Big).$$

- Denote $v_{i,s} = A_{i,s}^{-1}$. The variables $v_{i,s}$ are cluster X-coordinates of the initial cluster.

Truncated q-characters

- Take an order of initial cluster variables of $K_0(\mathcal{C}_\ell)$ and take the corresponding order of $v_{i,s}$.
- We take the outward normal vectors of the Newton polytope. Take these vectors as g-vectors, we obtain simple $U_q(\widehat{\mathfrak{g}})$-modules. We conjecture that these modules are prime.
- Moreover, we conjecture that every prime $U_q(\widehat{\mathfrak{g}})$-module corresponds to a facet of $\mathbf{N}^{(d)}_{\mathfrak{g},\ell}$ for some $d \geq 0$.

Example of Newton polytopes

- Let $\mathfrak{g}$ be of type A_2 and $\ell = 2$.
- Take the order of initial cluster variables: $Y_{1,-1}$, $Y_{1,-3}Y_{1,-1}$, $Y_{2,0}$, $Y_{2,-2}Y_{2,0}$.
- The corresponding order of $v_{i,s}$ is $v_{1,-2}, v_{1,-4}, v_{2,-1}, v_{2,-3}$.
- The truncated q-characters of KR modules are:

$$\begin{aligned}\widetilde{\chi}_q(L(Y_{1,-3})) &= \frac{1}{Y_{2,0}} + \frac{Y_{2,-2}}{Y_{1,-1}} + Y_{1,-3} \\ &= Y_{1,-3}(1 + v_{1,-2} + v_{1,-2}v_{2,-1}),\end{aligned}$$

Example of Newton polytopes

$$\widetilde{\chi}_q(L(Y_{1,-5})) = Y_{1,-5}(1 + v_{1,-4} + v_{1,-4}v_{2,-3}),$$

$$\widetilde{\chi}_q(L(Y_{2,-2})) = Y_{2,-2}(1 + v_{2,-1}),$$

$$\widetilde{\chi}_q(L(Y_{2,-4})) = Y_{2,-4}(1 + v_{2,-3} + v_{2,-3}v_{1,-2}),$$

$$\begin{aligned}\widetilde{\chi}_q(L(Y_{1,-3}Y_{1,-5})) = {} & Y_{1,-3}Y_{1,-5}(1 + v_{1,-2} + v_{1,-2}v_{1,-4} + v_{1,-2}v_{2,-1} \\ & + v_{1,-2}v_{2,-1}v_{1,-4} + v_{1,-2}v_{2,-1}v_{1,-4}v_{2,-3}),\end{aligned}$$

$$\widetilde{\chi}_q(L(Y_{2,-2}Y_{2,-4})) = Y_{2,-2}Y_{2,-4}(1 + v_{2,-1}v_{2,-3} + v_{2,-1}),$$

Example of Newton polytopes

The Newton polytope $\mathbf{N}^{(0)}_{\mathfrak{sl}_3,2}$ is given by the following half-spaces:

$$(-1,0,0,0)\cdot x+3\geq 0,\ (0,-1,0,0)\cdot x+2\geq 0,\ (0,0,-1,0)\cdot x+4\geq 0$$
$$(0,1,0,-1)\cdot x+2\geq 0,\ (0,0,1,-1)\cdot x+2\geq 0,\ (0,0,1,0)\cdot x+0\geq 0,$$
$$(0,0,0,1)\cdot x+0\geq 0,\ (1,-1,0,0)\cdot x+1\geq 0,\ (1,0,0,0)\cdot x+0\geq 0,$$
$$(1,0,-1,0)\cdot x+2\geq 0,\ (0,1,0,0)\cdot x+0\geq 0,\ (-1,0,0,1)\cdot x+2\geq 0,$$
$$(0,1,1,-1)\cdot x+1\geq 0.$$

The outward normal vectors of these facets correspond to the following prime modules respectively:

$$L(Y_{1,-1}),\ L(Y_{1,-1}Y_{1,-3}),\ L(Y_{2,0}),\ L(Y_{1,-5}Y_{2,-2}Y_{2,0}),$$
$$L(Y_{2,-2}),\ L(Y_{2,-4}Y_{2,-2}),\ L(Y_{2,-4}),\ L(Y_{1,-3}),\ L(Y_{1,-5}Y_{1,-3}),$$
$$L(Y_{1,-5}Y_{1,-3}Y_{2,0}),\ L(Y_{1,-5}),\ L(Y_{2,-4}Y_{1,-1}),\ L(Y_{1,-5}Y_{2,-2}).$$

Example of Newton polytopes

- Continue this procedure, we have that $\mathbf{N}^{(1)}_{\mathfrak{sl}_3,2}$ has 16 facets. The facets give exactly the 16 prime modules (not including the frozens) in $\mathcal{C}_2^{A_2}$.
- The Newton polytope $\mathbf{N}^{(d)}_{\mathfrak{sl}_3,2}$ ($d \geq 1$) also has 16 facets.

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned}\mathrm{Gr}(k,n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations}.\end{aligned}$$

- Scott 2006 showed that the coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$ has a cluster algebra structure.
- The algebra $\mathbb{C}[\mathrm{Gr}(k,n)]$ is called a Grassmannian cluster algebra.
- A certain quotient $\mathbb{C}[\mathrm{Gr}(k,n,\sim)]$ is isomorphic to $K_0(\mathcal{C}_\ell^{\mathfrak{sl}_k})$ (Hernandez-Leclerc 2010), $n = k + \ell + 1$.
- Dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n,\sim)]$ are parametrized by semistandard Young tableaux in $\mathrm{SSYT}(k,[n],\sim)$ (Chang-Duan-Fraser-L. 2020).

u-variables and u-equations

- Another application to physics is u-variables and u-equations.
- u-variables are certain rational fractions in Plücker coordinates originally defined by physicists Koba-Nielsen in 1969 in the case of $\mathrm{Gr}(2, n)$.
- In [Early-L. 2023], we give a general formula for u-variables for $\mathrm{Gr}(k, n)$ for any $k \leq n \in \mathbb{Z}_{\geq 1}$.

Grassmannian cluster categories

- Jensen, King, and Su 2016 gave an additive categorification of $\mathbb{C}[\mathrm{Gr}(k, n)]$ using Cohen-Macaulay modules.
- Denote by $\mathrm{CM}(B_{k,n})$ the category of Cohen-Macaulay $B_{k,n}$-modules. The category $\mathrm{CM}(B_{k,n})$ has an Auslander-Reiten quiver.

Cluster variables, rigid indecomposable modules, real prime modules, tableaux

- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable rigid indecomposable modules in $\mathrm{CM}(B_{k,n})$ [Jensen, King, Su 2016].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real modules in $\mathcal{C}_{\ell}^{\mathfrak{sl}_k}$ [Hernandez-Leclerc 2010, Qin 2017, Kashiwara-Kim-Oh-Park 2020, 2021].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real tableaux in $\mathrm{SSYT}(k,[n])$ [Chang-Duan-Fraser-L. 2020].
- We replace the modules at the vertices of the Auslander-Reiten quiver by the corresponding tableaux.

Auslander-Reiten quiver in the case of Gr$(3, 6)$

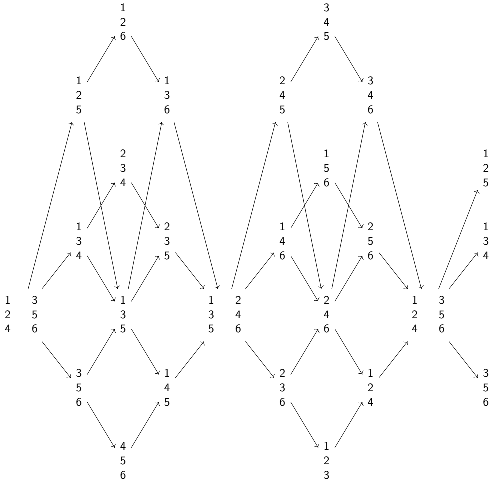

Figure: The Auslander-Reiten quiver for $\mathrm{CM}(B_{3,6})$ with vertices labelled by tableaux.

u-variables in the case of Gr$(3,6)$

The u-variables for Gr$(3,6)$ are

$$u_{126}=\frac{p_{136}}{p_{126}},\ u_{345}=\frac{p_{346}}{p_{345}},\ u_{125}=\frac{p_{126}p_{135}}{p_{125}p_{136}},\ u_{136}=\frac{\mathrm{ch}_{135,246}}{p_{136}p_{245}},\ u_{245}=\frac{p_{345}p_{246}}{p_{245}p_{346}},$$
$$u_{346}=\frac{\mathrm{ch}_{124,356}}{p_{346}p_{125}},\ u_{124,356}=\frac{p_{125}p_{134}p_{356}}{\mathrm{ch}_{124,356}p_{135}},\ u_{134}=\frac{p_{135}p_{234}}{p_{134}p_{235}},\ u_{135}=\frac{p_{136}p_{145}p_{235}}{p_{135}\mathrm{ch}_{135,246}},$$
$$u_{235}=\frac{\mathrm{ch}_{135,246}}{p_{235}p_{146}},\ u_{135,246}=\frac{p_{146}p_{245}p_{236}}{\mathrm{ch}_{135,246}p_{246}},\ u_{146}=\frac{p_{246}p_{156}}{p_{146}p_{256}},\ u_{246}=\frac{p_{346}p_{256}p_{124}}{p_{246}\mathrm{ch}_{124,356}},$$
$$u_{256}=\frac{\mathrm{ch}_{124,356}}{p_{256}p_{134}},\ u_{234}=\frac{p_{235}}{p_{234}},\ u_{156}=\frac{p_{256}}{p_{156}},\ u_{356}=\frac{p_{135}p_{456}}{p_{356}p_{145}},\ u_{145}=\frac{\mathrm{ch}_{135,246}}{p_{145}p_{236}},$$
$$u_{236}=\frac{p_{246}p_{123}}{p_{236}p_{124}},\ u_{124}=\frac{\mathrm{ch}_{124,356}}{p_{124}p_{356}},\ u_{456}=\frac{p_{145}}{p_{456}},\ u_{123}=\frac{p_{124}}{p_{123}},$$

where we use $\mathrm{ch}_{T_1,\ldots,T_r}$ to denote ch_T, and T_i's are columns of T.
Here $\mathrm{ch}_{124,356}=p_{124}p_{356}-p_{123}p_{456}$, and
$\mathrm{ch}_{135,246}=p_{145}p_{236}-p_{123}p_{456}$.

A general formula for u-variables

For every mesh

$$\begin{tikzcd} & T_1 \arrow[dr] & \\ S \arrow[ur] \arrow[r, dashed] \arrow[dr] & \vdots \arrow[r, dashed] & S' \\ & T_r \arrow[ur] & \end{tikzcd}$$

in the Auslander-Reiten quiver of $\mathrm{CM}(B_{k,n})$, we define the corresponding u-variable as

$$u_S = \frac{\prod_{i=1}^{r} \mathrm{ch}_{T_i}}{\mathrm{ch}_S \mathrm{ch}_{S'}}.$$

u-equations

- We conjecture that there exist unique integers $a_{T,T'}$ such that

$$u_T + \prod_{T' \in \mathrm{PSSYT}_{k,n}} u_{T'}^{a_{T,T'}} = 1,$$

  for all $T \in \mathrm{PSSYT}_{k,n}$, $\mathrm{PSSYT}_{k,n}$ is the set of all (non-frozen) prime tableaux in $\mathrm{SSYT}(k,[n])$.
- These equations are called u-equations.
- The following is an example of u-equation in the case of $\mathrm{Gr}(3,6)$:

$$u_{124,356} + u_{135}u_{136}u_{145}u_{146}u_{235}u_{236}u_{245}u_{246}u_{135,246}^2 = 1.$$

Thank you!