Quantum affine algebras and their applications to scattering amplitudes

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Quantum affine algebras

- A quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra that is a q-deformation of the universal enveloping algebra of an affine Lie algebra $\widehat{\mathfrak{g}}$.
- Hernandez and Leclerc proved that the Grothendieck ring $K_0(\mathcal{C}_\ell^{\mathfrak{g}})$ of certain subcategory $\mathcal{C}_\ell^{\mathfrak{g}}$ of the category of finite dimensional $U_q(\widehat{\mathfrak{g}})$-modules has a cluster structure. In the case of $\mathfrak{g} = \mathfrak{sl}_k$, $K_0(\mathcal{C}_\ell^{\mathfrak{sl}_k})$ is isomorphic to a quotient $\mathbb{C}[\mathrm{Gr}(k, n, \sim)]$ of the Grassmannian cluster algebra (some frozen variables are set to 1).
- In [CDFL19], we proved that the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k, n, \sim)]$ are parametrized by semistandard Young tableaux. Using results in representations of p-adic groups and representations of quantum affine algebras, we gave an explicit formula of elements $\mathrm{ch}(T)$ in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k, n, \sim)]$.

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned} \mathrm{Gr}(k,n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations}. \end{aligned}$$

- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{C}[\mathrm{Gr}(k,n)]$ $(i_1 < \cdots < i_k)$: for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, ..., kth rows and i_1th, ..., i_kth columns.
- Dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n)]$ is [CDFL2019]

$$\{\mathrm{ch}(T) : T \in \mathrm{SSYT}(k,[n])\},$$

where $\mathrm{ch}(T)$ is a polynomial in Plücker coordinates and is given by an explicit formula, $\mathrm{SSYT}(k,[n])$ is the set of rectangular tableaux with k rows and with entries in $[n]$.

Prime elements in the dual canonical basis

- $\mathrm{ch}(T)$ is called prime if $\mathrm{ch}(T) \neq \mathrm{ch}(T')\mathrm{ch}(T'')$ for any non-trivial tableaux T', T''.
- $\mathbb{C}[\mathrm{Gr}(2,5)]$ has 5 (non-frozen) prime elements $p_{13}, p_{24}, p_{14}, p_{25}, p_{35}$. They are all cluster variables.
- For general $\mathbb{C}[\mathrm{Gr}(k,n)]$, all cluster variables are prime but there are more prime elements than cluster variables.

Prime elements in the dual canonical basis

- How to classify all prime elements in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n)]$? This is a difficult question and it is only known in the case of $k = 2$. An element $\mathrm{ch}(T)$ in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(2,n)]$ is prime if and only if T is a one-column tableau, i.e. $\mathrm{ch}(T)$ is a Plücker coordinate (Chari-Pressley).
- When $\mathbb{C}[\mathrm{Gr}(k,n)]$ is of finite type, all prime elements are cluster variables and they can be obtained using mutations.

Prime elements in the dual canonical basis

- When $\mathbb{C}[\mathrm{Gr}(k,n)]$ is of infinite type, using tropical Grassmannians, certain prime elements in $\mathbb{C}[\mathrm{Gr}(4,8)]$, $\mathbb{C}[\mathrm{Gr}(4,9)]$, $\mathbb{C}[\mathrm{Gr}(3,9)]$, $\mathbb{C}[\mathrm{Gr}(3,10)]$, were obtained in the works by:
  - Nima Arkani-Hamed, Thomas Lam, Marcus Spradlin, 2021,
  - James Drummond, Jack Foster, Ömer Gürdoğan, Chrysostomos Kalousios, 2020,
  - Niklas Henke, Georgios Papathanasiou, 2020, 2021,
  - Dani Kaufman, Zachary Greenberg, 2021,
  - Lecheng Ren, Marcus Spradlin, Anastasia Volovich, 2021.
- In [Early-L. 2023] we use Newton polytopes to give a recursive way of constructing prime elements in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n)]$. We conjecture that all prime elements can be obtained in this way.

Prime elements in the dual canonical basis

- Let $\mathcal{T}_{k,n}^{(0)}$ be the set of all one-column tableaux which are obtained by cyclic shifts of the one-column tableau with entries $1, 2, \ldots, k-1, k+1$.
- For $d \geq 0$, we define recursively

$$\mathbf{N}_{k,n}^{(d)} = \mathrm{Newt}\left(\prod_{T \in \mathcal{T}_{k,n}^{(d)}} \mathrm{ch}_T(x_{i,j}) \right),$$

where $\mathcal{T}_{k,n}^{(d+1)}$ is the set of all tableaux which correspond to facets of $\mathbf{N}_{k,n}^{(d)}$, $\mathrm{ch}_T(x_{i,j})$ is the polynomial obtained by evaluating $\mathrm{ch}(T)$ on the web matrix (Speyer and Williams 2005).

From facets of Newton polytopes to tableaux

- The Newton polytope $\mathbf{N}_{k,n}^{(d)}$ can be described using certain equations and inequalities in its H-representation.
- Let F be a facet of the Newton polytope $\mathbf{N}_{k,n}^{(d)}$. The normal vector v_F of F is the coefficient vector in one of the inequalities in the H-representation of $\mathbf{N}_{k,n}^{(d)}$.
- If there is an entry of the vector v_F which is negative, then we add some vectors which are coefficients of the equations in the H-representation of $\mathbf{N}_{k,n}^{(d)}$ corresponding to frozen variables to v_F such that the resulting vector v'_F have non-negative entries.

From facets of Newton polytopes to tableaux

- The vector v'_F can be written as $v'_F = \sum_{i,j} c_{i,j} e_{i,j}$ for some positive integers $c_{i,j}$, where $e_{i,j}$ is the standard basis of $\mathbb{R}^{(k-1)\times(n-k)}$.
- We send the vector $e_{i,j}$ to a fundamental tableau $T_{i,j}$ which is defined to be the one-column tableau with entries $[j, j+k] \setminus \{i+j\}$.
- The tableau T_F corresponding to F is obtained from $\cup_{i,j} T_{i,j}^{\cup c_{i,j}}$ by removing all frozen factors (if any).

Example: Gr(3, 6)

- The web matrix for Gr(3, 6) is

$$M = \begin{bmatrix} 1 & 0 & 0 & x_{1,1}x_{2,1} & x_{1,1}x_{2,12} + x_{1,2}x_{2,2} & x_{1,1}x_{2,123} + x_{1,2}x_{2,23} + x_{1,3}x_{2,3} \\ 0 & 1 & 0 & -x_{2,1} & -x_{2,12} & -x_{2,123} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

  where we abbreviate for example $x_{2,23} = x_{2,2} + x_{2,3}$.
- Evaluating all Plücker coordinates on M and take their product, we obtain a polynomial p. The Newton polytope $\mathbf{N}_{3,6}^{(1)}$ is the Newton polytope defined by the vertices given by the exponents of monomials of p.

Example: $\mathrm{Gr}(3,6)$

- The H-representation of $\mathbf{N}_{3,6}^{(1)}$ is given by

$$(0,0,0,1,1,1)\cdot x-20=0,\ (1,1,1,0,0,0)\cdot x-10=0,\ (0,1,1,0,0,0)\cdot x-4\geq 0,$$
$$(0,0,1,0,0,0)\cdot x-1\geq 0,\ (0,0,0,0,1,1)\cdot x-11\geq 0,\ (0,0,0,0,0,1)\cdot x-4\geq 0,$$
$$(0,0,1,1,0,0)\cdot x-6\geq 0,\ (0,0,0,0,1,0)\cdot x-4\geq 0,\ (0,0,0,1,0,0)\cdot x-4\geq 0,$$
$$(1,0,0,0,0,0)\cdot x-1\geq 0,\ (1,0,0,0,1,0)\cdot x-6\geq 0,\ (1,1,0,0,1,1)\cdot x-16\geq 0,$$
$$(1,1,0,0,0,0)\cdot x-4\geq 0,\ (0,0,0,1,1,0)\cdot x-11\geq 0,\ (0,1,0,0,0,0)\cdot x-1\geq 0,$$
$$(1,0,0,0,1,1)\cdot x-14\geq 0,\ (0,1,0,0,0,1)\cdot x-6\geq 0,\ (1,1,0,0,0,1)\cdot x-11\geq 0,$$

where $(0,0,0,1,1,1)\cdot x$ is the inner product of the vectors $(0,0,0,1,1,1)$ and x.

Example: $\mathrm{Gr}(3,6)$

- For the facet F with the normal vector $v_F = (0,1,1,0,0,0)$ in the first line of the above, we have that $v_F = e_{1,2} + e_{1,3}$. The generalized roots $e_{1,2}$, $e_{1,3}$ corresponds to tableaux $\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline 6 \\ \hline\end{array}$ respectively. Removing the frozen factor $\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline\end{array}$ in $\begin{array}{|c|}\hline 2 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \cup \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline 6 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 5 & 6 \\ \hline\end{array}$, we obtain $T_F = \begin{array}{|c|}\hline 2 \\ \hline 5 \\ \hline 6 \\ \hline\end{array}$.

Example: Gr(3, 6)

facets, hyperplanes	tableaux	modules
$(0,0,0,1,0,0)$	[124]	$Y_{1,-1}$
$(0,0,0,1,1,0)$	[125]	$Y_{1,-3}Y_{1,-1}$
$(1,0,0,0,0,0)$	[134]	$Y_{2,0}$
$(1,0,0,0,1,0)$	[135]	$Y_{1,-3}Y_{2,0}$
$(1,0,0,0,1,1)$	[136]	$Y_{1,-5}Y_{1,-3}Y_{2,0}$
$(1,1,0,0,0,0)$	[145]	$Y_{2,-2}Y_{2,0}$
$(1,1,0,0,0,1)$	[146]	$Y_{1,-5}Y_{2,-2}Y_{2,0}$
$(0,0,0,0,1,0)$	[235]	$Y_{1,-3}$
$(0,0,0,0,1,1)$	[236]	$Y_{1,-5}Y_{1,-3}$
$(0,1,0,0,0,0)$	[245]	$Y_{2,-2}$
$(0,1,0,0,0,1)$	[246]	$Y_{1,-5}Y_{2,-2}$
$(0,1,1,0,0,0)$	[256]	$Y_{2,-4}Y_{2,-2}$
$(0,0,0,0,0,1)$	[346]	$Y_{1,-5}$
$(0,0,1,0,0,0)$	[356]	$Y_{2,-4}$
$(0,0,1,1,0,0)$	[[124],[356]]	$Y_{2,-4}Y_{1,-1}$
$(1,1,0,0,1,1)$	[[135],[246]]	$Y_{1,-5}Y_{2,-2}Y_{1,-3}Y_{2,0}$
$(0,0,0,1,1,1)$	[126]	$Y_{1,-5}Y_{1,-3}Y_{1,-1}$
$(1,1,1,0,0,0)$	[156]	$Y_{2,-4}Y_{2,-2}Y_{2,0}$

Quantum affine algebras

- The results in Grassmannian case correspond to representations of $U_q(\widehat{\mathfrak{sl}_k})$.
- The results in Grassmannian case can be generalized to general quantum affine algebras.

Quasi-homomorphisms of cluster algebras

- Chris Fraser 2017 introduced the concept of quasi-homomorphisms of cluster algebras (they have appeared in Melissa's talk).
- Let $\mathcal{A}$, $\mathcal{A}'$ be two cluster algebras defined over $\mathbb{Z}\mathbb{P}$, and $\mathbb{Z}\mathbb{P}'$ respectively, where $\mathbb{P}, \mathbb{P}'$ are semifields. An algebra homomorphism $f : \mathcal{A} \to \mathcal{A}'$ is called a quasi-homomorphism if $f(\mathbb{P}) \subset \mathbb{P}'$, and there is a seed Σ for $\mathcal{A}$ and a seed Σ' for $\mathcal{A}'$ such that f sends a cluster variable Σ to a cluster variable in Σ' (after removing frozen factors), f sends a cluster X-variable in Σ to a cluster X-variable in Σ', and the mutable part of the exchange matrix of $\mathcal{A}$ and mutable part of the exchange matrix of $\mathcal{A}'$ are the same.
- By definition, a quasi-homomorphism sends cluster variables to cluster variables (after removing frozen factors), and sends a cluster to a cluster.

Tropicalization of quasi-automorphisms of cluster algebras

- With James Drummond and Ömer Gürdoğan, we study tropicalization of quasi-automorphisms of cluster algebras.
- The tropicalization of a quasi-automorphism sends a g-vector to a g-vector.
- ([Drummond-Gürdoğan-L.]) The map $g \mapsto g'$ sends the g-vector $g(b)$ of a cluster variable b to the g-vector $g(f^{-1}(b))$ of a cluster variable $f^{-1}(b)$.
- I will explain the definition of tropicalization of a quasi-homomorphism using an example later.

Braid group actions

- Tropicalization of quasi-automorphisms gives a convenient way to compute quasi-automorphisms.
- Fraser 2020 defined a braid group action on $\mathbb{C}[\mathrm{Gr}(k,n)]$. Each generator σ_i of the braid group Br_d, $d = \gcd(k,n)$, is a quasi-automorphism on $\mathbb{C}[\mathrm{Gr}(k,n)]$.
- For each i, tropicalization of σ_i gives a map sending a g-vector to a g-vector, and sending a semistandard Young tableau to a semistandard Young tableau.

Tropicalization of quasi-automorphisms of cluster algebras

- Consider the case of $\mathrm{Gr}(3,6)$. The web matrix W (we use another version of web matrix) is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & \frac{x_{11}x_{21}+x_{21}+1}{x_{11}x_{21}} & \frac{x_{11}x_{12}x_{21}x_{22}+x_{12}x_{21}x_{22}+x_{12}x_{22}+x_{21}x_{22}+x_{22}+1}{x_{11}x_{12}x_{21}x_{22}} \\ 0 & 1 & 0 & -1 & -\frac{x_{11}+1}{x_{11}} & -\frac{x_{11}x_{12}+x_{12}+1}{x_{11}x_{12}} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- Evaluating cluster $\mathcal{X}$-coordinates on W, we obtain

$$(x_{11}, x_{12}, x_{21}, x_{22}).$$

Tropicalization of quasi-automorphisms of cluster algebras

- $\sigma_1(W)$ is equal to

$$\begin{pmatrix} 0 & -1 & 0 & \frac{x_{11}x_{21}+x_{21}+1}{x_{11}x_{21}} & \frac{x_{11}x_{12}x_{21}+x_{12}x_{21}+x_{12}+x_{21}+1}{x_{11}^2x_{12}x_{21}} & \frac{x_{11}x_{12}x_{21}x_{22}+x_{12}x_{21}x_{22}+x_{12}x_{22}+x_{21}x_{22}+x_{22}+1}{x_{11}x_{12}x_{21}x_{22}} \\ 1 & 1 & 0 & \frac{-x_{11}-1}{x_{11}} & \frac{-x_{11}x_{12}-x_{12}-1}{x_{11}^2x_{12}} & \frac{-x_{11}x_{12}-x_{12}-1}{x_{11}x_{12}} \\ 0 & 0 & 1 & 1 & \frac{1}{x_{11}} & 1 \end{pmatrix}.$$

- Evaluating cluster $\mathcal{X}$-coordinates on $\sigma_1(W)$, we obtain

$$\left(\frac{(x_{11}x_{21}+x_{21}+1)x_{12}}{x_{21}+1}, (x_{21}+1)x_{22}, \frac{x_{21}+1}{x_{11}x_{21}}, \frac{x_{11}}{x_{11}x_{21}+x_{21}+1}\right).$$

Tropicalization of quasi-automorphisms of cluster algebras

- Tropicalising the above vector, we obtain

$$(\widetilde{x}_{12} + \min(\widetilde{x}_{11} + \widetilde{x}_{21}, \widetilde{x}_{21}, 0) - \min(\widetilde{x}_{21}, 0), \min(\widetilde{x}_{21}, 0) + \widetilde{x}_{22}, \\ \min(\widetilde{x}_{21}, 0) - \widetilde{x}_{11} - \widetilde{x}_{21}, \widetilde{x}_{11} - \min(\widetilde{x}_{11} + \widetilde{x}_{21}, \widetilde{x}_{21}, 0)).$$

- The tropicalization of σ_1 sends a g-vector $(\widetilde{x}_{11}, \widetilde{x}_{12}, \widetilde{x}_{21}, \widetilde{x}_{22})$ to the above vector.
- There is a one to one correspondence between tableaux and g-vectors. Therefore tropicalization of σ_1 sends a tableau to a tableau.

Fixed points of quasi-automorphisms

- Let $\mathcal{A}$ be any cluster algebra of rank n and f is a quasi-automorphism on $\mathcal{A}$.
- We say that a g-vector $g \in \mathbb{Z}^n$ is a fixed point of f if $f(g) = g$.
- For a g-vector g which is fixed by a quasi-automorphism f on $\mathcal{A}$, we say that g is a stable fixed point if for every generic vector g' in $\mathbb{Z}^n$, the sequence $\sigma^j(g')$, $j = 1, 2, \ldots$, has a limit and the limit is $\frac{1}{r}g$ for some $r \in \mathbb{Z}_{\geq 1}$. Otherwise, we say that g is an unstable fixed point.

Fixed points of the maps given by braid group generators

- Denote by ρ the cyclic shift map. It is a cluster automorphism on $\mathbb{C}[\mathrm{Gr}(k,n)]$.
- Denote $\sigma_3 = \rho \circ \sigma_2 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3 in $\mathbb{C}[\mathrm{Gr}(3,9)]$ are:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}.$$

- Denote $\sigma_4 = \rho \circ \sigma_3 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3, σ_4 in $\mathbb{C}[\mathrm{Gr}(4,8)]$ are:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array},$$

  where the first and the third are the same, and the second and the fourth are the same.

Fixed points of the maps given by braid group generators

- Denote $\sigma_4 = \rho \circ \sigma_3 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3, σ_4 in $\mathbb{C}[\mathrm{Gr}(4,12)]$ are:

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&7&9\\\hline 4&8&11\\\hline 6&10&12\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&4&8\\\hline 5&9&10\\\hline 7&11&12\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&3&7\\\hline 2&5&9\\\hline 4&6&11\\\hline 8&10&12\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6&8\\\hline 5&7&10\\\hline 9&11&12\\\hline\end{array}.$$

- Denote $\sigma_5 = \rho \circ \sigma_4 \circ \rho^{-1}$. The stable fixed points of $\sigma_1, \ldots, \sigma_5$ in $\mathbb{C}[\mathrm{Gr}(5,10)]$ are:

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2&6\\\hline 4&8\\\hline 5&9\\\hline 7&10\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&7\\\hline 6&9\\\hline 8&10\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline 7&8\\\hline 9&10\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&4\\\hline 2&6\\\hline 3&7\\\hline 5&9\\\hline 8&10\\\hline\end{array}, \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&7\\\hline 6&8\\\hline 9&10\\\hline\end{array}.$$

Orbits of braid group actions

- We proved that the number of rank r prime non-real elements in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(3,9)]$ which can be obtained by applying the braid group action to the following three tableaux: $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$ is $3\phi(\frac{r}{3})$ if $r \pmod 3 = 0$, and is 0 if $r \pmod 3 \neq 0$, where $\phi(x)$ is the Euler totient function.
- We proved that the number of rank r prime non-real elements in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(4,8)]$ which can be obtained by applying the braid group action to the following two tableaux: $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$, $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$ is $2\phi(\frac{r}{2})$ if $r \pmod 2 = 0$, and is 0 if $r \pmod 2 \neq 0$.

Grassmannian string integrals

Arkani-Hamed, He, and Lam 2019 introduced Grassmannian string integrals:

$$I = (\alpha')^a \int_{\mathbb{R}^a_{>0}} \prod_{i,j} \frac{dx_{ij}}{x_{ij}} \prod_J p_J^{-\alpha' c_J},$$

where the second product runs over all Plücker coordinates p_J, α', c_J are some parameters, $a = (k-1)(n-k-1)$, x_{ij}'s are variables used in the web matrix (Speyer and Williams 2005).

Grassmannian string integrals

In [Early-L. 2023], we generalize the above integral: for every $d \geq 1$, we define

$$\mathbf{I}_{k,n}^{(d)} = (\alpha')^a \int_{\mathbb{R}^a_{>0}} \left(\prod_{(i,j)} \frac{dx_{i,j}}{x_{i,j}} \right) \left(\prod_T \mathrm{ch}_T^{-\alpha' c_T}(x_{i,j}) \right).$$

where the second product is over all tableaux T such that the face $\mathbf{F}_T$ corresponding to T is a facet of the Newton polytope $\mathbf{N}_{k,n}^{(d-1)}$, ch_T is given in [CDFL2019].
We expect that these integrals have applications in physics.

u-variables and u-equations

- Another application to physics is about u-variables and u-equations. u-variables and u-equations have appeared in Hugh's talk on Monday.
- u-variables are certain rational fractions in Plücker coordinates originally defined by physicists Koba-Nielsen in 1969 in the case of $\mathrm{Gr}(2, n)$.
- Arkani-Hamed, Frost, Plamondon, Salvatori, and Thomas have obtained general formulas for u-variables for cluster algebras from surfaces.
- In [Early-L.], we give a general formula for u-variables in the case of $\mathrm{Gr}(k, n)$.

Grassmannian cluster categories

- Jensen, King, and Su 2016 gave an additive categorification of $\mathbb{C}[\mathrm{Gr}(k,n)]$ using Cohen-Macaulay modules.
- Denote by $\mathrm{CM}(B_{k,n})$ the category of Cohen-Macaulay $B_{k,n}$-modules. The category $\mathrm{CM}(B_{k,n})$ has an Auslander-Reiten quiver.

Cluster variables, rigid indecomposable modules, real prime modules, tableaux

- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable rigid indecomposable modules in $\mathrm{CM}(B_{k,n})$ [Jensen, King, Su 2016].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real modules in $\mathcal{C}_\ell^{\mathfrak{sl}_k}$ [Hernandez-Leclerc 2010, Qin 2017, Kang-Kashiwara-Kim-Oh 2018, Kashiwara-Kim-Oh-Park 2019].
- Cluster variables in $\mathbb{C}[\mathrm{Gr}(k,n)]$ are in bijection with reachable prime real tableaux in $\mathrm{SSYT}(k,[n])$ [Chang-Duan-Fraser-L. 2020].
- We replace the modules at the vertices of the Auslander-Reiten quiver by the corresponding tableaux.

Auslander-Reiten quiver in the case of $\mathrm{Gr}(3,6)$

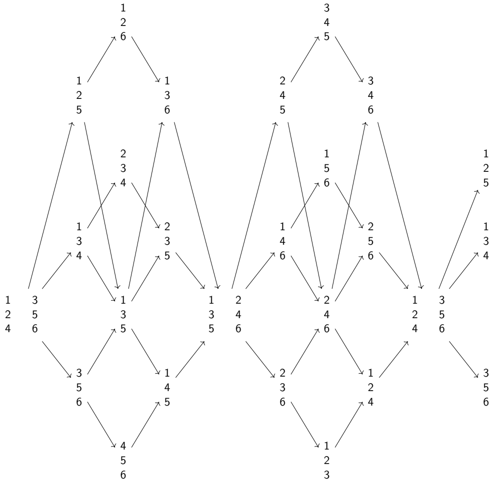

Figure: The Auslander-Reiten quiver for $\mathrm{CM}(B_{3,6})$ with vertices labelled by tableaux.

u-variables in the case of Gr$(3,6)$

The u-variables for Gr$(3,6)$ are

$$u_{126}=\frac{p_{136}}{p_{126}},\ u_{345}=\frac{p_{346}}{p_{345}},\ u_{125}=\frac{p_{126}p_{135}}{p_{125}p_{136}},\ u_{136}=\frac{\mathrm{ch}_{135,246}}{p_{136}p_{245}},\ u_{245}=\frac{p_{345}p_{246}}{p_{245}p_{346}},$$
$$u_{346}=\frac{\mathrm{ch}_{124,356}}{p_{346}p_{125}},\ u_{124,356}=\frac{p_{125}p_{134}p_{356}}{\mathrm{ch}_{124,356}p_{135}},\ u_{134}=\frac{p_{135}p_{234}}{p_{134}p_{235}},\ u_{135}=\frac{p_{136}p_{145}p_{235}}{p_{135}\mathrm{ch}_{135,246}},$$
$$u_{235}=\frac{\mathrm{ch}_{135,246}}{p_{235}p_{146}},\ u_{135,246}=\frac{p_{146}p_{245}p_{236}}{\mathrm{ch}_{135,246}p_{246}},\ u_{146}=\frac{p_{246}p_{156}}{p_{146}p_{256}},\ u_{246}=\frac{p_{346}p_{256}p_{124}}{p_{246}\mathrm{ch}_{124,356}},$$
$$u_{256}=\frac{\mathrm{ch}_{124,356}}{p_{256}p_{134}},\ u_{234}=\frac{p_{235}}{p_{234}},\ u_{156}=\frac{p_{256}}{p_{156}},\ u_{356}=\frac{p_{135}p_{456}}{p_{356}p_{145}},\ u_{145}=\frac{\mathrm{ch}_{135,246}}{p_{145}p_{236}},$$
$$u_{236}=\frac{p_{246}p_{123}}{p_{236}p_{124}},\ u_{124}=\frac{\mathrm{ch}_{124,356}}{p_{124}p_{356}},\ u_{456}=\frac{p_{145}}{p_{456}},\ u_{123}=\frac{p_{124}}{p_{123}},$$

where we use $\mathrm{ch}_{T_1,\ldots,T_r}$ to denote ch_T, and T_i's are columns of T. Here $\mathrm{ch}_{124,356}=p_{124}p_{356}-p_{123}p_{456}$, and $\mathrm{ch}_{135,246}=p_{145}p_{236}-p_{123}p_{456}$.

A general formula for u-variables

For every mesh

$$
\begin{tikzcd}
 & T_1 \arrow[dr] & \\
S \arrow[ur] \arrow[r] \arrow[dr] & \vdots \arrow[r] & S' \\
 & T_r \arrow[ur] &
\end{tikzcd}
$$

in the Auslander-Reiten quiver of $\mathrm{CM}(B_{k,n})$, we define the corresponding u-variable as

$$u_S = \frac{\prod_{i=1}^{r} \mathrm{ch}_{T_i}}{\mathrm{ch}_S \mathrm{ch}_{S'}}.$$

u-equations

- We conjecture that there exist unique integers $a_{T,T'}$ such that

$$u_T + \prod_{T' \in \mathrm{PSSYT}_{k,n}} u_{T'}^{a_{T,T'}} = 1,$$

  for all $T \in \mathrm{PSSYT}_{k,n}$, $\mathrm{PSSYT}_{k,n}$ is the set of all (non-frozen) prime tableaux in $\mathrm{SSYT}(k,[n])$.
- These equations are called u-equations.
- The following is an example of u-equation in the case of $\mathrm{Gr}(3,6)$:

$$u_{124,356} + u_{135}u_{136}u_{145}u_{146}u_{235}u_{236}u_{245}u_{246}u_{135,246}^2 = 1.$$

Thank you!