Transitive and Gallai colorings

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Gallai colorings of graphs

- A Gallai coloring of the complete graph K_n on n vertices is an edge-coloring which has no rainbow triangle, namely a triangle with edges of three different colors.
- The number of Gallai colorings of K_3 using 2-colors is 8. The number of Gallai colorings of K_3 using 3-colors is 21.
- The concept of Gallai colorings has been generalized to general graphs.

Gallai colorings of matroids

- A matroid is a mathematical structure that generalizes concepts from graph theory, linear algebra, etc.
- A matroid $M = (E, \mathcal{I})$ is a finite ground set E together with a collection of subsets of E, known as the independent sets, such that: (1) if $I \in \mathcal{I}$, $J \subset I$, then $J \in \mathcal{I}$, (2) if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$.
- $I \subset E$ is called dependent if $I \notin \mathcal{I}$. A circuit is an inclusionwise minimally dependent set of M. A basis is any maximal independent set.
- Let $k \in \mathbb{Z}_{\geq 0}$ and let M be a matroid on a finite set E. A Gallai k-coloring of M is a function $\epsilon : E \to [k]$ such that for any circuit X in M, $|\{\epsilon(e) : e \in X\}| < |X|$, [ABGMR23].

Transitive colorings of oriented matroids

- An oriented matroid is a special type of matroid where the circuits are signed sets.
- Oriented matroids can be viewed as a combinatorial abstraction of real hyperplanes arrangements, of point configurations over the field of real numbers, of convex polytopes, or of directed graphs.
- Let k be a positive integer, and let E be a finite set (or multiset) of vectors in a vector space over an ordered field (say, the field of real numbers). A transitive k-coloring of E is a function $\epsilon : E \to [k]$ such that, for any two disjoint subsets $S, T \subset E$,

$$\mathrm{span}_{>0}(S) \cap \mathrm{span}_{>0}(T) \neq \emptyset \quad \Rightarrow \quad \epsilon(S) \cap \epsilon(T) \neq \emptyset.$$

Examples of transitive and non-transitive colorings of directed graphs

$$\begin{tikzcd} \bullet \arrow[r, green] & \bullet \arrow[d, red] \\ \bullet \arrow[u] \arrow[r, red] & \bullet \end{tikzcd} \qquad \qquad \begin{tikzcd} \bullet \arrow[r, green] & \bullet \\ \bullet \arrow[u] \arrow[r, red] & \bullet \arrow[u, red] \end{tikzcd}$$

The coloring on the left hand side is transitive. The coloring on the right hand side is non-transitive.

Transitive colorings of the transitive tournament

Let $\overrightarrow{K_n}$ be the transitive tournament with vertex set $\{1, \ldots, n\}$ and edge set $\{(i,j) : i < j\}$. This is an acyclic orientation of the complete graph. An edge-coloring ϵ of $\overrightarrow{K_n}$ is transitive if and only if $\epsilon(i,k) \in \{\epsilon(i,j), \epsilon(j,k)\}$ for all $i < j < k$.

A generalization of Erdős, Simonovits and Sós's result

- The anti-Ramsey problem posed by Erdős, Simonovits and Sós (1973) asks for the maximal number k of colors such that there exists an edge-coloring of the complete graph K_n, with exactly k colors and without a rainbow complete subgraph K_s.
- Erdős, Simonovits and Sós proved, in particular, that the maximal number of edge colors of K_n without a rainbow triangle is $n-1$.
- A matroid has a Gallai coloring if and only if it is loopless (has no circuit of size 1).
- The rank $r(M)$ of a matroid is the size of any basis of the matroid M.
- We generalize Erdős, Simonovits and Sós's result and obtain: for any loopless matroid M, the maximal k such that there exists a Gallai coloring of M using exactly k colors is equal to $r(M)$, [ABGMR23].

Transitive colorings of oriented matroids

- An oriented matroid has a transitive coloring if and only if it is acyclic (i.e., has no positive circuit).
- For an acyclic oriented matroid M, the maximal k such that there exists a transitive coloring of M using exactly k colors is equal to $r(M)$.

Gallai partitions and transitive partitions

- A Gallai-partition of a complete graph K_n is a partition of the edges of K_n in which no triangle has edges in three distinct partition-classes. This concept was introduced by Körner, Simonyi and Tuza in 1992.
- Let M be a loopless matroid on a nonempty set E. A Gallai k-partition of M is a partition of E into k disjoint non-empty subsets, also called blocks, $B_1, \ldots, B_k$ such that for any circuit X in M, $|X \cap B_i| \geq 2$ for at least one value of i, [ABGMR23].
- Let M be an acyclic oriented matroid on a nonempty set E. A transitive k-partition of M is a partition of E into k disjoint non-empty subsets, also called blocks, $B_1, \ldots, B_k$ such that for any signed circuit $X = (X^+, X^-)$ in M, both $X^+ \cap B_i \neq \emptyset$ and $X^- \cap B_i \neq \emptyset$ for at least one value of i.

Polynomiality of the numbers of Gallai partitions and transitive partitions

For any loopless (respectively, acyclic oriented) matroid M on a nonempty set E there exists a polynomial $p_M(x) \in x\mathbb{Z}[x]$ such that, for any positive integer k, the number of Gallai (respectively, transitive) colorings of M using k colors is equal to $p_M(k)$. Specifically,

$$p_M(x) = \sum_{j \geq 1} a_j \cdot (x)_j$$

where $(x)_j = x(x-1)\cdots(x-j+1)$ and a_j is the number of Gallai (respectively, transitive) j-partitions of M.

Maximal Gallai and transitive partitions

- A Gallai (resp. transitive) partition is called maximal if the number of blocks is maximal, namely equal to the rank of the matroid.
- For every $n > 1$, the number of maximal Gallai partitions of the set of edges of the complete graph K_n is equal to the double factorial $(2n-3)!!$.
- For every $n > 1$, the number of maximal transitive partitions of the set of edges of the transitive tournament $\overrightarrow{K_n}$ is equal to the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

Quasisymmetric functions and symmetric functions

- A symmetric polynomial on n variables $x_1, \ldots, x_n$ is a function that is unchanged by any permutation of its variables.
- Quasi-symmetric functions are a generalization of symmetric functions introduced by I.M. Gessel in 1984.
- Take a totally ordered commutative alphabet $X = \{x_1, x_2, \ldots\}$. Monomials $X^v = x_1^{v_1} \cdots$ correspond to vectors $v = (v_1, \ldots)$ with finitely many non-zero entries. For such a vector v, we denote by $v_{\leftarrow}$ the vector obtained by removing all zero entries. A polynomial $p \in \mathbb{C}[X]$ is said to be quasi-symmetric if and only if for any v and w such that $v_{\leftarrow} = w_{\leftarrow}$, the coefficients of X^v and X^w in p are equal.
- $x_1x_2^2 + x_2x_3^2 + x_1x_3^2$ is a quasi-symmetric polynomial in three variables that is not symmetric.

Schur-positivity

- A symmetric function is called Schur-positive if all the coefficients in its expansion in the Schur basis are nonnegative (or polynomials with nonnegative coefficients).
- A fundamental quasi-symmetric function indexed by a subset $J \subset [n-1]$ is

$$\mathcal{F}_J(x) = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in J}} x_{i_1} \cdots x_{i_n}.$$

- For a set of combinatorial object, equipped with a map $\mathrm{Des} : A \to 2^{[n-1]}$, let $\mathcal{Q}(A) = \sum_{a \in A} \mathcal{F}_{\mathrm{Des}(a)}$.
- Gessel and Reutenauer in 1993 posed the problem: for which pairs (A, Des), $\mathcal{Q}(A)$ is symmetric and Schur-positive?

Schur-positivity in the case of complete graphs

- The descent set of a Gallai (resp. transitive) k-partition p of the complete graph K_n (resp. the transitive tournament $\overrightarrow{K_n}$) on the set of vertices $[n]$ is:

  $$\mathrm{Des}(p) = \{i : \text{the edge } (i, i+1) \text{ forms a singleton block in } p\}.$$

- The descent set of the Gallai 3-partition on the left below is $\{1, 3\}$ and the descent set of the Gallai 3-partition on the right below is empty.

Schur-positivity in the case of complete graphs

- Denote by $G_{n,k}$ the set of all Gallai k-partitions of K_n. Denote by $T_{n,k}$ the set of all transitive k-partitions of $\overrightarrow{K_n}$.
- For $1 \leq k < n$, the quasi-symmetric functions

$$\mathcal{Q}(G_{n,k}) := \sum_{p \in G_{n,k}} \mathcal{F}_{\mathrm{Des}(p)}, \quad \mathcal{Q}(T_{n,k}) := \sum_{p \in T_{n,k}} \mathcal{F}_{\mathrm{Des}(p)}$$

are symmetric and Schur-positive.

Representations of symmetric groups

- A partition λ of n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ such that $\sum_{i=1}^{\ell} \lambda_i = n$.
- There is a one-to-one correspondence between the set of irreducible representations of S_n and the set of partitions of n.
- Let (V, ρ) be a representation of G. The character of V is the function $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = \mathrm{tr}\rho_g$.
- Denote by ch the Frobenius characteristic map from class functions on S_n to symmetric functions, which is defined by $\mathrm{ch}(\chi^\lambda) = s_\lambda$ and extended by linearity, where s_λ is the Schur function.

Schur-positivity in the case of complete graphs

- For every $n > 1$,

$$\mathcal{Q}(T_{n,n-1}) = \mathrm{ch}(\chi^{(n-1,n-1)} \downarrow_{S_n}^{S_{2n-2}}),$$

  where $\chi^{(n-1,n-1)}$ is the irreducible S_{2n-2}-character indexed by the partition $(n-1, n-1)$ of $2n-2$.
- Let M be a perfect matching of n points $\{1, \dots, n\}$. If $(i, i+1) \in m$, then $(i, i+1)$ is called a short chord.
- For every $n > 1$,

$$\mathcal{Q}(G_{n,n-1}) = \mathrm{ch}((\sum_{r=0}^{n-1} a_r \chi^{(n-1+r,n-1-r)}) \downarrow_{S_n}^{S_{2n-2}}),$$

  where a_r is the number of perfect matchings of $2r$ points with no short chords.

Transitive algebras and Gallai algebras

- Let k be a positive integer, and let M be an oriented matroid on a finite set E, with set of signed circuits $\Gamma(M)$. The transitive k-algebra of M, denoted $\mathcal{T}_{M,k}$, is the commutative algebra over $\mathbb{C}$ generated by $\{x_e : e \in E\}$ subject to

$$x_e^k = 1, \quad \forall e \in E,$$
$$\prod_{e_1 \in X^+, e_2 \in X^-} (x_{e_1} - x_{e_2}) = 0, \quad \forall (X^+, X^-) \in \Gamma(M).$$

- Let k be a positive integer, and let M be a matroid on a finite set E, with set of circuits $\Gamma(M)$. Let $<$ be a linear order on E. The Gallai k-algebra of M, denoted $\mathcal{G}_{M,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_e : e \in E\}$ subject to

$$x_e^k = 1, \quad \forall e \in E,$$
$$\prod_{e_1, e_2 \in X, e_1 < e_2} (x_{e_1} - x_{e_2}) = 0, \quad \forall X \in \Gamma(M).$$

Enumeration via commutative algebras

Theorem (ABGMR23)

Let k be a positive integer. Then, for any finite oriented matroid M,

$$\dim \mathcal{T}_{M,k} = \#\{\textit{transitive } k\textit{-colorings of } M\},$$

and for any finite monoid M,

$$\dim \mathcal{G}_{M,k} = \#\{\textit{Gallai } k\textit{-colorings of } M\},$$

Transitive algebras for complete tournaments and Gallai algebras for complete graphs

- The transitive algebra $\mathcal{T}_{n,k} := \mathcal{T}_{\vec{K}_n,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_{ij} : 1 \leq i < j \leq n\}$ subject to

$$x_{ij}^k = 1,$$
$$(x_{im} - x_{ij})(x_{im} - x_{jm}) = 0, \quad \forall i < j < m.$$

- The Gallai algebra $\mathcal{G}_{n,k} := \mathcal{G}_{K_n,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_{ij} : 1 \leq i < j \leq n\}$ subject to

$$x_{ij}^k = 1,$$
$$(x_{ij} - x_{im})(x_{ij} - x_{jm})(x_{im} - x_{jm}) = 0, \quad \forall i < j < m.$$

Hilbert series and enumeration

The Hilbert series of a finitely generated algebra B is

$$\mathrm{Hilb}(B,q) := \sum_{j\geq 0}(\dim(B_{\leq j}) - \dim(B_{\leq j-1}))q^j,$$

where $B_{\leq j}$ is the degree j filtered component of B, where the filtered degree of each generator is 1.

Hilbert series and enumeration

- Denote $[k]_j = \prod_{i=0}^{j-1} \frac{q^{k-i}-1}{q-1}$, $k, j \geq 1$.
- The following are a few values of $\mathrm{Hilb}(\mathcal{T}_{n,k}, q)$:

$$\begin{aligned}
\mathrm{Hilb}(\mathcal{T}_{2,k}, q) &= [k]_1, \\
\mathrm{Hilb}(\mathcal{T}_{3,k}, q) &= 2q[k]_2 + [k]_1, \\
\mathrm{Hilb}(\mathcal{T}_{4,k}, q) &= 5q^3[k]_3 + q(5q+6)[k]_2 + [k]_1.
\end{aligned}$$

- The following are a few values of $\mathrm{Hilb}(\mathcal{G}_{n,k}, q)$:

$$\begin{aligned}
\mathrm{Hilb}(\mathcal{G}_{2,k}, q) &= [k]_1, \\
\mathrm{Hilb}(\mathcal{G}_{3,k}, q) &= q(q+2)[k]_2 + [k]_1, \\
\mathrm{Hilb}(\mathcal{G}_{4,k}, q) &= q^3(q^2+8q+6)[k]_3 + \\
&\quad + q(q^4+5q^3+10q^2+10q+5)[k]_2 + [k]_1.
\end{aligned}$$

In particular, $\mathrm{Hilb}(\mathcal{G}_{3,2}, 1) = 8$ and we recover the fact that there are 8 Gallai colorings on K_3 using 2 colors we mentioned before.

Conjectures about Hilbert series

For all $n > 1$, and $k \geq 1$,

$$\mathrm{Hilb}(\mathcal{T}_{n,k}, q) = \sum_{j=1}^{n-1} P_{n,j}(q) \cdot [k]_j,$$

where $P_{n,j}(q) \in \mathbb{Z}_{\geq 0}[q]$, $P_{n,n-1} = q^{\binom{n-1}{2}} C_{n-1}$, C_{n-1} is the Catalan number.

Conjectures about Hilbert series

- A Stirling permutation of order n is a permutation of the multi-set $\{1, 1, 2, 2, \ldots, n, n\}$ such that for all m, all entries between two copies of m are larger than m.
- Denote by $E(n, j)$ the second order Eulerian number which is the number of Stirling permutation of order n with j descents.
- For all $n > 1$, and $k \geq 1$,

$$\mathrm{Hilb}(\mathcal{G}_{n,k}, q) = \sum_{j=1}^{n-1} Q_{n,j}(q) \cdot [k]_j,$$

where $Q_{n,j}(q) \in \mathbb{Z}_{\geq 0}[q]$, and
$Q_{n,n-1} = q^{\binom{n}{2}-1} \sum_{j=0}^{n-1} E(n-1, j) q^{-j}$.

Thank you!