PROJECT 1: LINEAR PROGRAMMING

In this problem, we provide a short preview of linear programming, which allows us to study linear *inequalities* in the same manner that the methods of Chapter 1 study linear *equalities*. We will begin by exploring the solutions to

$$\begin{aligned} y &\geq 0 \\ z &\geq 0 \\ y+3z &\leq 21 \\ y+z &\leq 9 \\ 2y+z &\leq 14 \end{aligned}$$

Let R be the region in the (y,z) plane where these relations hold.

Question 1. Draw R.

It will be convenient to rewrite our equations using so-called *slack variables* to be a combination of linear *equalities* and conditions of the form variable ≥ 0. In this case, we introduce variables u, v and w defined by

$$\begin{aligned} u &= 21-y-3z \\ v &= 9-y-z \\ w &= 14-2y-z \end{aligned} \tag{1}$$

So R is the region where u, v, w, y and z are positive.

Question 2. Label the vertices of your drawing of R with their u, v, w, y and z values.

Question 3. Rewrite the linear equations (1) in an extended matrix, ordering the variables as (u,v,w,y,z). You should find that your equations are in reduced row echelon form.

Question 4. Reorder your variables as (y,v,w,u,z), (u,y,w,v,z) and (u,v,y,w,z). Put each of the resulting systems of equations into reduced row echelon form.

Question 5. Using the solutions to your previous computation, find the three solutions to Equations (1) which have, respectively, $u=z=0$, $v=z=0$ and $w=z=0$. Plot these solutions on your diagram of R. Which of these solutions correspond to solutions to the original inequalities?

The computations you did in the first parts of this question show how to go from the vertex $(y,z)=(0,0)$ of R to find a neighboring vertex where z remains 0, y stops being 0 and one of the other variables, u, v or w becomes 0.

Here is a larger set of equations and inequalities:

$$\begin{array}{rrrrrrrrrrrrr} x_1 & & & & + & x_5 & - & 3x_6 & - & 4x_7 & + & 2x_8 & = 5 \\ & x_2 & & & + & 3x_5 & + & 7x_6 & + & 2x_7 & - & 2x_8 & = 6 \\ & & x_3 & & - & 4x_5 & - & 2x_6 & - & 7x_7 & - & 2x_8 & = 2 \\ & & & x_4 & - & 3x_5 & + & 4x_6 & + & 3x_7 & + & 3x_8 & = 1 \\ \multicolumn{13}{r}{x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ \geq 0} \end{array} \tag{2}$$

Suppose we are at the point P where $x_5=x_6=x_7=x_8=0$.

Question 6. What are the values of x_1, x_2, x_3, x_4 at P?

Question 7. We want to move from P to a new point Q where the x_6, x_7 and x_8 coordinates stay 0, the x_5 coordinate becomes positive and one of x_1,x_2, x_3 and x_4 becomes zero, while the rest of x_1, x_2, x_3 and x_4 stay positive. What is this point Q? Explain how you found it.

Question 8. Reorder the columns of equations (2) by switching x_5 with the variable which became 0 when moving from P to Q. Apply row operations to put the resulting equations back into reduced row echelon form.

Questions 1-8 exhibit the basic steps of the *simplex method.* We have some region we want to explore, given by linear equations $A\vec{x} = \vec{b}$, and by inequalities x_1, x_2, $\dots$, $x_n \geq 0$. We start at some point P where $x_{k+1} = x_{k+2} = \cdots = x_n = 0$; these are known as the *basis variables.* The matrix A is in row reduced form with pivot columns at the non-basis variables and free columns at the basis variables. We chose one variable to remove from the basis and use the method sketched in Question 7 to put another variable into the basis. Then we rearrange our equations to have their pivot columns in the new non-basis variables.

We'll describe the computation in Questions 7-8 as saying that we started with basis variables $\{x_5, x_6, x_7, x_8\}$ and exchanged x_5 for a new basis variable.

What we have omitted so far is the discussion of *why* we would want to do this. Generally, we have an *objective function*: A quantity which we want to make smaller or larger. Let us suppose, for example, that we wanted to make x_1 larger.

Question 9. We continue discussing the equations of (2). We start at the point P where $x_5 = x_6 = x_7 = x_8 = 0$. Before we considered exchanging x_5 for a new basis variables. Did x_1 get larger or smaller as a result? If we exchanged x_6, x_7 or x_8, would x_1 become larger or smaller? How can we determine this without finding the exact coordinates of the new points?

If you've gotten this far, you should know how to:

- Start at a point with a basis of variables that are 0, forming the pivot columns of a row reduced matrix.
- Determine whether exchanging one of those basis variables will increase or decrease the objective function.
- Exchange a basis variable in order to increase the objective function, and put the equations into row reduced echelon form with respect to the new basis.

The *simplex algorithm* is to continue in this manner, always performing basis exchanges which improve the objective function, until no more exchanges are possible. We conclude with an example: Buzz Buzz Buzz Coffee has on hand 1 kg of coffee grounds, 1 gallon of milk and 10 cups of sugar. They can use these to make espressos, containing 8 grams of grounds and no milk or sugar; lattes, containing 10 grams of grounds, 0.03125 gallons of milk and 0.125 cups of sugar; or café cubano, containing 7.5 grams of grounds, no milk and 0.1250 cups of sugar. They will be able to sell all they produce, which they will sell at prices of \$2 for espressos, \$6 for lattes and \$4 for a café cubano.

Question 10. Let e, l and c be the number of espressos, lattes and cafés cubanos manufactured, and let g, m and s be the amounts of grounds, milk and sugar left over when they are done. Let p be the amount of money they take in. Record the linear equations relating e, l, c, g, m, s and p.

Question 11. Start at the point where no drinks are made (so $e = l = c = 0$). Exchange one of these variables, in order to increase p. Repeat the process of exchanging a basis variable to increase p until there are no exchanges which will make p larger. How many of each drink should be made?