EIGENVALUES AND OSCILLATIONS OF MECHANICAL SYSTEMS

Consider three weights, of mass m, sitting on a frictionless linear track.

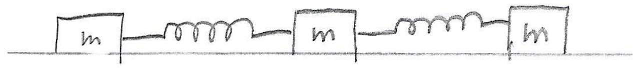

FIGURE 1. Three masses on a frictionless track, joined by springs

Mass 1 is joined to mass 2 by a spring with spring constant k and natural length 1, and likewise for masses 2 and 3. So the masses will stay still if they begin motionless in positions 1, 2 and 3. Let the positions of the masses at time t be $1+x_1(t)$, $2+x_2(t)$ and $3+x_3(t)$, so x_i is the displacement of mass i from its resting position.

So the motion of the springs is governed by the differential equation

$$\begin{aligned} m\ddot{x}_1(t) &= k(x_2(t)-x_1(t)) \\ m\ddot{x}_2(t) &= k(x_1(t)-x_2(t))+k(x_3(t)-x_2(t)) \\ m\ddot{x}_3(t) &= k(x_2(t)-x_3(t)) \end{aligned} \qquad (*)$$

Problem 1 Suppose that we had a solution of the form $(a_1\cos(\omega t), a_2\cos(\omega t), a_3\cos(\omega t))$. Find all possible values of ω and (a_1, a_2, a_3). (Hint: What does this have to do with eigenvalues?)

Problem 2 One possible solution is that the masses are sitting in positions $(1.1, 2.1, 3.1)$, not moving. Is this solution covered by your analysis from Part 1?

Problem 3 Let $(u_1(t), u_2(t), u_3(t))$, $(v_1(t), v_2(t), v_3(t))$ and $(w_1(t), w_2(t), w_3(t))$ be solutions to the equations $(*)$. Explain why, for any constants a, b and c, the linear combination

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = a\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} + b\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + c\begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}$$

is also a solution.

Problem 4 At time 0, we move the masses to positions 1.9, 2.3 and 3.2, holding them still, and then release them from rest. Give a solution of $(*)$ which has this starting behavior. You will need to solve 3 linear equations in 3 variables.

We now describe how one might use this sort of analysis to model the vibrations of a physical object, such as the head of a drum. Imagine a circular drum, as in Figure 2.

We approximate the head of the drum as a grid of nodes joined by springs, as in Figure 3. The black nodes, at the border of the drum, are tacked to the circular rim and cannot move, the white nodes can move up and down. For node i, let $x_i(t)$ be the height of that node above a flat plane, so, for the black nodes, $x_i(t)$ is always 0. If a white node i has neighbors a, b, c and d, then we have the differential equation

$$\frac{d^2}{(dt)^2}x_i(t) = k\Big((x_a(t)-x_i(t)) + (x_b(t)-x_i(t)) + (x_c(t)-x_i(t)) + (x_d(t)-x_i(t))\Big) \qquad (\dagger)$$

for some positive constant k.

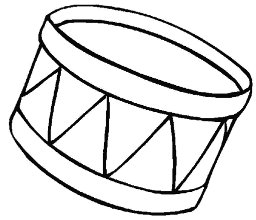

FIGURE 2. A drum

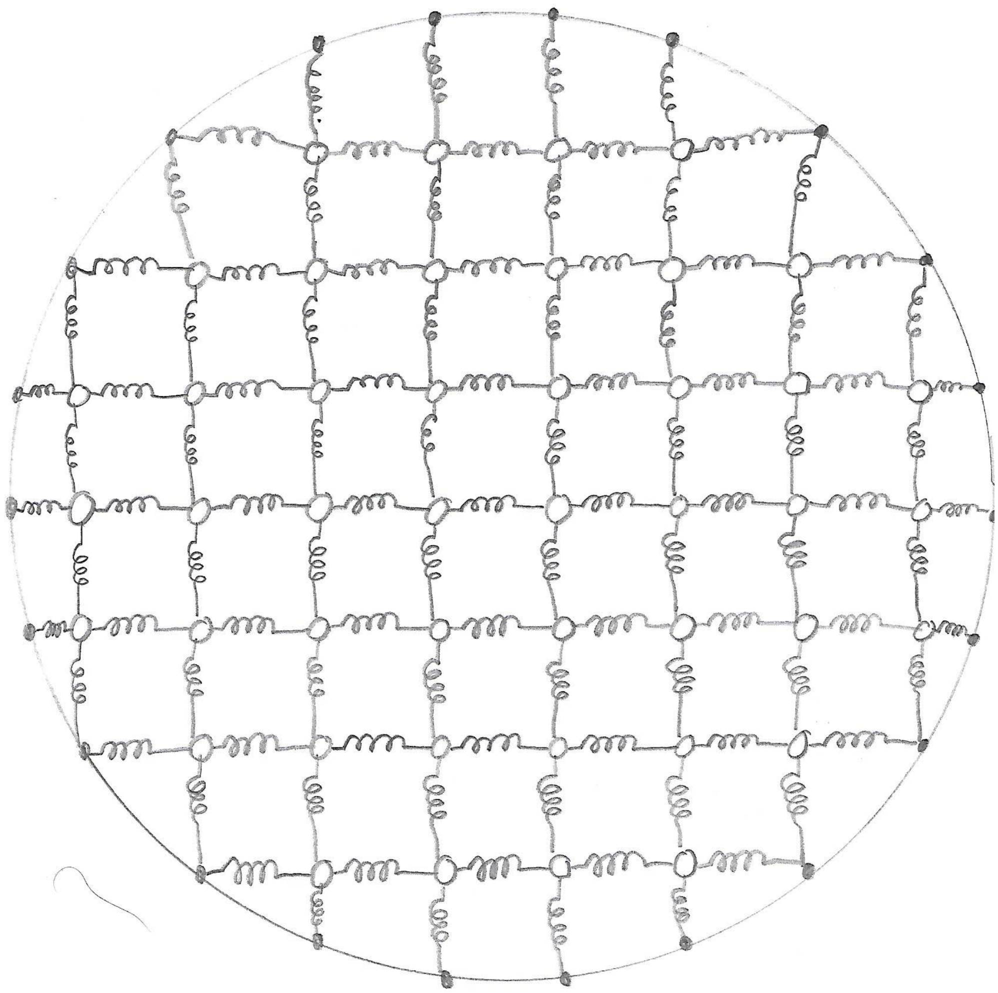

FIGURE 3. Approximating the drumhead by a grid of springs

Problem 5 Find the lowest three values of ω for which (†) has a solution of the form $x_i(t) = a_i(t)\cos(\omega t)$. Note: The lower frequencies are the ones which are most physically significant; the higher frequencies are less noticeable and are less robust to changing how we approximate the drumhead by a discrete spring model. In musical terms, the lowest frequency is the fundamental frequency and the higher frequencies are overtones.